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Control of Dividends, Capital Subscriptions, and Physical Inventories

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Manufacturers manage interrelated flows of material and cash. Material needs capital, and sales contribute cash. Therefore, it may be beneficial to coordinate operational and financial decisions. We study a dynamic model of coordination in an equity-financed firm in which inventory and financial decisions interact in the presence of demand uncertainty, financial constraints, and a risk of default. The criterion is to maximize the expected present value of dividends net of capital subscriptions. The optimal target inventory level and financial decision variables are nondecreasing functions of the levels of inventory and retained earnings. Some important attributes of an optimal policy remain the same regardless of whether default precipitates Chapter 7 or Chapter 11 bankruptcy. The optimal policy is myopic, and if pertinent cost functions are piecewise linear, it is characterized with simple formulas. We show that the methods of inventory theory are useful in analyzing models of operational and financial coordination.

Key words: inventory; production; stochastic; finance; corporate finance; dynamic programming; applications
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1. Operations and Finance

Business organizations are faced with the financial and physical task of managing flows of material and cash. In large firms, the task is decentralized into separate functional responsibilities, although the flows are interrelated because material needs capital and the sale of finished goods contributes to cash reserves. The same dichotomy into operations and finance occurs in management education and research. Although decentralization in professional practice was obligatory in the absence of enterprise-wide information systems, technology has advanced and in many firms it is technically feasible to coordinate financial and operational decisions. This paper contributes to understanding what coordination would entail.

Ease of analysis and concentration on special function are two reasons in practice to separate operational and financial decisions. In practice, the production manager has much to learn about sales, production/procurement, and inventories without being concerned directly with finance. But it is a practical question to decide when the interaction between finance and operations can be completely ignored, as a good enough approximation to reality, or when it is sufficiently important that the functions should be coordinated. Although these coordination issues are apt for firms of all sizes, it is particularly apparent that many small growing firms are severely cash constrained and could not survive without coordinating financial and operational decisions.

The large literature on asset pricing in financial economics asserts that the primary measure of the financial value of an investor-owned firm is the expected present value of the time stream of its dividends (see Lucas 1978; Brodie et al. 1995, p. 326; and Cochrane 2001), namely, its EPV. In contrast, the research methodologies and qualitative results in inventory theory respond to cost and/or profit criteria.

One may argue that operations models should *not* use the EPV criterion because of Modigliani and Miller's (1958) two propositions that specify conditions under which a firm's market value is independent of its capital structure and its dividend policy. The conditions include perfect capital markets and the absence of "frictions" such as bankruptcy costs, taxes, agency costs, and asymmetric information. There are many generalizations of the Modigliani and

Miller (1958) propositions (Allen and Michaely 1995), but bankruptcy costs would be a deal breaker! The cost of bankruptcy is a major element of our model.

Many firms do not operate in perfect markets and are influenced by the specter of bankruptcy. This has been evident for some large publicly traded firms during the past decade, but it is particularly true of those which are small and entrepreneurial, so asymmetry of information is the rule rather than the exception. Our modeling is partly motivated by firms such as these where prescriptions for operating behavior can be ineffective if they overlook the interaction of financial and operational decisions.

If a firm is leveraged, then its value to its shareholders is less than its total value. However, we model a firm that is financed entirely with equity and short-term debt and, therefore, we use the EPV criterion net of capital subscriptions. Even short-term debt is sufficient to illustrate the critical features of default without the considerable complications and extra realism of long-term lending.

There is a potential for liquidity crises in the firm facing production/procurement lags, uncertain demand, and financial constraints. Such crises can occur although the firm has no long-term debt. In young firms, as much as the stockholders might like to see their dividends flow and grow, the payouts may sometimes be negative in the sense that the stockholders may be called for capital subscriptions, namely, more cash to keep the enterprise solvent. At first, we consider this possibility by allowing the dividends to be negative. The model that we define and analyze in §§2–4 does not constrain dividends to be nonnegative. One may interpret a negative dividend for a small- or medium-sized business as an action by the owners to subscribe additional capital. So, the model corresponds to a small or medium-sized business whose owners may be obliged to subscribe additional capital. Later, in §6, we impose a nonnegativity constraint on dividends.

We make two kinds of contributions in highly stylized settings. First, we characterize optimal policies for coordinating operational and financial decisions. Second, we characterize the market value of a firm that adopts such policies.

1.1. Background Literature

Although there is a well-developed literature on models of production and inventory systems (see Graves et al. 1993), until recently, only a few models of operational decisions included financial considerations. An early analysis of financial considerations is based on a model of the dynamics of the growth of a dividend-paying nonfinancial firm as a controlled random walk between a reflecting and an absorbing barrier (Shubik and Thompson 1959). The reflecting barrier is created by dividend payments and the absorbing barrier by bankruptcy conditions that put the firm out of business. The separation of ownership and management in a game of economic survival provides the mathematical structure to reflect the potential differences in goals of management and ownership, in particular, concerning the roles of dividends and bankruptcy. The model in §2 reflects the strategic aspects of the trade-offs between bankruptcy and the paying of dividends. Alternatively, if the firm is penalized by an insolvency, but is in a position to continue to operate, this too can be modeled as a reflecting barrier; see Radner and Shepp (1996). Hadley and Whitin (1963) and Sherbrooke (1968) are early papers on inventory management with budgetary constraints. The treatment of the maintenance of cash safety levels as an inventory control problem (see Porteus 1972, Shubik and Sobel 1992) is another research connection between operations and finance.

Most of the fusion of operational and financial considerations is recent in origin. A series of papers model operational decisions in the presence of foreign exchange exposure (e.g., Kogut and Kulatilaka 1994, Huchzermeier and Cohen 1996, Dasu and Li 1997, Aytekin and Birge 2004, Dong et al. 2006). Papers in another series analyze capacity-expansion problems with financial constraints (e.g., Birge 2000, Van Mieghem 2003, Babich and Sobel 2004) and use approaches similar to ours.

Some recent work on the coordination of operational and financial decisions can be partitioned according to whether or not the models and criteria are influenced by the capital structure of the firm. Perhaps earliest among the papers that are orthogonal to capital structure considerations are Archibald et al. (2002), who optimized the probability of survival of a growing firm that manages an inventory, and Buzacott and Zhang (2004), who modeled a growing manufacturer that finances production both with loans secured by inventory and with unsecured loans. The latter paper demonstrates the importance of jointly considering production and financing decisions in a dynamic but deterministic setting and proposes a single-period newsvendor model to examine the incentives for a lender and a borrower to engage in asset-based financing.

Capital structure plays a role in the static model that is essentially common to Xu and Birge (2004, 2005) and Dada and Hu (2008). In these papers, a financially constrained firm coordinates its operating decisions and short-term borrowing. Xu and Birge (2004), assuming that the creditor is nonstrategic, comment that bankruptcy costs remove the firm from the Modigaliani–Miller world. As a result, the firm's borrowing capacity is limited, and it may have to produce less than the unconstrained optimum.

Comparative statics implies that the firm's optimal production quantity decreases as its debt increases. Xu and Birge (2004, p. 17) conclude that "a low-margin company should [select] a conservative output level and an aggressive financial decision, while [a] high-margin company" should do the opposite.

Xu and Birge (2005) numerically examined the dependence of the value of the firm on production unit cost and debt-equity ratio, and then compared their results with market data. They found that the model predicts lower debt-equity ratios than the data exhibit, and they attributed this difference to the absence of long-term debt in the model. They showed that a low-margin producer faces higher agency costs than a high-margin one. Dada and Hu (2008) assumed that the creditor *is* strategic and used game theory to show that (a) production is less than the unconstrained optimum without bankruptcy costs, and (b) the optimal production quantity increases as a function of equity.

The model that underlies these three papers is static and yields conclusions that differ from some of those in Hu and Sobel (2005), which is based on a dynamic model that distinguishes between long-term debt and short-term loans. In the latter paper, as the *long-term* debt increases, (i) the optimal inventory level increases and then decreases, and (ii) the optimal short-term loan decreases. As is typically true of dynamic newsvendor-like models without capacity constraints, the optimal production quantity (in any period after the first) is the previous period's demand. So inventory levels are driven by the amount of long-term debt, but production amounts do not depend on it at all.

In the valuation model in Xu and Birge (2006), the firm maximizes a combination of the EPV of the net cash flow to shareholders and multiples of other firm attributes, and it decides whether to default and liquidate the firm or to continue to produce. The contingent default opportunity raises the expected present value of the net cash flow to shareholders, so it yields higher equity valuations than traditional valuation and planning models. Xu and Birge (2006) conclude that valuations are understated if they stem from models that exclude the possibility of contingent defaults.

Other papers leverage the results in this paper but change the assumptions in §2. Brunet and Babich (2007) compute the signaling value of trade credit financing for the acquisition of goods. Hu and Sobel (2005) design the firm's capital structure before operations commence; the resulting mixture of equity and long-term debt affects the optimal base-stock levels for inventory and working capital. Zhang and Sobel (2010) have nonlinear costs of production, and, as a result, optimal inventory decisions become more complicated than the base-stock levels in this paper. Sobel and Turcic (2007) consider how to adapt to evolving market conditions that are modeled with a more flexible and realistic model of demand than in §2. That raises issues that cannot be addressed here, such as the change in the levels of inventory and working capital as the firm grows. That paper also has a formula for the financial value of coordinating operational and financial decisions.

1.2. Organization of This Paper

Section 2 specifies the model to which most of the paper is devoted. In particular, default results in a penalty and subsequent resumption of operations. The U.S. Bankruptcy Code has two routes for a corporation to enter bankruptcy.[1] "Chapter 7," which we term *wipeout* bankruptcy, entails a permanent halt of operations and an immediate liquidation of assets. "Chapter 11," which we term *reorganization* bankruptcy, consists of a restructuring of debt and a continuation of operations. Thus, the model in §2 corresponds to Chapter 11. Along with other assumptions, this results in a dynamic stochastic optimization problem in which the randomness is due to market demand.

Section 3 establishes key properties of a dynamic programming representation of the optimization problem. Section 3.1 reduces the dimensionality of the dynamic program, and §3.2 shows that the reduced dynamic program has a concave structure. Also, the optimal values of dividends, production, and short-term borrowing are nondecreasing functions of the levels of inventory and retained earnings, and the optimal level of inventory (residual retained earnings) is a nondecreasing (nonincreasing) function of the distribution of demand (in the sense of first-order stochastic dominance). Section 3.3 significantly simplifies the optimization by showing that the dynamic problem has a myopic optimum, and §3.4 observes that an optimal policy is consistent with the "pecking-order" principle of corporate finance.

Section 4 considers the important case in which the default penalty and the gross profit from sales (net of inventory costs) are piecewise linear functions. This leads to a nearly explicit solution in which the optimal level of goods inventory rises if there are increases in β, the discount factor, γ, the unit backorder cost, or ρ, the interest rate on short-term loans, or there are decreases in h, the unit holding cost, c, the unit procurement/production cost, or p_1, the unit default penalty. The optimal level of residual retained earnings rises if there are increases in γ, h, β, p_1, or ρ, or there is a decrease in c. As a result, the optimal

[1] http://www.uscourts.gov/bankruptcycourts/bankruptcybasics/process.html (accessed December 2, 2012).

short-term loan rises if c increases, or if there are decreases in γ, h, β, p_1, or ρ.

The next two sections investigate the extent to which the conclusions would remain valid if major features of the model were changed. In §5, default precipitates dissolution of the firm; so this section corresponds to Chapter 7 instead of Chapter 11 in the U.S. Bankruptcy Code. The primary conclusions are that major features of the optimal coordination of operations and finance transcend the particular form of bankruptcy; i.e., some important results in §§3 and 4 remain valid. Also under Chapter 7, the lifetime of the optimally coordinated firm has a geometric distribution. In §5.2, we alter the model by endowing the firm with the option to declare Chapter 7 bankruptcy even if it is not forced into it. As a consequence, the set of circumstances in which the firm should opt for bankruptcy is determined by a barrier limit, and the production quantities and dividends should continue to be selected according to order-up-to levels. However, the order-up-to levels become functions of retained earnings instead of being scalars. In §6, we preclude capital subscriptions, so dividends are constrained to be nonnegative. We find that some of the results in §3 remain valid and that inventories are lower, short-term loans are larger, and the default risk is higher than if capital subscriptions were permitted. We conclude with §8.

2. The Model

Consider a discrete-time multiperiod model in which a firm that lives in perpetuity decides each period how much money to borrow, how many units to produce, and how much dividend to issue before the realization of the period's demand. Although there may be several means of financing, we assume that only short-term loans are available. The specifics of the model are as follows.

In each period, the firm makes three decisions:

$b_n \equiv$ amount of money borrowed at the start of period and repaid at the end of period n;

$z_n \equiv$ quantity of goods procured/produced during period n; we assume that these goods are delivered immediately and are available to satisfy demand in period n; and

$v_n \equiv$ dividend issued in period n if $v_n > 0$; capital subscription if $v_n < 0$.

Notation for other variables:

$w_n \equiv$ amount of retained earnings at the beginning of period n;

$s_n \equiv$ the amount of internally generated working capital that is available at the beginning of period n (defined precisely in (2));

$x_n \equiv$ number of units in inventory (net of cumulative unsatisfied demand, if any) at the beginning of period n;

$y_n = x_n + z_n \equiv$ total amount of goods available to satisfy demand in period n;

$D_n \equiv$ new demand in period n; we assume that D, $D_1, D_2, \ldots$ are independent and identically distributed and nonnegative, and let F denote the distribution function of D;

$c \equiv$ unit procurement cost; we assume $c > 0$;

$r \equiv$ unit selling price; revenue is received in the period when demand is met;

$\gamma \equiv$ unit backorder cost; we assume $r > c + \gamma$; $r - \gamma$ is the revenue received when a backorder is met;

$g(y_n, D_n) \equiv$ sales revenue received on new demand that can be met in period n minus inventory-related costs, where $y_n = x_n + z_n$ is the total amount of product available to satisfy the current demand; for example, $g(y, d) = r\min\{y, d\} - h(y-d)^+ = ry - (r+h)(y-d)^+$, where d and h denote realization of demand and unit inventory cost, respectively;

$\rho \equiv$ interest rate on short-term loans; ρb_n is interest paid at the beginning of period n; the results would not significantly change if we assumed that interest were paid at the end of the period;

$\beta \equiv$ single-period discount factor $(0 < \beta < 1)$ (see the comments on β relative to ρ in the next paragraph); and

$p(w_n) \equiv$ default cost if $w_n < 0$ but it is convenient to define $p(\cdot)$ as a function on $\Re$. Although many of our results depend on $p(\cdot)$ being a convex decreasing function, the most important results do not depend on this assumption and we make no assumptions regarding $p(\cdot)$ now.

Borrowing would not occur for the sole purpose of immediately disbursing the loan as a dividend unless the interest rate is less than the opportunity cost of capital $(\rho < 1/\beta - 1)$. In that case, the value of the firm (in the model) would be unbounded because of the following argument. At the beginning of period n, suppose that the firm borrows \$1, pays interest of ρ, and distributes \$1 immediately as a dividend. At the beginning of period $n+1$, the firm obtains a capital subscription of $1+\rho$. The firm's liquidity at the beginning of period $n+1$ is unchanged, but the change in the shareholders welfare is $1 - \beta(1+\rho) > 0$ because $\rho < 1/\beta - 1$. That is, $1 - \beta(1+\rho) > 0$ implies that the firm should become an arbitrage machine, making unboundedly large short-term loans and capital subscriptions.

However, even if $1 - \beta(1+\rho) < 0$, it may be worthwhile for the firm to borrow to finance production. For example, if demand is deterministic or if there is a positive minimum level of demand, suppose that the firm borrows $c(1+\rho)$, spends c to produce a single unit, and receives r when it sells the unit later in the period. The net proceeds are $r - c(1+\rho)$, which would be positive while $1 - \beta(1+\rho) < 0$ if $r/c > 1 + \rho > 1/\beta$.

If $w_{n+1} < 0$, then the firm cannot fully repay period n's short-term loan, b_n. Although we call $p(\cdot)$ a "default" penalty, it encompasses three related phenomena. First, if the retained earnings go negative and the firm cannot reorganize, this could be the terminal bankruptcy costs. Second, if the firm can reorganize, it can do so by borrowing externally and paying legal fees and other financing costs. In this instance, the negotiated settlement may result in the lender receiving less than the loan (b_n). Third, it can raise money by subscriptions from current stockholders. Our current model illustrates the latter two choices (the terminal bankruptcy case will be dealt with in §5). Thus, the default penalty can be thought of as the amount of total corporate losses, including the utility losses of stockholders, because of an insolvency.

The sequence of decisions and events in period n is as follows:

1. Each period starts with computing the amount of retained earnings w_n and the current inventory level x_n.
2. The default penalty $p(w_n)$ is paid if the amount of retained earnings is negative.
3. Choose the levels of borrowing, production, and dividend, (b_n, z_n, v_n), subject to a liquidity constraint ((7) below).
4. The dividend and loan interest are paid, v_n and ρb_n, the production decision is implemented at a cost of cz_n, and the backordered demand is satisfied with a revenue of $(r-\gamma)x_n^-$.
5. The demand is realized, D_n, and sales revenue on current demand net of inventory costs, $g(y_n, D_n)$, is realized.
6. The firm repays as much of the principal of the loan, b_n, as its liquidity allows. See the details following Equation (3).

In the remainder of this paper, we assume that optimization problems have bounded solutions, and this leads to implicit assumptions. For example, see the discussion of an arbitrage "machine" earlier in this section.

The decision variables each period are z_n, v_n, and b_n, but it is more convenient in the analysis to replace z_n and v_n with these decision variables:

$$y_n \equiv x_n + z_n, \tag{1}$$

$$s_n \equiv w_n + (r-\gamma)x_n^- - p(w_n) - v_n - cz_n - \rho b_n. \tag{2}$$

We assume that $y_n \geq 0$, or $z_n \geq -x_n$. That means that the production quantity, z_n, is no less than the backordered demand, $-x_n$ if $x_n < 0$, namely, the backordered demand will always be satisfied first. This is a reasonable assumption because the margin that could be made on the backordered demand is positive and risk free.

We interpret s_n as the amount of internally generated working capital that is available at the beginning of period n. Specifically, s_n is the working capital after the dividend, loan interest, and production cost are paid; the revenue on backordered demand is received, and before the loan is received and revenue and inventory costs are realized. So $b_n + s_n$ is the total working capital available in period n, i.e., after event 4 and before event 5. By the end of the period, revenues and inventory costs have been realized, and as much of the loan is straightforwardly repaid as the firm's liquidity allows.

In general, the amount of the repayment is $\min\{[b_n+s_n+g(y_n, D_n)]^+, b_n\}$. The loan is immediately repaid in full if $b_n + s_n + g(y_n, D_n) \geq b_n$, i.e., if $w_{n+1} = s_n + g(y_n, D_n) \geq 0$. If $w_{n+1} = s_n + g(y, D_n) < 0$, then there is insufficient liquidity to repay part or all of the loan, the default penalty $p(w_{n+1})$ is levied, and the amount that is straightforwardly repaid is $[b_n + s_n + g(y_n, D_n)]^+ < b_n$.

We assume that demand is backordered if it exceeds the supply of goods; so $x_{n+1} = x_n + z_n - D_n$. As a result, standard arguments imply that the model corresponds to one in which there is a constant lag between the time at which goods are produced and the time at which they are available to satisfy demand.

The cash flow dynamics are specified by

$$w_{n+1} = w_n + (r-\gamma)x_n^- - p(w_n) - v_n - cz_n + g(y_n, D_n) - \rho b_n. \tag{3}$$

Equation (3) balances the cash flow and makes the implicit assumption that the short-term loan b_n is repaid at the end of period n (in this case $w_{n+1} \geq 0$) if there is adequate liquidity. Using the convenient notation of (1) and (2), the dynamics equations become

$$x_{n+1} = y_n - D_n, \tag{4}$$

$$w_{n+1} = s_n + g(y_n, D_n). \tag{5}$$

If excess demand were lost instead of backordered, (4) would be replaced by $x_{n+1} = (y_n - D_n)^+$ and all of the paper's qualitative results would be preserved.

We assume that the loan and production quantities are nonnegative:

$$b_n \geq 0 \quad \text{and} \quad z_n \geq 0. \tag{6}$$

The liquidity constraint is the inequality $w_n + (r-\gamma)x_n^- + (1-\rho)b_n \geq p(w_n) + v_n + cz_n$, which corresponds to

$$b_n + s_n \geq 0. \tag{7}$$

Inequality (7) prevents the expenditures in period n from exceeding the sum of retained earnings plus the loan proceeds. If $w_n + (r-\gamma)x_n^- + (1-\rho)b_n - p(w_n) - cz_n < 0$, then (7) forces v_n to be negative. In §6, we analyze the consequences of imposing the constraint that dividends are nonnegative, i.e., $v_n \geq 0$.

Given x_n and w_n, from (1) and (2) the decision variables in period n can be specified as y_n, s_n, and b_n instead of z_n, v_n, and b_n. Let β be the single-period discount factor ($0 < \beta < 1$), for $n = 1, 2, \ldots$ let $H_n \equiv (x_1, w_1, b_1, s_0, y_0, D_1, \ldots, x_{n-1}, w_{n-1}, b_{n-1}, s_{n-1}, y_{n-1}, D_{n-1}, x_n, w_n)$, and let

$$B = \sum_{n=1}^{\infty} \beta^{n-1} v_n \tag{8}$$

denote the present value of the dividends net of capital subscriptions. We treat the stockholders, who may be diverse, as a representative agent. A *policy* is a nonanticipative rule for choosing y_1, s_1, b_1, y_2, s_2, $b_2, \ldots$. That is, a policy is a rule that, for each n, chooses y_n, s_n, and b_n as a function of H_n. An *optimal* policy maximizes $E[B \mid x_1 = x, w_1 = w]$ for each $(x, w) \in \Re^2$. The goal is to characterize an optimal policy.

Consider a firm that perpetually salts away profits, so working capital tends to grow unboundedly large. It could be argued that the expected value of (8) would be a poor proxy for the value of that firm. In general, even in a firm without long-term debt, there is a distinction between the firm's value to the shareholders and its overall value. However, in this optimization model, a policy is suboptimal if it intends *not* to distribute as much as possible to the shareholders. Thus, we use (8) as a proxy for the value of the firm.

Our model, like all models, suppresses many details, and the inclusion of some of those details, such as shareholder income taxes, would complicate the analysis and affect the results. However, four kinds of additional details could be inserted in the model at the expense of expository clarity. These details would complicate the exposition but would not cause any significant change in the results. First, the dividend decision would be made less frequently than the borrowing decision, and the borrowing decision would be made less often than the production quantity decision. That is, borrowing decisions would be made only in periods 1, $1+r_b$, $1+2r_b, \ldots,$ and dividend decisions would be made only in periods 1, $1+r_d$, $1+2r_d, \ldots,$ where r_b and r_d are positive integers. Second, the right-hand side of (3) would include a credit for interest earned on retained earnings. Third, instead of the borrowing interest rate, ρ, being constant, the rate in period n would be a random variable whose distribution depends on state and decision variables. Similarly, the discount factor in period n would be the n-fold product of random variables whose distributions depend on state and decision variables. See Babich and Sobel (2004) for an analysis of a model with such dependencies. Similarly, our assumption that loan interest is prepaid at the beginning of a period could be replaced by the assumption that it is repaid at the end of the period. All of our results would remain valid, except in §4.2 where straightforward changes would be necessary. Fourth, the representative shareholder's attitude toward risk could be included in the model with an exponential utility function. This would slightly complicate the formulas in §4, but would preserve the paper's qualitative results.

Similarly, some of the model details could be changed without affecting the qualitative properties of the model, hence without altering the structure of an optimal policy. Some of these inconsequential changes include letting excess demand be lost instead of backordered, changing the time at which the revenue for backordered demand is received, imposing an upper bound on the inventory level, and introducing a delay between placing a production order and receiving the ordered goods.

3. Problem Simplification and Analysis

In this section we show that the problem has a more compact structure and a simpler optimal policy than might first appear. Although there seems to be two state variables (x_n and w_n) and three decision variables (b_n, s_n, and y_n), we reduce the problem to one with only a single state variable (x_n) and two decision variables (s_n and y_n), and its solution can be specified almost explicitly.

From (1), (2), and (4), $v_n = w_n + (r-\gamma)x_n^- - p(w_n) - s_n - cy_n + cx_n - \rho b_n$ and $x_n = y_{n-1} - D_{n-1}$ ($n > 1$). Substituting in (8) and rearranging terms yields

$$\begin{aligned} B = {} & (r-\gamma)x_1^- + cx_1 - \sum_{n=1}^{\infty} \beta^n c D_n \\ & + \sum_{n=1}^{\infty} \beta^{n-1}\big[w_n + \beta(r-\gamma)(y_n - D_n)^- - p(w_n) - s_n \\ & \qquad - (1-\beta)cy_n - \rho b_n\big]. \end{aligned} \tag{9}$$

Inserting (5) and $(y_n - D_n)^- = (y_n - D_n)^+ - y_n + D_n$ and rearranging terms produces

$$\begin{aligned} B = {} & (r-\gamma)x_1^- + cx_1 + w_1 - p(w_1) + (r-\gamma-c)\sum_{n=1}^{\infty} \beta^n D_n \\ & + \sum_{n=1}^{\infty} \beta^{n-1}\big\{-(1-\beta)s_n - [\beta(r-\gamma) + (1-\beta)c]y_n \\ & \qquad + \beta(r-\gamma)(y_n - D_n)^+ + \beta g(y_n, D_n) \\ & \qquad - \beta p[s_n + g(y_n, D_n)] - \rho b_n\big\}. \end{aligned} \tag{10}$$

For $(b, s, y) \in \Re^3$, let

$$\begin{aligned} K(b, s, y) = {} & -(1-\beta)s - [\beta(r-\gamma) + (1-\beta)c]y \\ & + \beta E\big\{(r-\gamma)(y-D)^+ + g(y, D) \\ & \qquad - p[s + g(y, D)]\big\} - \rho b. \end{aligned} \tag{11}$$

So

$$E(B) = (r-\gamma)x_1^- + cx_1 + w_1 - p(w_1) + (r-\gamma-c)E\left(\sum_{n=1}^{\infty}\beta^n D_n\right) + E\left(\sum_{n=1}^{\infty}\beta^{n-1}K(b_n, s_n, y_n)\right).$$

A policy maximizes $E(B \mid H_1)$ if and only if it

$$\text{maximizes} \quad E(B \mid H_1) - \left\{(r-\gamma)x_1^- + x_1 + w_1 - p(w_1) - \frac{\beta(\beta r - c)}{1-\beta}E(D)\right\},$$

so we utilize (10) and (11) and optimize the following criterion:

$$E\left[\sum_{n=1}^{\infty}\beta^{n-1}K(b_n, s_n, y_n)\right]. \tag{12}$$

A comparison of (11) and (12) with (9) reveals that $K(\cdot)$, the new single-period objective, does not depend on w_n or b_n. This suggests the possibility (which is confirmed in the following subsection) of reducing the dimensionality of a dynamic program that corresponds to (9). It is less apparent, but confirmed in §3.3, that the optimization of $K(\cdot)$ yields a myopic optimal policy in (8).

3.1. Reduction of Dimensionality

The assumption that $r > c+\gamma$ implies that it is always optimal to meet the backordered demand, i.e., $y_n \ge 0$. So *imposing* the constraint $y_n \ge 0$ for all n would be redundant. Therefore, the constraints on the decision variables are

$$y_n \ge x_n, \quad b_n + s_n \ge 0, \quad \text{and} \quad b_n \ge 0. \tag{13}$$

Thus, the optimization of the expected value of (8) subject to (6), (7), and $y_n \ge 0$ in each period n is equivalent to maximizing (12) subject to (13). However, a straightforward dynamic program for the former problem has a state consisting of a pair of scalars, whereas the latter has a single scalar state variable, namely, the inventory level. This reduces the computational effort to obtain numerical solutions, simplifies the characterization of an optimal policy, and justifies the following statement.

Proposition 1. *The optimization of the expected value of* (8) *subject to* (6) *and* (7) *corresponds to the following dynamic program*:

$$\psi(x) = \sup_{b,s,y}\{J(b,s,y)\colon y \ge x, b \ge 0, b+s \ge 0\}, \tag{14}$$

where

$$J(b,s,y) = K(b,s,y) + \beta E[\psi(y-D)]. \tag{15}$$

A finite-horizon recursion that corresponds to (14) and (15) is $\psi_{N+1}(\cdot) \equiv 0$ and

$$\psi_n(x) = \max_{b,s,y}\{J_n(b,s,y)\colon y \ge x, b \ge 0, b+s \ge 0\}, \tag{16}$$

where

$$J_n(b,s,y) = K(b,s,y) + \beta E[\psi_{n+1}(y-D)] \tag{17}$$

for each $n = 1, 2, \ldots, N$ and $x \in \Re$. Let $b_n(x)$, $s_n(x)$, and $y_n(x)$ be optimal values of b, s, and y, respectively, in (16). We deduce properties of (14) and (15) via the finite-horizon approximation ψ_n because ψ_1 converges pointwise to ψ as $N \to \infty$, and ψ inherits the essential properties of ψ_1.

3.2. Concavity

The following result gives conditions that imply that the marginal value of inventory increases as the planning horizon lengthens. This is an intuitive property of many dynamic concave resource allocation models (Mendelssohn and Sobel 1980). If the horizon is longer, then there is greater opportunity to make productive use of an additional unit of resource. Here and throughout the paper, "decrease" and "increase" are used in the weak sense.

Proposition 2. *Suppose that $p(\cdot)$ is a decreasing convex function on $\Re$ and $g(\cdot, d)$ is a concave function on $\Re$ for each $d \in \Re_+$.*

1. *The value function in* (16), *$\psi_n(\cdot)$, is a concave function on $\Re$ and $J_n(\cdot,\cdot,\cdot)$ is a concave function on $\Re^3$ for each n.*
2. *Let $\psi_n'(x)$ be the right-hand derivative of $\psi_n(x)$; then $\psi_n'(x) \le \psi_{n+1}'(x)$ for each n and x.*

Proof. For part 1, start with $\psi_{N+1}(\cdot) \equiv 0$ and perform an induction on n in (16) and (17). Part 2 can be established with lattice programming, e.g., Heyman and Sobel (2003, p. 398). □

Concavity of the dynamic program value function leads to sensitivity analysis results for the effects of changing the demand distribution. Let $s_n(x, F)$ and $y_n(x, F)$ make explicit the dependence of $s_n(x)$ and $y_n(x)$ on F, the distribution function of demand, D.

Corollary 1. *Under the assumptions of Proposition* 2, *if $F_1(a) \ge F_2(a)$ for all $a \in \Re$, then $y_n(x, F_1) \le y_n(x, F_2)$ and $s_n(x, F_1) \ge s_n(x, F_2)$ for all $x \in \Re$.*

Proof. See Veinott (1965). □

The partial ordering of demand in Corollary 1 is first-order stochastic dominance. So stochastically greater demand leads to higher product stock levels and lower cash stock levels. The former effect is intuitive. The rationale for lower cash stock levels is that higher demand yields greater end-of-period revenue, so the firm can accept a lower cash level at the beginning of the period. Also, Corollary 1 can be exploited in an algorithm. One uses the bounds to limit the search space at each iteration for a sequence of distribution functions that begins with a unit step function that jumps at zero and ends with the actual distribution function of demand.

We assume for each n and x that the maximum on the right-hand side of (16) is achieved, say, at $(b, s, y) = (b_n(x), s_n(x), y_n(x))$. From (2), $v_n = w_n + (r-\gamma)x_n^- - p(w_n) + cx_n - cy_n(x_n) - s_n(x_n) - \rho b_n(x_n)$. If the inventory level is x and the amount of retained earnings is w, the optimal dividend in period n is

$$v_n(x, w) = w + (r-\gamma)x^- - p(w) + cx - cy_n(x) - s_n(x) - \rho b_n(x). \tag{18}$$

Proposition 3. *Under the assumptions of Proposition 2, for each n, $y_n(x)$ is increasing and $z_n(x) = y_n(x) - x$ is decreasing with respect to $x \in \Re$. If $p(\cdot)$ is decreasing on $\Re$, then $v_n(x, \cdot)$ is increasing on $\Re$ for each $x \in \Re$.*

Proof. Adapt Corollary 8-5b in Heyman and Sobel (2004, p. 383). □

The monotonicity of the target inventory level and the production/purchase quantity is an attribute of optimal policies in many production/inventory models. It reflects the substitutability of current inventory and additional goods that are procured or produced. The monotonicity of the dividend originates from the fact that an increase in the "cash on hand," w, can only ease the liquidity constraint. If there is an increment in w, then there is at least a small increase in the dividend that would not preclude the selection of the order-up-to inventory level (y) and residual retained earnings (s) that were optimal at the original level of w.

It is convenient to define

$$G(y) = \sup_{b,s}\{K(b, s, y)\colon b \ge 0, b+s \ge 0\}, \tag{19}$$

and rewrite (14) and (15) as

$$\psi(x) = \sup_{y}\{G(y) + \beta E[\psi(y-D)]\colon y \ge 0, y \ge x,\}. \tag{20}$$

This dynamic program corresponds to the dynamic newsvendor model and, as a result, there is an optimal base-stock level policy for inventory replenishment. More generally, there is a myopic optimum.

We do *not* need the convexity and concavity assumptions in Proposition 2 for any of the remaining propositions in this section.

3.3. A Myopic Optimum

Let (b^*, s^*, y^*) maximize $K(b, s, y)$ subject to $b \ge 0$, $b + s \ge 0$, and $y \ge 0$:

$$K(b^*, s^*, y^*) = \sup_{b,s,y}\{K(b, s, y)\colon b \ge 0, b+s \ge 0, y \ge 0\}. \tag{21}$$

Under reasonable assumptions (see Proposition 2), this numerically easy nonlinear programming problem is a concave maximization problem with three variables and three polyhedral constraints. Let L denote the maximum of (12) subject to (13). Then,

$$L \le \frac{K(b^*, s^*, y^*)}{1-\beta}$$

with equality if $(b_n, s_n, y_n) = (b^*, s^*, y^*)$ is feasible in (12) (i.e., satisfies (13)) for all n. A simple condition is sufficient for feasibility. If $x_n \le y_n = y^*$ for some n, then $(b_n, s_n, y_n) = (b^*, s^*, y^*)$ is feasible because $b^* \ge 0$, $b^* + s^* \ge 0$, and $y^* \ge 0$ from (21). If $y_n = y^*$, then $x_{n+1} = y_n - D_n = y^* - D_n \le y^*$; so $y_{n+1} = y^*$ is feasible. Therefore, $x_1 \le y^*$ permits $(b_n, s_n, y_n) = (b^*, s^*, y^*)$ for all n. This argument yields the next result that is most significant if $x_1 \le y^*$.

Proposition 4. *If $x_k \le y^*$ for some k then $(b_n, s_n, y_n) = (b^*, s^*, y^*)$ for all $n \ge k$ is optimal.*

So $x_1 \le y^*$ permits $b_n = b^*$, $s_n = s^*$, and $y_n = y^*$ for all n. That is, if the initial inventory of goods is not excessive, then an optimal decision rule is determined by three scalars.

Let $\Gamma(s, y)$ be the set of values of demand D_n that do *not* precipitate default in period $n+1$ if $s_n = s$ and $y_n = y$, namely, $\Gamma(s, y) = \{d\colon s + g(y, d) > 0\}$. Let $q(s, y)$ be the probability that default does *not* occur in period $n+1$ if $s_n = s$ and $y_n = y$, i.e., $q(s, y) = P\{D \in \Gamma(s, y)\}$. When $b_n = b^*$, $s_n = s^*$, and $y_n = y^*$ for all n, each period the firm is exposed to the hazard that default will occur at the end of the period. Therefore, the intervals between successive defaults are independent and identically distributed geometric random variables with expected value $[1 - q(s^*, y^*)]^{-1}$.

Appendix A provides a proof that successive production quantities and dividends comprise a sequence of independent and identically distributed random vectors, and the intervals between defaults have a geometric distribution.

Proposition 5. *If $b_n = b^*$, $s_n = s^*$, and $y_n = y^*$ for all n, then* (i) *$(v_2, z_2), (v_3, z_3), \ldots$ are independent and identically distributed random vectors with the same joint distribution as the random vector*

$$\big((r-\gamma)(y^* - D)^- + g(y^*, D) - cD - p(s^* + g(y^*, D)) - \rho b^*, D\big),$$

and (ii) *the intervals between successive defaults are independent and identically distributed geometric random variables with parameter $1 - q(s^*, y^*)$.*

Here, the myopic policy (which is base-stock) stipulates that the firm produces just enough to raise the product stock to a target level y^* and issues dividends to bring the cash stock to a target level s^*.

Therefore, production just offsets the consumption of goods or demand, and the realized dividend equals the revenue net of production/inventory costs, default penalty, and loan interest payment. It is well known that production exactly offsets the most recent demand in a dynamic newsvendor model without a capacity constraint. However, the joint distribution of production and dividend and the probability distribution of the dividend are new results. A consequence of Proposition 5 is that each period's dividend would be a reflection of the previous period's actual financial performance (*not* its expected value) that depends on operational decisions (y^*), operations structure ($g(\cdot,\cdot)$), market characteristics (r), market events (D), and financial characteristics ($p(\cdot)$ and ρ).

We assume for each x that the maximum on the right-hand side of (14) is achieved, say, at $(b,s,y)=(b(x),s(x),y(x))$. Of course, if $x \le y^*$, then Proposition 4 implies that (b^*,s^*,y^*) achieves the maximum. It follows from definition (2) of s_n that in a period with inventory level x and retained earnings w, the optimal dividend is

$$v(x,w) = w+(r-\gamma)x^- - p(w)+cx-cy(x) - s(x)-\rho b(x). \tag{22}$$

Proposition 6. *Suppose that $x \le y^*$ and $w \ge 0$.* (a) *The optimal dividend function $v(\cdot,w)$ is differentiable, respectively, on $(-\infty,0)$ where its derivative is $-(r-\gamma-c)$ and on $(0,y^*)$ where its derivative is c.* (b) *If $p(w)=0$ when $w\ge 0$, then $v(x,\cdot)$ is differentiable on $[0,\infty)$ where its derivative is* 1.

Proof. If $x\le y^*$, then (22) becomes $v(x,w)=w-p(w)+(r-\gamma)x^-+cx-cy^*-s^*-\rho b^*$. If $w\ge 0$, then $p(w)=0$. □

Therefore, if the inventory level changes from x to $x+\Delta_I \le y^*$ and retained earnings is $w+\Delta_E\ge 0$ instead of $w\ge 0$, the dividend change is $\Delta_E + c\Delta_I$.

3.4. Pecking-Order Optimality

Now we show that it is optimal to borrow the smallest amount that satisfies the liquidity constraint; i.e., $b(x)=(-s(x))^+$ for all x is optimal. This is consistent with the well-known "pecking order" in elementary finance that advises a firm to use internal sources of funding before it uses external sources. More recently, "pecking-order theory" refers to asymmetry of information in the sense that financial decision makers in the firm have private information to which outsiders lack access. Therefore, outsiders, assuming that insiders behave rationally when they finance the firm in one way rather than another, can infer the direction in which stock is mispriced.

Appendix B provides a proof of the following result.

Proposition 7. *In* (14), $b(x)=(-s(x))^+$ *is optimal for all* $x\in\Re$.

4. Characterization of the Optimal Policy

This section characterizes the myopic optimal policy and provides an explicit solution for the following case in which the default penalty and the gross profit from sales (net of inventory costs) are piecewise linear functions:

$$p(x)=(p_1x)^- \quad\text{and}\quad g(y,d)=r\min\{y,d\}-h(y-d)^+. \tag{23}$$

It is convenient to rewrite the second equation as

$$g(y,d)=ry-(r+h)(y-d)^+. \tag{24}$$

Here, $p_1>0$ is the rate of default penalty. Notice that $p(\cdot)$ is decreasing and convex and $g(\cdot,d)$ is concave. These properties are essential in this section, which has the following organization. We exploit piecewise linearity to specialize (11), and then we examine the case in which borrowing is allowed (s is unconstrained in sign). Then we quantify the risk of default on the lender from which the short-term loans are borrowed.

The transformed single-period payoff (11), which uses Proposition 7, is concave and continuous:

$$\begin{aligned}
K(b,s,y) &= -(1-\beta)s-[\beta(r-\gamma)+(1-\beta)c]y\\
&\quad+\beta E\big[(r-\gamma)(y-D)^+ + ry-(r+h)(y-D)^+\\
&\qquad -p_1(s+ry-(r+h)(y-D)^+)^-\big]-\rho b\\
&= -(1-\beta)s+[\beta\gamma-(1-\beta)c]y-\rho b\\
&\quad -\beta E\big[(h+\gamma)(y-D)^+\\
&\qquad +p_1(s+ry-(r+h)(y-D)^+)^-\big]\\
&= \mathcal{K}(s,y)\\
&\equiv -(1-\beta)s+[\beta\gamma-(1-\beta)c]y-\rho(-s)^+\\
&\quad -\beta E\big[(h+\gamma)(y-D)^+\\
&\qquad +p_1(s+ry-(r+h)(y-D)^+)^-\big].
\end{aligned}\tag{25}$$

The challenging component of (25) is the expected value of the default penalty. We expand and discuss this component in Appendix C.

The solution in Appendix D to

$$K(b^*,s^*,y^*)=\sup\{K(b,s,y)\colon b\ge 0, b+s\ge 0, y\ge 0\} \tag{26}$$

leads to the major result in this section, which uses several definitions. Define

$$y_0^* = F^{-1}\bigg(\frac{\beta\gamma-(1-\beta)(c+h)}{\beta(h+\gamma)}\bigg),\qquad \hat{y}_0 = F^{-1}\bigg(\frac{1-\beta}{\beta p_1}\bigg); \tag{27}$$

$$y_2^* = F^{-1}\left(\frac{\beta\gamma - (1-\beta)(c+h) + \rho h}{\beta(h+\gamma)}\right),$$
$$\hat{y}_2 = F^{-1}\left(\frac{1-\beta-\rho}{\beta p_1}\right), \tag{28}$$

and let y_1^* be the solution to

$$\beta\gamma - (1-\beta)c - \beta(\gamma+h)F(y_1^*) - \beta p_1 h F\left(\frac{hy_1^*}{r+h}\right) = 0. \tag{29}$$

Note that y_1^* exists and is nonnegative if $\beta\gamma \ge (1-\beta)c$, and y_2^* and $\hat{y}_2$ are well defined if

$$\frac{1-\beta-\rho}{\beta p_1} \le 1. \tag{30}$$

Furthermore, because $F(\cdot)$ is a monotone increasing function, $y_0^* \le y_1^* \le y_2^*$ and $\hat{y}_0 \ge \hat{y}_2$. Let y_3^* be a solution to

$$F(y_3^*) = \frac{\beta\gamma - (1-\beta)c + \beta p_1 r}{\beta(\gamma+h) + \beta p_1(r+h)}. \tag{31}$$

Note that Equations (27)–(31) determine two threshold numbers for demand for each case i, where y_i^* is the optimal total goods available and $\hat{y}_i$ is the optimal no-default limit; namely, the optimal policy (s^*, y^*) is determined so that $(hy^* - s^*)/(r+h) = \hat{y}_i$ or, equivalently, the optimal working capital s^* is determined so that $s^* = hy^* - (r+h)\hat{y}_i$ for given y^*. That is, given the optimal policy, demand is satisfied if and only if $D \le y_i^*$, and there is no default if and only if $D \ge \hat{y}_i$.

Proposition 8. *Suppose that there are piecewise linear default costs and the firm may borrow if it wishes, so the objective function is* (25), *and that* $\beta\gamma \ge (1-\beta)c$.

1. *Suppose that* (30) *holds with a strict inequality.*

(a) *If*

$$hy_0^* - (r+h)\hat{y}_0 \ge 0, \tag{32}$$

then $y^* = y_0^*$ *and* $s^* = s_0^* = hy_0^* - (r+h)\hat{y}_0$, *where* y_0^* *and* $\hat{y}_0$ *are defined in* (27).

(b) *If*

$$hy_0^* - (r+h)\hat{y}_0 < 0 \le hy_2^* - (r+h)\hat{y}_2, \tag{33}$$

then $y^* = y_1^*$ *and* $s^* = s_1^* = 0$, *where* y_1^* *is defined in* (29).

(c) *If*

$$hy_2^* - (r+h)\hat{y}_2 < 0, \tag{34}$$

then $y^* = y_2^*$, $s^* = s_2^*$ *and* $b^* = -s^*$, *where* $s_2^* = hy_2^* - (r+h)\hat{y}_2$ *and* y_2^* *and* $\hat{y}_2$ *are defined in* (28).

(d) (i) *The optimal goods supply* y^* *increases when* γ, β, *or* ρ *increases, or when* h, c, *or* p_1 *decreases.*

(ii) *The optimal working capital* s^* *increases when* γ, h, β, p_1, *or* ρ *increases, or when* c *decreases.*

(iii) *The optimal amount to borrow* b^* *increases when* c *increases or when* γ, h, β, p_1, *or* ρ *decreases.*

2. *If*

$$\frac{1-\beta-\rho}{\beta p_1} \ge 1, \tag{35}$$

then $y^* = y_3^*$. *If* (35) *holds as an equality, then* $s^* = -ry^*$ *and* $b^* = ry^*$. *If* (35) *holds as a strict inequality, then* $s^* = -\infty$ *and* $b^* = \infty$.

Table 1 (y^*, s^*, b^*) **in Proposition 8**

		y^*	s^*	b^*
$1-\beta-\rho < \beta p_1$	(a) $hy_0^* - (r+h)\hat{y}_0 \ge 0$	y_0^*	s_0^*	0
	(b) $hy_0^* - (r+h)\hat{y}_0 < 0$ $0 \le hy_2^* - (r+h)\hat{y}_2$	y_1^*	$s_1^* = 0$	0
	(c) $hy_2^* - (r+h)\hat{y}_2 < 0$	y_2^*	s_2^*	$-s_2^*$
$1-\beta-\rho = \beta p_1$		y_3^*	$-ry_3^*$	ry_3^*
$1-\beta-\rho > \beta p_1$		y_3^*	$-\infty$	∞

Table 1 parses the cases in this proposition. The optimal production/inventory policy has the same structure as in a dynamic newsvendor model of a profit-maximizing firm without cash constraints, namely, a base-stock policy. That is, the production quantity in each period is so determined that the total available goods is maintained at the level y^*. However, the optimal base-stock level for a dividend-maximizing firm with cash constraints would generally be different from the one prescribed by a standard inventory model because overproduction not only implies higher holding costs or lower backorder costs but also affects the risk and the cost of default. In addition, an optimal base-stock policy for working capital, s^*, in conjunction with y^*, balances current dividends and future retained earnings. Also, if short-term borrowing is optimal, the firm should borrow just enough to bring the total working capital level, $b^* + s^*$, to zero.

If we were to increase the short-term loan b by one dollar, the marginal benefit would be $1-\beta-\rho$ and the marginal cost would be βp_1 if default were to occur next period. Thus, if $1-\beta-\rho \ge \beta p_1$, namely, (35), the net marginal value of a one dollar short-term loan is positive and, hence, the firm always has an incentive to borrow. This results in infinite borrowing in part 2 of the proposition. Otherwise, the firm does not borrow or borrows only a finite amount as in cases (a)–(c) in part 1.

For example, suppose that the newsvendor formulas in (28) determine the optimal base-stock policy (s^*, y^*). This is the case in 1(c) when short-term borrowing is optimal. The second formula in (28) sets the optimal cash stock level s^* (given the goods stock level y_2^*), or equivalently, it sets the optimal no-default limit for demand, namely, the lower bound $(hy^* - s^*)/(r+h)$. If the demand is lower than the limit, then default will occur, so this might be thought of as the case of "overage" in a newsvendor model. One dollar increase in s (one dollar decrease in b) will reduce both dividend and short-term loan (an interest payment of ρ) by one dollar in the current period while increasing the retained earnings by one dollar in the next period, regardless of whether or not default occurs. If default occurs, then one dollar increase in s will benefit the firm by decreasing the default cost by p_1 (in the next period). Thus, the "overage cost" is $C_o = \beta p_1 - (1-\beta-\rho)$, the "underage cost" is $C_u = 1-\beta-\rho$, and the optimal no-default limits for demand should be set so that the default probability equals the "critical ratio":

$$\frac{C_u}{C_u + C_o} = \frac{1-\beta-\rho}{\beta p_1}.$$

The first equation in (28) sets the optimal product stock level y^* in a similar fashion. Notice that a unit increase in y^* would increase the default cost by $p_1 h$ if the demand is lower than $F^{-1}((hy^* - s^*)/(r+h))$. This is the effect in addition to the usual trade-offs in a standard inventory model. Because the probability of default is already set by the critical ratio, this effect results in a marginal expected default cost $(1-\beta-\rho)h$ in the first critical ratio formula regardless of demand realizations.

Note that if y^* is y_0^*, y_1^*, or y_2^*, then $s^* + ry^* > 0$ implies that the optimal loan is less than the value of the product base-stock ($b^* = (-s^*)^+ < ry^*$), and default does not occur even when inventory is at some positive level. All the sensitivity analysis results are quite intuitive. Obviously, when the loan rate ρ is high, the firm will set the cash stock level high to reduce the need to borrow and the probability of default. It is interesting to observe that when ρ is high, the firm will also set the goods stock level high because a lower probability of default induces a higher goods stock level. A higher γ, h, β, or p_1, or a lower c implies a higher optimal cash stock level, s^*; hence, a smaller loan is needed to bring the total working capital to zero.

However, when $y^* = y_3^*$, some sensitivity analysis results do not hold. When $y^* = y_3^*$ and (35) holds in strict inequality, $s^* + ry^* < 0$ implies that the optimal loan is more than the value of the product base-stock ($b^* = -s^* > ry^*$) and default will occur with probability one. An increase in y will have no effect on the default probability. Consequently, all results in part 2 of Proposition 8 remain true except one, which will change to y^* is increasing in p_1 for $y^* = y_3^*$. However, lenders in reality are unlikely to grant a larger loan than the value of the product base stock.

Also, Corollary 1 has the following implication for the sensitivity analysis of demand when the assumptions of Proposition 2 are valid. If demand increases in the sense of first-order stochastic dominance, then y^* increases, s^* decreases, and b^* increases.

Lender's Risk. This paper focuses on the manufacturer and does not model the ramifications of default on the lender from which the short-term loans are borrowed. In particular, we do not examine whether the lender is fully repaid following a default and, if so, by whom and with what delay. Indeed, there is often more than one borrower and, especially in the United States, seniority is critical. Our abstraction rules out the importance of seniority conditions to borrowers. The inclusion of this complication would require a separate paper. Nevertheless, we can reach some conclusions regarding the lender's risk when the manufacturer behaves optimally.

Let A^* be the amount of $b^* > 0$ that the manufacturer directly and immediately repays in the same period when b^* is borrowed, and recall that F denotes the distribution function of demand, D. We assume that the loan covenant obliges the manufacturer to select y and s so that default does not occur if demand is sufficiently high to sell all of the inventory. That is, $s + ry - (r+h)(y-D)^+ \geq 0$ if $D \geq y$, which corresponds to $s + ry \geq 0$.

Appendix E provides a proof of the following result.

Proposition 9. *Assume that $s^* + ry^* \geq 0$ and that F is continuous at $(hy^* - s^*)/(r+h)$. Then in each period the probability of full and on-time repayment by the manufacturer is*

$$q(s^*, y^*) = P\{A^* = b^*\} = 1 - F\left(\frac{hy^* - s^*}{r+h}\right), \tag{36}$$

the mean amount repaid on time by the manufacturer is

$$E(A^*) = b^* - (hy^* - s^*)F\left(\frac{hy^* - s^*}{r+h}\right) + (r+h)\int_0^{(hy^*-s^*)/(r+h)} x\,dF(x), \tag{37}$$

and the mean fraction of the loan that is repaid on time by the manufacturer is

$$E(A^*)/b^* = 1 - \frac{hy^* - s^*}{b^*}F\left(\frac{hy^* - s^*}{r+h}\right) + \frac{r+h}{b^*}\int_0^{(hy^*-s^*)/(r+h)} x\,dF(x). \tag{38}$$

5. Bankruptcy Variations

Thus far, bankruptcy in the model is consistent with Chapter 11 and is accompanied by substantial costs of reorganizing the firm. In the first subsection, we consider an extreme alternative version of bankruptcy that is consistent with Chapter 7, namely, dissolution of the firm. In the second subsection, we insert the option for the firm to declare bankruptcy even if it is not forced into it.

5.1. Wipeout Bankruptcy

Here we maximize the expected present value of dividends prior to dissolution. There are three important insights in this subsection. First, key features of an optimal policy with reorganization bankruptcy remain valid with wipeout bankruptcy. These are the myopic optimum property and Propositions 1, 4, 5, and 7 remain valid (with minor changes). Second, the firm should be more shortsighted because its discount factor is reduced from β to $\beta q(s, y)$ (recall that $q(s, y)$ denotes the probability that bankruptcy does *not* occur in period $n+1$ if $s_n = s$ and $y_n = y$). In effect, the firm's choices influence its time preference, and conversely, the time preference also influences the firm's choices. Third, the lifetime of the optimally operated firm has a geometric probability distribution.

Let T denote the lifetime of the firm, so we let $T = \sup\{n\colon\, w_n > 0\}$ and maximize $E(B)$, where $B = \sum_{n=1}^{T} \beta^{n-1} v_n$. The same substitutions that lead from (8) to (10) yield

$$\begin{aligned} B = \sum_{n=1}^{T-1} \beta^{n-1}\{&\beta(r-\gamma)(y_n - D_n)^+ + \beta g(y_n, D_n) \\ &+ \beta(r-\gamma-c)D_n - (1-\beta)s_n \\ &- [(1-\beta)c + \beta(r-\gamma)]y_n - \rho b_n\} \\ -\beta^{T-1}&(s_T + cy_T + \rho b_T) + (r-\gamma)x_1^- + cx_1 + w_1. \end{aligned} \tag{39}$$

Therefore, an optimal coordinated operating and financial policy maximizes $E(B_0)$, where

$$\begin{aligned} E(B_0) = E\Bigg(\sum_{n=1}^{T-1} \beta^{n-1}\big[&\beta(r-\gamma)(y_n - D_n)^+ + \beta g(y_n, D_n) \\ &+ \beta(r-\gamma-c)D_n - (1-\beta)s_n \\ &- [(1-\beta)c + \beta(r-\gamma)]y_n\big] - \rho b_N\Bigg) \\ &- \beta^{T-1}(s_T + cy_T + \rho b_T). \end{aligned} \tag{40}$$

This model can be regarded as a generalization of an inventory process with a stopping time. Therefore, the results in Lovejoy (1992) yield bounds on the error that would result from using the policy identified in §3 rather than a policy that optimizes (40). However, we avoid the need for an approximation by showing that the model with wipeout bankruptcy satisfies the condition in Sobel (1981) and, therefore, has an optimal myopic solution.

Recall the definitions following Proposition 4 of $\Gamma(s, y)$ as the set of values of demand D_n that do *not* precipitate default in period $n+1$ if $s_n = s$ and $y_n = y$, and of $q(s, y)$ as the probability that default does *not* occur in period $n+1$ if $s_n = s$ and $y_n = y$. Because $s + ry$ is the level of the retained earnings when demand matches supply, it is the maximum amount of earnings the firm can achieve, and each unit of inventory will reduce that amount by $r + h$. It follows that if $s + ry < 0$, then the firm will default with certainty regardless of demand realizations; that is, $\Gamma(s, y) = \varnothing$ and $q(s, y) = 0$. If $s + ry = 0$, then any amount of inventory will result in default; so default does not occur if and only if supply matches demand perfectly. In this case, $\Gamma(s, y) = \{y\}$. Finally, if $s + ry > 0$, then the firm will remain solvent as long as the inventory is not too high, that is, if the demand is higher than $(hy - s)/(r + h)$. In this case, $\Gamma(s, y) = \{d\colon\, d > (hy - s)/(r + h)\}$ and $q(s, y) = 1 - F[(hy - s)/(r + h)]$. If demand is lower than the lower limit in $\Gamma(s, y)$, then default is precipitated by insufficient revenue and high holding costs.

Using this notation,

$$\begin{aligned} E(B_0) &= \sum_{n=1}^{\infty} \beta^{n-1}\big(-P[T=n]E[s_T + cy_T + \rho b_T \mid T=n] \\ &\qquad + P[T>n]E\big[\beta(r-\gamma)(y_n - D_n)^+ \\ &\qquad + \beta g(y_n, D_n) + \beta(\beta(r-\gamma)-c)D_n \\ &\qquad - (1-\beta)(s_n + cy_n) - \rho b_n \mid T>n\big]\big) \\ &= \sum_{n=1}^{\infty} \beta^{n-1}\Bigg\{-\prod_{k=1}^{n-1} q(s_k, y_k)(1-q(s_n, y_n))(s_n + cy_n + \rho b_n) \\ &\qquad + \prod_{k=1}^{n} q(s_k, y_k)E\big[\beta(r-\gamma)(y_n - D_n)^+ \\ &\qquad + \beta g(y_n, D_n) + \beta(\beta(r-\gamma)-c)D_n \\ &\qquad - (1-\beta)(s_n + cy_n) - \rho b_n \mid T>n\big]\Bigg\} \\ &= \sum_{n=1}^{\infty} \beta^{n-1} \prod_{k=1}^{n-1} q(s_k, y_k) K_0(b_n, s_n, y_n), \end{aligned} \tag{41}$$

where

$$\begin{aligned} &K_0(b, s, y) \\ &= q(s, y)E\big[\beta(r-\gamma)(y-D)^+ + \beta g(y, D) \\ &\qquad + \beta(\beta r - c)D - (1-\beta)(s + cy) - \rho b \mid D \in \Gamma(s, y)\big] \\ &\quad - (s + cy + \rho b)(1 - q(s, y)). \end{aligned} \tag{42}$$

When bankruptcy signified reorganization, the objective was (12). The only difference between (41) and (12) is that the single-period discount factor has been reduced from β to $\beta q(s, y)$. It follows that dynamic program (14) remains valid when (15) is replaced with

$$\begin{aligned} J(b, s, y) = K_0&(b, s, y) \\ &+ \beta q(s, y)E\big[\psi(y-D) \mid D \in \Gamma(s, y)\big]. \end{aligned} \tag{43}$$

Therefore, the myopic optimum property and Propositions 1, 4, 5, and 7 remain valid (with minor changes) when bankruptcy signifies dissolution of the firm.

The preceding observation leads to a testable hypothesis, namely, a geometric probability distribution for the lifetime of the firm. Let (s^*, y^*) globally maximize $K_0[(-s)^+, s, y]$ subject to $y \geq 0$. If $x_1 \leq y^*$, it is optimal for $(b_n, s_n, y_n) = ((-s^*)^+, s^*, y^*)$, $n = 1, \ldots, T$. Therefore,

$$P\{T = n\} = [1 - q(s^*, y^*)]q(s^*, y^*)^{n-1}.$$

Proposition 10. *If $x_1 \leq y^*$, it is optimal for $(b_n, s_n, y_n) = ((-s^*)^+, s^*, y^*)$, $n = 1, \ldots, T$, and, as a consequence, T has a geometric distribution with parameter $q(s^*, y^*)$.*

5.2. The Option to Declare Bankruptcy

The declaration of bankruptcy in reality is complex because of the minutia of judicial decisions based on the U.S. Bankruptcy Code, the relative seniority of the firm's various financial obligations, the status and liquidity of the markets in which the firm's assets would be disposed, and other considerations. So a serious model of the *decision* to declare bankruptcy should be structured very differently than the models in §2 or §5.1. In this subsection we take a small step in that direction to consider the effects on the coordination of operations and finance in a setting where the option to declare bankruptcy lurks in the background. We certainly do not propose its use to *plan* the declaration of bankruptcy.

Let the decision variable $\delta_n = 0$ if the firm chooses to continue operations in period n (if it has persisted until then), and let $\delta_n = 1$ if it chooses to declare bankruptcy in period n. We *re*define T as the last period in which the firm operates "normally" in the sense that $\delta_n = 0$ and $w_n > 0$, $n = 1, \ldots, T$, and either $w_{T+1} \leq 0$ or $\delta_{T+1} = 1$: $T = \sup\{n\colon \delta_n = 0 \text{ and } w_n > 0\}$.

There are legal constraints in the United States against borrowing to pay dividends in contemplation of bankruptcy. "Looting the till" here would consist of making a large loan, distributing a large (positive) dividend in anticipation of declaring bankruptcy the following period, and "walking away" from the default. To preclude this behavior, we impose an upper bound $U + cx^-$ on the loan. The rationale for this bound is (i) $U + cx^-$ is consistent with the firm's obligation (in the model) to produce at least enough goods in period n to meet its backlogged demand, and (ii) U models a loan covenant that precludes looting the till and walking away.

Let $\mathscr{L}$ be the dollar amount of obligations at bankruptcy that are senior to the equity interests, and recall the notation w_n for the retained earnings at the beginning of period n. If bankruptcy is declared in period n, the shareholders' limited liability implies that they receive $(w_n - \mathscr{L})^+$. Thus, we maximize $E[\sum_{n=1}^{T} \beta^{n-1} v_n + \beta^T (w_n - \mathscr{L})^+]$. For simplicity, we forgo an expansion and rearrangement analogous to (39) and (40) and observe that the optimization here corresponds to a dynamic program in which the state is a pair consisting of the retained earnings and inventory level, and the action is the four-tuple consisting of the decision to declare bankruptcy or not and the amounts of the short-term loan, the residual retained earnings, and the order-up-to inventory level.

The dynamic program is $\psi_{N+1}(\cdot, \cdot) \equiv 0$ and for each $n = 1, 2, \ldots, N$, $x \in \Re$, and $w \in \Re$, $\psi_n(w, x) = 0$ if $w \leq 0$ and

$$\psi_n(w, x) = \max\Big\{(w - \mathscr{L})^+, \max_{b,s,y,w}\{J_n(b, s, y)\colon y \geq (x)^+, b \geq 0, b + s \geq 0, b \leq U + cx^-\}\Big\} \quad \text{if } w > 0, \tag{44}$$

$$J_n(b, s, y, w) = w - p(w) + (r-\gamma)x^- + cx - cy - s - \rho b + \beta q(s, y) E\big(\psi_{n+1}[s + g(y, D), y - D] \mid D \in \Gamma(s, y)\big). \tag{45}$$

The term $(w - \mathscr{L})^+$ is the payoff from electing bankruptcy, and (44) is the expected present value of not electing bankruptcy. The first seven terms on the right-hand side of (45) are the dividend if bankruptcy is not elected, because $v_n = w_n + (r-\gamma)x_n^- - p(w_n) + cx_n - cy_n - s_n - \rho b_n$. The eighth term is the expected present value of continuation, beginning next period; the arguments of ψ_{n+1} are due to $w_{n+1} = s_n + g(y_n, D_n)$ and $x_{n+1} = y_n - D_n$.

This dynamic program can be viewed as an *optimal stopping problem* in which auxiliary decisions (b, s, and y) are made while continuation occurs and there are some *forced stopping states* (all pairs (w, x) where $w \leq 0$). Babich and Sobel (2004) is another optimal stopping problem with auxiliary decisions.

For economy of exposition, in the remainder of this subsection, we assume that $p(\cdot)$ is a decreasing convex function on $\Re$, and $g(\cdot, d)$ is a concave function on $\Re$ for each $d \geq 0$. Then an optimal policy in (44) has the following generalization of the *monotone optimal stopping* property (Derman and Sacks 1960). For each inventory level x, there is a critical level of retained earnings $w_n(x)$ such that it is optimal to declare bankruptcy ($\psi_n(w, x) = (w - \mathscr{L})^+$) at all states (w, x) with $w \leq w_n(x)$. This property stems from an analogue of Proposition 2 that is valid for (44) and (45).

The right-hand side of (45) includes the term $-p(w)$ in which $p(\cdot)$ is generally nonlinear, so w cannot be eliminated from the dynamic program in the manner in which it is eliminated in §3. Similarly, and for the same reason, w cannot be eliminated from $(w - \mathscr{L})^+$ on the right-hand side of (44). Thus, (44) has an additional state variable in comparison to (16) and an optimal policy is not generally myopic (unlike Proposition 4). Nevertheless, an optimal policy has several useful properties.

Proposition 11. *The variant of the model with the option to declare bankruptcy has the following properties*:

1. *The pecking-order property of the model in earlier sections (Proposition* 7*). So it is optimal to let* $b = (-s)^+$ *in* (44).
2. *As in earlier sections, the optimal choice of* s *and* y *in* (44) *is determined by two order-up-to levels. However,* $s_n(w)$ *and* $y_n(w)$ *are functions, rather than scalars as before.*
3. *As consequences, if it is optimal to continue rather than declare bankruptcy, then the production/procurement quantity should be* $z = y - x = y_n(w) - x$ *and the residual retained earnings should be* $s = \min\{s_n(w), U + cx^-\}$. *The residual retained earnings level is* $s_n(w)$ *if it is feasible, i.e., if* $s_n(w) \le U + cx^-$, *and otherwise it is* $U + cx^-$, *which is as close to* $s_n(w)$ *as feasible.*

6. Nonnegative Dividends

Large publicly traded firms cannot ordinarily obtain capital subscriptions from their stockholders who have limited liability. Therefore, in this section, we briefly analyze the model when capital subscriptions are precluded, i.e., with the additional constraint that dividends must be nonnegative. Because we add the constraint $v_n \ge 0$ to the formulation of the model in §2, the objective remains (12) with the constraints (13) augmented with

$$s_n + cy_n + \rho b_n \le w_n + (r - \gamma)x_n^- - p(w_n) + cx_n$$

because $v_n = w_n + (r - \gamma)x_n^- - p(w_n) - s_n - cy_n + cx_n - \rho b_n$. Because this constraint cannot be imposed without the knowledge of both w and x, it follows that both of them must be state variables. That is, one consequence of the constraint is an additional state variable in comparison to (16). The dynamic program for the model in §2 has a scalar state, the inventory level x, which must be augmented now with the amount of retained earnings at the beginning of a period, w. Instead of (16) and (17), the corresponding dynamic program is the following recursion with $\psi_{N+1}(\cdot\,,\cdot) \equiv 0$:

$$\psi_n(w, x) = \max_{b,s,y}\{J_n(b, s, y)\colon y \ge (x)^+, b \ge 0, b+s \ge 0, s + cy + \rho b \le w + (r-\gamma)x^- - p(w) + cx\}, \quad (46)$$

$$J_n(b, s, y) = K(b, s, y) + \beta E(\psi_{n+1}[s + g(y, D), y - D]) \quad (47)$$

for each $n = 1, 2, \ldots, N$, $x \in \Re$, and $w \in \Re$.

Let $b_n(w, x)$, $s_n(w, x)$, and $y_n(w, x)$ be the optimal values of b, s, and y, respectively, in (46). The following result corresponds to Proposition 7 and is proved in the same fashion. Again, it is optimal to borrow the smallest amount that satisfies the liquidity constraint.

Proposition 12. *The loan amount* $b_n(w, x) = (-s_n(w, x))^+$ *is optimal for all* $n = 1, 2, \ldots,$ *and* $(w, x) \in \Re^2$.

The next result is analogous to the concavity property in Proposition 2 and has a similar proof.

Proposition 13. *If* $p(\cdot)$ *is a decreasing convex function on* $\Re$ *and* $g(\cdot, d)$ *is a concave function on* $\Re$ *for each* $d \ge 0$, *then the value function in* (46), $\psi_n(\cdot\,,\cdot)$, *is a concave function on* $\Re^2$ *and* $J_n(\cdot\,,\cdot\,,\cdot)$ *in* (47) *is a concave function on* $\Re^3$ *for each* n.

Even with the nonnegativity of dividends in force, the next result shows that the optimal decision variables remain monotone (see Proposition 3).

Proposition 14. *Under the assumptions of Proposition* 13, *for each* n, $y_n(w, x)$, $z_n(w, x) = x - y_n(w, x)$, $v_n(w, x)$, *and* $s_n(w, x)$ *are increasing with respect to* $w \in \Re$ *and* $x \in \Re$. *So* $b_n(w, x)$ *is a decreasing function of* w *and* x.

Proof. Adapt Theorem 8-4 in Heyman and Sobel (2003, p. 378). □

The following result compares the optimal policy when capital subscriptions are allowable with the optimal policy when dividends are constrained to be nonnegative. Recall the notation $b_n(x)$, $s_n(x)$, and $y_n(x)$ for the optimal amounts of the short-term loan, residual retained earnings, and physical goods base-stock level when the inventory level is x and n periods remain in the planning horizon.

Proposition 15. *The following properties are valid for each* n, x, *and* w *under the assumptions of Proposition* 13.

1. *Comparing the optimal policies with and without capital subscriptions,* $y_n(x, w) \le y_n(x)$ *and* $s_n(x, w) \le s_n(x)$ *(so* $b_n(x, w) \ge b_n(x)$*).*
2. *As* w *grows,* $y_n(w, x) \to y_n(x)$ *and* $s_n(w, x) \to s_n(x)$ *(so* $b_n(w, x) \to b_n(x)$*).*

Proof. Combine the observation that constraint (47) becomes less binding as w grows with an adaptation of Theorem 8-4 in Heyman and Sobel (2003, p. 378). □

Therefore, a firm that optimally coordinates its operational and financial decisions but cannot mandate capital subscriptions has lower inventories and higher short-term loans than its counterpart that may obtain capital subscriptions if it wishes. Therefore, each period the former firm has a higher probability of default than the latter. The latter firm can turn to a capital subscription *or* a short-term loan, whereas the former firm can increase liquidity only with the loan, so its level of residual retained earnings is lower and its short-term loan is higher. Similarly, without recourse to capital subscriptions, the former firm has a lower base-stock level of physical goods because it is less prone to buy or produce goods.

A firm that can mandate only a limited capital subscription lies between one that has no access to capital subscriptions, and one that has unbounded access to them. The insertion of an upper bound on capital subscriptions again results in an additional state variable in the corresponding dynamic program. The structure of an optimal policy is intermediate between that of the model with unbounded access to capital subscriptions and that of the model where dividends must be nonnegative.

7. Testable Hypotheses

In this section we list some of the testable hypotheses that are suggested by formal results in the paper. Some of them, such as the first three, are prompted also by formal results in many other papers. Others, such as the fourth, fifth, and sixth, stem only (so far as the authors are aware) from this paper's formal results.

- Stochastically greater demand is accompanied by higher product stock levels and lower cash stock levels (Corollary 1).
- Higher inventories are accompanied by lower order quantities (Proposition 3).
- The variances of demand and the production quantity are the same (Proposition 5).
- The dividend has the probability distribution specified in Proposition 5.
- The interval between defaults is a geometric random variable (Propositions 5 and 10).
- Dividends increase linearly with inventory, at rate c (Proposition 6).
- Short-term loans are the minimum amount necessary to maintain liquidity (Propositions 7 and 12).
- Inventories increases when γ, β, or ρ increases, or when h, c, or p_1 decreases (Proposition 8).
- Retained earnings increases when γ, h, β, p_1, or ρ increases, or when c decreases (Proposition 8).
- Short-term loans are larger when c increases or when γ, h, β, p_1, or ρ decreases (Proposition 8).
- Market values of firms are concave functions of the levels of inventories and of retained earnings (Propositions 2 and 13).
- Firms that cannot mandate capital subscriptions have lower inventories, make larger short-term loans, and have higher likelihoods of default than their counterparts that may obtain capital subscriptions (Proposition 15).

We note also that as we provide a fully defined dynamic model, it is amenable for use as testable via experimental gaming.

It has been recognized in microeconomic theory that many of the theoretical constructs are difficult to test empirically directly because of both the paucity of clean data and the presence of a host of intervening variables. For this reason some economists have taken to building experimental games based on stripped down economic models to see if, at least in these highly simplified circumstances, the theory is validated. (For example, the players in the experimental game in Huber et al. (2010) select among equilibria even in the absence of exogenous uncertainty.) It may be worthwhile to build experimental games based on the stripped down models in papers such as this one to test whether actual behavior in experimental settings validates the theory.

8. Concluding Remarks

We formulate and analyze a dynamic stochastic model of the coordination of operational and financial decisions with the criterion of the expected present value of the time stream of dividends received by a representative share owner. The model is surprisingly tractable because it has a myopic policy that is optimal and, therefore, susceptible to analysis. So we can contrast the result with inventory policies based on models without financial considerations. Numerical examples in Li et al. (1997, 2009 revision) show that the opportunity cost of detaching the two functions, measured in dividends, can be significant. The examples also quantify the dependence of the opportunity cost on the default penalty. Because most of the paper employs a model of Chapter 11 bankruptcy and admits capital subscriptions, we also explore the effects of Chapter 7 bankruptcy and precluding capital subscriptions. Section 1.2 is a detailed enumeration of the primary insights in the paper.

In the new economics of industrial organization, there has been a tendency to emphasize the game theoretic study of market competition outside of the firm and agency problems caused by asymmetric information within the firm. In operations research, production and inventory systems are often studied in isolation from the other aspects of firm activity, by cutting out the feedback between production and finance or production and marketing (until recently). In the first instance, this ignores financial constraints and in the second instance it prespecifies the nature of the demand structure as a given of the model. In the actual application of formal mathematical models to the policy of a firm, the optimal value of a game theoretic model of competition remains unknown until we can analyze the associated dynamic game. Meanwhile, in contrast, production/inventory models are of value for many practical problems, even taken in isolation. Nevertheless, we observe here that for firms operating with thin budgets in new or otherwise volatile markets, the joining together of the production/inventory problem with the cash flow and financial problems of the firm may be relevant and worthwhile. Furthermore, we have demonstrated that although to do so is somewhat complicated, it is feasible.

Our results were obtained with a finite-horizon model, but they carry over to the analogous infinite-horizon model. Under reasonable assumptions, as $N \to \infty$ the functions in (16) and (17) (and in (46) and (47)) tend to be well-behaved limits that satisfy the obvious infinite-horizon analog of (16) and (17) (and (46) and (47)). If $x_1 \le y^*$, Proposition 5 states that the pairs of dividends and production quantities in successive periods (starting with the second period) are independent and identically distributed random vectors. Within a period, the dividend and production quantity are not independent.

A natural extension of this work is to add the pricing decisions of the firm as an important way of trying to correct inventory and cash flow problems. We leave this extra complication to a future investigation. The modern firm is a highly complex multiproduct institution. Even with modern computational methods, quantitative strategic analysis is at best crude. We suggest here that there is still the potential for considerable value-added investigation internal to the firm and the model investigated here is offered as an example to support this observation.

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Appendix A. Proof of Proposition 5

Utilizing (1) and (2) with $b_n = b^*$, $s_n = s^*$, and $y_n = y^*$ for $n = 1, 2, \ldots,$ if $n > 1$ then $x_n = y^* - D_{n-1}$, which causes $z_n = y_n - x_n = y^* - (y^* - D_{n-1}) = D_{n-1}$ and

$$v_n = w_n + (r-\gamma)x_n^- - p(w_n) - s^* - cz_n - \rho b^*. \tag{A1}$$

From (5), if $n > 1$ then $w_n = s^* + g(y^*, D_{n-1})$. Substitution in (A1) yields

$$\begin{aligned} v_n &= (r-\gamma)(y^* - D_{n-1})^- + g(y^*, D_{n-1}) - cD_{n-1} \\ &\quad - p(s^* + g(y^*, D_{n-1})) - \rho b^*. \end{aligned}$$

Appendix B. Proof of Proposition 7

From definition (11) and (16),

$$\begin{aligned} &\sup_{b,s}\Big\{-(1-\beta)s - [\beta(r-\gamma) + (1-\beta)c]y \\ &\quad + \beta E\big[(r-\gamma)(y-D)^+ + g(y,D) - p(s + g[y,D])\big] \\ &\quad - \rho b \colon b \ge 0, b+s \ge 0\Big\} \\ &= -[\beta(r-\gamma) + (1-\beta)c]y + \beta E[(r-\gamma)(y-D)^+ + g(y,D)] \\ &\quad + \sup_s\Big\{\sup_b\{-\rho b \colon b \ge 0, b+s\ge 0\} - (1-\beta)s \\ &\quad - \beta E\big(p[s + g(y,D)]\big)\Big\} \\ &= \sup_s\big\{-\rho(-s)^+ - (1-\beta)(s+cy) - \beta(r-\gamma)y \\ &\quad + \beta E\big[(r-\gamma)(y-D)^+ + g(y,D) - p(s + g[y,D])\big]\big\}. \end{aligned}$$

Appendix C. Expected Value of the Default Penalty in (25)

The challenging component of (25) is the expected value of the default penalty. The following expansion shows that it depends on the sign of $s + ry$, which is the level of the retained earnings when demand matches supply. Shortly we discuss the importance of $s + ry$. Let $f(\cdot)$ denote the density function of demand D. Then,

$$\begin{aligned} &E[p_1(s + ry - (r+h)(y-D)^+)^-] \\ &= \begin{cases} p_1 \displaystyle\int_0^{(hy-s)/(r+h)} (hy - s - (r+h)x) f(x)\,dx & \text{if } s + ry > 0, \\ p_1(r+h)\displaystyle\int_0^y (y-x) f(x)\,dx & \text{if } s + ry = 0, \\ p_1\Big(-s - ry + (r+h)\displaystyle\int_0^y (y-x) f(x)\,dx\Big) & \text{if } s + ry < 0. \end{cases} \end{aligned} \tag{C1}$$

To compute the expected default penalty cost (C1), note that

$$s + ry \ge 0 \quad \text{if and only if} \quad \frac{hy - s}{r+h} = y - \frac{s + ry}{r+h} \le y. \tag{C2}$$

Thus, if $s + ry \ge 0$,

$$\begin{aligned} &(s + ry - (r+h)(y-D)^+)^- \\ &= \begin{cases} 0 & \text{if } D > (hy-s)/(r+h), \\ hy - s - (r+h)D & \text{if } D \le (hy-s)/(r+h), \end{cases} \end{aligned}$$

and if $s + ry \le 0$,

$$\begin{aligned} &(s + ry - (r+h)(y-D)^+)^- \\ &= \begin{cases} -(s+ry) & \text{if } D > y, \\ -[s + ry - (r+h)(y-D)] & \text{if } D \le y. \end{cases} \end{aligned}$$

Appendix D. Solution of the Optimization Problem on Which Proposition 8 Is Based

The optimization problem is

$$K(b^*, s^*, y^*) = \sup\big\{K(b,s,y) \colon b \ge 0, b+s \ge 0, y \ge 0\big\}. \tag{D1}$$

The Kuhn–Tucker condition is as follows:

$$\frac{\partial K(b^*, s^*, y^*)}{\partial s} + \lambda_1^* = 0,$$

$$\frac{\partial K(b^*, s^*, y^*)}{\partial y} + \lambda_2^* = 0,$$

$$\frac{\partial K(b^*, s^*, y^*)}{\partial b} + \lambda_1^* + \lambda_3^* = -\rho + \lambda_1^* + \lambda_3^* = 0,$$

$$\lambda_2^* y^* = \lambda_1^*(b^* + s^*) = 0, \quad y^* \ge 0, \quad \text{and} \quad \lambda_3^* b^* = 0.$$

Note that $\lambda_1^* + \lambda_3^* = \rho$ implies that at least one of the λ_i^*s is positive. Therefore, either $b^* = 0$ or $b^* + s^* = 0$. So, the firm's borrowing policy is either not to borrow or to borrow just enough to bring the cash level to zero before the realization of revenue and inventory costs. In effect, this is an alternative proof of pecking-order optimality (Proposition 7) when there are piecewise linear functions for the default penalty and the gross profit from sales (net of inventory costs).

We consider the transformed single-period payoff as given in (25). It follows from (C1) that

$$\frac{\partial K}{\partial s} = -(1-\beta) + \begin{cases} \beta p_1 F[(hy-s)/(r+h)] & \text{if } s+ry > 0, \\ \beta p_1 & \text{if } s+ry < 0, \end{cases} \tag{D2}$$

and

$$\begin{aligned} \frac{\partial K}{\partial y} &= \beta\gamma - (1-\beta)c - \beta(\gamma+h)F(y) \\ &\quad + \begin{cases} -\beta p_1 hF[(hy-s)/(r+h)] & \text{if } s+ry > 0, \\ \beta p_1(r-(r+h)F[y]) & \text{if } s+ry < 0. \end{cases} \end{aligned} \tag{D3}$$

Note that K is not differentiable at $s + ry = 0$, and (D2) and (D3) provide left and right derivatives at that point. Recall that if $s + ry > 0$, then the firm will not default if demand is higher than the limit $D > (hy - s)/(r + h)$. On the other hand, $s + ry < 0$ represents a situation where the internal working capital, s, and/or the total goods supply, y, are set so low that the firm will default with certainty. The effects of changing the base-stock level y on the objective are different in the two cases. In both cases, a unit increase in the base-stock level y will increase the default cost by $p_1 h$ when demand is lower than the limit of no default or $D \notin \Gamma(s, y)$. However, an increase in y has an additional effect on the default cost in the latter case ($s + ry < 0$). Because the firm will default regardless of demand's realization, one more unit of supply will generate r more dollars of sales and, hence, reduce the default penalty by $p_1 r$ dollars as long as the unit can be sold (with a probability of $1 - F(y)$). On the other hand, one more unit of supply will increase the default cost by $p_1 h$ dollars when the unit cannot be sold (with a probability of $F(y)$).

Appendix E. Proof of Proposition 9

The loan is repaid in full and on time at the end of period n if and only if $w_{n+1} \geq 0$. From (24), the assumption that default does not occur unless demand is less than supply, and the discussion preceding (D1),

$$\begin{aligned} q(s^*, y^*) &= P\{w_{n+1} \geq 0\} = P\{s_n + g(y_n, D_n) \geq 0\} \\ &= P\{s^* + ry^* - (r+h)(y^* - D_n)^+ \geq 0\} \\ &= 1 - P\{s^* + ry^* - (r+h)(y^* - D_n)^+ < 0\} \\ &= 1 - P\{s^* - hy^* + (r+h)D_n < 0\} = 1 - F\left(\frac{hy^* - s^*}{r+h}\right), \end{aligned}$$

which confirms (36).

Notice that $A^* = (b^* - w_{n+1}^-)^+$ and $w_{n+1} < 0$ if and only if $D_n < (hy^* - s^*)/(r + h)$ in which case $w_{n+1} = s^* - hy^* + (r + h)D_n$. Also,

$$0 \leq a \leq \frac{hy^* - s^*}{r+h} \quad \text{implies} \quad b - hy^* + s^* + (r+h)a \geq 0.$$

Therefore, in confirmation of (37) and (38),

$$\begin{aligned} E(A^*) &= E(b^* - w_{n+1}^-)^+ \\ &= b^*\left[1 - F\left(\frac{hy^* - s^*}{r+h}\right)\right] \\ &\quad + \int_0^{(hy^*-s^*)/(r+h)} [b^* - hy^* + s^* + (r+h)x]^+ \, dF(x) \\ &= b^*\left[1 - F\left(\frac{hy^* - s^*}{r+h}\right)\right] \\ &\quad + \int_0^{(hy^*-s^*)/(r+h)} [b^* - hy^* + s^* + (r+h)x] \, dF(x). \end{aligned}$$

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