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Real Options and American Derivatives: The Double Continuation Region

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We study the nonstandard optimal exercise policy associated with relevant capital investment options and with the prepayment option of widespread collateralized-borrowing contracts like the gold loan. Option exercise is optimally postponed not only when moneyness is insufficient, but also when it is excessive. We extend the classical optimal exercise properties for American options. Early exercise of an American call with a negative underlying payout rate can occur if the option is moderately in the money. We fully characterize the existence, the monotonicity, the continuity, the limits, and the asymptotic behavior at maturity of the double free boundary that separates the exercise region from the double continuation region. We find that the finite-maturity nonstandard policy conspicuously differs from the infinite-maturity one.

Keywords: American options; valuation; optimal exercise; real options; gold loan; collateralized borrowing; asymptotic approximation of the free boundary; put–call symmetry
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1. Introduction

A number of significant decision-making problems in finance can be reformulated as American option problems with an endogenous negative interest rate. Two chief examples are the prepayment option in collateralized borrowing like the recently popular gold loans and a notable class of capital investment options. An endogenous negative interest rate for the American derivatives embedded into loans collateralized by tradable assets appears whenever the loan rate is above the risk-free rate. An endogenous negative interest rate in waiting-to-invest real options appears whenever the risk-adjusted expected growth rate of the project value is above the rate used by the firm to discount it.

We show that such decision-making problems can imply a nonstandard *double* continuation region: exercise is optimally postponed not only when the option is not enough in the money (the standard part of the continuation region), but also *when the option is too deep in the money* (the nonstandard part of the continuation region). For finite-maturity and perpetual American puts and calls with a negative interest rate in a diffusive setting, we provide a detailed analysis of the conditions that enable the double continuation region and a comprehensive characterization of the double free boundary entailed by such a continuation region.[1]

Our results add to the classical optimal exercise properties for American options. Given a positive risk-free rate r, it is well known that it is always suboptimal to exercise prior to maturity an American call on a tradable asset with payout rate π equal to zero (Merton 1973) and, more generally, an American contingent claim for which the net benefit of exercising immediately is nonpositive at all times (Detemple 2006). For example, consider the optimal exercise date t^* of the prepayment option embedded into a five-year loan collateralized by gold. To maximize intuition, assume the problem is deterministic. The loan amount is q, and the current gold price is G, so that the optimal exercise date boils down to

$$t^* = \arg\max_{0\le t\le 5} e^{-rt}(Ge^{(r-\pi)t} - qe^{\gamma t})^+,$$

[1] Our single underlying result of multiple continuation regions mirrors upside down the literature documenting multiple exercise regions in models with a single underlying asset; e.g., Chiarella and Ziogas (2005) and Detemple and Emmerling (2009).

where γ is the borrowing rate commanded by the loan contract. Focus on the in-the-money case ($G > q$). If γ had been zero, the standard Merton (1973) result of $t^* = 5$ would have applied as holding gold is burdened with the storage cost $-G\pi$ (the payout rate π is negative). A positive γ that dominates the risk-free rate ($\gamma > r$) introduces a prepayment incentive for the borrower. Such an incentive is overpowered by $-G\pi$ ($t^* > 0$) when gold is markedly dear, that is, when the degree of in-the-moneyness is huge. However, the storage cost is not overwhelming, and immediate prepayment does occur ($t^* = 0$) when the loan rate γ is sufficiently high and the degree of in-the-moneyness is moderate. Fix $r = 1\%$, $\pi = -1\%$, $\gamma = 7\%$, and $q = 1$. If $G = 7$, the prepayment option exercise is optimally delayed for three years ($t^* = 3.083$), whereas if $G = 2$ the borrower exercises immediately ($t^* = 0$). The deterministic decision-making example admits a neat restatement as an American option problem with a constant strike price q and an endogenous interest rate $\rho = r - \gamma$,

$$t^* = \arg\max_{0 \le t \le 5} e^{-\rho t}(Ge^{\mu t} - q)^+,$$

where $\mu = r - \pi - \gamma$ is the gold price's adjusted drift rate. The restatement streamlines the optimal exercise analysis. If $\rho = -6\%$, $\mu = -5\%$, and $q = 1$, the incentive to postpone exercise caused by the negative interest rate ρ wins over the aversion to delay induced by the drift μ toward the out-of-the-money region ($t^* = 3.083$) for $G = 7$, whereas the incentive is insufficient ($t^* = 0$) for $G = 2$.

Our findings contribute to the vast literature on American options; see, for instance, Broadie and Detemple (1996, 2004), Detemple and Tian (2002), Detemple (2006), and, more recently, Levendorskiĭ (2008) and Medvedev and Scaillet (2010). We study the existence, the monotonicity, the continuity, the limits, and the asymptotic behavior at maturity of both the upper and the lower free boundary for the American put problem via the variational inequality approach. We then translate such results into double-free-boundary statements for the American call problem via the American put–call symmetry (e.g., Carr and Chesney 1996, Detemple 2001).

In a gold loan, the precious metal is the collateral, which the borrower has the right to redeem at any time before or at the loan maturity. We show that, since gold is a tradable investment asset with storage (and insurance) costs and without earnings, a double continuation region can ensue: the exercise of a deep in-the-money redemption option may be optimally postponed by the borrower. This is a distinct addition to the existing literature on the optimal redeeming strategy of tradable securities used as loan collateral: Xia and Zhou (2007) focus on perpetual stock loans; Ekström and Wanntorp (2008) deal with margin call stock loans; Zhang and Zhou (2009) look into stock loans in the presence of regime switching; Liu and Xu (2010) consider capped stock loans, whose subtle variational-inequality issues are studied by Liang and Wu (2012); Dai and Xu (2011) examine the impact of the dividend-distribution criterion on the stock loan. The stock loan problem comes with a standard unique free boundary because the risk-neutral percentage drift of the underlying stock price equals the risk-free rate minus a nonnegative dividend yield.

By investigating the general American option problem with a negative interest rate with possibly finite maturities, our work thoroughly extends the specific perpetual-real-option analysis developed in Battauz et al. (2012). We examine capital investment options akin to, for instance, the option of entering the lucrative but challenging business of nuclear energy. Projects may have values with conspicuous growth rates even after risk adjustment (say rates above the discount rate used by the firm), but the overall cost of entering them is likely to increase even more conspicuously in the future (uranium is a scarce resource and demand for safety is definitely increasing). Such a hierarchy in the risk-adjusted growth/discount rates for the real option leads to the nonstandard optimal continuation policy. Our work focuses on mapping in detail the finite-maturity nonstandard optimal exercise policy (see §§2 and 3) and clearly shows that the perpetual early exercise region constitutes a rather poor proxy for the finite-maturity one (see the examples in §§4 and 5).

The rest of this paper is organized as follows. Sections 2 and 3 deal with the double continuation region for American puts and calls, respectively. Sections 4 and 5 discuss the double continuation region for the redemption option embedded in a gold loan and for an interesting class of real options. Section 6 concludes, and an appendix contains all of the proofs.

2. The American Put

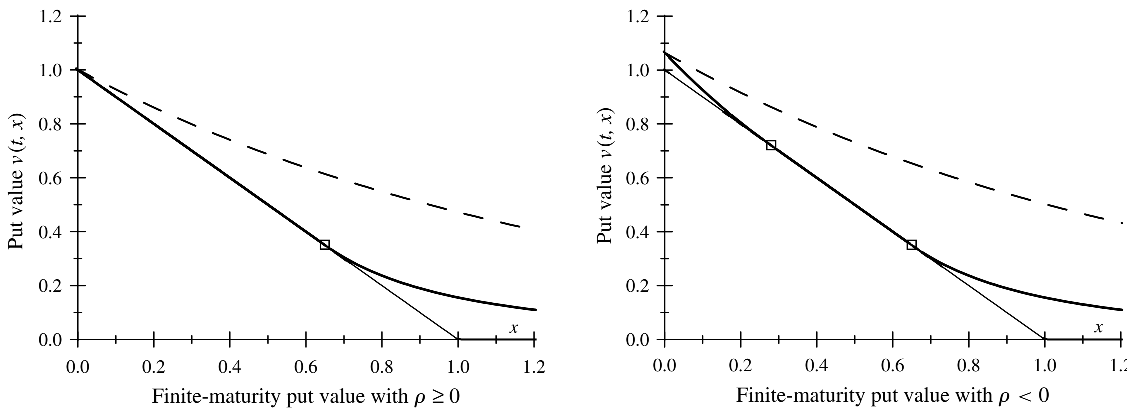
We consider an American put option written on the log-normal asset X, whose drift under the valuation measure is positive and denoted with μ. We denote the volatility with σ, the strike with K, and the interest rate with ρ. The put value at time t is given by

$$\operatorname*{ess\,sup}_{t \le \tau \le T} \mathbb{E}[e^{-\rho(\tau - t)}(K - X(\tau))^+ \mid \mathcal{F}_t] = v(t, X(t)),$$

where

$$v(t,x) = \sup_{0 \le \Theta \le T-t} \mathbb{E}\left[e^{-\rho\Theta}\left(K - x \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma B(\Theta)\right)\right)^+\right], \quad (1)$$

Figure 1 **The Value of the American Put Option $v(t,\cdot)$ (Thick Lines) and the Immediate Exercise Put Payoff (Thin Line)**



Note. The strike price is $K = 1$.

and B is a standard Brownian motion under the valuation measure. In §§2 and 3, expectations and distributions of stochastic processes refer all to the valuation measure, and, for the sake of simplicity, we will omit their dependence on the probability measure. If the option is perpetual, its value is

$$v_\infty(x) = \sup_{0\le\Theta} \mathbb{E}\left[e^{-\rho\Theta}\left(K - x\cdot\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)\Theta + \sigma B(\Theta)\right)\right)^+\right].$$

Regardless of the sign of ρ, the function v in (1) dominates the payoff function, is convex and decreasing with respect to (w.r.t.) x, is decreasing with respect to t, and is dominated by the perpetual put v_∞, that is,

$$(K-x)^+ \le v(t,x) \le v(t,0) \le v_\infty(x) \quad \text{for all } t\in[0,T] \text{ and } x\ge 0 \qquad (2)$$

(see, for instance, Karatzas and Shreve 1998, Broadie and Detemple 1997).

These properties interact with the sign of ρ to determine the shape of the free boundary and the "geometry structure" of the exercise region. More precisely, if $\rho \ge 0$, for any $t < T$, we have that $v(t,0) = \sup_{0\le\Theta\le T-t} \mathbb{E}[e^{-\rho\Theta}(K-0)^+] = (K-0)^+$. Since $v(t,x)$ coincides for $x=0$ with the immediate exercise payoff, convexity and (2) imply that either $v(t,x) > (K-x)^+$ for all $x>0$ (see the thick dashed line in the left-hand panel of Figure 1) or $v(t,x) = (K-x)^+$ for any x belonging to the interval whose extremes are 0 and

$$x^*(t) = \sup\{x\ge 0\colon v(t,x) = K-x\} \le K$$

(see the thick solid line in the left-hand panel of Figure 1). The value $x^*(t)$ is the unique *put critical price at t* with nonnegative interest rates. Detemple and Tian (2002) and Detemple (2006) show that this is true for a large class of diffusion processes with nonnegative stochastic interest rates.

On the contrary, if $\rho < 0$, then the value of the American option for $x = 0$ strictly dominates the immediate exercise payoff, because $v(t,0) = \sup_{0\le\Theta\le T-t} \mathbb{E}[e^{-\rho\Theta}(K-0)^+] = e^{-\rho\cdot(T-t)}\cdot K > K$. Then either early exercise is never optimal at date t, i.e., $v(t,x) > (K-x)^+$ for all $x>0$ (see the thick dashed line in the right-hand panel of Figure 1), or early exercise is optimal at time t for some $x' \in (0,K)$, i.e., $(K-x')^+ = v(t,x')$ (see the thick solid line in the right-hand panel of Figure 1). If x' is unique, then the exercise region collapses into a single point (the free boundary at time t). If x' is not unique, then by convexity and (2) the exercise region at time t is constituted by a connected segment defined by the extremes $l(t) \le u(t) \in [0,K]$ where[2]

$$l(t) = \inf\{x\ge 0\colon v(t,x) = (K-x)^+\}, \qquad (3)$$

$$u(t) = \sup\{x\ge 0\colon v(t,x) = (K-x)^+\}\wedge K, \qquad (4)$$

such that $v(t,x) = (K-x)^+$ for $l(t)\le x\le u(t)$, and $v(t,x) > (K-x)^+$ for $x < l(t)$ and $x > u(t)$. This implies that the continuation region at time t is split in two segments. Exercise is optimally postponed not only when the option is insufficiently in the money ($x > u(t)$), but also (surprisingly, at first sight) when the option is excessively in the money ($x < l(t)$). In the excessively in-the-money region ($x < l(t)$), moreover, the value of the American put decreases with a steeper slope than the immediate put payoff, i.e., $(\partial v/\partial x)(t,x) < -1$ (see the right-hand panel

[2] Whenever $t < T$, we have $\sup\{x\ge 0\colon v(t,x) = (K-x)^+\} \le K$, because $(K-x)^+ = 0$ and $v(t,x) > 0$ for $x \ge K$. On the contrary, for $t = T$, $\sup\{x\ge 0\colon v(T,x) = (K-x)^+\} = +\infty$. Hence, the cap at K in the definition of u is binding at T only.

of Figure 1). On the contrary, if $\rho \geq 0$, the derivative $(\partial v/\partial x)(t,x) \geq -1$ for all x. Thus, if the exercise region at date t is nonempty, it is the negativity of the interest rate that modifies its usual "geometry structure" (see Detemple and Tian 2002, Detemple 2006). Assumptions (6) and (7) in Proposition 2.2 are sufficient conditions for the nonemptiness of the exercise region in the perpetual case, and, consequently, in the finite-maturity case at *any* date t (see Theorem 2.3). In particular, Assumption (6) implies that the *dividend yield* $\delta = \rho - \mu$ is *negative*. Therefore, the negativity of both ρ and δ is crucial to determine the presence of the double continuation region. Clearly, the continuation region cannot be constituted by more than two nonconnected segments, because the *convex* function $v(t,\cdot)$ must lie above the payoff function $(K-\cdot)^+$.

Let us denote with $\mathscr{E}\mathscr{R} = \{(t,x) \in [0,T] \times [0,+\infty[: v(t,x) = (K-x)^+\}$ the early exercise region, and with $\mathscr{C}\mathscr{R} = \{(t,x) \in [0,T] \times [0,+\infty[: v(t,x) > (K-x)^+\}$ the continuation region. The function v in (1) can be expressed as the solution of the system of variational inequalities (for the related numerical solution, see, for instance, Bensoussan and Lions 1982, Jaillet et al. 1990, Feng et al. 2007, Kovalov et al. 2007):

$$\begin{cases} v(T,\cdot) = \pi(\cdot), \quad v(t,\cdot) \geq \pi(\cdot) \\ \qquad\qquad\qquad \text{for any } t \in [0,T], \\ \dfrac{\partial}{\partial t} v + \mathscr{L} v - \rho v \leq 0 \quad \text{on } (0,T) \times \Re^+, \\ \dfrac{\partial}{\partial t} v + \mathscr{L} v - \rho v = 0 \quad \text{on } \{(t,x) \in (0,T) \times \Re^+ : \\ \qquad\qquad\qquad v(t,x) > \pi(x)\}, \end{cases} \tag{5}$$

where $\pi(x) = (K-x)^+$, and $(\mathscr{L}v)(t,x) = \frac{1}{2}\sigma^2 x^2 \cdot (\partial^2/\partial x^2) v(t,x) + \mu x (\partial/\partial x) v(t,x)$. When interest rates are nonnegative, it is well known that (5) admits a smooth solution (see Jaillet et al. 1990). The same conclusion can be achieved even if the interest rate is negative, as shown in the next proposition.

Proposition 2.1 (Smoothness of the Put Value v, Negative Interest Rate). *The solution of* (5) *admits partial derivatives* $\partial v/\partial t$, $\partial v/\partial x$, $\partial^2 v/\partial x^2$ *that are locally bounded on* $[0,T) \times \Re^+$. *Moreover,* v *enjoys the smooth-fit property, i.e.,* $\partial v/\partial x$ *is continuous on* $[0,T) \times \Re^+$.

In the infinite-maturity case, the constant *double* free boundary can be explicitly computed by solving the differential equation implied by (5) in the continuation region and by applying the important *smooth-pasting* principle at the free boundary.[3] The result requires an ad hoc direct verification, because v_∞ violates the usual boundedness requirements. Indeed, when $\rho < 0$ and $x = 0$, the optimal exercise time is $\Theta^* = +\infty$, and the value of the American option is $v_\infty(0) = \mathbb{E}[e^{-\rho\Theta^*}(K-0)^+] = +\infty$. Battauz et al. (2012) work out a closed-form solution for the special case of a perpetual real-option problem. The following proposition adapts their statement to our current framework.

Proposition 2.2 (Perpetual Put, Negative Interest Rate). *If* $T = +\infty$,

$$\rho < 0, \quad \mu - \frac{\sigma^2}{2} > 0, \tag{6}$$

and

$$\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0, \tag{7}$$

then the perpetual American put option value is

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0, l_\infty), \\ K - x & \text{for } x \in [l_\infty, u_\infty], \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty, +\infty), \end{cases} \tag{8}$$

where $\xi_u < \xi_l$ *are the negative solutions of the equation*

$$\frac{1}{2}\sigma^2 \xi^2 + \left(\mu - \frac{\sigma^2}{2}\right)\xi - \rho = 0. \tag{9}$$

The critical asset prices are

$$l_\infty, u_\infty = K \frac{\xi_i}{\xi_i - 1} \quad \text{for } i = l, u, \tag{10}$$

and the constant A_l *and* A_u *are given by*

$$A_l = -\frac{(l_\infty)^{1-\xi_l}}{\xi_l} \quad \text{and} \quad A_u = -\frac{(u_\infty)^{1-\xi_u}}{\xi_u}. \tag{11}$$

Given a negative interest rate $\rho < 0$, the positive-drift condition (6) and the positive-discriminant condition (7) guarantee the existence of (negative) solutions of the Equation (9) and rule out the potential explosive effect of a negative interest rate on the put value function. If the interest rate is negative, the holder of the option may obtain an infinite expected gain by deferring indefinitely the exercise of the option. Such an incentive to indefinite deferment can be counteracted by a significant chance that the option goes out of the money as time goes by. This is the case if the growth rate of the underlying asset X is high enough compared to the absolute value of the negative interest rate, as stated by the condition (7): $|\rho| < (\mu - \sigma^2/2)^2/(2\sigma^2)$.

The function v_∞ defined in (8) enjoys the following properties in the continuation region: v is decreasing, it dominates the immediate payoff, and the process $\{v_\infty(X(t))e^{-\rho t}\}_t$ is a local martingale. The condition (7)

[3] See Battauz et al. (2012). For the standard case of nonnegative interest rates in models based on Lévy processes, see, e.g., Boyarchenko and Levendorskiĭ (2002a, b), Alili and Kyprianou (2005), and Jiang and Pistorius (2008).

also empowers the supermartingality of the process $\{v_\infty(X(t))e^{-\rho t}\}_t$ in the early exercise region.

Given a finite maturity and a negative interest rate, Theorem 2.3 provides an accurate description of the *double continuation region*, which is separated from the (single) early exercise region by a *double free boundary*. The upper free boundary enjoys all the properties it has in the standard case of nonnegative interest rates: it is increasing and continuous, and tends to the strike price at maturity. The lower free boundary is decreasing everywhere, and continuous everywhere but at maturity, where it exhibits a discontinuity. Our findings contribute to the extant literature on multiple free boundaries that separate the (single) continuation region from the multiple exercise region for certain American options with multiple underlying assets, e.g., Broadie and Detemple (1997).

Theorem 2.3 (Continuation Region and Free Boundary Characterization, Finite-Maturity Put, Negative Interest Rate). *If conditions* (6) *and* (7) *are satisfied, then for any* $t \in [0, T)$, *there exist*

$$\frac{\rho K}{\rho - \mu} \le l(t) < u(t) \le K \tag{12}$$

such that $(K-x)^+ = v(t,x)$ *for any* $x \in [l(t), u(t)]$ *and* $(K-x)^+ < v(t,x)$ *for any* $x \notin [l(t), u(t)]$.

The lower free boundary l: $[0,T] \to [0, l_\infty)$ *is decreasing, continuous for any* $t \in [0,T)$, $l(T^-) = (\rho K)/(\rho-\mu) > l(T) = 0$.

The upper free boundary u: $[0,T] \to (u_\infty, K]$ *is increasing, continuous for any* $t \in [0,T]$, *and* $u(T) = u(T^-) = K$.

The early exercise region is $\mathscr{ER} = \{(t,x) \in [0,T] \times [0,+\infty[\colon\ l(t) \le x \le u(t)\}$, *and the double continuation region is* $\mathscr{CR} = \{(t,x) \in [0,T] \times [0,+\infty[\colon\ 0 \le x < l(t)$ *or* $x > u(t)\}$, *where* $\{(t, l(t)); (t, u(t))\colon t \in [0,T]\}$ *is the double free boundary.*

Describing the free boundary close to maturity is of key importance for the American option holder. The asymptotic behavior of the free boundary of an American put option in the standard case of a positive interest rate and of a diffusive underlying asset has been studied by several authors, such as Barles et al. (1995) and, more recently, Evans et al. (2002) and Lamberton and Villeneuve (2003). In a diffusive framework with stochastic volatility and stochastic interest rates, Medvedev and Scaillet (2010) derive an accurate approximation formula for the American put price, by first introducing an explicit proxy for the exercise rule based on the normalized moneyness, and then by using proper asymptotic expansions for short-maturities.

In Theorem 2.4 we study the asymptotic behavior of the double free boundary at maturity in the case of a negative interest rate. When the interest rate dominates the nonnegative dividend yield of the underlying asset,[4] Evans et al. (2002) show that the free boundary of an American put option tends at maturity to the strike price in a *parabolic-logarithmic* form. In the case of a negative interest rate, the same asymptotic behavior at maturity is shown by the *upper free boundary*. As for the nonstandard *lower free boundary*, we prove that it converges at maturity monotonically decreasingly to its left-limit[5] $l(T^-) = \rho K/(\rho-\mu)$ in a *parabolic* form.

Theorem 2.4 (Asymptotic Behavior of the Free Boundaries at Maturity, Put, Negative Interest Rate). *If conditions* (6) *and* (7) *are satisfied, then for* $t \to T$, *the upper free boundary satisfies*

$$u(t) - K \sim -K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)\mu^2}}.$$

For $t \to T$, *the lower free boundary satisfies*

$$l(t) - \frac{\rho K}{\rho-\mu} \sim \frac{\rho K}{\rho-\mu}(-y^*\sigma\sqrt{(T-t)}),$$

where $y^* \in (-1, 0)$, $y^* \approx -0.638$, *is the number such that* $\phi(y) = \sup_{0\le\Theta\le1}\mathbb{E}[\int_0^\Theta (y + B(s))\,ds] = 0$ *for all* $y \le y^*$ *and* $\phi(y) > 0$ *for all* $y > y^*$.

In Figure 2 we plot the double free boundary for an American put option with a negative interest rate. The dashed part of the upper free boundary is obtained via binomial approximation. The solid lines correspond to the asymptotic approximation. (The binomial approximation of the lower free boundary coincides numerically with the parabolic asymptotic approximation for the entire life of the option.) Black dots (white circles) indicate the exercise (no exercise) region at T.

Conditions (6) and (7) are sufficient but not necessary for the existence of the double free boundary. In the next proposition, we provide a necessary time-dependent condition for the optimality of early exercise of the put option during the life of the option

[4] The introduction of jumps can produce effects akin to an additional dividend rate. See, e.g., Boyarchenko and Levendorskiĭ (2002a) and Levendorskiĭ (2004, 2008).

[5] The discontinuity of our nonstandard lower free boundary at T parallels the discontinuity of the (unique) free boundary at T in the standard case of a nonnegative interest rate that is dominated by the underlying payout rate (see, e.g., Evans et al. 2002, Ingersoll 1998). We here adapt the covered-put argument of Ingersoll (1998). Assume tradability, and consider the strategy of holding the underlying asset and the put at time $\tau = T - dt$ for a small positive dt. Recall that in our nonstandard case the interest rate ρ and the underlying payout rate $\rho - \mu$ are negative. The critical (lower) price $x^* = l(\tau)$ is the indifference point such that the value of unwinding the strategy at τ equals the present value of waiting to do so at T: $K = Ke^{-\rho dt} + x^*(\rho-\mu)dt$. It follows that $\lim_{dt\to0} x^* = K(\rho/(\rho-\mu))$. Notice that the covered-put argument does not apply to the upper free boundary $(u(T^-) = u(T) = K)$.

Figure 2 **The Double Free Boundary for a Put** $\rho = -4\%$, $K = 1.2$, $\sigma = 20\%$, $\mu = 8\%$, $T = 1$

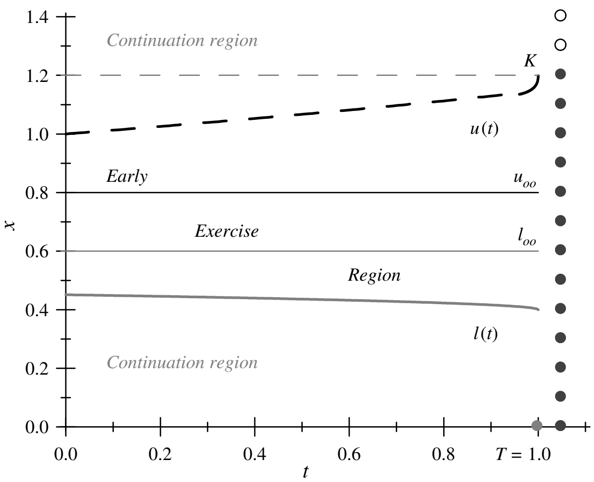
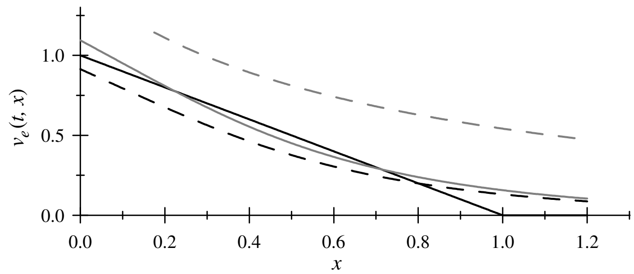


Figure 3 **The European Finite-Maturity Put Value** $v_e(t, x)$ **for** $T - t = 9$ **and** $K = 1$



Note. Black dashed line: $\rho = 1\%$, $\mu = 3\%$, $\sigma = 20\%$; gray solid line: $\rho = -1\%$, $\mu = 3\%$, $\sigma = 20\%$, gray dashed line: $\rho = -4\%$, $\mu = 3\%$, $\sigma = 40\%$.

when interest rates are negative. As a consequence, this condition is also necessary for the existence of a double free boundary with negative interest rates.

Proposition 2.5 (Necessary Condition for Early Exercise, Negative Interest Rate). *If $\rho < 0$ and $\mu > 0$, a necessary condition for the optimal exercise of the finite-maturity American put option at $t \in [0, T)$ is*

$$\mathcal{N}^{-1}(e^{\rho(T-t)}) - \mathcal{N}^{-1}(e^{(\rho-\mu)(T-t)}) \geq \sigma\sqrt{T-t}, \qquad (13)$$

where $\mathcal{N}^{-1}(\cdot)$ denotes the inverse of the standard normal cumulative distribution function.

Condition (13) requires μ, the growth rate of the underlying asset X, to be relatively high compared to the (negative) interest rate ρ in such a way that the distance between the two quantiles $\mathcal{N}^{-1}(e^{\rho(T-t)})$ and $\mathcal{N}^{-1}(e^{(\rho-\mu)(T-t)})$ is at least as big as $\sigma\sqrt{T-t}$. While working toward the common objective of limiting the relative strength of ρ versus μ, condition (13) is a requirement milder than the sufficient conditions (6) and (7).

The intuition behind Proposition 2.5 is visualized in Figure 3: If the time t value of the European put option, $v_e(t, x)$, strictly dominates the immediate payoff function (depicted in Figure 3 as a black solid line) *for all* $x \geq 0$, then there is no optimal early exercise at t, since the time t value of the American put option dominates $v_e(t, x)$, that is, $v(t, x) \geq v_e(t, x) > (K - x)^+$. If interest rates are nonnegative, i.e., $\rho \geq 0$, this can never happen, because at $x = 0$ we have that $v_e(t, 0) = Ke^{-\rho(T-t)} \leq (K - 0)^+ = K$, and by continuity $v_e(t, x)$ lies below $(K - x)^+$ on an entire segment of nonnegative underlying values (see the black dashed line in Figure 3). On the contrary, when interest rates are negative, i.e., $\rho < 0$, the time t value of the European put option when the underlying asset is 0 dominates the immediate payoff, because $v_e(t, 0) = Ke^{-\rho(T-t)} > (K - 0)^+ = K$. Hence, two alternatives are possible: either $v_e(t, x)$ dominates the immediate payoff function *for all* $x \geq 0$ (the gray dashed line in Figure 3), and consequently early exercise is never optimal at date t, or $v_e(t, x) < (K - x)^+$ for *some* $x > 0$ (the gray solid line in Figure 3), and early exercise *might* be optimal at date t. When $\rho < 0$, Equation (13) is equivalent to the existence of some $x > 0$ such that $v_e(t, x) \leq (K - x)^+$. Equation (13) is therefore a *minimal* necessary condition for the possibility of optimal early exercise at date t in case of negative interest rates that in turn implies the possible existence of a double continuation region.

3. The American Call

We consider an American call option written on the log-normal asset X, whose drift under the valuation measure is positive and denoted with μ. We denote the volatility with σ, the strike with K, and the interest rate with ρ. The call value at time t is given by

$$\operatorname*{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}[e^{-\rho(\tau-t)}(X(\tau) - K)^+ \mid \mathcal{F}_t] = v(t, X(t)),$$

where

$$v(t, x) = \sup_{0 \leq \Theta \leq T-t} \mathbb{E}\left[e^{-\rho\Theta}\left(x \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma B(\Theta)\right) - K\right)^+\right], \qquad (14)$$

and B is a standard Brownian motion under the valuation measure. We focus on the case $\rho < 0$.

If $\mu > 0$, the value of the perpetual call option

$$v(t, x) = v_\infty(x) = \sup_{0 \leq \Theta} \mathbb{E}\left[e^{-\rho\Theta}\left(x \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Theta + \sigma B(\Theta)\right) - K\right)^+\right]$$

is unbounded by Jensen's inequality.[6] By contrast, for $\rho, \mu < 0$, the function v in (14) can be bounded also in the perpetual case, as we show in Proposition 3.2. In the finite-maturity case, v in (14) can be characterized as the solution of the variational inequality (5) with $\pi(x) = (x-K)^+$. Regardless of the sign of ρ, the function v in (14) dominates the call payoff ($0 \le (x-K)^+ \le v(t,x)$ for any $t \in [0,T]$ and $x \ge 0$), and is convex and increasing with respect to x for any $t \in [0,T]$. These properties are inherited from the convexity and the monotonicity of the call payoff. From the definition of v in (14) as a supremum on the set of stopping times from 0 up to time to maturity, we can also deduce that, for any $x \ge 0$, the function $v(t,x)$ is decreasing with respect to t. Obviously, the finite-maturity option is dominated by the perpetual one: $v(t,x) \le v_\infty(x)$ for any $t \in [0,T]$ and $x \ge 0$. We also observe that the negative sign of ρ and μ (with the additional conditions (15) and (16)) prevents the function v_∞ to be dominated by the identity function, i.e., the standard inequality $v_\infty(x) \le x$ does not hold true, as opposite to the case depicted in Xia and Zhou (2007).

The mentioned properties of v in (14) imply that the early exercise region at time t is constituted by a connected segment defined by the extremes $l(t) \le u(t) \in [0,K]$, where

$$l(t) = \inf\{x \ge 0\colon v(t,x) = (x-K)^+\} \vee K$$
$$u(t) = \sup\{x \ge 0\colon v(t,x) = (x-K)^+\}$$

such that $v(t,x) = (x-K)^+$ for $l(t) \le x \le u(t)$ and $v(t,x) > (x-K)^+$ for $x < l(t)$ and $x > u(t)$. This entails that the continuation region at time t is split in two segments. We characterize the double continuation region, the early exercise region, and the double free boundary in Theorem 3.3. In Proposition 3.2 we give parameter value restrictions under which the American perpetual call option is finite even when interest rates are negative. We also provide explicit expressions for the constant double free boundary.

In the finite-maturity case, the lower free boundary enjoys all the property it has in the standard case, where interest rates are positive: it is decreasing and continuous, and tends to the strike price at maturity. The upper free boundary is increasing and continuous everywhere but at maturity, where it is infinite.

Proposition 3.2 and Theorem 3.3 are proved by building upon (respectively) Proposition 2.2 and Theorem 2.3 and by applying the American put–call symmetry (see Carr and Chesney 1996, Schroder 1999). The American put–call symmetry relates the price of an American call option to the price of an American put option by swapping the initial underlying price with the strike price and the dividend yield with the interest rate. As explained by Detemple (2001), such symmetry relies on the symmetry of the distribution of the log-price of X and on the symmetry of call and put payoffs. The change of numeraire allows us to derive such property also in our case, where *both* the *interest rate* ρ and the *dividend yield* $\delta = \rho - \mu$ are *negative*. For the ease of the reader, the following proposition remaps the American put–call symmetry to our framework.

Proposition 3.1 (American Put–Call Symmetry). *Consider the American call option with strike K, interest rate ρ, underlying drift μ, underlying volatility σ, and initial underlying value x, whose value at time $t \in [0,T]$ is denoted with $v(t,x) = v_{\text{call}}(t,x;K,\rho,\mu,\sigma)$ in* (14).

Consider the symmetric American put option with strike $K_{\text{put}} = x$, interest rate $\rho_{\text{put}} = \rho - \mu$, underlying drift $\mu_{\text{put}} = -\mu$, underlying volatility $\sigma_{\text{put}} = \sigma$, and initial underlying value $x_{\text{put}} = K$, whose value at time $t \in [0,T]$ is denoted with $v_{\text{put}}(t, x_{\text{put}}; K_{\text{put}}, \rho_{\text{put}}, \mu_{\text{put}}, \sigma_{\text{put}}) = v_{\text{put}}(t, K; x, \rho-\mu, -\mu, \sigma)$.

1. *The conditions*

$$\rho < \mu < -\frac{\sigma^2}{2} < 0, \tag{15}$$

$$\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2 > 0 \tag{16}$$

for ρ, μ, σ in the American call problem are equivalent to conditions (6) *and* (7) *for parameters ρ_{put}, μ_{put}, σ_{put} in the symmetric American put problem.*

2. (*Carr and Chesney* 1996; *Detemple* 2001, 2006). *The value of the American call coincides with the value of the symmetric American put*

$$\begin{aligned} v(t,x) &= v_{\text{call}}(t,x;K,\rho,\mu,\sigma) \\ &= v_{\text{put}}(t,K;x,\rho-\mu,-\mu,\sigma) \end{aligned} \tag{17}$$

for any $t \in [0,T]$.

3. *For any $t \in [0,T]$, let $l(t)$ (resp., $u(t)$) denote the lower (resp., upper) free boundary at time t for the American call option with strike K and parameters ρ, μ, and σ. Let $l_{\text{put}}(t)$ (resp., $u_{\text{put}}(t)$) denote the lower (resp., upper) free boundary at time t for the symmetric American put with strike $K_{\text{put}} = 1$ and parameters ρ_{put}, μ_{put}, and σ_{put}. If* (15) *and* (16) *are satisfied, then for any $t \in [0,T]$, we have*

$$l(t) = \frac{K}{u_{\text{put}}(t)} \quad \textit{and} \quad u(t) = \frac{K}{l_{\text{put}}(t)}. \tag{18}$$

We employ Proposition 3.1 to study the double free boundary for the American call option. Proposition 3.2 focuses on the perpetual case. Theorem 3.3 deals with the finite-maturity case, and Theorem 3.4 provides the asymptotic behavior of the upper and lower free boundaries at maturity.

[6] If $\mu > 0$, we have $v_\infty(x) \ge \sup_{0\le T} e^{-\rho T} \cdot (\mathbb{E}[x \cdot \exp((\mu - \sigma^2/2)T + \sigma B(T))] - K)^+ = \sup_{0\le T} e^{-\rho T}(x \cdot e^{\mu T} - K)^+ = +\infty$.

Proposition 3.2 (Perpetual Call, Negative Interest Rate). *If $T = +\infty$, and conditions* (15) *and* (16) *hold, then the perpetual American call option value is*

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0, l_\infty), \\ x - K & \text{for } x \in [l_\infty, u_\infty], \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty, +\infty), \end{cases}$$

where $\xi_l > \xi_u > 1$ are the positive solutions of the Equation (9). *The double free boundary is given by the constant l_∞, u_∞ defined in* (10), *with $A_l = (l_\infty)^{1-\xi_l}/\xi_l$ and $A_u = (u_\infty)^{1-\xi_u}/\xi_u$.*

Theorem 3.3 (Continuation Region and Free Boundary Characterization, Finite-Maturity Call, Negative Interest Rate). *Under conditions* (15) *and* (16), *for any $t \in [0, T)$, there exist*

$$l(t) \le l_\infty < u_\infty \le u(t)$$

such that $(x-K)^+ = v(t,x)$ for any $x \in [l(t), u(t)]$ and $(x-K)^+ < v(t,x)$ for any $x \notin [l(t), u(t)]$.

The lower free boundary l: $[0,T] \to [K, l_\infty)$ is decreasing, continuous for any $t \in [0,T]$, and $l(T) = l(T^-) = K$.

The upper free boundary u: $[0,T) \to (u_\infty, (\rho K)/(\rho - \mu)]$ is increasing, continuous for any $t \in [0,T)$, with $u(T^-) = (\rho K)/(\rho - \mu) > K$ and $u(T) = +\infty$.

The early exercise region is $\mathscr{ER} = \{(t,x) \in [0,T] \times [0,+\infty[\colon\ l(t) \le x \le u(t)\}$, and the double continuation region is $\mathscr{CR} = \{(t,x) \in [0,T] \times [0,+\infty[\colon\ 0 \le x < l(t)$ or $x > u(t)\}$, where $\{(t, l(t)); (t, u(t))\colon t \in [0,T]\}$ is the double free boundary.

Theorem 3.4 (Asymptotic Behavior of the Free Boundaries at Maturity, Call, Negative Interest Rate). *Under conditions* (15) *and* (16), *for $t \to T$ the upper free boundary satisfies*

$$u(t) - \frac{\rho K}{\rho - \mu} \sim y^* \sigma \sqrt{(T-t)}.$$

For $t \to T$, the lower free boundary satisfies

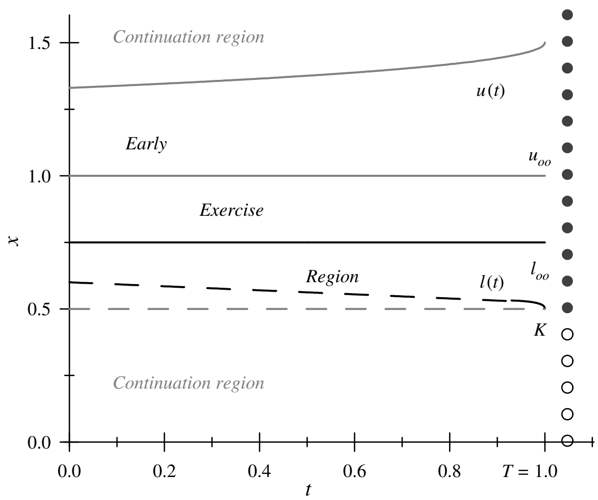
$$l(t) - K \sim K\sigma \sqrt{(T-t)\ln \frac{\sigma^2}{8\pi (T-t)\mu^2}},$$

where $y^* \approx -0.638$ is defined in Theorem 2.4.

In Figure 4 we plot the double free boundary for an American call option with a negative interest rate. The dashed part of the lower free boundary is obtained via binomial approximation. The solid lines correspond to the asymptotic approximation. Black dots (white circles) indicate the exercise (no exercise) region at T.

Conditions (15) and (16) are sufficient but not necessary for the existence of a double free boundary for the call option. A necessary condition for optimal exercise at t is $\mathcal{N}^{-1}(e^{-(\mu-\rho)(T-t)}) - \mathcal{N}^{-1}(e^{\rho(T-t)}) \ge \sigma\sqrt{T-t}$, which can be derived by applying the put–call symmetry (Proposition 3.1) to the necessary condition for the early exercise of put options established in Proposition 2.5.

Figure 4 **Double Free Boundary for a Call with** $\rho = -12\%$, $K = 0.5$, $\sigma = 20\%$, $\mu = -8\%$, $T = 1$



4. The Gold Loan

Collateralized borrowing has been seeing a huge increase after the financial crisis. Treasury bonds and stocks are the collateral usually accepted by financial institutions, but gold is increasingly being used around the world.[7] Major Indian nonbanking financial companies like Muthoot Finance and Manappuram Finance have been quite active in lending against gold collateral. As Churiwal and Shreni (2012) report in their survey of the Indian gold loan market, gold loans tend to have short maturities and rather high spreads (borrowing rate minus risk-free rate), even if significantly lower than without collateral. The prepayment option is common, permitting the redemption of the gold at any time before maturity. We emphasize that gold loans noticeably differ from stock loans, because gold is a tradable investment asset with storage/insurance costs and without earnings. This can lead to peculiar redemption policies that constitute an interesting application of our results in Proposition 3.2 and Theorems 3.3 and 3.4.

In a gold loan, the borrower receives at time 0 (the date of contract inception) the loan amount $q > 0$ using one mass unit (one troy ounce, say) of gold as collateral, which must be physically delivered to the lender.[8] This amount grows at the rate γ, where γ is a constant borrowing rate (higher than the risk-free rate r) stipulated in the contract, and the cost of reimbursing the loan at time t is thus given by $qe^{\gamma t}$. When paying back the loan, the borrower regains the gold, and the contract is terminated. We assume that the

[7] For example, see Cui and Hoyle (2011).

[8] It is not implausible that the lender's cost of storing the gold collateral is passed to the borrower by charging a higher borrowing rate, although we have no direct evidence for it (see, for example, Churiwal and Shreni 2012).

costs of storing and insuring gold holdings are $Gc > 0$ per unit of time, where G is the gold spot price. Consistently, the dynamics of G under the risk-neutral measure $\mathbf{Q}$ is assumed to be

$$\frac{dG(t)}{G(t)} = (r + c)\, dt + \sigma\, dW(t),$$

where r is the constant riskless rate, σ is the gold returns' volatility, and W is a Brownian motion under the risk-neutral measure $\mathbf{Q}$ (see, for instance, Hull 2011). Given a finite maturity T, the value of the redemption option at date 0 is

$$\begin{aligned} C(0, G(0)) &= \sup_{0 \le \tau \le T} \mathbb{E}^{\mathbf{Q}}[e^{-r\tau}(G(\tau) - qe^{\gamma\tau})^+] \\ &= \sup_{0 \le \tau \le T} \mathbb{E}^{\mathbf{Q}}[e^{-(r-\gamma)\tau}(X(\tau) - q)^+], \end{aligned}$$

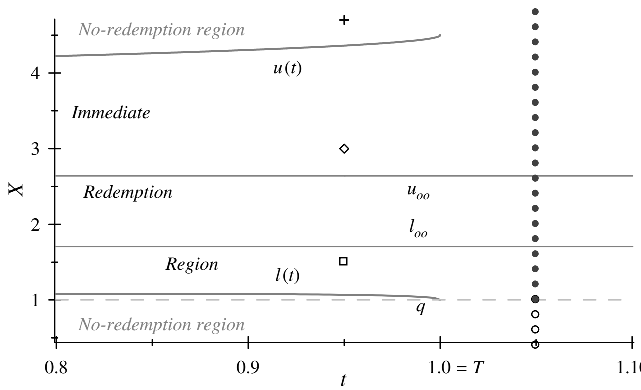
where $X(t) = G(t)e^{-\gamma t}$ is the *gold price deflated at the rate* γ. Therefore, the initial value of the redemption option of a gold loan is the initial value of an American call option in (14) on the lognormal underlying value X with parameters $\rho = r - \gamma < 0$, $\mu = r + c - \gamma$, and $K = q$.

Similarly, the value of the redemption option at any date $t \in [0, T]$ can be computed as $C(t, G(t)) = v(t, X(t))$, with v defined in (14). The percentage storage and insurance costs c are positive and usually below the spread $\gamma - r > 0$. Hence, we posit $\rho < \mu < 0$. If conditions (15) and (16) are also satisfied, i.e.,

$$r - \gamma < r - \gamma + c < -\frac{\sigma^2}{2} \quad \text{and}$$
$$\left(r - \gamma + c - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(r - \gamma) > 0,$$

a *double no-redemption region* appears in the perpetual case, as by Proposition 3.2. Using the same proposition, we can compute the perpetual constant free boundaries l_∞ and u_∞ in terms of the deflated gold price process $X(t) = G(t)e^{-\gamma t}$. For finite-maturity contracts, Theorem 3.4 provides the asymptotic approximation of the double free boundaries near maturity. Churiwal and Shreni (2012) show that maturities for gold loans are generally within 36 months. Borrowing rates typically range from 12% to 16% for banks, and from 12% to 24% for specialized institutions, whereas the yield on Indian short-term government bonds[9] has been hovering around 8%. Data from the Gold World Council[10] show that the daily log change in the gold spot price expressed

Figure 5 Double No-Redemption Region of a Gold Loan Near Maturity



Note. The parameter values are $T = 1$, $r = 8\%$, $c = 2\%$, $\gamma = 17\%$, $\sigma = 21.4\%$, and $q = 1$.

in Indian rupees has registered an annualized historical volatility of 21.4% over the period from January 3, 1979, to May 5, 2013. Average storage/insurance costs are about[11] 2%. By fixing $r = 8\%$, $c = 2\%$, $\gamma = 17\%$, and $\sigma = 21.4\%$, the mentioned parametric restrictions are met. Given quantities normalized by the loan amount ($q = 1$), Figure 5 visualizes the perpetual double free boundary ($l_\infty = 1.70$ and $u_\infty = 2.64$) and the proxied finite-maturity double free boundary ($l(t)$ and $u(t)$ for t close to the expiry date $T = 1$ expressed in years), as by Theorem 3.4. Figure 5 highlights that the two perpetual free boundaries are a poor proxy for the two finite-maturity free boundaries near expiration. For instance, if at $t = 0.95$ the deflated gold price X is equal to 3 (the point denoted with a diamond in Figure 5), the perpetual boundaries suggest to delay the gold loan redemption (the diamond lies outside the *perpetual* immediate-redemption region), though the asymptotic approximation of the double free boundary implies optimal immediate redemption (the diamond lies inside the immediate redemption region). Binomial-tree calculations show that the relative welfare loss associated with suboptimal delay is three basis points of the immediate-redemption value. A similar but lesser deep-in-the-money situation is represented in Figure 5 by the point denoted with a box ($X = 1.5$ at $t = 0.95$). The relative welfare loss from suboptimal delay in this case is of 23 basis points. Conversely, if the deflated gold price X is 4.7 at $t = 0.95$ (the point denoted with a cross in Figure 5), it is optimal to postpone the gold redemption even though the redemption option is quite deep in the money and very short lived. Black dots (white circles) indicate the redemption (no redemption) region at T.

[9] The source is the Government Securities Market Section of the Reserve Bank of India DataBase on The Indian Economy (http://dbie.rbi.org.in).

[10] See http://www.gold.org/investment/statistics/.

[11] The cost of leasing a bank safety locker and of insuring the jewelry kept in it adds up to about 2% of the sum assured (Mukerji 2012).

5. Capital Investment Options

This example closely follows the setup of Battauz et al. (2012), who consider exclusively the perpetual case. By contrast, we focus here on the finite-maturity case and on the behavior of the double free boundary near maturity. Uncertainty is described by the historical probability space $(\Omega, \mathbf{P}, (\mathcal{F}_t)_t)$. The present value V of the project and the investment cost I have lognormal dynamics under the historical probability measure $\mathbf{P}$ (for a classical review of risky investment, see Dixit and Pindyck 1993; for a recent survey, see Aase 2010). The firm's management decides when to disburse the irreversible investment cost I to undertake the project. Risk adjustment corresponds to choosing the valuation measure $\hat{\mathbf{P}}$ (equivalent to $\mathbf{P}$) by the firm's management. The discount rate $\hat{r}$ is also selected by the firm's management. The $\hat{\mathbf{P}}$-dynamics of V is

$$dV_t = V_t(\hat{\mu}_V\, dt + \sigma_V\, dW_t^{\hat{\mathbf{P}}} + \tilde{\sigma}_V\, d\tilde{W}_t^{\hat{\mathbf{P}}}),$$

where $\hat{\mu}_V$, σ_V, and $\tilde{\sigma}_V$ are real positive constants. The investment cost I has $\hat{\mathbf{P}}$-dynamics

$$dI_t = I_t(\hat{\mu}_I\, dt + \sigma_I\, dW_t^{\hat{\mathbf{P}}}),$$

where $\hat{\mu}_I$ and σ_I, are real positive constants, and $W^{\hat{\mathbf{P}}}$ and $\tilde{W}^{\hat{\mathbf{P}}}$ are $\hat{\mathbf{P}}$-independent Brownian motions.

Access to the project is possible only up to the date T. Thus, at any date $t \in [0, T]$ the management evaluates the t-dated *value of the option to invest*

$$\operatorname*{ess\,sup}_{t\le\tau\le T} \mathbb{E}^{\hat{\mathbf{P}}}[e^{-\hat{r}(\tau-t)}(V_\tau - I_\tau)^+ \mid \mathcal{F}_t]. \tag{19}$$

The real-option problem (19) can be reduced to a one-dimensional American put option by taking the process $V_t e^{\rho t}$ as the numeraire (see Battauz 2002, Carr 1995, Geman et al. 1995), where $\rho = -(\hat{\mu}_V - \hat{r})$ is the opposite of V's expected growth rate (under $\hat{\mathbf{P}}$) in excess of the discount rate $\hat{r}$. Indeed, denoting with $\mathbf{P}^V$ the probability measure associated to the numeraire $V_t e^{\rho t}$, whose Radon–Nikodym derivative with respect to the probability measure $\hat{\mathbf{P}}$ is $d\mathbf{P}^V/d\hat{\mathbf{P}} = (V_T e^{\rho T})/(V_0 e^{\hat{r}T})$, the problem (19) can be written as

$$\operatorname*{ess\,sup}_{t\le\tau\le T} \mathbb{E}^{\hat{\mathbf{P}}}[e^{-\hat{r}(\tau-t)}(V_\tau - I_\tau)^+ \mid \mathcal{F}_t] = V_t \cdot v(t, X_t), \tag{20}$$

with

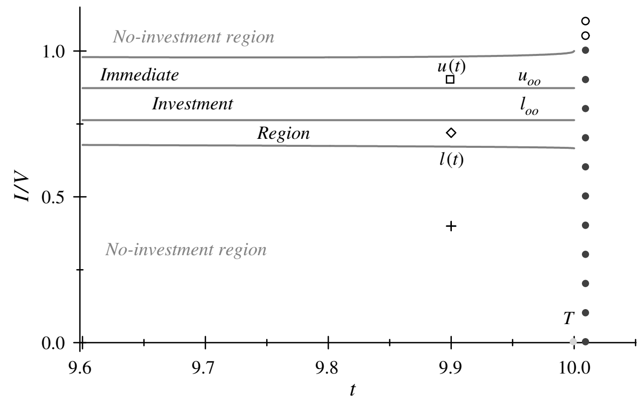
$$v(t, X_t) = \operatorname*{ess\,sup}_{t\le\tau\le T} \mathbb{E}^{\mathbf{P}^V}[e^{-\rho(\tau-t)}(1 - X_\tau)^+ \mid \mathcal{F}_t] \tag{21}$$

and $X_t = I_t/V_t$. The underlying asset of the put option in (21) is the lognormal cost-to-value ratio X, that under the probability measure $\mathbf{P}^V$ can be written as

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right),$$

where B_t is a $\mathbf{P}^V$-Brownian motion, and where $\sigma^2 = (\sigma_I - \sigma_V)^2 + \tilde{\sigma}_V^2$, $\mu = \hat{\mu}_I - \hat{\mu}_V$. The parameter $\rho = -(\hat{\mu}_V - \hat{r})$ plays in (21) the role of the interest rate. Consider now the case of a highly profitable project for which $\hat{\mu}_V > \hat{r}$. This case is usually neglected by the literature on real options, because it can lead to an explosive option value in the perpetual case (see Battauz et al. 2012 for a detailed discussion). In the finite maturity case, if $\mu = \hat{\mu}_I - \hat{\mu}_V < 0$, the option is optimally exercised only at maturity T. On the contrary, if $\mu = \hat{\mu}_I - \hat{\mu}_V > 0$, Theorem 2.3 shows that early exercise can be optimal, and that the early exercise region is surrounded by a double continuation region. Investments in nuclear plants are an interesting area of possible application. The business is extremely lucrative, but the overall cost of entering it is likely to increase markedly in the future (demand for nuclear plant safety is definitely rising). This may cause the cost of entering a nuclear energy project to grow at a higher expected rate than the value of the project itself, leading to $\mu = \hat{\mu}_I - \hat{\mu}_V > 0$.

Figure 6 **Double Free Boundary for a Capital Investment Option Near Maturity**



Note. The parameter values are $T = 10$, $\hat{r} = 3\%$, $\hat{\mu}_V = 5\%$, $\sigma_V = 7\%$, $\tilde{\sigma}_V = 3\%$, $\hat{\mu}_I = 6\%$, and $\sigma_I = 10\%$.

For instance, with $\hat{r} = 3\%$, $\hat{\mu}_V = 5\%$, $\sigma_V = 7\%$, $\tilde{\sigma}_V = 3\%$, $\hat{\mu}_I = 6\%$, and $\sigma_I = 10\%$ (see Figure 6),[12] we get $\rho = -(\hat{\mu}_V - \hat{r}) = -2\%$, $\sigma = 4.242\%$, and $\mu = 1\%$. Conditions (6) and (7) are met, and Proposition 2.2 delivers the two perpetual free boundaries, $l_\infty = 0.763$ and $u_\infty = 0.873$. Suppose that the option has 10 years to maturity, i.e., $T = 10$. Theorem 2.4 enables the investigation of the double free boundary near maturity. In Figure 6 the double free boundary is plotted for $t \in [9.6; 10]$, i.e., when only 4.8 months are left to expiration. At $t = 9.9$, if the cost-to-value ratio I/V is 0.72

[12] The seminal work of McDonald and Siegel (1986) also deals with risk for both the value V and the cost I. With the key difference of a distinct hierarchy for the discount and growth rates, the parameter values for the risk-adjusted processes of V and I employed in Figure 6 are in the same range as those used by McDonald and Siegel (1986). See, for example, their Tables I and II, p. 720.

(the diamond in Figure 6), immediate investment is optimal. The perpetual double free boundary is a poor proxy for the double free boundary near expiration and implies a delayed investment (the diamond lies outside the *perpetual* immediate investment region). Binomial-tree calculations show that the relative welfare loss associated with suboptimal delay is one basis point of the immediate-exercise value. A similar but lesser deep-in-the-money situation is depicted in Figure 6 by the box ($I/V = 0.9$ at $t = 9.9$). The relative welfare loss from suboptimal deferment in this case is of 15 basis points. Conversely, if the cost-to-value ratio I/V is 0.4 at $t = 9.9$ (the cross in Figure 6), the firm must postpone the investment, even if the investment option is quite deep in the money and definitely short lived. Black dots (white circles) indicate the investment (no investment) region at T.

6. Conclusions

American option problems with an endogenous negative interest rate are significant because they are reformulations of the option-like features of popular secured loans and of relevant capital budgeting problems. For finite-maturity and perpetual American puts and calls with a negative interest rate, we study the conditions that bring about a nonstandard double continuation region (option exercise is optimally delayed if moneyness is insufficient and, in a nonstandard fashion, if it is overly sufficient) and investigate the properties (existence, monotonicity, continuity, limits, and behavior close to maturity) enjoyed by the double free boundary that separates the early-exercise region from the double continuation region.

Our study extends the standard optimal exercise properties for American options and covers the exact necessary/sufficient conditions that empower optimal early exercise of an American call with a negative underlying payout rate. We also contribute to the extant literature on the optimal redeeming strategy of tradable securities used as loan collateral as we characterize the double continuation region implicit in the gold loan, a blooming form of collateralized borrowing. Real options that combine strong expected growth for the project values with a marked escalation of the investment costs provide another distinct area of application for our results.

Several promising avenues of further research emerge, with an interesting mix of economic and technical challenges. They include studying the impact on nonstandard optimal exercise policies of diffusive stochastic volatility, jump risk, and drift-parameter uncertainty. We plan to pursue them in future work.

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Appendix

Proof of Proposition 2.1. See the proofs of Theorem 3.6 and of Corollary 3.7 in Jaillet et al. (1990), and note that for $\rho < 0$, the discount factor is positive and bounded by $e^{-\rho T}$. $\square$

Proof of Proposition 2.2. See pages 20–21 of Battauz et al. (2013). $\square$

Proof of Theorem 2.3. The two branches l and u of the double free boundary are defined in (3) and (4). We start by proving inequality (12). Under our assumptions, the function v_∞ and the constants l_∞ and u_∞ are well defined and the strict inequality $l_\infty < u_\infty$ holds because of (7). The strict inequality $l(t) < u(t)$ for any $t \in [0, T]$ in (12) follows from the chain $l(t) \le l_\infty < u_\infty \le u(t)$. To show that $l(t) \le l_\infty$ and that $u(t) \ge u_\infty$ for any $t \in [0, T]$, it is sufficient to observe that $\{x\colon v_\infty(x) > (K-x)^+\} \supset \{x\colon v(t,x) > (K-x)^+\}$ for any fixed t. Hence, taking the complement sets, we get $\{x\colon v_\infty(x) = (K-x)^+\} \subset \{x\colon v(t,x) = (K-x)^+\}$. By passing to the infimum, this inclusion leads to $l_\infty \ge l(t)$, and by passing at the supremum, we get $u_\infty \le u(t)$.

Next, we prove that $l(t) \ge (\rho K)/(\rho - \mu)$ for any $t \in [0, T)$. We observe that any (t, x) in the exercise region $\mathscr{ER}$ satisfies the inequality $(\partial/\partial t)v + \mathscr{L}v - \rho v \le 0$ in (5). Since $v(t,x) = K - x$, the inequality simplifies to $-\mu x - \rho(K - x) = (\rho - \mu)x - \rho K \le 0$, that is, $x \ge (\rho K)/(\rho - \mu) > 0$ for any $(t, x) \in \mathscr{ER}$. By passing to the infimum over x for any fixed t in the previous inequality, we get that $l(t) \ge (\rho K)/(\rho - \mu)$.

We now prove the monotonicity properties of l and u. We first show that l is decreasing. Let $t' < t''$. We have $(K - l(t'))^+ \le v(t'', l(t')) \le v(t', l(t')) = (K - l(t'))^+$, where the first inequality follows from $v(t'', \cdot) \ge (K - \cdot)^+$, the second one from the fact that $v(\cdot, l(t'))$ is decreasing, and the last equality from the definition of $l(t')$. As a consequence, $v(t'', l(t')) = (K - l(t'))^+$, and therefore $l(t'') \le l(t')$.

To show that u is increasing, let $t' < t''$. We exploit the monotonicity properties of v, and we get $(K - u(t'))^+ = v(t', u(t')) \ge v(t'', u(t')) \ge (K - u(t'))^+$. Therefore, $v(t'', u(t')) = (K - u(t'))^+$, and, consequently, $u(t'') \ge u(t')$.

To prove that at maturity $l(T) = 0$ and $u(T) = K$, we observe that $l(T) = \inf\{x \ge 0\colon v(T,x) = (K-x)^+\} = \inf\{x \ge 0\} = 0$ and $u(T) = \sup\{x \ge 0\colon v(T,x) = (K-x)^+\} \wedge K = \sup\{x \ge 0\} \wedge K = K$.

We now show that $u(T^-) = K = u(T)$ and $l(T^-) = \rho K/(\rho - \mu) > 0 = l(T)$. By construction, $u(t) \le K$ for all $t \in [0, T]$, and hence $u(T^-) \le K$. Suppose by contradiction that $u(T^-) < K$. The set $(0, T) \times (u(T^-), K) \subset \mathscr{CR}$ and therefore

$(\mathcal{L}-\rho)v = -(\partial/\partial t)v \geq 0$. As $t \uparrow T$ we have $(\mathcal{L}-\rho)v \to (\mathcal{L}-\rho)(K-x) = (\rho-\mu)x - \rho K$ for $x \in (u(T^-), K)$. But then we have $(\rho-\mu)x - \rho K \geq 0$ for $x \in (u(T^-), K)$, and therefore $(\rho-\mu)u(T^-) - \rho K \geq 0 \Rightarrow u(T^-) \leq (\rho K)/(\rho-\mu)$, delivering the contradiction. Suppose now (by contradiction) that $l(T^-) > \rho K/(\rho-\mu)$. The set $(0,T)\times(0,l(T^-)) \subset \mathscr{CR}$ and hence $(\mathcal{L}-\rho)v = -(\partial/\partial t)v \geq 0$ for $x \in (\rho K/(\rho-\mu), l(T^-)) \subset (0, l(T^-))$. As $t \uparrow T$ we have $(\mathcal{L}-\rho)v \to (\mathcal{L}-\rho)(K-x) = (\rho-\mu)x - \rho K$ for $x \in (\rho K/(\rho-\mu), l(T^-))$, where the limit is in the sense of distributions. Hence, we have $(\rho-\mu)x - \rho K \geq 0$ for $x \in (\rho K/(\rho-\mu), l(T^-))$, that is, $(-\rho+\mu)x + \rho K \leq 0$ for $x \in (\rho K/(\rho-\mu), l(T^-))$, which delivers the contradiction because $x \geq \rho K/(\rho-\mu)$ implies $(-\rho+\mu)x + \rho K \geq (-\rho+\mu)\cdot(\rho K/(\rho-\mu)) + \rho K = 0$.

We finally deal with the continuity of the l and u. The argument for u is the same as the one used by Lamberton and Mikou (2008), so that we omit it. We show instead how to prove the continuity of l. Indeed, since l is decreasing, we have, for any sequence $t_n \downarrow t \in [0,T]$, that $\lim_{t_n\downarrow t} l(t_n) \leq l(t)$. Because of the definition of l, for any t_n, we have the equality $v(t_n, l(t_n)) = (K - l(t_n))^+$. By the continuity of v and of the put payoff we pass to the limit and we get $v(t, \lim_{t_n\downarrow t} l(t_n)) = (K - \lim_{t_n\downarrow t} l(t_n))^+$. This equality implies that $\lim_{t_n\downarrow t} l(t_n) \geq l(t)$, and right continuity is proved. To prove the *left* continuity, we observe that if for some $\bar{t} \in [0,T)$, we have $l(\bar{t}) = \rho K/(\rho-\mu)$, then $l(t) = \rho K/(\rho-\mu)$ for all $t \in [\bar{t}, T)$, because l is decreasing and bounded from below by the constant $\rho K/(\rho-\mu)$. With a small abuse of notation we denote with $[\bar{t}, T)$ the (possibly empty) set in which $l(t) = \rho K/(\rho-\mu)$. On $[\bar{t}, T)$ the function l is constant and therefore continuous. Let $t \in (0, \bar{t}]$ and take a generic sequence $t_n \uparrow t$. Since l is monotonically decreasing, the limit $l(t^-) = \lim_{t_n\uparrow t} l(t_n)$ exists, and $l(t^-) \geq l(t)$. Suppose by contradiction that the inequality is strict, i.e., $l(t^-) > l(t)$. Then the open set $(0,t)\times(l(t), l(t^-)) \subset \mathscr{CR}$, and therefore (5) implies $(\partial/\partial t)v + \mathcal{L}v - \rho v = 0$, that is, $\mathcal{L}v - \rho v = -(\partial/\partial t)v \geq 0$ on $(0,t)\times(l(t), l(t^-))$, where the inequality holds because v is decreasing with respect to t.

Conversely the open set $(t,T)\times(l(t), l(t^-)) \subset \mathscr{ER}$, and therefore (5) implies $0 \geq (\partial/\partial t)v + \mathcal{L}v - \rho v = \mathcal{L}v - \rho v = (\rho-\mu)x - \rho K$ on $(t,T)\times(l(t), l(t^-))$, where the equalities follow from $v(t,x) = K - x$ on $\mathscr{ER}$.

Hence, by continuity, we get $\mathcal{L}v - \rho v = (\rho-\mu)x - \rho K = 0$ for any $x \in (l(t), l(t^-))$, which is satisfied only for $l(t) = l(t^-) = x = \rho K/(\rho-\mu)$, delivering the contradiction. □

Proof of Theorem 2.4. To prove the asymptotic behavior of the upper free boundary, we exploit formula (1.5) on page 221 in Evans et al. (2002) with interest rate $r = \rho$ and dividend yield $D = \rho - \mu < \rho = r < 0$. Hence, we get $u(t) - K \sim -K\sigma\sqrt{(T-t)\ln(\sigma^2/(8\pi(T-t)\mu^2))}$ as $t \to T$. To prove the asymptotic behavior of the lower free boundary, we exploit Remark 2 in Lamberton and Villeneuve (2003), that in our framework, applied at $-y$ and with $\vartheta = T - t$ and $\lambda := l(T^-)e^{-\sigma y\sqrt{\vartheta}}$, implies $v(T-\vartheta, \lambda) = (K-\lambda)^+ + \vartheta^{3/2}|\rho|K\sigma\phi(y) + o(\vartheta^{3/2})$ for $y > y^*$, since $\frac{\partial}{\partial x}(-\rho K e^{-\rho t} + (\rho-\mu)e^{-(\rho-\mu+(\sigma^2/2))t+\sigma x})|_{(0;1/\sigma\ln(\rho K/(\rho-\mu)))} = \rho K\sigma < 0$. Since $\phi(y) > 0$, it follows that $v(T-\vartheta, \lambda) > (K-\lambda)^+$. Hence, $(T-\vartheta, \lambda) = (T-\vartheta, l(T^-)e^{-\sigma y\sqrt{\vartheta}}) \in \mathscr{CR}$, and for ϑ small enough it is equivalent to say that $\lambda = l(T^-)e^{-\sigma y\sqrt{\vartheta}} < l(T-\vartheta)$. Passing to the log and rearranging the terms, we get $\ln l(T^-) - \ln l(T-\vartheta) < \sigma y\sqrt{\vartheta}$ and therefore $\limsup_{t\to T}(l(T^-) - l(t))/(l(T^-)\sigma\sqrt{(T-t)}) \leq y$. Since the inequality holds for all $y > y^*$, we get $\limsup_{t\to T}(l(T^-) - l(t))/(l(T^-)\sigma\sqrt{(T-t)}) \leq y^*$. We now prove the opposite inequality for $y \leq y^*$. If $l(T-\vartheta) \leq l(T^-)e^{-\sigma y\sqrt{\vartheta}} \approx l(T^-)(1 - y\sigma\sqrt{\vartheta})$, for all $y \leq y^*$ and $\vartheta = T - t \to 0$, the proof is complete. Hence, suppose now that $l(T-\vartheta) > \lambda = l(T^-)e^{-\sigma y\sqrt{\vartheta}}$. This means that $(T-\vartheta, \lambda) \in \mathscr{CR}$. We exploit again Remark 2 in Lamberton and Villeneuve (2003) applied to $-y$ (instead of y) that implies

$$\varphi(\vartheta) = v(T-\vartheta, \lambda) = (K-\lambda)^+ + g(\vartheta) \quad \text{with } g(\vartheta) = o(\vartheta^{3/2}) > 0,$$

where the positivity of $g(\vartheta)$ follows from the fact that $\lambda \in \mathscr{CR}$. Proposition 2.1 allows us to find $\zeta \in (\lambda, l(T-\vartheta))$ such that

$$v(T-\vartheta, \lambda) - (K-\lambda) = \frac{(l(T-\vartheta)-\lambda)^2}{2}\frac{\partial^2 v}{\partial x^2}(T-\vartheta, \zeta). \tag{22}$$

Indeed, since v admits the continuous first-order derivative w.r.t. x and there exists $(\partial^2 v/\partial x^2)(T-\vartheta, x)$ for all $x \in (\lambda, l(T-\vartheta))$, we can apply a Taylor expansion with the Lagrange remainder for $x = \lambda$ and $\hat{x}_0 = l(T-\vartheta)$ to conclude that $v(T-\vartheta, x) = v(T-\vartheta, \hat{x}_0) + (\partial/\partial x)v(T-\vartheta, \hat{x}_0)(x - \hat{x}_0) + \frac{1}{2}(\partial^2 v/\partial x^2)(T-\vartheta, \zeta)(x-\hat{x}_0)^2$ for some $\zeta \in (x, \hat{x}_0) = (\lambda, l(T-\vartheta))$. Since $v(T-\vartheta, \hat{x}_0) = v(T-\vartheta, l(T-\vartheta)) = K - l(T-\vartheta)$ and $(\partial/\partial x)v(T-\vartheta, \hat{x}_0) = (\partial/\partial x)v(T-\vartheta, l(T-\vartheta)) = -1$, the Taylor expansion delivers (22).

Because $\zeta \in (\lambda, l(T-\vartheta))$, we have that $(T-\vartheta, \zeta) \in \mathscr{CR}$, and therefore $-(\partial/\partial\vartheta)v + \mathcal{L}v - \rho v = 0$ for $(t,x) = (T-\vartheta, \zeta)$. From this partial differential equation at $(t,x) = (T-\vartheta, \zeta)$, we derive that

$$\begin{aligned}
&\frac{1}{2}\sigma^2\zeta^2\frac{\partial^2 v}{\partial x^2}(T-\vartheta, \zeta)\\
&\quad= \frac{\partial}{\partial\vartheta}v(T-\vartheta, \zeta) - \mu\zeta\frac{\partial}{\partial x}v(T-\vartheta, \zeta) + \rho v(T-\vartheta, \zeta)\\
&\quad> \mu\lambda + \rho v(T-\vartheta, \lambda)
\end{aligned}$$

because v is increasing w.r.t. ϑ, $(\partial/\partial x)v(T-\vartheta, \zeta) \leq -1$, $\zeta > \lambda$, and $v(T-\vartheta, \zeta) < v(T-\vartheta, \lambda)$. The quantity $\mu\lambda + \rho v(T-\vartheta, \lambda)$ is positive, since $\mu\lambda + \rho v(T-\vartheta, \lambda) = \mu\lambda + \rho((K-\lambda) + g(\vartheta)) = \rho K(1 - e^{-\sigma y\sqrt{\vartheta}}) + \rho g(\vartheta) \sim \rho K\sigma y\sqrt{\vartheta} + o(\sigma y\sqrt{\vartheta}) > 0$. Therefore, we can write

$$\begin{aligned}
&(l(T-\vartheta)-\lambda)^2\\
&\quad= \frac{v(T-\vartheta, \lambda) - (K-\lambda)}{(1/2)(\partial^2 v/\partial x^2)(T-\vartheta, \zeta)} < \frac{g(\vartheta)}{(\mu\lambda + \rho v(T-\vartheta, \lambda))/(\sigma^2\zeta^2)}\\
&\quad= \frac{\sigma^2\zeta^2 g(\vartheta)}{\mu\lambda + \rho((K-\lambda)+g(\vartheta))} < C\frac{g(\vartheta)}{\mu\lambda + \rho((K-\lambda)+g(\vartheta))},
\end{aligned}$$

where $C > 0$. Hence,

$$\begin{aligned}
(l(T-\vartheta)-\lambda)^2 &< C\frac{g(\vartheta)}{\rho K(1 - e^{-\sigma y\sqrt{\vartheta}}) + \rho g(\vartheta)}\\
&\sim C\frac{o(\vartheta^{3/2})}{\rho K\sigma y\sqrt{\vartheta} + o(\sigma y\sqrt{\vartheta})}\\
&= C'\frac{o(\vartheta^{3/2})}{-\sigma y\sqrt{\vartheta} + o(\sigma y\sqrt{\vartheta})} = C' o(\sigma^2 y^2\vartheta),
\end{aligned}$$

where $C' > 0$. This implies that $l(T-\vartheta) - \lambda = l(T-\vartheta) - l(T^-)e^{-\sigma y\sqrt{\vartheta}} < o(-\sigma y\sqrt{\vartheta})$ as $\vartheta \to 0$, i.e.,

$$l(T-\vartheta) \le l(T^-)(1-\sigma y\sqrt{\vartheta}) + o(-\sigma y\sqrt{\vartheta}) \quad \text{as } \vartheta \to 0$$

for $y \le y^*$. In other words,

$$\begin{aligned} l(T^-) - l(t) &\ge l(T^-) - (l(T^-)(1 - y\sigma\sqrt{(T-t)})) \\ &= l(T^-)y\sigma\sqrt{(T-t)}, \end{aligned}$$

for all $y \le y^*$, and hence $l(T^-) - l(t) \ge l(T^-)y^*\sigma\sqrt{(T-t)}$. Therefore, we get

$$\liminf_{t\to T} \frac{l(T^-) - l(t)}{l(T^-)\sigma\sqrt{(T-t)}} \ge y^*,$$

and thus our proof is complete. □

Proof of Proposition 2.5. If the European put option v_e dominates the immediate payoff at t for all values of the underlying value x, then there is no optimal exercise for the American option at t. The distance between the European put option and the immediate payoff at (t, x) is $f(t, x) = v_e(t, x) - (K-x)^+$, where

$$v_e(t,x) = Ke^{-\rho(T-t)}\mathcal{N}(\bar{z}) - xe^{(\mu-\rho)(T-t)}\mathcal{N}(\bar{z} - \sigma\sqrt{T-t}), \qquad (23)$$

with $\mathcal{N}(y)$ denoting the distribution function of a standard normal random variable, and $\bar{z} = (\ln(K/x) - (\mu - \sigma^2/2)\cdot(T-t))(1/(\sigma\sqrt{T-t}))$. For any $t \in [0, T]$, the function $f(t,\cdot)$ is convex, reaching its minimum at $x_m < (0, K)$ such that $(\partial/\partial x)f(t, x_m) = 0$. Hence, $f(t, x_m) > 0$ is equivalent to the fact that the European option $v_e(t, x)$ dominates at t the immediate payoff *for any* $x > 0$. Therefore, x_m is the solution of the equation $(\partial/\partial x)f(t, x) = 0$ or $(\partial/\partial x)v_e(t, x) = -1$. We compute

$$\begin{aligned} \frac{\partial}{\partial x} v_e(t,x) &= Ke^{-\rho(T-t)} f_{\mathcal{N}}(\bar{z}) \frac{\partial \bar{z}}{\partial x} \\ &\quad - e^{(\mu-\rho)(T-t)}\mathcal{N}(\bar{z} - \sigma\sqrt{(T-t)}) \\ &\quad - xe^{(\mu-\rho)(T-t)} f_{\mathcal{N}}(\bar{z} - \sigma\sqrt{(T-t)}) \frac{\partial \bar{z}}{\partial x}, \end{aligned}$$

where $f_{\mathcal{N}}$ denotes the density of a standard normal random variable and $\partial\bar{z}/\partial x = -1/(x\sigma\sqrt{T-t})$. Hence,

$$\begin{aligned} \frac{\partial}{\partial x} v_e(t,x) &= \frac{e^{-\rho(T-t)}}{\sigma\sqrt{T-t}} \left(-\frac{K}{x} f_{\mathcal{N}}(\bar{z}) + \underbrace{e^{\mu(T-t)} f_{\mathcal{N}}(\bar{z} - \sigma\sqrt{(T-t)})}_{(K/x)f_{\mathcal{N}}(\bar{z})} \right) \\ &\quad - e^{(\mu-\rho)(T-t)}\mathcal{N}(\bar{z} - \sigma\sqrt{(T-t)}), \end{aligned}$$

delivering $(\partial/\partial x)v_e(t, x) = -e^{(\mu-\rho)(T-t)}\mathcal{N}(\bar{z} - \sigma\sqrt{(T-t)})$. Therefore, x_m is defined via the following equation $\mathcal{N}(\bar{z}_m - \sigma\sqrt{T-t}) = e^{-(\mu-\rho)(T-t)}$, where $\bar{z}_m = (\ln(K/x_m) - (\mu - \frac{\sigma^2}{2})(T-t))(1/(\sigma\sqrt{T-t})$. Finally,

$$\begin{aligned} v_e(t, x_m) &= Ke^{-\rho(T-t)}\mathcal{N}(\bar{z}_m) - x_m e^{(\mu-\rho)(T-t)} e^{-(\mu-\rho)(T-t)} \\ &= Ke^{-\rho(T-t)}\mathcal{N}(\bar{z}_m) - x_m, \end{aligned}$$

and hence $f(t, x_m) = v_e(t, x_m) - (K - x_m) = e^{-\rho(T-t)}K\mathcal{N}(\bar{z}_m) - K > 0$ if and only if $e^{-\rho(T-t)}\mathcal{N}(\bar{z}_m) - 1 > 0$. Therefore, the necessary condition for possible optimal exercise at t is $e^{-\rho(T-t)}\mathcal{N}(\bar{z}_m) - 1 \le 0$, i.e., $\bar{z}_m \le \mathcal{N}^{-1}(e^{\rho(T-t)})$. Since z_m is defined via $\mathcal{N}(\bar{z}_m - \sigma\sqrt{T-t}) = e^{-(\mu-\rho)(T-t)}$, we get $\bar{z}_m = \sigma\sqrt{T-t} + \mathcal{N}^{-1}(e^{-(\mu-\rho)(T-t)})$, that delivers (13). □

Proof of Proposition 3.1. For the proofs of points 1 and 2, we refer to Theorem 6 in Detemple (2001, p. 76), extending it to the case of a negative interest rate ρ as well as a negative *dividend yield* $\delta = \rho - \mu < 0$ for the call's underlying asset. Let $\rho_{\text{put}} = \rho - \mu$ and $\mu_{\text{put}} = -\mu$. Conditions (15) and (16) for ρ, μ are *equivalent* to Conditions (6) and (7) in Proposition 2.2 and in Theorem 2.3 for $\rho_{\text{put}} = \rho - \mu$ and $\mu_{\text{put}} = -\mu$.

To prove formulae (18) in point 3, we use formula (5) in Section III of Carr and Chesney (1996), which implies

$$\begin{aligned} &v_{\text{call}}(t, x; K, \rho, \mu, \sigma) \\ &\quad = \sqrt{xK}\, \frac{v_{\text{put}}(t, \hat{x}_{\text{put}}; \hat{K}_{\text{put}}, \rho-\mu, -\mu, \sigma)}{\sqrt{\hat{x}_{\text{put}}\hat{K}_{\text{put}}}}, \end{aligned} \qquad (24)$$

for $x/K = \hat{K}_{\text{put}}/\hat{x}_{\text{put}}$. We first show that formula (17) implies formula (24). In fact, take a $\beta > 0$ such that $\hat{K}_{\text{put}} = x/\beta$ is an unconstrained strike for the put option, and let $\hat{x}_{\text{put}} = x_{\text{put}}/\beta = K/\beta$. The put option with parameters $\hat{x}_{\text{put}}$ and $\hat{K}_{\text{put}}$ (and ρ_{put}, μ_{put}, and σ_{put} as before) has the same moneyness of the call option, because $\hat{K}_{\text{put}}/\hat{x}_{\text{put}} = x/K$. By formula (17), $v_{\text{call}}(t, x; K, \rho, \mu, \sigma) = v_{\text{put}}(t, K; x, \rho - \mu, -\mu, \sigma) = \beta \cdot v_{\text{put}}(t, \hat{x}_{\text{put}}; \hat{K}_{\text{put}}, \rho_{\text{put}}, \mu_{\text{put}}, \sigma_{\text{put}})$, where the last equality follows from the homogeneity property of the put option. Writing $\beta = \sqrt{\beta\cdot\beta} = \sqrt{x/\hat{K}_{\text{put}}\cdot K/\hat{x}_{\text{put}}}$, we arrive at (24). We then apply (24) to derive the expression of $u(t)$ as in formula (18). Since (6) and (7) in Proposition 2.2 and in Theorem 2.3 are satisfied, there exist two critical prices at time $t \in (0, T)$ for the American put option $v_{\text{put}}(t, \hat{x}_{\text{put}}; \hat{K}_{\text{put}}, \rho_{\text{put}}, \mu_{\text{put}}, \sigma_{\text{put}})$. Let $\hat{K}_{\text{put}} = 1$, and denote with $0 < l_{\text{put}}(t) < u_{\text{put}}(t)$ the lower and upper free boundary of the American put option $v_{\text{put}}(t, \hat{x}_{\text{put}}; 1, \rho_{\text{put}}, \mu_{\text{put}}, \sigma_{\text{put}})$. The parameters x, K, and $\hat{x}_{\text{put}}$ are constrained by the equality $x/K = 1/\hat{x}_{\text{put}}$. Formula (24) implies that $v_{\text{call}}(t, x; K, \rho, \mu, \sigma) = \sqrt{xK}(v_{\text{put}}(t, \hat{x}_{\text{put}}; 1, \rho-\mu, -\mu, \sigma)/\sqrt{1\cdot\hat{x}_{\text{put}}})$. Then $u(t)$ for the call can be written as $u(t) = \sup\{x \ge 0\colon v_{\text{call}}(t, x) = (x-K)^+\} = \sup\{K/\hat{x}_{\text{put}} \ge 0\colon \sqrt{xK}(v_{\text{put}}(t, \hat{x}_{\text{put}}; 1, \rho-\mu, -\mu, \sigma)/\sqrt{\hat{x}_{\text{put}}}) = ((K/\hat{x}_{\text{put}}) - K)^+\} = K\cdot(\inf\{\hat{x}_{\text{put}} \ge 0\colon (K/\hat{x}_{\text{put}})v_{\text{put}}(t, \hat{x}_{\text{put}}; 1, \rho-\mu, -\mu, \sigma) = (K/\hat{x}_{\text{put}})(1-\hat{x}_{\text{put}})^+\})^{-1} = K\cdot(\inf\{\hat{x}_{\text{put}} \ge 0\colon v_{\text{put}}(t, \hat{x}_{\text{put}}; 1, \rho-\mu, -\mu, \sigma) = (1-\hat{x}_{\text{put}})^+\})^{-1} = K\cdot(l_{\text{put}}(t))^{-1}$, which gives $u(t)$ in formula (18). The expression for $l(t)$ follows by similar arguments. □

Proof of Proposition 3.2 and Theorems 3.3 and 3.4. By point 1 of Proposition 3.1, $\rho_{\text{put}} = \rho - \mu$ and $\mu_{\text{put}} = -\mu$ satisfy conditions (6) and (7) in Proposition 2.2. Therefore, for the symmetric perpetual put option with $K_{\text{put}} = 1$, there exist two constant free boundaries $0 < l_\infty^{\text{put}} < u_\infty^{\text{put}}$ that lead to $u_\infty > l_\infty$ for the call option via Equations (18). This proves Proposition 3.2. Theorem 3.3 derives from Theorem 2.3 by applying Proposition 3.1. The asymptotic expressions of u and l at maturity in Theorem 3.4 derive from formulae (18) applied to the asymptotic expression found in Theorem 2.4 for the symmetric put with parameters as defined in Proposition 3.1. A Taylor approximation of the first order delivers the final expression. □

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