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Portfolio Choice with Illiquid Assets

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We present a model of optimal allocation to liquid and illiquid assets, where illiquidity risk results from the restriction that an asset cannot be traded for intervals of uncertain duration. Illiquidity risk leads to increased and state-dependent risk aversion and reduces the allocation to both liquid and illiquid risky assets. Uncertainty about the length of the illiquidity interval, as opposed to a deterministic nontrading interval, is a primary determinant of the cost of illiquidity. We allow market liquidity to vary from “normal” periods, when all assets are fully liquid, to “illiquidity crises,” when some assets can only be traded infrequently. The possibility of a liquidity crisis leads to limited arbitrage in normal times. Investors are willing to forgo 2% of their wealth to hedge against illiquidity crises occurring once every 10 years.

Keywords: asset allocation; liquidity; alternative assets; liquidity crises

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1. Introduction

Many financial assets are illiquid, and this lack of liquidity can often be traced to difficulty in finding a counterparty with whom to trade. In several markets, appropriate counterparties need to have specialized abilities and capital, which are in limited supply. Importantly, the waiting time until the next opportunity to trade is uncertain.¹ Furthermore, systematic variation in the level of market liquidity is possible as financial intermediaries receive negative shocks and withdraw from market making. In this paper, we investigate the effects of this illiquidity and illiquidity risk on asset allocation.

We develop a tractable model of illiquidity. An illiquid asset can only be traded contingent on the arrival of a randomly occurring trading opportunity—a liquidity event—modeled as an independent and identically distributed Poisson process. We interpret these random trading times as the reduced-form outcome of

a search process to find an appropriate counterparty (e.g., Diamond 1982). Since the illiquid asset cannot be traded for an *uncertain* period of time, the investor is exposed to risk that cannot be hedged.

Illiquidity risk affects the portfolio choice problem in two ways. First, liquid and illiquid wealth are imperfect substitutes. The investor’s immediate obligations—consumption or payout—can only be financed through liquid wealth. If the investor’s liquid wealth drops to zero, these obligations cannot be met until after the next liquidity event. Thus, the investor reduces her allocation to *both* the liquid and illiquid risky assets to reduce the probability that a state with zero liquid wealth—as opposed to only zero total wealth—is reached. This increase in effective risk aversion corresponds to real-world situations where investors or investment funds are insolvent, not because their assets under management have hit zero, but because they cannot fund their immediate obligations. The resulting underinvestment in illiquid assets relative to the Merton benchmark is substantial.² Second, since the investor’s

¹ Examples include structured credit products, private equity, and venture capital; small equity and bond issues; or large real estate and infrastructure projects. In some of these markets, the waiting time until the next opportunity to trade is uncertain because the number of participants is small. In other cases, for instance in private equity and venture capital limited partner investments, the time of exit and reinvestment is stochastic and depends on the initial public offering or mergers and acquisitions markets.

² A standard calibration indicates that if the expected time between liquidity events is once a year, the investor should cut her investment in the illiquid asset by 33% relative to an otherwise identical but fully liquid asset. Furthermore, the investor should be prepared for large, skewed changes in the relative value of illiquid to liquid holdings in her portfolio.

ability to fund intermediate consumption depends on her liquid wealth, fluctuations in the share of illiquid assets in the portfolio induce endogenous *time-varying* risk aversion.

We extend our baseline model of illiquidity to include time-varying arrival rates of liquidity events. Specifically, we allow for infrequently occurring illiquidity crises. There are two distinct regimes. The first regime represents “normal” times, when all assets are fully liquid, as in the Merton benchmark. The second regime represents a “drying up” of liquidity—a liquidity crisis—when one of the risky assets becomes illiquid and can only be traded at infrequent intervals. This model is thus applicable to a wide range of assets that are normally liquid but subject to occasional market freezes.³

The possibility of a liquidity crisis leads to limited arbitrage in normal times. We consider the case in which there are two perfectly correlated securities with different Sharpe ratios. In a portfolio choice model without illiquidity risk, this case presents an arbitrage opportunity: the investor takes positions of plus or minus infinity in the two different assets. When one asset is illiquid, the investor allocates a finite amount—and will not use leverage—in the “arbitrage trades,” even in normal times. Although both securities are fully liquid during normal times, arbitrage trades entail a hidden cost because in the event of a crisis they become imperfect substitutes. The inability to immediately delever at the onset of a crisis implies that potential arbitrageurs avoid leverage in normal times. Hence, investment in apparent arbitrage opportunities is limited by the wealth of arbitrageurs, even when realizing the arbitrage involves no short positions.

We derive the risk premium associated with a systematic liquidity crisis.⁴ Following an approach similar to the intertemporal capital asset pricing model (ICAPM) (Merton 1973), we examine the investor’s marginal value of wealth, taking the prices of other risky assets as given. A transition from the liquid into the illiquid state raises the marginal value of liquid wealth, implying a negative risk premium. Hence, assets that pay off in the onset of a liquidity crisis earn lower risk premia. For typical parameter values, the investor is

willing to pay 0.5%–2% per annum over the actuarial probability of a crisis to receive liquid funds at the onset of a deterioration of market liquidity. Hence, our model provides a theoretical justification of empirical specifications of the stochastic discount factor that load on measures of illiquidity.

Finally, we explore the determinants of the cost of illiquidity by varying the baseline model. First, we compute portfolio policies and the utility cost of illiquidity in a version of our model without intermediate consumption. We confirm that the suboptimal diversification that results from infrequent trade is not sufficient by itself to generate large utility costs of illiquidity. Absent the motive to smooth intermediate consumption, the effect of illiquidity on portfolio policies and welfare is minimal. Second, we compare welfare and portfolio policies between our setting and a version of our model with a deterministic rebalancing interval. The effect of illiquidity on portfolio choice is dramatically larger when the length of the illiquidity period is uncertain. Third, we verify that our quantitative results are not driven by the possibility that the investor reaches states with zero liquid wealth—and thus infinite marginal utility—by allowing the investor to pay a fixed cost to transact immediately. Last, we disentangle the effect of preference parameters; we find that the utility cost of illiquidity is highest for agents that unwilling to substitute across time (low elasticity of intertemporal substitution) but are willing to substitute across states (low risk aversion).

2. Illiquidity in Asset Markets

We motivate and quantify our notion of illiquidity based on a number of stylized facts, and we contrast our model with previous approaches in the literature.

STYLIZED FACT 1. *Most asset classes are illiquid, in the sense that trading is infrequent.*

Table 1 shows that most assets markets are characterized by long times between trades, low turnover, and trade in over-the-counter markets in which it is difficult to find counterparties. Except for “plain vanilla” fixed income securities and public equities, investors need to wait for indeterminate periods before they can rebalance illiquid assets, and sometimes the time between liquidity events can extend to decades. Even within the fixed income and public equity markets, there are subclasses that are illiquid. For instance, although the public equity market has a turnover of well over 100%, corporate bonds have a turnover of approximately 25%–35%. The average municipal bond trades only twice per year; the entire market has a turnover of less than 10% per year. Furthermore, transaction times for many over-the-counter equities, such as those traded on the pink sheet, NASDAQ, or

³ For example, Brunnermeier (2009), Gorton (2010), Tirole (2011), and others have highlighted market freezes as a stylized fact of the 2008–2009 financial crisis. This is not simply a question of a seller reducing prices to a level where a buyer is willing to step in. As Tirole (2011) and Krishnamurthy et al. (2014) comment, there were no bids, at any price, representing “buyers’ strikes” in certain markets where whole classes of investors simply exited previously liquid markets.

⁴ This “illiquidity risk premium” refers to the risk premium of an Arrow–Debreu security that pays off at the onset of a liquidity crisis. This risk premium is distinct from the “illiquidity premium” defined as the price discount of an illiquid security.

Table 1 Holding Periods and Turnover of Various Asset Classes

Asset class	Typical time between transactions	Annualized turnover	References
Public equities	Within seconds	> 100%	Turnover can be computed from NYSE and NASDAQ data at nyxdata.com and nasdaqtrader.com, respectively.
OTC (pink sheet) equities	Within a day, but many stocks over a week	~ 35%	Ang et al. (2012)
Corporate bonds	Within a day	25%–35%	Bao et al. (2011)
Municipal bonds	~6 months, with 5% of municipal bonds trading more infrequently than once per decade	<10%	Holding period statistics are computed from an updated version of the data set from Ang et al. (2010) and summarized in Ang and Green (2011). Turnover numbers are from U.S. Securities and Exchange Commission (2004).
Private equity	Funds last for 10 years; the median investment duration is 4 years; secondary trade before exit is relatively unusual	<10%	Private equity contracts are described by Metrick and Yasuda (2010); for duration, see Lopez-de Silanes et al. (2010). For estimates of “secondaries” in private equity, see LAVCA (2012) and Winchell (2010).
Residential housing	4–5 years, but ranges from months to decades	4%–6%	For median duration in residences see Hansen (1998) and Case and Shiller (1989); for comments on the range see Miller et al. (2011). Turnover numbers are computed by Dieleman et al. (2000).
Institutional real estate	8–11 years	~7%	See Fisher and Young (2000) and Collett et al. (2003) for holding periods and Ling et al. (2009) for turnover.
Institutional infrastructure	50–60 years for initial commitment, some as long as 99 years	Negligible	Beeferman (2008) and Bitsch et al. (2010)
Art	40–70 years	<15%	For holding periods, see Goetzmann (1993) and Kaplan (1997). Turnover can be inferred from the size of the art market estimated by Skaterschikov (2006) and estimated annual art sales.

OTCBB markets, are often longer than a week, with a turnover of approximately 35%.

In real estate markets, Levitt and Syverson (2008) report, for example, a typical time to sale between 110 and 135 days after the initial listing of a house. The standard deviation of the time to sale is even larger than the mean, and Levitt and Syverson note that some houses never sell. Lastly, typical holding periods for venture capital and private equity portfolios are 3 to 10 years. Although the investment horizon is nominally fixed, partnerships often return investor’s money prior to the partnership’s formal 10-year end. Furthermore, these times are stochastic. For example, in private equity, the median investment duration is four years with 16% returned before two years and 26% returned after six years (see, e.g., Lopez-de Silanes et al. 2010).⁵

In most of these markets, illiquidity is characterized by the need to find counterparties to trade. Our notion of illiquidity in which an illiquid asset can only trade when there is a liquidity event—when an appropriate counterparty is found—puts us squarely in the tradition of Diamond (1982). A number of authors have used models with search frictions to consider the impact of illiquidity risk. Duffie et al. (2005, 2007) consider

risk-neutral and constant absolute risk aversion (CARA) utility cases and restrict asset holdings at two levels: zero and one. In Vayanos and Weill (2008), agents can only go long or short one unit of the risky asset. Gârleanu (2009) and Lagos and Rocheteau (2009) allow for unrestricted portfolio choice, but Gârleanu considers only CARA utility and Lagos and Rocheteau focus on proving the existence of equilibrium with search frictions. In contrast to these models, we focus on the investors’ portfolio choice problem with constant relative risk aversion (CRRA) and Epstein–Zin utility, which allows us to consider more general preference specifications that allow for wealth effects.⁶ An alternative, and equally plausible, microfoundation for our notion of liquidity can be found in models with adverse selection. For instance, Daley and Green (2012, 2013) show that adverse selection can generate significant delays in trading times and time-varying costs of liquidity.

Our notion of liquidity is conceptually distinct from two other common conceptualizations previously studied in the portfolio choice literature. The first is the idea that many assets are expensive to trade: securities are partially marketable and can be traded at posted prices, but with transactions costs (e.g., Constantinides 1986, Grossman and Laroque 1990, Vayanos 1998, Lo et al. 2004). In these models, liquidity can *always* be generated by paying a cost. Our results imply that this

⁵ The turnover from trade of private equity investments on the secondary market is much lower. Although data on private equity portfolio turnover are not typically reported, Kensington, a Canadian private equity fund, reports a 2% turnover in 2008. Alpinvest, a large private equity fund-of-funds, reports flows that imply a turnover of approximately 6%. This compares with a typical turnover of over 70% for mutual funds (Wermers 2000).

⁶ Our work also relates to models with unhedgeable human capital risk (e.g., Heaton and Lucas 2000, Santos and Veronesi 2006). An important distinction is that our illiquid asset is infrequently traded, unlike human capital, which is never traded.

uncertain waiting time plays an important role in portfolio decisions.

The second type of liquidity studied in portfolio choice allows assets to be freely traded at posted prices, but only in limited quantities (e.g., Longstaff 2001) or at deterministic times (e.g., Kahl et al. 2003, Koren and Szeidl 2003, Schwartz and Tebaldi 2006, Longstaff 2009, De Roon et al. 2009, Dai et al. 2010). In these models, trade can always be generated at a known rate simply by waiting.⁷ Similarly, the margin-based asset pricing literature (e.g., Gârleanu and Pedersen 2011) develops models with differential borrowing costs and portfolio constraints but then allows for continuous trade within those constraints. In contrast, we show that illiquidity risk leads to investors behaving as if they were subject to portfolio constraints (e.g., not taking short positions in illiquid or potentially illiquid assets) as a response to the illiquidity friction.

STYLIZED FACT 2. *Illiquid asset classes are large.*

The illiquid asset markets listed in Table 1 are large and rival the size of the public equity market. For instance, the market capitalization of the NYSE and NASDAQ is approximately \$17 trillion. The estimated size of the residential real estate market is \$16 trillion, and the estimated size of the (direct) institutional real estate market is \$9 trillion.⁸ Furthermore, the share of illiquid assets in many investors' portfolios is very large. Kaplan and Violante (2010) show that individuals hold the majority of their net wealth in illiquid assets, with 91% and 81% of households' net portfolios tied up in illiquid positions, mostly housing, taking median and mean values, respectively. High net worth individuals in the United States allocate 10% of their portfolios to "treasure" assets like fine art and

jewelry, and the share of treasure assets rises to almost 20% in other countries.⁹ The share of illiquid assets in institutions' portfolios has also dramatically increased over the last 20 years. Pension funds increased their holdings in illiquid ("other") asset classes from 5% in 1995 to close to 20% in 2010, as reported in the "Global Pension Asset Study 2011" by Towers Watson. Data from the National Association of College and University Business Officers show that the (dollar-weighted) average share of illiquid "alternatives" in university endowment portfolios rose from 25% in 2002 to 52% in 2010.

STYLIZED FACT 3. *Normally liquid asset classes sometimes become illiquid.*

An important feature of feature of financial markets is that sometimes liquidity dries up in markets that are normally liquid. For instance, Krishnamurthy et al. (2014) document that in the market for money market funds, a usually liquid market, there were instances of "buyers' strikes" during the recent financial crisis, when investors were unwilling to trade at any price. Anderson and Gascon (2009) note that the commercial paper market froze not only in the 2008–2009 financial crisis, but it also froze in 1970 when the Penn Central railroad collapsed. In both cases, the Federal Reserve stepped in to help to restore liquidity.

These illiquidity crises occur regularly in many asset markets.¹⁰ Other examples include the repo market (Gorton and Metrick 2012); residential and commercial mortgage-backed securities (Gorton 2009, Acharya and Schnabl 2010, Dwyer and Tkac 2009); structured credit (Brunnermeier 2009); and the auction rate security (ARS) market (McConell and Saretto 2010), which became illiquid in 2008 and was still frozen in 2013.¹¹ Borio (2004) notes that liquidity also dries up during periods of severe market distress: the Latin American debt crisis in the 1980s, the Asian emerging market crisis in the 1990s, the Russian default crisis in 1998 (see also Elul 2008), and of course the financial crisis of 2008–2009. Major liquidity crises have occurred at least

⁷ The only other model that features random opportunities to trade is Rogers and Zane (2002), who solve a model with random trading opportunities and no liquid risky assets using asymptotic expansions near the Merton benchmark $1/\lambda \rightarrow 0$. However, Rogers and Zane do not prove that these expansions are valid. In contrast to Rogers and Zane, we solve the ordinary differential equations characterizing the investors' problem numerically and intentionally consider realistic cases where $1/\lambda$ is large, as is the case for many illiquid asset markets (see Table 1). The behavior of the model as $1/\lambda \rightarrow 0$ can be very different from the Merton benchmark. In particular, even as $1/\lambda \rightarrow 0$, the investor is still trading on a set of measure zero and hence would never take a short position in either liquid or illiquid wealth.

⁸ NYSE and NASDAQ market capitalizations are approximately \$12 trillion and \$5 trillion as of July 2012, according to nyxdata.com and nasdaqtrader.com. The estimated size of the U.S. residential real estate market is as of December 2011 and is estimated by Keely et al. (2012), down from a peak of \$23 trillion in 2006. The estimate of the U.S. institutional real estate market is by Florance et al. (2010), with the institutional real estate market losing \$4 trillion from 2006 to 2010. The direct real estate market dwarfs the traded real estate investment trust (REIT) market, with the FTSE NAREIT All Equity REITs Index having a total market capitalization of approximately \$500 billion at the end of June 2012.

⁹ Reported in "Profit or Pleasure? Exploring the Motivations Behind Treasure Trends," Wealth Insights, Barclays Wealth and Investment Management, 2012.

¹⁰ Practitioners refer to these events as "liquidity black holes." See Taleb (1997), Persaud (2001), and Morris and Shin (2010).

¹¹ As noted by U.S. Securities and Exchange Commission (2012): In 2008, the auction rate securities (ARS) market totaled approximately \$200 billion; in February 2008, the market froze because there were no bidders in the primary auctions, where floating interest rates are set. Because there was no secondary market, thousands of customers were unable to sell their ARS holdings. In 2011, there were no new issues of ARS. Other intermediated fund vehicles also became more illiquid during this time: hedge funds, for example, imposed "gates" provisions that did not allow investors to withdraw capital (see Ang and Bollen 2010).

once every 10 years, many occurring in tandem with large downturns in asset markets.

To allow for infrequent liquidity crises, we extend our baseline model to include two regimes. The first regime allows all assets to be traded continuously. The second, corresponding to a liquidity crisis, allows the illiquid assets to be traded only at liquidity events. In our model, investors underinvest in arbitrage opportunities during normal times because they carry illiquidity risk if a liquidity crisis arrives. Other models also generate limited arbitrage (see, e.g., Shleifer and Vishny 1997, Gromb and Vayanos 2002), but in these models investors underinvest in arbitrage opportunities because demand shocks to other investors can push prices farther away from fundamentals. Limited arbitrage occurs in our setting, in contrast, because of the risk of a market freeze. At the onset of a liquidity crisis, arbitrageurs cannot reduce leveraged positions to prevent states with zero consumption. Hence, they are unwilling to employ leverage in normal times, and arbitrage activity is limited by their wealth.

Last, our model provides a theoretical framework to study the pricing of liquidity risk. Recent empirical work has documented that several measures of liquidity are priced in the cross section of asset returns (e.g., Pastor and Stambaugh 2003, Acharya and Pedersen 2005, Sadka 2006, Korajczyk and Sadka 2008). Specifically, stocks with heterogeneous exposure to various measures of market liquidity earn different average stock returns, often controlling for their own level of liquidity. To the extent that these empirical measures of liquidity are correlated with the difficulty of finding a counterparty with whom to trade, our model provides a framework to quantify the magnitude of this liquidity. In the spirit of Merton's ICAPM, we derive the magnitude of this risk premium directly from the first-order conditions of an optimizing investor faced with the possibility of a liquidity crisis. Relative to Acharya and Pedersen (2005), who also propose a theoretical model with liquidity risk, the utility cost of illiquidity is endogenous in our model.

3. The Baseline Model

3.1. Information

The information structure obeys standard technical assumptions. There exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of two independent Brownian motions $Z_t = (Z_t^1, Z_t^2)$ and an independent Poisson process (N_t) ; \mathcal{P} is the corresponding measure and \mathcal{F} is a right-continuous increasing filtration generated by $Z \times N$.

3.2. Assets

There are three assets in the economy: a risk-free bond B , a liquid risky asset S , and an illiquid risky

asset P . The riskless bond B appreciates at a constant rate r :

$$dB_t = rB_t dt. \quad (1)$$

The second asset S is a risky asset whose price follows a geometric Brownian motion with drift μ and volatility σ :

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t^1. \quad (2)$$

The first two assets B and S are liquid, and holdings can be rebalanced continuously.

The third asset P is an illiquid risky asset; its fundamental value evolves according to a geometric Brownian motion with drift ν and volatility ψ :

$$\frac{dP_t}{P_t} = \nu dt + \psi \rho dZ_t^1 + \psi \sqrt{1 - \rho^2} dZ_t^2, \quad (3)$$

where ρ captures the correlation between the returns on the two risky assets. The illiquid asset P differs from the first two assets B and S because it can only be traded at stochastic times τ , which coincide with the arrival of a Poisson process with intensity λ . The parameter λ captures the severity of the illiquidity friction; the expected time between liquidity events is $1/\lambda$. When a trading opportunity arrives, the investor can trade at the price P_t without any other frictions.

In addition, the illiquid asset P cannot be pledged as collateral. Investors can issue non-state-contingent debt by taking a short position in the riskless bond B ; however, they cannot issue risky debt using the illiquid asset as collateral. If investors were allowed to do so, they could convert the illiquid asset into liquid wealth and thus implicitly circumvent the illiquidity friction.

3.3. The Investor

The investor has CRRA utility over sequences of consumption, C_t , given by

$$\mathbb{E} \left[\int_0^\infty e^{-\beta t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right], \quad (4)$$

where β is the subjective discount factor and $\gamma > 1$.¹²

The agent's wealth has two components, liquid and illiquid wealth. Liquid wealth includes the amount invested in the liquid risky asset and the risk-free asset. Illiquid wealth, which equals the amount invested in the illiquid asset, cannot be immediately consumed or converted into liquid wealth. The joint evolution of the

¹² The $\gamma = 1$ (log) case is qualitatively similar to the $\gamma > 1$ case. Quantitatively, the results for the log case can actually be more extreme because the low level of risk aversion results in a very high percentage of wealth invested in risky assets in the benchmark Merton economy. We discuss variation in risk aversion in §5.4, and a characterization of the Hamilton–Jacobi–Bellman equation in the log case is available from the authors on request.

investor's liquid wealth, W_t , and illiquid wealth, X_t , is given by

$$\frac{dW_t}{W_t} = (r + (\mu - r)\theta_t - c_t) dt + \theta_t \sigma dZ_t^1 - \frac{dI_t}{W_t}, \quad (5)$$

$$\frac{dX_t}{X_t} = \nu dt + \psi \rho dZ_t^1 + \psi \sqrt{1 - \rho^2} dZ_t^2 + \frac{dI_t}{X_t}. \quad (6)$$

The agent invests a fraction θ of her liquid wealth into the liquid risky asset, while the remainder $(1 - \theta)$ is invested in the bond. Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible trading strategies, θ , to those that satisfy the standard integrability conditions. All policies are appropriately adapted to \mathcal{F}_t . The agent consumes liquid wealth, so liquid wealth decreases at rate $c_t = C_t/W_t$. When a trading opportunity arrives, the agent can transfer an amount dI_t from her liquid wealth to the illiquid asset.

Finally, we assume the standard discount rate restriction, as in the Merton two-risky-asset model,

$$\begin{aligned} \beta &> (1 - \gamma)r \\ &+ \frac{1 - \gamma}{2\gamma(1 - \rho^2)} \left(\left(\frac{\mu - r}{\sigma} \right)^2 - 2\rho \left(\frac{\mu - r}{\sigma} \right) \left(\frac{\nu - r}{\psi} \right) \right. \\ &\quad \left. + \left(\frac{\nu - r}{\psi} \right)^2 \right), \end{aligned} \quad (7)$$

and the fact that the illiquid asset has at least as high a Sharpe ratio as the liquid asset,

$$\frac{\nu - r}{\psi} \geq \frac{\mu - r}{\sigma}. \quad (8)$$

3.4. Discussion of Assumptions

Our assumption that the illiquid asset P cannot be collateralized is motivated by the difficulty of finding a counterparty who is willing to lend cash using illiquid assets as collateral.¹³ Alternatively, we could reinterpret P as the fraction of illiquid wealth that cannot be collateralized. This interpretation is similar to the portfolio constraints analyzed in Gârleanu and Pedersen (2011).¹⁴ The key difference is that we introduce an *additional*

¹³ For instance, Krishnamurthy et al. (2014) find evidence suggesting that money market mutual funds, which are the main providers of repo financing, were unwilling to accept private asset-backed securities as collateral between the third quarter of 2008 and the third quarter of 2009. Even when illiquid assets like real estate, private equity, and even art can be used as collateral, an investor cannot borrow an amount equal to the whole value of the illiquid asset position, and thus our decomposition into liquid and illiquid assets is still valid.

¹⁴ In the case of real estate, we could interpret the illiquid asset P as the fraction of the value of the property that cannot be used as collateral against a mortgage or a home equity line. Our interpretation assumes that the amount that the asset can be collateralized does not vary over time and that the constraint is always binding. We could

stochastic trading friction on the noncollateralizable portion of wealth; namely that it can be traded only infrequently.

Our assumption of an infinite horizon for the investor is conservative; any effects of illiquidity are magnified with finite horizons. For instance, if opportunities to trade arise every 10 years, on average, then an investor with a one-year horizon views the illiquid asset as a very unattractive asset. Thus, the portfolio weights, effects on consumption policies, and certainty equivalent compensations for bearing illiquidity risk should all be viewed as conservative bounds for finite-horizon investors.

4. The Solution to the Baseline Model

Markets are not dynamically complete; hence we use dynamic programming techniques to solve the investor's problem. First, we establish some basic properties of the solution. Then, we compute the investor's value function and optimal portfolio and consumption policies.

4.1. The Value Function

The agent's value function is equal to the discounted present value of her utility flow,

$$F(W_t, X_t) = \max_{\{\theta, I, c\}} E_t \left[\int_t^\infty e^{-\beta(s-t)} U(C_s) ds \right]. \quad (9)$$

PROBLEM 1 (BASELINE). The investor performs the maximization in (9), subject to the two intertemporal budget constraints (5) and (6), with rebalancing ($dI_t \neq 0$) only when the Poisson process N_t^λ jumps.

Our first step is to establish that the investor does not use leverage, i.e., that liquidity risk eliminates any willingness by the investor either to short the illiquid asset or to fund long purchases of the illiquid asset using a net short position in liquid wealth.

PROPOSITION 1. Any optimal policies in Problem 1 will have $W > 0$ and $X \geq 0$ almost surely.

Thus, without loss of generality, we restrict our attention to solutions with $W_t > 0$ and $X_t \geq 0$.

Second, the value function is bounded below by the problem in which the illiquid asset does not exist, and it is bounded above by the problem in which the entire portfolio can be continuously rebalanced: the Merton one- and two-asset problems.

Third, the utility function is homothetic and the return processes have constant moments, and so it

extend the model to allow the investor to endogenously choose the amount of collateralized borrowing every period, up to a limit. This model is equivalent to a hybrid model of infrequent trading and transaction costs, with similar qualitative effects to the extension we introduce in §5.3.

must be the case that F is homogeneous of degree $1 - \gamma$:

$$F(W, X) = (W + X)^{1-\gamma} H(\xi), \quad \text{where } \xi \equiv \frac{X}{X + W}. \quad (10)$$

Thus, the investor's value function can be represented as a power function of total wealth times a function $H(\xi)$ of the fraction of her portfolio held in illiquid assets, ξ .

Our fourth step is to characterize the value function at the instant when the agent can rebalance between her liquid and illiquid wealth. When the Poisson process hits and the agent rebalances her portfolio, the value function will jump discretely. Denote the new, higher value function as $F^*(W_t, X_t)$, so that the total amount of the jump is $F^* - F$. At the Poisson arrival, the agent is free to make changes to her entire portfolio, and thus

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t]} F(W_t - I, X_t + I). \quad (11)$$

Since F^* must also be homogeneous of degree $1 - \gamma$, there exists a function H^* such that $F^* = (W + X)^{1-\gamma} H^*(\xi)$. Since trading the illiquid asset is costless when a liquidity event arrives, the investor rebalances her portfolio so that the ratio of illiquid to total wealth equals $\xi^* = \arg \max_{\xi} H(\xi)$; hence, H^* is a constant function and equal to $H(\xi^*)$.

We can now solve the baseline investor's problem.

PROPOSITION 2 (BASELINE). *In Problem 1, the investor's value function can be written as in (10), where $H(\xi)$ exists and is finite, continuous, and concave for $\xi \in [0, 1]$. The function $H(\xi)$ obtains its maximum for some $\xi \in [0, 1]$. Define $H^* = \max_{\xi} H(\xi)$ and $\xi^* = \arg \max_{\xi} H(\xi)$. When a trading opportunity occurs at time τ , the trader selects I_{τ} so that $X_{\tau}/(X_{\tau} + W_{\tau}) = \xi^*$. In addition, $H(\xi)$ is characterized between liquidity events by*

$$0 = \max_{c, \theta} \left[\frac{1}{1 - \gamma} c^{1-\gamma} (1 - \xi)^{1-\gamma} - \beta H(\xi) + \lambda (H^* - H(\xi)) \right. \\ \left. + H(\xi) A(\xi, c, \theta) + \frac{\partial H(\xi)}{\partial \xi} B(\xi, c, \theta) \right. \\ \left. + \frac{1}{2} \frac{\partial^2 H(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right], \quad (12)$$

where the functions A , B , and C are defined as

$$\begin{aligned} A(\xi, c, \theta) &\equiv (1 - \gamma) \\ &\cdot (r + (1 - \xi)((\mu - r)\theta - c) + \xi(\nu - r) \\ &\quad - \frac{1}{2}\gamma(\xi^2\psi^2 + (1 - \xi)^2\sigma^2\theta^2 + 2\xi(1 - \xi)\psi\rho\sigma\theta)), \\ B(\xi, c, \theta) &\equiv \xi(1 - \xi) \\ &\cdot (\nu - (r + (\mu - r)\theta - c) + \gamma\psi\theta\rho\sigma(2\xi - 1) \\ &\quad + \gamma\theta^2\sigma^2(1 - \xi) + \gamma\psi^2\xi), \\ C(\xi, c, \theta) &\equiv \xi^2(1 - \xi)^2(\theta^2\sigma^2 + \psi^2 - 2\psi\theta\rho\sigma). \end{aligned} \quad (13)$$

The investor's value function has two parts. The first part $(W + X)^{1-\gamma}$ captures the effect of total wealth on the continuation utility. The second component $H(\xi)$ captures the effect of wealth composition between liquid and illiquid wealth. We hold total wealth constant and plot the function $H(\xi)$ in panel A of Figure 1. We can interpret $H(\xi)$ as a composition penalty function: the investor has an optimal portfolio composition ξ^* to which she returns whenever she is able to rebalance. Between liquidity events, she experiences a welfare loss for two reasons as her portfolio composition deviates from the optimum. First, there is the standard effect from lack of optimal diversification. Second, there is an asymmetric effect arising from the fact that consumption is funded by liquid wealth only. Examining the slope of $H(\xi)$ as $\xi \rightarrow 1$, we see that this second effect is the main mechanism in our model.

4.2. Imperfect Substitutability of Liquid and Illiquid Wealth

In our model, liquid and illiquid wealth are imperfect substitutes. Illiquid wealth can be used to fund consumption only after the next trading time $t = \tau$. In contrast, liquid wealth can be used to fund consumption both before and after τ . To quantify this nonsubstitutability, consider a fictitious market that lets the investor exchange one unit of illiquid wealth for q units of liquid wealth. Between liquidity events, the investor would be indifferent in participating in this fictitious market as long as

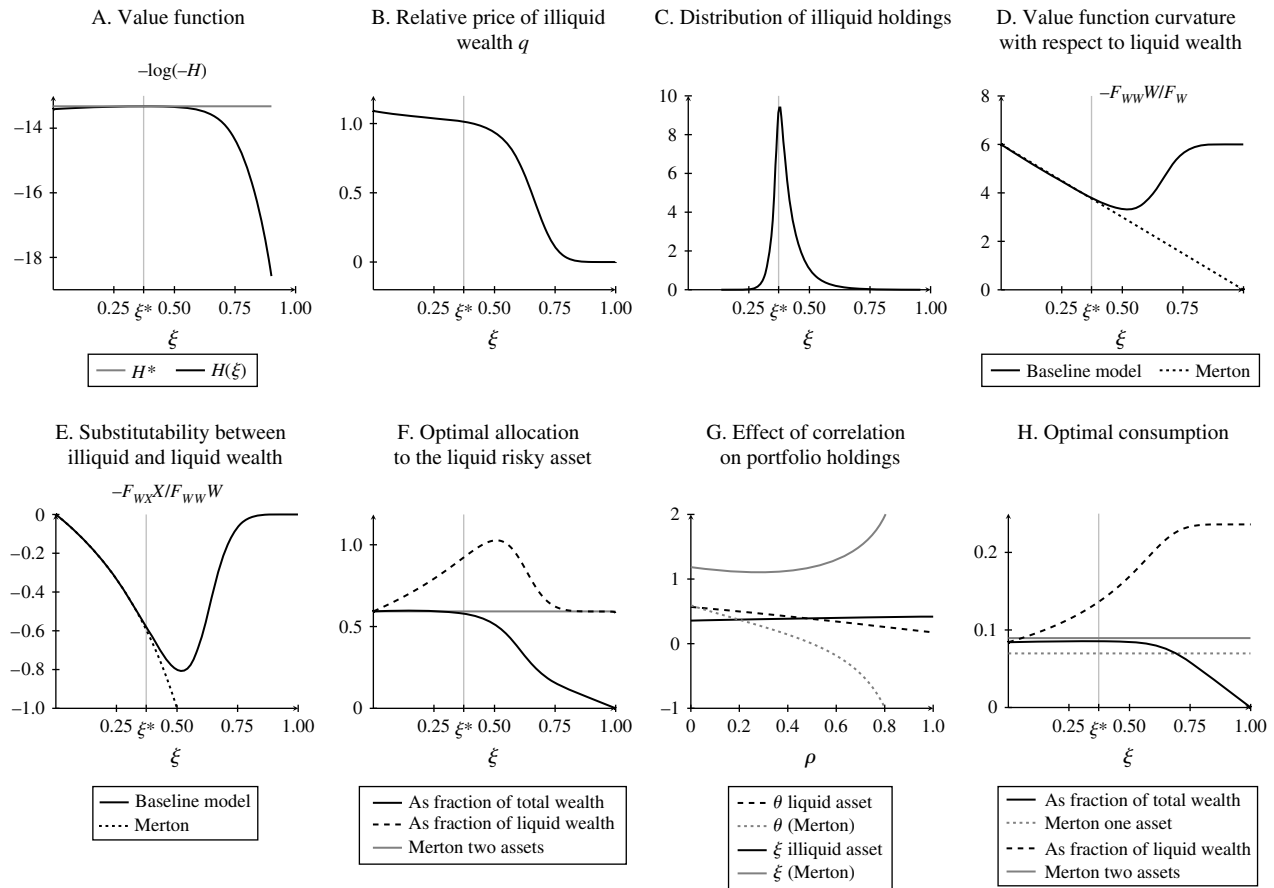
$$q = \frac{F_X}{F_W}. \quad (14)$$

When the investor has the opportunity to rebalance, $q = 1$. Between liquidity events, the relative price q differs from one, depending on whether the investor has too much or too little illiquid wealth X relative to her desired allocation. In panel B of Figure 1, we see that the relative price of illiquid wealth rapidly declines as the investor's allocation to illiquid assets ξ increases. When the illiquid endowment is large, $X \gg W$, liquid wealth W is only used to fund immediate consumption, while illiquid wealth is used to fund future consumption. In this case, variation in liquid wealth becomes unimportant for long-run consumption, and the value function becomes separable in X and W . Hence, although liquid and illiquid wealth may be correlated, that correlation becomes a secondary concern for portfolio allocation.

4.3. Parameter Values

In our numerical solutions, we select our parameter values so that the liquid asset can be interpreted as an investment in the aggregate stock market. We set the parameters of the liquid asset to be $\mu = 0.12$ and $\sigma = 0.15$ and set the risk-free rate to be $r = 0.04$.

Figure 1 Model Solution



Notes. Panel A plots the value function. The vertical solid gray line corresponds to the value of the optimal rebalancing point ξ^* . Panel B plots the relative price of illiquid wealth q . Panel C plots the stationary distribution of allocation to the illiquid asset as a fraction of total wealth, $\xi = X/(X + W)$. We obtain the stationary distribution using a long simulation of 10,000 years. Panels D and E plot the curvature of the value function with respect to liquid wealth, $-F_{ww}W/F_w$, and the elasticity of substitution in the value function between liquid and illiquid wealth, $-F_{wx}X/F_{ww}W$, respectively. The solid lines represent the case where $\lambda = 1$. The dotted lines correspond to the Merton case. In panel E, we assume $\rho = 0.6$. Panel F displays the optimal allocation to the liquid assets. Panel G examines the effect of correlation on risky asset holdings. We plot the optimal allocations to the liquid risky asset θ and the illiquid risky asset ξ as a fraction of total wealth at the rebalancing time, both as a function of ρ . The remainder is allocated to the riskless asset. For this panel we use $\nu = 0.2$. Panel H graphs consumption policy. In panels F and H, the gray horizontal line corresponds to the allocation to the risky asset in the one- and/or two-asset Merton economy. Unless noted otherwise, all figures use $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\lambda = 1$, $\sigma = \psi = 0.15$, and $\rho = 0$.

Table 2 shows that this set of parameters closely matches the performance of the S&P 500 before the financial crisis.¹⁵ We work mostly with the risk-aversion case $\gamma = 6$, which for an investor allocating money between only the S&P 500 and the risk-free asset produces an equity holding close to a classic 60% equity, 40% risk-free bond portfolio used by many institutional investors.

For most of our analysis, we take a conservative approach and set the parameters of the illiquid asset, $\nu = 0.12$ and $\psi = 0.15$, to be the same as those of the liquid asset. This has the advantage of isolating the effects of illiquidity rather than obtaining results attributable

to the higher Sharpe ratio of the illiquid asset. Furthermore, even for individual funds, this assumption is not unrealistic, at least for some illiquid asset classes.¹⁶ These parameters imply that our illiquid asset can be interpreted as any composite investment with the same Sharpe ratio as public equities, for example, a composite illiquid risky bond investment. Furthermore, to

¹⁵ Including 2008–2010 reduces the mean return of the S&P 500 to 0.10 and increases the volatility to 0.18, which are both close to our calibrated values.

¹⁶ Kaplan and Schoar (2005), Phalippou and Gottschalg (2009), and Driessen et al. (2012), for example, estimate private equity fund alphas, with respect to equity market indexes, at close to zero. Table 2 shows that the reported returns on a composite illiquid investment in private equity, buyout, and venture capital has similar characteristics to equity. For example, over the full sample (1981–2010), the mean log return on the illiquid investment is 0.11 with a volatility of 0.17. This is close to the S&P 500 mean and volatility of 0.10 and 0.18, respectively, over that period. Table 2 shows that the returns on liquid and illiquid investments are even closer in terms of means and volatilities before the financial crisis.

Table 2 Liquid and Illiquid Asset Returns

	1981Q3–2010Q2			1981Q3–2006Q4		
	Mean	SD	Corr	Mean	SD	Corr
Equity	0.103	0.182	1.000	0.125	0.157	1.000
Illiquid assets						
Private equity	0.103	0.229	0.629	0.110	0.231	0.605
Buyout	0.092	0.134	0.267	0.097	0.110	0.010
Venture capital	0.133	0.278	0.557	0.143	0.286	0.548
Illiquid investment	0.109	0.165	0.674	0.117	0.159	0.623

Notes. This table reports summary statistics on excess returns on liquid and illiquid assets. Liquid equity returns are total returns on the S&P 500. Data on private equity, buyout, and venture capital funds are obtained from Venture Economics and Cambridge Associates. We construct annual horizon log returns at the quarterly frequency. We compute log excess returns using the difference between log returns on the asset and year-on-year rollover returns on one-month treasury bills (T-bills) expressed as a continuously compounded rate. The column “Corr” reports the correlation of excess returns with equity. The illiquid investment is a portfolio invested with equal weights in private equity, buyout, and venture capital funds and is rebalanced quarterly.

isolate the effect of illiquidity, in the baseline case we assume that the two risky assets are uncorrelated, $\rho = 0$; we explore the effect of correlation by subsequently varying ρ between 0 and 1.

Regarding the severity of the illiquidity friction, we take a baseline case of $\lambda = 1$, implying on average one year between transactions. For comparison, individual private equity, buyout, and venture capital funds can have average investment durations of approximately four years, which corresponds to $\lambda = 1/4$; an appropriate horizon for a single large real estate investment by institutions is 10 years ($\lambda = 1/10$) (see, e.g., Table 1). Since λ is an important parameter, we take special care to show the portfolio and consumption implications for a broad range of λ . The economics behind the solution are immune to the particular parameter values chosen.

4.4. Optimal Portfolio Policies

In this section we characterize the investor’s optimal asset allocation and consumption policies. Although the investment opportunity set is constant, the optimal policies vary over time as a function of the value of illiquid assets held in the investor’s portfolio.

4.4.1. Participation. Before characterizing the optimal allocation, we first show the sufficient conditions for the investor to have a nonzero holding of the illiquid asset.

PROPOSITION 3. *An investor prefers holding a small amount of the illiquid asset to holding a zero position if and only if*

$$\frac{\nu - r}{\psi} \geq \rho \frac{\mu - r}{\sigma}. \quad (15)$$

The condition for participation is identical to the Merton two-asset case and depends only on the mean-variance properties of the two securities. Somewhat surprisingly, the degree of illiquidity λ does not affect the decision to invest a small amount in the illiquid asset because of the infinite horizon of the agent:

a trading opportunity will eventually arrive when the illiquid asset can be converted to liquid wealth and eventual consumption. However, although the conditions for participation are the same as the standard Merton case, the optimal holdings of the illiquid and liquid assets are very different, as we show below.

4.4.2. Illiquid Asset Holdings. Illiquidity induces underinvestment in the illiquid asset relative to the Merton case. In Table 3 we present the investor’s optimal rebalancing point ξ^* along with the long-run average level illiquid portfolio holdings $E[\xi]$ for different values of λ . For comparison, and in an abuse of notation, we report the consumption and portfolio policies for an investor able to continuously trade one ($\lambda = 0$) and two ($\lambda = \infty$) risky assets. The optimal holding of illiquid assets at $\lambda = 1$ upon arrival of a liquidity event is 0.37, which is lower than the optimal two-assets Merton holding at 0.60.

In addition to underinvestment in the illiquid asset, the inability to trade implies that the investor’s portfolio can deviate from optimal diversification for a long time. Panel C of Figure 1 plots the stationary distribution of an investor’s holding of the illiquid asset, ξ . For most of the time (the 20%–80% range), the share of wealth allocated in illiquid securities is 0.36–0.45, whereas in the 1%–99% range it is 0.30–0.65. Furthermore, the distribution of portfolio holdings is positively skewed, since illiquid wealth grows faster on average than liquid wealth because only the latter is used to fund consumption. As a result of this skewness, the investor chooses a rebalancing point *lower* than the mean of the steady-state distribution of portfolio holdings, that is, $\xi^* < E[\xi_t]$.

The degree of skewness is increasing in the illiquidity of the investment. When $\lambda = 1$, the mean holding is 0.41, compared to a rebalancing value of 0.37, while the distribution of portfolio holdings has a standard deviation of 6.3% and normalized skewness coefficient

Table 3 The Baseline Model

Avg. rebalancing interval ($1/\lambda$)	Optimal rebalance (ξ^*)	Illiquidity utility cost	Average policies		
			$E[\xi]$	$E[c(1 - \xi)]$	$E[\theta(1 - \xi)]$
0	0.593	—	0.593	0.089	0.593
1/50	0.535	0.018	0.541	0.088	0.583
1/10	0.493	0.028	0.511	0.087	0.578
1/4	0.475	0.036	0.485	0.086	0.571
1/2	0.442	0.045	0.461	0.083	0.568
1	0.373	0.067	0.409	0.081	0.558
2	0.251	0.103	0.299	0.075	0.546
4	0.132	0.165	0.212	0.069	0.536
10	0.048	0.222	0.214	0.059	0.489
∞	—	—	—	0.070	0.593

Notes. This table displays the effect of illiquidity on portfolio choice and welfare in the baseline model. The long-run average policies are computed by using a long simulation of 10,000 years. The cases $1/\lambda = \infty$ and $1/\lambda = 0$ correspond, with some abuse of notation, to the Merton one- and two-asset cases, respectively. The fraction of illiquid assets to total wealth is ξ , with optimal value ξ^* at the time of rebalancing. Consumption as a fraction of total wealth is $c(1 - \xi)$, and the allocation to the liquid asset as a function of total wealth is $\theta(1 - \xi)$. The table is computed by using the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$.

of 1.9. In the case when the investor can trade once every four years on average ($\lambda = 4$), the standard deviation of the investor's illiquid holdings increases to 12% and the skewness increases to 2.3.

4.4.3. Liquid Asset Holdings. In addition to underinvestment in the illiquid asset, illiquidity affects the investment in the liquid asset. The allocation to the liquid risk asset as a fraction of the investor's liquid holdings is equal to

$$\theta_t = \frac{\mu - r}{\sigma^2} \left(-\frac{F_W}{F_{WW} W_t} \right) + \rho \frac{\psi}{\sigma} \left(-\frac{F_{WX} X_t}{F_{WW} W_t} \right). \quad (16)$$

The allocation to the liquid asset as a function of the investor's total wealth is equal to $\theta(1 - \xi)$. There are two aspects of the optimal policy that merit attention.

First, even in the case where the liquid and illiquid asset returns are uncorrelated, $\rho = 0$, the allocation to the liquid asset differs from the Merton benchmark due to *time-varying effective risk aversion*. In panel D of Figure 1, we compare the curvature of the investor's value function with respect to liquid wealth $-F_{WW} W/F_W$ to that of a Merton investor. For low values of allocation to illiquid assets, ξ , the two behave in a similar fashion: as the share of liquid wealth W declines in the investor's total wealth $W + X$, so does the investor's aversion to gambles in W . However, when the investor's liquid wealth becomes sufficiently low, the two lines diverge, since liquid wealth is no longer viewed as a substitute for illiquid wealth. The investor in our problem becomes much more averse to taking gambles in terms of liquid wealth than a Merton investor. Furthermore, the investor's effective risk aversion not only increases but it varies over time as a function of her current allocation to illiquid assets, ξ .

Second, in the case where the liquid and illiquid asset are correlated, $\rho \neq 0$, the investor hedges changes in the

value of illiquid wealth. The hedging demands depend on the correlation between the liquid and illiquid asset returns, ρ , and the elasticity of substitution between liquid and illiquid wealth, $-F_{WX} X/F_{WW} W$. In panel E of Figure 1, we plot the second component for the demand for the liquid risky asset, $-F_{WX} X/F_{WW} W$, and contrast it to the term corresponding to a Merton investor for the case of $\rho = 0.6$. For low values of X relative to total wealth, the two lines are very similar, whereas they diverge dramatically as X increases relative to W . In our model, the term $-F_{WX} X/F_{WW} W$ converges to zero rather than minus infinity in the Merton case, implying *zero hedging demand at the limit*. The hedging motive disappears when illiquid securities compose the majority of the agent's portfolio since illiquid assets are not a substitute for liquid wealth. In this case, the investor chooses the allocation in liquid assets to smooth consumption rather than hedging fluctuations in her illiquid portfolio.

Panel F of Figure 1 plots the investor's optimal allocation to the liquid risky asset as a function of her current allocation in illiquid assets ξ . The investor partially compensates for the risk of being unable to trade the illiquid asset for a long period of time by underinvesting in the liquid risky asset. In Table 3 we summarize the average long-run holdings in the liquid risky asset for different degrees of illiquidity $1/\lambda$. Illiquidity negatively impacts the allocation to the liquid risky asset, but less so than the illiquid asset. In the case where $\lambda = 1$, the investor reduces her allocation to the liquid risky asset from 60% in the Merton benchmark to 56%, compared to a reduction from 60% to 37% for the illiquid security.

4.4.4. Effect of Correlation. To study the effect of correlation on portfolio policies, we focus on the interesting case where the two securities have different

Sharpe ratios; for this comparison, we set the expected return of the illiquid asset to $\nu = 0.2$. In the Merton case where both assets are fully liquid, varying ρ from zero to one leads to large swings in portfolio allocations. As ρ approaches one, the investor takes large offsetting positions in the two assets that tend toward plus or minus infinity. In panel G of Figure 1, we compare the target allocation ξ^* as a function of the correlation coefficient. We see that the effect of correlation is significantly muted relative to the Merton benchmark. The investor never shorts the liquid risky asset even when the correlation approaches one. From the investor's perspective, the liquid and illiquid assets are imperfect substitutes, since only the former can be used to fund short-term consumption. This imperfect substitutability decreases the desire to use the liquid risky asset to hedge price changes in the illiquid asset.

4.5. Consumption

The investor's optimal consumption choice satisfies

$$U'(C) = F_W(W, X). \quad (17)$$

In the short run, consumption is funded by liquid assets. Hence, the investor equates the marginal utility of consumption with the marginal value of *liquid*, rather than *total*, wealth. Using the form for the value function (10), the ratio of consumption to liquid wealth equals

$$c = [(1 - \gamma)H(\xi) - H'(\xi)\xi]^{-1/\gamma}(1 - \xi)^{-1}. \quad (18)$$

Panel H of Figure 1 plots the agent's optimal consumption to total wealth ratio, $c(1 - \xi)$, as a function of the current allocation to illiquid assets ξ . The investor always consumes a lower fraction of her total wealth than the two-asset Merton benchmark. Furthermore, in contrast to the Merton benchmark, the consumption-to-wealth ratio is time-varying, since the marginal value of liquid wealth varies with the current allocation to the illiquid security.

The investor's consumption policy sheds further light on the behavior of the marginal value of liquid wealth F_W . When the share of illiquid assets in the portfolio is small, the share of total wealth consumed is insensitive to portfolio composition ξ ; the investor smooths lifetime consumption by consuming a higher fraction of liquid wealth today as ξ increases. In contrast, as the share of illiquid assets ξ increases toward one, the investor's marginal value of liquid wealth increases, leading to a lower ratio of consumption to total wealth.

In Table 3 we compute the average consumption rate $E[c(1 - \xi)]$ for different values of λ . As we vary the expected time until the next trading opportunity $1/\lambda$ from one week ($1/\lambda = 1/50$) to 10 years, the fraction of total wealth consumed per year declines from 8.8% to 5.9%. Interestingly, when the average length of

the illiquidity period is sufficiently long ($\lambda \leq 1/4$), the investor consumes a lower fraction of her total wealth than an investor who is unable to trade the second asset at all. The presence of illiquidity risk constrains how an investor can fund consumption. Since consumption must be funded by liquid wealth, the greater the proportion of illiquid assets or the longer the times between liquidity events, the lower the optimal consumption.

4.6. The Cost of Illiquidity

To quantify the cost of illiquidity, we compute the fraction of initial wealth α the investor would be willing to give up at the instant of the liquidity event to be fully able to trade the illiquid asset

$$K_{M2}((W_t + X_t)(1 - \alpha))^{1-\gamma} = (W_t + X_t)^{1-\gamma}H(\xi^*). \quad (19)$$

The left side of Equation (19) is the value function of a Merton investor able to invest in two risky securities. We refer to α as the utility or certainty equivalent cost of illiquidity.

In Table 3 we compute the certainty equivalent cost of illiquidity for different values of λ . The cost of illiquidity can be substantial. Even when the investor can trade on average once a week ($\lambda = 50$), she would be willing to forgo 1.8% of her total wealth to make the second asset fully liquid. For higher degrees of illiquidity, the cost increases substantially; an investor trading an asset with an average of 10 years between trades ($\lambda = 0.1$), such as institutional real estate, would be willing to give up 22% of her total wealth to be able to continuously trade the illiquid asset. Here, we should emphasize that α is a conservative estimate of the cost of illiquidity. Since $H(\xi^*) \geq H(\xi_t)$, the investor would be willing to pay *at least* a fraction α at any point between liquidity events. Replacing $H(\xi^*)$ with its long-run average $E[H(\xi_t)]$ leads to higher costs.

5. Determinants of the Utility Cost of Illiquidity

In this section we explore the key determinants of the utility cost of illiquidity. First, we separate the cost due to suboptimal diversification from the inability to fund consumption; we show that smoothing consumption is more important than maintaining optimal diversification. Second, we disentangle the effect of illiquidity from illiquidity risk by comparing our setup to a model with deterministic periods of illiquidity; we find that the uncertainty about the frequency of trade magnifies the utility cost of illiquidity. Third, we verify that our results are not driven by the fact that marginal utility is infinite in the extreme states where liquid wealth—and consumption—drop to zero, by allowing the investor to pay a fixed cost to transact immediately. Last, we explore the impact of preference

parameters, disentangling the effect of the coefficient of risk aversion from the elasticity of intertemporal substitution; the utility cost of illiquidity is highest for investors who are willing to substitute across states but not across time.

5.1. The Effect of Consumption Smoothing

Illiquidity impedes consumption smoothing and optimal diversification. To separate these two effects, we consider an investor who only values consumption at some future stochastic terminal date τ .

PROBLEM 2 (NO INTERMEDIATE CONSUMPTION). The investor maximizes

$$F_{nc}(W_t, X_t) = \max_{\{C, I\}} E_t[U(C_{\hat{\tau}})], \quad (20)$$

where $\hat{\tau}$ is a stochastic retirement time that is exponentially distributed according to a Poisson process with arrival rate δ , subject to the budget constraints given by

$$\frac{dW_t}{W_t} = (r + (\mu - r)\theta_t) dt + \theta_t \sigma dZ_t^1 - \frac{dI_t}{W_t}, \quad (21)$$

and Equation (6), with $C_t = W_t + X_t$. Rebalancing ($dI_t \neq 0$) occurs only when the Poisson process N_t^A jumps.

The following proposition characterizes the solution to Problem 2.

PROPOSITION 4 (NO INTERMEDIATE CONSUMPTION). In Problem 2, the investor's value function can be written as $F_{nc}(W, X) = (W + X)^{1-\gamma} H_{nc}(\xi)$, where $H_{nc}(\xi)$ exists and is finite, continuous, and concave for $\xi \in [0, 1]$. The function $H_{nc}(\xi)$ obtains its maximum for some $\xi \in [0, 1]$. When a trading opportunity occurs at time τ , the trader selects I_τ so that $X_\tau/(X_\tau + W_\tau) = \xi_{nc}^*$, where H_{nc}^* and ξ_{nc}^* are defined as in Proposition 2. In addition, $H_{nc}(\xi)$ is characterized for $t \leq \hat{\tau}$ by

$$0 = \max_{\theta} \left[\delta \left(\frac{1}{1-\gamma} - H_{nc}(\xi) \right) + \lambda (H_{nc}^* - H_{nc}(\xi)) \right. \\ \left. + H_{nc}(\xi) A(\xi, 0, \theta) + \frac{\partial H_{nc}(\xi)}{\partial \xi} B(\xi, 0, \theta) \right. \\ \left. + \frac{1}{2} \frac{\partial^2 H_{nc}(\xi)}{\partial \xi^2} C(\xi, 0, \theta) \right], \quad (22)$$

where the functions A , B , and C are defined in (13).

The differential equation characterizing the solution to the problem without intermediate consumption is similar to our baseline model, up to a difference in the discount rate. To facilitate comparison with the baseline case, we consider the case where the investor's effective rate of impatience is equal to our baseline calibration, $\delta = \beta$.

Table 4 The Model Without Intermediate Consumption

Avg. rebalancing interval ($1/\lambda$)	Optimal rebalance (ξ^*)	Illiquidity utility cost	Average policies	
			$E[\xi]$	$E[\theta(1-\xi)]$
0	0.593	—	0.593	0.593
1/50	0.557	0.002	0.555	0.580
1/10	0.547	0.002	0.546	0.578
1/4	0.544	0.002	0.543	0.576
1/2	0.535	0.002	0.534	0.574
1	0.527	0.003	0.526	0.571
2	0.523	0.003	0.523	0.567
4	0.520	0.003	0.518	0.563
10	0.518	0.004	0.516	0.555
∞	—	—	—	0.593

Notes. This table displays the effect of illiquidity on portfolio choice and welfare in the case without intermediate consumption; see §5.1 for more details. The cases $1/\lambda = \infty$ and $1/\lambda = 0$ correspond, with some abuse of notation, to the Merton one- and two-asset cases, respectively. The long-run average policies are computed by using a long simulation of 10,000 years. The fraction of illiquid assets to total wealth is ξ , with optimal value ξ^* at the time of rebalancing. The allocation to the liquid asset as a function of total wealth is $\theta(1-\xi)$. The table is computed by using the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$.

In Table 4 we compute optimal policies and utility costs across different levels of illiquidity $1/\lambda$. Absent the motive to smooth intermediate consumption, the effects of illiquidity are quantitatively small. The illiquidity cost in terms of certainty equivalent wealth is below 40 basis points across all values of λ . Even for an average time between liquidity events of 10 years, the optimal holdings in the illiquid asset are 0.52, compared to 0.59 for the Merton case. This compares to 0.05 in Table 3 with intermediate consumption for the same average times between liquidity events. Thus, the inability to fully smooth consumption across states is the primary determinant of the cost of illiquidity, which confirms our intuition in §4.2.

5.2. Stochastic vs. Deterministic Trading Opportunities

To disentangle the effect of the length of the illiquid period from the uncertainty over its duration, we consider the case where the investor is allowed to rebalance her portfolio at fixed intervals, spaced T periods apart. The investor's problem is described below.

PROBLEM 3 (DETERMINISTIC LIQUIDITY). The investor maximizes (9), subject to the budget constraints (5) and (6), with rebalancing ($dI_t \neq 0$) only at the deterministic times $\tau = 0, T, 2T, \dots$.

The following proposition characterizes the solution to Problem 3.

PROPOSITION 5 (DETERMINISTIC LIQUIDITY). For Problem 3, the investor's value function can be written as $F_T(t, W, X) = (W + X)^{1-\gamma} H_T(t, \xi)$, where $H_T(t, \xi)$ exists and is finite, continuous, and concave in ξ for $\xi \in [0, 1]$.

Table 5 The Model With Deterministic Liquidity

Rebalancing interval (T)	Optimal rebalance (ξ^*)	Illiquidity utility cost	Average policies		
			$E[\xi]$	$E[c(1 - \xi)]$	$E[\theta(1 - \xi)]$
0	0.593	—	0.593	0.089	0.593
1/50	0.575	0.011	0.579	0.089	0.589
1/10	0.555	0.013	0.559	0.088	0.585
1/4	0.532	0.016	0.542	0.088	0.579
1/2	0.516	0.019	0.522	0.087	0.576
1	0.494	0.022	0.512	0.087	0.575
2	0.488	0.024	0.501	0.086	0.574
4	0.455	0.025	0.481	0.085	0.573
10	0.448	0.028	0.464	0.084	0.571
∞	—	—	—	0.070	0.593

Notes. This table displays the effect of illiquidity on portfolio choice and welfare in the case where trading opportunities arrive deterministically; see §5.2 for more details. The cases $1/\lambda = \infty$ and $1/\lambda = 0$ correspond, with some abuse of notation, to the Merton one- and two-asset cases, respectively. The long-run average policies are computed by using a long simulation of 10,000 years. The fraction of illiquid assets to total wealth is ξ , with optimal value ξ^* at the time of rebalancing. Consumption as a fraction of total wealth is $c(1 - \xi)$, and the allocation to the liquid asset as a function of total wealth is $\theta(1 - \xi)$. Unless otherwise noted, we use the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$.

The function $H_T(t, \xi)$ obtains its maximum in ξ for some $\xi \in [0, 1]$. We have $H_T(\tau, \xi) = \max_{\xi} \lim_{\epsilon \downarrow 0} H_T(\tau + \epsilon, \xi)$, and we define $\xi_T^* = \arg \max_{\xi} \lim_{\epsilon \downarrow 0} H_T(\tau + \epsilon, \xi)$ with $\tau = 0, T, 2T, \dots$, the repeated trading times. At $t = \tau$, the investor selects I_τ so that $X_\tau/(X_\tau + W_\tau) = \xi_T^*$.¹⁷ The function $H_T(t, \xi)$ is characterized by

$$0 = \max_{c, \theta} \left\{ \frac{1}{1 - \gamma} c^{1-\gamma} (1 - \xi)^{1-\gamma} - \beta H_T(t, \xi) + \frac{\partial H_T(t, \xi)}{\partial t} + H_T(t, \xi) A(\xi, c, \theta) + \frac{\partial H_T(t, \xi)}{\partial \xi} B(\xi, c, \theta) + \frac{1}{2} \frac{\partial^2 H_T(t, \xi)}{\partial \xi^2} C(\xi, c, \theta) \right\}, \quad (23)$$

where the functions A , B , and C are defined in (13).

Table 5 computes the optimal policies and the utility cost of illiquidity for different lengths of the illiquidity period. In contrast to the case with stochastic trading opportunities (see Table 3), varying the length of the deterministic illiquidity period has only a small effect on optimal policies. For example, when the time until the next trade is known in advance, varying the expected time until the next liquidity date from 1/50 to 10 years leads to a drop in the fraction of total wealth consumed per year from 8.9% to 8.4%. Similarly, the effects on welfare are small and are relatively insensitive to the length of the illiquidity period; the utility cost of illiquidity varies between 1.1% (1/50 years) and 2.8% (10 years).

Comparing Tables 5 and 3, we conclude that the uncertainty regarding the opportunity to trade is a

¹⁷ The limit statement reflects the fact that the value function is continuous in ξ but not in t at $t = \tau$: at $t = \tau + \epsilon$, ξ is a state variable, and so $H(\tau + \epsilon, \xi)$ is discretely less than $H(\tau, \xi)$, except for $\xi = \xi_T^*$.

major component of the utility cost of illiquidity. When the trading intervals are known in advance, lengthening the intervals of nontrading has a small impact on investor utility. The investor's main concern is to avoid states of the world where her liquid wealth—and therefore her consumption—drop to zero before the next opportunity to trade. If the investor can trade at deterministic intervals, this state can be avoided with probability one by investing an appropriate amount into the riskless asset and consuming a constant fraction. That is, the investor can hedge against deterministic illiquidity. Illiquidity risk represented by stochastic nontrading intervals—the unknown times until the next liquidity event—is unhedgeable and induces large portfolio choice effects.

5.3. Introducing Costly Liquidity

Our model can be interpreted as a setting where the cost of transacting is infinite, except at times when the Poisson process hits, in which case it is equal to zero. Effectively, we are assuming that sometimes there is no available counterparty with whom to trade, at any price. However, this assumption may be extreme: often, in addition to a decentralized market, there are certain trading partners who are always available, at a cost. Here, we explore a hybrid of our baseline model and a model with transaction costs. Specifically, the investor has two options if she wants to rebalance her portfolio: she can (i) wait for the arrival of the Poisson process, just like the baseline model, or (ii) pay a cost and rebalance freely. We assume that this fixed fee is independent of the trading amount and scales with the investor's total wealth.

PROBLEM 4 (FIXED TRANSACTION COST). The investor maximizes (9), subject to the budget constraints (5) and (6), with rebalancing ($dI_t \neq 0$) only when the Poisson process N_t^λ hits or upon the payment of a fixed fraction κ of total wealth $W + X$.

The following proposition characterizes the solution to Problem 4.

PROPOSITION 6 (FIXED TRANSACTION COST). In Problem 4, the investor's value function can be written as $F_{fc}(W, X) = (W + X)^{1-\gamma} H_{fc}(\xi)$, where $H_{fc}(\xi)$ exists and is finite, continuous, and concave in $\xi \in [\underline{\xi}, \bar{\xi}]$. Define H_{fc}^* and ξ_{fc}^* as in Proposition 2. When ξ hits the boundaries $[\underline{\xi}, \bar{\xi}]$, the trader pays $\kappa(W + X)$ and selects I_τ so that $X_\tau/(X_\tau + W_\tau) = \xi_{fc}^*$. In the no-trade region, $\xi \in [\underline{\xi}, \bar{\xi}]$, $H_{fc}(\xi)$ solves

$$0 = \max_{c, \theta} \left[\frac{1}{1 - \gamma} c^{1-\gamma} (1 - \xi)^{1-\gamma} - \beta H(\xi) + \lambda (H_{fc}^* - H(\xi)) + H(\xi) A(\xi, c, \theta) + \frac{\partial H(\xi)}{\partial \xi} B(\xi, c, \theta) + \frac{1}{2} \frac{\partial^2 H(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right], \quad (24)$$

Table 6 Allowing for Costly Liquidity

$\lambda \backslash \kappa$	A. Frequency cost paid				B. Optimal rebalance (ξ^*)				C. Illiquidity utility cost			
	1%	5%	10%	20%	1%	5%	10%	20%	1%	5%	10%	20%
50	0.000	0.000	0.000	0.000	0.535	0.535	0.535	0.535	0.017	0.017	0.017	0.017
10	0.000	0.000	0.000	0.000	0.505	0.505	0.505	0.505	0.024	0.024	0.024	0.024
4	0.001	0.000	0.000	0.000	0.496	0.496	0.496	0.496	0.025	0.025	0.025	0.025
2	0.002	0.000	0.000	0.000	0.483	0.483	0.483	0.483	0.026	0.026	0.026	0.026
1	0.007	0.001	0.001	0.001	0.460	0.454	0.451	0.446	0.029	0.034	0.037	0.038
1/2	0.034	0.006	0.004	0.003	0.435	0.409	0.397	0.38	0.034	0.040	0.043	0.048
1/4	0.063	0.023	0.014	0.009	0.418	0.367	0.341	0.308	0.039	0.053	0.061	0.073
1/10	0.101	0.046	0.033	0.021	0.405	0.332	0.291	0.242	0.043	0.065	0.081	0.103
0	0.131	0.077	0.059	0.046	0.395	0.302	0.248	0.193	0.046	0.078	0.102	0.135

$\lambda \backslash \kappa$	D. Average policies											
	$E[\xi]$				$E[c(1 - \xi)]$				$E[\theta(1 - \xi)]$			
	1%	5%	10%	20%	1%	5%	10%	20%	1%	5%	10%	20%
50	0.542	0.509	0.509	0.509	0.088	0.088	0.088	0.088	0.585	0.581	0.578	0.577
10	0.518	0.509	0.509	0.508	0.088	0.088	0.088	0.087	0.581	0.577	0.576	0.575
4	0.507	0.507	0.507	0.501	0.087	0.087	0.086	0.086	0.578	0.575	0.575	0.572
2	0.502	0.502	0.501	0.491	0.087	0.087	0.086	0.085	0.574	0.574	0.573	0.570
1	0.500	0.484	0.481	0.477	0.086	0.086	0.085	0.084	0.569	0.557	0.551	0.551
1/2	0.499	0.479	0.471	0.451	0.086	0.085	0.084	0.083	0.566	0.558	0.552	0.550
1/4	0.495	0.477	0.455	0.434	0.085	0.084	0.084	0.082	0.564	0.550	0.541	0.528
1/10	0.492	0.478	0.468	0.425	0.084	0.083	0.081	0.078	0.562	0.541	0.521	0.511
0	0.490	0.476	0.475	0.471	0.084	0.082	0.079	0.073	0.550	0.536	0.509	0.489

Notes. This table reports the optimal policies and the utility cost of illiquidity for different levels of illiquidity λ and transaction cost κ in the hybrid model with transaction costs. See §5.3 for more details. The long-run average policies are computed by using a long simulation of 10,000 years. The fraction of illiquid assets to total wealth is ξ , with optimal value ξ^* at the time of rebalancing. Consumption as a fraction of total wealth is $c(1 - \xi)$, and the allocation to the liquid asset as a function of total wealth is $\theta(1 - \xi)$. The table is computed by using the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$.

where the functions A , B , and C are defined in (13) and the boundaries of the no-trading region, $[\underline{\xi}, \bar{\xi}]$, solve $H(\underline{\xi}) = H(\bar{\xi}) = (1 - \kappa)^{1-\gamma} H_{\kappa}^*$.

We choose a fixed, rather than proportional, transaction cost for several reasons. First, the fixed cost of trading is the closest analogue to our baseline model, because it leads to very similar policy functions.¹⁸ Second, the fixed cost can be interpreted as the investor paying a fee to access a counterparty in the market for the illiquid asset, in the spirit of Abel et al. (2007). Effectively, we are endowing the investor with the option to pay a fee to temporarily increase the arrival

rate λ to infinity. Third, the numerical solution to the hybrid model with proportional transaction costs is more complex and numerically challenging. We expect, however, the underlying economics between fixed and proportional transaction costs to be similar: the investor always wishes to avoid high ξ states, which are states with low liquid wealth and therefore high marginal utility. The investor wishes to avoid these states by paying a cost, and whether the cost is fixed or proportional is secondary.

Table 6 computes the optimal policies and the utility cost of illiquidity for different levels of illiquidity λ and transaction cost κ . Comparing Tables 6 and 3, we see that the availability of the option to freely trade upon payment of the cost κ dampens the effect of illiquidity. For example, in the case where $\lambda = 1$, varying the level of the fixed cost from infinite to 1% of total wealth, the utility cost falls from 6.7% to 2.9%. In contrast to our baseline model, here the investor has the option to trade at will; hence she trades only when her illiquid allocation ξ exits the inaction region $[\underline{\xi}, \bar{\xi}]$, i.e., only when the cost of illiquidity is high. Doing so allows the investor to eliminate the occurrence of states when her liquid wealth—and therefore her consumption—drop to zero. This option is exercised more frequently if the probability of otherwise finding a counterparty, λ , or the exercise cost, κ , is low.

¹⁸ The portfolio policies take the same form in the baseline model and the fixed transaction cost model. Intuitively, the investor must wait for a liquidity event to trade in the baseline model. Then, the investor can rebalance his portfolio freely and choose allocation ξ^* . From Proposition 6, the fixed cost operates in a similar fashion. Once the wealth allocation process exits the set $[\underline{\xi}, \bar{\xi}]$, the investor can rebalance freely to any portfolio weight after paying the fixed cost and therefore rebalances to the same point ξ^* . Note that $\xi^* \in [\underline{\xi}, \bar{\xi}]$, so after hitting the boundary the investor rebalances once, and it will be some time before hitting the boundary again. In a proportional costs model, the investor's portfolio holdings also lie in a bounded set (see, e.g., Constantinides 1986). However, under proportional transaction costs, the investor transacts the minimal amount necessary to keep the portfolio weight within the boundary, which implies continuous trading.

In the last row of each panel in Table 6, we compute optimal policies and the utility cost of illiquidity in the case where $\lambda = 0$. Doing so allows us to compare our baseline setup with a pure transaction costs model. Our baseline model and the pure transaction costs model lead to similar qualitative predictions, and the investor trades infrequently. As a comparison of the utility costs across the two models, an investor facing liquidity events arriving on average once a year ($\lambda = 1$) would be willing to pay a fraction $\kappa = 2.6\%$ of her total wealth to an intermediary each time to transact freely. This extension illustrates that the substantial utility costs are not driven by the extreme behavior of marginal utility when liquid wealth drops to zero.

We conclude that endowing that investor with the option to trade at will, though at a cost, lowers the cost of illiquidity. If investors have access to a market maker who is able to absorb the illiquid asset for a fee, this hybrid model is more appropriate. However, in many situations, liquidity is not always available, even at a cost. Adverse selection or search frictions may lead to the inability to trade immediately at any price. In the financial crisis, many markets experienced liquidity freezes where no transactions were possible at any price (see Tirole 2011). In these cases, our baseline model is more appropriate than transaction costs notions of illiquidity.

5.4. Intertemporal Substitution and Risk Aversion

The curvature of the value function is an important determinant of the cost of illiquidity. In particular, the investor fears the probability that she reaches states with very low liquid wealth for two reasons. First, states with low liquid wealth are states with low consumption, and the investor likes to smooth consumption across states; the coefficient of risk aversion captures the magnitude of this preference. Second, in states where liquid wealth is low relative to total wealth, the investor faces a steeply increasing consumption profile. The investor dislikes these states because she wants to have smooth consumption paths over time; the elasticity of intertemporal substitution (EIS) governs this preference. A feature of time-separable preferences is that these two effects are linked. To investigate these two motives separately, we consider the case where the investor has recursive preferences.

PROBLEM 5 (EPSTEIN–ZIN). The investor maximizes

$$F_{ez}(W_t, X_t) = \max_{\{\theta, I, c\}} E_t \left[\int_t^\infty f(C_s, F_{ez}(W_s, X_s)) ds \right], \quad (25)$$

where the aggregator f is defined following Duffie and Epstein (1992) as

$$f(C, I) \equiv \frac{\beta}{1-\zeta} \left(\frac{C^{1-\zeta}}{((1-\gamma)I)^{(\gamma-\zeta)/(1-\gamma)}} - (1-\gamma)I \right), \quad (26)$$

subject to the budget constraints (5) and (6), with rebalancing ($dI_t \neq 0$) only when the Poisson process N_t^λ jumps.¹⁹

The following proposition characterizes the solution to Problem 5.

PROPOSITION 7 (EPSTEIN–ZIN). *For Problem 5, the investor's value function can be written as $F_{ez}(W, X) = (W + X)^{1-\gamma} H_{ez}(\xi)$, where $H_{ez}(\xi)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. The function $H_{ez}(\xi)$ obtains its maximum for some $\xi \in [0, 1)$. Define H_{ez}^* and ξ_{ez}^* as in Proposition 2. When a trading opportunity occurs at time τ , the trader selects I_τ so that $X_\tau/(X_\tau + W_\tau) = \xi_{ez}^*$. In addition, $H_{ez}(\xi)$ is characterized by*

$$0 = \max_{c, \theta} \left\{ \frac{\beta}{1-\zeta} (c^{1-\zeta} (1-\xi)^{1-\zeta} ((1-\gamma)H_{ez}(\xi))^{(\zeta-\gamma)/(1-\gamma)} - (1-\gamma)H_{ez}(\xi)) \right. \\ \left. + \lambda(H_{ez}^* - H_{ez}(\xi)) + H_{ez}(\xi)A(\xi, c, \theta) \right. \\ \left. + \frac{\partial H_{ez}(\xi)}{\partial \xi} B(\xi, c, \theta) \right. \\ \left. + \frac{\partial^2 H_{ez}(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right\}, \quad (27)$$

where the functions A , B , and C are defined in (13).

To quantify the impact of risk aversion and the elasticity substitution on portfolio policies, we compute optimal policies and utility costs for different preference parameters and levels of illiquidity $1/\lambda$. We show the results in Table 7. In panel A, we vary the coefficient of risk aversion γ and the likelihood of trading λ , holding the EIS fixed. As risk aversion increases, the investor's policies (ξ^*), her rate of consumption of liquid wealth (c), and the fraction of her liquid wealth invested in the liquid asset (θ) all converge to the frictionless benchmark. For instance, in the case where $\lambda = 10$, an investor with a risk aversion of $\gamma = 3$ allocates 80% of her total wealth to the illiquid asset, compared to 118% for a Merton investor. In contrast, if the investor had a risk aversion of $\gamma = 15$, she would allocate only 22% of her total wealth to the illiquid asset, compared to 24% for a Merton investor. An investor with high risk aversion chooses to invest less in risky assets; hence, for her, the cost of illiquidity is small. Indeed, as we see in panel A(ii), the utility cost of illiquidity decreases with risk aversion.

In panel B, we vary the inverse elasticity of intertemporal substitution (IEIS, ζ) and the likelihood of trading λ , holding risk aversion fixed. Changing the IEIS has a quantitatively small impact on portfolio policies.

¹⁹ See Duffie and Epstein (1992) for more details. Here, β is the subjective discount rate, γ is the coefficient of risk aversion, and ζ is the inverse of the elasticity of intertemporal substitution. The case of power utility corresponds to $\gamma = \zeta$.

Table 7 The Effects of Preference Parameters on Optimal Policies and Welfare

λ	A. Risk aversion— γ															
	3				6				10				15			
	(i) Optimal rebalance				(ii) Utility cost				(iii) $E(c(1 - \xi))$				(iv) $E(\theta(1 - \xi))$			
∞	1.185	0.593	0.356	0.237	—	—	—	—	0.129	0.090	0.074	0.069	1.185	0.593	0.356	0.237
10	0.799	0.493	0.336	0.220	0.109	0.029	0.020	0.005	0.117	0.086	0.072	0.068	1.122	0.572	0.349	0.255
4	0.666	0.475	0.291	0.201	0.114	0.037	0.031	0.021	0.117	0.086	0.064	0.065	1.113	0.571	0.349	0.238
2	0.530	0.442	0.285	0.199	0.132	0.041	0.035	0.030	0.114	0.082	0.068	0.065	1.097	0.561	0.343	0.234
1	0.409	0.373	0.280	0.197	0.172	0.067	0.041	0.035	0.110	0.081	0.072	0.065	1.076	0.552	0.335	0.233
1/2	0.289	0.246	0.192	0.149	0.256	0.107	0.051	0.041	0.105	0.080	0.069	0.063	1.069	0.548	0.331	0.224
1/4	0.178	0.132	0.101	0.075	0.295	0.165	0.106	0.073	0.093	0.075	0.065	0.060	1.045	0.544	0.332	0.225
1/10	0.061	0.047	0.035	0.025	0.325	0.222	0.158	0.116	0.088	0.068	0.061	0.057	1.020	0.539	0.328	0.222
0	—	—	—	—	—	—	—	—	0.090	0.070	0.062	0.058	1.185	0.593	0.356	0.237

λ	B. Inverse elasticity of intertemporal substitution— ζ															
	2/3				1.5				3				6			
	(i) Optimal rebalance				(ii) Utility cost				(iii) $E(c(1 - \xi))$				(iv) $E(\theta(1 - \xi))$			
∞	0.593	0.593	0.593	0.593	—	—	—	—	0.109	0.096	0.092	0.090	0.593	0.593	0.593	0.593
10	0.501	0.500	0.499	0.493	0.010	0.023	0.026	0.029	0.108	0.095	0.089	0.087	0.573	0.572	0.572	0.572
4	0.495	0.490	0.489	0.475	0.019	0.027	0.029	0.037	0.108	0.095	0.089	0.087	0.572	0.571	0.571	0.571
2	0.433	0.439	0.441	0.442	0.036	0.039	0.040	0.041	0.108	0.094	0.089	0.086	0.569	0.569	0.568	0.568
1	0.391	0.398	0.383	0.373	0.040	0.047	0.052	0.067	0.107	0.093	0.088	0.085	0.561	0.558	0.558	0.557
1/2	0.244	0.245	0.248	0.251	0.077	0.093	0.100	0.103	0.103	0.089	0.083	0.080	0.542	0.544	0.545	0.545
1/4	0.114	0.121	0.129	0.131	0.139	0.154	0.161	0.165	0.103	0.084	0.077	0.074	0.534	0.534	0.534	0.534
1/10	0.026	0.038	0.045	0.048	0.177	0.203	0.216	0.222	0.099	0.078	0.069	0.065	0.508	0.510	0.511	0.511
0	—	—	—	—	—	—	—	—	0.118	0.088	0.076	0.070	0.593	0.593	0.593	0.593

Notes. This table shows the effects on welfare and portfolio choice of preference parameters—the coefficient of relative risk aversion γ and the inverse of the elasticity of intertemporal substitution ζ . See §5.4 for more details. The long-run average policies are computed by using a long simulation of 10,000 years. The fraction of illiquid assets to total wealth is ξ , with optimal value ξ^* at the time of rebalancing. Consumption as a fraction of total wealth is $c(1 - \xi)$, and the allocation to the liquid asset as a function of total wealth is $\theta(1 - \xi)$. Unless noted otherwise, the table is computed by using the following parameter values: $\zeta = 6$, $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, and $\rho = 0$.

As we see in panels B(i) and B(iv), varying the IEIS from 2/3 to 6 has essentially no impact on the allocation to the liquid asset and a small impact on the allocation to the illiquid asset. However, as we see in panel B(iii), varying the IEIS has an impact on the investor's optimal consumption policy. When the investor's desire to smooth consumption across states increases (IEIS = 6), her optimal fraction of wealth consumed declines faster with λ than when her IEIS is low. Hence, we find that high IEIS magnifies the utility cost of illiquidity to the investor, as we see in panel B(ii).

We conclude that the utility cost of illiquidity is higher for agents with low intertemporal elasticity of substitution and low risk aversion. Holding portfolio allocations constant, the cost of illiquidity is higher for more risk-averse investors who are also reluctant to substitute across time. However, this utility cost depends on the fraction of wealth invested in illiquid securities, and the amount of investment in the illiquid risky asset is decreasing in risk aversion.

6. Liquidity Crises

Here, we extend the model to a setting where financial markets are normally liquid but transition to infrequent

liquidity crises. These liquidity crises are temporary, and they adversely affect the liquidity of otherwise liquid securities. We consider two applications of the extended model. First, we show that the possibility of a deterioration in market liquidity leads to limited arbitrage in normal times. Second, using the investor's marginal value of wealth, we derive the price of illiquidity risk.

6.1. A Model with Systematic Liquidity Risk

The level of market liquidity depends on two states, $S_t \in \{I, L\}$. In state L , corresponding to "normal" times, all assets are perfectly liquid, as in the Merton benchmark. In state I , the "crisis" state, the investor needs to wait for the arrival of a trading opportunity to trade the now illiquid asset P , as in the model in §3. The state of market liquidity, S_t , follows a continuous-time Markov process with a transition probability matrix between time t and $t + dt$ given by

$$P = \begin{pmatrix} 1 - \chi^L dt & \chi^L dt \\ \chi^I dt & 1 - \chi^I dt \end{pmatrix}. \quad (28)$$

Hence, the frequency and average duration of a liquidity crisis are χ_I and $1/\chi_L$, respectively.

The investor's problem is described below.

PROBLEM 6 (LIQUIDITY CRISES). The investor maximizes (9) subject to the budget constraints (5) and (6). The state of the economy ($S_t \in \{I, L\}$) evolves as in (28). If $S_t = L$, trade in both assets is continuous; if $S_t = I$, the investor can rebalance ($dI_t \neq 0$) only when the Poisson process N_t^λ jumps.

Our first result is that even in normal times, $S = L$, the investor will not short the *potentially* illiquid asset or have a short position in liquid wealth.

PROPOSITION 8. Any optimal policies in Problem 6 will have $W > 0$ and $X \geq 0$ almost surely for both $S = I$ and $S = L$.

The possibility of a liquidity crisis affects portfolio policies in normal times. The transition from liquid to illiquid is a surprise event and occurs without the opportunity to rebalance. Consequently, the portfolio restrictions from the illiquid state (see Proposition 1) also apply in the liquid state.

The following proposition characterizes the solution to Problem 6.

PROPOSITION 9 (LIQUIDITY CRISES). For Problem 6, the investor's value function can be written as $F_{LC}(W, X, S) = (W + X)^{1-\gamma} H_{LC}(\xi, S)$, where $H_{LC}(\xi)$ exists and is finite, continuous, and concave in ξ for $\xi \in [0, 1]$. $H_{LC}(\xi)$ obtains its maximum for some $\xi \in [0, 1]$. The function H_{LC} is given by

$$H_{LC}(\xi, S) = \begin{cases} H_I(\xi) & S = I, \\ H_L^* & S = L, \end{cases} \quad (29)$$

where the function $H_I(\xi)$ satisfies the Hamilton–Jacobi–Bellman equation

$$0 = \max_{c, \theta} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} (1-\xi)^{1-\gamma} - \beta H_I(\xi) + \lambda (H_L^* - H_I(\xi)) \right. \\ \left. + \chi^L (H_L^* - H_I(\xi)) + H_I(\xi) (1-\gamma) A(\xi, c, \theta) \right. \\ \left. + \frac{\partial H_I(\xi)}{\partial \xi} B(\xi, c, \theta) + \frac{1}{2} \frac{\partial^2 H_I(\xi)}{\partial \xi^2} C(\xi, c, \theta) \right\}, \quad (30)$$

and the functions A , B , and C are defined in (13). The constants H_L^* and H_I^* solve

$$0 = \max_{c, \theta, \xi} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} (1-\xi)^{1-\gamma} - \beta H_L^* + \chi^I (H_I(\xi) - H_L^*) \right. \\ \left. + A\left(\xi, \frac{c}{1-\xi}, \frac{\theta}{1-\xi}\right) H_L^* \right\}, \quad (31)$$

$$H_I^* = \max_{\xi} H_I(\xi). \quad (32)$$

The policies $\{c_I^*, \theta_I^*\}$ and $\{c_L^*, \theta_L^*, \xi_L^*\}$ maximize (30) and (31), respectively. The policy ξ_I^* maximizes (32).

In this case, the investor's value function depends not only on her wealth composition ξ , but also on the condition of market liquidity. In normal times, $S = L$, the investor can freely rebalance between both risky assets, and thus the function $H_L(\xi)$ is a constant. During a liquidity crisis, $S = I$, the investor's problem is similar to the problem analyzed in §3; therefore, her value function H_I depends on the ratio of illiquid to total wealth ξ . When a liquidity crisis occurs, the investor is constrained to hold her current allocation to the now illiquid security until the next opportunity to trade. Hence, the investor's optimal portfolio holdings in the liquid state are affected by the possibility of a liquidity crisis.

In Figure 2, we compare portfolio policies across regimes for different values of the frequency χ_I , average duration $1/\chi_L$, and severity $1/\lambda$ of liquidity crisis. As we see, the investor reduces her allocation in illiquid asset not only during a crisis, but also during normal times. Similarly, the investor reduces her consumption rate in both regimes. Both of these effects are increasing in χ_I , $1/\chi_L$, and $1/\lambda$. In addition, the investor holds fewer liquid risky assets during the liquidity crisis; her portfolio allocation in liquid risky assets is the same as the Merton benchmark in normal times, assuming that the two assets S and P are uncorrelated.

In summary, the possibility of a liquidity crisis leads to underinvestment in assets that are currently fully liquid but whose liquidity can dry up during a crisis. The same mechanism leads to limited arbitrage, which we explore below.

6.2. Limits to Arbitrage

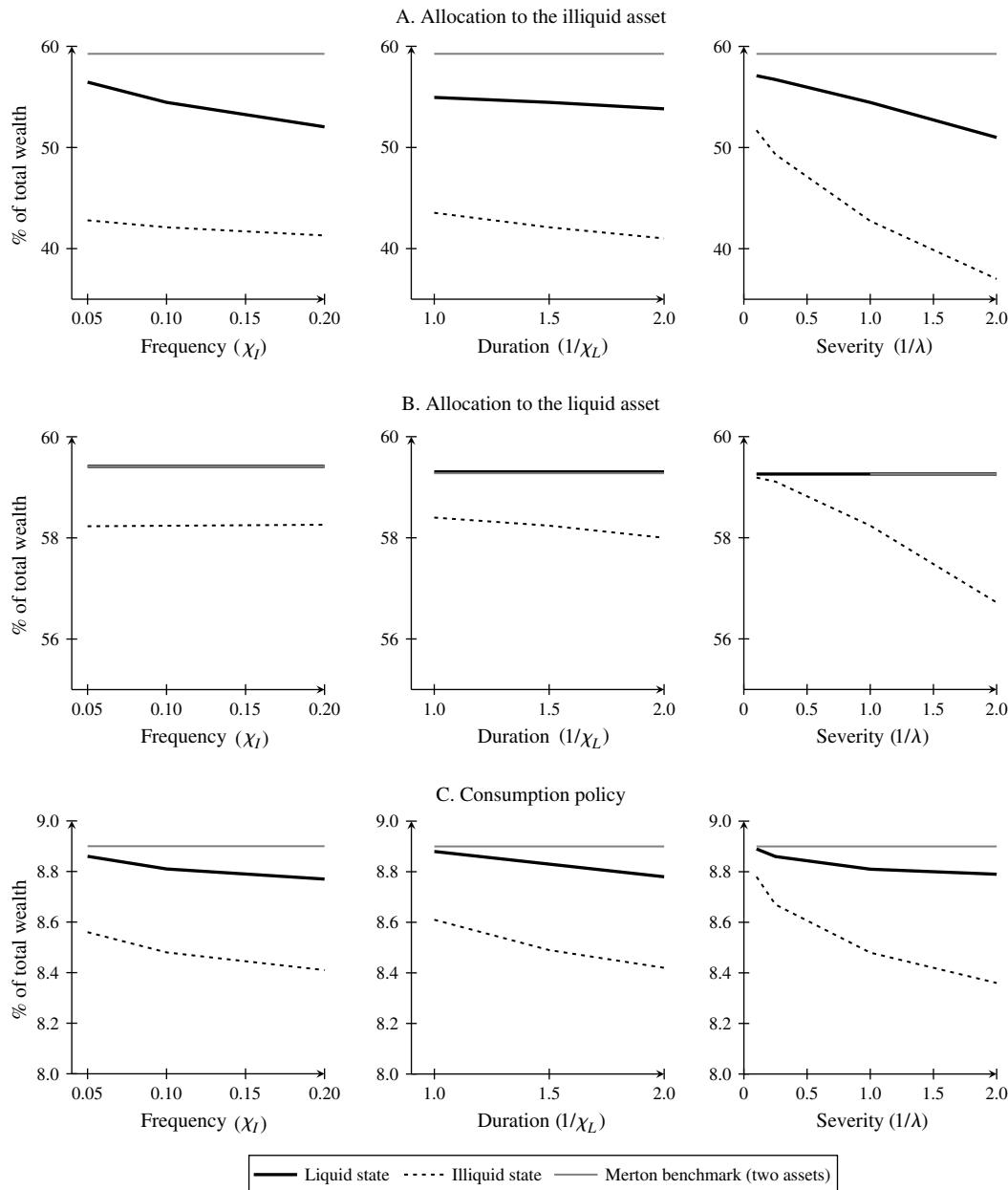
In the absence of any trading friction, the existence of two perfectly correlated securities with different Sharpe ratios implies an arbitrage opportunity. Faced with this arbitrage, the investor should construct a zero-investment portfolio that has a positive payoff and take an infinite position in this strategy. In our setting, the investor is reluctant to fully invest in arbitrage opportunities that involve potentially illiquid securities—even if both securities are currently fully liquid and taking advantage of this arbitrage involves no short positions in risky assets.

COROLLARY 1 (LIMITS TO ARBITRAGE). If $|\rho| = 1$ and $(\nu - r)/\psi \neq (\mu - r)/\sigma$, the investor's portfolio policies θ_L^* and ξ_L^* are finite and satisfy

$$\frac{\nu - r}{\psi} - \rho \frac{\mu - r}{\sigma} = -\frac{\chi^I}{\psi} \frac{H_I'(\xi_L^*)}{H_L^*(1-\gamma)}, \quad (33)$$

$$\xi_L^* \in [0, 1], \quad (34)$$

$$\theta_L^* = \frac{\mu - r}{\gamma \sigma^2} - \rho \frac{\psi}{\sigma} \xi_L^*. \quad (35)$$

Figure 2 Optimal Policies Across Liquidity Regimes

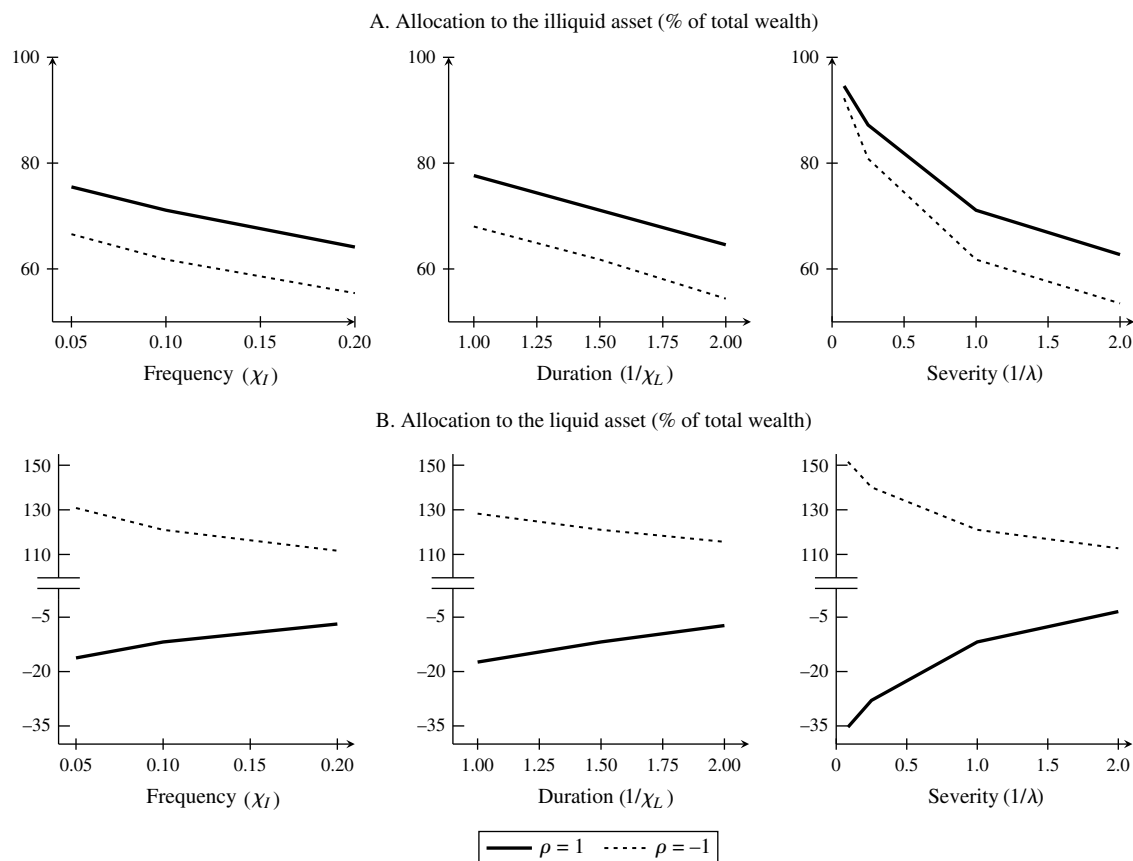
Notes. This figure compares portfolio (panels A and B) and consumption (panel C) policies across the liquid (solid line) and illiquid (dotted line) regimes for different frequency (χ_I), average duration ($1/\chi_L$), and severity ($1/\lambda$) of liquidity crises. Unless noted otherwise, the curves are plotted with the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, $\rho = 0$, $\chi_I = 0.1$, $\chi_L = 1/1.5$, and $\lambda = 1$.

Limits to arbitrage arise naturally in our setting. In the event of a crisis, the investor is not able to continuously rebalance her position. Hence, even if both securities are currently fully liquid, undertaking the arbitrage exposes the investor to illiquidity risk. In Equation (33), the investor will increase her holdings of the illiquid asset until the marginal welfare loss in the illiquid state—determined by $H'_I(\xi_L^*)$ —times the probability of that state occurring is proportional to the difference in the Sharpe ratios between the liquid

and illiquid assets. Examining the investor's allocation to the liquid asset (35), we see two components: the first part depends on the Sharpe ratio of the liquid risky asset; the second part hedges fluctuations to her wealth due to the investment in the potentially illiquid security. Since ξ_L^* is finite, her overall portfolio is *not* riskless.

The arbitrageur's investment in this apparent arbitrage opportunity is limited by her wealth. The investor will never take a levered position in the illiquid asset

Figure 3 Limits to Arbitrage



Notes. This figure compares portfolio policies of liquid assets (panel A) and illiquid assets (panel B) in the liquid regime in the case of an apparent arbitrage opportunity. We vary the frequency (χ_I), average duration ($1/\chi_L$) and severity ($1/\lambda$) of liquidity crises. Unless noted otherwise, the curves are plotted with the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = 0.12$, $\nu = 0.14$, $r = 0.04$, $\sigma = \psi = 0.15$, $\chi_I = 0.1$, $\chi_L = 1/1.5$, and $\lambda = 1$.

during a liquidity crisis (see Proposition 1) since doing so would lead to states with zero consumption. The inability to reduce leverage at the onset of a liquidity crisis (since the illiquid asset cannot be traded immediately) dissuades her from leveraging her *potentially illiquid* investment in normal times, as we see in Equation (34). Consequently, the amount of resources the agent commits to an “arbitrage opportunity” will be bounded above by her level of wealth, leading to limited arbitrage.

To quantify the magnitude of limited arbitrage in our setting, we compute the optimal portfolio policies θ_L^* and ξ_L^* in a setting with an arbitrage opportunity. We assume the two risky assets are perfectly correlated, $|\rho| = 1$, and we set the mean return to the potentially illiquid asset to be two percentage points higher than the mean return of the liquid asset, $\nu = 0.14$. As we see in Figure 3, the leverage constraint (34) is not binding. Even when the severity of the liquidity crisis is small (the potentially illiquid asset can be traded on average once a month), the investor only allocates 95% of her wealth to the potentially illiquid security; as the

severity of the crisis increases to $1/\lambda = 2$ years, the investor allocates just 60% of her wealth in the apparent arbitrage. An increase in the frequency, or the average duration, of a crisis has similar qualitative results.

We conclude that the possibility of a liquidity crisis leads to limited arbitrage in normal times. Our mechanism is distinct from existing models; limits to arbitrage arise even in a state of the world where all securities are fully liquid, there are no transaction costs, and riskless profits are possible. To the extent that there is a negative relation between the degree of aggressiveness in arbitrageurs’ strategies and the degree of mispricing, our model suggests that mispricing is worse in markets where liquidity crises are more likely, last longer, and are more severe.

6.3. The Pricing of Illiquidity Risk

A model with systematic liquidity crises allows us to quantify the illiquidity *risk* premium, i.e., the expected return differential between two *liquid* securities with heterogeneous return exposure to illiquidity crises. Using an approach similar to Merton’s ICAPM, we

derive the price of the risk of a liquidity crisis. To compute this risk premium, we examine the effect of a rise of illiquidity on the investor's marginal value of liquid wealth F_W , taking the processes governing asset returns as given. During normal times (i.e., in the liquid state $S = L$), the investor's marginal value of wealth process $\pi_t \equiv F_W$ evolves according to

$$\frac{d\pi_t}{\pi_t} = [\dots]dt + [\dots]dZ_t^1 + [\dots]dZ_t^2 + \frac{F_W(W, X, I) - F_W(W, X, L)}{F_W(W, X, L)} \bigg|_{\xi=\xi_L^*} dN_t^I, \quad (36)$$

where N_t^I is a Poisson count process such that $dN_t^I = 1$ at the onset of a liquidity crisis, $S = I$. The last term in (36) corresponds to the increase in the investor's marginal utility at the onset of a financial crisis. This term determines the market price of illiquidity risk.

The marginal utility process (36) provides a theoretical justification for empirical specifications of the stochastic discount factor that include measures of systematic illiquidity risk (e.g., Pastor and Stambaugh 2003, Sadka 2006, Korajczyk and Sadka 2008). In particular, several of the illiquidity measures considered in the literature are likely to be correlated with an increase in the difficulty of finding counterparties with whom to trade. For instance, Korajczyk and Sadka (2008) show comovement among several measures of liquidity; these measures include the average turnover in the stock market. A more direct test of the stochastic discount factor (36) would include the common component of illiquidity measures across asset markets. Hu et al. (2013) provide some empirical evidence along these lines.

To quantify the magnitude of the illiquidity risk premium, we introduce a derivative security in zero net supply that allows the investor to hedge a deterioration in market liquidity, which we call illiquidity protection. By purchasing illiquidity protection, the investor pays an annual premium equal to $\hat{\chi}_I$ to receive a cash payment of \$1 in the event of a liquidity crisis. The following corollary computes the cost of illiquidity insurance that would induce zero demand for the derivative security.

COROLLARY 2. *The annual premium for liquidity protection is equal to*

$$\hat{\chi}_I = \chi_I \frac{F_W(W, X, I)}{F_W(W, X, L)} \bigg|_{\xi=\xi_L^*}. \quad (37)$$

Corollary 2 shows that the cost of illiquidity insurance is equal to the probability of a liquidity crisis times the increase in the marginal value of liquid wealth on the event of a crisis. Since liquid wealth becomes more valuable during a liquidity crisis than during normal

times, the investor is willing to pay a higher rate than the objective probability χ_I to obtain some protection against a liquidity crisis.

The illiquidity risk premium is related to the utility cost of illiquidity. In panel A of Figure 4, we compare the risk premium $\hat{\chi}_I - \chi_I$ of a liquidity crisis across different values of χ_I , χ_L , and λ . We see that the investor is willing to pay a substantial premium over the subjective probability χ_I to obtain liquidity during a crisis. For example, the investor would be willing to pay an excess premium of 80 basis points (bps) per year to obtain liquidity in the event of a once-in-a-decade liquidity crisis ($\chi_I = 0.1$), with average duration of two years ($\chi_L = 0.5$), during which liquidity events arrive on average once a year ($\lambda = 1$).

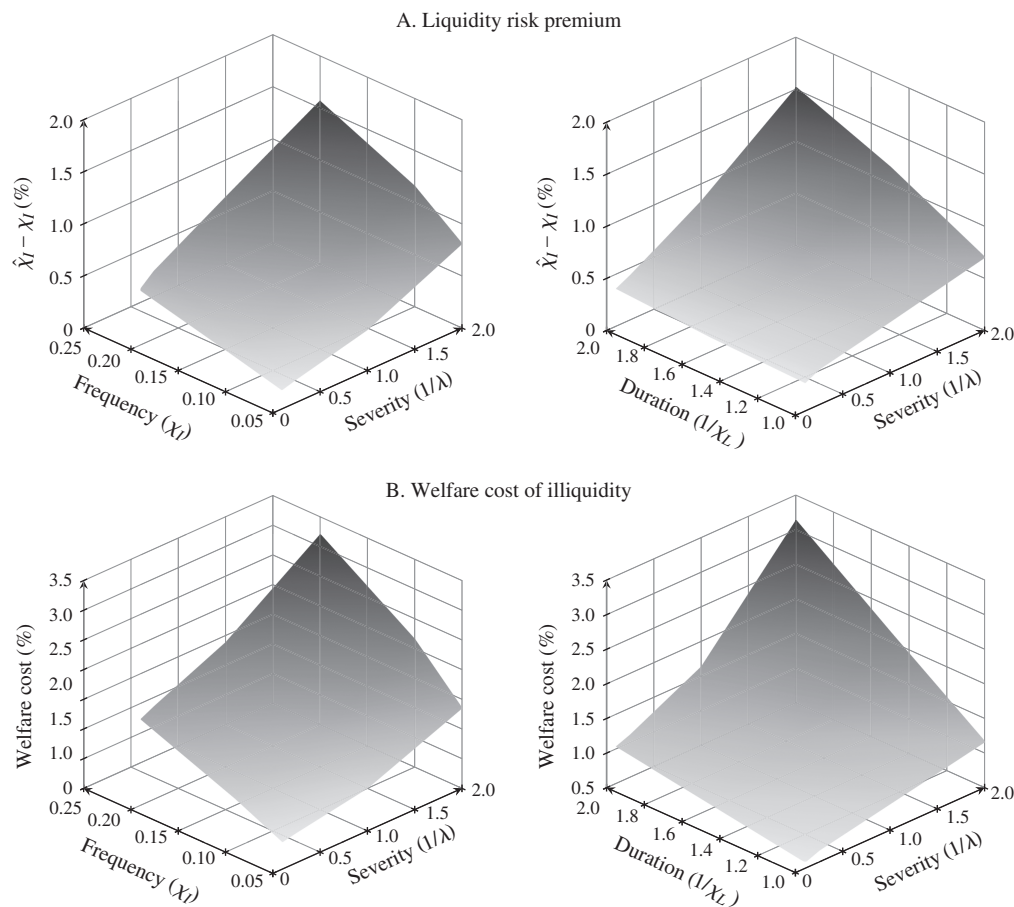
In panel B of Figure 4, we compute the utility cost of a liquidity crisis, defined as the fraction of total wealth the investor would pay ex ante to eliminate the possibility of a crisis. Although illiquidity crises are temporary, they still lead to substantial utility costs. For instance, the investor would be willing to forgo 2% of her wealth to eliminate the possibility of a once-in-a-decade liquidity crisis ($\chi_I = 0.1$), with average duration of two years ($\chi_L = 0.5$), during which the investor can rebalance on average once a year ($\lambda = 1$). Comparing panels A and B, we see that the illiquidity risk premium is related to the utility cost of illiquidity. Changes in the model specifications or parameters that amplify the utility cost of illiquidity (e.g., those considered in §5) also lead to a higher illiquidity risk premium.

The risk premium of the security offering illiquidity protection helps us to understand differences in average return among similar, liquid securities that have different price behavior during a liquidity crisis. A classic example is the swap–treasury spread. Both securities are very liquid and have similar exposures to interest rate and credit risk; yet, swaps have been historically priced cheaper than treasuries. This difference in price could be due to their differential price reaction to changes in the level of market illiquidity. For instance, during the flight-to-quality episodes that followed the financial crisis of 2008, the swap spread increased and stayed high for some time. Similar behavior was observed for mortgage spreads, especially for the riskiest parts of the mortgage market. Our model provides a framework to understand the spread differential during a liquidity crisis by relating the price of illiquidity risk to the marginal utility of financial market participants.

7. Conclusion

We study the effect of illiquidity risk on portfolio choice by extending the Merton (1971) framework to allow for infrequent and stochastic trading opportunities. Relative to the Merton economy, illiquidity leads to

Figure 4 Liquidity Risk Premium and the Welfare Cost of Illiquidity



Notes. This figure shows the risk premium associated with liquidity insurance $\hat{\chi}_L - \chi_L$ (panel A) and the welfare cost of illiquidity (panel B) for different frequency (χ_L), average duration ($1/\chi_L$), and severity ($1/\lambda$) of liquidity crises. Unless noted otherwise, the curves are plotted with the following parameter values: $\gamma = 6$, $\beta = 0.1$, $\mu = \nu = 0.12$, $r = 0.04$, $\sigma = \psi = 0.15$, $\rho = 0$, $\chi_L = 0.1$, and $\chi_L = 1/1.5$.

a large reduction in the allocation to both illiquid and liquid assets, lower consumption rates, and time-varying effective relative risk aversion. There are two main drivers of these results. First, consumption is financed through liquid assets. Investors care about both liquid and illiquid wealth and, as illiquid wealth becomes larger, the investor endogenously acts in a more risk-averse fashion, fearing states with low liquid wealth. Second, the fact that the duration of the illiquidity period is uncertain greatly amplifies the cost of illiquidity. In contrast to models with deterministic trading dates, the investor cannot hedge against the likelihood that her liquid wealth, and therefore her intermediate consumption, drop to zero.

We study the pricing of liquidity crises by allowing the risk of illiquidity to vary over time. Motivated by the behavior of many asset markets that exhibit periodic pronounced periods where liquidity “dries up,” we extend the model to allow for multiple liquidity regimes. We consider the case where all assets are fully liquid during normal times, but there exist temporary regimes during which the illiquid asset can be traded

only infrequently. Our model allows for the possibility of “arbitrage opportunities” that occur when all assets are perfectly liquid but agents do not take advantage of them because of the possibility that liquidity may evaporate. Our calibration implies that agents would be willing to pay an illiquidity risk premium of 2% to insure against illiquidity crises occurring once every 10 years.

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Appendix. Proofs and Derivations

A.1. Proof of Proposition 1

The value function is bounded below by the problem in which the illiquid asset does not exist, and it is bounded above by the problem in which the entire portfolio can be continuously rebalanced: the Merton one- and two-asset problems. Hence, there exist constants K_{M1} and K_{M2} such that

$$K_{M1} W^{1-\gamma} \leq F(W, X) \leq K_{M2} (W + X)^{1-\gamma} \leq 0. \quad (38)$$

Combining (38) and (10) yields that $H(\xi)$ exists and is finite for $\xi \in [0, 1]$.

Consumption is of liquid wealth only, and the illiquid asset cannot be pledged, so $W_t \leq 0$ implies zero consumption before the next trading day, leaving the objective function (4) at $-\infty$. For $|\rho| < 1$, $X_t < 0$ implies that under any admissible investment and consumption policy, there is a positive probability that at the next trading time $W_\tau + X_\tau \leq 0$, violating limited liability, implying zero consumption, and leaving the objective function (4) at $-\infty$. For $\rho = 1$, $X_t < 0$ is ruled out by assuming that the illiquid asset has a weakly higher Sharpe ratio than the liquid asset (8). For $\rho = -1$, the investor invests positive amounts in the illiquid asset X_t and the liquid risky asset.

A.2. Proof of Proposition 2

The arguments in the proof of Proposition 1 are sufficient to show that $H(1) = -\infty$. Concavity of $H(\xi)$ on $\xi \in [0, 1]$ follows from Lemma 1 (below), and continuity on $\xi \in [0, 1]$ from inspection. That $H(\xi)$ obtains its maximum for some $\xi \in [0, 1]$ follows from concavity, continuity, and $H(1) = -\infty$.

LEMMA 1. $H(\xi)$ is concave on $\xi \in [0, 1]$.

PROOF. Define $Q = X + W$ to be total wealth, and let $\{Q_0, X_0^1\}$ and $\{Q_0, X_0^3\}$ be two initial values with the associated optimal policies $\{C^1, \pi^1\}$ and $\{C^3, \pi^3\}$ where $\pi = \theta W$. For $\kappa \in (0, 1)$, we consider a middle initial value $\{Q_0, X_0^2 = \kappa X_0^1 + (1 - \kappa) X_0^3\}$ and the associated (possibly optimal) policies $\{C^2 = \kappa C^1 + (1 - \kappa) C^3, \pi^2 = \kappa \pi^1 + (1 - \kappa) \pi^3\}$, which are feasible because of the linearity of the budget constraint. From (5) and (6), we have

$$dQ_t = [rQ_t + (\mu - r)\pi_s + (\nu - r)X_t - C_t] dt + [\pi_s \sigma + \psi \rho X_t] dZ_t^1 + \psi X_t \sqrt{1 - \rho^2} dZ_t^2 \quad (39)$$

for any time t . Thus, from the construction of our initial values and optimal policies, we have $Q_t^2 = \kappa Q_t^1 + (1 - \kappa) Q_t^3$. Next, consider the objective function

$$E\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right] = E\left[\int_0^\tau e^{-\beta t} U(C_t) dt + e^{-\beta \tau} Q_\tau^{1-\gamma} H^*\right]. \quad (40)$$

Because $U(C)$ is increasing and concave, we have $U(C_t^2) > \kappa U(C_t^1) + (1 - \kappa) U(C_t^3)$. From Jensen's inequality and $H^* < 0$, we have $Q_\tau^{2^{1-\gamma}} H^* > \kappa Q_\tau^{1^{1-\gamma}} H^* + (1 - \kappa) Q_\tau^{3^{1-\gamma}} H^*$. Thus,

$$E^2\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right] > \kappa E^1\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right] + (1 - \kappa) E^3\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right],$$

and so the value function is concave in X for fixed Q . Since $\xi = X/Q$, this is sufficient to show that the value function is concave in ξ for fixed Q , so H is concave. \square

To continue, we observe that the principle of optimality implies the Hamilton–Jacobi–Bellman equation between rebalancing times:

$$0 = \max_{c, \theta} \left[\frac{1}{1 - \gamma} (cW)^{1-\gamma} - \beta F + F_W W(r + (\mu - r)\theta - c) + F_X X\nu + \lambda(F^* - F) + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 + \frac{1}{2} F_{XX} X^2 \psi^2 + F_{WX} WX\psi\sigma\rho\theta \right], \quad (41)$$

and substituting in (10) yields the stated ordinary differential equation. A standard verification argument completes the proof.

A.3. Proof of Proposition 3

An investor prefers holding a small amount of the illiquid asset to holding a zero position if and only if $F_X(W, X = 0) \geq F_W(W, X = 0)$.

We begin by showing that $(\nu - r)/\psi - \rho((\mu - r)/\sigma) \leq 0$ implies $F_X(W, X = 0) \leq F_W(W, X = 0)$.

Assume that we have $\{W_0, X_0 = \epsilon\}$, which gives rise to an optimal portfolio policy in number of shares equal to $\zeta_t = (\theta_t W_t)/S_t$ along paths for $t \in [0, \tau]$, where τ is the next trading time. $\{W_0, X_0 = \epsilon\}$ also gives rise to a consumption policy C_t along those same paths. Then, total discounted wealth at the next trading time equals

$$e^{-r\tau} (W_\tau + X_\tau) = W_0 + \epsilon + \int_0^\tau e^{-rt} [\zeta_t (\mu - r) S_t + (\nu - r) X_t - C_t] dt + \int_0^\tau e^{-rt} [\zeta_t \sigma S_t + \psi \rho X_t] dZ_t^1 + \int_0^\tau e^{-rt} [\psi \sqrt{1 - \rho^2} X_t] dZ_t^2.$$

Now consider the starting point $\{\hat{W}_0 = W_0 + \epsilon, \hat{X}_0 = 0\}$ and use the previous consumption policy state by state (feasible because consumption is of liquid wealth). The portfolio policy is now $\hat{\zeta}_t = \zeta_t + (\psi \rho X_t)/(\sigma S_t)$. Then,

$$e^{-r\tau} (\hat{W}_\tau + \hat{X}_\tau) = W_0 + \epsilon + \int_0^\tau e^{-rt} \left[\zeta_t (\mu - r) S_t + \frac{\psi \rho X_t}{\sigma} (\mu - r) - C_t \right] dt + \int_0^\tau e^{-rt} [\zeta_t \sigma S_t + \psi \rho X_t] dZ_t^1.$$

The drift in the second ($\{\hat{W}_0 = W_0 + \epsilon, \hat{X}_0 = 0\}$) minus the drift in the first ($\{W_0, X_0 = \epsilon\}$) equals

$$\int_0^\tau e^{-rt} \left[\psi \rho X_t \frac{\mu - r}{\sigma} - (\nu - r) X_t \right] dt,$$

which is positive if $\psi \rho((\mu - r)/\sigma) - (\nu - r) \geq 0$. Thus, the second initial condition produces higher expected wealth and lower volatility, path by path, with a possibly suboptimal portfolio and consumption strategy. Since the value function at rebalancing (F^*) is increasing and concave, this proves that $\rho((\mu - r)/\sigma) - (\nu - r)/\psi \geq 0$ implies $F(W_0 + \epsilon, 0) \geq F(W_0, \epsilon)$.

Next we will show that $(\nu - r)/\psi - \rho((\mu - r)/\sigma) \geq 0$ implies $F_X(W, X = 0) \geq F_W(W, X = 0)$. Consider a deviation in which

a trader starting with $\{W_0, 0\}$ is able to move an amount ϵ into X and then to withdraw it at the next trading day. This results in higher utility if

$$0 \leq -F_W(W_0, 0)\epsilon + E\left[e^{-\beta\tau} F_W(W_\tau, 0)\epsilon \frac{X_\tau}{X_0}\right],$$

with W_t following the optimal portfolio and consumption policies (as a function of W_t) for $X_t = 0$. Plugging in the value function at $X = 0$, we obtain

$$1 \leq E\left[e^{-\beta\tau} \left(\frac{W_\tau}{W_0}\right)^{-\gamma} \frac{X_\tau}{X_0}\right].$$

Direct calculations show that this is true if $(\nu - r)/\psi - \rho((\mu - r)/\sigma) \geq 0$, and hence $F_X(W, X=0) \geq F_W(W, X=0)$.

A.4. Sketch of Proof for Propositions 4, 5, 7, and 9

The characterization of the value function in each of these economies closely follows the characterization in the baseline (Problem 1) economy. Each value function is bounded above by an economy in which the illiquid asset is fully liquid and below by an economy in which the illiquid asset does not exist. Concavity and the other properties of the $H(\cdot)$ functions can be observed when using arguments analogous to those in the proof of Proposition 2. The Hamilton–Jacobi–Bellman equation follows from the principle of optimality, and a standard verification theorem can be used to complete the characterizations. The solution to the model with fixed transaction costs follows standard arguments. See Stokey (2008) for a textbook treatment.

A.5. Proof of Proposition 8

The arguments in the proof of Proposition 1 are sufficient to show that for $|\rho| < 1$, the objective function is at $-\infty$ if either $W \leq 0$ or $X < 0$ for $S = I$. For $S = L$, we observe that the state will shift to $S = I$ without the possibility of rebalancing; as a result, if either $W \leq 0$ or $X < 0$, the objective function in the liquid state is also equal to $-\infty$. For $\rho = 1$, $X < 0$ is ruled out by (7). For $\rho = -1$, the investor invests positive amounts in the illiquid asset X_t and the liquid risky asset.

A.6. Proof of Corollary 1

In the liquid state, the investor's optimal portfolio policies θ_L^* and ξ_L^* satisfy the first-order conditions

$$0 = H_L^*(1 - \gamma)(\mu - r) - H_L^*(1 - \gamma)\gamma\theta_L^*\sigma^2 - H_L^*(1 - \gamma)\rho\psi\sigma\xi_L^*,$$

$$0 = \chi^I H_L^*(\xi_L^*) + H_L^*(1 - \gamma)(\nu - r) - H_L^*(1 - \gamma)\psi^2\xi_L^* - H_L^*(1 - \gamma)\gamma\rho\psi\theta_L^*\sigma.$$

Setting $\rho = 1$, dividing the first equation by $\sigma H_L^*(1 - \gamma)$ and the second by $\psi H_L^*(1 - \gamma)$, and then subtracting the first equation from the second leads to (33) and (35) with $\rho = 1$. Setting $\rho = -1$, dividing the first equation by $\sigma H_L^*(1 - \gamma)$ and the second by $\psi H_L^*(1 - \gamma)$, and then adding the first equation to the second leads to (33) and (35) with $\rho = -1$.

A.7. Proof of Corollary 2

Consider a derivative security Y that pays a fixed rate of return κ when the aggregate state S switches from L to I . Denoting by dN_t^I the Poisson count process that denotes the arrival of a liquidity crisis, the price of this security evolves according to

$$\frac{dY_t}{Y_t} = (r + \mu_Y - \kappa\chi^I) dt + \kappa dN_t^I.$$

The investor is indifferent between participating in the market for security Z and her current portfolio policy as long as the excess return μ_Y is equal to

$$\begin{aligned} \mu_Y dt &= -\text{cov}\left(\frac{dY_t}{Y_t}, \frac{dF_W}{F_W}\right) \\ &= \kappa\chi^I \frac{H_L^*(1 - \gamma) - H_I(\xi_L^*)(1 - \gamma) + \xi_L^* H_I'(\xi_L^*)}{H_L^*(1 - \gamma)} dt, \end{aligned} \quad (42)$$

where we have used the investor's marginal value of wealth process (36). There exists a fictitious probability measure \mathbb{Q} under which the security Y has an expected excess return equal to zero,

$$E_t^{\mathbb{Q}}\left[\frac{dY_t}{Y_t}\right] = (r + \mu_Y - \kappa\chi^I + \kappa\hat{\chi}^I) dt = r dt.$$

Rearranging, using (42), and solving for the risk-neutral crisis probability $\hat{\chi}^I$ yields

$$\hat{\chi}_t^I = \chi^I \frac{F_W(W, X, I)}{F_W(W, X, L)} = \chi^I \frac{H_I(\xi_L^*)(1 - \gamma) - \xi_L^* H_I'(\xi_L^*)}{H_L^*(1 - \gamma)}.$$

Under that measure, the present value of the expected payments p has to equal the expected payoff in the event of a liquidity crisis

$$\begin{aligned} E_t^{\mathbb{Q}} \int_t^\tau e^{-r(s-t)} p ds &= E_t^{\mathbb{Q}} [e^{-r(\tau-t)}] \\ &\Rightarrow \int_t^\tau e^{-(r+\hat{\chi}_t^I)(s-t)} p ds = \int_\tau^\infty e^{-(r+\hat{\chi}_t^I)(\tau-t)} \hat{\chi}_t^I d\tau \\ &\Rightarrow p = \hat{\chi}_t^I. \end{aligned}$$

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