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Pricing in Queues Without Demand Information

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We consider revenue optimization in an $M/M/1$ queue with price and delay sensitive customers, and we study the performance of demand-independent pricing that does not require any arrival rate information. We formally characterize the optimal demand-independent price and its performance relative to pricing with precise arrival rate knowledge. We find that demand-independent pricing can perform remarkably well and its performance improves as customers become more delay sensitive. In particular, for uniformly distributed customer valuations, under a large set of parameters, we find that demand-independent prices can capture more than 99% of the optimal revenue. We also study social optimization and find that demand-independent pricing can perform quite well; however, the performance is better under revenue optimization.

Keywords: static pricing; dynamic pricing; revenue optimization; robust optimization
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1. Introduction

Customers seeking service typically must endure waits in addition to paying for the service. Consequently, when deciding whether to queue up, customers weigh the value they would gain from the service against the total cost incurred, which includes the monetary cost and the nonmonetary waiting cost. Because the set of customers who seek the service typically varies over time, ideally one expects the service provider to indulge in demand contingent pricing so that when demand is high, a higher price is chosen. An example of such pricing is in the context of road tolls, for instance, Route 1 from Jerusalem to Tel Aviv in Israel and Route 91 in Orange County in the United States, both have toll lanes in which the price of access varies based on traffic conditions. (The toll on Route 91 is set by the hour of the day, and an online schedule of tolls is available at http://www.91expresslanes.com/schedules.asp.)

It is obvious that such dynamic pricing schemes must generate more value, either in terms of revenue or social welfare, compared with one in which a one-for-all price is used. However, implementing dynamic prices can be difficult for several reasons, and some establishments such as amusement parks set the same price irrespective of demand. Thus, a natural question arises: How valuable is it to change prices in response to demand, or in other words, how bad can setting a single, demand-independent price be?

Our goal in this paper is to answer the above question. Clearly, the value of demand contingent pricing depends on the variability of the possible demand. So, one way to measure the performance of demand-independent prices is to consider the demand distribution that leads to the worst relative performance of such pricing. Another, more conservative, approach could be a maximin approach in which the decision maker does not use any demand information, and hence selects a price that will make the worst performance (under all possible demand values) as good as possible. In this paper, we take the latter approach and find, somewhat surprisingly, that demand-independent pricing can perform quite well.

We use a simplified model to study the value of demand-independent pricing in an environment in which the overall demand or arrival rate may change over time. In this simplified model, we consider a stationary demand process for the service firm and use an "adversarial" formulation to study the performance of setting a single price without using any knowledge of the arrival rate, with that of a single price set based on this knowledge. Here, the adversary selects the parameters of the stationary demand process and the manager solves a corresponding optimization problem in anticipation. The stationarity of the demand process allows us to use underlying steady-state analysis, and in this sense allows tractability. This stationarity assumption is in the spirit of point-wise stationary approximations (see Green and Kolesar 1991, Whitt 1991) and if changes in the demand process occur at a slow pace, then this would indeed provide a good approximation to the true system dynamics.

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The formal model is as follows. Our service firm operates as an $M/M/1$ queue (i.e., Poisson arrival process, exponential service distribution, a single server, and first-come, first-served scheduling policy). Customers have heterogeneous service valuations that are modeled as independent and identically distributed (i.i.d.) random variables; the corresponding distribution is assumed to be known to the system manager. The customers know their valuation but do not observe the queue length, and decide whether to join or not only based on their assessment of the expected wait times. Further, once customers make their decision to join the queue, it is irrevocable. Such queues are called *unobservable queues* (see, e.g., Hassin and Haviv 2003). This customer model has been used extensively in the literature and dates back to Littlechild (1974) and Mendelson (1985); some recent papers that use this model are Cachon and Feldman (2011) and Kumar and Randhawa (2010). Focusing on the firm's revenue maximization problem, we study the value of demand-independent pricing, in which the price is set without using *any* knowledge of the potential arrival rate, with respect to demand-dependent pricing, in which the price is set with precise knowledge of the potential arrival rate.

The main contributions of this paper are as follows. We explicitly compute a worst-case performance bound for demand-independent pricing relative to demand-dependent pricing and characterize the demand-independent price that ensures a performance that equals or exceeds this lower bound, irrespective of the demand. This analysis is performed by proving that the worst-case performance is realized in one of two extreme scenarios, one with demand approaching zero and another with demand approaching the highest value possible. We show that both these extreme scenarios lend to a fairly simple analysis and so provide a tractable characterization of the worst-case bound and the corresponding demand-independent price that achieves this bound. Using these analytically derived performance bounds, we find that demand-independent pricing can perform surprisingly well. For uniformly distributed customer valuations, under a large set of parameters, demand-independent pricing can capture more than 99% of the optimal revenue and more than 85% of the optimal social welfare. We also find that the performance of demand-independent pricing improves as customers become more delay sensitive.

In summary, the performance bounds derived in this paper provide a baseline to evaluate prices that do not change in response to demand in a queueing system. Thus, the results of this paper imply that when customers are quite sensitive to delay, setting a single price that is based on the a priori available demand information can potentially perform well enough to the extent that there may be no need to change prices dynamically in response to demand.

2. Connections to Literature

The notion of charging a fixed price that does not change in response to demand has been investigated to some extent in the literature. In the classical setting of selling a set of goods over a finite time horizon in which the selling price can be varied over the time horizon, Gallego and van Ryzin (1994) develop explicit bounds on the performance of using a fixed or static price over the entire selling horizon, which is obtained using the mean demand, and prove that it can perform quite close to optimal dynamic pricing (in which prices change dynamically over the time horizon). In the context of Internet service provision, Paschalidis and Tsitsiklis (2000) study revenue and welfare maximization using a Markov decision process framework in which customers are only price sensitive, and prove that static pricing, which is again set based on the mean demand, can be asymptotically optimal in some limiting regimes; furthermore, using numerical experiments, they find that such static pricing rules perform quite well in general.

This issue has also been studied in settings in which prices cannot be changed over time and a single price is used for the entire time horizon. Here, a static price refers to a price set before the realization of the demand using its (forecasted) distribution, whereas a dynamic price refers to one set based on the actual demand realization. Such an approach is taken in the thesis by Scheidmann (2000), which numerically studies the value of such a pricing scheme in the context of revenue optimization in an $M/M/1$ queue. For uniformly distributed customer valuation distributions and uniformly distributed arrival rates, the author finds that static pricing performs almost as well as dynamic pricing. This work was perhaps the first to investigate the value of static pricing in systems in which customers queue for service. Recently, Cachon and Feldman (2010) find that when selling to strategic customers, the commitment offered by such a static price may help generate more revenues than a pricing scheme that depends on the realized demand.

Our work focuses on pricing without using the knowledge of some of the model primitives and thus relates to robust optimization methods that have been widely used in revenue management (see, e.g., Lim and Shanthikumar 2007, Lan et al. 2008, Ball and Queyranne 2009, Perakis and Roels 2010). In this paper, we assume that although the system manager does not use any information on the market size, he does use his knowledge about customers' valuations distributions. Farias and Van Roy (2010) make a similar assumption, but use dynamic pricing to learn the uncertain customer arrival rate. Examples of recent work on dynamic pricing that do not assume any knowledge about customer valuation distributions are Besbes and Zeevi (2009, 2011) and Eren and Maglaras (2010). Afèche

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and Ata (2013) study Bayesian dynamic pricing in an observable $M/M/1$ queue in which customer valuations are known but their patience costs are not known and are learned via the pricing policy.

We point out that we use what is referred to as a "full price" criterion because in our model waiting costs are monetized and considered as an additional price a customer needs to pay to receive service. This value is then compared with the monetary reward obtained by service. This approach seems to have initiated in Naor (1969), and is assumed throughout most of the literature dealing with decision making in queues, and in particular, is explicitly or implicitly assumed in most of the models reviewed in Hassin and Haviv (2003).

3. Framework

3.1. Model

We consider an $M/M/1$ queue in which each arriving customer places a *random* utility on her service completion. (Hereafter, the system manager will be referred to with masculine pronouns and customers with feminine pronouns.) Let $F(r)$ be the cumulative distribution function of this random utility and denote $1-F(r)$ by $\bar{F}(r)$. We further assume that this distribution has support $\mathcal{V}=[0, v]$ for some $v>0$ and is associated with a continuously differentiable density denoted by f; the corresponding random variable is denoted by V. The utility is assumed to be i.i.d. across customers and independent of all other uncertainty in the system.

Let λ denote the maximum, or potential, arrival rate of customers, and let γ denote the equilibrium, or effective, arrival rate. The expected waiting time (in queue) of a customer is denoted by $W(\gamma)=\gamma/(\mu(\mu-\gamma))$, where μ is the service rate. Suppose the manager sets a price $p\le v$ for obtaining service, then a customer joins if and only if $V\ge p+hW(\gamma)$, where h is the cost of waiting per time unit, and is also referred to as the *customer delay sensitivity*. Hence, $\bar{F}(p+hW(\gamma))$ is the probability that a customer joins the system and the effective arrival rate γ satisfies the following relationship:

$$\gamma=\lambda\bar{F}(p+hW(\gamma)), \quad \text{where } \gamma\ge 0. \tag{1}$$

This equation leads to a unique solution for the effective arrival rate, which we denote by $\gamma(\lambda, p)$.

The system manager's goal is to select the monopoly price that maximizes the firm's revenue. (We discuss social welfare maximization in §5.2.) The firm's revenue as a function of the price and potential arrival rate is given by

$$R(\lambda, p)\equiv p\gamma(\lambda, p). \tag{2}$$

Thus, the optimal demand-dependent price is given by

$$p^*(\lambda)\equiv \arg\max_{p\in\mathcal{V}} R(\lambda, p), \tag{3}$$

and the optimal objective value equals $R^*(\lambda)\equiv R(\lambda, p^*(\lambda))$. We also define the optimal demand-dependent price for the case in which the arrival rate diminishes to zero, or the optimal congestion-free price, as $p^*(0)\equiv \arg\max_{p\in\mathcal{V}} p\bar{F}(p)$.

The goal of this paper is to study the performance of demand-independent prices relative to that of demand-dependent prices. To do so, we introduce the following optimization problem:

$$\max_{p\in\mathcal{V}} \inf_{0<\lambda<\Lambda} \frac{R(\lambda, p)}{R^*(\lambda)}, \tag{4}$$

where $\Lambda>0$ represents a bound on the possible potential arrival rates. This optimization problem comprises an "adversary" who selects the "worst" arrival rate corresponding to any price chosen by the manager and the manager's objective is to select a single price that maximizes the value obtained using demand-independent pricing as a fraction of that obtained using demand-dependent pricing, or equivalently, to select a single price that minimizes his worst-case regret measured as the percentage loss by using demand-independent pricing, over all the arrival rates possible. The optimal objective value of this optimization problem is a lower bound on the performance of demand-independent pricing, and we refer to the price that solves this optimization problem as the optimal demand-independent price. We point out that the role of (4) is to establish a worst-case performance bound, and as such, the role of the optimal demand-independent price is limited because for any given problem, if the firm has additional information on arrival rates, such as their forecasted values, then these should be used in computing the best demand-independent price.

We next perform some preliminary analysis that will aid in solving (4). In particular, in §3.2.1, we derive a regularity condition that ensures that the optimal demand-dependent price is increasing in the arrival rate, which implies that the price that solves (4) lies in the set $[p^*(0), p^*(\Lambda)]$. In §3.2.2, we characterize the revenue and optimal demand-dependent price for the limiting case in which the arrival rate grows without bound, which will play an important role in the solution to (4).

3.2. Preliminary Analysis

3.2.1. Conditions for $p^*(\lambda)$ To Be Increasing. One expects the demand-dependent price to increase in the potential demand. However, it turns out that unless we impose some regularity conditions on the primitives, the price may be nonmonotone, which would complicate our analysis. To ensure the prices are

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increasing in potential demand, we make the following assumption:

Assumption 1. 1. *The density of the customer valuation distribution $f(r)$ is nonincreasing for all $r \in [p^*(0), v]$.*
2. *The customer valuation distribution has nondecreasing hazard rate, i.e., $(d/dr)(f(r)/\bar{F}(r)) \geq 0$ for all $r \in \mathcal{V}$.*

This assumption is easily seen to be satisfied by distributions that have increasing hazard rates and nonincreasing density functions, such as uniform, triangular, and exponential distributions. In the class of distributions with nonmonotone density functions, this assumption is satisfied by Weibull distributions with shape parameter between 1 and 2, and gamma distributions with shape parameter between 1 and 3.8. Although Assumption 1.1 is difficult to verify in general, for the class of unimodal (inverted U-shaped) density functions, the second portion of Proposition 1 (stated below) proves that Assumption 1.1 is in fact necessary for $p^*(\lambda)$ to be an increasing function.

The following result formally proves that Assumption 1 ensures that $p^*(\lambda)$ is well defined and increasing in λ.

Proposition 1. *If Assumption 1 holds, then $p^*(\lambda)$ is unique for each $\lambda > 0$, and increasing. In fact, if $f'(p^*(0)) > 0$, then there exists $\epsilon > 0$ such that $p^*(\lambda)$ is decreasing for $\lambda < \epsilon$. In particular, the condition $f'(p^*(0)) \leq 0$ is necessary for $p^*(\lambda)$ to be increasing in λ.*

Assumption 1, and in turn the property that $p^*(\cdot)$ is increasing, simplify our analysis considerably. In §5.1, we revisit this assumption and study some distributions that do not satisfy it. (See §1 of the electronic companion, available as supplemental material at http://dx.doi.org/10.1287/msom.2014.0479, for an example of a distribution with nonmonotone $p^*(\cdot)$.)

3.2.2. Limiting Case: $\lambda \to \infty$. We note that the stability of the system, in particular, the condition $\lambda\bar{F}(p+hW(\gamma(\lambda,p))) < \mu$, implies that $p+hW(\gamma(\lambda,p)) \to v$ as $\lambda \to \infty$. Then, using $W(\gamma(\lambda,p)) = (\gamma(\lambda,p))/(\mu(\mu-\gamma(\lambda,p)))$, it follows that

$$\lim_{\lambda\to\infty}\left(p+h\frac{\gamma(\lambda,p)}{\mu(\mu-\gamma(\lambda,p))}\right)=v,$$

and we obtain that the limiting effective arrival rate to the system is given by

$$\lim_{\lambda\to\infty}\gamma(\lambda,p)=\mu\frac{v-p}{v-p+h/\mu}=\mu\frac{v}{v+h/\mu}\bar{G}(p),$$

where $\bar{G}(p) = ((v+h/\mu)/v)((v-p)/(v-p+h/\mu))$. Observe that $\bar{G}(p)$ corresponds to a tail distribution because $\bar{G}$ is decreasing and satisfies $\bar{G}(0)=1$ and $\bar{G}(v)=0$. We also note that as h/μ decreases to zero, the distribution corresponding to $\bar{G}$ converges to a point mass at the valuation v, and that as h/μ grows without bound, this distribution converges to a uniform distribution on $[0,v]$, i.e., we have $\lim_{h\to\infty}\bar{G}(p)=1-p/v$. We next use the fact that $\bar{G}$ can be written entirely in terms of $h/(\mu v)$ and p/v to plot $\bar{G}$ for different values of $h/(\mu v)$ and p/v in Figure 1(a).

We can use $\bar{G}$ to write out the limiting revenue as

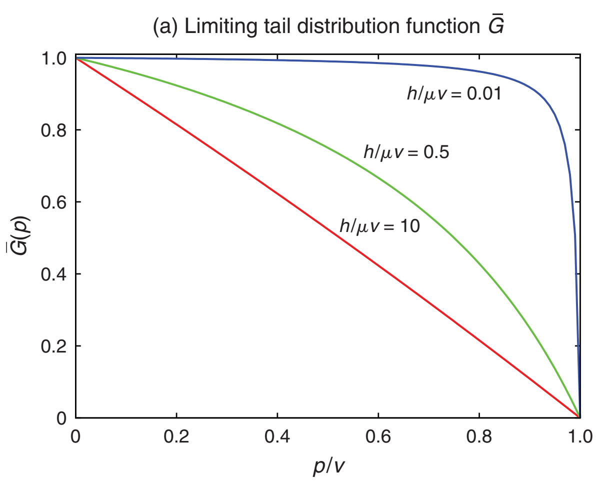
$$\lim_{\lambda\to\infty}R(\lambda,p)=p\mu\left(\frac{v-p}{v-p+h/\mu}\right)=p\bar{G}(p)\mu\frac{v}{v+h/\mu}.$$

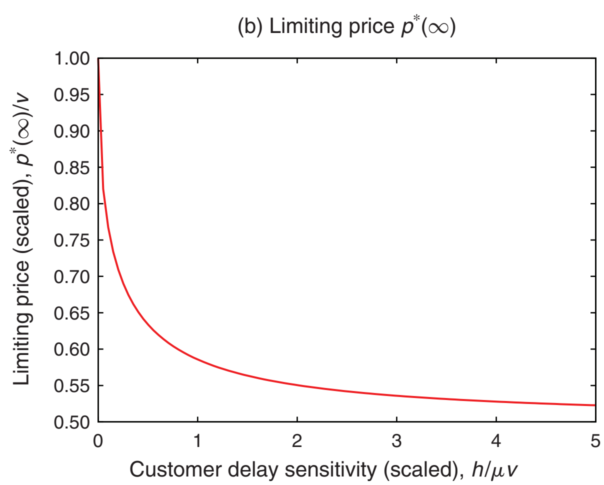
Clearly, we have

$$\arg\max_{p\in\mathcal{V}} p\bar{G}(p)=v+\frac{h}{\mu}-\sqrt{\frac{h}{\mu}\left(v+\frac{h}{\mu}\right)}.$$

This leads us to the following result.

Figure 1 **(Color online) Limiting Case of $\lambda \to \infty$**





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LEMMA 1 (OPTIMAL PRICES AS λ GROWS WITHOUT BOUND). *For any $p \in \mathcal{V}$, we have*

$$p^*(\infty) \equiv \lim_{\lambda\to\infty} p^*(\lambda) = v + \frac{h}{\mu} - \sqrt{\frac{h}{\mu}\left(v+\frac{h}{\mu}\right)} \quad \textit{and}$$
$$R^*(\infty) \equiv \lim_{\lambda\to\infty} R^*(\lambda) = \left(v + 2\frac{h}{\mu} - 2\sqrt{\frac{h}{\mu}\left(v+\frac{h}{\mu}\right)}\right)\mu. \tag{5}$$

Figure 1(b) plots the optimal limiting price $p^*(\infty)$ as a function of the customer delay sensitivity. Notice that when customers are not sensitive to delay, this price takes the value of v and as the delay sensitivity grows without bound, it converges to $v/2$, the optimal price corresponding to the limiting distribution of $\bar{G}$, i.e., the uniform distribution on $[0, v]$. It is also interesting to note that $p^*(\infty)$ decreases quite rapidly toward its asymptotic value.

4. Performance of Demand-Independent Pricing

4.1. Unbounded Potential Arrival Rate

We begin by studying the following conservative version of (4) in which $\Lambda = \infty$, i.e., the potential arrival rate is unbounded:

$$\max_{p\in\mathcal{V}} \inf_{\lambda>0} \frac{R(\lambda,p)}{R^*(\lambda)}. \tag{6}$$

We analyze this case first because it turns out to be quite simple, and a similar solution approach applies to the general case in which $\Lambda < \infty$, as well. Note that the optimal value function of (6) trivially provides us with a lower bound on the performance of demand-independent pricing for the case of bounded potential arrival rate.

4.1.1. Analysis. We begin the analysis by noting the following property of the revenue function, which simplifies (6) tremendously.

PROPOSITION 2. *If Assumption 1 holds, then $R(\lambda,p)/R^*(\lambda)$ is quasi-concave in λ for each $p \in \mathcal{V}$. This implies that we have*

$$\max_{p\in\mathcal{V}} \inf_{\lambda>0} \frac{R(\lambda,p)}{R^*(\lambda)} = \max_{p\in\mathcal{V}} \min\left\{\lim_{\lambda\to 0}\frac{R(\lambda,p)}{R^*(\lambda)}, \lim_{\lambda\to\infty}\frac{R(\lambda,p)}{R^*(\lambda)}\right\}. \tag{7}$$

This result implies that if Assumption 1 holds, then the adversary's optimization problem has a corner solution of $\lambda \to 0$ or $\lambda\to\infty$. Defining

$$Z(p) \equiv \lim_{\lambda\to 0}\frac{R(\lambda,p)}{R^*(\lambda)}$$

and

$$I(p,v) \equiv \lim_{\lambda\to\infty}\frac{R(\lambda,p)}{R^*(\lambda)}$$

as the limiting performance ratios in (7), the next result characterizes these functions and provides us with some useful properties that will aid in solving (6).

LEMMA 2 (PERFORMANCE AS λ APPROACHES 0 AND ∞). 1. *For any $p \in \mathcal{V}$, we have*

$$Z(p) = \lim_{\lambda\to 0}\frac{R(\lambda,p)/\lambda}{R^*(\lambda)/\lambda} = \frac{p\bar{F}(p)}{p^*(0)\bar{F}(p^*(0))} \quad \textit{and} \tag{8}$$

$$I(p,v) = \lim_{\lambda\to\infty}\frac{R(\lambda,p)}{R^*(\lambda)} = \frac{p((v-p)/(v-p+h/\mu))}{v+2(h/\mu)-2\sqrt{(h/\mu)(v+h/\mu)}}. \tag{9}$$

2. *If Assumption 1 holds, then $Z(p)$ is decreasing in p for $p \in (p^*(0), p^*(\infty)]$ with $Z(p^*(0)) = 1$ and $Z'(p^*(0)) = 0$.*
3. *$I(p,v)$ is increasing in p for $p \in [p^*(0), p^*(\infty))$ with $I(p^*(\infty), v) = 1$ and $(\partial/\partial p) I(p,v)|_{p=p^*(\infty)} = 0$, and $I(p,v)$ is decreasing in v.*

Proposition 2 implies that for any price $p \in \mathcal{V}$, the worst-case performance of demand-independent pricing equals either the term $Z(p)$ given in (8), which characterizes the performance as the potential arrival rate approaches zero, or the term $I(p,v)$ given in (9), which characterizes the performance as the potential arrival rate grows without bound. Because $Z(\cdot)$ is a decreasing function and $I(\cdot, v)$ is an increasing function (see parts 2 and 3 of Lemma 2), it follows that the functions $Z(p)$ and $I(p,v)$ must be equal for *exactly* one price between $p^*(0)$ and $p^*(\infty)$, and this implies that the price p, which maximizes the worst-case performance is obtained by solving

$$Z(p) = I(p,v), \quad p\in[p^*(0), p^*(\infty)]. \tag{10}$$

Substituting the expressions for Z and I and collecting all terms that depend on p on the left-hand side leads to the following equivalent relation:

$$\left(1+\frac{h}{\mu(v-p)}\right)\bar{F}(p) = \frac{p^*(0)\bar{F}(p^*(0))}{v+2(h/\mu)-2\sqrt{(h/\mu)(v+h/\mu)}}, \quad p\in[p^*(0), p^*(\infty)]. \tag{11}$$

We refer to the price that satisfies this relation as the optimal demand-independent price, $\bar{p}^*$. The following result formally characterizes the worst-case performance of demand-independent pricing:

THEOREM 1. *If Assumption 1 holds, then the worst-case performance of demand-independent pricing is given by*

$$\max_{p\in\mathcal{V}} \inf_{\lambda>0} \frac{R(\lambda,p)}{R^*(\lambda)} = I(\bar{p}^*, v). \tag{12}$$

The optimal demand-independent price $\bar{p}^*$ solves (11) and achieves the performance stated above.

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This result proves that if the system manager sets the price $\bar{p}^*$, then irrespective of the potential arrival rate, he is guaranteed, at the least, the fraction $I(\bar{p}^*, v)$ of the value of demand-dependent pricing.

4.1.2. Uniform Valuation Distributions: An Illustration. We next illustrate the performance of demand-independent pricing for the case of uniformly distributed customer valuations, i.e., $F \sim U[0, v]$. It is clear that Assumption 1 holds here. In this case, we have $p^*(0) = v/2$ and $p^*(0)\bar{F}(p^*(0)) = v/4$, which when substituted in (11) gives us the simple characterization

$$\bar{p}^* = \frac{1}{4}\left(3v + 2\frac{h}{\mu} - 2\sqrt{\frac{h}{\mu}\left(\frac{h}{\mu} + v\right)}\right).$$

Substituting this in (12), we obtain

$$\max_{p \in \mathcal{V}} \inf_{\lambda > 0} \frac{R(\lambda, p)}{R^*(\lambda)} = \frac{3}{4} + \sqrt{1+\nu}(\sqrt{\nu} - 2\nu(\sqrt{1+\nu} - \sqrt{\nu})),$$

$$\text{where } \nu \equiv \frac{h}{\mu v}.$$

Figure 2 illustrates this bound and the optimal demand-independent price for different values of $h/(\mu v)$. We note that demand-independent pricing seems to perform remarkably well. In fact, the bound exceeds 99% when the ratio $h/(\mu v)$ exceeds 1 and asymptotes to 100% when the ratio $h/(\mu v)$ grows without bound. It is only for low values of customer delay sensitivity that the performance deteriorates. The lowest value of the bound equals 75% and is achieved when the customers are delay insensitive, i.e., $h = 0$. Nevertheless, we highlight that the computed bound represents the worst performance possible. In practice, one expects to observe much better performance.

Let us next try to understand the reason behind this excellent performance. Consider any fixed potential arrival rate. Then, as the customer delay sensitivity increases, we expect the firm's revenue at any fixed price to decrease. Figure 3(a) demonstrates this for the case in which the potential arrival rate equals 2 and $v = 1$. Notice that another effect of increasing customer delay sensitivity is that the impact of a change in price on the firm's revenue reduces. That is, the revenue function becomes increasing flat as the customer delay sensitivity increases. This phenomenon by itself suggests that the relative performance of a price different from the optimal demand-dependent price should improve as the customer delay sensitivity increases.

It turns out that there is another factor in play here. Consider the optimal demand-dependent price and its variation with the potential arrival rate. If customers are not sensitive to delay, i.e., $h = 0$, and the potential arrival rate is infinite, then it would be optimal to set the price at the maximum possible valuation, v. However, as the customer delay sensitivity increases, the optimal demand-dependent price decreases. Thus, the "spread" between the optimal demand-dependent prices for the cases of zero and infinite demand is the highest when $h = 0$ and decreases as h increases (see, Figure 3(b) for an illustration). Intuitively, as this spread in optimal demand-dependent prices decreases, the performance of a carefully selected demand-independent price should improve.

The optimal demand-independent price uses both the aforementioned phenomena to its advantage and hence is able to perform very well. Note that the fact that the performance approaches 100% when the ratio $h/(\mu v)$ grows without bound follows from the discussion in §3.2.2, in particular, that the limiting function $\bar{G}$ converges to a uniform distribution, and thus $p^*(\infty)$ converges to $p^*(0)$. In fact, one would obtain this near 100% performance for large delay sensitivity for any distribution with $p^*(0) = 1/2$.

Figure 2 **(Color online) Performance of Demand-Independent Pricing Under Revenue Optimization for Customer Valuations That Are Uniformly Distributed on** $[0, v]$

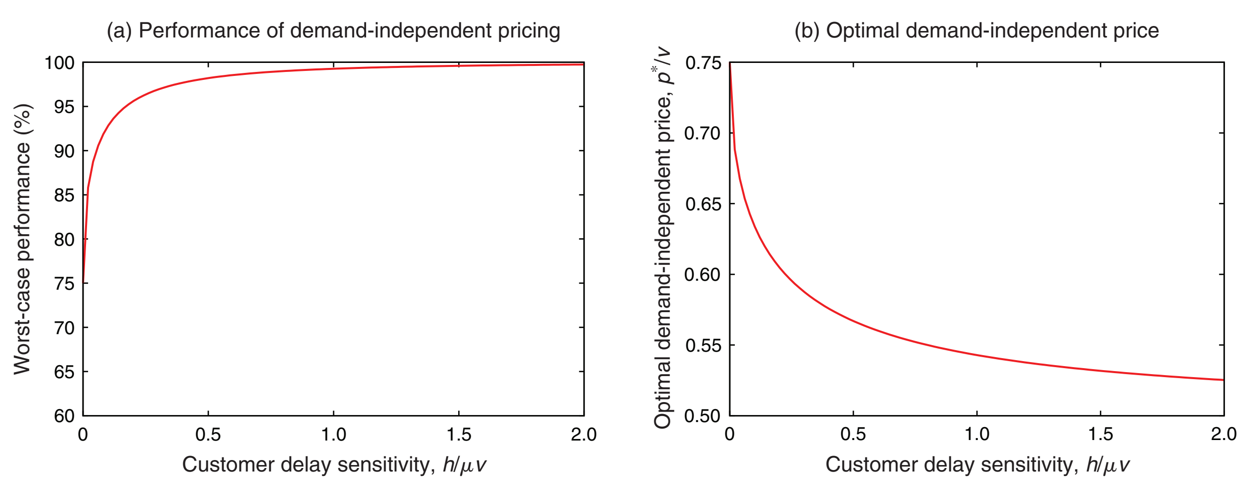

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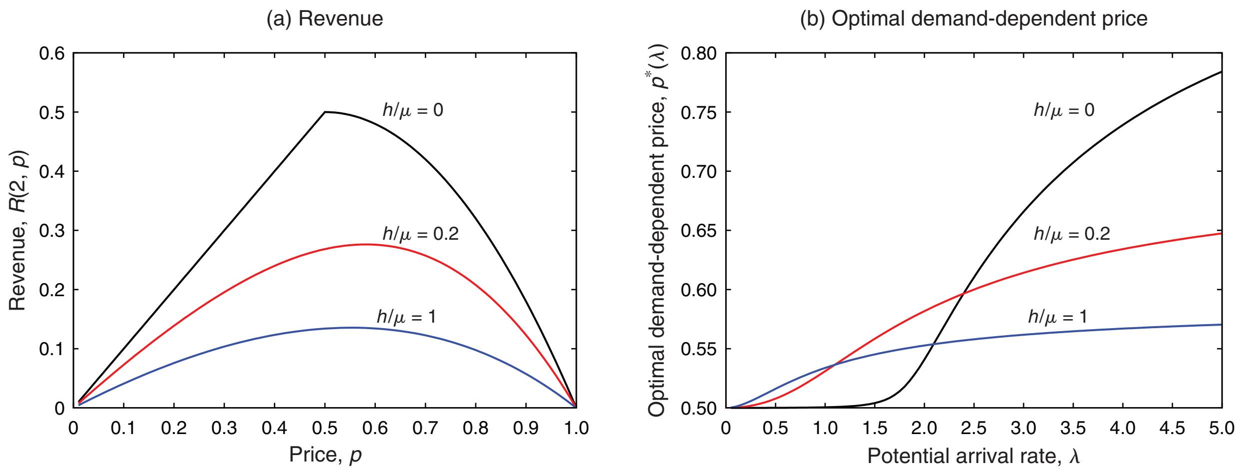
Figure 3 **(Color online) Effect of Increase in Customer Delay Sensitivity When Customer Valuations Are Uniformly Distributed on** [0, 1]



4.1.3. Performance for Other Distributions. We have seen that demand-independent pricing works extremely well for uniformly distributed customer valuations for all arrival rates. In this section, we investigate how demand-independent pricing fares for other distributions. First, consider the case of a triangular distribution with $f(r) = (2/v)(1 - r/v)$ on $[0, v]$. Noting that this distribution satisfies Assumptions 1, we can apply Theorem 1 to compute the worst-case performance bound. This bound takes the lowest value of 61.5% when $h/(\mu v) = 0$ and asymptotes to 96.6% as $h/(\mu v)$ grows without bound with performance greater than 90% for $h/(\mu v) \geq 0.5$. (Figure EC.1 in the electronic companion displays this bound as a function of delay sensitivity.) Thus, we find that demand-independent pricing performs slightly worse for customer valuations that are distributed according to this triangular distribution as compared with the uniform distribution.

Intuitively, there seems to be two reasons behind this deterioration in performance for the triangular distribution. The first reason is as follows: Notice that in computing the worst-case performance bound, we focus on the cases in which the potential arrival rate either becomes negligible or grows without bound. In the latter case, $p^*(\infty)$ depends on the distribution only though v, however, for the case in which the arrival rate is negligible, we find that $p^*(0) = 1/3$ for the triangular distribution compared with $p^*(0) = 1/2$ for the uniform distribution. Hence, the spread between the optimal demand-dependent prices for the cases of zero and infinite demand is higher in the case of the triangular distribution. The second reason pertains to the distribution function F. As the potential arrival rate grows without bound, because of the limited capacity, the valuations of the customers who actually get served approaches the maximum value possible, v. Thus, the less likely such valuations are, the less reasonable this scenario becomes, and consequently, the worse demand-independent pricing performs. Indeed, for the triangular distribution, we have $\bar{F}(r) = (1 - r/v)^2$, which is much smaller than the tail distribution function for the uniform distribution function, $(1 - r/v)$, in the vicinity of the maximum valuation v.

To formalize this intuitive reasoning, we rewrite (11), that characterizes the optimal demand-independent price, as follows:

$$\frac{\bar{F}(p)}{p^*(0)\bar{F}(p^*(0))} = \frac{((v-p)/(v-p+h/\mu))}{v + 2(h/\mu) - 2\sqrt{(h/\mu)(v+h/\mu)}}, \quad p \in [p^*(0), p^*(\infty)]. \tag{13}$$

This representation collects all terms that depend on the distribution $\bar{F}$ in the left-hand side, and consequently the right-hand-side term is identical for all distributions with support $[0, v]$. This allows us to compare the performance of demand-independent pricing across different distributions by comparing the left-hand-side term in (13) as follows:

Proposition 3. *For two customer valuation distributions F_1 and F_2 with the same support $[0, v]$ for some $v > 0$ that satisfy Assumption 1, denoting $p_i^*(\cdot)$ as the optimal demand-dependent price for distribution F_i for $i = 1, 2$, if*

$$\frac{\bar{F}_1(p)}{p_1^*(0)\bar{F}_1(p_1^*(0))} \geq \frac{\bar{F}_2(p)}{p_2^*(0)\bar{F}_2(p_2^*(0))}, \quad \text{for all } p \in [p_1^*(0), p_1^*(\infty)], \tag{14}$$

then the worst-case performance bound for demand-independent pricing is higher for distribution F_1 compared with that for distribution F_2.

This result is a modification of the first-order stochastic dominance ranking that accounts for the revenue. In particular, noting that $I(\cdot, v)$ is identical for distributions with the same support, (14) ranks the function $Z(\cdot)$ between the two distributions. It follows that

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the distribution with the higher $Z(\cdot)$ would have the better performance bound. Returning to the case of uniform and triangular distributions, it is easy to verify that the uniform distribution dominates the triangular distribution in the sense of (14), and thus Proposition 3 proves that demand-independent pricing performs better for the uniform distribution.

The insights obtained thus far suggest that if the spread in prices were quite large and customer valuations in the vicinity of the maximum value v were very unlikely, then the performance bound for demand-independent pricing could be quite poor. This is demonstrated in the following example.

Example 1. Consider exponentially distributed customer valuations with unit mean truncated to $[0, v]$. It is easy to see that for any fixed price p, $I(p, v)$ approaches zero as v increases without bound. It is only if p increases without bound with v that $I(p, v)$ can have a nonzero limit. However, then $Z(p)$ converges to zero. Thus, it follows that the worst-case performance bound for demand-independent pricing under this truncated exponential distribution approaches 0 as v grows without bound.

So, we observe that demand-independent pricing that proposes a single price for unbounded potential arrival rates can perform very well for some distributions, such as uniform and triangular, but may not work well for other distributions such as exponential distributions. This motivates the need to refine demand-independent pricing by placing a bound on the potential arrival rates possible. This topic is investigated next.

4.2. Bounded Potential Arrival Rate

4.2.1. Analysis. We now solve optimization problem (4) with $\Lambda < \infty$. If Assumption 1 holds, then the adversary's minimization problem is once again solved at extrema values of the arrival rate, namely, λ decreasing to zero or λ approaching Λ. However, because $\Lambda < \infty$, we cannot characterize the ratio $R(\Lambda, p)/R^*(\Lambda)$ exactly. Nevertheless, we construct a bound on this ratio, which is tight as $\Lambda \to \infty$. To do so, we define

$$v(\Lambda) \equiv p^*(\Lambda) - \frac{h}{2\mu} + \frac{1}{2}\sqrt{\frac{h}{\mu}\left(4p^*(\Lambda) + \frac{h}{\mu}\right)}, \tag{15}$$

where $p^*(\Lambda)$ is the optimal demand-dependent price for the arrival rate Λ as defined in (3). We then use the limiting function $I(p, v(\Lambda))$ to obtain the following bound:

Proposition 4. *If Assumption* 1 *holds, then for any* $p \in [p^*(0), p^*(\Lambda)]$, *we have*

$$\frac{R(\Lambda, p)}{R^*(\Lambda)} \geq I(p, v(\Lambda)). \tag{16}$$

Further, this bound becomes tight as Λ grows without bound, in particular, $I(p, v(\Lambda))$ decreases in a monotone fashion and converges to $I(p, v)$ as Λ increases without bound.

Intuitively, the use of the limiting function $I(p, v(\Lambda))$ in (16) is equivalent to truncating the customer valuation distribution to the set $[0, v(\Lambda)]$ and then using the limit (as the arrival rate grows without bound) of the corresponding ratio of revenues as a lower bound. The definition of the "truncation point" $v(\Lambda)$ in (15) ensures that we have

$$\begin{aligned} p^*(\Lambda) &= \arg\max_{p \in [0, v(\Lambda)]} p\left(\frac{v(\Lambda) - p}{v(\Lambda) - p + h/\mu}\right) \\ &= v(\Lambda) + \frac{h}{\mu} - \sqrt{\frac{h}{\mu}\left(v(\Lambda) + \frac{h}{\mu}\right)}, \end{aligned} \tag{17}$$

which implies that $I(p^*(\Lambda), v(\Lambda)) = 1$ so that (16) is tight for $p = p^*(\Lambda)$. Further, note that as $\Lambda \to \infty$, we have $p^*(\Lambda) \to p^*(\infty)$, and correspondingly, we can verify that $v(\Lambda) \to v$ so that we recover the analysis for the case of unbounded potential arrival rate.

Using the bound (16), we define the "best" demand-independent price in this setting $\bar{p}^*(\Lambda)$ as the solution to

$$Z(p) = I(p, v(\Lambda)), \quad p \in [p^*(0), p^*(\Lambda)], \tag{18}$$

which after substituting the expressions for Z and I yields the following equivalent relation that is analogous to (11):

$$\begin{aligned} &\left(1 + \frac{h}{\mu(v(\Lambda) - p)}\right)\bar{F}(p) \\ &\quad = \frac{p^*(0)\bar{F}(p^*(0))}{v(\Lambda) + 2(h/\mu) - 2\sqrt{(h/\mu)(v(\Lambda) + h/\mu)}}, \\ &\qquad\qquad p \in [p^*(0), p^*(\Lambda)]. \end{aligned} \tag{19}$$

Note that because we are using only a bound on the performance (see (16)) instead of the exact value, the corresponding demand-independent price is not necessarily optimal as it was previously. Nevertheless, we obtain the following bound on worst-case performance:

Theorem 2. *If Assumption* 1 *holds, then we have*

$$\max_{p \in \mathcal{V}} \inf_{0 < \lambda < \Lambda} \frac{R(\lambda, p)}{R^*(\lambda)} \geq I(\bar{p}^*(\Lambda), v(\Lambda)),$$

and the demand-independent price $\bar{p}^*(\Lambda)$, that solves (19), *satisfies this performance bound.*

As a remark, we mention that the proof of the first part of Proposition 4 (the relation (16)) allows for the customer valuation distribution to have unbounded support. Hence, Theorem 2 applies to all such distributions also, as long as they satisfy Assumption 1 on their support.

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4.2.2. Numerical Study. To see the effect of bounded potential arrival rates, we perform a numerical experiment using the three customer valuation distributions previously used: (a) exponential distribution with unit mean (untruncated), (b) $U[0,1]$ distribution, and (c) triangular distribution with $f(r) = 2(1-r)$ on $[0,1]$. In each case, we vary the upper bound on the potential arrival rate Λ to take the values 1, 3 and 10 while fixing $\mu = 1$. We also vary the customer delay sensitivity by setting $h = 0, 0.2, 1$ and 2. Table 1 displays the lower bounds for the performance of demand-independent pricing obtained using Theorem 2. We note that demand-independent pricing performs very well in all cases. For the uniform distribution, good performance was expected in view of the results for the unbounded potential arrival rates. However, surprisingly, even at the delay sensitivity of $h = 0.2$, we obtain a performance bound of 96.7% for the case $\Lambda = 10$. It is only for $h = 0$ and large values of Λ that we approach the 75% bound obtained previously.

The performance for the exponential distribution is also remarkable because we had observed in §4.1.3 that as the potential arrival rate grows without bound, the performance of demand-independent pricing for exponential distributions diminishes to zero. From the table, we observe that the performance for the exponential distribution for the case $h = 0$ is quite similar to that for the uniform distribution. Though the performance is worse at higher delay sensitivities, it is at least 79.7% for all the values considered, including the cases in which the potential arrival rate equals 10 times the service rate and zero delay sensitivity. The bound on the potential arrival rate allows us to bound the customer valuations by $v(\Lambda)$ and thus tightens the performance bound of demand-independent pricing.

As a closing point regarding Table 1, the reader may wonder whether *any* fixed price would generate similar performance. The short answer to this question is *no*, there is indeed value to computing the best demand-independent price. (See §1 of the electronic companion for further details.)

5. Discussion

5.1. The Role of Assumption 1

Assumption 1 simplified our analysis considerably. In particular, under this assumption, the adversary's minimization problem always has a corner solution. If this assumption does not hold, then we may have interior solutions to the adversary's minimization problem, which are not analytically tractable. However, we still expect the basic intuition as to why demand-independent pricing performs well to carry over to such distributions. To see whether one can still expect good performance for such distributions, we perform a numerical study for distributions that do not satisfy Assumption 1. For this study, we focused on the family of beta distributions, primarily because of the flexibility of this distribution class in modeling various different structural forms of the density function and hazard rate function. In particular, we chose the following: (a) beta(1/2, 1/2) that has U-shaped hazard rate and density functions; (b) beta(2, 1) that has increasing hazard rate and density functions; and (c) beta(2, 2) that has increasing hazard rate and inverted-U-shaped density functions, where the distribution beta(α, β) has the density function $(\Gamma(\alpha+\beta)/(\Gamma(\alpha)\Gamma(\beta)))x^{\alpha-1}(1-x)^{\beta-1}$ for $x \in [0,1]$. Table 2 displays the lower bounds for the performance of demand-independent pricing for these distributions. The results seem quite similar to those obtained for the uniform distribution and we find that demand-independent pricing performs quite well even though Assumption 1 does not hold for any of these distributions.

Table 1 **Performance for Bounded Potential Arrival Rates**

h	$\lambda \in (0,1)$	$\lambda \in (0,3)$	$\lambda \in (0,10)$
	(a) Exponential distribution		
0	100%	99.6%	79.7%
0.2	99.7%	96.1%	84.9%
1	98.4%	94.0%	85.9%
2	97.3%	92.8%	85.6%
	(b) Uniform distribution		
0	100%	93.8%	80.3%
0.2	99.9%	98.5%	96.7%
1	99.9%	99.6%	99.4%
2	99.9%	99.8%	99.8%
	(c) Triangular distribution		
0	100%	96.5%	78.2%
0.2	99.5%	96.9%	92.6%
1	98.8%	97.2%	95.5%
2	98.4%	97.3%	96.2%

Table 2 **Performance for Three Distributions That Do Not Satisfy Assumption 1**

h	$\lambda \in (0,1)$	$\lambda \in (0,3)$	$\lambda \in (0,10)$
	(a) Beta(1/2, 1/2) distribution		
0	100%	97%	85.3%
0.2	99.6%	99%	92.3%
1	99.8%	99.7%	99.5%
2	99.6%	99.5%	99.5%
	(b) Beta(2, 1) distribution		
0	100%	89%	81.3%
0.2	99.9%	98.8%	97.9%
1	99.8%	99.6%	99.5%
2	99.7%	99.6%	99.6%
	(c) Beta(2, 2) distribution		
0	100%	89.8%	77%
0.2	99.1%	97.5%	94.6%
1	99.6%	98.8%	96.8%
2	99.6%	99.4%	97.9%

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It turns out that some analysis is still possible for distributions that do not satisfy Assumption 1. In particular, if customers are not sensitive to delay, i.e., $h=0$, then the extreme cases of the potential arrival rate being negligible and infinite remain the optimizers of the adversary's minimization problem, and hence are amenable to analysis. The following result computes the performance bound for this setting.

Proposition 5. *If customers are delay insensitive, i.e., $h=0$, then for any customer valuation distribution F with support $[0,v]$, we have*

$$\max_{p\in\mathcal{V}}\inf_{\lambda>0}\frac{R(\lambda,p)}{R^*(\lambda)}=\frac{1}{v}\bar{F}^{-1}\left(\frac{p^*(0)}{v}\bar{F}(p^*(0))\right)\geq\min\left(\frac{median(F)}{v},\frac{1}{2}\right). \tag{20}$$

The case $h=0$ can be considered as that of "monopoly pricing" because in this case, customers are simply price takers and in this sense this result is quite general. For distributions that satisfy Assumption 1, the worst-case performance bound for demand-independent pricing improves with delay sensitivity (for a formal proof, see Proposition 1 in §2.4 of the electronic companion), and hence the bounds in (20) provide an absolute guarantee. Notice that the second relation (20) provides a simpler (weaker) bound that only depends on the median of the distribution, and hence is extremely easy to compute.

5.2. Demand-Independent Pricing for Social Optimization

In this section we study the value of demand-independent pricing in solving a social planner's problem of maximizing the social welfare. The social welfare equals the surplus of the joining customers ignoring the prices they pay because these cause no social losses, and is given by

$$S(\lambda,p)\equiv\lambda\int_{p+hW(\gamma(\lambda,p))}^{v}(r-hW(\gamma(\lambda,p)))f(r)\,dr. \tag{21}$$

We denote the optimal demand-dependent price as a function of the potential arrival rate λ as $p_S^*(\lambda)\equiv\arg\max_{p\in\mathcal{V}}S(\lambda,p)$ and the optimal social welfare as $S^*(\lambda)\equiv S(\lambda,p_S^*(\lambda))$. We will focus on solving

$$\max_{p\in\mathcal{V}}\inf_{\lambda>0}\frac{S(\lambda,p)}{S^*(\lambda)}. \tag{22}$$

We make the following assumption here that is similar to Assumption 1, though slightly stronger.

Assumption 2. *$f(r)/\bar{F}(r)$ is nondecreasing and $f(r)$ is nonincreasing on $[0,v]$.*

Notice that while this assumption continues to hold for the uniform, triangular, and exponential distributions, which have nondecreasing hazard rate and density functions, it no longer holds for Weibull and gamma distributions that have nonmonotone density functions. In this sense, this assumption is stronger than Assumption 1.

We now proceed with analyzing (22). The approach here is similar to that for revenue optimization. That is, we will prove that $S(\cdot,p)/S^*(\cdot)$ is quasi-concave so that we only need to consider the limiting cases $\lambda\to 0,\infty$. It turns out that the case $\lambda\to\infty$ is identical to that of revenue optimization, that is, we have $R(\lambda,p)/\lambda - S(\lambda,p)/\lambda\to 0$ as $\lambda\to\infty$. This is intuitive—as the arrival rate grows without bound, the surplus obtained by any customer joining the system must shrink to zero, otherwise the system would not remain stable, and further, because the joining rate of customers is bounded by the service rate, the aggregate surplus would also shrink to zero. However, the case $\lambda\to 0$ is somewhat different here. In particular, in this case we obtain that $p_S^*(\lambda)\to 0$ because in the absence of congestion, the social planner would set a zero price. Thus, we modify the Z function as follows:

$$Z_S(p)\equiv\lim_{\lambda\to 0}\frac{S(\lambda,p)}{S^*(\lambda,p)}=\frac{\int_p^v rf(r)\,dr}{\mathbb{E}\,V}, \tag{23}$$

where $\mathbb{E}\,V=\int_0^v rf(r)\,dr$ denotes the expected customer utility per service. The optimal demand-independent social price $\bar{p}_S^*$ is then given by the solution to

$$Z_S(p)=I(p,v),\quad p\in[0,v]. \tag{24}$$

The following result formally characterizes the worst-case performance of demand-independent pricing for social optimization:

Theorem 3. *If Assumption 2 holds, then the worst-case performance of demand-independent pricing is given by*

$$\max_{p\in\mathcal{V}}\inf_{\lambda>0}\frac{S(\lambda,p)}{S^*(\lambda)}=I(\bar{p}_S^*,v). \tag{25}$$

The optimal demand-independent price $\bar{p}_S^*$ solves (24) and achieves the performance stated above.

Figure 4 plots the performance of demand-independent pricing for uniformly distributed customer valuations for social optimization. We notice that demand-independent pricing seems to perform well, in particular, we observe that its worst-case performance varies from 61.8% when $h=0$ and asymptotes to 88.9% as h grows without bound. Clearly, its performance is worse compared with revenue optimization. One of the reasons for this is that although $p_S^*(\infty)$ is identical to that for revenue optimization, $p_S^*(0)=0$, which leads to a greater spread in the prices between the two limiting cases. The following proposition formally states this result.

Proposition 6. *If Assumption 2 holds, then the worst-case performance bound (over unbounded potential arrival*

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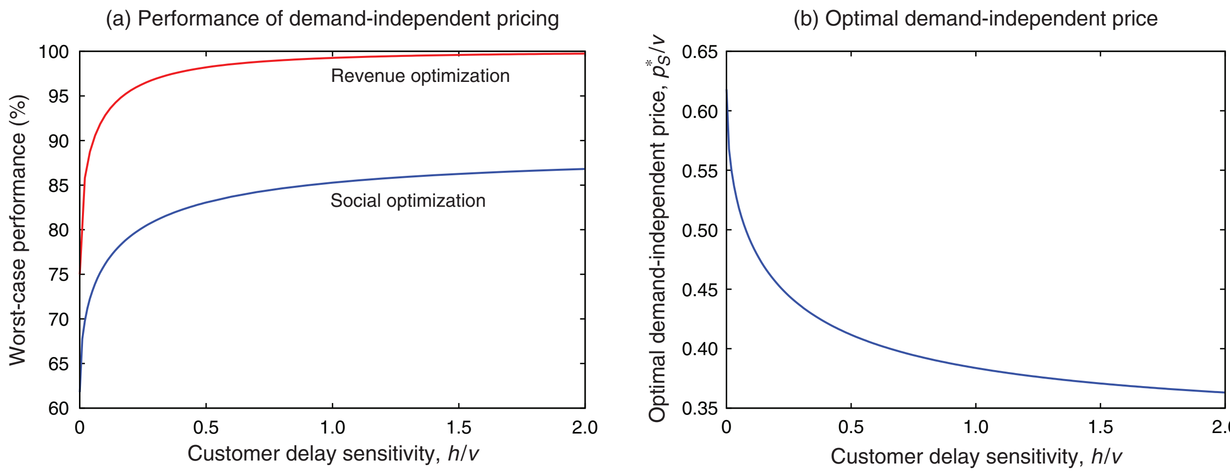
Figure 4 (Color online) Performance of Demand-Independent Pricing Under Social Optimization for Customer Valuations That Are Uniformly Distributed on $[0, v]$



rates) for demand-independent pricing is higher under revenue optimization compared with that under social optimization.

6. Conclusion

This paper shows using an adversarial framework that demand-independent pricing can perform quite well in a queueing system. The analysis in the paper uses a simplified steady-state model as a proxy for an environment in which the demand changes over time, and compares a static price with a demand-dependent (or dynamic, though slowly changing) price. This approximation ignores some potential temporal effects that may occur in such systems, for instance, time-varying prices could potentially be used to communicate demand information to the customers, who could then change their arrival patterns correspondingly. Incorporating these effects would be a worthy endeavor. Another dimension that is worth considering is the model of congestion. In this paper, congestion is modeled using an $M/M/1$ queueing model, and this sheds some light on the role of congestion in the context of demand-dependent pricing. However, it would be quite interesting to consider general congestion functions that may incorporate networks of resources.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/msom.2014.0479.

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