Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

The Impact of Modeling on Robust Inventory Management Under Demand Uncertainty

Oğuz Solyalı, Jean-François Cordeau, Gilbert Laporte

MANAGEMENT SCIENCE
Vol. 62, No. 4, April 2016, pp. 1188–1201
ISSN 0025-1909 (print) | ISSN 1526-5501 (online)

http://dx.doi.org/10.1287/mnsc.2015.2183

The Impact of Modeling on Robust Inventory Management Under Demand Uncertainty

Oğuz Solyalı
Business Administration Program, Middle East Technical University, Northern Cyprus Campus, Kalkanlı, Mersin 10, Turkey, solyali@metu.edu.tr

Jean-François Cordeau, Gilbert Laporte
Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation (CIRRELT), Montreal, Quebec H3C 3J7, Canada; and HEC Montréal, Montréal, Quebec H3T 2A7, Canada
{jean-francois.cordeau@hec.ca, gilbert.laporte@cirrelt.ca}

This study considers a basic inventory management problem with nonzero fixed order costs under interval demand uncertainty. The existing robust formulations obtained by applying well-known robust optimization methodologies become computationally intractable for large problem instances due to the presence of binary variables. This study resolves this intractability issue by proposing a new robust formulation that is shown to be solvable in polynomial time when the initial inventory is zero or negative. Because of the computational efficiency of the new robust formulation, it is implemented on a folding-horizon basis, leading to a new heuristic for the problem. The computational results reveal that the new heuristic is not only superior to the other formulations regarding the computing time needed, but also outperforms the existing robust formulations in terms of the actual cost savings on the larger instances. They also show that the actual cost savings yielded by the new heuristic are close to a lower bound on the optimal expected cost.

Data, as supplemental material, are available at http://dx.doi.org/10.1287/mnsc.2015.2183.

Keywords: robust optimization; inventory management; lot sizing; integer programming
History: Received September 26, 2010; accepted February 2, 2015, by Dimitris Bertsimas, optimization. Published online in *Articles in Advance* October 14, 2015.

1. Introduction

Robust optimization (RO) has recently emerged as a powerful approach to optimization problems involving uncertain parameters with ambiguous probability distributions. It incorporates uncertain parameters into the optimization models in a tractable way, does not require full knowledge of the probability distribution of uncertain parameters, and finds the best solution considering all possible realizations of uncertain data while ensuring feasibility regardless of their realization.

Several modeling frameworks are available within the RO paradigm because uncertainty can be handled in a number of ways. Since RO has proved to be an effective approach to optimization problems with uncertain parameters, the aim of this study is not to compare RO approaches against dynamic programming and stochastic programming, but rather to study the effect of modeling in robust optimization on the practical performance (i.e., actual cost savings and computing time) yielded by the robust formulations. Our analysis will be performed on a basic inventory management problem using different RO approaches.

The work of Soyster (1973) was the first robust optimization study to address linear programming (LP) problems with uncertain parameters taking their worst-case values in a given uncertainty set. Later, Ben-Tal and Nemirovski (1998, 1999, 2000), El-Ghaoui and Lebret (1997), and El-Ghaoui et al. (1998) considered ellipsoidal uncertainty sets for uncertain convex optimization problems (e.g., second-order cone programming (SOCP) and semidefinite programming (SDP) problems). However, this approach does not easily extend to discrete optimization problems because solution methods for SOCP and SDP problems with integer variables are not well developed. To fill this gap, besides controlling the degree of conservativeness, Bertsimas and Sim (2004) proposed the budget of uncertainty approach, which has the desirable property that the robust counterpart of the uncertain problem preserves the complexity of its nominal (i.e., without uncertainty) problem. Thus, this approach naturally extends to discrete optimization problems (see Bertsimas and Sim 2003) with only additional variables and constraints in the robust counterpart of an uncertain problem. Bertsimas and Sim (2003) showed that 0–1 discrete optimization problems with uncertain objective coefficients can be tackled by solving $m+1$ nominal problems, where m is the number of binary variables. For polynomially solvable 0–1 discrete optimization problems (e.g., the shortest path problem or the matching problem), this means their robust counterparts are also polynomially solvable. Atamtürk (2006) proposed strong robust formulations for mixed 0–1 programming problems under objective coefficient uncertainty. Instead of solving $m+1$ nominal problems as in Bertsimas and Sim (2003), he developed a tight robust formulation for the polynomially solvable 0–1 discrete optimization problems yielding integral solutions. Kouvelis and Yu (1997) considered a robust decision-making framework for discrete optimization problems in which the realization of uncertain data was represented by a set of scenarios, and the aim was to minimize either the worst-case performance or the maximum regret under the given set of scenarios. However, in this robustness approach, the robust counterparts of many polynomially solvable 0–1 discrete optimization problems become *NP*-hard problems.

All of the above-mentioned solution frameworks were developed for single-stage decision-making problems, ignoring the fact that some decisions can be made after observing the realization of some uncertain parameters in multistage decision-making problems. Considering this fact, Ben-Tal et al. (2004) developed the adjustable robust counterpart (ARC) for uncertain multistage LP problems in which some decision variables, called nonadjustable, are determined a priori (i.e., at time 0), whereas others, called adjustable, are set after having observed the realization of some uncertain parameters. Since the ARC of an uncertain LP problem is intractable except for some restricted cases, Ben-Tal et al. (2004) proposed, as a tractable approximation to the ARC, the affinely adjustable robust counterpart (AARC), in which adjustable variables are expressed as an affine function of realized uncertain parameters. Recently, Chen and Zhang (2009) improved the AARC by proposing the so-called extended AARC (EAARC). However, the EAARC of an uncertain LP problem is an SOCP problem, so it does not fit well when integrality restrictions exist. For network flow and design problems under demand uncertainty, Atamtürk and Zhang (2007) proposed a two-stage robust discrete optimization approach in which the first-stage decisions are made a priori, whereas the second-stage decisions are made after observing the demand. Unlike Ben-Tal et al. (2004), Atamtürk and Zhang (2007) did not restrict the second-stage decision variables to be affine functions of uncertain demand.

Robust inventory management problems under demand uncertainty, being intrinsically multistage problems due to their multiperiod planning horizons, have been solved by many researchers using the robust optimization approaches just described. Ben-Tal et al. (2004, 2005) formulated two different inventory management problems as uncertain LP problems and considered their AARCs, which give better objective function values than their pure robust counterparts. Bertsimas and Thiele (2006) modeled a single-installation and a multi-installation inventory management problem with a tree network structure as mixed integer linear programming problems (resp., linear programming problems) in the presence (resp., absence) of fixed order costs using the robustness approach developed by Bertsimas and Sim (2004). Bienstock and Özbay (2008) solved the same single-installation inventory management problem without fixed order costs as Bertsimas and Thiele (2006), in addition to the problem with base-stock policy, using a Benders-like two-phase decomposition algorithm. Ben-Tal et al. (2009) modeled a multi-installation inventory management problem with a serial structure as an uncertain LP problem and considered its globalized robust counterpart, which is an extension of the AARC. Unlike the above-mentioned studies, See and Sim (2010) considered a single-installation inventory management problem with nonzero order lead times and characterized uncertain demand by covariance and directional deviations besides the usual mean and support information. They considered static, linear, and truncated linear replenishment policies and formulated the problem as an SOCP model, which approximates the objective function with good upper bounds (UBs). Note that all these papers, except Bertsimas and Thiele (2006), consider zero fixed order costs (i.e., there are no integer decision variables).

In this study, we consider the same basic single-installation inventory management problem as Bertsimas and Thiele (2006) with nonzero fixed order costs, where the aim is to determine when and how much to order under interval demand uncertainty over a planning horizon of T time periods, so that the worst-case total cost across all feasible demand scenarios is minimized. Note that this problem is called the uncapacitated lot-sizing problem with backlogging (ULSB) in the literature when the demand is known (see Pochet and Wolsey 2006, Chap. 10). We introduce a new modeling approach to robust inventory management by making use of a well-known reformulation of the ULSB. Aiming to study the impact of modeling in robust inventory management under demand uncertainty, with and without budget of uncertainty, we present several robust formulations constructed by using the main known robust optimization approaches (i.e., pure robust, budget of uncertainty robust, and affinely adjustable robust counterparts, as well as robust counterpart with an approximate objective function) and compare these with the new modeling approach.

The contributions of this study are as follows:

• Whereas, to the best of our knowledge, all studies in the robust inventory management literature use a variant of standard inventory flow balance equations in their robust formulations, we consider a reformulation approach and propose a new robust formulation with budget of uncertainty.

• In contrast to the existing robust formulations obtained by applying prominent RO methodologies, the proposed robust formulation is computationally efficient even for large problem instances. Specifically, we show that the new robust formulation with a budget of uncertainty can be solved in polynomial time by solving $O(T^2)$ LP problems when the initial inventory is zero or negative. We further propose another reformulation-based robust formulation, which we prove to be polynomially solvable in $O(T^4)$ time. Because of the computational efficiency of the new robust formulation, we propose a folding-horizon implementation of it that results in a new heuristic for the problem.

• We adapt to our problem the robust formulation for the linear replenishment policy of See and Sim (2010), which works with an approximate objective function and results in an SOCP formulation with binary variables. We computationally test this formulation for the first time in the RO literature.

• The computational results on test instances reveal that the new heuristic not only outperforms the others in terms of computing time needed, but also exhibits a superior performance in terms of the actual cost savings compared to pure, budget of uncertainty, and affinely adjustable robust formulations. Our approach also outperforms the robust formulation with an approximate objective function on the larger instances. The computational results also show that the actual cost savings yielded by the new heuristic are close to a lower bound (LB) on the optimal expected cost.

• Our computational results provide insights into the performance of different RO formulations.

The remainder of this paper is organized as follows. We give the detailed problem description in §2. We present a summary of the existing robust formulations in §3, and the new robust formulation and resulting folding-horizon heuristic in §4. We present the computational results obtained by all formulations and heuristics on test instances and conclude the paper in §§5 and 6, respectively.

2. Problem Description: The Robust Single-Installation Inventory Management Problem

To show the impact of modeling in robust inventory management, we chose the single-installation inventory management problem, which is a basic building block of more complex inventory management problems. The single-installation (with a single item) inventory management problem can be described as follows.

Over a finite horizon, the installation faces dynamic stochastic demand d_k in each period $k \in \mathcal{T} = \{1, \dots, T\}$. The probability distribution of the random variable d_k is unknown. The only information available concerning d_k is that it can take a value from the interval $[\bar{d}_k - \hat{d}_k, \bar{d}_k + \hat{d}_k]$, where $\bar{d}_k$ is the point estimate (nominal value) and $\hat{d}_k$ is the maximum deviation for the demand during period k. The installation can place an order u_k to its supplier at the beginning of each period k to fulfill some of the demand. Whenever an order is placed, a variable order cost c_k for each unit ordered and a fixed order cost f_k independent of the order size are incurred. It is assumed, without loss of generality, that the orders arrive instantaneously (i.e., the lead time is zero). The inventory level I_{k+1} at the beginning of period $k+1$ is equal to the inventory level I_k at the beginning of period k, plus the ordered amount u_k, minus the demand d_k in period k. If $I_k + u_k > d_k$, then the excess amount is held in inventory with a holding cost h_k for each unit held in inventory at the end of period k. If $I_k + u_k < d_k$, then the excess demand is backlogged (i.e., it is not met on time) and is either satisfied when the stock becomes available or never satisfied. Each unit backlogged at the end of a period incurs a unit backlogging cost p_k. We assume that $p_k > c_k$ so that it is likely to be more economical to order stock until the last period. We also assume that all parameters are nonnegative. Note that the inventory level I_{k+1} at the beginning of period $k+1$ can be rewritten as

$$I_{k+1} = I_1 + \sum_{t=1}^{k} (u_t - d_t) \quad k \in \mathcal{T}, \tag{1}$$

where I_1 is the known initial inventory level. Equation (1) is the standard equation used in the robust inventory management literature to determine the inventory levels and the inventory holding and backlogging costs.

There are no capacity limits on the order sizes that can be placed to the supplier, and no limits on the amount of inventory that can be carried. The problem, referred to as the robust single-installation inventory management problem (RSP), is to determine order quantities and their timing. The objective is min-max, i.e., minimizing the worst-case total cost composed of inventory holding and backlogging as well as fixed and variable order costs across all feasible demand scenarios.

3. Robust Formulations: Summary of Existing Methods

In this section, we present first the pure and the budget of uncertainty robust formulations for the RSP, and then the affinely adjustable robust formulation and the robust formulation with an approximate objective function for the same problem.

3.1. Pure and Budget of Uncertainty Robust Formulations

Here we present the pure robust formulation (PR) and the closely related budget of uncertainty robust formulation (BUR) of Bertsimas and Thiele (2006).

In the pure robust formulation, the optimal solution is certainly feasible provided that the realized demands are within their given supports. Whereas such a solution is quite robust, it may also be overconservative. To control the degree of robustness and conservatism, Bertsimas and Thiele (2006) define "budgets of uncertainty" Γ_t, obeying the relation $\Gamma_t \leq \Gamma_{t-1}+1$ for $t \in \mathcal{T}$, with $\Gamma_0 = 0$, which allows only some of the demand figures to simultaneously deviate from their nominal values. This approach guarantees the feasibility of the optimal solution if no more than Γ_t ($t \in \mathcal{T}$) demands of periods 1 through t jointly deviate from their nominal values, and it provides a probabilistic guarantee of feasibility (with respect to a constraint) depending on the value Γ_t and on the number of uncertain parameters (see Bertsimas and Sim 2004). Note that when $\Gamma_t = t$ for $t \in \mathcal{T}$, the budget of uncertainty robust formulation is equivalent to the pure robust formulation. All the decisions are given a priori (i.e., at time 0) in both robust formulations.

Let y_k be the inventory holding/backlogging cost at the end of period $k \in \mathcal{T}$, and let v_k be 1 if an order is placed in period $k \in \mathcal{T}$ and 0 otherwise. Then, the uncertain formulation for the RSP is as follows:

$$\begin{aligned} \min\ & \sum_{k=1}^{T}(c_k u_k + f_k v_k + y_k) \\ \text{s.t.}\ & y_k \geq h_k\left(I_1 + \sum_{t=1}^{k}(u_t - d_t)\right), \quad k \in \mathcal{T},\ d \in D^k, \\ & y_k \geq -p_k\left(I_1 + \sum_{t=1}^{k}(u_t - d_t)\right), \quad k \in \mathcal{T},\ d \in D^k, \\ & 0 \leq u_k \leq M v_k, \quad v_k \in \{0,1\},\ k \in \mathcal{T}, \end{aligned} \tag{2}$$

where D^k is the uncertainty set denoting the vector of demands $d = (d_1, \ldots, d_k)$, and $M = \sum_{k=1}^{T}(\bar{d}_k + \hat{d}_k)$.

The above formulation with $D^k = D^k_{\mathrm{PR}} = \{\bar{d}_t + \hat{d}_t z_t \colon |z_t| \leq 1,\ \forall 1 \leq t \leq k\}$, where z_t denotes the scaled deviation of d_t, yields the uncertain pure formulation, whereas the one with $D^k = \{\bar{d}_t + \hat{d}_t z_t \colon |z_t| \leq 1,\ \forall 1 \leq t \leq k,\ \sum_{t=1}^{k}|z_t| \leq \Gamma_k\}$ yields the uncertain budget of uncertainty formulation.

Because (2) is a semi-infinite optimization problem that is intractable, we equivalently reformulate it by ensuring feasibility for any possible realization of demands in the given uncertainty set as follows:

$$y_k \geq \max_{d \in D^k}\left[h_k\left(I_1 + \sum_{t=1}^{k}(u_t - d_t)\right)\right], \quad k \in \mathcal{T}, \tag{3}$$

$$y_k \geq \max_{d \in D^k}\left[-p_k\left(I_1 + \sum_{t=1}^{k}(u_t - d_t)\right)\right], \quad k \in \mathcal{T}. \tag{4}$$

In the case of the uncertain pure formulation, inequalities (3) and (4) yield the pure robust formulation:

$$\begin{aligned} \text{(PR)}\quad \min\ & \sum_{k=1}^{T}(c_k u_k + f_k v_k + y_k) \\ \text{s.t.}\ & y_k \geq h_k\left(I_1 + \sum_{t=1}^{k}(u_t - \bar{d}_t + \hat{d}_t)\right), \quad k \in \mathcal{T}, \\ & y_k \geq -p_k\left(I_1 + \sum_{t=1}^{k}(u_t - \bar{d}_t - \hat{d}_t)\right), \quad k \in \mathcal{T}, \\ & 0 \leq u_k \leq M v_k, \quad v_k \in \{0,1\}\ k \in \mathcal{T}. \end{aligned} \tag{5}$$

In the case of the budget of uncertainty formulation, Bertsimas and Thiele (2006) must deal with the following auxiliary problem in (3) and (4) for each $k \in \mathcal{T}$ when deriving the budget of uncertainty robust formulation:

$$\begin{aligned} A_k = \max\ & \sum_{t=1}^{k}\hat{d}_t z_t \\ \text{s.t.}\ & \sum_{t=1}^{k} z_t \leq \Gamma_k, \\ & 0 \leq z_t \leq 1, \quad 1 \leq t \leq k. \end{aligned} \tag{6}$$

Note that a "budget of uncertainty" Γ_k is considered for each cumulative demand up to period $k \in \mathcal{T}$ in (6). Using the robust optimization methodology of Bertsimas and Sim (2004) derived by associating the first two sets of constraints in (6) with dual variables q and r, respectively, and using strong duality, we obtain the budget of uncertainty robust formulation, which was introduced by Bertsimas and Thiele (2006):

$$\text{(BUR)}\quad \min\ \sum_{k=1}^{T}(c_k u_k + f_k v_k + y_k) \tag{7}$$

s.t.

$$y_k \geq h_k\left(I_1 + \sum_{t=1}^{k}(u_t - \bar{d}_t) + q_k\Gamma_k + \sum_{t=1}^{k} r_{tk}\right), \quad k \in \mathcal{T}, \tag{8}$$

$$y_k \geq p_k\left(-I_1 - \sum_{t=1}^{k}(u_t - \bar{d}_t) + q_k\Gamma_k + \sum_{t=1}^{k} r_{tk}\right), \quad k \in \mathcal{T}, \tag{9}$$

$$q_k + r_{tk} \geq \hat{d}_t, \quad k \in \mathcal{T}, 1 \leq t \leq k, \tag{10}$$

$$q_k \geq 0, r_{tk} \geq 0, \quad k \in \mathcal{T}, 1 \leq t \leq k, \tag{11}$$

$$0 \leq u_k \leq M v_k, \quad v_k \in \{0,1\}, k \in \mathcal{T}. \tag{12}$$

Note that $q_k \Gamma_k + \sum_{t=1}^{k} r_{tk}$ is the objective function of the dual of (6), and constraints (10) and (11) are the constraints of the dual of (6). We formulate the following remarks on the BUR.

Remark 1. As also stated by Bienstock and Özbay (2008), one can replace $q_k \Gamma_k + \sum_{t=1}^{k} r_{tk}$ with A_k in constraints (8) and (9) and eliminate constraints (10) and (11) from the BUR, which leads to a robust formulation without any additional variables and constraints. This is because the optimal objective function value A_k of formulation (6) for each $k \in \mathcal{T}$ can be calculated in advance independently of the optimal solution values of the u, v, and y variables. Also note that the optimal solution of formulation (6) for each $k \in \mathcal{T}$ gives the realization of the uncertain demands, which is independent of the optimal solution of the BUR.

Remark 2. For the case of static cost parameters, Bertsimas and Thiele (2006) show that the optimal robust policy can be found by solving an equivalent nominal problem with modified demand:

$$d'_k = \bar{d}_k + \frac{p-h}{p+h}(A_k - A_{k-1}), \quad k \in \mathcal{T}, \tag{13}$$

where $A_0 = 0$, $A_k = q_k^* \Gamma_k + \sum_{t=0}^{k} r_{tk}^*$ for $k \in \mathcal{T}$, and q^* and r^* are the optimal q and r variables in the BUR formulation. The objective function value of the optimal robust policy is equal to that of the nominal problem with modified demand, plus the extra term $[2ph/(p+h)] \sum_{k=1}^{T} A_k$. Furthermore, base-stock levels ((s,S) if $f > 0$ and (S,S) if $f = 0$) defining the optimal robust policy depend on modified demand values d'_k ($k \in \mathcal{T}$). Thus, the associated robust formulation can be solved in closed form.

Note that Remarks 1 and 2 can easily be extended to the network case (i.e., several installations with an arborescent/distribution structure) addressed in Bertsimas and Thiele (2006).

3.2. Affinely Adjustable Robust Formulation

Here we present the affinely adjustable robust formulation derived using the methodology of Ben-Tal et al. (2004). The affinely adjustable robust formulation decides on the time of orders a priori but adapts the order quantities based on the realized demand. Specifically, we take the binary v_k variables as nonadjustable variables determined at time 0 (as also discussed by See and Sim 2010 in the context of an SOCP) and the u_k and y_k variables as adjustable variables defined as affine functions of the demands by letting $u_k = u_{k0} + \sum_{t=1}^{k-1} u_{kt} d_t$ and $y_k = y_{k0} + \sum_{t=1}^{k} y_{kt} d_t$. Then, the affinely adjustable counterpart of (2) is as follows:

$$\min \sum_{k=1}^{T} \left(c_k \left(u_{k0} + \sum_{t=1}^{k-1} u_{kt} d_t \right) + f_k v_k + y_{k0} + \sum_{t=1}^{k} y_{kt} d_t \right)$$

$$\text{s.t.} \quad y_{k0} + \sum_{t=1}^{k} y_{kt} d_t \geq h_k \left(I_1 + \sum_{i=1}^{k} \left(u_{i0} + \sum_{t=1}^{i-1} u_{it} d_t - d_i \right) \right), \quad k \in \mathcal{T}, d \in D_{\text{PR}}^k,$$

$$y_{k0} + \sum_{t=1}^{k} y_{kt} d_t \geq -p_k \left(I_1 + \sum_{i=1}^{k} \left(u_{i0} + \sum_{t=1}^{i-1} u_{it} d_t - d_i \right) \right), \quad k \in \mathcal{T}, d \in D_{\text{PR}}^k,$$

$$0 \leq u_{k0} + \sum_{t=1}^{k-1} u_{kt} d_t \leq M v_k, \quad v_k \in \{0,1\}, \; k \in \mathcal{T}. \tag{14}$$

Since (14) is a semi-infinite optimization problem, it is reformulated as a tractable mixed integer programming (MIP) problem, in a way similar to what was done in (3) and (4), by ensuring feasibility of its constraints for any demand realization. Unlike (3) and (4), where uncertain parameters have coefficients of -1 or 1, because some variables are defined as affine functions of the demands, some uncertain parameters have continuous variables as coefficients in (14), which may lead to additional variables and constraints in the tractable MIP problem. Ben-Tal et al. (2004) reformulate an inequality of the form

$$\lambda_0 + \sum_{t \in \mathcal{T}} \lambda_t d_t \leq 0, \quad d \in \mathcal{D}_{\text{PR}}^T, \tag{15}$$

where λ_0 is a constant and λ_t ($t \in \mathcal{T}$) denotes an expression involving some variables, as

$$\begin{aligned} &\lambda_0 + \sum_{t \in \mathcal{T}} (\lambda_t \bar{d}_t + \mu_t \hat{d}_t) \leq 0, \\ &-\mu_t \leq \lambda_t \leq \mu_t, \quad t \in \mathcal{T}. \end{aligned} \tag{16}$$

Using this reformulation methodology, we obtain the following affinely adjustable robust formulation:

(AAR)

$$\min \sum_{k=1}^{T} \left(c_k \left(u_{k0} + \sum_{t=1}^{k-1} u_{kt} \bar{d}_t \right) + f_k v_k + y_{k0} + \sum_{t=1}^{k} y_{kt} \bar{d}_t + \hat{d}_k n_k \right) \tag{17}$$

$$\text{s.t.} \quad -n_t \leq y_{tt} + \sum_{k=t+1}^{T} (c_k u_{kt} + y_{kt}) \leq n_t, \quad t \in \mathcal{T}, \tag{18}$$

$$y_{k0} + \sum_{t=1}^{k} y_{kt} \bar{d}_t \geq h_k \left(I_1 + \sum_{i=1}^{k} \left(u_{i0} + \sum_{t=1}^{i-1} u_{it} \bar{d}_t - \bar{d}_i \right) \right) + \sum_{t=1}^{k} \hat{d}_t s_{kt}, \quad k \in \mathcal{T}, \tag{19}$$

$$-s_{kt} \le \sum_{i=t+1}^{k} h_k u_{it} - h_k - y_{kt} \le s_{kt}, \quad k \in \mathcal{T},\ 1 \le t \le k, \tag{20}$$

$$y_{k0} + \sum_{t=1}^{k} y_{kt}\bar{d}_t \ge -p_k\left(I_1 + \sum_{i=1}^{k}\left(u_{i0} + \sum_{t=1}^{i-1} u_{it}\bar{d}_t - \bar{d}_i\right)\right) + \sum_{t=1}^{k}\hat{d}_t s'_{kt}, \quad k \in \mathcal{T}, \tag{21}$$

$$-s'_{kt} \le -\sum_{i=t+1}^{k} p_k u_{it} + p_k - y_{kt} \le s'_{kt}, \quad k \in \mathcal{T},\ 1 \le t \le k, \tag{22}$$

$$u_{k0} + \sum_{t=1}^{k-1}(\bar{d}_t u_{kt} - \hat{d}_t x_{kt}) \ge 0, \quad k \in \mathcal{T}, \tag{23}$$

$$-x_{kt} \le u_{kt} \le x_{kt}, \quad k \in \mathcal{T},\ 1 \le t < k, \tag{24}$$

$$u_{k0} + \sum_{t=1}^{k-1}(\bar{d}_t u_{kt} + \hat{d}_t x_{kt}) \le M v_k, \quad v_k \in \{0,1\},\ k \in \mathcal{T}, \tag{25}$$

where $M = \sum_{k=1}^{T}(\bar{d}_k + \hat{d}_k)$.

Note that objective (17) and constraints (18) are obtained by treating the objective function of (14) as a constraint (i.e., $\min \xi$ s.t. $\xi \ge \sum_{k=1}^{T}(c_k(u_{k0} + \sum_{t=1}^{k-1} u_{kt}d_t) + f_k v_k + y_{k0} + \sum_{t=1}^{k} y_{kt}d_t)$, $d \in D_{\mathrm{PR}}^{T}$).

3.3. The Robust Formulation with an Approximate Objective Function

We now present the robust formulation we derived from See and Sim (2010). Unlike the RSP, See and Sim (2010) considered an inventory management problem that involves nonzero lead time, zero fixed cost, a capacity over order quantities, and a factor-based demand model. As a result, we adapt this robust formulation to our problem. The main idea of this formulation is to approximate the expected value functions in the objective function of the stochastic problem with good upper bounds.

The factor-based demand model adopted by See and Sim (2010) is an affine function of zero mean random factors z'_t defined as follows:

$$d_k = d_{k0} + \sum_{t=1}^{N} d_{kt} z'_t, \quad k \in \mathcal{T}, \tag{26}$$

where N is the number of random factors, which are realized in sequence over time.

In our case of independently distributed demand, the factor-based demand model takes the form

$$d_k = d_{k0} + d_{kk} z'_k + \sum_{t=1}^{k-1} d_{kt} z'_t, \quad k \in \mathcal{T}, \tag{27}$$

where $d_{k0} = \bar{d}_k$, $d_{kk} = 1$, $d_{kt} = 0$ for $1 \le t < k$, $z'_t \in [-\hat{d}_t, \hat{d}_t]$ for $t \in \mathcal{T}$, and the random factors $z'_1, \ldots, z'_t$ are known at the end of period t.

In contrast to See and Sim (2010), where the truncated linear replenishment policy (TLRP) performs best in terms of average-case performance, the robust formulation for the linear replenishment policy (LRP) always yields better results than the robust formulation for the TLRP in our computational experiments. This is because there are much more second order cone constraints in the formulation for the TLRP than for the LRP, which means that obtaining a good TLRP is very difficult within the time limit using a state-of-the-art solver in the presence of binary variables. Hence we adapt the robust formulation for the LRP to our problem in the following.

In the same way as AAR, the robust formulation with an approximate objective function decides on the time of orders a priori, but adapts the order quantities based on the realized demand. Defining y'_k as the inventory level at the beginning of period k, the binary v_k variables are nonadjustable variables determined at time 0, whereas the u_k and y'_k variables are adjustable variables defined as affine functions of the random demand factors (i.e., $u_k = u_{k0} + \sum_{t=1}^{k-1} u_{kt} z'_t$ and $y'_k = y'_{k0} + \sum_{t=1}^{k-1} y'_{kt} z'_t$). Then, the robust formulation for the LRP is as follows:

$$\min \sum_{k=1}^{T}\left(c_k u_{k0} + f_k v_k + \sum_{i=1}^{3}(h_k \gamma^1_{i,k+1} + p_k \gamma^2_{i,k+1})\right) \tag{28}$$

$$\text{s.t. } y'_{k+1,t} = y'_{kt} + u_{kt} - d_{kt}, \quad k \in \mathcal{T},\ 0 \le t \le k, \tag{29}$$

$$0 \le u_{k0} + \sum_{t=1}^{k-1} u_{kt} z'_t \le M v_k, \quad v_k \in \{0,1\},\ k \in \mathcal{T},\ z'_t \in [-\hat{d}_t, \hat{d}_t], \tag{30}$$

$$Y^i_{1,k+1,0} + \max_{z'_t}\sum_{t=1}^{k} Y^i_{1,k+1,t} z'_t \le \gamma^i_{1,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{31}$$

$$0 \le \gamma^i_{1,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{32}$$

$$\max_{z'_t}\sum_{t=1}^{k} z'_t(-Y^i_{2,k+1,t}) \le \gamma^i_{2,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{33}$$

$$Y^i_{2,k+1,0} \le \gamma^i_{2,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{34}$$

$$\tfrac{1}{2}Y^i_{3,k+1,0} + \tfrac{1}{2}\left((Y^i_{3,k+1,0})^2 + \sum_{t=1}^{k}\sigma^2_{z'_t}(Y^i_{3,k+1,t})^2\right)^{1/2} \le \gamma^i_{3,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{35}$$

$$Y^1_{1,k+1,t} + Y^1_{2,k+1,t} + Y^1_{3,k+1,t} = y'_{k+1,t}, \quad k \in \mathcal{T},\ 0 \le t \le k, \tag{36}$$

$$Y^2_{1,k+1,t} + Y^2_{2,k+1,t} + Y^2_{3,k+1,t} = -y'_{k+1,t}, \quad k \in \mathcal{T},\ 0 \le t \le k, \tag{37}$$

where $M = \sum_{k=1}^{T}(\bar{d}_k + \hat{d}_k)$, $y'_{10} = I_1$, $d_{k0} = \bar{d}_k$, $d_{kt} = 0$ for $1 \le t < k$, $d_{kk} = 1$, $y'_{kk} = u_{kk} = 0$, and $\sigma^2_{z'_t} = (2\hat{d}_t)^2/12$. Note that $\sigma^2_{z'_t}$ is the variance of z'_t, which is uniformly distributed between $-\hat{d}_t$ and $\hat{d}_t$.

As done for AAR, using the reformulation methodology (15) and (16), we reformulate model (28)–(37) as a tractable SOCP formulation with binary constraints:

(SS) min (28)

s.t. (29), (32), (34)–(37),

$$u_{k0} - \sum_{t=1}^{k-1} \hat{d}_t x_{kt} \geq 0, \quad k \in \mathcal{T}, \tag{38}$$

$$-x_{kt} \leq u_{kt} \leq x_{kt}, \quad 1 \leq t < k,\ k \in \mathcal{T}, \tag{39}$$

$$u_{k0} + \sum_{t=1}^{k-1} \hat{d}_t x_{kt} \leq M v_k, \quad v_k \in \{0,1\},\ k \in \mathcal{T}, \tag{40}$$

$$Y^i_{1,\,k+1,\,0} + \sum_{t=1}^{k} \hat{d}_t m^i_{k+1,\,t} \leq \gamma^i_{1,\,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{41}$$

$$-m^i_{k+1,\,t} \leq Y^i_{1,\,k+1,\,t} \leq m^i_{k+1,\,t}, \quad 1 \leq t \leq k,\ k \in \mathcal{T},\ i = 1,2, \tag{42}$$

$$\sum_{t=1}^{k} \hat{d}_t n^i_{k+1,\,t} \leq \gamma^i_{2,\,k+1}, \quad k \in \mathcal{T},\ i = 1,2, \tag{43}$$

$$-n^i_{k+1,\,t} \leq -Y^i_{2,\,k+1,\,t} \leq n^i_{k+1,\,t}, \quad 1 \leq t \leq k,\ k \in \mathcal{T},\ i = 1,2. \tag{44}$$

Note that (38)–(40), (41) and (42), and (43) and (44) are the robust counterparts of (30), (31), and (33), respectively.

4. A New Robust Formulation

We now propose a new robust formulation that represents the RSP differently from the PR, BUR, AAR, and SS formulations. As mentioned earlier, the RSP is actually the robust counterpart of the ULSB for which tight reformulations, such as facility location (FL) and shortest path (SP) reformulations, are known. We make use of the FL formulation because of its simplicity compared with the SP formulation, in proposing a new robust formulation for the RSP. The main idea is to disaggregate u_t variables for the demand of each period by specifying when to place an order to satisfy the demand of a specific period.

An important difference between the robust problem and the deterministic ULSB is the impact of the positive initial inventory. In the case of the deterministic ULSB, the positive initial inventory can be treated as zero because one can obtain an equivalent problem with zero initial inventory by deducting the initial inventory from the deterministically known demands until the initial inventory is depleted, whereas this cannot be performed in the case of the robust problem since the demand realizations are not known a priori. Thus, the positive initial inventory should explicitly be considered in the formulation. On the other hand, in the case of backlogged demand (i.e., negative initial inventory, $I_1 < 0$), we can easily treat initial inventory as zero by defining a dummy demand $d_0 = -I_1$ at period 0, which is going to be backlogged, and solving an equivalent problem with $I_1 = 0$ and periods $k \in \mathcal{T} \cup \{0\}$.

Using the disaggregation idea of the FL formulation, let w_{0k} be the fraction of the customer demand in period k that is supplied by the positive initial inventory, let w_{tk} be the fraction of the customer demand in period k that is ordered in period t, and let $w_{T+1,\,k}$ be the fraction of customer demand in period k that is unsatisfied. Defining g_{tk} as the unit cost of satisfying customer demand in period k by ordering in period t, the new uncertain formulation we propose for the RSP (assuming $I_1 \geq 0$) is as follows:

$$\min \left\{ \sum_{k=1}^{T} f_k v_k + \sum_{t=1}^{T+1} \sum_{k=1}^{T} g_{tk} d_k w_{tk} + \sum_{k=1}^{T} (h_k I_1 - H_{kT} d_k w_{0k}) \right\} \tag{45}$$

$$\text{s.t.} \quad \sum_{t=0}^{T+1} w_{tk} = 1, \quad k \in \mathcal{T}, \tag{46}$$

$$w_{tk} \leq v_t, \quad t \in \mathcal{T},\ k \in \mathcal{T}, \tag{47}$$

$$\sum_{k=1}^{T} d_k w_{0k} \leq I_1, \quad d_k \in [\bar{d}_k - \hat{d}_k,\ \bar{d}_k + \hat{d}_k], \tag{48}$$

$$w_{tk} \geq 0,\ v_k \in \{0,1\}, \quad 0 \leq t \leq T+1,\ k \in \mathcal{T}, \tag{49}$$

where $H_{kT} = \sum_{l=k}^{T} h_l$, $g_{tk} = c_t + \sum_{l=t}^{k-1} h_l$ if $t \leq k$, $g_{tk} = c_t + \sum_{l=k}^{t-1} p_l$ if $k < t$, and $g_{T+1,k} = \sum_{l=k}^{T} p_l$.

The objective (45) is the total of ordering costs, inventory holding/shortage costs, and the cost due to the initial inventory. Constraints (46) ensure that the sum of the fraction of demand in period k that is supplied by the initial inventory, the fraction of demand in period k that is ordered from period 1 through T, plus the fraction of demand in period k that is left unmet is equal to one. Constraints (47) stipulate that a fixed order cost is incurred in period t if any amount is ordered in that period. Constraints (48) ensure that the total amount supplied by the initial inventory does not exceed its level. Constraints (49) impose the nonnegativity and integrality of the variables.

Since uncertain demand parameters appear both in the objective (45) and constraints (48), we apply the robust optimization methodology of Bertsimas and Sim (2004) to these, and obtain the following nonlinear robust formulation

$$\text{(NR)}\quad \min\Bigg\{\sum_{k=1}^{T} f_k v_k+\sum_{k=1}^{T}\bar{d}_k\bigg(\sum_{t=1}^{T+1} g_{tk}w_{tk}-H_{kT}w_{0k}\bigg)+H_{1T}I_1 + \max\bigg\{\sum_{k=1}^{T}\hat{d}_k\bigg|\sum_{t=1}^{T+1} g_{tk}w_{tk}-H_{kT}w_{0k}\bigg|z_{0k}\colon \sum_{k=1}^{T} z_{0k}\le\Gamma_T,\ 0\le z_{0k}\le 1,\ k\in\mathcal{T}\bigg\}\Bigg\} \tag{50}$$

s.t. (46), (47), (49),

$$\sum_{k=1}^{T}\bar{d}_k w_{0k}+\max\bigg\{\sum_{k=1}^{T}\hat{d}_k w_{0k}z_{1k}\colon \sum_{k=1}^{T} z_{1k}\le\Gamma_T,\ 0\le z_{1k}\le 1,\ k\in\mathcal{T}\bigg\}\le I_1. \tag{51}$$

Using the strong duality theorem, we obtain the following equivalent linear model:

$$\text{(NR)}\quad \min\Bigg\{\sum_{k=1}^{T} f_k v_k+\sum_{k=1}^{T}\bar{d}_k\bigg(\sum_{t=1}^{T+1} g_{tk}w_{tk}-H_{kT}w_{0k}\bigg) + H_{1T}I_1+\theta_0\Gamma_T+\sum_{k=1}^{T}\alpha_{0k}\Bigg\} \tag{52}$$

s.t. (46), (47), (49),

$$\theta_0+\alpha_{0k}\ge\hat{d}_k\beta_k,\quad k\in\mathcal{T}, \tag{53}$$

$$-\beta_k\le\sum_{t=1}^{T+1} g_{tk}w_{tk}-H_{kT}w_{0k}\le\beta_k,\quad k\in\mathcal{T}, \tag{54}$$

$$\sum_{k=1}^{T}\bar{d}_k w_{0k}+\theta_1\Gamma_T+\sum_{k=1}^{T}\alpha_{1k}\le I_1, \tag{55}$$

$$\theta_1+\alpha_{1k}\ge\hat{d}_k w_{0k},\quad k\in\mathcal{T}, \tag{56}$$

$$\theta_0\ge 0,\ \theta_1\ge 0,\ \alpha_{0k}\ge 0,\ \alpha_{1k}\ge 0,\quad k\in\mathcal{T}. \tag{57}$$

We have three important observations to formulate on the NR formulation. First, the solution of NR defines a policy like AAR; that is, the solution explicitly defines the time of orders at time 0 and the fraction of each period's demand that is to be satisfied regardless of the demand realization. Thus, the v_k variables indicating the time of orders are decided a priori, whereas the w_{tk} variables are adaptable to demand realization if $t > k$. Unlike the w_{tk} variables for $t > k$, where the demand in period k has already realized, meaning that the amount to order in period t is known, the w_{tk} variables for $t \le k$ do not explicitly give the amount to order in period t since demand in period k is still unknown at t. Nevertheless, we can compute the amounts to order in each period t by inserting the values of the w_{tk} variables into (50) and solving the inner maximization problem of (50), which can easily be done by solving a continuous knapsack problem. Note that the solution of the inner maximization problem for a given w, denoted by z^*_{0k}, yields the worst-case demand realization. In particular, the worst-case demand realization in period k would be $\bar{d}_k + s_k\hat{d}_k z^*_{0k}$, where s_k is equal to 1 if $\sum_{t=1}^{T+1} g_{tk}w_{tk} \ge H_{kT}w_{0k}$, and -1 otherwise. When z^*_{0k} are inserted into the z_{1k} variables in the maximization problem of (51), this is feasible to the maximization problem of (51), which means that we have a lower bound for the maximization problem of (51). Thus, this solution is also feasible for (51). Using the obtained worst-case demand realization, one can easily compute the amount to order in period t as $u_t = \sum_{k=1}^{t-1}\tilde{d}_k w_{tk} + \sum_{k=t}^{T}(\bar{d}_k + s_k\hat{d}_k z^*_{0k})w_{tk}$, where $\tilde{d}_k$ denotes the realized demand in period k.

Second, NR is a computationally attractive formulation since it can be solved in polynomial time when the initial inventory level is zero or negative, as discussed in the sequel. Third, unlike the BUR formulation, NR requires only a global budget of uncertainty Γ_T, instead of a funnel of budgets of uncertainty $(\Gamma_1, \ldots, \Gamma_T)$. Although a funnel of budgets could be used in formulating NR, we could not extend the polynomial solvability property of the current NR to the resulting formulation, and we have not observed any advantage to using a funnel of budgets over a global budget of uncertainty regarding the average-case performance.

In the presence of nonzero fixed costs, the formulation for the RSP in Bertsimas and Thiele (2006) is a mixed integer programming problem, which is not polynomially solvable. This is because demand uncertainty normally implies data uncertainty in the constraints of a formulation. However, we have transferred demand uncertainty from the constraints to the objective function of the formulation, and there can thus be an opportunity to find a polynomial time algorithm when the initial inventory is nonpositive. Next, we show how NR can be solved in polynomial time when $I_1 \le 0$. (As discussed before, the case with $I_1 < 0$ can be treated as the case with $I_1 = 0$.) Letting $I_1 = 0$ in NR, it is easy to show that the w variables automatically take a binary value, and thus we obtain the following robust formulation:

$$\text{(NR-0)}\quad \min\Bigg\{\sum_{k=1}^{T} f_k v_k+\sum_{t=1}^{T+1}\sum_{k=1}^{T}\bar{d}_k g_{tk}w_{tk} + \theta_0\Gamma_T+\sum_{k=1}^{T}\alpha_{0k}\Bigg\} \tag{58}$$

$$\text{s.t.}\quad \sum_{t=1}^{T+1} w_{tk}=1,\quad k\in\mathcal{T}, \tag{59}$$

$$w_{tk} \le v_t, \quad t \in \mathcal{T}, k \in \mathcal{T}, \tag{60}$$

$$\theta_0 + \alpha_{0k} \ge \hat{d}_k \sum_{t=1}^{T+1} g_{tk} w_{tk}, \quad k \in \mathcal{T}, \tag{61}$$

$$\theta_0 \ge 0,\ \alpha_{0k} \ge 0,\ w_{tk} \in \{0,1\},\ v_k \in \{0,1\},\quad 1 \le t \le T+1,\ k \in \mathcal{T}. \tag{62}$$

Note that $\alpha_{0k} = \max\{\hat{d}_k \sum_{t=1}^{T+1} g_{tk} w_{tk} - \theta_0, 0\}$ in an optimal solution. Thus, we can rewrite the expression for α_{0k} as follows:

$$\begin{aligned}\alpha_{0k} &= \max\left\{\hat{d}_k \sum_{t=1}^{T+1} g_{tk} w_{tk} - \theta_0, 0\right\}\\ &= \sum_{t=1}^{T+1} \max\{\hat{d}_k g_{tk} w_{tk} - \theta_0, 0\} \quad \text{(due to (59))}\\ &= \sum_{t=1}^{T+1} \max\{\hat{d}_k g_{tk} - \theta_0, 0\} w_{tk}.\end{aligned} \tag{63}$$

We can then restate NR-0 using (63) and letting $\hat{g}_{tk} = \hat{d}_k g_{tk}$ as

$$\text{(NR-0)} \quad \min\left\{\sum_{k=1}^{T} f_k v_k + \sum_{t=1}^{T+1}\sum_{k=1}^{T} \bar{d}_k g_{tk} w_{tk} + \theta_0 \Gamma_T + \sum_{k=1}^{T}\sum_{t=1}^{T+1} \max\{\hat{g}_{tk} - \theta_0, 0\} w_{tk}\right\}$$

$$\begin{aligned}\text{s.t. } & (59), (60),\\ & \theta_0 \ge 0,\ w_{tk} \in \{0,1\},\ v_k \in \{0,1\},\\ & 1 \le t \le T+1,\ k \in \mathcal{T}.\end{aligned} \tag{64}$$

Similar to Bertsimas and Sim (2003), it is easy to show that θ_0 takes a value in $\{0, \hat{g}_{11}, \ldots, \hat{g}_{T+1,T}\}$ in an optimal solution. Therefore, solving NR-0 for each distinct value of θ_0 and selecting the one with the best objective function value yields the optimal solution.

Theorem 1. *NR-0 can be solved in polynomial time by solving at most $T(T+1)+1$ LP problems.*

Proof. Since the LP relaxation of the FL formulation for the ULSB is integral, the proof follows from the above discussion. □

To propose a polynomial time algorithm for the RSP with a budget of uncertainty, which does not require solving LP problems, we make use of the SP formulation proposed for the ULSB and propose a new robust formulation referred to as the SPR formulation (see the appendix for details). The result is given in the following theorem.

Theorem 2. *The SPR formulation is polynomially solvable in $O(T^4)$ time.*

The results obtained for the RSP with a budget of uncertainty can easily be extended to the robust problem without backlogging. Hence, a reformulation-based robust formulation of the latter problem is also polynomially solvable in $O(T^4)$ time.

As will be seen in the sequel, the NR formulation can be solved to optimality within a few seconds even for large-scale instances with positive initial inventory. Thus, to improve its performance, NR can be implemented through a folding horizon approach, which we denote by NR-FH. Under this approach, the NR formulation is sequentially solved for each period, and only the decisions made for the current period are implemented, whereas the decisions concerning the following periods are reoptimized when solving the next problem. In this approach, when NR is solved for a specific period i (covering periods from i to T), a fixed order cost is incurred if an order is placed in period i and the amount to order in period i is calculated as $u_i = -\tilde{I}_i w_{i,i-1}\rho + \sum_{k=i}^{T}(\bar{d}_k + s_k \hat{d}_k z^*_{0k}) w_{ik}$, where $\rho = 1$ if the realized inventory level $\tilde{I}_i$ at the beginning of period i is negative (i.e., $\tilde{I}_i = \tilde{I}_{i-1} + u_{i-1} - \tilde{d}_{i-1} < 0$), and 0 otherwise. We next present the NR formulation that is sequentially solved for each period i $(1 \le i \le T)$ in NR-FH:

$$(\text{NR}^i) \quad \min\left\{\sum_{k=i}^{T} f_k v_k + \sum_{k=i}^{T} \bar{d}_k \left(\sum_{t=i}^{T+1} g_{tk} w_{tk} - H_{kT} w_{0k}\right) + \theta_0 \Gamma_{T-i+1} + \sum_{k=i}^{T} \alpha_{0k} - \sum_{t=i}^{T+1} \rho \tilde{I}_i g_{t,i-1} w_{t,i-1}\right\}$$

$$\begin{aligned}\text{s.t. } & (1-\rho) w_{0k} + \sum_{t=i}^{T+1} w_{tk} = 1, \quad i \le k \le T,\\ & \sum_{t=i}^{T+1} w_{t,i-1} = 1 \quad \text{if } \tilde{I}_i < 0,\\ & w_{tk} \le v_t, \quad i \le t \le T, i \le k \le T,\\ & \rho w_{t,i-1} \le v_t, \quad i \le t \le T,\\ & \theta_0 + \alpha_{0k} \ge \hat{d}_k \beta_k, \quad i \le k \le T,\\ & -\beta_k \le \sum_{t=i}^{T+1} g_{tk} w_{tk} - H_{kT} w_{0k} \le \beta_k, \quad i \le k \le T,\\ & \sum_{k=i}^{T} \bar{d}_k w_{0k} + \theta_1 \Gamma_{T-i+1} + \sum_{k=i}^{T} \alpha_{1k} \le (1-\rho)\tilde{I}_i\\ & \theta_1 + \alpha_{1k} \ge \hat{d}_k w_{0k}, \quad i \le k \le T,\\ & \theta_0 \ge 0, \theta_1 \ge 0, \alpha_{0k} \ge 0, \alpha_{1k} \ge 0, w_{0k} \ge 0,\\ & \qquad v_k \in \{0,1\}, \quad i \le k \le T,\\ & w_{tk} \ge 0, \quad i \le t \le T+1, i-1 \le k \le T,\end{aligned} \tag{65}$$

where $w_{t0} = 0$ for $i \le t \le T+1$.

5. Computational Results

We performed computational experiments on test instances to assess the average-case performance of solutions yielded by the PR, BUR, AAR, SS, NR, and NR-FH formulations. The average-case performance of formulations was assessed by considering the average performance of the solution yielded by each formulation for an instance over a given number of simulations of realized demand. To evaluate how far the average-case performance of different formulations is from the optimal expected objective function values, we also computed a lower bound on the optimal expected objective function value for each test instance. We also tested the computational effectiveness of these formulations with respect to the CPU time needed to solve them to optimality. All these formulations were solved by CPLEX 12.5 with its default settings using a single thread, and all computational experiments were performed on a 2.4 GHz Workstation with 48 GB RAM running on Windows 7 (64 bit).

For the computational experiments, we generated test instances using settings similar to those used by Bertsimas and Thiele (2006) as follows. The length of the planning horizon was set to two different values: $T \in \{20, 50\}$. The unit order cost was taken as $c_k = 1$. The fixed cost was set to $f_k = 1{,}000$. Two opposite relations were used for unit holding and unit backlogging costs: $h_k < p_k$ and $h_k > p_k$. The unit backlogging cost was set to $p_k \in \{5, 7, 9\}$ when $h_k = 4$, and the unit holding cost was set to $h_k \in \{5, 7, 9\}$ when $p_k = 4$. Both zero initial inventory $I_1 = 0$ and nonzero initial inventory $I_1 = 130$ were considered. The uncertain demand has a mean $\bar{d}_k = 100$ and standard deviation $\sigma = 10$ or 20. In our robust optimization framework, the uncertain demand d_k can take any value from the interval $[\bar{d}_k - 2\sigma,\ \bar{d}_k + 2\sigma]$, i.e., $\hat{d}_k = 2\sigma$. To simulate the performance of solutions yielded by the PR, BUR, AAR, SS, NR, and NR-FH formulations, different demand scenarios with different underlying demand distributions and different degrees of correlation among demands were considered. Specifically, realized demand values were generated as (see, e.g., Federgruen and Tzur 1999) $d_1 = \epsilon_1$ and $d_t = \alpha d_{t-1} + (1-\alpha)\epsilon_t$, where ϵ_t is an independent random variable that is uniformly or truncated normally distributed from the interval $[\bar{d}_k - 2\sigma, \bar{d}_k + 2\sigma]$, and $\alpha \in \{0, 0.25, 0.75\}$. Note that the demands are independent when $\alpha = 0$, whereas demands are weakly and strongly correlated, respectively, when $\alpha = 0.25$ and $\alpha = 0.75$. Thus, combining all parameter settings, we generated 48 core instances and 288 instances (48 instances with two probability distributions and three correlation coefficients) for the average-case performance assessment, in total. To estimate the average performance of solutions provided by the different formulations, we generated 100 simulations of realized demand (see, e.g., Ben-Tal et al. 2004) for each instance. Note that these 100 sample paths for demand are identical across all formulations. For BUR, NR, and NR-FH, the budgets of uncertainty Γ_k were generated using the algorithm by Bertsimas and Thiele (2006), which attempts to compute the optimal budgets of uncertainty minimizing the expected value of the objective function value. For NR and NR-FH, we also tested the method of Bertsimas and Sim (2004) to determine the budgets of uncertainty Γ_k. Specifically, we chose Γ_{T-k+1} for the formulation at period k ($k \in \mathcal{T}$) such that the realized cost will be larger than the objective function value of the formulation with probability of at most δ%, where $\delta \in \{1, 5, 10\}$. Note that such a Γ value corresponds to at most δ% probability of infeasibility of constraint (48) for a given optimal solution of the formulation (see Bertsimas and Sim 2004, Theorem 3). However, the average-case performance obtained with this method was worse than that obtained by the former method on the majority of the instances. Thus, we did not report the results obtained with the method of Bertsimas and Sim (2004). The LB on the optimal expected objective function value was found by solving a deterministic version of the problem for each simulation of realized demand, assuming all demand is known in advance, and then taking the average objective function value over 100 simulations of realized demand.

We first tested the computational effectiveness of the formulations. We present the average results obtained by all formulations in Table 1. We imposed a time limit of two hours for all formulations. The seven columns in Table 1 show the formulation, the length of the planning horizon, the initial inventory level, the time spent to solve the instances in seconds, the number of nodes explored after the root node in the branch-and-bound tree, the percentage gap between the best known upper (z_{UB}) and lower (z_{LB}) bounds (i.e., Gap% $= 100(z_{\mathrm{UB}} - z_{\mathrm{LB}})/z_{\mathrm{LB}}$) obtained by each formulation, and the number of optimally solved instances out of 72 instances for each combination of T and I_1, respectively. Note that the results provided in "Seconds," "Nodes," and "Gap%" columns were found by taking the average over 72 instances for each combination of T and I_1.

Table 1 shows that all instances with $T = 20$ could be solved to optimality within a few seconds by PR, BUR, and NR, and within a few minutes by AAR, whereas SS could optimally solve just 8.33% of the instances with $I_1 = 0$ within 1.97 hours and less than half of the instances with $I_1 > 0$ within 1.85 hours. NR could optimally solve larger instances within a second on average (at most two seconds), whereas AAR and SS (resp., PR and BUR) could not solve any (resp., many) of the instances with $T = 50$ to optimality within the two-hour time limit. Also, the remaining percentage gap between the best UB and LB is still

Table 1 **Effectiveness of the Formulations**

Formulation	T	I_1	Seconds	Nodes	Gap%	#Opt
PR	20	0	0.5	2,238.7	0.00	72
		>0	0.6	2,658.4	0.00	72
	50	0	4,104.3	14,456,491.5	0.34	42
		>0	4,317.1	15,076,808.7	0.27	36
BUR	20	0	0.7	2,816.3	0.00	72
		>0	0.6	1,965.1	0.00	72
	50	0	3,913.1	13,786,640.2	1.41	36
		>0	3,377.9	10,704,624.5	0.67	54
AAR	20	0	424.4	71,708.8	0.00	72
		>0	306.5	55,807.4	0.00	72
	50	0	7,200.0	12,627.6	74.16	0
		>0	7,200.0	13,222.9	74.97	0
SS	20	0	7,075.1	95,892.3	13.46	6
		>0	6,644.9	91,755.5	7.22	31
	50	0	7,200.0	777.1	276.93	0
		>0	7,200.0	738.2	281.00	0
NR	20	0	0.3	7.9	0.00	72
		>0	0.3	55.7	0.00	72
	50	0	0.5	30.6	0.00	72
		>0	0.8	294.1	0.00	72
NR-FH	20	0	75.1	—	—	—
		>0	74.3	—	—	—
	50	0	837.2	—	—	—
		>0	864.3	—	—	—

quite large for AAR, and especially for SS. Because of the computational inefficiency of the PR, BUR, AAR, and SS formulations, only NR was implemented through a folding horizon approach, which is denoted by NR-FH. For PR, BUR, AAR, SS, and NR, the formulations were solved only once, and the obtained policies or solutions were implemented directly. As can be seen in the last four rows of Table 1, NR-FH can be executed quite fast even for the instances with $T = 50$, where NR-FH was solved in 15 minutes on average.

We compared all the formulations with each other with respect to their best solutions (the ones found within the two-hour time limit) and their practical performance with respect to the same demand realizations. The key results presented in Table 2 are as follows:

- NR-FH and SS yield the best average-case performance and outperform all other formulations on most instances. In particular, NR-FH outperforms all formulations on instances with $T = 50$ and all formulations except SS on instances with $T = 20$. SS, on the other hand, outperforms all formulations on instances with $T = 20$.
- AAR, being the third best, outperforms PR, BUR, and NR on most instances and performs slightly better than SS on instances with $T = 50$.
- BUR performs better than PR on most instances and better than NR on the majority of instances. NR, on the other hand, performs better than PR on most instances.
- PR performs poorly compared to other formulations on almost all instances, regardless of the initial inventory level or of the length of the time horizon.
- The presence of a nonzero initial inventory adversely affects the performance of PR, BUR, and NR-FH compared to the cases without an initial inventory. On the other hand, the performance of AAR, SS, and NR becomes slightly better in the presence of an initial inventory.
- The main benefit of NR is computational in that it does not perform well when not executed on a folding horizon, whereas with a folding-horizon implementation (i.e., NR-FH) it is very attractive and yields the best average-case performance on most large-scale instances.

Next we determined how far from LB the average-case performance is by simply deriving the efficiency

Table 2 **Average-Case Comparison of Formulations**

	$T = 20$								$T = 50$						
$I_1 = 0$	PR	BUR	AAR	SS	NR	NR-FH	Best	$I_1 = 0$	PR	BUR	AAR	SS	NR	NR-FH	Best
PR	—	0	1	0	18	0	0	PR	—	0	0	0	12	0	0
BUR	72	—	6	0	48	5	0	BUR	72	—	0	1	49	6	0
AAR	71	66	—	0	69	16	0	AAR	72	72	—	29	70	17	11
SS	72	72	72	—	72	67	67	SS	72	71	43	—	72	18	12
NR	54	24	3	0	—	3	0	NR	60	23	2	0	—	6	0
NR-FH	72	67	56	5	69	—	5	NR-FH	72	66	55	54	66	—	49
PR	—	4	1	0	13	0	0	PR	—	0	0	0	0	0	0
BUR	68	—	3	0	36	5	0	BUR	72	—	0	3	39	8	0
AAR	71	69	—	1	72	17	0	AAR	72	72	—	44	72	27	19
SS	72	72	71	—	72	69	69	SS	72	69	28	—	68	20	14
NR	59	36	0	0	—	2	0	NR	72	33	0	4	—	6	0
NR-FH	72	67	55	3	70	—	3	NR-FH	72	64	45	52	66	—	39

Notes. Except for the column "Best," each entry represents the number of times the formulation indicated in the row yielded a better average-case performance than the one in the column. Each entry in the column "Best" indicates the number of times the formulation written in the row yielded the best average-case performance.

of a robustness approach as $LB/Z(\cdot)$, where $Z(\cdot)$ is the average-case performance yielded by a robustness approach. Note that a higher efficiency value, which can be at most one, indicates a better performance. We compared all the formulations with each other with respect to their efficiency in Table 3, where the first column shows the instances included, the second column shows the length of the planning horizon, the third column shows the initial inventory level, the fourth column shows the variability of demand, and the remaining columns show the average efficiencies of PR, BUR, AAR, SS, NR, and NR-FH, respectively. Because NR-FH does not perform so well on all instances with $h=4$ and $p=9$, we provide the average efficiencies on all instances excluding those with $h=4$ and $p=9$ in the last eight rows of Table 3. The key results of Table 3 are as follows:

- NR-FH, SS, and AAR exhibit a performance superior to that of the other formulations with regard to efficiency. They mostly provide actual cost savings that are very close to the optimal expected cost on most instances.
- NR-FH shows a superior (resp., competitive) performance on the instances with $T=50$ (resp., $T=20$) except for those with $h=4$ and $p=9$ compared to all formulations (resp., SS).
- SS performs the best on the instances with $T=20$. The efficiency of SS is the most stable of all.
- The AAR formulation also performs very well. Specifically, it performs the best when $T=50$ and $\sigma=10$.
- When the variability of uncertain demand increases (i.e., when σ increases from 10 to 20), the efficiency gap between the NR-FH, SS, and AAR formulations and the NR, BUR, and PR formulations increases. In other words, the NR, BUR, and PR formulations are more adversely affected by an increase in the variability of demand than are the NR-FH, SS, and AAR formulations with respect to efficiency.
- Excluding instances with $h=4$ and $p=9$ positively affects the average efficiency of all formulations.
- The initial inventory, demand distributions, and correlation values do not significantly affect the efficiency results.

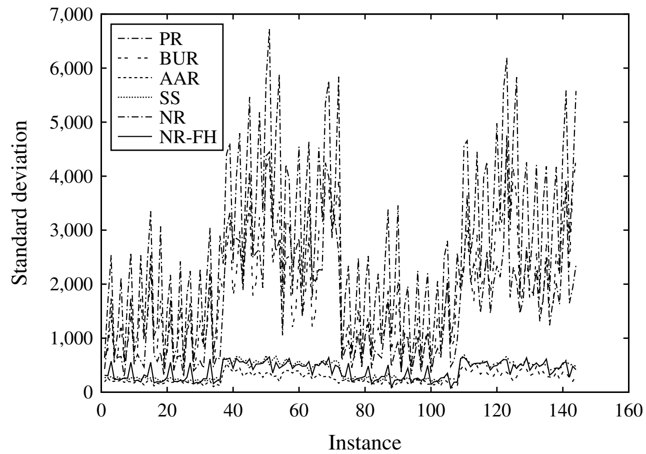
To see the variability of the formulations, the standard deviations of objective function values (over 100 simulations of realized demand) are plotted and presented in Figure 1. These results clearly reveal that AAR has the smallest variability, followed by NR-FH and SS. The remaining formulations yield high variability, with NR being slightly less variable than BUR and PR.

Table 3 Average Efficiency of the Formulations

Instance	T	I_1	σ	PR	BUR	AAR	SS	NR	NR-FH
All	20	0	10	0.836	0.912	0.955	0.971	0.904	0.949
			20	0.693	0.819	0.915	0.944	0.777	0.927
		>0	10	0.827	0.909	0.957	0.974	0.904	0.948
			20	0.690	0.814	0.911	0.948	0.820	0.926
	50	0	10	0.676	0.871	0.944	0.943	0.857	0.914
			20	0.485	0.731	0.910	0.913	0.705	0.898
		>0	10	0.672	0.866	0.942	0.932	0.864	0.909
			20	0.485	0.730	0.909	0.915	0.712	0.897
All′	20	0	10	0.864	0.921	0.958	0.975	0.917	0.971
			20	0.724	0.831	0.919	0.949	0.787	0.942
		>0	10	0.855	0.917	0.962	0.977	0.918	0.969
			20	0.724	0.826	0.916	0.953	0.834	0.942
	50	0	10	0.712	0.881	0.945	0.945	0.870	0.966
			20	0.517	0.742	0.913	0.915	0.718	0.935
		>0	10	0.709	0.876	0.944	0.933	0.882	0.964
			20	0.517	0.741	0.914	0.917	0.748	0.936

Note. All′ indicates all instances except those with $h=4$ and $p=9$.

Figure 1 Standard Deviation of Objective Function Values on Instances with $T=20$



6. Conclusions

We have studied the impact of modeling in robust inventory management problems under interval demand uncertainty. We have derived robust formulations by applying prominent robust optimization methodologies, and we have proposed a new robust model obtained by reformulating the problem. Our results show that the new robust formulation is polynomially solvable when the initial inventory is zero or negative, and also indicate that the heuristic in its folding-horizon implementation of the new robust formulation is superior to the pure, budget of uncertainty, and affinely adjustable robust formulations. It is also competitive with the robust formulation using an approximate objective function with respect to the average-case performance. Moreover, the new formulation and the new heuristic are far more efficient than the others in terms of computing time: instances with

50 periods could be optimally solved within a few seconds with the new formulation, or within 15 minutes with the new heuristic, whereas the majority of these instances could not be solved to optimality within two hours with other formulations, which also exhibited large gaps between the best known upper and lower bounds.

The reformulation idea is important since all inventory management problems considered in the literature can be reformulated using the same facility location idea (see Pochet and Wolsey 2006). Considering that the problem studied in this paper is a basic building block of more complex inventory management problems, like problems with a network structure as in Bertsimas and Thiele (2006), the reformulation-based robust formulations can provide well-performing policies for more complex inventory management problems within reasonable computing times. Another important implication of this study is the realization that reformulation-based robust formulations, as opposed to standard robust formulations obtained by applying classical robust optimization methodologies, can yield different and better-performing policies or solutions to problems under parameter uncertainty.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2015.2183.

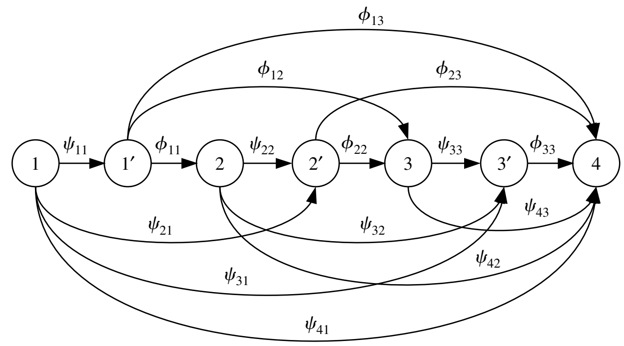
Acknowledgments

This work was partly supported by the Canadian Natural Sciences and Engineering Research Council [Grants 227837-09 and 39682-10]. This support is gratefully acknowledged.

Appendix

In this appendix, we show how the SPR formulation can be solved in polynomial time. We simplify the SPR formulation given in Pochet and Wolsey (2006) and define new variables to account for the unmet demand case, as was done in NR. We first define the parameters and variables used in the SPR formulation, and we then describe the associated shortest path network representation. Let ϕ_{tk} be 1 if the total customer demand from periods t through k is satisfied in period t ($1 \le t \le k \le T$) and 0 otherwise, and let ψ_{tk} be 1 if the total demand of the customer from periods k through $t-1$ is satisfied in period t ($1 \le k \le t \le T$) and 0 otherwise. Also, let $\psi_{T+1,k}$ be 1 if the total demand of the customer from periods k ($k \in \mathcal{T}$) through T is not satisfied and 0 otherwise. Define D_{tk} as the total demand from periods t through k, i.e., $D_{tk} = \sum_{j=t}^{k} d_j$; define e_{tk} as the total cost of satisfying the customer demand from periods t through k in period t, i.e., $e_{tk} = c_t D_{tk} + \sum_{j=t}^{k-1} h_j D_{j+1,k}$; and let b_{tk} be the total cost of satisfying demand of the customer from period k through $t-1$ in period t, i.e., $b_{tk} = c_t D_{k,t-1} + \sum_{j=k}^{t-1} p_j D_{kj}$. In the associated network, nodes represent the time periods, whereas arcs represent the order decisions. There are two nodes for each period t, denoted by t and t'', and a dummy node $T+1$. There are three types of arcs on which the flow is represented by the variables ϕ_{tk}, ψ_{tk} for $t \le T$, and $\psi_{T+1,k}$. Variables ϕ_{tk} represent the flow from node t'' to $k+1$ ($1 \le t \le k \le T$) with cost e_{tk}, ψ_{tk} the flow from node k to t'' ($1 \le k \le t \le T$) with cost b_{tk}, and $\psi_{T+1,k}$ the flow from node k ($1 \le k \le T$) to $T+1$ with cost $b_{T+1,k}$. A unit flow is sent to the network through node 1. An example shortest path network representation for an instance with $T=3$ is presented in Figure A.1. The objective function of the SPR formulation for the nominal ULSB is $\sum_{k=1}^{T} f_k v_k + \sum_{t=1}^{T+1}\sum_{k=1}^{t-1} b_{tk}\psi_{tk} + \sum_{t=1}^{T}\sum_{k=t}^{T} e_{tk}\phi_{tk}$, which involves expressions where d_k ($k \in \mathcal{T}$) is encountered several times. As a result, we should extract d_k to be able to write the robust counterpart. After some algebraic manipulations, we rewrite the objective function as $\sum_{k=1}^{T} f_k v_k + \sum_{k=1}^{T} d_k (\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt})$. Then, the SPR formulation for the RSP with budget of uncertainty is as follows:

Figure A.1 **The Shortest Path Network Representation for an Example Instance with $T=3$**



$$\min\left\{\sum_{k=1}^{T} f_k v_k + \sum_{k=1}^{T}\bar{d}_k\left(\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt}\right) + \max\left\{\sum_{k=1}^{T}\hat{d}_k z_k\left(\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt}\right): \sum_{k=1}^{T} z_{0k} \le \Gamma_T,\ 0 \le z_{0k} \le 1,\ k \in \mathcal{T}\right\}\right\} \quad (66)$$

$$\text{s.t.} \quad \sum_{k=1}^{T+1}\psi_{k1} = 1, \quad \text{node } 1, \quad (67)$$

$$\sum_{k=t}^{T+1}\psi_{kt} - \sum_{k=1}^{t-1}\phi_{k,t-1} = 0, \quad \text{node } t,\ 2 \le t \le T, \quad (68)$$

$$-\sum_{k=1}^{t}\psi_{tk} + \sum_{k=t}^{T}\phi_{tk} = 0, \quad \text{node } t'',\ 1 \le t \le T, \quad (69)$$

$$\sum_{k=t}^{T}\phi_{tk} \le v_t, \quad 1 \le t \le T, \quad (70)$$

$$\phi_{tk} \in \{0,1\}, \quad 1 \le t \le k \le T, \quad (71)$$

$$\psi_{tk} \in \{0,1\}, \quad 1 \le k \le t \le T+1,\ k \le T, \quad (72)$$

$$v_t \in \{0,1\}, \quad 1 \le t \le T. \quad (73)$$

Using the strong duality theorem, we obtain the following equivalent model:

$$\min\left\{\sum_{k=1}^{T} f_k v_k + \sum_{k=1}^{T} \bar{d}_k\left(\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt}\right) + \theta_0\Gamma_T + \sum_{k=1}^{T}\alpha_{0k}\right\} \quad (74)$$

s.t. (67)–(73),

$$\theta_0 + \alpha_{0k} \geq \hat{d}_k\left(\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt}\right), \quad 1 \leq k \leq T, \quad (75)$$

$$\theta_0 \geq 0, \ \alpha_{0k} \geq 0, \quad k \in \mathcal{T}. \quad (76)$$

Note that summing up constraint (67), constraints (68) from $t = 2$ to $t = k$, and constraints (69) from $t = 1$ to $t = k$ gives

$$\sum_{r=1}^{k}\sum_{t=k}^{T}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k}\psi_{rt} = 1, \quad 1 \leq k \leq T. \quad (77)$$

We can rewrite α_{0k} using (71), (72), and (77) as

$$\begin{aligned}
\alpha_{0k} &= \max\left\{\hat{d}_k\left(\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt}\right) - \theta_0, 0\right\} \\
&= \max\left\{\hat{d}_k\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} - \theta_0, 0\right\} + \max\left\{\hat{d}_k\sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt} - \theta_0, 0\right\} \\
&= \sum_{r=1}^{k}\sum_{t=k}^{T}\max\{\hat{d}_k g_{rk}\phi_{rt} - \theta_0, 0\} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k}\max\{\hat{d}_k g_{rk}\psi_{rt} - \theta_0, 0\} \\
&= \sum_{r=1}^{k}\sum_{t=k}^{T}\max\{\hat{d}_k g_{rk} - \theta_0, 0\}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k}\max\{\hat{d}_k g_{rk} - \theta_0, 0\}\psi_{rt}.
\end{aligned} \quad (78)$$

Using (78), we can rewrite the SPR formulation as follows:

$$\text{(SPR)} \quad \min\left\{\sum_{k=1}^{T} f_k v_k + \sum_{k=1}^{T}\bar{d}_k\left(\sum_{r=1}^{k}\sum_{t=k}^{T} g_{rk}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k} g_{rk}\psi_{rt}\right) + \theta_0\Gamma_T + \sum_{k=1}^{T}\left(\sum_{r=1}^{k}\sum_{t=k}^{T}\max\{\hat{g}_{rk} - \theta_0, 0\}\phi_{rt} + \sum_{r=k+1}^{T+1}\sum_{t=1}^{k}\max\{\hat{g}_{rk} - \theta_0, 0\}\psi_{rt}\right)\right\}$$

$$\text{s.t. } (67)\text{–}(73), \ \theta_0 \geq 0. \quad (79)$$

Proof of Theorem 2. For any $\theta_0 \geq 0$ value, SPR can be recast as a shortest path problem on a network with $O(T^2)$ nodes and $O(T^2)$ acyclic directed arcs, which can be solved in $O(T^2)$ time. As already discussed, θ_0 takes a value in $\{0, \hat{g}_{11}, \ldots, \hat{g}_{T+1,T}\}$ in an optimal solution. Thus, solving SPR as a shortest path problem for each distinct value of θ_0 (at most $T(T+1)+1$ distinct values) and selecting the one with minimum cost gives the optimal solution. □

References

Atamtürk A (2006) Strong formulations of robust mixed 0-1 programming. *Math. Programming, Ser. B* 108:235–250.

Atamtürk A, Zhang M (2007) Two-stage network flow and design under demand uncertainty. *Oper. Res.* 55:662–673.

Ben-Tal A, Nemirovski A (1998) Robust convex optimization. *Math. Oper. Res.* 23:769–805.

Ben-Tal A, Nemirovski A (1999) Robust solutions to uncertain programs. *Oper. Res. Lett.* 25:1–13.

Ben-Tal A, Nemirovski A (2000) Robust solutions of linear programming problems contaminated with uncertain data. *Math. Programming* 88:411–424.

Ben-Tal A, Golany B, Shimrit S (2009) Robust multi-echelon multi-period inventory control. *Eur. J. Oper. Res.* 199:922–935.

Ben-Tal A, Golany B, Nemirovski A, Vial JP (2005) Retailer-supplier flexible commitments contracts: A robust optimization approach. *Manufacturing Service Oper. Management* 7:248–271.

Ben-Tal A, Goryashko A, Guslitzer E, Nemirovski A (2004) Adjustable robust solutions of uncertain linear programs. *Math. Programming* 99:351–376.

Bertsimas D, Sim M (2003) Robust discrete optimization and network flows. *Math. Programming, Ser. B* 98:49–71.

Bertsimas D, Sim M (2004) The price of robustness. *Oper. Res.* 52:35–53.

Bertsimas D, Thiele A (2006) A robust optimization approach to inventory theory. *Oper. Res.* 54:150–168.

Bienstock D, Özbay N (2008) Computing robust basestock levels. *Discrete Optim.* 5:389–414.

Chen X, Zhang Y (2009) Uncertain linear programs: Extended affinely adjustable robust counterparts. *Oper. Res.* 57:1469–1482.

El-Ghaoui L, Lebret H (1997) Robust solutions to least-square problems to uncertain data matrices. *SIAM J. Matrix Anal. Appl.* 18:1035–1064.

El-Ghaoui L, Oustry F, Lebret H (1998) Robust solutions to uncertain semidefinite programs. *SIAM J. Optim.* 9:33–52.

Federgruen A, Tzur M (1999) Time partitioning heuristics: Application to one warehouse, multi-item, multi-retailer lot-sizing problems. *Naval Res. Logist.* 46:463–486.

Kouvelis P, Yu G (1997) *Robust Discrete Optimization and Its Applications* (Kluwer, Norwell, MA).

Pochet Y, Wolsey LA (2006) *Production Planning by Mixed Integer Programming* (Springer, New York).

See CT, Sim M (2010) Robust approximation to multiperiod inventory management. *Oper. Res.* 58:583–594.

Soyster AL (1973) Convex programming with set-inclusive constraints and applications to inexact linear programming. *Oper. Res.* 21:1154–1157.