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Compact Bid Languages and Core Pricing in Large Multi-item Auctions

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We introduce an auction design framework for large markets with hundreds of items and complex bidder preferences. Such markets typically lead to computationally hard allocation problems. Our new framework consists of compact bid languages for sealed-bid auctions and methods to compute second-price rules such as the Vickrey–Clarke–Groves or bidder-optimal, core-selecting payment rules when the optimality of the allocation problem cannot be guaranteed. To demonstrate the efficacy of the approach for a specific, complex market, we introduce a compact bidding language for TV advertising markets and investigate the resulting winner-determination problem and the computation of core payments. For realistic instances of the respective winner-determination problems, very good solutions with a small integrality gap can be found quickly, although closing the integrality gap to find marginally better solutions or prove optimality can take a prohibitively large amount of time. Our subsequent adaptation of a constraint-generation technique for the computation of bidder-optimal core payments to this environment is a practically viable paradigm by which core-selecting auction designs can be applied to large markets with potentially hundreds of items. Such auction designs allow bidders to express their preferences with a low number of parameters, while at the same time providing incentives for truthful bidding. We complement our computational experiments in the context of TV advertising markets with additional results for volume discount auctions in procurement to illustrate the applicability of the approach in different types of large markets.

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1. Introduction

Electronic markets allow market participants to express rich information about their preferences for different goods or services beyond a price quote for individual items only. It is easy for an auctioneer to elicit complementarities, synergies, or volume discounts for large volumes of items. More comprehensive information about cost structures or utility functions of market participants can increase allocative efficiency and lead to higher economic welfare.

In recent years, a growing body of literature in the management sciences is devoted to the design of such smart markets (Gallien and Wein 2005), with combinatorial auctions (CAs) emerging as a pivotal example (Cramton et al. 2006). A CA allows bidders to bid on combinations of items, offering protection against the well-known "exposure problem" present in simultaneous auctions for heterogeneous items, in which a bidder is exposed to winning too

few complementary goods to realize synergies at a high price or winning too many items that she considers substitutes at a high cost. Logistics markets have used combinatorial auctions for several years (Caplice 2007), industrial procurement is a large field of application (Bichler et al. 2006), energy exchanges are using bundle bidding in day-ahead markets (Meeus et al. 2009), and more recently spectrum auctions across the world have started using combinatorial auction designs (Cramton 2013). In all these cases, the market design has a profound effect on bidder behavior and efficiency, and many market designs originated from academic research.

Mechanism design and auction theory provide a basic framework to think about strategies and efficiency of auctions, but the design of multi-item markets has led to many new problems complementary to those discussed in microeconomics. For example, the computational complexity of allocation problems

in multi-item markets has been a topic of interest in operations research and computer science Lehmann et al. (2006). The information systems literature has made contributions on decision support, pricing, and information feedback (Xia et al. 2004, Adomavicius and Gupta 2005, Bichler et al. 2009), the analysis of bidder behavior (Scheffel et al. 2011, Adomavicius et al. 2012), and the design of markets for specific domains (Guo et al. 2007, Bapna et al. 2008).

Combinatorial auctions have been used for increasingly large markets. For example, in some spectrum auctions there are around 100 licenses for sale—i.e., 2^{100} packages, which is on the order of 1.267×10^{30}.[1] As a comparison, 3×10^{23} is the number of stars in the observable universe. It is clear that larger bidders can only specify a small proportion of their bids of interest. Note that the winner-determination problem for auctions with a fully expressive XOR bid language treats missing package bids as if a bidder had no value for the package. Recent lab experiments have shown that this "missing bids problem" can lead to substantial efficiency losses, even with a much lower number of possible packages compared with a simultaneous multiround auction where bids can only be submitted on individual items (Bichler et al. 2013). In a simultaneous multiround auction, the bids are additive (OR bid language), and for each package there is an estimate of the valuations for this package, which is just the sum of the bids on the individual items. This allows for higher efficiency in larger markets than some combinatorial auction designs, even though bidders cannot express their complementarities without the risk of winning only parts of a bundle of interest and having to pay more than this subset of items is worth to the bidder. CA designs, therefore, face a natural trade-off between the efficiency gains of allowing bids on packages and the efficiency losses resulting from missing bids. This observation has caused a debate on the design of spectrum auctions, but the debate goes beyond this application and asks the question how large markets with many items can be designed such that bidders are incentivized and able to express their preferences truthfully and that auctioneers achieve allocations with high efficiency.

Our paper provides a contribution to the design of large markets with dozens or hundreds of items. In general, the term "large markets" can be used to refer to those in which the number of packages in a fully enumerative XOR bid language is more than a few hundred bids, making it clear that bidders cannot be expected to submit bids on all possible packages.

Simplification was suggested as a theoretical concept to reduce the message space without losing efficient equilibria (Milgrom 2010), but apart from this, there has been little research in the design of large markets. We outline a framework that includes the design of compact bid languages and computational techniques to determine VCG and bidder-Pareto-optimal core payments, which provide incentives for bidders in a sealed-bid auction to submit their preferences truthfully.

There are many potential applications of this approach, ranging from the allocation of a large number of resources with potentially varying quality, such as the capacity on flexible manufacturing machines, the procurement of a large number of raw materials, or the sale of both TV and Internet banner advertisement with varying quality in terms of customer reach. To prove the computational applicability to realistic markets, we focus on two particular markets of interest: (1) multi-item procurement auctions with economies of scale and discounts for large quantities of each item and (2) markets for TV advertising (hereinafter, TV ad) slots. We emphasize that the techniques we propose are quite general and applicable to a number of markets, but because they involve heuristics, it is necessary to show that the methods function effectively on a few realistic implementations in terms of size and complexity.

The procurement context involves several types of raw materials that need to be procured, where the procurement manager needs hundreds or thousands of tons of each. (For example, Bichler et al. 2006 report on real-world procurement auctions with 10 to 90,000 line items. See also Sandholm 2007 for another discussion of advanced procurement auctions in recent decades). Such markets could be organized as a combinatorial procurement auction where bidders can win one out of many package bids they submit. But with only 10 material types (items) and six units of each, however, a bidder faces more than 284 million packages to consider under an XOR bidding language, which requires a unique bid for any package that might be won to verify full efficiency. Similarly, markets for TV ad slots involve the sale of air time in hundreds of different time slots, weekly or biweekly, where bidding advertisers have differing preferences over slots, based on varying audience demographics and firm-specific needs for sufficient ad reach or coverage. Although these are only examples of large markets, the size of these markets, in terms of the large number of distinct interrelated goods and heterogeneity of bidder preferences, prohibits the application of existing CA designs, making it a good test bed for the methods proposed here.

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[1] The spectrum auction in Canada in 2014 included 97 licenses (Government of Canada 2009). Planned spectrum auctions in Canada and the United States have even more items for sale.

1.1. Compact Bid Languages and Allocation Rules

First, we will discuss compact bid languages as a remedy for efficiency losses resulting from missing bids. Several generic logic-based bid languages have been discussed in the literature on CAs (Boutilier and Hoos 2001). But for large markets like the ones discussed here, even these could require too many bids to be submitted. Often, prior knowledge about bidder preferences and market nuances allow for compact bid languages with a very low number of parameters that bidders need to specify to describe their preferences. For the procurement markets discussed here, the various discount policies that are regularly used in pricing can be elements of a bid language as described in Goossens et al. (2007) or Bichler et al. (2011), who substantially extend the expressiveness of a bid language for markets with economies of scale and scope. These bid languages follow established market practices, and bidders do not need to change their established discount policies. In a similar way, we will introduce a bid language for TV ad markets, which is natural to media agencies, allowing them to express their preferences with a few parameters only by describing substitutes in a succinct way. Such *domain-specific* bid languages require adequate optimization models to compute cost-minimal allocations in procurement or revenue-maximal allocations in forward TV ad auctions.

Advanced mixed-integer programming solvers allow for the computation of allocations of large TV ad markets with hundreds of ad slots and procurement markets with dozens of items and several quantity schedules to near optimality. Although such near-optimal solutions can typically be found in minutes, finding (or proving) the exact solution might take hours or even be intractable. This is a widespread pattern in combinatorial optimization. An integrality gap of a few percent would be considered acceptable in the types of large-scale private-sector markets that we discuss in this paper. Even the recent design of incentive auctions for the Federal Communications Commission in the United States includes allocation problems that are too large and difficult to be solved to full optimality.[2]

1.2. Payment Rules

Second, we will discuss payment rules to encourage truthful bidding in large markets. The celebrated Vickrey–Clarke–Groves (VCG) payment rule charges each bidder the harm they cause to other bidders and ensures that the dominant strategy for a bidder is to bid her true valuation of the items. The VCG outcome can be "outside the core" leading to low revenue

and possibilities for shill bidding among other problems (Ausubel and Milgrom 2006). Intuitively, VCG provides discounts to ensure that an *individual* cannot benefit from *unilateral* deviation from truth-telling, but the resulting discounts can be so large that payments are absurdly low and remain manipulable by *groups* of bidders. For example, Ausubel and Milgrom (2006) provide a classic setup where VCG payments total zero for two bidders, despite a competitor's bid to pay the seller a large amount for their combined winnings, and they show that these payments of zero can be achieved through group manipulation or the use of false-name (i.e., shill) bids. This occurs when the first two bidders bid for their disjoint respective bundles of interest, with a losing competitor offering exactly the same amount for the union of these bundles.

Core-selecting auctions were introduced in recent years (Day and Raghavan 2007) to combat these weaknesses of VCG. As a payment paradigm for multi-item markets in general, they were designed to balance the incentives of bidders to reveal bids truthfully (achieved by making the bidders pay the least amount possible) against the perceived fairness of payments (such that payments are adequately large to preclude any set of losing bids from become winning). This auction design computes prices that are "in the core" with respect to submitted bids, stating roughly that no coalition of bidders could claim that their bids offered a mutually preferable outcome that would also raise seller revenue. Thus in the example above, the winners will always combine to pay at least the same amount in a core-selecting auction.

The game-theoretical properties of bidder-Pareto-optimal core (BPOC) auctions have been discussed extensively in the recent years (Day and Milgrom 2007, Goeree and Lien 2015), and core-selecting auction rules have been adopted for spectrum license auctions around the world, including Australia, Austria, Canada, Denmark, Ireland, Portugal, the Netherlands, and the Unites Kingdom. The approach in these spectrum auctions has been to use a combinatorial clock auction (CCA) with bidding in an iterative auction, in response to rising price clocks for each item, and finishing with a sealed-bid core-selecting auction using all bids from these iterative rounds, as well as additional combinatorial bids submitted in a sealed-bid round subject to activity rules. Thus, even if one were to argue to use the CCA format as is used in spectrum auctions for our applications (although we do not) the auctioneer would still need to run the winner-determination and core-pricing algorithm, and the algorithmic contributions of this paper would still be relevant as the auction gets large.

Although BPOC payment rules are not strategy-proof, the incentives for manipulation can be considered minimal in most large-scale markets, where

[2] http://www.fcc.gov/topic/incentive-auctions (accessed February 25, 2015).

typically neither the number of bidders nor their exact preferences are known. Note that existing game-theoretical models assume that bidders are all interested in only a single package and that all bidders know which packages their competitors bid on, in order to keep the analysis tractable (Goeree and Lien 2015). Although these analyses are insightful and illustrate situations where bidders would not bid truthful in equilibrium in a BPOC auction, such information is rarely available in large real-world markets. The number of packages that bidders could bid on can serve as a proxy for how much information would be needed by a bidder to profitably manipulate a market. Still, a simple pay-as-bid rule sets strong incentives for bid shading, while the benefits of bid shading are greatly reduced under VCG or BPOC payment rules in large markets with little or no prior distributional information. Recent lab experiments comparing a BPOC payment rule with a pay-as-bid payment rule provide evidence for this hypothesis (Bichler et al. 2014).

1.3. Relationship to Approximation Mechanisms

Recent research in computer science has explored whether strategy-proofness can be maintained by giving up on optimal social welfare and using approximation algorithms with provable approximation ratios on the quality of the allocation as an allocation rule (Lavi 2007). Unfortunately, in spite of the theoretical value of results in this field, the approximation ratios of algorithms for most combinatorial optimization problems are often not acceptable for real-world market design, and often no such approximation algorithms are available for specific problems. The approximation ratio of approximation algorithms to solve the winner-determination problem in CAs with general valuations is $O(\sqrt{N})$ (Halldorsson et al. 2000), where N is the number of items. No strategy-proof approximation mechanism can have a better ratio than this algorithmic bound. This means in an auction with 25 items only, the solution can be five times worse than the optimal solution in the worst case. Randomized approximation mechanisms with the same approximation ratio have already been found (Lavi and Swamy 2011, Dobzinski et al. 2012). However, the best deterministic truthful approximation guarantee known for general combinatorial auctions is $O(N/\sqrt{\log N})$ (Holzman et al. 2004). Note that much of the literature on approximation mechanisms relies on randomized mechanisms, which also lead to somewhat weaker notions of truthfulness than strategy-proofness with deterministic mechanisms.

We consider this literature as complementary to our research. Although there are no provable guarantees to solve certain problem sizes of combinatorial optimization problems, experiments typically lead to high confidence about the problem sizes that can be solved in due time in practice. There is a great deal of literature with various benchmark problems analyzing the empirical hardness of certain optimization problems in operations research,[3] which practitioners rely on for scheduling, vehicle routing, or other types of resource allocation problems. We do this as well. We also give up on strategy-proofness in the strong sense. Strategy-proofness is a powerful but also restrictive concept, which is why we instead focus on the weaker notion of core-selecting payments.

It is worth noting that the VCG mechanism is no longer strategy-proof if the allocation does not necessarily maximize social welfare. The simple proof showing that the VCG mechanism leads to a dominant-strategy equilibrium for each individual bidder (see, e.g., Shoham and Leyton-Brown 2008, p. 276) relies on the argument that the auctioneer chooses the allocation that maximizes the *coalitional value* based on the reported bids of all bidders. So, if the allocation cannot be computed optimally, then the VCG mechanism also loses this strong game-theoretical properties. We refer to coalitional value as the result of the allocation problem assuming bidders report their true valuations.

Still, the basic concept of a second-price rule can encourage truthful bidding because shading one's bids might not increase profit, but it might increase the risk of losing in the auction or getting a less desired outcome. Note that the information a bidder would need to manipulate grows exponentially with the number of items in a combinatorial auction. With many bidders and many items but little distributional information about all possible combinations, profitable manipulation becomes almost impossible. The amount of information required by a bidder to profitably manipulate in a specific auction could well serve as an alternative way to characterize markets, different from the game-theoretical solution concepts that are typically used in auction theory. On one hand, dominant strategies restrict the auction designer to the VCG mechanism (Green and Laffont 1979), only applicable with optimal allocation rules. On the other hand, Nash equilibria are computationally hard to compute in general (Daskalakis et al. 2009), and Bayes–Nash equilibria of combinatorial auctions require a very large number of distributional assumptions, rendering this solution concept intractable in large markets as discussed in our paper. Given the lack of sufficient information about other bidders' valuations or the specific packages they are interested in, and the hardness of computing Bayes–Nash equilibrium strategies in large markets, we argue that a second-price rule

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[3] See "OR Library" at http://people.brunel.ac.uk/~mastjjb/jeb/info.html for different types of discrete optimization problems.

such as in BPOC payments offers a compelling compromise, encouraging truthful bidding with substantial discounts, rather than guaranteeing it.

The application of second-price payment rules such as BPOC or VCG rules with near-optimal rather than exact solutions to the allocation problem in our framework is not without challenges, however. For example, the coalitional value of a coalition without one of the winners (required to compute the VCG payments) might return a higher value than the coalitional value with all bidders. Our adaptation of a constraint-generation technique for the computation of BPOC payments by Day and Raghavan (2007) to large-scale markets is a new, practically viable paradigm by which core-selecting auction designs with good incentive properties can be applied to large markets. It is the combination of the compact bid language and the payment rule that allows bidders to express their complementarities, but at the same time, it provides incentives for truthful bidding. Our experiments help understand how the near optimality of the allocations impacts the payments of bidders. Overall, this can be a recipe for many large-scale markets beyond the ones discussed in this paper.

1.4. Contributions and Outline

In summary, our contributions are as follows: First, motivated by work with industrial partners, we propose a compact bid language for the TV ad market. The TV ads application will be our leading example because it provides a rich test bed to demonstrate our ideas with a realistic valuation model and a type of winner-determination problem that would benefit most readily from the approach. The bid language allows the expression of preferences for a large number of packages with only a few parameters. The winner-determination problem in such markets is $\mathcal{NP}$-hard and cannot always be solved optimally. However, as is typical in many combinatorial optimization problems, near-optimal solutions can be found within a few minutes for limited problem sizes, which is promising, but previous algorithms for computing core payments break down when solutions are not exact. We therefore propose two algorithms to deal with markets where the winner determination might not be solved to optimality. The approaches are evaluated in an extensive set of experiments and their properties are characterized. In addition to the TV ad market, we also analyze the two algorithms in the context of volume-discount auctions to show that the basic framework and the results carry over to other large markets. Here we draw on a compact bid language for procurement markets with economies of scale introduced by Bichler et al. (2011).

Overall, we show that the dynamic reuse algorithm we develop is not too slow relative to the quicker

trimming algorithm, so the time cost of computing more accurate and more fair outcomes should not be out of reach for this or similar applications. Although the trimming algorithm is quicker and generates more revenue for a fixed set of bids, its inferior efficiency and incentive properties make its use harder to justify, particularly for government applications where efficiency concerns naturally dominate. The market design can serve as a template for other large markets with many items and complex bidder preferences, an area of research with many applications but little attention in the literature so far.

In §2, we discuss related work. Section 3 introduces the market design including the allocation and the pricing rules. In §4.2, we propose two algorithms to deal with nonoptimal solutions to the winner-determination problem. Section 5 summarizes the results of our experiments, and §6 concludes with a summary and future outlook. A complete glossary of symbols is also provided in the appendix for quick reference.

2. BPOC Auctions

In this section, we want to provide further background on auctions with bidder-Pareto-optimal payment rules because they are central to this paper. The concept of the core has a long history in economics, and indeed a mechanism that selects a core outcome based on submitted preferences was the foundation for the 2012 Nobel Prize in Economics, although that stable-matching market involved allocations that do not allow for monetary transfers. The extension of these ideas to the auction context (with payments) began indirectly in the work of Parkes and Ungar (2000), Ausubel and Milgrom (2002), with explicit computation of core outcomes and formal treatment of the core-selecting approach coming later in Day and Milgrom (2007), Day and Raghavan (2007), and Day and Cramton (2012). Core-selecting auctions have been suggested as an alternative to the VCG mechanism, which suffers from a number of problems such as low seller revenue (Ausubel and Milgrom 2006). VCG solutions outside the core, where a subset of bidders has indicated to be able to pay more than what the winners paid, is often seen as undesirable.

Day and Raghavan (2007) showed that under semi-sincere bidding strategies and perfect information, every BPOC price vector forms a Nash equilibrium. Thus, assigning BPOC payments based on bids being true values only makes a bidder pay what she should have bid in equilibrium if she had expertly anticipated the true values (and bids) of others. Day and Milgrom (2007) show that bidder-Pareto optimality implies optimal incentives for truthful revelation over

all core-selecting auctions, among other supporting results, including decreased vulnerability to false-name bidding and collusive behavior relative to other auction formats discussed in the literature, in particular the VCG mechanism. Day and Raghavan also note that *total-payment minimizing* BPOC points are further resistant to certain forms of collusion with side payments, and this minimum revenue condition has been implemented in all core-selecting spectrum auctions to date. Selecting a BPOC point is further supported by the fact that, if the truth-revealing VCG vector is in the core, then any BPOC algorithm will produce VCG as its output.

A core-selecting auction only provides a dominant strategy if the VCG outcome is in the core. Goeree and Lien (2015) actually show that no Bayesian incentive-compatible core-inducing auction exists when the VCG outcome is not in the core. In specific settings, where the VCG outcome is outside the core, the equilibrium bidding strategy is to shade bids below one's true valuation, speculating that the reduced bid can lower one's payment without affecting the bundle of goods awarded. Simple threshold problems where multiple local bidders only interested in one item compete against a global bidder, who is interested in all items, provide an illustrative example. Local bidders could try to free ride on each other. However, Bayesian analyses of such markets assume that bidders are interested in a single item and that they know what other bidders bid on and have commonly known prior distributions available about other bidders' valuations for their bundles of interest. In large TV ad markets, such information is not available to bidders and bidders are multiminded. Often bidders do not even know how many competitors exist in the market, making speculation quite risky. The same assumptions hold in procurement markets such as the volume-discount auctions analyzed in this paper. In such situations, manipulation comes at a high risk of winning nothing.

A potential alternative to the approach proposed here is the use of proxy agents bidding in multiple rounds of an ascending auction until an equilibrium is reached, as in the ascending proxy auction (Ausubel and Milgrom 2002) or iBundle (Parkes and Ungar 2000). Unfortunately, this approach requires a very large number of auction rounds (unless the problem size is quite small) and the auctioneer needs to solve a winner-determination problem in each round (Schneider et al. 2010). By trying to avoid unnecessary winner-determination optimizations, constraint generation after a sealed-bid auction is a more effective and practical method for price generation, especially when considering cases of hard winner-determination problems.

3. Compact Bid Languages and Allocation Problems

The TV ad slot market will serve as the main example in our paper, which shares many of the features of other large markets for the sale of spectrum licenses, in logistics, or in industrial procurement. Also, we will briefly review volume-discount auctions designed for industrial procurement to illustrate that the framework outlined in this paper can easily be applied to other large-scale markets. For this latter setting, we are able to find exact solutions in our computational experiments for smaller instances, allowing for direct benchmarking against optimality. These benchmarks are provided to demonstrate the approach, but clearly we propose the near-optimal approach to be relevant in practice to larger instances where exact optimality is out of reach.

3.1. TV Ad Markets

In what follows, we provide a brief overview of the essential requirements. Parts of the advertisement capacity of a typical TV station are sold via specially negotiated, large, long-term contracts of about a year and are not considered in our study. We focus instead on the sale of the remaining ad slot inventory to specific marketing campaigns that run in the short term, which in Europe are typically sold via posted prices, in advance of airing. Prices for different slots can range from 6,000€ to 50,000€ for a duration of 30 seconds and are set by the TV station based on historical demand. Buyers are large media agencies that purchase a set of slots with the intent to procure the best slots for each of their customers' campaigns. The number of agencies in a particular market depends on the country and the particular station, but a typical short-term market for a TV station in Germany, for example, consists of approximately 50 media agencies, booking slots for several hundred customers in a particular channel.

Because the amount of airtime filled by long-term customers varies, the length of a slot available in the short-term market can vary between 2 and 5 minutes, while the length of an ad also varies considerably, lasting up to 1 minute. For a particular channel in the markets we investigated, there are on the order of 150 short-term slots available during the program per week.

Different slots have a different reach for different customer segments or the population overall. The reach of a particular slot varies over time, but there are estimates based on historical panel data available to clients of the media agencies. Clients use reach per segment (based on gender, age, or other demographics) or per population to determine their willingness to pay for different slots. Clearly, the value of some slots, such as those during the finals of the national

soccer league, may be difficult to estimate and their valuation varies considerably depending on the target market of an advertiser. Apart from these high-value slots, there is also typically a segment of low-value slots, which are also difficult to price because the demand is hard to predict. This difficulty in demand or valuation prediction together with limited supply suggests that an auction market would outperform the existing posted-price mechanism.

The allocation of TV ad slots can be modeled as a multiple knapsack problem, in which each time slot i in the set $I = \{1, 2, \ldots, N\}$ is treated as a knapsack with a maximum capacity/duration of c_i, which cannot be exceeded. As mentioned above, each slot can potentially hold a number of ads, although some may have been previously allocated to larger customers, so we assume that c_i reflects only short-term capacity in the current market, making for a potentially heterogeneous list of c_i values, even for a TV station with slots of the same size when considering all ads aired. We also assume that each slot i has a reservation value or minimum price per time unit r_i, which reflects the station's ability to off-load excess capacity at a low price to existing customers if needed. Station call signs and other brief announcements can also be used to fill any excess unused time.

Each bidding advertiser k in the set $K = \{1, 2, \ldots, \mathcal{K}\}$ has an ad of duration d_k to be shown repeatedly (at most once per time slot) if she wins a bundle of time slots in the auction. To ensure adequate reach, each bidder specifies an abstract "priority vector" or "weight vector" W_k, containing an arbitrary weight value w_{ik} for each time slot. These "weights" conveying "strength of priority" could specifically represent the expected viewership, expected viewership of a particular demographic, viewership weighted by expected sales, etc., reflecting the advertiser's performance metric of choice. She can then bound the total priority value in the auction outcome to be greater than or equal to a minimum amount to qualify bids of various levels.

Thus, after specifying the priority vector and ad duration, a bidder places one or more tuples $(w_j^{\min}, b_j)$ containing the desired sum of priority values $w_j^{\min}$ necessary to justify a monetary bid b_j. At most, one of the bids placed by a bidder can win, making the bidding language an XOR of "weight threshold levels." For example, if the bidder sets the priority weights w_{ik} at the expected viewership of each slot i, the XOR structure lets her set an exact price for any particular price point of interest. She can set a price for a total of $w_j^{\min} = 1$ million viewers, a price for $w_j^{\min} = 2$ million viewers, etc., regardless of which slots are chosen to reach this total viewership. This price-point structure reflects the ability of the language to represent the fundamental complementarity in this type of market;

a small number of ad slots (or small reach, etc.) may have little or no value, but several of them together are worth more than the sum of the parts.

The set J_k contains all bid indexes j by a bidder k, and the superset J is defined as $J := \bigcup_{k \in K} J_k$. We assume these bids are submitted in a sealed-bid format, consistent with the timing of Google's auction, in which bids were submitted once to a proxy. In such markets, it is not practical for media agencies to participate in an ascending auction every week or two. After the bids are submitted, the market is cleared at a particular point in time, and the allocation is determined for some period for a time (e.g., two weeks) in the future.

Formulation WD maximizes the value of accepted bids given that ad durations do not exceed capacity in any slot (1a), the bid values are not less than the seller's reservation values (1b), the priority threshold level $w_j^{\min}$ of a bid j is met if and only if that bid is accepted (1c,d), and at most one bid j is accepted for each bidder k (1e). Decision variables x_{ij} and y_j indicate time if slot i is assigned to bid j and bid j itself is accepted, respectively, while M is a sufficiently large positive constant parameter. WD(K) indicates that all bidders $k \in K$ are included. Later we refer to WD(C) for a coalition $C \subseteq K$, which indicates the same formulation but with all bids made $k \notin C$ set to zero. Overall, we use the term *coalitional value* to describe the objective function value of formulation WD.

$$\text{WD}(K) = \max \ \sum_{j \in J} b_j y_j \tag{WD}$$

$$\text{subject to} \ \sum_{j \in J} d_k x_{ij} \leq c_i \quad \forall i \in I, \tag{1a}$$

$$d_k \sum_{i \in I} r_i x_{ij} \leq b_j \quad \forall j \in J, \tag{1b}$$

$$\sum_{i \in I} w_{ik} x_{ij} \leq M y_j \quad \forall j \in J, \tag{1c}$$

$$w_j^{\min} - \sum_{i \in I} w_{ik} x_{ij} \leq M(1 - y_j)$$
$$\forall j \in J, \tag{1d}$$

$$\sum_{j \in J_k} y_j \leq 1 \quad \forall k \in K, \tag{1e}$$

$$x_{ij} \in {0, 1} \quad \forall i \in I, j \in J, \tag{1f}$$

$$y_j \in {0, 1} \quad \forall j \in J. \tag{1g}$$

The priority vector W_k provides quite a bit of flexibility to the bidders in expressing their preferences over ad slots, and we propose that this novel bidding language could be relevant in a number of other areas. Indeed, the language captured in this formulation is quite general and includes the "k-of-singletons" expressions described in Hoos and Boutilier (2000), which were shown to be difficult to express succinctly with more fundamental logical operators and result

in hard optimizations. For example, a bidder in the ad slot auction might want her ad to be on the air at least five times within one week between 8 P.M. and 10 P.M. That is, all ad slots between 8 P.M. and 10 P.M. are substitutes, but the bidder needs at least five of them, a complementarity valuation for a sufficient volume from a group of substitutes. The priority-vector format would then have weights equal to one for the selected set of substitute times and $w_j^{\min} = 5$ playing the role of the k term in Hoos and Boutilier (2000).

THEOREM 1. *The decision version of the* WD *problem is strongly* $\mathcal{NP}$-*complete.*

The proof is by reduction from the multiple knapsack problem, and it can be found in the appendix. The decision version of the multiple knapsack problem is strongly $\mathcal{NP}$-complete (Chekuri and Khanna 2006). Whereas weakly $\mathcal{NP}$-complete problems may admit efficient solutions in practice as long as their inputs are of relatively small magnitude, strongly $\mathcal{NP}$-complete problems do not admit efficient solutions in such cases. Unless $\mathcal{P} = \mathcal{NP}$, there is no fully polynomial-time approximation scheme (FPTAS) for strongly $\mathcal{NP}$-complete problems (Garey and Johnson 1979). Even if we cannot hope for FPTAS, we can get near-optimal solutions with standard mixed-integer programming solvers for practically relevant problem sizes as we will show.

3.2. Volume-Discount Auctions

Volume discounts are in widespread use in markets with economies of scale. Davenport and Kalagnanam (2000) were among the first authors to discuss volume-discount bids in an auction. Their bid language requires suppliers to specify continuous supply curves for each item. They apply discounts only to additional units above a threshold of a specific price interval. In contrast to these incremental volume-discount bids, Goossens et al. (2007) proposed tiered bids, which they refer to as total-quantity bids. The latter are valid for the entire volume of goods purchased after a certain quantity threshold. For example, a supplier charges \$4 per unit for up to 1,500 units, but only \$2.50 per unit for the entire quantity if the purchasing manager were to buy 1,500–2,000 units. In practice, suppliers employ various types of such discount policies in different settings. In addition to volume-discount bids and total-quantity bids one can find lump-sum rebates on total spend, and such discounts can be based on the quantity or spend of one or a few items that are being auctioned off. Bichler et al. (2010) introduced a comprehensive bid language that allows for different types of discount policies, including volume-discount bids, total-quantity bids, and lump-sum rebates. They propose a mixed-integer program to solve problems of up to 30 suppliers,

30 items, and five quantity schedules to near optimality in less than 10 minutes. Because of space constraints, we refer the interested reader to Bichler et al. (2010) for a detailed description of the bid language and the experimental setup and results. An analysis of various heuristics and metaheuristics to solve the problem can be found in Hass et al. (2013). Even though such near-optimal solutions were always possible with these problem sizes, proving the optimality of a solution was typically intractable and even after hours there would be a small integrality gap. This phenomenon is widespread in combinatorial optimization overall. In our experiments, we will use the compact bid language introduced by Bichler et al. (2010) and their experimental setup to compute VCG and BPOC payments for near-optimal allocations.

4. Payment Rules

If we only aim for near-optimal solutions, not for exact solutions to the winner-determination problem, some computational issues can arise. For example, the objective function value of the best allocation with one winner excluded might be higher than that of the best allocation with all bidders included when computing VCG payments. We will first revisit BPOC payments, before we discuss different algorithms how to compute these payments with near-optimal solutions to the winner-determination problem. We will use the terms payments and prices interchangeably.

4.1. Bidder-Pareto-Optimal Core Payments

We will determine BPOC payments in the following treatment and compare them with the VCG payments in our experiments. The technique of using constraint generation to find the coalitions defining the core was designed to work in any context where the winner-determination problem could be solved exactly. Here, we quickly reiterate that approach before extending it to situations of near-optimal winner determination in the next section.

The approach discussed in the literature is to find core prices by iteratively creating new price vectors p^t for the winning coalition W and then checking at each iteration t whether there is an alternative outcome that generates strictly more revenue for the seller and for which every bidder in this new outcome weakly prefers to the current outcome. If such a coalition exists, the alternative winning coalition C is called a *blocking coalition*, and a constraint is added to a partial representation of the core in payment space until no further blocking coalitions can be found. To discover the most violated blocking coalition C^t relative to the

Figure 1 (Color online) A Typical Instance Showing the Reduction of the Integrality Gap Over Time
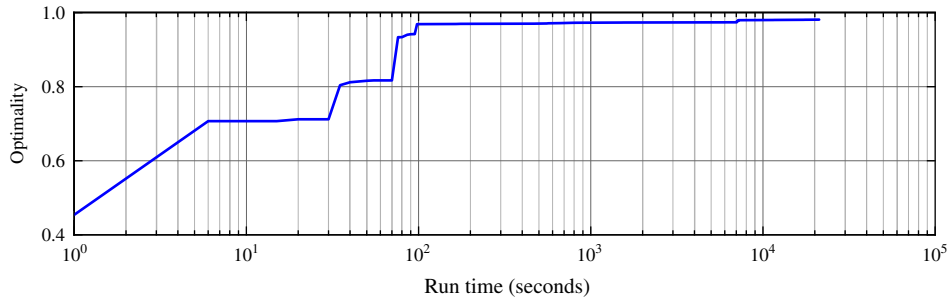
current payments at iteration t, the WD is extended as in the separation problem SEP^t.

$$z(p^t) = \max \sum_{j \in J} b_j y_j - \sum_{k \in W}(b_k^* - p_k^t)\gamma_k \qquad (\text{SEP}^t)$$

$$\text{subject to} \quad (1a)-(1g),$$

$$\sum_{j \in J_k} y_j \leq \gamma_k \quad \forall k \in W,$$

$$\gamma_k \in [0,1] \quad \forall k \in W.$$

Here, W is the set of winners from the solution of WD(K), and b_k^* represents bidder k's winning bid. If the sum of the current payments p^t is less than the solution to (SEP^t), then a violated core constraint has been found, and we must add a constraint to our partial representation of the core. Following Day and Raghavan (2007), this partial representation is given in the following linear program to find equitable bidder-Pareto-optimal (EBPO) payments. The program is then solved to find the next tentative set of payments p^{t+1} until no further constraints can be found.

$$\theta(\epsilon) = \min \sum_{k \in W} p_k + \epsilon m \qquad (\text{EBPO}^t)$$

$$\text{subject to} \sum_{k \in W \setminus C^\tau} p_k \geq z(p^\tau) - \sum_{k \in W \cap C^\tau} p_k^t$$
$$\forall \tau \leq t, \quad (\text{EBPO}^t.1)$$

$$p_k - m \leq p_k^{vcg} \quad \forall k \in W,$$

$$p_k \leq b_k^* \quad \forall k \in W,$$

$$p_k \geq p_k^{vcg} \quad \forall j \in W.$$

As in SEP^t, b_k^* is the winning bid for k, the parameters $p_k^{vcg} = b_k^* - (\text{WD}(K) - \text{WD}(K_{-k}))$ represent VCG payments, and m represents a maximum deviation from VCG, which is minimized as a secondary objective after minimizing total payments.[4] We then use the value of each p_k in the solution for the next iteration (i.e., set $p_k^{t+1} = p_k$).

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[4] In practice, these two minimizations can be handled as separate optimizations, but they are presented here as a single optimization using a sufficiently small ϵ for the sake of concise exposition.

4.2. Core Payments with Near-Optimal Allocations

For many combinatorial optimization problems, good solutions often can be found quickly, even though finding a provably optimal solution may take a very long time. Figure 1 shows a typical example of WD with 336 slots and 50 bidders, where a feasible solution with 95% optimality could be reached within a few minutes. This is a common phenomenon in many combinatorial optimization problems.

Without the ability to guarantee an optimal solution quickly enough for a practical application, one would naturally consider a provably high-quality solution that can be found quickly. Most industrial mixed-integer programming solvers (e.g., CPLEX, Gurobi) provide absolute and relative worst-case optimality gap parameters, allowing the optimization routine to terminate if the optimality gap (the difference between the best feasible solution and the theoretical bound) falls below some target or is a small enough percentage of the best feasible solution, respectively. For now, we leave the exact specification of how a "good enough" approximate solution is qualified, but motivated by Figure 1, the reader may assume a 5% optimality gap or an at-least-95%-optimal solution for concreteness. We will thus write WD^a for any qualified approximation of a WD value and consider an implementation using these approximations in place of true WD values. Similarly, we will write $z^a(p^t)$ for separation problem values found using near-optimal solutions.

Problems can arise, however, during the VCG and core price calculation when accepting these approximate or near-optimal solutions. For example, under truly optimal solutions, with the standard assumption of free disposal, WD(K_{-k}) is always at most the value of WD(K). But with a series of near-optimal computations this is not guaranteed, opening up the possibility that one might compute an approximate VCG payment with $b_k^* - (\text{WD}^a(K) - \text{WD}^a(K_{-k})) > b_k^*$. Similarly, under near-optimal computation the coalitional value of SEP^t can be higher than the value of the WD. If this happens, the newly generated constraint added to EBPO^t can cause an infeasibility if

$\sum_{k \in W \setminus C^t} b_k^* < z^a(p^t) - \sum_{k \in W \cap C^t} p_k^t$. Two different solutions are presented to address this problem that can potentially arise during computations.

4.2.1. The TRIM Algorithm. With known b_k^* values determining individual rationality (IR) constraints (i.e., payments must not exceed bids), a natural first approach is to adjust each WD-based result so that it fits into the IR region.

For the VCG prices, this technique makes use of the fact that

$$b_k^* \geq p_k^{vcg} \geq 0 \quad \forall k \in W \tag{2}$$

whereas for the (constant) right-hand side (RHS) of the constraints in the EBPO^t, we must always have:

$$\sum_{k \in W \setminus C^\tau} b_k^* \geq z(p^\tau) - \sum_{k \in W \cap C^\tau} p_k^\tau \quad \forall \tau \leq t \tag{3}$$

Thus, our first algorithm[5] is to simply trim the infeasibilities based on known bids, represented in Algorithm 1 in the two steps using min functions.

Algorithm 1 (Core Price Calculation—TRIM)
Solve: WD$^a(K)$;
for $k \in W$ **do**
  Solve: WD$^a(K_{-k})$;
  $p_k^{vcg} \leftarrow \min(b_k^*, b_k^* - (WD^a(K) - WD^a(K_{-k})))$;
$p^1 \leftarrow p^{vcg}$;
$\theta^0 \leftarrow \sum_{k \in W} p_k^{vcg}$;
while *true* **do**
  Solve: SEP^t
  if $z^a(p^t) \leq \theta^{t-1}$ **then**
    Break: 'core' price vector found;
  Generate RHS of new constraint:
    $\alpha^t \leftarrow \min(\sum_{k \in W \setminus C^t} b_k^*, z^a(p^t) - \sum_{k \in W \cap C^t} p_k^t)$;
  Add constraint $\sum_{k \in W \setminus C^t} p_k \geq \alpha^t$ to EBPO;
  $p^t, \theta^t \leftarrow$ Solve: EBPO;
  if $(C^t, \theta^t) = (C^{t-1}, \theta^{t-1})$ **then**
    Break: no better price vector possible;
  Iterate: $t \leftarrow t + 1$.

4.2.2. The REUSE Algorithm. An alternative to trimming infeasibilities is based on the observation that whenever an infeasibility is found, the validity of expressions (2) and (3) imply that an update can be made to an approximate WD value, from a previously best-known feasible solution to a new tentatively optimal feasible solution. To implement this change, the storage of any value based on a winner-determination solution can no longer be treated as constant and must be regenerated at each iteration

based on current WD^a values. This includes VCG price estimations and the RHS values for generated core constraints, whose definitions must be reformulated based on current WD^a values.

Thus, our second approach is to store a list of all discovered WD$^a(C)$ values, reusing all coalitions found so far and reformulating the entire separation problem and EBPO problem at each iteration, noting that the set of relevant core constraints, and indeed the set of winners itself, may be changing as new information becomes available. Whenever we run WD^a (the first time, to compute each VCG price, and inside each run of SEP), we get a new collection of feasible bids, representing a coalition of bidders, and we check these values against our current list of coalitions and WD^a values. If the coalition has not been found before, our list is extended to include it as among the "potentially important" coalitions to consider. If any superset coalition has been listed previously but with a lower coalitional value, we can update it to the current WD$^a(C)$ value, as a new better approximation has been found.

Because we will now store the blocking coalitions C^t and its value instead of $z(p^t)$ after each SEP^t has been solved, we are forced to work with a reformulation of core constraints based on WD^a values rather than separation levels. For a general winner-determination problem (i.e., with respect to any alternative bidding language), the core constraints can be expressed with the alternative, equivalent expression that can be derived by substitution:

$$\sum_{k \in W \setminus C^\tau} p_k \geq WD(C^\tau) - \sum_{k \in W \cap C^\tau} b_k^* \quad \forall \tau \leq t \quad (\text{EBPO}^{t'}.1)$$

Using this formulation of the core, we can generate a constraint in EBPO for any C^τ found so far using the current best-approximation WD$^a(C^\tau)$ in place of WD(C^τ). For bidding languages with only one relevant bid b_k per bidder (as is the case in the scenarios presented in §5), this constraint can be further simplified, resulting in the following formulation (4) in place of EBPO$^{t'}$.1.

$$\sum_{k \in W \setminus C^\tau} p_k \geq \sum_{k \in C^\tau \setminus W} b_k \quad \forall \tau \leq t \tag{4}$$

This new set of constraints provides an intuitively pleasing interpretation of core constraints in the TV ad context: Any subset of winners pays enough to match the counteroffer including a set of losing bidders that would otherwise benefit the seller, a direct analogue to second prices.

Although it is not guaranteed that each stored coalition C^t provides a potential maximally violated coalition at the end of the constraint generation process, the addition of all constraints found at any point

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[5] In all algorithmic implementations that follow, we assume that all feasible integer solutions are stored by the optimizer and used to generate bounds on subsequent optimizations using alternate objective functions.

drastically improves the overall performance of the algorithm when compared with having to completely rebuild a set of blocking coalitions after a change in W. That is, it is better to reuse potentially redundant constraints than to start over, looking for relevant constraints from scratch each time the set of winners is updated. Also, because the core pricing is computed as an LP (without integer constraints), it is not computationally expensive to have redundant constraints.

The formulation of the separation problem as an altered WD problem has the additional benefit that all feasible solutions remain feasible for a WD or SEP^t instance. MIP solvers store feasible (integer) bases internally, and if the separation problem is implemented as the same problem instance with some changes to objective coefficients, all feasible bases (stored as a branch-and-bound tree) remain feasible and thus provide immediate bounds on the SEP^t problem, making efficient use of all information found by the solver. This makes it progressively more difficult to find relevant feasible solutions. Therefore, in our experimental evaluation, we allowed for longer time limits on the optimization routine for later SEP^t instances.

Function (Core Price Calculation—REUSE—Update Coalitions(*optwinners*, *bidsum*))
for $C \in$ *Coalitions* **do**
   if $C \supseteq$ *optwinners* and val$(C) <$ *bidsum* **then**
      val$(C) \leftarrow$ *bidsum*;
      if $C = K$ **then**
         $W \leftarrow$ *optwinners*;
         EBPO $\leftarrow$ null;
         reset $\leftarrow$ *true*;
for $k \in W$ **do**
   $p_k^{vcg} \leftarrow b_k^* -$ val$(K) +$ val(K_{-k}).

Algorithm 2 as presented below keeps a list *Coalitions*, each element being a list of winners under some feasible integer solution to WD. For each coalition $C \in$ *Coalitions*, we also store the best known value val(C), which can be revised as the algorithm progresses. Furthermore, for ease of exposition, these algorithms refer to the winning bidders from the most recent optimization run as *optwinners*, and the sum of the (actual, i.e., unaltered) bids of these winners is given as *bidsum*.

Algorithm 2 (Core Price Calculation—REUSE)
Solve: $WD^a(K)$;
Coalitions $\leftarrow \{K\}$;
val$(K) \leftarrow$ *bidsum*;
$W \leftarrow$ *optwinners*;
$t \leftarrow 0$;
EBPO $\leftarrow$ null;
reset $\leftarrow$ *false*;
while *true* **do**

   if EBPO $=$ *null* **then**
      ComputeVCG:
      for $k \in W$ **do**
         Solve: $WD^a(K_{-k})$;
         Coalitions $\leftarrow$ *Coalitions* $\cup \{K_{-k}\}$;
         val$(K_{-k}) \leftarrow WD^a(K_{-k})$;
         UpdateCoalitions(*optwinners*, *bidsum*);
         if *reset* $=$ *true* **then**
            Break k loop;
      $p^t \leftarrow p^{vcg}$;
      $\theta^t \leftarrow \sum_{k \in W} p_k^t$;
   if *reset* $=$ *true* **then**
      reset $\leftarrow$ *false*;
      Continue;
   Solve: $SEP^t(p^t)$;
   if $z(p^t) \leq \theta^t$ **then**
      Break: 'core' price vector found;
   Coalitions $\leftarrow$ *Coalitions* $\cup \{$*optwinners*$\}$;
   val$($*optwinners*$) \leftarrow$ *bidsum*;
   UpdateCoalitions(*optwinners*, *bidsum*);
   if *reset* $=$ *true* **then**
      reset $\leftarrow$ *false*;
      Continue;
   if $EBPO =$ *null* **then**
      build EBPO with constraints $EBPO^{t\prime}.1$ for all
      $C \in$ *Coalitions* with val(C) as a best
      approximation of WD(C);
   else
      add constraint $EBPO^{t\prime}.1$ to EBPO with
      $C =$ *optwinners* and with val(C) as a
      best approximation of WD(C);
   $p^{t+1}, \theta^{t+1} \leftarrow$ Solve: EBPO;
Iterate: $t \leftarrow t + 1$.

The difference in the quality of the TRIM and REUSE approaches, in terms of closeness to core-selecting prices, can be described as follows. Let ζ_C^t represent the amount that the final prices p^t and allocation x^t violate the core-defining constraint (with respect to submitted bids) indexed by coalition C. Let $trim^t$ denote the amount "trimmed" in the final iteration of the TRIM algorithm—i.e., $trim^t = \max(0, z^\alpha(p^t) - \sum_{k \in W \cap C^t} p_k^t - \sum_{k \in W \setminus C^t} b_k^*)$. Finally, let gap^t represent the final absolute optimality gap when solving SEP^t. Theorem 2 provides simple bounds on possible deviation from optimality-based core-selecting prices. This result indicates that the optimality gap (in absolute rather than relative terms) of the final separation measures the potential for violation of core selection under a near-optimal approach, with any trimming performed by the TRIM algorithm translating one-to-one into further potential for core violation.

Theorem 2. *For a fixed set of bids, $\zeta_C^t \leq gap^t \ \forall C \subseteq K$ under REUSE, while $\zeta_C^t \leq gap^t + trim^t \ \forall C \subseteq K$ under TRIM.*

PROOF. Core constraints are most often written as $\sum_{k \in C} \pi_k \geq WD(C)$, where C must include the seller and π represents each player's payoff. Suppose such a constraint is violated and replace payoffs with surplus for bidders (i.e., $b_k^* - p_k$) and total payments for the seller. We get that for some positive value ζ_C^t:

$$\sum_{k \in C} p_k^t + \zeta_C^t = WD(C) - \sum_{k \in C} (b_k^* - p_k^t).$$

Under the REUSE algorithm, this expression becomes

$$z^a(p^t) + \zeta_C^t = z(p^t, C)$$

because the final separated cut must be tight. Here $z(p^t, C)$ represents the true value of the separation objective function evaluated at the feasible solution implied by WD(C). Because that same feasible solution was a candidate when solving SEP^t approximately, however, by the definition of the optimality gap we must have

$$z^a(p^t) + gap^t \geq z(p^t, C)$$

with the desired result for REUSE following by substitution. The result follows analogously for TRIM, with the difference that the second line becomes

$$z^a(p^t) - trim^t + \zeta_C^t = z(p^t, C). \quad \square$$

5. Experimental Evaluation

In this section, we examine the solution quality of the presented algorithms under constrained computation time. Using the simulations provided below, we analyzed a number of primary attributes to measure overall performance, such as allocative efficiency, revenue, and speed. To directly compare the quality of the generated prices, a series of secondary metrics were computed. These values allow a comparison of how much a bidder could possibly gain by shading her bid by comparing the ratios of the bids with the BPOC prices and to the VCG prices, respectively.

- Primary metrics
  1. The efficiency in terms of the coalitional value achieved by the set of winners computed after a restricted time compared with that of the optimal allocation (E).
  2. The overall auctioneer's revenue based on the computed payments compared with that in the optimal allocation (R).
  3. The duration of the computation (D).
- Secondary metrics
  4. The ratio of the BPOC payments p_k to the bids b_k (core/bid).
  5. The ratio of the VCG payments p_k^{vcg} to the bids b_k (VCG/bid).
  6. The ratio of the VCG payments p_k^{vcg} to the BPOC payments p_k (VCG/core).

Figure 2 provides values of different instances for the maximum revenue and the sum of payments achieved with TRIM and REUSE. These absolute values are difficult to compare because the instances are based on different value draws. We are interested in the comparison between the performance of the algorithms across different instances, so we require a baseline for a sensible comparison of different payment schemes. A potential baseline is the optimal revenue of the winner-determination problem, against which we could compare the value of the winning coalition after a restricted solving time, the VCG payments, and the solutions by TRIM or REUSE. In the volume discount auction, we could select instance sizes for which we could compute the optimal solution with more time allotted. Achieving optimality took a prohibitively large amount of time for the TV ad market experiments, however. All instances could be solved to near optimality, but not to optimality in several hours. For these experiments, we used the objective function value or optimal revenue of the best linear programming relaxation (LPR) of the winner determination problem as an upper bound for the optimal integer solution.

An example of a typical instance of WD is shown in Table 1. REUSE runs longer and generates less

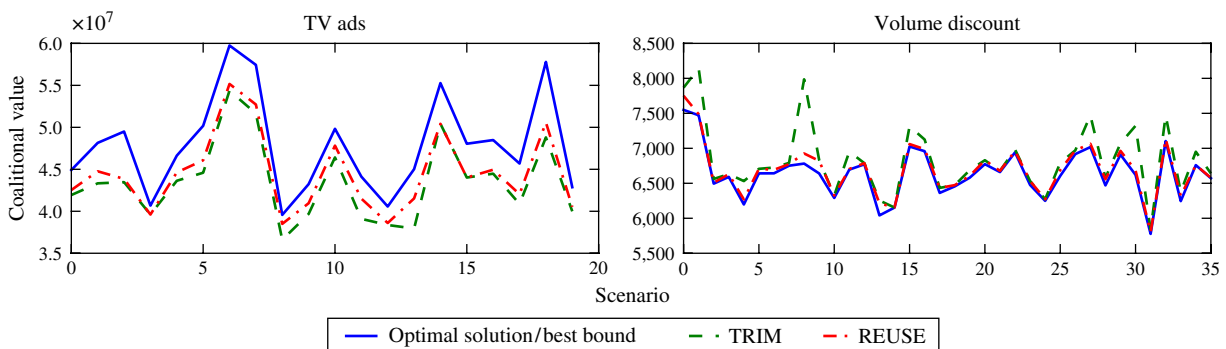**Figure 2** **(Color online) Coalitional Values in Different Experiments**

Table 1   Comparison of a Representative Experiment

Example instance	REUSE	TRIM	Optimal revenue (LPR)
Coalitional value	46,073,899	44,590,749	50,387,546
Revenue	36,569,158	42,766,735	—
Run time (h)	2.8	2.0	—
Median(p_k/b_k)	0.79	1.00	—
Median(p_k^{vcg}/b_k)	0.58	1.00	—
Median(p_k^{vcg}/p_k)	0.78	1.00	—

Table 2   Parameters for the Experiments

	Name	Parameters $\{\mu; \sigma\}$ or $\{\lambda\}$	Distribution
I	Number of slot	336	—
J	Number of bids	50	—
K	Number of bidders	50	—
c_i	Slot duration	$\{60; 30\}$	Normal
r_i	Slot reserve price steps (in €/s) [1, 2, 5, 10, 50, 75]	$\{1.2\}$	Poisson
d_k	Ad duration	$\{20; 10\}$	Normal
β_j	Bid base price (in €/s)	$\{50; 25\}$	Normal
$w_{j,\,rel}^{min}$	Min $\sum$ of campaign priorities (in %)	$\{30; 20\}$	Normal
—	Correlation of priority to slot reserve price	—	Linear
—	Distribution of priorities around the priority/price value	—	Normal

revenue than TRIM, but it is more efficient. The TRIM technique, on the other hand, often results in pay-as-bid pricing where bids and payments coincide. We will see that this pattern also emerges in a larger set of experiments.

All experiments were run on Dual Socket Octo Core AMD Opteron 2.4 GHz computers running the Linux operating system with 8 GB DDR2 RAM. All optimization problems were solved with the Gurobi 5.5 mixed-integer programming solver using the default parameters. The time limit for each single optimization was set to 300 seconds for TRIM and REUSE, unless stated otherwise. An optimization describes the process of solving a single mixed-integer program such as the initial WD, the WD_{-j} needed to compute the VCG payments and the discovery of blocking coalitions in SEP^t. Overall, this can lead to a solution time of two to three hours for each experiment because many of these optimization problems need to be solved for a single experiment. A time limit of 300 seconds for one optimization problem allowed us to conduct a larger number of experiments and get statistically significant results. For significance tests, we will provide the *p*-values of a Wilcoxon signed-rank test throughout.

5.1. Research Design for the TV Ad Market
For the generation of sample instances for the TV ad market, we could draw on data from a booking system of an industry partner. This provides us with a distribution of prices paid in this market. We will briefly summarize the main characteristics of the generated data. The distributions of all relevant random variables in the experiments can be found in Table 2.

• A typical campaign duration is from one to four weeks, averaging two weeks.

• An advertisement slot is 120 seconds long, but it can be prefilled before the auction starts because of the existing booking system (effectively reducing capacity available in a slot).

• The duration of an ad is at most 40 seconds long.

• Up to 50 different bidders (media agencies) are interested in placing ads during the average campaign time span.

• Each bidder has its own budget and target customer group that defines the slots she is interested in.

• The reserve price per second during a particular time is set by the TV station, which puts different slots into sets with different reserve prices.

Although the reported evaluation concentrates on a biweekly market (336 slots), a series of experiments with 168 slots (one week) and 504 slots (three weeks) was also performed to verify the robustness of the results presented here. The integrality gap was small in these cases as well (Table 3). We will therefore only report the detailed results for experiments with the biweekly market and 336 slots. The following parameters and distributions were used for the random variables of our experiments (see Table 2).

The normal distributions are truncated to an interval $[0; 2\mu]$. The Poisson distribution models the frequency of the six discrete reservation prices [1, 2, 5, 10, 50, 75], which follows the empirical distribution that we observed in the field. The bid base price β_j can be interpreted as how much a bidder would spend at a maximum to obtain the right to reach one priority point with her ad for one second. The actual bid price for a campaign is then computed as $b_j = d_k \beta_j w_j^{min}$. This means, for the bid price we multiply the duration of the ad, the base price of the bidder for one viewer, and the minimum reach or viewership the bidder wants to achieve. Based on the parameters of Table 2, we generated 20 scenarios used in our experiments.

Table 3   Average Integrality Gaps for 168, 336, and 504 Ad Slots After 300 and 3,600 Seconds

Slots	Time limit (%)	
	300 s	3,600 s
168	0.04	0.01
336	0.10	0.01
504	0.20	0.02

Table 4   **Efficiency E, Revenue R, and Duration D in Minutes for the TV Ad Market Experiments**

Iteration	TRIM			REUSE			REUSE+			Optimal (LPR)		
	E	R	D	E	R	D	E	R	D	E	R	D
1	0.93	0.78	115	0.95	0.65	290	0.96	0.60	580	1.00	—	—
2	0.90	0.83	105	0.93	0.70	225	0.93	0.72	890	1.00	—	—
3	0.88	0.68	95	0.89	0.67	205	0.91	0.65	480	1.00	—	—
4	0.97	0.65	95	0.97	0.66	310	0.98	0.65	410	1.00	—	—
5	0.94	0.78	105	0.96	0.63	245	0.96	0.65	460	1.00	—	—
6	0.89	0.85	85	0.92	0.73	150	0.94	0.68	450	1.00	—	—
7	0.91	0.76	75	0.92	0.66	100	0.93	0.71	340	1.00	—	—
8	0.90	0.76	100	0.92	0.70	245	0.92	0.67	490	1.00	—	—
9	0.93	0.92	105	0.97	0.71	205	0.98	0.71	390	1.00	—	—
10	0.92	0.81	105	0.95	0.66	245	0.96	0.66	560	1.00	—	—
11	0.93	0.79	90	0.96	0.65	185	0.97	0.66	400	1.00	—	—
12	0.89	0.85	110	0.94	0.68	295	0.95	0.65	480	1.00	—	—
13	0.95	0.80	85	0.95	0.75	120	0.96	0.77	330	1.00	—	—
14	0.84	0.84	90	0.92	0.68	220	0.92	0.69	590	1.00	—	—
15	0.91	0.66	95	0.91	0.64	195	0.93	0.63	680	1.00	—	—
16	0.92	0.76	90	0.92	0.77	300	0.93	0.74	610	1.00	—	—
17	0.92	0.78	100	0.93	0.73	170	0.94	0.67	470	1.00	—	—
18	0.89	0.79	95	0.92	0.64	300	0.93	0.60	430	1.00	—	—
19	0.84	0.83	60	0.88	0.74	90	0.90	0.71	370	1.00	—	—
20	0.93	0.79	95	0.95	0.72	225	0.95	0.73	410	1.00	—	—
μ	0.91	0.79	95	0.93	0.69	216	0.94	0.68	491	1.00	—	—
σ	0.03	0.06	12	0.03	0.04	65	0.02	0.04	129	0.00	—	—

5.2. Results of the TV Ad Market Experiments

We will first report the efficiency E, revenue R, and duration D in minutes in Table 4. The REUSE algorithm is able to improve the winning coalition if a coalition of bidders with higher revenue is found, and therefore the REUSE efficiency is higher than TRIM's (p-value < 0.001). However, the actual revenue R generated by the payments from the REUSE algorithm is consistently lower than that of TRIM (p-value < 0.001), despite the fact that a higher coalitional value was achieved. REUSE+ describes the results of running the REUSE algorithm with a time limit of 3,600 seconds for the first winner-determination problem and 600 seconds for every subsequent optimization problem. This helps understand the impact of allowing for a longer computation time. This impact is low as the numbers in Table 4 show, illustrating that even twice the computation time has little impact on efficiency and revenue.

To understand why the overall revenue is significantly higher for TRIM in spite of the lower efficiency, we compared the ratios between the bids submitted and the resulting VCG and BPOC payments. Table 5 provides an overview of the average secondary metrics across all experiments. The ratios were all significantly higher for TRIM than for REUSE (p-value < 0.001).

In addition to these aggregate secondary metrics, we provide a more detailed summary in Figure 3, where a single frame groups an algorithm and a metric. For each algorithm and metric, we aggregated the individual (i.e., bidder-wise) ratios for all bidders in small box plots. In each of the frames of Figure 3, the light gray area of the box plot marks the interquartile range for a specific metric and the line the median for one of the 20 experiments. This provides an overview of the ratio distribution for all 50 bidders. Finally, the solid line across the box plots in each frame marks the overall mean of all ratios in all scenarios. What follows is a brief interpretation of these values.

The core/bid ratio for TRIM shows that the core prices are close to the bid prices submitted and different from the results seen in the second row in Figure 3 for the REUSE algorithm. Multiple factors influence this high ratio for TRIM: As seen in §4, the VCG payment vector sets the lower bound for the BPOC payment computation. Because of the possibility of switching to coalitions with a higher coalitional value, the VCG payment vector computed by the REUSE algorithm is always at most as high as the TRIM algorithm relative to the coalitional value of the respective winner coalition. If the coalitional value computed with TRIM $\text{WD}^a(W)$ is smaller than all $\text{WD}^a(W_{-k})$ for all winners k, the VCG payments are equal to the winning bid prices. In contrast, REUSE would switch the winning coalition in such a case,

Table 5   **Average Ratios for Secondary Metrics**

Secondary metrics	TRIM			REUSE		
	μ	σ		μ	σ	
Core/bid	0.88	0.18	▲	0.69	0.25	▼
VCG/bid	0.75	0.31	▲	0.39	0.29	▼
VCG/core	0.84	0.30	▲	0.53	0.37	▼

Note. ▼ and ▲ values are significantly lower or higher, respectively, compared with the other algorithm.

Figure 3   (Color online) Secondary Metrics of the TV Ad Market Experiments
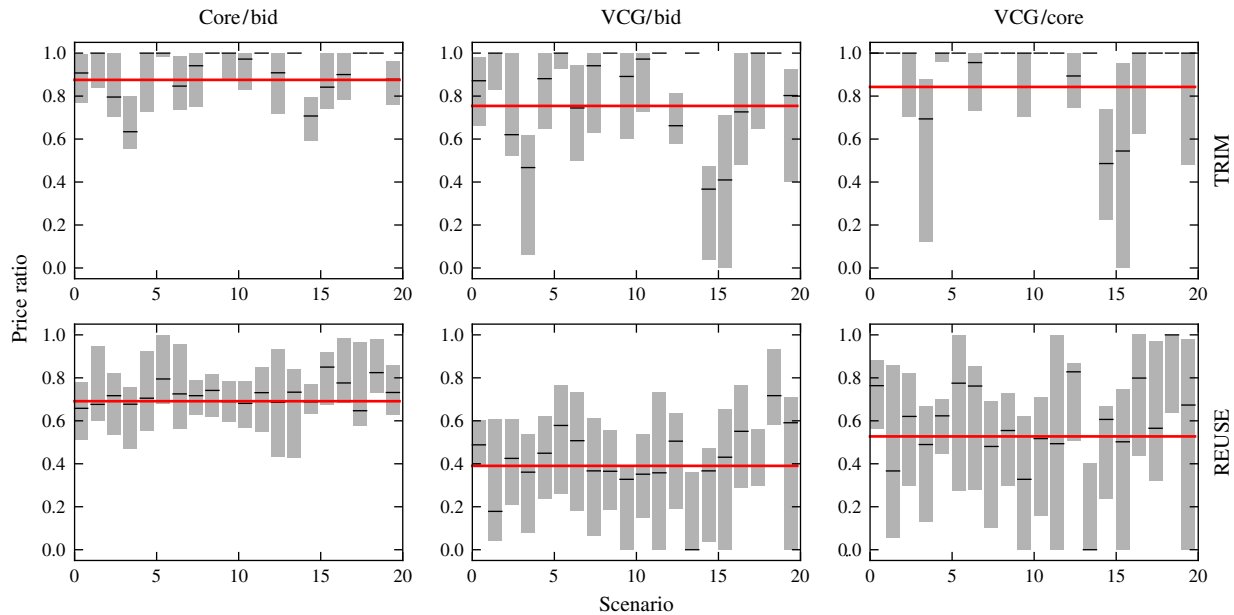
effectively increasing the difference between $\mathrm{WD}^a(W)$ and $\mathrm{WD}^a(W_{-k})$. This in turn increases the VCG discount of individual bidders and hence decreases the VCG payments for each bidder.

Similarly, if a blocking coalition is found during the BPOC payment computation in TRIM, where the coalitional value is higher than the coalitional value of the winning coalition at this point, then this coalition remains blocking even if all BPOC payments of the winners are at their bid price. In contrast, REUSE would switch the winning coalition, effectively raising the upper bound on the BPOC payments, in addition to lowering all VCG payments as described in the last paragraph. In the plots describing the VCG/bid ratio and those describing the core/bid ratio, the values for TRIM are higher than those for REUSE. In many TRIM instances, the median ratios are 1.0—i.e., for most winners the VCG and BPOC payments correspond to their bid price.

The overall duration D needed to compute using the REUSE algorithm takes significantly longer than when using TRIM (p-value < 0.001), even if each optimization run is restricted to the same limit of 300 seconds. The REUSE algorithm updates the winning coalition 3.5 times on average, whereas the TRIM algorithm will always maintain the initial coalition. Updating the winning coalition also initiates new VCG computations, which explains why the REUSE algorithm takes 63% longer than the TRIM algorithm.

5.3. Research Design for the Volume Discount Auctions

To generate bid data for the volume discount auction format, we draw on the cost function and the experiments base on Bichler et al. (2011). Being a

procurement auction, all primary and secondary metric ratios (i.e., the E, R, core/bid, VCG/bid, VCG/core) have to be reversed to allow an easier comparison between the two auction formats. The multiproduct cost function $c_s(x_1, \ldots, x_{|I|})$ used by Bichler et al. draws on econometric literature and enables a systematic evaluation of markets with different economies of scale and scope. Based on these cost functions, incremental volume discount bids are generated approximating the cost curve. We could fortunately use the very same bid generation as described in Bichler et al. (2011). These bids serve as an input to the winner determination problem.

We will briefly introduce the cost function and the main parameters. There are $s \in S$ suppliers competing for a fixed quantity W_i of one or more items $i \in I$, and x_i describes the quantity produced of each item.

$$c_s(x_1, \ldots, x_{|I|}) = \sum_{i \in I} B_{i,s} \lceil x_i/z_i \rceil + \sum_{i \in I} \beta_{i,s}(x_i/\gamma_{i,s})^\rho.$$

The function allows us to model very different shapes with convex and concave sections. The item-specific stepwise fixed cost of supplier s for item i is denoted by $B_{i,s}$. The parameter z_i models the capacity bound, after which an additional machine or plant is needed, adding an additional $B_{i,s}$ fixed costs. Note that with the inclusion of stepwise fixed costs, the cost functions are no longer continuous. The term $\beta_{i,s}$ describes the slope of a variable cost function for product i, and the exponent ρ is the nonlinear element in the cost function, representing economies (or diseconomies) of scale. The distribution for ρ was truncated at zero such that only positive values were drawn. Parameter $\gamma_{i,s}$ moderates the economies of

Table 6   Parameters of the Cost Curve $c_s(x_1, \ldots, x_I)$

Parameter	Description	μ	σ_{item}	$\sigma_{supplier}$
$B_{i,s}$	Per item (stepwise) fixed costs	100.0	0.0	0.0
z_i	Capacity of production line (%)	100	0	0
ρ	Power of the variable cost function	0.0	0.1	1.0
$\beta_{i,s}$	Slope of the variable cost function	1,000.0	0.0	0.0
$\gamma_{i,s}$	Slope delay of the variable cost function	100.0	0.0	0.0

scale. A brief summary of all relevant variables and their distributions can be found in Table 6.

An example of an average cost function based on Table 6 can be found in Figure 4. The winner determination problem minimizes total costs of the procurement manager and is formally described in Bichler et al. (2011). This mixed-integer program allows for various allocation constraints. The only constraint used in our experiments was that a supplier could only win a certain percentage of the volume of each item (20%, 40%, 60%, and 80%), but nothing in between. This requirement can be found in the field where procurement managers try to avoid odd quantity splits.

5.4. Results of the Volume Discount Auction Experiments

In each of our experiments, 14 bidders submit volume discount bids for eight items. We chose this problem size because it is a realistic problem size, but at the same time, the exact solution can be computed within a few hours at most. No time constraints were imposed. This allows us to report the ratio of the optimal (OPT) to the near-optimal integer solution as efficiency E, in contrast to the TV ads problem where we could only use the LPR, because the optimal integer solution of the winner-determination problem proved intractable for all but unrealistically small problem instances. While the optimal allocation was indeed found for all instances, 4 out of the 40 experiments were aborted because of out of memory exceptions during the BPOC payment computation, as seen in lines 37–40 of Table 7. However, even in such cases, the integrality gap of the near-optimal solution was low, on the order of 4% at most.

Table 7   Efficiency E, Revenue R, and Duration D in Minutes for the Volume Discount Auction Experiments

Iteration	TRIM E	R	D	REUSE E	R	D	OPT E	R	D
1	0.96	0.90	2.21	0.97	0.85	3.06	1.00	0.88	100
2	0.92	0.90	2.56	1.00	0.77	2.22	1.00	0.86	195
3	0.99	0.83	3.06	1.00	0.77	2.56	1.00	0.85	47
4	0.99	0.82	2.21	0.99	0.82	2.39	1.00	0.84	56
5	0.95	0.86	2.72	0.99	0.74	2.72	1.00	0.79	62
6	0.99	0.80	2.41	0.99	0.80	2.04	1.00	0.85	42
7	0.99	0.77	2.21	0.99	0.73	2.22	1.00	0.78	62
8	0.99	0.82	2.90	1.00	0.80	2.22	1.00	0.84	86
9	0.85	0.85	2.38	0.98	0.82	2.56	1.00	0.91	61
10	0.97	0.81	2.56	0.97	0.80	3.23	1.00	0.80	125
11	0.99	0.78	2.04	0.99	0.78	1.87	1.00	0.79	44
12	0.96	0.91	2.21	1.00	0.80	2.22	1.00	0.82	42
13	1.00	0.76	2.89	1.00	0.78	3.06	1.00	0.87	70
14	0.97	0.83	2.22	0.97	0.81	2.22	1.00	0.80	43
15	1.00	0.74	2.89	1.00	0.74	2.73	1.00	0.77	37
16	0.96	0.88	2.89	1.00	0.78	2.22	1.00	0.80	108
17	0.98	0.88	2.55	1.00	0.85	2.39	1.00	0.89	144
18	0.99	0.74	2.55	0.99	0.75	2.55	1.00	0.76	77
19	1.00	0.77	2.21	1.00	0.79	2.55	1.00	0.82	53
20	0.98	0.78	2.38	1.00	0.74	2.90	1.00	0.80	68
21	0.99	0.82	2.21	0.99	0.82	2.21	1.00	0.83	77
22	1.00	0.79	3.57	1.00	0.77	2.72	1.00	0.81	183
23	0.99	0.81	3.57	1.00	0.73	2.55	1.00	0.83	53
24	0.99	0.79	2.56	0.99	0.79	2.39	1.00	0.82	52
25	1.00	0.77	2.90	1.00	0.73	2.55	1.00	0.81	46
26	0.97	0.90	2.56	0.99	0.86	2.38	1.00	0.86	37
27	0.99	0.80	2.56	0.99	0.82	2.38	1.00	0.82	41
28	0.94	0.85	3.06	0.99	0.80	3.23	1.00	0.81	89
29	0.98	0.78	2.21	0.99	0.77	2.21	1.00	0.76	73
30	0.98	0.91	3.23	0.99	0.85	2.38	1.00	0.85	75
31	0.90	0.90	2.56	0.99	0.85	2.56	1.00	0.83	57
32	0.99	0.75	2.38	1.00	0.75	2.73	1.00	0.78	35
33	0.95	0.86	2.72	1.00	0.78	2.21	1.00	0.82	71
34	0.98	0.84	2.38	0.99	0.81	2.89	1.00	0.82	41
35	0.97	0.76	2.89	1.00	0.71	3.07	1.00	0.77	68
36	0.99	0.78	2.55	1.00	0.73	2.38	1.00	0.79	50
37	0.99	0.76	2.62	0.99	0.77	2.63	1.00	—	—
38	0.99	0.77	2.63	0.99	0.81	2.80	1.00	—	—
39	0.96	0.85	2.93	0.99	0.79	3.12	1.00	—	—
40	0.96	0.88	2.97	0.98	0.79	2.95	1.00	—	—
μ	0.97	0.82	2.63	0.99	0.79	2.56	1.00	0.82	71
σ	0.03	0.05	0.36	0.01	0.04	0.34	0.00	0.04	38

As in the previous section, the experiments with TV ads, the efficiency of REUSE is significantly higher than for the TRIM algorithm (p-value < 0.001). Additionally, the efficiency of TRIM and REUSE with a time limit of 300 seconds per optimization

Figure 4   (Color online) Average Costs for One Unit Produced and Increasing Quantity Sold
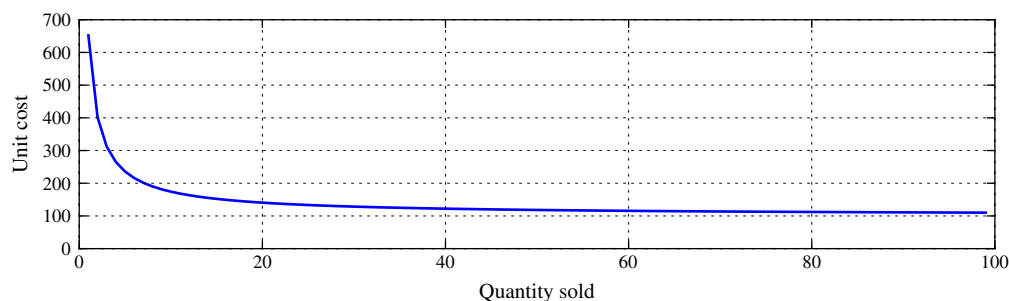
Table 8   Average Ratios for Secondary Metrics

Secondary metrics	TRIM			REUSE			OPT	
	μ	σ		μ	σ		μ	σ
Bid/core	0.85	0.15	▲	0.81	0.15	▼	0.82	0.11
Bid/vcg	0.80	0.18	▲	0.72	0.16	▼	0.82	0.11
Core/vcg	0.93	0.13	▲	0.90	0.14	▼	1.00	0.00

Note. ▼ and ▲ values are significantly lower or higher, respectively, compared with the competing BPOC algorithm.

are significantly lower than the optimal efficiency (*p*-value < 0.001). Also the results on revenue are in line with the TV ad market experiments: The revenue achieved with REUSE is significantly lower than with TRIM.

The secondary metrics are illustrated in Table 8 and Figure 5 similar to what we have reported for the TV ad market experiments in §5.1. However, now we are also able to compare the ratios of payment vectors using near-optimal solutions with those if the problems are solved optimally. Note that in the optimal solution the core payment vector coincides with the VCG payments. We conjecture that this is because we

did not have economies of scope in our cost functions. Table 8 shows that again the ratios for TRIM are higher than those for REUSE. The difference is again significant (*p*-value < 0.001 for bid/core and bid/VCG, *p*-value of 0.02 for core/VCG), but not as high as in the TV ad market experiments. This is because the integrality gap was lower for the instance sizes chosen.

A comparison with the price vectors based on the optimal solution, OPT, shows that the core payments achieved with TRIM are indeed very high. The difference of bid/core between TRIM and OPT was significant (*p*-value < 0.001). The difference between the bid/core ratios of OPT and reuse REUSE was not significant (*p*-value of 0.4). The bid/VCG ratios of TRIM and OPT were not significantly different (*p*-value of 0.965). However, the core/VCG ratio was significantly higher for OPT than for TRIM and REUSE (*p*-value < 0.03). This is because there was no difference in OPT, i.e., both payment vectors coincided, while there was a difference with the near-optional winner determination in TRIM and REUSE.

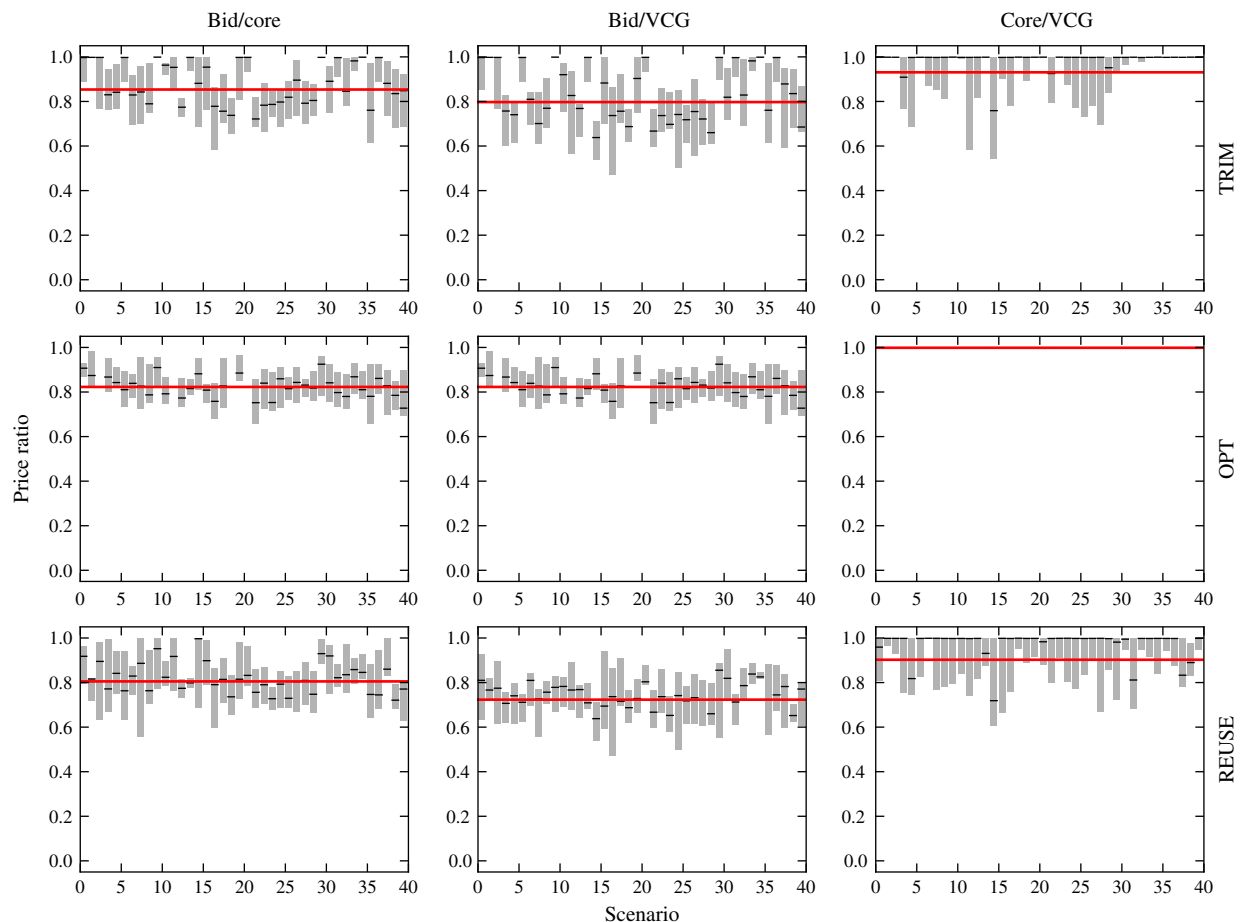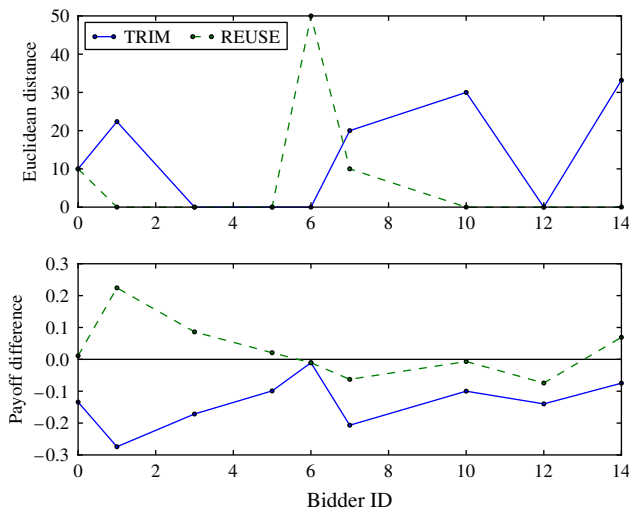Figure 5 again provides a more detailed view of the secondary metrics. As we have seen in the TV ad mar-

Figure 5   (Color online) Secondary Metrics of the Volume Discount Auction Experiments

Figure 6 (Color online) Allocative Distance and Relative Payoff Difference Between TRIM or REUSE and OPT for All Bidders in a Single Auction Instance

Table 9 Summary of the Primary Metrics Comparing TRIM and REUSE

		TRIM			REUSE			Baseline
		μ			μ			μ
TV ad market								LPR
Efficiency	E	0.914	▼	∘	0.928	▲	∘	1.000
Revenue	R	0.788	▲	—	0.680	▼	—	—
Run time (minutes)	D	95	▼	—	222	▲	—	—
Volume discount auction								OPT
Efficiency	E	0.988	▼	∘	0.994	▲	∘	1.000
Revenue	R	0.807	▲	∘	0.786	▼	∘	0.818
Run time (minutes)	D	3		∘	3		∘	54

Note. ▼ and ▲ indicate significant difference compared with the competing BPOC algorithm; ∘ indicates significant difference from the baseline.

ket experiments, bidders' final BPOC and VCG payments are nearer to the bid price than necessary on an aggregate level, often as low as their bid prices for the volume discount auctions if the TRIM algorithm is used.

The availability of optimal solutions in these experiments allows us to compare how the allocation and the payments for individual bidders would differ in OPT, TRIM, and REUSE. These differences are described in Figure 6 for a specific auction. The upper part of the figure describes how the final allocation of TRIM or REUSE differs from the one in OPT by treating each bidder's allocation as a vector and computing the Euclidean distance between the different allocations. For example, suppose there were just two different items, A and B, and two different allocations X_{i1} and X_{i2} describing the quantity of each item that the bidder i has to provide. Then, for $X_{i1} = (10, 10)$ and $X_{i2} = (0, 15)$, the Euclidean distance is

$$\sqrt{((10-0)^2 + (10-15)^2)} \approx 11.2.$$

The lower part of Figure 6 shows how the payoff in TRIM or REUSE differs from the one in OPT. To gain insight about the magnitude of the payoff difference, the values are normalized against the average payment the auctioneer has to provide to the winners. In this example, many bidders have a lower payoff in TRIM because the payments in this reverse auction are lower in TRIM, which is indicated by a high bid/core ratio. For REUSE, the individual payoff can even surpass the payoff obtained when computations are solved to optimality. Also, a change in the allocative efficiency does not necessarily cause drastic changes in every bidder's payoff, as we see a payoff difference close to zero for bidder 6, for example. The

computational hardness of these problems is such that these differences from the optimal benchmark can in some cases not be avoided, but the overall efficiency of the near-optimal solutions is still very high in all experiments.

Table 9 provides a summary of the primary metrics to compare TRIM and REUSE, showing that the main results are the same in the TV ad market experiments and the volume discount auction experiments. Overall, if speed and revenue are primary concerns, then TRIM may be the right approach. In other situations, where incentives for truthful bidding and high efficiency are a concern, the REUSE algorithm is preferable.

6. Conclusions

The design of large-scale markets where bidders have complex preferences has been given little attention in the literature as of yet. In several countries, regulators sell dozens or hundreds of licenses to telecom companies. The incentive auctions in the US are another example where complex bidder preferences and allocation constraints lead to computationally hard allocation problems. Similar examples can be found in many other domains including the sale of TV ads to media agencies or multi-item and multiunit industrial procurement auctions. Much research in market design has focused on ascending combinatorial auctions with a fully expressive XOR bid language, and such designs have recently been used for selling spectrum (Cramton 2013, Bichler et al. 2013) and in logistics and procurement (Bichler et al. 2006). Such designs do not scale to large markets because of the exponential growth in the number of package bids that can be submitted.

We describe an auction design framework using compact bid languages and payment rules that incentivize truthful bidding. In markets where bidders have independent private values, which is the standard assumption in auction theory, this can yield

highly efficient allocations. Compact bid languages can often draw on domain specifics and allow bidders to describe their preferences with a low number of parameters that they have to specify, as the TV ad market and the volume discount auctions in this paper illustrate. Commercial off-the-shelf mixed-integer programming solvers can now solve large and realistic instances of such problems to near optimality on standard hardware, which allows us to use such bid languages in real-world markets. Such compact bid languages, however, defy the ask pricing rules typically used in ascending combinatorial auctions (Scheffel et al. 2011), but they can easily be used in sealed-bid auctions.

In sealed-bid auctions, second price rules such as VCG or BPOC payment rules can be used to provide incentives for truthful bidding. In many markets, auctioneers would prefer core pricing to VCG mechanisms to avoid noncore outcomes where the bids of losing bidders are higher than the payments of the winners. With the introduction of core-selecting auctions for spectrum licenses in recent years, stakeholders have developed software to determine winners and core prices based on the use of integer programming to solve a series of winner-determination problems. Extending the use of this software to larger and more complex markets (such as the TV ads and procurement contexts we address here) cannot be accomplished by merely specifying time limits or optimality-gap thresholds to the solver engine because it could for the more simple case of a single optimization problem. Doing so would often result in an infeasible pricing problem. This general problem exists for all larger markets with near-optimal winner determination.

We compared two potential algorithms for dealing with these infeasibilities, finding one faster and higher revenue method (for a fixed set of bids) and one slower but more efficient method. Our results show that the former TRIM algorithm may be suited to a fast-clearing market in which speculation to lower bids is offset by uncertainty about the competition. For other applications, such as government spectrum auctions the goal of public efficiency might outweigh the computational costs and suggest an advantage for the latter REUSE algorithm.

Further study may improve the application of core-selecting auction algorithms to large and complex markets like the TV ad and volume-discount markets, but we have provided the first steps to the extension of the core-selecting auction paradigm beyond provably optimal winner-determination settings. The paper shows that the overall auction design framework using compact bid languages and second-price payment rules provides a computationally feasible approach to achieve high efficiency in large-scale markets with dozens or hundreds of items.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2014.2076.

Appendix

PROOF OF THEOREM 1. The reduction is from the decision version of the strongly $\mathcal{NP}$-hard multiple knapsack problem: given a set of n items and a set of m knapsacks ($m \leq n$), with a profit b_j and a weight d_j for each item j, and a capacity c_i of each knapsack i, can you select m disjoint subsets of items such that the total profit of the selected items exceeds a given target profit T, with each subset assigned to a knapsack, and the total weight of any subset not exceeding the capacity of the assigned knapsack?

To see that this problem is a special instance of the WD problem, let the minimum price per unit $r_i = 0$; also let each bidder only bid with a single bid (item) j with a bid price of b_j, and each bidder's priority vector $W_k = \{1, \ldots, 1\}$ with a $w_j^{\min} = 1$. This means, he wants his ad with a length (weight) d_j to be assigned to one out of all slots (knapsacks) i with a duration (capacity) c_i. The multiple knapsack decision problem can be answered affirmatively if and only if this specific WD instance has an optimal objective value greater than or equal to T. The problem is in $\mathcal{NP}$ because it is straightforward to check for a given solution, whether it is correct. □

List of Symbols

b_j	bid amount for bid j
b_k^*	bid amount of bidder k's winning bid
C	coalition of bidders. $C \subseteq K$
C^t	most violated coalition relative to the current payment vector at iteration t
I	set of goods offered/requested.
i	index of a single good i
J	set of bids containing all bid indexes
j	index of a single bid j
J_k	set of bids containing all bid indexes j by a bidder k
K	set of bidders
k	index of a single bidder k
p^t	payment vector at iteration t
p_k^t	payment amount for bidder k at iteration t
p_k	shorthand notation for p_k^t with the currently highest t
p_k^{vcg}	VCG payment amount for bidder k
$w_j^{\min}$	minimum sum of weight values to justify a monetary bid b_j
W_k	weight vector for bidder k
w_{ik}	weight value for time slot i and bidder k
x_{ij}	decision variable indicating that bid j is assigned to time slot i
y_j	decision variable indicating that the bid j is accepted
$z(p^t)$	computed value of SEP^t
LPR	solution of the linear programming relaxation of the winner determination problem
OPT	optimal solution of the winner determination problem
SEP^t	core separation problem

WD   winner determination problem, optimally solved
WD^a winner determination problem, approximately solved

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