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Search for Information on Multiple Products

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We develop a framework for continuous search for information on a choice set of multiple alternatives and apply it to consumer search in a product market. When a consumer considers purchasing a product in a product category, the consumer can gather information sequentially on several products. At each moment, the consumer can choose which product to gather more information on and whether to stop gathering information and purchase one of the products or to exit the market with no purchase. Given costly information gathering, consumers end up not gathering complete information on all the products and need to make decisions under imperfect information. Under the assumption of constant informativeness of search, we solve for the optimal search, switch, and purchase or exit behavior in such a setting, which is characterized by an optimal consideration set and a purchase threshold structure. This paper shows that a product is only considered for search or purchase if it has a sufficiently high expected utility. Given multiple products in the consumer's consideration set, the consumer only stops searching for information and purchases a product if the difference between the expected utilities of the top two products is greater than some threshold. Comparative statics show that negative information correlation among products widens the purchase threshold, and so does an increase in the number of the choices. Under our rational consumer model, we show that choice overload can occur when consumers search or evaluate multiple alternatives before making a purchase decision. We also find that it is optimal for a monopolistic seller of multiple products to facilitate information search for low-valuation consumers and obfuscate information for those with high valuations.

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1. Introduction

When a consumer considers purchasing a product in a product category, she can gather information sequentially on several products.[1] Take the purchase of a car as an example. A consumer has some initial expected utilities for the cars in the market. She decides to start searching for information on one of the cars and keeps gaining further information. Without having complete information on that car, she might decide to switch, search for information on some other cars, and so on. At some point the consumer may decide to stop searching and purchase one of the cars or stop searching and leave the market without making any purchase. This paper investigates what is a consumer's optimal search, switch, and purchase or exit strategy. Two essential features of this problem are important to highlight: First, a consumer would never gain full information on any of the products given finite search but will have to make a purchase or exit decision with imperfect information. Second, searching for information is costly to the consumer, so she will want to limit the extent of the search. These search costs could involve the physical cost of traveling to a store, the opportunity cost of time spent searching for information, or the psychological cost of processing information.

This general problem, in addition to applying to the case of a consumer searching for information to choose one product, applies to any setting where a decision maker searches sequentially for information on multiple options. Search is costly and gradual, and any potential benefit is realized at the end of the search process. Individuals have to make this type of decision quite frequently: politicians seeking better public policies, managers choosing promising research and development projects, individuals looking for jobs, employers recruiting suitable job candidates, and firms considering alternative suppliers. In the consumer setting, the choice of almost any product or service can be seen through this perspective, from the choice of a car, to that of a house, a coat, a restaurant for dinner, telephone service, etc. Proliferation of product information on the Internet and social media has

[1] Throughout this paper, the consumer is referred to as "she."

made more visible and quantifiable the importance of modeling gradual search for information and purchase under imperfect information. While bearing in mind the generality of the problem, we take consumer search in a product category as the leading example in the presentation below.

Although the problem considered is central to choice in a market environment, it is quite underresearched when all its dimensions are included. For the simpler case where all information about an alternative could be learned in one search occasion, there is a large literature on optimal search and some of its market implications (e.g., McCall 1970, Diamond 1971, Rothschild 1974, Weitzman 1979). This literature, however, does not consider the possibility of gradual revelation of information throughout the search process. There is also some literature on gradual learning when a single alternative is considered (e.g., Roberts and Weitzman 1981, Moscarini and Smith 2001, Branco et al. 2012) and the choice there is between adopting the alternative or not. When faced with more than one alternative, as is the case considered in this paper, the problem becomes more complicated. This is because opting for one alternative in a choice set means giving up potential high payoffs from other alternatives about which the consumer has yet to learn more information. This paper can then be seen as combining these two literatures, with gradual search for information on multiple products.[2]

Another related literature is the one on the multiarmed bandit problem (e.g., Gittins 1979, Whittle 1980, Bergemann and Välimäki 1996, Bolton and Harris 1999), where a decision maker learns about different options by trying them one for each period while earning some stochastic rewards along the way. This problem has an elegant result that the optimal policy is to choose the arm with the highest Gittins index, which for each arm depends only on what the decision maker knows about that arm until then. However, the problem considered here is different from the bandit problem in one major aspect. In the case of gradual search for information considered here, a consumer optimally decides when to stop searching and make a purchase. Therefore, the decision horizon is endogenous and optimally determined by the decision maker. By contrast, multiarmed bandit problems generally presume an exogenously given decision horizon, which could be either finite or infinite. In fact, it has been shown that when a decision maker is allowed to choose the optimal stopping time, in general, the optimal policy does not include choosing the product with the highest Gittins index (Glazebrook 1979, Bergman 1981).[3] There is also a literature in computer science trying to find algorithms close to the optimal policy with multiarmed bandit problems with a limited budget (e.g., Guha and Munagala 2007, Hoffman et al. 2013), which is close to gradual search for information if the shadow price of the budget constraint is interpreted as the search cost of the consumer. Also, in experimental neuroeconomics, similar models have been proposed to study how people process information and make value-based decisions using eye-tracking data, but only heuristics have been used (e.g., Krajbich et al. 2010, Krajbich and Rangel 2011).

In this paper, we present a framework where we compute the optimal policy for a consumer searching for information across multiple products. We consider a continuous setting where information about the product being searched changes according to a Brownian motion (interpreted as gathering information on different attributes). This implies that the informativeness of search does not decrease over the extent of search. In this setting, we completely characterize the optimal policy of a consumer in the search for information in closed form, by an *optimal consideration set* and *purchase threshold structure*. Given a set of products, a consumer will not consider them all for search or purchase under the optimal policy, because search is costly. We show that a consumer will optimally construct her consideration set by a simple rule: for a product to have a positive probability of being considered for purchase and to remain in the consideration set, its expected utility has to exceed a threshold. Unlike heuristics (e.g., Hauser and Wernerfelt 1990, Feinberg and Huber 1996, Hauser 2014) or simultaneous search (e.g., Liu and Dukes 2013) in previous studies, the consideration set in our model is based on the optimal decision rule to a rational consumer model under sequential search. Given a consumer's consideration set, we further show that if the cost and informativeness of search are the same across products, a consumer always searches for information on the product with the highest expected utility. Given that there are multiple products in the consumer's consideration set, she should keep searching for information on the product until the difference in her expected utilities of the top two products is sufficiently large. This reflects the idea of a consumer continuing to search for information until one of the

[2] Fudenberg et al. (2015b) study a similar problem with a different utility function and prior beliefs, and they also assume that by paying a search cost, the expected utilities of both alternatives are updated. Another related setting is considered by Callander (2011), where the search for the best alternative from a structured continuum of alternatives is done by trial and error and where the mapping from choices to outcomes is represented as the realized path of a Brownian motion.

[3] In the appendix, we summarize the intuition on the role of the Gittins index in multiarmed bandit problems and present a counterexample where the Gittins index policy is not the optimal policy in the gradual search for information case considered here. The setting considered here also enables us to solve the optimal search problem with information updates correlated across products, which has not been possible in the multiarmed bandit problem (Gittins et al. 1989).

products clearly distinguishes itself as the best choice. This purchase threshold structure formalizes one's intuition that consumers are looking at the *relative* instead of absolute value of a product compared with the alternative options.

We also consider the case with different costs and informativeness of search across products and find that the purchase threshold structure is different. We show that a consumer should first search on the product with the highest informativeness or lowest search cost when both products have the same expected utility, and she should search on that product only when her expected utility of the alternative product is sufficiently large. By searching for information on the product with the highest informativeness of attributes, the consumer learns more information per search cost incurred.

On the basis of the optimal search policy, we compute the purchase likelihood of a product given a consumer's initial expected utilities of all products and the probability of no purchase at all. We find that a higher expected utility of one product may lead, under some conditions, to lower sales of all products combined. To understand this point, consider the case with two products. A product with a high expected utility is definitely bought if the alternative product has a low expected utility (such that the consumer does not search for information on any product). Suppose now that the expected utility of the alternative is increased. This encourages the consumer to continue searching on the two products. It is possible that positive information realizes after search, in which case the consumer can still buy at most one product. It is also possible that negative information realizes on both products, in which case no product will be bought at all. Therefore, as the expected utility of the alternative product gets higher, the total sales may decrease. Along the same logic, we find that *choice overload* can occur: more choices available may lead a consumer to search more, which might lead to lower purchase likelihood. We also find that higher information availability or a lower consumer search cost leads to lower sales of products with high expected utilities. Therefore, a seller of multiple products should obfuscate information (e.g., increase search costs) on products with high expected utilities, or for high-valuation consumers. This finding parallels recent studies on obfuscation of price information from consumer search (e.g., Gabaix and Laibson 2006, Ellison and Ellison 2009), though under a rather different setting and rationale.

Information can be correlated across products: after a consumer obtains some information on one product, she may get some partial inferences on the alternatives without searching them. We consider the case of information correlation across products and show that with positive correlation, the consumer requires a smaller difference in expected utilities of the products to choose one of the products, and a bigger difference for negative correlation. The rationale behind this result is that if information is positively correlated across products, it is more difficult to get a big difference between expected utilities across products, and a small difference can make a consumer choose to purchase one of them. Consumers get higher expected utilities with negatively correlated products as a result of a greater chance of one of the products leading to a higher expected payoff. We focus mostly on the two-product case, but we also present results for the case with more than two products. We find that more choices of products will widen a consumer's purchase threshold.

The reminder of this paper is organized as follows. In the next section we present a basic model of the two-product case, where products have the same informativeness of attributes and search costs. Section 3 presents a consumer's optimal search policy in that case, and §4 presents results on the probabilities of purchase and no purchase. Section 5 considers the case of correlated information across products, and §6 presents what happens when the informativeness of attributes or search costs are different across products. Section 7 considers the case with more than two products. In §8 we present numerical simulations of a multiproduct monopoly's pricing decisions given the consumer search behavior. In §9 we consider discounting, the possibility of a finite mass of product attributes, and the possibility of decreasing informativeness of attributes for each product. Section 10 concludes. All proofs are presented in the appendix.

2. Basic Model

2.1. Consumer Problem

A consumer gathers information sequentially on n products before making a purchase decision. Each product has many attributes that are uncertain to the consumer a priori. Consider cars, for example. Consumers obtain information, such as brand, model, safety design, fuel efficiency, warranty, and numerous other attributes before deciding which car to buy. Specifically, we model a product as a collection of T attributes. A consumer's utility of product i, U_i, is the sum of the utility derived from each attribute of the product:[4]

$$U_i = v_i + \sum_{t=1}^{T} x_{it}, \tag{1}$$

where v_i is the consumer's initial expected utility, which is known before search, and x_{it} is the utility of

[4] Fudenberg et al. (2015a) studied a similar utility function and discussed its equivalent representations.

attribute i, which is unknown before search. Without loss of generality, we assume that $\mathrm{E}[x_{it}] = 0$.[5] It is also assumed that x_{it} is independent identically distributed across attribute t for product i.

The independence assumption is based on the fact that only unexpected information changes one's belief, along the same line of Samuelson's (1965) celebrated proof that properly anticipated stock prices fluctuate randomly. The identically distributed assumption implies that information revealed per search action stays constant over time, which facilitates the analysis and allows for the search problem to be stationary when $T \to \infty$. In the real world, consumers may start with the most important attributes, and the longer a consumer spends searching for information, the less information per search she may expect to get. Simply put, a consumer may become more and more certain as she gets more and more information. We abstract from this possibility in the main model. We further discuss this issue in §6, where we consider the case that different products can differ in information per search, and in §9, where we consider numerically the case in which the informativeness of each attribute decreases as more attributes are checked. Allowing for constant informativeness of attributes permits us to focus on the situation where purchase decisions are done without full information. The search literature with one-step search (e.g., McCall 1970, Diamond 1971, Rothschild 1974) takes one extreme by assuming that a consumer learns everything by one search action, in which case the information per search is a step function decreasing to zero after one search action. This model takes the other extreme by assuming that the information gained per step of search stays constant over time, and that after each step of search the variance of what is unknown remains unchanged. This model identifies the critical effect of making purchase decisions without full information and can be seen as approximating situations where consumers have to make purchase decisions when there is substantial information about the products that is still unknown given the search costs, and the consumers make these decisions when checking product attributes that are of similar importance (potentially after the consumer having already checked the most crucial attributes).

Each time a consumer checks one attribute of product i, the consumer pays a search cost c_i, where we assume that the search costs for different products can be different but are the same across attributes for the same product. Search costs are sunk once paid.

[5] Suppose that $\mathrm{E}[x_{it}] \neq 0$; then we can redefine $x'_{it} = x_{it} - \mathrm{E}[x_{it}]$ and $v'_i = v_i + \mathrm{E}[x_{it}]$. Then we can rewrite $U_i = v'_i + x'_{1i} + \cdots + x'_{it} + \cdots$, where now $\mathrm{E}[x'_{it}] = 0$.

After checking t attributes on product i, a consumer's expected utility of the product u_i is[6]

$$u_i(t) = \mathrm{E}_t[U_i] = v_i + \sum_{s=1}^{t} x_{is} + \mathrm{E}_t\left[\sum_{s=t+1}^{T} x_{is}\right] = v_i + \sum_{s=1}^{t} x_{is}. \quad (2)$$

Given initial expected utilities v_i, search costs c_i, and distribution of x_{it} for $i = 1, \ldots, n$ and $t = 1, \ldots, T$, a consumer's optimal search problem is to dynamically decide which product to search and when to stop searching, and which product to buy or to buy none of them. To make the search problem tractable, we consider the case where each attribute is increasingly subdivided into smaller attributes, and the search cost of each smaller attribute converges to zero at the rate that attributes are subdivided, such that in the limit we have a continuous-attribute analog of the discrete-attribute model, where the information in each attribute has infinitesimal importance and the number of attributes go to infinity (for a similar formulation, see also Bolton and Harris 1999, Moscarini and Smith 2001). This enables us to get a sharp characterization of consumers' optimal search problem in closed form. As in our previous example, safety design, as a broad category, can consist of many minute attributes described in sellers' descriptions, in thousands of online customer reviews, etc. Another way of thinking of an infinitesimal attribute of a product is as a quantum of valuable information that can be discovered by a consumer by an infinitesimal search. Specifically, under the continuous-attribute formulation, a consumer's utility and conditional expected utility of product i are, respectively,

$$U_i = v_i + \int_{t=0}^{T} dB_i(t) = v_i + B_i(T), \quad (3)$$

$$u_i(t) = \mathrm{E}_t[U_i] = v_i + B_i(t), \quad (4)$$

where $B_i(t)$ is a Brownian motion with zero drift and volatility σ_i^2, where σ_i characterizes the *informativeness* of the consumer's search on product i.[7] The continuous fluctuation of a consumer's expected utility over search reflects the continuous flow of information amassed. The last assumption that we make is that the mass of attributes is infinite, $T \to \infty$, which allows the problem to be stationary. We consider the case with finite T numerically in §9.[8]

In this section, we develop a basic model of optimal search on multiple products, and we develop generalizations in §§5–9. Let us consider a consumer who

[6] The notation $\mathrm{E}_t[\cdot]$ is short for expectation conditioning on observed realized utilities $x_{i1}, \ldots, x_{it}$.

[7] Given that the x_{it} are independently distributed, by the law of large numbers, we have that the change of expected utility follows a Brownian motion. For a detailed exposition of translating a discrete-attribute model to a continuous-attribute model, see Branco et al. (2012).

[8] Alternatively, the solution that we present can be seen as the limit of the optimal solution for finite T when $T \to \infty$.

has two products under consideration for purchase (i.e., $n = 2$) but is interested in buying at most one of them. Before making a purchase decision, the consumer optimally chooses which product to search for information on over time. Let us name the two products as products 1 and 2. The two products are *homogeneous* in that they have the same informativeness of search σ, as well as the same unit search cost c. Heterogeneous products are discussed in §6.

We normalize the consumer's reservation utility without any purchase to be zero. At any point during the search process, the consumer has five choices: to search product 1, to search product 2, to purchase product 1 and leave the market, to purchase product 2 and leave the market, and to exit the market without making any purchase. She makes the decision based on her current expected utilities of the two products, u_1 and u_2.[9] It is assumed that the information updates for the two products are uncorrelated. Specifically, when a consumer searches information on product i, her expected utility of product i gets updated to $u_i + du_i$, with $du_i = dB_i(t)$, whereas her expected utility of the alternative remains unchanged. We relax the assumption to consider correlated information updates in §5. It is straightforward to show that u_1 and u_2 are sufficient statistics of the past observations; therefore, we can define $V(u_1, u_2)$ to be the consumer's maximum expected utility when she follows the optimal search policy in the future, given her current expected utilities u_1 and u_2. In the language of dynamic programming, u_1 and u_2 are state variables, and $V(u_1, u_2)$ is known as the value function. Given that there is an infinite mass of attributes to be checked, we have that $V(u_1, u_2)$ does not depend on t explicitly.

Note first that the maximum expected utility $V(u_1, u_2)$ is nondecreasing in either of the expected utilities u_1 or u_2, as expected. We state this result in the following lemma (the proof is provided in the appendix).

Lemma 1. *A consumer's maximum expected utility $V(u_1, u_2)$ is nondecreasing in her current expected utilities of the two products u_1 and u_2.*

We now consider the dynamic problem of consumer search.

2.2. Dynamics

Let us define the *search strategy* of a consumer as the mapping from her current expected utilities of the two products to her action. To determine a consumer's optimal search strategy, we need to solve her maximum expected utility $V(u_1, u_2)$ for all u_1 and u_2. We characterize $V(u_1, u_2)$ by considering the following two cases below.

In one case, if a consumer's optimal decision is to leave the market immediately, with or without a purchase, her maximum expected utility can be obtained directly as

$$V(u_1, u_2) = \max\{0, u_1, u_2\}. \tag{5}$$

If her expected utilities of both products are negative, the consumer will exit without any purchase; otherwise, she will purchase the product with higher expected utility.

Consider now the other case, in which it is optimal for the consumer to continue searching for information. Given the continuation of search, a consumer determines which product to search on by expected utility maximization and pays some search cost. Let us consider an infinitesimal search dt. A consumer's current maximum expected utility $V(u_1, u_2)$ should satisfy the following equation:

$$V(u_1, u_2) = -c\,dt + \max\{\mathrm{E}_{t_1}[V(u_1 + du_1, u_2)], \\ \mathrm{E}_{t_2}[V(u_1, u_2 + du_2)]\}, \tag{6}$$

where t_i is the mass of attributes of product i that has been already searched. The first term on the right-hand side is the search cost in time dt. The second term is the maximization between the expected utility from searching for information on product 1 and that from searching for information on product 2. Let us do a Taylor expansion of $\mathrm{E}_{t_1}[V(u_1 + du_1, u_2)]$ to get

$$\begin{aligned}\mathrm{E}_{t_1}&[V(u_1+du_1, u_2)]\\ &= \mathrm{E}_{t_1}\left[V(u_1, u_2) + V_{u_1} du_1 + \frac{1}{2} V_{u_1u_1} du_1^2 + o(du_1^2)\right]\\ &= V(u_1, u_2) + \frac{\sigma^2}{2} V_{u_1u_1} dt + o(dt),\end{aligned} \tag{7}$$

where V_{u_1} and $V_{u_1u_1}$ are the first- and second-order partial derivatives with respect to u_1, respectively, and $o(dt)$ represents the terms that converge to zero faster than dt. In writing the second equality above, we have used the fact that $\mathrm{E}_{t_1}[du_1] = \mathrm{E}_{t_1}[dB_1(t_1)] = 0$ and $\mathrm{E}_{t_1}[du_1^2] = \mathrm{E}_{t_1}[dB_1(t_1)^2] = \sigma^2 dt$, which are due to Ito's lemma. Similarly, we can do a Taylor expansion of $\mathrm{E}_{t_2}[V(u_1, u_2 + du_2)]$ and substitute into Equation (6) to obtain

$$\begin{aligned}V(u_1&, u_2)\\ &= -c\,dt + \max\left\{V(u_1, u_2) + \frac{\sigma^2}{2} V_{u_1u_1} dt,\right.\\ &\qquad \left. V(u_1, u_2) + \frac{\sigma^2}{2} V_{u_2u_2} dt\right\} + o(dt).\end{aligned} \tag{8}$$

By canceling out the same terms and dividing by dt on both sides of the equation, we obtain the following equality:

$$\max\{V_{u_1u_1}, V_{u_2u_2}\} = \frac{2c}{\sigma^2}. \tag{9}$$

The partial differential equation (PDE) (9) completely characterizes a consumer's search behavior when she

[9] We drop the argument t of $u_i(t)$ below, when there is no confusion.

is willing to continue searching for information. The consumer optimally chooses to search product 1 if and only if

$$V_{u_1u_1} = \frac{2c}{\sigma^2} \geq V_{u_2u_2}, \tag{10}$$

and similarly for product 2. This optimality condition shows that a consumer optimally chooses which product to search on based on the curvature instead of the slope of her value function. This reflects the essence of information seeking: positive and negative information can occur with equal odds, and therefore one should focus on the second-order derivative.

Equation (9) determines $V(u_1, u_2)$ when it is optimal for a consumer to continue searching for information; Equation (5) determines $V(u_1, u_2)$ when it is optimal for a consumer to stop searching. Now we need to determine a boundary that separates the two regimes. Within the boundary, it is optimal for a consumer to continue searching, with $V(u_1, u_2)$ determined by Equation (9). Beyond the boundary, it is optimal for the consumer to stop searching for information and exit the market with or without a purchase, where $V(u_1, u_2)$ is given by Equation (5).

2.3. Boundary Conditions

Intuitively, when a consumer's expected utility of product i is rather high, she will stop searching for information and purchase product i immediately. This is the upper boundary separating searching and purchasing. On the other hand, when a consumer's expected utilities of both products are rather low, she will stop searching for information and exit the market without any purchase. This is the lower boundary condition separating searching and exiting. Bearing these ideas in mind, we can construct the boundary conditions.

Let us define $\overline{U}_i(u_j)$ as the *purchase boundary* for product i given the expected utility u_j for product j. Given u_j, when u_i is so high that it reaches $\overline{U}_i(u_j)$, the consumer will be indifferent between continuing searching for information and stopping to purchase product i. Correspondingly, we have the following *value-matching* condition at the purchase boundary:

$$V(u_1, u_2)|_{u_i=\overline{U}_i(u_j)} = \overline{U}_i(u_j), \quad i, j = 1, 2, i \neq j. \tag{11}$$

The left-hand side is the utility a consumer expects if she continues searching for information, whereas the right-hand side is the expected utility a consumer can obtain right away by purchasing product i. The following lemma formalizes our intuition that as a consumer's expected utility of the alternative gets higher, the product under search must provide a correspondingly higher expected utility to incentivize the consumer to stop searching and purchase the product.

Lemma 2. *The purchase boundary of product i, $\overline{U}_i(u_j)$, is nondecreasing in a consumer's expected utility of its alternative, u_j.*

Equation (11) can be treated as the definition of the purchase boundary $\overline{U}_i(\cdot)$ but, per se, does not suffice to determine the locus of the boundary. The missing element is the *smooth-pasting* condition (e.g., Dixit 1993, p. 30). We make a technical assumption that $\overline{U}_i(\cdot)$ is continuous and piecewise differentiable. The smooth-pasting condition at the boundary of $u_i = \overline{U}_i(u_j)$ is then

$$V_{u_k}(u_1, u_2)|_{u_i=\overline{U}_i(u_j)} = \begin{cases} 1 & \text{if } k = I, \\ 0 & \text{if } k \neq I, \end{cases} \quad k = 1, 2;\ i \neq j = 1, 2. \tag{12}$$

The value-matching condition can be thought of as a zero-order condition, and smooth pasting can be seen as the first-order condition across the boundary. The appendix provides further intuition on the smooth-pasting conditions. Equations (11) and (12) together constitute the complete set of conditions to determine the upper boundary $\overline{U}_i(u_j)$.

Now let us turn our attention to the lower boundary conditions. Let us define $\underline{U}_i(u)$ as the *exit boundary* for product i. Given u_j, when u_i is so low that it touches $\underline{U}_i(u_j)$, the consumer will be indifferent between continuing searching and exiting the market with or without a purchase. Correspondingly, we have the following value-matching condition at the lower boundary of $u_i = \underline{U}_i(u_j)$:

$$V(u_1, u_2)|_{u_i=\underline{U}_i(u_j)} = \max\{0, u_j\}, \quad i \neq j = 1, 2. \tag{13}$$

Similarly, we also need the following smooth-pasting conditions at the lower boundary:

$$V_{u_k}(u_1, u_2)|_{u_i=\underline{U}_i(u_j)} = 0, \quad k = 1, 2;\ i \neq j = 1, 2. \tag{14}$$

Equations (13) and (14) together constitute the complete set of conditions to determine the exit boundary $\underline{U}_i(u)$.

Since the two products have the same search costs and informativeness of search, they are symmetric in the search strategy space. Therefore, the purchase and exit boundaries should be the same for the two products, which are denoted as $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$, respectively, in the discussion that follows.[10]

This completes the mathematical formulation of a consumer's optimal search problem. If a consumer's optimal decision is to stop searching and make a purchase decision, her maximum expected utility $V(u_1, u_2)$ is given by Equation (5). If a consumer's optimal decision is to continue searching for information, her maximum expected utility $V(u_1, u_2)$ can be solved by combining Equation (9) with boundary conditions (11)–(14). Correspondingly, the optimal

[10] When the two products have different search costs and informativeness of search, purchase and exit boundaries differ for different products. We analyze this case with heterogeneous products in §6.

search strategy can then be inferred from $V(u_1, u_2)$ by Equations (5) and (10).

Technically, solving Equation (9) under boundary conditions (11)–(14) is not as straightforward as solving a boundary value problem of a PDE because of the following two complexities: (1) although Equation (9) appears to be a common parabolic PDE, there is a maximization operator in the equation, and (2) the purchase and exit boundaries are not given. A consumer needs to decide not only which product to search, which is characterized by the PDE, but also when to stop searching and make a purchase decision, which is characterized by the boundaries. We must solve the PDE and determine the boundaries simultaneously. This is a so-called *problem with ambiguous boundary conditions* (see Peskir and Shiryaev 2006). We present an analytical solution to the problem in the next section.

3. Optimal Search for Information

In this section we solve the problem of optimal search on two products analytically and characterize the comparative statics. Let us define $a \equiv \sigma^2/(4c)$, which serves as a natural scale for a consumer's expected utilities of the two products.[11] Let us also introduce the product logarithm function (also known as the Lambert W function): $W(z)$ defined as the upper branch of the inverse function of $z(W) = We^W$. The following theorem presents the solution, with the proof in the appendix.

Theorem 1. *There exists a unique solution $V(u_1, u_2)$ along with boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$, which satisfies Equations (5), (9), and (11)–(14). The value function is obtained as*

$$V(u_1,u_2)=\begin{cases}\dfrac{1}{4a}[\overline{U}(u_2)-u_1]^2+u_1 \\ \qquad \text{if } u_2\le u_1\le \overline{U}(u_2) \text{ and } u_1\ge \underline{U}(u_2), \\ \dfrac{1}{4a}[\overline{U}(u_1)-u_2]^2+u_2 \\ \qquad \text{if } u_1\le u_2\le \overline{U}(u_1) \text{ and } u_2\ge \underline{U}(u_1), \\ u_1 \quad \text{if } u_1>\overline{U}(u_2), \\ u_2 \quad \text{if } u_2>\overline{U}(u_1), \\ 0 \quad \text{otherwise},\end{cases} \tag{15}$$

and the purchase and exit boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$ are given as

$$\overline{U}(u)=\begin{cases} u+[1+W(e^{-(2u/a+1)})]a & \text{if } u\ge -a, \\ a & \text{otherwise;}\end{cases} \tag{16}$$

$$\underline{U}(u) = -a \quad (\text{relevant when } u \le -a). \tag{17}$$

[11] The term $\sigma^2/(4c)$ is the optimal purchase boundary in the single product case (Branco et al. 2012).

Note that the value function takes different forms in different regions. It actually belongs to the class of the so-called *viscosity solution*, a generalization of the classical concept of a solution to the PDE, to allow for discontinuities and singularities (see Crandall et al. 1992). The value function is quadratic in u_i and $\overline{U}(u_j)$ in each region for $i \neq j \in \{1, 2\}$. Note also that the value function, as well as the boundary conditions, is highly nonlinear, expressed in terms of product logarithm functions. Figure 1 presents the value function $V(u_1, u_2)$ as well as the payoff from search, which is defined by $V(u_1, u_2) - \max\{u_1, u_2, 0\}$, i.e., the difference between the maximum expected utility when search is allowed and that when search is not allowed.

We first note that the payoff from search is always nonnegative. Although information is ex ante neutral, search indeed benefits consumers, because consumers have the option to learn the products first before committing to buy a potentially poor fit. Similar to a stock covered by its put option, search provides an upside possibility while protecting consumers from a downside risk. We also find that the payoff from search peaks at $u_1 = u_2 = 0$, which is the point where a consumer's three options—purchase 1, purchase 2, and exit without purchase—are most undistinguished. A consumer benefits most from search when she is most uncertain about which option to take without search. It is not hard to show that

$$\lim_{u\to\infty} V(u, u) - u = \frac{a}{4}. \tag{18}$$

It implies that a consumer can always benefit from search no matter how high her current expected utilities are, as long as the two alternatives are not easily distinguished from each other.

Given a consumer's maximum expected utility $V(u_1, u_2)$, a consumer's optimal search strategy can be correspondingly determined, as presented in Figure 2. As delimited by solid lines, a consumer's expected utility space is segmented into five regions, corresponding to her optimal choice of five actions given her expected utilities of the two products.

As shown by Figure 2, roughly speaking, when u_1 is significantly greater than u_2, a consumer will purchase product 1 immediately and leave the market without any search; when u_1 is slightly greater than u_2, a consumer will search for more information on product 1 so as to distinguish between the two products; and when u_1 and u_2 are both very low, a consumer will leave the market without any purchase. The following theorem completely characterizes a consumer's optimal search strategy rigorously.

Theorem 2. *Suppose that both products have the same cost and informativeness of search. Then, only products with expected utilities above $-a$ constitute a consumer's consideration set for search and purchase. Given two products*

Figure 1 (Color online) Maximum Expected Utility (Left) and Payoff From Search (Right) Given a Consumer's Current Expected Utilities

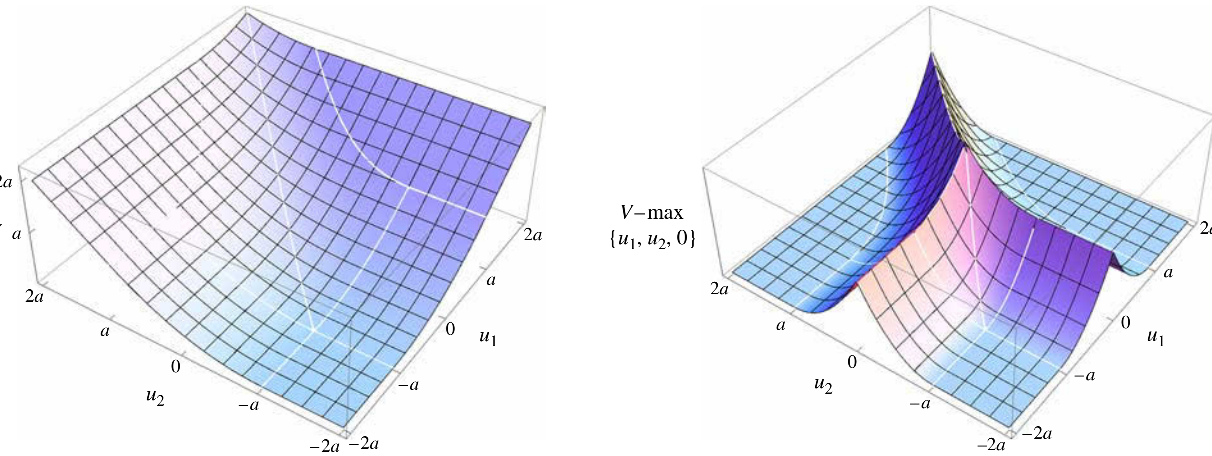
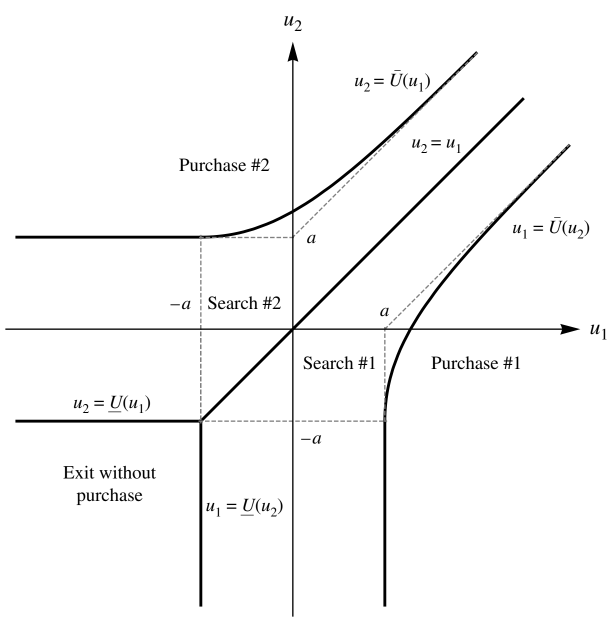


Figure 2 Optimal Search Strategy on Two Products



in her consideration set, the consumer always searches for information on the one with higher expected utility. She stops searching and purchases the product if the difference in her expected utilities of the two products is above the purchase threshold of $[1+W(e^{-(2u/a+1)})]a$, *where* u *is her expected utility of the alternative.*[12]

Throughout this paper, when talking about "purchase threshold," we always mean the threshold imposed on the difference between the expected utilities of the two products. Note that a consumer's purchase threshold narrows as her expected utility of the alternative u increases, and it converges to a relatively quickly. Therefore, a consumer with high expected utilities stops searching and purchases the product if her expected utility of the product exceeds that of the alternative by a. To summarize, we have the following corollary.

Corollary 1. *The purchase threshold on the expected utility difference between the two products decreases as the expected utility of the alternative product increases, and it converges to* a.

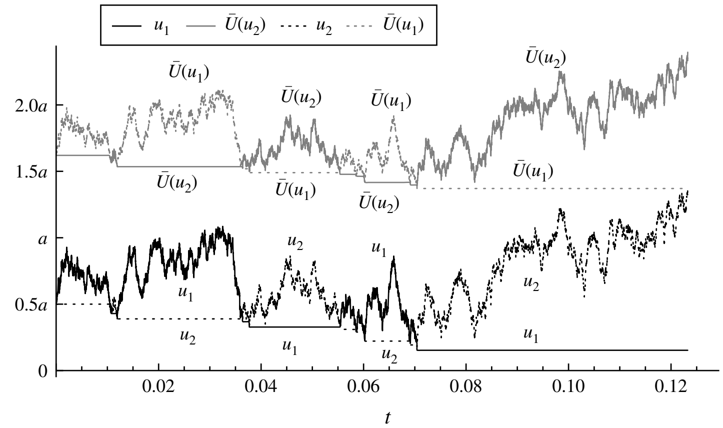
Given a consumer's optimal search strategy, Figure 3 presents a simulation example of a consumer's dynamic search process. The consumer's initial expected utilities are $(0.5a, 0.5a)$. She starts by searching on product 1, then switches to search on product 2 shortly afterward, and then switches back and forth several times before she finally decides to purchase product 2. The left panel in Figure 3 records the evolution of her expected utilities $u_1(t)$, $u_2(t)$, as well as her purchase boundaries $\overline{U}(u_2(t))$ and $\overline{U}(u_1(t))$ over time. It shows that when the consumer searches on one product, her expected utility of this product follows a Brownian motion, and her expected utility of the alternative stays constant. The right panel shows the trajectory of her expected utilities in the utility space.

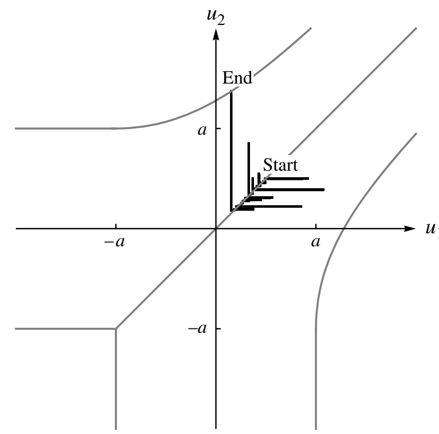
The comparative statics are summarized in Proposition 1. We defer the proofs to §5, where we prove the proposition under a more general model setting.

Proposition 1. *Given a consumer's expected utility of the alternative product as* u, *her purchase threshold of the product increases in* a, *i.e., increases in the informativeness of search* σ *and decreases in the search costs* c. *Given a consumer's expected utilities of the two products as* u_1 *and* u_2, *her maximum expected utility* $V(u_1, u_2)$ *increases in* a, *i.e., increases in the informativeness of search* σ *and decreases in the search costs* c. *As* a *goes to infinity,* $V(u_1, u_2)$ *goes to infinity; as* a *goes to zero,* $V(u_1, u_2)$ *converges to* $\max\{u_1, u_2, 0\}$.

[12] If there is only one product in the consumer's consideration set, one can obtain from Branco et al. (2012) that the consumer stops searching for information and purchases the product when u hits a and stops searching for information and exits the market when u hits $-a$.

Figure 3 An Example of a Consumer's Optimal Search Process





As search costs decrease, or informativeness of search increases, the purchase threshold gets higher, and consequently, a consumer searches more and correspondingly gets more benefit from information. Finally, note that the solution presented, and correspondingly, our basic model, is extremely parsimonious in parameterization, with essentially only one parameter, a, given the complexity of the problem.

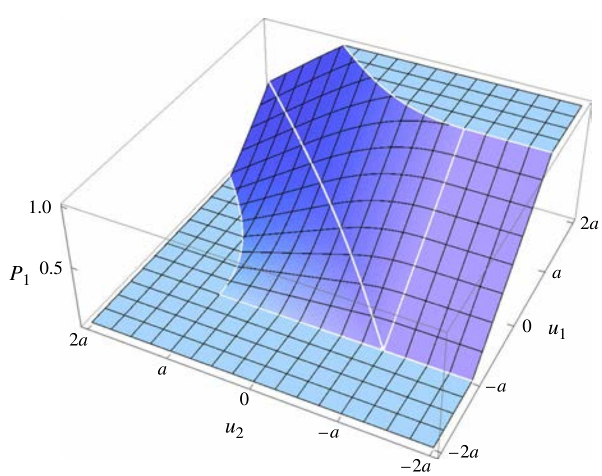
4. Purchase Likelihood

Given a consumer's optimal search strategy, we can infer her purchase likelihood of each product, starting from any initial state (u_1, u_2). Let us define the purchase likelihood of product i as $P_i(u_1, u_2)$. Then, according to symmetry, the purchase likelihood of product 2 starting from (u_1, u_2) would be $P_2(u_2, u_1) = P_1(u_1, u_2)$. The function $P_1(u_1, u_2)$ can be calculated by invoking the optional stopping theorem (see Williams 1991, p. 100) and solving an ordinary differential equation (see details in the appendix):

$$P_1(u_1,u_2)=\begin{cases}0 \quad \text{if } u_1\le -a \text{ or } u_2\ge \overline{U}(u_1),\\ 1-\dfrac{\overline{U}(u_2)-u_1}{2a}\\ \quad \text{if } u_2\le u_1<\overline{U}(u_2) \text{ and } u_1>-a,\\ \dfrac{\overline{U}(u_1)-u_2}{\overline{U}(u_1)-u_1}-\dfrac{\overline{U}(u_1)-u_2}{2a}\\ \quad \text{if } -a<u_1<u_2<\overline{U}(u_1),\\ 1 \quad \text{if } u_1\ge \overline{U}(u_2).\end{cases} \tag{19}$$

The left panel in Figure 4 presents an illustration of $P_1(u_1, u_2)$. From the figure, we can see the intuitive result (proof is straightforward and thus omitted) that a consumer is more likely to buy one product if her expected utility of the product is higher or her expected utility of the alternative is lower.

Figure 4 (Color online) Purchase Likelihood of Product 1 (Left) and of Either Product (Right)



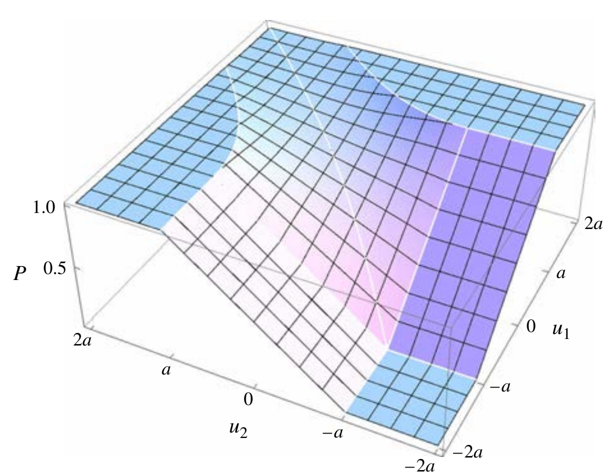
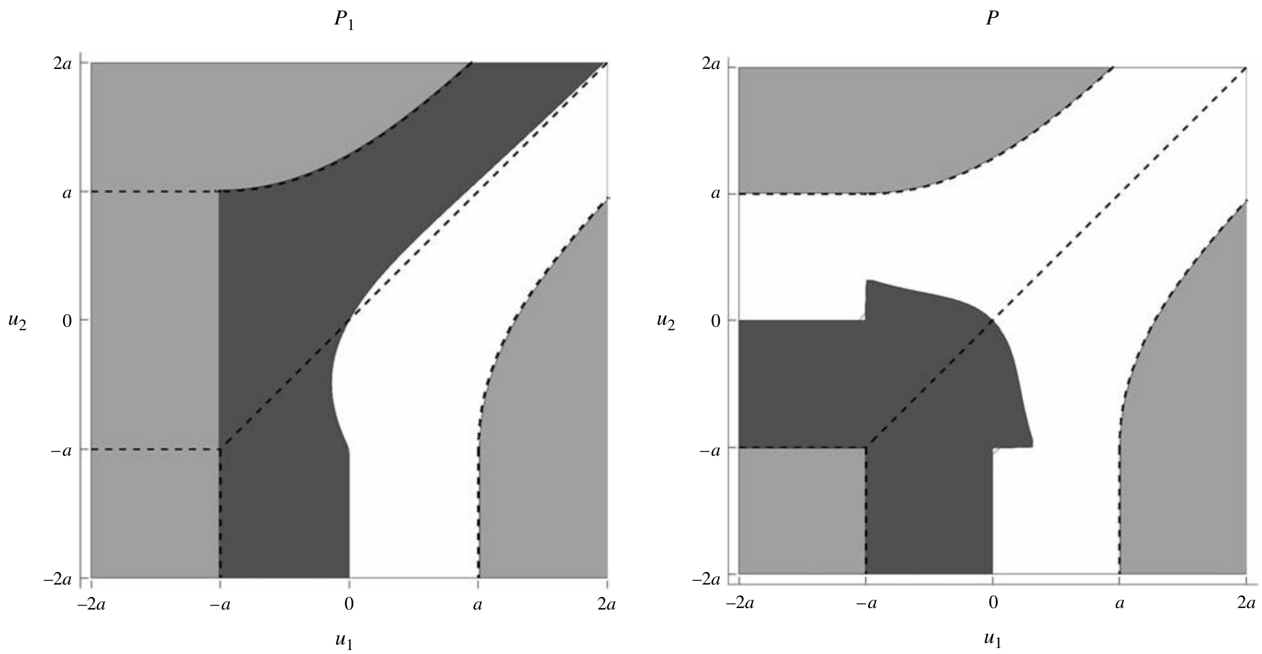


Figure 5 **Comparative Statics of Purchase Likelihoods $P_1(u_1, u_2)$ and $P(u_1, u_2)$**



Let us define the purchase likelihood of either product to be $P(u_1, u_2) \equiv P_1(u_1, u_2) + P_2(u_1, u_2)$. The right panel in Figure 4 presents an illustration of $P(u_1, u_2)$. It is interesting to note that $P(u_1, u_2)$ does not always increase with u_1 or u_2. This means that a higher expected utility of one product may lead to a lower purchase likelihood of the two products combined. This will never happen in a classical setup without considering consumer search behavior. To understand the intuition, let us consider a special case. Given a consumer's expected utilities of the two products as u_1 and u_2, if u_2 is high enough such that the difference between u_2 and u_1 is greater than the purchase threshold, the consumer will purchase product 2 immediately. In this case, the purchase likelihood is 1. Now suppose that for some reason (for example, promotions), the seller increases the consumer's expected utility of product 1. As a result, the difference between u_2 and u_1 is now below the purchase threshold. In this case, the consumer will optimally search for more information before making a purchase decision. After search, it is possible that the consumer will like the products more, in which case, she will buy at most one of them; it is also possible that the consumer will get some negative information on both products and decide to buy nothing. In general, the purchase likelihood will be lower than one after the increase of u_1. At an aggregated level, a higher expected utility of 1 product might decrease the total sales. By the same argument, we can show that more available alternatives may decrease the purchase likelihood.[13] In this way, we provide a rational explanation to consumer *choice overload*,[14] under the circumstance that the consumer engages in gradual evaluations or information search before making a choice. More options to choose from may lead a consumer to exert a greater effort level to distinguish the best from the rest and result in a lower probability of choosing anything. It is also noteworthy that more alternatives will never decrease a consumer's ex ante welfare in this case, because a consumer can simply ignore the added alternatives; however, it is possible that more alternatives decrease a consumer's ex post welfare.[15]

It is also interesting to study the comparative statics of the consumers' purchase likelihood. We describe the results in Figure 5, which characterizes how search costs and informativeness of search influence a consumer's purchase likelihoods (see proofs in the appendix). The left panel plots the sign of $\partial P_1(u_1, u_2)/\partial a$ as a function of u_1 and u_2, and the right panel plots the sign of $\partial P(u_1, u_2)/\partial a$. Grayness indicates the sign: if the sign is positive, it is dark gray; if the sign is zero, it is light gray; and if the sign is negative, it is white. Thus, the purchase likelihood increases with informativeness and decreases with search costs in the dark gray area; it decreases with informativeness and increases with search costs in the white area; it stays constant in the light gray area. The

[13] Introduction of a new product can be equivalently viewed as increasing its expected utility from negative infinity to some positive level.

[14] For lab and field experiments on choice overload, see, e.g., a meta-analytic review by Scheibehenne et al. (2010). See also Kuksov and Villas-Boas (2010) for an alternative explanation of choice overload.

[15] Different from our setting, Fudenberg and Strzalecki (2015) consider a dynamic logit model with choice aversion where a consumer may prefer to have a smaller choice set ex ante.

dashed lines in both plots replicate the boundaries of the optimal search strategy shown in Figure 2.

Figure 5 can lead to the following observations.[16] First, when a consumer's expected utilities of the two products are positive, her purchase likelihood of the product with high (low) expected utility decreases (increases) when informativeness of search increases or search costs decrease. Otherwise, her purchase likelihood of the product with positive (negative) expected unity decreases (increases) when informativeness of search increases or search costs decrease. Therefore, it is not always a wise decision for the seller to facilitate consumer search by increasing the informativeness of search or decreasing search costs. In particular, higher informativeness of search or lower search costs may lead to lower purchase likelihood of the high-valuation products.

Second, when a consumer's expected utilities of at least one of the products is relatively high, her purchase likelihood of the two products combined decreases when the informativeness of search increases or search costs decrease. To summarize, increasing information availability and facilitating consumer searching behavior will deteriorate sales for consumers who have a high valuation of at least one product and enhance sales for consumers who have a low valuation for both products. Therefore, a seller should carefully manage the information accessibility of its products, even though information is ex ante neutral. If both products are from the same seller, who cares about the total sales, then he should obfuscate product information from consumer search if currently consumers already have a relatively high valuation of either the two products.

5. Products with Correlated Information

Two houses in the same neighborhood share similar characteristics in transportation accessibility, quality of schools, crime statistics, climate, etc. Two car models under the same brand share similar information in engine technology, driving performance, safety design, warranty, etc. In general, two products under purchase consideration may share common attributes. When a consumer searches for information on one product, she will get some partial information on the other at the same time. Sometimes, however, positive information from one product speaks negatively of the other. For example, when searching for information on electric vehicles, consumers may get reviews of disadvantages of traditional gasoline vehicles. That is, information can be correlated either positively or negatively between the two products under consideration.

Possible information correlation among products has so far not been considered in our basic model, given in §2. In this section, we extend our basic model to study the problem of optimal search on two products with correlated information. In particular, instead of assuming uncorrelated utility updates, we consider the following utility updating dynamics in a consumer's search process. When a consumer searches for information on product 1, she gets a utility update for product 1 as $du_1 = dB_1(t_1)$; meanwhile, she also gets some partial information on product 2, with utility update $du_2 = \rho\, du_1$. Similarly, when a consumer spends dt in searching for information on product 2, she gets utility updates du_2 for product 2 and $du_1 = \rho\, du_2$ for product 1. The constant ρ characterizes the information correlation between the two products. Intuitively, searching one product should not consistently reveal more information about others; hence, it is stipulated that $|\rho| < 1$. When $\rho = 0$, we go back to our basic model without interproduct information correlation. As above, we can construct the Bellman equation as well as the boundary conditions for the problem of optimal search on two informationally correlated products.

By taking dt ahead, we have the following iterative relationship:

$$V(u_1, u_2) = -c\,dt + \max\{\mathrm{E}_{t_1}[V(u_1 + du_1, u_2 + \rho\, du_1)], \mathrm{E}_{t_2}[V(u_1 + \rho\, du_2, u_2 + du_2)]\}. \quad (20)$$

Similarly, we can reduce the equation above as the following partial differential equation:

$$\max\{V_{u_1u_1} + \rho^2 V_{u_2u_2}, V_{u_2u_2} + \rho^2 V_{u_1u_1}\} + 2\rho V_{u_1u_2} = \frac{2c}{\sigma^2}. \quad (21)$$

Despite the slightly increased complexity, one can still obtain that a consumer optimally chooses to search product 1, if and only if

$$V_{u_1u_1} \geq V_{u_2u_2}, \quad (22)$$

and vice versa for product 2, as long as $|\rho| < 1$.

As for boundary conditions, it turns out that Equations (11)–(14) still apply here exactly. It may appear straightforward at first glance, but the smooth-pasting condition for the general case here with $\rho \neq 0$ is not a trivial result. One should note that we now have a constrained multidimensional Brownian motion: a consumer's expected utility can only move along the direction with a slope equal to either ρ or $1/\rho$. We provide the derivation of the smooth-pasting conditions in the appendix. The following theorem presents the solution for the value function.

[16] The "protrusion" in the right panel of Figure 5 can be understood by considering the case with only one product. It can be shown that given a consumer's expected utility of u, her purchase likelihood is $P(u) = \frac{1}{2}(1 + u/a)$ if $-a < u < a$, $P(u) = 0$ if $u \leq -a$, and $P(u) = 1$ if $u \geq a$. One can easily verify that $\partial P(u)/\partial a$ is discontinuous at $u = -a$. Now in the case of two products, one can similarly show that $\partial P_i(u_i, u_j)/\partial a$ is discontinuous at $u_i = -a$ $(i = 1, 2)$, and thus $\partial P(u_i, u_j)/\partial a$ is discontinuous at $u_i = -a$ $(i = 1, 2)$.

Theorem 3. *There exists a unique solution $V(u_1, u_2)$ along with boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$ that satisfies Equations (5) and (21) under boundary conditions (11)–(14). The value function is*

$$V(u_1,u_2)=\begin{cases}(1/(4a))[\hat{U}(u_1,u_2)-u_1]^2+u_1\\ \qquad \text{if } u_2\le u_1\le \overline{U}(u_2) \text{ and } u_1\ge \underline{U}(u_2),\\ (1/(4a))[\hat{U}(u_2,u_1)-u_2]^2+u_2\\ \qquad \text{if } u_1\le u_2\le \overline{U}(u_1) \text{ and } u_2\ge \underline{U}(u_1),\\ u_1 \quad \text{if } u_1>\overline{U}(u_2),\\ u_2 \quad \text{if } u_2>\overline{U}(u_1),\\ 0 \quad \text{otherwise},\end{cases} \tag{23}$$

where $\hat{U}(u_i, u_j)$, with support on $\{(u_i, u_j) \mid u_j \le u_i \le \overline{U}(u_j)$ and $u_i \ge -a\}$, is defined as

$$\hat{U}(u_i,u_j)\equiv\begin{cases}\dfrac{u_j-\rho u_i}{1-\rho}+(1-\rho)\\ \quad\cdot\left[1+W\left(\dfrac{1+\rho}{1-\rho}e^{-\frac{2(u_j-\rho u_i)}{(1-\rho)^2(1+\rho)a}-\frac{1-2\rho-\rho^2}{1-\rho^2}}\right)\right]a\\ \qquad\qquad \text{if } u_j\ge \rho u_i-(1-\rho)a,\\ a \qquad\qquad \text{otherwise.}\end{cases} \tag{24}$$

The purchase and exit boundaries $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$ are given as

$$\overline{U}(u)=\begin{cases}u+(1-\rho^2)\\ \quad\cdot\left[W\left(e^{-\frac{2u}{(1-\rho^2)a}-\frac{1-4\rho+\rho^2}{1-\rho^2}}\right)+1\right]a\\ \qquad \text{if } u\ge -(1-2\rho)a,\\ a \quad \text{otherwise;}\end{cases} \tag{25}$$

$$\underline{U}(u)=-a \quad (\text{relevant when } u\le -a). \tag{26}$$

The value function above is similar to its counterpart in the uncorrelated case in Theorem 1, except that $V(u_1, u_2)$ is no longer quadratic in the purchase boundary $\overline{U}(u_i)$; rather, it is quadratic in $\hat{U}(u_i, u_j)$. In fact, $\hat{U}(u_i, u_j)$ is also related to the concept of purchase boundary. Given a consumer's current expected utilities of the two products $u_1 \ge u_2$, Theorem 3 states that she will search for information on product 1. During the search process, she gets new information on product 1 as well as some partial new information on product 2. If she has accumulated enough positive information on product 1, she will purchase product 1 at some point. The term $\hat{U}(u_1, u_2)$ is her expected utility of product 1 at the boundary when she is indifferent between continuing searching for information on product 1 and purchasing product 1, given that she starts from (u_1, u_2). The model is still quite parsimonious, parameterized by a and ρ only. Figure 6 presents an illustration of the value function $V(u_1, u_2)$.

With information spillovers between products, a consumer's optimal search strategy is similar to the case without information correlation. Interproduct information correlation impacts both a consumer's consideration set and the purchase threshold. The following theorem characterizes a consumer's optimal search strategy. The corresponding corollary describes the optimal search strategy when the expected utilities of the two products are relatively high.

Theorem 4. *With information correlated between two products, a consumer considers a product for search and purchase if and only if her expected utility of the product is above $-a+\max\{\rho(u+a), 0\}$, where u is her expected utility of the alternative product, and ρ is the information correlation coefficient. Given two products in her consideration set, the consumer always searches for information on the product with higher expected utility. She stops searching for information and purchases the product if the difference in her expected utilities of the two products is above the purchase threshold of $(1-\rho^2)W(e^{-2u/((1-\rho^2)a)-(1-4\rho+\rho^2)/(1-\rho^2)})a+(1-\rho)^2a$.*

Figure 6 **(Color online) Maximum Expected Utility of Two Products with Correlated Information**

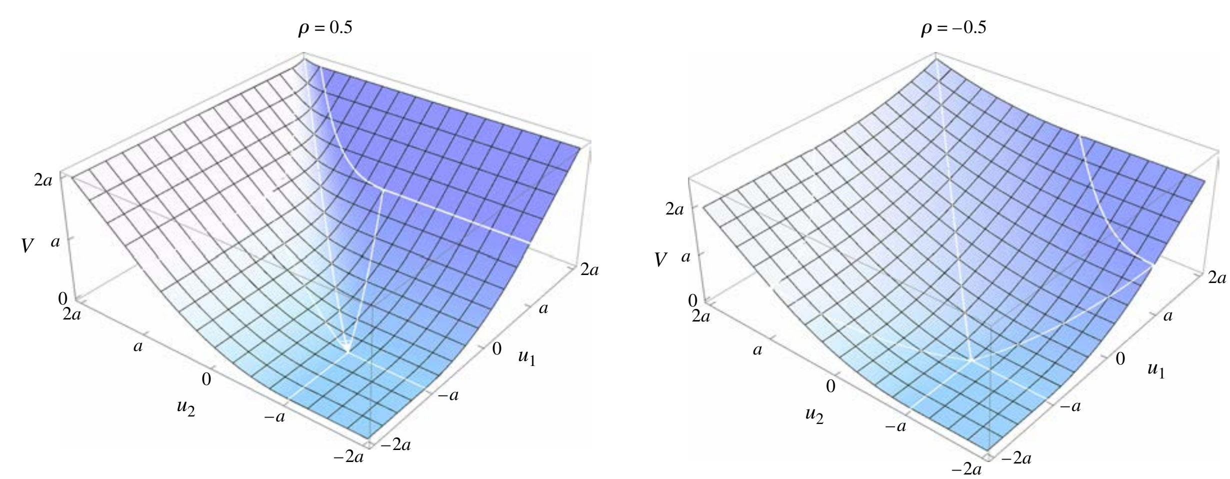
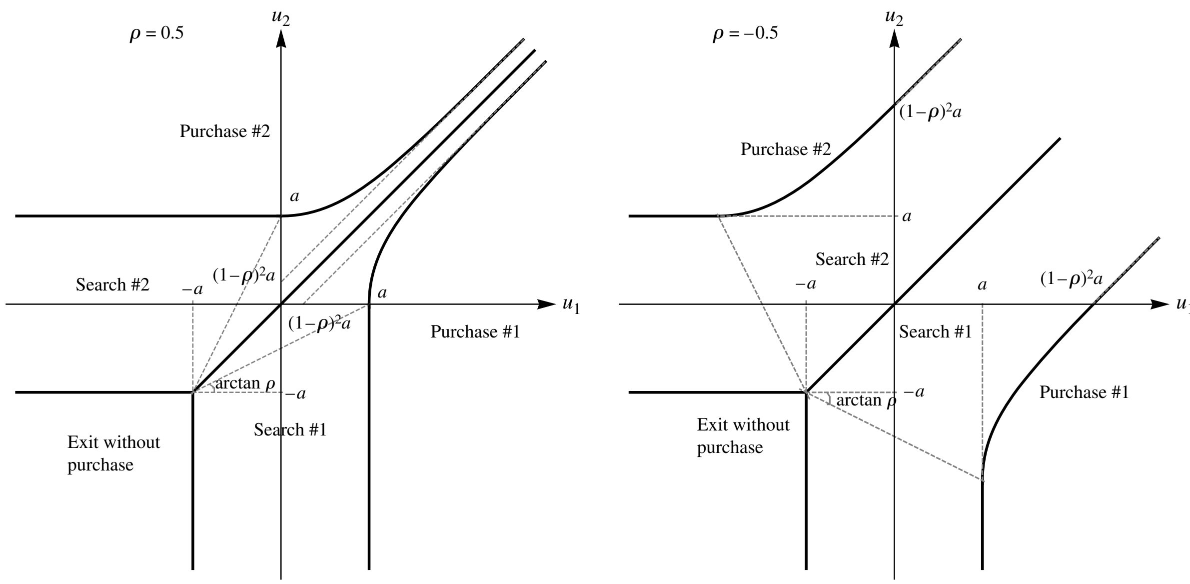


Figure 7 Optimal Search Strategy on Two Products with Correlated Information



Corollary 2. *The purchase threshold on the expected utility difference between the two products decreases as the expected utility of the alternative product increases, and it converges to* $(1-\rho)^2 a$.

Figure 7 illustrates a consumer's optimal search strategy given her current expected utilities of the two products, under both positive and negative information correlation.

The comparative statics are summarized in Proposition 2.

Proposition 2. *Given a consumer's expected utility of the alternative product u, her purchase threshold of the product increases in a and decreases in the information correlation ρ. Given a consumer's expected utilities of the two products u_1 and u_2, her maximum expected utility $V(u_1, u_2)$ increases in a and decreases in the information correlation ρ.*

As information correlation gets higher, a consumer will impose a narrower purchase threshold on the difference between her expected utilities of the two products. Therefore, two products with positive information correlation compete with each other more fiercely: a small informational advantage can render a consumer to choose one product over the other. Interestingly, a consumer expects higher expected utility when searching for information over two products with negative information correlation. In fact, negative information correlation benefits consumers by playing a role of insurance. During the search process, as a consumer is downgrading one product, she favors the other product more at the same time. This increases a consumer's likelihood of purchase and thus her expected utility.

For a firm selling two products, it would then be better to sell products with negative correlation in attribute fit than positive correlation, as products with a negative correlation lead to a greater probability of one of the products being bought by any given consumer. Furthermore, in terms of obfuscation strategies, obfuscation would be even more beneficial in the case of positive correlation if the expected valuations are high, because bad information on one product also means a negative shock on the other product. On the other hand, the firm would tend to reduce obfuscation and facilitate search in the case of negatively correlated products, because in that case bad news about one product means good news about the other product.

6. Heterogeneous Products

Another natural extension to our basic model is to consider heterogeneous products, where searching cost c_i and informativeness coefficient σ_i are different across products. We restrict our discussion to two products with uncorrelated information only.

The problem formulation is similar to the homogeneous case. Given c_i and σ_i for product i $(i = 1, 2)$, Equation (9) now would be

$$\max\{-2c_1 + \sigma_1^2 V_{u_1 u_1}, -2c_2 + \sigma_2^2 V_{u_2 u_2}\} = 0. \tag{27}$$

A consumer optimally chooses to search product 1 if and only if

$$V_{u_1 u_1} = \frac{2c_1}{\sigma_1^2} \quad \text{and} \quad V_{u_2 u_2} \le \frac{2c_2}{\sigma_2^2}, \tag{28}$$

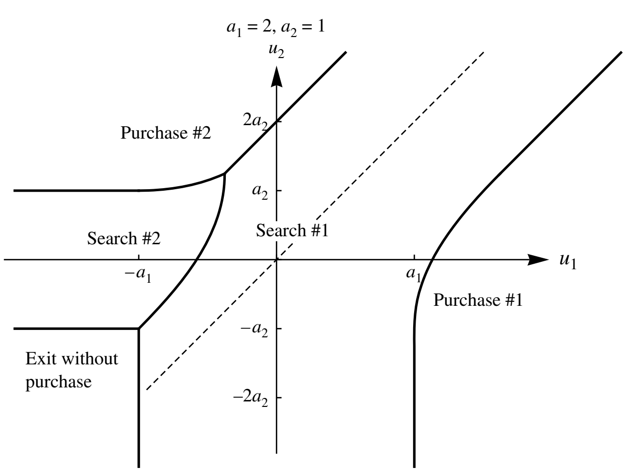
and vice versa for product 2. The boundary conditions (11)–(14) apply directly here by recognizing that the purchase boundary $\overline{U}_i(u)$ and exit boundary $\underline{U}_i(u)$ are specific for each product i.

By defining $a_i \equiv \sigma_i^2/(4c_i)$ $(i=1,2)$, the optimal search problem is, in fact, completely characterized by only two parameters: a_1 and a_2. From a mathematical perspective, the optimal search problem with heterogeneous products is a nontrivial extension of that in the homogeneous case. The new complexity comes from the difficulty of pinning down the boundary between searching product 1 and searching product 2. Nevertheless, the problem can still be solved analytically. The purchase boundary $\overline{U}_i(u)$ now cannot be written down explicitly, and instead it is given implicitly. Theorem 9, provided in the appendix, presents the solution of $V(u_1, u_2)$. The consumer's optimal search strategy is described by the following theorem.

Theorem 5. *Let a consumer's expected utility of product i be u_i. Product i will be considered for search and purchase if and only if $u_i \geq -a_i$. Suppose that two products are in a consumer's consideration set with $a_1 > a_2$. If $u_2 \geq -(\sqrt{a_1 a_2}/2)\ln((\sqrt{a_1}-\sqrt{a_2})/(\sqrt{a_1}+\sqrt{a_2}))$, , the consumer will keep searching for information only on product 1 until either she purchases product 1 when u_1 exceeds u_2 by a_1 or she purchases product 2 when u_2 exceeds u_1 by a_1. Otherwise, a consumer will keep searching for information on product i if her expected utility of product i plus the purchase threshold of product i exceeds that of the alternative product, i.e., $u_i + \overline{U}_i(u_j) \geq u_j + \overline{U}_j(u_i)$, until either she switches to search for information on the alternative when $u_i + \overline{U}_i(u_j) < u_j + \overline{U}_j(u_i)$, or she purchases product i when her expected utility of product i exceeds that of the alternative by some threshold.*

Figure 8 presents a consumer's optimal search strategy for $a_1 = 2$ and $a_2 = 1$. We find that the optimal consideration set still applies for the case with heterogeneous products. When a consumer's expected utility of product i is lower than $-a_i$, she will never consider this product. However, the purchase threshold structure is new and different. Consumers who have high expected utilities for both products only search for information on one product, the one with the highest a_i. Let $i^* = \arg\max_i a_i$. During the search process, the consumer imposes a constant purchase threshold of a_{i^*} on the expected utility difference of the two products. When her expected utility of product i^* exceeds that of the alternative by a_{i^*}, she purchases product i^* right away; otherwise, when her expected utility of product i^* is below that of the alternative by a_{i^*}, she purchases the alternative product right away. Therefore, the alternative product only serves as a reservation option, and the consumer will never search for information on it. With sufficiently high expected utilities of the two products, a consumer will not exit the market without a purchase, so her primary objective is to decide which product is a better choice. To achieve this goal, it is optimal for her to search on the product with the highest information per search cost, which is exactly the one with the highest a_i.

Figure 8 **Optimal Search Strategy on Two Heterogeneous Products**



Note that in this case, the purchase threshold is greater for the product that has the highest informativeness of search than for the other product. Therefore, it is easier to get an immediate purchase when the product with the lowest informativeness of search has a high expected valuation and the alternative product has a sufficiently low expected valuation than when the product with the highest informativeness has a high expected valuation and the alternative product has a sufficiently low expected valuation; that is, to get an immediate purchase, it is easier to reduce the expected utility of the product with the highest informativeness of search than to reduce the expected utility of the product with lowest informativeness of search. This would then lead to a benefit for the firm to try to sell the product with the lowest informativeness of search, if both expected valuations are relatively high. On the other hand, if the expected valuations are relatively low, the product with the highest informativeness of search has a greater advantage, because the exit threshold is lower.

The result that consumers with sufficiently high expected utilities only search for information on one of the products (the one that delivers more information per search cost) should be interpreted with caution. As preluded, this result depends on the assumption of identical distribution of utilities of attributes, which in our continuous-time model is equivalent to the assumption that the informativeness stays constant during the search process. In a setting where informativeness decreases as a consumer accumulates more information, the result above will no longer hold. Intuitively, a consumer will search first on the product that provides more information per search cost initially, but then after some time, the informativeness of that product decreases, and then the consumer will optimally switch to search for information on the alternative product, which now provides higher information per search cost.

The following proposition also comes from Theorem 9 in the appendix. It states that a consumer prefers to search for information on the product with lower search costs or higher informativeness of search, given her expected utilities of the two products being equal. Low search costs and high informativeness of search can prioritize a product with low expected utility being searched.

Proposition 3. *Given her expected utilities of the two products being equal, it is optimal for a consumer to search product i if $a_i > a_j$, i.e., if the search cost for product i is smaller or the informativeness of search for product i is greater than that for the other product.*

7. More Than Two Products

In this section, we extend our basic model of optimal search on two products to the case of more than two products. We first solve the problem of optimal search on three products analytically and find that, in general, the consideration set and purchase threshold structures extend robustly to the case with three products. This case allows us also to obtain new insights regarding the purchase threshold.

Let us consider the optimal search problem with three products that have the same informativeness of search and search costs, without information correlation. At any time, a consumer optimally chooses which product to search on, based on her current expected utilities of the three products as (u_1, u_2, u_3). A consumer's maximum expected utility is defined as $V(u_1, u_2, u_3)$. If a consumer's optimal decision is to stop searching and make a purchase decision right away, we have

$$V(u_1, u_2, u_3) = \max\{u_1, u_2, u_3, 0\}. \tag{29}$$

If the consumer chooses to continue searching for information, we have the following, where we have used the cyclic indexing rule with $u_i \equiv u_{i \bmod 3}$ for $i > 3$, and where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$:

$$\max\{V_{u_1u_1}, V_{u_2u_2}, V_{u_3u_3}\} = \frac{1}{2a};$$

Value matching at upper boundary:

$$V(u_1, u_2, u_3)|_{u_i=\overline{U}(u_{i+1},\, u_{i+2})} = \overline{U}(u_{i+1}, u_{i+2}), \quad i = 1, 2, 3;$$

Smooth pasting at upper boundary:

$$V_{u_j}(u_1, u_2, u_3)|_{u_i=\overline{U}(u_{i+1},\, u_{i+2})} = \delta_{ij}, \quad i, j = 1, 2, 3;$$

Value matching at lower boundary:

$$V(u_1, u_2, u_3)|_{u_i=\underline{U}(u_{i+1},\, u_{i+2})} = \max\{0, u_{i+1}, u_{i+2}\}, \quad i = 1, 2, 3;$$

Smooth pasting at lower boundary:

$$V_{u_j}(u_1, u_2, u_3)\big|_{u_i=\underline{U}(u_{i+1},\, u_{i+2})} = 0, \quad i, j = 1, 2, 3. \tag{30}$$

The function $\overline{U}(u_i, u_j)$ is the purchase boundary. Given u_i and u_j, when u_k hits $\overline{U}(u_i, u_j)$, the consumer will purchase product k right away. The function $\underline{U}(u_i, u_j)$ is the exit boundary, defined accordingly. The following results present the solution to the optimal search problem with three products.

Theorem 6. *There exists a unique solution $V(u_1, u_2, u_3)$, which satisfies Equations (29) and (30):*

$$V(u_1, u_2, u_3) = \begin{cases} \dfrac{1}{4a}[\overline{U}(u_{i+1}, u_{i+2}) - u_i]^2 + u_i & \text{if } -a, u_{i+1}, u_{i+2} \le u_i \le \overline{U}(u_{i+1}, u_{i+2}),\ i = 1, 2, 3, \\ u_i & \text{if } u_i > \overline{U}(u_{i+1}, u_{i+2}),\ i = 1, 2, 3, \\ 0 & \text{otherwise}. \end{cases} \tag{31}$$

The purchase and exit boundaries, $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$, for product k, with $u_k > \max\{u_i, u_j\}$, are given as

$$\overline{U}(u_i, u_j) = \begin{cases} u_{i\vee j} + \left[1 + W\left(e^{-2-(u_i+u_j+u_{i\vee j})/a} \cdot \dfrac{1 + 4W\left(\left(\frac{1}{2}\right)e^{-7/4-9u_{i\wedge j}/(4a)}\right)}{6 \times 2^{1/3} W\left(\left(\frac{1}{2}\right)e^{-7/4-9u_{i\wedge j}/(4a)}\right)^{4/3}}\right)\right]a & \text{if } u_i, u_j \ge -a, \\ \overline{U}(u_i) & \text{if } u_i \ge -a > u_j, \\ \overline{U}(u_j) & \text{if } u_j \ge -a > u_i, \\ a & \text{otherwise}; \end{cases} \tag{32}$$

$$\underline{U}(u_i, u_j) = -a \quad (u_i, u_j \le -a), \tag{33}$$

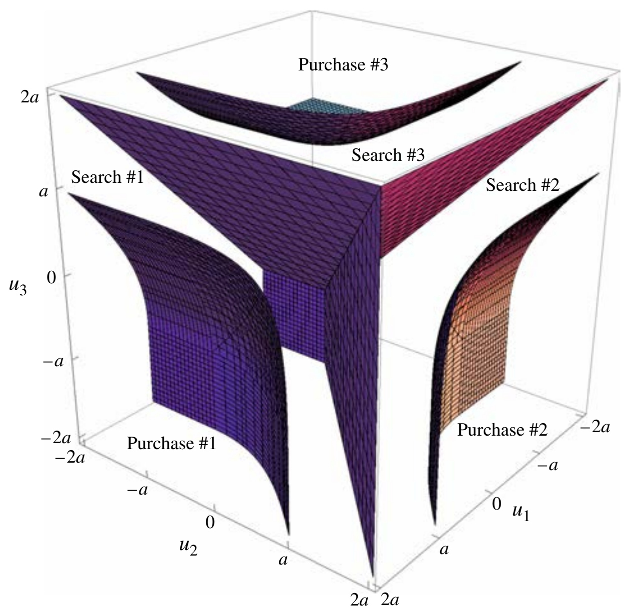
where $u_{i\vee j} \equiv \max\{u_i, u_j\}$ and $u_{i\wedge j} \equiv \min\{u_i, u_j\}$.

The solution structure for the three-product case looks similar to the one for the two-product case. The maximum expected utility $V(u_1, u_2, u_3)$ is still quadratic in the purchase boundary. In fact, this can be shown to be true for any number of products. However, the purchase boundary $\overline{U}(u_i, u_j)$ now becomes more complicated. We provide intuition on $\overline{U}(u_i, u_j)$ below. A consumer's optimal search strategy is characterized by the following theorem, also illustrated in Figure 9.

Theorem 7. *Only products with expected utility above $-a$ constitute a consumer's consideration set for search and purchase. Given three products in her consideration set, the consumer always searches for information on the one with the highest expected utility. She stops searching and makes a purchase if the difference in her expected utilities of the top two is above some purchase threshold, which depends on the consumer's expected utilities of the alternatives.*

The following corollary presents the monotonicity and asymptotics of the purchase threshold with respect to the expected utilities of the alternative products.

Figure 9 (Color online) Optimal Search Strategy with Three Products



COROLLARY 3. *Suppose $u_k > u_{i\vee j}$. The purchase threshold of product k with respect to the other two alternatives, $\bar{U}(u_i, u_j) - u_{i\vee j}$ decreases with $u_{i\vee j}$ and increases with $u_{i\wedge j}$, satisfies*

$$\bar{U}(u_i, u_j) - u_{i\vee j} \to \left[1 + W\left(\frac{1}{3}e^{1/3-2(u_{i\vee j}-u_{i\wedge j})/a}\right)\right]a, \quad \text{as } u_i, u_j \to +\infty. \tag{34}$$

Recall that in the two-product case, a consumer imposes a purchase threshold on the difference between her expected utilities of the two products, and the purchase threshold gets narrower as her expected utility of the alternative product gets higher, and it converges to a. Now with three products, we show that a consumer imposes a purchase threshold on the difference between her expected utilities of the *top* two products, and the purchase threshold still gets narrower as her expected utility of the *second* alternative product gets higher, but it gets *wider* as her expected utility of the *third* alternative product gets higher. As the expected utility of the third alternative gets higher, a consumer has a higher "reservation"; therefore she needs to see a bigger difference between the top two to convince her to buy one product over the other. Moreover, the asymptotics show that as the expected utility of the second alternative goes to infinity, the purchase threshold converges to $[1 + W((\frac{1}{3})e^{1/3-2\Delta u/a})]a$, which is greater than a. Consequently, more alternatives widen a consumer's purchase threshold, as more alternatives provoke more search efforts, and a consumer needs to see a bigger difference between the top two to convince her to buy one product over the other.

The problem of optimal search for information on four or more products can be stated and obtained in a similar way, with increased computational complexity. Yet it is interesting to revisit Bergman's (1981) findings for the case of an infinite number of products with equal initial expected utilities. For this case, Bergman shows that the optimal search strategy is to search for information on the product with the highest Gittins index.

Consider a consumer's expected utilities of an infinite number of products being equal as u_0 initially. If she has an outside option with value K, the maximum expected utility of search for information when only one product is available can be obtained as follows:

$$V(u_0; K) = \frac{1}{4a}(a + K - u_0)^2 + u_0. \tag{35}$$

The Gittins index for a product can then be obtained as the value of the outside option that equates the maximum expected utility of choosing one arm (i.e., searching information on one product) with the value of the outside option, $V(u_0; K) = K$ (Whittle 1980). Solving for K, we obtain the Gittins index $K = u_0 + a$.

The consumer's optimal search strategy is then to continue searching for information on one product until her expected utility of the product either decreases below u_0 or increases above $u_0 + 2a$. In the former case, the consumer picks another product to search information on. In the latter case, she purchases the product and leaves the market; that is, given an infinite number of products with equal initial expected utilities, a consumer imposes a constant purchase threshold of $2a$ on the difference of her expected utilities between the product under search and the remaining unsearched products. By contrast, a high-valuation consumer imposes a purchase threshold of a for two products and a purchase threshold of $\frac{4}{3}a$ for three products (if the two other products have the same expected utility). The purchase threshold widens as a consumer takes more products under consideration, as she has more options to acquire a higher payoff, but that purchase threshold on the difference of her expected utilities is bounded from above by $2a$.

8. Firm's Pricing Decision

In this section, we present some numerical simulations on a multiproduct monopoly's pricing decisions given that consumers search for product information before making a purchase decision. Consider a seller of two products, based on our basic model. We assume that consumers observe the seller's prices before engaging in any search. Consumers are homogeneous in their initial valuations of the two products, q_1 and q_2. Consumers' initial expected utility of product i is thus, $v_i = q_i - p_i$. Because all consumers' preferences are aligned, the two products can be considered as ex ante vertically differentiated.[17] It is interesting to notice that we are able to study the vertical differentiation problem under ex ante homogeneous consumers, because consumers will become heterogeneous in their valuations after search.

Without loss of generality, we assume the marginal costs of both products to be zero.[18] The seller chooses prices to maximize the expected total profit,

$$\max_{p_1, p_2}\{p_1P_1(q_1-p_1, q_2-p_2)+p_2P_2(q_1-p_1, q_2-p_2)\}, \tag{36}$$

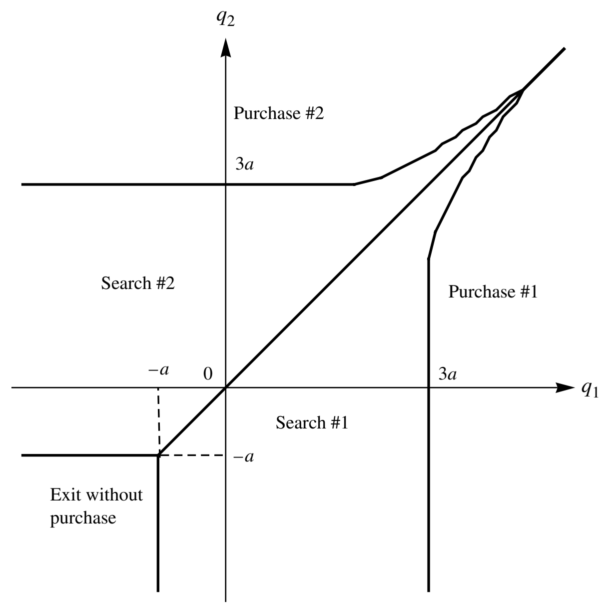
where $P_i(u_1, u_2)$, defined as in §4, is the purchase likelihood of product i given a consumer's current expected utilities of the two products, u_1 and u_2. Let us denote the optimal prices as p_1^* and p_2^*. Without solving the profit optimization problem, we can show the following lemma, with the proof in the appendix.

Lemma 3. *If $q_1 > q_2 \geq -a$, we have $q_1 - p_1^* \geq q_2 - p_2^*$.*

Under the optimal pricing policy, consumers expect higher expected utility from the product with the higher valuation. From Theorem 2, we know that a consumer always searches on the product with higher expected utility; therefore, given the optimal pricing policy, homogeneous consumers always search on the product with the higher valuation. Figure 10 presents a numerical simulation of a consumer's optimal search strategy in her valuation space, under the seller' optimal pricing policy.[19]

Compared with Figure 2, a clear feature is that consumers with high valuations will be incentivized to purchase directly without any search. Consistent with our previous observations in §4, high-valuation consumers' search behavior will harm the seller's profit, and thus they are deterred from search by the firm offering a sufficiently low price such that those consumers choose to purchase immediately without search. Figure 11 shows the seller's optimal price for product 1 and the maximum profit.[20] The optimal price for product 2 can be obtained by symmetry, $p_2^*(q_1, q_2) = p_1^*(q_2, q_1)$. We find that with $q_1 > q_2 \gg 0$ and $q_1 \simeq q_2$, we have $q_1 - p_1^* \simeq q_2 - p_2^* + 2a$; i.e., $p_1^* - p_2^* \simeq q_1 - q_2 - 2a \simeq -2a < 0$. This implies that when a consumer's valuations of the two products q_1 and q_2 are relatively high and close to each other, the seller deters her search behavior and incentivizes her to purchase immediately by setting a lower price for her favored product and imposing a price difference between the two products.

Figure 10 **Homogeneous Consumers' Optimal Search Strategy on Two Products, Given a Monopolistic Seller's Optimal Pricing Policy**



We can also check how the optimal prices and maximum profits vary with a. We find that $\partial\pi^*(q_1, q_2)/\partial a$ is similar to $\partial P(u_1, u_2)/\partial a$, shown in Figure 5. The seller's profit increases with search costs, whereas it decreases with informativeness of search if and only if q_1 and q_2 are relatively high. Therefore, in the case that a seller's objective is to maximize profit instead of sales, we obtain again our previous managerial implications that a seller should deter search for high-valuation consumers while facilitating search for low-valuation consumers.

9. Discounting, Finite Mass of Attributes, and Decreasing Informativeness

9.1. Discounting

In this section, we consider three more extensions to the basic model: discounting, finite mass of attributes, and decreasing informativeness of attributes. We have so far implicitly assumed that a consumer searches fairly fast, and there is no time discounting in the search process. In some cases a consumer can search for information for longer time horizons, and it may be interesting in those cases to consider discounting the consumer's future search efforts as well as the payoff

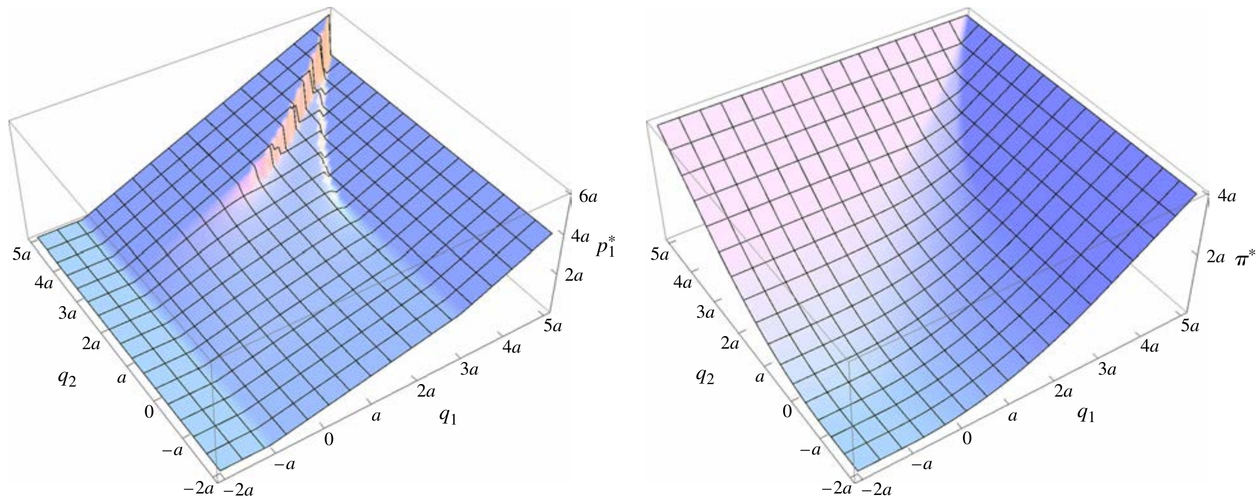
[17] For consumer search on horizontally differentiated products, see, e.g., Wernerfelt (1994).

[18] In the case with marginal cost for product i, $g_i > 0$, we can redefine $q_i' = v_i - g_i$ and $p_i' = p_i - g_i$, and then we get back to the profit optimization problem with zero marginal costs.

[19] We cannot solve the optimization problem (36) analytically. This problem involves a constrained nonconvex global optimization problem that makes it hard to obtain analytical solutions. We explain our approach in the appendix.

[20] When a product is neither searched nor purchased, its price is not uniquely determined. In this case, we stipulate the price to be its infimum. See the appendix for more details.

Figure 11 (Color online) A Monopolistic Seller's Optimal Price for Product 1, p_1^*, and Maximum Profit, π^*



from purchase. To incorporate discounting, we can reformulate Equation (6) as

$$V(u_1, u_2) = -c\,dt + e^{-rdt}\max\{\mathrm{E}_{t_1}[V(u_1 + du_1, u_2)], \mathrm{E}_{t_2}[V(u_1, u_2 + du_2)]\}, \quad (37)$$

where r is the time-discounting factor. Using the same technique as above, we can rewrite the above equation as the following partial differential equation:

$$\max\{V_{u_1u_1}, V_{u_2u_2}\} = \frac{2c}{\sigma^2} + \frac{2r}{\sigma^2}V, \quad (38)$$

which looks almost the same as Equation (9), except that now we have an extra term $(2r/\sigma^2)V$ on the right-hand side of the equation. The boundary conditions are still exactly given by Equations (11)–(14), which, together with Equations (38) and (5), constitute the mathematical problem of optimal search with time discounting. Theorem 10 in the appendix completely characterizes the optimal solution, where the value function $V(u_1, u_2)$ can be explicitly expressed as a function of the purchase boundary $\overline{U}(u)$, which is no longer in quadratic form as in the basic model. However, $\overline{U}(u)$ cannot be expressed explicitly. It is determined by an ordinary differential equation with a boundary condition. The following theorem characterizes a consumer's optimal search strategy. (The proof is straightforward given Theorem 10 and thus omitted.)

Theorem 8. *Only products with expected utilities above* $\sqrt{c^2/r^2 + \sigma^2/2r} - c/r - (\sigma/\sqrt{2r})\ln[\sqrt{r\sigma^2/(2c^2)} + \sqrt{r\sigma^2/(2c^2)+1}]$ *constitute a consumer's consideration set for search and purchase. Given two products in her consideration set, the consumer always searches for information on the one with higher expected utility. She stops searching and purchases the product if the difference in her expected utilities of the two products is above some purchase threshold, which depends on her current expected utility of the alternative.*

From the theorem above, we find that the way for a consumer to optimally constitute her consideration set is almost the same as in the basic model, except that the consumer now has a higher bar for selection. In fact, we can show that $\sqrt{c^2/r^2 + \sigma^2/(2r)} - c/r - (\sigma/\sqrt{2r})\ln[\sqrt{r\sigma^2/(2c^2)} + \sqrt{r\sigma^2/(2c^2)+1}]$ increases with r. The more impatient a consumer is, the higher a bar she will impose on the expected utilities when selecting products into her consideration set. The purchase threshold structure is almost the same (consumers still search on the product with higher expected utility), but the asymptotics are different, as shown by the following corollary (with the proof in the appendix).

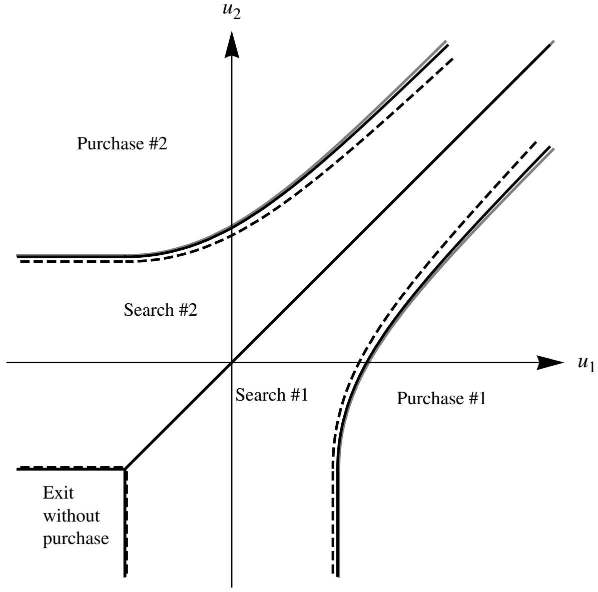
Corollary 4. *With time discounting $r > 0$, the purchase threshold on the expected utility difference between the two products decreases as the expected utility of the alternative product increases, and it converges to zero.*

As before, the purchase threshold decreases with the expected utility of the alternative but now converges to zero instead of a positive constant as in the basic model. This is easy to understand from Equation (38); with time discounting, a consumer essentially bears two kinds of costs: an explicit search cost modeled by c and an implicit cost due to delays of the purchase rV. Therefore, impatient high-valuation consumers will search less before making a purchase. Figure 12 illustrates a consumer's optimal search strategy with time discounting, which seems to suggest that discounting does not affect the optimal search strategy too much.

9.2. Finite Mass of Attributes

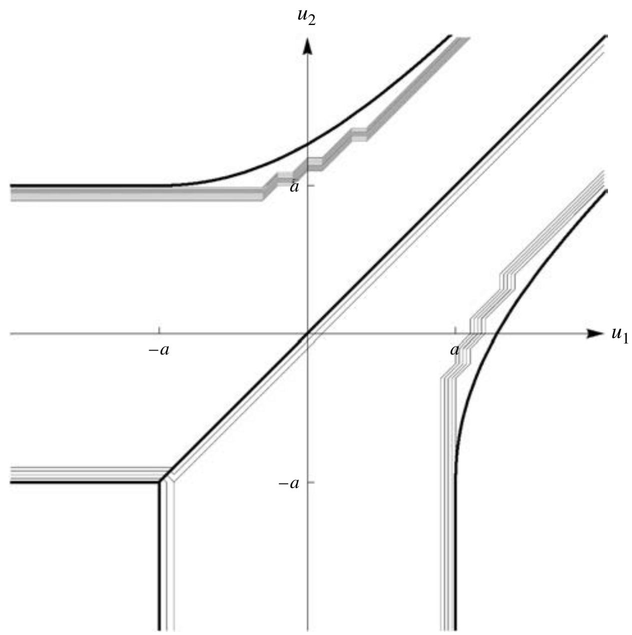
Consider now the possibility of a finite mass of attributes; i.e., T is finite. Given finite T, the optimal search problem becomes intractable analytically, but we can use numerical simulations to consider the consumers' optimal search behavior. With finite T, as a

Figure 12 Optimal Search Strategy on Two Products with Time Discounting



Notes. The black and dashed lines represent the cases with $r = 0.1$ and $r = 0.5$, respectively. The original case of $r = 0$ is presented by the gray lines.

Figure 13 Optimal Search Strategy on Two Products with Finite Mass of Attributes



Note. The gray lines represents the numerical solution with finite T, and the black lines represents the analytical solution with infinite T.

consumer searches attributes of the different products, the consumer becomes less demanding on the difference of expected utilities to make a choice. At the beginning of the search process it is also interesting to consider how the optimal search process for finite mass of T compares with the case of infinite T. Figure 13 presents a comparison of the optimal strategies between the analytical solution with infinite T and a numerical solution with finite T for $T = 10$, $c = 1$, and $\sigma = 10$. We can see that even for a relatively big a (large σ, small c) and relatively small T, our analytical solution with infinite T seems to approximate the numerical solution with finite T relatively well.

9.3. Decreasing Informativeness

A natural framework to incorporate decreasing informativeness is to model a consumer search process as sequential costly acquisitions of independent noisy signals of the unknown true product utility. As above, a consumer's utility of product i is denoted by U_i, unknown to the consumer. At any time t, a consumer's current belief of U_i follows $\mathrm{N}(u_i, \sigma_i^2)$. Now, σ_i^2 is no longer a constant. In fact, if a consumer spends an infinitesimal time dt to search for information on product i, she pays a search cost $c_i dt$ and gets a noisy signal $\tilde{U}_i \mid U_i \sim N(U_i, \kappa_i^2/dt)$, where κ_i^2 is a measure of the noisiness of the signal. Upon receiving the signal, the consumer updates her belief of product i's utility, by Bayes' rule, as

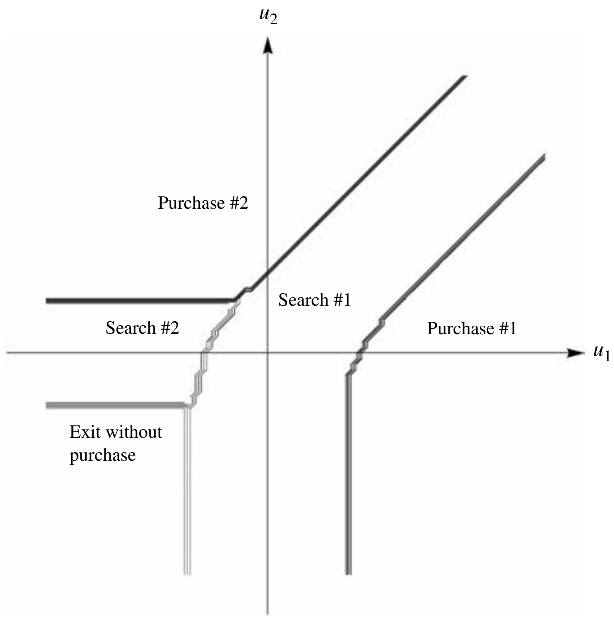
$$\mathrm{N}\left(\frac{1/\sigma_i^2}{1/\sigma_i^2 + dt/\kappa_i^2} u_i + \frac{dt/\kappa_i^2}{1/\sigma_i^2 + dt/\kappa_i^2} \tilde{U}_i, \frac{1}{1/\sigma_i^2 + dt/\kappa_i^2}\right).$$

To simplify the notation, let us define $s_i \equiv 1/\sigma_i^2$ and $k_i \equiv 1/\kappa_i^2$. Let us consider a model of two products with zero information correlation. A consumer's maximum expected utility is denoted as $V(u_1, u_2, s_1, s_2)$, which now depends on not only her current expected utility of each product but also the variance, or the uncertainty of her current belief. Similarly, a consumer's optimal search problem can be formulated by the following iterative relationship:

$$\begin{aligned}
&V(u_1, u_2, s_1, s_2)\\
&= \max\Bigg\{0, u_1, u_2, -c_1 dt\\
&\quad + \mathrm{E}_t\left[V\left(\frac{s_1}{s_1 + k_1 dt} u_1 + \frac{k_1 dt}{s_1 + k_1 dt} \tilde{U}_1, u_2, s_1 + k_1 dt, s_2\right)\right],\\
&\quad -c_2 dt\\
&\quad + \mathrm{E}_t\left[V\left(u_1, \frac{s_2}{s_2 + k_2 dt} u_2 + \frac{k_2 dt}{s_2 + k_2 dt} \tilde{U}_2, s_1, s_2 + k_2 dt\right)\right]\Bigg\}\\
&= \max\Bigg\{0, u_1, u_2, V(u_1, u_2, s_1, s_2) + \left[k_1 V_{s_1} + \frac{k_1}{2s_1^2} V_{u_1 u_1} - c_1\right] dt,\\
&\quad V(u_1, u_2, s_1, s_2) + \left[k_2 V_{s_2} + \frac{k_2}{2s_2^2} V_{u_2 u_2} - c_2\right] dt\Bigg\}.
\end{aligned} \tag{39}$$

The conditional expectation E_t in the first equality above is with respect to $\tilde{U}_i \sim \mathrm{N}(u_i, \sigma_i^2 + \kappa_i^2/dt)$ from the consumer's perspective. The second equality above is due to Taylor expansions, and $o(dt)$ terms have been omitted in the limit. As before, we can formulate the above problem as an ambiguous-boundary PDE

Figure 14 Optimal Search Strategy on Two Products with Decreasing Informativeness



problem. However, now we have two more arguments s_1 and s_2 besides u_1 and u_2, which makes the problem difficult to solve analytically. The problem can still be solved numerically.

Figure 14 shows a consumer's optimal search strategy at some time point with $c_1/k_1 = c_2/k_2 = 1$, and the consumer's current variances of the two products' utilities, σ_1^2 and σ_2^2, are not equal, given by $s_1 = 0.5$, $s_2 = 1$. We can see that, in general, Figure 14 is similar to Figure 8 in terms of the structure of the boundaries. We still have the optimal consideration set and the purchase threshold structures. However, because the parametric frameworks are different, we cannot compare the locus of the boundaries in the two figures directly. As expected with decreasing informativeness, we can also get that when a product is searched, the purchase threshold for that product falls, and that the boundary separating "Search 1" and "Search 2" moves in the direction of being more likely for the other product to be searched next.

10. Conclusion

Gradual search for information is important for understanding numerous economic activities with imperfect competition and market frictions. We consider this possibility, presenting a parsimonious model on continuous search for information on a choice set of multiple options. Although this paper has taken consumer search in a product market as the leading example, the model can be applied generally to other cases of gradual search for information on multiple alternatives.

This paper solves for the optimal search, switch, and purchase or exit behavior in such a setting, which is characterized by an optimal consideration set and purchase thresholds. A consumer always searches for information on the product with the highest expected utility if the informativeness of search per search cost is the same across products, and she only stops to make a purchase if her expected utility of a product is sufficiently greater than those of the alternatives. Positive correlation across products narrows the purchase threshold, whereas negative correlation widens it. More product alternatives also widen the purchase threshold. With heterogeneous products, if the informativeness of search is constant through time, the consumer only searches on the product with the highest informativeness of search or lowest search costs if her expected utility of the alternative is sufficiently high, and she will always first search for information on that product when both products have the same expected utility. The model also presents several implications that are empirically testable.

Understanding consumers' search behavior for information also helps to explain some seemingly puzzling results: more alternatives might lead to a lower purchase likelihood when consumers engage in search for information. Also, information availability decreases sales of products for high-valuation consumers, whereas it increases sales for low-valuation consumers. Therefore, sellers of multiple products may want to facilitate information search for low-valuation consumers while obfuscating information for high-valuation consumers.

The setup considered may motivate further studies on the economics of search for information. One interesting possibility to consider is to allow consumers to search on multiple products at the same time, known as parallel search (Vishwanath 1988). It would also be interesting to investigate what happens in terms of vertical differentiation under oligopolistic competition when there is a correlation of information across products.

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Appendix

Multiarmed Bandits, Gittins Index, and Search for Information

In the standard multiarmed bandit problem, a decision maker chooses which arm to pull in each period, where the reward obtained in the pulled arm is stochastic with unknown distribution for each arm (with independent distributions across arms). Gittins (1979) show that the optimal policy in this problem is to pull the arm that has the highest dynamic allocation index, as defined by Gittins, also known as the Gittins index, which is obtained for each arm and depends only on the information on the distribution of rewards that the decision maker has at that time for that arm. One interpretation of the Gittins index (provided by Whittle 1980) is that it is the value at which the decision maker would be indifferent between getting that value for sure or playing the arm with the option of taking that retirement value at any time. That is, if we define $V(I_i, K_i)$ as the value of playing arm i, with information I_i about arm i, with the possibility of retiring and getting K_i, then the Gittins index can be seen as the K_i such that $K_i = V(I_i, K_i)$. Bergman (1981) showed that, in general, the optimal policy in the problem of gradual search for information does not involve playing the arm with the highest index (i.e., highest Gittins index).

To see this, we present a counterexample (adapted from Bergman 1981 for rational expectations in search for information). Suppose that there are two possible products that a consumer can purchase, A and B. Prior to any search, the expected value of product A is 10, and the expected value of product B is 4. The first time a consumer searches on any of the products, she does not learn anything about the value of any product. The second time the consumer searches for information on product A, the consumer learns that the value of product A is either 20 or 0 with equal probability. The second time that the consumer searches for information on product B, the consumer learns that the value of product B is either 18 or −10 with equal probability. The search cost of each time the consumer searches for information is 1. The Gittins index for product A is obtained by making $K_A = -2 + 20/2 + K_A/2$, which yields $K_A = 16$. The Gittins index for product B is obtained by making $K_B = -2 + 18/2 + K_B/2$, which yields $K_B = 14$; that is, the Gittins index policy would suggest to check information on product A first, which yields an expected payoff of $-2 + 20/2 + \frac{1}{2}(-2 + 18/2) = 11.5$. However, by checking information on product B first, the consumer is able to get the higher expected payoff of $-2 + 18/2 + 10/2 = 12$. By checking product B first, the consumer keeps product A in reserve, which she can choose to buy without checking further.

As discussed in §7, Bergman (1981) shows that the Gittins index policy is optimal when there are an infinite number of products that are ex ante equal in distribution. In this case, the Gittins index is a direct extension to the "reservation price" in classical sequential search (McCall 1970), to allow for multiple search actions on one alternative.

Proof of Lemma 1. According to the symmetry between u_1 and u_2, it suffices to show that, for all u_2, $V(u_1'', u_2) \geq V(u_1', u_2)$ for all $u_1'' > u_1'$. In fact, let $x(u_1, u_2)$ be the consumer's optimal action given her current expected utilities of the two products as u_1 and u_2. Any other strategy, including $x'(u_1, u_2) \equiv x(u_1 + u_1' - u_1'', u_2)$, must be suboptimal to x. By the definition of x', we know that to follow strategy x' starting from (u_1'', u_2) will always generate the same action sequence as to follow strategy x starting from (u_1', u_2). Any search process will end up with purchasing product 1, purchasing product 2, or exiting market without any purchase. Because the same action sequence is followed for both random searching processes starting from (u_1'', u_2) and (u_1', u_2), they will end up with the same choice of actions. In any case, the consumer will be no worse off by following x' in the search process starting from (u_1'', u_2), because $u_1'' > u_1'$. As a result, $V(u_1', u_2)$, as the expected utility by following x for the search process starting from (u_1', u_2), will be no larger than the expected utility by following x' for the search process starting from (u_1'', u_2), which in turn is no larger than $V(u_1'', u_2)$ according to the suboptimality of x'. □

Proof of Lemma 2. To simplify notation, we drop the subscript i in $\overline{U}_i(u)$. For all $u'' > u'$, we know $V(\overline{U}(u'), u'') \geq V(\overline{U}(u'), u') = \overline{U}(u')$, according to the monotonicity of $V(u_1, u_2)$ by Lemma 1. So given a consumer's expected utility of product 2 as u'', when her expected utility of product 1 reaches $\overline{U}(u')$, she has the maximum expected utility of continuing to search for information of $V(\overline{U}(u'), u'')$, which is greater than the expected utility of purchasing product 1 right away, $\overline{U}(u')$. Her optimal decision is then to continue searching until she hits a higher expected utility of product 1 of $\overline{U}(u'')$. Therefore, we have $\overline{U}(u'') \geq \overline{U}(u')$. □

Derivation of the Smooth-Pasting Condition in Equation (12)

We prove the smooth-pasting condition at the purchase boundary of product 1. The proof for product 2 can be constructed similarly according to symmetry. Let us consider an extra search in dt on product 1 at the boundary $(\overline{U}(u_2), u_2)$. The corresponding utility update du_1 can be positive or negative, with equal odds. If $du_1 \geq 0$, the consumer will purchase product 1 immediately and leave the market; otherwise, if $du_1 < 0$, the consumer's expected utility of product 1 decreases, and she will stay in the market searching for more information. Therefore, her value function upon the extra search on product 1 would be

$$\begin{aligned} V_1(\overline{U}(u_2), u_2) &\equiv -c\,dt + \frac{1}{2}(\overline{U}(u_2) + \mathrm{E}[du_1 \mid du_1 \geq 0]) \\ &\quad + \frac{1}{2}\mathrm{E}[V(\overline{U}(u_2) + du_1, u_2 \mid du_1 < 0)] \\ &= V(\overline{U}(u_2), u_2) + \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}[1 - V_{u_1}(\overline{U}(u_2), u_2)] \\ &\quad + o(\sqrt{dt}), \end{aligned} \tag{40}$$

where we use the fact that $\mathrm{E}[du_1 \mid du_1 \geq 0] = -\mathrm{E}[du_1 \mid du_1 < 0] = \sigma\sqrt{dt/(2\pi)}$.

On the other hand, let us consider a consumer who spends dt in searching for information on product 2 at the boundary $(\overline{U}(u_2), u_2)$. If $du_2 = dB_2(t_2) \geq 0$, according to Lemma 2, the consumer's purchase threshold for product 1 increases, so she will stay in the market continuing the search for information; otherwise, if $du_2 < 0$, the consumer will purchase product 1 immediately. Therefore, her value function upon the extra search on product 2 would be

$$\begin{aligned} &V_2(\overline{U}(u_2), u_2) \\ &\quad \equiv -c\,dt + \frac{1}{2}\mathrm{E}[V(\overline{U}(u_2), u_2 + du_2 \mid du_2 < 0)] + \frac{1}{2}\overline{U}(u_2) \\ &\quad = V(\overline{U}(u_2), u_2) - \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}V_{u_2}(\overline{U}(u_2), u_2) + o(\sqrt{dt}). \end{aligned} \tag{41}$$

A consumer chooses which product to search for information on based on expected utility maximization. Therefore, her value function upon the extra search should satisfy

$$V(\overline{U}(u_2), u_2) = \max\{V_1(\overline{U}(u_2), u_2), V_2(\overline{U}(u_2), u_2)\}. \quad (42)$$

By substituting the expression of $V_1(\overline{U}(u_2), u_2)$ and $V_2(\overline{U}(u_2), u_2)$ into the above equation, we have

$$\max\{1 - V_{u_1}(\overline{U}(u_2), u_2), V_{u_2}(\overline{U}(u_2), u_2)\} = 0. \quad (43)$$

Meanwhile, by taking the derivative of both sides of Equation (11) with respect to u_2, we have

$$\overline{U}'(u_2)[1 - V_{u_1}(\overline{U}(u_2), u_2)] = V_{u_2}(\overline{U}(u_2), u_2). \quad (44)$$

Combining the above two equations, we obtain $V_{u_1}(\overline{U}(u_2), u_2) = 1$ and $V_{u_2}(\overline{U}(u_2), u_2) = 0$. □

Proof of Theorem 1. The solution is not easy to obtain, but it is fairly straightforward to verify that it satisfies Equation (9) along with all boundary conditions (11)–(14). Actually, as a viscosity solution, $V(u_1, u_2)$ takes different forms in different regions in our solution. We also need a set of conditions at the boundaries separating different regions. Say $V(u_1, u_2)$ takes the form of $V^1(u_1, u_2)$ in region 1 and $V^2(u_1, u_2)$ in region 2. At each "internal" boundary $\mathscr{C}$ separating regions 1 and 2, we need to impose the following:

(1) Value-matching condition:
$$V^1(u_1, u_2)|_{\{u_1, u_2\} \in \mathscr{C}} = V^2(u_1, u_2)|_{\{u_1, u_2\} \in \mathscr{C}};$$

(2) Smooth-pasting condition:
$$V^1_{u_i}(u_1, u_2)|_{\{u_1, u_2\} \in \mathscr{C}} = V^2_{u_i}(u_1, u_2)|_{\{u_1, u_2\} \in \mathscr{C}} \ (i = 1, 2).$$

One can verify that $V(u_1, u_2)$ in Equation (15) satisfies the two conditions above at all internal boundaries: $\mathscr{C}_1 \equiv \{(u_1, u_2) \mid u_1 = u_2 \geq -a\}$, $\mathscr{C}_2 \equiv \{(u_1, u_2) \mid u_1 = -a, -a \leq u_2 \leq a\}$, and $\mathscr{C}_3 \equiv \{(u_1, u_2) \mid u_2 = -a, -a \leq u_1 \leq a\}$.

The uniqueness of the solution is guaranteed by the generic uniqueness of viscosity solution to the Hamilton–Jacobi–Bellman equation (9) (Bardi and Capuzzo-Dolcetta 2008, p. 6). □

Derivation of Purchase Likelihood in Equation (19)

If $u_1 \geq \overline{U}(u_2)$, the consumer will purchase product 1 right away; therefore, $P_1(u_1, u_2) = 1$. If $u_1 \leq -a$ or $u_2 \geq \overline{U}(u_1)$, the consumer will never purchase product 1; therefore, $P_1(u_1, u_2) = 0$. Otherwise, if $-a < u_1 < \overline{U}(u_2)$ and $u_2 < \overline{U}(u_1)$, there are two cases, depending on the value of u_2.

In the first case with $u_2 \leq -a$, the consumer will search on product 1 only. Given her current expected utility u_1, she will either hit $-a$ first or hit a first. According to the optional stopping theorem, we have $u_1 = P_1(u_1, u_2)a + [1 - P_1(u_1, u_2)](-a)$; i.e.,

$$P_1(u_1, u_2) = \frac{1}{2} + \frac{u_1}{2a}, \quad -a < u_1 \leq a, \ u_2 \leq -a. \quad (45)$$

In the second case, $u_2 > -a$. When $u_1 \geq u_2$, the consumer searches on product 1 and either hits $\overline{U}(u_2)$ first or hits u_2 first. Let us define the probability of hitting $\overline{U}(u_2)$ first as $q_1(u_1, u_2)$. Then, by invoking the optional stopping theorem, we similarly get

$$q_1(u_1, u_2) = \frac{u_1 - u_2}{\overline{U}(u_2) - u_2}. \quad (46)$$

According to symmetry, the probability of hitting $\overline{U}(u_1)$ first, starting from (u_1, u_2) with $u_1 < u_2$, would be $q_1(u_2, u_1)$. Let us further define $P_0(u)$ as the probability of exiting the market without any purchase, given current expected utilities as (u, u). Let us consider an infinitesimal search on product 1 at (u, u), with utility update du. By conditioning on du, we have the following equality:

$$\begin{aligned} &P_0(u) \\ &= \frac{1}{2}\Pr[\text{exit} \mid du \geq 0] + \frac{1}{2}\Pr[\text{exit} \mid du < 0] \\ &= \frac{1}{2}[1 - q_1(u + du, u)]P_0(u) + \frac{1}{2}[1 - q_1(u, u - du)]P_0(u - du) \\ &= P_0(u) - \frac{du}{2}\left[P_0'(u) - \left(\frac{\partial q_1(u, u)}{\partial u_2} - \frac{\partial q_1(u, u)}{\partial u_1}\right)P_0(u)\right], \end{aligned} \quad (47)$$

where the last equality is obtained by doing a Taylor expansion of $q_1(u + du, u)$, $q_1(u, u - du)$, and $P_0(u - du)$. Then we have

$$\frac{P_0'(u)}{P_0(u)} = \frac{\partial q_1(u, u)}{\partial u_2} - \frac{\partial q_1(u, u)}{\partial u_1} = -\frac{2}{a[1 + W(e^{-(2u/a+1)})]}. \quad (48)$$

Combining the differential equation above with the initial condition $P_0(-a) = 1$, we can solve $P_0(u)$ as

$$P_0(u) = W(e^{-(2u/a+1)}). \quad (49)$$

Starting from (u_1, u_2) with $u_1 \geq u_2$, the consumer searches for information on product 1. With probability $q_1(u_1, u_2)$, she hits the boundary $\overline{U}(u_2)$ first and purchases product 1 right away. With probability $1 - q_1(u_1, u_2)$, she hits u_2 first. And then starting from (u_2, u_2), she eventually purchases product 1 with probability $\frac{1}{2}[1 - P_0(u_2)]$. Therefore, we have

$$P_1(u_1, u_2) = q_1(u_1, u_2) + [1 - q_1(u_1, u_2)]\frac{1}{2}[1 - P_0(u_2)], \quad -a < u_2 < u_1 < \overline{U}(u_2). \quad (50)$$

Similarly, starting from (u_1, u_2) with $u_1 < u_2$, the consumer searches on product 2. With probability $1 - q_1(u_2, u_1)$, she hits u_1 first. And then starting from (u_1, u_1), she eventually purchases product 1 with probability $\frac{1}{2}[1 - P_0(u_1)]$:

$$P_1(u_1, u_2) = [1 - q_1(u_2, u_1)]\frac{1}{2}[1 - P_0(u_1)], \quad -a < u_1 < u_2 < \overline{U}(u_1). \quad (51)$$

By combining all the scenarios above, we have Equation (19). □

Comparative Statics of Purchase Likelihoods in Figure 5

We prove the comparative statics of $P_1(u_1, u_2)$ first and then those of $P(u_1, u_2)$. We only focus on the region where $-a < u_1 < \overline{U}(u_2)$ and $-a < u_2 < \overline{U}(u_1)$. For other regions, the proof is straightforward and thus omitted.

To prove the comparative statics of $P_1(u_1, u_2)$, we consider two cases. In the first case with $u_2 > u_1$, we have

$$P_1(u_1, u_2) = \frac{\overline{U}(u_1) - u_2}{\overline{U}(u_1) - u_1} - \frac{\overline{U}(u_1) - u_2}{2a}. \tag{52}$$

Given u_1 and u_2,

$$\frac{\partial P_1(u_1, u_2)}{\partial a} = -\frac{u_1 + u_2}{2a^2} + \frac{2u_1(u_2 - u_1)(\overline{U}(u_1) - u_1 - a)}{(\overline{U}(u_1) - u_1)^3 a} + \frac{u_2}{(\overline{U}(u_1) - u_1)a}. \tag{53}$$

If $u_1 \geq 0$, it is easy to verify that

$$\frac{\partial P_1(u_1, u_2)}{\partial a} > 0$$

$$\Leftrightarrow \quad u_2 > \frac{u_1(\overline{U}(u_1) - u_1)^3 + 4au_1^2(\overline{U}(u_1) - u_1 - a)}{2a(\overline{U}(u_1) - u_1)^2 - (\overline{U}(u_1) - u_1)^3 + 4au_1(\overline{U}(u_1) - u_1 - a)}. \tag{54}$$

Otherwise, if $u_1 < 0$, one can show that $\partial P_1(u_1, u_2)/\partial a > 0$. In fact, $\partial P_1(u_1, u_2)/\partial a$ is a linear function of u_2. It suffices to verify that

$$\left.\frac{\partial P_1(u_1, u_2)}{\partial a}\right|_{u_2=0} = -\frac{u_1}{2a^2} - \frac{2u_1^2(\overline{U}(u_1) - u_1 - a)}{(\overline{U}(u_1) - u_1)^3 a} > 0, \tag{55}$$

$$\left.\frac{\partial P_1(u_1, u_2)}{\partial a}\right|_{u_2=\overline{U}(u_1)} = -\frac{u_1 + \overline{U}(u_1)}{2a^2} + \frac{2u_1(\overline{U}(u_1) - u_1 - a)}{(\overline{U}(u_1) - u_1)^2 a} + \frac{\overline{U}(u_1)}{(\overline{U}(u_1) - u_1)a} > 0. \tag{56}$$

In the second case with $u_2 \leq u_1$, we have

$$P_1(u_1, u_2) = 1 - \frac{\overline{U}(u_2) - u_1}{2a}. \tag{57}$$

Given u_1 and u_2,

$$\frac{\partial P_1(u_1, u_2)}{\partial a} = -\frac{u_1 + u_2}{2a^2} + \frac{u_2}{(\overline{U}(u_2) - u_2)a} > 0 \quad \Leftrightarrow \quad u_1 < u_2 + \frac{2au_2}{\overline{U}(u_2) - u_2}. \tag{58}$$

Now let us turn to the comparative statics of $P(u_1, u_2)$. Because of symmetry, we only need to consider the case with $u_1 \geq u_2$. We have

$$P(u_1, u_2) = 1 - \frac{\overline{U}(u_2) - u_1}{a} + \frac{\overline{U}(u_2) - u_1}{\overline{U}(u_2) - u_2}. \tag{59}$$

Given u_1 and u_2,

$$\frac{\partial P(u_1, u_2)}{\partial a} \propto 2au_2(u_1 - u_2) - (u_1 + u_2)a^2 - (u_1 + u_2) \cdot (\overline{U}(u_2) - u_2 - a)(\overline{U}(u_2) - u_2 + a). \tag{60}$$

If $u_1 \geq 0$, it is easy to verify that

$$\frac{\partial P(u_1, u_2)}{\partial a} > 0$$

$$\Leftrightarrow \quad u_1 < -\frac{u_2 a(2u_2 + a) + u_2(\overline{U}(u_2) - u_2 - a)(\overline{U}(u_2) - u_2 + a)}{a^2 + (\overline{U}(u_2) - u_2 - a)(\overline{U}(u_2) - u_2 + a) - 2au_2}. \tag{61}$$

Otherwise, if $u_1 < 0$, one can show that $\partial P(u_1, u_2)/\partial a > 0$. In fact, $\partial P(u_1, u_2)/\partial a$ is a linear function of u_1. It suffices to verify that for $-a \leq u_2 < 0$ we have

$$\left.\frac{\partial P(u_1, u_2)}{\partial a}\right|_{u_1=0} \propto -3au_2^2 - u_2(\overline{U}(u_2) - u_2 - a)(\overline{U}(u_2) - u_2 + a) > 0, \tag{62}$$

$$\left.\frac{\partial P(u_1, u_2)}{\partial a}\right|_{u_1=-a} \propto a^3 - 3a^2u_2 - 2au_2^2 - (u_2 - a)(\overline{U}(u_2) - u_2 - a) \cdot (\overline{U}(u_2) - u_2 + a) > 0. \quad \square \tag{63}$$

Smooth-Pasting Conditions for Correlated Products

We derive the smooth-pasting condition (12) for product 1, with the two products informationally correlated. We focus on the case with $0 < \rho < 1$ below. (The case of $\rho < 0$ can be obtained similarly.) Similarly to the proof of the case with $\rho = 0$, let us consider an extra infinitesimal search at the boundary $(\overline{U}(u_2), u_2)$. By searching for information on product 1 for extra time dt, the consumer earns an extra utility update du for product 1 and $\rho\, du$ for product 2. The utility update du can be either greater or less than zero with the same probability $\frac{1}{2}$. Let us first consider the scenario with $du \geq 0$. Now the consumer has a higher expected utility of product 1 with $du_1 = du$, which may drive the consumer to purchase product 1 and leave the market right away. However, at the same time, the consumer's expected utility of product 2 also increases by $du_2 = \rho\, du$, which rises the $\overline{U}(u_2)$ by $\rho \overline{U}'(u_2)\, du$. As a result, it is also possible for her to continue staying in the market. The choice between immediate purchase and continuation of search depends on the comparison between the utility update $du_1 = du$ and the update $\rho \overline{U}'(u_2)\, du$.

Consequently, if $\overline{U}'(u_2) < 1/\rho$, the consumer will purchase the product 1 and leave the market with utility $\overline{U}(u_2) + \mathrm{E}[du \mid du \geq 0]$; otherwise, if $\overline{U}'(u_2) \geq 1/\rho$, the consumer stays in the market to continue searching for information with expected utility $\mathrm{E}[V(\overline{U}(u_2) + du, u_2 + \rho du \mid du \geq 0)]$.

Similarly, we have the following assertions for the case with $du < 0$. If $\overline{U}'(u_2) \leq 1/\rho$, the consumer will continue searching for information with expected utility $\mathrm{E}[V(\overline{U}(u_2) + du, u_2 + \rho du \mid du < 0)]$; otherwise, if $\overline{U}'(u_2) > 1/\rho$, the consumer purchases the product 1 and leaves the market with expected utility $\overline{U}(u_2) + \mathrm{E}[du \mid du < 0]$.

To summarize, if $\overline{U}'(u_2) < 1/\rho$, the consumer's expected utility of searching extra dt on product 1 is

$$\begin{aligned} &V_1(\overline{U}(u_2), u_2) \\ &\equiv -c\,dt + \frac{1}{2}(\overline{U}(u_2) + \mathrm{E}[du \mid du \geq 0]) \\ &\quad + \frac{1}{2}\mathrm{E}[V(\overline{U}(u_2) + du, u_2 + \rho du \mid du < 0)] \\ &= V(\overline{U}(u_2), u_2) + \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}[1 - V_{u_1}(\overline{U}(u_2), u_2) - \rho V_{u_2}(\overline{U}(u_2), u_2)] \\ &\quad + o(\sqrt{dt}). \end{aligned} \tag{64}$$

If $\bar{U}'(u_2) > 1/\rho$, the consumer's expected utility of searching extra dt on product 1 is

$$\begin{aligned} &V_1(\bar{U}(u_2), u_2) \\ &\equiv -c\,dt + \frac{1}{2}(\bar{U}(u_2) + \mathrm{E}[du \mid du < 0]) \\ &\quad + \frac{1}{2}\mathrm{E}[V(\bar{U}(u_2) + du, u_2 + \rho du \mid du \geq 0)] \\ &= V(\bar{U}(u_2), u_2) - \frac{\sigma}{2}\sqrt{\frac{dt}{2\pi}}[1 - V_{u_1}(\bar{U}(u_2), u_2) - \rho V_{u_2}(\bar{U}(u_2), u_2)] \\ &\quad + o(\sqrt{dt}). \end{aligned} \tag{65}$$

Finally, if $\bar{U}'(u_2) = 1/\rho$, the consumer's expected utility of searching extra dt on product 1 is

$$\begin{aligned} V_1(\bar{U}(u_2), u_2) &\equiv -c\,dt + \frac{1}{2}\mathrm{E}[V(\bar{U}(u_2) + du, u_2 + \rho du \mid du < 0)] \\ &\quad + \frac{1}{2}\mathrm{E}[V(\bar{U}(u_2) + du, u_2 + \rho du \mid du \geq 0)] \\ &= V(\bar{U}(u_2), u_2) + o(\sqrt{dt}), \end{aligned} \tag{66}$$

which is the same as Equation (9).

On the other hand, when the consumer searches product 2 for extra dt at the boundary $(\bar{U}(u_2), u_2)$, we apply the same analysis above and conclude that the consumer's expected utility of searching extra dt on product 2 is

$$V_2(\bar{U}(u_2), u_2) \equiv \begin{cases} V(\bar{U}(u_2), u_2) + \dfrac{\sigma}{2}\sqrt{\dfrac{dt}{2\pi}}[\rho - \rho V_{u_1}(\bar{U}(u_2), u_2) \\ \quad - V_{u_2}(\bar{U}(u_2), u_2)] + o(\sqrt{dt}) \quad \text{if } \bar{U}'(u_2) < \rho, \\ V(\bar{U}(u_2), u_2) - \dfrac{\sigma}{2}\sqrt{\dfrac{dt}{2\pi}}[\rho - \rho V_{u_1}(\bar{U}(u_2), u_2) \\ \quad - V_{u_2}(\bar{U}(u_2), u_2)] + o(\sqrt{dt}) \quad \text{if } \bar{U}'(u_2) > \rho, \\ V(\bar{U}(u_2), u_2) + o(\sqrt{dt}) \qquad \text{otherwise.} \end{cases} \tag{67}$$

So far, we have the expected utility of searching extra dt on product 1 and 2. The consumer will choose to search for information on the product with greater expected utility, so her expected utility at the boundary $(\bar{U}(u_2), u_2)$ is $\max\{V_1(\bar{U}(u_2), u_2), V_2(\bar{U}(u_2), u_2)\}$. At the same time, the consumer's expected utility at $(\bar{U}(u_2), u_2)$ is given exactly by $V(\bar{U}(u_2), u_2)$. To make the above two expressions identical, the $\sqrt{dt}$-order term must vanish. We obtain the following set of equations:

$$\begin{cases} \max\{1 - V_{u_1} - \rho V_{u_2}, \rho - \rho V_{u_1} - V_{u_2}\} = 0 \\ \quad \text{if } \rho > \bar{U}'(u_2) \geq 0, \\ \max\{1 - V_{u_1} - \rho V_{u_2}, 0\} = 0 \quad \text{if } \bar{U}'(u_2) = \rho, \\ \max\{1 - V_{u_1} - \rho V_{u_2}, -\rho + \rho V_{u_1} + V_{u_2}\} = 0 \\ \quad \text{if } 1/\rho > \bar{U}'(u_2) > \rho, \\ \max\{0, -\rho + \rho V_{u_1} + V_{u_2}\} = 0 \quad \text{if } \bar{U}'(u_2) = 1/\rho, \\ \max\{-1 + V_{u_1} + \rho V_{u_2}, -\rho + \rho V_{u_1} + V_{u_2}\} = 0 \\ \quad \text{if } \bar{U}'(u_2) > 1/\rho. \end{cases} \tag{68}$$

To simplify notation, we have dropped $(\bar{U}(u_2), u_2)$ in writing the value function V. Let us first have a look at the case with $\rho > \bar{U}'(u_2) \geq 0$. The first equation in (68) implies either $\rho - \rho V_{u_1} - V_{u_2} = 0 \geq 1 - V_{u_1} - \rho V_{u_2}$ or $1 - V_{u_1} - \rho V_{u_2} = 0 \geq \rho - \rho V_{u_1} - V_{u_2}$. In the latter case with $\rho - \rho V_{u_1} - V_{u_2} = 0$, we have $V_{u_1} = 1 - (1/\rho)V_{u_2}$. Then $0 \geq 1 - V_{u_1} - \rho V_{u_2} = (1/\rho - \rho) \cdot V_{u_2} \geq 0$. So we must have $1 - V_{u_1} - \rho V_{u_2} = 0$ in either case. Therefore, the first equation in (68) is equivalent to $1 - V_{u_1} - \rho V_{u_2} = 0$. With a similar argument, we can show that the above set of equations can be equivalently rewritten as

$$\begin{cases} V_{u_1} + \rho V_{u_2} = 1 & \text{if } \rho > \bar{U}'(u_2) \geq 0, \\ V_{u_1} + \rho V_{u_2} \geq 1 & \text{if } \bar{U}'(u_2) = \rho, \\ V_{u_1} = 1 \text{ and } V_{u_2} = 0 & \text{if } 1/\rho > \bar{U}'(u_2) > \rho, \\ \rho V_{u_1} + V_{u_2} \leq \rho & \text{if } \bar{U}'(u_2) = 1/\rho, \\ \rho V_{u_1} + V_{u_2} = \rho & \text{if } \bar{U}'(u_2) > 1/\rho. \end{cases} \tag{69}$$

Now, by taking the derivative of both sides of Equation (11) with respect to u_2, we have $(1 - V_{u_1})\bar{U}'(u_2) = V_{u_2}$. If $\rho > \bar{U}'(u_2) \geq 0$ and $V_{u_1} \neq 1$, we have $\bar{U}'(u_2) = V_{u_2}/(1 - V_{u_1}) = 1/\rho > \rho$ by Equation (69), which is a contradiction. Therefore, if $\rho > \bar{U}'(u_2) \geq 0$, there must be $V_{u_1} = 1$ and $V_{u_2} = 0$. Similarly, we can show that if $\bar{U}'(u_2) > 1/\rho$, or $\bar{U}'(u_2) = \rho$ or $\bar{U}'(u_2) = 1/\rho$, there must be $V_{u_1} = 1$ and $V_{u_2} = 0$, too. In summary, we have obtained Equation (12) for the general case with $0 < \rho < 1$. □

Proof of Proposition 2. We prove the comparative statics for the purchase threshold first. From Equation (25), we know that we only need to show that when $u \geq -(1 - 2\rho)a$, $\bar{U}(u) = u + (1 - \rho^2)W(e^{-2u/((1-\rho^2)a) - (1-4\rho+\rho^2)/(1-\rho^2)})a + (1 - \rho)^2 a$ increases in a and decreases in ρ.

In fact, when $u \geq -(1 - 2\rho)a$, we have the inequalities $0 < W(e^{-2u/((1-\rho^2)a) - (1-4\rho+\rho^2)/(1-\rho^2)}) \leq 1$. Let us introduce the notation $w \equiv 1 + W(e^{-2u/((1-\rho^2)a) - (1-4\rho+\rho^2)/(1-\rho^2)})$. Then, we have $1 < w \leq 2$, and $\ln(w - 1) \leq 0$. We have

$$\begin{aligned} \frac{\partial \bar{U}(u)}{\partial a} &= (1 - \rho)^2 + (1 - \rho^2)W(e^{-2u/((1-\rho^2)a) - (1-4\rho+\rho^2)/(1-\rho^2)}) \\ &\quad + \frac{2u}{a}\left[1 - \frac{1}{1 + W(e^{-2u/((1-\rho^2)a) - (1-4\rho+\rho^2)/(1-\rho^2)})}\right] \\ &= (\rho^2 - 2\rho)\frac{2 - w}{w} + 1 - (1 - \rho^2)\frac{w - 1}{w}\ln(w - 1) \\ &\geq -\frac{2 - w}{w} + 1 - (1 - \rho^2)\frac{w - 1}{w}\ln(w - 1) \\ &= 2\frac{w - 1}{w} - (1 - \rho^2)\frac{w - 1}{w}\ln(w - 1) > 0. \end{aligned} \tag{70}$$

Similarly, one can show that

$$\frac{\partial \bar{U}(u)}{\partial \rho} = -2(1 - \rho)a\frac{2 - w}{w} + 2\rho a\frac{w - 1}{w}\ln(w - 1) \leq 0. \tag{71}$$

Now, we start proving the comparative statics for the maximum expected utility $V(u_1, u_2)$. The function $V(u_1, u_2)$ is continuous and symmetric with respect to $u_1 = u_2$. It suffices to show that when $\rho u_1 - (1 - \rho)a \leq u_2 \leq u_1 \leq \bar{U}(u_2)$, $V(u_1, u_2)$ increases in a and decreases in ρ.

Let us introduce the notation

$$\tilde{w} \equiv 1 + W\left(\frac{1 + \rho}{1 - \rho}\right)e^{-2\frac{u_2 - \rho u_1}{(1-\rho)^2(1+\rho)a} - \frac{1-2\rho-\rho^2}{1-\rho^2}}.$$

When $\rho u_1-(1-\rho)a\le u_2\le u_1\le \bar{U}(u_2)$, we have $u_1\le \hat{U}(u_1,u_2)$ and $1<\tilde{w}\le 2/1-\rho$. We have

$$\begin{aligned}\frac{\partial V(u_1,u_2)}{\partial a}&=\frac{\hat{U}(u_1,u_2)-u_1}{4a(1-\rho)}\cdot\biggl[(u_1-u_2)+(1-\rho)^2a\tilde{w}\\&\qquad+\frac{4(u_2-\rho u_1)}{1+\rho}\Bigl(1-\frac{1}{\tilde{w}}\Bigr)\biggr]\\&\propto (u_1-u_2)+(1-\rho)^2a\tilde{w}+\frac{4(u_2-\rho u_1)}{1+\rho}\Bigl(1-\frac{1}{\tilde{w}}\Bigr)\\&=(u_1-u_2)+2(1-\rho)^2a+\frac{4\rho(1-\rho)a}{1+\rho}\\&\quad-2(1-\rho)^2a\Bigl(1-\frac{1}{\tilde{w}}\Bigr)\ln\Bigl[\frac{1-\rho}{1+\rho}(\tilde{w}-1)\Bigr]\\&\quad-\Bigl[(1-\rho)^2a\tilde{w}+\frac{4\rho(1-\rho)a}{1+\rho}\frac{1}{\tilde{w}}\Bigr].\end{aligned}\tag{72}$$

In the last equality, each term in the first line is nonnegative, whereas the term in the second line is negative. Let us define an auxiliary function $h(\tilde{w})\equiv(1-\rho)^2a\tilde{w}+(4\rho(1-\rho)a/(1+\rho))(1/\tilde{w})$. It is easy to show that $h(\tilde{w})$ is unimodal with a minimum at $\tilde{w}^*=2\sqrt{\rho/(1-\rho^2)}<2/(1-\rho)$. Thus, for $\tilde{w}\in[1,2/(1-\rho)]$, $h(\tilde{w})$ must reach a maximum at either $\tilde{w}=1$ or $\tilde{w}=2/(1-\rho)$. At the same time, $h(2/(1-\rho))-h(1)=(1-\rho)^2a\ge 0$. Then, for $\tilde{w}\in(1,2/(1-\rho)]$, $h(\tilde{w})$ must reach a maximum at $\tilde{w}=2/(1-\rho)$. Consequently, $h(\tilde{w})\le h(2/(1-\rho))$, for all $\tilde{w}\in(1,2/(1-\rho)]$. Therefore, we have

$$\begin{aligned}&\frac{\partial V(u_1,u_2)}{\partial a}\\&\ge(u_1-u_2)+2(1-\rho)^2a+\frac{4\rho(1-\rho)a}{1+\rho}\\&\quad-2(1-\rho)^2a\Bigl(1-\frac{1}{\tilde{w}}\Bigr)\ln\Bigl[\frac{1-\rho}{1+\rho}(\tilde{w}-1)\Bigr]-h\Bigl(\frac{2}{1-\rho}\Bigr)\\&=(u_1-u_2)-2(1-\rho)^2a\Bigl(1-\frac{1}{\tilde{w}}\Bigr)\ln\Bigl[\frac{1-\rho}{1+\rho}(\tilde{w}-1)\Bigr]\ge 0.\end{aligned}\tag{73}$$

Similarly, we can show that

$$\begin{aligned}&\frac{\partial V(u_1,u_2)}{\partial \rho}\\&=-\frac{\hat{U}(u_1,u_2)-u_1}{2(1+\rho)(1-\rho)^3a\tilde{w}}\biggl\{[(u_1-u_2)+2(1-\rho)^2a(2-\tilde{w})]\\&\qquad\cdot\Bigl(\frac{2}{1-\rho}-\tilde{w}\Bigr)-2\rho(1-\rho)a(\tilde{w}-1)\ln\Bigl[\frac{1-p}{1+p}(\tilde{w}-1)\Bigr]\biggr\}\\&\le-\frac{\hat{U}(u_1,u_2)-u_1}{2(1+\rho)(1-\rho)^3a\tilde{w}}\biggl\{[(u_1-u_2)+2(1-\rho)^2a(2-\tilde{w})]\\&\qquad\cdot\Bigl(\frac{2}{1-\rho}-\tilde{w}\Bigr)-2\rho(1-\rho)a(\tilde{w}-1)\Bigl[\frac{1-p}{1+p}(\tilde{w}-1)-1\Bigr]\biggr\}\\&=-\frac{\hat{U}(u_1,u_2)-u_1}{2(1+\rho)(1-\rho)^3a\tilde{w}}\Bigl[(u_1-u_2)+\frac{(1-\rho)^3}{1+\rho}\Bigl(\frac{2}{1-\rho}-\tilde{w}\Bigr)\Bigr]\\&\quad\cdot\Bigl(\frac{2}{1-\rho}-\tilde{w}\Bigr)\le 0.\quad\square\end{aligned}\tag{74}$$

Optimal Search with Heterogeneous Products

Theorem 9. *There exists a unique solution $V(u_1,u_2)$ along with boundaries $\bar{U}_i(\cdot)$ and $\underline{U}_i(\cdot)$ $(i=1,2)$ that satisfies Equations (5), (28), and (11)–(14). The value function is*

$$V(u_1,u_2)=\begin{cases}\dfrac{1}{4a_1}[\bar{U}_1(u_2)-u_1]^2+u_1 & \text{if } u_2+\bar{U}_2(u_1)-\bar{U}_1(u_2)\le u_1\\ & \quad\le\bar{U}(u_2)\text{ and } u_1\ge\underline{U}(u_2),\\ \dfrac{1}{4a_2}[\bar{U}_2(u_1)-u_2]^2+u_2 & \text{if } u_1+\bar{U}_1(u_2)-\bar{U}_2(u_1)\le u_2\\ & \quad\le\bar{U}(u_1)\text{ and } u_2\ge\underline{U}(u_1),\\ \max\{0,u_1,u_2\} & \text{otherwise.}\end{cases}\tag{75}$$

Without loss of generality, assume $a_1>a_2$. The purchase boundary $\bar{U}_1(\cdot)$ is given as

$$\bar{U}_1(u)=\begin{cases}u+a_1 & \text{if } u>u^*,\\ u+\dfrac{a_1-a_2Z_1(u)}{1-Z_1(u)} & \text{if } -a_2<u\le u^*,\\ a_1 & \text{otherwise,}\end{cases}\tag{76}$$

where $u^*\equiv-(\sqrt{a_1a_2}/2)\ln((\sqrt{a_1}-\sqrt{a_2})/(\sqrt{a_1}+\sqrt{a_2}))>0$. The purchase boundary $\bar{U}_2(\cdot)$ is supported in $(-\infty,u^*-a_1]$ and is given as

$$\bar{U}_2(u)=\begin{cases}u+\dfrac{a_1Z_2(u)-a_2}{Z_2(u)-1} & \text{if } -a_1<u\le u^*-a_1,\\ a_2 & \text{if } u\le -a_1.\end{cases}\tag{77}$$

The functions $Z_1(u)<1$ and $Z_2(u)>1$ are defined implicitly by the following two equations, respectively:

$$\frac{\sqrt{a_2/a_1}-\sqrt{Z_1(u)}}{1-Z_1(u)}+\frac{1}{2}\ln\frac{1-\sqrt{Z_1(u)}}{1+\sqrt{Z_1(u)}}=\frac{u}{\sqrt{a_1a_2}}+\sqrt{\frac{a_2}{a_1}}+\frac{1}{2}\ln\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}};\tag{78}$$

$$\frac{\sqrt{Z_2(u)}-\sqrt{a_1/a_2}}{Z_2(u)-1}+\frac{1}{2}\ln\frac{\sqrt{Z_2(u)}-1}{\sqrt{Z_2(u)}+1}=\frac{u}{\sqrt{a_1a_2}}+\sqrt{\frac{a_1}{a_2}}+\frac{1}{2}\ln\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}.\tag{79}$$

The exit boundaries $\underline{U}_i(\cdot)$ $(i=1,2)$ are given as

$$\underline{U}_1(u)=\begin{cases}-a_1 & \text{if } u\le -a_2,\\ u-a_1 & \text{if } u\ge u^*;\end{cases}\tag{80}$$

$$\underline{U}_2(u)=-a_2\ (\text{relevant when } u\le -a_1).\tag{81}$$

Proof. It is straightforward to verify that the solution satisfies Equations (5), (28), and (11)–(14). The more difficult part comes from the verification of the value-matching and smooth-pasting conditions at internal boundaries.[21] There are four internal boundaries: $\mathscr{C}_1\equiv\{(u_1,u_2)\mid \bar{U}_1(u_2)+u_1=u_2+\bar{U}_2(u_1)\text{ and }-a_2\le u_2\le u^*\}$, $\mathscr{C}_2\equiv\{(u_1,u_2)\mid u_1=-a_1,-a_2\le u_2\le a_2\}$, $\mathscr{C}_3\equiv\{(u_1,u_2)\mid u_2=-a_2,-a_1\le u_1\le a_1\}$, and $\mathscr{C}_4\equiv\{(u_1,u_2)\mid u_2=u^*,u^*-a_1\le u_1\le u^*+a_1\}$. Verifications of the

[21] See the proof of Theorem 1 for explanation of internal boundaries.

boundary conditions at $\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_4$ are straightforward and thus omitted here. We focus on the value-matching and smooth-pasting conditions at boundary $\mathscr{C}_1$ below, which is the boundary separating "search product 1" from "search product 2."

Given $-a_2 \le u_2 \le u^*$ implied from $\mathscr{C}_1$, the purchase boundaries can be written as

$$\bar{U}_1(u) = u + \frac{a_1 - a_2 Z_1(u)}{1 - Z_1(u)}, \tag{82}$$

$$\bar{U}_2(u) = u + \frac{a_1 Z_2(u) - a_2}{Z_2(u) - 1}, \tag{83}$$

where $Z_i(u)$ $(i = 1, 2)$ are given in Equations (78) and (79). It is straightforward to show that the left-hand sides of Equations (78) and (79), as a function of $Z_1(u)$ and $Z_2(u)$, respectively, are monotonic. Therefore, $Z_1(u)$ and $Z_2(u)$ are well defined. One can verify that $\bar{U}_1(u)$ and $\bar{U}_2(u)$ satisfy the following ordinary differential equations subject to the boundary conditions:[22]

$$\bar{U}_1'(u) = \frac{\sqrt{a_1a_2(a_1 + u - \bar{U}_1(u))(a_2 + u - \bar{U}_1(u))} - a_1a_2}{a_2(u - \bar{U}_1(u))}, \quad \bar{U}_1(-a_2) = a_1; \tag{84}$$

$$\bar{U}_2'(u) = \frac{\sqrt{a_1a_2(a_1 + u - \bar{U}_2(u))(a_2 + u - \bar{U}_2(u))} - a_1a_2}{a_1(u - \bar{U}_2(u))}, \quad \bar{U}_2(-a_1) = a_2. \tag{85}$$

Given $(u_1, u_2) \in \mathscr{C}_1$, our objective is to verify that u_1 and u_2 satisfy the following value-matching and smooth-pasting conditions:

$$\frac{1}{4a_1}(\bar{U}_1(u_2) - u_1)^2 + u_1 = \frac{1}{4a_2}(\bar{U}_2(u_1) - u_2)^2 + u_2, \tag{86}$$

$$-\frac{1}{2a_1}(\bar{U}_1(u_2) - u_1) + 1 = \frac{1}{2a_2}(\bar{U}_2(u_1) - u_2)\bar{U}_2'(u_1), \tag{87}$$

$$-\frac{1}{2a_2}(\bar{U}_2(u_1) - u_2) + 1 = \frac{1}{2a_1}(\bar{U}_1(u_2) - u_1)\bar{U}_1'(u_2). \tag{88}$$

By substituting the expressions of $\bar{U}_1'(u)$ and $\bar{U}_2'(u)$ in Equations (84) and (85) into the three equations above, one can show that they are not independent—only two of the three equations are independent. By substituting the expressions of $\bar{U}_1(u)$ and $\bar{U}_2(u)$ in Equations (82) and (83), we can rewrite the three equations equivalently as follows:

$$\sqrt{Z_1(u_2)} = \frac{\sqrt{a_1a_2} - \sqrt{(u_1 - u_2)(a_1 - a_2 + u_1 - u_2)}}{a_2 - u_1 + u_2}, \tag{89}$$

$$\sqrt{Z_2(u_1)} = \frac{\sqrt{a_1a_2} + \sqrt{(u_1 - u_2)(a_1 - a_2 + u_1 - u_2)}}{a_1 + u_1 - u_2}. \tag{90}$$

To reiterate, our equivalent objective now is to verify that given $(u_1, u_2) \in \mathscr{C}_1$, u_1 and u_2 satisfy Equations (89) and (90). In fact, because $(u_1, u_2) \in \mathscr{C}_1$, we know that $\bar{U}_1(u_2) + u_1 = u_2 + \bar{U}_2(u_1)$, which implies

$$Z_1(u_2)Z_2(u_1) = 1, \tag{91}$$

which implies

$$\ln\frac{1 - \sqrt{Z_1(u_2)}}{1 + \sqrt{Z_1(u_2)}} = \ln\frac{\sqrt{Z_2(u_1)} - 1}{\sqrt{Z_2(u_1)} + 1}.$$

On the basis of this fact, we take $u = u_2$ in Equation (78) and $u = u_1$ in Equation (79) and subtract these two equations to get

$$\frac{\sqrt{a_2/a_1} - \sqrt{Z_1(u_2)}}{1 - Z_1(u_2)} - \frac{\sqrt{Z_2(u_1)} - \sqrt{a_1/a_2}}{Z_2(u_1) - 1} = \frac{u_2 - u_1}{\sqrt{a_1a_2}} + \sqrt{\frac{a_2}{a_1}} - \sqrt{\frac{a_1}{a_2}}. \tag{92}$$

By combining and solving Equations (91) and (92), we actually prove that $Z_1(u_2)$ and $Z_2(u_1)$ satisfy Equations (89) and (90). □

Proof of Corollary 3. The monotonicity of $\bar{U}(u_i, u_j) - u_{i\vee j}$ with respect to $u_{i\vee j}$ and $u_{i\wedge j}$ is straightforward to show by taking derivatives and thus omitted here. Suppose $u_1 > u_2 \to +\infty$; then

$$\begin{aligned}
&\bar{U}(u_1, u_2) \\
&= \lim_{u_1 > u_2 \to +\infty} u_1 + \left[1 + W\left(e^{-2-(2u_1+u_2)/a}\frac{1 + 4W\left(\left(\frac{1}{2}\right)e^{-7/4-9u_2/(4a)}\right)}{6 \times 2^{1/3}W\left(\left(\frac{1}{2}\right)e^{-7/4-9u_2/(4a)}\right)^{4/3}}\right)\right]a \\
&= u_1 + \left[1 + W\left(e^{-2-2\Delta u/a} \cdot \lim_{u_2\to+\infty}\frac{e^{-3u_2/a}\left[1 + 4W\left(\left(\frac{1}{2}\right)e^{-7/4-9u_2/4a}\right)\right]}{6 \times 2^{1/3}W\left(\left(\frac{1}{2}\right)e^{-7/4-9u_2/(4a)}\right)^{4/3}}\right)\right]a \\
&= u_1 + \left[1 + W\left(e^{-2-2\Delta u/a} \cdot \lim_{x \equiv e^{-u_2/a}\to 0}\frac{x^3\left[1 + 4W\left(\left(\frac{1}{2}\right)e^{-7/4}x^{9/4}\right)\right]}{6 \times 2^{1/3}W\left(\left(\frac{1}{2}\right)e^{-7/4}x^{-9/4}\right)^{4/3}}\right)\right]a \\
&= u_1 + \left[1 + W\left(e^{-2-2\Delta u/a}\lim_{x\to 0}\frac{x^3 + o(x^3)}{3e^{-7/3}x^3 + o(x^3)}\right)\right]a \\
&= u_1 + \left[1 + W\left(\frac{1}{3}e^{1/3-2(u_1-u_2)/a}\right)\right]a. \quad \square
\end{aligned} \tag{93}$$

Proof of Lemma 3. We prove by contradiction. Suppose $q_1 > q_2 \ge -a$, but $q_1 - p_1^* < q_2 - p_2^*$. From the expression of $P_i(u_1, u_2)$, we can easily get $P_1(q_1 - p_1^*, q_2 - p_2^*) < P_2(q_1 - p_1^*, q_2 - p_2^*)$. Let us define

$$p_1' \equiv q_1 - q_2 + p_2^*, \tag{94}$$

$$p_2' \equiv \max\{q_2 - q_1 + p_1^*, 0\}. \tag{95}$$

By definition $p_1', p_2' \ge 0$. Let us first consider the case $q_2 - q_1 + p_1^* > 0$. Then we have $P_1(q_1 - p_1', q_2 - p_2') = P_1(q_2 - p_2^*, q_1 - p_1^*) = P_2(q_1 - p_1^*, q_2 - p_2^*)$, where the second equality is due to the symmetry of $P_i(u_1, u_2)$. Similarly, we have $P_2(q_1 - p_1', q_2 - p_2') = P_1(q_1 - p_1^*, q_2 - p_2^*)$. Let us denote the profit under the pricing

[22] In fact, we obtain $\bar{U}_1(u)$ and $\bar{U}_2(u)$ in Equations (82) and (83) by solving these ordinary differential equations.

policy $p_i = p_i^*$ as π^*, and that under the pricing policy $p_i = p_i'$ as π'. We have

$$\begin{aligned}\pi' - \pi^* &= [p_1' P_1(q_1 - p_1', q_2 - p_2') + p_2' P_2(q_1 - p_1', q_2 - p_2')] \\ &\quad - [p_1^* P_1(q_1 - p_1^*, q_2 - p_2^*) + p_2^* P_2(q_1 - p_1^*, q_2 - p_2^*)] \quad (96)\\ &= [(q_1 - q_2 + p_2^*) P_2(q_1 - p_1^*, q_2 - p_2^*) \\ &\quad + (q_2 - q_1 + p_1^*) P_1(q_1 - p_1^*, q_2 - p_2^*)] \\ &\quad - [p_1^* P_1(q_1 - p_1^*, q_2 - p_2^*) + p_2^* P_2(q_1 - p_1^*, q_2 - p_2^*)] \quad (97)\\ &= (q_1 - q_2)[P_2(q_1 - p_1^*, q_2 - p_2^*) - P_1(q_1 - p_1^*, q_2 - p_2^*)] \\ &> 0. \quad (98)\end{aligned}$$

In the second case with $q_2 - q_1 + p_1^* \geq 0$ and $p_2' = 0$, it is easy to show that the first equality (97) above will instead take $\geq$, because $P_1(u_1, u_2)$ decreases with u_2. Therefore, we still have $\pi' > \pi^*$. This contradicts the optimality of p_i^*. □

Numerical Profit Optimization in Equation (36)

If $q_i \leq -a$, $u_i = q_i - p_i \leq -a$, product i will never be considered. In this case, for optimal pricing of a single product, the profit optimization problem is straightforward and is given by Branco et al. (2012). By symmetry, the only case we need to consider is that $q_1 > q_2 \geq -a$. In this case, Lemma 3 implies that $u_1 = q_1 - p_1^* > q_2 - p_2^* = u_2$. There are two cases. In the first case, when q_1 is much greater than q_2, and correspondingly, $u_1 \geq \bar{U}(u_2)$, the consumer will purchase product 1 immediately without any search. In this case, the seller's objective is to maximize p_1. We know that

$$p_1 = q_1 - u_1 \leq q_1 - \bar{U}(u_2) = q_1 - \bar{U}(q_2 - p_2) \leq q_1 - a. \quad (99)$$

The equal sign in the above equality holds when $p_2 \geq q_2 + a$. Therefore, the optimal price $p_1^* = q_1 - a$ and $p_2^* \in \{p_2\colon p_2 \geq q_2 + a\}$. In the second case, q_1 is greater than u_2 but not by a lot, and correspondingly, $\bar{U}(u_2) \geq u_1 > u_2$. By Equation (19), we have

$$P_1(u_1, u_2) = 1 - \frac{\bar{U}(u_2) - u_1}{2a}, \quad (100)$$

$$P_2(u_1, u_2) = \frac{\bar{U}(u_2) - u_1}{\bar{U}(u_2) - u_2} - \frac{\bar{U}(u_2) - u_1}{2a}. \quad (101)$$

By substituting these purchase likelihood functions into the optimization problem (36), we can numerically obtain the optimal prices by solving the first-order necessary conditions.

Optimal Search with Time Discounting

Theorem 10. *There exists a unique solution $V(u_1, u_2)$ along with boundaries $\bar{U}(\cdot)$ and $\underline{U}(\cdot)$ that satisfies Equations (5), (38), and (11)–(14). The value function is obtained as*

$$\begin{aligned}V(u_1,u_2) &= \left(\bar{U}(u_2) + \frac{c}{r}\right)\cosh\left[\frac{\sqrt{2r}}{\sigma}(\bar{U}(u_2) - u_1)\right] \\ &\quad - \frac{\sigma}{\sqrt{2r}}\sinh\left[\frac{\sqrt{2r}}{\sigma}(\bar{U}(u_2) - u_1)\right] - \frac{c}{r}, \\ &\qquad u_2 \leq u_1 \leq \bar{U}(u_2),\ u_1 \geq \underline{U}(u_2), \\ V(u_1,u_2) &= \left(\bar{U}(u_1) + \frac{c}{r}\right)\cosh\left[\frac{\sqrt{2r}}{\sigma}(\bar{U}(u_1) - u_2)\right] \\ &\quad - \frac{\sigma}{\sqrt{2r}}\sinh\left[\frac{\sqrt{2r}}{\sigma}(\bar{U}(u_1) - u_2)\right] - \frac{c}{r}, \\ &\qquad u_1 \leq u_2 \leq \bar{U}(u_1),\ u_2 \geq \underline{U}(u_1), \\ V(u_1,u_2) &= u_1, \quad u_1 > \bar{U}(u_2), \\ V(u_1,u_2) &= u_2, \quad u_2 > \bar{U}(u_1), \\ V(u_1,u_2) &= 0, \quad \textit{otherwise}, \end{aligned} \quad (102)$$

and the purchase and exit boundaries $\bar{U}(\cdot)$ and $\underline{U}(\cdot)$ are given as

$$\bar{U}(u) = \begin{cases} X(u) & \text{if } u \geq \underline{U}, \\ \bar{U} & \text{otherwise}, \end{cases} \quad (103)$$

$$\underline{U}(u) = \underline{U} \ (\text{relevant when } u \leq \underline{U}), \quad (104)$$

where $\bar{U}$ and $\underline{U}$ are the purchase and exit boundaries, respectively, for the optimal search problem with only one product:

$$\bar{U} = \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}, \quad (105)$$

$$\underline{U} = \bar{U} - \frac{\sigma}{\sqrt{2r}} \ln\left[\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{\frac{r\sigma^2}{2c^2} + 1}\right]. \quad (106)$$

Here, $X(u)$ is given by the following ordinary differential equation with a boundary condition:

$$\sqrt{\frac{r}{2}}\sigma \coth\left[\frac{\sqrt{2r}}{\sigma}(X(u) - u)\right] = (1 + X'(u))(c + rX(u)), \quad (107)$$

$$X(\underline{U}) = \bar{U}. \quad (108)$$

The solution is by construction, and the proof is omitted here.

Proof of Corollary 4. We first show that

$$\lim_{u \to +\infty} (\bar{U}(u) - u) = 0. \quad (109)$$

In fact, we only need to show that $\lim_{u\to+\infty}(X(u) - u) = 0$, where $X(u)$ is defined in Theorem 10. By definition, we know $X(u) \geq u$. By Lemma 2, we know $X'(u) \geq 0$. Therefore, as $u \to +\infty$, $(1 + X'(u))(c + rX(u)) \to +\infty$, which implies $\coth[(\sqrt{2r}/\sigma)(X(u) - u)] \to 0$ by Equation (107). This implies that $X(u) - u \to 0$.

Next, to show that $\bar{U}(u) - u$ decreases with u, we need to prove that

$$\bar{U}'(u) \leq 1. \quad (110)$$

We only need to show that $X'(u) \leq 1$ for $u \geq \underline{U}$, where $u \geq \underline{U}$ is defined in Theorem 10. In fact, by contradiction, suppose there exists $u_0 \geq \underline{U}$ such as $X'(u_0) > 1$. By Equation (107), we have

$$X'(u) = \sqrt{\frac{r}{2}}\sigma \frac{\coth[(\sqrt{2r}/\sigma)(X(u) - u)]}{c + rX(u)} - 1. \quad (111)$$

Taking derivatives on both sides of the equation above, we have

$$\begin{aligned}X''(u) &= -\frac{r(X'(u) - 1)}{(c + rX(u))\sinh^2[(\sqrt{2r}/\sigma)(X(u) - u)]} \\ &\quad - \frac{r^{3/2}\sigma}{\sqrt{2}(c + rX(u))^2}\coth\left[\frac{\sqrt{2r}}{\sigma}(X(u) - u)\right]X'(u). \quad (112)\end{aligned}$$

As $X'(u_0) > 1$, we have $X''(u_0) < 0$ by the equation above. This implies that for any small positive number ε, $X'(u_0 - \varepsilon) \simeq X'(u_0) - X''(u_0)\varepsilon > X'(u_0) > 1$, which in turn implies that $X''(u_0 - \varepsilon) < 0$ by using the expression of $X''(u)$ above. By mathematical induction, we can show that for all $u_0 \geq u \geq \underline{U}$, we should have $X'(u) > 1$ and $X''(u) < 0$. However, we know that $X'(\underline{U}) = 0$, which is a contradiction. □

References

Bardi M, Capuzzo-Dolcetta I (2008) *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations* (Birkhäuser, Boston).

Bergemann D, Välimäki J (1996) Learning and strategic pricing. *Econometrica* 64(5):773–705.

Bergman SW (1981) Acceptance sampling: The buyer's problem. Ph.D. thesis, Yale University, New Haven, CT.

Bolton P, Harris C (1999) Strategic experimentation. *Econometrica* 67(2):349–374.

Branco F, Sun M, Villas-Boas JM (2012) Optimal search for product information. *Management Sci.* 58(11):2037–2056.

Callander S (2011) Searching and learning by trial and error. *Amer. Econom. Rev.* 101(6):2277–2308.

Crandall MG, Ishii H, Lions PL (1992) User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* 27(1):1–67.

Diamond PA (1971) A model of price adjustment. *J. Econom. Theory* 3(2):156–168.

Dixit A (1993) *Art of Smooth Pasting*, Vol. 55 (Harwood Academic, Chur, Switzerland).

Ellison G, Ellison SF (2009) Search, obfuscation, and price elasticities on the Internet. *Econometrica* 77(2):427–452.

Feinberg FM, Huber J (1996) A theory of cutoff formation under imperfect information. *Management Sci.* 42(1):65–84.

Fudenberg D, Strzalecki T (2015) Dynamic logit with choice aversion. *Econometrica* 83(2):651–691.

Fudenberg D, Iijima R, Strzalecki T (2015a) Stochastic choice and revealed perturbed utility. Working paper, Harvard University, Cambridge, MA.

Fudenberg D, Strack P, Strzalecki T (2015b) Stochastic choice and optimal sequential sampling. Working paper, Harvard University, Cambridge, MA.

Gabaix X, Laibson D (2006) Shrouded attributes, consumer myopia, and information suppression in competitive markets. *Quart. J. Econom.* 121(2):505–540.

Gittins JC (1979) Bandit processes and dynamic allocation indices. *J. Roy. Statist. Soc. Ser. B* 41(2):148–164.

Gittins JC, Glazebrook KD, Weber R (1989) *Multi-Armed Bandit Allocation Indices*, 2nd ed. (John Wiley & Sons, Chichester, UK).

Glazebrook KD (1979) Stoppable families of alternative bandit processes. *J. Appl. Probab.* 16(4):843–854.

Guha S, Munagala K (2007) Approximation algorithms for budgeted learning problems. *Proc. 39th Annual ACM Sympos. Theory Comput.* (Association for Computing Machinery, New York), 104–113.

Hauser JR (2014) Consideration-set heuristics. *J. Bus. Res.* 67(8): 1688–1699.

Hauser JR, Wernerfelt B (1990) An evaluation cost model of consideration sets. *J. Consumer Res.* 16(4):393–408.

Hoffman MW, Shahriari B, de Freitas N (2013) Exploiting correlation and budget constraints in Bayesian multi-armed bandit optimization. Preprint, arXiv:1303.6746.

Krajbich I, Rangel A (2011) Multialternative drift-diffusion model predicts the relationship between visual fixations and choice in value-based decisions. *Proc. Natl. Acad. Sci. USA* 108(33): 13852–13857.

Krajbich I, Armel C, Rangel A (2010) Visual fixations and the computation and comparison of value in simple choice. *Nature Neurosci.* 13(10):1292–1298.

Kuksov D, Villas-Boas JM (2010) When more alternatives lead to less choice. *Marketing Sci.* 29(3):507–524.

Liu L, Dukes A (2013) Consideration set formation with multiproduct firms: The case of within-firm and across-firm evaluation costs. *Management Sci.* 59(8):1871–1886.

McCall JJ (1970) Economics of information and job search. *Quart. J. Econom.* 84(1):113–126.

Moscarini G, Smith L (2001) The optimal level of experimentation. *Econometrica* 69(6):1629–1644.

Peskir G, Shiryaev A (2006) *Optimal Stopping and Free-Boundary Problems* (Birkhäuser Verlag, Basel, Switzerland).

Roberts K, Weitzman ML (1981) Funding criteria for research, development, and exploration projects. *Econometrica* 49(5): 1261–1288.

Rothschild M (1974) Searching for the lowest price when the distribution of prices is unknown. *J. Political Econom.* 82(4):689–711.

Samuelson PA (1965) Proof that properly anticipated prices fluctuate randomly. *Indust. Management Rev.* 6(2):41–49.

Scheibehenne B, Greifeneder R, Todd PM (2010) Can there ever be too many options? A meta-analytic review of choice overload. *J. Consumer Res.* 37(3):409–425.

Vishwanath T (1988) Parallel search and information gathering. *Amer. Econom. Rev.* 78(2):110–116.

Weitzman ML (1979) Optimal search for the best alternative. *Econometrica* 47(3):641–654.

Wernerfelt B (1994) Selling formats for search goods. *Marketing Sci.* 13(3):298–309.

Whittle P (1980) Multi-armed bandits and the Gittins index. *J. Roy. Statist. Soc. Ser. B* 42(2):143–149.

Williams D (1991) *Probability with Martingales* (Cambridge University Press, Cambridge, UK).