Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Analysis of Tailored Base-Surge Policies in Dual Sourcing Inventory Systems

Ganesh Janakiraman, Sridhar Seshadri, Anshul Sheopuri

MANAGEMENT SCIENCE
Vol. 61, No. 7, July 2015, pp. 1547–1561
ISSN 0025-1909 (print) | ISSN 1526-5501 (online)

http://dx.doi.org/10.1287/mnsc.2014.1971

Analysis of Tailored Base-Surge Policies in Dual Sourcing Inventory Systems

Ganesh Janakiraman
Naveen Jindal School of Management, University of Texas at Dallas, Richardson, Texas 75080, ganesh@utdallas.edu

Sridhar Seshadri
Indian School of Business, Gachibowli, Hyderabad 500032, India, sridhar_seshadri@isb.edu

Anshul Sheopuri
Business Insight Group, IBM Thomas J. Watson Research Center, New York, New York 10532, sheopuri@us.ibm.com

We study a model of a firm managing its inventory of a single product by sourcing supplies from two supply sources, a regular supplier who offers a lower unit cost and a longer lead time than a second, emergency, supplier. A practically implementable policy for such a firm is a tailored base-surge (TBS) policy [Allon G, Van Mieghem JA (2010) Global dual sourcing: Tailored base-surge allocation to near- and offshore production. *Management Sci.* 56(1):110–124] to manage its inventory. Under this policy, the firm procures a constant quantity from the regular supplier in every period and dynamically makes procurement decisions for the emergency supplier. Allon and Van Mieghem describe this practice as using the regular supplier to meet a *base* level of demand and the emergency supplier to manage demand *surges*, and they conjecture that this practice is most effective when the lead time difference between the two suppliers is large. We confirm these statements in two ways. First, we show the following analytical result: when demand is composed of a *base* demand random component plus a *surge* demand random component, which occurs with a certain small probability, the best TBS policy is close to optimal (over all policies) in a well-defined sense. Second, we also numerically investigate the cost effectiveness of the best TBS policy on a test bed of problem instances. The emphasis of this investigation is the study of the effect of the lead time difference between the two suppliers. Our study reveals that the cost difference between the best TBS policy and the optimal policy decreases dramatically as the lead time of the regular supplier increases. On our test bed, this cost difference decreases from an average (over the test bed) of 21% when the lead time from the regular supplier is two periods (the emergency supplier offers instant delivery) to 3.5% when that lead time is seven periods.

Keywords: inventory/production systems; dual sourcing; multiple suppliers; optimal policies
History: Received January 5, 2012; accepted December 9, 2013, by Martin Lariviere, operations management. Published online in *Articles in Advance* September 10, 2014.

1. Introduction

We study a periodically reviewed, single location inventory system experiencing stochastic demand and having access to two supply sources, one with a lower unit cost ("R" for regular) and the other with a shorter lead time ("E" for express or emergency). We focus attention on tailored base-surge allocation policies (we will refer to this class of policies as TBS), which work as follows: source a constant quantity from supplier R in every period and use a base stock (order-up-to) policy for supplier E. TBS policies have practical appeal, as discussed in Allon and Van Mieghem (2010) (hereafter referred to as AVM), who also mention that similar policies called "standing order policies" have been proposed earlier by Rosenshine and Obee (1976) and Janssen and de Kok (1999). TBS policies also have the same spirit as an industry practice called "base-commitment" contracts[1] (Simchi-Levi et al. 2008).

[1] See, for example, the discussion on p. 138 of the supply chain text by Simchi-Levi et al. (2008).

Our work is most closely related to AVM (who also coined the name TBS). The focus of their study was the optimization *within* TBS policies, i.e., the optimal choice of the constant quantity to be sourced from R and the order-up-to level to be used for E. As far as the motivation for these policies is concerned, AVM argue that these policies are easy to manage and that they are intuitive—R is used to handle a constant *base* level of demand whereas the use of E is *tailored* to meet any *surge* in demand above the base. They also conjecture that such a policy is effective when the lead time difference between the two suppliers is high. We validate these statements in the following two ways:

1. *Analytical results*: When demand comes from a two-point distribution and when the probability mass at the smaller demand level is sufficiently large, the best TBS policy is optimal (see §5.1). Moreover, when demand is composed of a sum of two random variables, the first of which represents *base demand* and the second of

which represents *surge demand*, which only occurs with a sufficiently small probability, TBS policies work well. More precisely, the relative difference between the cost of the best TBS policy and the optimal cost is smaller than 1.2 times the ratio of the standard deviation of base demand to the expected surge demand. In other words, when the surge probability is small but the expected surge, given that there is a surge, is large, the best TBS policy is close to optimal (see §5.2).

2. *Numerical results*: We conduct a numerical investigation of the cost-effectiveness of the best TBS policy (see §6). On our test bed of problem instances, this effectiveness increases dramatically as the lead time of the regular supplier increases from two to seven periods (while the lead time of the emergency supplier is zero). More specifically, the relative difference between the cost of the best TBS policy and the optimal cost over all policies decreases from 21% to 3.5%.

The remainder of the paper is organized as follows. We describe our model in detail and present our notation in §2. A review of the related literature is given in §3. Section 4 contains preliminary results, which are used in our technical analysis. The main analytical results are presented in §5, and the numerical results are presented in §6.

2. Model and Notation

Let $\mathcal{D}$ refer to the dual sourcing inventory system. The lead time from R is l^R periods and that from E is l^E periods. We use δ to denote the lead time difference; i.e., $\delta = l^R - l^E$. Let θ denote any feasible ordering policy. The sequence of events in every period t under this policy is the following: (1) The order placed in period $t - l^R$ from R for $q^{\theta,R}_{t-l^R}$ units is delivered, and if $l^E > 0$, the order placed in period $t - l^E$ from E for $q^{\theta,E}_{t-l^E}$ units is delivered. The net inventory, defined as the amount of inventory on hand less the amount of backordered demand, at this instant is x^θ_t. (2) Ordering decisions are made: $q^{\theta,R}_t \geq 0$ units are ordered from R and $q^{\theta,E}_t \geq 0$ units are ordered from E. If $l^E = 0$, these $q^{\theta,E}_t \geq 0$ units are immediately delivered. (3) Demand, d_t, is realized. (4) The cost c^θ_t for this period is charged according to the following expressions:

$$c^\theta_t = h\cdot(x^\theta_t - d_t)^+ + b\cdot(d_t - x^\theta_t)^+ + c\cdot q^{\theta,E}_t \quad \text{if } l^E > 0, \quad \text{and}$$
$$c^\theta_t = h\cdot(x^\theta_t + q^{\theta,E}_t - d_t)^+ + b\cdot(d_t - x^\theta_t - q^{\theta,E}_t)^+ + c\cdot q^{\theta,E}_t \quad \text{if } l^E = 0.$$

Here, h and b represent the per-unit holding and backorder costs, respectively, and c is the unit-cost premium charged by E over R. (We do not include a unit procurement cost for R because the difference in the unit costs of E and R, and not these unit costs themselves, exclusively determines the optimal policy.[2]) *We assume that $c < b\cdot(l^R - l^E)$, failing which the problem is trivial in the sense that single sourcing from R would then be optimal.* (This is because the maximum benefit of ordering a unit from E instead of R is the reduction in the backorder cost associated with a unit of demand, which is bounded above by $b\cdot(l^R - l^E)$.) Demands in different periods are assumed to be independently and identically distributed. Let D denote the random demand in any period; we use F to denote its distribution function. We denote the mean and standard deviation of D by μ and σ, respectively, both of which are assumed to be finite. Throughout the paper, we assume that demand is not deterministic; i.e., $\sigma > 0$.

The performance measure we use in this paper is C^θ, the long-run average cost of policy θ, as defined below:

$$C^\theta = \limsup_{T\to\infty} \frac{\sum_{t=1}^T E[c^\theta_t]}{T}.$$

The optimal long-run average cost is given by

$$C^{\mathcal{D},*} = \inf_\theta C^\theta.$$

In several places in the paper, it will be necessary to show the dependence of the optimal cost on the problem parameters; we will then use the notation $C^{\mathcal{D},*}(h, b, c, l^E, l^R, F)$.

3. Literature Review

Our paper is sharply focused on the analysis of the effectiveness of the simple TBS policies in dual sourcing inventory systems relative to the optimal policies that have a complicated structure. Consequently, we confine ourselves to specifically discussing those results in the literature that are critical to understanding the value of our work. We refer the reader to AVM for a detailed review of earlier papers, which suggested the use of TBS policies. For an extensive review of the literature on multiple-supplier inventory systems, see Minner (2003). Other more recent papers in which these systems

[2] A more detailed explanation follows. Consider a cost model in which the cost incurred in period t under policy θ is $\tilde{c}^\theta_t$, where $\tilde{c}^\theta_t$ is identical to c^θ_t except that it includes a cost c^R for procuring a unit from R and that the cost for procuring a unit from E is c^E. Any policy θ with a finite long-run average cost will be such that the long-run average quantity procured from E and R combined per period is exactly equal to the mean demand per period, μ. Using this fact, it can be verified that the difference in the long-run average costs of two policies θ and θ' under the cost function $\tilde{c}^\theta_t$ is the same as the difference under c^θ_t when $c = c^E - c^R$. Thus, the optimal policy under both cost models is the same. Moreover, under the cost function $\tilde{c}^\theta_t$, the quantity $c^R\cdot\mu$ can be viewed as a "sunk cost" that every policy has to incur per period, and disregarding that quantity (which is done under the cost model, c^θ_t) while comparing the cost of a TBS policy with the optimal cost describes the relative performance of the TBS policy conservatively.

have been studied (Sheopuri et al. 2010, Song and Zipkin 2009, Veeraraghavan and Scheller-Wolf 2008) also provide and discuss several related references. Several other policies have been proposed and tested in these papers. We limit our attention in this discussion to explaining what is known in the literature about optimal policies in dual sourcing systems, why simple policies such as TBS policies are valuable, and what we know about TBS policies.

Sheopuri et al. (2010) show that the problem of finding an optimal policy in a dual sourcing system is a generalization of the problem of finding an optimal policy in a single-supplier system with a positive lead time when excess demand is lost. Their argument is based on considering the following special case of the dual sourcing problem: Assume that the backorder cost is prohibitively high in a dual sourcing system. Assume also that the lead time from E is -1; that is, we are allowed to place an order on the emergency supplier after observing demand in a period, and this order is delivered instantly and the resulting inventory is available to satisfy the demand that arose in that period. Under these assumptions, it becomes optimal to use the emergency supplier exclusively to clear any demand in a period that we are unable to meet from inventory at the beginning of that period. Thus, the quantity sourced from the emergency supplier can be thought of as the lost sales incurred by the "regular supply system," and the emergency procurement cost is the lost-sales penalty cost. This lost-sales inventory problem with lead times is considered a challenging problem in the sense that the optimal policy does not admit any neat structure such as that of order-up-to policies (Karlin and Scarf 1958). (This has spurred an active line of research studying the effectiveness of heuristic policies; see Zipkin 2008a, Huh et al. 2009.) Thus, optimal policies in dual sourcing systems, in general, are not simple policies like base stock policies. In fact, the optimal ordering quantities from the two suppliers are functions of a state vector (of inventories) whose length is the difference in the lead times of the two suppliers. Thus, the problem of computing the optimal policy by solving the dynamic program suffers from the curse of dimensionality. Equally important, the likelihood of practicing managers implementing that policy is diminished by the complexity (or lack of a transparent insight) of its mechanics. This is why there has been significant interest among inventory researchers in proposing and evaluating simple policies for these systems. The TBS policy we study is a prominent choice for such a simple policy. Another prominent choice is the *dual index* policy studied by Veeraraghavan and Scheller-Wolf (2008), among others. We have explained earlier the motivation for TBS policies. A more detailed argument in support of these policies is provided in AVM.

The dual sourcing model studied by AVM is a continuous time model in which demand is a counting process and the supplies from both sources follow renewal processes. The capacity rates from both sources are decision variables. Furthermore, given a pair of capacity rates for the two sources, their TBS policy works as follows: R supplies continuously at its capacity rate (this *base* supply process is analogous to the constant quantity ordered from R in every period in our discrete time TBS policy), and E supplies whenever the total inventory position in the system falls below a base stock level (our ordering policy for E is identical). The authors study the optimization of the two capacity rates and the base stock level for E. The optimal base stock level is obtained from the newsvendor formula using the distribution of the steady-state *overshoot* (the amount by which the inventory position exceeds the base stock level). The dynamics of this overshoot process is identical to that of a $GI/GI/1$ queue. The optimization of the capacity rates does not allow for a closed-form solution, though. Consequently, they perform an asymptotic analysis of this optimization as the expected demand rate grows infinitely large (the coefficient of variation of interarrival times is held constant). A key result of this analysis is that the optimal capacity rate from the regular supplier becomes close to the expected demand rate in this regime. A consequence of this result is that under the optimal choice of capacity rates, the overshoot process resembles a queue in heavy traffic. This enables the authors to perform a heavy traffic analysis to derive closed-form expressions for asymptotically optimal capacity rates for the two suppliers and an asymptotically accurate expression for the optimal expected cost of the system. These expressions are simple square root formulas that are intuitive and provide clear insights about the trade-offs between costs and lead times.

The model studied in our paper is different from that studied in AVM (e.g., their model is in continuous time and their supply processes are stochastic, whereas ours is in discrete time and supply is deterministic); the most important distinction between our work and theirs is the following: AVM's focus is on optimizing *within the class of TBS policies*. Our focus is on analyzing the effectiveness of the best TBS policy relative to the *optimal policy over all feasible policies*.

4. Preliminary Results

This section is devoted to deriving preliminary results, which we use in §5 for proving our main analytical results. In addition, Theorem 1 is useful in §6 for numerically computing the best TBS policy.

4.1. Tailored Base-Surge Policies: Optimization

A tailored base-surge policy is specified by two parameters, $Q < \mu$ and S.[3] In every period, an order for Q units is placed on R. The ordering decision for E follows an order-up-to rule with target level S—here, the inventory position that is raised to the target is the *expedited inventory position*, which includes the net inventory and all the outstanding orders from both suppliers that will be delivered within the next l^E periods. It should be noted that in some periods, the expedited inventory position before ordering will exceed the target level S, in which case no order is placed from E. The quantity by which the expedited inventory position exceeds S in such a period is called the *overshoot*.

Let $C^{\mathscr{D},Q,S}(h,b,c,l^E,l^R,F)$ denote the long-run average cost of the TBS policy with parameters Q (the quantity ordered from R in every period) and S (the order-up-to level for E). Let

$$S^*(Q) = \arg\min_S C^{\mathscr{D},Q,S}(h,b,c,l^E,l^R,F).$$

For a given Q, let $O_\infty(Q)$ denote the steady-state *overshoot* random variable, which is defined as the steady-state version of the stochastic process $\{O_t\}$ that follows the recursion

$$O_{t+1} = \max(0, O_t - D_t + Q). \tag{1}$$

The existence of the stationary distribution denoted by $O_\infty(Q)$ is guaranteed by Loynes' lemma (Loynes 1962). Let $D[1,t]$ denote the demand over t periods. Let

$$G(y) = h\cdot E[(y - D[1,l^E+1])^+] + b\cdot E[(D[1,l^E+1]-y)^+]$$

denote the holding and shortage cost incurred in a period, given the expedited inventory position a lead time earlier is y. Then, we can write

$$C^{\mathscr{D},Q,S}(h,b,c,l^E,l^R,F) = c\cdot(\mu - Q) + E[G(S+O_\infty(Q))].$$

Therefore, $S^*(Q)$, the optimal order-up-to level or base stock level for a given Q, is the solution to the newsvendor equation $E[G'(S+O_\infty(Q))] = 0$; that is,

$$P(S^*(Q) + O_\infty(Q) \ge D[1,l^E+1]) = \frac{b}{b+h}.$$

Whereas this equation gives a closed-form expression for the best order-up-to level for a given constant order quantity Q, the optimal value of Q itself can be found by a simple technique such as bisection search using the following new result. (The proofs of all the analytical results that follow can be found in the appendix.)

Theorem 1. *The function* $C^{\mathscr{D},Q,S^*(Q)}(h,b,c,l^E,l^R,F)$ *is convex in* Q.

[3] If $Q \ge \mu$, the system is not "stable" in the sense that the expected inventory on hand approaches ∞.

Since the quantity received from R is the same in every period under any TBS policy, both $S^*(Q)$ and $C^{\mathscr{D},Q,S^*(Q)}$ are independent of the regular lead time, l^R. However, the optimal cost (over all admissible policies), $C^{\mathscr{D},*}$, is nondecreasing in l^R.[4] Thus, the relative performance of the best TBS policy (in fact, any TBS policy) improves as the regular lead time increases. In §6, we use a numerical investigation to study this improvement when all other parameters are held constant (i.e., demand distribution, cost parameters) while the regular lead time alone increases. In §5, we complement that study by analytically characterizing demand distributions and cost parameter values under which the best TBS policy is near optimal.

4.2. Upper Bounding the Cost of the Best TBS Policy

In this section, we will derive an analytical upper bound on the cost of the best TBS policy. This bound will be useful later when we derive a bound on the cost performance of the best TBS policy relative to the optimal policy.

The standard, multiperiod, newsvendor system with backordering is useful in the development of this bound. More specifically, $\mathscr{B}(h,b,l^E,F)$ is a system where there is a single supplier providing deliveries (with zero procurement costs) with a replenishment lead time of l^E periods; the unit holding and shortage cost parameters are h and b, respectively; and the demand distribution is F for every period. The optimal policy in $\mathscr{B}$ is to set the inventory position after ordering in every period to be S^E, the newsvendor level with a demand distribution of $D[1,l^E+1]$; that is, S^E solves $P(S^E \ge D[1,l^E+1]) = b/(b+h)$. We use $C^{\mathscr{B},*}(h,b,l^E,F)$ to denote the optimal long-run average cost of this system and $C^{\mathscr{B},S}(h,b,l^E,F)$ to denote the long-run average cost in $\mathscr{B}$ under the order-up-to S policy.

Next, consider the TBS policy of ordering Q units from R in every period and using an order-up-to S^E policy with E. Now, the difference between the cost of using this policy in $\mathscr{D}$ and the optimal cost in $\mathscr{B}$ can be bounded above by bounding the expected steady-state overshoot random variable. The dynamics of the overshoot (see (1)) are the same as those of the waiting time in a single-server queue. One of Kingman's bounds (Kingman 1970) is then used to bound the expected steady-state overshoot. We note that AVM also use a similar bound in their analysis.

[4] The proof of the claim that $C^{\mathscr{D},*}$ is nondecreasing in l^R is as follows. Consider the optimal policy in a system with a regular lead time of $k+1$ periods. That same policy is also admissible in a system with a regular lead time of k periods by simply delaying every order from R deliberately by one period. Thus, the optimal cost in the former system is also achievable in the latter system, and the optimal cost in the latter system is, by definition, smaller than this cost.

Lemma 2. *The infinite horizon average cost of the best TBS policy with a given Q is bounded above as follows*:

$$C^{\mathscr{D},\,Q,\,S^*(Q)}(h,b,c,l^E,l^R,F) \le C^{\mathscr{B},\,*}(h,b,l^E,F)+h\cdot\frac{\sigma^2}{2\cdot(\mu-Q)}+c\cdot(\mu-Q).$$

We are now ready to establish the main result of this section. We present an upper bound on the cost of the best TBS policy by minimizing both sides of the inequality in the statement of Lemma 2. This bound is used in our worst-case analysis of TBS policies.

Theorem 3. *The expected cost per period of the best TBS policy is bounded above as follows*:

$$\min_Q C^{\mathscr{D},\,Q,\,S^*(Q)}(h,b,c,l^E,l^R,F) \le C^{\mathscr{B},\,*}(h,b,l^E,F)+\sigma\cdot\sqrt{2\cdot h\cdot c}.$$

We close this section with a second upper bound on the cost of the best TBS policy. This bound is simply the cost of procuring exclusively from E optimally, which is achieved using an order-up-to policy. Thus, this policy is a TBS policy with $Q=0$, i.e., zero sourcing from R. Therefore, its cost is an upper bound on the cost of the best TBS policy.

Theorem 4. *The expected cost per period of the best TBS policy is bounded above as follows*:

$$\min_Q C^{\mathscr{D},\,Q,\,*}(h,b,c,l^E,l^R,F) \le C^{\mathscr{B},\,*}(h,b,l^E,F)+c\cdot\mu.$$

A quick comparison of the two upper bounds derived in Theorems 3 and 4 (which corresponds to the TBS policy with $Q=0$) provides us with a simple insight about TBS policies: in environments in which the demand uncertainty is not excessive (mathematically speaking, when σ/μ is not high), a nontrivial TBS policy (i.e., $Q>0$) dominates the policy of using E exclusively. Thus the simple version of dual sourcing (i.e., using TBS policies) provides value.

4.3. Effect of Variability on Optimal Cost

We show in this section that if demand becomes more variable (in the formal sense of *convex ordering*), the optimal cost of the system increases. Although this result is obvious in the case of the single-period newsvendor model, a proof is necessary for more sophisticated, multiperiod models such as the dual sourcing model here. A more recent example of such a proof for another multiperiod model is that of Zipkin (2008b) for lost-sales inventory systems. The result is interesting in its own right since it confirms our intuition about the effect of variability on costs. Moreover, it is used in the proof of an important result in §5.2.

We say $\tilde{F} \le_{cx} F$ ("$\tilde{F}$ is smaller than F in the convex order") if $E[g(\tilde{D})] \le E[g(D)]$ for any convex function g and two random variables D and $\tilde{D}$ with distributions F and $\tilde{F}$, respectively.

Theorem 5. *Assume that $\tilde{F} \le_{cx} F$. Then, the optimal long-run average cost when the demand distribution is $\tilde{F}$ is smaller than the optimal long-run average cost when the demand distribution is F. That is,*

$$C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,\tilde{F}) \le C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,F).$$

We now present two related results, which are both intuitive and useful in our analysis in the following section. The first result is that the optimal cost is unaffected if the demand distribution is shifted by a constant, and the second result is that if the demand distribution is scaled by a constant, then the optimal cost is also scaled by the same constant.

Lemma 6. *Consider three distributions F, $\tilde{F}$, and $\hat{F}$ (say, representing random variables D, $\tilde{D}$, and $\hat{D}$, respectively) such that $\tilde{F}(x)=F(x-A)$ (i.e., $\tilde{D}\sim D+A$) and $\hat{F}(x)=F(x/B)$ (i.e., $\hat{D}=D\cdot B$), where $A\ge 0$ and $B>0$ are constants. Then, $C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,\tilde{F})=C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,F)$ and $C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,\hat{F})=B\cdot C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,F)$.*

An immediate corollary of this lemma is the following.

Corollary 7. *Consider two distributions F and $\check{F}$ (say, representing random variables D and $\check{D}$, respectively) such that $\check{F}(x)=F((x-A)/B)$ (i.e., $\check{D}=B\cdot D+A$), where $A\ge 0$ and $B>0$ are constants. Then,*

$$C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,\check{F})=B\cdot C^{\mathscr{D},\,*}(h,b,c,l^E,l^R,F).$$

5. Main Analytical Results

In this section, we first study in §5.1 the special case of a two-point demand distribution. We characterize conditions on the cost parameters and the probability describing the two-point distribution under which a TBS policy is an optimal policy (over all admissible policies). Subsequently, in §5.2, we build on that analysis to study a more general demand distribution, which we call a *base-surge* distribution. In that case, we derive a bound on the ratio of the cost of the best TBS policy to the optimal cost under some assumptions.

5.1. Two-Point Distributions

In this section, we study the case in which demand comes from a two-point distribution. Let D, the random variable representing demand in a period, have two possible values, d_{Low} and d_{High}, and let it take these values with probabilities p_{Low} and $p_{\text{High}}=1-p_{\text{Low}}$, respectively. We now present an assumption that ensures that the probability of a low demand is sufficiently high and then explain its consequence.

Assumption 1. *The probability of low demands, p_{Low}, exceeds $\gamma/(\gamma+1)$, where $\gamma := (c+b\cdot(l^E+1)+h\cdot(l^R+1))/h$.*

Consider first the special case in which $d_{\text{Low}} = 0$ and $d_{\text{High}} = 1$. Consider a starting state in which there are no backorders and there is no inventory anywhere in the system. The expected time until the arrival of a customer is $p_{\text{Low}}/(1-p_{\text{Low}})$ since the demand process is a Bernoulli process (which is memoryless). Now, if we order a unit from R in anticipation of the first customer's arrival, this unit will be held as inventory on hand from the time it is delivered to the time that the customer arrives (assuming the latter happens after the former). Thus, the expected holding cost incurred as a result of this unit exceeds

$$h \cdot (p_{\text{Low}}/(1-p_{\text{Low}}) - (l^R + 1)).$$

(Note that, since $l^R > l^E$, the expected holding cost incurred by ordering a unit from E in anticipation of a customer's arrival is even higher.) On the other hand, waiting until a customer arrives and immediately ordering a unit from E to satisfy that customer's demand upon delivery results in a procurement cost plus backordering cost of

$$c + b \cdot (l^E + 1).$$

Therefore, if the condition

$$h \cdot (p_{\text{Low}}/(1-p_{\text{Low}}) - (l^R + 1)) \geq c + b \cdot (l^E + 1) \qquad (2)$$

is satisfied, it is never optimal to procure a unit when there is no customer who is already backordered; moreover, since $c < b \cdot (l^R - l^E)$, it is optimal to procure a unit from E (as opposed to R) as soon as a customer arrives. This policy is essentially a TBS policy with $Q = 0$ and $S = 0$, and the condition in (2) is equivalent to Assumption 1. An identical argument, along with Lemma 6, shows that a TBS policy with $Q = d_{\text{Low}}$ and $S = d_{\text{Low}} \cdot (l^E + 1)$ is optimal under any general two-point distribution satisfying that assumption. The following theorem states this formally, and we supplement the intuition above with a rigorous proof in the appendix.

Theorem 8. *For a two-point distribution, F, satisfying Assumption* 1, *a TBS policy with $Q = d_{\text{Low}}$ and $S = d_{\text{Low}} \cdot (l^E + 1)$ is optimal.*

What Theorem 8 demonstrates is that when demand is usually at a *base level* and occasionally rises to a *surge level*, the TBS policy of using R to constantly meet the base level of demand is optimal *over all admissible policies* as long as the probability of a surge is sufficiently low. To show this result, we have modeled the base and surge levels as *demand points* (i.e., degenerate distributions) rather than the more general model of associating a probability distribution with each of these two levels. We explore such a generalization in the next subsection.

5.2. Base-Surge Distributions

In this section, we model demands using *base-surge* distributions, which we describe as follows. Let X and Y be two nonnegative, independent random variables. Here, X is the base demand distribution and Y is the surge demand distribution. Surges are rare, and we model this using a probability p that there will be no surge (equivalently, a probability $1-p$ of a surge). Thus, the demand in a period is given by $D = X + Z$ (Z is independent of X), where $Z = 0$ with probability p and $Z = Y$ with probability $(1-p)$. We derive a bound on the ratio of the cost of the best TBS policy to the optimal cost *over all admissible policies* when p is sufficiently large, under a reasonable assumption on the cost parameters. We will use σ_X to denote the standard deviation of X and μ_Y to denote the expectation of Y.

Theorem 9. *Under Assumption* 1, *the ratio of the cost of the best TBS policy and the optimal cost* (*over all admissible policies*) *is smaller than*

$$1 + \frac{\sigma_X \cdot (\sqrt{2 \cdot h \cdot c} + \sqrt{h \cdot b \cdot (l^E + 1)})}{(c + b \cdot (l^E + 1)) \cdot (1-p) \cdot \mu_Y}.$$

Moreover, when the cost parameters h, b, and c are such that $h \leq c$ and $c < b \cdot (l^E + 1)$, this ratio is bounded above by $1 + ((1+\sqrt{2})/2) \cdot (\sigma_X/((1-p) \cdot \mu_Y))$.

This bound suggests that when the probability of a surge is small but the expected surge (given that there is a surge) is large relative to the uncertainty in base demand, the best TBS policy is near optimal.

We conclude this subsection with a remark on what this analysis implies for managing inventory and sourcing decisions for *service parts*. For a typical service part, the demand distribution is characterized by a large probability mass at zero and some probability distribution conditional on demand being nonzero. We can now use Theorem 9 to make the following observation.

Remark 1. Let the demand distribution F be such that $F(0) = p$. Let $\tilde{F}$ represent the demand distribution conditional on the demand being strictly positive; i.e., $\tilde{F}(0) = 0$ and $\tilde{F}(x) = P(D \leq x \mid D > 0)$. For any $\tilde{F}$, the policy of never ordering from R and ordering up to zero from E (i.e., reacting to all positive demand by ordering the demanded quantity from E) is optimal if the probability of zero demand, i.e., p, is large enough for Assumption 1 to be satisfied. This is a trivial TBS policy with $Q = 0$ and $S = 0$. Thus, in this case, single sourcing is optimal.

6. Numerical Results

In this section, we evaluate the cost effectiveness of the best TBS policy over a test bed of problem instances by numerically computing, on each instance, both the

Table 1 **Two-Point Distribution:** $P(D=1)=2/3$ **and** $P(D=4)=1/3$

	I^R	Opt	TBS	% diff		I^R	Opt	TBS	% diff		I^R	Opt	TBS	% diff
$h=20$	2	60.0	60.0	0.0	$h=20$	2	71.1	82.1	15.5	$h=20$	2	71.1	103.0	44.8
$b=80$	3	60.0	60.0	0.0	$b=80$	3	71.5	82.1	14.8	$b=80$	3	75.4	103.0	36.6
$c=20$	4	60.0	60.0	0.0	$c=50$	4	74.7	82.1	9.9	$c=100$	4	83.3	103.0	23.6
	5	60.0	60.0	0.0		5	77.4	82.1	6.1		5	89.1	103.0	15.6
	6	60.0	60.0	0.0		6	79.8	82.1	2.9		6	89.7	103.0	14.8
	7	60.0	60.0	0.0		7	81.1	82.1	1.3		7	91.8	103.0	12.2
$h=20$	2	60.0	60.0	0.0	$h=20$	2	82.2	87.2	6.1	$h=20$	2	82.2	112.0	36.2
$b=180$	3	60.0	60.0	0.0	$b=180$	3	84.0	87.2	3.9	$b=180$	3	96.7	112.0	15.8
$c=20$	4	60.0	60.0	0.0	$c=50$	4	85.0	87.2	2.6	$c=100$	4	103.1	112.0	8.6
	5	60.0	60.0	0.0		5	85.7	87.2	1.8		5	103.7	112.0	8.0
	6	60.0	60.0	0.0		6	86.2	87.2	1.2		6	105.4	112.0	6.3
	7	60.0	60.0	0.0		7	86.7	87.2	0.6		7	107.2	112.0	4.4

Table 2 **Unimodal Symmetric Distribution:** $P(D=0)=0.125$, $P(D=1)=0.2$, $P(D=2)=0.35$, $P(D=3)=0.2$, **and** $P(D=4)=0.125$

	I^R	Opt	TBS	% diff		I^R	Opt	TBS	% diff		I^R	Opt	TBS	% diff
$h=20$	2	49.1	52.7	7.4	$h=20$	2	54.9	68.1	23.9	$h=20$	2	56.9	84.9	49.3
$b=80$	3	50.9	52.7	3.5	$b=80$	3	58.9	68.1	15.6	$b=80$	3	64.4	84.9	31.8
$c=20$	4	51.7	52.7	1.9	$c=50$	4	61.3	68.1	11.1	$c=100$	4	69.5	84.9	22.1
	5	52.0	52.7	1.3		5	62.9	68.1	8.2		5	72.9	84.9	16.5
	6	52.2	52.7	0.9		6	64.2	68.1	6.1		6	75.2	84.9	12.8
	7	52.2	52.7	0.9		7	65.1	68.1	4.6		7	76.9	84.9	10.4
$h=20$	2	56.9	61.4	8	$h=20$	2	65.0	76.2	17.3	$h=20$	2	69.7	94.5	35.6
$b=180$	3	58.8	61.4	4.6	$b=180$	3	68.6	76.2	11.1	$b=180$	3	76.8	94.5	23.1
$c=20$	4	59.6	61.4	3.1	$c=50$	4	70.9	76.2	7.5	$c=100$	4	81.4	94.5	16.1
	5	60.0	61.4	2.4		5	72.4	76.2	5.3		5	84.4	94.5	11.9
	6	60.2	61.4	2.1		6	73.3	76.2	4.1		6	86.4	94.5	9.3
	7	60.4	61.4	1.8		7	73.9	76.2	3.1		7	88.1	94.5	7.2

Table 3 **Right-Skewed Distribution:** $P(D=0)=0.125$, $P(D=1)=0.5$, $P(D=2)=0.125$, $P(D=3)=0.125$, **and** $P(D=4)=0.125$

	I^R	Opt	TBS	% diff		I^R	Opt	TBS	% diff		I^R	Opt	TBS	% diff
$h=20$	2	53.1	56.9	7.1	$h=20$	2	58.3	70.2	20.4	$h=20$	2	63.0	87.5	38.9
$b=80$	3	54.4	56.9	4.6	$b=80$	3	62.6	70.2	12.3	$b=80$	3	68.9	87.5	27.1
$c=20$	4	55.1	56.9	3.3	$c=50$	4	64.8	70.2	8.4	$c=100$	4	72.8	87.5	20.3
	5	55.5	56.9	2.4		5	66.4	70.2	5.7		5	75.8	87.5	15.4
	6	55.8	56.9	1.8		6	67.4	70.2	4.2		6	77.8	87.5	12.5
	7	56.0	56.9	1.5		7	68.1	70.2	3.2		7	79.5	87.5	10.1
$h=20$	2	62.7	66.3	5.6	$h=20$	2	73.1	80.9	10.6	$h=20$	2	78.2	100.0	27.9
$b=180$	3	64.3	66.3	3.1	$b=180$	3	76.3	80.9	6.0	$b=180$	3	83.9	100.0	19.2
$c=20$	4	65.0	66.3	2.0	$c=50$	4	77.8	80.9	3.9	$c=100$	4	88.3	100.0	13.2
	5	65.3	66.3	1.4		5	78.7	80.9	2.7		5	91.0	100.0	9.9
	6	65.5	66.3	1.1		6	79.3	80.9	2.0		6	93.0	100.0	7.5
	7	65.7	66.3	0.9		7	79.6	80.9	1.5		7	94.4	100.0	5.9

long-run average cost of that policy and the optimal cost (by solving the *infinite horizon, average-cost* dynamic program). The ratio of the two costs mentioned above is the performance measure of interest here. The goal is to understand how this ratio depends on the $b/(b+h)$ ratio, the lead time difference between R and E, and the emergency procurement cost c. We also want to understand whether these patterns are consistent across a variety of demand distributions.

Our initial test bed consists of the problems defined by the following parameter choices. The lead time from E is zero, the lead time from R is varied between two and seven periods, and $h=20$ throughout. (As an illustrative example, consider a review period of one month and a holding cost of $20 per unit per month. At a 20% annual cost of capital, this corresponds to a nominal or regular unit cost of $1,200.) The value of b is either 80 or 180 corresponding to $b/(b+h)$ ratios of 80% or 90%, respectively. The value of c is either 20, 50, or 100. (In the example, our choices of b, i.e., $80 per unit per month and $180 per unit per month, translate to 6.7% and 15% of the regular unit cost per

Table 4 **Left-Skewed Distribution:** $P(D=0)=0.125$, $P(D=1)=0.125$, $P(D=2)=0.125$, $P(D=3)=0.5$, **and** $P(D=4)=0.125$

	l^R	Opt	TBS	% diff		l^R	Opt	TBS	% diff		l^R	Opt	TBS	% diff
$h=20$	2	44.7	48.1	7.4	$h=20$	2	51.5	65.6	27.4	$h=20$	2	52.7	85.5	62.2
$b=80$	3	46.1	48.1	4.3	$b=80$	3	58.1	65.6	12.8	$b=80$	3	63.6	85.5	34.4
$c=20$	4	47.0	48.1	2.3	$c=50$	4	60.8	65.6	7.8	$c=100$	4	70.0	85.5	22.2
	5	47.4	48.1	1.4		5	62.3	65.6	5.2		5	74.8	85.5	14.4
	6	47.6	48.1	1.0		6	63.0	65.6	4.1		6	77.6	85.5	10.2
	7	47.6	48.1	0.9		7	63.5	65.6	3.3		7	79.6	85.5	7.4
$h=20$	2	51.8	56.7	9.4	$h=20$	2	56.7	72.5	27.9	$h=20$	2	62.0	91.7	47.7
$b=180$	3	53.7	56.7	5.6	$b=180$	3	63.4	72.5	14.2	$b=180$	3	69.4	91.7	32.1
$c=20$	4	54.6	56.7	3.9	$c=50$	4	66.4	72.5	9.2	$c=100$	4	76.3	91.7	20.1
	5	55.2	56.7	2.8		5	68.1	72.5	6.4		5	79.8	91.7	14.8
	6	55.6	56.7	2.1		6	69.2	72.5	4.7		6	82.4	91.7	11.2
	7	55.8	56.7	1.7		7	69.9	72.5	3.6		7	84.4	91.7	8.6

Table 5 **Bimodal Distribution:** $P(D=0)=0.1$, $P(D=1)=0.35$, $P(D=2)=0.1$, $P(D=3)=0.1$, **and** $P(D=4)=0.35$

	l^R	Opt	TBS	% diff		l^R	Opt	TBS	% diff		l^R	Opt	TBS	% diff
$h=20$	2	62.1	63.6	2.5	$h=20$	2	69.9	84.7	21.1	$h=20$	2	69.9	108.4	55.0
$b=80$	3	62.8	63.6	1.2	$b=80$	3	76.8	84.7	10.3	$b=80$	3	82.6	108.4	31.2
$c=20$	4	63.2	63.6	0.7	$c=50$	4	79.8	84.7	6.2	$c=100$	4	88.6	108.4	22.4
	5	63.2	63.6	0.6		5	81.2	84.7	4.3		5	93.3	108.4	16.2
	6	63.3	63.6	0.5		6	82.2	84.7	3.0		6	96.6	108.4	12.2
	7	63.3	63.6	0.4		7	82.9	84.7	2.1		7	99.0	108.4	9.4
$h=20$	2	63.2	64.0	1.3	$h=20$	2	80.6	88.1	9.3	$h=20$	2	89.5	115.1	28.6
$b=180$	3	63.5	64.0	0.8	$b=180$	3	84.2	88.1	4.7	$b=180$	3	98.2	115.1	17.3
$c=20$	4	63.6	64.0	0.6	$c=50$	4	85.6	88.1	2.9	$c=100$	4	103.3	115.1	11.4
	5	63.6	64.0	0.5		5	86.4	88.1	1.9		5	106.6	115.1	8.0
	6	63.7	64.0	0.5		6	87.0	88.1	1.3		6	108.8	115.1	5.8
	7	63.7	64.0	0.5		7	87.3	88.1	0.9		7	110.2	115.1	4.5

Table 6 **Uniform Distribution:** $P(D=0)=0.2$, $P(D=1)=0.2$, $P(D=2)=0.2$, $P(D=3)=0.2$, **and** $P(D=4)=0.2$

	l^R	Opt	TBS	% diff		l^R	Opt	TBS	% diff		l^R	Opt	TBS	% diff
$h=20$	2	59.1	61.7	4.5	$h=20$	2	66.7	80.7	21.0	$h=20$	2	68.0	102.2	50.2
$b=80$	3	60.3	61.7	2.3	$b=80$	3	72.1	80.7	11.9	$b=80$	3	78.0	102.2	31.0
$c=20$	4	60.8	61.7	1.5	$c=50$	4	75.2	80.7	7.3	$c=100$	4	84.4	102.2	21.1
	5	61.1	61.7	1.1		5	76.9	80.7	5.0		5	88.6	102.2	15.3
	6	61.2	61.7	0.8		6	78.1	80.7	3.3		6	91.6	102.2	11.5
	7	61.3	61.7	0.7		7	79.0	80.7	2.2		7	93.7	102.2	9.0
$h=20$	2	67.1	67.8	1.1	$h=20$	2	78.0	89.2	14.3	$h=20$	2	83.1	112.0	34.8
$b=180$	3	67.5	67.8	0.4	$b=180$	3	82.5	89.2	8.1	$b=180$	3	92.4	112.0	21.1
$c=20$	4	67.6	67.8	0.3	$c=50$	4	84.8	89.2	5.2	$c=100$	4	97.9	112.0	14.4
	5	67.6	67.8	0.3		5	86.4	89.2	3.2		5	101.2	112.0	10.7
	6	67.6	67.8	0.3		6	87.4	89.2	2.0		6	103.4	112.0	8.2
	7	67.6	67.8	0.3		7	88.1	89.2	1.2		7	105.1	112.0	6.5

unit per month, respectively. Our choices for the unit cost premium charged by the faster supplier, i.e., \$20, \$50, and \$100, range from 1.7% to 8.3% of the unit cost.) Using p_i for $P(D=i)$, the following six distributions are chosen:

1. *Two-point*: $p_0=0$, $p_1=2/3$, $p_2=0$, $p_3=0$, and $p_4=1/3$.
2. *Unimodal symmetric*: $p_0=0.125$, $p_1=0.2$, $p_2=0.35$, $p_3=0.2$, and $p_4=0.125$.
3. *Right-skewed*: $p_0=0.125$, $p_1=0.5$, $p_2=0.125$, $p_3=0.125$, and $p_4=0.125$.
4. *Left-skewed*: $p_0=0.125$, $p_1=0.125$, $p_2=0.125$, $p_3=0.5$, and $p_4=0.125$.
5. *Bimodal*: $p_0=0.1$, $p_1=0.35$, $p_2=0.1$, $p_3=0.1$, and $p_4=0.35$.
6. *Uniform*: $p_0=0.2$, $p_1=0.2$, $p_2=0.2$, $p_3=0.2$, and $p_4=0.2$.

(The reason for limiting our investigation over distributions with support on $\{0,1,2,3,4\}$ is the computational effort required to compute the optimal cost by dynamic programming.) For each of these instances, in Tables 1–6, we report the optimal cost ("Opt"), the cost

Table 7 Lead Time Differences Between R and E Held Constant at 3

				$(l^E, l^R) = (1, 4)$			$(l^E, l^R) = (2, 5)$			$(l^E, l^R) = (3, 6)$		
	h	b	c	Opt	TBS	% diff	Opt	TBS	% diff	Opt	TBS	% diff
Uniform	20	80	20	74.9	77.1	3.0	86.3	88.9	3.0	96.0	98.2	2.3
$P(D=0)=0.2$	20	80	50	83.9	93.5	11.4	94.1	104.3	10.9	102.6	113.9	11.1
$P(D=1)=0.2$	20	80	100	88.2	114.2	29.4	96.5	123.9	28.4	104.4	132.3	26.7
$P(D=2)=0.2$	20	180	20	88.5	90.7	2.4	104.0	106.1	2.1	116.0	118.2	1.9
$P(D=3)=0.2$	20	180	50	100.3	108.0	7.7	113.4	122.9	8.4	124.9	134.8	8.0
$P(D=4)=0.2$	20	180	100	107.0	130.1	21.6	118.6	142.6	20.2	129.4	154.7	19.5
Two-point	20	80	20	72.0	72.7	1.0	89.8	90.9	1.2	95.7	97.1	1.4
$P(D=1)=2/3$	20	80	50	83.6	90.4	8.1	93.9	106.4	13.3	103.4	113.8	10.0
and	20	80	100	93.8	112.2	19.5	95.8	124.5	30.0	107.3	133.9	24.8
$P(D=4)=1/3$	20	180	20	92.0	96.7	5.1	102.2	102.2	0.0	119.4	125.2	4.8
	20	180	50	102.8	110.6	7.6	115.3	123.4	7.0	128.2	138.5	8.0
	20	180	100	107.7	129.6	20.4	123.6	144.8	17.2	131.5	157.6	19.8
Unimodal symmetric	20	80	20	62.1	63.5	2.3	71.5	73.7	3.1	79.8	82.5	3.3
$P(D=0)=0.125$	20	80	50	69.5	78.1	12.2	77.8	86.9	11.7	85.4	94.8	11.0
$P(D=1)=0.200$	20	80	100	72.9	95.3	30.7	80.9	103.8	28.3	87.5	110.7	26.5
$P(D=2)=0.350$	20	180	20	74.3	75.7	1.9	86.2	88.7	2.8	97.5	100.6	3.2
$P(D=3)=0.200$	20	180	50	83.2	91.3	9.6	94.5	102.4	8.3	104.4	113.6	8.8
$P(D=4)=0.125$	20	180	100	88.6	108.6	22.6	99.5	120.1	20.8	107.7	129.6	20.4
Right-skewed	20	80	20	65.4	66.6	1.8	75.3	76.5	1.6	84.3	85.2	1.1
$P(D=0)=0.125$	20	80	50	73.9	80.0	8.2	82.6	89.8	8.6	90.5	98.2	8.4
$P(D=1)=0.500$	20	80	100	78.3	97.3	24.1	86.1	106.2	23.3	93.0	113.6	22.1
$P(D=2)=0.125$	20	180	20	79.1	80.1	1.2	93.7	95.3	1.7	104.3	106.2	1.8
$P(D=3)=0.125$	20	180	50	89.8	94.4	5.2	102.0	108.5	6.4	112.2	119.3	6.3
$P(D=4)=0.125$	20	180	100	96.8	112.5	16.2	107.6	125.4	16.6	117.1	135.7	15.9
Left-skewed	20	80	20	60.8	62.8	3.3	72.6	75.5	4.1	80.5	83.3	3.4
$P(D=0)=0.125$	20	80	50	69.8	79.7	14.3	78.5	90.2	14.8	85.0	98.1	15.5
$P(D=1)=0.125$	20	80	100	71.8	99.5	38.6	79.1	108.0	36.7	86.4	116.0	34.2
$P(D=2)=0.125$	20	180	20	70.4	72.5	3.0	82.9	85.4	3.0	93.9	96.8	3.0
$P(D=3)=0.500$	20	180	50	78.9	87.9	11.4	91.6	100.8	10.0	101.9	112.4	10.3
$P(D=4)=0.125$	20	180	100	84.0	106.7	27.0	94.8	119.5	26.0	104.6	131.3	25.5
Bimodal	20	80	20	78.7	81.6	3.6	90.1	92.6	2.8	100.7	103.5	2.8
$P(D=0)=0.10$	20	80	50	88.2	98.1	11.3	99.0	109.6	10.7	107.7	119.7	11.2
$P(D=1)=0.35$	20	80	100	92.2	119.9	30.1	101.9	131.8	29.4	109.4	140.8	28.6
$P(D=2)=0.10$	20	180	20	93.8	95.5	1.9	107.9	110.7	2.6	121.3	124.3	2.5
$P(D=3)=0.10$	20	180	50	106.2	115.2	8.4	118.9	127.7	7.4	131.3	141.5	7.7
$P(D=4)=0.35$	20	180	100	112.0	138.1	23.4	124.6	150.1	20.5	135.8	163.9	20.7

of the best TBS policy ("TBS"), and the percentage difference between the two ("% diff"). Some observations based on these results follow.

In each of the tables, we see that the performance of the best TBS policy improves as l^R increases (i.e., as the lead time difference between R and E increases). The improvement from a one-period increase to l^R is quite dramatic when l^R is small. In all our instances with $c=20$ or $c=50$, the cost of the best TBS policy is at most 4.6% more than the optimal cost when $l^R=7$. That is, TBS policies emerge as an effective choice for such lead time differences. Next, we would expect that TBS policies become a less effective choice when c is large. (This is because when c is large, using an order-up-to policy with R and not placing any orders on E, is a preferred policy. That policy passes the demands to R as its orders. The TBS policy, on the other hand, sources a constant amount from R every period.) This is observed in our numerical results. Our next observation is on the service level (more precisely, the $b/(b+h)$ ratio). As this level increases, the policy of sourcing exclusively from E (more generally, the policy of using R to source the deterministic part of demand, if any, and using E to manage uncertainty) becomes relatively more attractive. Since this policy is a special case of TBS policies, we expect the effectiveness of the best TBS policy to improve as service levels increase. Broadly speaking, such a trend can be observed in our results by comparing the percentages corresponding to $b=180$ versus that for $b=80$ in each of our tables. Another effect we notice in our results is that the best TBS policy becomes more effective for higher levels of demand uncertainty. More specifically, the average percentage cost gap between the best TBS policy relative to the optimal decreases with the standard deviation of demand on our test bed. This is likely because when demand uncertainty is high, the optimal policy devotes a larger fraction of the supply sourced to E, thus approaching the policy of sourcing only from E—this policy is the TBS policy with $Q=0$.

Finally, we also present some results of tests to investigate whether the observations made above on the performance of TBS policies are affected if the lead time from E were not zero. To this end, we include Table 7, which shows the results for problems in which the difference between the two lead times is held constant at 3 (i.e., $l^R - l^E = 3$) while l^E is varied from 1 to 3. This table[5] seems to indicate that there is no noticeable effect of l^E on the performance of TBS policies relative to the optimal for a fixed value of $l^R - l^E$. It appears that, on average (over the instances tested), an increase in l^E is accompanied by approximately the same percentage increase in both the cost of the best TBS policy and the optimal cost, thus leaving the percentage difference between the two mostly unchanged.

Acknowledgments

The authors sincerely thank the department editor, associate editor, and three referees for their exceptionally useful and prompt reviews.

Appendix. Proofs

Proof of Theorem 1. Let $C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F)$ denote the optimal cost over all admissible policies that order a constant amount Q from R in every period (these policies include policies other than order-up-to policies for E). We first claim that among policies that order a constant amount $Q<\mu$ from R in every period, there exists an optimal policy of the order-up-to type for E. That is,

$$C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F)=C^{\mathcal{D},Q,S^*(Q)}(h,b,c,l^E,l^R,F).$$

The proof of this claim is as follows: The inventory process in a dual sourcing system that orders Q from supplier R in every period is identical to that in a single sourcing system with lead time l^E and in which the random variable representing demand in any period is given by $D-Q$. It is well known that the optimal policy in such a single sourcing system (even though the random demand, $D-Q$, in a period need not be nonnegative) is an order-up-to policy. This fact implies the claim.

We will suppress the dependence of $C^{\mathcal{D},Q,*}$ on the parameters (h,b,c,l^E,l^R,F) in this proof. We will show the result by demonstrating the following inequality:

$$\frac{C^{\mathcal{D},Q_1,*}+C^{\mathcal{D},Q_2,*}}{2}\geq C^{\mathcal{D},(Q_1+Q_2)/2,*}\quad\text{for all } Q_1 \text{ and } Q_2. \tag{3}$$

Let θ_1 (θ_2) denote the optimal policy within the class of policies that order the constant quantity Q_1 (Q_2) from R in every period. That is, $C^{\mathcal{D},\theta_1}=C^{\mathcal{D},Q_1,*}$ and $C^{\mathcal{D},\theta_2}=C^{\mathcal{D},Q_2,*}$. Let us now consider a third policy, θ_3, defined as follows: it orders the constant quantity $(Q_1+Q_2)/2$ from R every period and, in every period t, orders from E the average of the quantities ordered by θ_1 and θ_2. That is, $q_t^{\theta_3,R}=(Q_1+Q_2)/2$ and $q_t^{\theta_3,E}=(q_t^{\theta_1,E}+q_t^{\theta_2,E})/2$ for all t. This implies that the procurement cost under θ_3 in every period is the average of the procurement costs under θ_1 and θ_2. Let us now examine the holding and shortage costs under θ_3. Assuming the same starting state under all three policies, it is easy to see from the definition of θ_3 that $x_t^{\theta_3}=(x_t^{\theta_1}+x_t^{\theta_2})/2$ for all t. The holding and shortage cost in period t is a convex function of the net inventory at the beginning of period t. This convexity and the equality above, along with our earlier observation about procurement costs, imply that $c_t^{\theta_3}\leq(c_t^{\theta_1}+c_t^{\theta_2})/2$. Therefore, we obtain $C^{\mathcal{D},(Q_1+Q_2)/2,*}\leq C^{\mathcal{D},\theta_3}\leq(C^{\mathcal{D},\theta_1}+C^{\mathcal{D},\theta_2})/2=(C^{\mathcal{D},Q_1,*}+C^{\mathcal{D},Q_2,*})/2$. This completes the proof of (3). □

[5] A similar investigation was conducted when $l^R-l^E=2$ with almost identical results.

Proof of Lemma 2. In this proof, we will use $\mathcal{B}$ to denote $\mathcal{B}(h,b,l^E,F)$ and $\mathcal{D}$ to denote $\mathcal{D}(h,b,c,l^E,l^R,F)$. Let $S^{\mathcal{B},*}$ denote the optimal order-up-to level in $\mathcal{B}$. Note that the order-up-to policy with this target level achieves a long-run average cost of $C^{\mathcal{B},*}(h,b,l^E,F)$ in $\mathcal{B}$.

Let us now study $\mathcal{D}$ under the TBS policy with a constant order quantity of Q from R and the order-up-to policy with target level $S^{\mathcal{B},*}$ for E. Let us use θ to denote this policy. Observe that the expected holding and shortage cost in $\mathcal{D}$ per period under θ is

$$\begin{aligned}&h\cdot E[(S^{\mathcal{B},*}+O_\infty(Q)-D[1,l^E+1])^+]\\&\qquad+b\cdot E[(S^{\mathcal{B},*}+O_\infty(Q)-D[1,l^E+1])^-].\end{aligned}$$

Using the facts that, for all (x,y) such that $y\geq 0$, $(x+y)^+\leq x^++y$ and $(x+y)^-\leq x^-$, we see that this cost is bounded above by

$$\begin{aligned}&h\cdot E[(S^{\mathcal{B},*}-D[1,l^E+1])^+]+b\cdot E[(S^{\mathcal{B},*}-D[1,l^E+1])^-]\\&\qquad+h\cdot E[O_\infty(Q)].\end{aligned}$$

Notice that the sum of the first two terms in this expression is nothing but $C^{\mathcal{B},*}(h,b,l^E,F)$. Thus, we know that the long-run average cost in $\mathcal{D}$ under θ is bounded above by

$$C^{\mathcal{B},*}(h,b,l^E,F)+h\cdot E[O_\infty(Q)]+c\cdot(\mu-Q), \tag{4}$$

where the last term captures the procurement cost (since Q units are procured from R every period, $\mu-Q$ is the average amount procured from E per period).

Next, recall that $O_\infty(Q)$ represents the steady state of the stochastic process $O_{t+1}=(O_t+Q-D_t)^+$, which is identical to the waiting time recursion in a $GI/GI/1$ queue with D representing the interarrival time and Q the service time. We know from Inequality (9) of Kingman (1970) that the expected waiting time in a $GI/GI/1$ queue is bounded above by $(\sigma_s^2+\sigma_t^2)/[2(\bar{t}-\bar{s})]$, where σ_s (σ_t) is the standard deviation of the service (interarrival) time and $\bar{t}$ ($\bar{s}$) is the mean interarrival (service) time. In our case, $\sigma_s=0$, $\sigma_t=\sigma^2$, $\bar{s}=Q$, and $\bar{t}=\mu$. Substituting these quantities into Kingman's bound above, we obtain

$$E[O_\infty(Q)]\leq\frac{\sigma^2}{2\cdot(\mu-Q)}. \tag{5}$$

Combining (4) and (5), we obtain the following upper bound on the cost of θ and therefore on the cost of the best TBS policy for a given Q: $C^{\mathcal{B},*}(h,b,l^E,F)+h\cdot(\sigma^2/(2\cdot(\mu-Q)))+c\cdot(\mu-Q)$. □

Proof of Theorem 3. Let $\tilde{Q}$ minimize the upper bound expression given in Lemma 2. It is easy to verify that $\tilde{Q} = \mu - \sigma \cdot \sqrt{h/2c}$, which is nonnegative if $\mu \geq \sigma \cdot \sqrt{h/2c}$. In this case, the cost of the best TBS policy is bounded above by the cost of the TBS policy that uses $Q = \tilde{Q}$; it can be verified by substituting Q with $\tilde{Q}$ in Lemma 2 that the cost of this policy is bounded above by $C^{\mathcal{B},*}(h, b, l^E, F) + \sigma \cdot \sqrt{2 \cdot h \cdot c}$. In the case when $\mu \leq \sigma \cdot \sqrt{h/2c}$, the cost of the best TBS policy is bounded above by the cost of the TBS policy that uses $Q = 0$. The cost of that policy is

$$\begin{aligned} C^{\mathcal{B},*}(h,b,l^E,F)+c\cdot\mu &\leq C^{\mathcal{B},*}(h,b,l^E,F)+\sigma\cdot\sqrt{h\cdot c/2} \\ &\leq C^{\mathcal{B},*}(h,b,l^E,F)+\sigma\cdot\sqrt{2\cdot h\cdot c}. \quad \square \end{aligned}$$

Proof of Theorem 5. To avoid excessive notation, we find it convenient to present the elements of the proof rather than those details of these elements that can be found in references that we provide. Although the performance measure used in this paper is the long-run average cost, our proof requires considering a finite horizon, discounted-cost dynamic program first; an infinite horizon, discounted-cost dynamic program next; and finally, the average-cost dynamic program.

Let us first consider the finite horizon dynamic program that is characterized by a cost function (say, $f_t^{\alpha,T}(\mathbf{x})$) when the demand distribution is F and $\mathbf{x}$ represents the state of the system, i.e., information on how much inventory is present on hand, amount on backorder if any, and amounts on order from both supply sources scheduled to arrive in the next several periods (because of the lead times). Here, $\alpha \in (0, 1)$ denotes a discount factor to capture the time value of money and T denotes the length of a finite planning horizon. Similarly, let $\tilde{f}_t^{\alpha,T}(\mathbf{x})$ denote the corresponding cost function when the demand distribution is $\tilde{F}$. We define $f_{T+1}^{\alpha,T}(\mathbf{x}) = \tilde{f}_{T+1}^{\alpha,T}(\mathbf{x}) = 0$ for all $\mathbf{x}$. It is a standard exercise in dynamic programming to show that $f_t^{\alpha,T}$ and $\tilde{f}_t^{\alpha,T}$ are convex functions. Let us assume by induction that $f_{t+1}^{\alpha,T}(\mathbf{x}) \geq \tilde{f}_{t+1}^{\alpha,T}(\mathbf{x})$ for all $\mathbf{x}$. We use the following facts: (a) the single-period cost is a convex function of demand, for any given $\mathbf{x}$; (b) the state transformation (i.e., how the state in period $t+1$ depends on the state in period t and the demand in period t) is linear in $(\mathbf{x}, d)$, where d is the realization of demand; (c) $F \geq_{cx} \tilde{F}$; and (d) the functions $f_t^{\alpha,T}$ and $\tilde{f}_t^{\alpha,T}$ are convex. We can now use standard arguments, the above inequality, and facts (a)–(d) to show that

$$f_t^{\alpha,T}(\mathbf{x}) \geq \tilde{f}_t^{\alpha,T}(\mathbf{x}) \quad \text{for all } \mathbf{x}. \tag{6}$$

Next, we appeal to standard dynamic programming convergence arguments[6] (as $T \to \infty$) along with (6) to conclude that the infinite horizon, discounted-cost functions $f_1^{\alpha,\infty}(\mathbf{x})$ and $\tilde{f}_1^{\alpha,\infty}(\mathbf{x})$ also satisfy the inequality

$$f_1^{\alpha,\infty}(\mathbf{x}) \geq \tilde{f}_1^{\alpha,\infty}(\mathbf{x}) \quad \text{for all } \mathbf{x}. \tag{7}$$

[6] See Schal (1993) and Hernandez-Lerma and Lassere (1996) for general results of this type and Huh et al. (2011) for how these results apply to a general class of inventory problems. More specifically, Sheopuri et al. (2010) comment on the implication of these results to the dual sourcing problem studied here.

Furthermore, using the vanishing discount approach in the dynamic programming literature, we know that the functions $f_1^{\alpha,\infty}(\mathbf{x})$ and $\tilde{f}_1^{\alpha,\infty}(\mathbf{x})$ converge to the optimal long-run average costs $C^{\mathcal{D},*}(h, b, c, l^E, l^R, F)$ and $C^{\mathcal{D},*}(h, b, c, l^E, l^R, \tilde{F})$, respectively, as the discount factor α approaches 1. This convergence along with (7) implies the desired inequality $C^{\mathcal{D},*}(h, b, c, l^E, l^R, F) \geq C^{\mathcal{D},*}(h, b, c, l^E, l^R, \tilde{F})$. $\square$

Proof of Lemma 6. We begin with the proof of the second result since it is straightforward. Consider any policy θ that is used in the system that faces the demand distribution F. We modify that policy to construct another policy $\hat{\theta}$ as follows: $q_t^{\hat{\theta},R} = B \cdot q_t^{\theta,R}$ and $q_t^{\hat{\theta},E} = B \cdot q_t^{\theta,E}$ for all t. Then, since the random demand in period t under the distribution $\hat{F}$ is $B \cdot D_t$ for all t, it is easy to verify that the long-run average cost under $\hat{\theta}$ and $\hat{F}$ is exactly B times the long-run average cost under θ and F. Since θ was an arbitrary, feasible policy, this implies that

$$C^{\mathcal{D},*}(h, b, c, l^E, l^R, \hat{F}) \leq B \cdot C^{\mathcal{D},*}(h, b, c, l^E, l^R, F).$$

An argument symmetric to the above leads to the opposite inequality, thus yielding the desired equality. This completes the proof of the second result.

We now proceed to prove the first result. We first appeal to a result from Janakiraman and Seshadri (2014), which states that

$$C^{\mathcal{D},*}(h, b, c, l^E, l^R, F) = C^{\hat{D},*}(h, b, c, l^E, l^R, F),$$

where $\hat{D}$ is a dual sourcing system that is identical to $\mathcal{D}$ with the exception that we are allowed to place negative orders on R. In other words, the result states that the option of ordering negative quantities from R confers no cost advantage. Since the distribution F is arbitrary, the equality

$$C^{\mathcal{D},*}(h, b, c, l^E, l^R, \tilde{F}) = C^{\hat{D},*}(h, b, c, l^E, l^R, \tilde{F})$$

also holds. Thus, to prove the desired result, we only have to prove $C^{\hat{D},*}(h, b, c, l^E, l^R, \tilde{F}) = C^{\hat{D},*}(h, b, c, l^E, l^R, F)$, which we proceed to prove next. Thus, in the remainder of this proof, negative orders from R are allowed.

Since our interest is limited to the long-run average-cost performance measure, we assume convenient starting states in the two systems without loss of generality. We assume that the system facing the demand distribution F starts period 1 with an "empty pipeline" (i.e., no outstanding orders) and a net inventory of zero. We assume that the system facing the demand distribution $\tilde{F}$ starts period 1 with A units of inventory on hand and an outstanding order for A units scheduled to arrive from R in each of the next $l^R - 1$ periods—that is, periods $2, 3, \ldots, l^R$. (Notice that since the latter system receives a demand of exactly A units in excess of that received by the former system in each period, the starting states we have chosen in the two systems are essentially the "same.")

Consider any policy θ, which is used in the system that faces the demand distribution F. We modify that policy to construct another policy, $\tilde{\theta}$, as follows: $q_t^{\tilde{\theta},R} = A + q_t^{\theta,R}$ and $q_t^{\tilde{\theta},E} = q_t^{\theta,E}$ for all t. That is, in every period, $\tilde{\theta}$ follows θ identically for orders from E and orders A units in excess of that ordered by θ from R. It is easy to see that the long-run average cost under $\tilde{\theta}$ and $\tilde{F}$ is exactly equal to the long-run average cost under θ and F. Since θ was an arbitrary, feasible policy, this implies that

$$C^{\hat{D},*}(h,b,c,l^E,l^R,\tilde{F}) \le C^{\hat{D},*}(h,b,c,l^E,l^R,F).$$

It only remains to establish the opposite inequality, which we focus on next.

Let us now consider any policy $\tilde{\theta}$, which is used in the system that faces the demand distribution $\tilde{F}$. We modify that policy to construct another policy θ as follows:

$$q_t^{\theta,R} = q_t^{\tilde{\theta},R} - A \quad \text{and} \quad q_t^{\theta,E} = q_t^{\tilde{\theta},E} \quad \forall\, t \ge 1.$$

It is now easy to verify that the long-run average cost under θ and F is equal to that under $\tilde{\theta}$ and $\tilde{F}$. Since $\tilde{\theta}$ is an arbitrary, feasible policy, this implies that

$$C^{\hat{D},*}(h,b,c,l^E,l^R,F) \le C^{\hat{D},*}(h,b,c,l^E,l^R,\tilde{F}).$$

The desired equality follows from this inequality and its opposite, which was proved earlier. This completes the proof of Lemma 6. $\square$

A Preliminary Result Used in the Proof of Theorem 8

Recall that, in the discussion preceding the statement of Theorem 8, we had argued intuitively that the TBS policy with $Q=0$ and $S=0$ is optimal when $d_{\text{Low}}=0$ and $d_{\text{High}}=1$. We first prove this claim formally before proving the theorem for the more general case of arbitrary, nonnegative values of d_{Low} and d_{High} such that $d_{\text{Low}} < d_{\text{High}}$.

Lemma 10. *Under Assumption* 1, *the TBS policy with* $Q=0$ *and* $S=0$ *is optimal when* $d_{\text{Low}}=0$ *and* $d_{\text{High}}=1$. *Moreover, the optimal cost and the cost of this TBS policy equal* $\beta\cdot(1-p_{\text{Low}})$, *where* $\beta := c + b\cdot(l^E+1)$.

Proof. Assume that the system starts with zero backorders and no inventory anywhere in the system. This is without loss of generality, since our focus is on the long-run average-cost performance measure. Let $\hat{\theta}$ denote the TBS policy with $Q=0$ and $S=0$. This policy does not incur any holding costs. Moreover, every customer is backordered for exactly l^E+1 periods and is satisfied by a unit ordered from E at a cost of c. Thus, the long-run average cost of $\hat{\theta}$ is

$$C^{\hat{\theta}} = \beta\cdot\mu = \beta\cdot(1-p_{\text{Low}}).$$

We now claim that, for any feasible policy, θ, the long-run average cost C^θ is at least as large as $C^{\hat{\theta}}$; that is,

$$C^\theta \ge \beta\cdot\mu. \tag{8}$$

This claim implies the desired result that $\hat{\theta}$ is an optimal policy. We now proceed to prove (8).

Consider any feasible policy, θ. Let $C^\theta[1,T]$ denote the costs incurred under θ during the first T periods. Similarly, we use $H^\theta[1,T]$ to denote the holding costs and $B^\theta[1,T]$ to denote the sum of procurement costs and backordering costs under θ during that interval. That is,

$$C^\theta[1,T] = H^\theta[1,T] + B^\theta[1,T].$$

The long-run average cost C^θ can be written as

$$C^\theta = \limsup_{T\to\infty} \frac{E[C^\theta[1,T]]}{T}.$$

We now claim that to show (8), it is sufficient to show the following statement:

$$\begin{aligned}&\text{For any } \epsilon>0,\ \text{there exists } T(\epsilon)<\infty\\ &\quad \text{s.t. } E[C^\theta[1,T+T(\epsilon)]] \ge \beta\cdot\mu\cdot(1-\epsilon)\cdot T \quad \text{for all } T.\end{aligned} \tag{9}$$

The proof of the fact that (9) implies (8) is the following:

$$\begin{aligned} C^\theta &= \limsup_{T\to\infty} \frac{E[C^\theta[1,T]]}{T}\\ &= \limsup_{T\to\infty} \frac{E[C^\theta[1,T+T(\epsilon)]]}{T+T(\epsilon)}\\ &= \limsup_{T\to\infty} \left(\frac{E[C^\theta[1,T+T(\epsilon)]]}{T}\cdot\frac{T}{T+T(\epsilon)}\right)\\ &\ge \limsup_{T\to\infty} \left(\beta\cdot\mu\cdot(1-\epsilon)\frac{T}{T+T(\epsilon)}\right) \quad \text{(from (9))}\\ &= \beta\cdot\mu\cdot(1-\epsilon).\end{aligned}$$

Since ϵ is an arbitrary constant, this implies that $C^\theta \ge \beta\cdot\mu$, which is the desired inequality (8). Thus, it only remains to prove that the statement in (9) is true.

We proceed to prove (9). Since the demand in every period is zero or one, we refer to a unit of demand as a customer in this proof. We use $\mathcal{F}_t$ to denote the demand history up until the beginning of period t, that is, the realization of $(D_1, D_2, \ldots, D_{t-1})$.

Let n_t^θ denote the number of units that are ordered (from either R or E) in period t *without a corresponding customer ready at the beginning of that period*. Although a mathematical expression for n_t^θ is not used in the proof, we find it useful to present one in order to help the reader understand the meaning of n_t^θ more precisely. For example, if there are zero customers backordered at the beginning of period t, then $n_t^\theta = q_t^{\theta,R} + q_t^{\theta,E}$. More generally, we have

$$n_t^\theta = \max(0, q_t^{\theta,R} + q_t^{\theta,E} - (IP_{t,\text{start}}^{\theta,R})^-),$$

where $IP_{t,\text{start}}^{\theta,R}$ is the total inventory position in the system at the start of period t before any ordering decisions are made. Thus, $(IP_{t,\text{start}}^{\theta,R})^-$ is the number by which backordered demand exceeds committed supply. Therefore, if the total order quantity in period t, i.e., $q_t^{\theta,R} + q_t^{\theta,E}$ exceeds $(IP_{t,\text{start}}^{\theta,R})^-$, their difference is the number of units that are ordered (from either R or E) in period t without a corresponding customer ready at the beginning of that period. Let

$$N_t^\theta = \sum_{u=1}^{t} n_u^\theta.$$

We now derive a lower bound on the expected number of periods each of these units will stay in inventory and use that to derive a lower bound on $E[C^\theta[1,T+T(\epsilon)]]$ for a suitably chosen $T(\epsilon)$ and complete the proof of (9). We accomplish this by assembling several ideas.

Let ζ denote a geometric random variable with success probability $(1-p_{\text{Low}})$. That is, for any t, given $\mathcal{F}_t$, the distribution of the number of periods until the next customer arrives is ζ. Note that $E[\zeta] = p_{\text{Low}}/(1-p_{\text{Low}})$.

Let

$$\tau = E[(\zeta - (l^R+1))^+].$$

The meaning of this quantity is the following. Consider a unit that is ordered from R without a *corresponding* customer ready at that time. Then, this unit incurs a holding cost from the time it is delivered and *at least*[7] until the next customer arrives. Let

$$\tau(u) = E[\min\{(\zeta-(l^R+1))^+, (u-(l^R+1))^+\}] \quad \text{for any } u \geq 1.$$

This quantity can be understood as follows: consider a unit that is ordered from R in some period t without a *corresponding* customer ready at that time. Then, $\tau(u)$ is a lower bound on the expected number of periods this unit is held in inventory during the interval $[t, t+u]$.

It is easy to verify that $\tau(u)$ is an increasing function of u and that

$$\lim_{u\to\infty} \tau(u) = \tau.$$

Therefore, for any $\epsilon > 0$, we know that $\tau(u) \geq \tau\cdot(1-\epsilon)$ for sufficiently large u—we choose such a value of u as $T(\epsilon)$; more formally,

$$T(\epsilon) = \min\{u \in \mathbb{N}\colon\ \tau(u) \geq \tau\cdot(1-\epsilon)\}.$$

Next, observe from the definition of τ that

$$\tau \geq E[\zeta] - (l^R+1) = p_{\text{Low}}/(1-p_{\text{Low}}) - (l^R+1).$$

This implies that

$$\tau(T(\epsilon)) \geq (p_{\text{Low}}/(1-p_{\text{Low}}) - (l^R+1))\cdot(1-\epsilon).$$

Recall that Assumption 1 (see the discussion on that assumption in §5.1) is equivalent to (2); that is,

$$h\cdot(p_{\text{Low}}/(1-p_{\text{Low}}) - (l^R+1)) \geq c + b\cdot(l^E+1).$$

Combining the last two inequalities, we obtain

$$h\cdot\tau(T(\epsilon)) \geq (c+b\cdot(l^E+1))\cdot(1-\epsilon) = \beta\cdot(1-\epsilon). \tag{10}$$

We are now ready to examine $E[C^\theta[1, T+T(\epsilon)]]$, the left-hand side of (9). We have

$$\begin{aligned} E[C^\theta[1, T+T(\epsilon)]] &= E[H^\theta[1, T+T(\epsilon)]] \\ &\quad + E[B^\theta[1, T+T(\epsilon)]]. \end{aligned} \tag{11}$$

We will first derive a lower bound on the holding cost and then do the same for the procurement-plus-backorder cost.

[7] We say "at least" because when a unit is released from R in a period without having a corresponding customer ready, this unit might have to wait in inventory for a period longer than τ if a unit is subsequently released from E and is delivered before the former unit. We also note that we use discrete terms such as "units" in this part of the proof purely for the sake of expositional clarity. We are not assuming that order quantities are integers.

From the definition of n_t^θ and the discussion surrounding it, it can be observed that the expected holding cost $E[H^\theta[1, T+T(\epsilon)]]$ exceeds

$$\begin{aligned} &\sum_{t=1}^{T} E[\text{holding costs incurred in } [t, T+T(\epsilon)] \text{ by the } n_t^\theta \text{ units ordered in } t] \\ &\quad = \sum_{t=1}^{T} E_{\mathcal{F}_t}[E[\text{holding costs incurred in } [t, T+T(\epsilon)] \text{ by the } n_t^\theta \text{ units ordered in } t \mid \mathcal{F}_t]]. \end{aligned}$$

From the definition of $\tau(T(\epsilon))$, we can see that, given $\mathcal{F}_t$, the expected holding costs incurred in $[t, T+T(\epsilon)]$ by the n_t^θ units ordered in t exceeds $n_t^\theta\cdot h\cdot\tau(T(\epsilon))$. Thus, we have

$$\begin{aligned} E[H^\theta[1, T+T(\epsilon)]] &\geq \sum_{t=1}^{T} E[n_t^\theta\cdot h\cdot\tau(T(\epsilon))] \\ &= E[N_T^\theta]\cdot h\cdot\tau(T(\epsilon)) \\ &\geq E[N_T^\theta]\cdot\beta\cdot(1-\epsilon) \quad \text{(from (10))}. \end{aligned}$$

We now bound the procurement and backorder costs, i.e., $E[B^\theta[1, T+T(\epsilon)]]$. Let $D[1, T] := \sum_{t=1}^{T} D_t$. First, we use the definition of N_T^θ to make the following observation: of the $D[1, T]$ customers who arrive in the first T periods, at least $(D[1, T] - N_T^\theta)^+$ customers do not find a "corresponding unit" waiting for them immediately upon arrival. Given our assumption that $c < b\cdot(l^R - l^E)$, this implies that

$$\begin{aligned} E[B^\theta[1, T+T(\epsilon)]] &\geq E[(D[1, T] - N_T^\theta)^+]\cdot(c+b\cdot(l^E+1)) \\ &= E[(D[1, T] - N_T^\theta)^+]\cdot\beta \\ &\geq E[(D[1, T] - N_T^\theta)]\cdot\beta \\ &\geq E[(D[1, T] - N_T^\theta)]\cdot\beta\cdot(1-\epsilon) \\ &\qquad\qquad \text{(since } \epsilon > 0\text{)}. \end{aligned}$$

Summing the lower bounds on the holding costs and the procurement-cum-backorder costs, we obtain

$$\begin{aligned} E[C^\theta[1, T+T(\epsilon)]] &\geq E[N_T^\theta]\cdot\beta\cdot(1-\epsilon) \\ &\quad + E[(D[1, T] - N_T^\theta)]\beta\cdot(1-\epsilon) \\ &= \beta\cdot\mu\cdot(1-\epsilon)\cdot T. \end{aligned}$$

Thus, we have shown (9). This completes the proof of the lemma. □

Proof of Theorem 8. Let F denote the two-point distribution with a mass of p_{Low} at 0 and a mass of $(1-p_{\text{Low}})$ at 1. Let $\check{F}$ denote the two-point distribution with a mass of p_{Low} at d_{Low} and a mass of $(1-p_{\text{Low}})$ at d_{High}. Then, an application of Corollary 7 leads to the following relationship:

$$C^{\mathcal{D},*}(h, b, c, l^E, l^R, \check{F}) = (d_{\text{High}} - d_{\text{Low}})\cdot C^{\mathcal{D},*}(h, b, c, l^E, l^R, F).$$

We know from Lemma 10 that $C^{\mathcal{D},*}(h, b, c, l^E, l^R, F) = \beta\cdot(1-p_{\text{Low}})$. Thus, we have

$$C^{\mathcal{D},*}(h, b, c, l^E, l^R, \check{F}) = (d_{\text{High}} - d_{\text{Low}})\cdot\beta\cdot(1-p_{\text{Low}}).$$

It only remains to show that $C^\theta = (d_{\text{High}} - d_{\text{Low}})\cdot\beta\cdot(1-p_{\text{Low}})$, where θ is the TBS policy with $Q = d_{\text{Low}}$ and $S = (l^E+1)\cdot d_{\text{Low}}$. We proceed to establish this claim.

Assume without loss of generality (as a result of the consideration of long-run average costs) that the system starts period 1 with d_{Low} units of inventory on hand and exactly d_{Low} units to be delivered k periods from now for every $k \in \{1, 2, \ldots, l^R - 1\}$. Intuitively, this is the ideal state for the system to start from if the demand in every period is d_{Low}. In this state, the expedited inventory position at the beginning of period 1 is exactly $d_{\text{Low}} \cdot (l^E + 1)$—thus, no order is placed on E in period 1 under θ. In fact, it is easy to see that the first strictly positive order is placed on E by θ in the first period t ($t \geq 2$) such that $D_{t-1} = d_{\text{High}}$. Moreover, the expedited inventory position at the beginning of such a period t is $d_{\text{Low}} \cdot (l^E + 1) - (d_{\text{High}} - d_{\text{Low}})$, and therefore, θ orders the quantity $d_{\text{High}} - d_{\text{Low}}$ from E to raise the expedited inventory position to its target level $d_{\text{Low}} \cdot (l^E + 1)$. Furthermore, it can also be verified that, subsequently, a strictly positive order is placed on E by θ in a period if and only if the demand in the previous period is d_{High}; that order quantity will also be $d_{\text{High}} - d_{\text{Low}}$. Thus, the expected procurement cost in a period under θ is $(1-p) \cdot c \cdot (d_{\text{High}} - d_{\text{Low}})$. By design, θ never holds inventory on hand since the demand in any interval of $(l^E + 1)$ periods is at least $d_{\text{Low}} \cdot (l^E + 1)$, which is equal to the expedited inventory position after ordering in any period (note that there is no overshoot under policy θ since $Q = d_{\text{Low}} \leq D_t$ for all t). So θ does not incur holding costs. In other words, θ is equipped to handle a demand of d_{Low} in every period. On the other hand, in every period (say, t) that the demand is d_{High}, the "excess demand" $d_{\text{High}} - d_{\text{Low}}$ is backordered, and this amount stays on backorder for exactly $(l^E + 1)$ periods (until the order placed on E in period $t+1$ arrives, i.e., until period $t + 1 + l^E$). Thus, the expected backorder cost incurred by θ in a period is $(1 - p_{\text{Low}}) \cdot b \cdot (d_{\text{High}} - d_{\text{Low}}) \cdot (l^E + 1)$. Summing the procurement and backorder costs, we see that the long-run average cost under θ is $C^\theta = (1 - p_{\text{Low}}) \cdot (d_{\text{High}} - d_{\text{Low}}) \cdot (c + b \cdot (l^E + 1))$, which equals $(d_{\text{High}} - d_{\text{Low}}) \cdot \beta \cdot (1 - p_{\text{Low}})$. This completes the proof. □

Proof of Theorem 9. The first statement we are required to prove is that

$$\frac{\min_Q C^{\mathcal{D}, Q, *}(h, b, c, l^E, l^R, F)}{C^{\mathcal{D}, *}(h, b, c, l^E, l^R, F)} \leq 1 + \frac{\sigma_X \cdot (\sqrt{2 \cdot h \cdot c} + \sqrt{h \cdot b \cdot (l^E + 1)})}{(c + b \cdot (l^E + 1)) \cdot (1 - p) \cdot \mu_Y}. \tag{12}$$

Let F_X, F_Y, and F_Z denote the distribution functions of the random variables, X, Y, and Z, respectively. Let Q_X minimize $C^{\mathcal{D}, Q, *}(h, b, c, l^E, l^R, F_X)$. Clearly,

$$\min_Q C^{\mathcal{D}, Q, *}(h, b, c, l^E, l^R, F) \leq C^{\mathcal{D}, Q_X, *}(h, b, c, l^E, l^R, F). \tag{13}$$

Next, we claim that

$$\begin{aligned} C^{\mathcal{D}, Q_X, *}&(h, b, c, l^E, l^R, F) \\ &\leq \min_Q C^{\mathcal{D}, Q, *}(h, b, c, l^E, l^R, F_X) + c \cdot (1-p) \cdot \mu_Y \\ &\quad + C^{\mathcal{B}, *}(h, b, l^E, F_Z). \end{aligned} \tag{14}$$

The proof is the following: Let $S_X = \arg\min_S C^{\mathcal{D}, Q_X, S}(h, b, c, l^E, l^R, F_X)$, i.e., the optimal order-up-to level for E given that the demand distribution is F_X and a constant order of Q_X is placed on R every period. Let $S_Z = \arg\min C^{\mathcal{B}, S}(h, b, l^E, F_Z)$. Let $I_{X,t}$ denote the expedited inventory position in $\mathcal{D}(h, b, c, l^E, l^R, F_X)$ at the beginning of period t (after ordering) under the (Q_X, S_X) policy. Assume that $I_{X,1} = S_X$ and that those outstanding orders that are not a part of this expedited inventory position are each of size Q_X (this is without loss of generality because our interest is in the long-run average costs of the inventory systems analyzed). Similarly, let $I_{Z,t}$ denote the expedited inventory position in $\mathcal{B}(h, b, l^E, F_Z)$ at the beginning of period t (after ordering) under the order-up-to S_Z policy. Assuming that $I_{Z,1} = S_Z$, we have the trivial identity $I_{Z,t} = S_Z$ for all t. Consider the following policy θ in $\mathcal{D}(h, b, c, l^E, l^R, F)$, defined as follows. The constant quantity Q_X is ordered from R in every period. The order quantity from E under θ (in period t) is the sum of the quantity ordered in $\mathcal{D}(h, b, c, l^E, l^R, F_X)$ under the (Q_X, S_X) policy (in period t) and the order quantity in $\mathcal{B}(h, b, l^E, F_Z)$ (in period t) under the order-up-to S_Z policy. Let $I_{X+Z,t}$ denote the expedited inventory position in $\mathcal{D}(h, b, c, l^E, l^R, F)$ at the beginning of period t after ordering under policy θ. Under the assumption that $I_{X+Z,1} = S_X + S_Z$, the definition of θ implies the following relationship:

$$I_{X+Z,t} = I_{X,t} + I_{Z,t} \quad \text{for all } t.$$

Let us define $e_{X,t}$, $e_{Z,t}$ and $e_{X+Z,t}$ to be the net inventories at the end of period t in the three systems, $\mathcal{D}(h, b, c, l^E, l^R, F_X)$ (under the (Q_X, S_X) policy), $\mathcal{B}(h, b, l^E, F_Z)$ (under the order-up-to S_Z policy), and $\mathcal{D}(h, b, c, l^E, l^R, F)$ (under policy θ). These quantities also have a similar relationship; that is,

$$e_{X+Z,t} = e_{X,t} + e_{Z,t} \quad \text{for all } t.$$

This relationship along with the subadditivity of the newsvendor cost function $h(e)^+ + b(e)^-$ implies that the holding and shortage cost in $\mathcal{D}(h, b, c, l^E, l^R, F)$ (under policy θ) is smaller than the sum of the holding and shortage costs in $\mathcal{D}(h, b, c, l^E, l^R, F_X)$ (under the (Q_X, S_X) policy) and in $\mathcal{B}(h, b, l^E, F_Z)$ (under the order-up-to S_Z policy). Moreover, since the policy θ orders Q_X from R in every period while the expected demand is $E[X+Z]$, the procurement cost in $\mathcal{D}(h, b, c, l^E, l^R, F)$ is equal to the procurement cost in $\mathcal{D}(h, b, c, l^E, l^R, F_X)$ (under the (Q_X, S_X) policy) plus $c \cdot E[Z]$, which is nothing but $c \cdot (1-p) \cdot \mu_Y$. These observations about the holding, shortage, and procurement costs yield the following inequality:

$$\begin{aligned} C^{\mathcal{D}, \theta}(h, b, c, l^E, l^R, F) &\leq C^{\mathcal{D}, Q_X, S_X}(h, b, c, l^E, l^R, F_X) \\ &\quad + c \cdot (1-p) \cdot \mu_Y + C^{\mathcal{B}, S_Z}(h, b, l^E, F_Z). \end{aligned}$$

The following three relationships follow directly from definitions: (i) $C^{\mathcal{D}, Q_X, *}(h, b, c, l^E, l^R, F) \leq C^{\mathcal{D}, \theta}(h, b, c, l^E, l^R, F)$ (since θ is just one specific feasible policy that orders Q_X from R in every period), (ii) $\min_Q C^{\mathcal{D}, Q, *}(h, b, c, l^E, l^R, F_X) = C^{\mathcal{D}, Q_X, S_X}(h, b, c, l^E, l^R, F_X)$, and (iii) $C^{\mathcal{B}, *}(h, b, l^E, F_Z) = C^{\mathcal{B}, S_Z}(h, b, l^E, F_Z)$. Combining these relationships with the previous inequality, we obtain the claimed inequality of (14).

Next, observe that using an order-up-to zero policy in $\mathcal{B}(h, b, l^E, F_Z)$ leads to an average cost of $b \cdot (l^E+1) \cdot E[Z] = b \cdot (l^E+1) \cdot (1-p) \cdot \mu_Y$. This implies the fact that $C^{\mathcal{B},*}(h, b, l^E, F_Z) \le b \cdot (l^E+1) \cdot (1-p) \cdot \mu_Y$. Using this fact in (14), we obtain

$$C^{\mathcal{D}, Q_X,*}(h,b,c,l^E,l^R,F) \le \min_Q C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F_X) + (c+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y. \quad (15)$$

We know from Theorem 3 that

$$\min_Q C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F_X) \le C^{\mathcal{B},*}(h,b,l^E,F_X) + \sigma_X \cdot \sqrt{2\cdot h\cdot c}.$$

Since F_X has a standard deviation of σ_X, we can use a well-known distribution-free newsvendor result (Gallego and Moon 1993) that $C^{\mathcal{B},*}(h,b,l^E,F_X) \le \sigma_X \cdot \sqrt{l^E+1} \cdot \sqrt{h\cdot b}$. Thus,

$$\min_Q C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F_X) \le \sigma_X \cdot \left(\sqrt{2\cdot h\cdot c} + \sqrt{h\cdot b\cdot(l^E+1)}\right). \quad (16)$$

We can now use (15) to obtain

$$C^{\mathcal{D},Q_X,*}(h,b,c,l^E,l^R,F) \le \sigma_X\cdot\left(\sqrt{2\cdot h\cdot c}+\sqrt{h\cdot b\cdot(l^E+1)}\right) + (c+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y. \quad (17)$$

This inequality along with (13) implies that

$$\min_Q C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F) \le \sigma_X\cdot\left(\sqrt{2\cdot h\cdot c}+\sqrt{h\cdot b\cdot(l^E+1)}\right) + (c+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y. \quad (18)$$

The inequality above provides an upper bound on the cost of the best TBS policy. Next, we derive a lower bound on the optimal cost over all admissible policies.

Our lower bound derivation makes use of Theorem 5 on convex ordering. Let $\tilde{F}$ denote the distribution of a random variable $\tilde{D}$ defined as $\tilde{D} = \mu_X + Z$. It is easy to see that $\tilde{F} \le_{cx} F$. Thus, $C^{\mathcal{D},*}(h,b,c,l^E,l^R,F) \ge C^{\mathcal{D},*}(h,b,c,l^E,l^R,\tilde{F})$. Let us now examine $C^{\mathcal{D},*}(h,b,c,l^E,l^R,\tilde{F})$. Since $\tilde{D}$ is the sum of a deterministic quantity μ_X and the random variable Z, the optimal policy will procure μ_X units from R every period to handle the deterministic part of demand. So the only procurement, holding, and shortage costs incurred are those incurred on the stochastic part of demand, Z. Note that Z has a probability mass of, at least, p at zero. The mean of Z is $(1-p)\cdot\mu_Y$. We can now use an analysis that is identical to that presented in §5.1 to see that the optimal policy, when the demand is given by the random variable Z $(\mu_X + Z)$, is to order zero (μ_X) from R in every period and order-up-to zero from E in every period. The cost of this policy, i.e., the optimal cost, is $(c+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y$. To summarize, we have now established that

$$C^{\mathcal{D},*}(h,b,c,l^E,l^R,F) \ge (c+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y. \quad (19)$$

Combining (18) and (19), we obtain

$$\frac{\min_Q C^{\mathcal{D},Q,*}(h,b,c,l^E,l^R,F)}{C^{\mathcal{D},*}(h,b,c,l^E,l^R,F)} \le 1 + \frac{\sigma_X\cdot\left(\sqrt{2\cdot h\cdot c}+\sqrt{h\cdot b\cdot(l^E+1)}\right)}{(c+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y},$$

which is the desired result in (12).

We now proceed to establish the second result. When $h \le c$ and $c < b\cdot(l^E+1)$, the right-hand side in the above inequality (i.e., (12)) is bounded above by

$$1 + \frac{\sigma_X\cdot(\sqrt{2}+1)\cdot\sqrt{h\cdot b\cdot(l^E+1)}}{(h+b\cdot(l^E+1))\cdot(1-p)\cdot\mu_Y},$$

which, in turn, is smaller than

$$1 + \frac{1+\sqrt{2}}{2}\cdot\frac{\sigma_X}{(1-p)\cdot\mu_Y}.$$

This is so because the arithmetic mean exceeds the geometric mean. This completes the proof of the second desired result. □

References

Allon G, Van Mieghem JA (2010) Global dual sourcing: Tailored base-surge allocation to near- and offshore production. *Management Sci.* 56(1):110–124.

Gallego G, Moon I (1993) The distribution free newsboy problem: Review and extensions. *J. Oper. Res. Soc.* 44(8):825–834.

Hernandez-Lerma O, Lassere JB (1996) *Discrete-Time Markov Control Processes: Basic Optimality Criteria* (Springer-Verlag, New York).

Huh WT, Janakiraman G, Nagarajan M (2011) Average cost single-stage inventory models: An analysis using a vanishing discount approach. *Oper. Res.* 59(1):143–155.

Huh WT, Janakiraman G, Muckstadt JA, Rusmevichientong P (2009) Asymptotic optimality of order-up-to policies in lost sales inventory systems. *Management Sci.* 55(3):404–420.

Janakiraman G, Seshadri S (2014) Dual sourcing inventory systems: On optimal policies and the value of costless returns. Working paper, University of Texas at Dallas, Richardson.

Janssen F, de Kok T (1999) A two-supplier inventory model. *Internat. J. Production Econom.* 59(1–3):395–403.

Karlin S, Scarf H (1958) Inventory models of the Arrow-Harris-Marschak type with time lag. Arrow KJ, Karlin S, Scarf H, eds. *Studies in the Mathematical Theory of Inventory and Production* (Stanford University Press, Redwood City, CA), 155–178.

Kingman JFC (1970) Inequalities in the theory of queues. *J. Roy. Statist. Soc. Ser. B* 32(1):102–110.

Loynes RM (1962) The stability of a queue with nonindependent inter-arrival and service. *Proc. Cambridge Philos. Soc.* 58:497–520.

Minner S (2003) Multiple-supplier inventory models in supply chain management: A review. *Internat. J. Production Econom.* 81–82(January):265–279.

Rosenshine M, Obee D (1976) Analysis of a standing order inventory system with emergency orders. *Oper. Res.* 24(6):1143–1155.

Schal M (1993) Average optimality in dynamic programming with general state space. *Math. Oper. Res.* 18(1):163–172.

Sheopuri A, Janakiraman G, Seshadri S (2010) New policies for the stochastic inventory control problem with two supply sources. *Oper. Res.* 58(3):734–745.

Simchi-Levi D, Kaminsky P, Simchi-Levi E (2008) *Designing and Managing the Supply Chain: Concepts, Strategies, and Cases*, 3rd ed. (McGraw-Hill Irwin, New York).

Song J-S, Zipkin P (2009) Inventories with multiple supply sources and networks of queues with overflow bypasses. *Management Sci.* 55(3):362–372.

Veeraraghavan S, Scheller-Wolf A (2008) Now or later: A simple policy for effective dual sourcing in capacitated systems. *Oper. Res.* 56(4):850–864.

Zipkin P (2008a) Old and new methods for lost-sales inventory systems. *Oper. Res.* 56(5):1256–1263.

Zipkin P (2008b) On the structure of lost-sales inventory models. *Oper. Res.* 56(4):937–944.