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Mean Field Equilibria of Dynamic Auctions with Learning

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We study learning in a dynamic setting where identical copies of a good are sold over time through a sequence of second-price auctions. Each agent in the market has an unknown independent private valuation that determines the distribution of the reward she obtains from the good; for example, in sponsored search settings, advertisers may initially be unsure of the value of a click. Though the induced dynamic game is complex, we simplify analysis of the market using an approximation methodology known as *mean field equilibrium* (MFE). The methodology assumes that agents optimize only with respect to long-run average estimates of the distribution of other players' bids. We show a remarkable fact: In a mean field equilibrium, the agent has an optimal strategy where she bids truthfully according to a *conjoint valuation*. The conjoint valuation is the sum of her current expected valuation, together with an overbid amount that is exactly the expected marginal benefit of one additional observation about her true private valuation. Under mild conditions on the model, we show that an MFE exists, and that it is a good approximation to a rational agent's behavior as the number of agents increases. Formally, if every agent except one follows the MFE strategy, then the remaining agent's loss on playing the MFE strategy converges to zero as the number of agents in the market increases. We conclude by discussing the implications of the auction format and design on the auctioneer's revenue. In particular, we establish the revenue equivalence of standard auctions in dynamic mean field settings, and discuss optimal selection of reserve prices in dynamic auctions.

Keywords: mean field equilibrium; conjoint valuation; dynamic auction markets
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1. Introduction

Auctions are observed as a market mechanism in a wide range of economic transactions: sponsored search markets run by Google and Yahoo!, online marketplaces such as eBay and Amazon, crowdsourcing, procurement auctions for public service contracts, licensing auctions (e.g., for mining or oil tracts), etc. Nearly all of these examples are characterized by two important features. On one hand, the auction format is typically relatively straightforward to describe, consisting of repetitions of a simple one-shot auction format. On the other hand, despite the simplicity of the mechanism itself, such markets can give rise to complex dynamic incentives for bidders. As a result, many basic questions become quite challenging: determining optimal bidding strategies for bidders; characterizing dynamic equilibrium behavior among the bidders; and determining optimal choices of market parameters for the auctioneer, such as auction format and reserve prices.

In this paper, we address these challenges in the context of dynamic auctions with *learning bidders*. Dynamic auctions are often characterized by incomplete information among the bidders regarding their and other bidders' preferences. As a concrete example, consider online sponsored search auctions (Edelman et al. 2007, Varian 2007). These repeated auctions operate on a per keyword basis; in a typical scenario, advertisers bidding on a particular keyword estimate their underlying valuation based on the conversion rate from an ad click to revenue (e.g., from sales). As the advertisers win more ad placement through the auction, they *learn* this conversion rate, which informs their bidding decision in the auction (Ghose and Yang 2009, Rey and Kannan 2010, Sculley et al. 2009). Similar incentives arise in other applications as well (e.g., in licensing auctions).

In these settings, the bidders face a trade-off between *exploration*, where they bid higher to obtain more information about their value, and *exploitation*, where they bid optimally given their current information.

The exploration–exploitation trade-off has significant ramifications for market operation and design. First, it complicates the design of optimal bidding strategies. In sponsored search markets, for example, identifying optimal learning strategies would lead to better design of bidding agents that incorporate advertisers' uncertainty about their conversion rate. Second, as the bidders are playing a complex dynamic game, it can be intractable to characterize equilibrium behavior among many interacting bidders, and in particular to determine how bidders' uncertainty affects the distribution of bids seen over time. Third, as a consequence, we lose the ability to guide market operation and design. For instance, auctioneers usually set reserve prices in such markets to increase their revenue. As we later demonstrate, setting a reserve without regard to learning among the bidders may cause unwarranted restriction of allocation, and ultimately yield *lower* revenue.

Before continuing, we note that the existing theoretical literature on modeling and analysis of dynamic mechanisms focuses heavily on the *dynamic mechanism design problem*, where the objective is to design mechanisms that achieve given objectives such as efficient allocation (Athey and Segal 2013, Bergemann and Välimäki 2010). Such mechanisms typically align incentives for participants to report their private information truthfully, and as a consequence strategic behavior is eliminated (simplifying the analysis of the market). However, in return for this "simple" description of bidder behavior, such optimal dynamic mechanisms are typically extremely complex in design and difficult to implement. As a result, they are rarely deployed in practical settings. There are many nontechnical considerations (e.g., ability to explain the auction to participants, protection from liability, etc.) that lead market operators to choose simple mechanisms (Ausubel and Milgrom 2006, Milgrom 2004); we do not attempt to be exhaustive here. By contrast to the dynamic mechanism design literature, our work focuses on repetitions of a *simple* one-shot mechanism, and addresses the resulting *complex* dynamic incentives.

In this paper, as described in detail in §2, we study a setting where identical copies of a good are sold through a sequence of second-price auctions over time; as an example, a copy of the good may denote a click on an advertiser's ad in sponsored search settings. (We also analyze repetitions of other standard one-shot auction formats.) Each agent in the market has an independent private valuation that determines the distribution of the reward she obtains from the good; the private valuation may denote an advertiser's conversion rate in sponsored search. Although agents are initially unaware of their own private valuation, every time an agent wins an auction and obtains a copy of the good, her realized reward from the good incrementally informs her about her valuation. The strategic interactions among the agents along with their beliefs about their valuation influence their bids in the auction. Thus, we naturally obtain a dynamic game among the agents in our model.

The standard game-theoretic tool used to analyze such dynamic games is the equilibrium concept known as *perfect Bayesian equilibrium* (PBE). However, there are two central problems with this approach. First, such equilibria are *intractable*: the state space complexity is enormous (since bidders must maintain beliefs over all that is unknown to them). Second, and partly in consequence, such equilibria are *implausible*: PBE requires each bidder to accurately forecast and estimate *exactly* how her competitors will respond to any bid she makes today.

The complexity of PBE motivates us to consider an approximation methodology that we refer to as *mean field equilibrium* (MFE). (See Adlakha et al. 2015, Huang et al. 2007, Jovanovic and Rosenthal 1988, Lasry and Lions 2007, and Tembine et al. 2009.) MFE is inspired by a *large market* approximation: with a large number of bidders, tracking and forecasting the exact behavior of individual bidders is impractical and implausible. In an MFE, individuals take a simpler view of the world. They postulate that fluctuations in the empirical distribution of other agents' bids have "averaged out," and thus optimize holding the bid distribution of other agents fixed. MFE requires a consistency check: the postulated bid distribution must arise from the optimal strategies agents compute. The benefit of analyzing a large market using MFE is that for the agents to optimize their behavior, it is sufficient for them to just maintain beliefs about their *own* private valuation. This reduces the dimension of the system state that each agent needs to track, simplifying the analysis tremendously.

Furthermore, we believe that MFE corresponds more closely to an equilibrium concept that might be applicable in practice, particularly in settings with a large number of bidders. For example, in sponsored search auction markets, bidders generally do not have access to complete information about the bid history for auctions they participated in. Rather, bidders are usually provided with various tools by the auctioneer to aid in strategizing how to bid; e.g., Google provides the advertisers with a bid simulator that simulates how often an ad would get displayed and clicked upon on making a particular bid (Friedman 2009). The bid simulator bases its predictions on aggregated historical data that gives a "bid landscape" of competitors' bids on the same category or keyword (i.e., the distribution of bids). Bid landscapes inherently assume stationarity in the market, at least for a limited time horizon of interest; thus, bidders are reacting to average information about their competitors. It is reasonable to expect that for many bidders, therefore, their own decision of how to bid will *not* explicitly forecast opponents' reactions, and instead will assume that these reactions have averaged out in any forecasting about future auction outcomes. This type of example illustrates how the rationality assumptions in MFE might naturally arise in practice.

Our main contributions address the challenges raised above.

1. *Characterizing optimal strategies for bidders*: *conjoint valuations.* We first determine, in §4.1, an agent's optimal strategy in the large market. In Proposition 1, we show that in the large market model, the optimal strategy of an agent takes a remarkably simple form: given her current belief about her valuation, the optimal strategy is to bid according to a conjoint valuation. The conjoint valuation is the sum of her current expected valuation, together with an overbid. This overbid denotes an agent's value for learning about her true private valuation, and we show that it is exactly the expected marginal benefit to one additional observation about her valuation. Thus, the conjoint valuation presents a structurally simple and *plausible* strategy that captures how an agent in the large market balances the trade-off between exploration and exploitation. We note that similar trade-offs have been observed in the design of efficient direct mechanisms in similar dynamic environments (Bergemann and Välimäki 2010).

2. *Consistency and validity of the mean field model*: *existence of MFE and an approximation theorem.* We next show in §4.2 that the mean field model is consistent by proving the existence of an MFE. This involves showing that the stationary distribution of a market where each agent follows the mean field strategy turns out to be the market distribution that each agent had assumed to solve their decision problem. In §6, we extend this result under mild conditions to a dynamic auction setting consisting of repetitions of a fixed *standard auction*; for example, standard auctions include second-price, first-price, and all-pay auctions. Thus, we obtain, in fairly general settings, the existence of informationally simple equilibria in mean field models, where agents make bids taking into consideration only their own belief about their valuation and the bid distribution in the market.

We next tackle the issue of whether an MFE, which rests on a large market assumption, accurately captures a rational agent's behavior in a finite market. We prove in §5 that indeed an MFE is asymptotically a good approximation to agent behavior in a finite market. Formally, we show that if in a finite market, every agent except one follows the MFE strategy, then the remaining agent's loss on playing the MFE strategy converges to zero as the number of agents in the market increases. Our main approach is to establish correlation decay in the interacting particle system induced by the meetings of bidders in successive auctions. We achieve this by carefully tracking the interactions among the agents, using the notion of interaction sets defined in Graham and Méléard (1994). This result justifies formally the use of an MFE to analyze agent behavior in a finite market as the number of agents increases.[1]

3. *Market design*: *auction format and reserve prices.* Finally, to illustrate the power of our approach, we leverage the analytical and computational simplicity of MFE to address "second best" market design: how should an auctioneer choose market parameters to maximize revenue, with the constraint of relatively "simple" repeated auction mechanisms?

In static settings, an implication of the revenue equivalence theorem is that an auctioneer's expected revenue in any standard auction is the same. This implies that an auctioneer seeking to maximize her revenue has to either restrict allocation (Myerson 1981) or increase participation (Bulow and Klemperer 1996). In a dynamic setting, the main difference is that now changing the auction format not only affects an agent's payment in each auction, but also affects her incentive to learn more about her valuation. Nevertheless, in §6, we prove the *revenue equivalence of standard auctions* in dynamic mean field settings. We show that for every MFE of a repeated second-price auction market, there exists an MFE of a repeated standard auction market that yields the auctioneer the same expected revenue. This result shows that changing the one-shot auction format within the class of standard auctions will not increase the seller's expected revenue.

We then consider the possibility of increasing revenue by choosing a reserve price. In static auctions, setting a reserve has the effect of extracting greater revenues from high-valuation bidders, at the expense of shutting out bidders with lower valuations. In dynamic auctions with learning, however, a reserve has an added negative effect on revenue: it reduces bidders' incentives to learn their valuation. Thus, there may potentially be a high penalty if the auctioneer ignores learning. In §7, we develop benchmarks to evaluate this penalty, by comparing the MFE where an auctioneer anticipates bidders' learning behavior, against one where the auctioneer is oblivious to bidders' learning. The computational tractability of MFE allows us to

[1] We emphasize, however, that MFE may be a useful approximation even when the number of agents is not large, simply because it more accurately captures the information available to bidders when they optimize (e.g., in sponsored search auctions, bidders are responding to bid landscape information).

evaluate these benchmarks: we numerically observe that the auctioneer can incur a potentially significant loss by ignoring the learning cost while setting the reserve price.

We make two significant modeling assumptions to aid our analysis and to obtain the preceding results. First, we assume that in each auction, a subset of the agents is chosen independently and uniformly to participate. Although this assumption is restrictive, technically it guarantees that as the market size grows, the influence exerted by any particular agent on any other agent vanishes. This ensures that MFE is a good approximation to agent behavior in large finite markets.

Second, we assume that agents have finite (geometrically distributed) lifetimes in the market, and the arrivals and the departures of agents in the market are perfectly matched: for every agent that leaves the market, a new one arrives, both having the same label. This allows us to study a stationary setting in which learning is nevertheless constantly occurring; it is also a nice way to capture the fact that bidder valuations may evolve over time. Our model can easily be generalized to incorporate stochastic variations in arrivals and departures; see §2.5.

1.1. Related Work

As noted in the introduction, one inspiration for our model comes from sponsored search auction markets, and there is an extensive literature analyzing these in a static setting (Edelman et al. 2007, Varian 2007). In the dynamic setting, Babaioff et al. (2009) and Devanur and Kakade (2009) study the mechanism design problem of a sponsored search auctioneer who has to allocate ads among advertisers; here it is the auctioneer who is learning ad quality, rather than bidders learning private valuations (as in our model). Li et al. (2010) study a single agent's decision problem in a repeated generalized second-price auction setting where the agent does not know her click-through rate, and show that agents will overbid; however, they do not study the interaction of multiple bidders (i.e., their model is not game theoretic).

Varian (2009) performs empirical analysis on bid data to estimate the valuation of the advertisers, and thus computes the value to the advertisers from participating in the auction. Our model provides a structural foundation for a dynamic complement to Varian's analysis: estimation of bidder valuations in our model would allow for estimation of how much agents overbid because of learning.

Another line of literature that is closely related to our work is that of dynamic mechanism design; see, e.g., Athey and Segal (2013), Bapna and Weber (2005), Bergemann and Välimäki (2010), Cavallo et al. (2009), and Nazerzadeh et al. (2013), as well as Bergemann and Said (2010) for a survey. Bergemann and Välimäki (2010) study the problem of designing efficient auctions in a dynamic setting where, among other possibilities, the agents have unknown private values. The authors show that, in the efficient dynamic auction, the item is awarded to the bidder with the highest Gittins index, taking into account both the current expected value and the value of information. In our work, we capture this value of information through the notion of conjoint valuation. In general, this line of literature considers the *design* of either efficient or revenue-maximizing dynamic auctions when agents have unknown and possibly dynamic private values. In contrast to these, our work analyzes agent behavior in a *fixed*, individually rational, repeated auction mechanism.

Mean field equilibrium models have received extensive attention recently. The origins of mean field models can be traced back to statistical physics; in the context of stochastic dynamic games, mean field equilibrium and related concepts have been studied in economics as well as control and operations research (Adlakha et al. 2015, Huang et al. 2007, Jovanovic and Rosenthal 1988, Lasry and Lions 2007, Weintraub et al. 2008, Yin et al. 2010), and applied to analysis of dynamic oligopoly models (Hopenhayn 1992; Weintraub et al. 2008, 2011), dynamic search models (Duffie et al. 2009), and large markets (Bodoh-Creed 2013, Peters and Severinov 1997), among other applications. Our approach to our existence and approximation theorems is inspired in part by this literature.

Mean field models for dynamic economic models take inspiration from work on both large markets and boundedly rational agents. We do not attempt to exhaustively survey the extensive literature in both these areas; rather, we focus on few closely related works. Wolinsky (1988) considers a large dynamic market where buyers and sellers are matched and trade occurs through first-price auctions. McAfee (1993) studies a large dynamic model with many competing sellers and buyers. In equilibrium, each seller chooses a second-price auction with reserve price to sell the good, while the buyers randomize over different sellers. Backus and Lewis (2010) consider an auction market with boundedly rational, unit demand buyers where multiple similar products with random supply are sold through a sequence of second-price auctions. In all these models, all market participants know their private value (or cost) exactly, and as such there is no learning involved in the model.

Finally, a related line of work considers equilibrium learning schemes for repeated auctions with incomplete information; see, e.g., Hon-Snir et al. (1998) and Ashlagi and Monderer (2006). In both these settings, the bidders know their own valuation, and learn the equilibrium bidding strategies of the competitors. In contrast, in our work, the bidders learn their unknown private valuation, and the (mean field) equilibrium bidding strategy is common knowledge.

2. Model

We describe a dynamic auction model where each bidder has an independent private valuation for the auctioned good; however, this private valuation is initially unknown to the bidder. To completely specify the model, we need to describe how the market operates, the learning model employed by the agents, and the dynamics of the market. We conclude with a section discussing and summarizing the assumptions made in our model.

2.1. Market

We consider a setting with n agents participating in a dynamic market where multiple identical copies of a good are auctioned over time through a sequence of second-price auctions. Because a static second-price auction has a truthful dominant strategy equilibrium, this modeling choice ensures that the strategic behavior in our model arises only from learning and the dynamics of the market. In §6, we extend our model to repetitions of more general *standard auctions*.

We assume that in each auction, the participation is restricted to some (randomly chosen) subset of agents. We implement the restricted participation assumption as follows: in each auction, $\alpha > 0$ agents are chosen independently and uniformly at random to bid for a single copy of the good. We refer to α as the *auction thickness*. For ease of exposition, we assume that the auction thickness is a fixed positive integer, although our results remain unchanged even if we assume α to be a bounded positive random variable.

We assume the sequence of second-price auctions takes place according to the jump times of a Poisson process $N^{(n)} = \{N_t^{(n)}: t \geq 0\}$ with rate n/α; an auction occurs each time the process $N^{(n)}$ jumps. Because the agents are chosen uniformly at random, this modeling choice ensures that the random times when a single agent i is chosen to participate are distributed as the jump times of a Poisson process of rate 1. Let τ_i^k, $k \geq 1$, denote these participation times for an agent i, i.e., at time τ_i^k, agent i is chosen to participate in an auction for the kth time. Let $S_{i,k}$ denote the set of agents other than agent i in the auction at time τ_i^k. Denote the bid made by agent i in the auction at time τ_i^k as b_{i,τ_i^k}. Because the market organizer uses a second-price auction, agent i wins if her bid b_{i,τ_i^k} is larger than the bids made by all the agents in $S_{i,k}$. We define $w_{i,k}$ as the indicator variable denoting whether agent i won in the auction at time τ_i^k; i.e., if agent i wins in the auction, then $w_{i,k} = 1$, otherwise $w_{i,k} = 0$.

On winning the auction at time τ_i^k, an agent i obtains the auctioned good and makes a payment equal to the largest bid made by agents in $S_{i,k}$, denoted by $b_{-i,\tau_i^k} \triangleq \max_{j \in S_{i,k}} b_{j,\tau_i^k}$. On obtaining the good, the agent receives a random reward, which we denote by $x_{i,k}$. For example, in a sponsored search market, the reward may be the (random) profit an advertiser obtains from a user arriving at their website after an ad click. On the other hand, on losing the auction, the agent does not obtain the good and has to make no payment; for convenience, we define $x_{i,k} = 0$ if agent i loses in the auction at time τ_i^k, i.e., if $w_{i,k} = 0$.

Associated with each agent i is an unknown private valuation $v_i \in [0, 1]$. For an agent i, conditional on the event $w_{i,k} = 1$, the distribution of the reward $x_{i,k}$ is stationary and depends on her valuation v_i. We assume that the rewards $x_{i,k}$ are distributed randomly and independently across all agents in the market; further, we assume given v_i, the rewards $x_{i,k}$ are independent over time. In particular, we make the following distributional assumption on $x_{i,k}$:

$$\text{if } w_{i,k} = 0, \text{ then } x_{i,k} = 0;$$

$$\text{if } w_{i,k} = 1, \text{ then } x_{i,k} = \begin{cases} 1 & \text{with probability } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the valuation v_i denotes the probability with which agent i obtains a fixed positive reward of \$1 when she wins her auction. The uniform normalization of rewards across agents is a model simplification; a more general model can allow the maximum reward to vary across agents. However, this model extension would entangle the effect of learning with the effect of the maximum reward on the agents' bids.

2.2. Learning Model

Note that the agents do not know their own valuation. As the agents win auctions and observe their rewards, they learn more about their valuation. We now describe the learning model employed by the agents.

We assume that the agents are Bayesians who model their incomplete information by updating beliefs about their valuation using Bayes' rule as new information (in the form of $w_{i,k}$ and $x_{i,k}$) arrives. In particular, we assume that the agents employ a beta prior belief model: the initial belief of an agent i is specified by the beta distribution $B(m_{i,0}, n_{i,0})$, where $m_{i,0}, n_{i,0} \geq 1$ are positive real parameters. We extend our model to more general belief structures satisfying certain continuity assumptions in Appendix EC.1 in the electronic companion (available at http://dx.doi.org/10.1287/mnsc.2014.2018).

Note that the beta distribution with parameters $m, n \geq 1$ has a density on $[0, 1]$ given by

$$f_{(m,n)}(x) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} x^{m-1}(1-x)^{n-1}, \quad \text{for } x \in [0, 1],$$

where $\Gamma(\cdot)$ denotes the gamma function. Beta distributions are widely used in Bayesian statistics for estimation and as learning schemes. In particular, we choose the beta prior model for two primary reasons.

First, because of conjugacy with Bernoulli rewards, the class of beta distribution is closed under the application of Bayes' rule. Second, by varying the parameters $(m_{i,0}, n_{i,0})$, one can model different degrees of certainty among the agents regarding their valuation: as either $m_{i,0}$ or $n_{i,0}$ increases, the agent becomes more certain about her valuation, i.e., the variance of her belief about her valuation decreases. We denote by $\mathcal{S} = \{(x, y) \in \mathbb{R}^2: x \geq 1, y \geq 1\}$ the set of possible beliefs for an agent, and with a mild abuse of notation, refer to $s \in \mathcal{S}$ as an agent's belief when her belief is given by the beta distribution with parameters s.

The transitions in the belief of an agent i can be represented as follows. Because agent i does not receive any new information about her valuation in the time between two consecutive auctions, her belief in this time interval remains unchanged. Hence, we only keep track of her belief at times τ_i^k: $k \geq 1$. If the agent does not win the auction at time τ_i^k, i.e., if $w_{i,k} = 0$, then she receives no reward, and hence does not get any new information regarding her valuation. In this case, she does not update her belief about her valuation. On the other hand, if the agent i wins in the auction at time τ_i^k, i.e., $w_{i,k} = 1$, then she updates her belief via Bayes' rule, after observing her realized reward $x_{i,k}$. It is straightforward to verify that if the realized reward $x_{i,k}$ is 1, then using Bayes' rule, the posterior belief of the agent is given by $B(m_{i,k}, n_{i,k})$, where $m_{i,k} = m_{i,k-1} + 1$ and $n_{i,k} = n_{i,k-1}$. On the other hand, if the realized reward is zero, then Bayes' rule entails that the posterior belief is given by $B(m_{i,k}, n_{i,k})$, where $m_{i,k} = m_{i,k-1}$ and $n_{i,k} = n_{i,k-1} + 1$. Thus, after k auctions, the belief of the agent i regarding her valuation is given by $B(m_{i,k}, n_{i,k})$, where $m_{i,k} - m_{i,0}$ is the number of times in the first k auctions that the agent won and received a positive reward, and $n_{i,k} - n_{i,0}$ is the number of times in the first k auctions that the agent won but received zero reward. Letting $s_{i,k} = (m_{i,k}, n_{i,k})$, we see that the transitions in the belief of an agent can be succinctly represented as

$$s_{i,k} = \begin{cases} s_{i,k-1} & \text{if } w_{i,k} = 0, \\ s_{i,k-1} + x_{i,k} e_1 + (1 - x_{i,k}) e_2 & \text{if } w_{i,k} = 1, \end{cases} \tag{1}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

For $s \in \mathcal{S}$, let $\mu(s) \triangleq \mathbf{P}(x_{i,k} = 1 \mid s)$ denote an agent's belief of getting a positive reward on obtaining the good, if her belief about her valuation is s. Note that $\mu(s)$ also denotes an agent's expected valuation. It is straightforward to see that $\mu(x, y) = x/(x + y)$ for $(x, y) \in \mathcal{S}$.

To complete the description, we need to specify how an agent's initial belief is set. We require that each agent i's valuation v_i and her initial belief $s_{i,0}$ are jointly distributed according to a distribution Ψ over $[0, 1] \times \mathcal{S}$, independent from the rest of the market. Though it is not crucial for our results, we assume that the distribution Ψ is consistent with each agent's belief: given a belief $s \in \mathcal{S}$, the conditional distribution $\Psi(\cdot \mid s)$ of an agent's valuation is exactly the beta distribution indexed by s. In the rest of the paper, we make the following technical assumption on the measure Ψ.

Assumption 1. *The measure Ψ has a continuous density ψ on $[0, 1] \times \mathcal{S}$.*

2.3. Arrivals and Departures (Regeneration)

We next describe the dynamics of the agent population. The model we consider assumes agents arrive and depart over time. We make this modeling choice primarily so that there is continual learning in the market and we can study the effect of learning in a steady-state model. To the contrary, suppose the agents persist in the market forever. Then we expect that after a sufficient number of observations, agents would learn their valuations almost exactly. Such a model precludes studying the effect of learning in a stationary setting. Moreover, studying learning in a transient model is significantly more complex, as the effect of learning is confounded with effects of the initial configuration and the dynamics of the entire market.

By contrast, arrivals and departures ensure that even in steady state, all agents are learning. Moreover, it is reasonable to expect that any realistic model of a dynamic market should involve the arrival and attrition of market participants over time. Our model reflects this natural process.

Since we are interested in a stationary setting, we assume that the rate of arrival and departures of the agents balance, so that in expectation the market size remains constant. We implement this assumption through the following arrival and departure process. After each auction, with probability $1 - \beta$ and independently across agents, each participating agent departs the market (never to return). Here $0 < \beta < 1$. Furthermore, for each agent i that leaves the market, a new agent immediately arrives at the market; we assign the new agent the same label i, but sample a new valuation v_i and belief s_i, jointly distributed as Ψ.

One additional advantage of assuming that arrivals and departures are exactly matched is that we can equivalently interpret arrivals and departures as changes in the agents' valuations. More precisely, we could instead assume that the agents' valuations and beliefs receive an independent shock after a geometrically distributed time interval with parameter β, at which point they obtain a new valuation and belief distributed according to Ψ. We call this event a *regeneration* for the agent. In the sequel, for mathematical simplicity and without loss of generality, we interpret the arrivals and departures as regenerations for the agents.

2.4. The Dynamic Market Game

In this section, we view the market described above as a dynamic game, and formally define perfect Bayesian equilibrium.[2]

We denote an agent i's *history* at time t by $h_{i,t}$. Agent i's history at time t includes her initial belief about her valuation, how many auctions she has participated in until time t, the times these auctions took place, the bids she made in these auctions, whether she won in these auctions, and if so, her realized rewards, and the payments she had to make. When an agent regenerates, she forgets her past history, and obtains a new history that reflects only her initial belief about her valuation.

We define the *market history* at time t to be the vector of joint histories of all agents up to time t. Similarly, we define the *infinite market history* to be the vector of joint histories of all agents over the entire infinite horizon. At each time t, an agent in the market holds a *belief* $\nu_i(h_{i,t})$ about the future evolution of the market; this is a probability distribution over all possible infinite market histories that are consistent with $h_{i,t}$. (In the interests of clarity we deliberately suppress measure-theoretic details here.)

A (pure) *strategy* δ_i for an agent i is a mapping that at any time t, given her history $h_{i,t}$, specifies a bid $\delta_i(h_{i,t})$ if she is selected to participate in an auction at time t. We let $\delta = (\delta_1, \ldots, \delta_n)$ denote the strategy profile where agent i follows strategy δ_i, and let $\delta_{-i} = (\delta_j\colon j \neq i)$ denote the strategies of all agents except agent i.

We assume that after any history, the agents seek to maximize their *total expected payoff* in the market. Formally, given δ and ν_i, the *value function* for agent i at time t after any history $h_{i,t}$ is defined as

$$V_i(h_{i,t}, \delta) \triangleq \mathbf{E}_{\nu_i(h_{i,t}),\delta}\left[\sum_{k=1}^{\infty}(x_{i,k} - b_{-i,\tau_i^k})\mathbf{I}\{w_{i,k} = 1, \tau_i^k \leq T_i^t\}\right], \quad (2)$$

where T_i^t is the first time after time t when agent i regenerates. Here the expectation is with respect to the agent's belief $\nu_i(h_{i,t})$ after history $h_{i,t}$. Recall that b_{-i,τ_i^k} is defined as $b_{-i,\tau_i^k} = \max_{j\in S_{i,k}} b_{j,\tau_i^k}$.

We are now ready to describe the equilibrium notion of the dynamic game, perfect Bayesian equilibrium. A strategy profile δ and belief structures ν_i for each agent i are said to be a PBE if the following two conditions hold:

1. For each agent i, after any history $h_{i,t}$, we have $\delta_i \in \arg\max_{\delta_i'} V_i(h_{i,t}, \delta_i', \delta_{-i})$.
2. For each agent i not undergoing regeneration after an auction at time t, her belief $\nu_i(h_{i,t})$ after the auction is obtained from the belief $\nu_i(h_{i,t^-})$ through Bayes' rule, whenever possible.

[2] Note that most of the formalism in this section will not be used in the remainder of the paper.

The first condition requires that given the strategies of all the other agents, and the belief structure of agent i, the strategy δ_i is optimal for the agent after any history. The second condition requires that the beliefs of an agent in the market remain consistent across time as long as she does not undergo regeneration.

From the definition, we observe that the equilibrium notion of PBE requires each agent in the market to maintain and update her belief about the entire future dynamics of the market, not just over her valuation. With a large number of agents in the market, PBE requirement imposes a very strong requirement on the rationality of each agent. Furthermore, this strong demand does not come with the benefit of tractability; an agent's decision problem in a finite market is fairly complicated. In general, each agent's equilibrium strategy would depend on the entire history of the market as well as all other agents' strategies. As a result, a closed-form characterization of agents' bids or a structural analysis of an equilibrium is essentially impossible.

By contrast, in a market with a large number of agents, it becomes reasonable to conjecture that the distribution of the "price" $b_{-i,t}$ seen by an agent i in an auction becomes stationary. Furthermore, because of the large number of agents, it is very unlikely that any two randomly chosen agents would have participated in an auction together. This suggests that in a large market, any two randomly chosen agents are independent of each other. Finally, because of the size of the market, it is plausible to assume that the actions of a single agent do not influence the future market dynamics. These features greatly simplify analysis of the resulting game; in particular, the state space of an agent *collapses* to just her current posterior belief over her valuation, instead of her entire past history. We formally develop these ideas in the next section.

2.5. Assumptions: Summary and Discussion

In this section, we discuss in more detail the assumptions that go into our model of the dynamic auction market.

2.5.1. Agent Participation. Perhaps the most important assumptions in our model are that (1) participation in any given auction is restricted to a subset of agents, and (2) the set of agents that participate is drawn independently and uniformly at random from the population. Restricting participation reflects the reality of the advertising markets for two reasons. First, advertisers bid on a category of keywords, which may be different across different advertisers, and in each specific auction on a keyword, only a subset of advertisers participate. Furthermore, search engines adopt various matching schemes, for example *broad match*, to match relevant advertisers to keywords. Second, in such markets, advertisers typically set budgets over their campaign lifetime, and search engines employ *throttling policies* where each advertiser is selectively chosen to participate in the auction so as to smooth the budget consumption over the campaign lifetime (Goel et al. 2010). Although we do not explicitly model budgets in our setting, we believe our approach is an approximation to such practical concerns. (Note that although we assume the number of participants in any auction to be constant, this is an assumption made largely for technical convenience; we can easily extend the model to settings where the number of participants in any given auction is drawn randomly.)

On the other hand, the assumption that this subset of agents is chosen independently and uniformly at random is a much stronger assumption, and one essential to the development of our mean field approach. This model precludes, for example, a setting where multiple advertisers share similar preferences over the features of the user responsible for the search query (say demographics, location, etc.)—since in such a model certain groups of advertisers would be more likely to participate repeatedly in the auctions together. Incorporating such dependencies in participation among different advertisers requires a more careful model detailing advertiser preferences over auction features, which our current model does not capture. Nevertheless, we believe our model provides a first order approximation to the concerns of participants in repeated auctions.

2.5.2. Valuations. We assume (unknown) independent private values. In practice, since each advertiser is facing the same user (i.e., same *eyeball*), there can be substantial correlations in the value each advertiser attaches to a click. In our model, we have essentially assumed that this common value component across the advertisers has been learned, and only the unknown, possibly advertiser-dependent aspect is being learned by the advertisers over their lifetime in the market.

We also assume that the rewards are normalized to 0 and 1. This assumption essentially states that the advertiser knows the (mean) reward from a conversion. This is reasonable if the conversion is defined to be the sale of a product that the advertiser is selling. However, in more general settings, where conversion can mean brand awareness, or potential future sales, this reward is nontrivial, and our assumption requires that the advertiser has effectively estimated this reward from such a conversion. In practice, an advertiser may estimate the conversion rate and the reward from a conversion simultaneously; we do not consider this more general learning problem in our model.

Finally, we have assumed that agents hold beta beliefs over their valuations. We assume this belief structure purely for technical convenience, and our results hold for more general belief models (we discuss general belief models in more detail in Appendix EC.1); the only difficulty in considering a general belief model may be in computing the belief updates as an agent sees a stream of rewards.

Assumption 1—that Ψ, the joint distribution over initial valuation and belief, has a continuous density—is a technical assumption made to ensure that equilibria exist in our model.

2.5.3. Arrivals and Departures. Recall that we assumed agents arrive and depart so that we can study learning in auctions in a steady-state setting. Furthermore, we believe this assumption realistically reflects the arrival and attrition of market participants in any practical dynamic market.

We make the assumption that arrivals and departures are exactly matched primarily for technical convenience; our model can incorporate stochastic variance of arrivals and departures, and hence a stochastic market size, as long as the rate of arrival is at least as high as that of the departure, since otherwise, in steady state the market vanishes.

Finally, note that we assume each agent participates in a geometric number of auctions, with the same parameter β across agents. This assumption is made mainly for analytical convenience, and can be relaxed in many ways. For example, we can allow different agents to participate in a geometric number of auctions with different parameters: if the maximum value $\bar{\beta}$ across all agents of the parameter β satisfies $\bar{\beta} < 1$, then our results carry forward without any significant change. If some agents persist in the market forever, i.e., when $\bar{\beta} = 1$, the value function of such agents is not well defined. However, note that in steady state one would expect such agents to know their private values exactly, and hence bid their values truthfully in each (second-price) auction. If we assume this behavior for such agents, our analysis can be modified to carry through as well.

3. The Mean Field Model

In this section, we describe the mean field model consisting of a continuum of agents and define a mean field equilibrium. To specify the definition, we need to describe an agent's decision problem in a mean field model, as well as characterize the bid distribution that arises in the steady state of the model. As we see below, these entities are coupled together intricately, and describing one requires the other. To do this effectively, we present a single agent's decision problem in the mean field model in §3.1, assuming the bid distribution in the market is fixed. Then, in §3.2, we describe the distribution of the valuation and belief of the agent in such a market given the agent employs a stationary strategy. Finally, in §3.3, we combine the two to define a mean field equilibrium.

3.1. Agent's Decision Problem

In this section, we begin by viewing the mean field model with a continuum of agents from the perspective of the single agent i. In such a model, the agent never faces the same agent twice in the auctions she participates in. So, from the perspective of agent i, her competitors' bids in each auction are all distributed independently, and in steady state, identically. Let g denote this distribution. (We assume that g has support $[0,1]$.) Then, the probability that agent i wins with a bid $x \in [0,1]$ in an auction she participates in at time t is

$$q(x \mid g) \triangleq \mathbf{P}(b_{-i,t} \leq x) = g(x)^{\alpha-1}, \tag{3}$$

since $\alpha - 1$ other bidders participate in the same auction. (Recall that α is the auction thickness.) Furthermore, the expected payment if she bids x is

$$\begin{aligned} p(x \mid g) &\triangleq \mathbf{E}[b_{-i,t}\mathbf{I}\{b_{-i,t} \leq x\}] \\ &= xq(x \mid g) - \int_0^x q(u \mid g)\,du. \end{aligned} \tag{4}$$

(The last result is standard in static auction theory, and can be derived using integration by parts.)

Recall the decision problem (2) faced by an agent i. The agent has to choose bids $b_{i,t}$ over time to maximize her total expected payoff, defined as

$$V_i(h_{i,t}, \delta) = \mathbf{E}_{\nu_i(h_{i,t}),\delta}\left[\sum_{k=1}^{\infty}(x_{i,k} - b_{-i,\tau_i^k})\mathbf{I}\{w_{i,k} = 1, \tau_i^k \leq T_i^t\}\right].$$

We now make the following observations in this model:

1. After each auction, each bidder departs (independent of all other sources of randomness) with probability $1-\beta$.
2. At the kth auction that agent i participates in, given her history h_{i,τ_i^k}, if she wins the auction, then $x_{i,k}$ has conditional expectation $\mu(s_{i,k-1})$, where $s_{i,k-1}$ is the belief of the agent after the $(k-1)$th auction.

It follows from the preceding discussion that we can reduce the total expected payoff of an agent to

$$V_i(h_{i,t}, \delta) = \mathbf{E}_{\nu_i(h_{i,t}),\delta}\left[\sum_{k=1}^{\infty}\beta^{k-1}(q(b_{i,\tau_i^k} \mid g)\mu(s_{i,k-1}) - p(b_{i,\tau_i^k} \mid g))\right].$$

From the preceding expression, we see that in a large market, it suffices for the agent to maintain and update her belief over her valuation, ignoring the rest of the dynamics of the market, as long as she knows the stationary bid distribution $g(\cdot)$. Thus, in a large stationary market, the (payoff-relevant) state space of an agent i collapses, and can be summarized by her belief $s_{i,t}$ at each time t.

Furthermore, the decision problem the agent faces is a stationary, discrete time, infinite horizon expected discounted reward maximization problem where she has to choose bids $b_{i,k} \triangleq b_{i,\tau_i^k}$ in a repeated sequence of second-price auctions. Thus, the value function for an agent i with belief s is well defined, and given by

$$V(s \mid g) \triangleq \sup_{\xi} \mathbf{E}_{\xi}\left[\sum_{k=1}^{\infty}\beta^{k-1}(q(b_{i,k} \mid g)\mu(s_{i,k-1}) - p(b_{i,k} \mid g)) \,\middle|\, s_{i,0} = s\right], \tag{5}$$

where the supremum is over all stationary Markov policies ξ for the preceding problem.

In developing the large market model, we assume the function $g(\cdot)$ has support on $[0,1]$; in other words, we restrict attention to large markets where an agent never bids above \$1. Furthermore, we assume the function $g(\cdot)$ is continuous on $[0,1]$. Thus, $g(\cdot)$ is a nondecreasing continuous function over $[0,1]$, with $g(0) \geq 0$ and $g(1) = 1$. Let $\mathcal{G}$ denote the set of all such functions. It is straightforward to check that both $q(\cdot \mid g)$ and $p(\cdot \mid g)$ are nondecreasing and continuous for each $g \in \mathcal{G}$.

3.2. Steady State Distribution of a Single Agent's State

We next study the transitions in the valuation and belief of a single agent in the mean field model with fixed bid distribution, assuming the agent employs a stationary strategy. While considering the transitions, we also take into account any regenerations that the agent might undergo. This allows us to characterize the steady-state distribution of the valuation and belief of the agent in the mean field model with a fixed bid distribution.

Fix a bid distribution $g \in \mathcal{G}$, and let $\xi(\cdot)$ denote a stationary policy for the decision problem specified by (5). In this scenario, the transitions in the agent i's belief $s_{i,t}$ and her valuation $v_{i,t}$ can be specified by the following transition probability kernel:

$$\begin{aligned} &\mathbf{P}((v_{i,t}, s_{i,t}) \in (A,B) \mid (v_{i,t-1}, s_{i,t-1}) = (v,s)) \\ &\quad = \beta v q(\xi(s) \mid g)\mathbf{I}\{v \in A, s + e_1 \in B\} \\ &\qquad + \beta(1-v)q(\xi(s) \mid g)\mathbf{I}\{v \in A, s + e_2 \in B\} \\ &\qquad + \beta(1 - q(\xi(s) \mid g))\mathbf{I}\{v \in A, s \in B\} \\ &\qquad + (1-\beta)\Psi(A,B), \end{aligned} \tag{6}$$

for all Borel sets $A \subseteq [0,1]$ and $B \subseteq \mathcal{S}$. This transition kernel defines a Markov chain over the space $[0,1] \times \mathcal{S}$. The first and second terms correspond to the event that an agent wins her auction and does not regenerate; the first term corresponds to the event that she receives a positive reward, whereas the second term corresponds to the event that she does not. The third term corresponds to the event that an agent loses her auction and does not regenerate. The last term corresponds to the event that an agent regenerates after the auction (note that in this case it is irrelevant whether an agent won or lost).

Because of the geometric regeneration of agents over time, we can show that the chain has a unique invariant distribution over $[0,1]\times\mathcal{S}$, which we denote by $\Phi(\cdot \mid g, \xi)$. See Lemma EC.2 in Appendix EC.2 for more details.

3.3. Mean Field Equilibrium

We are now ready to define mean field equilibrium. Suppose all agents initially conjecture the bid distribution to be g, and optimize to obtain their bidding strategy; this strategy induces dynamics in the overall bid distribution. Mean field equilibrium requires a consistency check: the steady-state bid distribution of these dynamics should itself be g.

Formally, fix $g \in \mathcal{G}$ as the bid distribution, and let each agent follow a stationary policy $\xi(\cdot)$. Let $\Phi = \Phi(\cdot \mid g, \xi)$ denote the steady-state distribution of an agent's valuation and belief. Since an agent with belief s makes a bid $\xi(s)$, we can write the cumulative distribution function of the agent's bids in steady state as $\Phi([0,1], \xi^{-1}[0,x])$ for $x \in [0,1]$. For the market to be consistent, we require that this cumulative distribution function is *exactly* $g(\cdot)$, i.e., for all $x \in [0,1]$,

$$g(x) = \Phi([0,1], \xi^{-1}[0,x]). \tag{7}$$

This consistency condition leads to the definition of mean field equilibrium of a large market.

Definition 1 (Mean Field Equilibrium). Let $g \in \mathcal{G}$ be a bid distribution, and let $\xi(\cdot)$ denote a stationary policy for an agent facing decision problem (5). We say the pair (g, ξ) constitutes a *mean field equilibrium* if

1. the policy $\xi(\cdot)$ is optimal for the decision problem (5), given bid distribution g; and
2. the consistency condition (7) holds, for $\Phi = \Phi(\cdot \mid g, \xi)$.

It is important to note that in a large market that is not in equilibrium, the bid distribution would also change as the market evolves. Thus, the transitions defined in (6) do not reflect the true market dynamics out of equilibrium, as it assumes a fixed bid distribution. However, in equilibrium, the bid distribution is fixed because of the consistency condition, and hence the process captures market dynamics in equilibrium.

4. Conjoint Valuations and Existence of MFE

In this section, we start our analysis of the mean field model by first focusing on a single agent's problem when the bid distribution g is fixed. We show that working with a mean field model makes the analysis tractable and allows us to characterize an agent's optimal bid. In §4.1, we derive a key result of our paper: given g and her current posterior s, an optimal strategy for an agent is to bid her conjoint valuation. As we show, this is her present expected valuation, together with the marginal future gain from one additional observation.

In §4.2, we use the characterization of an agent's optimal strategy to show that, under mild conditions on the distribution Ψ, an MFE exists.

4.1. Conjoint Valuation

Using standard dynamic programming arguments as in Maitra (1968), we can show the value function in (5) satisfies Bellman's equation:

$$\begin{aligned} V(s \mid g) = \sup_{b\in[0,1]} \big\{ & q(b \mid g)\mu(s) - p(b \mid g) \\ & + \beta q(b \mid g)\mu(s) V(s+e_1 \mid g) \\ & + \beta q(b \mid g)(1-\mu(s)) V(s+e_2 \mid g) \\ & + \beta(1-q(b \mid g)) V(s \mid g)\big\}. \end{aligned} \tag{8}$$

The right-hand side is derived as follows. In the discrete time optimization problem in (5), a single period corresponds to a single auction. In the preceding expression, the first two terms in the supremum correspond to the expected profit earned in the current auction with a bid b. Future profit is only earned if the bidder does not regenerate; hence the coefficient β on all remaining terms. In the third term, the bidder wins the auction and receives a positive reward, so the posterior updates to $s+e_1$. In the fourth term on the second line, the bidder wins the auction, but does not receive a positive reward, so the posterior updates to $s+e_2$. (Note that in these two terms, we take an expectation over the private valuation given the posterior s, so we obtain $\mu(s)$ as the probability of earning a reward given that the agent won.) In the final term, the bidder does not win the auction, so her posterior stays the same.

We now rearrange the right-hand side as follows. For any $g \in \mathcal{G}$, define $\xi(\cdot \mid g)\colon \mathcal{S} \to \mathbb{R}$ as

$$\begin{aligned} \xi(s \mid g) = {} & \mu(s) + \beta\mu(s) V(s+e_1|g) \\ & + \beta(1-\mu(s)) V(s+e_2 \mid g) - \beta V(s \mid g). \end{aligned} \tag{9}$$

Now observe that we can write (8) as

$$V(s|g) = \frac{1}{1-\beta} \sup_{b\in[0,1]} \{q(b|g)\xi(s|g) - p(b|g)\}. \tag{10}$$

Note that since an agent can always bid 0 and get zero payoff, we have $V(s \mid g) \ge 0$ for all $s \in \mathcal{S}$. This observation, together with (9), implies that $\xi(s \mid g) \ge 0$. To see this, suppose on the contrary that $\xi(s \mid g) < 0$ for some $s \in \mathcal{S}$. It follows from (10) that $V(s \mid g) = 0$. But then (9) implies that $\xi(s \mid g) \geq 0$, a contradiction. Hence, we have $\xi(s \mid g) \geq 0$ for all s.

This implies that the optimization problem (10) has the same form as the expected profit maximization problem of a bidder in a single second-price auction with valuation $\xi(s \mid g) \geq 0$, playing against $\alpha - 1$ bidders whose bids are independently and identically distributed (i.i.d.), drawn from the distribution g. Since bidding one's valuation is an optimal strategy regardless of competitors' bids in the static second-price auction, we conclude that bidding $\xi(s \mid g)$ at posterior s is optimal for the agent.

The above discussion yields the following proposition.

Proposition 1. *For any $g \in \mathcal{G}$, $\xi(\cdot \mid g)$ is nonnegative. Furthermore, the stationary strategy $\xi(\cdot \mid g)$, where an agent bids $\xi(s \mid g)$ if her belief is given by s, is optimal for the agent's decision problem* (5).

We refer to $\xi(s \mid g)$ as the conjoint valuation of an agent at posterior s, given g. Examining (9), it combines two elements: the first is her current posterior expectation of her private valuation, $\mu(s)$. The second is the *expected marginal future gain* from one additional observation regarding her valuation—i.e., the future value of learning. Thus, we get a remarkably simple structural description of an agent's optimal strategy: at each time period, she bids her conjoint valuation, optimally combining both exploitation and exploration.

4.2. Existence of MFE

We have now characterized the optimal behavior of an agent in a large market with a given bid distribution. To show the existence of an MFE, we need to show that there is a bid distribution that satisfies the consistency condition in the definition of an MFE. To that end, we make the following assumption:

Assumption 2. *The distribution Ψ has a compact support.*

We can now state our main result.

Theorem 1. *Under Assumption* 2, *the dynamic auction market has a mean field equilibrium (g, ξ) with $g \in \mathcal{G}$, and $\xi(\cdot) = \xi(\cdot \mid g)$.*

Proof Sketch. The proof is technical, and provided with complete details in the electronic companion. Here we present a brief sketch. Consider the function $\mathcal{F}(\cdot \mid g)$ denoting the steady state bid distribution induced by the agents' optimal strategies when each agent conjectures the bid distribution $g \in \mathcal{G}$ in solving their decision problem (8). Formally, we have $\mathcal{F}(x \mid g) \triangleq \mathbf{P}_\Phi(\xi(\tilde{s}) \leq x)$, for all $x \in [0, 1]$, where $\xi = \xi(\cdot \mid g)$ and $\Phi = \Phi(\cdot \mid g, \xi)$. Observe that if g is a fixed point of the map $\mathcal{F}$, i.e., if $\mathcal{F}(x \mid g) = g(x)$ for all $x \in [0, 1]$, then g is an MFE bid distribution. Thus, to show the existence of an MFE, we need to verify that the map $\mathcal{F}$ has a fixed point. We show this by using Schauder fixed point theorem. To apply the fixed point theorem, it suffices to show that (1) $\mathcal{F}$ as a map on $\mathcal{G}$ is continuous, and (2) its range $\mathcal{F}(\mathcal{G})$ is compact.

Showing the continuity of $\mathcal{F}$ involves showing the continuity of the conjoint valuation $\xi(\cdot \mid g)$ and the invariant distribution as maps on $\mathcal{G}$. This requires a careful analysis of the Bellman operator in (8) and the transition kernel (6). It is to show the latter compactness condition that we require Assumption 2. In particular, using Assumption 2, we bound the derivative of $\mathcal{F}(\cdot \mid g)$, uniformly for all $g \in \mathcal{G}$. Using the Arzelà-Ascoli theorem, we conclude that $\mathcal{F}(\mathcal{G})$ is compact. Taken together with the Schauder fixed point theorem, this proves the existence of an MFE. $\square$

The main restriction imposed by Assumption 2 is that there is a minimum level of uncertainty that a newly arriving agent has about their private valuation. In particular, agents do not arrive at the market with beliefs about their private valuation with variance that are arbitrarily low. Although this assumption is slightly restrictive, this can be relaxed: our proof works under a milder "light-tail" condition on the density of Ψ. In particular, as the belief parameters (s_1, s_2) increase, the density ψ should decrease at a rate faster than approximately $\beta^{s_1+s_2}$. The extension is straightforward but technically cumbersome, and we omit the details. Moreover, our model and results can be easily generalized to a case where certain fraction of the newly arriving agents know their private valuations exactly.

5. Approximation

We now show that an MFE is a good approximation to agent behavior in an equilibrium of a finite market. Formally, we show that if all other agents except agent i follow a strategy prescribed by a mean field equilibrium, then the maximum additional payoff agent i obtains on deviating from the mean field equilibrium strategy converges to zero as the number of agents in the market increases.

Let $(g(\cdot), \xi(\cdot))$ constitute an MFE for this market. Suppose all agents except agent 1 follow the mean field equilibrium strategy ξ. We study the decision problem for agent 1, and allow her to use any history-dependent strategy she likes. Let $\Phi = \Phi(\cdot \mid g, \xi)$. Let $V^{(n)}(s, \delta_1)$ denote the value function for agent 1 when her initial belief is $s_{1,0} = s \in \mathcal{S}$, she is following a (possibly history-dependent) policy δ_1 and all others are following policy ξ, and the initial valuations and beliefs of all other agents are drawn as $n-1$ i.i.d. samples from Φ.

We have the following theorem establishing the desired approximation property. We use ideas from the theory of propagation of chaos for our result. In particular, we analyze the market as a stochastic interacting particle system, with the agents as particles, and an auction as an interaction. "Propagation of chaos" refers to asymptotic independence of any finite subcollection of particles as the system becomes large; once this property holds, then effectively the assumptions under which the MFE strategy is optimal hold approximately.

Theorem 2. *Let (g, ξ) be an MFE. In a market with n agents, suppose all agents other than agent 1 follow the MFE strategy ξ. Moreover, suppose the distribution of the initial valuation and belief of each agent is given by Φ, independent across the agents. Then, for any sequence of strategies $\{\delta_1^{(n)}: n \geq 1\}$ for agent 1 and for all initial beliefs $s \in \mathcal{S}$ that agent 1 holds about her valuation, we have*

$$\limsup_{n\to\infty} (V^{(n)}(s, \delta_1^{(n)}) - V^{(n)}(s, \xi)) \leq 0.$$

Furthermore, for $\alpha \geq 4$, large enough $\beta < 1$, and all large enough n, we have

$$V^{(n)}(s, \delta_1^{(n)}) - V^{(n)}(s, \xi) \leq \frac{10}{1-\beta} \cdot \left[\frac{4\alpha(\alpha-1)^4}{(1-\beta)(\alpha-2)^2(\alpha-3)\ln(1/\beta)^2(n-1)}\right]^{\frac{\ln(1/\beta)}{8(\alpha-2)+(1+\omega)\ln(1/\beta)}},$$

where $\omega > 0$ is an arbitrary positive constant.

Note that a propagation of chaos approach inherently does not employ any features of the *payoffs* of the agents. It is worth noting that this proof technique is reflected in the convergence rate: the bounds worsen as the number of agents interacting in any given auction increases. We expect that the convergence rates can be improved substantially if the payoff information is exploited as well.

Before proceeding with the proof, we describe a construction called *interaction sets* (inspired by Graham and Méléard 1994) that will aid in the proof. At any time t, the interaction set $A_{i,t}^{(n)}$ denotes the set of agents that have influenced the state and belief of agent i, either directly through participating in an auction with agent i, or indirectly by participating in an auction with some agent who later influenced agent i. Formally, we define the interaction sets inductively as follows.

For each agent i, we define $A_{i,0}^{(n)} = \{(0, i)\}$. This reflects the fact that at time 0, no agent has interacted with any other agent in the market, and hence any two agents in the market are independent. As long as there is no auction occurring in the market, the interaction sets do not change. Let an auction occur at time t, involving agents $i_1, \ldots, i_\alpha$. Then, the interaction sets associated with these agents change as

$$A_{i,t}^{(n)} = A_{i_1,t^-}^{(n)} \cup A_{i_2,t^-}^{(n)} \cdots \cup A_{i_\alpha,t^-}^{(n)} \cup \{(t, i_1, \ldots, i_\alpha)\},$$

for all $i \in \{i_1, \ldots, i_\alpha\}$. The interaction sets associated with agents not participating in the auction do not change. Here $A_{i,t^-}^{(n)}$ denotes the interaction set associated with agent i just prior to the auction. Thus, once the auction occurs, the interaction set of each participating agent expands to include the interaction sets of every other participant. Furthermore, the interaction sets include the fact that these agents participated in the auction at time t, captured by the last set on the right-hand side.

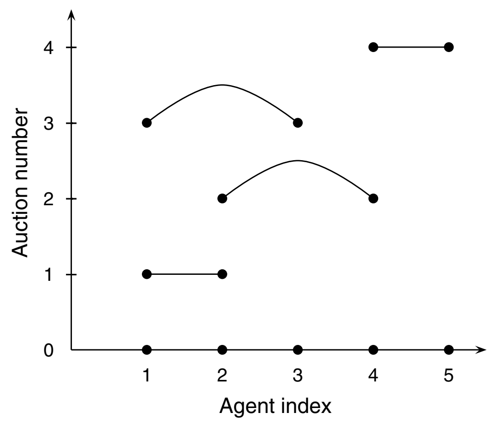
As mentioned previously, interaction sets capture the influence relationships among the agents. In particular, any two agents i and j with disjoint interaction sets have had no influence upon each other and their states and beliefs are independent. On the other hand, if the interaction sets $A_{i,t}^{(n)}$ and $A_{j,t}^{(n)}$ have nonempty intersection, then the two agents' states and beliefs are dependent at time t. Our proof technique relies on the fact that after any finite time, as the number of agents approaches infinity, the interaction sets of any finite collection of agents become disjoint with high probability, and hence the states of these agents become independent.

We briefly note that naively, one might expect that the "interaction set" of two agents could simply be defined as the transitive closure of the relationship between two agents defined by participation in the same auction; according to this definition, i and k would be in the same set at time t if there is some chain of agents $i = i_0, \ldots, i_\ell = k$ such that each successive pair of agents in the chain participated in some auction with each other prior to time t. Unfortunately, with this definition, it is straightforward to construct examples such that after a finite time, a *constant fraction* of the agents are in the same interaction set—precluding the asymptotic independence we require in the proof. Interaction sets avoid this complication because they also encode a *temporal* element of interaction: an agent only "interacts" with others whose *past* behavior (directly or indirectly) has influenced her *current* state.

To illustrate our definition of interaction sets, we consider in Figure 1 the first few auctions of a market with $n = 5$ agents where the auction thickness is $\alpha = 2$. At time $t = 0$, the interaction set for each agent i is given by $\{(0, i)\}$. In auction 1, agents 1 and 2 interact, and their interaction sets become $A_{1,1} = A_{2,1} = \{(0,1), (0,2), (1,1,2)\}$. (We suppress the superscript on $A_{i,t}^{(n)}$ for the example.) The interaction sets of other agents remain unchanged. Similarly, at auction 4, agents 4 and 5 interact and their interaction sets become $A_{4,4} = A_{5,4} = \{(0,1), (0,2), (0,4), (0,5), (1,1,2), (2,2,4), (4,4,5)\}$. The interaction sets can be visualized pictorially as in Figure 2.

Proof Sketch of Theorem 2. We outline here the main steps of the proof. See the appendix for the complete proof.

Figure 1 First Four Auctions Among Five Agents



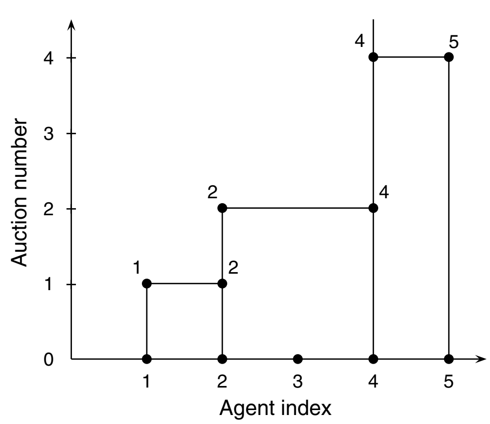
First, note that as the geometric lifetimes of the agents induce a discounting over their payoffs, it is sufficient to analyze the first k auctions for a sufficiently large k. The proof then proceeds in two main steps:

1. The first step involves showing that, with high probability as n increases, the agents faced by agent 1 in the first k auctions have had no interactions among themselves or with agent 1 prior to the auction. This correlation decay result follows by showing that their interaction sets are pairwise disjoint with high probability using techniques from Graham and Méléard (1994).

2. The second step involves showing that as n increases, the distribution of the valuation and belief of an agent in the market converges in total variation to the distribution Φ. This follows by showing that the interaction set of an agent is, with high probability, similar to that of an agent in the large market. Because the agent in the large market has her valuation and belief distributed as Φ, the result follows.

The two steps together show that with high probability as n increases, the agents faced by agent 1 in the first k auctions are all independent and have valuation and belief distributed as Φ. This is the same scenario that occurs in the large market, and hence, with high probability, the decision problem faced by agent 1 for the first k auctions in the finite market is the same as that faced by an agent in the large market. The result then follows from the optimality of the strategy $\xi(\cdot \mid g)$ for the large market. $\square$

Figure 2 Interaction Set of Agent 4 After Auction 4



The preceding result establishes that, asymptotically, no unilateral deviation from the MFE strategy is profitable at time 0. However, this does not preclude a situation where, asymptotically, an agent prefers to deviate after some finite history h_t at time $t > 0$. In other words, the preceding result does not guarantee the asymptotic *perfectness* of the mean field equilibrium as the number of agents increases, i.e., whether an agent would continue to follow the mean field strategy after any history in the market, if all her competitors continue to do so. We address this concern in Theorem EC.1 in Appendix EC.4 in the electronic companion. Formally, we show that, with high probability as n increases, after any history where all agents had been following the mean field strategy, no agent can gain significantly higher payoffs by subsequently deviating unilaterally to any other (possibly history-dependent) strategy. In this sense, the mean field equilibrium is approximately *perfect*.

6. Dynamic Revenue Equivalence of Standard Auctions

Until now, we considered a market model involving a sequence of second-price auctions. However, the choice of repeated auction format is a key decision variable for the auctioneer. How does this choice influence the auctioneer's revenue?

In this section we answer this question by analyzing a model variation where the auction format employed is a general *standard auction*; these are auctions where (1) a bidder with the highest bid gets the auctioned good, and (2) a bidder bidding zero makes no payment. In addition to the second-price auction, other common examples include the first-price auction (where the winner pays her bid) and the all-pay auction (where all bidders pay their bid, regardless of whether they won). Examples of nonstandard auctions include the second-price auction with (positive) reserve price, a lottery among the two highest bids, etc. We show the revenue equivalence of standard auctions in dynamic mean field settings: the auctioneer's choice of the auction format in a large market, so long as it is standard, does not influence her expected revenue from the agents.[3] We observe here that we obtain this result in

[3] We emphasize here that, just as in the static setting, the auctioneer can influence her revenue by adopting a nonstandard auction format. We investigate this "revenue maximization" problem for the auctioneer in the context of the optimal choice of reserve in second-price auctions in §7.5.

spite of the possible multiplicity of MFE; we prove the equivalence of the various standard auction formats for each individual MFE. Note that this does not preclude a scenario where different MFE of a repeated standard auction market yield different expected revenue.

Before we analyze the dynamic model with learning, we briefly consider a static auction A involving α agents. To begin, suppose the bids of all agents except an agent i are drawn i.i.d. from a cumulative distribution g_A. Because the agent winning the auction has the highest bid, the bid distribution g_A completely determines the probability that agent i wins the auction with a bid b; letting $q_A(b \mid g_A)$ be this probability, we have

$$q_A(b \mid g_A) = g_A(b)^{\alpha-1}, \quad \text{for all } b \geq 0.$$

Similarly, the rules of the auction A determine the *payment function* $p_A(b \mid g_A)$, which specifies the expected payment agent i has to make in the auction as a function of her bid b.

Now suppose the agents have their private values drawn i.i.d. from a distribution with cumulative distribution function $h \in \mathcal{G}$. Assume the static auction A has a symmetric Bayesian Nash equilibrium with a strictly increasing strategy $\eta_A(\cdot \mid h)$. The distribution h and the equilibrium $\eta_A(\cdot \mid h)$ together determine the equilibrium bid distribution as $g_A(\cdot) = h(\eta_A^{-1}(\cdot \mid h))$. We present without proof the following standard result, the *payoff equivalence theorem* (Milgrom 2004), that relates the functions $q_A(\cdot \mid g_A)$ and $p_A(\cdot \mid g_A)$ with the functions $q(\cdot \mid h)$ and $p(\cdot \mid h)$—the probability of winning and expected payment in the second-price auction (see (3) and (4)).

Lemma 1 (Static Payoff Equivalence of Standard Auctions). *Suppose each agent has her private value drawn independently from a distribution $h \in \mathcal{G}$, and the resulting static auction A has a symmetric Bayesian Nash equilibrium with strictly increasing strategy $\eta_A(\cdot \mid h)$. Then, for $g_A(\cdot) = h(\eta_A^{-1}(\cdot \mid h))$, the functions $q_A(\cdot \mid g_A)$ and $p_A(\cdot \mid g_A)$ satisfy $q_A(\eta_A(x \mid h) \mid g_A) = q(x \mid h)$ and $p_A(\eta_A(x \mid h) \mid g_A) = p(x \mid h)$ for all $x \in [0, 1]$.*

A key corollary of the payoff equivalence theorem is the *revenue equivalence of standard auctions* (Milgrom 2004), which that an auctioneer receives the same expected revenue in the symmetric Bayesian Nash equilibrium of any static standard auction. As we now show, this result naturally extends to dynamic auctions with learning, if we work with mean field equilibria. For convenience, we denote the dynamic model where a standard auction format A is repeated over time as M_A, and the corresponding large market as LM_A. The model considered in the previous sections will be denoted as M_{SP}, and the corresponding large market as LM_{SP} (where the subscript SP denotes "second price").

The first main result of this section relates, under some conditions, an MFE of the large market LM_{SP} to an MFE of the large market LM_A.

Theorem 3. *Let (g, ξ) denote an MFE of the market LM_{SP}, where g denotes the bid distribution and ξ denotes the strategy of the agents. Suppose the static auction A with α agents, where the agents' valuations are drawn i.i.d. from g, has a symmetric Bayesian Nash equilibrium with strictly increasing equilibrium strategy denoted by $\eta_A(\cdot \mid g)$. Then the market LM_A has an MFE (g_A, ξ_A), where*

1. *the bid distribution g_A satisfies $g_A(\cdot) = g(\eta_A^{-1}(\cdot \mid g))$, and*
2. *the strategy ξ_A satisfies $\xi_A(\cdot) = \eta_A(\xi(\cdot) \mid g)$.*

Proof Sketch. The proof is in the electronic companion in Appendix EC.5.1; here we briefly sketch the argument. The proof follows in two steps, both relying heavily on the static payoff equivalence in Lemma 1. First, we consider the decision problem faced by an agent in the market LM_A with bid distribution g_A (as defined in the theorem statement), and show that it is exactly the same as that faced by an agent in the market LM_{SP} with bid distribution g; see Lemma EC.18. We achieve this by constructing for any policy Π for the agent's decision problem in the market LM_{SP}, a policy Π_η for the agent's decision problem in the market LM_A that obtains the same expected payoff (and vice versa).

Second, we complete the argument by showing that the market LM_A, where each agent employs ξ_A and the market LM_{SP}, where each agent employs ξ, both have the same invariant distribution. This is achieved by showing, using Lemma 1, that the transition probability kernel for the state transitions of a single agent in both the markets are the same. □

The preceding result establishes that the maximum revenue an auctioneer can obtain from an MFE of a repeated standard auction market is at least as large as that in an MFE of a repeated second-price auction. We now consider the converse problem of relating an MFE of the market LM_A to an MFE of the market LM_{SP}. We study the converse under a technical regularity assumption, stated next.

Definition 2. A standard auction format A is *regular* if, for any continuous bid distribution g_A in the market, the expected payment function $p_A(b \mid g_A)$ is continuous, nondecreasing, and nonnegative for $b \geq 0$.

Note that the preceding definition is not very restrictive, since the payment function is an expectation over other players; in particular, the first-price, second-price, and all-pay auctions are all regular.

The second main theorem of the section establishes a partial converse to Theorem 3.

Theorem 4. *Let A denote a regular standard auction format, and let (g_A, ξ_A) be an MFE for the market LM_A, where $g_A(\cdot)$ is a continuous bid distribution, and ξ_A is an optimal strategy for the agents when the bid distribution is g_A. Then, there exists an MFE (g, ξ) with $\xi = \xi(\cdot \mid g)$ for the market LM_{SP} such that the auctioneer's revenue in the MFE (g, ξ) is the same as that in the MFE (g_A, ξ_A).*

We provide the proof in the electronic companion in Appendix EC.5.2. The proof of the preceding theorem is significantly more complicated than the proof of Theorem 3. To see the complexity, consider the relationships (1) and (2) identified in Theorem 3. Note that if we are *given* an MFE of the repeated second-price auction market LM_{SP}, then passing the MFE through these two relationships yields an MFE of the repeated auction market LM_A.[4] However, things are not so straightforward if we are given an MFE of LM_A, identified by g_A and ξ_A. The issue is that we must simultaneously find a symmetric BNE η_A of the static auction A, and a bid distribution g, such that (1)–(2) in Theorem 3 hold. This amounts to a complex fixed point condition, and thus is not amenable to analysis. A key issue is that in the market LM_A, we have no guarantee that the optimal dynamic strategy in any MFE is obtained through simple application of a static BNE of the auction A. Indeed, this is why it should be surprising that a dynamic revenue equivalence result is possible for these markets at all.

Instead, we prove the theorem by explicitly constructing an MFE for the market LM_{SP} using the structural characterization from Theorem 1. Given an MFE (g_A, ξ_A) of the market LM_A, the main idea of the construction is that we can analogously define a conjoint valuation $C_A(\cdot)$ for the MFE (g_A, ξ_A) based on the corresponding value function V_A, and let g denote the distribution of $C_A(s)$ where the belief s is distributed according to the steady-state distribution Φ_A. It turns out that $C_A(s)$ and g together constitute an MFE of LM_{SP}.

As a consequence of the preceding two theorems, we obtain the revenue equivalence of standard auctions in dynamic mean field settings.

Corollary 1 (Dynamic Revenue Equivalence of Standard Auctions). *An auctioneer's maximum revenue per auction over all MFE of the market LM_A is the same for all regular standard auctions A.*

This preceding result shows that even in the presence of learning among the agents, the auctioneer's choice of an auction format does not affect the revenue obtained in a large market, so long as the auction format is regular and standard. Thus, the auctioneer is forced to pursue other methods, such as increasing the auction thickness or imposing a reserve price, to increase the revenue. In the next section, we use a computational approach to investigate the effect of such decisions on the auctioneer's revenue.

[4] This is similar to a standard argument in static auction theory for direct revelation mechanisms, using the revelation principle.

7. Computational Insights: Comparative Statics and Reserve Prices

In this section, we use a computational approach to gain additional insight into the structure of MFE in repeated second-price auction markets. We first describe an algorithm to estimate a mean field equilibrium in the beta belief model. Although proving the convergence of the algorithm is an open problem, we find computationally that it converges within tolerance in a reasonable number of iterations.

Next, we use our algorithm to study MFE, with a particular eye to the effect of market parameters on market performance. We first carry out comparative statics, with respect to the market thickness α and survival probability β. We then turn our attention to a model with reserve prices. Since the dynamic revenue equivalence result suggests changing the auction format will not increase revenue, we investigate instead whether setting a reserve price would help the auctioneer. We use a computational approach to study the opportunity cost (in terms of lost revenue) to the auctioneer if she ignores the fact that bidders are learning over time; our investigation suggests this cost can be potentially very high.

7.1. Parameters

Recall that in the beta belief model, the set $\mathcal{S}$ is given by $\{(s_1, s_2) \in \mathbb{R}^2 \colon s_1 \geq 1, s_2 \geq 1\}$, with each $s = (s_1, s_2) \in \mathcal{S}$ corresponding to the belief given by the beta distribution with parameters (s_1, s_2). A newly arriving agent in the market has her valuation and belief distributed according to the distribution Ψ over $[0, 1] \times \mathcal{S}$. In our computations, we choose Ψ with the following density:

$$\psi(v, s_1, s_2) = \begin{cases} 0 & \text{if } s_1 > 2 \text{ or } s_2 > 2, \\ \dfrac{v^{s_1-1}(1-v)^{s_2-1}}{B(s_1, s_2)} & \text{otherwise.} \end{cases}$$

We can interpret the distribution Ψ as follows. We first choose the belief of a newly arriving agent by selecting s uniformly at random from the set $[1, 2] \times [1, 2]$. Once the belief of the agent is chosen, we choose her valuation v according to the beta distribution with parameters (s_1, s_2).

We choose the survival probability β to lie in the range $[0.9, 0.99]$. This implies that the expected number of auctions that an agent participates in before her valuations and belief reset lies in the range from 10 to 100. Finally, we choose the auction thickness α, denoting the number of agents participating in any single auction, to lie in the set $\{5, 10, 15\}$.

7.2. Algorithm for Computing MFE

The computational model assumes a large market with a continuum of agents. We use the following recursion to compute an MFE approximately.

1. Initialize g_0 as the uniform distribution, $\xi_0(\cdot) = \mu(\cdot)$, and $\phi_0 = \psi$. Set $n = 1$.
2. Update the density of the market distribution ϕ_n using (6):

$$\begin{aligned}\phi_n(v,s) = {}&(1-\beta)\psi(v,s)\\ &+\beta q(\xi_{n-1}(s-e_1)\,|\,g_{n-1})\mu(s-e_1)\phi_{n-1}(v,s-e_1)\\ &+\beta q(\xi_{n-1}(s-e_2)\,|\,g_{n-1})\\ &\cdot(1-\mu(s-e_2))\phi_{n-1}(v,s-e_2)\\ &+\beta(1-q(\xi_{n-1}(s)\,|\,g_{n-1}))\phi_{n-1}(v,s).\end{aligned}$$

3. Compute the new bid distribution: $g_n(x) = \mathbf{P}_{\phi_n}(\xi_{n-1}(Y) \le x)$.
4. Update the strategy as $\xi_n(\cdot) = \xi(\cdot \,|\, g_n)$.
5. If $\|g_n - g_{n-1}\|_\infty > \epsilon$, set $n = n+1$ and repeat from step 2. Else stop.

One way to interpret the preceding recursion is as the evolution of a large market where each agent updates her strategy myopically. We start with a large market, with each agent's state and belief independently distributed as ϕ_0, and each agent using the policy ξ_0. We suppose that every time an agent participates in an auction, she conjectures the current bid distribution g_n *will remain constant indefinitely*; she optimizes, and bids *once* according to the resulting optimal strategy. The bid distribution then changes (if the market is not at equilibrium), and the process repeats. We continue this recursion until two consecutive bid distributions remain sufficiently close, in the sense that the sup norm of their difference stays within some small threshold. It is straightforward to check that if the preceding recursive procedure converges, then the bid distribution and the policy together constitute a mean field equilibrium of the market. This is a form of myopic learning dynamic executed by the bidders over time, in response to the current "bid landscape" g_n presented by the mechanism. (See Adlakha and Johari 2013 for a similar learning dynamic in a different stochastic game setting.)

7.3. Details of the Computation

We performed the computations on an 8-core Opteron server with 31.49 GB RAM. Because the distributions involved are on the continuous real space, we discretized the relevant spaces to perform the computations. In particular, we restricted the bid distribution to a discrete distribution over $\{0.0125k\colon 0 \le k \le 80\}$. Similarly, we discretized the space in which private valuations lie to $\{0.04k\colon 0 \le k \le 25\}$. Finally, we discretized and truncated the space of beliefs to $\mathcal{S} = \{0.04 \cdot k \cdot l\colon 0 \le k \le 25,\ 1 \le l \le 80\}$.

To determine the stopping condition, we chose the value of ϵ to be 0.0025. We found that, typically, the recursive procedure converged in fewer than 50 iterations.

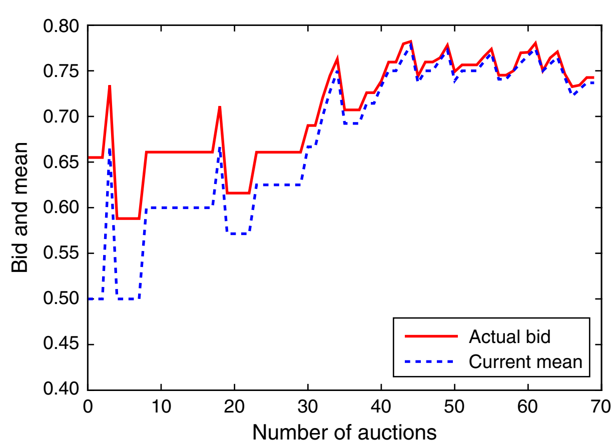
7.4. Results and Comparative Statics

In this section we describe some basic observations from our numerical analysis, as well as insight into how the market varies with changes in parameters.

7.4.1. Evolution of Overbid. Figure 3 depicts the evolution of an agent's optimal bid $\xi(\cdot)$ and her expected reward $\mu(\cdot)$ in the market over time. From the figure, we notice that the agent always overbids in the equilibrium, and the amount of overbid decreases over time, as the agent obtains more refined information about her valuation. However, from Figure 4, we see that this decrease need not be monotone over time. To see why, note that if the agent's expected reward were fixed, then as she obtains better information, her overbid should decrease monotonically; however, in reality, the agent's expected reward also changes as new information arrives, leading to nonmonotonicity of the overbid.

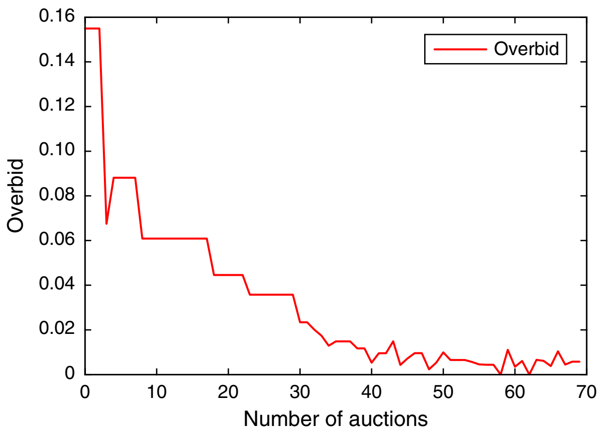
7.4.2. Effect of Auction Thickness on Bid Distribution. Next, we compute the bid distribution g in a mean field equilibrium for different values of the auction thickness α. As the number of agents in an auction increases, two countervailing effects can arise. First, since the competition in each auction has increased, each agent would have to bid more aggressively to win in the auctions. This has the effect of moving the bid distribution to the right. On the other hand, if all other agents bid high, then an agent on winning an auction has to make higher payments. This reduces the marginal benefit of learning her own valuation, thereby reducing an agent's conjoint valuation. Since the agent bids her conjoint valuation, this causes the bid distribution to move to the left. In equilibrium, we

Figure 3 (Color online) Evolution of the Optimal Bid of an Agent



Note. Here $\alpha = 5$ and $\beta = 0.95$.

Figure 4 (Color online) Evolution of the Overbid



Note. Here $\alpha = 5$ and $\beta = 0.95$.

do not know a priori which of these two effects, if any, would dominate.

From Figure 5, we observe that the bid distribution becomes more concentrated in the middle as auction thickness increases. One possible explanation for this is that in the market with larger auction thickness, an agent with a mediocre conjoint valuation typically wins fewer auctions, as there is a higher chance of facing an agent with a higher conjoint valuation. Because of this, her beliefs about her valuation do not change often. This explains the concentration in the middle of the bid distribution as auction thickness increases. In contrast, in a market with smaller α, these agents have a higher chance of victory in an auction, causing their beliefs to resolve into either a high or a low valuation, resulting in a spread in the bid distribution.

7.4.3. Effect of Survival Probability on Bid Distribution. Finally, we discuss the effect of varying β while keeping the auction thickness fixed. We observe from Figure 6 that as β increases, the equilibrium bid

Figure 5 (Color online) Equilibrium Bid Distribution for Different α, for $\beta = 0.95$

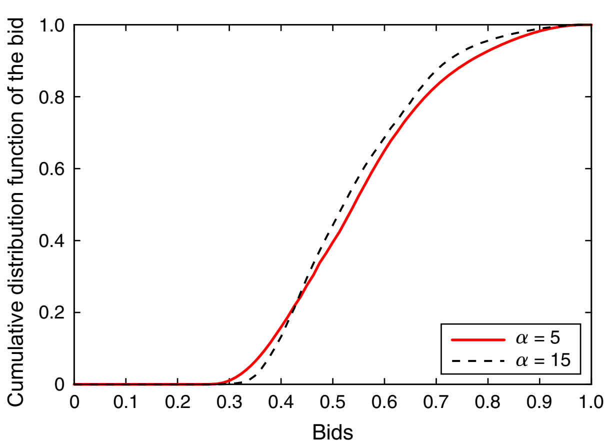
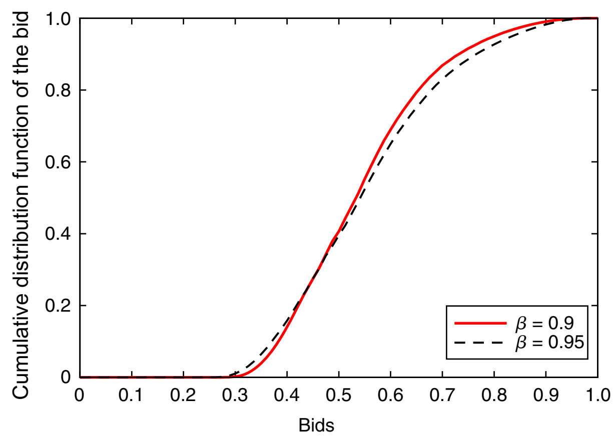


Figure 6 (Color online) Equilibrium Bid Distribution for Different β, for $\alpha = 5$



distribution spreads more toward the extreme values. This is mainly because as β increases, the lifespan of an agent in the market increases, giving her more opportunities to learn her valuation.

7.5. Reserve Price

We now investigate the auctioneer's problem of choosing a revenue-maximizing reserve price. In static auctions, setting a reserve price leads to two effects: (1) it relinquishes revenue from those bidders with low valuation, by excluding them from winning; and (2) it extracts more revenue from those bidders with high valuation. A high reserve excludes too many bidders from participation, and a low reserve does not extract enough revenue from the high-valued bidders. Thus, an auctioneer in a static auction has to balance between these two effects while choosing an optimal reserve.

In the presence of learning, setting a reserve price leads to an additional effect: it imposes a *learning cost*. This learning cost is realized in two ways. First, by direct exclusion, a reserve precludes those agents with low expected valuations from learning. Second, and more importantly, a reserve reduces the incentives of agents with high expectations to learn more about their private valuation. As we show below, ignoring this learning cost can lead to potentially high lost revenue for the auctioneer.

We study a behavioral model for the agents where for each reserve r, they behave as in an MFE corresponding to that reserve, denoted (g_r, ξ_r). This interaction between the auctioneer and the agents can be analyzed in two ways, as follows.

1. *The Nash Approach.* In this approach, the auctioneer ignores the effect of the reserve on the agents' strategies. In particular, the auctioneer does not anticipate that changing the reserve would also change agent behavior in the market.

This interaction between the auctioneer and the agents leads to a game, where the agents play according to (g_r, ξ_r) given the reserve r, and the auctioneer assumes that the MFE bid distribution $g = g_r$ is fixed and sets a reserve r that maximizes her revenue $\Pi(r, g)$. We denote by r_{NASH} a Nash equilibrium of this game. Note that at r_{NASH}, by definition, assuming the corresponding MFE bid distribution is fixed, the auctioneer does not desire to change the reserve.

2. *The Stackelberg Approach.* In the second approach, the auctioneer anticipates that changing the reserve would change the agents' behavior to the MFE corresponding to that reserve. Taking this dynamic into consideration, the auctioneer chooses a reserve that maximizes her revenue $\Pi(r, g_r)$ over all r; by an abuse of notation we write $\Pi(r) \triangleq \Pi(r, g_r)$. We denote by r_{OPT} the maximizer of this problem. This choice of r corresponds to a Stackelberg equilibrium of the game described earlier between the auctioneer and the agents.

Note that if the agents know their valuations exactly (there is no learning), then both approaches yield the same optimal reserve for the auctioneer. When there is learning, however, the Stackelberg approach yields, in general, a strictly higher revenue. The Stackelberg approach may thus be interpreted as a model where the auctioneer considers the learning cost in her reserve price decision, whereas in the Nash approach, she ignores the learning cost.

We compare the two approaches through numerical computations using the algorithm described earlier for different values of the distribution Ψ. In particular, we consider the following specific form for the density ψ:

$$\psi_L(v, s_1, s_2) = \begin{cases} \dfrac{2v^{s_1-1}(1-v)^{s_2-1}}{(L-1)^2 B(s_1, s_2)} & (s_1, s_2) \in [1, L]^2,\ s_1 + s_2 \le L+1, \\ 0 & \text{otherwise,} \end{cases}$$

for different values of L. In other words, for a fixed value of L, we assume that a newly arriving agent has a beta prior with parameters that are distributed uniformly on a triangle with vertices $(1,1)$, $(1,L)$, and $(L,1)$. The parameter L controls the level of uncertainty in the agent population regarding their private valuation. More precisely, as L increases, a typical newly arrived agent has more certainty about her valuation.

Before we state the results of our computation, we briefly describe the metric we use to compare the Nash and Stackelberg approaches. If the agents' valuations were common (public) knowledge, then, in each of the auctions, the auctioneer can extract the total surplus by allocating the good to the agent with the highest private valuation at a price equal to her private valuation. Thus, under full information, the maximum revenue $\Pi_{\max}$ an auctioneer can extract is given by the total surplus according to the underlying valuation distribution. Note that this underlying distribution is given by $F \triangleq \Psi(\cdot, \mathcal{S})$, the marginal of Ψ over the private valuation. We thus have

$$\Pi_{\max} \triangleq \mathbf{E}_\Psi\left[\max_{1\le i\le \alpha} v_i\right] = \int_0^1 x\, dF^\alpha(x),$$

where $v_1, \ldots, v_\alpha$, distributed independently and identically according to F, denote the private valuations of the agents participating in any given auction.

With this benchmark in place, we use the following metric to evaluate any reserve price r:

$$m(r) \triangleq \frac{\Pi(r) - \Pi(0)}{\Pi_{\max} - \Pi(0)}.$$

The metric $m(r)$ captures the increase $\Pi(r) - \Pi(0)$ in the auctioneer's revenue on setting a reserve r as a fraction of the maximum increase possible under full information. The main motivation behind using the metric $m(\cdot)$ is that it is invariant to affine transformations in the distribution of the private valuations, i.e., $m(\cdot)$ does not change if private valuations are scaled or shifted by a common constant. This allows us to focus only on the relative performance of r_{NASH} and r_{OPT}.

The results of our computations are summarized in Table 1. We observe that, depending on the parameter values, $m(r_{\mathrm{OPT}})$ varies from 12% to 20%, i.e., setting the reserve price optimally increases the auctioneer's revenue by about 12%–20% of the maximum increase possible under full information. Furthermore, we see that in comparison to setting the reserve price as r_{NASH}, setting the reserve price optimally obtains 3%–4% more of the maximum increase. Thus, from the results, we make two observations. First, the auctioneer can extract a large fraction of the maximum possible increase in revenue by setting a constant reserve price. Second, and more importantly, given that an auctioneer has decided to impose a reserve in the auction, the benefit of setting the reserve optimally by including the learning cost can be quite significant.

Table 1 Comparison of the Two Approaches to Setting a Reserve Price

L	r_{NASH}	r_{OPT}	$\Pi(0)$	$\Pi_{\max}$	$\Pi(r_{\mathrm{NASH}})$	$\Pi(r_{\mathrm{OPT}})$	$m(r_{\mathrm{NASH}})$ (%)	$m(r_{\mathrm{OPT}})$ (%)
3	0.3375	0.4	0.4025	0.6151	0.4212	0.4296	8.80	12.75
5	0.3375	0.4125	0.3931	0.6353	0.4134	0.4202	8.38	11.19
10	0.3625	0.45	0.3737	0.6498	0.4086	0.4158	12.64	15.25
15	0.375	0.4625	0.3634	0.6561	0.4082	0.4155	15.30	17.80
20	0.3875	0.4625	0.3581	0.6600	0.4090	0.4177	16.86	19.74

Note. All computations have been performed for $\alpha = 2$, $\beta = 0.95$, and $\epsilon = 0.002$.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2014.2018.

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Appendix. Approximation: Proof of Theorem 2

In this section, we provide the proof of Theorem 2. We follow the same notation as in §5.

Proof of Theorem 2. Because the geometric lifetimes of the agents induces a discounting over their payoffs, it is sufficient to analyze the first k auctions for a sufficiently large k. Given $\epsilon > 0$, choose $\kappa = \ln(\epsilon)/\ln(\beta) + 1$. This implies that $\beta^T < \epsilon$ for all $T \geq \kappa$.

Next, note that since auctions are held according to a Poisson process with rate n/α, and at each time α agents are chosen uniformly at random to participate in the auction, agent 1 participates in the auctions at jump times of a Poisson process with rate 1. Choose $\tau > 0$ such that $P(X_\tau \geq \kappa) > 1 - \epsilon$, where X_τ is a Poisson process with rate 1. This implies that at time τ, with high probability, agent 1 must have participated in at least κ auctions. Let $E^{(n)}$ denote the event that agent 1 participated in at least κ auctions until time τ. Denote the times of the first κ of these auctions by τ_σ for $\sigma = 1, \ldots, \kappa$. Note that by choice of τ, we have $\mathbf{P}(E^{(n)}) > 1 - \epsilon$.

Let $D^{(n)} \subseteq E^{(n)}$ denote the event that $S_{1,\tau_\rho} \cap S_{1,\tau_\sigma} = \varnothing$ for all distinct $\rho, \sigma \leq \kappa$. Thus, on the event $D^{(n)}$, agent 1 never faces any other agent more than once in the first κ auctions. Because the agents are chosen uniformly at random (and independently) from the population to participate in the auctions, it is straightforward to show that for large n, the event $D^{(n)}$ occurs with very high probability.

Next, let $C^{(n)} \subseteq D^{(n)}$ denote the event that for all $\sigma = 1, \ldots, \kappa$ and for each distinct pair j, k with $j \in S_{1,\tau_\sigma}$ and $k \in (\bigcup_{\rho \leq \sigma} S_{1,\tau_\rho}) \cup \{1\}$, the interaction sets $A^{(n)}_{j,\tau_\sigma^-}$ and $A^{(n)}_{k,\tau_\sigma^-}$ are disjoint. Thus, on event $C^{(n)}$, the agents faced by agent 1 in each of the first κ auctions were neither influenced by nor had an influence on agent 1 (either directly or indirectly) until they met agent 1 in an auction. Moreover, these agents have had no prior influence on each other. Lemma 5 proves that under the event $C^{(n)}$, the distribution of the valuation and belief at time τ_σ of each agent in S_{1,τ_σ} converges in total variation, as n increases, to the product distribution $\Phi^{\otimes(\alpha-1)}$ uniformly for all $\sigma = 1, \ldots, \kappa$. This implies for large enough n, there is an event $B^{(n)} \subseteq C^{(n)}$ with $\mathbf{P}((B^{(n)})^c \mid C^{(n)})$ equal to half the total variation distance, such that on the event $B^{(n)}$, the conditional distribution of the valuation and belief of each agent faced by agent 1 in the first κ auctions, given the history agent 1 has seen so far, is given by Φ independently across these agents. Because these agents use the strategy ξ, this implies that on the event $B^{(n)}$, the bid distribution seen by agent 1 at each time τ_σ for $\sigma = 1, \ldots, \kappa$ is given by g. Thus, on the event $B^{(n)}$, for the first κ auctions, agent 1 faces the same decision problem as an agent in the large market.

This implies that agent 1 must do quite well on the event $B^{(n)}$ by following the strategy $\xi(\cdot)$. In particular, as the reward obtained by an agent and the payment she makes in an auction are both bounded above by 1, this implies that on event $B^{(n)}$, agent 1's total expected payoff in the first κ auctions on following any strategy is bounded above by $V(s \mid g) + \beta^\kappa/(1-\beta) \leq V(s \mid g) + \epsilon/(1-\beta)$. Subsequent to the first κ auctions, the difference in the total expected payoff obtained by agent 1 on choosing any other strategy again has to be bounded above by $\beta^\kappa/(1-\beta) \leq \epsilon/(1-\beta)$. Thus, on event $B^{(n)}$, the maximum total expected payoff that agent 1 can achieve by choosing any (possibly history-dependent) strategy is bounded above by $V(s \mid g) + 2\epsilon/(1-\beta)$. Finally, on the complementary event $(B^{(n)})^c$, the difference in the total expected payoff obtained by agent 1 on choosing any other strategy over that obtained by the strategy $\xi(\cdot)$ can be at most $2/(1-\beta)$.

From the preceding argument, we obtain that by following any strategy $\delta_1^{(n)}$, agent 1's total expected payoff $V^{(n)}(s, \delta_1^{(n)})$ is bounded above by

$$V^{(n)}(s, \delta_1^{(n)}) - V(s \mid g) \leq \frac{2\epsilon}{1-\beta} + \frac{2}{1-\beta}\mathbf{P}((B^{(n)})^c).$$

On the contrary, suppose agent 1 was using the mean field equilibrium strategy ξ. In this case, on the event $B^{(n)}$, agent 1 performs exactly as an agent in the large market for the first κ auctions. Since the total loss to an agent in any auction is bounded, we obtain by similar arguments as before that

$$V^{(n)}(s, \xi) - V(s \mid g) \geq -\frac{2\epsilon}{1-\beta} - \frac{2}{1-\beta}\mathbf{P}((B^{(n)})^c).$$

Using these two inequalities, we have

$$V^{(n)}(s, \delta_1^{(n)}) - V^{(n)}(s, \xi) \leq \frac{4\epsilon}{1-\beta} + \frac{4}{1-\beta}\mathbf{P}((B^{(n)})^c).$$

Next, recall that $B^{(n)} \subseteq C^{(n)} \subseteq D^{(n)} \subseteq E^{(n)}$. Thus, we have from the union bound,

$$\begin{aligned}\mathbf{P}((B^{(n)})^c) &\leq \mathbf{P}((B^{(n)})^c \mid C^{(n)}) + \mathbf{P}((C^{(n)})^c \mid D^{(n)}) \\ &\quad + \mathbf{P}((D^{(n)})^c \mid E^{(n)}) + \mathbf{P}((E^{(n)})^c).\end{aligned}$$

Letting $\tau = \max(4/\ln(1/\beta), 1/\ln(1/\beta) + 2)\ln(1/\epsilon)$ and using the bounds from Lemmas 2, 3, 4, and 5, we obtain for $\alpha \geq 4$,

$$\begin{aligned}\mathbf{P}((B^{(n)})^c) &\leq \frac{\kappa\alpha(\alpha-1)^4}{4(n-1)(\alpha-2)^2(\alpha-3)}(e^{2(\alpha-2)\tau} - 1) \\ &\quad + \frac{\kappa^2(\alpha-1)^4}{(n-1)(\alpha-2)^2}(e^{2(\alpha-2)\tau} - 1) \\ &\quad + \frac{(\kappa-1)(\kappa-2)(\alpha-1)^2}{2(n-\alpha+1)} + \epsilon.\end{aligned}$$

This implies that

$$\begin{aligned}&V^{(n)}(s, \delta_1^{(n)}) - V^{(n)}(s, \xi) \\ &\leq \frac{8\epsilon}{1-\beta} + \frac{\kappa\alpha(\alpha-1)^4}{(1-\beta)(n-1)(\alpha-2)^2(\alpha-3)}(e^{2(\alpha-2)\tau} - 1) \\ &\quad + \frac{4\kappa^2(\alpha-1)^4}{(1-\beta)(n-1)(\alpha-2)^2}(e^{2(\alpha-2)\tau} - 1) + \frac{2(\kappa-1)(\kappa-2)(\alpha-1)^2}{(1-\beta)(n-\alpha+1)}.\end{aligned}$$

For small enough $\epsilon > 0$, let $n_0 = ((4\kappa^2\alpha(\alpha-1)^4)/(\epsilon(1-\beta) \cdot (\alpha-2)^2(\alpha-3)))(e^{2(\alpha-2)\tau} - 1)$. Then, we have for all $n > n_0$,

$$V^{(n)}(s, \delta_1^{(n)}) - V^{(n)}(s, \xi) \leq \frac{10\epsilon}{1-\beta}.$$

Because ϵ is arbitrary, this proves the convergence.

Finally, to prove the convergence rate, observe that $e^{\tau} = \max((1/\epsilon)^{4/\ln(1/\beta)}, (1/\epsilon)^{1/\ln(1/\beta)+2})$. Thus, we have

$$\begin{aligned} n_0 &= \frac{4\kappa^2\alpha(\alpha-1)^4}{\epsilon(1-\beta)(\alpha-2)^2(\alpha-3)}(e^{2(\alpha-2)\tau}-1) \\ &\le \frac{4\kappa^2\alpha(\alpha-1)^4}{(1-\beta)(\alpha-2)^2(\alpha-3)} \\ &\quad \cdot \max\left(\left(\frac{1}{\epsilon}\right)^{\frac{8(\alpha-2)}{\ln(1/\beta)}+1}\left(\frac{1}{\epsilon}\right)^{\frac{2(\alpha-2)}{\ln(1/\beta)}+4(\alpha-2)+1}\right). \end{aligned}$$

Thus, for small enough ϵ, large enough β and for $\alpha \ge 4$, we have

$$\begin{aligned} n_0 &\le \frac{4\alpha(\alpha-1)^4}{(1-\beta)(\alpha-2)^2(\alpha-3)}\left(\frac{1}{\epsilon}\right)^{\frac{8(\alpha-2)}{\ln(1/\beta)}+1}\left(\frac{\ln(\epsilon)}{\ln(\beta)}+1\right)^2 \\ &\le \frac{4\alpha(\alpha-1)^4}{(1-\beta)(\alpha-2)^2(\alpha-3)\ln(1/\beta)^2}\left(\frac{1}{\epsilon}\right)^{\frac{8(\alpha-2)}{\ln(1/\beta)}+1+\omega}, \end{aligned}$$

for some $\omega > 0$. This implies that,

$$\epsilon \le \left(\frac{4\alpha(\alpha-1)^4}{(1-\beta)(\alpha-2)^2(\alpha-3)\ln(1/\beta)^2(n_0-1)}\right)^{\frac{\ln(1/\beta)}{8(\alpha-2)+(1+\omega)\ln(1/\beta)}}.$$

Hence, for $\alpha \ge 4$, large enough β, and all large enough n_0, we have

$$\begin{aligned} &V^{(n_0)}(s, \delta_1^{(n_0)}) - V^{(n_0)}(s,\xi) \\ &\le \frac{10}{1-\beta}\left(\frac{4\alpha(\alpha-1)^4}{(1-\beta)(\alpha-2)^2(\alpha-3)\ln(1/\beta)^2(n_0-1)}\right)^{\frac{\ln(1/\beta)}{(8(\alpha-2)+(1+\omega)\ln(1/\beta)}}, \end{aligned}$$

for all small enough $\omega > 0$. □

Lemma 2. *For any $\epsilon > 0$, define $\kappa = \ln(\epsilon)/\ln(\beta) + 1$, and $\tau = \max(4/\ln(1/\beta), 1/\ln(1/\beta) + 2)\ln(1/\epsilon)$. Let $E^{(n)}$ denote the event that agent 1 has participated in at least κ auctions until time τ. Then, we have $\mathbf{P}(E^{(n)}) \ge 1 - \epsilon$.*

Proof. Fix $\epsilon > 0$. Let $\{X_t : t \ge 0\}$ denote a Poisson process with rate 1. Using a Chernoff bound argument, it can be shown that $\mathbf{P}(X_t \le x) \le (e^{-t}(et)^x)/x^x$, for all $t \ge x$. Note that with τ as defined in the statement of the lemma, we have $\tau \ge \kappa$. Thus, we obtain

$$\mathbf{P}(E^{(n)}) = \mathbf{P}(X_\tau \ge \kappa) = 1 - \mathbf{P}(X_\tau \le \kappa - 1) \ge 1 - \frac{e^{-\tau}(e\tau)^{\kappa-1}}{(\kappa-1)^{\kappa-1}}.$$

Define $\mu \triangleq \max(4, 1 + 2\ln(1/\beta))$. Note that, we have $\tau = \mu(\kappa - 1)$. Then, we have

$$\begin{aligned} \mathbf{P}(E^{(n)}) &\ge 1 - \frac{e^{-\tau}(e\tau)^{\kappa-1}}{(\kappa-1)^{\kappa-1}} \ge 1 - e^{-\mu(\kappa-1)}(e\mu)^{\kappa-1} \\ &= 1 - (e^{1-\mu}\mu)^{\kappa-1} \ge 1 - (e^{0.5(1-\mu)})^{\kappa-1}, \end{aligned}$$

where the last inequality follows from the fact that since $\mu \ge 4$, we have $\mu \le e^{0.5(\mu-1)}$. Finally, it is straightforward to verify that $0.5(\mu-1)(\kappa-1) > \ln(1/\epsilon)$, and hence $\mathbf{P}(E^{(n)}) \ge 1 - e^{\ln(\epsilon)} = 1 - \epsilon$. □

Lemma 3. *Let $D^{(n)} \subseteq E^{(n)}$ denote the event that for all $\rho, \sigma \le \kappa$, the sets S_{1,τ_ρ} and S_{1,τ_σ} are disjoint. Then, we have*

$$\mathbf{P}(D^{(n)} \mid E^{(n)}) \ge 1 - \frac{(\kappa-1)(\kappa-2)(\alpha-1)^2}{2(n-\alpha+1)}.$$

In particular, we have $\mathbf{P}(D^{(n)} \mid E^{(n)}) \to 1$ as $n \to \infty$.

Proof. Under the event $D^{(n)}$, in each of the first κ auctions, agent 1 faces a new set of $\alpha - 1$ competitors. Because the competitors are chosen uniformly at random (and independently) from the agent population, we have

$$\mathbf{P}(D^{(n)} \mid E^{(n)}) = \prod_{m=1}^{\kappa}\binom{n-1-(m-1)(\alpha-1)}{\alpha-1}\binom{n-1}{\alpha-1}^{-1}.$$

Here, for each $m \in \{1, \ldots, \kappa\}$, the corresponding term denotes the probability that in the mth auction, agent 1 does not face any competitor she faced in the previous $m-1$ auctions. Expanding and rearranging, we obtain that

$$\begin{aligned} \mathbf{P}(D^{(n)}|E^{(n)}) &= \prod_{m=1}^{\kappa}\prod_{\ell=1}^{\alpha-1}\left(1 - \frac{(m-1)(\alpha-1)}{(n-\ell)}\right) \\ &\ge \prod_{m=1}^{\kappa}\prod_{\ell=1}^{\alpha-1}\left(1 - \frac{(m-1)(\alpha-1)}{(n-\alpha+1)}\right) \\ &\ge 1 - \frac{(\kappa-1)(\kappa-2)(\alpha-1)^2}{2(n-\alpha+1)}, \end{aligned}$$

where, in the final inequality, we have used the fact that $(1-x)(1-y) \ge 1 - x - y$ for nonnegative x and y. □

Lemma 4. *Let $C^{(n)}$ denote the event that for all $\sigma = 1, \ldots, \kappa$ and for each distinct pair j, k with $j \in S_{1,\tau_\sigma}$ and $k \in (\bigcup_{\rho \le \sigma} S_{1,\tau_\rho}) \cup \{1\}$, the interaction sets $A^{(n)}_{j,\tau_\sigma^-}$ and $A^{(n)}_{k,\tau_\sigma^-}$ are disjoint. Then $\mathbf{P}(C^{(n)} \mid D^{(n)}) \to 1$ as $n \to \infty$. More precisely, we have for $\alpha \ge 3$,*

$$\mathbf{P}(C^{(n)} \mid D^{(n)}) \ge 1 - \frac{1}{n-1}\frac{\kappa^2(\alpha-1)^4}{(\alpha-2)^2}(e^{2(\alpha-2)\tau} - 1).$$

We need the following construct for the proof. For $i \ne 1$, let $F^{(n)}_{i,t}$ denote the restricted interaction set of agent i obtained by ignoring all interactions involving agent 1. Formally, $F^{(n)}_{i,t}$ for $i \ne 1$ evolves exactly as $A^{(n)}_{i,t}$ except that if agent i and agent 1 participate in an auction at time t, then the interaction set $F^{(n)}_{i,t}$ remains unaltered.

Proof of Lemma 4. The proof proceeds in two steps. First, in Step 1, we show that if for some $\sigma \le \kappa$ and some distinct pair j, k with $j \in S_{1,\tau_\sigma}$ and $k \in (\bigcup_{\rho \le \sigma} S_{1,\tau_\rho}) \cup \{1\}$ the sets $A^{(n)}_{j,\tau_\sigma^-}$ and $A^{(n)}_{k,\tau_\sigma^-}$ have nonempty intersection, then there exists a distinct pair of agents $\ell, m \in \bigcup_{\rho \le \sigma} S_{1,\tau_\rho}$ whose restricted interaction sets $F^{(n)}_{\ell,\tau_\sigma}$ and $F^{(n)}_{m,\tau_\sigma}$ have nonempty intersection. This implies that to prove the lemma, it suffices to show that, with high probability, for all distinct $\ell, m \in \bigcup_{\sigma \le \kappa} S_{1,\tau_\sigma}$ the sets $F^{(n)}_{\ell,\tau_\kappa}$ and $F^{(n)}_{m,\tau_\kappa}$ are disjoint. Second, in Step 2, we associate the preceding restricted interaction sets with the interaction sets in a market with $n-1$ agents, and use a result from Graham and Méléard (1994) to prove that the preceding restricted interactions sets are disjoint with high probability.

Step 1. First, suppose for some τ_σ and for some $j \in S_{1,\tau_\sigma}$, the sets $A^{(n)}_{1,\tau_\sigma^-}$ and $A^{(n)}_{j,\tau_\sigma^-}$ are not disjoint. Because agent j

has not directly interacted with agent 1 until time t, it must mean that there is an agent k who has influenced both agent 1 and agent j. This implies that there is an agent $\ell \in \bigcup_{\rho\leq\sigma-1} S_{1,\tau_\rho}$ such that agent k is either same as agent ℓ, or agent k has influenced both agent ℓ and agent j. Since this interaction does not involve agent 1, we obtain that there is an agent $\ell \in \bigcup_{\rho\leq\sigma-1} S_{1,\tau_\rho}$ such that the sets $F^{(n)}_{\ell,\tau_\sigma}$ and $F^{(n)}_{j,\tau_\sigma}$ have nonempty intersection.

Next, suppose for each τ_σ and for each $j\in S_{1,\tau_\sigma}$, we have that $A^{(n)}_{1,\tau_\sigma^-}$ and $A^{(n)}_{j,\tau_\sigma^-}$ are disjoint. This implies that for any agent in $\bigcup_{\sigma\leq\kappa} S_{1,\tau_\sigma}$, until agent 1 met the agent in an auction, she never had an influence on or was influenced by that agent. In this case, if for some $\sigma\leq\kappa$ and for some distinct j, k with $j\in S_{1,\tau_\sigma}$ and $k\in\bigcup_{\rho\leq\sigma} S_{1,\tau_\rho}$, the interaction sets $A^{(n)}_{j,\tau_\sigma^-}$ and $A^{(n)}_{k,\tau_\sigma^-}$ were not disjoint, then these two agents must have influenced each other through interactions not involving agent 1. This implies that the interaction sets $F^{(n)}_{j,\tau_\sigma}$ and $F^{(n)}_{k,\tau_\sigma}$ have nonempty intersection.

Step 2. Let $S_1=\bigcup_{\sigma=1}^{\kappa} S_{1,\tau_\sigma}$ denote the set of all agents faced by agent 1 until time τ_κ. Now, consider the market conditioned on the times $\tau_1,\ldots,\tau_\kappa$, on $S_{1,\tau_\rho}\cap S_{1,\tau_\sigma}=\varnothing$ for all $\rho,\sigma\leq\kappa$, and on S_1. In the rest of the market, auctions are held at the jump times of a Poisson process with rate $n/\alpha-1$, and in each of these auctions, α agents other than agent 1 are chosen uniformly to participate. Furthermore, the rest of the market, insofar as the identity of the matched agents are concerned, is independent of the conditioned information. Thus, conditioning on the times $\tau_1,\ldots,\tau_\kappa$, on $S_{1,\tau_\rho}\cap S_{1,\tau_\sigma}=\varnothing$ for all $\rho,\sigma\leq\kappa$, and on S_1, we see that the process $\{F^{(n)}_{i,t}\colon i\neq 1\}$ is distributed exactly as the process $\{\tilde{A}^{(n-1)}_{i,t}\colon 2\leq i\leq n\}$, the interaction sets in an independent copy of the market with $n-1$ agents where auctions are held at rate $n/\alpha-1$. Thus, we obtain

$$\begin{aligned}\mathbf{P}(F^{(n)}_{k,\tau_\kappa}\cap F^{(n)}_{\ell,\tau_\kappa}=\varnothing \text{ for all } k,\ell\in S_1)\\ =\mathbf{P}(\tilde{A}^{(n-1)}_{k,\tau_\kappa}\cap \tilde{A}^{(n-1)}_{\ell,\tau_\kappa}=\varnothing \text{ for all } k,\ell\in S_1).\end{aligned}$$

Using Theorem 4.1 of Graham and Méléard (1994), we conclude that latter probability approaches 1 as $n\to\infty$. More precisely, we obtain for $\alpha\geq 3$,

$$\begin{aligned}&\mathbf{P}(F^{(n)}_{k,\tau_\kappa}\cap F^{(n)}_{\ell,\tau_\kappa}=\varnothing \text{ for all } k,\ell\in S_1)\\ &\quad=\mathbf{P}(\tilde{A}^{(n-1)}_{k,\tau_\kappa}\cap \tilde{A}^{(n-1)}_{\ell,\tau_\kappa}=\varnothing \text{ for all } k,\ell\in S_1)\\ &\quad\geq 1-\frac{(n-\alpha)(\alpha-1)}{(n-1)(n-2)}\\ &\qquad\cdot\frac{|S_1|(|S_1|-1)(\alpha-1)}{[\alpha-2+(n-\alpha)(\alpha-1)/((n-1)(n-2))](\alpha-2)}\\ &\qquad\cdot(e^{2(\alpha-2)\tau}-1)\\ &\quad\geq 1-\frac{1}{n-1}\frac{\kappa^2(\alpha-1)^4}{(\alpha-2)^2}(e^{2(\alpha-2)\tau}-1),\end{aligned}$$

where we have used the fact that $|S_1|=\kappa(\alpha-1)$. The statement of the lemma now follows from Step 1. □

Lemma 5. *On event $C^{(n)}$, the distribution of the valuation and the belief at time σ of each agent in S_{1,τ_σ} for $\sigma=1,\ldots,\kappa$ converges uniformly in total variation, as n increases, to the product distribution $\Phi^{\otimes\kappa(\alpha-1)}$. More precisely, we have for $\alpha\geq 4$, conditioned on $C^{(n)}$,*

$$\begin{aligned}&\|\mathcal{L}(\{Z_i(\tau_\sigma)\colon i\in S_{1,\tau_\sigma},\sigma=1,\ldots,\kappa\})-\Phi^{\otimes\kappa(\alpha-1)}\|_{TV}\\ &\quad\leq\frac{\kappa\alpha(\alpha-1)^4}{2(n-1)(\alpha-2)^2(\alpha-3)}(e^{2(\alpha-2)\tau}-1),\end{aligned}$$

where $Z_i(t)$ is the valuation and the belief of the agent i at time t, $\|\cdot\|_{TV}$ is the total variation norm, and $\mathcal{L}(X)$ denotes the distribution of a random variable X.

Proof. Recall that on the event $C^{(n)}$, the interaction sets $A^{(n)}_{j,\tau_\sigma^-}$ and $A^{(n)}_{k,\tau_\sigma^-}$ are disjoint for any $j\in S_{1,\tau_\sigma}$ and $k\in(\bigcup_{\rho\leq\sigma} S_{1,\tau_\rho})\cup\{1\}$. This implies that on the event $C^{(n)}$, the random variables $\{Z_i(\tau_\sigma)\colon i\in S_{1,\tau_\sigma},\sigma=1,\ldots,\kappa\}$ are independent. Thus, we obtain that, on the event $C^{(n)}$,

$$\begin{aligned}&\|\mathcal{L}(\{Z_i(\tau_\sigma)\colon i\in S_{1,\tau_\sigma},\sigma=1,\ldots,\kappa\})-\Phi^{\otimes\kappa(\alpha-1)}\|_{TV}\\ &\quad=\Bigg\|\bigotimes_{\substack{i\in S_{1,\tau_\sigma}\\ \sigma=1,\ldots,\kappa}}\mathcal{L}(\{Z_i(\tau_\sigma)\})-\Phi^{\otimes\kappa(\alpha-1)}\Bigg\|_{TV}\\ &\quad\leq\sum_{\substack{i\in S_{1,\tau_\sigma}\\ \sigma=1,\ldots,\kappa}}\|\mathcal{L}(\{Z_i(\tau_\sigma)\})-\Phi\|_{TV}.\end{aligned}$$

From Theorem 5.1 of Graham and Méléard (1994), we obtain that for $\alpha\geq 4$,

$$\begin{aligned}&\|\mathcal{L}(Z_i(\tau_\sigma))-\Phi\|_{TV}\\ &\quad\leq\frac{(n-\alpha)(\alpha-1)}{(n-1)(n-2)}\\ &\qquad\cdot\frac{\alpha(\alpha-1)^2}{(n-1)\left[2(\alpha-2)+\frac{(n-\alpha)(\alpha-1)}{(n-1)(n-2)}\right](\alpha-2)(\alpha-3)}\\ &\qquad\cdot(e^{2(\alpha-2)\tau_\sigma}-1).\end{aligned}$$

The result then follows by observing that $|\bigcup_{\sigma=1,\ldots,\kappa} S_{1,\tau_\sigma}|=\kappa(\alpha-1)$ on the event $C^{(n)}$. □

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