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Prioritization via Stochastic Optimization

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We take a novel approach to decision problems involving binary activity-selection decisions competing for scarce resources. The literature approaches such problems by forming an optimal portfolio of activities. However, often practitioners instead form a rank-ordered list of activities and select those with the highest priority. We account for both viewpoints. We rank activities considering both the uncertainty in the problem parameters and the optimal portfolio that will be obtained once the uncertainty is revealed. We use stochastic integer programming as a modeling framework, and we apply our approach to a facility location problem and a multidimensional knapsack problem. We develop two sets of cutting planes to improve computation.

Data, as supplemental material, are available at http://dx.doi.org/10.1287/mnsc.2013.1865.

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1. Introduction

Resource-constrained activity-selection problems (RCASPs) involve binary activity-selection decisions, which compete for scarce resources. The bulk of the current literature approaches RCASPs by forming an optimal portfolio of activities that meets the resource constraints. However, practitioners in industry and government often avoid the optimal-portfolio approach, instead forming a rank-ordered list of activities and funding those that have the highest priority. We propose a novel prioritization approach that takes both viewpoints into account.

Brown et al. (2006, p. 531) characterize how the military would approach a problem of selecting projects to harden critical infrastructure against a terrorist attack and how the operations research community views that approach:

> The budget for this purpose will always be limited, but often not pre-specified. The military typically draws up a prioritized list of "defended assets" in need of protection, along with a list of potential protective measures, and presents these to policy makers. The latter parties make the final decisions after balancing costs, effectiveness, and intangibles, and after determining the budget.... However, a prioritized list of defended assets has a serious flaw for our applications. Such a list creates a "preferred set" of $n+1$ assets by adding one asset to the preferred set of size n. But, we know that an optimal set of size n and an optimal set of size $n+1$ may have nothing in common.

In discussing typical approaches used in industry to select projects in the context of capital budgeting, Savage et al. (2006) state the following:

> It is common when choosing a portfolio of capital investment projects to rank them from best to worst, then start at the top of the list and go down until the budget has been exhausted. This flies in the face of modern portfolio theory, which is based on the interdependence of investments.

We agree with Brown et al. (2006) and Savage et al. (2006) that simplistic ranking schemes, which individually score each candidate activity, can be inferior. In a capital budgeting problem, ranking candidate projects based on their profit or benefit-investment ratio ignores structural and stochastic dependencies that may exist among the projects. On the other hand, forming an optimal portfolio assuming project costs and resource availabilities are known may yield a portfolio that is fragile with respect to changes in these parameters.

We take both viewpoints into account. Our approach prioritizes the activities recognizing their structural and stochastic dependencies and recognizing that the activities ultimately implemented, after the stochastic parameters are realized, will act as a portfolio. Prioritization is of interest when some problem parameters are random and we must commit to a ranking of the activities before these parameters are realized. Prioritization involves optimally placing activities into a priority list before the uncertainty is revealed and, after realizing the uncertainty, making an activity selection consistent with the priority list. Taken from another perspective, prioritization requires the portfolio of activities to be *nested*.

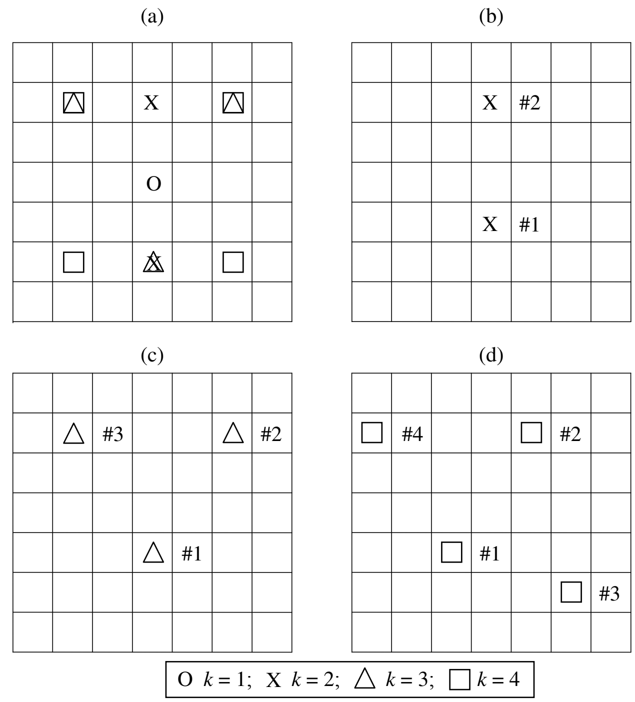
Figure 1 illustrates our approach using the k-median problem (e.g., Mirchandani and Francis 1990). Customers are at the corners of the grids, and facilities may be located at the centers, amounting to 64 customers and 49 potential facility locations. The goal is to choose k locations to minimize the sum of Euclidean distances that the customers must travel to reach their closest facility. Figure 1(a) gives optimal solutions for deterministic values of $k = 1$, 2, 3, and 4.

Figures 1(b)–1(d) concern a version in which k is stochastic, and we incrementally install facilities until k is exhausted. Suppose k takes values 1 or 2 with equal probability. If we first solve this problem for $k = 1$ and greedily locate a facility at the center grid, as indicated by the "O" in Figure 1(a), then under the realization of $k = 2$, we find ourselves in an awkward position because we cannot relocate the initial facility. Figure 1(b) shows an optimal solution to the problem where there is 50% chance we will locate an additional facility, after locating this initial facility. We locate the first facility at the location indicated by the first "X" in Figure 1(b). If $k = 1$ is realized, we are close to the optimal solution of the 1-median problem. If $k = 2$, we install the facility at the second "X," obtaining a solution close to that of the 2-median problem. Figures 1(c) and 1(d) illustrate the same idea when k is equally likely to be 1, 2, 3, and 1, 2, 3, 4, respectively.

In spite of its common use in practice, prioritization has received little attention in the academic literature. Mettu and Plaxton (2003) and Plaxton (2006) develop constant factor approximation algorithms for a variant of the k-median problem, in which they seek to minimize the competitive ratio over all k. See Lin et al. (2006) for further related work. Dean et al. (2008) consider a prioritized knapsack problem in which the items' values and the knapsack size are deterministic but the items' sizes are random. They seek a priority list that maximizes the expected value of items successfully inserted, develop constant-factor approximation algorithms, and further study the benefit of adaptivity, i.e., the benefit from being able to revise the remaining priority list based on the residual knapsack capacity. Dean et al. (2005) analyze the benefit of adaptivity in a more general class of stochastic packing problems.

Witzgall and Saunders (1988) and Hochbaum (2009) consider a knapsack-constrained version of the selection problem of Balinski (1970) and Rhys (1970). Constructing the entire efficient frontier that trades off benefit and cost is NP-hard. However, the concave

Figure 1 **Prioritizing the k-Median Problem**



Notes. Part (a) shows optimal solutions to four k-median problems with $k = 1$–4. In parts (b)–(d), k is uncertain and we prioritize. In part (b), we have a 50% chance of having either one or two facilities. In parts (c) and (d), we are equally likely to have $k = 1$–3 and $k = 1$–4, respectively. Of course, symmetry allows multiple optimal solutions. For example, in part (b) the vertical locations of the $k = 2$ solution could instead be the symmetric horizontal locations.

envelope of the efficient frontier can be formed efficiently, and from the perspective of prioritization, solutions at kink points of that concave envelope are nested as the budget grows. See Nehme and Morton (2010) for work that relates this form of nestedness to submodularity of the benefit and cost functions. Seref et al. (2009) address the problem of finding most improving incremental solutions on a set of well-known problems, such as minimum spanning tree, minimum cost flow, and maximum flow problems. In this approach, incremental modifications are iteratively made to a prespecified feasible solution. See Allen et al. (2003) for related work in a telecommunications network, where the goal is to iteratively modify the initial solution to obtain a near-optimal solution to the original problem. In both incremental optimization and nested solutions on the concave envelope of an efficient frontier, the incremental points are an output of the procedure. Viewed from this perspective, our procedure takes the budget increments, for example, as input in the form of budget scenarios.

Koç et al. (2009) apply the prioritization approach to capital budgeting in the nuclear power industry, and Koç (2010) develops a branch-and-price algorithm for the same class of models. The contributions of this paper are as follows: In §2, we develop a more general mathematical model for prioritization, beginning with the most natural notion of prioritizing activities. In §3, we apply that model to prioritizing the stochastic k-median problem from Figure 1, where we can easily visualize solutions, and we investigate both the value and the cost of prioritizing optimally. The former is the optimality gap between forming an optimal priority list and employing a natural heuristic. The latter is computed via the expected value of perfect information. In §4, we introduce priority lists that satisfy a notion of total order (versus the less stringent orders of activity prioritization), and we compare the two formulations theoretically and computationally on both the facility location model and a multidimensional knapsack model. We then provide a novel alternative formulation in §5 that prioritizes scenarios instead of prioritizing activities. In §5, we further establish an equivalence between prioritizing activities and scenarios and give guidance on their relative merit for computation. In §6, we introduce two sets of valid inequalities, which further improve computational tractability, as we demonstrate again using both multidimensional knapsack and facility location problem instances. In §7, we conclude.

2. Activity Prioritization

We state the RCASP and then proceed with its stochastic version and our prioritization approach. Here we prioritize activities, and in §5 we prioritize scenarios. Consider the following notation and the associated integer programming (IP) formulation:

Indices and sets:

$i \in I$ activities

Data:

$\mathbf{A}$ matrix of nonnegative resource-consumption coefficients
$\mathbf{b}$ vector of nonnegative resource coefficients
$\mathbf{c}_x = (c_{xi})_{i\in I}$ cost coefficients for binary activity-selection variables
$\mathbf{c}_y$ cost coefficients for remaining variables
C constraint set that links the activity-selection decisions with the remaining decisions

Decision variables:

$\mathbf{x} = (x_i)_{i\in I}$ 1 if activity i is selected; 0 otherwise
$\mathbf{y}$ remaining decision variables

Formulation:

$$\min_{\mathbf{x},\mathbf{y}} \quad \mathbf{c}_x\mathbf{x} + \mathbf{c}_y\mathbf{y} \tag{1a}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} \le \mathbf{b}, \tag{1b}$$
$$\mathbf{x} \in \{0,1\}^{|I|}, \tag{1c}$$
$$(\mathbf{x},\mathbf{y}) \in C. \tag{1d}$$

We minimize cost in (1a), and constraint (1b) models resource limits. Constraint (1c) restricts $\mathbf{x}$ to be binary, and set C links decisions $\mathbf{x}$ and $\mathbf{y}$. The k-median problem sketched in the discussion surrounding Figure 1(a) is an example of an RCASP (1):

Indices and sets:

$i \in I$ candidate facility locations
$j \in J$ customers

Data:

d_{ij} Euclidean distance from customer j to facility i
k budget in terms of total number of facilities

Decision variables:

x_i 1 if location i is selected; 0 otherwise
y_{ij} 1 if customer j is assigned to facility at location i; 0 otherwise

Formulation:

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{j\in J}\sum_{i\in I} d_{ij}y_{ij} \tag{2a}$$
$$\text{s.t.} \quad \sum_{i\in I} x_i \le k, \tag{2b}$$
$$x_i \ge y_{ij}, \quad i \in I,\ j \in J, \tag{2c}$$
$$\sum_{i\in I} y_{ij} = 1, \quad j \in J, \tag{2d}$$
$$x_i \in \{0,1\}, \quad i \in I, \tag{2e}$$
$$y_{ij} \in [0,1], \quad i \in I,\ j \in J. \tag{2f}$$

In (2a), we minimize the total Euclidean distance that customers must travel. Constraint (2b) ensures we locate at most k facilities. Constraint (2c) limits its customer assignments to open facilities. Constraint (2d) ensures each customer is assigned to a facility. Because each facility has ample capacity for all customers, the last two sets of constraints yield binary solutions. The location selection decisions, $\mathbf{x}$, and the customer assignment decisions, $\mathbf{y}$, of model (2) correspond to the activity-selection decisions and the remaining decisions of RCASP (1), respectively.

The problem data, as given in the RCASP model (1), are deterministic. Its optimal solution, in terms of the $\mathbf{x}$ variables, is a portfolio of activities. For reasons we outline in §1 and further explain in §3, our prioritization approach is of interest when the parameters $(\mathbf{A}, \mathbf{b}, \mathbf{c}_x, \mathbf{c}_y, C)$ are random, and we must commit to a ranking of the binary activity-selection decisions before their realizations are known.

Let $(\mathbf{A}^\omega, \mathbf{b}^\omega, \mathbf{c}_x^\omega, \mathbf{c}_y^\omega, C^\omega)$, $\omega \in \Omega$, denote the parameter realizations with probability mass q^ω, where $|\Omega|$ is finite. A *priority list* is a many-to-one assignment of activities to priority levels such that each priority level is assigned at least one activity. *Prioritized activity selection* has two requirements: Under any scenario $\omega \in \Omega$, a lower-priority activity cannot be selected unless all higher-priority activities are selected, and either all or none of the activities on the same priority level are selected. Activity prioritization places activities into a priority list before the uncertainty is revealed and, after realizing the uncertainty, makes an activity selection consistent with the priority list. The second requirement regarding "all or none of the activities" is without loss of optimality. An optimal prioritization will place multiple activities on a priority level only if there is no scenario in which we would benefit from selecting a strict subset of those activities. (We return to this issue in §4.)

In the deterministic model (1), we minimize the cost of the portfolio of activities we select. In activity prioritization we form a priority list, which determines the portfolio of activities selected under each scenario $\omega \in \Omega$, to minimize expected cost. To prioritize, we formulate a two-stage stochastic integer program. The first-stage decision forms the priority list. The second-stage decision uses that list to form a portfolio of activities and determines other decision variables under each scenario $\omega \in \Omega$. We will also discuss the "wait-and-see solution," which can be carried out when prioritization is unnecessary. In this case, we can wait until a scenario $\omega \in \Omega$ is revealed and then solve model (1) under $(\mathbf{A}^\omega, \mathbf{b}^\omega, \mathbf{c}_x^\omega, \mathbf{c}_y^\omega, C^\omega)$ to obtain the best decision $(\mathbf{x}^\omega, \mathbf{y}^\omega)$ for that scenario.

Let $\mathcal{L}_l \subseteq I$, $l = 1, \ldots, L$, denote the priority levels, where $\mathcal{L}_l \neq \varnothing$, $l = 1, \ldots, L$, $\bigcup_{l=1}^{L} \mathcal{L}_l = I$, and $\mathcal{L}_l \cap \mathcal{L}_{l'} = \varnothing$, $l \neq l'$, $l, l' = 1, \ldots, L$. We denote a priority list $\mathcal{L} = [\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_L]$, where activities in $\mathcal{L}_1$ have higher priority than those in $\mathcal{L}_2$, etc. Let $F^\omega(\mathcal{L})$ be the optimal value of the RCASP under scenario $\omega \in \Omega$, subject to the prioritized activity-selection restrictions imposed by $\mathcal{L}$:

$$
\begin{aligned}
F^\omega(\mathcal{L}) \equiv \min_{\mathbf{x}^\omega, \mathbf{y}^\omega} \quad & \mathbf{c}_x^\omega \mathbf{x}^\omega + \mathbf{c}_y^\omega \mathbf{y}^\omega && \text{(3a)}\\
\text{s.t.} \quad & \mathbf{A}^\omega \mathbf{x}^\omega \le \mathbf{b}^\omega, && \text{(3b)}\\
& \mathbf{x}^\omega \in \{0,1\}^{|I|}, && \text{(3c)}\\
& (\mathbf{x}^\omega, \mathbf{y}^\omega) \in C^\omega, && \text{(3d)}\\
& x_i^\omega \ge x_{i'}^\omega, \quad i \in \mathcal{L}_l,\ i' \in \mathcal{L}_{l'},\ l < l',\\
& \qquad\qquad l, l' = 1, \ldots, L, && \text{(3e)}\\
& x_i^\omega = x_{i'}^\omega, \quad i, i' \in \mathcal{L}_l,\ l = 1, \ldots, L. && \text{(3f)}
\end{aligned}
$$

Model (3) is identical to (1), except that the parameters and decisions now depend on $\omega \in \Omega$, and we have added two constraints to enforce the two requirements of prioritized activity selection. Under constraint (3e), a lower-priority activity cannot be selected unless all higher-priority activities are selected. Constraint (3f) requires that either all or none of the activities on the same priority level are selected. We define activity prioritization as follows:

$$
\begin{aligned}
\min_{\mathcal{L}, L} \quad & \left[F(\mathcal{L}) \equiv \sum_{\omega \in \Omega} q^\omega F^\omega(\mathcal{L}) \right] && \text{(4a)}\\
\text{s.t.} \quad & \mathcal{L} = [\mathcal{L}_1, \ldots, \mathcal{L}_L], && \text{(4b)}\\
& \mathcal{L}_l \subseteq I, \quad l = 1, \ldots, L, && \text{(4c)}\\
& \mathcal{L}_l \neq \varnothing, \quad l = 1, \ldots, L, && \text{(4d)}\\
& \mathcal{L}_l \cap \mathcal{L}_{l'} = \varnothing, \quad l \neq l',\ l, l' = 1, \ldots, L, && \text{(4e)}\\
& \bigcup_{l=1,\ldots,L} \mathcal{L}_l = I. && \text{(4f)}
\end{aligned}
$$

Model (4) minimizes the expected cost of prioritized activity selection, i.e., the sum of weighted costs of RCASP across scenarios $\omega \in \Omega$. The solution to model (4) is an optimal priority list, and the corresponding solution to model (3) is the prioritized selection of activities under scenario $\omega \in \Omega$. The timing of decisions and observations of uncertainty is key to understanding the prioritization model (4). First, the priority list is formed via $\mathcal{L}$. Next, the values of the problem parameters, $(\mathbf{A}^\omega, \mathbf{b}^\omega, \mathbf{c}_x^\omega, \mathbf{c}_y^\omega, C^\omega)$, $\omega \in \Omega$, are realized. We then effectively work down the priority list selecting the activities via $\mathbf{x}^\omega$, until a point where either the budget is exhausted or the cost no longer decreases by selecting more activities.

Model (4) minimizes over $\mathcal{L}$ and L, but in what follows we occasionally suppress L in referring to a priority list. Also, we assume the set of activities, I, is an ordered set so that we can write restrictions such as $i < i' \in I$. With an eye toward computation, we give an IP formulation for activity-prioritization model (4), which uses the additional decision variable $s_{ii'}$ that takes the value of 1 if activity i has no lower priority than i' and 0 otherwise:

$$
\begin{aligned}
\min_{\mathbf{s}, \mathbf{x}, \mathbf{y}} \quad & \sum_{\omega \in \Omega} q^\omega (\mathbf{c}_x^\omega \mathbf{x}^\omega + \mathbf{c}_y^\omega \mathbf{y}^\omega) && \text{(5a)}\\
\text{s.t.} \quad & s_{ii'} + s_{i'i} \ge 1, \quad i < i',\ i, i' \in I, && \text{(5b)}\\
& s_{ii'} \in \{0,1\}, \quad i \neq i',\ i, i' \in I, && \text{(5c)}\\
& x_i^\omega \ge x_{i'}^\omega + s_{ii'} - 1, \quad i \neq i',\ i, i' \in I,\ \omega \in \Omega, && \text{(5d)}\\
& \mathbf{A}^\omega \mathbf{x}^\omega \le \mathbf{b}^\omega, \quad \omega \in \Omega, && \text{(5e)}\\
& \mathbf{x}^\omega \in \{0,1\}^{|I|}, \quad \omega \in \Omega, && \text{(5f)}\\
& (\mathbf{x}^\omega, \mathbf{y}^\omega) \in C^\omega, \quad \omega \in \Omega. && \text{(5g)}
\end{aligned}
$$

Model (5) is a two-stage stochastic integer program. First-stage variables $\mathbf{s}$ form the priority list, second-stage variables $\mathbf{x}^\omega$ select the portfolio of activities for each $\omega \in \Omega$, and second-stage variables $\mathbf{y}^\omega$ handle the remaining decisions.

Model (5) minimizes expected cost. Given a pair of activities i, $i' \in I$, constraints (5b) and (5c) ensure either they have the same priority (i.e., $s_{ii'} = s_{i'i} = 1$) or one has higher priority than the other. If i has higher priority than i', we have $s_{ii'} = 1$ and $s_{i'i} = 0$, and constraint (5d) for pair (i, i') then reads $x_i^\omega \geq x_{i'}^\omega$ and $x_{i'}^\omega \geq x_i^\omega - 1$. The latter is redundant, so this amounts to constraint (3e). If i and i' have the same priority, we have $s_{ii'} = s_{i'i} = 1$, and constraint (5d) yields $x_i^\omega = x_{i'}^\omega$, which is same as constraint (3f). The last three sets of constraints replicate (3b)–(3d).

The activity-prioritization model (4) is NP-hard, in general, if the underlying deterministic RCASP (1) is NP-hard because solving the former amounts to solving the latter in the case of a single scenario. Although this provides some justification for an integer programming formulation like model (5), other approaches are possible. Greedy solution procedures produce nested solutions and, hence, can be used to form priority lists. In some settings, we have performance guarantees for greedy algorithms. For example, when maximizing a submodular function subject to a cardinality constraint, we can obtain a greedy solution with a value that is within a constant factor, $(1 - e^{-1})$, of the optimal value (Nemhauser et al. 1978). In such a case, this a priori performance guarantee also holds for the prioritized solution obtained by the greedy algorithm. Section 1 points to related work in the literature yielding such performance guarantees.

3. An Application: Site Prioritization

Using the ideas in §2, we formalize the site-prioritization problem. In the next section, we develop a prioritized version of a multidimensional knapsack problem. Figure 1(a) shows optimal solutions of an instance of problem (2) under $k = 1$–4. To form the priority lists of Figures 1(b)–1(d), we apply the general prioritization model (5) to the k-median problem (2). With k^ω denoting the budget under scenario ω and with obvious extensions of the notation above, we obtain the following:

$$\min_{\mathbf{s},\mathbf{x},\mathbf{y}} \quad \sum_{\omega \in \Omega} q^\omega \sum_{j \in J} \sum_{i \in I} d_{ij} y_{ij}^\omega \tag{6a}$$

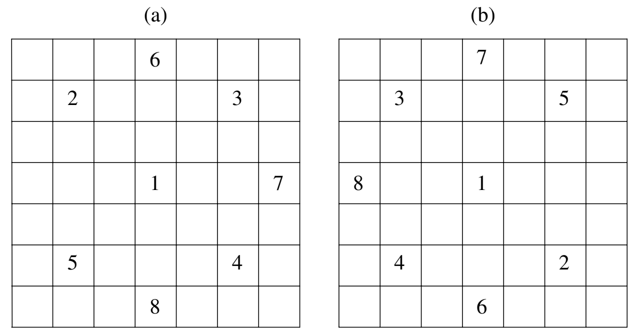
$$\text{s.t.} \quad s_{ii'} + s_{i'i} \geq 1, \quad i < i', \, i, i' \in I, \tag{6b}$$

$$s_{ii'} \in \{0, 1\}, \quad i \neq i', \, i, i' \in I, \tag{6c}$$

$$x_i^\omega \geq x_{i'}^\omega + s_{ii'} - 1, \quad i \neq i', \, i, i' \in I \; \omega \in \Omega, \tag{6d}$$

$$\sum_{i \in I} x_i^\omega \leq k^\omega, \quad \omega \in \Omega, \tag{6e}$$

Figure 2 Greedy Solutions for the Stochastic k-Median Problem



Notes. Parts (a) and (b) show two alternative greedy solutions. We obtain each solution by first finding an optimal solution to the 1-median problem, then to the 2-median problem with one of the locations fixed to the solution obtained from the 1-median problem, and so forth.

$$x_i^\omega \geq y_{ij}^\omega, \quad i \in I, \, j \in J, \, \omega \in \Omega, \tag{6f}$$

$$\sum_{i \in I} y_{ij}^\omega = 1, \quad j \in J, \, \omega \in \Omega, \tag{6g}$$

$$x_i^\omega \in \{0, 1\}, \quad i \in I, \, \omega \in \Omega, \tag{6h}$$

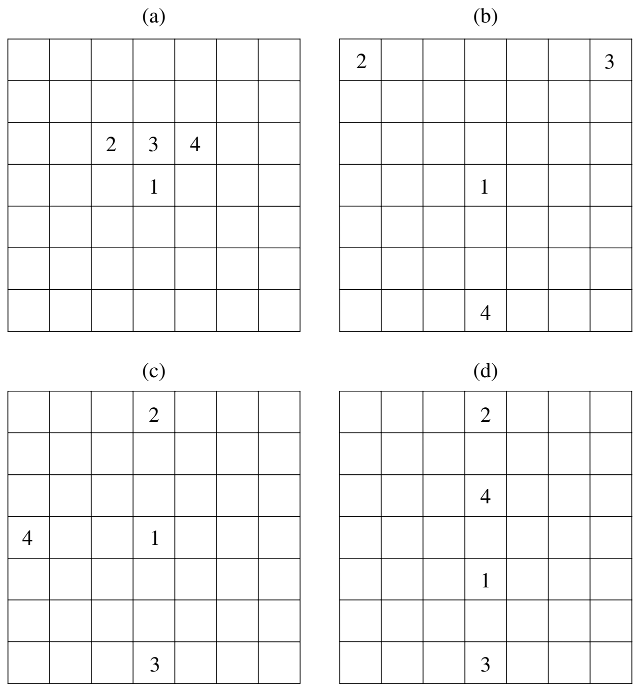
$$y_{ij}^\omega \in [0, 1], \quad i \in I, \, j \in J, \, \omega \in \Omega. \tag{6i}$$

The variables and constraints for model (5) read similarly in model (6), so we do not repeat that discussion here. Optimal solutions for $k = 1, 2$, $k = 1, 2, 3$, and $k = 1, 2, 3, 4$ with equal probabilities are given in Figures 1(b)–1(d). The corresponding optimal values are 178.99, 162.79, and 148.55.

A natural approach to building a heuristic priority list uses the following greedy algorithm. Suppose the budget, k^ω, has realizations 1 and 2 with equal probability. We first solve the 1-median problem, obtaining the location marked by 1 in Figure 2(a). We then solve the 2-median problem, fixing the location of the first facility, and we obtain the location marked by 2 in Figure 2(a). If instead $k = 1, 2, 3$ with equal probability, the heuristic solution gives the same first two locations and the third location is marked by 3. Figure 2(a) gives the greedy heuristic's solution when the budget is probabilistic and takes values $k^\omega = 1, \ldots, k^{\max}$, $k^{\max} \leq 8$. The objective function value of any such solution is the weighted sum of total Euclidean distances under each budget scenario. If the budget takes equally likely values of 1–8, the heuristic priority list in Figure 2(a) has the expected cost $(194.78 + 167.4 + 140.01 + 115.1 + 90.19 + 85.62 + 81.04 + 76.46)/8 = 118.82$. If we instead use model (6), we obtain a solution with an optimal value 115.47.

Because of symmetry, the greedy heuristic gives alternative solutions. Figures 2(a) and 2(b) give two such solutions for $k^{\max} = 1$–8. These two solutions happen to have identical values for expected total distance. If we minimize the expected maximum distance instead, we see that some alternative solutions

Figure 3 **Greedy Solutions and Optimal Solution for Model (7) for Stochastic k**



Notes. Parts (a)–(c) show three alternative greedy solutions for model (7), where k^ω takes equally likely values of 1–4. Part (d) gives the corresponding optimal prioritization solution.

have far better objective function values than others. Model (7) gives the deterministic RCASP that minimizes the maximum distance over all customers:

$$\min_{\mathbf{x},\,\mathbf{y}} \quad \max_{j\in J} \sum_{i\in I} d_{ij} y_{ij} \tag{7a}$$

$$\text{s.t.} \quad \text{(2b)–(2f)}. \tag{7b}$$

Figures 3(a)–3(c) give three alternative greedy solutions for the stochastic version of model (7). These solutions are formed similarly to the greedy solutions of Figure 2: We first solve model (7) with $k=1$. Then we solve the model with $k=2$, but with the first location fixed, and so forth. There are multiple optimal solutions to these models, and breaking ties in different ways leads to the three solutions shown in the figure. The objective function values of the prioritization models for these alternative solutions are 4.95, 4.66, and 4.38, respectively. It is no surprise that the "unintelligent" solution of Figure 3(a) has the worst objective function value, followed by better solutions of Figures 3(b) and 3(c). Figure 3(d) gives the corresponding optimal priority list with optimal value 4.18. The optimality gaps of the alternative solutions are 18.42%, 11.48%, and 4.78%.

Table 1 summarizes the greedy heuristic's performance and the cost of prioritization, relative to the optimal value of a prioritized solution. We consider prioritization of both models (2) and (7), as indicated in the first column. The first column of the table also indicates the problem size in terms of the number of grids (n_G). For each problem size, we consider seven instances, represented by columns 2–8. In each instance, all budget scenarios are equally likely. For each instance, Table 1 lists three values: the middle value gives the optimal value of the corresponding prioritization model; the lower value gives the percentage optimality gap of a solution from the greedy heuristic, as compared with the optimal prioritization solution; and the upper value gives the expected value with perfect information or the *cost of prioritization*, defined as the percentage difference of the optimal prioritization solution from the wait-and-see solution. Under the wait-and-see solution, we can select facility locations after observing the realization of the budget, k^ω, and hence prioritization is unnecessary. As we indicate above, the greedy heuristic gives alternative solutions with different solution values. Table 1 selects one such solution arbitrarily.

One prominent pattern in Table 1 is the differing performance of the heuristic under models (2) and (7). When minimizing expected total distance, the heuristic produces optimality gaps of less than 5% but is less successful in minimizing expected maximum distance. The primary reason for this seems to be the greedy nature of the heuristic, coupled with the extreme nature of the maximum-distance objective. After centrally locating the first facility, the objective function cannot be reduced when placing a single second facility at any of the $n_G^2 - 1$ potential locations.

We obtain wait-and-see solutions by solving model (5) without prioritization constraints (5b)–(5d), as applied to models (2) and (7). The cost of prioritization is usually less than 5% in model (2), except when the number of grids is very small. This cost in model (7) is much greater, with a similar observation of having the largest costs for the smallest grid set. Similar to the performance of the greedy heuristic, the cost of prioritization tends to be higher in model (7) because of the extreme nature of the maximum-distance objective. Of course, the wait-and-see solutions can be obtained only under perfect information. Under imperfect information on the budget k for installing facilities, we seek the prioritized solution with optimal expected cost, as opposed to using the greedy heuristic, which would further increase the cost of prioritization.

4. Prioritization with Total Order Restriction

The concept of a priority list may give the impression that there is a one-to-one matching between priorities and activities, with each activity receiving a unique

Table 1 **Optimal and Greedy Solutions for Prioritization of Models (2) and (7), and Corresponding Wait-and-See Solutions**

	$k^{\max}$						
n_G	2	3	4	5	6	7	8
Model (2)							
3	5.90%	8.78%	9.88%	8.53%	7.45%	6.63%	5.91%
	22.38	20.70	18.79	17.30	16.30	15.59	15.05
	0%	0.47%	2.27%	1.97%	1.74%	1.56%	1.41%
5	3.44%	4.25%	4.68%	4.01%	3.83%	3.79%	4.00%
	75.88	68.85	63.08	58.35	54.76	51.73	49.12
	0%	1.58%	3.24%	4.13%	4.28%	3.32%	2.81%
7	3.71%	4.85%	5.80%	5.24%	4.76%	4.58%	4.38%
	178.99	162.79	148.55	137.15	128.34	121.28	115.47
	1.17%	2.83%	3.88%	3.17%	2.99%	2.97%	2.90%
9	2.43%	4.25%	4.87%	4.40%	4.03%	3.90%	3.92%
	348.03	317.99	290.66	268.10	250.68	236.85	225.26
	1.76%	2.86%	3.83%	3.37%	3.31%	3.25%	2.34%
Model (7)							
3	14.59%	10.23%	23.33%	34.33%	33.71%	31.03%	27.95%
	2.12	1.94	1.85	1.80	1.65	1.52	1.41
	0%	9.28%	7.29%	6.01%	12.20%	19.63%	26.13%
5	9.75%	7.00%	14.72%	20.89%	22.27%	20.09%	18.32%
	3.54	3.21	3.04	2.94	2.80	2.63	2.50
	0%	10.25%	16.20%	13.40%	14.45%	18.47%	21.86%
7	13.01%	11.55%	20.42%	26.28%	27.99%	28.32%	26.49%
	4.95	4.57	4.34	4.18	3.98	3.71	3.45
	0%	8.33%	7.56%	8.64%	3.01%	7.63%	11.59%
9	10.42%	8.79%	17.09%	21.04%	20.62%	24.75%	19.14%
	6.36	5.82	5.55	5.35	5.03	5.09	4.97
	0%	0%	0%	0.81%	4.35%	5.25%	2.1%

Notes. Each cell contains three values. The middle value is the optimal value to the prioritized version of model (2) or (7). The lower value is the optimality gap of a corresponding greedy solution in percentage terms. The upper value is the difference between the optimal value of the prioritized solution and the value of the wait-and-see solution in percentage terms.

ranking. The formulation given by constraints (4b)–(4f), however, allows a many-to-one assignment of activities to priorities. This section modifies models (4) and (5), requiring each activity to receive a unique ranking. We compare the many-to-one and one-to-one formulations both theoretically and computationally.

Given a priority list, $\mathcal{L}$, we define a *refinement*, $\bar{\mathcal{L}}$, as a priority list that satisfies the following: (i) each priority level in $\bar{\mathcal{L}}$ is a subset of some priority level in $\mathcal{L}$, and (ii) for any two activities i, $i' \in I$ with $i \in \mathcal{L}_l$, $i' \in \mathcal{L}_{l'}$ and $l < l'$, $\bar{\mathcal{L}}$ has the property that $i \in \bar{\mathcal{L}}_{\bar{l}}$, $i' \in \bar{\mathcal{L}}_{\bar{l}'}$ with $\bar{l} < \bar{l}'$. For example, $[\{1,2\},\{3\}]$, $[\{1\},\{2\},\{3\}]$, and $[\{2\},\{1\},\{3\}]$ are refinements of $[\{1,2\},\{3\}]$, whereas $[\{3\},\{1,2\}]$ and $[\{3\},\{2\},\{1\}]$ are not because the latter two do not satisfy the second condition for activity pairs $(1,3)$ and $(2,3)$. The following proposition considers the relative objective function values of a priority list and its refinement.

Proposition 1. *Let $\mathcal{L}$ and $\bar{\mathcal{L}}$ be two feasible priority lists for the activity-prioritization model* (4), *and let $\bar{\mathcal{L}}$ be a refinement of $\mathcal{L}$. Then, $F(\bar{\mathcal{L}}) \le F(\mathcal{L})$.*

Proof. It suffices to show $F^{\omega}(\bar{\mathcal{L}}) \le F^{\omega}(\mathcal{L})$, $\omega \in \Omega$, where $F^{\omega}(\cdot)$ is the optimal value of model (3). For a pair of activities i, $i' \in I$, we have either constraint (3e) or (3f), depending on whether i and i' are on different priority levels or the same level, respectively. Consider the following two cases:

Case I. Activities i and i' are on the same priority level of $\bar{\mathcal{L}}$. This implies i and i' are on the same priority level of $\mathcal{L}$ as well because of condition (ii) in the definition of refinement. For activity pair i and i', we have constraint (3f) in the definitions of both $F^{\omega}(\mathcal{L})$ and $F^{\omega}(\bar{\mathcal{L}})$.

Case II. Activities i and i' are on different priority levels of $\bar{\mathcal{L}}$. Assume without loss of generality that i has higher priority than i'. This means we have constraint (3e) in the definition of $F^{\omega}(\bar{\mathcal{L}})$. Because of condition (ii) in the definition of refinement, either i has higher priority than i' in $\mathcal{L}$ or they have the same priority. In the former case, for this pair of activities, we have constraint (3e) in $F^{\omega}(\mathcal{L})$'s definition; in the latter case, we have constraint (3f). That is, in the former case, we have the same constraint as in the definition of $F^{\omega}(\bar{\mathcal{L}})$, and in the latter case, we have a more restrictive constraint because we replace a greater-than type constraint with an equal-to constraint. Thus, $F^\omega(\mathscr{L}) \geq F^\omega(\bar{\mathscr{L}})$ because model (3) under $\bar{\mathscr{L}}$ is a relaxation of model (3) under $\mathscr{L}$. □

A *total order* (Rosen 2006, Chap. 6) is a permutation of the elements of I. That is, a total order is a priority list, $\mathscr{L} = [\mathscr{L}_1, \ldots, \mathscr{L}_L]$, with $|\mathscr{L}_l| = 1$, $l = 1, \ldots, L$, and $L = |I|$. Thus, a total order can be viewed as the "most-refined" priority list. That is, if $\mathscr{L}$ is a total order and $\bar{\mathscr{L}}$ is a refinement of $\mathscr{L}$, then $\mathscr{L} = \bar{\mathscr{L}}$. This observation, together with Proposition 1, implies that activity prioritization with a total order restriction has the same optimal value as that without this restriction. Consider the following model, corollary, and IP formulation:

$$\min_{\mathscr{L}, L} \quad \left[F(\mathscr{L}) \equiv \sum_{\omega \in \Omega} q^\omega F^\omega(\mathscr{L}) \right] \tag{8a}$$

$$\text{s.t.} \quad \text{(4b)–(4f)}, \tag{8b}$$

$$|\mathscr{L}_l| = 1, \quad l = 1, \ldots, L. \tag{8c}$$

Corollary 1. *Models* (4) *and* (8) *have the same optimal value.*

Formulation:

$$\min_{\mathbf{s},\mathbf{x},\mathbf{y}} \quad \sum_{\omega \in \Omega} q^\omega (\mathbf{c}_x^\omega \mathbf{x}^\omega + \mathbf{c}_y^\omega \mathbf{y}^\omega) \tag{9a}$$

$$\text{s.t.} \quad s_{i_1 i_2} + s_{i_2 i_3} + \cdots + s_{i_{p-1} i_p} + s_{i_p i_1} \leq p - 1,$$
$$i_1 \neq i_2 \neq \cdots \neq i_p, \quad i_1, \ldots, i_p \in I,\ p \in \{2, 3\}, \tag{9b}$$

$$\text{(5b)–(5g)}. \tag{9c}$$

Compared with model (5), model (9) has a modified definition of the $\mathbf{s}$ variables. Here, $s_{ii'}$ indicates whether activity i has "higher" priority than i' rather than "no lower" priority than i', as reflected by the addition of *cycle-elimination* constraints (9b). Having cycle-elimination constraints (9b) for $p = 3$ implies the cycle-elimination constraints for $p > 3$, as any cycle, $i_1, \ldots, i_{p-1}, i_p, i_1$, of size p ($p > 3$) is implied by a combination of a cycle, $i_1, \ldots, i_{p-2}, i_p, i_1$, of size $p - 1$ and a cycle, $i_{p-2}, i_{p-1}, i_p, i_{p-2}$, of size 3. Specifically, adding constraints (9b) for cycles $i_1, \ldots, i_{p-2}, i_p, i_1$ and $i_{p-2}, i_{p-1}, i_p, i_{p-2}$ together yields $s_{i_1 i_2} + s_{i_2 i_3} + \cdots + s_{i_{p-2} i_p} + s_{i_p i_1} + s_{i_{p-2} i_{p-1}} + s_{i_{p-1} i_p} + s_{i_p i_{p-2}} \leq p$, which implies $s_{i_1 i_2} + s_{i_2 i_3} + \cdots + s_{i_{p-2} i_{p-1}} + s_{i_{p-1} i_p} + s_{i_p i_1} \leq p - 1$. The implication comes from the inequality $s_{i_{p-2} i_p} + s_{i_p i_{p-2}} \geq 1$, i.e., constraint (5b) for i_{p-2}, i_p. See Potts (1985) for a similar discussion. Cycle-elimination constraints (9b) for p prevent the priority list from having a priority level of cardinality p, and thus they require the priority list to be a total order.

Proposition 1 allows us to iteratively refine an optimal priority list of model (4), without loss of optimality, until we obtain a total order. Similarly, given an optimal total order, $\mathscr{L}^*$, and a corresponding solution vector, $(\mathbf{x}^*, \mathbf{y}^*)$ with $\mathbf{x}^* = (x_i^{*\omega})_{i \in I,\, \omega \in \Omega}$, to model (8), we can aggregate any two of its priority levels, l, l', if two activities exist, i, i', such that $i \in \mathscr{L}_l^*$, $i' \in \mathscr{L}_{l'}^*$ and $x_i^{*\omega} = x_{i'}^{*\omega}$ for all $\omega \in \Omega$. This can be done without loss of optimality because the aggregated priority list, together with $(\mathbf{x}^*, \mathbf{y}^*)$, still satisfies constraints (3e) and (3f) and has the same objective function value. Thus, a solution to either model (4) or model (8) can be transformed to a solution to the other model with the same objective function value.

We can, therefore, use either model (5) or (9) in solving activity prioritization, so we seek to understand which is more efficient. Model (9) has $\binom{|I|}{3}$ additional structural constraints. (Constraint (9b) for $p = 2$ can be combined with constraint (5b) into an equality constraint.) Typically, an IP formulation with more constraints is tolerated, even preferred, if it has a tighter linear programming (LP) relaxation. To compare the LP relaxations of these two formulations, we first give two lemmas characterizing the LP relaxation of model (5) and then give the main result that compares the optimal values of models (5) and (9). In Lemmas 1 and 2, in Theorem 1, and in further discussions below, we speak of the *LP relaxation* of models (5) and (9). We abuse this term slightly in that we mean a continuous relaxation of constraints (5c) and (5f)—i.e., we assume constraints (5g) hold as stated even though the sets C^ω, $\omega \in \Omega$, need not be polyhedral. In what follows, we also use the positive-part operator, defined as $(\cdot)^+ = \max\{\cdot, 0\}$.

Lemma 1. *Let* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}})$ *with* $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i \in I,\, \omega \in \Omega}$ *be feasible to the LP relaxation of model* (5), *and let* $\alpha_{ii'} = \max_{\omega \in \Omega}\{\bar{x}_i^\omega - \bar{x}_{i'}^\omega\}$, $i \neq i'$, $i, i' \in I$. *Then the following inequalities hold*:

$$\alpha_{i_1 i_2}^+ + \alpha_{i_2 i_1}^+ \leq 1, \tag{10}$$

$$\alpha_{i_1 i_3}^+ \geq \alpha_{i_1 i_2}^+ + \alpha_{i_2 i_3}^+ - 1, \tag{11}$$

for $i_1, i_2, i_3 \in I$ *that are all distinct; i.e.,* $i_1 \neq i_2 \neq i_3$.

Proof. Let $i_1 \neq i_2 \in I$ be given. A solution vector, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}})$, feasible to the LP relaxation of model (5) satisfies constraint (5d); i.e.,

$$\bar{x}_{i_1}^\omega \geq \bar{x}_{i_2}^\omega + \bar{s}_{i_1 i_2} - 1, \quad \omega \in \Omega.$$

Rearranging and maximizing over the scenarios, we obtain

$$\max_{\omega \in \Omega}\{\bar{x}_{i_2}^\omega - \bar{x}_{i_1}^\omega\} \leq 1 - \bar{s}_{i_1 i_2} \quad \Rightarrow \quad \alpha_{i_2 i_1} \leq 1 - \bar{s}_{i_1 i_2}.$$

We also have $0 \leq 1 - \bar{s}_{i_1 i_2}$ by the LP relaxation of constraint (5c). Thus, $\alpha_{i_2 i_1}^+ \leq 1 - \bar{s}_{i_1 i_2}$. We similarly obtain $\alpha_{i_1 i_2}^+ \leq 1 - \bar{s}_{i_2 i_1}$. Summing these two inequalities yields $\alpha_{i_1 i_2}^+ + \alpha_{i_2 i_1}^+ \leq 2 - (\bar{s}_{i_1 i_2} + \bar{s}_{i_2 i_1}) \leq 1$, where the last inequality follows from constraint (5b).

Let all-distinct $i_1, i_2, i_3 \in I$ be given, and let $\omega^* \in \Omega$ be a maximizing scenario for $\alpha_{i_2 i_3}$; i.e., $\alpha_{i_2 i_3} = \bar{x}_{i_2}^{\omega^*} - \bar{x}_{i_3}^{\omega^*}$. Then, by the definition of the max- and positive-part operators, $\bar{x}_{i_2}^{\omega^*} - \bar{x}_{i_1}^{\omega^*} \le \alpha_{i_2 i_1} \le \alpha_{i_2 i_1}^+$. By inequality (10), we have $\alpha_{i_1 i_2}^+ + \alpha_{i_2 i_1}^+ \le 1$. Combining the last two inequalities, we have $\bar{x}_{i_1}^{\omega^*} \ge \bar{x}_{i_2}^{\omega^*} + \alpha_{i_1 i_2}^+ - 1$. Using the definition of the max- and positive-part operators again, we have $\bar{x}_{i_1}^{\omega^*} - \bar{x}_{i_3}^{\omega^*} \le \alpha_{i_1 i_3} \le \alpha_{i_1 i_3}^+$. Combining the last two inequalities, we have $\alpha_{i_1 i_3}^+ \ge \bar{x}_{i_1}^{\omega^*} - \bar{x}_{i_3}^{\omega^*} \ge \bar{x}_{i_2}^{\omega^*} + \alpha_{i_1 i_2}^+ - 1 - \bar{x}_{i_3}^{\omega^*} = \alpha_{i_1 i_2}^+ + \alpha_{i_2 i_3} - 1$, where the last equality follows from the definition of $\alpha_{i_2 i_3}$ (without application of the positive-part operator). Thus, to complete this step of the proof, it suffices to show $\alpha_{i_1 i_3}^+ \ge \alpha_{i_1 i_2}^+ + 0 - 1$. Again from inequality (10), $\alpha_{i_1 i_2}^+ + \alpha_{i_2 i_1}^+ \le 1$. Eliminating $\alpha_{i_2 i_1}^+$ from the left-hand side of the inequality, and adding $\alpha_{i_1 i_3}^+$ to the right-hand side, we obtain the desired result. $\square$

Lemma 2. *Assume the hypotheses of Lemma* 1. *Then the following inequalities hold*:

$$\alpha_{i_1 i_3}^+ \le \alpha_{i_1 i_2}^+ + \alpha_{i_2 i_3}^+, \tag{12}$$

$$\alpha_{i_1 i_2}^+ + \alpha_{i_2 i_3}^+ + \alpha_{i_3 i_1}^+ - (\alpha_{i_1 i_3}^+ + \alpha_{i_3 i_2}^+ + \alpha_{i_2 i_1}^+) \ge -1, \tag{13}$$

for $i_1, i_2, i_3 \in I$ that are all distinct.

Proof. Let all-distinct $i_1, i_2, i_3 \in I$ be given, and let $\omega \in \Omega$. Then,

$$\begin{aligned}\bar{x}_{i_1}^{\omega} - \bar{x}_{i_3}^{\omega} &= (\bar{x}_{i_1}^{\omega} - \bar{x}_{i_2}^{\omega}) + (\bar{x}_{i_2}^{\omega} - \bar{x}_{i_3}^{\omega}) \\ &\le \max_{\omega \in \Omega}\{\bar{x}_{i_1}^{\omega} - \bar{x}_{i_2}^{\omega}\} + \max_{\omega \in \Omega}\{\bar{x}_{i_2}^{\omega} - \bar{x}_{i_3}^{\omega}\} \\ &= \alpha_{i_1 i_2} + \alpha_{i_2 i_3} \\ &\le \alpha_{i_1 i_2}^+ + \alpha_{i_2 i_3}^+.\end{aligned}$$

Maximizing the left-hand side over $\omega \in \Omega$ and noting the right-hand side is nonnegative yields (12).

Inequality (12) for i_2, i_3, i_1 yields $\alpha_{i_2 i_1}^+ \le \alpha_{i_2 i_3}^+ + \alpha_{i_3 i_1}^+$. Inequality (11) for i_1, i_3, i_2 yields $\alpha_{i_1 i_2}^+ \ge \alpha_{i_1 i_3}^+ + \alpha_{i_3 i_2}^+ - 1$. Combining these two yields (13). $\square$

Theorem 1. *The LP relaxations of models* (5) *and* (9) *have the same optimal value.*

Proof. Model (5) is a relaxation of model (9), so it suffices to show that the optimal value of the LP relaxation of model (5) is at least that of model (9). To this end, we prove that given an optimal solution to the LP relaxation of model (5), $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$, we can find a solution vector $\hat{\mathbf{s}}^*$ with $\hat{\mathbf{s}}^* = (\hat{s}_{ii'}^*)_{i, i' \in I}$ such that $(\mathbf{x}^*, \mathbf{y}^*, \hat{\mathbf{s}}^*)$ is feasible to the LP relaxation of model (9). This will prove our claim since $\hat{\mathbf{s}}^*$ does not contribute to the objective function.

Let $i \ne i' \in I$. Given $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ with $\mathbf{x}^* = (x_i^{*\omega})_{i \in I, \omega \in \Omega}$, let $\alpha_{ii'}^+ = \max\{0, \max_{\omega \in \Omega}\{x_i^{*\omega} - x_{i'}^{*\omega}\}\}$. We define $\hat{s}_{ii'}^* = \alpha_{ii'}^+ + (1 - \alpha_{ii'}^+ - \alpha_{i'i}^+)/2 = (1 + \alpha_{ii'}^+ - \alpha_{i'i}^+)/2$. By construction $0 \le \alpha_{ii'}^+ \le 1$, which implies $-1 \le \alpha_{ii'}^+ - \alpha_{i'i}^+ \le 1$. Thus, we have $0 \le \hat{s}_{ii'}^* \le 1$, and the LP relaxation of constraint (5c) is satisfied. By construction, $\hat{s}_{ii'}^* + \hat{s}_{i'i}^* = 1$, and constraint (5b) is satisfied.

We have $\hat{s}_{ii'}^* = (1 + \alpha_{ii'}^+ - \alpha_{i'i}^+)/2 \le 1 - \alpha_{i'i}^+ \le 1 - \alpha_{i'i} = 1 - \max_{\omega \in \Omega}\{x_{i'}^{*\omega} - x_i^{*\omega}\} \le 1 - x_{i'}^{*\omega} + x_i^{*\omega}$, $\omega \in \Omega$. The first inequality follows from inequality (10), the second inequality comes from dropping the positive-part operator, and the third follows from the definition of the max operator. Thus, given $i \ne i' \in I$, $x_i^{*\omega} \ge x_{i'}^{*\omega} + \hat{s}_{ii'}^* - 1$, $\omega \in \Omega$, and constraint (5d) is satisfied. The solution vector $(\mathbf{x}^*, \mathbf{y}^*)$ already satisfies constraints (5e)–(5g), and thus it remains to show that $\hat{\mathbf{s}}^*$ satisfies constraint (9b).

As we indicate above, we have $\hat{s}_{ii'}^* + \hat{s}_{i'i}^* = 1$, $i \ne i' \in I$, and thus constraint (9b) is satisfied for $p = 2$. For $p = 3$, let all-distinct $i_1, i_2, i_3 \in I$ be given. $\hat{s}_{i_1 i_3}^* + \hat{s}_{i_3 i_2}^* + \hat{s}_{i_2 i_1}^* = 3/2 + (\alpha_{i_1 i_3}^+ + \alpha_{i_3 i_2}^+ + \alpha_{i_2 i_1}^+)/2 - (\alpha_{i_1 i_2}^+ + \alpha_{i_2 i_3}^+ + \alpha_{i_3 i_1}^+)/2 \ge 3/2 + (-1)/2 = 1$, where the inequality follows from (13). Substituting $\hat{s}_{ii'}^* = 1 - \hat{s}_{i'i}^*$ in the last inequality, we obtain $\hat{s}_{i_1 i_2}^* + \hat{s}_{i_2 i_3}^* + \hat{s}_{i_3 i_1}^* \le 2$. $\square$

Since the optimal values of the LP relaxations of models (5) and (9) are equal, we expect to be able to solve model (5) more quickly because of its smaller size. To see whether this expectation holds in practice, we compare models (5) and (9) on a set of multidimensional knapsack problem instances. We use (Cplex 2011, version 11.1) on a Dell Poweredge 2,950 computer with Intel (Xenon) 3.73 GHz processor and with 8 GB of RAM, and we report the results in Table 2. We first formulate model (5)'s prioritization of the multidimensional knapsack problem, skipping the straightforward deterministic RCASP model. Application of model (9) can be performed similarly. The notation and formulation of the prioritization model follow:

Indices and sets:

$i, i' \in I$	items
$t \in T$	knapsack dimensions
$\omega \in \Omega$	scenarios

Data:

a_i^{ω}	profit of item i under scenario ω
b_t^{ω}	capacity of dimension t under scenario ω
c_{it}^{ω}	resource consumption of item i for dimension t under scenario ω
q^{ω}	probability of scenario ω

Decision variables:

$s_{ii'}$	1 if item i has no lower priority than i'; 0 otherwise
x_i^{ω}	1 if item i is selected under scenario ω; 0 otherwise

Formulation:

$$\max_{\mathbf{s},\mathbf{x}} \quad \sum_{\omega\in\Omega} q^{\omega}\sum_{i\in I} a_i^{\omega} x_i^{\omega} \tag{14a}$$

$$\text{s.t.} \quad s_{ii'}+s_{i'i}\geq 1, \quad i<i',\, i,i'\in I, \tag{14b}$$

$$s_{ii'}\in\{0,1\}, \quad i\neq i',\, i,i'\in I, \tag{14c}$$

$$x_i^{\omega}\geq x_{i'}^{\omega}+s_{ii'}-1, \quad i\neq i',\, i,i'\in I,\, \omega\in\Omega, \tag{14d}$$

$$\sum_{i\in I} c_{it}^{\omega} x_i^{\omega}\leq b_t^{\omega}, \quad t\in T,\, \omega\in\Omega, \tag{14e}$$

$$x_i^{\omega}\in\{0,1\}, \quad i\in I,\, \omega\in\Omega. \tag{14f}$$

The problem sets for the underlying multidimensional knapsack problem are taken from the OR Library (Beasley 2011). We use the first two collections of instances from the library, named mknap1 and mknap2. From mknap1, we use instances (10, 10), (15, 10), (20, 10), (28, 10), (39, 5) and (50, 5), where the first number represents the number of items and the second represents the number of dimensions; and from mknap2, we use instances (60, 5), (70, 5), (80, 5), and (90, 5). The profit, budget, and resource consumption uncertainties for the test instances are made stochastic in a manner detailed in Koç (2010).

The first row in Table 2 gives the number of scenarios, and the first column gives the item–dimension pairs. Each cell has two values, with the upper value corresponding to model (5) and the lower to model (9). Each value is the average over two problem instances. The values with the percentage sign give the optimality gaps after one hour of computation. These percentage values are *geometric* averages. The values without the percentage sign give the running times in terms of CPU seconds to solve the instances to within 0.01% optimality. These values are the *arithmetic* averages over the two instances. Some of the cells are empty, which means after one hour we lack an optimality gap for at least one instance because we cannot find a feasible solution. We mostly solve the smaller instances to within 0.01% optimality and spend only an hour on the larger ones. This is not strict, however. We can solve some of the instances to within 0.01% optimality in a few hours using one model, but not the others. For these instances, we report only one-hour optimality gaps under both models.

From Table 2, we see that the problem instances with 20 or fewer items do not strictly favor either of the models. Those with 28 or more items, however, favor model (5) over model (9). This conclusion provides evidence of what we have inferred from the LP relaxations and the sizes of the two formulations. In §§5 and 6, we return to the test instances of Table 2.

In the remainder of this paper, we focus on the general form of a priority list rather than the total-order form. The general form may seem more restrictive because it requires the set of activities on a priority level to be selected, or not, as an entire set. The total-order formulation has only a single activity on each priority level, and hence distinct activities are not bound together in this fashion. We prefer the general form over the total-order formulation for two main reasons. First, in the general model an optimal priority list will place multiple activities on the same priority level only if there is no (modeled) scenario under which we would benefit from selecting a subset of the activities on that priority level. As a result, such a solution yields important insight rather than arbitrarily breaking a tie and prioritizing one activity over another. As an example in which an optimal priority list can have multiple activities on one level, consider the site-prioritization problem in Figure 1 in which the budget takes values 2 and 4 with equal probability. Second, as backed by the theoretical and computational results of this section, enforcing a total-order restriction increases model size without tightening the value of the LP relaxation.

Table 2 **Computational Comparison of Models (5) and (9)**

No. of items–no. of dimensions	No. of scenarios: 3	9	18	27	54	81
10–10	0	1	4	17	174	1.92%
	0	1	4	15	121	0%
15–10	0	8	27	5,778	0.87%	5.66%
	0	7	35	6,239	1.49%	6.45%
20–10	0	48	717	1.06%	3.20%	6.10%
	3	87	872	1.49%	3.02%	6.31%
28–10	1	150	1,293	1.42%	1.93%	
	5	312	3,482	1.82%	2.91%	
39–5	29	0%	1.49%			
	341	2.05%	2.61%			
50–5	34	0.95%	3.41%			
	651	2.24%	4.38%			
60–5	9	0.43%	2.02%	2.70%		
	329	2.21%				
70–5	10	0%	1.17%	3.64%		
	981	0.81%				
80–5	24	0%	2.14%			
	1,481	0.43%				
90–5	17	0%				
	2,831	0.45%				

Notes. The first row gives the number of scenarios, and the first column gives the number of items and dimensions. The percentage values are the optimality gaps after solving the instances for one hour of CPU time. The plain values are the CPU times in seconds to solve the instances to within 0.01% optimality. The upper value in a cell corresponds to model (5) and the lower to model (9).

5. Scenario Prioritization

We indicate in the introduction that one way to view prioritization is to require the activities to be implemented in a nested manner. In this section we further explore this perspective and introduce another formulation for prioritization. In the activity-prioritization model (4), we place activities on a priority list and let each scenario select its optimal set of activities, making sure that the activity-selection decision respects the priority list. Here we consider a complementary formulation, where we place the scenarios on a priority list and let each activity select its optimal set of scenarios, making sure that the scenario-selection decision respects the priority list. The notion of prioritizing a scenario-selection decision is similar to prioritizing activity selection: An activity cannot select a lower-priority scenario unless all higher-priority scenarios are selected, and that activity must select either all or none of the scenarios on each priority level. As we show in this section, the scenario-prioritization model is equivalent to the activity-prioritization model of §2. For computational reasons, it is useful to have both models because the effort required to solve the models differs, depending on the relative number of activities and scenarios.

Let $\mathcal{K}_k \subseteq \Omega$, $k = 1, \ldots, K$, denote the priority levels, where $\mathcal{K}_k \neq \varnothing$, $k = 1, \ldots, K$, $\bigcup_{k=1}^{K} \mathcal{K}_k = \Omega$, and $\mathcal{K}_k \cap \mathcal{K}_{k'} = \varnothing$, $k \neq k'$, $k, k' = 1, \ldots, K$. Then $\mathcal{K} = [\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_K]$ denotes a priority list for scenarios, where scenarios in $\mathcal{K}_1$ have higher priority than those in $\mathcal{K}_2$, etc. Let $G(\mathcal{K})$ be the optimal value of the expected cost of RCASP, subject to the prioritized scenario-selection restriction:

$$
\begin{aligned}
G(\mathcal{K}) \equiv \min_{\mathbf{x}, \mathbf{y}} \quad & \sum_{\omega \in \Omega} q^{\omega} (\mathbf{c}_x^{\omega} \mathbf{x}^{\omega} + \mathbf{c}_y^{\omega} \mathbf{y}^{\omega}) && \text{(15a)}\\
\text{s.t.} \quad & \mathbf{A}^{\omega} \mathbf{x}^{\omega} \leq \mathbf{b}^{\omega}, \quad \omega \in \Omega, && \text{(15b)}\\
& \mathbf{x}^{\omega} \in \{0, 1\}^{|I|}, \quad \omega \in \Omega, && \text{(15c)}\\
& (\mathbf{x}^{\omega}, \mathbf{y}^{\omega}) \in C^{\omega}, \quad \omega \in \Omega, && \text{(15d)}\\
& x_i^{\omega} \geq x_i^{\omega'}, \quad i \in I,\ \omega \in \mathcal{K}_k,\ \omega' \in \mathcal{K}_{k'},\\
& \qquad k < k',\ k, k' = 1, \ldots, K, && \text{(15e)}\\
& x_i^{\omega} = x_i^{\omega'}, \quad i \in I,\ \omega, \omega' \in \mathcal{K}_k,\\
& \qquad k = 1, \ldots, K. && \text{(15f)}
\end{aligned}
$$

We minimize expected cost in (15a). Constraints (15b)–(15d) match those in (1b)–(1d), except they now depend on $\omega \in \Omega$. Constraint (15e) imposes the first condition of prioritized scenario selection: An activity selected under a lower-priority scenario must also be selected under a higher-priority scenario. Constraint (15f) imposes the second condition: An activity is either selected, or not, under both scenarios on the same priority level. Based on $G(\mathcal{K})$ defined by model (15), we define the scenario-prioritization model as follows:

$$
\begin{aligned}
\min_{\mathcal{K}, K} \quad & G(\mathcal{K}) && \text{(16a)}\\
\text{s.t.} \quad & \mathcal{K} = [\mathcal{K}_1, \ldots, \mathcal{K}_K], && \text{(16b)}\\
& \mathcal{K}_k \subseteq \Omega, \quad k = 1, \ldots, K, && \text{(16c)}\\
& \mathcal{K}_k \neq \varnothing, \quad k = 1, \ldots, K, && \text{(16d)}\\
& \mathcal{K}_k \cap \mathcal{K}_{k'} = \varnothing, \quad k \neq k',\ k, k' = 1, \ldots, K, && \text{(16e)}\\
& \bigcup_{k=1,\ldots,K} \mathcal{K}_k = \Omega. && \text{(16f)}
\end{aligned}
$$

Model (16) selects the priority list for scenarios to minimize the expected cost of the RCASP, subject to the prioritized scenario-selection constraints. Similar to the discussion surrounding model (4), the minimization in model (16) is taken over both $\mathcal{K}$ and K, but we suppress K when it is not relevant. We prove that models (4) and (16) are equivalent, i.e., a solution to one can be transformed to a solution to the other with the same objective function value. This result enables us to solve either model, whichever is simpler, and obtain a solution to the other.

Theorem 2. *Consider model* (4) *with* $F^{\omega}(\cdot)$ *defined in* (3) *and model* (16) *with* $G(\cdot)$ *defined in* (15). *Let* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\bar{\mathbf{x}}^{\omega}, \bar{\mathbf{y}}^{\omega})_{\omega \in \Omega}$, *where* $(\bar{\mathbf{x}}^{\omega}, \bar{\mathbf{y}}^{\omega})$ *satisfies* (3b)–(3d). *There exists a priority list* $\bar{\mathcal{K}}$ *for the set of scenarios,* Ω, *such that* $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ *satisfies* (15e)–(15f) *and* $\bar{\mathcal{K}}$ *satisfies* (16b)–(16f) *if and only if there exists a priority list* $\bar{\mathcal{L}}$ *for the set of activities,* I, *such that* $(\bar{\mathcal{L}}, \bar{\mathbf{x}})$ *satisfies* (3e)–(3f), $\omega \in \Omega$ *and* $\bar{\mathcal{L}}$ *satisfies* (4b)–(4f). *Furthermore, constructing* $\bar{\mathcal{L}}$ *from* $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ *and constructing* $\bar{\mathcal{K}}$ *from* $(\bar{\mathcal{L}}, \bar{\mathbf{x}})$ *can be performed in* $O(|I||\Omega|)$ *time.*

Proof. Suppose the hypothesis for $(\bar{\mathcal{K}}, \bar{\mathbf{x}}, \bar{\mathbf{y}})$ holds. It suffices to find a priority list $\bar{\mathcal{L}}$ for the set of activities, satisfying constraints (4b)–(4f), such that $(\bar{\mathcal{L}}, \bar{\mathbf{x}})$ is feasible to constraints (3e)–(3f). Let $\bar{K}$ denote the number of priority levels in $\bar{\mathcal{K}}$. Let $S_k = \{i \in I \mid \bar{x}_i^{\omega} = 1 \text{ for some } \omega \in \bar{\mathcal{K}}_k\}$, $k = 1, \ldots, \bar{K}$, and let $S_0 = I$, $S_{\bar{K}+1} = \varnothing$. By constraints (15e) and (15f), we have $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_{\bar{K}} \supseteq S_{\bar{K}+1}$. Consider Algorithm 1 to construct $\bar{\mathcal{L}}$.

Algorithm 1 (Construct $\bar{\mathcal{L}}$ from $\bar{\mathcal{K}}$)

Input: $\bar{\mathcal{K}} = [\bar{\mathcal{K}}_1, \ldots, \bar{\mathcal{K}}_{\bar{K}}]$, where $\bar{K}$ is the number of priority levels; $\bar{\mathbf{x}} = (\bar{x}_i^{\omega})_{i \in I, \omega \in \Omega}$.
Output: $\bar{\mathcal{L}} = [\bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{\bar{L}}]$, where $\bar{L}$ is the number of priority levels.
$S_k \leftarrow \{i \in I \mid \bar{x}_i^{\omega} = 1 \text{ for some } \omega \in \bar{\mathcal{K}}_k\}$, $k = 1, \ldots, \bar{K}$.
  $S_0 \leftarrow I$. $S_{\bar{K}+1} \leftarrow \varnothing$.
$t \leftarrow \bar{K}$, $j \leftarrow 1$.
repeat
  if $S_t \backslash S_{t+1} \neq \varnothing$ **then**
    $\bar{\mathcal{L}}_j \leftarrow S_t \backslash S_{t+1}$.
    $j \leftarrow j + 1$.
  end if
  $t \leftarrow t - 1$.
until $t = -1$
$\bar{L} \leftarrow j$, and $\bar{\mathcal{L}} \leftarrow [\bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{\bar{L}}]$.

By construction, $\bar{\mathcal{L}}$ satisfies constraints (4b)–(4f). Consider two activities i, i', where $i \in \bar{\mathcal{L}}_l$, $i' \in \bar{\mathcal{L}}_{l'}$. We must show that if $l = l'$, constraint (3f) is satisfied, and otherwise constraint (3e) is satisfied.

- If $l = l'$, by the construction of $\bar{\mathcal{L}}$, $\exists\, t \in \{0, \ldots, \bar{K}\}$ such that $i, i' \in S_k$ for $k = 0, \ldots, t$, and $i, i' \notin S_k$ for $k = t+1, \ldots, \bar{K}+1$. That is, $\bar{x}_i^\omega = \bar{x}_{i'}^\omega = 1$ for $\omega \in \bigcup_{k=1}^{t} \bar{\mathcal{K}}_k$, and $\bar{x}_i^\omega = \bar{x}_{i'}^\omega = 0$ for $\omega \in \bigcup_{k=t+1}^{\bar{K}} \bar{\mathcal{K}}_k$. Hence, $\bar{x}_i^\omega = \bar{x}_{i'}^\omega$, $\forall\, \omega \in \Omega$.
- Suppose, without loss of generality, that $l < l'$. Then by the construction of $\bar{\mathcal{L}}$, $\exists\, t > t' \in \{0, \ldots, \bar{K}\}$ such that $i, i' \in S_k$ for $k = 0, \ldots, t'$, $i \in S_k$, $i' \notin S_k$ for $k = t'+1, \ldots, t$, and $i, i' \notin S_k$ for $k = t+1, \ldots, \bar{K}+1$. That is, $\bar{x}_i^\omega = \bar{x}_{i'}^\omega = 1$ for $\omega \in \bigcup_{k=1}^{t'} \bar{\mathcal{K}}_k$, $\bar{x}_i^\omega = 1$, $\bar{x}_{i'}^\omega = 0$ for $\omega \in \bigcup_{k=t'+1}^{t} \bar{\mathcal{K}}_k$, and $\bar{x}_i^\omega = \bar{x}_{i'}^\omega = 0$ for $\omega \in \bigcup_{k=t+1}^{\bar{K}} \bar{\mathcal{K}}_k$. Hence, $\bar{x}_i^\omega \geq \bar{x}_{i'}^\omega$, $\forall\, \omega \in \Omega$.

Suppose the hypothesis for $(\bar{\mathcal{L}}, \bar{\mathbf{x}}, \bar{\mathbf{y}})$ holds. It suffices to find a priority list $\bar{\mathcal{K}}$ for the set of scenarios, satisfying constraints (16b)–(16f), such that $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ is feasible to constraints (15e)–(15f). Let $\bar{L}$ denote the number of priority levels in $\bar{\mathcal{L}}$. Let $S_l = \{\omega \in \Omega \mid \bar{x}_i^\omega = 1$ for some $i \in \bar{\mathcal{L}}_l\}$, $l = 1, \ldots, \bar{L}$, and $S_0 = \Omega$, $S_{\bar{L}+1} = \varnothing$. By constraints (3e) and (3f), we have $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_{\bar{L}} \supseteq S_{\bar{L}+1}$. Consider Algorithm 2 to construct $\bar{\mathcal{K}}$.

Algorithm 2 (Construct $\bar{\mathcal{K}}$ from $\bar{\mathcal{L}}$)

Input: $\bar{\mathcal{L}} = [\bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{\bar{L}}]$, where $\bar{L}$ is the number of priority levels; $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i \in I,\, \omega \in \Omega}$.
Output: $\bar{\mathcal{K}} = [\bar{\mathcal{K}}_1, \ldots, \bar{\mathcal{K}}_{\bar{K}}]$, where $\bar{K}$ is the number of priority levels.
$S_l \leftarrow \{\omega \in \Omega \mid \bar{x}_i^\omega = 1,$ for some $i \in \bar{\mathcal{L}}_l\}$, $l = 1, \ldots, \bar{L}$.
  $S_0 \leftarrow \Omega$. $S_{\bar{L}+1} \leftarrow \varnothing$.
$t \leftarrow \bar{L}$, $j \leftarrow 1$.
repeat
  if $S_t \backslash S_{t+1} \neq \varnothing$ **then**
    $\bar{\mathcal{K}}_j \leftarrow S_t \backslash S_{t+1}$.
    $j \leftarrow j+1$.
  end if
  $t \leftarrow t-1$.
until $t = -1$
$\bar{K} \leftarrow j$, and $\bar{\mathcal{K}} \leftarrow [\bar{\mathcal{K}}_1, \ldots, \bar{\mathcal{K}}_{\bar{K}}]$.

By construction, $\bar{\mathcal{K}}$ satisfies constraints (16b)–(16f). Consider two scenarios ω, ω', where $\omega \in \bar{\mathcal{K}}_k$, $\omega' \in \bar{\mathcal{K}}_{k'}$. We must show that if $k = k'$, constraint (15f) is satisfied, and otherwise, constraint (15e) is satisfied.

- If $k = k'$, by the construction of $\bar{\mathcal{K}}$, $\exists\, t \in \{0, \ldots, \bar{L}\}$ such that $\omega, \omega' \in S_l$ for $l = 0, \ldots, t$, and $\omega, \omega' \notin S_l$ for $l = t+1, \ldots, \bar{L}+1$. That is, $\bar{x}_i^\omega = \bar{x}_i^{\omega'} = 1$ for $i \in \bigcup_{l=1}^{t} \bar{\mathcal{L}}_l$, and $\bar{x}_i^\omega = \bar{x}_i^{\omega'} = 0$ for $i \in \bigcup_{l=t+1}^{\bar{L}} \bar{\mathcal{L}}_l$. Hence, $\bar{x}_i^\omega = \bar{x}_i^{\omega'}$, $\forall\, i \in I$.
- Suppose, without loss of generality, that $k < k'$. Then by the construction of $\bar{\mathcal{K}}$, $\exists\, t > t' \in \{0, \ldots, \bar{L}\}$ such that $\omega, \omega' \in S_l$ for $l = 0, \ldots, t'$, $\omega \in S_l$, $\omega' \notin S_l$ for $l = t'+1, \ldots, t$, and $\omega, \omega' \notin S_l$ for $l = t+1, \ldots, \bar{L}+1$. That is, $\bar{x}_i^\omega = \bar{x}_i^{\omega'} = 1$ for $i \in \bigcup_{l=1}^{t'} \bar{\mathcal{L}}_l$, $\bar{x}_i^\omega = 1$, $\bar{x}_i^{\omega'} = 0$ for $i \in \bigcup_{l=t'+1}^{t} \bar{\mathcal{L}}_l$, and $\bar{x}_i^\omega = \bar{x}_i^{\omega'} = 0$ for $i \in \bigcup_{l=t+1}^{\bar{L}} \bar{\mathcal{L}}_l$. Hence, $\bar{x}_i^\omega \geq \bar{x}_i^{\omega'}$, $\forall\, i \in I$.

Constructing the sets S_k, $k = 1, \ldots, \bar{K}$, at the beginning of Algorithm 1 takes $O(|I|\bar{K})$ time, as does the rest of the algorithm. Similarly, Algorithm 2 takes $O(|\Omega|\bar{L})$ time. Since $\bar{K} \leq |\Omega|$ and $\bar{L} \leq |I|$, both algorithms can be performed in $O(|I||\Omega|)$ time. □

We develop an IP formulation for model (16), analogous to the development of model (5) for model (4). This model is very similar to model (5), except we now prioritize scenarios instead of activities. For this formulation, it is convenient to have the set of scenarios, Ω, to be an ordered set so that it makes sense to write restrictions such as $\omega < \omega' \in \Omega$. The notation and formulation read similarly; hence, we do not discuss in detail, except the additional decision variable $s_{\omega\omega'}$, which takes the value of 1 if scenario ω has no lower priority than ω' and 0 otherwise:

$$\min_{\mathbf{s},\mathbf{x},\mathbf{y}} \quad \sum_{\omega \in \Omega} q^\omega (\mathbf{c}_x^\omega \mathbf{x}^\omega + \mathbf{c}_y^\omega \mathbf{y}^\omega) \tag{17a}$$

$$\text{s.t.} \quad s_{\omega\omega'} + s_{\omega'\omega} \geq 1, \quad \omega < \omega',\ \omega, \omega' \in \Omega, \tag{17b}$$

$$s_{\omega\omega'} \in \{0, 1\}, \quad \omega \neq \omega',\ \omega, \omega' \in \Omega, \tag{17c}$$

$$x_i^\omega \geq x_i^{\omega'} + s_{\omega\omega'} - 1, \quad i \in I,\ \omega \neq \omega',\ \omega, \omega' \in \Omega, \tag{17d}$$

$$\text{(5e)–(5g)}. \tag{17e}$$

We can develop an IP formulation for the scenario-prioritization models (16) and (17) with a total-order restriction, as we have done for activity-prioritization models (4) and (5) in §4. We do not pursue this path for the reasons we discuss at the end of §4. We instead compare the optimal values of the LP relaxations of models (5) and (17). This is useful because we already have Algorithms 1 and 2 to efficiently transform a solution to one model to a solution to the other model. Hence, we have the freedom of solving the more efficient formulation and then obtaining a solution to the other model, if desired. We first give two lemmas and then compare the LP relaxations of the two models with a theorem.

Lemma 3. *If a solution vector* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}})$ *with* $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i \in I,\, \omega \in \Omega}$, $\bar{\mathbf{s}} = (\bar{s}_{ii'})_{i,\, i' \in I}$ *is feasible to the LP relaxation of model* (5), *then*

$$\alpha_{ii'} + \alpha_{i'i} \leq 1, \quad i \neq i',\ i, i' \in I, \tag{18}$$

holds, where $\alpha_{ii'} = \max_{\omega \in \Omega}\{\bar{x}_i^\omega - \bar{x}_{i'}^\omega\}$. *Conversely, if a solution vector* $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ *with* $0 \leq \bar{x}_i^\omega \leq 1$, $i \in I$, $\omega \in \Omega$, *is feasible to constraints* (5e) *and* (5g), *and inequality* (18) *holds for the* $\bar{\mathbf{x}}$ *vector, there exists a vector* $\bar{\mathbf{s}} = (\bar{s}_{ii'})_{i,\, i' \in I}$ *such that* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}})$ *is feasible to the LP relaxation of model* (5).

Lemma 4. *If a solution vector* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}})$ *with* $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i\in I,\,\omega\in\Omega}$, $\bar{\mathbf{s}} = (\bar{s}_{\omega\omega'})_{\omega,\,\omega'\in\Omega}$ *is feasible to the LP relaxation of model* (17), *then*

$$\alpha_{\omega\omega'} + \alpha_{\omega'\omega} \le 1, \quad \omega \ne \omega',\ \omega, \omega' \in \Omega, \tag{19}$$

holds, where $\alpha_{\omega\omega'} = \max_{i\in I}\{\bar{x}_i^\omega - \bar{x}_i^{\omega'}\}$. *Conversely, if a solution vector* $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ *with* $0 \le \bar{x}_i^\omega \le 1$, $i \in I$, $\omega \in \Omega$, *is feasible to constraints* (5e) *and* (5g), *and inequality* (19) *holds for the* $\bar{\mathbf{x}}$ *vector, there exists a vector* $\bar{\mathbf{s}} = (\bar{s}_{\omega\omega'})_{\omega,\,\omega'\in\Omega}$ *such that* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}})$ *is feasible to the LP relaxation of model* (17).

The first half of both Lemmas 3 and 4 can be proven similarly to the proof of inequality (10). The second halves can be established as in the proof of Theorem 1, albeit with a different definition of the $\bar{\mathbf{s}}$ variables. We note that, given $i \ne i' \in I$, $\omega \ne \omega' \in \Omega$, we have $\max\{\alpha_{ii'}, \alpha_{i'i}\} \ge 0$ and $\max\{\alpha_{\omega\omega'}, \alpha_{\omega'\omega}\} \ge 0$. Assuming, without loss of generality, $\alpha_{ii'} \ge 0$ and $\alpha_{\omega\omega'} \ge 0$, we set $\bar{s}_{ii'} = \alpha_{ii'}$, $\bar{s}_{i'i} = 1 - \alpha_{ii'}$, $\bar{s}_{\omega\omega'} = \alpha_{\omega\omega'}$, $\bar{s}_{\omega'\omega} = 1 - \alpha_{\omega\omega'}$, and proceed as in the proof of Theorem 1.

Theorem 3. *The LP relaxations of models* (5) *and* (17) *have the same optimal value.*

Proof. Suppose $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ is an optimal solution to the LP relaxation of model (5). Then, $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies constraints (5e), (5g), and $0 \le x_i^{*\omega} \le 1$, $i \in I$, $\omega \in \Omega$. By the first part of Lemma 3,

$$\begin{aligned}
&\alpha_{ii'} + \alpha_{i'i} \le 1, \quad i \ne i',\ i, i' \in I \\
&\quad\Leftrightarrow\quad \max_{\omega\in\Omega}\{x_i^{*\omega} - x_{i'}^{*\omega}\} + \max_{\omega\in\Omega}\{x_{i'}^{*\omega} - x_i^{*\omega}\} \le 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad i \ne i',\ i, i' \in I \quad (20a)\\
&\quad\Leftrightarrow\quad x_i^{*\omega} - x_{i'}^{*\omega} + x_{i'}^{*\omega'} - x_i^{*\omega'} \le 1, \\
&\qquad\qquad\qquad\qquad\qquad \omega, \omega' \in \Omega,\ i \ne i',\ i, i' \in I \quad (20b)\\
&\quad\Leftrightarrow\quad x_i^{*\omega} - x_{i'}^{*\omega} + x_{i'}^{*\omega'} - x_i^{*\omega'} \le 1, \\
&\qquad\qquad\qquad\qquad\qquad \omega \ne \omega',\ \omega, \omega' \in \Omega,\ i, i' \in I \quad (20c)\\
&\quad\Leftrightarrow\quad x_i^{*\omega} - x_i^{*\omega'} + x_{i'}^{*\omega'} - x_{i'}^{*\omega} \le 1, \\
&\qquad\qquad\qquad\qquad\qquad \omega \ne \omega',\ \omega, \omega' \in \Omega,\ i, i' \in I \quad (20d)\\
&\quad\Leftrightarrow\quad \max_{i\in I}\{x_i^{*\omega} - x_i^{*\omega'}\} + \max_{i\in I}\{x_i^{*\omega'} - x_i^{*\omega}\} \le 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \omega \ne \omega',\ \omega, \omega' \in \Omega \quad (20e)\\
&\quad\Leftrightarrow\quad \alpha_{\omega\omega'} + \alpha_{\omega'\omega} \le 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \omega \ne \omega',\ \omega, \omega' \in \Omega. \quad (20f)
\end{aligned}$$

Equivalence (20a) is the definition of $\alpha_{ii'}$, and (20b) follows from the definition of the max operator. In (20c) we exclude or include the scenario indices such that $\omega = \omega'$ and the activity indices such that $i = i'$. This is done with equivalence because, for $i = i'$ or $\omega = \omega'$, the left-hand side of the inequality is zero. We rearrange terms in (20d), again use the max operator in (20e), and finally use the definition of $\alpha_{\omega\omega'}$ in (20f). By the second part of Lemma 4, we can find a solution vector $\hat{\mathbf{s}}^*$ such that $(\mathbf{x}^*, \mathbf{y}^*, \hat{\mathbf{s}}^*)$ is feasible to model (17). Hence, the optimal value of the LP relaxation of model (17) is no worse than that of model (5). We can carry out the same argument in the opposite direction, proving the theorem. □

The constraint sets that link prioritization and activity selection in models (5) and (17) are (5d) and (17d), which have cardinality $|I|(|I|-1)|\Omega|$ and $|\Omega|(|\Omega|-1)|I|$, respectively. The number of respective $\mathbf{s}$ variables in these two models is $|I|(|I|-1)$ and $|\Omega|(|\Omega|-1)$. Thus, for $|I| < |\Omega|$ model (5) is smaller than model (17), and vice versa. Having also proven that optimal values of their LP relaxations are equal, we expect the smaller model to be more efficient. To see if this holds in practice, we test both models on the k-median problem (2) instances from Table 1 and on the multidimensional knapsack problem (14) instances from Table 2. For both problem instances, we use an identical experimental setting as in Table 2. We report the results for the multidimensional knapsack problem instances in the left half of Table 3. The results are interpreted as in Table 2 except now the upper and the lower values in a cell correspond to models (5) and (17), respectively. We indeed observe the expected behavior. For $|I| < |\Omega|$, model (5) performs better, and for $|\Omega| < |I|$, model (17) performs better. In Table 3, to illustrate the relative performance of models (5) and (17), we show running times longer than one hour for some larger problem instances for which we only show one-hour optimality gaps in Table 2. We report the results of the k-median problem instances in the left half of Table 4. We see that the same observations regarding the relative benefits of both models hold in the k-median problem instances as well.

The activity-prioritization model (4) and the scenario-prioritization model (16) differ in that, if $(\mathscr{L}, L)$ is fixed in the former, evaluating the objective function involves computations that separate by scenario. However, fixing $(\mathscr{K}, K)$ in the latter does not lead to separable computations by scenario. This observation has implications for the potential application of decomposition algorithms. For the two-stage stochastic integer program (5) for prioritizing activities, a resource directed decomposition algorithm (e.g., Carøe and Tind 1998, Laporte and Louveaux 1993, Sen and Higle 2005) or a branch-and-price algorithm (e.g., Lulli and Sen 2004) could exploit this structure, leading to scenario subproblems. Although more subtle, the scenario-prioritization model (17) is also amenable to decomposition, if the underlying deterministic problem involves loosely coupled activities. In this case, it is possible to employ "geographical" decomposition algorithms, yielding activity subproblems (Koç and Kalagnanam 2012, Shiina and Birge 2004, Silva and Wood 2006).

6. Prioritization Cuts

To improve solution times further, we develop two sets of cutting planes for models (5) and (17), which we call *prioritization cuts*. Let $Y^\omega = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \{0,1\}^{|I|}, \mathbf{A}^\omega \mathbf{x} \le \mathbf{b}^\omega, (\mathbf{x}, \mathbf{y}) \in C^\omega\}$, and $X^\omega = \{\mathbf{x} \mid \exists \mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) \in Y^\omega\}$. For $\mathbf{x} \in X^\omega$, let

$$h^\omega(\mathbf{x}) \equiv \mathbf{c}_x^\omega \mathbf{x} + \min_{\{\mathbf{y} \mid (\mathbf{x},\mathbf{y}) \in C^\omega\}} \mathbf{c}_y^\omega \mathbf{y}. \tag{21}$$

Model (21) is feasible by construction since $\mathbf{x} \in X^\omega$ and thus has a minimizer, which we denote $\mathbf{y}^\omega(\mathbf{x})$. Given a solution, $\mathcal{K}$, to model (16) and a corresponding optimal solution, $(\mathbf{x}^\omega, \mathbf{y}^\omega)_{\omega \in \Omega}$, to model (15), we have $G(\mathcal{K}) = \sum_{\omega \in \Omega} q^\omega h^\omega(\mathbf{x}^\omega)$, where $G(\cdot)$ is defined in model (16). Finally, for $\mathbf{x} \in \{0,1\}^{|I|}$, we let $S(\mathbf{x})$ denote the set of selected activities, i.e., $S(\mathbf{x}) = \{i \in I \mid x_i = 1\}$.

We say scenario ω is *improving* if, for any two vectors $\mathbf{x}, \bar{\mathbf{x}} \in X^\omega$ with $S(\mathbf{x}) \supseteq S(\bar{\mathbf{x}})$, we have $h^\omega(\mathbf{x}) \le h^\omega(\bar{\mathbf{x}})$. We let $\Omega_V \subseteq \Omega$ denote the set of improving scenarios. Given two scenarios $\omega, \omega' \in \Omega$, we say ω *dominates* ω' if $\mathbf{x} \in X^{\omega'}$ implies $\mathbf{x} \in X^\omega$. We let Ω_D denote the set of dominating scenario pairs, i.e., $\Omega_D = \{(\omega, \omega') \in \Omega \times \Omega \mid \omega \text{ dominates } \omega'\}$.

For a vector $\mathbf{x} \in \{0,1\}^{|I|}$, we let $\mathbf{x}_{(i,i')}$ denote a new vector $\bar{\mathbf{x}} \in \{0,1\}^{|I|}$, which is the same as $\mathbf{x}$, except that its ith element is set to 1 and its i'th element is set to 0, i.e., $\bar{x}_j = x_j$ for $j \notin \{i, i'\}$, $\bar{x}_i = 1$, $\bar{x}_{i'} = 0$. We say activity i *dominates* activity i' if, for any vector $\mathbf{x} \in X^\omega$ with $x_i = 0$, $x_{i'} = 1$, we have $\mathbf{x}_{(i,i')} \in X^\omega$ and $h^\omega(\mathbf{x}_{(i,i')}) \le h^\omega(\mathbf{x})$, for all $\omega \in \Omega$. We let I_D denote the set of dominating activity pairs, i.e., $I_D = \{(i, i') \in I \times I \mid i \text{ dominates } i'\}$.

We illustrate these ideas using the facility-prioritization problem (6) and the prioritized multidimensional knapsack problem (14). In the latter problem, given two scenarios, ω dominates ω' if the resource-consumption coefficients under ω are at most those under ω' and if the budgets under ω are at least those under ω', i.e., $c_{it}^\omega \le c_{it}^{\omega'}$, $i \in I$, $t \in T$, and $b_t^\omega \ge b_t^{\omega'}$, $t \in T$. Scenario ω is improving if the profits under ω are nonnegative, i.e., $a_i^\omega \ge 0$, $i \in I$. In the facility-prioritization problem, scenarios with larger budgets dominate scenarios with smaller budgets. That is, given two scenarios, ω, ω', scenario ω dominates scenario ω' if $k^\omega \ge k^{\omega'}$. All scenarios in the facility-prioritization problem are improving because opening more facilities cannot increase the total distance the customers travel. In the prioritized multidimensional knapsack problem, given two activities, i, i', if the profit of activity i is at least that of i' under all scenarios, and the resource consumption of activity i is no more than that of i' under all scenarios, then i dominates i', i.e., $a_i^\omega \ge a_{i'}^\omega$, and $c_{it}^\omega \le c_{i't}^\omega$, $\omega \in \Omega$. In the instances of the facility-prioritization problem that we describe in §3, we do not have any dominating activity pairs. We give two propositions based on this notion.

Proposition 2. *There exists an optimal solution, $\bar{\mathcal{K}}$, to model* (16) *and a corresponding optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, to model* (15) *with $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i \in I, \omega \in \Omega}$ such that $\bar{\mathbf{x}}$ satisfies the set of inequalities*

$$\bar{x}_i^{\omega'} \ge \bar{x}_i^{\omega''}, \quad i \in I, \tag{22}$$

for all $(\omega', \omega'') \in \Omega_D$ such that $\omega' \in \Omega_V$.

Proof. Let $\mathcal{K}^*$ be an optimal solution to model (16), and let $(\mathbf{x}^*, \mathbf{y}^*) = (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega})_{\omega \in \Omega}$ be the corresponding optimal solution to model (15). Given two scenarios, $\omega', \omega'' \in \Omega$, having $S(\mathbf{x}^{*\omega'}) \supseteq S(\mathbf{x}^{*\omega''})$ implies having $x_i^{*\omega'} \ge x_i^{*\omega''}$, $i \in I$. Hence, if for all $(\omega', \omega'') \in \Omega_D$ with $\omega' \in \Omega_V$ we have $S(\mathbf{x}^{*\omega'}) \supseteq S(\mathbf{x}^{*\omega''})$, the proof is complete. Suppose for a pair $(\omega', \omega'') \in \Omega_D$ with $\omega' \in \Omega_V$ we have $S(\mathbf{x}^{*\omega'}) \subset S(\mathbf{x}^{*\omega''})$. Then, we construct a new solution, $\bar{\mathcal{K}}$, to model (16), and a corresponding solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega)_{\omega \in \Omega}$, to model (15) such that

(i) $S(\bar{\mathbf{x}}^\omega) = S(\mathbf{x}^{*\omega})$, $\omega \in \Omega \backslash \{\omega'\}$, and $S(\bar{\mathbf{x}}^{\omega'}) = S(\mathbf{x}^{*\omega''})$;

(ii) $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies constraints (15b)–(15d), $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ satisfies constraints (15e)–(15f), and $\bar{\mathcal{K}}$ satisfies constraints (16b)–(16f); and

(iii) $\bar{\mathcal{K}}$ has no worse objective function value than $\mathcal{K}^*$, i.e., $G(\bar{\mathcal{K}}) \le G(\mathcal{K}^*)$.

Repeating this argument for $(\omega', \omega'') \in \Omega_D$ with $\omega' \in \Omega_V$ such that $S(x^{*\omega'}) \subset S(x^{*\omega''})$ proves the proposition.

Consider a new solution $(\bar{\mathcal{K}}, \bar{\mathbf{x}}, \bar{\mathbf{y}})$ with $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega)_{\omega \in \Omega}$ formed as follows. Set $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) = (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega})$ for $\omega \in \Omega \backslash \{\omega'\}$, and $(\bar{\mathbf{x}}^{\omega'}, \bar{\mathbf{y}}^{\omega'}) = (\mathbf{x}^{*\omega''}, \mathbf{y}^{\omega'}(\mathbf{x}^{*\omega''}))$.

(i) This holds by construction.

(ii) Proving that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies constraints (15b)–(15d) is the same as showing that $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) \in Y^\omega$, which holds for scenarios $\omega \in \Omega \backslash \{\omega'\}$ because for these scenarios we have $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) = (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega})$. For scenario ω', $(\bar{\mathbf{x}}^{\omega'}, \bar{\mathbf{y}}^{\omega'}) \in Y^{\omega'}$, where $(\bar{\mathbf{x}}^{\omega'}, \bar{\mathbf{y}}^{\omega'}) = (\mathbf{x}^{*\omega''}, \mathbf{y}^{\omega'}(\mathbf{x}^{*\omega''}))$, by the following observations:

- $\mathbf{x}^{*\omega''} \in X^{\omega''}$ and scenario ω' dominates scenario ω''. Thus, $\mathbf{x}^{*\omega''} \in X^{\omega'}$.
- $\mathbf{y}^{\omega'}(\mathbf{x}^{*\omega''})$ is the optimizer of model (21) for scenario ω' and for $\mathbf{x} = \mathbf{x}^{*\omega''}$, and thus $(\mathbf{x}^{*\omega''}, \mathbf{y}^{\omega'}(\mathbf{x}^{*\omega''})) \in Y^{\omega'}$.

By constraints (15e)–(15f), there exists an ordering, $\omega_1, \ldots, \omega_{|\Omega|}$, so that $S(\mathbf{x}^{*\omega_1}) \supseteq S(\mathbf{x}^{*\omega_2}) \supseteq \cdots \supseteq S(\mathbf{x}^{*\omega_{|\Omega|}})$. By item (i) above, we have this nested structure for $S(\bar{\mathbf{x}}^\omega)_{\omega \in \Omega}$ as well, possibly after reordering the scenarios. Algorithm 3 constructs $\bar{\mathcal{K}}$ so that $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ satisfies constraints (15e)–(15f) and $\bar{\mathcal{K}}$ satisfies constraints (16b)–(16f).

Algorithm 3 (Construct $\mathcal{K}$ from the nested sets, $S(\mathbf{x}^{\omega_1}) \supseteq \cdots \supseteq S(\mathbf{x}^{\omega_{|\Omega|}})$)

Input: $S(\mathbf{x}^{\omega_1}) \supseteq \cdots \supseteq S(\mathbf{x}^{\omega_{|\Omega|}})$.
Output: $\mathcal{K} = [\mathcal{K}_1, \ldots, \mathcal{K}_K]$, where K is the number of priority levels.
$t \leftarrow 1$, $j \leftarrow 1$.
repeat
  $\mathcal{K}_j \leftarrow \mathcal{K}_j \cup \{\omega_t\}$.
  if $S(\mathbf{x}^{\omega_t}) \backslash S(\mathbf{x}^{\omega_{t+1}}) \neq \varnothing$ **then**
    $j \leftarrow j+1$.
  end if
  $t \leftarrow t+1$.
until $t = |\Omega|$
$\mathcal{K}_j \leftarrow \mathcal{K}_j \cup \{\omega_{|\Omega|}\}$. $K \leftarrow j$. $\mathcal{K} \leftarrow [\mathcal{K}_1, \ldots, \mathcal{K}_K]$.

(iii) $G(\mathcal{K}^*) = \sum_{\omega\in\Omega} q^\omega h^\omega(\mathbf{x}^{*\omega})$, and $G(\bar{\mathcal{K}}) = \sum_{\omega\in\Omega} q^\omega h^\omega(\bar{\mathbf{x}}^{\omega})$. Since $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) = (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega})$, $\omega \in \Omega\backslash\{\omega'\}$, we have $G(\bar{\mathcal{K}}) - G(\mathcal{K}^*) = q^{\omega'}[h^{\omega'}(\bar{\mathbf{x}}^{\omega'}) - h^{\omega'}(\mathbf{x}^{*\omega'})] = q^{\omega'}[h^{\omega'}(\mathbf{x}^{*\omega''}) - h^{\omega'}(\mathbf{x}^{*\omega'})]$. Hence, it suffices to show that $h^{\omega'}(\mathbf{x}^{*\omega''}) \leq h^{\omega'}(\mathbf{x}^{*\omega'})$. Since $\mathbf{x}^{*\omega''} \in X^{\omega''}$ and ω' dominates ω'', we have that $\mathbf{x}^{*\omega''} \in X^{\omega'}$. We also have $S(\mathbf{x}^{*\omega''}) \supset S(\mathbf{x}^{*\omega'})$. Finally, since ω' is improving, we must have $h^{\omega'}(\mathbf{x}^{*\omega''}) \leq h^{\omega'}(\mathbf{x}^{*\omega'})$. □

Proposition 3. *There exists an optimal solution, $\bar{\mathcal{K}}$, to model* (16) *and a corresponding optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, to model* (15) *with $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i\in I,\,\omega\in\Omega}$ such that, for all $(i, i') \in I_D$, $\bar{\mathbf{x}}$ satisfies the set of inequalities*

$$\bar{x}_i^\omega \geq \bar{x}_{i'}^\omega, \quad \omega \in \Omega. \tag{23}$$

Proof. Let $\mathcal{K}^*$ be an optimal solution to model (16), and let $(\mathbf{x}^*, \mathbf{y}^*) = (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega})_{\omega\in\Omega}$ be the corresponding optimal solution to model (15). Suppose there exists a pair $(i, i') \in I_D$ for which $\mathbf{x}^*$ does not satisfy the set of inequalities in (23). Then, we construct a new solution, $\bar{\mathcal{K}}$, to model (16), and a corresponding solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega)_{\omega\in\Omega}$, to model (15) such that

(i) $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies constraints (15b)–(15d), $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ satisfies constraints (15e)–(15f), and $\bar{\mathcal{K}}$ satisfies constraints (16b)–(16f);

(ii) $\bar{\mathcal{K}}$ has no worse objective function value than $\mathcal{K}^*$, i.e., $G(\bar{\mathcal{K}}) \leq G(\mathcal{K}^*)$; and

(iii) $\bar{\mathbf{x}}$ satisfies the set of inequalities in (23) for pair (i, i').

Repeating this argument for all pairs $(i, i') \in I_D$ for which $\mathbf{x}^*$ does not satisfy the set of inequalities in (23) proves the proposition.

By constraints (15e)–(15f), there exists an ordering, $\omega_1, \ldots, \omega_{|\Omega|}$, on the set of scenarios such that $S(\mathbf{x}^{*\omega_1}) \supseteq S(\mathbf{x}^{*\omega_2}) \supseteq \cdots \supseteq S(\mathbf{x}^{*\omega_{|\Omega|}})$. Because the set of inequalities in (23) are not satisfied for pair (i, i'), there must exist a nonempty subset of scenarios under which activity i is not selected, but activity i' is selected. Because of the nested structure of the sets $S(\mathbf{x}^{*\omega_j})$, $j = 1, \ldots, |\Omega|$, this nonconforming subset of scenarios must constitute a subset of the ordering $\omega_1, \omega_2, \ldots, \omega_{|\Omega|}$, without an intermediary conforming scenario that satisfies the set of inequalities in (23). In other words, there must exist t, t' with $0 \leq t < t' \leq |\Omega|$ such that $i, i' \in S(\mathbf{x}^{*\omega_j})$ for $j = 1, \ldots, t$, $i \notin S(\mathbf{x}^{*\omega_j})$, $i' \in S(\mathbf{x}^{*\omega_j})$ for $j = t+1, \ldots, t'$, and $i, i' \notin S(\mathbf{x}^{*\omega_j})$ for $j = t'+1, \ldots, |\Omega|$. Consider a new solution $(\bar{\mathcal{K}}, \bar{\mathbf{x}}, \bar{\mathbf{y}})$ with $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega)_{\omega\in\Omega}$ constructed as follows:

$$(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) = \begin{cases} (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega}) & \text{for } \omega \in \Omega\backslash\{\omega_{t+1}, \ldots, \omega_{t'}\}, \\ (\mathbf{x}^{*\omega}_{(i,i')}, \mathbf{y}^\omega(\mathbf{x}^{*\omega}_{(i,i')})) & \text{for } \omega \in \{\omega_{t+1}, \ldots, \omega_{t'}\}. \end{cases}$$

(i) Proving that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies constraints (15b)–(15d) is the same as showing that $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) \in Y^\omega$, which holds for scenarios $\omega \in \Omega\backslash\{\omega_{t+1}, \ldots, \omega_{t'}\}$ obviously and for scenarios $\omega \in \{\omega_{t+1}, \ldots, \omega_{t'}\}$, where $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) = (\mathbf{x}^{*\omega}_{(i,i')}, \mathbf{y}^\omega(\mathbf{x}^{*\omega}_{(i,i')}))$, by the following observations:

- $\mathbf{x}^{*\omega} \in X^\omega$, $x_i^{*\omega} = 0$, $x_{i'}^{*\omega} = 1$, and i dominates i'. Thus, $\mathbf{x}^{*\omega}_{(i,i')} \in X^\omega$.
- $\mathbf{y}^\omega(\mathbf{x}^{*\omega}_{(i,i')})$ is the optimizer of model (21) for scenario ω and for $\mathbf{x} = \mathbf{x}^{*\omega}_{(i,i')}$, and thus $(\mathbf{x}^{*\omega}_{(i,i')}, \mathbf{y}^\omega(\mathbf{x}^{*\omega}_{(i,i')})) \in Y^\omega$.

The new solution, $(\bar{\mathbf{x}}^\omega)_{\omega\in\Omega}$, differs from the old one, $(\mathbf{x}^{*\omega})_{\omega\in\Omega}$, only in scenarios $\omega \in \{\omega_{t+1}, \ldots, \omega_{t'}\}$. In these scenarios, we set $\bar{x}_i^\omega = 1$ and $\bar{x}_{i'}^\omega = 0$. This preserves the nestedness of the sets $S(\bar{\mathbf{x}}^\omega)_{\omega\in\Omega}$. In fact, this preserves the ordering of the scenarios as well. Algorithm 3 constructs $\bar{\mathcal{K}}$ so that $(\bar{\mathcal{K}}, \bar{\mathbf{x}})$ satisfies constraints (15e)–(15f), and $\bar{\mathcal{K}}$ satisfies constraints (16b)–(16f).

(ii) $G(\mathcal{K}^*) = \sum_{\omega\in\Omega} q^\omega h^\omega(\mathbf{x}^{*\omega})$, and $G(\bar{\mathcal{K}}) = \sum_{\omega\in\Omega} q^\omega \cdot h^\omega(\bar{\mathbf{x}}^\omega)$. Since $(\bar{\mathbf{x}}^\omega, \bar{\mathbf{y}}^\omega) = (\mathbf{x}^{*\omega}, \mathbf{y}^{*\omega})$, $\omega \in \Omega\backslash\{\omega_{t+1}, \ldots, \omega_{t'}\}$, we have $G(\bar{\mathcal{K}}) - G(\mathcal{K}^*) = \sum_{\omega\in\{\omega_{t+1},\ldots,\omega_{t'}\}} q^\omega[h^\omega(\bar{\mathbf{x}}^\omega) - h^\omega(\mathbf{x}^{*\omega})] = \sum_{\omega\in\{\omega_{t+1},\ldots,\omega_{t'}\}} q^\omega[h^\omega(\mathbf{x}^{*\omega}_{(i,i')}) - h^\omega(\mathbf{x}^{*\omega})]$. Hence, it suffices to show that $h^\omega(\mathbf{x}^{*\omega}_{(i,i')}) \leq h^\omega(\mathbf{x}^{*\omega})$ for $\omega \in \{\omega_{t+1}, \ldots, \omega_{t'}\}$. Since $\mathbf{x}^{*\omega} \in X^\omega$, $x_{i'}^{*\omega} = 1$, $x_i^{*\omega} = 0$, and i dominates i', we must have $h^\omega(\mathbf{x}^{*\omega}_{(i,i')}) \leq h^\omega(\mathbf{x}^{*\omega})$.

(iii) By construction, we have $i, i' \in S(\bar{\mathbf{x}}^{\omega_j})$ for $j = 1, \ldots, t$, $i \in S(\bar{\mathbf{x}}^{\omega_j})$, $i' \notin S(\bar{\mathbf{x}}^{\omega_j})$ for $j = t+1, \ldots, t'$, and $i, i' \notin S(\bar{\mathbf{x}}^{\omega_j})$ for $j = t'+1, \ldots, |\Omega|$. Hence, $x_i^\omega \geq x_{i'}^\omega$, $\omega \in \Omega$. □

An immediate corollary to both propositions and Theorem 2 is that the inequalities in (22) and (23) hold also for models (3) and (4).

Corollary 2. *There exists an optimal solution, $\bar{\mathcal{L}}$, to model* (4) *and a corresponding optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, to model* (3) *with $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i\in I,\,\omega\in\Omega}$ such that $\bar{\mathbf{x}}$ satisfies the set of inequalities in* (22) *for all $(\omega', \omega'') \in \Omega_D$ such that $\omega' \in \Omega_V$. Furthermore, there exists an optimal solution, $\bar{\mathcal{L}}$, to model* (4) *and a corresponding optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, to model* (3) *with $\bar{\mathbf{x}} = (\bar{x}_i^\omega)_{i\in I,\,\omega\in\Omega}$ such that $\bar{\mathbf{x}}$ satisfies the set of inequalities in* (23) *for all $(i, i') \in I_D$.*

We do not use the term *valid inequality* for prioritization cuts (22) and (23) because they may rule out

Table 3 **Computational Comparison of Activity Prioritization, Scenario Prioritization, and the Prioritization Cuts on Multidimensional Knapsack Problem Instances**

No. of items–no. of dimensions	Without prioritization cuts (no. of scenarios)						With prioritization cuts (no. of scenarios)					
	3	9	18	27	54	81	3	9	18	27	54	81
10-10	0	1	4	17	174	1.92%	0	1	2	3	13	67
	0	1	22	638	181k	12.17%	0	1	2	6	32	772
15-10	0	8	27	5,778	72k	5.66%	0	2	3	162	227	2.60%
	0	6	107	185k	246k		0	1	3	352	901	3.42%
20-10	0	48	717	8,558	3.20%	6.10%	0	6	7	2,123	0.51%	2.43%
	0	15	1,015	30k	6.25%		0	2	7	603	1.67%	2.82%
28-10	1	150	1,293	1.42%	1.93%		1	12	580	0.77%	0.71%	1.97%
	0	19	860	1.51%	2.81%		1	4	33	5,238	1.12%	2.03%
39-5	29	14k	1.49%				2	63	1,800	2.02%	1.80%	2.18%
	1	220	0.75%				1	8	97	1.08%	2.12%	2.35%
50-5	34	203k	3.41%				5	123	0.62%	2.11%	2.22%	2.30%
	1	3,895	0.74%				4	17	336	1.22%	2.13%	2.21%
60-5	9	11k	2.02%	2.70%			10	591	0.92%	1.80%	1.93%	2.26%
	0	63	1.22%	3.44%			7	26	972	1.42%	1.91%	1.94%
70-5	10	81k	1.17%	3.64%			17	4,233	0.47%	0.68%	1.09%	1.02%
	0	203	0.50%	1.12%			10	66	360	0.62%	1.12%	0.99%
80-5	24	23k	2.14%				29	4,127	0.62%	0.82%	1.11%	1.03%
	0	46	0.47%	6.34%			20	76	1,358	0.72%	1.07%	1.08%
90-5	17	2,578					35	3,218	0.33%	0.44%	0.92%	0.56%
	0	23	0.10%	0.57%			31	122	265	5,752	0.80%	0.62%

Notes. This table consists of two parts. The right and left halves give results with and without employing prioritization cuts, respectively. Values in both halves are interpreted as in Table 2, except that upper and lower values in a cell now correspond to models (5) and (17), respectively.

feasible, and even some optimal, solutions. What we guarantee is that there remains at least one optimal solution that is feasible. Israeli and Wood (2002) refer to such inequalities as super-valid inequalities.

We can extract the prioritization cuts (22) and (23) from the problem data before we solve the problem, and they are valid for all nodes of the branch-and-bound tree. To avoid increasing the problem size unnecessarily, we can place the prioritization cuts in a pool and iteratively add only those that are violated by the LP-relaxation solution. This is the approach we use for cuts (22) in model (5) and cuts (23) in model (17). There is a simpler procedure to use cuts (22) in model (17) and cuts (23) in model (5). With cuts of the form $x_i^{\omega'} \geq x_i^{\omega''}$, $i \in I$, we can fix variable $s_{\omega'\omega''} = 1$ in model (17) so that constraint (17d) for pair (ω', ω'') reads $x_i^{\omega'} \geq x_i^{\omega''}$, $i \in I$. Similarly, with cuts of the form $x_i^{\omega} \geq x_{i'}^{\omega}$, $\omega \in \Omega$, we can fix variable $s_{ii'} = 1$ in model (5) so that constraint (5d) for pair (i, i') reads $x_i^{\omega} \geq x_{i'}^{\omega}$, $\omega \in \Omega$.

The right halves of Tables 3 and 4 show the effect of the prioritization cuts (22) and (23). These right halves of the tables are interpreted as the left halves and can be compared cell by cell. Using prioritization cuts in small problem instances can increase solution times as we see in some of the results for the three-scenario instances in Table 3. As the number of scenarios grows, the benefit of the prioritization cuts is amplified, yielding ratios of CPU times up to 5,600 in model (17) and up to 1,600 in model (5). We also observe improvements in instances in which we only compute for one hour. We can solve some of the instances to optimality in an hour that we could not solve in this time without using the prioritization cuts. Similarly, we can obtain one-hour optimality gaps for several instances that we could not without the prioritization cuts. We observe similar improvements in Table 4 as well.

7. Conclusions

Resource-constrained activity-selection problems have applications in many real-life decision-making processes, such as capital budgeting and facility location problems. The current literature frequently approaches these problems by forming an optimal portfolio of activities and tends to discredit ranking schemes because they ignore structural and stochastic dependencies among the activities. Practitioners in industry and government, on the other hand, often form a priority list of activities and discredit the optimal-portfolio approach because, if the problem parameters change, the set of activities that was once optimal no longer remains optimal. Even worse, the new optimal set of activities may exclude some of the previously optimal activities, which they may have already been selected. Considering both viewpoints, we have proposed a new approach that prioritizes the activities recognizing structural and stochastic

Table 4 **Computational Comparison of Activity Prioritization, Scenario Prioritization, and the Prioritization Cuts on *k*-Median Problem Instances**

	Without prioritization cuts (no. of scenarios/k^{max})							With prioritization cuts (no. of scenarios/k^{max})						
n_G	2	3	4	5	6	7	8	2	3	4	5	6	7	8
3	0.1	0.1	0.2	0.2	0.3	0.4	0.6	0	0.1	0.2	0.2	0.3	0.1	0.3
	0	0	0.2	0.2	0.3	0.3	0.4	0	0	0.1	0.1	0.1	0.3	0.1
5	0.4	3	8	16	31	61	391	0.3	1	2	3	6	15	28
	0.1	0.7	2	3	4	8	22	0.1	0.3	0.6	0.7	1	3	3
7	3	22	61	133	1,093	726	1.9%	1	7	17	28	57	109	312
	0.6	3	12	14	29	63	88	0.5	2	4	5	11	16	30
9	15	141	752	0.97%	1.03%	2.74%	3.45%	9	44	129	208	611	1,935	0.64%
	3	30	42	104	208	2,234	2,779	0.4	7	23	37	12	164	246

Note. This table is interpreted as Table 3, except that the underlying RCASP model is (2) with instances from Table 1.

dependencies among them. The approach ranks activities considering both the uncertainty in the problem parameters and the optimal portfolio that will be obtained once the uncertainty is revealed.

We have provided two primary mathematical models to prioritize resource-constrained activity-selection problems. One model takes an activity-prioritization perspective and a second model takes a scenario-prioritization perspective. We have showed that each of these two models can be more computationally efficient than the other depending on the problem parameters. We have developed a third model that forms a fully ordered priority list, but we have argued that we prefer our two primary formulations for reasons involving both modeling and computation. We have also developed two sets of cutting planes for both of our main formulations and showed their computational value.

We have illustrated our approach using a facility location model and a multidimensional knapsack problem, but our approach applies to a wide class of resource-constrained activity-selection problems, such as infrastructure development, capacity expansion, and capital improvement problems. More specifically, the decision-making process in building a new infrastructure under uncertainty can utilize our approach in distinguishing high-priority projects from low-priority ones. For example, the problem of improving recreational areas in a territory could use this approach because total funding ultimately received from a local government is not known in advance. A retail company could also use such an approach in constructing or expanding its chain of stores in a territory because the number of stores ultimately opened cannot be predicted ahead of the time.

Our exposition has largely focused on RCASP models with linear objective functions of the form $\mathbf{c}_x\mathbf{x} + \mathbf{c}_y\mathbf{y}$ and with linear resource constraints of the form $\mathbf{Ax} \le \mathbf{b}$. Model (7) and the discussion surrounding Figure 3 is an exception, however, that does not have a linear objective function. Fortunately, the paper's results, prior to §6 on prioritization cuts, do not hinge on linearity of the objective function or resource constraints and extend naturally, e.g., to a nonlinear objective function of the form $\mathbf{f}(\mathbf{x}, \mathbf{y})$. The results in §6 also extend to handle nonlinearities but require some notational changes. For example, nonlinear constraints can be embedded in $(\mathbf{x}, \mathbf{y}) \in C$, and we would modify the definition of Y^ω to $\{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \{0, 1\}^{|I|}, (\mathbf{x}, \mathbf{y}) \in C^\omega\}$. Similarly, under a nonlinear objective function, we would redefine $h^\omega(\mathbf{x})$ for $\mathbf{x} \in X^\omega$ as $h^\omega(\mathbf{x}) \equiv \min_{\{\mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in C^\omega\}} \mathbf{f}^\omega(\mathbf{x}, \mathbf{y})$.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2013.1865.

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