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We study a joint inventory and pricing problem in a single-stage system with a positive lead time. We consider both additive and multiplicative demand forms. This problem is, in general, intractable due to its computational complexity. We develop a simple heuristic that resolves this issue. The heuristic involves a myopic pricing policy that generates each period's price as a function of the initial inventory level and a base-stock policy for inventory replenishment. In each period, the firm monitors its so-called price-deflated inventory position and places an order to reach a target base-stock level. The price-deflated inventory position weights the on-hand and pipeline inventory according to a factor that reflects the sensitivity of price to the net inventory level. To assess the effectiveness of our heuristic, we construct an upper bound to the exact system. The upper bound is based on an information-relaxation approach and involves a penalty function derived from the proposed heuristic. A numerical study suggests that the heuristic is near-optimal. The heuristic approach can be applied to a wide variety of inventory systems, such as systems with fixed ordering costs or fixed batch sizes. The heuristic enables us to explore the use of price as a lever to balance supply and demand. Our findings indicate that a responsive strategy (that effectively reduces the replenishment lead time) leads to a more stable pricing policy and that the value of dynamic pricing increases with lead time.

1. Introduction

This paper studies a joint inventory and pricing problem for systems with a positive lead time. Specifically, we consider a periodic-review system in which demand in each period is stochastic and depends on the pricing decision. Two price-dependent demand forms are considered: additive and multiplicative. Unfulfilled demand at the end of each period is fully backlogged, and linear holding and backorder costs are charged. The price and the replenishment quantity are simultaneously determined at the beginning of each period, and the order quantity is received after a positive delivery lead time. The objective is to maximize the total expected discounted profit over a finite horizon.

In practice, companies often integrate inventory and pricing decisions to match demand with supply more efficiently. For instance, a company may offer a discounted price when there is excess inventory or raise the price when the inventory level is low. The problem of joint control of price and inventory has attracted significant attention in the field—many papers have characterized the optimal joint pricing and replenishment policy in various settings involving both the additive and multiplicative demand forms. However, almost all of those papers assume a zero delivery lead time. Given the increasing number of firms that source from low-cost countries at the expense of longer lead times, the determination of an efficient policy for firms that experience a longer procurement lead time is of interest. For example, Murphy (2012) reports on Abercrombie and Fitch's shift from air to ocean delivery to save in shipping costs. The teen-apparel retailer sources products from China and this shift has resulted in a significant increase in the delivery lead time—from days to weeks. As described in the article, the longer lead time increases the chances of supply and demand mismatches, requiring frequent price discounts to liquidate inventory. To illustrate the relation between prices and inventory, Figure 1 exhibits prices and inventory levels for two products (from two different product categories) sold through amazon.com.[1]

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[1] The authors thank Seyed Emadi for suggestions on how to track inventory levels at amazon.com.

Figure 1   (Color online) Tracking Inventory Levels and Prices
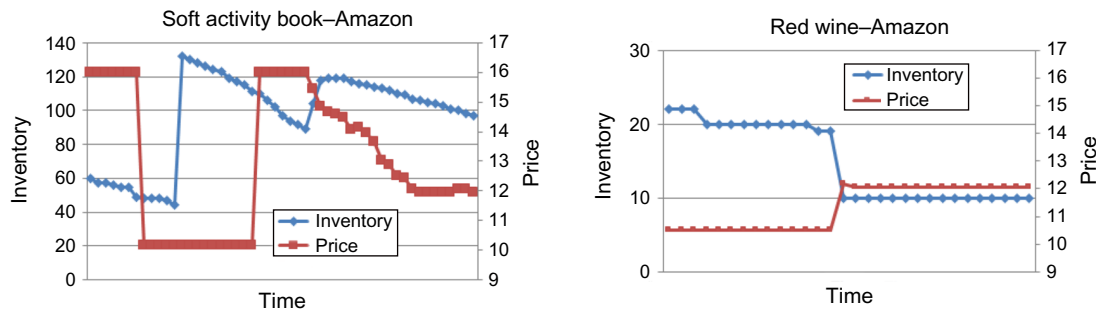
The activity book[2] is made in China, while the red wine[3] is elaborated in California. The graphs show how prices fluctuate over time and how these price adjustments correlate with the prevailing inventory levels—in particular, the graphs suggest that the price tends to decrease when the inventory level is relatively high.

It is therefore crucial to understand how to coordinate inventory and pricing decisions when a lead time is present. Unfortunately, it is difficult to characterize the optimal joint pricing and replenishment policy in settings with a positive lead time. It is well known that the problem is intractable due to its computational complexity. Indeed, to characterize the optimal policy in systems with a positive lead time, one has to keep track of the pricing decisions in each of the lead time periods, giving rise to the curse of dimensionality.

In this paper, we propose a heuristic to determine a close-to-optimal joint pricing and inventory replenishment policy for systems with a positive lead time. The idea of the heuristic stems from analytical findings which suggest that the shape of the *myopic price* is close to that of the optimal pricing policy. (The myopic price is the price that maximizes the single-period profit based on the initial inventory level in that period.) Therefore, as a first step, we propose the myopic price as the heuristic pricing policy. Although the myopic price simplifies the pricing policy, it does not reduce the computational complexity for the inventory policy. To resolve this, we further propose a linear approximation of the myopic expected demand (the expected demand evaluated at the myopic price) as a function of the initial inventory level. This approximation is supported by the fact that the myopic expected demand tends to be linear over a wide range of initial inventory levels. Substituting this linear approximation into the profit function allows us to aggregate all inventory states (i.e., net inventory and pipeline

inventory) into a single state variable, which we refer to as *price-deflated inventory position*. The price-deflated inventory position plays the same role as the inventory position (= net inventory + pipeline inventory) in the traditional inventory problem with exogenous prices. However, unlike the inventory position, the price-deflated inventory position is defined as a weighted sum of the net inventory and pipeline inventory, where the weights are related to the sensitivity of the myopic price to the initial inventory level. This new state variable assigns more weight to the inventory that is farther away from the system. We then prove that a base-stock policy is optimal in this approximated system.

Based on these results, we propose a heuristic policy that decouples the pricing and inventory decisions. For the pricing policy, we use the myopic price that depends on the initial inventory level. For the inventory policy, a base-stock policy is implemented. That is, at the beginning of each period, the system places an order to bring the price-deflated inventory position to the target base-stock level if the price-deflated inventory position is lower than the base-stock level, and it does not order, otherwise. To test the effectiveness of our heuristic for long lead times, we develop a theoretical upper bound to the optimal profit in the exact system. The upper bound is based on the information-relaxation approach proposed by Brown et al. (2010). One key enabler of their method is the design of a problem-specific penalty function that effectively compensates for the relaxed information set and that allows for a tractable computation of the upper bound. We construct the penalty function based on our proposed heuristic and show how to efficiently solve the resulting upper bound problem. We also discuss the application of our heuristic to more general inventory-pricing models, including a model with fixed ordering costs, a model with fixed batch sizes, the infinite-horizon model, and a model with Markov-modulated demand. After introducing the myopic price and the linear-demand approximation in these settings, the resulting problems have known optimal policies which can be used as the heuristic policies for inventory replenishment. In all

[2] http://www.amazon.com/gp/product/B0001NEAD4 (last accessed November 4, 2015).

[3] http://www.amazon.com/Red-Verjus-Fusion-750-ml/dp/B0029AFEHI (last accessed November 4, 2015).

settings, the ordering decision is based on the price-deflated inventory position.

We examine the effectiveness of our heuristic in an extensive numerical study. For systems with a lead time of up to three periods, we compare the exact optimal profit to the heuristic profit and find that the average percentage error is 0.54% for the case of additive demand and 0.70% for the case of multiplicative demand. These results suggest that the heuristic policy is near-optimal. (For the same instances, the percentage error between the optimal profit and the upper bound is 1.60% for additive demand and 1.81% for multiplicative demand.) For instances with longer lead times, we compare the heuristic profit with that of the upper bound. When applied to a system with fixed ordering costs, the proposed heuristic is also very effective: For a study involving instances with lead times of up to three periods, the average percentage error compared to the optimal expected profit is 0.54% for additive demand and 1.02% for multiplicative demand. In settings with zero lead time, Federgruen and Heching (1999) characterize the optimal policy when there are no fixed ordering costs and Chen and Simchi-Levi (2004a, b) characterize the optimal policy for systems with fixed order costs (see §2 for further details on these policies). Our heuristic delivers close-to-optimal decisions in systems with zero lead time as well. From a computational perspective, obtaining the exact optimal policy in settings with zero lead time requires solving a dynamic program with two decision variables whereas our heuristic involves a single-period problem for the price and a single-variable dynamic program for the inventory policy.

The contribution of this study is threefold. First, the heuristic approach is innovative and widely applicable. The myopic pricing rule and subsequent linear approximation of the myopic demand function reduce the optimization to a standard inventory problem, with the ordering decision based on the price-deflated inventory position. The approach can be applied to many inventory systems of interest. For example, we present a brief case study applied to a large distributor of components in the aerospace industry—the company replenishes inventory using fixed batches. The second contribution is computational. As stated, the proposed heuristic transforms the original problem with L states into a traditional inventory problem with a single-dimensional state variable (the price-deflated inventory position). To our knowledge, this is the first heuristic that generates a simple and effective solution for a joint inventory and pricing problem with a positive lead time. Finally, the heuristic allows us to derive insights regarding the impact of lead time on pricing decisions. As reported in the literature (see, e.g., Federgruen and Heching 1999, Chen et al. 2010,

Feng and Chen 2011), the benefit of dynamic pricing in systems with zero lead time is limited. In contrast, when lead time is positive, the system is less responsive to demand through its replenishment process and therefore the value of dynamic pricing can be significant. This is even more so when there are fixed ordering costs. We find that a shorter lead time induces a more stable pricing policy. Intuitively, when lead time is shorter, it is easier to predict future demand and therefore control the inventory available at the beginning of each period. This translates into a more stable pricing policy. Our findings suggest that increased responsiveness (e.g., through a reduction in the procurement/production lead time) reduces the need to adjust demand through prices in order to balance supply and demand. In settings with a positive lead time, we also find that price discounts may be offered when there is a significant amount of inventory in transit, even if the on-hand inventory is relatively low. Thus, pipeline information can be of critical value to pricing decisions.

2. Literature Review

The study of joint inventory and pricing problems can be traced back to Whitin (1955) and Mills (1959), who study this problem in a single-period model. Petruzzi and Dada (1999) provide a comprehensive review on this stream of literature. In a multiperiod setting, Federgruen and Heching (1999) study a single-stage system with zero lead time. The authors prove that a so-called list-price-base-stock policy is optimal for both the nonstationary finite-horizon and stationary infinite-horizon problems. This policy involves two parameters—a list price and a base-stock level. When the inventory level at the beginning of a period is below the base-stock level, an order is placed to reach the target level and the price is set equal to the list-price. Otherwise, no order is placed and price is discounted so that a higher initial inventory level leads to a deeper discount. The authors also suggest a heuristic for systems with a positive lead time and additive demand. In their heuristic, the price is fixed within the lead time, which allows them to reduce the state space of the problem. Chen and Simchi-Levi (2004a, b) study a similar single-stage, multiperiod model with fixed ordering costs. The authors prove that an (s, S, p) policy and an (s, S, A, p) policy are optimal under additive demand and a general form of demand (involving both a multiplicative and an additive demand term), respectively. Chen et al. (2006) and Song et al. (2009) extend the optimality of the (s, S, p) and (s, S, A, p) policies to lost-sales models with additive and multiplicative demands, respectively, under mild assumptions on the demand function and randomness. Huh and Janakiraman (2008) consider a

multidimensional demand control approach in both backorder and lost-sales models. Chen et al. (2010) study optimal pricing and replenishment decisions in a system with fixed costs and zero lead time. Demand follows a Brownian motion with mean depending on the price. The authors characterize conditions under which dynamic pricing is most effective. Chen and Iyengar (2012) propose an approach to generate policies for the joint inventory and pricing problem, which involves solving a collection of linear programs. However, it only applies to systems with zero lead time. We refer to Chen and Simchi-Levi (2012) for a recent review of this stream of work.

The literature studying joint dynamic pricing and replenishment decisions with a positive lead time is scarce. Pang et al. (2012) study this problem under additive demand. The authors show that the problem is $L^{\natural}$-concave, which guarantees the existence of a state-dependent optimal policy. However, they do not characterize the structure of the optimal policy. From a computational perspective, it remains difficult to obtain the global optimal solution due to the high dimensionality of the state space. Nevertheless, they show that the initial inventory level has a larger impact on the optimal pricing decision than the inventory in the pipeline. This finding supports the use of a myopic pricing policy in our heuristic. Yu (2012) discusses another approach to solve this problem for the case of additive demand, with additional conditions on the ordering behavior.

Another related stream of research considers joint inventory and pricing decisions with uncertain supply. Li and Zheng (2006) study a setting with stochastic production yield and demand. Feng (2010) considers a model with uncertain capacity supply. In both papers, the demand function is additive and the optimal replenishment policy is of a threshold type, i.e., a positive amount is ordered only when the inventory level is below a critical point (for comprehensive reviews of the literature on dynamic pricing and inventory control, see Yano and Gilbert 2003, Elmaghraby and Keskinocak 2003, Chan et al. 2004).

In the inventory management literature, it is fairly common to examine the effectiveness of a heuristic by comparing its performance with that of a theoretical bound. For example, Chen and Zheng (1998) compare a heuristic (r, nQ) policy with a lower bound to the optimal cost for serial systems with fixed order costs. Shang and Zhou (2010) employ the same approach to verify the effectiveness of a heuristic (s, T) policy for a serial system. In this paper, we assess the effectiveness of the proposed heuristic by comparing its resulting profit to that of an upper bound. To our knowledge, this is the first theoretical bound developed for the dynamic joint pricing and inventory problem.

The remainder of this paper is organized as follows. In §3, we describe the basic model and provide preliminary results. We then introduce the heuristic in §4 and the upper bound in §5. Section 6 discusses the application of the heuristic to other inventory models. Section 7 reports a numerical study to examine the effectiveness of the heuristic and to derive insights on the role of dynamic pricing in settings with a positive lead time. All proofs are relegated to Online Appendix A (online appendices available as supplemental material at http://dx.doi.org/10.1287/mnsc.2015.2227). Additional results are provided in Online Appendix B.

3. Model and Preliminaries

We consider a single-stage system with a positive delivery lead time of L periods. The net inventory level at the beginning of period t before replenishment is x_t and the pipeline inventory is represented by a vector $\mathbf{w_t} = (w_{1,t}, \ldots, w_{L-1,t})$, where $w_{l,t}$ denotes the replenishment quantity to be delivered in l periods, $l = 1, \ldots, L-1$. Let α denote the discount factor. At the beginning of each period t, the replenishment quantity q_t and the selling price p_t are determined simultaneously in order to maximize the total expected discounted profit through the end of a finite horizon with T periods. Next, stochastic demand is realized and unfulfilled demand at the end of each period is fully backlogged.[4] Demand in each period depends on the current period's price p_t. Price is allowed to change bidirectionally (decrease or increase) from period to period.

We consider both additive and multiplicative demand forms.

Additive demand: $D_t(p_t, \epsilon_t) = \mathscr{D}_t(p_t) + \epsilon_t$, where ϵ_t is a random variable with $\mathsf{E}[\epsilon_t] = 0$.

Multiplicative demand: $D_t(p_t, \epsilon_t) = \mathscr{D}_t(p_t)\epsilon_t$, where ϵ_t is a nonnegative random variable with $\mathsf{E}[\epsilon_t] = 1$.

We assume that $\mathscr{D}_t(p_t)$ is strictly decreasing in p_t and the inverse demand function $\mathscr{D}_t^{-1}(d_t) \stackrel{\triangle}{=} p_t(d_t)$ exists, where d_t represents the expected demand. The problem can be equivalently formulated as selecting the optimal replenishment quantity q_t and the expected demand d_t in each period t (as opposed to the order quantity and the price p_t). To that end, we

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[4] In a traditional inventory setting (with exogenous demand), the backorder model closely approximates the corresponding lost-sales system when the service level is high. Huh et al. (2009) show that the base-stock policy is asymptotically close to the optimal policy in the lost-sales model when the backorder cost increases to infinity. High service levels are common not only in retailing (Nagarajan and Rajagopalan 2009), but also for manufacturers and distributors (Bossert and Willems 2007, Humair et al. 2013).

define the demand function in terms of the expected demand d_t and the random variable ϵ_t:

$$D_t(d_t, \epsilon_t) = \begin{cases} d_t + \epsilon_t & \text{additive demand,} \\ d_t \epsilon_t & \text{multiplicative demand.} \end{cases}$$

The feasible action space for the expected demand d_t is given by Ω_t.

Let c_t denote the unit purchasing cost, h_t the unit holding cost, and b_t the unit backorder cost in period t. We assume that $\alpha c_{t+1} \geq h_t$. The retailer pays a total purchasing cost $c_t q_t$ and incurs an inventory-related cost given by

$$G_t(x_t, d_t) \triangleq \mathsf{E}\big[h_t(x_t - D_t(d_t, \epsilon_t))^+ + b_t(D_t(d_t, \epsilon_t) - x_t)^+\big]$$

in each period t.

ASSUMPTION 1. *The revenue collected in each period depends on the demand volume instead of the sales amount, i.e., consumers pay upon arrival when backorders happen.*

This is a standard assumption in the entire literature on joint pricing and inventory decisions. Based on this assumption, the expected revenue in period t only depends on the expected demand volume d_t, i.e.,

$$R_t(d_t) \triangleq \mathsf{E}[p_t D_t(p_t, \epsilon_t)] = \mathsf{E}[\mathscr{D}_t^{-1}(d_t) D_t(d_t, \epsilon_t)]$$
$$= p_t(d_t) \mathsf{E}[D_t(p_t, \epsilon_t)] = p_t(d_t) d_t.$$

Furthermore, we make the following assumption regarding the shape of $R_t(d_t)$.

ASSUMPTION 2. *$R_t(d_t)$ is concave in d_t with*

$$\lim_{d_t \to 0^+} R'_t(d_t) > 0 \quad and \quad \lim_{d_t \to \infty} R'_t(d_t) \leq 0.$$

Assumption 2 is quite general and is satisfied by the demand functions considered in the literature. We refer to Huang et al. (2013, Table 1) for a list of commonly used price-dependent demand functions.

We index time counting forward, i.e., $t = 1$ represents the beginning of the planning horizon and $t = T$ represents the last period; $t = T + 1$ represents the end of the planning horizon. The sequence of events in each period t is as follows: (1) If $t \geq L + 1$, then the replenishment order placed L periods ago, namely q_{t-L}, is delivered; (2) the expected demand volume d_t (or the selling price p_t) and the replenishment quantity q_t are determined simultaneously; (3) demand occurs; (4) cost and revenue are calculated at the end of the period.

Let $V_t(x_t, \mathbf{w_t})$ denote the maximum expected discounted profit from period t until the end of the planning horizon with initial state vector $(x_t, \mathbf{w_t})$. We assume that, at the end of the horizon, the on-hand inventory $(x_{T+1})^+$ can be salvaged at the purchase cost c_{T+1} or that any backorder $(x_{T+1})^-$ has to be filled from

an external source at a cost c_{T+1}. In other words, the terminal value function is $V_{T+1}(x_{T+1}, \mathbf{w_t}) = c_{T+1} x_{T+1}$. (This terminal value function is commonly used in the inventory literature.) The optimal value function can be expressed as follows: For $t = 1, 2, \ldots, T$,

$$\begin{aligned} V_t(x_t, \mathbf{w_t}) &= \max_{q_t \geq 0,\, d_t \in \Omega_t} \big\{ R_t(d_t) - c_t q_t - G_t(x_t, d_t) \\ &\quad + \alpha \mathsf{E} V_{t+1}(x_{t+1}, \mathbf{w_{t+1}}) \big\} \\ &= \max_{q_t \geq 0,\, d_t \in \Omega_t} \big\{ J_t(x_t, \mathbf{w_t}, d_t, q_t) \big\}, \end{aligned} \tag{1}$$

where the state dynamics are $(x_{t+1}, \mathbf{w_{t+1}}) = (x_t + w_{1,t} - D_t(d_t, \epsilon_t)), w_{2,t}, \ldots, w_{L-1,t}, q_t)$.

PROPOSITION 1. *Under both the additive and multiplicative demand forms, $V_t(x_t, \mathbf{w_t})$ is jointly concave in $(x_t, \mathbf{w_t})$ and $J_t(x_t, \mathbf{w_t}, d_t, q_t)$ is jointly concave in $(x_t, \mathbf{w_t}, d_t, q_t)$ for all t.*[5]

From Proposition 1, the optimal joint decisions, denoted by (d_t^*, q_t^*), can be obtained from the first order conditions of $J_t(x_t, \mathbf{w_t}, d_t, q_t)$. The optimal decisions in each period depend on the L-dimensional state vector $(x_t, \mathbf{w_t})$, i.e., $(d_t^*, q_t^*) = (d_t^*(x_t, \mathbf{w_t}), q_t^*(x_t, \mathbf{w_t}))$. Computing this state-dependent optimal policy is computationally infeasible due to the curse of dimensionality. This is in contrast to the traditional inventory problem with exogenous demand, in which the inventory states (pipeline inventory and on-hand inventory) can be aggregated into a single inventory state variable—the inventory position. Thus, the objective of this paper is to provide a simple heuristic procedure for the joint pricing and replenishment problem with a positive lead time.

4. Heuristic
The development of the heuristic consists of three steps. In §4.1, we introduce the myopic expected demand policy and characterize its properties. We then construct a linear approximation to the myopic expected demand in §4.2. In §4.3, we study the remaining inventory problem in which demand is replaced by the linear approximation.

4.1. Myopic Expected Demand
The dynamic program in (1) is equivalent to the following recursion: Let $\bar{V}_{T+1}(x_{T+1}, \mathbf{w_{T+1}}) = 0$. For $t = 1, 2, \ldots, T$,

$$\begin{aligned} &\bar{V}_t(x_t, \mathbf{w_t}) \\ &= \max_{q_t \geq 0,\, d_t \in \Omega_t} \big\{ R_t(d_t) - c_t(q_t + x_t) \\ &\quad + \alpha \mathsf{E}[c_{t+1}(x_t + w_{1,t} - D_t(d_t, \epsilon_t))] \\ &\quad - G_t(x_t, d_t) + \alpha \mathsf{E}\bar{V}_{t+1}(x_{t+1}, \mathbf{w_{t+1}}) \big\}. \end{aligned} \tag{2}$$

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[5] Pang et al. (2012) show the concavity of the exact problem for the additive demand case. Here, we show the result for both demand forms.

From the transformed dynamic program in (2), the single-period profit is

$$R_t(d_t) - c_t(q_t + x_t) + \alpha c_{t+1}(x_t + w_{1,t} - d_t) - G_t(x_t, d_t).$$

Let $d_t^M(x_t)$ be the demand that maximizes the single-period profit. Because $-c_t(q_t + x_t)$ and $\alpha c_{t+1}(x_t + w_{1,t})$ do not affect the demand decision, $d_t^M(x_t)$ is the solution to the following problem:

$$\max_{d_t \in \Omega_t} \left\{ R_t(d_t) - G_t(x_t, d_t) - \alpha c_{t+1} d_t \right\}.$$

We refer to the resulting solution as the myopic demand policy, which is a function of the net inventory level x_t only, ignoring the effect of pipeline inventory. Pang et al. (2012) show that the initial inventory level has the largest impact on the optimal price for the case of additive demand. In addition, Proposition 5 in Online Appendix B shows that the myopic demand policy shares a similar structure to that of the exact optimal policy. This result provides support for the use of the myopic policy to approximate the optimal pricing policy.

Note that the myopic demand decision accounts for the cost implications of this decision in future periods. More precisely, the price decision in period t impacts the corresponding demand in that period. The system will in turn need to replenish the inventory consumed by demand in period t. The cost of doing so is reflected by the term $\alpha c_{t+1} d_t$ in the single-period profit function. Thus, the myopic demand function incorporates the impact of future purchase costs on the current period's pricing decision.

We next derive some properties of the myopic demand function. Define $f^{-1}(\cdot)$ as the inverse function of $f(x) = a$ such that $f^{-1}(a) = x$.

PROPOSITION 2. (i) *The myopic expected demand $d_t^M(x_t)$ is nondecreasing in x_t;*
(ii) $d_t^- \le d_t^M(x_t) \le d_t^+$, *where*

$$d_t^- = \max\{0, R_t'^{-1}(b_t + \alpha c_{t+1})\}$$

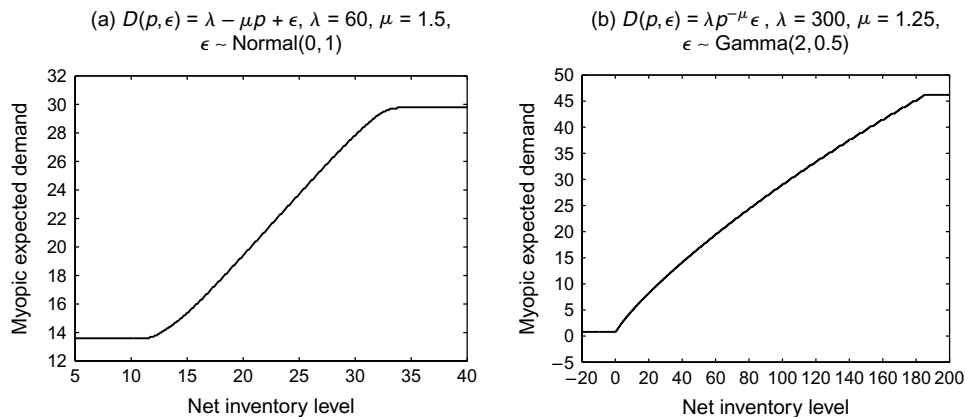*and* $d_t^+ = R_t'^{-1}(-h_t + \alpha c_{t+1})$.

Figure 2 illustrates the shape of the myopic expected demand function for both the additive and the multiplicative demand forms. Figure 2(a) shows the case of a linear mean demand $\mathcal{D}(p) = \lambda - \mu p$ under the additive form, whereas Figure 2(b) illustrates the case of an iso-elastic mean demand $\mathcal{D}(p) = \lambda p^{-\mu}$ under the multiplicative form. A similar shape of the myopic expected demand function applies to other mean demand functions satisfying Assumption 2.

As shown in Figure 2, the myopic demand function has an asymptotic slope of zero when x is either small or large and it is strictly increasing in x when x falls in an intermediate range of initial inventory levels. Intuitively, when x increases, the system chooses a higher expected demand to reduce the inventory holding cost. However, due to the concavity of the revenue function, a higher expected demand leads to a reduction of marginal revenue. Thus, when x is sufficiently large, the relative marginal revenue reduction that results from an increase of the expected demand outweighs the benefit in the associated inventory cost reduction. Consequently, the expected demand curve is flat when the initial inventory level x is large. A similar logic applies when x takes small values.

4.2. Linear Approximation

Under the myopic pricing policy, we transform the original problem into an inventory problem where prices depend on inventory through the myopic expected demand functions. Despite this simplification, the state of the resulting inventory problem continues to be L-dimensional. To resolve this issue, we propose a linear approximation $\tilde{d}_t(x_t)$ for the myopic demand $d_t^M(x_t)$, where

$$\tilde{d}_t(x_t) = \delta_t x_t + \kappa_t.$$

We next discuss the derivation of the parameters δ_t and κ_t. For simplicity, we omit the time index t in the remainder of this section.

Figure 2   Myopic Expected Demand, $c = 2$, $h = 1$, $b = 20$

(a) $D(p, \epsilon) = \lambda - \mu p + \epsilon$, $\lambda = 60$, $\mu = 1.5$, $\epsilon \sim$ Normal$(0, 1)$

(b) $D(p, \epsilon) = \lambda p^{-\mu} \epsilon$, $\lambda = 300$, $\mu = 1.25$, $\epsilon \sim$ Gamma$(2, 0.5)$

The myopic demand function is fairly flat when x is either small or large but tends to be linearly increasing when x falls in an interval of intermediate values. As a result, we define two points, $(x^-, d^M(x^-))$ and $(x^+, d^M(x^+))$, to construct the linear function that approximates the myopic expected demand for an intermediate range of inventory levels. To that end, let x^- be the solution to $d^M(x) = \lceil d^- \rceil$ and x^+ the solution to $d^M(x) = \lfloor d^+ \rfloor$, where d^- and d^+ are defined in Proposition 2 and $\lceil x \rceil$ (resp., $\lfloor x \rfloor$) is the smallest (resp., largest) integer greater (resp., lower) than or equal to x. Below we introduce the linear approximation for the cases of additive and multiplicative demand separately, and derive properties of the linear approximation in each case.

PROPOSITION 3 (ADDITIVE DEMAND). *Define* $\hat{x} = (x^+ + x^-)/2$. *Let the parameters of the linear approximation be* $\delta = d^{M'}(\hat{x})$ *and* $\kappa = -\delta\hat{x} + d^M(\hat{x})$. *(i) For any* x_1, $x_2 \in [x^-, x^+]$, *we have that* $|d^{M'}(x_1) - d^{M'}(x_2)| \to 0$ *as* $b \to \infty$. *(ii) The slope of* $\tilde{d}(x)$ *is positive with* $0 < \delta < 1$.

Proposition 3(i) shows that the myopic demand function $d^M(x)$ is approximately linear in the interval $[x^-, x^+]$ for large values of the backorder cost (and, therefore, for relatively high service levels). Proposition 3(ii) shows that the slope of this linear demand approximation is less than one. We later use this property to interpret the heuristic procedure.

PROPOSITION 4 (MULTIPLICATIVE DEMAND). *Let* x^0 *be the largest intersection point between the diagonal line* $y = x$ *and the function* $d^M(x)$. *Define* $\hat{x} = \max\{(x^+ + x^-)/2, x^0\}$. *Let the parameters of the linear approximation be* $\delta = d^{M'}(\hat{x})$ *and* $\kappa = -\delta\hat{x} + d^M(\hat{x})$. *Then, the slope of* $\tilde{d}(x)$ *is positive with* $0 < \delta < 1$ *if the following conditions hold:* *(1)* $uf(u)$ *decreases in* u; *and (2) the revenue function* R *satisfies* $R''' \geq 0$.

Conditions (1) and (2) are satisfied, for example, by the Gamma distribution and the iso-elastic mean demand function, respectively. In our numerical experiments, we generally observe that $\delta < 1$ for other distributions and mean demand functions as well. Figures 3(a) and (b) illustrate the linear approximation under the additive and multiplicative demand forms, respectively.

4.3. State Space Reduction and Heuristic Policy
In this section, we describe how to determine the heuristic replenishment and pricing policy. We first substitute each period's expected demand by the linear approximation $\tilde{d}_t(x_t) = \delta_t x_t + \kappa_t$ in the dynamic program (1). Since $\tilde{d}_t(x_t)$ can take negative values, we extend the definition of the revenue function to ensure that the problem is concave. When the price function is defined for negative values, we just let $\tilde{R}_t(d_t) = d_t p_t(d_t)$, $d_t \in (-\infty, +\infty)$. Otherwise, we let

$$\tilde{R}_t(d_t) = \begin{cases} d_t p_t(d_t) & \text{if } d_t \geq \xi, \\ d_t R_t'(\xi) & \text{if } d_t < \xi, \end{cases}$$

where ξ is a small positive value. In either case, $\tilde{R}_t(d_t)$ is concave in d_t.

Before proceeding with the dynamic programming recursion for the optimal inventory policy under the linear approximation, we define *new* system variables:

$\bar{x}_t$ = price-deflated inventory position at the beginning of period t before ordering

$$= \nu_{0,t} x_t + \sum_{l=1}^{L-1} \nu_{l,t}(w_{l,t} - \kappa_{t+l-1}) - \kappa_{t+L-1}, \quad (3)$$

where $\nu_{l,t} = \prod_{k=l}^{L-1}(1 - \delta_{t+k})$, $l = 0, 1, \dots, L-1$. (For the case of multiplicative demand, $\nu_{l,t} = \prod_{k=l}^{L-1}(1 - \delta_{t+k}\epsilon_{t+k})$, so we require an additional approximation

Figure 3   **Linear Approximation,** $c = 2$, $h = 1$, $b = 20$

(a) $D(p, \epsilon) = \lambda - \mu p + \epsilon$, $\lambda = 60$, $\mu = 1.5$, $\epsilon \sim \text{Normal}(0, 1)$

(b) $D(p, \epsilon) = \lambda p^{-\mu}\epsilon$, $\lambda = 300$, $\mu = 1.25$, $\epsilon \sim \text{Gamma}(2, 0.5)$
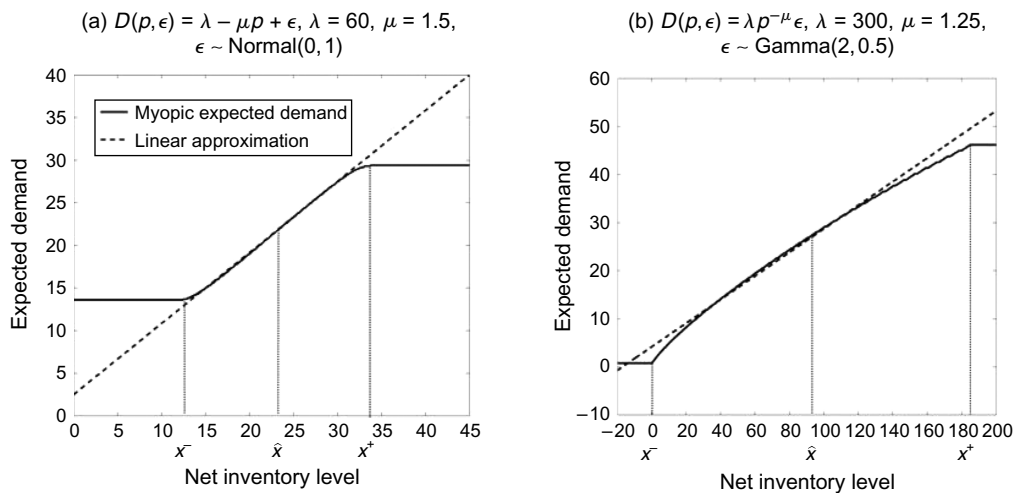
by setting $\epsilon_{t+k} = 1$, the mean value of these random coefficients.) In addition,

$$\bar{y}_t = \bar{x}_t + q_t,$$

$\epsilon[t, t+L) = $ total weighted errors in periods t, $t+1, \ldots, t+L-1$,

$$= \begin{cases} \displaystyle\sum_{l=0}^{L-1} \nu_{l+1,t} \epsilon_{t+l} & \text{additive demand,} \\[2em] \displaystyle\sum_{l=0}^{L-1} \nu_{l+1,t} \kappa_{t+l}(\epsilon_{t+l} - 1) & \\ & \text{multiplicative demand.} \end{cases}$$

The optimal inventory decision in the resulting approximate dynamic program can be obtained from the following recursion (the details of this derivation are provided in the proof of Theorem 1): $\widetilde{V}_{T-L+1}(\bar{x}_{T-L+1}) = \alpha^L c_{T+1} \bar{x}_{T-L+1}$, and

$$\tilde{V}_t(\bar{x}_t) = c_t \bar{x}_t + \max_{\bar{y}_t \geq \bar{x}_t} \tilde{J}_t(\bar{y}_t), \qquad (4)$$

where

$$\tilde{J}_t(\bar{y}_t) = \mathsf{E}\big[\alpha^L \widetilde{R}_{t+L}(\tilde{d}_{t+L}(x_{t+L})) - c_t \bar{y}_t$$
$$- \alpha^L G_{t+L}(x_{t+L}, \tilde{d}_{t+L}(x_{t+L})) + \alpha \tilde{V}_{t+1}(\bar{x}_{t+1})\big],$$
$$\bar{x}_{t+1} = (1 - \delta_{t+1})(\bar{y}_t - \epsilon[t, t+1)) - \kappa_{t+L},$$
$$x_{t+L} = \bar{x}_t + q_t - \epsilon[t, t+L), \quad t = 1, \ldots, T-L.$$

We next derive the optimal policy for the transformed inventory problem in (4), obtained by implementing the linear approximation of the myopic demand function.

Theorem 1. *The value function $\tilde{V}_t(\bar{x}_t)$ in (4) is concave in $\bar{x}_t$ and $\tilde{J}_t(\bar{y}_t)$ is concave in $\bar{y}_t$ for all t. Let $\bar{s}_t = \arg\max \tilde{J}_t(\bar{y}_t)$. The optimal inventory policy for the problem in (4) is a base-stock policy with parameters $\{\bar{s}_t\}_{t=1}^T$.*

The joint inventory and pricing heuristic is as follows. In each period t, given the state $(x_t, \mathbf{w}_t)$:

(i) Set the price equal to the myopic price, i.e., $p_t^M = p_t(d_t^M(x_t))$;

(ii) Calculate $\bar{x}_t$ as in (3) and let $\bar{s}_t = \arg\max \tilde{J}_t(\bar{y}_t)$. An order quantity $q_t = \bar{s}_t - \bar{x}_t$ is placed if $\bar{x}_t \leq \bar{s}_t$, and no order is placed otherwise.

We now discuss the physical meaning of $\bar{x}_t$. Demand in each lead-time period is composed of two terms, controllable and noncontrollable. The controllable term refers to the demand determined by the pricing decisions, while the noncontrollable part refers to the remaining random terms. The value of $\bar{x}_t$, given in (3), can be interpreted as the expected inventory position at the beginning of period $t+L$, which is equal to the sum of the net inventory x_t and all the pipeline inventory terms $w_{l,t}$ minus the total controllable demand during the lead time. It follows from

Figure 4   Heuristic Replenishment Decision
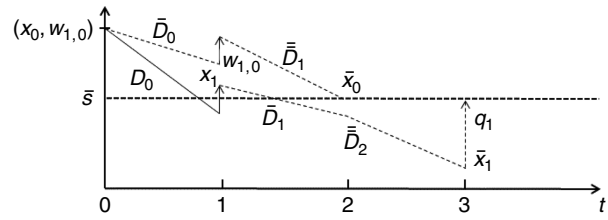
the property of the slope δ_t that the weights $\nu_{l,t}$ in $\bar{x}_t$ satisfy $\nu_{0,t} < \nu_{1,t} < \cdots < \nu_{L-1,t}$. This suggests that the price-deflated inventory position assigns a lower weight to the inventory that is closer to the system. To see this, consider the inventory state at the beginning of period t, $(x_t, \mathbf{w}_t)$. Since x_t is present in the system before the arrival of the pipeline inventory $\mathbf{w}_t$, the amount x_t can be used to satisfy the entire controllable demand over the lead time. In contrast, the pipeline inventory will be used to satisfy only a portion of that lead time demand. Thus, the proportion of x_t that is expected to be available at the end of the lead time will be smaller than that of the pipeline inventory. This implies that the weights of the pipeline inventory in $\bar{x}_t$ are progressively larger.

The order quantity q_t that arises from the optimal policy in Theorem 1 equals one period of future controllable demand plus the sum of past forecasting errors during the lead time. Figure 4 illustrates this ordering behavior for a system with stationary parameters and $L = 2$. We assume that the system starts with $\bar{s}_0 = \bar{x}_0$ so no order is placed in period 0. At the beginning of period 0, the manager observes x_0 and $w_{1,0}$ and anticipates demand according to the linear approximation $\tilde{d}_0(x_0)$. Let $\bar{D}_0 = \tilde{d}_0(x_0)$. Based on x_0, $w_{1,0}$, and $\bar{D}_0$, the manager can further anticipate the net inventory level at the beginning of period 1, given by $x_0 + w_{1,0} - \bar{D}_0$, and therefore anticipate a demand quantity $\bar{D}_1 = \tilde{d}_1(x_0 + w_{1,0} - \bar{D}_0)$ in period 1. One period later, demand D_0 is realized, so the manager can update $\bar{D}_1$ by a new estimate $\bar{\bar{D}}_1 = \tilde{d}_1(x_0 + w_{1,0} - D_0)$ that accounts for the actual initial net inventory level in period 1. At that point, the manager also estimates the demand in period 2 based on the linear approximation of demand in that period. It follows that $q_1 = \bar{\bar{D}}_2 + (D_0 - \bar{D}_0) + (\bar{D}_1 - \bar{\bar{D}}_1)$. That is, the order quantity in each period equals the estimate of next-period's demand (through the linear approximation of demand in that period) plus a sum of correction terms of past demand estimations given the updated demand realized in the latest period.

5.   Upper Bound

In this section, we derive an upper bound to the optimal expected profit of the exact problem. The

idea behind the upper bound is based on the duality approach proposed in Brown et al. (2010). The construction of the upper bound involves relaxing the information set (i.e., the information available to the manager) and imposing a penalty cost to compensate for the information relaxation. A key feature of Brown et al. framework is that it requires the development of a problem-specific penalty cost that closes the gap created by the relaxed information set and that results in a problem that can be solved efficiently. We next propose a penalty function for our problem and show how to solve for the resulting upper bound.

First, we relax the information set by considering the joint pricing and replenishment problem under deterministic demand, i.e., the manager knows the demand sample path when making the pricing and ordering decision. We define the sample-path dependent counterparts of the functions introduced in §3. In particular, given a sample path $\{\epsilon\}_1^T$, let $R_t(d_t \mid \epsilon_t) = p_t(d_t)D_t(d_t, \epsilon_t)$ and $G_t(x_t, d_t \mid \epsilon_t) = h_t(x_t - D_t(d_t, \epsilon_t))^+ + b_t(D_t(d_t, \epsilon_t) - x_t)^+$. Define the recursion $V_{T+1} = c_{T+1}x_{T+1}$, and

$$
\begin{aligned}
V_t&(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) \\
&= \max_{q_t \geq 0, d_t \in \Omega_t} \big\{ R_t(d_t \mid \epsilon_t) - c_t q_t - G_t(x_t, d_t \mid \epsilon_t) \\
&\qquad\qquad + \alpha V_{t+1}(x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \big\} \\
&= \max_{q_t \geq 0, d_t \in \Omega_t} J(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T),
\end{aligned}
$$

where $(x_{t+1}, \mathbf{w_{t+1}}) = (x_t + w_{1,t} - D_t(d_t, \epsilon_t), w_{2,t}, \ldots, w_{L-1,t}, q_t)$. Due to Jensen's inequality and the concavity of the value function, $\mathsf{E}V_t(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) \geq V_t(x_t, \mathbf{w_t})$ for $t = 1, \ldots, T+1$. The expectation on the left-hand side is the average value of $V_t(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T)$ over all sample paths. Intuitively, a higher profit is achieved if demand is revealed before making the pricing and ordering decisions.

Proposition 2.2 in Brown et al. (2010) shows how to generate effective penalties for the upper bound. Following their result, we construct the penalty cost function for each period t and each sample path $\{\epsilon\}_t^T$ as a difference of the form

$$
W_t(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T) - W_t(x_t, \mathbf{w_t}, q_t, d_t),
$$

for a suitable function W_t. The upper bound is then obtained by taking the average over all sample paths of

$$
\begin{aligned}
V_t^D&(x_t, \mathbf{w_t} \mid \{\epsilon\}_t^T) \\
&= \max_{q_t \geq 0, d_t \in \Omega_t} \big\{ R_t(d_t, \epsilon_t) - c_t q_t - G_t(x_t, d_t, \epsilon_t) \\
&\qquad - [W_t(x_t, \mathbf{w_t}, q_t, d_t \mid \{\epsilon\}_t^T) - W_t(x_t, \mathbf{w_t}, q_t, d_t)] \\
&\qquad\qquad + \alpha V_{t+1}^D(x_{t+1}, \mathbf{w_{t+1}} \mid \{\epsilon\}_{t+1}^T) \big\}, \quad (5)
\end{aligned}
$$

where $(x_{t+1}, \mathbf{w_{t+1}}) = (x_t + w_{1,t} - D_t(d_t, \epsilon_t), w_{2,t}, \ldots, w_{L-1,t}, q_t)$. (The superscript D refers to the deterministic value function that includes the penalty cost.)

The penalty function needs to capture as much of the value of demand information as possible. To that end, we use the value function obtained from our heuristic in (4), with the appropriate corresponding definitions to denote the dependence on a given sample path. That is, the penalty function is constructed using $\tilde{J}_t(\bar{y}_t \mid \{\epsilon\}_t^T)$ and $\tilde{J}_t(\bar{y}_t)$, where $\tilde{J}_t(\bar{y}_t \mid \{\epsilon\}_t^T)$ is the sample-path dependent counterpart of $\tilde{J}_t(\bar{y}_t)$, and is defined as

$$
\begin{aligned}
\tilde{J}_t(\bar{y}_t \mid \{\epsilon\}_t^T) &= \alpha^L \tilde{R}_{t+L}(\tilde{d}_{t+L}(x_{t+L}) \mid \epsilon_{t+L}) - c_t \bar{y}_t \\
&\quad - \alpha^L G_{t+L}(x_{t+L}, \tilde{d}_{t+L}(x_{t+L}) \mid \epsilon_{t+L}) \\
&\quad + \alpha \tilde{V}_{t+1}(\bar{x}_{t+1} \mid \{\epsilon\}_{t+1}^T).
\end{aligned}
$$

Because each of these functions is concave, their difference is not necessarily concave. We therefore further approximate $\tilde{J}_t(\bar{y}_t \mid \{\epsilon\}_t^T)$ and $\tilde{J}_t(\bar{y}_t)$ by their first-order Taylor expansions around $\bar{s}_t = \arg\max \tilde{J}_t(\bar{y}_t)$. More specifically, let

$$
\begin{aligned}
W_t(\bar{x}_t, q_t \mid \{\epsilon\}_t^T) &= \tilde{J}_t(\bar{s}_t \mid \{\epsilon\}_t^T) + \frac{\partial \tilde{J}_t(\bar{y}_t \mid \{\epsilon\}_t^T)}{\partial \bar{y}_t}\bigg|_{\bar{y}_t = \bar{s}_t} (\bar{y}_t - \bar{s}_t), \\
W_t(\bar{x}_t, q_t) &= \tilde{J}_t(\bar{s}_t). \quad (6)
\end{aligned}
$$

The penalty function depends on the vector $(\bar{x}_t, q_t)$ as it uses the myopic price for the pricing decision and the price-deflated inventory position $\bar{x}_t$ to determine the order quantity q_t.

THEOREM 2. *The average value of $V_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_t^T)$ taken over all sample paths is an upper bound to the optimal profit $V_1(x_1, \mathbf{w_1})$ of the joint pricing and replenishment problem given in* (1).

Calculating the optimal value $V_1^D(x_1, \mathbf{w_1} \mid \{\epsilon\}_1^T)$ involves solving a multistate dynamic program. This problem is, in fact, related to the dynamic lot-sizing problem studied in Wagner and Whitin (1958). We refer to Appendix B for details of an algorithm to calculate the upper bound.

6. Application to Other Inventory Models

In §§3 and 4, we developed a heuristic for the joint pricing and replenishment problem with a positive lead time. The heuristic is based on a myopic pricing rule and a linear approximation of the myopic demand function, which transforms the original problem into a standard inventory problem with a modified—price-deflated—inventory position. This approach can be applied to more general settings with different cost structures and time horizon. Using

the linear approximation introduced earlier, one can transform the dynamic inventory and pricing problem into a traditional inventory problem for which the optimal policy is known. This optimal inventory policy serves as the heuristic replenishment policy.

Fixed Ordering Costs. Chen and Simchi-Levi (2004a, b) show that an (s, S, A, p) policy is optimal in a setting with fixed ordering costs and zero lead time. Under an (s, S, A, p) policy, an order is placed when the inventory level at the beginning of the period $x < s$ or when $x \in A \subset [s, (s + S)/2]$; otherwise, no order is placed. Our heuristic approach can be applied to the same model with positive lead time. The resulting approximate dynamic program is similar to that in (4) by adding a fixed cost term $KI(q_t)$, where $I(u) = 1$ if $u > 0$ and $I(u) = 0$, otherwise. An (s, S) policy is optimal for the resulting approximate dynamic program. The heuristic joint pricing and replenishment policy is to charge the myopic price in every period and monitor the price-deflated inventory position to replenish the inventory according to an (s, S) policy. (Details of the derivation are presented in Online Appendix B.) In a numerical study presented in §7, we show that this heuristic policy is again very effective.

Fixed Batches. In practice, companies often replenish inventory using fixed batch sizes. (The fixed batch size is typically a result of fixed truck capacity or pallet/package sizes.) For the infinite-horizon model, it is well known that an (r, nQ) policy is optimal for both single-stage and serial systems with fixed batch sizes (see, e.g., Chen 2000)—under this policy, the firm monitors the inventory position and places an order (of size equal to an integer multiple of Q) when the inventory position falls below the reorder level r. Huh and Janakiraman (2012) show that the (r, nQ) policy is optimal for the finite horizon model. Following their result, our heuristic extends to a system with fixed batch sizes. The heuristic is executed by using the myopic price and implementing the (r, nQ) policy according to the price-deflated inventory position in each period.

Infinite Time Horizon. As stated, the approximated dynamic program obtained by using a linear approximation of the myopic demand function has the same structure as the traditional inventory model. Following the analysis of monotone convergence in Iglehart (1963), one can show that the optimal value function converges to a stationary function, which implies that the optimal policy converges to a stationary base-stock policy. Thus, in an infinite horizon setting, our heuristic uses the myopic price as the heuristic pricing policy and a stationary base-stock level as the replenishment policy (based on the price-deflated inventory position). In our numerical experiments with stationary parameters, the optimal base-stock level generally converges to a stationary value within three to four periods.

World-Driven Demand. In our model, we assume that demands are i.i.d. between periods. We can generalize the demand process by assuming that demand depends on the state of an exogenous Markov chain, i.e., $D_t(p_t, \epsilon_t, i)$, where i is the demand state. Yin and Rajaram (2007) study this model for the case of $L = 0$. The heuristic presented in this paper can be applied to a system with a positive lead time. The approximate dynamic program after the linear demand transformation is similar to an inventory model with Markov-modulated demand. It can be shown that the optimal inventory policy is a state-dependent base-stock policy. (We refer to Iglehart and Karlin 1962 or Zipkin 2000, §9.7 for a detailed analysis.) In each period t, the manager first examines the state of the Markov chain. Then, for each state, there is a corresponding myopic pricing function as well as a state-dependent base-stock level. The heuristic is executed by using the state-dependent myopic price and implementing a state-dependent base-stock policy, with orders based on the price-deflated inventory position.

7. Numerical Study

In this section, we present the results of our numerical study. We first report the results of a brief case study based on data obtained from a large distributor. We then explore the performance of our heuristic and finally discuss insights related to the impact of a positive lead time on the joint pricing and inventory decisions.

7.1. Case Study

We begin our numerical exploration of the proposed heuristic with a brief case study based on data from a large distributor of consumables in the aerospace industry. This case study illustrates how the proposed heuristic may be applied in practice. We have purchasing, pricing, sales, and inventory data for several spare parts for the years 2006–2009. We focus on a rivet (an "A" product under the ABC inventory classification scheme) with a unit purchasing cost $c = \$0.20$. The company uses a single supplier for this item and inventory is reviewed on a monthly basis. The lead time is 18 weeks, i.e., roughly four months (we therefore set $L = 4$ in our numerical experiment). The company replenishes inventory using fixed batches and follows an (r, nQ)-type policy. The batch size is $Q = 400,000$. The annual interest rate is assumed to be the WACC (weighted average cost of capital), which is about 10%–15%. Therefore, the holding cost is estimated to be $h = \$0.00306$. For all "A" products, the company operates under a fill rate of 98%. Based on this fill rate, we derive a backorder cost by solving $b/(h + b) = 98\%$, resulting in $b = \$0.14994$.

To determine the dependency between price and demand, we run a simple regression using the average quantity sold per month over the four-year period and the corresponding average per-unit selling price observed in each month. Based on these data, we consider two possible demand functions: (1) $D(p, \epsilon) = 184,676 - 348,029p + \epsilon$, where $\epsilon \sim \text{Normal}(0, 20, 103)$; and (2) $D(p, \epsilon) = 610,279e^{-7.038p} + \epsilon$, where $\epsilon \sim \text{Normal}(0, 17354)$. For both demand functions, we examine the performance of our heuristic by comparing the resulting heuristic profit with the exact optimal profit through a planning horizon of 20 periods. The table below presents the results, which suggest that the heuristic performs well in this setting.

	Linear	Exponential
Exact optimal profit	21,014	22,109
Heuristic	20,578	21,736
Relative error (%)	2.07	1.69

7.2. Performance of the Heuristic

We now examine the effectiveness of our heuristic by comparing the resulting heuristic profit to the exact optimal solution for $L \leq 3$ and to the profit derived from the upper bound in all experiments.

We consider a total of 3,816 problem instances, including a set of 3,216 instances with stationary parameters (536 instances with each $L \in \{1, 2, 3, 4, 5, 6\}$) and a set of 600 problem instances with non-stationary parameters. For additive demand, which accounts for half of the problem instances, we consider the linear mean demand function $\mathscr{D}(p) = \lambda - \mu p$. For multiplicative demand, which accounts for the other half, we consider the iso-elastic mean demand function $\mathscr{D}(p) = \lambda p^{-\mu}$. Both demand functions are commonly used in the dynamic pricing literature (e.g., Federgruen and Heching 1999, Petruzzi and Dada 1999). We have also experimented with the power-linear $\mathscr{D}(p) = (\lambda - \mu p)^{\theta}$, algebraic $\mathscr{D}(p) = (\theta + \lambda p)^{-\mu}$, and exponential $\mathscr{D}(p) = \lambda e^{-\mu p}$ mean demand functions. In all cases, the performance of the heuristic has been consistent with that reported later in this section for the linear and iso-elastic mean demand functions. We restrict attention to experiments with lead times of up to six periods, as this seems to be reasonable in practice.[6] The planning horizon is set at $T = 20$ and $\alpha = 0.95$. For

the instances with stationary parameters, we consider scenarios with $c \in \{1.5, 2, 2.5\}$, $h \in \{0.4, 1\}$, and $b \in \{10, 20, 50, 90\}$. For additive demand, we set $\lambda \in \{60, 90, 120\}$, $\mu \in \{0.5, 1, 1.5\}$, and $\epsilon \sim \text{Normal}(0, 1)$; for multiplicative demand, we set $\lambda \in \{500, 700, 900\}$, $\mu \in \{1.1, 1.25, 1.5\}$, and $\epsilon \sim \text{Gamma}(2, 0.5)$. These system parameters cover a wide range of scenarios, including instances with service levels $(= b/(b + h))$ ranging from 90% to 99%.

For instances with $L \leq 3$, we compute the exact optimal solution and use simulation to generate the expected optimal profit. For all instances, we compute the expected profit generated by the upper bound. We compute our heuristic policy for all instances and use simulation to obtain the corresponding expected profits. The simulation is conducted by randomly generating 10,000 sample paths for each problem instance. For each sample path, we first calculate the average value of inventory states over all periods and then compute the profit associated with that sample path by taking the initial on-hand and pipeline inventory states equal to those average values calculated in the first round.[7] We calculate the following percentage ratios to evaluate the performance of the proposed heuristic:

$$\begin{aligned} (\{\text{upper bound, optimal}\} \text{ expected profit} \\ - \text{heuristic expected profit}) \\ \cdot (\{\text{upper bound, optimal}\} \text{ expected profit})^{-1} \times 100\%. \end{aligned}$$

This ratio represents the percentage error with respect to the optimal (or upper bound) profit. Table 1 provides a summary of the results.

The average percentage error between the heuristic profit and the optimal profit for $L \leq 3$ is 0.62%, while the maximum gap is 1.96%. Moreover, our heuristic significantly outperforms the heuristic for positive lead time proposed by Federgruen and Heching (1999). Their heuristic assumes that a period's price is maintained over the subsequent lead-time periods. Based on this assumption, the inventory states can be aggregated into a single variable and the heuristic policy takes the form of a list-price base-stock policy. In their paper, this heuristic is examined in a setting with additive demand; we extend their heuristic to the multiplicative demand case in our numerical study. For $L \leq 2$, the average gap between the heuristic in Federgruen and Heching (1999) and the optimal profit is 8.93%, with a maximum gap of 37.17%.

---

[6] The lead time for products distributed by the company documented in §7.1 rarely exceeds —four to five periods. Huang and Van Mieghem (2014) discuss inventory management at a company that experiences a procurement lead time of three months and uses a review period of three months, therefore taking $L = 1$. Farasynm et al. (2011) investigate the supply chain for P&G and report that lead times range from one to eight weeks with review periods between one and four weeks. Wieland et al. (2012) investigate the

supply chain for Intel and report that the most upstream echelon has both a lead time and review period of one month, while the second echelon and the most downstream echelon have both lead times and review periods of one week, i.e., $L = 1$ in all cases.

[7] With stationary parameters, the inventory states tend to be stationary after the initial warm-up periods.

Table 1	Average Percentage Error—Stationary Cases				
	Comparison to exact system (%)		Comparison to upper bound (%)		
	Additive	Multiplicative	Additive	Multiplicative	Average
$L = 1$	0.23	0.43	0.96	1.03	0.99
$L = 2$	0.41	0.61	1.47	1.71	1.59
$L = 3$	0.99	1.07	2.38	2.69	2.54
$L = 4$	—	—	3.79	3.87	3.83
$L = 5$	—	—	5.13	5.17	5.15
$L = 6$	—	—	6.55	6.62	6.58

Table 2	Average Percentage Error—Nonstationary Cases			
	Comparison to exact system (%)		Comparison to upper bound (%)	
	Additive	Multiplicative	Additive	Multiplicative
$L = 1$	0.25	0.50	0.72	1.31
$L = 2$	0.60	0.76	1.63	1.82
$L = 3$	1.31	1.78	2.56	3.13
$L = 4$	—	—	3.64	4.10
$L = 5$	—	—	4.28	4.58
$L = 6$	—	—	5.71	5.78

In general, the heuristic performs slightly better for the additive demand case,[8] and tends to deteriorate when the lead time becomes longer, as the system experiences more variability. Nevertheless, the average percentage error remains small, with an average gap of 1.03% relative to the exact solution for $L = 3$. For long lead times, we compare the heuristic profit with the upper bound profit. While the percentage gap increases as L becomes larger, this increase occurs at a decreasing rate. More specifically, the incremental change in the average percentage gap systematically decreases from 1.6 (=1.59%/0.99%) as L increases from 1 to 2, to 1.3 (=6.58%/5.15%) as L increases from 5 to 6. Moreover, the gap between the optimal profit and the heuristic profit is less than one half of that between the upper bound and the heuristic for $L \leq 3$. This suggests that the heuristic profit must be reasonably close to the optimal profit for scenarios with longer lead times. We also find that the performance of the heuristic tends to deteriorate as λ decreases. When λ is smaller, demand is lower for any given value of the price, implying a lower revenue as well, so the pricing decision becomes more critical. On the other hand, the performance deteriorates as μ increases. When μ is larger, customers are more sensitive to price. As a result, the performance of the heuristic is more sensitive to deviations in the pricing decision (myopic pricing versus optimal) as the price sensitivity μ increases.

To explore the impact of demand uncertainty on the performance of the heuristic, we have conducted all numerical experiments with $L = 2$ and the following random terms: $\epsilon \sim \text{Normal}(0, 5)$ for additive demand and $\epsilon \sim \text{Gamma}(1.5, 1/1.5)$ for multiplicative demand. As expected, higher demand uncertainty leads to larger gaps. However, the heuristic still performs very well, with an average percentage error of 1.06% for additive demand and of 1.23% for multiplicative demand.

We also report the results of the numerical study for settings with nonstationary parameters. We consider base cases with $c = 2$, $h = 1$, $b = 50$, and $\lambda = 60$,

$\mu \in \{0.5, 1\}$ for additive demand, and $\lambda = 500$, $\mu \in \{1.25, 1.5\}$ for multiplicative demand. The cost and demand parameters are varied one at a time, following one of six patterns: increasing; decreasing; jump up/jump down (the cost takes a constant value for the first half of the planning horizon, then increases [decreases] to a higher [lower] level and remains at that level); and seasonal up/seasonal down (the cost increases [decreases] gradually in the first half of the planning horizon and then decreases [increases] gradually in the second half). For each instance with nonstationary parameters, we set the initial state to equal that used for the (stationary) base case. Table 2 reports the profit gap between the heuristic and the exact system for $L \in \{1, 2, 3\}$, and the profit gap with the upper bound for all values of L. The numerical results suggest that the heuristic is also very effective in nonstationary systems.

As discussed in Section 6, the heuristic can be applied in systems with fixed costs. Table 3 reports the average performance for a subset of the numerical experiments with $L \leq 3$. We consider the following: $c = \{1, 5, 2\}$, $h = 1$, $b = \{20, 50\}$, and $K = \{50, 100\}$; $\lambda = \{60, 90\}$, $\mu = \{1, 1.5\}$ for additive demand; and $\lambda = \{300, 500\}$, $\mu = \{1.25, 1.5\}$ for multiplicative demand. As noted from the table, the heuristic is also very effective when the system operates under fixed ordering costs.

To further examine the performance of the heuristic, we compare the prices and order quantities from our heuristic and the optimal policy. See Figures 5 and 6 for the cases of additive demand and multiplicative demand, respectively. The decisions under the heuristic are very close to the optimal decisions. The larger gaps occur for very low (negative) and very high net inventory levels under additive demand

[8] The multiplicative demand case involves an additional approximation discussed in §4.3.

Table 3	Average Percentage Error—Fixed Ordering Cost			
	Additive demand (%)		Multiplicative demand (%)	
	$K = 50$	$K = 100$	$K = 50$	$K = 100$
$L = 1$	0.31	0.38	0.67	0.82
$L = 2$	0.49	0.59	0.90	1.03
$L = 3$	0.68	0.80	1.25	1.44

Figure 5 **Policy Comparison—Additive Demand** $D(p, \epsilon) = \lambda - \mu p + \epsilon$, $\lambda = 60$, $\mu = 1.5$, $\epsilon \sim \text{Normal}(0, 1)$, $L = 2$, $w_{1,0} = 15$, $T = 20$, $c = 2$, $h = 1$, $b = 20$
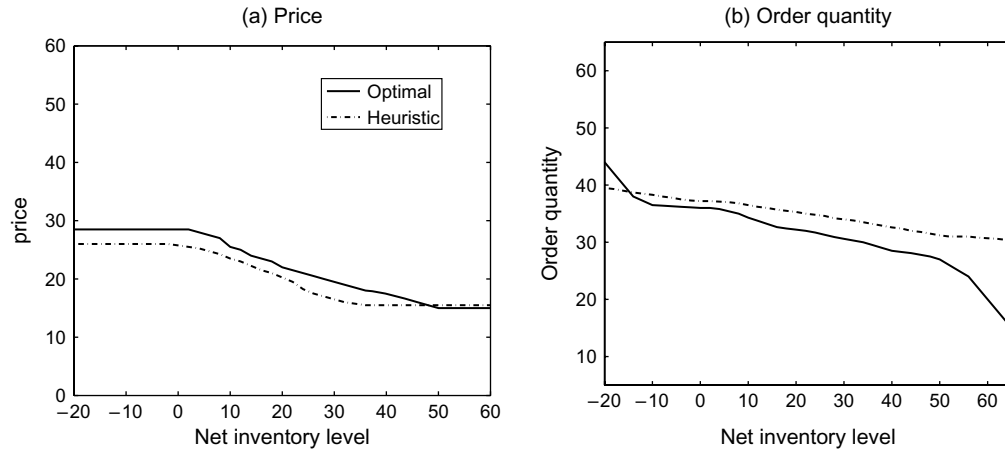
and for very low inventory levels for multiplicative demand. These gaps are mainly the result of the linear approximation used to reduce the state space. Because these larger gaps occur for inventory levels that lead to relatively high backorder or inventory costs, they are less likely to occur and therefore do not greatly affect the performance of the heuristic. We also note that the myopic price tends to be lower than the exact optimal price. The myopic price is generated by solving a single-period problem. Thus, the system tries to sell as much inventory as possible since the leftover inventory does not carry over to the next period in the myopic problem. Consequently, the myopic demand tends to be larger, leading to a lower myopic price.

To more accurately disentangle the effects of the myopic pricing policy and of the linear approximation on the performance gap between the heuristic and the optimal solution, we have run a set of numerical instances for an intermediate system in which the transformed inventory problem was generated by

using the exact myopic demand function (instead of its linear approximation). Because we ignore the linear approximation, the computational burden of this intermediate heuristic is similar to that of the original problem. Comparing the profit gaps between the optimal solution, the heuristic solution, and the solution to this intermediate system, we find that the main source of error in the heuristic indeed arises from the linear approximation and the resulting approximate inventory policy (and not from the use of the myopic price).

7.3. Managerial Insights
In this section, we explore insights regarding the impact of lead time on pricing decisions.

7.3.1. Pricing and Responsiveness. We first examine the value of responsiveness (e.g., through the implementation of a quick response strategy) and the use of price controls to balance supply and demand. We base our findings on the numerical study with
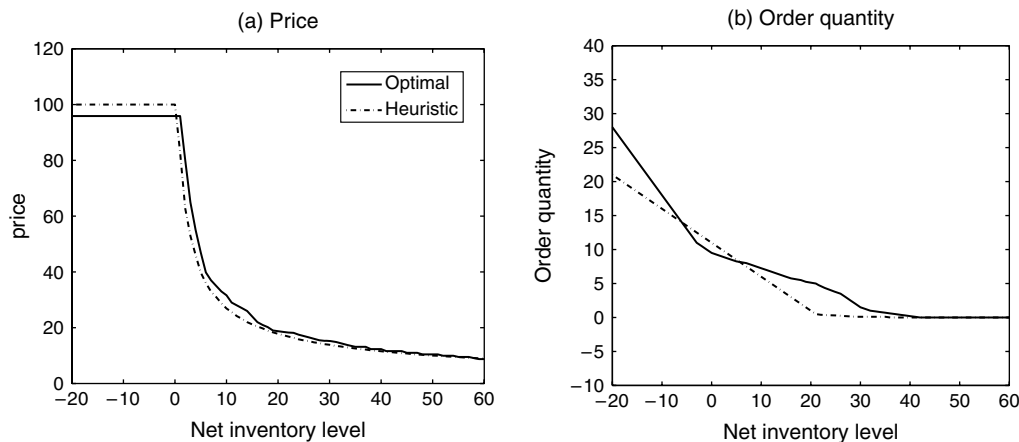
Figure 6 **Policy Comparison—Multiplicative Demand** $D(p, \epsilon) = \lambda p^{-\mu} \epsilon$, $\lambda = 300$, $\mu = 1.25$, $\epsilon \sim \text{Gamma}(2, 0.5)$, $L = 2$, $w_{1,0} = 5$, $T = 20$, $c = 2$, $h = 1$, $b = 20$

Table 4   Value of Dynamic Pricing

	Additive demand (%)					Multiplicative demand (%)				
	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$K=0$	0.10	0.30	0.59	0.94	1.47	0.90	2.80	4.42	6.41	8.05
$K=50$	0.54	0.84	1.24	1.78	2.54	1.78	3.56	5.06	6.97	9.01
$K=100$	0.94	1.45	2.16	2.95	3.96	2.39	4.02	5.49	7.44	9.61

nonstationary parameters. In particular, we consider settings in which the unit purchasing cost c_t follows an increasing, decreasing, jump up, or jump down pattern. We find that price is more sensitive to changes in the purchasing cost as lead time increases. More precisely, the range of prices charged over the planning horizon increases as lead time increases. This is consistent with the empirical study in Kesavan et al. (2014). As their study indicates, a firm with high inventory turnover (i.e., short lead times at a given level of sales) tends to have fewer price changes. We also examine the effect of price sensitivity μ (of the mean demand function) on the range of prices charged over the planning horizon. As the price sensitivity increases, the length of the price-range decreases. A large value of μ means that demand is more sensitive to price changes. Therefore, a relatively smaller adjustment in price is sufficient to control demand as the cost (and therefore the order quantity and inventory levels) changes.

In general, the value of dynamic pricing increases with lead time. To see this, we consider a subset of problem instances under the additive demand form and a linear demand function $\lambda - \mu p$, with $\lambda = 60$, $\mu = 1$, and $\epsilon \sim \text{Normal}(0, 1)$, and a subset under the multiplicative form and iso-elastic mean demand function $\lambda p^{-\mu}$, with $\lambda = 500$, $\mu = 1.25$, and $\epsilon \sim \text{Gamma}(2, 0.5)$. We compare the profit associated with the dynamic pricing and replenishment heuristic to the profit that arises under the best static pricing policy (under which the price is maintained constant throughout the selling season and inventory is replenished according to the corresponding optimal policy). Table 4 reports the corresponding gaps for various values of the lead time and fixed cost parameters. As noted in the table, the value of dynamic pricing (relative to static pricing) increases with lead time and can be substantial when lead time is long. As lead time becomes longer, pricing becomes a more useful lever to balance supply and demand due to the delay in receiving shipments.

7.3.2. Price Discounts. In settings with zero lead time, the list-price policy defined in Federgruen and Heching (1999) suggests that the selling price is discounted when the on-hand inventory exceeds the period's base-stock level. In a setting with positive lead time, it may be beneficial to offer a discounted price even if the on-hand inventory is relatively low (below the base-stock level). To see this, we have conducted a numerical study for a subset of the (stationary) parameters and $L = 0$. Let $\bar{s}^{L=0}$ denote the corresponding heuristic base-stock level. (The heuristic policy is a good approximation of the list-price policy defined in Federgruen and Heching 1999—the average profit gap between our heuristic and the exact system when $L = 0$ is 0.21%.) The heuristic price is nearly constant for inventory levels below $\bar{s}^{L=0}$ and, consistent with the notion of a list price, it decreases for inventory levels above the base-stock level $\bar{s}^{L=0}$. We next study how a change in the lead time impacts the replenishment policy. The result shows that the base-stock level increases with the lead time.

Theorem 3. *Consider two systems with different lead times $L_1 \leq L_2$, identical stationary parameters, and $\alpha = 1$. Let the terminal condition be defined as $\tilde{V}_{T+1} = c\bar{x}_{T-L_i+1}$, $i = 1, 2$, and let $\bar{s}^i$ denote the corresponding base-stock level derived from Theorem 1. Then, $\bar{s}^1 \leq \bar{s}^2$.*

As the lead time increases, Theorem 3 shows that the heuristic base-stock level increases as well. However, the heuristic price is independent of L. This implies that, when lead time is positive, the price may need to be discounted even if the net inventory is below the base-stock level, in contrast to a setting with zero lead time. In the case of a positive lead time, it may be necessary to discount the price even if the on-hand inventory is below the base-stock level in anticipation of the inventory in transit that will arrive in subsequent periods.

7.3.3. Form of Demand Uncertainty. We use the numerical study to explore the impact of demand uncertainty on prices. The additive and multiplicative forms exhibit a different dependence on the random terms. Specifically, demand variance is independent of price under additive demand, but it is decreasing in price under multiplicative demand. This results in different findings regarding the change of prices driven by changes in demand uncertainty under the different demand forms. Under additive demand, we find that the range of prices charged over the planning horizon increases with the variance, i.e., when demand is more variable, prices fluctuate more in

order to balance supply and demand. Under multiplicative demand, the range of prices shifts upwards as demand variability increases, i.e., relatively higher prices are charged in a more variable system. In this case, an increase in price leads to a reduction in mean demand, counteracting the increased variance of the random term.

7.3.4. Fixed Ordering Cost. We finally explore the value of dynamic pricing in systems with fixed ordering costs. (Chen et al. 2010 study a related question in settings with $L = 0$.) To that end, we consider the set of numerical experiments reported in Table 4. We find that the value of dynamic pricing over the best static pricing policy is increasing in the fixed ordering cost. Indeed, a higher fixed cost leads to a reduced frequency of orders, so the system is less responsive to changes in demand through its replenishment process. This is exacerbated when lead time is long. As a result, in settings with fixed ordering cost and positive lead time, dynamic pricing plays an important role in influencing market demand.

8. Conclusion

In this paper, we study the joint inventory and pricing control problem for a single-stage system with a positive lead time. Demand is price sensitive and is either of the additive or of the multiplicative form. A replenishment decision and a selling price are determined simultaneously in each period. Deriving the optimal policy in this system is computationally intractable. We develop a simple and effective heuristic for this problem in which the decisions about the pricing and the replenishment strategies are decoupled. First, the price (or, equivalently, the expected demand) decision is determined by solving a series of single-period problems, leading to a myopic pricing policy. We then propose a linear approximation to the myopic policy as a function of the net inventory level. This approximation allows us to reduce the dimension of the state space. The single state variable, referred to as price-deflated inventory position, is a weighted sum of inventories in the system. The weights are determined by the slopes of the linear approximation, which measure the sensitivity of price to the inventory level. We show that a base-stock policy is optimal in the approximated system. We also derive an upper bound to the exact system and show how to efficiently compute the resulting expected profit. The heuristic also applies to other inventory settings, such as settings with fixed ordering costs or fixed batch sizes.

We evaluate the performance of the heuristic by comparing it to the exact system (for $L \leq 3$) and to the upper bound. Under both the additive and multiplicative forms, the heuristic is near-optimal when

compared to the profit of the exact system. The performance of the heuristic tends to deteriorate as demand variability increases, as the mean demand price sensitivity increases, and as the lead time increases. Nevertheless, the relative gaps with the upper bound suggest that the heuristic performs very well for longer lead times as well. Linear holding and back-order cost functions are essential to the applicability of the heuristic. A variation of the heuristic would be required for other nonlinear functional forms of the inventory cost function. In general, we find that the heuristic is effective under various forms of the mean demand function.

Finally, we discuss the impact of lead time on the pricing and inventory decisions. We find that, under both types of demand functions, price becomes an efficient lever to balance supply and demand as lead time or fixed ordering costs increase. This is in contrast to settings with zero lead time in which there is generally a limited benefit to dynamic pricing. We also find that, when lead time is positive, price discounts may be offered in anticipation of the upcoming inventory in transit.

Supplemental Material

Acknowledgments

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