Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Multicommodity Production Planning: Qualitative Analysis and Applications

Iara Ciurria-Infosino, Daniel Granot, Frieda Granot, Arthur F. Veinott Jr.deceased

M&SOM MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 17, No. 4, Fall 2015, pp. 589–607
ISSN 1523-4614 (print) | ISSN 1526-5498 (online)

informs
http://dx.doi.org/10.1287/msom.2015.0549

Multicommodity Production Planning: Qualitative Analysis and Applications

Iara Ciurria-Infosino
Resource Analysis Branch, U.S. Air Force/A9RP, Pentagon, Washington, DC 20330,
iara.c.infosino.civ@mail.mil

Daniel Granot, Frieda Granot
Sauder School of Business, University of British Columbia, Vancouver, British Columbia V6T 1Z2, Canada,
{daniel.granot@sauder.ubc.ca, frieda.granot@sauder.ubc.ca}

Arthur F. Veinott Jr.
deceased
Formerly at the Department of Management Science and Engineering, Stanford University, Stanford, California

We develop a qualitative analysis theory for the convex-cost dynamic multicommodity production planning problem, which can be used, without performing any computational work, to provide invaluable insight to managers when faced with the task of deciding how to respond to changes in problem environment. We first formulate the problem as a multicommodity flow problem with parameters associated with each arc–commodity pair. We then reduce the problem to an equivalent single-commodity flow problem and develop a complete characterization of conformality among production, sales, and inventory activities in various instances of the problem. By combining the conformality characterizations with the monotonicity theory of Granot and Veinott [Granot F, Veinott AF Jr (1985) Substitutes, complements and ripples in network flow. *Math. Oper. Res.* 10:471–497] for single-commodity problems, we study the effects of changes in problem environment on optimal production, sales, and inventory schedules in the multicommodity problem. Numerous applications are presented and analyzed.

Keywords: convex multicommodity flows; qualitative analysis; monotonicity; substitutes and complements; dynamic production planning problem
History: Received: September 11, 2014; accepted: April 20, 2015. Published online in *Articles in Advance* August 17, 2015.

1. Introduction

In this paper we develop qualitative analysis theory for the important class of minimum convex-cost dynamic multicommodity production planning problems, which are concerned with finding production, sales, and inventory levels of κ products over n periods, which minimize a convex objective function. The convexity of the objective function may stem, for example, from diseconomies in production cost; an increase in inventory unit cost beyond a certain amount of stock; or upper and lower bounds on production, sales, and inventory of each product at each period or for all products per period.

As is well known, sensitivity analysis is an essential aspect of decision making. At the modeling stage, it may reveal the sensitivity of the model to various problem parameters and the need for collecting additional information. At the solution stage, sensitivity analysis may suggest, for example, the benefit of allocating additional resources to various activities. In linear programming, sensitivity analysis—say, of resource i—reveals the marginal value of this resource and the range of values of resource i for which this marginal value is valid. Note, however, that sensitivity analysis results in linear programming hold ceteris paribus. That is, they are valid essentially only when the values of all other problem parameters remain unchanged.

Qualitative analysis results are not as specific as sensitivity analysis results in linear programming. However, they are broader and are more robust. They hold for convex optimization problems, for which the sensitivity analysis tools of linear programming may not be available. Thus, they are particularly useful for large nonlinear optimization problems, such as large-scale convex multicommodity problems, for which reoptimization could be computationally expensive. Further, although they do not provide, for example, the exact rate of change in the objective function value resulting from a parameter change but only the direction of change (i.e., increase or decrease), they are valid for all values of problem parameters. Moreover, they also provide information such as the relative magnitude of the changes in the values of decision variables as a result of changes in parameter values.

To illustrate, consider a minimum convex-cost dynamic multicommodity production planning problem, where revenues from sales are incorporated in the objective function with a negative sign, and suppose we would like to analyze the effect of increasing the lower bound on total production in period j, which may stem, for example, from increased sales commitments in that period. Intuitively, introducing such a constraint will increase total production in period j and will also increase the total cost. But what would be the effect on total revenue? Would it increase or decrease? Would market share of the various products in the various periods increase or decrease? And could the changes in total production in the various periods, or changes in production or sales of the various products in the various periods, exceed the change in total production in period j?

The qualitative analysis theory we develop in this paper for the minimum convex-cost dynamic multicommodity production planning problem reveals that for the above example, under some mild assumptions, the total revenue increases, market share of all products in all periods increases, total production in period j will be equal to the newly introduced lower bound, and changes in all variables are bounded from above by the change in total production in period j. Moreover, these conclusions are valid for any instance of problem parameters, as long as problem feasibility is maintained.

In general, to develop the qualitative results for the minimum convex-cost dynamic multicommodity production planning problem, we formulate this problem as a κ-commodity n-period network flow problem with parameters associated with each arc–commodity pair. The parameter associated with an arc–commodity pair could be an upper or lower bound on the flow of the commodity in the arc or, more generally, a parameter of the commodity flow cost on the arc. The total cost is arc–commodity additive, and each arc–commodity cost is $+\infty$ or real valued, convex in the flow of the commodity, and submodular in the flow–parameter pair. The goal is to determine when it is possible to predict, for all such cost functions and without computation, the direction of change in the optimal value of a commodity and bounds on the magnitude of this change when a parameter associated with another (possibly the same) commodity is changed.

The dynamic multicommodity production planning problem is one of the most fundamental problems in operations management and has received significant attention in the literature. Indeed, from early contributions by, e.g., Modigliani and Hohn (1955), Bowman (1956), Manne (1957), Karush (1958), Wagner and Whitin (1958), and Zangwill (1969), this problem and some of its variants have generated a huge literature; see, e.g., Buffa and Taubert (1972), Hax and Candea (1984), Shapiro (1993), Zipkin (2000), Graves (2002), and Pochet and Wolsey (2003). This literature is mostly concerned with the design of effective formulations, computational complexity, and the development of efficient solution methods; see, e.g., Ahuja et al. (1993), Graves (2002), and Pochet and Wolsey (2003).

Lloyd Shapley appears to be the first to study qualitative analysis in network optimization problems. Shapley (1961) investigated when two arcs are substitutes or complements in the maximum flow problem, and subsequently, Shapley (1962) extended the analysis to variations in supply and demand in the classical assignment problem. Gale and Politof (1981) extended Shapley's work and investigated the effects of changes in upper and lower bounds and costs in minimum linear cost network models.

In economics, the study of the qualitative behavior of optimal solutions has a long history, including work by Hicks (1939) and Hicks and Allen (1934a, b). Already in 1947, Paul Samuelson (see Samuelson 1971) posed the problem of whether it is possible to predict the direction of change of certain variables in an economic model given the direction of change of some other variable(s). See Topkis (1998) for an extensive study of supermodularity, monotonicity, and related development in economics.

Veinott (1964) could be viewed as a precursor for the study of qualitative analysis of the production planning problem. He shows therein that for the strictly convex-cost single-product dynamic production planning problem, the optimal production level in a given period is a nondecreasing function of (i) the requirements in any period, (ii) the upper and lower production capacity limits in the given period, and (iii) the upper and lower storage limits in the given period and all succeeding periods. Granot and Veinott (1985) carried out a thorough qualitative analysis of convex-cost network optimization problems that encompasses, as a special case, the dynamic single-commodity production planning problem. Recently, Ciurria-Infosino et al. (2014) carried out qualitative analysis in multicommodity problems on suspension tree graphs (i.e., graphs G with a special node, whose removal and the removal of all arcs incident to it in G reduces G to a tree graph). Since the underlying graph of the dynamic multicommodity production planning problem is a very special suspension tree graph, the results in Ciurria-Infosino et al. (2014) are valid for the dynamic multicommodity production planning problem. However, many more results will be derived in this paper by exploiting special features of the dynamic multicommodity production planning problem, as will be elaborated in the sequel.

The plan of the paper is as follows. In §2 we formulate the minimum convex-cost dynamic multicommodity production planning problem and discuss our assumptions. We further show how to reduce the dynamic multicommodity production planning problem to an equivalent single-commodity network flow problem. Arc–commodity pairs in the multicommodity problem correspond to arcs in the equivalent single-commodity problem. In §3 we derive a complete characterization of conformality among production, sales, and inventory activities of the original production planning problem. (The definition of conformality is provided in §3.) By combining this characterization with the monotonicity theory of Granot and Veinott (1985), we obtain qualitative results for the *original* multicommodity production planning problem. In §4 we study several interesting instances of the multicommodity production planning problem. By exploiting the special structure of both the cost functions and the corresponding underlying graphs, we provide, for all these instances, a complete characterization of conformality and derive insights into a variety of scenarios without performing any computational work whatsoever. In §5 we summarize the results. All proofs are in the appendix.

2. Minimum Convex-Cost Dynamic Multicommodity Production Planning

We first formulate the convex-cost dynamic multicommodity production planning problem and discuss our assumptions.

2.1. Formulation

2.1.1. Decision Variables and Constraints. Let $\hat{K} = \{1, 2, \ldots, \kappa\}$ denote a set of κ commodities called *primitive*. In addition, there is a $(\kappa+1)$th fictitious commodity called *aggregate*, denoted by K. The *aggregate production*, *sales*, and *inventories* in a period are the sums of the productions, sales, and inventories of the primitive commodities in those periods, respectively. Let $\mathcal{K}$ denote the set of all $\kappa+1$ commodities.

Let p_i^k, s_i^k, and h_i^k respectively denote the amounts of commodity $k \in \mathcal{K}$ produced (or ordered), sold, and held (stored) at period i, $1 \le i \le n$. For each primitive commodity k, negative values of production, sales, and storage signify disposal, returns, and back orders, respectively. These interpretations carry over to the corresponding aggregate quantities and are meaningful only if the corresponding primitive commodities have the same sign to ensure no cancellation of positive and negative quantities.

For convenience, we assume that there are no initial or ending inventories of any commodity; i.e., $h_0^k \equiv h_n^k \equiv 0$ for each $k \in \mathcal{K}$. Production, sales, and inventories of each commodity k satisfy the usual stock-conservation constraints:

$$p_i^k - s_i^k + h_{i-1}^k - h_i^k = 0, \quad 1 \le i \le n \text{ and } k \in \mathcal{K}. \tag{1}$$

Let $p^k = (p_i^k)$, $s^k = (s_i^k)$, and $h^k = (h_i^k)$ be the *production*, *sales*, and *inventory schedules of commodity k*, respectively, if they satisfy (1). Call $x^k \equiv (p^k, s^k, h^k)$ *a schedule of commodity* $k \in \mathcal{K}$ if p^k, s^k, and h^k are the production, sales, and inventory schedules of that commodity, respectively. A *schedule* is a vector $x \equiv (x^k)$ of schedules of each commodity such that the aggregate production, sales, and inventory schedules are sums of corresponding quantities of primitive commodities; that is,

$$x^K = \sum_{k \in \hat{K}} x^k. \tag{2}$$

2.1.2. Costs and Parameters. We assume throughout that all costs are convex. When dealing with the total cost of a given schedule, we include all real costs (e.g., of production), the negative of revenues (from sales), as well as fictitious costs (e.g., penalty from over- or underproduction) associated with this schedule. Moreover, we may choose to incorporate in the cost function discount factors on profits or losses. Various costs associated with maintaining the machines/warehouses, sales restrictions resulting from government policies, etc., can also be incorporated. We assume that the total cost of a given schedule can be expressed as the sum of the costs associated with the individual activities. Explicitly, we suppose that there are $+\infty$ or real-valued convex costs associated with production, sales, and storage of each commodity (primitive or aggregate) in each period. We allow the costs to be infinite to reflect infeasible values of the variables. For example, if production of a commodity must be nonnegative in a period, then the associated cost of negative production (or disposal) of that commodity in that period will be assumed to be $+\infty$. Similarly, if the total sales of all commodities cannot exceed an upper bound in a period, then the associated cost of sales of the aggregate commodity at that period beyond the upper bound will be $+\infty$ and 0 otherwise. This device permits us to avoid discussing constraints on variables separate from their costs. Note further that convex nonlinear costs of aggregate commodities may also arise, for example, when limited resources can be used interchangeably for all commodities.

To study the effects of changes in problem environment, we associate a parameter with production, sales, and storage costs of each commodity at each period that is a vector of real numbers. The elements of the vector may represent parameters of the cost function or an upper or lower bound on the associated variable. For example, consider a price setter who sets a price r_j^k when he is facing a downward sloping demand curve $s_j^k = \sigma_j^k - \beta_j^k r_j^k$ for commodity k in period j. His revenue from sales of s_j^k units of commodity k in period j is $r_j^k s_j^k = (\sigma_j^k - s_j^k)s_j^k/\beta_j^k$, and the sales parameter, σ_j^k, is chosen to be the y-intercept of the demand function. Similarly, we can associate with production of commodity k at period j the vector parameter $(\ell_j^k, u_j^k, r_j^k, o_j^k, t_j^k)$, where ℓ_j^k and u_j^k respectively denote the lower and upper bounds on production of commodity k at period j and r_j^k, o_j^k, and t_j^k denote rates for regular time, overtime, and subcontracting time for commodity k at period j, respectively. Associating a vector parameter with a cost function permits us to discuss several parameters associated with the same variable in a unified way. By varying the values of parameters, we can simulate events such as changes in production costs (e.g., an increase in some or all of the wage rates r_j^k, o_j^k, and t_j^k of commodity k in period j while satisfying $r_j^k \le o_j^k \le t_j^k$), strikes and supply disruptions (e.g., a decrease in u_j^k for commodity k in period j), increased demand for commodity k in period j (e.g., an increase in the y-intercept, σ_j^k, of the demand function), and an increase in sales commitments or a decrease in storage space. Our objective in this paper is to evaluate the qualitative effects of these and similar events on optimal solutions in the dynamic multicommodity production planning problem.

We assume that there is a $+\infty$ or real-valued cost $P_i^k(p_i^k, \pi_i^k)$ of producing p_i^k units of commodity k in period i, given an associated production parameter π_i^k for commodity k in that period. Furthermore, there is a $-\infty$ or real-valued revenue from selling s_i^k units of commodity k in period i, given an associated sales parameter σ_i^k for commodity k in that period. Denote by $S_i^k(s_i^k, \sigma_i^k)$ the sales costs associated with sales of s_i^k units minus the corresponding sales revenue. The sales cost may account for expenses that are incident to sales—for example, handling, packing (labor and/or material), and shipments—or express constraints on the amount s_i^k to be sold. Finally, there is a $+\infty$ or real-valued cost $H_i^k(h_i^k, \eta_i^k)$ of storing h_i^k units of commodity k in period i when η_i^k is the storage parameter of commodity k in that period. In each case, the commodity may, of course, be the aggregate one.

The cost of a schedule $x = (x^k)$ is

$$\sum_{k\in\mathcal{K}} \sum_{i=1}^{n} \left(P_i^k(p_i^k, \pi_i^k) + S_i^k(s_i^k, \sigma_i^k) + H_i^k(h_i^k, \eta_i^k)\right). \qquad (3)$$

The *κ-commodity n-period production planning* (κnPP) problem is to choose a schedule, $x = (x^k)$, that minimizes (3). If $\kappa = 1$, the κnPP problem is reduced to a *single-commodity n-period production planning* (PP) problem.

Observe that any "interaction" among primitive commodities is assumed to be expressed by the costs of aggregate production, sales, and/or inventories. For example, an upper bound on total production in a period is modeled via a cost function of aggregate production at this period; the value 0 is attained if total production at this period does not exceed the upper bound and $+\infty$ otherwise. Also, the additivity of the cost function (3) permits us to incorporate a linear cost of an aggregate commodity into the cost of the corresponding primitive commodities and thus eliminate the aggregate commodity from the problem. Hence, the κ-commodity production planning problem is separable into κ single-commodity production planning problems when all aggregate cost functions are linear. Therefore we assume in this paper that the cost function associated with an aggregate quantity is identically zero or nonlinear.

2.1.3. Convexity of the Cost Functions. We assume throughout that there are scale diseconomies; i.e., the cost function is convex in the schedule.

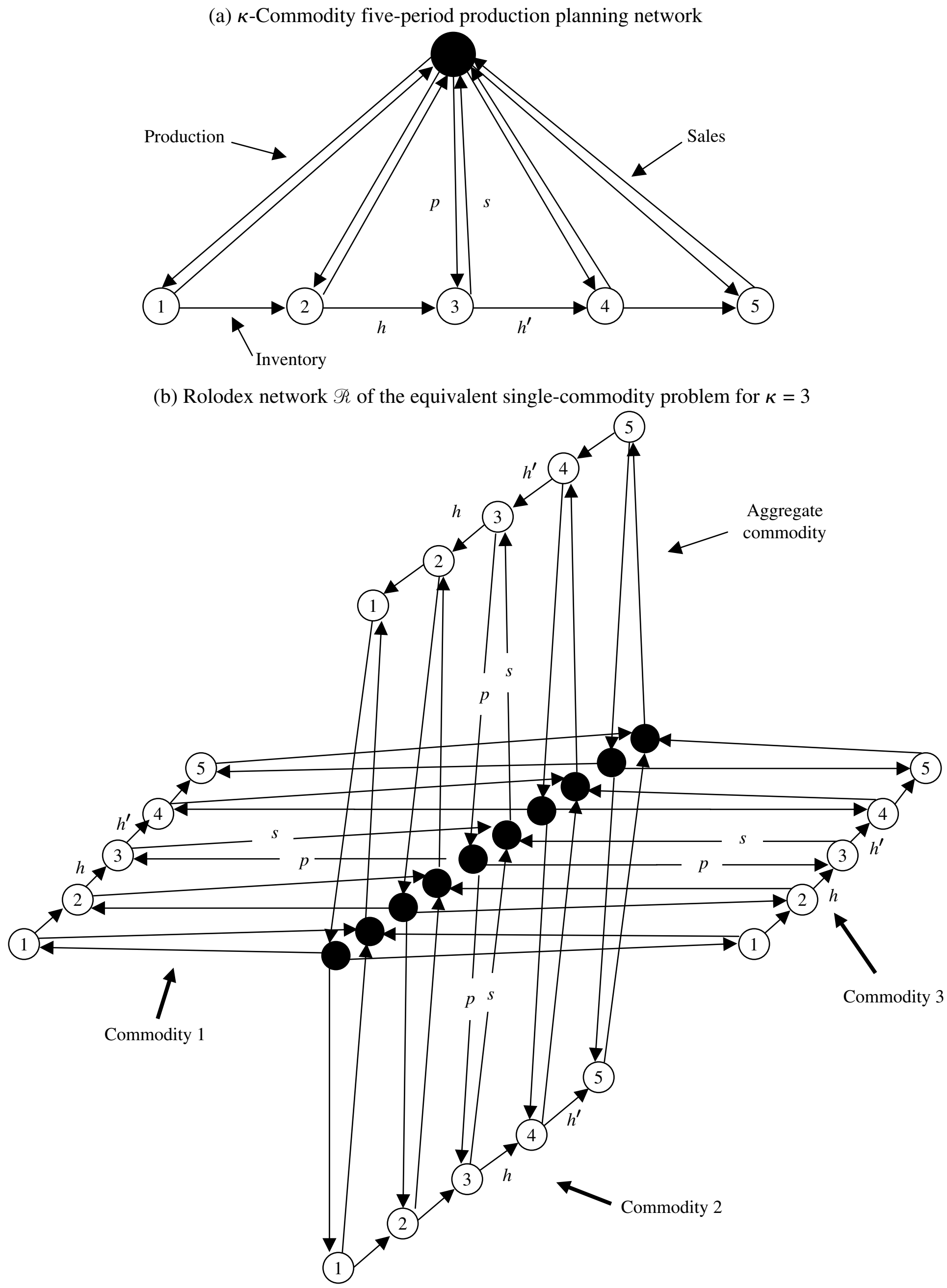
Convex Costs for Primitive Commodities. Scale diseconomies in the production of a commodity may arise when production is fixed, there are upper or lower bounds on production, or there are alternative sources of supply, each with linear costs and limited capacity. Convex production costs also arise when production at a plant in excess of normal capacity must be deferred to a future shift, to overtime, or to subcontracting, with attendant increases in unit labor costs. Thus, a piecewise convex production cost may take into account different wages for overtime or shift premiums, or different skill levels on different shifts. Scale diseconomies in negative sales revenue arise where the sales revenue function, i.e., the product of price and sales, is concave in sales. This can happen when forecasted future sales are presumed fixed or where sales are variable and the firm is a price taker or setter. Convexity of the sales cost functions S_i^k may also arise from upper or lower bounds on the variable s_i^k. The upper bounds may reflect market strategy, limited demand for the commodity, or shipment limitations, for example. Finally, convexity assumptions on inventory costs may arise from upper or lower bounds on inventories or from an increase in the unit cost of storage beyond a certain amount of stock.

Convex Costs for the Aggregate Commodity. Nonlinear convex aggregate production costs appear when certain equipment, materials, working capital, funds, and/or labor can be used interchangeably for all commodities, and where there are limitations on the their supply. Upper bounds on aggregate production may reflect such restrictions. Positive lower bounds on aggregate production may arise from prior commitments, for example, for employment with a prohibitively high cost for idle time. Nonlinear convex aggregate production costs could also arise where there are limited supplies at each of several sources for the interchangeable resources referred to above, and the cost of each is linear.

2.2. Reduction of the Dynamic Multicommodity Production Planning Problem to an Equivalent Single-Commodity Production Planning Problem

2.2.1. Formulation as a Multicommodity Network Flow Problem. A *multicommodity flow* is a collection of primitive (single-)commodity flows together with an aggregate commodity flow that is the sum of the primitive commodities flows. The κnPP problem is a multicommodity network flow problem with zero supply at each node for each commodity in the associated underlying graph, as illustrated in Figure 1(a) for $n = 5$. (We use h and h' to denote incoming and outgoing inventories to a generic node in all graphs, respectively. Similarly, we use in all graphs p and s to denote production and sales activities, respectively, for all primitive and aggregate commodities). To see this, observe that flow conservation at each node i for each commodity $k \in \mathcal{K}$ expresses the

Figure 1 Reduction of the Three-Commodity Five-Period Production Planning Problem to a Single-Commodity Network Flow Problem



stock-conservation constraint (1) in period i for that commodity. The (redundant) flow-conservation constraint at the black node for each commodity $k \in \mathcal{K}$ expresses the fact that total production of that commodity in all periods equals total sales of that commodity in those periods. Constraint (2) defines the aggregate commodity flow as the sum of the primitive commodity flows. A schedule of any commodity is thus a circulation, and a schedule is a multicommodity circulation. The *dynamic multicommodity flow* problem associated with the κnPP problem is to find a multicommodity circulation that minimizes the cost function (3) over all circulations that have finite costs. For additional material regarding the formulation and applications of the multicommodity network flow problem, see, e.g., Muckstadt and Roundy (1987), Anily (1991), Ahuja et al. (1993), Roundy (1993), and Anily and Tzur (2006).

2.2.2. Reduction to a Single-Commodity Network Flow Problem on a Rolodex Graph. The special structure of the graph underlying the κnPP problem can be used to reduce it to an equivalent PP problem, thereby generalizing earlier results of this type (Veinott 1969, pp. 272–274; Evans 1978, Soun and Truemper 1980). Such a reduction was recently carried out by Ciurria-Infosino et al. (2014) on suspension tree graphs and, for completeness, is presented here. Specifically, consider the directed graph underlying the κnPP problem (displayed in Figure 1(a) for $\kappa = 3$ and $n = 5$). Remove from this graph the node incident to the production and sales arcs (the black node) and add one node, referred to as the *junction node*, at the end of each arc incident to it. Duplicate the graph obtained for each commodity (primitive and aggregate) and join the $\kappa + 1$ copies by coalescing, for each period i, the $\kappa + 1$ newly added junction nodes incident to the sales (respectively, production) arcs. This construction results in the directed rolodex graph $\mathcal{R}$ underlying the equivalent PP problem to the original κnPP problem. Figure 1(b) displays $\mathcal{R}$ for a three-commodity five-period production planning problem. Observe that three arcs corresponding to the three primitive production (respectively, sales) arcs leave (respectively, enter) each junction node incident to the production (respectively, sales) arcs, and one aggregate production (respectively, sales) arc enters (respectively, leaves) this junction node. The term "rolodex" arises from viewing the graph as living on a Rolodex business card file with cards joined at the spine of the file. The subgraph associated with each commodity (primitive or aggregate) lives on a single card, and the cards are joined at the black junction nodes on the spine. The flow-conservation constraints at the nodes not on the spine are stock-conservation constraints (1). Thus, the flow on each card of the rolodex graph is a schedule of a commodity. The single-commodity network flow problem equivalent to the dynamic multicommodity production planning problem is thus to find a circulation on the resulting rolodex network $\mathcal{R}$ that minimizes the flow-cost function (3) over all feasible circulations. We denote both the rolodex network and its underlying graph by $\mathcal{R}$.

3. Qualitative Results: All Nonlinear Aggregate Costs

In this section we derive qualitative analysis results for the case when there are nonlinear costs of aggregate production, sales, and inventory in every period. Our main objective is to identify all pairs of activities such that there exists an optimal solution in which the optimal level of one activity is monotone in the parameter of the other activity.

We first recall a few graph-theoretic definitions. We frequently apply graph-theoretic concepts defined for undirected graphs to directed graphs. When this is done, it is understood that the concept applies to the induced undirected graph formed by replacing each arc in the directed graph by an undirected arc (i.e., edge) joining the same nodes.

A *simple path* in an (undirected) graph is an alternating sequence of distinct nodes and edges such that each edge joins the nodes immediately preceding and following it. If the beginning and ending nodes of a simple path are joined, we obtain a *simple cycle*. For a positive integer n, an (undirected) graph with two or more nodes and without loops is *n-connected* if every pair of distinct nodes is joined by at least n internally node-disjoint paths. We use *connected* for 1-connected. An *edge-cutset* of a connected graph is a set of edges whose deletion disconnects the graph.

We now recall some results of Granot and Veinott (1985), which will be used for the qualitative analysis of the multicommodity production planning problem. The ripple theorem (Granot and Veinott 1985, pp. 476, 480) provides conditions under which the magnitude of changes as a result of one parameter ("source") decreases ("ripple down") as one gets "farther" away from the source; i.e., it provides conditions for the diminishing effects of change. It implies that the change in the optimal flow in arc a resulting from changing the parameter of arc b is not smaller than the corresponding change in arc d if every simple cycle that contains b and d also contains a. The convexity assumption on the cost functions is enough to guarantee that the ripple theorem holds. The monotonicity theorem (Granot and Veinott 1985, p. 484) provides conditions when the optimal flow in one arc is monotone in the parameters of another arc. It is based on substitutes and complements relations among arcs (see the definition below) in the network, and it asserts that an optimal flow in arc a increases (respectively, decreases) in the parameter of arc b if arcs a and b are complements (respectively, substitutes). The smoothing theorem (Granot and Veinott 1985, p. 490) explores the effect of a change in an arc parameter on an optimal flow in the same arc. It provides conditions when an optimal value of an arc flow does not increase faster than the change in its corresponding parameter. We will refer to the monotonicity, smoothing, and ripple theorems as the *monotonicity theory*.

To ensure that the optimal value of an activity—production, sales, or inventory—is monotone in the parameter of another activity, two types of assumptions are needed. One concerns the nature of the cost function of the activity and the other the relative position of the corresponding arcs in the graph, i.e., their possible conformality. We first develop a full characterization of conformality between arcs in the rolodex graph $\mathcal{R}$, which, coupled with the single-commodity monotonicity theory, allows us to obtain monotonicity results for the κnPP problem.

3.1. Substitutes and Complements in 2-Connected Graphs

An arc in a 2-connected graph is called a *complement* (respectively, a *substitute*) of a second arc if every simple cycle containing both arcs orients them in the same (respectively, opposite) way. Thus, in a directed graph, a pair of arcs are complements (respectively, substitutes) if every simple cycle orients both arcs either from head to tail or from tail to head (respectively, one of them from head to tail and the other from tail to head). An arc is called *conformal* with a second arc if the former is either a complement or substitute of the second. All three relations are symmetric. Note that two distinct arcs incident to a common node are conformal. They are complements if the common node is the head of one and the tail of the other, and they are substitutes otherwise. Theorem 1 below provides a complete characterization of all pairs of conformal arcs in $\mathcal{R}$.

Note that since the underlying graph of the κnPP problem (e.g., Figure 1(a)) is a special case of a suspension tree graph, Theorem 1 is a simplified version of Theorem 4 in Ciurria-Infosino et al. (2014). For completeness, we provide a simpler proof of Theorem 1, which employs the special structure of the rolodex graph associated with the κnPP problem.

Theorem 1 (Characterization of Conformality in the Rolodex Graph $\mathcal{R}$). *Consider a κnPP problem ($\kappa \geq 2$) with $\mathcal{R}$ as its associated rolodex graph. Then, two activities corresponding to arcs a and b in $\mathcal{R}$ are conformal if and only if one of the following conditions holds:* (i) $n = 1$, (ii) *a and b are incident to a common node, or* (iii) $\kappa = 2$ *and a and b correspond to inventories of distinct commodities in the same period.*

Once it has been determined that two arcs are conformal, we can determine whether they are complements or substitutes by checking their relative orientation in any simple cycle of the rolodex graph.

3.2. Submodularity and Double Submodularity of the Cost Functions

The second assumption needed for the monotonicity result to hold is that the cost functions on each arc are convex and lower semicontinuous in the flow. In addition, it requires that the cost functions be submodular in the flow–parameter pair. Roughly speaking, the submodularity of the cost function in the flow–parameter pair expresses the fact that the marginal cost of a flow is decreasing in the parameter. Formally, a $+\infty$ (respectively, $-\infty$) or real-valued function f on a lattice L in $\Re^n$ ordered by the usual less than or equal to relation is called *submodular* if

$$f(r \wedge s) + f(r \vee s) \leq f(r) + f(s), \quad \text{for all } r, s \in L, \quad (4)$$

where $r \wedge s$ and $r \vee s$ are the greatest lower bound and the least upper bound in L of the pair r, s of points in L, respectively. A twice-continuously differentiable function on a given rectangle in the plane is submodular if and only if its mixed partial derivative is nonpositive thereon. Call f *supermodular* if $-f$ is submodular. If both the function $f(\xi, \tau)$, with argument ξ and parameter τ, and its dual $f^\sharp(\xi, \tau)$, given by $f^\sharp(\xi, \tau) \equiv f(\tau - \xi, \tau)$, are submodular, $f(\xi, \tau)$ is called *doubly submodular*. The classes of submodular and doubly submodular functions are closed under addition and multiplication by nonnegative numbers.

For the following monotonicity and smoothing theorems for single-commodity problems, let t_a denote a parameter associated with arc a and assume that t_a lies in a lattice T_a. Let $t \equiv (t_a)$ be a parameter vector that lies in a given subset T of $\times_a T_a$ and assume that for each $t \in T$ there is a bounded, nonempty set $X(t)$ of minimum-cost-feasible (i.e., finite-cost) flows $x = x(t)$.

Theorem 2 (Monotonicity Theorem for Single-Commodity Problems; Granot and Veinott 1985, p. 484). *In a 2-connected graph, suppose that the flow cost $c_a(\cdot, \tau)$ in arc a is convex and lower semicontinuous for each $\tau \in T$ and $c_a(\cdot\,, \cdot)$ is submodular for each arc a. Then for each $t \in T$, there is an optimal flow $x(t)$ for which $x_a(t)$ is nondecreasing (respectively, nonincreasing) in t_b if a and b are complements (respectively, substitutes). If $c_a(\cdot\,, \cdot)$ is supermodular for each arc a, the results are reversed.*

We say (Granot and Veinott 1985, p. 488) that t and t' are monotonically step-connected if it is possible to proceed from t to t' by making a sequence of feasible single-coordinate parameter changes that always move toward t'.

Table 1 **Monotonicity of Optimal Schedules in Parameters of the κnPP Problem**

	π_j^ℓ	σ_j^ℓ	η_j^ℓ
p_i^k	$i=j\ \&\begin{cases} k=\ell \text{ or } K\in\{k,\ell\}\Rightarrow\uparrow \\ k\neq\ell \text{ or } K\notin\{k,\ell\}\Rightarrow\downarrow\end{cases}$	$i=j\ \&\ k=\ell\Rightarrow\uparrow$	$k=\ell\ \&\begin{cases} i=j \Rightarrow\uparrow \\ i=j+1\Rightarrow\downarrow\end{cases}$
s_i^k		$i=j\ \&\begin{cases} k=\ell \text{ or } K\in\{k,\ell\}\Rightarrow\uparrow \\ k\neq\ell \text{ or } K\notin\{k,\ell\}\Rightarrow\downarrow\end{cases}$	$k=\ell\ \&\begin{cases} i=j \Rightarrow\downarrow \\ i=j+1\Rightarrow\uparrow\end{cases}$
h_i^k			$k=\ell\ \&\ \lvert i-j\rvert\le 1\Rightarrow\uparrow$ $k\neq\ell\ \&\ i=j\ \&\ \kappa=2\ \&\begin{cases} K\in\{k,\ell\}\Rightarrow\uparrow \\ K\notin\{k,\ell\}\Rightarrow\downarrow\end{cases}$

Theorem 3 (Smoothing Theorem for Single-Commodity Problems; Granot and Veinott 1985, p. 490). *Assume, in addition to the hypotheses of Theorem 1, that the arc parameters t_a are real and the arc costs c_a are doubly submodular for all a. Then, $t_b - x_a(t)$ has the same monotonicity property in t_b as the optimal flow $x(t)$ described previously. Moreover, there exists an optimal flow $x(\cdot)$ satisfying $\|x(t')-x(t)\|_\infty \le \|t'-t\|_1$ for all monotonically step-connected t, t' in T.*

By applying the monotonicity theorem to the κnPP problem, we may obtain the direction of change of optimal production, inventory, and sales of each commodity in each period as a result of changes in either the production, inventory, or sales parameter. The monotonicity results are summarized in Table 1, where we indicate with arrows the direction at which the optimal value of one activity changes as the parameter of another activity increases. Of course, "↑" here means nondecreasing, and "↓" means nonincreasing. For example, the production of commodity k in period i, say, is nondecreasing in its own parameter π_i^k and in the aggregate production parameter π_i^K and is nonincreasing in the production parameter π_i^ℓ of a distinct primitive commodity ℓ in period i.

Naturally, Table 1 also provides conformality relations among the various activities. Indeed, we can view the down arrows (respectively, up arrows) in Table 1 as indicating that activities p_i^k, s_i^k, and h_i^k and the activity associated with parameters π_j^k, σ_j^k, or η_j^k are substitutes (respectively, complements) in $\mathcal{R}$. For example, sales of commodity k in period i, s_i^k, and sales of commodity ℓ in period j are conformal if and only if $i=j$. They are substitutes if the commodities are distinct ($k\neq\ell$) and neither of them is the aggregate commodity ($K\notin\{k,\ell\}$); they are complements if the commodities are the same or if one of them is the aggregate commodity.

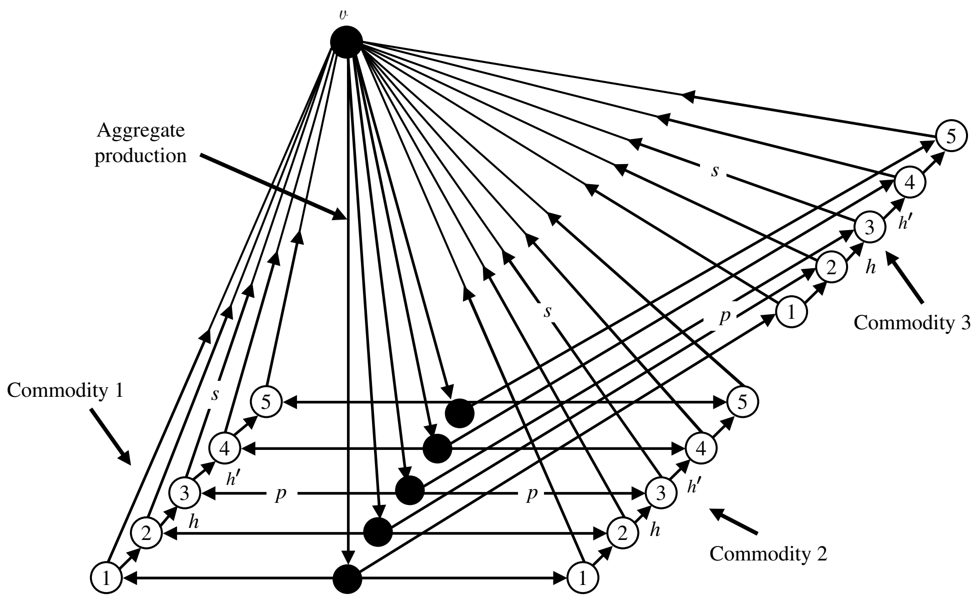
4. Qualitative Results with Some Nonlinear Aggregate Costs

It is important to note that if some arcs in the rolodex graph $\mathcal{R}$ are contracted, additional arcs may become conformal, and thus additional insight can be derived. In this section we study the qualitative variation of optimal schedules with the associated parameters for special cases of the κnPP problem. Each case corresponds to an instance of the κnPP problem where some aggregate costs are identically zero. Specifically, we study models where there is no change regarding the costs of primitive commodities but where nonlinear aggregate costs are present only for production, or only for sales, or only for production and sales, and so on. Thus, we consider cases where there are possible "interactions" among primitive commodities corresponding only for production, or only for sales, or only for production and sales, etc. Let $\mathcal{R}_p$, $\mathcal{R}_s$, $\mathcal{R}_h$, $\mathcal{R}_{ps}$, $\mathcal{R}_{ph}$, and $\mathcal{R}_{sh}$ denote the contracted rolodex graphs derived from $\mathcal{R}$. For example, $\mathcal{R}_p$ is derived from $\mathcal{R}$ by contracting each arc associated with aggregate sales and aggregate inventories so that the end nodes of the arc being contracted are coalesced into a node with zero supply. It corresponds to the case where there are zero costs of aggregate sales and inventories. As $\mathcal{R}\equiv\mathcal{R}_{psh}$, for consistency, we use $\mathcal{R}_{psh}$ in the sequel. Figures 2 and 4–6 illustrate various contracted rolodex graphs obtained from $\mathcal{R}$. Note that none of the contracted rolodex graphs is a rolodex graph corresponding to a suspension tree graph. Thus, Ciurria-Infosino et al. (2014, Theorem 4) does not imply the conformality characterizations derived in this section.

4.1. Aggregate Production Costs

We first consider the case in which there are nonlinear costs of aggregate production but zero costs of aggregate sales and inventories. Figure 2 illustrates the corresponding contracted rolodex graph $\mathcal{R}_p$ for a three-commodity five-period production planning problem. Observe that conservation of flow at nodes corresponding to a primitive commodity, or at the "black nodes" other than ν, is the same as in Figure 1(b). Conservation of flow at node ν reflects the obvious fact that total aggregate production in all periods equals total aggregate sales in all periods.

Figure 2 Contracted Rolodex Graph $\mathcal{R}_p$ of the Three-Commodity Five-Period Production Planning Problem with Aggregate Production Costs



Theorem 4 (Characterization of Conformality in the Contracted Rolodex Graph $\mathcal{R}_p$). *The activities corresponding to arcs a and b in the contracted rolodex graph $\mathcal{R}_p$ are conformal if and only if at least one of the following conditions holds*: (i) $n = 1$, (ii) *a and b are incident to a common node*, *or* (iii) $n = 2$ *or* $\kappa = 2$, *and the deletion of* a, b *and the top black node* ν *disconnects* $\mathcal{R}_p$.

From Theorem 4, one can determine all conformal activities in Figure 2. For example,

- Aggregate productions in distinct periods are substitutes.
- Productions of any two distinct primitive commodities in the same period are substitutes.
- Sales of two primitive commodities are substitutes if either the commodities or the periods are distinct.
- Inventories of the same primitive commodity in two consecutive periods are complements.
- At any period, aggregate production and sales of a primitive commodity are complements.
- At any period, production and sales of the same primitive commodity are complements.

All monotonicity and, hence, conformality relations among activities in $\mathcal{R}_p$ derived from Theorem 4, as well as subsequent cases considered in the sequel, are summarized in Tables 2–4 in §5.

We now apply the monotonicity theory to several practical problems.

4.1.1. Changes in Production Costs. Consider first the effects of changes in the marginal cost of production on the optimal production, sales, and inventory schedules of each commodity. An increase in the marginal cost of production may occur, e.g., from increasing wage rates or increasing the unit costs of limited alternative sources of supply. A decrease in the marginal cost of production may arise, e.g., from making technological improvements, expanding the production capacity, and adding workers.

Recall that the absence of costs for aggregate sales and aggregate inventories permits us to have negative sales and negative inventories of each primitive commodity. To ensure that the aggregate production cost is meaningful, we assume in the sequel that production of each primitive commodity is nonnegative whenever there are two or more commodities.

Example 1 (Across-the-Board Increase in Wage Rates). Suppose a *temporary* fixed percent increase in wage rates in, say, period j is introduced. (Henceforth, a *temporary* change, such as in period j in this case, refers to a change that is introduced only in period j and is not in effect in the ensuing periods. If a temporary change in period j becomes *permanent*, it is meant that the change is applied in period j and thereafter.) How should optimal production, storage, and sales decisions of each commodity respond? Intuitively, aggregate production in period j will decrease. But would production and sales of each primitive commodity also decrease in period j? What would be the effect on sales of the primitive commodities in all periods other than j? If aggregate sales in period j decreases, is it possible that the sales of some primitive commodities will strictly increase in other periods? Could aggregate production in some periods other than j decrease? Below, we apply the qualitative analysis tools developed so far to provide answers to questions such as these.

To investigate the qualitative effects of across-the-board increase in wage rates, let w_j and c_j respectively denote the wage and other costs that make up the aggregate production cost P_j^K in period j. Thus,

$$P_j^K(p_j^K, \pi_j^K) = \pi_j^K w_j(p_j^K) + c_j(p_j^K), \tag{5}$$

where π_j^K is the index of wage rates; i.e., $100(\pi_j^K - 1)$ is the percent increase in wage rates in period j. The current wage rate in period j corresponds to $\pi_j^K = 1$, and an increase therein corresponds to $\pi_j^K > 1$. Observe that $P_j^K(\cdot, \cdot)$ is supermodular if $w_j(\cdot)$ is nondecreasing on $\Re_+$ and $w(z) = 0$ for $z \leq 0$, and $P_j^K(\cdot, \pi_j^K)$ is convex if $w_j(\cdot)$ and $c_j(\cdot)$ are convex and $\pi_j^K \geq 0$. In that event, increasing the wage rate in period j—that is, increasing π_j^K and noting that P_j^K is supermodular—has the following effects on optimal production, sales, and inventory decisions. (Henceforth, when we describe the monotonicity results, "increasing" should be understood as nondecreasing, and "decreasing" should be understood as nonincreasing.)

- Sales of each primitive commodity decrease (i.e., prices rise) in every period (since aggregate production in period j and sales of a primitive commodity in every period are complements, and $P_j^K(p_j^K, \pi_j^K)$ in (5) is supermodular).
- Production of each primitive commodity decreases in period j (since the aggregate production in period j and production of any primitive commodity in period j are complements).
- Aggregate production decreases in period j (since each activity is self-complementary, and $P_j^K(p_j^K, \pi_j^K)$ is supermodular) and increases in every other period (since aggregate productions in distinct periods are substitutes). Moreover, conservation of flow at node ν implies that total aggregate production in all periods decreases; hence the decrease in period j is greater than the total increase in the other periods.
- From the ripple theorem, it follows that the change in aggregate production in period j exceeds all other changes.

If the increase in period j becomes permanent, then the effects on optimal production, sales, and inventory decisions are as follows:

- The above conclusions remain valid before period j, but not in period j or thereafter, except that sales of each primitive commodity does fall in every period in this case as well.
- Total aggregate production from all periods decreases. So even though aggregate production need not decrease in period j or thereafter, total aggregate production during those periods falls. The size of this reduction exceeds the total decrease of sales of all primitive commodities in all periods by the amount that total aggregate production increases before period j.

Note that in the above example, π_j^K is a single parameter. However, the qualitative theory can be applied when π_j^K is a vector parameter. For example, it can be shown that precisely identical conclusions to those above can be drawn when there is a temporary increase in the wage rates r_j on regular time, or o_j on overtime, or t_j on subcontracting in period j. In this event, the aggregate production cost P_j^K in period j is given by

$$P_j^K(p_j^K, \pi_j^K) = r_j p_j^K + (o_j - r_j)(p_j^K - m_j)^+ + (t_j - o_j)(p_j^K - M_j)^+ + c_j(p_j^K), \tag{6}$$

where z^+ denotes the positive part of z, $\pi_j^K \equiv (r_j, o_j, t_j)$; the function c_j comprises the other aggregate production costs in period j; and $0 \leq m_j \leq M_j$ are the upper bounds on regular and regular plus overtime levels of production in period j, respectively.

Example 2 (Guaranteed Total Wages). Suppose a guaranteed total wages (GTWs) constraint is temporarily imposed in period j. Such a requirement could be modeled, for example, by increasing the lower bound on aggregate production in period j from 0 to $\pi_j^K > 0$, or by imposing a floor, $w_j(\pi_j^K)$, on wage costs in period j. Naturally, introducing a lower bound on aggregate production in period j would result with an increase in total production in period j as well as an increase in overall cost. But could total production in period j exceed π_j^K? Would total revenue increase? Could aggregate production increase in some periods other than j? Would the impact of imposing floor $w_j(\pi_j^K)$ on wage costs be similar to that of an across-the-board increase in wage rates, as discussed in Example 1?

When the introduction of a GTW is modeled by increasing the lower bound on aggregate production in period j, the aggregate production cost P_j^K in period j is given by

$$P_j^K(p_j^K, \pi_j^K) = P_j^K(p_j^K) + \delta_+(p_j^K - \pi_j^K), \tag{7}$$

where $\delta_+(\cdot)$ denotes the indicator function of the subset $\Re_+$ of $\Re$. (An *indicator function* of a set L in $\Re^n$ is a function $\delta_L(\cdot)$ whose value is 0 on L and $+\infty$ otherwise.) Since $\delta_+(\cdot)$ is convex, the function $P_j^K(\cdot, \cdot)$ is doubly submodular and $P_j^K(\cdot, \pi_j^K)$ is convex if $P_j^K(\cdot)$ is convex. Therefore, since $P_j^K(\cdot, \cdot)$ is submodular—rather than supermodular as was the case in Example 1—the qualitative effects of establishing a temporary or permanent GTW constraint in period j are precisely opposite to those of an across-the-board increase in wage rates.

We note that since $P_j^K(\cdot, \cdot)$ here is also doubly submodular, the smoothing theorem applies as well, and more can be said. Let $x(t) \equiv (p(t), s(t), h(t))$ denote a suitable minimum-cost schedule given that t is the lower bound on aggregate production in period j. Clearly, $x(0) = x(t)$ for $t \equiv p_j^K(0) \wedge \pi_j^K$. Then, from the smoothing theorem, increasing t by the amount $(\pi_j^K - t)^+$ has the effect of increasing $p_j^K(t)$ by at most $(\pi_j^K - t)^+$, which implies $p_j^K(\pi_j^K) \le \pi_j^K$. On the other hand, $p_j^K(\pi_j^K) \ge \pi_j^K$ by hypothesis, so $p_j^K(\pi_j^K) = \pi_j^K$. Summing up, the optimal schedule responds to an introduction of a GTW constraint in period j as follows.

- $p_j^K(\pi_j^K) = \pi_j^K$; i.e., the total production in period j is equal to the newly introduced lower bound on production in this period.
- The magnitude of each individual change in each variable is bounded from above by $\lambda_j \equiv (\pi_j^K - p_j^K(0))^+$ (since λ_j is the change in the activity whose parameter was changed and, by the ripple theorem, this change is at least as large as the change in any other activity).
- Sales of each primitive commodity increase in every period.
- Production of each primitive commodity increases in period j.
- Aggregate production in period j satisfies $p_j^K(\pi_j^K) = \pi_j^K$ and $0 \le p_i^K(\pi_j^K) \le p_i^K(0)$ for $i \ne j$. The total decrease of aggregate production in all periods other than j is λ_j minus the increase of total sales.

Assume now that the GTW constraint becomes permanent after period j, and let $x(\pi_j^K)$ denote a suitable minimum-cost schedule given a common lower bound π_j^K for aggregate production in period j and thereafter. Then, $x(\pi_j^K)$ is related to $x(0)$ in the following ways.

- The minimum-cost aggregate production $(p_1(\pi_j^K), \ldots, p_n(\pi_j^K))$ is such that

$$0 \le p_i^K(\pi_j^K) \le p_i^K(0), \quad i = 1, 2, \ldots, j-1, \quad \text{and}$$
$$\pi_j^K \le p_i^K(\pi_j^K) \le \pi_j^K \vee p_i^K(0), \quad i = j, j+1, \ldots, n.$$

- No change in a variable is greater than $\lambda_{jn} \equiv \sum_{i=j}^n \lambda_i$.
- Sales of each primitive commodity increase in every period, and the total sales of all commodities in all periods increase by at most λ_{jn}.
- Aggregate production decreases in every period before period j. Since total aggregate production in all periods increases by the total amount aggregate sales rise, total aggregate production from period j to period n increases by more than total aggregate sales. Since the increase of aggregate production in all periods is bounded from above by λ_{jn}, then the total decrease of aggregate sales before period j is bounded from above by λ_{jn} minus the increase of total aggregate sales.

As previously noted, an alternative modeling of a temporary GTW requirement in period j is to put a floor, $w_j(\pi_j^K)$, on wage costs but not on production levels. Then,

$$P_j^K(p_j^K, \pi_j^K) = w_j(p_j^K \vee \pi_j^K) + c_j(p_j^K), \tag{8}$$

where the function c_j denotes the nonwage costs associated with aggregate production in period j. In this case, $P_j^K(\cdot\,, \cdot)$ is doubly submodular and $P_j^K(\cdot, \pi_j^K)$ is convex if $w_j(\cdot)$ and $c_j(\cdot)$ are convex and $w_j(\cdot)$ is nondecreasing. The results for the first modeling of GTW remain valid for this alternative interpretation, with the exception that the lower bound π_j^K for $p_i^K(\pi_j^K)$ is no longer valid.

Thus, qualitative analysis reveals that although an across-the-board increase in wage rates and an increase in guaranteed total wages both increase total labor costs, there are optimal schedules for which, perhaps surprisingly, the two alternatives have precisely opposite effects. For example, an across-the-board increase in wage rates causes optimal sales of each commodity in each period to decrease, whereas increasing guaranteed total wages causes them to increase.

Example 3 (Capacity Expansion and Strikes). Suppose capacity is temporarily expanded in period j. The effect is to increase the upper bound π_j^K on aggregate production in period j. Then,

$$P_j^K(p_j^K, \pi_j^K) = P_j^K(p_j^K) + \delta_+(\pi_j^K - p_j^K). \tag{9}$$

The function $P_j^K(\cdot\,, \cdot)$ is doubly submodular, and $P_j^K(\cdot, \pi_j^K)$ is convex if $P_j^K(\cdot)$ is convex. Therefore, the qualitative impact of a temporary or permanent increase in capacity is opposite to that of across-the-board increase in wage rates. The effects of a temporary or permanent strike in period j that prevents any production can be studied by reducing to zero the upper bound π_j^K on aggregate production. Therefore, the effects of a temporary or permanent strike in period j are precisely opposite to that of an expansion of capacity.

4.1.2. Changes in Sales Costs or Inventory Costs. Changes in the sales parameter of a primitive commodity, say, ℓ, can arise for different reasons. For a price taker, e.g., a seller in a competitive market, the change may reflect an increase (or decrease) in the market price. For a price setter, e.g., monopolist, the change may describe a rightward (or leftward) translation in the demand curve. In both cases, price setting and price taking, the change may also express an increase (or decrease) in an upper or lower bound on sales. A third possibility is an increase (or decrease) in forecast sales in an environment with fixed sales. We now evaluate the effect of one of these possibilities.

Example 4 (Rightward Translation of Demand Curve of a Price Setter). Assume that for a price setter, sales s_j^ℓ of commodity ℓ in period j is determined by its price $r_j^\ell \geq 0$ through the downward sloping demand curve $s_j^\ell = \sigma_j^\ell - \beta_j^\ell r_j^\ell$, where $\sigma_j^\ell, \beta_j^\ell > 0$, and $0 \leq r_j^\ell \leq \sigma_j^\ell/\beta_j^\ell$; i.e., $0 \leq s_j^\ell \leq \sigma_j^\ell$. Therefore, since $r_j^\ell = (\sigma_j^\ell - s_j^\ell)/\beta_j^\ell$, the sales cost S_j^ℓ of commodity ℓ in period j is given by

$$S_j^\ell(s_j^\ell, \sigma_j^\ell) = (-s_j^\ell/\beta_j^\ell)(\sigma_j^\ell - s_j^\ell) + \delta_L(s_j^\ell) + c_j^\ell(s_j^\ell), \qquad (10)$$

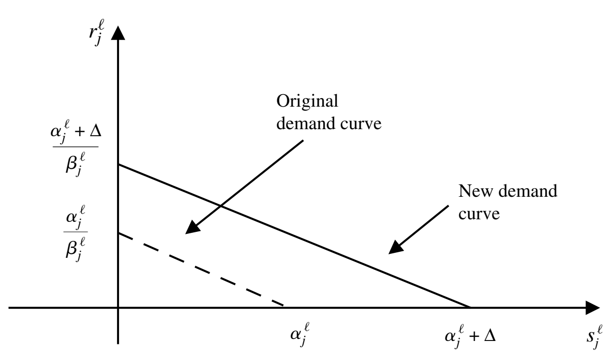
where δ_L denotes the indicator function of the interval $L = [0, \sigma_j^\ell]$, and c_j^ℓ denotes the other sales costs for commodity ℓ in period j. As illustrated in Figure 3, an increase of demand, i.e., a rightward translation of the demand curve, may be represented by an increase Δ in the parameter σ_j^ℓ.

What would be the impact of such an increase on demand for commodity ℓ in period j? If production of commodity ℓ in period j increases, could production of other commodities increase in some periods? Would inventory of commodity ℓ in all periods i, $i \leq j-1$, increase so as to satisfy the increased demand for commodity ℓ in period j? Could aggregate production decrease in some periods? Could sales of other commodities increase in some periods?

Since the function $S_j^\ell(\cdot, \cdot)$ is submodular and $S_j^\ell(\cdot, \sigma_j^\ell)$ is convex if $c_j^\ell(\cdot)$ is convex, we can conclude the following:

- Aggregate production increases in every period.
- Sales of commodity ℓ increase in period j and decrease in every other period, whereas sales of all other primitive commodities decrease in every period. However, the total sales of all commodities in all periods increase by the amount that total aggregate production increases.
- Changes in production and inventories can only be predicted for commodity ℓ, for which production in period j and inventory in period $j-1$ increase, whereas the inventory in period j decreases.

Figure 3 **Rightward Translation of the Demand Curve**



- The sales increase of commodity ℓ in period j exceeds all other changes.

If, in addition, the increase in market price in period j is permanent, then the following apply:

- Aggregate production still increases in every period.
- Sales of all commodities decrease in every period, but the direction of change in sales of commodity ℓ in periods $j, j+1, \ldots, n$ is unpredictable.
- Total sales of all commodities in all periods increase. So, even though the sales of commodity ℓ need not increase in period j or thereafter, the total sales of commodity ℓ in periods j and thereafter increase. Moreover, this increase surpasses the total decrease of sales (of commodity ℓ before period j, plus the total decrease of sales of every other commodity) by the amount that total aggregate production increases.

Observe further that since the function $S_j^\ell(\cdot, \cdot)$ in (10) is doubly submodular, then, by applying the smoothing theorem, we can conclude, in addition, the following.

- The magnitude of each individual change is bounded above by the increase of the parameter σ_j^ℓ. This allows us to get upper bounds on important changes; e.g., if the shift in the demand is temporary, then the increase in total sales of all commodities and the increase in total aggregate production are each bounded from above by the increase of the parameter σ_j^ℓ.
- The price r_j^ℓ set by the price setter is increasing in the parameter σ_j^ℓ of the demand curve.

Example 5 (Changes in Storage Capacity or Admissible Backlog Levels). Limitations on storage space or admissible backlog levels may be introduced into the model through an upper bound on the inventory in question. For example, an increase in the upper bound on inventory of a primitive commodity ℓ in period j can be studied as an increase in the parameter η_j^ℓ of a doubly submodular inventory cost H_j^ℓ. The effects of that on optimal schedules are as follows.

- No change is greater than the increase Δ_j^ℓ in the parameter η_j^ℓ, nor greater than the change in the inventory of commodity ℓ in period j.
- Inventories increase in periods $j, j-1$, and $j+1$. Production in period j and sales in period $j+1$ increase, whereas sales in period j and production in period $j+1$ decrease.
- If $\kappa = 2$, the inventory of another primitive commodity, say, k, in period j decreases. If $j = 1$ (respectively, $j = n-1$), then production of commodity k in period 1 (respectively, n) decreases (respectively, increases).

Assume now that the increase in inventory upper bound of commodity ℓ becomes permanent after period $j \le n-2$. Then the following applies:

• The magnitude of each individual change is bounded above by $\sum_{i=j}^{n-1} \Delta_i^\ell$, but the direction of change can be predicted only if $j \ge n-3$. If $j = n-3$, then inventory of commodity ℓ increases in period $n-2$. If $j = n-2$, then inventory of commodity ℓ increases in periods $n-2$ and $n-1$.

4.2. Aggregate Sales Costs

The contracted rolodex graph $\mathcal{R}_s$ of the κnPP problem with nonlinear aggregate sales costs and zero costs of aggregate production and inventory is identical to $\mathcal{R}_p$ with the exception that sales and production arcs are interchanged and their directions reversed. Thus, $\mathcal{R}_s$ has exactly the same conformality characterizations—parts (i)–(iii) in Theorem 4—as $\mathcal{R}_p$. Moreover, the conformality results for $\mathcal{R}_s$ can be obtained from those for $\mathcal{R}_p$ by interchanging production and sales everywhere, and whenever the conformality is between production or sales and inventories, complements and substitutes should also be interchanged. Thus, the effect of changes in production and sales parameters of a primitive or aggregate commodity in $\mathcal{R}_s$ is similar to the effect of these changes in $\mathcal{R}_p$, as can be verified by the reader.

4.3. Aggregate Inventory Costs

We consider now the case in which there are nonlinear costs of aggregate inventories and zero costs of aggregate production and sales. The contracted rolodex graph $\mathcal{R}_h$ is illustrated in Figure 4 for a three-commodity five-period production planning problem.

Theorem 5 (Characterization of Conformality in the Rolodex Graph $\mathcal{R}_h$). *The activities corresponding to arcs a and b in $\mathcal{R}_h$ are conformal if and only if one of the following four conditions holds*: (i) *a and b are incident to a common node*; (ii) $n = 2$; (iii) $\kappa = 2$, *and a and b are on the boundary of a common face*; *or* (iv) $\kappa \ge 3$, *and a and b can be made incident to a common node by deleting a production or sales arc in the first or last period and then contracting two arcs in series.*

As always, it follows from the ripple theorem that the change in an optimal value of an activity is greatest for the activity whose parameter was changed. The following is additional insight that can be derived from the ripple theorem:

• If we change a parameter of a primitive commodity ℓ, then for any other primitive commodity $j, j \ne \ell$, the resulting change in the inventory of the first (respectively, next to the last) period is greater than the change in production or sales in the first (respectively, last) period. (This result holds since any simple cycle that contains the arc whose parameter was changed and the arc corresponding to either production or sales of commodity j in period 1, for example, also contains the arc corresponding to the inventory of commodity j in period 1.)

• The above result also holds for the primitive commodity ℓ whose parameter was changed, except if the parameter changed corresponds to production or sales in the first or last period. Assuming, in this case, that the change is in the production parameter of commodity ℓ in period 1, then the greatest change is in p_1^ℓ, and every other change, except the change in s_1^ℓ, is smaller than the change in h_1^ℓ.

Combining the conformality results with the monotonicity theory, we can determine the direction of changes in optimal schedules resulting from changes in production, sales, and storage parameters.

4.3.1. Changes in Production Costs.

A temporary increase in the parameter π_j^ℓ of a submodular production cost function P_j^ℓ of primitive commodity ℓ in period j (e.g., a decrease in the production cost) includes the following effects on optimal schedules.

• Production of commodity ℓ in period j increases, whereas production of every other primitive commodity in period j decreases.

• Sales of each commodity increase in period j.

• Inventory of commodity ℓ and aggregate inventory increase in period j and decrease in period $j-1$.

• If $\kappa = 2$, production (respectively, sales) of commodity ℓ decreases (respectively, increase) in periods $j-1$ and $j+1$.

Assume now that there is a temporary increase in the parameters of the submodular production cost function of each primitive commodity in period j. Moreover, assume that it is feasible to increase the κ parameters one at a time. Then, the following apply.

• Sales of every commodity increase in period j.

• Aggregate inventory increases in period j and decreases in period $j-1$.

4.3.2. Changes in Sales Costs.

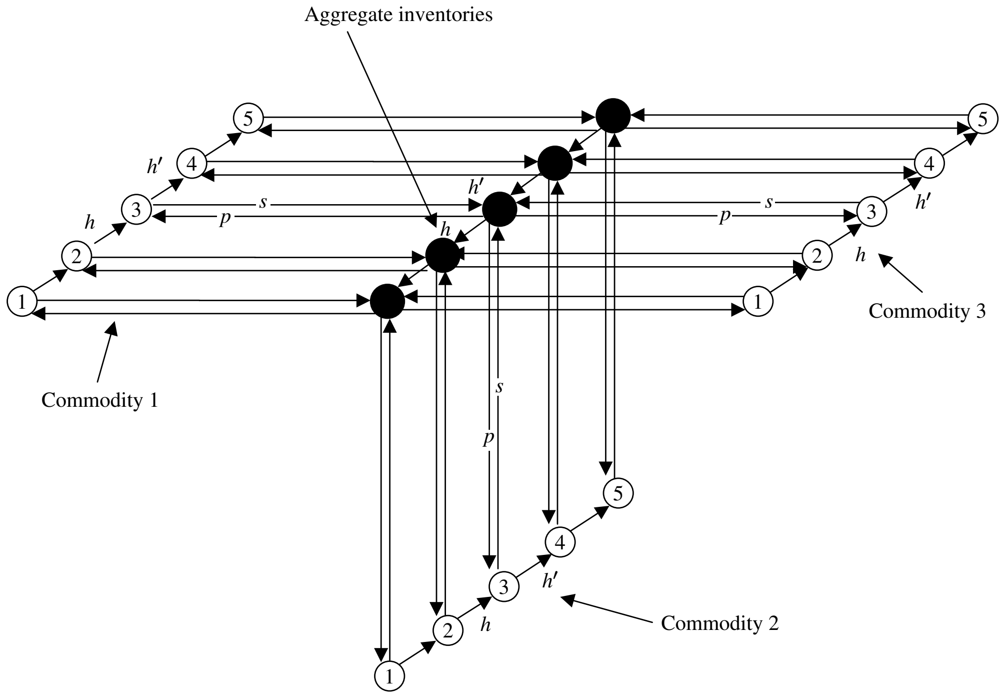
The effects of changes in sales costs on optimal schedules can be obtained from the effects of changes in production costs by interchanging production and sales everywhere. In addition, whenever the statement is about inventories, the direction of the change must also be reversed.

4.3.3. Changes in Inventory Costs.

Assume, for example, that there is a temporary increase in the parameter η_j^ℓ of the submodular inventory cost function H_j^ℓ of primitive commodity ℓ in period j. Then, the effects on optimal schedules are the following.

• Production (respectively, sales) of commodity ℓ increases (respectively, decrease) in period j and decreases (respectively, increases) in period $j+1$.

Figure 4 Contracted Rolodex Graph $\mathcal{R}_h$ of the Three-Commodity Five-Period Production Planning Problem with Aggregate Inventory Costs



- Inventories of commodity ℓ increase in periods j, $j-1$, and $j+1$.
- If $\kappa = 2$, all inventories of commodity ℓ increase, whereas all inventories of the other primitive commodity decrease.
- If the increase in η_j^ℓ becomes permanent, then we can only say that if $\kappa = 2$, the inventories of primitive commodity ℓ (respectively, k) increase (respectively, decrease) in every period.

If there is a change in inventory costs in period j that affects all commodities, then consider the following two cases. First, the change may be in the inventory parameter of a submodular or supermodular aggregate inventory cost function in period j. A percent decrease in the rent of a common storage space in period j is an example of a supermodular cost function. In this case, the effects are as follows.

- For $\kappa \geq 3$, a decrease in η_j^K has the same effects on optimal schedules as described above for an increase in η_j^ℓ of a submodular cost function.
- If $\kappa = 2$ and the decrease in η_j^K is temporary, all inventories increase in period j.

Second, the change may represent an increase, say, in the inventory parameters of the (submodular) inventory cost functions of each primitive commodity in period j. If it is feasible to increase the κ inventory parameters in period j one at a time, the only general statement we can make is this:

- If $\kappa = 2$, and the increase is temporary, then aggregate inventory increases in period j.

4.4. Aggregate Production and Sales Costs

We now briefly consider the case in which there are nonlinear costs of aggregate production and sales and zero costs of aggregate inventories. The corresponding contracted rolodex graph $\mathcal{R}_{ps}$ is illustrated in Figure 5 for a three-commodity five-period case. Theorem 6 below provides a complete characterization of conformality in $\mathcal{R}_{ps}$ and can be used to derive the qualitative results for this scenario.

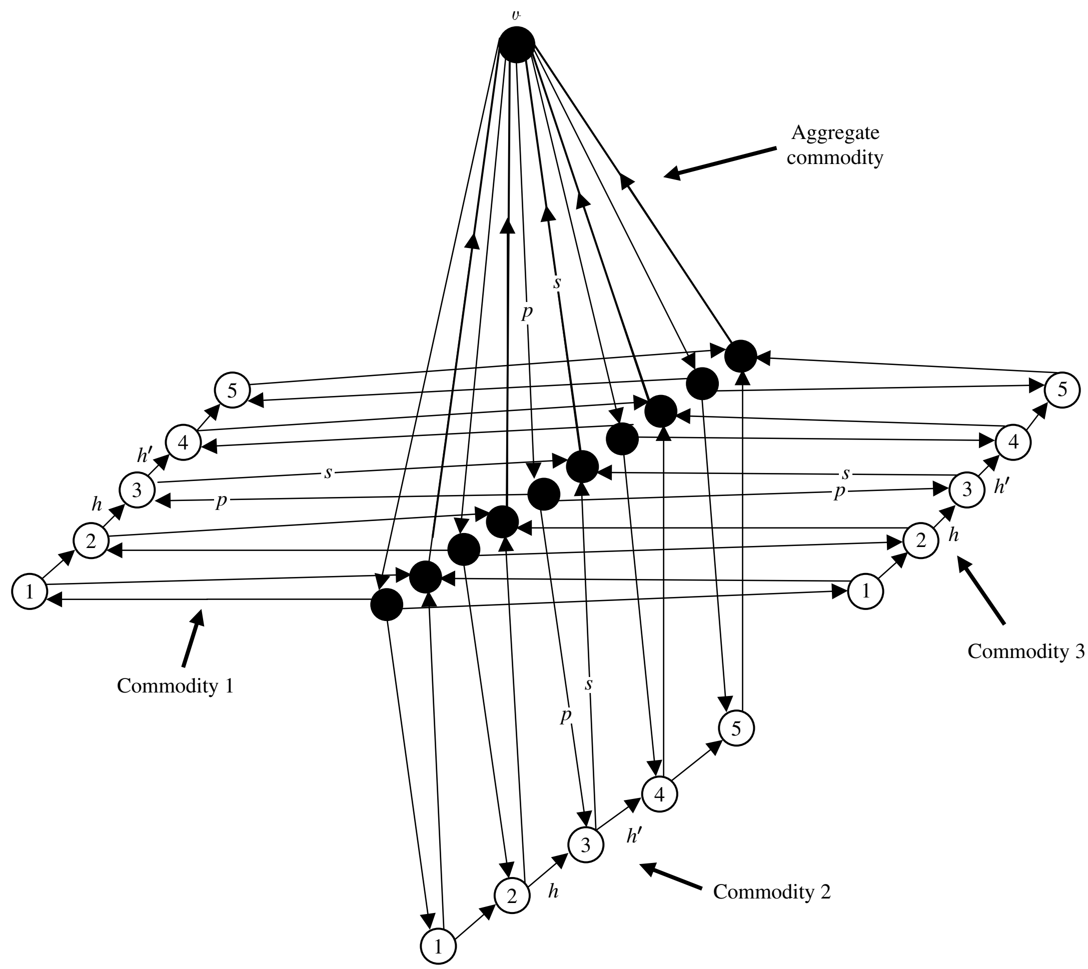
Theorem 6 (Characterization of Conformality in the Contracted Rolodex Graph $\mathcal{R}_{ps}$). *The activities corresponding to arcs a and b in $\mathcal{R}_{ps}$ are conformal if and only if one of the following three conditions holds*: (i) $n = 1$, (ii) *a and b are incident to a common node, or* (iii) $\kappa = 2$, *and the deletion of a and b together with the top black node ν disconnects $\mathcal{R}_{ps}$.*

4.5. Aggregate Production and Inventory Costs

The contracted rolodex graph $\mathcal{R}_{ph}$ when there are nonlinear costs of aggregate production and inventories and zero costs of aggregate sales is illustrated in Figure 6 for a three-commodity five-period production planning problem. It has the following conformality properties.

Theorem 7 (Characterization of Conformality in the Rolodex Graph $\mathcal{R}_{ph}$). *The activities corresponding to arcs a and b in $\mathcal{R}_{ph}$ are conformal if and only if one of the following conditions holds*: (i) $n = 1$; (ii) *a and b are incident to a common node*; (iii) $\kappa = 2$, $n = 2$, *and a and b are on the boundary of a common face*; *or* (iv) $\kappa = 2$,

Figure 5 **Contracted Rolodex Graph $\mathcal{R}_{ps}$ of the Three-Commodity Five-Period Production Planning Problem with Aggregate Production and Sales Costs**



a and b are associated with the same primitive commodity, and they form a quadrilateral with two inventory arcs.

Theorem 7, when combined with the monotonicity theory, can be used to evaluate the effects of changes in production, sales, and inventory costs on optimal schedules in the presence of aggregate production and inventory costs.

4.6. Aggregate Sales and Inventory Costs

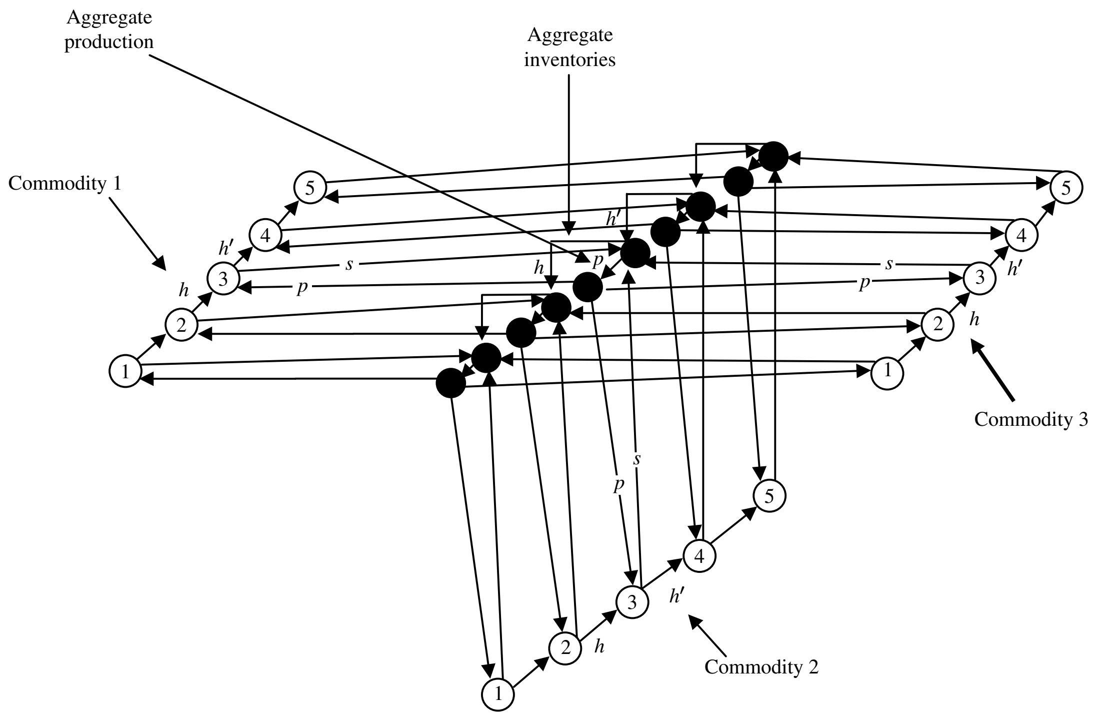
The contracted rolodex graph $\mathcal{R}_{sh}$ of the κnPP problem with nonlinear aggregate sales and inventory costs has exactly the same properties—parts (i)–(iv) in Theorem 7—as $\mathcal{R}_{ph}$, with the following exceptions: in the proof, $\mathcal{R}_{sh} \equiv \mathcal{R}_s$ replaces $\mathcal{R}_{ph} \equiv \mathcal{R}_p$ when $n = 1$, and $\mathcal{R}_{sh}$ replaces $\mathcal{R}_{ph}$ everywhere.

5. Summary and Discussion of the Qualitative Results

We have developed in this paper a qualitative analysis theory for the fundamental convex-cost κ-commodity n-period production planning problem, which can be used to provide important insights to managers who would like to assess the effects of changes in problem environment. Our insights stem from our ability to predict the direction of change in optimal values of some decision variables—and, at times, to even calculate bounds on the magnitude of these changes—when some problem parameters are changed. Our theory is based on conformality relations between production, sales, and inventory activities and the structure of the cost functions. By exploiting the topologies of the graphs underlying various important instances of the κnPP problem, we were able to derive complete characterizations of complement or substitute (i.e., conformality) relations among production, sales, and inventory activities in each case. Combining the conformality characterizations with the monotonicity theory of Granot and Veinott (1985) allowed us to gain profound insight into the effect of environmental changes on optimal decisions without having to carry out any computational work.

Tables 1–4 summarize the various monotonicity results that can be obtained from Theorem 1 and

Figure 6 **Contracted Rolodex Graph $\mathscr{R}_{ph}$ of the Three-Commodity Five-Period Production Planning Problem with Aggregate Production and Inventory Costs**



Theorems 4–7 when applied to $\mathscr{R}_p$, $\mathscr{R}_s$, $\mathscr{R}_{ps}$, and so on. The conventions we use are the following.

First, to indicate that a condition is true for the rolodex graph $\mathscr{R}_p$, we simply write p in the tables. Moreover, the condition is valid for all rolodex graphs that are indexed by a subset of the indicated letters provided that the graph has arcs corresponding to both variables that are being analyzed. For example, the effect of a change in the production, p_i^k, of commodity k in period i stemming from a change in the parameter η_j^K of the inventory of the aggregate commodity in period j, $j \neq i$, can only be predicted if aggregate inventory or both aggregate inventory and sales are present—that is, when the contracted

Table 2 **Monotonicity Results for Same Commodity, Distinct Periods**

$i \neq j$	k	π_j^k		σ_j^k		η_j^k	
p_i^k	K	p, s	$\Rightarrow\downarrow$	p, s	$\Rightarrow\uparrow$	$p, s, h \ \& \ j = i-1$	$\Rightarrow\downarrow$
	$\neq K$	$s, h \ \& \ \kappa = 2 \ \& \ \|i-j\| = 1$ $p, h \ \& \ \kappa = 2 \ \& \ n = 2$ s $p \ \& \ n = 2$ $h \ \& \ \begin{cases} n \leq 3 \text{ or} \\ i+j = 3,\ 2n-1 \end{cases}$ $h \ \& \ \kappa = 2 \ \& \ \|i-j\| = 1, n-1$	$\Rightarrow\downarrow$	$h \ \& \ \begin{cases} n \leq 3 \text{ or} \\ i+j = 3,\ 2n-1 \end{cases}$ $h \ \& \ \kappa = 2 \ \& \ \|i-j\| = 1, n-1$	$\Rightarrow\uparrow$	$p, s, h \ \& \ j = i-1$ $h \ \& \ \begin{cases} j = i-1 \text{ or} \\ i = j+2 = n \end{cases}$ $h \ \& \ \kappa = 2 \ \& \ \begin{cases} j = i-1 \text{ or} \\ i = n \geq 2 \end{cases}$ $\Rightarrow\downarrow$ $h \ \& \ i = j-1 = 1$ $h \ \& \ \kappa = 2 \ \& \ i = 1$ $\Rightarrow\uparrow$	
h_i^k	K	$p, s, h \ \& \ j = i+1$	$\Rightarrow\downarrow$	$p, s, h \ \& \ j = i+1$	$\Rightarrow\uparrow$	$p, s, h \ \& \ \|i-j\| = 1$	$\Rightarrow\uparrow$
	$\neq K$	$p, s, h \ \& \ j = i+1$ $h \ \& \ \begin{cases} j = i+1 \text{ or} \\ j = i+2 = n \end{cases}$ $h \ \& \ \kappa = 2 \ \& \ \begin{cases} j = i+1 \text{ or} \\ j = n \geq 2 \end{cases}$ $\Rightarrow\downarrow$ $h \ \& \ j = i-1 = 1$ $h \ \& \ \kappa = 2 \ \& \ j = 1$ $\Rightarrow\uparrow$		$p, s, h \ \& \ j = i+1$ $h \ \& \ \begin{cases} j = i+1 \text{ or} \\ j = i+2 = n \end{cases}$ $h \ \& \ \kappa = 2 \ \& \ \begin{cases} j = i+1 \text{ or} \\ j = n \geq 2 \end{cases}$ $\Rightarrow\uparrow$ $h \ \& \ j = i-1 = 1$ $h \ \& \ \kappa = 2 \ \& \ j = 1$ $\Rightarrow\downarrow$		$p, s, h \ \& \ \|i-j\| = 1$ $h \ \& \ \kappa = 2$	$\Rightarrow\uparrow$

Table 3 Monotonicity Results for Distinct Commodities, Same Period

$k \neq \ell, K$	ℓ	π_i^ℓ	σ_i^ℓ	η_i^ℓ
p_i^k	K	$p, s, h \Rightarrow\uparrow$	$s, h \Rightarrow\uparrow$	$s, h \Rightarrow\uparrow$
	$\neq K$	$p, s, h \Rightarrow\downarrow$	$h \Rightarrow\uparrow$	$\left.\begin{array}{l} p, h \ \& \ \kappa = 2 \ \& \ i = 1 \\ h \ \& \ i = 1 \end{array}\right\} \Rightarrow\downarrow$
h_i^k	K			$\left.\begin{array}{l} p, s, h \ \& \ \kappa = 2 \\ h \ \& \ i = 1, n-1 \end{array}\right\} \Rightarrow\uparrow$
	$\neq K$	$\left.\begin{array}{l} p, h \ \& \ \kappa = 2 \ \& \ i = 1 \\ h \ \& \ i = 1 \end{array}\right\} \Rightarrow\downarrow$	$\left.\begin{array}{l} s, h \ \& \ \kappa = 2 \ \& \ i = 1 \\ h \ \& \ i = 1 \end{array}\right\} \Rightarrow\uparrow$	$\left.\begin{array}{l} p, s, h \ \& \ \kappa = 2 \\ h \ \& \ i = 1, n-1 \end{array}\right\} \Rightarrow\downarrow$

Table 4 Monotonicity Results for Distinct Commodities, Distinct Periods

$i \neq j, k \neq \ell, K$	ℓ	π_j^ℓ	σ_j^ℓ	η_j^ℓ
p_i^k	K		$s \Rightarrow\uparrow$	$s, h \ \& \ j = i-1 \Rightarrow\downarrow$
	$\neq K$	$s \Rightarrow\downarrow$ $\left.\begin{array}{l} p, h \ \& \ \kappa = 2 \ \& \ n = 2 \\ h \ \& \ n = 2 \\ h \ \& \ \kappa = 2 \ \& \ i, j \in \{1, n\} \end{array}\right\} \Rightarrow\uparrow$	$\left.\begin{array}{l} h \ \& \ n = 2 \\ h \ \& \ \kappa = 2 \ \& \ i, j \in \{1, n\} \end{array}\right\} \Rightarrow\downarrow$	$\left.\begin{array}{l} p, h \ \& \ \kappa = 2 \ \& \ i = j+1 = n \\ h \ \& \ i = j+1 = n \\ h \ \& \ \kappa = 2 \ \& \ i = n \geq 2 \end{array}\right\} \Rightarrow\uparrow$ $h \ \& \ \kappa = 2 \ \& \ i = 1 \Rightarrow\downarrow$
h_i^k	K			
	$\neq K$	$\left.\begin{array}{l} p, h \ \& \ \kappa = 2 \ \& \ i+1 = j = n \\ h \ \& \ i+1 = j = n \\ h \ \& \ \kappa = 2 \ \& \ j = n \geq 2 \end{array}\right\} \Rightarrow\uparrow$ $h \ \& \ \kappa = 2 \ \& \ j = 1 \Rightarrow\downarrow$	$\left.\begin{array}{l} s, h \ \& \ \kappa = 2 \ \& \ i+1 = j = n \\ h \ \& \ i+1 = j = n \\ h \ \& \ \kappa = 2 \ \& \ j = n \geq 2 \end{array}\right\} \Rightarrow\downarrow$ $h \ \& \ \kappa = 2 \ \& \ j = 1 \Rightarrow\uparrow$	$h \ \& \ \kappa = 2 \Rightarrow\downarrow$

rolodex graph is either $\mathcal{R}_h$ or $\mathcal{R}_{sh}$. In this case, the monotonicity of p_i^k in η_j^K for $k \neq K$ and $i \neq j$ in Table 4 is denoted as "$s, h \ \& \ j = i-1 \Rightarrow \downarrow$." This means that p_i^k is decreasing in η_j^K in $\mathcal{R}_{sh}$ and $\mathcal{R}_h$ if $j = i-1$. (Note that we have excluded $\mathcal{R}_s$ because in this graph, there is no arc corresponding to η_j^K since this arc has zero flow cost in $\mathcal{R}_s$.)

Second, an empty entry means that no monotonicity result can be found for that particular pair of activities. In addition, observe that each row corresponding to sales can be obtained from the corresponding production row by interchanging sales with production everywhere and interchanging up arrows and down arrows only if one of the variables or parameters is associated with inventory. For example, in Table 4, p_i^k is $\downarrow$ in π_j^ℓ $(l \neq k)$ in $\mathcal{R}_s$; therefore s_i^k is $\downarrow$ in σ_j^ℓ $(l \neq k)$ in $\mathcal{R}_p$. On the other hand, in Table 4, p_i^k is $\downarrow$ in η_j^K in $\mathcal{R}_{sh}$ and $\mathcal{R}_h$ if $j = i-1$; so, s_i^k is $\uparrow$ in η_j^K in $\mathcal{R}_{ph}$ and $\mathcal{R}_h$ if $j = i-1$.

There are several important avenues for future research. For example, one could attempt to carry out a qualitative analysis for other important classes of problems, such as network reliability, or to extend the qualitative analysis to other classes of problems that cannot be cast as network optimization problems.

Acknowledgments

The authors gratefully acknowledge many constructive comments by the editor, the associate editor, and the referees, which have led to a significant improvement of the paper. I. Ciurria-Infosino was partially supported by the Brazilian government through the Universidade Estadual de Campinas (UNICAMP) and Conselho Nacional de Desenvolvimento Cientiflco e Technológico (CNPq), as well as by General Motors. D. Granot was partially supported by the Natural Sciences and Engineering Research Council [Grant 5-84181]. F. Granot was partially supported by the Natural Sciences and Engineering Research Council [Grant 5-83998]. A. F. Veinott Jr. was partially supported by the National Science Foundation [Grant ENG ECS83-12356] and by General Motors.

This paper is dedicated to the memory of Professor Pete Veinott, a colleague, a mentor, and a friend.

Appendix

Proof of Theorem 1. For $n = 1$, $\mathcal{R}$ is a series-parallel graph, and thus each pair of arcs is conformal (Dirac 1952, Duffin 1965). (Recall our convention that graph-theoretic concepts that we often use apply to the undirected graph induced by the corresponding directed graph. For example, by (the directed graph) $\mathcal{R}$ being a series-parallel graph, it is understood that the undirected graph induced by $\mathcal{R}$ is series parallel. A 2-connected undirected graph is series parallel if and only if it can be reduced to a simple cycle with two edges by sequentially contracting an edge that is incident to a node of degree 2 or by deleting an edge if it shares its end nodes with another edge.) Clearly, (ii) is valid. To show (iii), note that the rolodex graph $\mathcal{R}$ of a κnPP problem with $\kappa, n \geq 2$ is not series parallel. Actually, it is not even planar because it contains a subgraph, $\mathcal{Q}$, homeomorphic to the bipartite graph $K_{3,3}$. (Two graphs are homeomorphic if one is obtained from the other by the following two operations: (i) replacing an edge with a two-edge path or (ii) eliminating a node of degree 2 and replacing the two edges incident to it with a single edge.) To construct $\mathcal{Q}$, take, for example, three copies of node 1 and let them form one of the two node sets of the bipartition. The other node set in the bipartition is formed by three junction nodes that are connected to each copy of node 1 through three internally node-disjoint paths. Therefore, not every pair of node-disjoint arcs in $\mathcal{R}$ is conformal. Next, we show that if $\kappa = 2$, and a and b correspond to inventories of distinct commodities in the same period, then they are conformal. To do that, observe that a, b, and the third inventory arc in the same period, say, d, form an edge-cutset in $\mathcal{R}$. Any simple cycle that uses a and b but not d induces a direction, and if a and b are not conformal, there must exist another simple cycle that orients precisely one of a or b, but not the other, in an opposite direction. However, since $\{a, b, d\}$ forms an edge-cutset, such a cycle must use d twice, and therefore, it is not a simple cycle; thus we conclude that a and b are conformal. For $\kappa = 2$ and all other cases of two node-disjoint arcs, or for $\kappa \geq 3$ and every pair of node-disjoint arcs, one can construct two simple cycles orienting precisely one of the two arcs in opposite directions. For example, one such simple cycle can be chosen to be contained in just two cards of the rolodex graph $\mathcal{R}$, and the other cycle is then constructed using three cards. $\square$

Proof of Theorem 4. For $n = 1$, the graph $\mathcal{R}_p$ is series parallel, and thus every pair of arcs is conformal. Clearly, if a and b are incident to a common node, they are conformal. We next show that for $n = 2$ or $\kappa = 2$, a and b are conformal if and only if deleting a, b, and ν disconnects $\mathcal{R}_p$. Assume first that $\kappa = 2$, and let $a = (i, j)$ and $b = (\ell, m)$. Consider the graph $\mathcal{R}_p^\nu$ derived from $\mathcal{R}_p$ by removing node ν and all arcs incident to it in $\mathcal{R}_p$. Since $\mathcal{R}_p^\nu$ is 2-connected, there exists a simple cycle, $\mathcal{C}$, therein that traverses, say, a from i to j and b from m to ℓ. Now, if $\{a, b\}$ is a two-edge-cutset in $\mathcal{R}_p^\nu$, nodes j and m belong to one of the two connected components, say, W_1, created after the removal of $\{a, b\}$ from $\mathcal{R}_p^\nu$, and i and ℓ belong to the other connected component, denoted by W_2. If a and b are not conformal, there must exist two pairwise node-disjoint simple paths, P_1 and P_2, in $\mathcal{R}_p$, one between j and ℓ and the other between i and m. However, since $j, m \in W_1$ and $i, \ell \in W_2$, such paths must traverse ν, implying that P_1 and P_2 are not pairwise node disjoint, and thus a and b are conformal. If, on the other hand, $\{a, b\}$ is not an edge-cutset in $\mathcal{R}_p^\nu$, there exists a simple path, P, in $\mathcal{R}_p^\nu$ between, for example, j and ℓ, which does not traverse a and b. By augmenting this path with a, b, (i, ν), and (m, ν), we obtain a simple cycle that orients b in the opposite direction than it is oriented in $\mathcal{C}$, implying that a and b are not conformal, and the proof for $\kappa = 2$ is complete. A similar proof holds for $n = 2$. To complete the proof, we note that for $\kappa \geq 3$ or $n \geq 3$ and any pair of arcs a and b not incident to a common node, one can find two simple cycles traversing both a and b but orienting precisely one of them in opposite directions. For example, one such simple cycle can be chosen to be contained in two cards of $\mathcal{R}_p$, and the other simple cycle is constructed using three cards of $\mathcal{R}_p$. $\square$

Proof of Theorem 5. Clearly, (i) holds. For (ii), $\mathcal{R}_h$ is series parallel if and only if $n = 2$, in which case, every pair of arcs is conformal. For $\kappa = 2$, $\mathcal{R}_h$ is planar, and so, if a and b are on the boundary of a common face, they are conformal (Perl and Shiloach 1978). (More formally, a pair of node-disjoint edges of a planar graph is conformal if—and provided the graph is 3-connected, only if—they lie on the boundary of a common face of some planar representation of the graph; see Perl and Shiloach 1978.) Thus, for the "only if" part of (iii), one needs to show that if a and b are not on the boundary of a common face, one can always find two simple cycles that orient precisely one of them, but not the other, in opposite directions. For example, let $a = p_i^1$ and $b = h_j^2$, $j \geq i$, $1 < i < n$. Then, one can choose the first cycle, $\mathcal{C}^1$, to start at aggregate inventory node i, traverse a and continue, clockwise, along the outer boundary, traversing b, and returning to a via p_i^2. The second simple cycle, $\mathcal{C}^2$, can be obtained by traversing a and continuing, counterclockwise, along the outer boundary, traversing b in the opposite direction to that of $\mathcal{C}^1$, and returning to a via p_{j+1}^2 and aggregate inventory arcs. To show (iv), let $\mathcal{R}_h'$ be derived from $\mathcal{R}_h$ after the elimination of production or sales (but not both) arcs for all primitive commodities in the first and last periods and contraction of either the inventory or the remaining production or sales arc in those periods. Since such contraction does not effect conformality, arcs a and b in (iv), which became incident to a common node in $\mathcal{R}_h'$, are conformal in $\mathcal{R}_h$. (For example, production of commodity ℓ in period 1 is a complement of inventories of commodity ℓ in periods 1 and 2 and a substitute of inventories of all other primitive commodities in period 1.) For $\kappa \geq 3$ and any pair of node-disjoint arcs, one can easily find two simple cycles traversing both of them but orienting precisely one of them in opposite directions. Thus, for $\kappa \geq 3$, only arcs incident to a common node are conformal. $\square$

Proof of Theorem 6. Note that the graph $\mathcal{R}_{ps}$ is series parallel if and only if $n = 1$, and it is nonplanar and 3-connected if $n \geq 2$. Furthermore, condition (iii) holds if and only if a and b correspond to inventories of primitive commodities in the same period. The remainder of the proof is similar to that of Theorem 4, and the details are left to the reader. $\square$

Proof of Theorem 7. Observe that $\mathcal{R}_{ph}$ is series parallel if and only if $n = 1$. In this case, $\mathcal{R}_{ph} \equiv \mathcal{R}_p$. The graph $\mathcal{R}_{ph}$ is planar if and only if either $n = 1$ or $n = 2$ and $\kappa = 2$. The graph $\mathcal{R}_{ph}$ is 3-connected if $n \geq 2$ and $\kappa \geq 3$, and it becomes

3-connected after series contractions if $n = 1$. Note further that condition (iv) is satisfied if a and b correspond to sales of the same commodity in consecutive periods—say, i and $i+1$. They form a quadrilateral with the inventory arc of the primitive commodity and the aggregate inventory arc in period i. Conformality of a and b follows from the fact that the three inventories in period i form an edge-cutset in $\mathcal{R}_{ph}$. The remaining details of the proof are left to the reader. □

References

Ahuja RK, Magnanti TL, Orlin JB (1993) *Network Flows: Theory, Algorithms, and Applications* (Prentice-Hall, Upper Saddle River, NJ).

Anily S (1991) Multi-item replenishment and storage problem (MIRSP): Heuristics and bounds. *Oper. Res.* 39:233–243.

Anily S, Tzur M (2006) Algorithms for the multi-item multi-vehicles dynamic lot sizing problem. *Naval Res. Logist.* 53:157–169.

Bowman EH (1956) Production scheduling by the transportation method of linear programming. *J. Oper. Res. Soc. Amer.* 4: 100–103.

Buffa ES, Taubert WH (1972) *Production-Inventory Systems Planning and Control*, revised ed. (Richard D. Irwin, Homewood, IL)

Ciurria-Infosino I, Granot F, Veinott AF Jr (2014) Substitutes, complements and ripples in multicommodity flows on suspension graphs. *Networks* 64:65–75.

Dirac GA (1952) A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.* 27:85–92.

Duffin RJ (1965) Topology of series-parallel networks. *J. Math. Anal. Appl.* 10:303–318.

Evans JR (1978) A single-commodity transformation for certain multi-commodity networks. *Oper. Res.* 26:673–680.

Gale D, Politof T (1981) Substitutes and complements in network flow problems. *Discrete Appl. Math.* 3:175–186.

Granot F, Veinott AF Jr (1985) Substitutes, complements and ripples in network flow. *Math. Oper. Res.* 10:471–497.

Graves SC (2002) Manufacturing planning and control. Pardalos P, Resende M, eds. *Handbook of Applied Optimization* (Oxford University Press, New York), 728–746.

Hax AC, Candea D (1984) *Production and Inventory Management* (Prentice-Hall, Upper Saddle River, NJ).

Hicks JR (1939) *Value and Capital*, 2nd ed. (Oxford University Press, Oxford, UK).

Hicks JR, Allen RGD (1934a) A reconsideration of the theory of value. Part I. *Economica* 1:52–76.

Hicks JR, Allen RGD (1934b) A reconsideration of the theory of value. Part II. A mathematical theory of individual demand functions. *Economica* 1:196–219.

Karush W (1958) On a class of minimum-cost problems. *Management Sci.* 4:136–153.

Manne AS (1957) A note on the Modigliani-Hohn production smoothing model. *Management Sci.* 3:371–379.

Modigliani F, Hohn F (1955) Production planning over time and the nature of expectation and planning horizon. *Econometrica* 23:46–66.

Muckstadt JA, Roundy RO (1987) Multi-item, one-warehouse, multi-retailer distribution systems. *Management Sci.* 33:1613–1621.

Perl Y, Shiloach Y (1978) Finding two disjoint paths between two pairs of vertices in a graph. *J. ACM* 25:1–9.

Pochet Y, Wolsey LA (2003) *Production Planning by Mixed Integer Programming* (Springer, New York).

Roundy RO (1993) Efficient, effective lot-sizing for multi-product, multi-stage production systems. *Oper. Res.* 41:371–386.

Samuelson PA (1971) *Foundations of Economic Analysis* (Antheneum, New York). [Orig. pub. 1947 (Harvard University Press, Cambridge, MA.]

Shapiro JF (1993) Mathematical programming models and methods for production planning and scheduling. Graves SC, Rinnooy Kan AHG, Zipkin PH, eds. *Logistics of Production and Inventory* Handbooks in Operations Research and Management Science, Vol. 4 (Elsevier Science, Amsterdam), 371–443.

Shapley LS (1961) On network flow functions. *Naval Res. Logist. Quart.* 8:154–158.

Shapley LS (1962) Complements and substitutes in the optimal assignment problem. *Naval Res. Logist. Quart.* 9:45–48.

Soun Y, Truemper K (1980) Single commodity representation of multicommodity networks. *SIAM J. Algebraic Discrete Methods* 1:348–358.

Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press, Princeton, NJ).

Veinott AF Jr (1964) Production planning with convex costs: A parametric study. *Management Sci.* 10:441–460.

Veinott AF Jr (1969) Minimum concave-cost solution of Leontief substitution models of multi-facility inventory systems. *Oper. Res.* 17:262–291.

Wagner HM, Whitin TH (1958) Dynamic version of the economic lot size model. *Management Sci.* 5:89–96.

Zangwill W (1969) A backlogging model and a multi-echelon model of a dynamic lot size production system. *Management Sci.* 15:506–527.

Zipkin PH (2000) *Foundations of Inventory Management* (McGraw-Hill, New York).