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Managing Rentals with Usage-Based Loss

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Motivated by innovative rental business models, we study a rental system with random loss of inventory due to use. We utilize a discrete-time model in which the inventory level is chosen before the start of a finite rental season, and customers not immediately served are lost. Demand, rental durations, and rental unit lifetimes are stochastic; sample path coupling allows us to derive structural results that hold under limited distributional assumptions. Considering different "recirculation" rules (i.e., which unit to select to meet a demand), we prove the concavity of the expected profit function and identify the optimal recirculation rule under two different models of a rental unit's state: the number of times rented out or its condition. We develop two upper bounds on the number of lost rental units and two heuristics for the inventory decision. Numerical study shows the following: (1) accounting for rental unit loss can increase the expected profit by 7% for a single season; (2) the optimal inventory level in response to increasing loss probability is nonmonotonic; (3) both heuristics perform well; and (4) choosing the optimal recirculation rule over a commonly used policy can increase the profit-maximizing service level by up to six percentage points.

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1. Introduction

Advances in online commercial models have produced a new generation of innovative businesses built upon renting goods. For an increasing variety of products, the promise of flexibility and affordability has led to rental businesses specializing in just about every aspect of our business and personal lives. Besides traditional rental products such as movies, cars, and hotel rooms, less common goods available to rent range from bicycles to jets, cribs to coffins, and furniture to camping gear. According to IBISWorld industry analysts, the annual revenue of 15 different rental industries in the United States each exceeded $1 billion in 2013, while the annual revenue of each of the car, heavy equipment, and industrial equipment rental industries surpassed $25 billion.

Luxury goods have received particular attention as fertile ground for rental businesses that make those goods available to new customer classes. For example, Rent the Runway is a company that allows customers to rent high-fashion dresses for either four or eight days at approximately 10% of the retail price of a dress (Wortham 2009). Customers can view the selection of dresses and their availability through a website and receive style and fit advice from Rent the Runway consultants and customer reviews. Dresses are shipped to customers and returned by mail. The critical decision about the number of dresses that will comprise Rent the Runway's seasonal rental inventory must be made shortly after preseason fashion shows, which are several months in advance of the rental season (Binkley 2011).

Choosing the number of rental units to procure before the start of a rental season without the possibility of replenishment during the season is an important problem that many rental businesses face. And once procured, deriving maximal revenue from these items is crucial: for online start-up businesses, how efficiently their capital-intensive rental inventory is managed influences the need to raise additional capital and determines key metrics (e.g., the average number of rental cycles that can be performed by each rental unit) presented to potential investors. In addition, the availability of inventory can affect the reputation of the rental service and customer retention rates, and failing to rent to a customer because the rental unit was damaged by a previous customer can result in a challenging customer service encounter. Furthermore, inventory management that accounts for loss relates to key strategic decisions. For example, Rent the Runway has recognized its laundry operation, crucial to ensuring quick inventory turnaround times, as a core competency and brought it in-house. Also, while customers are not permitted to purchase the high-fashion dresses at the end of a rental, the company has introduced a subscription rental service that allows customers to buy other clothing items instead of returning them.

Despite the seemingly fundamental nature of this problem, operations management literature offers very little analytical support for rental systems having lost sales and discrete time periods—natural assumptions for many rental systems. In this paper, we analyze a single-product rental system using a discrete-time framework. We focus on the effect of usage-based loss of rental units over a finite rental horizon during which no additional rental units may be ordered, e.g., when long procurement lead times or high administrative costs prohibit in-season reordering. In particular, we consider each rental unit to have a random lifetime, which is characterized by a general probability distribution on the number of times the unit can be rented before becoming lost. Our goal is to understand the role of this uncertainty arising from the usage-based loss of rental units on the management of rental inventory.

In addition to Rent the Runway, whose dresses are susceptible to both destructive incidents and wearing out over time, other rental systems face the challenge of losing inventory that can be difficult to replace in the middle of the rental season. For example, a Paris-based bicycle sharing program that began with 20,600 bicycles in 2007 had more than 8,000 bikes stolen and another 8,000 bikes severely damaged and in need of replacement within two years (Erlanger and De La Baume 2009). Inventory loss can also occur when customers exercise an option to purchase a product. Users of Redbox, an automated movie and game rental kiosk, rent a DVD for $1.50 a day. If the DVD is not returned in 17 days, then the customer pays $25.50 for the accrued daily rental charge and owns the DVD. Another example is Rent-A-Center, a company with over $3 billion in revenue in 2012 and which rents furniture, appliances, and electronics to customers who can own the item if it is rented beyond a certain duration. In its 2012 annual report, Rent-A-Center states that approximately 25% of its rental agreements result in customer ownership.

Existing work supporting capacity planning for rental businesses relies primarily on queueing models. Although Poisson or compound Poisson arrival processes may adequately represent demands for some rental businesses, better choices may exist for modeling demand in rental systems characterized by discrete rental time slots. For example, business travelers occupy a hotel room for a discrete number of days and are more likely to begin renting a hotel room on a Monday night than a Saturday night. At Rent the Runway, whose customers primarily rent dresses for events on Fridays and Saturdays, a discrete-time demand model with a period of one week more accurately represents a customer demand pattern than a Poisson arrival process. Similarly, Rent the Runway's customers are largely time sensitive, which makes a model such as the Netflix DVD-by-mail service, in which customers wait for an available rental unit, less applicable. Extending discrete-time inventory theory to include loss of rental inventory offers a modeling advantage for a rental system like Rent the Runway. We do so by developing a model that makes no distributional assumptions and captures (a) operational details such as random rental unit lifetimes (with constant, increasing, or decreasing failure rates) and random rental durations; (b) very general demand models with features such as seasonality, autocorrelations, and forecast uncertainty; and (c) recirculation rules that are used in practice for choosing among available rental units to satisfy demands.

We make the following contributions regarding the inventory management of rental systems:

1. *Model and framework*. To the best of our knowledge, we are the first to consider the loss of rental units according to distributions over the number of times that each unit can be rented before loss. Thus, our model includes either a state variable that represents the number of times that a rental unit has been rented out (i.e., a "count-based" model) or a state variable that represents a rental unit's condition (i.e., a "condition-based" model). It also accommodates an arbitrary demand process and general distributions for the lifetime and duration of each rental unit.

2. *Structural results*. (a) We establish the concavity of the expected profit function in the initial inventory of rental units for geometric lifetime distributions. Not surprisingly, this structural property holds independent of the rental unit recirculation rule as the loss probability is constant over time.

(b) For general lifetime distributions, it becomes necessary to consider the recirculation rule for allocating rental units to satisfy customer demand for both count-based or condition-based models.

(c) *Count-based model*: We establish the concavity of the expected profit function for the "static priority" recirculation rule; i.e., the units to be rented are prioritized according to a list that does not change over the rental horizon. We show that the concavity of the expected profit function also holds for a policy that spreads the rental load evenly over all units, allocating the rental unit that has been rented out the fewest number of times. Referring to this recirculation rule as the "even spread" policy, we prove its optimality when rental unit loss probabilities are nondecreasing in the number of times that the unit has been rented.

(d) *Condition-based model*: We demonstrate analogous results for the condition-based model, showing the concavity of the expected profit function for the "best-first" policy in which the rental unit in the best condition receives the highest allocation priority. Also, we prove that the best-first policy is optimal when the state transition probability matrix is totally positive of order 2, a condition that implies that the rental unit failure rate is increasing as its condition worsens.

3. *Managerial insights from numerical study*. (a) Failing to account for usage-based loss of rental inventory leads to a significant reduction in the expected profit. For a 5% probability of loss each time a unit is rented, we find that ignoring the loss of rental units reduces the expected profit by 7.3% and 33.0% for a half-year and a full-year rental horizon, respectively. We provide two simple upper bounds on the number of lost rental units and two heuristics for choosing the number of rental units to procure that account for rental unit loss.

(b) The optimal response to the increasing loss probability is to first increase the number of rental units, then decrease the number of rental units, and finally stock zero rental units. One of the heuristics (SR2) that focuses on achieving a target service rate accounts for this relationship and performs well in numerical testing.

(c) For a rental unit lifetime distribution with increasing loss probability, the rental unit recirculation rule plays an important role determined by the rate at which the loss probability increases. We focus on the count-based model, as similar results apply for the condition-based model, and compare the even spread policy to the static priority recirculation rule. Choosing the even spread policy increases the optimal initial inventory level, which in turn leads to a corresponding increase of up to six percentage points in the profit-maximizing service level.

The remainder of this paper is organized as follows. Section 2 reviews the rental inventory management literature. Section 3 introduces our rental inventory model. We establish the structural properties of this model for geometric lifetime distributions in §4 and for general lifetime distributions in §5, where we further identify the optimal rental unit recirculation rule under certain conditions. The numerical analysis follows in §6. We conclude with a summary of findings in §7. An online supplement (available as supplemental material at http://dx.doi.org/10.1287/msom.2016.0576) lists the notation that we use throughout the paper for easy reference and contains all proofs as well as additional numerical tests for the bounds and heuristics described in §6.2.

2. Literature Review

Early research on rental inventory management exclusively uses queueing models as a foundation for analysis. The initial advances in queueing theory by Takács (1962) and Riordan (1962) for the telephone trunking problem—finding the stationary probabilities of a multiserver pure loss system—sparked two seminal papers on the problem of sizing a fleet of rental equipment. Tainiter (1964) formulates an optimization problem for $M/G/c/c$ and $G/M/c/c$ rental systems based on the limiting distributions of the system states derived by Takács (1962). The decision variable is the capacity of the rental system, and the problem is studied both asymptotically and over a finite horizon. Whisler (1967), on the other hand, shows that the optimal policy structure for a rental system with lost sales, periodic reordering, and nonstationary state transition probabilities—as in Riordan (1962)—has upper critical values above which inventory should be discarded and lower critical values below which inventory should be ordered. Our work differs from these studies by its focus on the inventory decision prior to the rental season, the challenge of handling random usage-based loss of rental units, and stochastic rental duration, and the use of a discrete-time model for demand representation.

The early research on rental inventory management with lost sales was followed by an extensive study of the $M/M/c$ queueing model with backlogged demands. Specifically, the problem was posed as finding the optimal number of servers to employ in a multiserver queuing system, where servers represent rental units and service time corresponds to the rental duration (see Huang et al. (1977), Jung and Lee (1989), Green et al. (2001), Zhang et al. (2012)). Motivated by the time-specific nature of customers' rental needs, however, we restrict our focus to lost sales models in this paper. Table 1 compares our rental inventory model to the other rental inventory models that also make the assumption of lost sales. In addition to the continuous-time rental inventory models of Tainiter (1964) and Whisler (1967), Papier and Thonemann (2008) build on the $M/M/c/c$ queueing model in Harel (1988), which develops approximations as well as lower and upper bounds for the lost sales rate as a function of the system capacity. Extending this model to account for a compound Poisson arrival process, Papier and Thonemann (2008) conduct a stationary queueing analysis to obtain structural results for a fleet sizing problem and provide an approximation suitable for implementation. Adelman (2008) also uses an approximation of the $M/M/c/c$ or $M/G/c/c$ queueing model to study the capacity decision for a rental system. However, our work is different from this stream of research due to our consideration of a discrete-time rental model with a finite rental season, random usage-based inventory loss, and an arbitrary demand model possessing the ability to capture any distributional characteristic.

Table 1 Comparison of Selected Lost Sales Rental Inventory Models

	Continuous time			Discrete time		
	Tainiter (1964)	Whisler (1967)	Papier and Thonemann (2008)	Cohen et al. (1980)	Baron et al. (2011)	Our paper
Inventory decision	One time	Repeated	One time	Repeated	One time	One time
Time horizon	Finite	Finite	Infinite	Finite	Finite	Finite
Demand process	i.i.d. interarrival times	i.i.d. interarrival times	Compound Poisson stationary; also nonstationary	General i.i.d.	Arbitrary	Arbitrary
Rental duration	General i.i.d.	General i.i.d.	General i.i.d.	Deterministic	General i.i.d. with a restricted return process	General i.i.d.
Inventory loss	N/A	N/A	N/A	Constant decay	N/A	Usage-based random loss

In contrast to the continuous-time queueing models, Cohen et al. (1980) use a discrete-time model to represent a return process to a blood bank with the goal of determining an optimal order-up-to level in every period. Reflecting hospitals' tendency to order significantly more units of blood than needed, a constant percentage of the quantity rented by hospitals is returned to the blood bank and the rest is consumed after a rental duration of a fixed number of periods. A constant percentage of the inventory leftover at the blood bank is, on the other hand, considered to have decayed. The problem of finding the optimal inventory level under a periodic review policy is formulated as a dynamic program, and an approximate solution is provided. In comparison, we examine the one-time preseasonal ordering problem and consider the loss of inventory as random, instead of being a constant proportion. Furthermore, we do not require the assumption of an independent and identically distributed (i.i.d.) demand process, and we allow randomness in the rental duration.

Closest to our model is that by Baron et al. (2011), who determine the optimal preseason order quantity for a video rental store with lost sales but no inventory loss. In particular, Baron et al. (2011) consider a return process that is monotone; i.e., the percentage of the rental units rented in period t and returned by period n is always greater than or equal to the percentage of the units rented in period $t+1$ and returned by the same period n. Their key result is the concavity of the expected number of rentals in the number of rental units procured. Our approach based on the forward differences of the state equations allows us to relax this assumption and accommodate general rental duration distributions. More importantly, we are the first to establish this structural result for a rental system with random usage-based inventory loss. We also address the issue of rental unit inventory allocation, which arises only in our rental inventory model as a result of accounting for random lifetimes of the rental units.

While we focus on allocating rental units based on their state, others focus on allocation policies for choosing among different rental customers based on customer class. Miller (1969) analyzes the admission decision to an Erlang loss queue when customers are classed based on the revenue received through service, and Savin et al. (2005) show the impact of class-based allocation policies on optimal fleet size. Papier and Thonemann (2010) and Levi and Shi (2011) build on this model to consider customers that request reservations in advance. Gans and Savin (2007) consider how to price admission to the queue, and Tang and Deo (2008) study the pricing decision when customers have uncertain return times. How to prioritize rental customers who pay a monthly rental subscription fee and are heterogeneous in their rental duration is also the subject of research by Bassamboo et al. (2009) and Jain et al. (2015).

To analyze models in which rental unit lifetimes do not follow a geometric distribution, we use sample path analysis in a very general setting to prove the two main results of our work: the concavity of the expected profit function and the optimal rental unit recirculation rule. This approach has been used in various settings to model complexities of production and inventory systems. Examples include the number of customers and their utilities for a model with dynamic substitution by Mahajan and van Ryzin (2001) and the processing times for multistation production lines by Muth (1979) and Tayur (1993). Also, our proofs of concavity bear similarities to that of Shanthikumar and Yao (1987) in their study of systems with multiserver stations.

3. Rental Model: A Sample Path Approach

In this section we describe a sample path approach to modeling a rental inventory system that allows us to analyze the system—including rules for recirculating rental units—under general assumptions about the demand process. Motivated by the problem of selecting the number of rental units to procure before the start of a finite rental season, we begin our analysis with the following two-stage model of a single-product, discrete-time rental inventory system with lost sales. In the first stage, the size of the rental inventory is chosen to be y. A critical aspect of rental inventory planning is to account for the loss of rental units. Misuse by customers, customer options to purchase rented items, or simply the deterioration of the rental unit's quality over time present reasons for why a unit would be lost.

Each rental unit is procured before the start of the season and has a value at the end of the rental season that depends on whether the rental unit is lost during the season. A rental unit's loss may decrease its salvage value due to damage or may increase its value if it is sold to a customer (or if a penalty is charged to the customer.) Hence, the unit procurement cost accounts for not just the purchase price, but is adjusted to also include the salvage value for a dress in "good" condition and the cost of holding the item for the duration of the rental season.

In the second stage, demands occur over N periods, and the units purchased in the first stage are rented to satisfy the customer demands. Each customer is assumed to rent a single unit, and the rental has a random rental duration. Thus, if the duration of some rental is a periods, fulfilling the demand requires that one unit of the inventory is withdrawn for the period in which the demand is received and for the $a-1$ succeeding periods.

The duration of the customer rental that is ith demand served by the rental unit m over that rental unit's lifetime is a random variable denoted by $A_{m,i}$ for $i \geq 1$ and $m = 1, 2, \ldots, y$. We define the realization of this random variable on a sample path as $a_{m,i}$. Each rental lasts for any number of periods between a minimum of $A_{\min}$ and a maximum of $A_{\max}$; i.e., $a_{m,i} \in \{A_{\min}, A_{\min}+1, \ldots, A_{\max}\}$. We assume that $A_{m,i}$ is independent and identically distributed according to a general probability mass function characterized by $h_a := \mathbb{P}\{a_{m,i} = a\}$, $a = A_{\min}, A_{\min}+1, \ldots, A_{\max}$.

Each rental unit becomes lost after a random number of rentals (i.e., its "lifetime" $\mathbf{l}_m$), with a loss probability $f_{i,a}$ corresponding to the ith rental served by some rental unit given a rental duration a. Upon completion of its $\mathbf{l}_m$th rental, unit m satisfies no further demands, although it might still have a salvage value that is earned at the end of the horizon. The marginal lifetime distribution $\ell_i = \Pr(\mathbf{l}_m = i)$ can then be defined recursively as

$$\ell_i = \left(1 - \sum_{j}^{i-1} \ell_j\right) \sum_{a=A_{\min}}^{A_{\max}} h_a f_{i,a}.$$

We note that our model accommodates correlation between the loss probability and the duration of each rental.

Demand d_n is received in period $n \in \{1, 2, \ldots, N\}$. Taken together, the demands $d_1, d_2, \ldots, d_N$, the rental unit lifetimes $\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_y$, and the rental durations $a_{m,i}$, $i \geq 1$ and $m = 1, 2, \ldots, y$, comprise a sample path which we denote by $\boldsymbol{\xi}$. When rental unit loss probabilities change based on the number of times rented, we must also specify the recirculation rule γ to fully characterize the system's operation. The recirculation rule determines which rental unit is chosen, given a set of available rental units for each customer served, and may affect the system's profit by changing the pattern at which rental units are lost over the rental horizon.

Next, we define variables to track the system's performance in each period and over the entire horizon. For the number of rentals, we have the notation

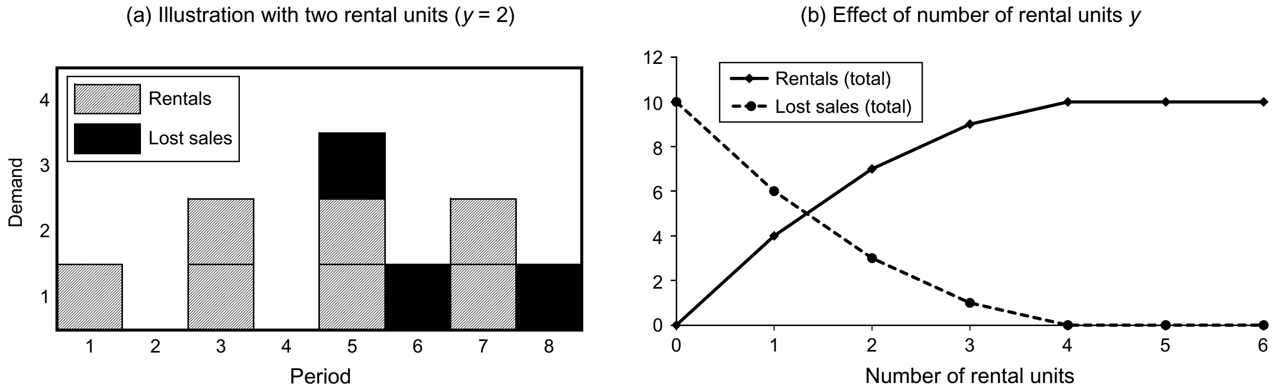
- $R^{\gamma}_{a,n}(y, \boldsymbol{\xi})$ as the number of rentals of duration a that begin in period n for a policy γ, sample path $\boldsymbol{\xi}$, and initial inventory level y;
- $R^{\gamma}_{n}(y, \boldsymbol{\xi})$ as the number of rentals that begin in period n for a policy γ, sample path $\boldsymbol{\xi}$, and initial inventory level y, i.e., $R^{\gamma}_{n}(y, \boldsymbol{\xi}) := \sum_{a=A_{\min}}^{A_{\max}} R^{\gamma}_{a,n}(y, \boldsymbol{\xi})$;
- $\mathcal{R}^{\gamma}_{a}(y, \boldsymbol{\xi})$ as the number of rentals over the entire horizon with a duration of a periods for a policy γ, sample path $\boldsymbol{\xi}$, and initial inventory level y, i.e., $\mathcal{R}^{\gamma}_{a}(y, \boldsymbol{\xi}) := \sum_{n=1}^{N} R^{\gamma}_{a,n}(y, \boldsymbol{\xi})$; and
- $\mathcal{R}^{\gamma}(y, \boldsymbol{\xi})$ as the number of rentals over the entire horizon for a policy γ, sample path $\boldsymbol{\xi}$, and initial inventory level y, i.e., $\mathcal{R}^{\gamma}(y, \boldsymbol{\xi}) := \sum_{a=A_{\min}}^{A_{\max}} \sum_{n=1}^{N} R^{\gamma}_{a,n}(y, \boldsymbol{\xi})$.

Some or all of these four variables are defined analogously for three other measures of system performance. We use the notation L and $\mathcal{L}$ for the number of lost sales, the notation W for the number of returns, and the notation Z and $\mathcal{Z}$ for the number of lost rental units. To account for the return of rental units that are rented near the end of the horizon, we define $W^{\gamma}_{N+1}(y, \boldsymbol{\xi}), W^{\gamma}_{N+2}(y, \boldsymbol{\xi}), \ldots, W^{\gamma}_{N+A_{\max}}(y, \boldsymbol{\xi})$ as the returns and $Z^{\gamma}_{N+1}(y, \boldsymbol{\xi}), Z^{\gamma}_{N+2}(y, \boldsymbol{\xi}), \ldots, Z^{\gamma}_{N+A_{\max}}(y, \boldsymbol{\xi})$ as the lost units in each of the corresponding periods. To explicitly state the number of lost rental units in any period, we define $Z^{\gamma}_{n}(y, \boldsymbol{\xi}) := \sum_{a=A_{\min}}^{A_{\max}} (R^{\gamma}_{n-a,a}(y, \boldsymbol{\xi}) - W^{\gamma}_{n,a}(y, \boldsymbol{\xi}))$ as the number of rental units that would have been returned in period n but were lost. The total number of lost rental units is denoted by $\mathcal{Z}^{\gamma}(y, \boldsymbol{\xi}) := \sum_{n=A+1}^{N+A} Z^{\gamma}_{n}(y, \boldsymbol{\xi})$. A complete characterization of these variables is provided in the online appendix.

The rental system operates for period n of the second stage as follows:

1. $W^{\gamma}_{n}(y, \boldsymbol{\xi})$ rental units are returned, whereas $Z^{\gamma}_{n}(y, \boldsymbol{\xi})$ units are lost. After returns are received but before rentals are made, the total inventory available to rent out in period n is $I^{\gamma}_{n}(y, \boldsymbol{\xi}) := y - \sum_{t=1}^{n-1} R^{\gamma}_{t}(y, \boldsymbol{\xi}) + \sum_{t=1}^{n} W^{\gamma}_{t}(y, \boldsymbol{\xi})$.
2. The demand D_n is realized as d_n. If $d_n \leq I^{\gamma}_{n}(y, \boldsymbol{\xi})$, then d_n units are rented out. Otherwise, $I^{\gamma}_{n}(y, \boldsymbol{\xi})$ units are rented out. More succinctly, $R^{\gamma}_{n}(y, \boldsymbol{\xi}) := d_n \wedge I^{\gamma}_{n}(y, \boldsymbol{\xi})$, where $a \wedge b$ denotes the minimum

Figure 1 **Number of Rentals and Lost Sales for Example 1 with a Rental Duration of Two Periods ($A = 2$)**



of a and b. The rental unit recirculation rule determines which rental unit is allocated to satisfy each unit of demand, and consequently may influence $W_n^\gamma(y, \boldsymbol{\xi})$ and $Z_n^\gamma(y, \boldsymbol{\xi})$.

3. Excess demand $L_n^\gamma(y, \boldsymbol{\xi}) := (d_n - I_n^\gamma(y, \boldsymbol{\xi})) \vee 0$ is lost, where $a \vee b$ denotes the maximum of a and b. This expression can be alternatively written as $L_n^\gamma(y, \boldsymbol{\xi}) = d_n^\gamma - R_n^\gamma(y, \boldsymbol{\xi})$.

Therefore, given the sample path $\boldsymbol{\xi}$, the dynamics of the rental system's operation can be represented recursively as follows, where $I_0^\gamma(y, \boldsymbol{\xi}) = y$ and $R_t^\gamma(y, \boldsymbol{\xi}) = 0$ for $t \leq 0$:

$$\begin{aligned} I_{n+1}^\gamma(y, \boldsymbol{\xi}) &= I_n^\gamma(y, \boldsymbol{\xi}) - R_n^\gamma(y, \boldsymbol{\xi}) + W_{n+1}^\gamma(y, \boldsymbol{\xi}), \\ R_{n+1}^\gamma(y, \boldsymbol{\xi}) &= d_{n+1} \wedge I_{n+1}^\gamma(y, \boldsymbol{\xi}), \\ L_{n+1}^\gamma(y, \boldsymbol{\xi}) &= d_{n+1} - R_{n+1}^\gamma(y, \boldsymbol{\xi}). \end{aligned} \tag{1}$$

Example 1. Figure 1 illustrates an example rental system with a demand sequence of $\{d_1, \ldots, d_8\} = \{1, 0, 2, 0, 3, 1, 2, 1\}$ for eight periods ($N = 8$). Each rental lasts for a deterministic two periods; i.e., a unit that is rented in period n will next be available to be rented again in period $n + 2$. In this example, we assume that there is no rental unit loss. Thus, $W_n^\gamma(y, \boldsymbol{\xi}) = R_{n-2}^\gamma(y, \boldsymbol{\xi})$ and the recirculation rule does not matter, as rental units have infinite lifetimes. If the system would operate with only one rental unit (i.e., $y = 1$), then that unit would be rented in periods 1, 3, 5, and 7 for a total of four rentals, while six units of the demand would be lost. Figure 1(a) shows how the demand is divided into rentals and lost sales for a system with $y = 2$ rental units. Thus, the addition of the second rental unit allows an additional unit of demand to be satisfied in periods 3, 5, and 7, so that there are now 7 units of fulfilled demand and 3 units of lost sales. Figure 1(b) shows how the number of rentals and lost sales change with the number of rental units y. We observe that the number of rentals is concave in y and that the number of lost sales is convex in y on this sample path. In other words, the number of additional rentals produced by one additional rental unit (i.e., the slope of the rentals curve) is decreasing in y.

One way to model rental unit loss is to consider geometrically distributed rental unit lifetimes. The memoryless property of the geometric distribution leads to a constant probability of rental unit loss over time. However, if a rental unit does indeed have a higher probability of wearing out over time, then a rental unit lifetime distribution with an increasing failure rate (i.e., a loss probability increasing with the number of times the unit has been rented) would be a suitable choice. Bicycles, cars, and large equipment are examples of assets for which an increasing loss probability is likely. Furthermore, lifetimes that are deterministic—when enforced by safety regulations that require their disposal after a certain number of uses—can be analyzed as a special case of an increasing loss probability.

Next, we define the revenues and costs in our models. A reward r_a is earned every time a unit is rented for a duration of a periods, and c is the unit cost of a lost sale (in addition to the loss of r_a). The salvage value of a rental unit that is lost during the rental season may differ from the salvage value of a unit that is still functional at the end of the season. We incorporate salvage value into the procurement costs, defining k^S as the procurement cost for a unit that can be still rented at the end of the rental season (i.e., "survives") and k^L as the procurement cost for a unit that is lost. The relation $k^S \leq k^L$ indicates that the unit that is lost has been damaged and has lost a portion of its value. The relation $k^S \geq k^L$ may, on the other hand, represent the purchase of the rental unit by the customer who is renting it, as discussed in §1 for the rental companies Redbox and Rent-A-Center. Note that we assume k^L is independent of the "age" of the item when it retires. Our model can easily be extended to allow for this, or to allow for the salvage value as a random variable, and the results then hold using the expected salvage value.

To account for the cost of inventory loss in the objective function of our rental inventory model, the reduction in the salvage value of a lost rental unit $(k^L - k^S)$ is multiplied by the number of lost rental units and subtracted from the revenue as part of the profit function, which we denote by $\Pi^\gamma(y, \boldsymbol{\xi})$. Consequently, we obtain the profit function on any sample path as follows:

$$\Pi^\gamma(y,\boldsymbol{\xi}) = \sum_{a=A_{\min}}^{A_{\max}} r_a \mathscr{R}_a^\gamma(y,\boldsymbol{\xi}) - c\mathscr{L}^\gamma(y,\boldsymbol{\xi}) - k^S y - (k^L - k^S)\mathscr{Z}^\gamma(y,\boldsymbol{\xi}).$$

We are now ready to formulate the rental inventory optimization problem as the maximization of the expected profit function $\pi^\gamma(y) := \mathbb{E}[\Pi^\gamma(y, \boldsymbol{\xi})]$ subject to $y \geq 0$. We investigate the concavity of this expected profit function in the initial inventory of y rental units for geometric lifetime distributions in §4 and for general lifetime distributions in §5.

When rental duration is random, the convexity of the total number of lost sales, $\mathscr{L}(y, \boldsymbol{\xi})$, in y and thus the concavity of the total number of rentals, $\mathscr{R}(y, \boldsymbol{\xi})$, in y might not hold for every sample path $\boldsymbol{\xi}$. As an example, we consider the addition of two rental units to our inventory system, where the first additional unit fulfills one customer demand with a very long duration and the second additional unit fulfills several customer demands with short rental durations. In this case, the number of additional customer demands satisfied by one extra rental unit is not necessarily nonincreasing in y. Therefore, we proceed by analyzing the structural properties of the expected number of lost sales and the expected number of rentals in the following sections. We are the first to consider this modeling aspect simultaneously with random loss of rental inventory. It is worth noting that the random rental duration accounts for each customer's decision to keep the rental unit for a different number of periods, but it can also include the random service time needed to repair the rental unit depending on its condition upon return.

4. Rental Inventory Loss with Geometric Lifetime Distributions

This section considers a model in which each rental unit m experiences a loss probability of p with each rental. Specifically, the random variable $\mathbf{l}_m$, which denotes the number of times the unit $m \in \{1, 2, \ldots, y\}$ is rented before retiring from the rental inventory, follows a geometric distribution with an expected value of $1/p$. We assume that $\sum_{a=A_{\min}}^{A_{\max}} r_a h_a + c \geq p(k^L - k^S)$. This condition implies that the expected benefit, $\sum_{a=A_{\min}}^{A_{\max}} r_a h(a) + c$, of converting a lost sale into a rental is greater than or equal to the expected cost, $p(k^S - k^L)$, of the rental unit loss. Because of the constant loss probability, the recirculation rule has no effect on $\mathbb{E}[L_n^\gamma(y, \boldsymbol{\xi})]$, $\mathbb{E}[R_n^\gamma(y, \boldsymbol{\xi})]$, or $\mathbb{E}[W_n^\gamma(y, \boldsymbol{\xi})]$ for any n. Therefore, we omit the superscript in the notation used in this section.

We establish the concavity of the expected profit function by first presenting a condition related to the rental return process for which the expectation of the number of lost sales, $\mathbb{E}[\mathscr{L}(y, \boldsymbol{\xi})] := \sum_{n=1}^{N} \mathbb{E}[L_n(y, \boldsymbol{\xi})]$, is convex. Correspondingly, the expectation of the number of rentals, $\mathbb{E}[\mathscr{R}(y, \boldsymbol{\xi})] := \sum_{n=1}^{N} \mathbb{E}[R_n(y, \boldsymbol{\xi})]$, is concave in the initial inventory of y rental units for this condition.

Lemma 1. *If the expected number of rental units returned over the rental horizon* $\mathbb{E}[\sum_{t=1}^{n} \sum_{a=A_{\min}}^{A_{\max}} W_{a,t}(y, \boldsymbol{\xi})]$ *is concave and nondecreasing in* y *for* $n = 1, 2, \ldots, N$, *then the expected number of lost sales* $\mathbb{E}[\mathscr{L}(y, \boldsymbol{\xi})]$ *is convex and nonincreasing while the expected number of rentals* $\mathbb{E}[\mathscr{R}(y, \boldsymbol{\xi})]$ *is concave and nondecreasing in* y.

We prove Lemma 1 using recursive substitution of the forward differences of the state Equations (1). After eliminating all $\Delta R_n(y, \boldsymbol{\xi})$ terms, we can see that the concavity of the return process is a sufficient condition for the concavity of the expected number of rentals. We then use an argument by induction and Lemma 1 to show that the expected number of returns is indeed concave in y, which implies that the expected profit function is also concave in y.

Proposition 1. *When rental unit lifetimes are geometrically distributed, the expected profit* $\pi(y)$ *is concave in* y *for any rental unit recirculation rule and* $y \geq 0$.

5. Rental Inventory Loss with General Lifetime Distributions

When the lifetimes of the rental units follow a general distribution, the number of rental units returned in any period n may depend on the policy used to choose among available rental units to satisfy the demand in previous periods. Specifically, in any period n a sequence of $R_n(y, \boldsymbol{\xi})$ allocation decisions must be made. For each decision we allow $\mathscr{A}$, $\varnothing \subset \mathscr{A} \subseteq \{1, 2, \ldots, y\}$, to denote the set of available rental units from which the allocated rental unit, m^γ, is chosen according to some recirculation rule γ.

Once again, we investigate whether it is possible to establish the concavity of the expected profit in the initial inventory of y rental units. Because we have not yet found a direct algebraic proof, we compare sample paths via coupling, as described in Chapter 4 of Lindvall (1992). A coupling approach allows us to compare the value of an additional rental unit in two systems that differ only in the number of rental units. Because of the rental unit lifetime distributions and the recirculation rule, analysis of the change in the expected number of rentals would otherwise be extremely difficult.

Our approach uses the following steps:

1. Establish demand values $d_1, d_2, \ldots, d_N$, which do not require any distributional assumptions.

2. Operate the system with y rental units, each of which has a lifetime $\mathbf{l}_m$, $m = 1, \ldots, y$ to indicate the number of customers that can be served by each rental unit before it retires from circulation. The ith demand, $i \geq 1$, served by rental unit m has a duration of $a_{m,i}$ periods.

3. Add an additional rental unit—the $(y+1)$st unit to the system—that has a lifetime $\mathbf{l}'$ and serves demands drawn sequentially from the durations $\{a'_1, a'_2, \ldots\}$. To be clear, the new system has rental units with lifetimes $\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_y, \mathbf{l}'$.

4. To the system described in Step 2 (i.e., ignoring Step 3), add a $(y+1)$st unit that has a lifetime $\mathbf{l}_{y+1}$ and serves demands drawn sequentially from the durations $\{a_{y+1,1}, a_{y+1,2}, \ldots\}$.

5. To the system described in Step 4, add an additional rental unit—the $(y+2)$nd unit—so that the system has rental units with lifetimes $\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_y, \mathbf{l}_{y+1}, \mathbf{l}'$. This additional rental unit has the same lifetime $\mathbf{l}'$ and serves demands with same durations $\{a'_1, a'_2, \ldots\}$ as the additional unit added to the system in Step 3.

For notational convenience, we define $\boldsymbol{\xi}(y)$ as the sample path consisting of the demands $d_1, d_2, \ldots, d_N$ of all N periods, the rental unit lifetimes $\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_y$, and the demand durations $\{a_{m,1}, a_{m,2}, \ldots\}$ for $m = 1, 2, \ldots, y$, as well as the lifetime $\mathbf{l}'$ and rental durations $\{a'_1, a'_2, \ldots\}$ for an additional rental unit. For example, $\boldsymbol{\xi}(y)$ and $\boldsymbol{\xi}(y+1)$ contain all of the sample path information necessary to analyze the systems described in Steps 3 and 5, respectively.

We consider two types of decisions for rental unit allocations. First, we examine a "count-based" rental unit state in which the allocation decision is based on the number of times that each unit has been rented. Then, we study a "condition-based" rental unit state in which the allocation decision is based on the current state of each rental unit. Each of these models may be relevant for Rent the Runway. Specifically, the dress's physical condition may not be observed—requiring a count-based model—if it is not carefully inspected or if the cause of a dress failure is difficult to observe as the dress's condition degrades. For instance, a zipper may be more likely to fail over time even if indications of impending failure may not be observed. On the other hand, a dress's physical condition may be observed if it relates to the condition of the fabric. Satin dresses are susceptible to developing minor damage to individual threads due to their loose weaves, and the repeated ironing of silk taffeta dresses may cause them to lose their ideal appearance around pleats and seams. A condition-based model would then be more appropriate for this setting.

5.1. Count-Based Rental Unit State

For the analysis in this section, we assume that $\sum_{a=A_{\min}}^{A_{\max}} r_a h_a + c \geq (k^L - k^S)\ell_i$ for $1 \leq i \leq N/A$. This condition implies that the expected benefit of an additional rental to a customer (i.e., $\sum_{a=A_{\min}}^{A_{\max}} r_a + c$) is greater than or equal to the reduction in the salvage value due to loss multiplied by the loss probability for a rental unit of any age i.

We denote by $\mathscr{C}$ the set of all policies for choosing an available rental unit to satisfy a demand based on knowledge of the number of times that each unit has been rented. At the time of the decision, the number of times that each rental unit has been rented is η_m for $m = 1, 2, \ldots, y$; for the chosen rental unit m^γ, the rental unit is removed from $\mathscr{A}$, and η_m is increased by one. If the rental unit is lost (i.e., $\mathbf{l}_m = \eta_m$), the rental unit is not returned to $\mathscr{A}$. Otherwise, it is returned to $\mathscr{A}$ at the beginning of period $n + a_{m^\gamma, \eta_m}$.

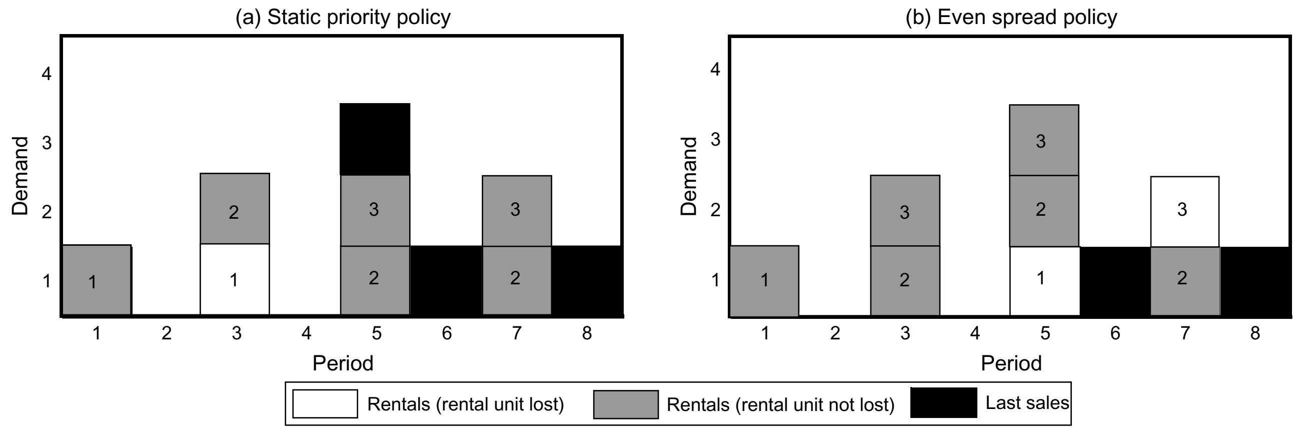
In this section, we examine two recirculation rules: The even spread policy (denoted by "ES") and the static priority policy (denoted by "SP"). In the even spread policy, each demand is served by an available rental unit that has been rented out the fewest number of times among all available rental units. We note that if units have an increasing hazard rate, the priority is assigned to rental units least likely to fail (i.e., become lost) under the even spread policy. The static priority policy, on the other hand, allocates rental units according to a priority list that does not change over the course of the rental horizon. These two policies can be defined as selecting some rental unit m^{ES} or m^{SP} such that

$$m^{\mathrm{ES}} \in \arg\min_{m \in \mathscr{A}} \eta_m,$$

$$m^{\mathrm{SP}} \in \arg\max_{m \in \mathscr{A}} \eta_m.$$

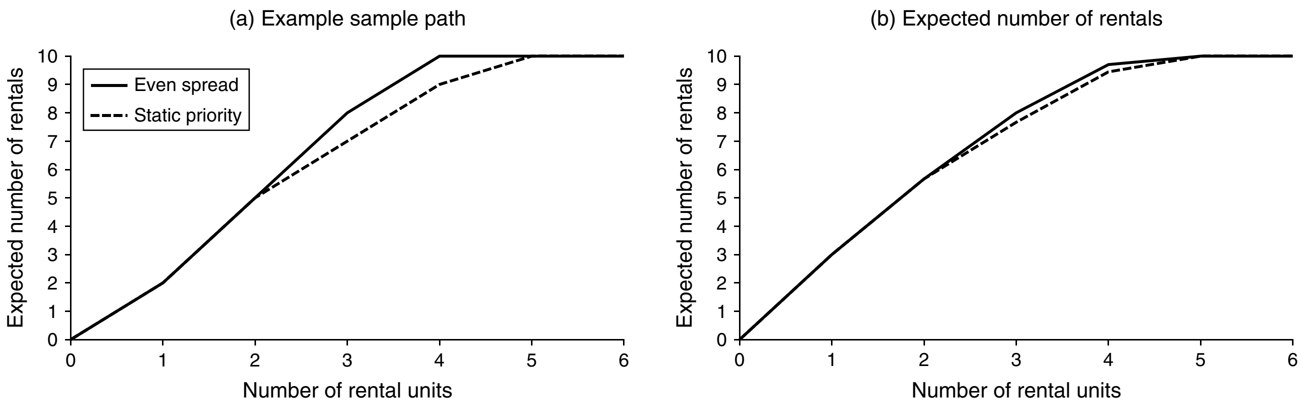
Example 2. Figure 2 shows how different rental unit recirculation schemes can affect the number of rentals and lost sales for an example sample path $\boldsymbol{\xi}$. Using the same demand values as in Example 1, we now state the lifetimes of available rental units as $\{\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_5\} = \{2, 4, 3, 4, 2\}$. For $y = 1$ and $y = 2$ with $\mathbf{l}_1 = 2$ and $\mathbf{l}_2 = 4$, both the even spread and static priority policies satisfy the same number of demands; i.e., $\mathscr{R}(1, \boldsymbol{\xi}) = 2$ and $\mathscr{R}(2, \boldsymbol{\xi}) = 5$. However, when $y = 3$, the even spread recirculation rule enables one more rental over the rental horizon than the static priority rule. Under the static priority rule, rental unit 1 is lost after serving a demand in period 3, whereas under the even spread rule it is lost after serving a demand in period 5. This allows one extra demand to be served in period 5 for the even spread rule because it has one more rental unit available than the static priority rule. Figure 3(a) shows that the even spread rule also

Figure 2 Rental Unit Recirculation Schemes for Example 2 with $y = 3$



Note. The number inside each box identifies the rental unit that satisfies a demand.

Figure 3 Effect of the Number of Rental Units and Rental Unit Recirculation Rule for Example 2



serves one more demand than the static priority rule when $y = 4$ and that both policies serve all 10 units of demand when $y \geq 5$. Figure 3(a) also demonstrates that the number of rentals is not necessarily concave in y; i.e., the addition of rental unit 1 with lifetime $\mathbf{l}_1 = 2$ satisfies fewer additional units of demand than the addition of rental unit 2 with lifetime $\mathbf{l}_2 = 4$.

In Figure 3(b), we use the demand values from Example 2, but instead let the lifetime of each rental unit be a discrete uniform random variable between 2 and 4 (i.e., $\ell_2 = \ell_3 = \ell_4 = 1/3$), and estimate the expected number of rentals with a simulation executed for a sufficiently large number of replications so that the standard error of the experiment is negligible. Even though concavity is violated on individual sample paths, the expected number of rentals is revealed to be a concave function of the number of rental units. The even spread and static priority policies result in the same number of rentals regardless of the sample path for $y \leq 2$ and $y \geq 5$. However, the expected number of rentals for the even spread policy exceeds that of the static priority policy by 0.33 when $y = 3$ and by 0.26 when $y = 4$. When $y = 3$, the even spread policy results in at least one more rental than the static priority policy on 44.1% of the sample paths, and the static priority policy exceeds the even spread policy on 10.9% of all sample paths. As an example of a sample path of rental unit lifetimes in which the static priority policy outperforms the even spread policy, the static priority policy produces one more rental than the even spread policy when $y = 3$ and $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\} = \{4, 3, 2\}$.

Because the static priority rule is easier to analyze, we begin by proving that policy's structural properties and then consider the even spread policy.

5.1.1. The Static Priority Recirculation Rule. The static priority recirculation rule, denoted by the superscript SP, guides the selection of rental units to satisfy demands according to a constant priority list; that is, when a rental unit is needed to satisfy demand, the one with the highest priority among the set of available rental units is chosen. Therefore, when the rental duration is deterministic, the static priority recirculation rule is the same as the policy selecting the rental unit that has been rented the most.

Proposition 2. *In a rental system with general i.i.d. rental unit lifetime distributions, the expected profit $\pi^{\mathrm{SP}}(y)$ is concave in the initial inventory of y rental units for the static priority recirculation rule.*

To prove this property of the expected profit function, we utilize first forward differences of the state equations and sample path coupling to compare the expected value of an additional rental unit for a system with initial inventory level y to the expected value of an additional rental unit for a system with initial inventory level $y+1$. By assigning the lowest priority to the additional rental unit, we are able to analyze the system without changing any existing allocations of rental units to customers.

5.1.2. The Even Spread Recirculation Rule. We now consider the even spread recirculation policy, which satisfies a demand with the rental unit that has been rented the fewest number of times. We assume that ties are broken by some static priority list for allocating rental units. When the loss probability for rental units is nondecreasing in the number of times rented, the even spread recirculation priority corresponds to a hazard rate ordering.

We first investigate the concavity of the expected profit function in the initial rental inventory (Proposition 3). We then demonstrate the optimality of the even spread policy to maximize the expected profit when the loss probability of each rental unit increases with the number of times that the unit has been rented (Proposition 4).

Proposition 3. *In a rental system with general i.i.d. rental unit lifetime distributions, the expected profit $\pi^{\mathrm{ES}}(y)$ is concave in the initial inventory of y rental units under the even spread recirculation rule.*

The proof of Proposition 3 poses new challenges compared to Proposition 2 because the additional rental units may change which demand is served by the existing rental units. Therefore, when comparing the effect of an additional unit on systems with y and $y+1$ rental units, we restrict rental units from serving certain customer demands in the system with y units so that corresponding rental units serve the same customer demands in the two systems. We then show that the benefit of relaxing this restriction is nonincreasing in y.

We next prove that when the rental unit loss probability is increasing in the number of times that the unit has been rented, the even spread policy is the optimal rental unit recirculation rule to maximize the expected profit.

Proposition 4. *If the loss probability of each rental unit increases with the number of times that the unit has been rented, then the profit from the even spread recirculation rule is stochastically larger than that of any other count-based recirculation rule; i.e.,* $\Pi^{\mathrm{ES}}(y,\boldsymbol{\xi}) \geq_{\mathrm{st}} \Pi^{\gamma}(y,\boldsymbol{\xi})$, $\gamma \in \mathscr{C}$.

Our key argument in this proof is a pairwise interchange argument in which iteratively switching each allocation that violates the even spread policy to conform to the even spread policy increases the expected number of rentals. We require additional notation to compare sample paths in our argument, which we describe along with an overview of the steps of the proof:

1. Find the first allocation decision over the rental horizon that violates the even spread policy. We denote this existing policy with the superscript V for "violating." Assume that this violating decision occurs in some period n. Specifically, a rental unit j is allocated to demand when some other rental unit i is available and $\sum_{t=1}^{n-1} R^V_{(t,j)}(y,\boldsymbol{\xi}) > \sum_{t=1}^{n-1} R^V_{(t,i)}(y,\boldsymbol{\xi})$. The availability of rental units i and j implies that $\sum_{t=1}^{n-1} R^V_{(t,j)}(y,\boldsymbol{\xi}) < \mathbf{l}_j$ and $\sum_{t=1}^{n-1} R^V_{(t,i)}(y,\boldsymbol{\xi}) < \mathbf{l}_i$.
2. Consider a switched forward allocation path of units i and j in periods $n, n+1, \ldots, N$ so that rental unit i is allocated instead of rental unit j. We refer to this allocation with the superscript S for "switched."
3. Replace values in $\boldsymbol{\xi}$ related to the lifetimes and rental durations after period n for units i and j with two new partial sample path vectors $\xi_{(1)}$ and $\xi_{(2)}$. Specifically, we generate two sets of random durations $(a_{(1),1}, a_{(1),2}, \ldots)$ and $(a_{(2),1}, a_{(2),2}, \ldots)$ for demands after period n served by two different rental units and use inverse probability mass function values $\omega_{(1)}$ and $\omega_{(2)}$ for the conditional lifetime distributions of the two rental units to generate lifetimes $\mathbf{l}_{(1)}$ and $\mathbf{l}_{(2)}$.
4. Calculate the number of rentals over the entire horizon under four scenarios (with corresponding notation for the total number of rentals used for convenience): (1) $\mathscr{R}^V(\xi_{(1)}, \xi_{(2)})$ for the violating allocation with $\xi_{(1)}$ applied to rental unit i and $\xi_{(2)}$ to rental unit j; (2) $\mathscr{R}^V(\xi_{(2)}, \xi_{(1)})$ for the violating allocation with $\xi_{(2)}$ applied to unit i and $\xi_{(1)}$ to unit j; (3) $\mathscr{R}^S(\xi_{(1)}, \xi_{(2)})$ for the switched allocation with $\xi_{(1)}$ applied to unit i and $\xi_{(2)}$ to unit j; and (4) $\mathscr{R}^S(\xi_{(2)}, \xi_{(1)})$ for the switched allocation with $\xi_{(2)}$ applied to unit i and $\xi_{(1)}$ to unit j.
5. Compare scenarios to observe that $\mathbb{E}[\mathscr{R}^S(y)] \geq \mathbb{E}[\mathscr{R}^V(y)]$, which implies that $\mathbb{E}[\Pi^S(y)] \geq \mathbb{E}[\Pi^V(y)]$ under the stated assumptions on the cost parameters.
6. Go to Step 1 and repeat until the switched allocation is equivalent to the even spread allocation.

5.2. Condition-Based Rental Unit State

We now study a different model of rental units in which each rental unit m has a known state $s_m \in \{1, 2, \ldots, S\}$ that may change after each time that the unit is rented. On a sample path $\boldsymbol{\xi}$, we define $s_{m,i}$ as the state of rental unit m after it is rented for the ith time, $m \in \{1, 2, \ldots, y\}$ and $i \in \{1, 2, \ldots, \mathbf{l}_m\}$. The initial state of each rental unit is defined as $s_{m,0} = 1$, and a rental unit's retirement from recirculation corresponds to $s_{m,\mathbf{l}_m} = S$. A transition probability matrix P_a governs the evolution of each rental unit's state upon each instance in which the unit is rented with duration a. We define $P_a(i, j)$ as the probability that a rental unit transitions from state i to state j after a rental of duration a with $i, j \in \{1, 2, \ldots, S\}$. We also assume that $\sum_{a=A_{\min}}^{A_{\max}} r_a h_a + c \geq (k^L - k^S) \sum_{a=A_{\min}}^{A_{\max}} P_a(i, S)$ for $i = 1, 2, \ldots, S-1$ so that the expected value of offering a rental is never negative. For convenience, we define an overall transition probability matrix P with $P(i, j) := \sum_{a=A_{\min}}^{A_{\max}} h_a P_a(i, j)$.

One simple recirculation policy based on the observed rental unit state is to allocate the rental units in increasing order of their state s_m. In other words, the rental unit that is in the best condition is given the highest allocation priority. We label this policy as the "best-first" policy, denoted by the superscript BF. Similarly, the "worst-first" policy, which we denote as WF, gives the highest priority to the rental unit in the worst condition for which it can still be rented out. Specifically, for each policy, the rental unit selected from the set of available rental units $\mathcal{A}$ obeys

$$m^{\mathrm{BF}} \in \arg\min_{m \in \mathcal{A}} s_m,$$
$$m^{\mathrm{WF}} \in \arg\max_{m \in \mathcal{A}} s_m.$$

For both policies, we show that the expected number of rentals is concave in the initial inventory level.

Proposition 5. *For the best-first and worst-first recirculation rules, the expected profit $\pi^\gamma(y)$, $\gamma \in \{\mathrm{BF}, \mathrm{WF}\}$, is concave in the initial inventory of y rental units.*

We next consider the optimal rental unit recirculation policy when rental unit selection decisions are based on the rental unit condition. We assume that the transition matrix is totally positive of order 2; i.e., that $P(i, j)P(i', j') \geq P(i, j')P(i', j)$ for all $i < i'$, $j < j'$. Brown and Chaganty (1983) show that this property implies that the first passage time from state 1 to some state $C_j = \{i\colon i > j\}$ has an increasing failure rate for $j = 1, \ldots, S-1$. In the proposition below, we use $\mathcal{D}$ to represent the set of all recirculation rules for choosing a rental unit to allocate solely based on the condition and availability of each rental unit.

Proposition 6. *If the transition matrix P is totally positive of order 2, then the profit from the best-first policy is stochastically larger than that of all other condition-based recirculation rules; i.e., $\Pi^{\mathrm{BF}}(y, \boldsymbol{\xi}) \geq_{\mathrm{st}} \Pi^\gamma(y, \boldsymbol{\xi})$, $\gamma \in \mathcal{D}$.*

6. Numerical Study: Rent the Runway

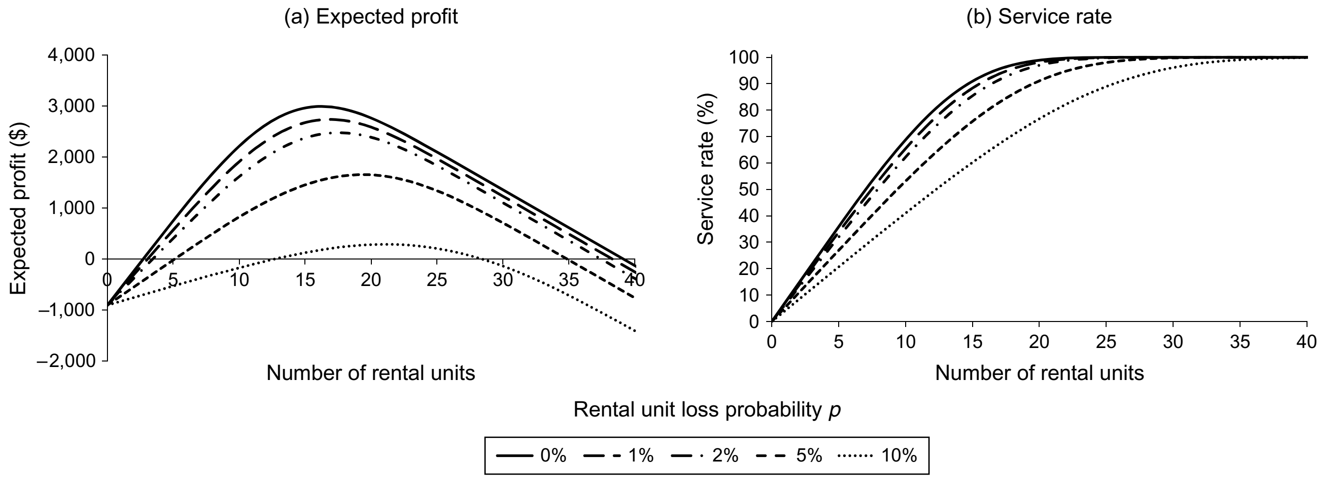
Motivated by the high-fashion dress rental business Rent the Runway, we introduce the model parameters representing a rental system with usage-based loss of inventory in §6.1. We discuss the impact of the rental inventory loss on the optimal procurement decision in §6.2, and the effect of the rental unit recirculation rule on rental inventory management in §6.3. All numerical testing is performed via sample average approximation, as described in Kleywegt et al. (2002).

6.1. Rental Model Parameters

The product we consider is a "middle-tier" dress as described in Eisenmann and Winig (2012); i.e., a full-price rental provides a net revenue of \$59, which is the difference between \$90 in revenue and \$31 in costs of cleaning, shipping, packaging, and credit card processing. However, customers are allowed to rent a second style for \$25 and a second size for free; thus, a unit may not achieve \$59 in net revenue every time it is rented. We assume that these three scenarios for a rental—renting as the primary dress with net revenue of \$59, renting as the secondary dress with net revenue of \$20, and renting as the free second size with net cost of \$5—occur with probabilities 50%, 20%, and 30%, resulting in an expected net revenue of $r = \$32$ per rental.

Eisenmann and Winig (2012) report that Rent the Runway purchases a middle-tier dress with a retail price of \$750 for \$226. We assume an annual unit holding cost that is equal to 20% of the purchase price of the dress to account for the cost of storage and the cost of capital. At the end of a fashion season, dresses in a variety of conditions are sold in New York City at what is known as a "sample sale." Based on websites such as Yannetta (2013) that report on these sales, we let a dress in good condition sell for 80%–85% off of the \$750 retail price and a dress in bad condition (i.e., a dress that retires from the rental inventory) to sell for 95% off of the retail price. Adjusting these sample sale prices for staging and transaction costs, we assume a dress that does not retire from the rental inventory by the end of the season to have a salvage value of \$100 and a dress that retires from the rental inventory to have a salvage value of \$30. We calculate procurement costs separately for these two types of dresses by combining their purchase prices, holding costs, and salvage values. For a 26-week horizon, the cost of procuring a dress is $k^S := \$149$, which consists of a purchase price of \$226, a holding cost of \$23, and a salvage value of \$100. A dress retiring from the rental inventory incurs an additional penalty of \$70, resulting in a procurement cost of $k^L := \$219$. Finally, we choose $c := \$5$ as a customer goodwill penalty for the loss of a sale. With these parameters, a dress must be rented five times (seven times, on average, if the

Figure 4 **The Optimal Inventory Level Increases with the Loss Probability for a 26-Week System, and Ignoring Inventory Loss Significantly Reduces Expected Profit Compared to the Optimal Inventory Level**



possibility of loss exists) to break even based on the ratio k^S/c (k^L/c).

With each period corresponding to a week, we consider Poisson distributed demand with a mean of $\lambda = 7$ per week and a rental horizon of $N = 26$ weeks, which corresponds to one of two major fashion seasons each year. We will also consider a longer rental horizon of $N = 52$ for a dress that could be in style for two consecutive seasons. We model each rental duration as lasting for a constant of $A = 2$ periods; i.e., the rented dress will be unavailable during the weekend for which it is rented and the weekend either preceding or following that weekend, depending on the day of the week on which the rental begins. A more granular representation of the rental duration in terms of the individual days is certainly possible. However, we believe that weekly periods adequately represent the system under the assumption that customers of Rent the Runway rent dresses primarily for weekend events.

6.2. Rental Inventory Loss

We first investigate the importance of accounting for the possibility of usage-based loss when choosing the initial inventory of rental units. Allowing the lifetime of each rental unit to follow a geometric distribution with a loss probability of $p \in \{0, 0.01, 0.02, 0.05, 0.10\}$, we illustrate the expected profit as a function of the initial inventory of rental units for the short rental horizon of $N := 26$ periods in Figure 4. The optimal solution for a model that does not include inventory loss—essentially, that of Baron et al. (2011)—is provided as a benchmark. Consistent with Proposition 1, we observe the expected profit function to be concave in the number of rental units to procure in the beginning of the rental season. In the system with no inventory loss ($p = 0$), we identify the optimal solution as 16 units with a corresponding service rate of 93%; i.e., the percentage of customers that are served. However, when there is the possibility of inventory loss, i.e., $p > 0$, we find the optimal number of rental units to increase in the rental unit loss probability. Specifically, for a 5% loss probability, the optimal policy is to add three rental units to the initial inventory. Hence, ignoring inventory loss and using 16 rental units instead of the optimal 19 rental units results in a reduction of 7.3% in the expected profit. Furthermore, the service rate would only be 79.4% instead of the 88.7% corresponding to the optimal number of rental units for the system with $p = 5\%$.

Figure 5 shows that the impact of ignoring inventory loss is more dramatic for the longer rental horizon covering 52 weeks than for the shorter rental horizon with 26 weeks. This can be explained by the availability of fewer rental units to rent toward the end of the longer rental horizon. The comparison of Figure 5 with Figure 4 reveals more asymmetry in the expected profit as a function of the initial inventory of rental units for the longer horizon. More specifically, the slope of the expected profit function for a lower value of the number of rental units is steeper because each rental unit averts more lost sales in a long horizon than in a short horizon. Furthermore, the higher optimal service rate for the system with the longer horizon than the system with the shorter horizon reflects the higher value of a marginal rental unit. In other words, the consequence of having too few rental units is more severe in the longer horizon.

For rental systems considered in Figures 4 and 5, the optimal policy is to always add more rental inventory to account for the loss of rental units; i.e., the profit-maximizing inventory level is increasing in p for $p \in \{0, 0.01, 0.02, 0.05, 0.10\}$. However, if the loss probability is sufficiently high, then the units will not be rented enough to justify having any stock at all,

Figure 5 Accounting for Inventory Loss Is More Important in Terms of Effect on Expected Profit for a 52-Week System

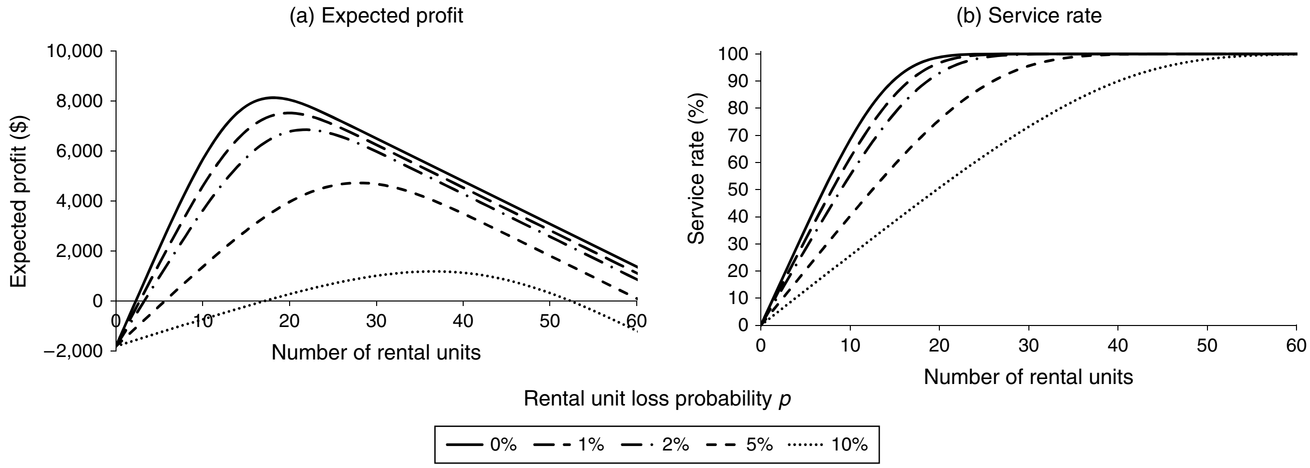
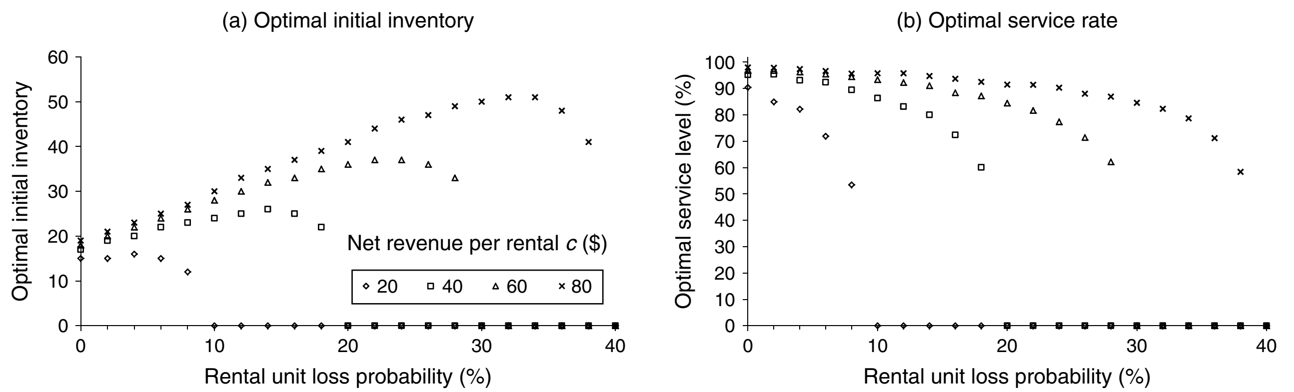


Figure 6 The Optimal Inventory Level Is Nonmonotonic in the Rental Unit Loss Probability p



which means that the optimal policy is to not stock any rental units. Figure 6 illustrates such a policy by considering $r \in \{20, 40, 60, 80\}$ for the net revenue per rental to represent varying levels of profitability per rental, $N = 26$ weeks for a rental horizon, and $\lambda = 7$ for the mean demand. We observe that the optimal response to an increasing inventory loss probability is to initially increase the inventory of rental units until we reach a certain value of the loss probability p associated with the optimal number of rental units $y^\star$ to procure in the beginning of the rental horizon. As p continues to increase, the optimal number of rental units decreases and the optimal service level also appears to be nonincreasing in the loss probability. Eventually, a loss probability $\hat{p}(r)$ is reached such that $y^\star = 0$ for all $p \geq \hat{p}(r)$. Naturally, the optimal service level and $\hat{p}(r)$ increase with r because a dress with a higher net revenue per rental requires fewer rentals to be profitable.

We also provide two simple bounds on the number of lost rental units and two heuristics for choosing the number of rental units to procure when the loss probability is p and $k^S \leq k^L$ so that a penalty to the salvage value is associated with loss. Both of the bounds also translate to easy-to-implement policies and may represent the approach that a manager could use to compensate for usage-based loss by adding extra rental units. Allowing the subscript 0 to denote variables for a system with no usage-based loss, a manager who has chosen the optimal initial inventory level y_0^* when $p = 0\%$—possibly through techniques described by Baron et al. (2011) or by approximating the system as an $M/G/c/c$ queue—can simply increase the initial inventory level by the approximate number of lost rental units.

First, we present an upper bound (UB1) based on the assumption that all demand is served using the relationship $\mathbb{E}[\mathcal{R}(y, \boldsymbol{\xi})] \leq \mathbb{E}[\sum_{n=1}^{N} D_n]$, where $D_1, D_2, \ldots, D_N$ are random variables representing the demand over the horizon. Because the expected number of lost rental units is $\mathbb{E}[\mathcal{L}(y, \boldsymbol{\xi})] = p\,\mathbb{E}[\mathcal{R}(y, \boldsymbol{\xi})] \leq p\,\mathbb{E}[\sum_{n=1}^{N} D_n]$, we have $y^{\text{UB1}} = y_0^* + p\,\mathbb{E}[\sum_{n=1}^{N} D_n]$ as an upper bound on the optimal inventory level. A second bound (UB2) is derived from assuming that all rental units have 100% utilization resulting from demand that is sufficiently large in each period. Denoting the expected rental duration by $\bar{a}$, the expected maximum number of customers that a rental unit can serve is $\lceil N/\bar{a} \rceil$, where $\lceil \cdot \rceil$ is the ceiling

Table 2 Performance of Simple Policies Based on Estimates of the Number of Lost Rental Units and Service Rate

Loss probability p	0%	1%	2%	5%	10%
y^*	16	17	18	19	21
y^{UB1}	16	18	20	25	34
$1-(\pi^*-\pi^{\text{UB1}})/\pi_0^*$ (%)	100.0	99.3	97.2	89.4	70.9
y^{UB2}	16	18	20	26	37
$1-(\pi^*-\pi^{\text{UB2}})/\pi_0^*$ (%)	100.0	99.3	97.2	85.8	57.4
y^{SR1}	16	17	18	20	25
$1-(\pi^*-\pi^{\text{SR1}})/\pi_0^*$ (%)	100.0	100.0	100.0	99.9	97.3
y^{SR2}	16	17	17	19	18
$1-(\pi^*-\pi^{\text{SR2}})/\pi_0^*$ (%)	100.0	100.0	100.0	100.0	98.6

Note. Performance is scaled by π_0^* to allow for comparisons when π^* is close to 0.

function. Thus, $\mathbb{E}[\mathcal{R}(y,\boldsymbol{\xi})]\le y\lceil N/\bar{a}\rceil$ and $\mathbb{E}[\mathcal{Z}(y,\boldsymbol{\xi})]=p\mathbb{E}[\mathcal{R}(y,\boldsymbol{\xi})]\le py\lceil N/\bar{a}\rceil$. The upper bound on the number of rental units to procure is then $y^{\text{UB2}}=y_0^*+py_0^*\lceil N/\bar{a}\rceil$.

If all demand is assumed to be satisfied (UB1), the expected number of rentals is 182, and the expected number of lost rental units is 182p. If all rental units are fully utilized (UB2), each rental unit will be rented out exactly 13 times, which results in a maximum number of rentals of 208 and a maximum expected number of lost rental units of 208p. Table 2 shows the initial inventory levels and their performance in relation to the optimal policy. Values in Table 2 are scaled by π_0^* to enable comparisons when the system's profit is near zero. As expected, given that the policies are upper bounds on the number of lost rental units, the initial inventory level for both bounds exceeds that of the optimal policy. For example, when $p=5\%$, the initial inventory should be increased from 16 to 19 for the optimal policy due to inventory loss. However, under UB1, the initial inventory is increased to 25, and the profit achieved compared to the optimal profit is 89.4% of the value of π_0^*. Under UB2, the initial inventory is increased to 26, and the corresponding profit achievement percentage is 85.8%.

We next present the SR1 heuristic, which focuses on preserving the service rate (i.e., number of rentals) instead of establishing bounds on the number of lost rental units. In the example with $p=0\%$ from the previous paragraph, the $y_0^*=16$ rental units in the optimal solution are expected to satisfy approximately $\mathbb{E}[\mathcal{R}_0(y_0^*)]=169.5$ units of demand, or a mean of 10.6 units of demand served per rental unit. Because of loss, the 16 rental units might not be able to serve all 169.5 units of demand. Therefore, we add rental units with the expectation that a rental unit that is not lost serves $\mathbb{E}[\mathcal{R}_0(y_0^*)]/y_0^*$ units of demand and that a rental unit that is lost serves up to $\mathbb{E}[\mathcal{R}_0(y_0^*)]/y_0^*$ units of demand. With loss probability p, we then approximate the expected number of rentals served by one rental unit as $\beta(p)$, defined as

$$\beta(p)=\sum_{i=1}^{\lfloor\mathbb{E}[\mathcal{R}_0(y_0^*)]/y_0^*\rfloor}(1-p)^{i-1}+(\mathbb{E}[\mathcal{R}_0(y_0^*)]/y_0^*-\lfloor\mathbb{E}[\mathcal{R}_0(y_0^*)]/y_0^*\rfloor)\cdot(1-p)^{\lfloor\mathbb{E}[\mathcal{R}_0(y_0^*)]/y_0^*\rfloor},$$

where $\lfloor\cdot\rfloor$ represents the floor function. So that approximately $\mathbb{E}[\mathcal{R}_0(y_0^*)]$ units of demand are served over the horizon,

$$y^{\text{SR1}}=\frac{\mathbb{E}[\mathcal{R}_0(y_0^*)]}{\beta(p)}.$$

When $p=5\%$ in the example above, we expect each rental unit to serve 8.4 units of demand. The number of rental units needed to serve 169.5 units of demand is then 20.2 rental units. As shown in Table 2, we find this heuristic to substantially improve upon the bounds that were proposed in the previous paragraph for larger values of p. For $p=5\%$ in which $y^*=19$, we then have $y^{\text{SR1}}=20$, which achieves 99.9% of the optimal profit.

We also present a second heuristic, SR2, based on SR1 but accounting for the observation from Figure 6 that the optimal service rate decreases with the loss probability. We estimate the break-even loss probability $\hat{p}(r)$ as $\tilde{p}(r)$, defined as

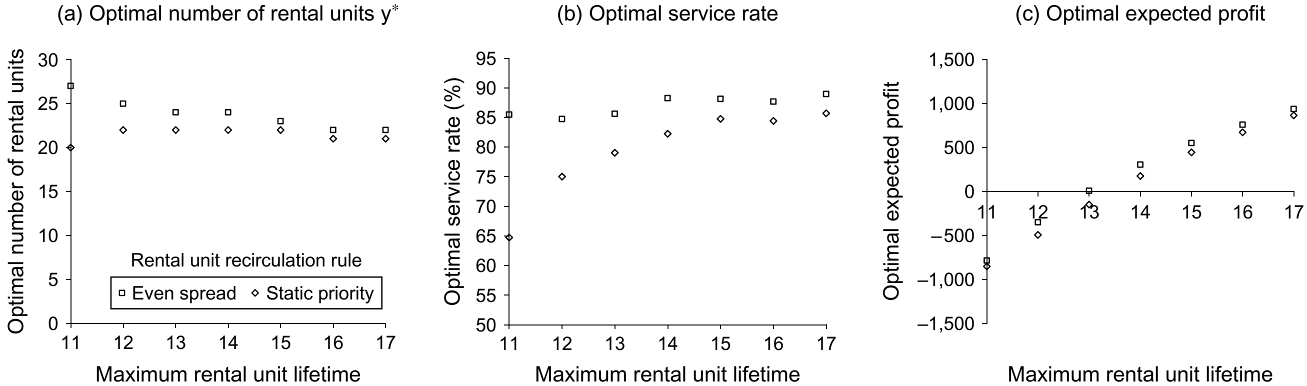
$$(r+c)\beta(\tilde{p}(r))=k^S+(k^L-k^S)(1-(1-p)^{\beta(\tilde{p}(r))}),$$

in which the benefit of serving $\beta(\tilde{p}(r))$ customers equals the sum of the cost to procure a rental unit and the potential cost of loss after $\beta(\tilde{p}(r))$ rentals. We then choose the number of rental units to procure by modifying the target number of customers to serve for $p\in[0,\tilde{p}(r)]$ as

$$y^{\text{SR2}}=\frac{\sqrt{\mathbb{E}^2[\mathcal{R}_0(y_0^*)](1-p^2/\tilde{p}^2(r))}}{\beta(p)},$$

with y^{SR2} rounded to the nearest whole number. To model the optimal inventory level as a decreasing function of p in a simple fashion, we use the formula for an ellipse centered on the origin with two vertices at points $(0,\mathbb{E}[\mathcal{R}_0(y_0^*)])$ and $(\tilde{p}(r),0)$. For $p=5\%$, the SR2 heuristic recommends procuring 19 rental units as in the optimal solution. The break-even loss probability is estimated to be 14.6%, and the SR2 heuristic recommends procuring only 18 rental units when $p=10\%$ and zero rental units for $p\ge 14.6\%$. We provide additional numerical tests in the online appendix that show that the SR2 heuristic either outperforms or provides very similar performance to the SR1 heuristic for the scenarios corresponding to Figures 5 and 6.

Figure 7 Choosing the Even Spread Policy Over the Static Priority Policy Increases the Optimal Initial Inventory Level, Service Rate, and Expected Profit



6.3. Rental Unit Recirculation Rules

Different recirculation rules employed during the rental horizon may result in different numbers of units available near the horizon's end. We expect that the importance of the rental unit recirculation policy varies according to factors such as the horizon length, rental unit lifetime distribution, and demand characteristics. Of concern to us is a horizon that is short enough that some rental units are still functional by the end of the last time period but long enough that some rental units have already retired from the rental inventory during the season. In this section, we compare the even spread and static priority policy for the count-based model. (Similar managerial insights and results can be obtained for the best-first and worst-first rules of the condition-based model.)

Executives at Rent the Runway indicate that the policy used in practice more closely resembles the static priority policy than the even spread policy: out of convenience, dresses that have just returned from cleaning after a rental may be selected to satisfy the next rental. However, because individual units are not tracked, there may be an element of randomness in dress selection as workers select a dress to rent out. The goal of this section is to quantify the effect of using the even spread policy for rental unit recirculation over the static priority recirculation rule. For an adequate representation of the role of the rate at which the loss probability is increasing, we consider the lifetime of a rental unit to be a discrete uniform random variable that takes values between 1 and $A_{\max} \in \{10, 11, \ldots, 20\}$ rentals. As before, we consider a rental horizon of 26 periods and a mean demand of seven units, with all other parameters remaining the same.

The rental system illustrated in Figure 7 is only profitable when $A_{\max} \geq 13$ for the even spread policy and when $A_{\max} \geq 14$ for the static priority policy due to the costs incurred when rental units are lost. Consistent with Proposition 4, the even spread policy achieves a higher expected profit than the static priority policies. This performance difference can be explained by the nature of the even spread policy to delay the failure of rental units until later periods; thus, the even spread policy satisfies more demand in later periods compared to the static priority recirculation policy.

The optimal number of rental units for the even spread recirculation rule, as well as the corresponding service rate, exceeds that of the static priority rule. Choosing the even spread rule instead of the static priority rule allows for rental units to be profitably added, thereby increasing the service rate. For example, the optimal initial inventory level is two units higher for even spread policy than the static priority policy when $A_{\max} = 14$, and the service rate is six percentage points higher for the even spread policy.

7. Conclusion

We develop a discrete-time rental model with random usage-based loss of inventory that also includes arbitrarily distributed customer demands and random rental durations and identify structural properties for this model. The concavity of the expected profit function in the initial inventory of rental units is shown to hold for geometrically distributed rental unit lifetimes regardless of the rental unit recirculation rule. When rental unit lifetimes are generally distributed, we also show the concavity of the expected profit function in the initial inventory of rental units for simple rental unit recirculation rules that are count-based or condition-based. We further demonstrate the optimality of the even spread policy in the count-based setting and the best-first policy in the condition-based setting when the loss probability of each rental unit increases with the number of times it is rented.

Several important insights emerge from a numerical analysis of our rental inventory management solutions for a high-fashion dress rental business. First, we find that the possibility of inventory loss during the rental season can significantly affect profitability, even with a small probability of loss each time that a unit is rented. Choosing the number of rental units to procure in the beginning of the rental season by ignoring the effect of rental inventory loss can reduce the expected profit by 7%. We provide bounds on the optimal inventory level and heuristics for choosing the number of rental units to procure that perform well. Second, we examine how the optimal inventory policy responds to the increasing loss probability. We show that the optimal policy is to first procure additional rental units, then decrease the number of rental units to be procured, and eventually procure zero rental units. Finally, we consider rental unit lifetime distributions with loss probabilities that are increasing in the number of rentals. For horizon lengths and lifetime distributions in which the recirculation rule affects the expected profit, we show that choosing the even spread policy allows for more inventory to be profitably obtained and can increase the service level by up to six percentage points.

Future research directions in the study of rental inventory management include multiple products with stock-out based substitution, advance reservation acceptance policies, in-season reordering and maintenance decisions, and rental pricing.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/msom.2016.0576.

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