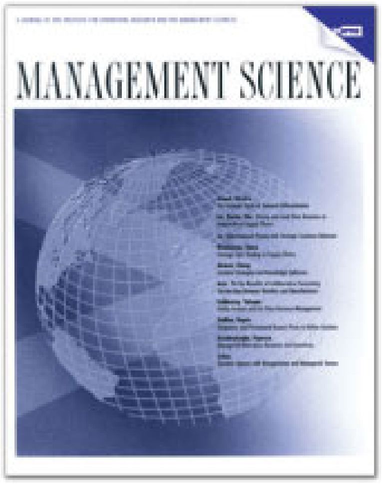

Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MANAGEMENT SCIENCE
Vol. 61, No. 2, February 2015, pp. 474–484
ISSN 0025-1909 (print) | ISSN 1526-5501 (online)

http://dx.doi.org/10.1287/mnsc.2014.1948

Timing of Product Allocation: Using Probabilistic Selling to Enhance Inventory Management

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This paper examines probabilistic selling (PS) as an inventory-management mechanism, paying special attention to the impact of the timing of product assignment to buyers of probabilistic goods. In practice, sellers tend to offer probabilistic products only after major demand uncertainty has been resolved. By deferring product assignments, a firm is able to obtain more information about demand for each specific product before deciding which product to assign to consumers. However, our analysis demonstrates that PS can be an effective inventory-management mechanism even if the firm allocates products before knowing which product will be more popular and, thus, scarcer. Interestingly, we show that it can be more profitable for the firm to allocate products to consumers before, rather than after, learning the true demand for a product because, although early allocation imposes higher inventory costs (as a result of larger required inventory levels), it also enables the firm to charge higher prices. Our results also reveal that, when introducing probabilistic goods, the firm should order less inventory (relative to the case where probabilistic goods are not offered) if costs are very low but more inventory otherwise. Finally, we show that PS, as an inventory-management mechanism, can create a win–win situation, both improving profit and increasing social welfare.

Keywords: inventory production; uncertainty; marketing; retailing and wholesaling; pricing
History: Received May 11, 2012; accepted February 19, 2014, by J. Miguel Villas-Boas, marketing. Published online in *Articles in Advance* July 17, 2014.

1. Introduction

Probabilistic selling (PS) is a marketing strategy that is becoming more commonplace. In today's travel industry, in addition to "opaque" travel services offered by Priceline and Hotwire (the two industry pioneers selling probabilistic goods), various discounted probabilistic services are available to travelers, from both service providers (e.g., "blind booking" airline tickets on Germanwings.com) and online travel agents (e.g., "undercover" hotels on LastMinuteTravel.com, "secret carrier" flights on KAYAK.com, and "mystery hotels" on SuperBreak.com). When purchasing probabilistic services from these Internet sites, some specific attributes of the service (e.g., the itinerary of the flight, the location of the hotel, even the vacation destination) are not revealed to buyers before payment. Also, an increasing number of online retailers (e.g., SwimOutlet.com, BustedTees.com, AgonSwim.com, Tmall.com, Buloso.com), including some very prominent ones (e.g., Amazon), have begun to offer discounted "grab bag" apparel, toys, and shoes, where certain product attributes (e.g., color, style, design, pattern) are not known to the consumer prior to purchase.

As more firms adopt PS and more consumers experience probabilistic goods, further academic attention is needed to deepen our understanding of this relatively new business concept and to provide managerial insights. In this paper, we advance the theory of probabilistic selling in two ways. First, we integrate the demand-side decision (pricing) and the supply-side decision (inventory) into the PS business model and examine how PS can be used to enhance inventory management. Difficulty in predicting future demand for a specific product can severely impede inventory efficiency and reduce profits: inventory holding costs and unprofitable liquidations occur when an inventory order exceeds realized demand, and loss of potential sales and consumer goodwill arises when demand exceeds the inventory order. Analyses in the extant research on probabilistic goods (e.g., Fay and Xie 2008, 2010) have mainly focused on the pricing decision and consider inventory as an exogenous variable.[1] This assumption is reasonable for some markets in which

[1] One exception is Rice et al. (2014), who examine PS's advantage as a price-discrimination tool compared with *markdown selling*. By focusing on a fully covered market even in the absence of a probabilistic good, their model demonstrates that PS can never result in either more sales or larger inventory orders. In contrast, our model allows the degree of market coverage to depend on the selling mechanism chosen by the firm, thus enabling sales and inventory orders potentially either

changes in capacity (e.g., expanding the size of a hotel) are very costly. However, in many industries, such as apparel, household goods, and electronics, inventory orders are placed on a regular basis, e.g., before each selling season. For these firms, inventory management can significantly impact profitability. Hence, it is important, both from a theoretical and from a practical perspective, to understand how the seller should adjust its inventory decisions when adopting the PS strategy.

Our second contribution is to offer insights on how the seller's timing of product assignment impacts inventory decisions and profitability. Under the PS strategy, sellers need to decide when to assign the specific product to buyers of probabilistic goods (e.g., whether to delay assignments substantially after the purchase). We propose that the primary benefit of delaying product assignment is that it provides an opportunity for the seller to learn more about demand.[2] From the perspective of inventory management under demand uncertainty, it would seem to be more profitable if product allocation to consumers were made after, rather than before, the seller has obtained more information about market demand. In this paper, we challenge the generality of this conventional wisdom. Specifically, we capture the impact of allocation timing on the inventory-management decision and the profitability of PS by examining two extreme cases: (1) *PS-Early*, where product assignment is made before the seller knows which product will be more popular, and (2) *PS-Late*, where the seller observes all demand before making allocation decisions. In most real-world examples of PS, firms wait until there are mismatches between inventory and demand before they offer units of their product to consumers (e.g., by allocating hotel rooms for sale through Hotwire). Thus, PS-Late is a close approximation of reality for many current implementations of PS. Although PS-Early does not occur in reality, we use this case to raise the issue of the timing of product allocation and to examine whether moving up the selling time can be potentially beneficial.

To explore the impact of waiting for decreased demand uncertainty, we compare the cases of PS-Early and PS-Late. It is reasonable to assume that delaying product assignment may negatively affect consumers' willingness to pay since consumers most likely prefer to know their assignments earlier. For example, when purchasing an airline ticket, a consumer may need to know the time of arrival in order to make complementary purchases and/or reservations such as lodging, car rental, and concert/theatre tickets. In our analysis, however, to focus on other, less intuitive economic forces behind the time-of-allocation decision, we intentionally exclude this potential disadvantage from the model and assume that delays, in and of themselves, will *not* reduce consumer valuations as a result of scheduling inconvenience or opportunity costs caused by waiting. If we are able to show the profit advantage of early over late allocation under this assumption (and we can), the advantage of allocating early would be even stronger in a more realistic setting, i.e., when the negative effect of customers' distaste for waiting is considered.

A key finding from our analysis of PS as an inventory-management mechanism is that late allocation is *not* always beneficial to sellers facing limited capacity and uncertain demand. This finding is intriguing because under PS-Early, the seller does not use information about demand to assign products. Lack of information makes it more difficult to shift demand toward the less popular product; i.e., buyers of the probabilistic good are more likely to receive the ex post popular (and thus scarcer) product. One might expect that delaying assignments would increase profit since the seller can avoid providing units of the scarcer product at a discount. This profit advantage from late allocation can be particularly strong if consumers do not incur costs from waiting (as assumed in our model). However, we show that late assignments induce a countervailing effect—a loss of commitment to make assignments truly random. When assignments are made prior to the firm learning about its demand, it is intuitive and believable that the product allocation will be randomized. However, when assignments are delayed, consumers anticipate that the seller will use its acquired information when assigning products. Thus, expecting to receive the less popular item under late assignment, consumers value the probabilistic good more under early allocation, and the seller can thereby maintain higher prices. If costs are sufficiently low, the demand-side gain from making early assignment can exceed the supply-side loss. It is interesting to note that the profit advantage of PS-Early is fundamentally driven by removing the seller's informational advantage, i.e., contracting with the buyer regarding product assignment when both the seller and buyer are uninformed rather than at a time when the seller has a significant information advantage. The economic literature has suggested that information advantages can hurt sellers in auction markets (e.g., Bazerman and Samuelson 1983, Thaler 1988). Our finding demonstrates that such a dark side of seller information advantage can also occur in posted-price markets.

Another interesting finding is that PS can create a win–win situation: introducing a probabilistic good

to decrease or to increase when a probabilistic good is introduced. Also, the current paper examines a different issue: how the firm's timing of product allocation may impact the pricing, inventory decisions, and profit of PS.

[2] Researchers have identified several ways to defer allocation decisions until the seller has more information about demand, including "flexible goods" (Gallego and Phillips 2004), "contingent pricing" (Biyalogorsky and Gerstner 2004), "callable" products (Gallego et al. 2008), and "upgradeable" tickets (Biyalogorsky et al. 2005).

can both improve a seller's profit and increase social welfare. In particular, the PS strategy improves market efficiency by expanding service to otherwise unserved consumers and by reducing the number of units that go unsold.

2. Model

2.1. Assumptions

A seller produces two products, $j = 1, 2$. Demand is modeled by using a Hotelling model, in which the value for one's ideal product is normalized to 1, the fit-cost-loss coefficient equals 1, and the consumer's location on the Hotelling line is x_i. Let v_{ji} be the value of product j to consumer i: $v_{1i} = 1 - x_i$ and $v_{2i} = x_i$. Each consumer buys at most one unit of one good, choosing the product that maximizes the expected surplus. Demand uncertainty is captured in the model by having the seller be uncertain about which product is the more popular (each is equally likely to be more popular). Specifically, with probability $\frac{1}{2}$, x_i is distributed uniformly on the interval $[0, \frac{1}{2}]$, and with probability $\frac{1}{2}$, x_i is distributed uniformly on the interval $[\frac{1}{2}, 1]$.

When making inventory and pricing decisions, the seller does not know which of the two products will be more popular. Specifically, prior to the selling season, the seller orders K_j units of each product at a cost of $c_1 = c_2 = c$ per unit, where $0 < c < 1$. To keep the model simple, we assume that any units that go unsold have no value. However, incorporating positive salvage values would not qualitatively alter this paper's insights.[3] Furthermore, prices for each product (P_j) are set at the beginning of the selling period. Because of a priori identical expectations, the optimal prices and orders are symmetric: $P_1 = P_2 \equiv P$; $K_1 = K_2 \equiv K$. Next, demand is realized. If demand exceeds the available supply, inventory is allocated randomly, with each willing buyer having an equal probability of receiving a unit of the scarce product.

2.2. Traditional Selling

We begin our analysis by considering the case of traditional selling (TS), in which the seller does not offer probabilistic products. Demand for the popular good is

$$D_{\text{popular}} = \begin{cases} 1 & \text{if } P^{\text{TS}} \le \frac{1}{2}, \\ 2(1 - P^{\text{TS}}) & \text{if } \frac{1}{2} < P^{\text{TS}} < 1, \\ 0 & \text{if } P^{\text{TS}} \ge 1. \end{cases} \tag{1}$$

[3] Positive salvage values lessen the negative impact of having excess inventory. This reduces the magnitude of the profit advantage of PS-Early over traditional selling (when it exists) and increases the magnitude of the profit advantage of PS-Early over PS-Late (when it exists).

Since the same price is charged for each product, no consumer prefers to purchase the unpopular one. However, if $P^{\text{TS}} < \frac{1}{2}$, $2(\frac{1}{2} - P^{\text{TS}})$ consumers would be willing to buy the unpopular good if the popular one stocks out. Thus, for $P^{\text{TS}} < \frac{1}{2}$, demand for the unpopular good equals $\min[2(\frac{1}{2} - P^{\text{TS}}), 1 - K]$.

Let D_j be the realized demand for product j, and its specification depends on whether it is the popular or the unpopular product, as defined above. Given that each product's sales cannot exceed the number of units ordered, profit is given by

$$\Pi^{\text{TS}} = P^{\text{TS}} \min[D_1, K] + P^{\text{TS}} \min[D_2, K] - 2cK. \tag{2}$$

The seller chooses price (P^{TS}) and inventory order (K) to maximize the profit in (2). Table 1 summarizes the optimal solution. As shown in Table 1, under TS, it is optimal to set prices such that the demand for the popular good equals the available capacity. At this price, no consumers purchase the less popular item. Thus, units of the unpopular good go unsold. This suggests that demand uncertainty increases the true cost of each sold unit. As a result, the firm is only able to earn positive profit under TS if $c < \frac{1}{2}$.

2.3. The Probabilistic Selling Strategy with Early Allocation (PS-Early)

We now examine PS-Early, the probabilistic selling strategy with early allocation. The seller orders K^{early} units of each product. From this inventory, the firm sells each specific product at a price P^{early} and also a probabilistic good at a price P_o^{early}. The seller does not know which good is more popular when assigning the probabilistic good. Thus, the seller divides the demand for the probabilistic good equally across the two products. Since each product is equally likely to be received, a consumer's valuation for the probabilistic good is $(v_{1i} + v_{2i})/2$. Substituting the valuations given in §2.1, the expected value of the probabilistic good equals $\frac{1}{2}$ for all consumers. Thus, the optimal price of the probabilistic good is $P_o^{\text{early}} = \frac{1}{2}$. A lower price does not generate greater market coverage, and a higher price eliminates all demand for the probabilistic good (and thus makes PS-Early and TS equivalent strategies). Furthermore, note that selling at this price is profitable only if $c < \frac{1}{2}$.

Each consumer has four choices: (a) buy product 1, (b) buy product 2, (c) buy the probabilistic good, or (d) buy nothing; each consumer chooses the option that yields the highest expected surplus. Let D_j^{early} and D_o^{early} represent the demand for the specific products and the probabilistic good, respectively. The demand function for the specific goods (D_j^{early}) will be the same as under TS (i.e., Equation (1)) since the probabilistic good does not create any consumer surplus. Any consumer who does not wish to purchase a specified

Table 1 Optimal Solutions Under Traditional Selling and Probabilistic Selling

	Traditional selling[a]	PS-Early[a]	PS-Late[b]
Price	$P^{TS}=\frac{1+2c}{2}$	$P^{\text{early}}=\frac{3+2c}{4}$,	$P^{\text{late}}=\frac{5+4c}{8}$,
		$P_o=\frac{1}{2}$	$P_o=\begin{cases}\frac{3-4c}{8} & \text{if } c<\frac{1}{4},\\ \frac{1+4c}{8} & \text{if } \frac{1}{4}\le c<\frac{3}{4}\end{cases}$
Inventory order (same for both products 1 and 2	$1-2c$	$\frac{3-2c}{4}$	$\frac{3-4c}{4}$
Unsold units	$1-2c$	$\frac{1-2c}{2}$	$\begin{cases}\frac{1-4c}{2} & \text{if } c<\frac{1}{4},\\ 0 & \text{if } \frac{1}{4}\le c<\frac{3}{4}\end{cases}$
Total sales	$1-2c$	1	$\begin{cases}1 & \text{if } c<\frac{1}{4},\\ \frac{3-4c}{2} & \text{if } \frac{1}{4}\le c<\frac{3}{4}\end{cases}$
Profit	$\frac{(1-2c)^2}{2}$	$\frac{4c^2-12c+5}{8}$	$\frac{16c^2-24c+9}{16}$
Composition of probabilistic good	N/A	Both types	Unpopular good only

[a]These solutions assume that $c<\frac{1}{2}$; if $c\ge\frac{1}{2}$, it is not possible to earn a positive profit.
[b]This solution assumes that $c<\frac{3}{4}$; if $c\ge\frac{3}{4}$, it is not possible to earn a positive profit.

good (or cannot purchase because of a stockout) will buy the probabilistic good. Let the optimal number of sales of the popular product (at P^{early}) be X_{Pop}, and let the optimal number of sales for probabilistic goods be X_{PG}. To meet demand, the firm needs $X_{\text{Pop}}+(X_{\text{PG}}/2)$ units of the popular good. Since the firm does not know which good is more popular, the firm must order $K^{\text{early}}=X_{\text{Pop}}+(X_{\text{PG}}/2)$ units of both goods. Ordering more than this amount of inventory would be suboptimal since it would increase costs without increasing revenue (for a given X_{Pop} and X_{PG}). Furthermore, the firm could not obtain X_{PG} units of sales of the probabilistic good if it ordered less inventory. Also note that the optimal price, P^{early}, will be no less than $\frac{1}{2}$, since the firm can obtain a revenue of $\frac{1}{2}$ per unit by selling the unit as the probabilistic good. The seller's objective function is

$$\max\ \Pi^{\text{early}}_{X_{\text{Pop}},X_{\text{PG}},P^{\text{early}}}=\max\left[P^{\text{early}}X_{\text{Pop}}+\frac{X_{\text{PG}}}{2}-2c\left(X_{\text{Pop}}+\frac{X_{\text{PG}}}{2}\right)\right]$$

$$\text{s.t.}\quad 0\le X_{\text{PG}}\le 1-X_{\text{Pop}}\quad\text{and}\quad X_{\text{Pop}}=2(1-P^{\text{early}}),\qquad(3)$$

where the constraint on X_{PG} ensures that the sales of the probabilistic good are nonnegative and cannot exceed the demand for the probabilistic good, whereas the second constraint ensures that the chosen price attracts the number of consumers needed to sell X_{Pop} units of the popular good. By solving (3), we obtain the optimal values of X_{Pop}, X_{PG}, K^{early}, and P^{early}. Details of this analysis are provided in the appendix. Table 1 summarizes the optimal inventory, prices, and the resulting profit under PS-Early. In equilibrium, the market is completely covered. All units of the popular good are sold, but units of the unpopular good will go unsold.

2.4. The Probabilistic Selling Strategy with Late Allocation (PS-Late)

Under the PS-Late strategy, the seller delays assignments of the probabilistic good until it learns the demand for each product. Specifically, the seller orders K^{late} units of each product prior to observing demand. Also prior to learning demand, the seller sets prices: P^{late} for each specific product and P_o^{late} for the probabilistic good. After observing all specified good and probabilistic good sales, the seller decides which specific good to allocate to each probabilistic good purchaser. Thus, the key difference between PS-Late and PS-Early is that, with PS-Late, the probabilistic good assignments are made *after* the seller has learned which product is more popular.[4] Let X_{PG} be the total sales of the probabilistic good, and let K_P be the number of units of

[4] An alternative way to implement PS-Late would be first to sell the specific products and then offer the probabilistic goods. The

the popular good that are allocated to the probabilistic good buyers (where all other probabilistic good buyers receive the unpopular good). We consider two potential inventory allocation arrangements and allow the seller to choose the one that maximizes its profit: (1) the seller assigns the unpopular good to all buyers of the probabilistic good (i.e., no units of the popular good are sold at the discounted price, $K_P = 0$), and (2) the seller covers the entire market at the lowest possible cost (i.e., $K^{\text{late}} = \frac{1}{2}$). These two implementations align with the motivation voiced by practitioners and the extant literature that supply-side flexibility allows for the disposal of excess inventory. These arrangements also capture the fundamental characteristic of PS-Late, which is that, by delaying assignment, the seller obtains flexibility to use information about demand in order to manage inventories, but that this flexibility comes at the cost of consumers' recognizing that they do not have a 50/50 chance of receiving their preferred product (as discussed in §1).[5]

Prior to purchase, each consumer knows the values for the preferred and less preferred good. For notational convenience, assume that product 1 is preferred. Since all consumers prefer the same product, the expected surplus from purchasing the probabilistic good is $CS^{\text{PG}} = (K_P/D_{\text{PG}})(1-x_i) + ((D_{\text{PG}} - K_P)/D_{\text{PG}})x_i - P_o^{\text{late}}$. The first (second) term multiplies the probability of receiving the popular (unpopular) good by the consumer's valuation of the popular (unpopular) good. The third term accounts for the cost of obtaining a unit of the probabilistic good. Taking the derivative of CS^{PG} with respect to (w.r.t.) x_i: $\partial CS^{\text{PG}}/\partial x_i = (D_{\text{PG}} - 2K_P)/D_{\text{PG}} > 0$. Thus, the probabilistic good is most attractive to consumers with weaker preferences, i.e., farther from the endpoint. Let $\hat{X}$ be the x_i such that $CS^{\text{PG}} = 0$. Therefore, demand for the probabilistic good is $D_{\text{PG}} = 2(\frac{1}{2} - \hat{X})$.

The seller faces the following objective function:

$$\max \quad \Pi^{\text{late}}_{K^{\text{late}},\, K_P,\, P_o^{\text{late}},\, P^{\text{late}}} = \max \Big[P^{\text{late}}[2(1-P^{\text{late}})] + P_o^{\text{late}}\big(2\big(\tfrac{1}{2} - \hat{X}\big)\big) - 2cK^{\text{late}} \Big]$$

$$\text{s.t.} \quad \frac{K_P}{D_{\text{PG}}}(1-\hat{X}) + \frac{D_{\text{PG}} - K_P}{D_{\text{PG}}}\hat{X} - P_o^{\text{late}} = 0,$$
$$1 - P^{\text{late}} \le \hat{X} \le \tfrac{1}{2}, \quad K^{\text{late}} \ge 2(1-P^{\text{late}}) + K_P, \quad \text{and}$$
$$K^{\text{late}} \ge 2\big(\tfrac{1}{2} - \hat{X}\big) - K_P. \tag{4}$$

The first constraint defines the marginal consumer who will purchase the probabilistic good. The second constraint ensures that consumers purchase at most one product and that the demand for the probabilistic good is nonnegative. The third (fourth) constraint ensures that the amount of capacity is sufficient to meet the demand for the popular (unpopular) product, including sales to buyers of the probabilistic good.

The appendix details the derivation of the optimal solution to the maximization problem given by (4). This analysis reveals that the optimal price P^{late} is set so that all units of the popular good will be sold at the full price. Thus, only units of the unpopular good are left for buyers of the probabilistic good; i.e., the optimal solution has $K_P = 0$ even though our model allows the seller to choose $K_P > 0$. The price P_o^{late} is set so that probabilistic good sales will exhaust the inventory of the unpopular product. The last column in Table 1 reports the specific prices and inventory orders under this optimal solution, as well as the resulting profit.

An interesting observation from Table 1 is that, unlike TS and PS-Early, which both involve having unsold units, PS-Late may result in full inventory utilization. However, this only occurs when the unit cost is high ($\frac{1}{4} \le c < \frac{3}{4}$). If unit cost is low ($c < \frac{1}{4}$), some units of inventory will remain unsold under PS-Late. This is because, with high unit costs, it is optimal for the firm to avoid excess inventory by choosing $K^{\text{late}} < \frac{1}{2}$, although with such limited inventory the firm is unable to fully cover the market. In contrast, with low unit costs, it is optimal for the firm to invest in excess capacity by choosing $K^{\text{late}} > \frac{1}{2}$, which leads to a fully covered market at the cost of residual inventory.

3. Results

3.1. Probabilistic Selling: Early Allocation vs. Late Allocation

Our key concern here is to examine how allocation timing impacts a firm's ability to manage inventory in a market with uncertain demand. One might expect that the enhanced ability to utilize the inventory of the unpopular good under PS-Late would make it a more effective inventory-management tool than PS-Early. However, a formal comparison of the profit from PS-Late versus PS-Early (using Table 1) reveals that each strategy can be optimal under different conditions.

optimization problem that is given by (4) would also capture this implementation (which was termed "opaque selling" by Jerath et al. 2010).

[5] If credibility were not an issue, consumers would believe a seller's claim that the probabilistic good is equally likely to be each product (and the seller would keep this promise). Thus, PS-Late could replicate the results under PS-Early and would thus always be (weakly) superior. However, it seems unlikely that consumers would believe that a seller who has information about which product is more popular would not use this information when making allocation decisions. (If one were not going to use the information, then why wait?) In contrast, PS-Early is a mechanism for committing to equal assignment (since the firm does not know which product is more popular when making allocation decisions).

Proposition 1 (Effect of Allocation Timing on Prices, Inventory, and Profit Under Probabilistic Selling).

(a) *Early allocation earns a higher profit than late allocation if costs are sufficiently low* ($c < \tilde{c}$), *but the opposite holds if otherwise* ($c > \tilde{c}$), *where* $\tilde{c} \approx 0.353$.

(b) *Compared with late allocation, early allocation leads to*[6]

(i) *higher prices for both the specified goods and for the probabilistic good*,

(ii) *higher inventory orders*, *and*

(iii) *a larger number of unsold units*.

The finding in Proposition 1, part (a) that PS can be more profitable with early than with late allocation is somewhat counterintuitive. It would seem that PS-Late offers a more efficient way of disposing of excess inventory, since the firm can wait to decide on product assignments for those who purchase the probabilistic good and thus ensure that it does not assign scarce units of the popular good to such consumers. In contrast, under PS-Early, half of those who buy the probabilistic good will receive the popular product. The high inventory cost of PS-Early can be seen by noting from Table 1 that a larger number of units will go unsold under PS-Early than under PS-Late.

Given its higher inventory cost, why is PS-Early *not* a dominated strategy for a firm making inventory decisions in the presence of demand uncertainty? It is intriguing to realize that the same information disadvantage of PS-Early creates both a supply-side weakness (more unsold units) and a demand-side strength (higher revenue from the sold units). The demand-side strength of PS-Early comes from two sources: First, the price of the probabilistic good is higher when there is early allocation because the seller's inability to intentionally assign the unpopular product to buyers ensures a higher expected value for the probabilistic product under PS-Early than under PS-Late.[7] Second, under PS-Early, the seller obtains higher prices for the specific goods. Under both PS-Early and PS-Late, prices are set such that the marginal consumer of the probabilistic good earns zero surplus. Thus, there is no cannibalization of specified-good sales. However, under PS-Late, relatively low demand for the probabilistic good gives the firm an incentive to boost demand for the specified good by reducing the specified-good price. Under PS-Early, because the probabilistic good can capture all consumers who do not want to buy the specified good, this incentive is weaker, and thus the firm maintains high prices for the specified goods.

As shown in Proposition 1, the optimal PS strategy is jointly determined by both PS-Late's supply-side strength and PS-Early's demand-side strength. PS-Late enables the firm to incur smaller inventory costs. In general, when per-unit costs are high, the inventory savings are very important to profit, and PS-Late is more profitable. However, when costs are low, PS-Early enables the firm to manage inventories in a more profitable manner than can be done under PS-Late.

3.2. Probabilistic Selling vs. Traditional Selling

Let Π^{PS} represent the profit from offering the probabilistic good; i.e., $\Pi^{PS} = \max[\Pi^{early}, \Pi^{late}]$. Proposition 2 compares the profit under PS to that under TS.

Proposition 2 (Profit Advantage of Probabilistic Selling).

(a) *Offering probabilistic goods* (1) *enlarges the market settings in which selling is profitable and* (2) *increases the magnitude of profit* (*relative to the case in which probabilistic goods are not offered*). *Specifically*, (1) $\Pi^{TS} > 0$ *iff* $c < \frac{1}{2}$, *but* $\Pi^{PS} > 0$ *iff* $c < \frac{3}{4}$; *and* (2) $\Pi^{PS} - \Pi^{TS} > 0$.

(b) *In markets in which both TS and PS are profitable, at any given unit cost*, *PS enables the firm to enjoy higher margins and to incur fewer units of unsold inventory* (*relative to TS*).

(c) *The advantage from higher margins is largest under PS-Early, but the advantage from fewer units of inventory is largest under PS-Late.*

Part (a) of Proposition 2 shows that PS weakens the condition required for a seller to make a positive profit, part (b) identifies the key reasons for these results, and part (c) underscores the specific strengths of the two different PS strategies. In the absence of probabilistic goods, many units of the unpopular good will go unsold. If unit costs are sufficiently large ($c \geq \frac{1}{2}$), the firm is unable to earn a positive profit since the losses from unsold units outweigh the gains from sales of the popular product. PS enables the firm to generate additional revenue by selling units of the unpopular good (that would have otherwise gone unsold) to buyers of the probabilistic good. However, the format of PS suggested in the prior literature (Fay and Xie 2008); i.e., PS-Early, would not be profitable for $c \geq \frac{1}{2}$. Interestingly, as we show in this paper, PS-Late requires a weaker profitability condition ($c < \frac{3}{4}$). Thus, by adjusting the allocation timing, the advantage of PS extends beyond the situations identified in the extant literature, i.e., from only those situations where $c < \frac{1}{2}$ to those where $c < \frac{3}{4}$.

Using the prices given in Table 1, we see that $P^{early} > P^{late} > P^{TS}$, thus indicating that both versions of PS enable the seller to charge higher prices for the popular good. Furthermore, using the expressions given in

[6] These comparisons are valid over the region in which both PS-Early and PS-Late can earn positive profit, i.e., $c < \frac{1}{2}$.

[7] In the current model, we assume that delaying product assignment does not impose any disutility on consumers. If one incorporated the disutility from time delay, the price premium for the probabilistic good under PS-Early would be even greater, and thus the relative advantage of PS-Early would be even larger.

Table 1, we find that the number of unsold units is larger under TS than under PS-Early, which in turn is larger than the number of unsold units under PS-Late. Thus, although PS-Early and PS-Late have advantages over TS in regard to both profit margin and inventory management, the relative magnitude of the advantage differs across the two PS mechanisms. PS-Early produces the greatest gains in regard to profit margins, whereas PS-Late produces the greatest gains in terms of limiting the amount of unsold inventory.

We now address the following question: When switching from TS to PS, should the seller order more or less inventory? To answer this question, we compare the optimal inventory orders under PS and TS.

Proposition 3 (Inventory Decision: Probabilistic Selling vs. Traditional Selling). *When introducing a probabilistic good, the seller should adjust its inventory decision.*

(a) *The optimal inventory order is smaller under PS than under TS if costs are sufficiently small* ($c < 1/6$) *but larger otherwise.*

(b) *Under PS, the optimal inventory level is higher when the seller makes early, rather than late, allocation if* $c < \frac{1}{2}$. (*For* $\frac{1}{2} < c < \frac{3}{4}$, *the optimal inventory is positive for PS-Late but* 0 *for PS-Early.*)

Part (a) of Proposition 3 shows that, depending on product unit costs, introducing a probabilistic good can either increase or decrease the optimal inventory order. Demand asymmetry is smaller under PS because consumers with weak preferences will buy the probabilistic good rather than their preferred good. The reduction in demand asymmetry results in two countervailing effects on order quantity. First, under TS, for a given level of total sales, the retailer would need more inventory (relative to PS) in order to meet the demand for the popular good. Second, that TS is less successful at generating demand for the unpopular good reduces the total market coverage under TS (relative to PS). Specifically, when costs are high, potential overstocking of the unpopular product strongly affects the retailer (since it is costly to acquire units of this good). Thus, under TS, the retailer orders relatively few units of both goods so as to avoid having too much of the unpopular good go unsold. However, under PS, the firm would order a larger amount of both goods since the demand for the unpopular good is larger under PS (because the seller can dispose of a less popular product through sales of the probabilistic good). In contrast, when costs are low, the seller will order many units of both products under TS to ensure that it has sufficient inventory to meet the demand for the popular product. However, the firm need not order as much inventory under PS because it can sell the probabilistic product to consumers who only slightly prefer the popular product.

Part (b) of Proposition 3 indicates that, for $c < \frac{1}{2}$, inventory levels are higher when the seller makes early, rather than late, allocation. PS-Late enables more efficient utilization of inventory since the seller can assign the less popular product to all buyers of the probabilistic good. In contrast, under PS-Early, some units of the popular product will be used to satisfy the demand for the probabilistic good. If costs are high ($\frac{1}{2} < c < \frac{3}{4}$), PS-Early does not yield positive profits, and thus the inventory order is zero. In contrast, in this cost region, PS-Late is a profitable strategy (and thus $K^{\text{late}} > 0$).

3.3. Importance of Endogenizing Inventory Orders

Proposition 3 shows that the optimal inventory order under PS differs from that under TS, thus suggesting that it is important for a retailer to adjust its inventory orders appropriately to take full advantage of the PS strategy. This point is worth emphasizing, since a retailer may be tempted to implement PS by taking its inventory orders as given and simply adding the probabilistic product to its product offerings. Although such additions can be profitable (as was shown in Fay and Xie 2008), integrating inventory decisions with a PS strategy can significantly increase the profit advantage of PS. For example, if $c \sim U[0, \frac{1}{2}]$, endogenizing inventory orders, on average, increases the profit advantage of PS by 23% (for details, see the appendix). Interestingly, we find that integration of inventory decisions with a PS strategy makes the PS-Early strategy relatively more attractive. Specifically, when inventory orders are kept at the levels that were optimal under TS, PS-Late is more profitable than PS-Early if $c > \frac{1}{4}$. However, when inventory orders are adjusted to their optimal level under PS, PS-Late is more profitable than PS-Early if $c > \tilde{c}$, where $\tilde{c} > \frac{1}{4}$. Thus, optimizing inventory decisions allows the seller to gain more from adopting PS-Early. The reason for this is as follows. When faced with significant unit costs, inventory orders under TS are very small. And, with this scarce inventory, the firm is reluctant to allocate the probabilistic good to consumers unless it is certain that these customers are receiving the less popular item. However, early allocation would be optimal (and would substantially improve profit) if additional inventory were ordered. For example, suppose $c = 0.3$. Under TS, the seller orders 0.4 units of each item and earns a profit of 0.08. At this inventory level, PS-Late is more profitable than PS-Early, and PS-Late generates a profit of 0.2. However, if the inventory order were revised appropriately (to $K^{\text{PS}} = 0.6$) and the seller switched to PS-Early, profit would equal 0.22, which is larger than the profit under PS-Late ($K^{\text{late}} = 0.45$ leads to $\Pi^{\text{late}} = 0.2025$). These findings underscore the importance of extending PS from a pricing mechanism to an inventory-management mechanism.

3.4. Effect of Probabilistic Selling on Market Efficiency

We now explore the impact of PS on total social welfare (the value of consumption less production costs). Proposition 4 summarizes the effects of PS on welfare when the seller faces demand uncertainty.

Proposition 4 (Welfare: Probabilistic Selling vs. Traditional Selling).

(a) *Adopting a PS strategy enhances social welfare if the unit cost of acquiring inventory is not too low. Specifically,* $W^{PS} > W^{TS}$ *if* $c > \frac{1}{18}$.

(b) *Social welfare under PS is higher when the seller makes late rather than early allocation to consumers.*

Part (a) of Proposition 4 reveals that, as long as the unit cost is not too low, introducing a probabilistic product can create a win–win situation, benefiting both the firm and consumers. PS leads to market expansion and better inventory management by generating greater demand for the unpopular product, which can significantly enhance efficacy and lead to higher total welfare. The exception occurs in markets with very low unit costs. Here, market coverage under TS is high, and thus PS expands the market by only a few units, making the benefit from market expansion small. Furthermore, the benefit of enhanced inventory management is also small when unit costs are low.

Part (b) of Proposition 4 indicates that PS-Late yields higher social welfare than PS-Early because the former enables the seller to reduce the number of unsold units (relative to PS-Early) and thus to better utilize inventory (see Proposition 1, part (b)). Although the total value created by the sales of the probabilistic good is less under PS-Late (since all consumers receive their less preferred product), these sales are less costly to generate than under PS-Early because additional inventory is not needed to satisfy demand for the probabilistic good; i.e., all sales come from units that would otherwise have gone unsold.

4. Concluding Comments

4.1. Main Findings and Contributions

This paper extends the research on probabilistic selling from three perspectives. First, our model formally extends this novel strategy from a pricing tool to an inventory-management mechanism. Our results illustrate that PS is a promising new mechanism to manage inventory decision making for sellers who face demand variations across their product line but do not know demand for each product. Specifically, PS enables a seller to improve capacity utilization by reducing the number of leftover units that need to be salvaged. Our results also provide specific guidance on how the seller should revise its inventory policy when moving from traditional selling to PS. Usually, the seller should carry greater inventory for its products; however, less inventory can be required if unit costs are very low.

Second, this paper augments the existing models of PS by analyzing how allocation timing impacts the profitability of the PS strategy. Our model uncovers a counterintuitive finding: when facing the challenge of demand uncertainty, a seller can be worse off by delaying product allocation to consumers (even if such delay will not cause consumer disutility as a result of scheduling inconvenience and opportunity costs). Our analysis reveals the dual effects of early allocation (i.e., assigning products to consumers before the seller learns demand): (1) a higher inventory cost (since the seller is unable to assign the unpopular product to all buyers who paid discounted prices) and (2) a higher revenue from each sold unit (since the seller can charge higher prices for its products). Our results also offer insights into how firms might optimally implement a PS strategy, making early allocation if costs are relatively low but late allocation if costs are sufficiently high.

Finally, this paper extends the advantages of PS from a purely profit perspective (used exclusively in the extant research) to include a social welfare perspective. Our results reveal that PS can create a win–win situation, enhancing both a seller's profit and market efficiency. This outcome is possible because (a) more consumers are served, and (b) inventory for the popular product is reserved for consumers with the highest valuations for this product.

4.2. Areas for Future Research

This paper demonstrates that PS makes inventory more productive by reducing the degree of asymmetry in demand. We suspect that similar intuition could be applied to settings where the seller chooses capacity rather than inventory (e.g., the number of airplanes to buy). For service industries, such a formal study of how PS impacts capacity choices would be particularly relevant. Future research could also utilize a dynamic framework, e.g., by allowing for replenishment or price adjustments within the selling period. Finally, we have developed and analyzed a stylized model to obtain theoretical insights into the impact of PS on inventory decisions. Future research is necessary, however, to develop inventory-management algorithms for retailers. To develop such tools, it will be important to model demand and the effects on demand of selling a probabilistic good empirically, which may depend on additional industry characteristics not considered here. It would also be important to generalize the model to incorporate different degrees and types of demand uncertainty, such as situations in which it is unknown if any of the products will be popular with consumers.

Acknowledgments

The authors thank the Robert H. Brethen Operations Management Institute and J. C. Penney Company Inc. for providing financial support for this research project.

Appendix

Proof of the Strategies Illustrated in Table 1. *Traditional Selling:* Since the demand for the unpopular product is positive only if $P^{\text{TS}} < \frac{1}{2}$, there are two relevant regions to consider: $P^{\text{TS}} \geq \frac{1}{2}$ and $P^{\text{TS}} < \frac{1}{2}$. For $P^{\text{TS}} \geq \frac{1}{2}$, profit, as given in (2), is maximized when $K = D_1$, since capacity above demand has no impact on revenue but increases costs. Substituting for K into (2), we have $\Pi^{\text{TS}} = 2(P^{\text{TS}} - 2c)(1 - P^{\text{TS}})$. Taking the derivative with respect to P^{TS}: $\partial\Pi^{\text{TS}}/\partial P^{\text{TS}} = 2(1 - 2P^{\text{TS}} + 2c) \equiv 0 \Rightarrow P^{\text{TS}} = (1 + 2c)/2$. This price yields positive demand only if $c < \frac{1}{2}$. At this price, profit is $\Pi_a^{\text{TS}} = (1 - 2c)^2/2$. For $P^{\text{TS}} < \frac{1}{2}$,

$$\Pi^{\text{TS}} = \begin{cases} 2K(P^{\text{TS}} - c) & \text{if } K \leq 1 - 2P^{\text{TS}}, \\ P^{\text{TS}}(K + 1 - 2P^{\text{TS}}) - 2cK & \text{if } K > 1 - 2P^{\text{TS}}. \end{cases}$$

Since the top expression is increasing in K, maximum profit in this range is achieved when $K = 1 - 2P^{\text{TS}}$. Substituting for K, we find $\partial\Pi^{\text{TS}}/\partial P^{\text{TS}} = 2(1 - 4P^{\text{TS}} + 2c) \equiv 0 \Rightarrow P^{\text{TS}} = (1 + 2c)/4$. Note that $P^{\text{TS}} < \frac{1}{2}$ only if $c < \frac{1}{2}$. At this price, profit is $\Pi_b^{\text{TS}} = \frac{1}{4} - c(1 - c) < \Pi_a^{\text{TS}}$. For $K > 1 - 2P^{\text{TS}}$, solving the simultaneous equations $\partial\Pi^{\text{TS}}/\partial P^{\text{TS}} \equiv 0$ and $\partial\Pi^{\text{TS}}/\partial K \equiv 0$ results in the solution ($P^{\text{TS}} = 2c$, $K = 8c - 1$). At this solution, $\Pi_c^{\text{TS}} = 2c(1 - 4c) < \Pi_a^{\text{TS}}$. Thus, profit is maximized at the price $P^{\text{TS}} = (1 + 2c)/2$, which yields the profit $\Pi_a^{\text{TS}} = (1 - 2c)^2/2$.

PS-Early. Taking the derivative of Π^{early} w.r.t. X_{PG}: $\partial\Pi^{\text{early}}/\partial X_{\text{PG}} = \frac{1}{2} - c$. If $c > \frac{1}{2}$, Π^{early} is decreasing in X_{PG}, and thus it is optimal to set $X_{\text{PG}} = 0$, i.e., not offer the probabilistic good. If $c < \frac{1}{2}$, Π^{early} is increasing in X_{PG}, and thus the first constraint in (3) is binding; i.e., $X_{\text{PG}} = 1 - X_{\text{Pop}}$. Rearranging the second constraint, $P^{\text{early}} = 1 - X_{\text{Pop}}/2$. Substituting for X_{PG} and P^{early}, the optimization problem in (3) becomes $\max\{\Pi^{\text{early}}_{X_{\text{Pop}}}\} = (1 - X_{\text{Pop}}/2)X_{\text{Pop}} + ((1 - X_{\text{Pop}})/2) - 2c(X_{\text{Pop}} + (1 - X_{\text{Pop}})/2)$. Setting $\partial\Pi^{\text{early}}/\partial X_{\text{Pop}} \equiv 0$, we find $X_{\text{Pop}} = \frac{1}{2} - c$. Using this value for X_{Pop}, we find $X_{\text{PG}} = \frac{1}{2} + c$ and $P^{\text{early}} = (3 + 2c)/4$. The resulting profit is $\Pi^{\text{early}} = (4c^2 - 12c + 5)/8$.

PS-Late. First, consider the implementation in which all buyers of the probabilistic good receive the unpopular product; i.e., $K_P = 0$. In solving the maximization problem given by Equation (4), the market will be either fully covered, i.e., $\hat{X} = 1 - P^{\text{late}}$; or incompletely covered, i.e., $\hat{X} > 1 - P^{\text{late}}$. If the market is fully covered ($\hat{X} = 1 - P^{\text{late}}$), the consumer located at $\hat{X}$ ($< \frac{1}{2}$) must be indifferent between buying the flexible good and buying the preferred item. Thus, the firm's objective function given in (4) becomes maximizing $\Pi^{\text{late}} = 2P^{\text{late}}\hat{X} + P_o^{\text{late}}(1 - 2\hat{X}) - 2cK^{\text{late}}$ s.t. $K^{\text{late}} = \max[2(1 - P^{\text{late}}), 1 - 2\hat{X}]$. The constraint guarantees that the firm has sufficient inventory of each product to meet the market demand. This constraint will hold with equality since greater inventory levels will not impact revenue but will increase costs. Since profit is increasing in both prices, the firm chooses the highest price possible that will induce participation by all consumers: $\hat{X} - P_o^{\text{late}} \equiv 0 \Rightarrow P_o^{\text{late}} = \hat{X}$ and $1 - \hat{X} - P^{\text{late}} \equiv 0 \Rightarrow P^{\text{late}} = 1 - \hat{X}$. Substituting back into the objective function, the firm's problem becomes

$$\max_{\hat{X}} \Pi^{\text{late}} = 2\hat{X}(1 - \hat{X}) + \hat{X}(1 - 2\hat{X}) - 2c\max[2\hat{X}, 1 - 2\hat{X}]. \quad (5)$$

For $\hat{X} \leq \frac{1}{4}$, Π^{late} is increasing in $\hat{X}$. Thus, profit is maximized at $\hat{X} = \frac{1}{4}$, which yields a profit of $\Pi^{\text{late}} = \frac{1}{2} - c$. For $\hat{X} > \frac{1}{4}$, Π^{late} is maximized at $\hat{X} = (3 - 4c)/8$. The solution, which meets the requirement that $\hat{X} > \frac{1}{4}$ only if $c < \frac{1}{4}$, generates a profit of $\Pi^{\text{late}} = (16c^2 - 24c + 9)/16 > \frac{1}{2} - c\ \forall\, c \geq 0$.

Now consider the case of incomplete market coverage ($\hat{X} > 1 - P^{\text{late}}$). The consumer located at $\mathbf{x}_i = 1 - P^{\text{late}}$ is indifferent between buying the preferred good and not purchasing anything. The consumer located at $x_i = P_o^{\text{late}}$ is indifferent between buying the probabilistic good and not purchasing anything. Since the profit from (4) is increasing at both prices, the firm sets prices to exhaust capacity: $2(1 - P^{\text{late}}) \equiv K^{\text{late}} \Rightarrow P^{\text{late}} = 1 - K^{\text{late}}/2$ and $2(\frac{1}{2} - P_o^{\text{late}}) \equiv K^{\text{late}} \Rightarrow P_o^{\text{late}} = (1 - K^{\text{late}})/2$. Substituting in for prices and maximizing (4) yields an optimal capacity of $K^{\text{late}} = (3 - 4c)/4$, which results in a profit of $\Pi^{\text{late}} = (16c^2 - 24c + 9)/16$. This solution is only valid if $0 < K^{\text{late}} < \frac{1}{2}$, i.e., there is a positive amount of capacity and selling this capacity results in incomplete market coverage. Substituting in for K^{late}, we find that this solution is only valid if $\frac{1}{4} < c < \frac{3}{4}$.

Second, consider the implementation in which $K^{\text{late}} = \frac{1}{2}$. We focus on the case where $K_P > 0$. (The case of $K_P = 0$ was examined in the preceding section, andthe constraint was added that $K^{\text{late}} = \frac{1}{2}$ cannot improve upon the profits derived in the preceding analysis.) Under this implementation of PS-Late, the market is fully covered, and buyers of the probabilistic good have a positive probability of receiving the popular good. Since the expected value of the probabilistic good is increasing in K_P, the firm has the incentive to choose K_P so that the entire inventory for the popular good is exhausted: $2(1 - P^{\text{late}}) + K_P \equiv K^{\text{late}} = \frac{1}{2} \Rightarrow K_P = 2P^{\text{late}} - \frac{3}{2}$. At $\hat{X}$, the consumer is just indifferent between purchasing her preferred product, purchasing the probabilistic good, and not making any purchase. Thus, we have $1 - \hat{X} - P^{\text{late}} \equiv 0 \Rightarrow \hat{X} = 1 - P^{\text{late}}$. Demand for the probabilistic good is $D_{\text{PG}} = 1 - 2(1 - P^{\text{late}}) = 2P^{\text{late}} - 1$. Using the first constraint from Equation (4) and substituting for K^{late}, $\hat{X}$, and K_P, we can identify the price of the probabilistic good:

$$\frac{4P^{\text{late}} - 3}{2(2P^{\text{late}} - 1)}P^{\text{late}} + \frac{1}{2(2P^{\text{late}} - 1)}(1 - P^{\text{late}}) - P_o^{\text{late}} = 0$$

$$\Longrightarrow\ P_o^{\text{late}} = P^{\text{late}} - \frac{1}{2}.$$

Substituting $K^{\text{late}} = \frac{1}{2}$, $P_o^{\text{late}} = P^{\text{late}} - \frac{1}{2}$ and $\hat{X} = 1 - P^{\text{late}}$ into Π^{late} from (4) yields $\Pi^{\text{late}} = \frac{1}{2} - c$. Note that this is the same profit as one of the two possible outcomes under the first form of implementation ($K_P = 0$). Thus, for all $c \in [0, \frac{3}{4}]$, the maximum profit obtainable under either implementation of PS-Late is $\Pi^{\text{late}} = (16c^2 - 24c + 9)/16$.

Proof of Proposition 1. Using Table 1, when there is demand uncertainty, the following calculations can be made:

$$\Pi^{\text{early}} - \Pi^{\text{late}} = \frac{1 - 8c^2}{16}\begin{cases} > 0, & \text{if } c < 1/\sqrt{8} \equiv \tilde{c} \approx 0.353, \\ < 0, & \text{if } c > \tilde{c}; \end{cases} \quad (6)$$

$$K^{\text{early}} - K^{\text{late}} = c/2 > 0; \quad (7)$$

$$Unsold^{\text{early}} - Unsold^{\text{late}} = \begin{cases} c/2 > 0, & \text{if } c < \frac{1}{4}, \\ \frac{1}{2} - c > 0, & \text{if } \frac{1}{4} \leq c < \frac{1}{2}; \end{cases} \quad (8)$$

$$P^{\text{early}} - P^{\text{late}} = \tfrac{1}{8} > 0; \quad (9)$$

$$P_o^{\text{early}} - P_o^{\text{late}} = \begin{cases} \dfrac{1 + 4c}{8} > 0, & \text{if } c < \frac{1}{4}, \\ \dfrac{3 - 4c}{8} > 0, & \text{if } \frac{1}{4} \leq c < \frac{1}{2}. \end{cases} \quad (10)$$

Proof of Proposition 2.

$$\Pi^{PS}-\Pi^{TS}=\begin{cases}\dfrac{(1+6c)(1-2c)}{8}>0, & \text{if } c<\tilde{c},\\ \dfrac{1-8c(2c-1)}{16}>0, & \text{if } \tilde{c}<c<\frac{1}{2},\\ \dfrac{16c^2-24c+9}{16}>0, & \text{if } \frac{1}{2}<c<\frac{3}{4}.\end{cases}\tag{11}$$

For this discussion, which explains the rationale for Proposition 2, it is important to note the following:

$$\mathit{Unsold}^{TS}-\mathit{Unsold}^{PS}=\begin{cases}\dfrac{1-2c}{2}>0, & \text{if } c<\tilde{c},\\ 1-2c>0, & \text{if } \tilde{c}<c<\frac{1}{2}.\end{cases}\tag{12}$$

Proof of Proposition 3.

$$K^{TS}-K^{PS}=\begin{cases}\dfrac{1-6c}{4}\begin{cases}>0, & \text{if } c<\frac{1}{6},\\ <0, & \text{if } c>\frac{1}{6},\end{cases} & \text{if } c<\tilde{c};\\ \frac{1}{4}-c<0, & \text{if } \tilde{c}<c<\frac{1}{2}.\end{cases}\tag{13}$$

Proof of the Results of §3.3 (*Endogenous Inventory Orders*). Define Δ_{Adjusted} as the advantage of PS over TS when inventory orders are chosen optimally: $\Delta_{\text{Adjusted}}=\Pi^{PS}(K^{PS})-\Pi^{TS}(K^{TS})$; Δ_{Adjusted} is given by Equation (11). Let Δ_{constant} be the advantage of PS when inventory orders are kept at the levels that were optimal under TS: $\Delta_{\text{constant}}=\Pi^{PS}(K^{TS})-\Pi^{TS}(K^{TS})$. To derive Δ_{constant}, we must calculate $\Pi^{PS}(K^{TS})$, where K^{TS} is given in Table 1: $K^{TS}=1-2c$. We start by deriving Π^{PS} when the seller uses the early allocation strategy. We consider two cases: $K^{TS}<\frac{1}{2}$ (which occurs if $c>\frac{1}{4}$) and $K^{TS}\geq\frac{1}{2}$ (which occurs if $c\leq\frac{1}{4}$). If $K^{TS}<\frac{1}{2}$, it is not possible to completely cover the market. Let X_{MAX} represent the maximum number of sales the firm is willing to make of each specific product. At the optimum, we must have $0\leq X_{\text{MAX}}\leq 2(1-P)$. Larger values of X_{MAX} would not be binding, since X_{MAX} would exceed the demand for the popular good. Given X_{MAX}, the number of probabilistic good sales will be $X_{PG}=2(K-X_{\text{MAX}})$. Thus, profit is

$$\Pi^{\text{early}}=P(X_{\text{MAX}})+\frac{2(K-X_{\text{MAX}})}{2}-2cK.\tag{14}$$

Taking the derivative of Π^{early} w.r.t. X_{MAX}: $\partial\Pi^{\text{early}}/\partial X_{\text{MAX}}=P-1<0$. Thus, profit is maximized at the lower bound; i.e., $X_{\text{MAX}}=0$. Substituting $K=1-2c$ and $X_{\text{MAX}}=0$ into (14), this yields a profit of

$$\Pi^{\text{early}}=(1-2c)^2.\tag{15}$$

Second, consider $K^{TS}\geq\frac{1}{2}$. Here, it is possible to cover the whole market. Note that at $c=\frac{1}{4}$, $K=\frac{1}{2}$, and the solution given in the previous paragraph leads to full market coverage. Thus, for $K^{TS}>\frac{1}{2}$, the firm must also choose to fully cover the market. The maximization problem becomes

$$\max\ \Pi^{\text{early}}_{X_{\text{MAX}},P}=\max\left[P(X_{\text{MAX}})+\frac{1-X_{\text{MAX}}}{2}-2cK\right]$$
$$\text{s.t.}\quad 0\leq X_{\text{MAX}}\leq 2(1-P)\quad\text{and}\quad K\geq X_{\text{MAX}}+\frac{1-X_{\text{MAX}}}{2}.\tag{16}$$

The second constraint can be rewritten as $X_{\text{MAX}}\leq 2K-1$. The derivative of Π^{early} w.r.t. X_{MAX} is $\partial\Pi^{\text{early}}/\partial X_{\text{MAX}}=P-\frac{1}{2}>0$. Thus, we must have a corner solution in which $X_{\text{MAX}}=\min[2(1-P),2K-1]$. If $2(1-P)\leq 2K-1$, profit is maximized at $P=\frac{3}{4}$. The inequality will hold only if $c\leq\frac{1}{8}$. The resulting profit is

$$\Pi^{\text{early}}=\tfrac{5}{8}-2c(1-2c).\tag{17}$$

If $c>\frac{1}{8}$, limited capacity will constrain how many units of the popular good can be sold. In particular, we have $X_{\text{MAX}}=2K-1$. Since profit is now increasing in P, the firm sets P so that demand exactly equals X_{MAX}: $2(1-P)=2K-1\Rightarrow P=(3-2K)/2$. Substituting $X_{\text{MAX}}=2K-1$ and this price into (16), we find profit is

$$\Pi^{\text{early}}=\frac{1-8c^2}{2}.\tag{18}$$

Now, we derive $\Pi^{PS}(K^{TS})$ when the seller uses the late allocation strategy. For $c>\frac{1}{4}$, $K^{TS}<\frac{1}{2}$, and thus the market is not fully covered. Therefore, buyers of the probabilistic good will receive the less popular product and prices are set to just exhaust the inventory: $P^{\text{late}}=1-K^{TS}/2$ and $P_o^{\text{late}}=(1-K^{TS})/2$. At these prices, profit is

$$\Pi^{\text{late}}=K^{TS}\left(\tfrac{3}{2}-K^{TS}-2c\right)=\tfrac{1}{2}-c.\tag{19}$$

From (19), Π^{late} is strictly larger than Π^{early} from (18), thus proving Proposition 4, part (b). Summarizing the analysis,

$$\Pi^{PS}(K^{TS})=\begin{cases}\frac{5}{8}-2c(1-2c), & \text{if } 0<c\leq\frac{1}{8},\\ \dfrac{1-8c^2}{2}, & \text{if } \frac{1}{8}<c<\frac{1}{4},\\ \frac{1}{2}-c, & \text{if } \frac{1}{4}\leq c<\frac{1}{2}.\end{cases}\tag{20}$$

Using Equations (11) and (19), along with Table 1, we can calculate the following:

$$\Delta_{\text{Adjusted}}-\Delta_{\text{constant}}=\begin{cases}\dfrac{c(1-7c)}{2}>0, & \text{if } 0<c\leq\frac{1}{8},\\ \dfrac{(1-6c)^2}{8}>0, & \text{if } \frac{1}{8}<c<\frac{1}{4},\\ \dfrac{1-8c(1-2c)}{8}>0, & \text{if } \frac{1}{4}\leq c<\frac{1}{2}.\end{cases}\tag{21}$$

Using the assumption that $c\sim U[0,\frac{1}{2}]$, we calculate the following:

$$\frac{\int_{c=0}^{1/2}(\Delta_{\text{Adjusted}}-\Delta_{\text{constant}})f(c)\,dc}{\int_{c=0}^{1/2}\Delta_{\text{constant}}f(c)\,dc}=\frac{5}{22}=0.22727.\tag{22}$$

Proof of Proposition 4. Under TS, the seller has sufficient capacity of both goods to meet the entire demand at the equilibrium price, P^{TS}. Thus, the expected welfare under TS equals

$$W^{TS}=2\int_{x_i=0}^{1-P^{TS}}(1-x_i)\,dx_i-2cK^{TS}=\frac{3(1-2c)^2}{4}.\tag{23}$$

Under PS-Early, the optimal price of the probabilistic good is $\frac{1}{2}$, which also equals the expected value of the probabilistic product for all consumers. Therefore, the expected welfare under PS-Early equals

$$\begin{aligned} W^{\text{early}} &= 2\int_{x_i=0}^{1-p^{\text{early}}} (1-x_i)\,dx_i \\ &\quad + \tfrac{1}{2}\left[1-2\int_{x_i=0}^{1-p^{\text{early}}} dx_i\right] - 2cK^{\text{early}} \\ &= \frac{(1-2c)(11-6c)}{16}. \end{aligned} \tag{24}$$

Under PS-Late, buyers' valuations of the probabilistic good depend on their location on the Hotelling line. Therefore, the expected welfare under PS-Late equals

$$\begin{aligned} W^{\text{late}} &= 2\left[\int_{x_i=0}^{1-p^{\text{late}}} (1-x_i)\,dx_i + \int_{x_i=p_o^{\text{late}}}^{1/2} x_i\,dx_i\right] - 2cK^{\text{late}} \\ &= \frac{3(4c-3)^2}{32}. \end{aligned} \tag{25}$$

Comparing welfare under TS and PS (noting that PS will involve PS-Early $c < \tilde{c}$ and PS-Late otherwise),

$$W^{\text{PS}} - W^{\text{TS}} = \begin{cases} \dfrac{c(5-9c)}{4} - \frac{1}{16} \begin{cases} <0, & \text{if } c < \frac{1}{18}, \\ >0, & \text{if } c > \frac{1}{18}, \end{cases} & \text{if } c < \tilde{c}; \\[2ex] \dfrac{3(1+8c(1-2c))}{32} > 0, & \text{if } \tilde{c} < c < \frac{1}{2}. \end{cases} \tag{26}$$

Comparing the welfare under PS-Early and PS-Late,

$$W^{\text{late}} - W^{\text{early}} = \frac{5-8c(2-3c)}{32} > 0. \tag{27}$$

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