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Bullish/bearish/neutral strategies under short sale restrictions

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ABSTRACT

This study investigates the effects of short sale restrictions by extending the model of Dridi and Germain (2004) and infers informed traders' strategies and the relation between order imbalance and price thereunder. The results are generally in line with the empirical evidence documented in the literature and are summarized as follows: First, seller-initiated trading incurs a greater price reaction. Second, short sale restrictions shift the skewness of asset returns. Third, the restrictions can stimulate investors to acquire information or increase each individual trader's order flow under the bullish and neutral signals as well as the bearish signal, which is yet to be explored empirically.

1. Introduction

During the global financial crisis of 2008–2009, more than 30 countries, including the United States, restricted short sales, which were regarded as a source of stock price decline (Beber and Pagano, 2013). However, in the finance literature, there is much debate about whether short sales restrictions can boost stock prices. For example, Miller (1977) insists that short sale restrictions may increase stock prices. He assumes that investors have different views on stock prices. Under this assumption, pessimistic investors pursue a selling strategy (including the short sale strategy), while optimistic investors adopt a buying strategy. Thus, short sale restrictions increase the portion of optimistic investors pursuing a buying strategy, which will increase stock prices and even generate upward biases in stock prices. On the other hand, there are studies that show short sale restrictions may induce downward biases in stock prices. For example, Bai et al. (2006) show that short sale restrictions may induce downward biases in stock prices under a rational expectation equilibrium framework owing to traders' hedging needs. Gallmeyer and Hollifield (2008) also show that short sale restrictions may decrease stock prices if the optimist's intertemporal elasticity of substitution is less than one. Additionally, according to Nezafat et al. (2015), short sale restrictions reduce investors' incentive to acquire information and the demand on assets, which can reduce stock prices. In summary, these studies about the effects of short sale restrictions show two conflicting results for stock prices.

In this study, unlike previous studies, we focus on microstructural relations between trading behavior and stock price under short sale restrictions. Although Diamond and Verrecchia (1987) also investigate market microstructural issues that short sale restrictions may cause by adopting Glosten and Milgrom (1985), their main focus is on the bid-ask spread and the speed of convergence. Our paper focuses more on properties related to the size of order flows.

Our model setting is based on the approach of Dridi and Germain (2004), which is an extension of Kyle (1985). Under Kyle's model, an informed investor and a market maker strategically decide the order flow and the price. Therefore, this model enables us to investigate the trading behavior of informed traders, as well as the relation between the order imbalance and the price of an asset. Dridi and Germain (2004) inherit the basic properties and assumptions in Kyle's model, but unlike Kyle, they assume that informed investors know only whether the price of a security will increase or decrease in the future, not the security's true value. In other words, informed investors obtain a bullish or bearish signal. Under this additional assumption, they show that stock price is nonlinearly related to order imbalance, consistent with the

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empirical results of Kempf and Korn (1999) and Gabaix et al. (2003).[1] In this paper, we investigate short sale restrictions in a model setting similar to that of Dridi and Germain. The details of our model setting regarding the information informed investors have and short sale restrictions are as follows. We assume that informed investors have a directional signal about the future stock return. First, we assume that informed investors get a bullish/bearish signal, as Dridi and Germain (2004) do. However, the signals seem to be too simple to reflect reality; therefore, we extend the setting and assume that informed investors may get a neutral signal as well as a bullish/bearish signal. A neutral signal means information that is not useful for directional trading or information that the stock price will not change. This assumption is closer to what we can observe in reality. Frequently, individual stock prices do not change much, and the information investors gather is not always valuable in directional trading. Accordingly, most financial analysts express their opinions with a buy, sell, or hold recommendation, and sentiment indices are categorized as bullish, bearish, or neutral.[2]

Regarding short sale restrictions, we assume that only some of informed investors are prohibited from short selling. It differs from the assumption of Bai et al. (2006) or Nezafat et al. (2015) that all informed investors are prohibited from making short sales. We believe that our setting is more general in the sense that it reflects the reality that the severity of short sale restrictions is not the same among investors. For example, some classes of investors like mutual funds and pension funds either face explicit or implicit short-selling constraints through their fund agreements as Almazan et al. (2004) and Nezafat et al. (2015) state. On the other hand, other classes of investors like hedge funds may not impose any short sale restrictions at all. Our model embraces this different degree of short-sale constraints among investors. Furthermore, for some periods, due to the short-sale ban, no investors may engage in short sales. Our model can handle this change in the degree of short-sale restrictions and look at the implications of the change. In addition, our model includes the no short-selling case and no restriction case as special cases.

We first consider equilibriums under which the total number of informed investors is given exogenously to make our research simple. Next, we endogenize the number of informed investors. Investors collect private information only when the cost does not exceed its expected return. Thus, the number of informed investors is determined endogenously via decisions about information acquisition.

On the basis of the settings described above, we derive the optimal strategy for informed investors and the equilibrium price function, and provide some empirical implications as follows. First, when informed investors get a bullish/bearish signal and the number of informed investors is fixed, the effect of short sale restrictions can be similar to that of a reduction in the number of informed traders in a restriction-free economy. Accordingly, the profit of informed investors increases and the expected price variance increases. These implications are consistent with the empirical results documented by Beber and Pagano (2013), Boehmer et al. (2013), and Boehmer and Wu (2013) in that short sale restrictions make the market inefficient. Though the inefficiency under the restrictions is widely known from both empirical and theoretical studies, our study contributes to the literature by showing the increased inefficiency based on the link between trading behavior and price.

Second, the presence of restriction-bound investors causes the net order flow of the total market to be positive, which means buyers initiate more than sellers do, on average, consistent with empirical results of Choe and Lee (2012) and Sifat and Mohamad (2015). Consequently, a sell order becomes more informative than a buy order, and the absolute return of sell orders is larger than that of buy orders, consistent with the empirical findings of Chordia et al. (2002).

Third, when informed investors can obtain a neutral signal rather than only a bullish or bearish signal, the aforementioned results are still valid, but there are two additional implications. One is that informed investors can make profits with buy orders even when they have neutral signals. The other is that restrictions increase the skewness of returns, which is consistent with Bris et al. (2007), Chang et al. (2007), and Saffi and Sigurdsson (2011). Diamond and Verrecchia (1987) briefly mention this phenomenon without any formal derivation, but we prove that the restrictions change the skewness of returns and show that they increase the skewness of returns numerically.

Fourth, when we allow the number of informed investors to be determined endogenously, investors' decisions on information acquisition become dependent on the severity of short sale restrictions. When short sale restrictions are limited to a small number of investors, investors do not acquire information. However, when many investors are restricted—in other words, when the restrictions become severe—more investors acquire information to profit from the market inefficiency caused by short sale restrictions. This result is in contrast with the result of Nezafat et al. (2015), which shows that short sale constraints reduce information acquisition because restrictions hinder the taking advantage of bearish signals.

The paper is organized as follows. Section 2 presents the baseline model setting and assumptions. Section 3 provides propositions and their implications. Section 4 extends the baseline model to a bullish/bearish/neutral economy. Section 5 describes the equilibrium with information acquisition. Section 6 concludes this study.

2. Baseline model

The basic structure of our model is identical to that of Dridi and Germain (2004). We briefly review their assumptions. The true value of a stock is $\tilde{v}$, with distribution $N(0, \sigma_v^2)$. There are three types of market participants: uninformed traders, risk-neutral informed traders, and competitive risk-neutral market makers. Uninformed traders place orders on the basis of liquidity needs. The total order flow of uninformed traders is $\tilde{u}$, with distribution $N(0, \sigma_u^2)$. There are N different informed traders. Although Kyle (1985) assumes informed traders forecast the realization of $\tilde{v}$ perfectly, it is generally hard to believe that they know the exact future value. Hence, Dridi and Germain (2004) assume that informed traders know only the sign of $\tilde{v}$. Each informed trader $i = 1, \ldots, N$ places an order of $\tilde{x}_i$ shares based on the signal, the sign of $\tilde{v}$. The total order flow of informed and uninformed traders is represented as $\tilde{w}$:

$$\tilde{w} = \tilde{u} + \sum_{i=1}^{N} \tilde{x}_i \tag{1}$$

Market makers observe $\tilde{w}$ and determine the price, $\tilde{p}$. Equilibrium is obtained if

$$\tilde{p} = E[\tilde{v}|\tilde{w}] \tag{2}$$

and

$$\tilde{x}_i = arg\max_x E[(\tilde{v} - \tilde{p})x|sign(\tilde{v})] \tag{3}$$

[1] This is in contrast to Kyle's linear relation between stock price and order imbalance.

[2] For example, the *AAII Investor Sentiment Survey* measures the percentages of investors' views by classifying them as bullish, bearish, or neutral on the stock market, and the *Investors Intelligence* index is a sentiment measure based on market newsletters and is bullish, bearish, or neutral. These measures are used for measuring investors' market sentiments in the literature (e.g., Brown and Cliff, 2004, 2005; Han, 2008).

Neither Kyle (1985) nor Dridi and Germain (2004) consider short sale restrictions. However, short sales can be restricted directly, as in the 2008 financial crisis, and also indirectly, through short sale costs. Here, we investigate the strategy of informed traders under short sale restrictions. According to empirical results, short sales precede bad news or stock price drops (e.g., Diether et al., 2009; Christophe et al., 2010; Karpoff and Lou, 2010; Blau and Wade, 2012; Chakrabarty and Shkilko, 2013; Callen and Fang, 2015). Therefore, we presume short sales are made by informed traders and short sale restrictions hinder the sell orders from $\underline{N}$ informed investors. For convenience, let $i \in \{1, \ldots, N-\underline{N}\}$ be an informed trader who is free of restrictions, and let $i \in \{N-\underline{N}+1, \ldots, N\}$ be an informed trader who is bound by the restrictions. Under these assumptions, we obtain several propositions, presented in Section 3.

3. Results and implications

Informed traders' strategies and market makers' pricing schedules in our model are given by the following proposition.

Proposition 1.

(1) *Strategy of informed traders.*

If an informed trader is restriction-free, that is, $i \in \{1, \ldots, N-\underline{N}\}$, *then*

$$\tilde{x}_i = \begin{cases} \gamma^+ = \dfrac{\sigma_u}{\sqrt{N-0.5\underline{N}}}, & \text{if } \tilde{v} > 0 \\ -\gamma^- = -\dfrac{\sigma_u}{\sqrt{N-0.5\underline{N}}}, & \text{if } \tilde{v} < 0 \end{cases} \tag{4}$$

If an informed trader is restriction-bound, that is, $i \in \{N-\underline{N}+1, \ldots, N\}$, *then*

$$\tilde{x}_i = \begin{cases} \underline{\gamma}^+ = \dfrac{\sigma_u}{\sqrt{N-0.5\underline{N}}}, & \text{if } \tilde{v} > 0 \\ 0, & \text{if } \tilde{v} < 0 \end{cases} \tag{5}$$

(2) *Equilibrium price function.*

$$\begin{aligned} &p(w, N, \underline{N}) \\ &= \frac{2\sigma_v}{\sqrt{2\pi}} \tanh\left(\frac{\sqrt{N-0.5\underline{N}}}{\sigma_u}\left(w - \frac{\sigma_u \underline{N}}{2\sqrt{N-0.5\underline{N}}}\right)\right) \\ &= p\left(w - \frac{\sigma_u \underline{N}}{2\sqrt{N-0.5\underline{N}}}, N-0.5\underline{N}, 0\right) \end{aligned} \tag{6}$$

The proof is given in the Appendix.

According to this proposition, for every case, the order flows of informed investors increase in the presence of short sale restrictions; that is,

$$\begin{aligned} \gamma^-(N, \underline{N}) = \gamma^+(N, \underline{N}) = \underline{\gamma}^+(N, \underline{N}) > \gamma^-(N, 0) \\ = \gamma^+(N, 0) \text{ if } \underline{N} > 0 \end{aligned} \tag{7}$$

where $\gamma^-(N, \underline{N})$, $\gamma^+(N, \underline{N})$ and $\underline{\gamma}^+(N, \underline{N})$ denote the values of γ^-, γ^+, and $\underline{\gamma}^+$ when the total numbers of informed investors and restricted informed investors are N and $\underline{N}$, respectively.

The intuition behind this is as follows. The restrictions prevent some informed investors from selling when there is bad news. Accordingly, restriction-free informed traders feel more comfortable increasing their order flow γ^- to increase their profits because their information is relatively less disclosed. At first glance, the increases of γ^+ and $\underline{\gamma}^+$ may not be intuitive, since the number of informed traders who can buy does not decrease. However, since market makers perceive the strength of a signal relative to the expected total order flow, they infer the signal not by the trading size but, rather, by the variation in the informed traders' total order flow. Therefore, it is natural to increase buy orders upon a good signal if there is a reduction in total sell orders upon a bearish signal. This property of total order flow is put forth in the following corollary.

Corollary 2. *Variation of informed traders' order flows.*

Let $\underline{N}$ be a positive integer.

(1) The total buy order quantities of informed traders in a bullish market increase as short sale restrictions become severe; that is, when $\underline{N}_1 > \underline{N}_2$,

$$\sum_{i=1}^{N} x_i(N, \underline{N}_1; \tilde{v} > 0) > \sum_{i=1}^{N} x_i(N, \underline{N}_2; \tilde{v} > 0) \tag{8}$$

(2) The total sell order quantities of informed traders in a bearish market decrease as short sale restrictions become severe; that is, when $\underline{N}_1 > \underline{N}_2$,

$$\left|\sum_{i=1}^{N} x_i(N, \underline{N}_1; \tilde{v} < 0)\right| < \left|\sum_{i=1}^{N} x_i(N, \underline{N}_2; \tilde{v} < 0)\right| \tag{9}$$

(3) The difference in informed traders' total order flows between bullish and bearish markets decreases as short sale restrictions become severe; that is, when $\underline{N}_1 > \underline{N}_2$,

$$\begin{aligned} &\sum_{i=1}^{N} x_i(N, \underline{N}_1; \tilde{v} > 0) - \sum_{i=1}^{N} x_i(N, \underline{N}_1; \tilde{v} < 0) \\ &< \sum_{i=1}^{N} x_i(N, \underline{N}_2; \tilde{v} > 0) - \sum_{i=1}^{N} x_i(N, \underline{N}_2; \tilde{v} < 0) \end{aligned} \tag{10}$$

(4) The average of informed investors' total order flows increases as short sale restrictions become severe; that is, when $\underline{N}_1 > \underline{N}_2$,

$$E\left[\sum_{i=1}^{N} \tilde{x}_i(N, \underline{N}_1)\right] > E\left[\sum_{i=1}^{N} \tilde{x}_i(N, \underline{N}_2)\right] = 0 \tag{11}$$

Proof. The proof is trivial.

This corollary shows that the restrictions increase the expected total order flow (the fourth statement of the corollary) and reduce the variation of total order flow (the third statement of the corollary). The increase in the expected total order flow is consistent with the empirical findings in Choe and Lee (2012) and Sifat and Mohamad (2015), which document that the average order imbalances increase during short sale ban periods in the Korean and UK stock markets, respectively. As mentioned above, the main interest of informed traders and market makers lies in the variation of their order flows rather than their expected order flows. Accordingly, when deciding on the price level, market makers will take into account that the variation of total order flow is reduced under the restrictions. Since reduced variation of total order flow implies reduced spread of information, statement (3) of Corollary 2 implies that the market becomes less efficient under the restrictions, which is consistent with the empirical results documented by Beber and Pagano (2013). The following proposition, examining the profit of informed traders, provides another indirect reason for the market inefficiency due to the short sale restrictions.

Proposition 3. *The (expected) profit of an informed investor.*

The profit of a restriction-free informed investor is

$$\begin{aligned} \pi(N, \underline{N}) &= \int_0^\infty \frac{4\frac{\sigma_u \sigma_v}{\pi\sqrt{N-0.5\underline{N}}}}{e^{\frac{\left(t-\sqrt{N-0.5\underline{N}}\right)^2}{2}} + e^{\frac{\left(t+\sqrt{N-0.5\underline{N}}\right)^2}{2}}} dt \\ &= \pi(N-0.5\underline{N}, 0) \end{aligned} \tag{12}$$

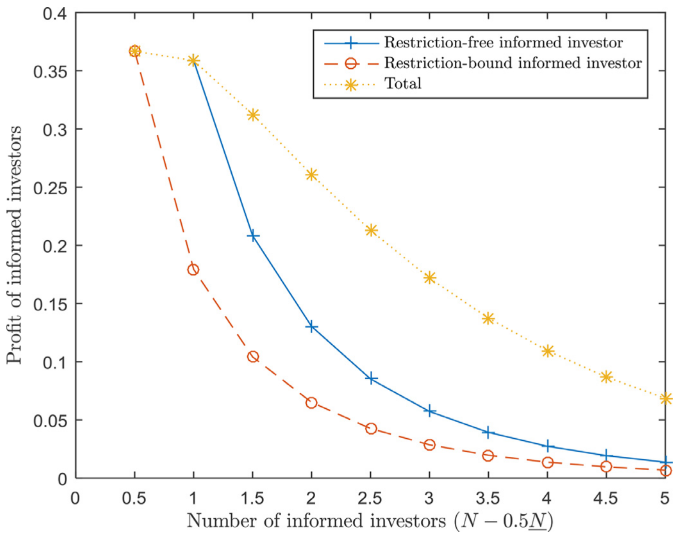

Fig. 1. Numbers of informed investors and profits. This figure shows the individual and total profits of informed investors under the parameters $\sigma_u = \sigma_v = 1$. Because the profits are functions of $N - 0.5\underline{N}$, we depict them as a two-dimensional graph rather than as a three-dimensional graph of $f(N, \underline{N})$. The solid line represents the profit of a restriction-free investor, the dashed line represents the profit of a restriction-bound investor, and the dotted line represents the total profit of all investors. In addition, when $N - 0.5\underline{N} = 0.5$, there are no restriction-free informed investors, which is why the solid line is not connected to the point where $N - 0.5\underline{N} = 0.5$.

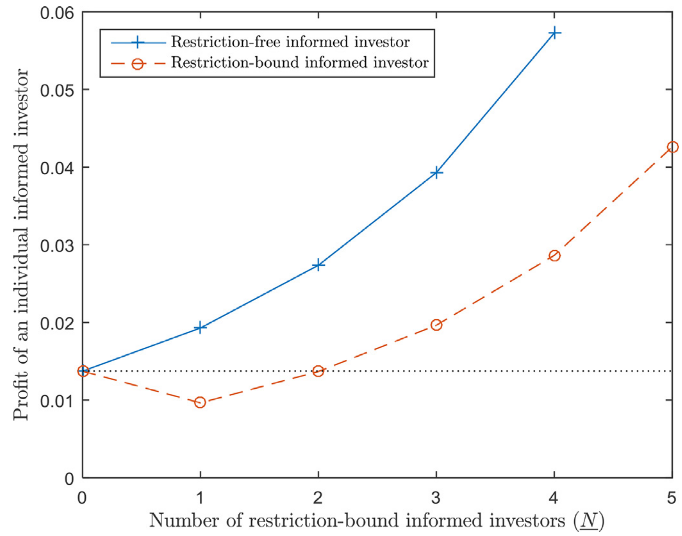

Fig. 2. Number of restriction-bound informed investors and the profit of an individual informed investor. This figure shows the profit of an individual informed investor as a function of $\underline{N}$, the number of restriction-bound informed investors, under the parameters $\sigma_u = \sigma_v = 1$ and $N = 5$. The solid line represents the profit of a restriction-free investor, the dashed line represents the profit of a restriction-bound investor, and the dotted line represents the profit of an individual with no restriction.

When $\underline{N} > 0$, the profit of a restriction-bound informed investor is

$$\underline{\pi}(N, \underline{N}) = \frac{1}{2}\pi(N, \underline{N}) \tag{13}$$

Therefore, the total profit of informed investors is

$$(N - 0.5\underline{N})\pi(N - 0.5\underline{N}, 0) \tag{14}$$

Individual profits $\pi(y, 0)$ and $\underline{\pi}(y, 0)$ are decreasing functions in y when $y > 0$. In addition, $y\pi(y, 0)$, the total profit for informed investors, is decreasing on the domain $\{\frac{z}{2} : z \in \mathbb{N}\}$.[3] In addition, the profits, $\pi(y, 0)$, $\underline{\pi}(y, 0)$, and $y\pi(y, 0)$, converge to zero as y goes to infinity.

Proof. The proof is given in the Appendix.

As in the case of the order flows and price in Proposition 1, an individual informed investor's profit and the total profit of informed investors are functions of $N - 0.5\underline{N}$. In addition, they are decreasing in $N - 0.5\underline{N}$. We confirm these findings in Fig. 1 and deduce several implications of short sale restrictions.

First, because $\pi(N - 0.5\underline{N}, 0)$ is increasing in $\underline{N}$, a short sale restriction increases the profit of restriction-free informed investors. Second, a short sale restriction is not always bad for restriction-bound informed investors, although their profits are lower than those of restriction-free informed investors, as shown in Fig. 2.

Fig. 2 illustrates the profits of informed investors with and without short sale restrictions as a function of $\underline{N}$, given $N = 5$. As described in Proposition 3, the profit of a restriction-free informed investor increases in $\underline{N}$ for the entire domain. By contrast, the profit of a restriction-bound informed investor shows a different pattern because it is half of the profit of a restriction-free informed investor, as shown in Eq. (13). When the restrictions are mild (such as $\underline{N} = 1$ in Fig. 2), they do not much change the profit of each restriction-free informed investor. Therefore, half of the profit of a restriction-free informed investor under these mild restrictions is less than the profit under no restriction ($\underline{N} = 0$). However, as the restrictions become severe ($\underline{N} = 3,\ 4,\ 5$), the profit of a restriction-free informed investor increases to even more than double the profit under no restriction. Therefore, the profit of a restriction-bound informed investor becomes larger than the profit of an informed investor when all face no restriction ($\underline{N} = 0$).[4]

Concerning total profit, restrictions always increase the total profit of informed investors, as we can infer from the fact that $y\pi(y, 0)$ is decreasing in y. This increase in total profits and in individual profit results from the reduced competition among informed investors due to short sale restrictions, and so short sale restrictions reduce competition among informed traders; in other words, short sale restrictions make the market inefficient. The inefficiency due to short sale restrictions is also shown in Proposition 4 using the conditional variance as a proxy of market efficiency.

Proposition 4. *$E[var[\tilde{v}|\tilde{p}]]$ is increasing in $\underline{N}$.*

Proof. The proof is given in the Appendix.

When the market becomes more efficient, the current price more accurately reflects the liquidation value, and, therefore, the conditional variance of the liquidation value decreases. From this point of view, Proposition 4 more directly tells us that short sale restrictions make the market inefficient by showing that the conditional variance increases as the restrictions increase. This implication is consistent with empirical findings of Boehmer et al. (2013) who show that price volatility during a short ban increases relative to the volatility before the short ban, and Boehmer and Wu (2013) who show that the standard deviation of the noise of a price decreases after the uptick rule restricting short sales is eliminated.

As Corollary 2 and Propositions 3 and 4 show, short sale restrictions hinder the efficiency of the market for the following three reasons: They (1) reduce the variation of total order flow and

[3] Although $y\pi(y, 0)$ is a decreasing function on the real number with $y > 1$, it is not always so when $y > 0$. However, because N and $\underline{N}$ are integers, the domain of the function $x\pi(x, 0)$ is $\{\frac{z}{2} : z \in \mathbb{N}\}$, and so the function is decreasing on the domain.

[4] We revisit this topic in Section 5 when considering the information acquisition cost.

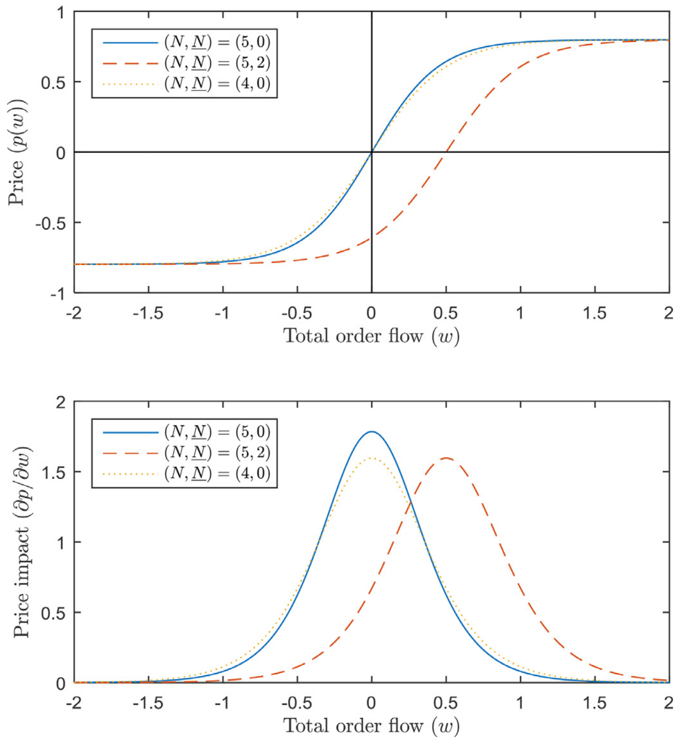

Fig. 3. Price and price impact as a function of order imbalance. The top graph represents prices and the bottom graph represents the price impact under the parameters $\sigma_u = \sigma_v = 1$.

make it hard for market participants to observe hidden information, which reduces the spread of information, (2) reduce competition among informed investors owing to the reduced number of investors who can exploit the bearish signal, and (3) increase the conditional variance of the liquidation value.

Next, we investigate the price impact from the total order flow of all investors.

Proposition 5. *Price impact.*

We have the following equation:

$$\frac{\partial p(w, N, \underline{N})}{\partial w} = \frac{2\sigma_v}{\sqrt{2\pi}\sigma_u} \frac{\sqrt{N - 0.5\underline{N}}}{\cosh^2\left(\frac{w}{\sigma_u}\sqrt{N - 0.5\underline{N}} - \frac{1}{2}\underline{N}\right)} = \frac{\partial p\left(w - \frac{\sigma_u \underline{N}}{2\sqrt{N - 0.5\underline{N}}}, N - 0.5\underline{N}, 0\right)}{\partial w} \tag{15}$$

Proof. The proof is trivial.

Because the function p is symmetric around $\frac{\sigma_u \underline{N}}{2\sqrt{N-0.5\underline{N}}}$ rather than zero, the price impact is asymmetric around zero. Fig. 3 shows this relation. The top panel represents the relation between the total order flow and price, as in Proposition 1, and the bottom panel represents the relation between the total order flow and the price impact, as in Proposition 5. In the case of N=5, compared to the non-restrictive case ($\underline{N} = 0$), the restrictions ($\underline{N} = 2$) shift both the price and the price impact to the right. In addition, the shape of the function with parameters $(N, \underline{N}) = (5, 2)$ is equal to that with $(N, \underline{N}) = (4, 0)$. This confirms that the function is determined by the value of $N - 0.5\underline{N}$.

According to Chordia et al. (2002), who investigate the relation between order imbalances and returns, the price reaction is greater

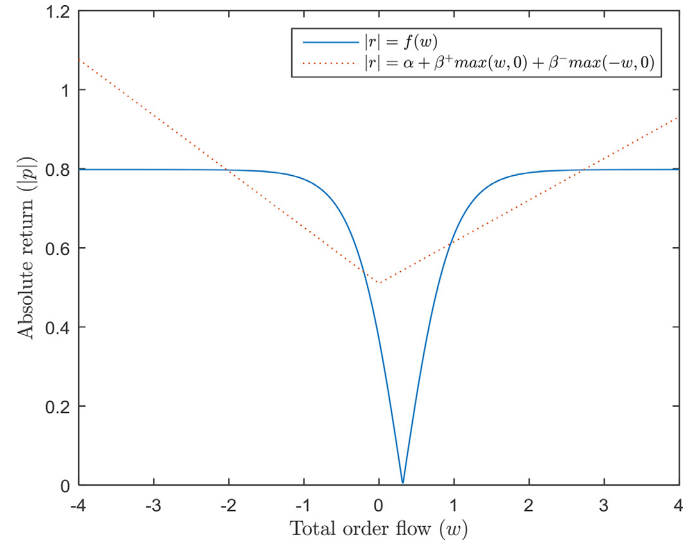

Fig. 4. Order imbalance and the absolute return. This figure shows the relation between order imbalance and the absolute return under the parameters $\sigma_u = \sigma_v = 1$, $N = 5$, and $\underline{N}$=2. The solid line represents the original curve of the absolute return, and the dotted line represents the fitted line of Eq. (16) in the sense of least squares error.

for seller-initiated trading, that is, $\beta^- > \beta^+$, in Eq. (16):[5]

$$|return_t| = \alpha + \beta^+ \max(w_t, 0) + \beta^- max(-w_t, 0) \tag{16}$$

At first glance, the bottom panel of Fig. 3 seems to contradict the empirical result of Chordia et al. (2002). However, β^+ and β^- of Eq. (16) are related to $\frac{|p|}{|w|}$ rather than to $\frac{\partial p}{\partial w}$, because the return is equivalent to the price in our model. In addition, Fig. 4 shows that the absolute price is greater when the order flow is negative than when it is positive, which is consistent with Chordia et al. (2002). From this result, we can infer and prove Proposition 6.

Proposition 6. *Price impact.*

The conditional absolute price is

$$E[|\tilde{p}(w, N, \underline{N}) \parallel \tilde{w} > 0] \leq E[|\tilde{p}(w, N, \underline{N}) \parallel \tilde{w} < 0] \tag{17}$$

Equality holds only if $\underline{N} = 0$, that is, there is no restriction.

Proof. The proof is given in the Appendix.

Brennan et al. (2012) also show that the price reaction is greater for seller-initiated trades in a different framework, that is, Eq. (18):

$$return_t = \alpha + \beta^+ \max(w_t, 0) + \beta^- min(w_t, 0) \tag{18}$$

However, our model predicts that β^- will be less than or equal to β^+ under Eq. (18).[6] Thus, unlike in the previous discussion, the price reactions to buyer-initiated trades are greater than those to seller-initiated trades under the restrictions of Brennan et al. (2012).

A possible explanation for this seemingly contradictory result is as follows: Brennan et al. (2012) conduct their tests at the transaction order level rather than at the accumulated level. Because there are transactions every second, traders and market makers may not expect a positive order flow on average, and do not react with a price decline for a small positive order flow. Therefore, the α of the transaction level can be ignored, and the β^- and

[5] Although the work of Chordia et al. (2002) is not about short sale restrictions, the cost of short sales restricts short sales indirectly in their study, as Diamond and Verrecchia (1987) describe.

[6] We conduct a numerical analysis that shows β^- is equal to β^+ under the baseline model and less than β^+ under the extended model discussed in Section 4.

β^+ of the authors' test are related to $-p(w|w<0)$ and $p(w|w>0)$, respectively, in our model. Then, β^- is greater than β^+, as in Proposition 6.

4. Bullish/bearish/neutral information and short sales

A possible extension of the baseline model is the addition of an informatively non-valuable state, that is, a neutral state. The baseline model includes bullish and bearish states only. However, it is natural that informed traders may not be able to determine whether the market is bullish or bearish when the change in a security's true value is not significant and the signal is therefore weak. In this section, we assume informed traders fail to obtain information on the direction of $\tilde{v}$ when $-\underline{v}<\tilde{v}<\underline{v}$, for some positive value of $\underline{v}$. Accordingly, they recognize a bullish signal only when $\tilde{v}>\underline{v}$ and they obtain a bearish signal only when $\tilde{v}<-\underline{v}$. Therefore, if an informed trader is restriction-free, that is, $i\in\{1,\ldots,\ N-\underline{N}\}$, then

$$\tilde{x}_i=\begin{cases}\gamma^+, & if\ \tilde{v}>\underline{v}\\ \gamma^0, & if\ -\underline{v}<\tilde{v}<\underline{v}\\ -\gamma^-, & \text{otherwise}\end{cases} \tag{19}$$

If an informed investor is restriction-bound,[7]

$$\tilde{x}_i=\begin{cases}\gamma^+, & if\ \tilde{v}>\underline{v}\\ \max\left(\gamma^0,0\right), & if\ -\underline{v}<\tilde{v}<\underline{v}\\ 0, & \text{otherwise}\end{cases} \tag{20}$$

For convenience, let us define G^+, G^0, and G^- as follows:

$$\sum_{i=1}^{N}\tilde{x}_i=\begin{cases}G^+, & if\ \tilde{v}>\underline{v}\\ G^0, & if\ -\underline{v}<\tilde{v}<\underline{v}\\ G^-, & \text{otherwise}\end{cases} \tag{21}$$

Then, we can show the following proposition.

Proposition 7. *Properties under the bullish/bearish/neutral strategy.*

If there are restriction-bound informed investors and the probability of a neutral state is non-zero, then

(1) $G^0\neq 0$,
(2) $\frac{G^++G^-}{2}\neq G^0$,
(3) $(\frac{G^++G^-}{2}-G^0)\cdot G^0>0$,
(4) $G^0\cdot SKEW(\tilde{p})>0$.

In addition, if there is no restriction, then each of the inequalities becomes an equality.

Proof. The proof is given in the Appendix.

When there is no restriction, regardless of the existence of a neutral state, the order flow of each informed trader is symmetric around zero, as in the baseline model. However, under restrictions, the addition of a neutral state makes a difference. One difference is that informed traders' total order flow is not symmetric around the expected total order flow owing to statements (1) and (2) of Proposition 7.

More interestingly, statement (3) of Proposition 7 implies that informed traders can profit even in a non-informative state, that is, a neutral state. For example, if $\frac{G^++G^-}{2}>G^0$, which implies that G^0 is less than the order flow expected from informed investors,[8] then $G^0>0$. Therefore, in the neutral state, even though informed investors place buy orders, market makers may perceive the orders as a bearish signal and set a negative price, on average. Therefore, the informed investors make positive profits, on average.

In addition, while the skewness of returns is always zero under the baseline model, regardless of restrictions, the restrictions make the skewness of returns non-zero under the bullish/bearish/neutral information set.[9] While Hong and Stein (2003) predict that returns will be less skewed under restrictions, empirical evidence, including that of Xu (2007), often shows that restrictions increase the skewness of returns, which is consistent with our result when G^0 is positive.[10] We cannot prove whether G^0 is positive, but we conjecture that it is. The intuition is as follows.

From the baseline model, we can conjecture that an average informed trade order is tilted toward a buy order under short sale restrictions. Without informed investors' trades, the order imbalance will be zero, on average, which market makers will interpret as a negative signal. Thus, if informed investors do not trade or place orders when they have neutral information, the price will be negative, on average. If the informed investors' order flow in a neutral state is less than the unconditional expected value of informed order flows from the market makers' point of view, it will still be a negative signal to the market makers. In this case, the price will be negative, on average. Therefore, by placing buy orders that are greater than zero but less than the amount that keeps the order imbalance below the unconditional expected value of informed order flows, informed investors can profit, even in a neutral state. That is, since they buy at a negative price but the expected true value is zero in the neutral state, they profit, on average. This makes the median of the return negative and its skewness positive.

Because we cannot show the positivity of skewness analytically, we provide a numerical solution for informed traders' orders under short sale restrictions in Fig. 5. The top panel of Fig. 5 describes G^+, G^0, and G^- as functions of $\underline{N}$. Consistent with our conjecture, G^0 is positive for each positive $\underline{N}$. Combined with Proposition 7, this means that the skewness of the return is positive for each positive $\underline{N}$. The middle panel of Fig. 5 also describes γ^+, γ^0, and $-\gamma^-$ as functions of $\underline{N}$. As in the baseline model, γ^+ and γ^- are increasing functions of $\underline{N}$. However, γ^+ is less than γ^-, which is in stark contrast with the baseline model. This may be because informed investors place buy orders even in neutral states.

Meanwhile, in the top panel of Fig. 5, we can observe that the difference between total order at the bullish signal and that at the bearish signal narrows as the number of restriction-bound informed investors become larger; this is the same as the result in Section 3 when investors get a bullish/bearish signal. This implies that the release of information from informed investors decreases and the market becomes less efficient as restrictions become severe. The bottom panel of Fig. 5 also shows the inefficiency under short sale restrictions 5 because the total profit of informed investors increases as restrictions become severe. Therefore, the result for inefficiency under short sale restrictions is also valid even when investors may get a neutral signal as well as a bullish or bearish signal. Fig. 6 confirms the other result of Section 3. Similar to the result in Section 3, even in the case that investors get a

[7] Proposition 1 shows that γ_i^+ is identical for all informed investors. This can be observed again under this model.

[8] Let $P=\Pr\{\tilde{v}>\underline{v}\}$; then, $E[\sum_{i=1}^{N}\tilde{x}_i]=P(G^++G^-)+(1-2P)G^0=G^0+2P(\frac{G^++G^-}{2}-G^0)>G^0$.

[9] The initial stock price, the expected liquidation value, is equal to zero, if we assume that informed traders have no additional signal the very first time. Therefore, a return is equivalent to a price in our model.

[10] Although skewness increased after the short sale ban was lifted in the Chinese market (Chang et al., 2014), Bris et al. (2007), Chang et al. (2007), and Saffi and Sigurdsson (2011) show with international data that skewness is reduced when short sales are allowed or when the lending supply increases. In addition, Duffee (1995) and Albuquerque (2012) show that firm returns display positive skewness, while index returns display negative skewness. These findings are also related to our result that restrictions increase skewness, because our model is about a single security and restrictions always exist, such as those for mutual fund and pension fund managers.

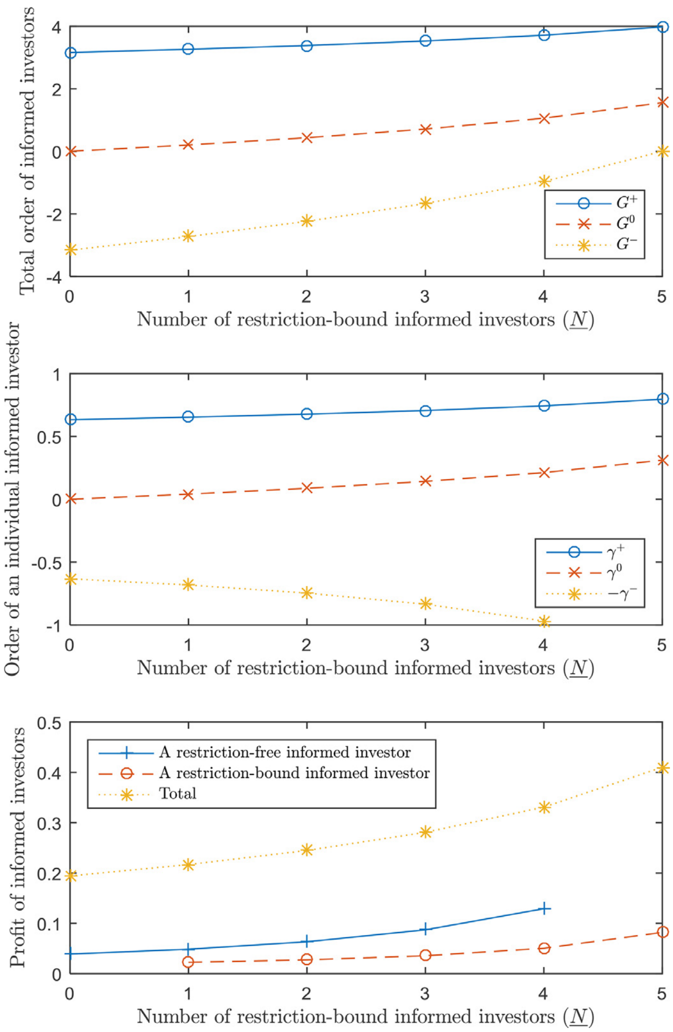

Fig. 5. Bullish/bearish/neutral strategy under restriction. The top graph represents the total order flow of all informed investors, the middle graph represents the order flow of an individual informed investor, and the bottom graph represents the profits of informed investors as functions of $\underline{N}$ under a bullish/bearish/neutral information structure. The parameters in this figure are $\sigma_u = \sigma_v = 1$, $N = 5$, and $P = 1/3$. Each set $(\gamma^+, \gamma^0, -\gamma^-)$ or (G^+, G^0, G^-), given $\underline{N}$, is a numerical solution to the system of first-order conditions, given in the Appendix.

bullish/bearish/neutral signal, the price reaction on seller-initiated trading is still larger than that on buyer-initiated trading. Therefore, all the qualitative results of Section 3 hold even when we extend the model to the bullish/bearish/neutral economy.

5. Information acquisition

While previous sections assume the number of informed traders is given exogenously, this section allows the number of informed investors to be determined endogenously as they acquire information on stock returns by paying costs. When the information acquisition requires cost c, each investor will acquire the information as long as c does not exceed the expected profit from the information. Therefore, when there is no short sale restriction, the number of informed investors is given as follows:

$$\bar{N}^* = \max\{z \in \{0, 1, 2, \ldots\} : \pi(z, 0) \geq c\} \tag{22}$$

where c is the cost of information acquisition.[11] Now, let us consider an equilibrium under the restrictions. According to the findings in previous sections, if there are both restriction-free and restriction-bound traders, restriction-bound investors have less incentive to take information because their profit is less than that of restriction-free investors. Accordingly, if there are too many restriction-free investors and the information cost is nontrivial, then restriction-bound investors may not engage in information acquisition activity. On the other hand, if there are only a few restriction-free informed investors and thus the speed of information dissemination is not fast enough, it is possible that the expected profit from information will exceed the cost of information acquisition even for restriction-bound investors. Thus, the number of informed restriction-bound investors resulting from the optimal behavior of investors should be the function of the number of restriction-free investors and the cost of information acquisition. In other words, when the restrictions limit some investors from short sales and only $\bar{N}$ investors are restriction-free, the numbers of restriction-free and restriction-bound informed investors, $(\bar{N}^*, \underline{N}^*)$, are represented as functions of $\bar{N}$ and cost c as follows:[12,13]

$$\bar{N}^*\left(\bar{N}, c\right) = \max\left\{z \in \{0, 1, 2, \ldots\} : z \leq \bar{N} \text{ and } \pi(z, 0) \geq c\right\} \tag{23}$$

$$\underline{N}^*\left(\bar{N}, c\right) = \max\left\{z \in \{0, 1, 2, \ldots\} : \ \underline{\pi}\left(\bar{N}^* + z, z\right) \geq c\right\} \tag{24}$$

Then, the numbers of total informed traders under no restriction and complete restriction are represented by $\bar{N}^*(\infty, c)$ and $\underline{N}^*(0, c)$, respectively. In particular, in the case of the bullish/bearish economy, we can show the following proposition regarding the number of informed investors determined endogenously from the optimal decisions of investors:

Proposition 8. *The number of informed investors under the bullish/bearish economy.*

When investors can receive a bullish/bearish signal and the cost to acquire information is c, the numbers $\bar{N}^*(\infty, c)$ and $\underline{N}^*(0, c)$

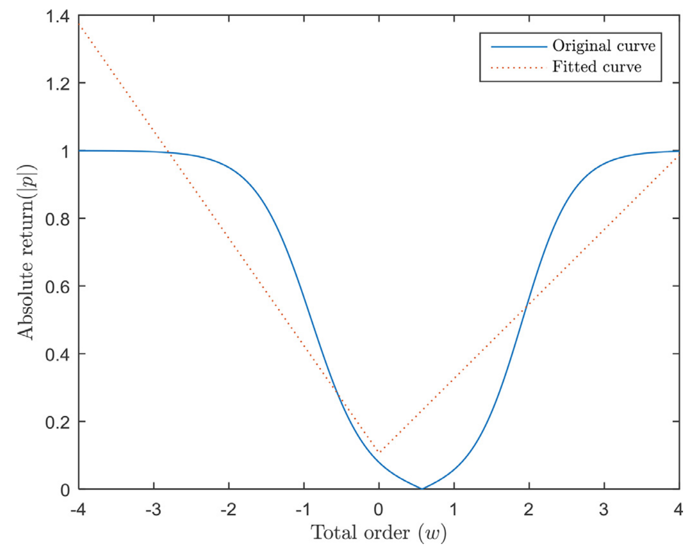

Fig. 6. Order imbalance and the absolute return under the bullish/bearish/neutral strategy. This figure shows the relation between order imbalance and the absolute return under the bullish/bearish/neutral economy with the parameters $\sigma_u = \sigma_v = 1$, $N = 5$, $\underline{N} = 2$, and $P = 1/3$. The solid line represents the original curve of the absolute return and the dotted line represents the fitted line of Eq. (16) in the sense of least squares error.

[11] Recall that $\pi(N, \underline{N})$ is the profit for a restriction-free informed investor when there are $N - \underline{N}$ restriction-free and $\underline{N}$ restriction-bound informed investors.

[12] Recall that $\underline{\pi}(N, \underline{N})$ is the profit for a restriction-bound informed investor when there are $N - \underline{N}$ restriction-free and $\underline{N}$ restriction-bound informed investors.

[13] Note that although $\bar{N}^*(\bar{N}, c)$ is determined from the condition $\pi(\bar{N}^*, 0) \geq c$, $\bar{N}^*$ finally satisfies the condition $\pi(\bar{N}^* + \underline{N}^*, \underline{N}^*) \geq c$ due to Eq. (24) because $\pi(\bar{N}^* + \underline{N}^*, \underline{N}^*) \geq \underline{\pi}(\bar{N}^* + \underline{N}^*, \underline{N}^*)$.

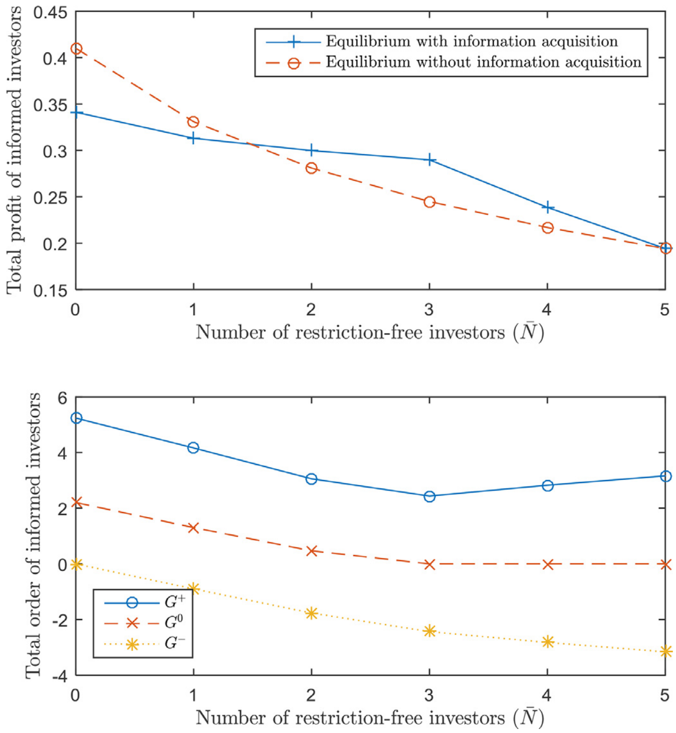

Fig. 7. Information acquisition and efficiency under short sale restrictions. The top graph represents the profit of all informed investors as a function of $\bar{N}$ under a bullish/bearish/neutral information structure. The parameters in this figure are $\sigma_u = \sigma_v = 1$ and $P = 1/3$. In addition, we assume $c = \pi(5, 0) \approx 0.039$, which is the profit of an informed investor when there are only five restriction-free informed investors. Then it compares the equilibrium with and without the information acquisition process. The solid line represents equilibrium with information acquisition and the dashed line represents equilibrium when the total number of informed investors is 5. The bottom graph represents the total order flow of all informed investors under equilibrium with the information acquisition process as functions of $\underline{N}$. Note that the number of restriction-free investors ($\bar{N}$) decreases to 0 as the restrictions become severe.

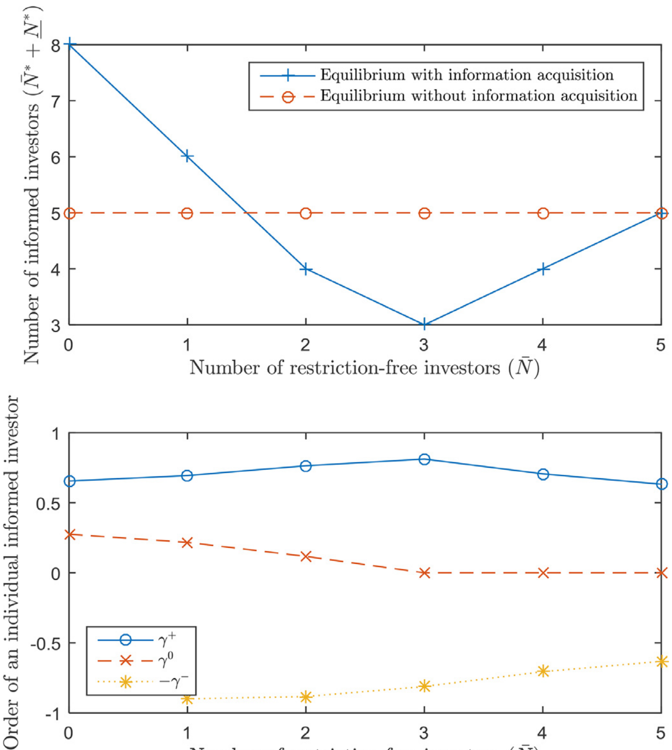

Fig. 8. Information acquisition and number of informed investors under short sale restrictions. The top graph represents the number of informed investors as a function of $\bar{N}$ under a bullish/bearish information structure. Additionally, the bottom graph represents the order flow of an individual informed investor under equilibrium with the information acquisition process as functions of $\underline{N}$. The parameters in this figure are the same as those in Fig. 7.

have the following relation:

$$\bar{N}^*(\infty, c) \le \underline{N}^*(0, c) \tag{25}$$

Proof. The proof is given in the Appendix.

In contrast with Nezafat et al. (2015), Proposition 8 states that the full restriction encourages investors to acquire information. This may be against intuition in the sense that restrictions make bearish signals useless and therefore reduce the profit of each informed investor. However, short sale restrictions limit the usage of bearish signals, which makes the market very inefficient. In this case, more investors can try to exploit this market inefficiency by acquiring information, even without using the bearish signals through short sales. Proposition 8 represents the property under the bullish/bearish economy for an analytic representation. Now, let us investigate efficiency and the number of investors from Figs. 7 and 8, respectively, under the bullish/bearish/neutral framework.

The top panel of Fig. 7 describes the total profit of informed investors as a function of the number of restriction-free investors.[14] The solid line shows the equilibrium when investors can buy information with cost $c = \pi(5, 0)$ (hereafter, equilibrium with information acquisition).[15] For a comparison, we also present the equilibriums when the number of informed investors is exogenously given as 5 (hereafter, equilibrium without information acquisition), which is a result under the model setting of Section 4. It shows that both the solid line and the dashed line are decreasing in $\bar{N}$. This means that the market becomes more efficient as the restrictions are relaxed, even under the equilibrium with information acquisition. The bottom panel supports this result by showing that the variation of total order flow is increasing in $\bar{N}$. In other words, when the restrictions are relaxed, information spreads out more quickly as the variation of the order flow increases. Therefore, Fig. 7 confirms all the market-wide results of Sections 3 and 4 even when we introduce decision-making about information acquisition.[16]

Now, let us examine the number of informed traders for each case in the top panel of Fig. 8. It shows that the number of informed investors is reduced when a slight restriction ($\bar{N} = 3,\ 4$) is introduced from no restriction ($\bar{N} = 5$). This is because inefficiency due to the non-availability of a bearish signal is not enough for the restricted investors to bear the information acquisition cost and exploit the inefficiency. However, as the restrictions become severe, the market becomes more inefficient and the profit for informed investors increases. Therefore, more restriction-bound investors start to buy information to exploit the market inefficiency. Finally, as Proposition 8 describes, when all investors are restricted from shorting stocks ($\bar{N} = 0$), the number of informed investors can increase more than that under no restriction.

The bottom panel of Fig. 8 shows the number of restriction-free investors, which is different from the result in previous sections.

[14] Note that, unlike Figs. 5 and 6, Fig. 7 describes the equilibrium as functions of the number of restriction-free investors ($\bar{N}$) rather than of the number of restriction-bound informed investors ($\underline{N}$).

[15] In this example, $\bar{N} = 5$ is equivalent to $\bar{N} = \infty$, which represents no restriction.

[16] All the other results, in addition to those related to the inefficiency discussed above, hold because the other result, which is asymmetric price reaction to the order imbalance, is not affected by the consideration of information acquisition.

In the previous sections, individual investors' orders, which are γ^+, γ^0, and γ^-, are maximized when the restrictions are most severe. However, the bottom panel of Fig. 8 shows that buy orders at the severest restrictions decline compared to orders at mild restrictions. The reason for this is the endogeneity of the number of informed investors. Unlike in the previous sections, as the inefficiency attracts more informed investors, competition among informed investors limits the size of buy orders.[17]

In sum, the consideration of information acquisition does not qualitatively change the results of previous sections. However, when the information acquisition decision becomes endogenous, we can show that information acquisition may increase when the restriction is severe because investors try to exploit the inefficiency caused by short sale restrictions.

6. Conclusion

Since the 2008 financial crisis, short sale restrictions have been a major concern in finance. We incorporate short sale restrictions into the framework of Dridi and Germain (2004) and obtain some implications that generally seem to be consistent with the empirical observations documented in the literature. First, we show that short sale restrictions lead to market inefficiency. Second, under the restrictions, stock prices react asymmetrically to order imbalance under short sale restrictions. This finding is consistent with an empirical observation of Chordia et al. (2002), in that absolute returns are greater when the order imbalance is negative. Third, when informed traders distinguish a neutral state from bullish and bearish states, restrictions increase the skewness of returns, which is in line with the prior literature (e.g., Saffi and Sigurdsson, 2011).

In addition to these empirically supported results, we also find empirically unexplored results that the restriction can stimulate investors to acquire information. Additionally, if the entrance of new informed traders is limited, each informed trader increases trading size, even under bullish and neutral signals. These results are empirically testable but remain unconfirmed in the literature.

The literature about short sale restrictions, including this study, assumes that the restrictions are exogenously given. However, in reality, short sale restrictions are prevalent only when stock prices drastically drop, as in the period of 2008 credit crisis. This can be reflected with a two-period model: The first period is free of restrictions, and the restriction is invoked at the second period only when the stock price drastically decreases in the first period. Obtaining the result from the fully rational equilibrium under this two-period model seems to be challenging.

However, short sale restrictions only occur in the second period, and this second period is different from our model only in that the initial price becomes a random variable. In other words, "liquidation value at the second period minus equilibrium price at the first period" replaces the liquidation value of the current model. Therefore, only the assumption that the liquidation value is normally distributed is violated. Accordingly, the result regarding the inefficiency or greater impact of sell orders under short sale restrictions does not seem to be changed under the new model.

On the other hand, the result for skewness can be affected because the distribution is changed, and the model can provide new implications about the normal periods. Thus, we believe that endogenizing restrictions can provide new, meaningful implications. However, this is out of the scope of this paper, and we leave these issues for future research.

[17] In other words, if, for any reason, the entrance of new informed investors is limited as in the previous sections, each individual informed investor could increase orders even under the bullish signal.

Acknowledgements

This paper extends chapter 3 of Lee's doctoral dissertation. We are grateful to the anonymous referees and the editor (Carol Alexander) for valuable comments.

Appendix

Proof of Proposition 1. This proof is similar to that of Dridi and Germain (2004).

Let the order flow of a restriction-free informed trader $i \in \{1, \ldots, N - \underline{N}\}$ be

$$\tilde{x}_i = \begin{cases} \sigma_u g_i^+, & \text{if } \tilde{v} > 0 \\ -\sigma_u g_i^-, & \text{if } \tilde{v} < 0 \end{cases}; \tag{A1}$$

let the order flow of a restriction-bound informed trader $i \in \{N - \underline{N} + 1, \ldots, N\}$ be

$$\tilde{x}_i = \begin{cases} \sigma_u \underline{g}_i^+, & \text{if } \tilde{v} > 0 \\ 0, & \text{if } \tilde{v} < 0 \end{cases}; \tag{A2}$$

and let the market makers' conjecture of informed traders' strategy be

$\tilde{x}_i = \sigma_u g_i^{c+}$ or $\sigma_u \underline{g}_i^{c+}$ by type of investor if $\tilde{v} > 0$
$\tilde{x}_i = -\sigma_u g_i^{c-}$ or 0 by type of investor, if $\tilde{v} < 0$

Then, the conditional probability density function under the market makers' conjecture is given by

$$f(v, w) = \begin{cases} \dfrac{1}{\sigma_u \sigma_v} n\left(\dfrac{v}{\sigma_v}\right) n\left(\dfrac{w}{\sigma_u} - s\left(g^{c+}, \underline{g}^{c+}\right)\right), & \text{if } v > 0 \\ \dfrac{1}{\sigma_u \sigma_v} n\left(\dfrac{v}{\sigma_v}\right) n\left(\dfrac{w}{\sigma_u} + s(g^{c-})\right), & \text{if } v < 0 \end{cases} \tag{A3}$$

Because

$$\begin{aligned} f(w) &= \int_{-\infty}^{0} f(v, w) dv + \int_{0}^{\infty} f(v, w) dv \\ &= \frac{1}{2\sigma_u} n\left(\frac{w}{\sigma_u} + s\left(g^{c-}\right)\right) + \frac{1}{2\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^{c+}, \underline{g}^{c+}\right)\right) \end{aligned} \tag{A4}$$

and

$$f_{(\tilde{v}|\tilde{w})}(v, w) = \frac{f(v, w)}{f(w)}, \tag{A5}$$

the conditional probability density function is given by

$$f_{(\tilde{v}|\tilde{w})}(v, w) = \begin{cases} \dfrac{2}{\sigma_v} n\left(\dfrac{v}{\sigma_v}\right) P_u(w, N, \underline{N}), & \text{if } v > 0 \\ \dfrac{2}{\sigma_v} n\left(\dfrac{v}{\sigma_v}\right) P_d(w, N, \underline{N}), & \text{if } v < 0 \end{cases} \tag{A6}$$

Accordingly, the conditional value, or the market makers' price schedule, is given by

$$E[\tilde{v}|\tilde{w}] = p^*(P_u - P_d) \tag{A7}$$

where

$$p^* = \frac{2\sigma_v}{\sqrt{2\pi}} \tag{A8}$$

$$s\left(g^{c-}\right) = \sum_{i=1}^{N-\underline{N}} g_i^{c-}, \quad s\left(g^{c+}, \underline{g}^{c+}\right) = \sum_{i=1}^{N-\underline{N}} g_i^{c+} + \sum_{i=N-\underline{N}+1}^{N} \underline{g}_i^{c+} \tag{A9}$$

$$P_u(w, N, \underline{N}) = \frac{n\left(\frac{w}{\sigma_u} - s\left(g^{c+}, \underline{g}^{c+}\right)\right)}{n\left(\frac{w}{\sigma_u} + s(g^{c-})\right) + n\left(\frac{w}{\sigma_u} - s\left(g^{c+}, \underline{g}^{c+}\right)\right)} \tag{A10}$$

$$P_d(w, N, \underline{N}) = \frac{n\left(\frac{w}{\sigma_u} + s(g^{c-})\right)}{n\left(\frac{w}{\sigma_u} + s(g^{c-})\right) + n\left(\frac{w}{\sigma_u} - s\left(g^{c+}, \underline{g}^{c+}\right)\right)} \tag{A11}$$

Then, the expected agent reward for agent $i \in \{N-\underline{N}+1, N\}$ is[18]

$$\begin{aligned}
&\sigma_u \underline{g}_i^+ \int_{-\infty}^{\infty}\int_0^{\infty} v f(v,w)\,dv\,dw \\
&= \sigma_u \underline{g}_i^+ \int_0^{\infty} v \frac{1}{\sigma_v} n\left(\frac{v}{\sigma_v}\right) dv \int_{-\infty}^{\infty} \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw \\
&= \sigma_u \underline{g}_i^+ \int_0^{\infty} v \frac{1}{\sigma_v} n\left(\frac{v}{\sigma_v}\right) dv \\
&= \frac{1}{2} p^* \sigma_u \underline{g}_i^+
\end{aligned} \tag{A12}$$

In addition, because $\tilde{p}(w) = E[\tilde{v}|w]$, the expected cost for agent $i \in \{N-\underline{N}+1, N\}$ is

$$\begin{aligned}
&\sigma_u \underline{g}_i^+ \int_{-\infty}^{\infty}\int_0^{\infty} \tilde{p} f(v,w)\,dv\,dw \\
&= \sigma_u \underline{g}_i^+ \int_{-\infty}^{\infty}\int_0^{\infty} E[\tilde{v}|w] f(v,w)\,dv\,dw \\
&= \sigma_u \underline{g}_i^+ \int_{-\infty}^{\infty}\int_0^{\infty} p^*(P_u - P_d) \frac{1}{\sigma_u\sigma_v} n\left(\frac{v}{\sigma_v}\right) n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dv\,dw \\
&= \frac{p^*}{2}\sigma_u \underline{g}_i^+ \int_{-\infty}^{\infty} (P_u - P_d) \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw
\end{aligned} \tag{A13}$$

Therefore, the expected profit for agent $i \in \{N-\underline{N}+1, N\}$ is

$$\begin{aligned}
\pi_i &= \sigma_u \underline{g}_i^+ \int_{-\infty}^{\infty}\int_0^{\infty} (v - \tilde{p}) f(v,w)\,dv\,dw \\
&= \frac{1}{2}\sigma_u \underline{g}_i^+ p^* - \frac{1}{2}\sigma_u \underline{g}_i^+ p^* \int_{-\infty}^{\infty} (P_u - P_d) \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw \\
&= \frac{1}{2}\sigma_u \underline{g}_i^+ p^* \int_{-\infty}^{\infty} (P_u + P_d) \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw \\
&\quad - \frac{1}{2}\sigma_u \underline{g}_i^+ p^* \int_{-\infty}^{\infty} (P_u - P_d) \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw \\
&= \sigma_u \underline{g}_i^+ p^* \int_{-\infty}^{\infty} P_d \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw
\end{aligned} \tag{A14}$$

Let $t = \frac{w}{\sigma_u} - s(g^+, \underline{g}^+)$; then,

$$\begin{aligned}
\pi_i &= \int_{-\infty}^{\infty} \frac{\sigma_u \underline{g}_i^+ p^* n\left(t + s(g^{c-}) + s\left(g^+, \underline{g}^+\right)\right) n(t)\,dt}{n(t + s(g^{c-}) + s(g^+, \underline{g}^+)) + n\left(t - s\left(g^{c+}, \underline{g}^{c+}\right) + s\left(g^+, \underline{g}^+\right)\right)} \\
&= \frac{\sigma_u \underline{g}_i^+ p^*}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\exp\left(\frac{t^2}{2}\right) + A\left(\underline{g}_i^+\right) \exp\left(\frac{\left(t + s\left(g^{c+}, \underline{g}^{c+}\right) + s(g^{c-})\right)^2}{2}\right)} dt
\end{aligned} \tag{A15}$$

where

$$A\left(\underline{g}_i^+\right) = \exp\left(\left(s\left(g^{c+}, \underline{g}^{c+}\right) + s\left(g^{c-}\right)\right)\left(s\left(g^+, \underline{g}^+\right) - s\left(g^{c+}, \underline{g}^{c+}\right)\right)\right) \tag{A16}$$

Then, the first-order condition is

$$\begin{aligned}
\frac{\sqrt{2\pi}}{\sigma_u p^*} \frac{\partial \pi_i}{\partial \underline{g}_i^+} &= \int_{-\infty}^{\infty} \frac{1}{\exp\left(\frac{t^2}{2}\right) + A\left(\underline{g}_i^+\right) \exp\left(\frac{(t + s(g^{c+}, \underline{g}^{c+}) + s(g^{c-}))^2}{2}\right)} dt \\
&\quad - \underline{g}_i^+ \int_{-\infty}^{\infty} \frac{\left(s\left(g^{c+}, \underline{g}^{c+}\right) + s\left(g^{c-}\right)\right) A\left(\underline{g}_i^+\right) \exp\left(\frac{(t + s(g^{c+}, \underline{g}^{c+}) + s(g^{c-}))^2}{2}\right)}{\left(\exp\left(\frac{t^2}{2}\right) + A\left(\underline{g}_i^+\right) \exp\left(\frac{(t + s(g^{c+}, \underline{g}^{c+}) + s(g^{c-}))^2}{2}\right)\right)^2} dt \\
&= 0
\end{aligned} \tag{A17}$$

Because, under equilibrium, $g_i^{c+} = g_i^+$, $\underline{g}_i^{c+} = \underline{g}_i^+$, and $g_i^{c-} = g_i^-$, $\underline{g}_i^+$ satisfies the following condition:

$$\underline{g}_i^+ = \frac{\int_{-\infty}^{\infty} \frac{1}{\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{(t + s(g^+, \underline{g}^+) + s(g^-))^2}{2}\right)} dt}{\int_{-\infty}^{\infty} \frac{(s(g^+, \underline{g}^+) + s(g^-)) \exp\left(\frac{(t + s(g^+, \underline{g}^+) + s(g^-))^2}{2}\right)}{\left(\exp(\frac{t^2}{2}) + \exp\left(\frac{(t + s(g^+, \underline{g}^+) + s(g^-))^2}{2}\right)\right)^2} dt} \tag{A18}$$

Similarly, the expected profit for an agent $i \in \{1, \ldots, N-\underline{N}\}$ is given by

$$\begin{aligned}
\pi_i &= \sigma_u g_i^- p^* \int_{-\infty}^{\infty} P_u \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} + s\left(g^-\right)\right) dw \\
&\quad + \sigma_u g_i^+ p^* \int_{-\infty}^{\infty} P_d \frac{1}{\sigma_u} n\left(\frac{w}{\sigma_u} - s\left(g^+, \underline{g}^+\right)\right) dw
\end{aligned} \tag{A19}$$

By substituting $t = \frac{w}{\sigma_u} + s(g^-)$ and $t = \frac{w}{\sigma_u} - s(g^+, \underline{g}^+)$ and the equilibrium conditions into the first-order conditions of Eq. (A19), we obtain

$$g_i^- = \frac{\int_{-\infty}^{\infty} \frac{1}{\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(-t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)} dt}{\int_{-\infty}^{\infty} \frac{\left(s\left(g^+, \underline{g}^+\right) + s(g^-)\right) \exp\left(\frac{\left(-t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)}{\left(\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(-t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)\right)^2} dt} \tag{A20}$$

and

$$g_i^+ = \frac{\int_{-\infty}^{\infty} \frac{1}{\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)} dt}{\int_{-\infty}^{\infty} \frac{\left(s\left(g^+, \underline{g}^+\right) + s(g^-)\right) \exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)}{\left(\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)\right)^2} dt} \tag{A21}$$

By symmetry,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)}{\left(\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)\right)^2} dt \\
&= \int_{-\infty}^{\infty} \frac{\exp\left(\frac{\left(-t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)}{\left(\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(-t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)\right)^2} dt \\
&= \int_{-\infty}^{\infty} \frac{\exp\left(\frac{t^2}{2}\right)}{\left(\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)\right)^2} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\exp\left(\frac{t^2}{2}\right) + \exp\left(\frac{\left(t + s\left(g^+, \underline{g}^+\right) + s(g^-)\right)^2}{2}\right)} dt
\end{aligned} \tag{A22}$$

Hence,

$$\begin{aligned}
\underline{g}_i^+ &= g_i^+ = g_i^- = \frac{2}{s\left(g^+, \underline{g}^+\right) + s\left(g^-\right)} \\
&= \frac{2}{(N - \underline{N})(g^+ + g^-) + \underline{N}\underline{g}^+} \\
&= \sqrt{\frac{1}{N - 0.5\underline{N}}}
\end{aligned} \tag{A23}$$

□

Proof of Proposition 3. The proof of this proposition follows from Proposition 2 of Dridi and Germain (2004) and the proof of Proposition 1 above.

[18] The probability density function of an informed investor is the same as that under the market makers' conjecture, except that γ replaces γ^c.

Proof of Proposition 4. Because p is an increasing function of w, *conditional on* w is equivalent to saying *conditional on* p. Therefore, we use w instead of p as a conditioning variable. The forms of conditional moments are as follows:

$$\begin{aligned} E[\tilde{v}^n|w] &= \int_{-\infty}^{\infty} v^n f(v|w,N,\underline{N})dv \\ &= P_d(w,N,\underline{N})\int_{-\infty}^{0} v^n \frac{2}{\sigma_v} n\left(\frac{v}{\sigma_v}\right)dv \\ &\quad + P_u(w,N,\underline{N})\int_{0}^{\infty} v^n \frac{2}{\sigma_v} n\left(\frac{v}{\sigma_v}\right)dv \end{aligned} \quad \text{(A24)}$$

Therefore,

$$E[\tilde{v}|w] = p^*(P_u - P_d)$$

$$E\left[\tilde{v}^2|w\right] = \sigma_v^2$$

and

$$E[var[\tilde{v}|w]] = \sigma_v^2 - p^{*2}E\left[(P_u - P_d)^2\right] \quad \text{(A25)}$$

By letting $t = \frac{w}{\sigma_u} - \frac{\underline{N}}{2\sqrt{N-0.5\underline{N}}}$ and $x = \sqrt{N-0.5\underline{N}}$, we obtain

$$\begin{aligned} &E\left[(P_u - P_d)^2\right] \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{(n(t-x)-n(t+x))^2}{(n(t-x)+n(t+x))^2}(n(t-x)+n(t+x))dt \\ &= 1 - 2\int_{-\infty}^{\infty} \frac{n(t-x)n(t+x)}{n(t-x)+n(t+x)}dt \end{aligned} \quad \text{(A26)}$$

Hence,

$$\begin{aligned} \frac{d}{dx}E[var[\tilde{v}|w]] &= -\frac{p^{*2}}{\pi}\int_{-\infty}^{\infty} \frac{xe^{-t^2-x^2}}{n(t-x)+n(t+x)}dt \\ &\quad - \frac{p^{*2}}{\pi}\int_{-\infty}^{\infty} t\frac{(n(t-x)-n(t+x))e^{-t^2-x^2}}{(n(t-x)+n(t+x))^2}dt \end{aligned} \quad \text{(A27)}$$

The above form is negative because $t(n(t-x)-n(t+x))$ is positive for all t. Therefore, $E[var[\tilde{v}|w]]$ is decreasing in $N - 0.5\underline{N}$. □

Proof of Proposition 6. Note that

$$p(w,N,\underline{N}) = \tanh\left(\frac{\sqrt{N-0.5\underline{N}}}{\sigma_u}\left(w - \frac{\sigma_u\underline{N}}{2\sqrt{N-0.5\underline{N}}}\right)\right) \quad \text{(A28)}$$

There is an α, $0 \le \alpha \le 1$, such that

$$\begin{aligned} &E[|p(\tilde{w})||\tilde{w} > 0] \\ &= \alpha E\left[|p(\tilde{w})||\tilde{w} > \frac{\sigma_u\underline{N}}{\sqrt{N-0.5\underline{N}}}\right] \\ &\quad + (1-\alpha)E\left[|p(\tilde{w})||0 < \tilde{w} < \frac{\sigma_u\underline{N}}{\sqrt{N-0.5\underline{N}}}\right] \end{aligned} \quad \text{(A29)}$$

Because $p(w)$ is symmetric around $(w, p(w)) = (\frac{\sigma_u\underline{N}}{2\sqrt{N-0.5\underline{N}}}, 0)$ and is an increasing function,

$$|p(w_1)| < |p(w_2)| \quad \text{(A30)}$$

when $0 < w_1 < \frac{\sigma_u\underline{N}}{\sqrt{N-0.5\underline{N}}}$, $w_2 > \frac{\sigma_u\underline{N}}{\sqrt{N-0.5\underline{N}}}$.

Therefore, the following holds:

$$\begin{aligned} E[|p(\tilde{w})||\tilde{w} > 0] &\le E\left[|p(\tilde{w})||\tilde{w} > \frac{\sigma_u\underline{N}}{\sqrt{N-0.5\underline{N}}}\right] \\ &= E[|p(\tilde{w})||\tilde{w} < 0] \end{aligned} \quad \text{(A31)}$$

Equality holds only if $\underline{N} = 0$. □

For the proof of Proposition 7, the following two lemmas are helpful.

Lemma A1. Define a function $f(w)$ as follows:

$$\begin{aligned} f(w) &= \frac{2Pn\left(w-\bar{G}\right) - 2Pn\left(w+\bar{G}\right)}{Pn\left(w+\bar{G}\right) + Pn\left(w-\bar{G}\right) + (1-2P)n\left(w-\bar{G}^0\right)}n\left(w-\bar{G}^0\right) \\ &= \frac{2Pn\left(w-\bar{G}\right) - 2Pn\left(w+\bar{G}\right)}{\frac{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)}{n\left(w-\bar{G}^0\right)} + (1-2P)} \end{aligned} \quad \text{(A32)}$$

Then, $\bar{G}^0\int_{-\infty}^{\infty} f(w)dw > 0$ if $\bar{G}^0 \ne 0$.

Proof of Lemma A1. Assume that $\bar{G}^0 > 0$. Then, $n(|w| - \bar{G}^0) > n(-|w| - \bar{G}^0)$ holds. Accordingly, $f(|w|) \ge |f(-|w|)| = -f(-|w|)$ holds. Therefore, $\int_{-\infty}^{\infty} f(w)dw > 0$. When $\bar{G}^0 < 0$, all inequalities are changed in the opposite direction. □

Lemma A2. Let the function $g(w) = f(w)(w - \bar{G}^0)$. Then, $\int_{-\infty}^{\infty} g(w)dw > 0$ if $|\bar{G}^0| < \bar{G}$.

Proof of Lemma A2. Assume that $\bar{G}^0 > 0$. Then, $g(w) < 0$ if and only if $w \in (0, \bar{G}^0)$. Note that $n(\bar{G}^0 + \alpha - \bar{G}) > n(\bar{G}^0 - \alpha - \bar{G}) > n(\bar{G}^0 - \alpha + \bar{G}) > n(\bar{G}^0 + \alpha + \bar{G})$ if $0 < \alpha < \bar{G}^0 < \bar{G}$. Therefore, we have $f(\bar{G}^0 + \alpha) > f(\bar{G}^0 - \alpha) > 0$ and $g(\bar{G}^0 + \alpha) > |g(\bar{G}^0 - \alpha)| > g(\bar{G}^0 - \alpha)$ for $0 < \alpha < \bar{G}^0$. Therefore, $\int_0^{2\bar{G}^0} g(w)dw > 0$ and $\int_{-\infty}^{\infty} g(w)dw > 0$. In the case of $\bar{G}^0 < 0$, $\int_{-\infty}^{\infty} g(w)dw > 0$ can be proven similarly. □

Proof of Proposition 7. Because the order flow of each informed trader is a multiple of σ_u, without loss of generality, let $\sigma_u = 1$. Then, we have the following first-order condition:[19]

$$\int_{-\infty}^{\infty} (P_m + 2P_d)n\left(w - G^+\right)\left(G^+w - \left(G^+\right)^2 + N\right)dw = 0 \quad \text{(A33)}$$

$$\int_{-\infty}^{\infty} (P_m + 2P_u)n\left(w - G^-\right)\left(G^-w - \left(G^-\right)^2 + N - \underline{N}\right)dw = 0 \quad \text{(A34)}$$

$$\int_{-\infty}^{\infty} (P_u - P_d)n\left(w - G^0\right)\left(G^0w - \left(G^0\right)^2 + N\right)dw = 0 \quad \text{(A35)}$$

where

$$P_u(w,N,\underline{N}) = \frac{Pn(w-G^+)}{Pn(w-G^-) + Pn(w-G^+) + (1-2P)n\left(w-G^0\right)} \quad \text{(A36)}$$

$$P_d(w,N,\underline{N}) = \frac{Pn(w-G^-)}{Pn(w-G^-) + Pn(w-G^+) + (1-2P)n\left(w-G^0\right)} \quad \text{(A37)}$$

$$P_m(w,N,\underline{N}) = \frac{(1-2P)n\left(w-G^0\right)}{Pn(w-G^-) + Pn(w-G^+) + (1-2P)n\left(w-G^0\right)} \quad \text{(A38)}$$

and

$$P = \Pr(\tilde{v} > \underline{v}) \quad \text{(A39)}$$

---

[19] Eq. (A35) could be $\int_{-\infty}^{\infty}(P_u - P_d)n(w - G^0)(G^0w - (G^0)^2 + N - \underline{N})dw = 0$ rather than $\int_{-\infty}^{\infty}(P_u - Pdnw - G0G0w - G02 + Nd = 0$, if $G0 < 0$. However, Proposition 7 holds in this case as well.

For convenience, let $\bar{G} = \frac{G^+ - G^-}{2}$ and $\bar{G}^0 = G^0 - \frac{G^+ + G^-}{2}$. Then, Eqs. (A33)–(A35) can be rewritten as follows:

$$\int_{-\infty}^{\infty}\left(\frac{(1-2P)n\left(w-\bar{G}^0\right)+2Pn\left(w+\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)\times n\left(w-\bar{G}\right)\left(w-\bar{G}+\frac{N}{G^+}\right)dw=0 \tag{A40}$$

$$\int_{-\infty}^{\infty}\left(\frac{(1-2P)n\left(w-\bar{G}^0\right)+2Pn\left(w-\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)\times n\left(w+\bar{G}\right)\left(w+\bar{G}+\frac{N-\underline{N}}{G^-}\right)dw=0 \tag{A41}$$

$$\int_{-\infty}^{\infty}\left(\frac{2Pn\left(w-\bar{G}\right)-2Pn\left(w+\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)\times n\left(w-\bar{G}^0\right)\left(G^0\left(w-\bar{G}^0\right)+N\right)dw=0 \tag{A42}$$

Proof of Proposition 7.1. If $G^0 = 0$, Eq. (A42) changes to

$$\int_{-\infty}^{\infty}\left(\frac{2Pn\left(w-\bar{G}\right)-2Pn\left(w+\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)\times n\left(w-\bar{G}^0\right)dw=0 \tag{A43}$$

First, if $\bar{G}^0 \neq 0$, Eq. (A43) contradicts the result of Lemma A1. Second, if $\bar{G}^0 = 0$, G^+ is equal to $-G^-$ because $G^0 = 0$. Accordingly,

$$\int\left(\frac{(1-2P)n\left(w-\bar{G}^0\right)+2Pn\left(w+\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)\times n\left(w-\bar{G}\right)\left(w-\bar{G}\right)$$
$$=-\int\left(\frac{(1-2P)n\left(w-\bar{G}^0\right)+2Pn\left(w-\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)\times n\left(w+\bar{G}\right)\left(w+\bar{G}\right)$$

$$\int\left(\frac{(1-2P)n\left(w-\bar{G}^0\right)+2Pn\left(w+\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)n\left(w-\bar{G}\right)$$
$$=\int\left(\frac{(1-2P)n\left(w-\bar{G}^0\right)+2Pn\left(w-\bar{G}\right)}{Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)}\right)n\left(w+\bar{G}\right)$$

Therefore, Eqs. (A40) and (A41) imply $\frac{N}{G^+} = -\frac{N-\underline{N}}{G^-}$. This contradicts $G^+ = -G^-$ when $\underline{N} \neq 0$. □

Proposition 7.2 is verified by the proof of Proposition 7.3.

Proof of Proposition 7.3. From Lemmas A1 and A2 and Eq. (A42), $\bar{G}^0 G^0 < 0$ when $\underline{N} \neq 0$. Accordingly, Proposition 7.3 holds because $\bar{G}^0 = G^0 - \frac{G^+ + G^-}{2}$. □

Proof of Proposition 7.4. The nth moments of $\tilde{p}$ have the following form:

$$E\left[\tilde{p}(\tilde{w},N,\underline{N})^n\right] = p^{*n}\int_{-\infty}^{\infty}\frac{\left(Pn\left(w-\bar{G}\right)-Pn\left(w+\bar{G}\right)\right)^n}{\left(Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)\right)^{n-1}}dw \tag{A44}$$

Therefore, first $E[\tilde{p}(\tilde{w},N,\underline{N})] = 0$ holds. In addition, if $w > 0$ and $\bar{G}^0 < 0$, then

$$\frac{\left(Pn\left(w-\bar{G}\right)-Pn\left(w+\bar{G}\right)\right)^3}{\left(Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)\right)^2}$$
$$>\frac{\left(Pn\left(w-\bar{G}\right)-Pn\left(w+\bar{G}\right)\right)^3}{\left(Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w+\bar{G}^0\right)\right)^2}$$
$$=\frac{-\left(Pn\left(-w-\bar{G}\right)-Pn\left(-w+\bar{G}\right)\right)^3}{\left(Pn\left(-w+\bar{G}\right)+Pn\left(-w-\bar{G}\right)+(1-2P)n\left(-w-\bar{G}^0\right)\right)^2}$$

Therefore, from Proposition 7.3, when $G^0 > 0$,

$$E\left[(\tilde{p}-E[\tilde{p}])^3\right] = p^{*3}\int_{-\infty}^{\infty}\frac{\left(Pn\left(w-\bar{G}\right)-Pn\left(w+\bar{G}\right)\right)^3}{\left(Pn\left(w+\bar{G}\right)+Pn\left(w-\bar{G}\right)+(1-2P)n\left(w-\bar{G}^0\right)\right)^2}dw > 0$$

In the case of $G^0 < 0$, $E[(\tilde{p}-E[\tilde{p}])^3] < 0$ can be proven similarly. □

Proof of Proposition 8. From Proposition 3, we have the following relation:

$$z\underline{\pi}(z,z) = \frac{z}{2}\pi(z,z) = \frac{z}{2}\pi\left(\frac{z}{2},0\right) \geq z\pi(z,0) \tag{A45}$$

This yields:

$$\underline{\pi}\left(\bar{N}^*(\infty,c),\bar{N}^*(\infty,c)\right) \geq \pi\left(\bar{N}^*(\infty,c),0\right) \geq c \tag{A46}$$

The second equality is from the definition of the function $\bar{N}^*(\bar{N},c)$. In addition, because $\underline{N}^*(0,c) = \max\{x : \underline{\pi}(x,x) \geq c\}$ and $\underline{\pi}(x,x)$ is decreasing function of x, we have

$$\underline{N}^*(0,c) \geq \bar{N}^*(\infty,c) \tag{A47}$$

□

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