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Failure and Rescue in an Interbank Network

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This paper is concerned with systemic risk in an interbank market, modelled as a directed graph of interbank obligations. This builds on the modelling paradigm of Eisenberg and Noe [Eisenberg L, Noe TH (2001) Systemic risk in financial systems. *Management Sci.* 47(2):236–249] by introducing costs of default if loans have to be called in by a failing bank. This immediately introduces novel and realistic effects. We find that, in general, many different clearing vectors can arise, among which there is a greatest clearing vector, arrived at by letting banks fail in succession until only solvent banks remain. Such a collapse should be prevented if at all possible. We then study situations in which consortia of banks may have the means and incentives to rescue failing banks. This again departs from the conclusions of the earlier work of Eisenberg and Noe, where in the absence of default losses there would be no incentive for solvent banks to rescue failing banks. We conclude with some remarks about how a rescue consortium might be constructed.

Key words: contagion; interbank network; bank failure; merger
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1. Introduction

Over the last 10 or so years, there has been a growth of interest in the general phenomenon of the spread of bank failure through a network of interbank obligations. As the Asian banking crisis of the late 1990s, and the more recent banking crisis of 2007–2008 have shown, the banking system generally can be very vulnerable to deterioration of assets, loss of liquidity, and loss of confidence. The modelling effort to date is largely restricted to static models, reflecting the urgent nature of the crises; traditional solutions such as the issuance of new equity or bonds cannot be achieved in the time scales available, typically just a few days. Among such studies of static failure models, there are two main types. The first (see, for example, Eisenberg and Noe 2001, Cifuentes et al. 2005) supposes that the network of interbank obligations is given to us and then attempts to understand how it might fail, where it is weak, and what steps could be taken to strengthen it. The second (see, for example, Nier et al. 2007, Beyeler et al. 2007) supposes some regular or symmetric structure[1] for the network of interbank obligations and tries to derive general results for such a network. The second approach lends itself better to elegant theory, but we view its value as being largely indicative; in practice, we are *not* sampling from some distribution of possible bank networks, nor are we working with a symmetric network of interbank obligations—there is an *actual* network to be saved, and it will be a complicated, ugly object.

Our contribution therefore will be of the first type, modelling the system as a directed graph with edge weights, representing the indebtedness of each bank to the others. Each bank is owed money by external borrowers, and initially all banks are solvent. But the value of the banks' loan portfolio is subject to variation, and it may come about that one bank's loan portfolio falls in value so that it is technically insolvent. We investigate the consequences of such a default on the system.

The model we present here is an extension of a model proposed by Eisenberg and Noe (2001). They assume that a defaulting bank calls in its loans, realizing the face value in full. In such a default situation, however, it is more likely that the bank will only realize some proportion of the face value that is strictly less than 1. Allowing for this makes the model much more realistic, as we will see. When a bank defaults, it does not repay its obligations in full, which may precipitate other collapses. We provide a detailed description of this domino effect. Furthermore, we investigate circumstances under which banks have incentives to bail out other banks in the network.

We proceed as follows. In §2 we describe the model for the interbank market, which extends the model

[1] The regular structure may be stochastic in nature, as in Nier et al. (2007).

by Eisenberg and Noe (2001) by introducing default costs.

In §3 we study how the default of banks affects the system. We prove the existence of a clearing vector that represents a vector of payments to settle all liabilities in the system within a simultaneous clearing mechanism. Moreover we propose an algorithm (see Definition 3.6) for computing clearing vectors very efficiently.

Section 4 contains the main theoretical results of this paper on merger decisions in the interbank market. We show in Theorem 4.10 that, on one hand, if there are no default costs, there is no incentive for rescue consortia to form. On the other hand, in Theorem 4.11 we provide a sufficient condition for the existence of a rescue consortium, assuming strictly positive default costs. We provide a condition on the relationship between default costs and costs for mergers. Based on these results, Corollary 4.12 shows that under strictly positive default costs, but in the absence of costs for mergers, rescue consortia exist under mild regularity conditions. In §5 we provide some examples and illustrate our theoretical results in these situations.

In many realistic situations there is a benefit to be derived from the solvent banks in a system rescuing the insolvent banks, and it is indeed in the interests of the solvent banks to do this. Why then does it seem so difficult in times of crisis for the system to act in its own interest and mend itself? There are, of course, many aspects to this question: How is a rescue to be coordinated? Would a rescue consortium be able to satisfy itself as to the risks involved in a rescue? Why would a bank join a rescue consortium and take on losses if by waiting other banks might take the losses instead? In §6 we discuss some of these questions and present some measures which can be used to assess the state of a financial system and particular its risk of contagion.

Finally, §7 concludes. The appendix contains all proofs of the theoretical results.

1.1. Related Literature

Contagion in financial networks has been frequently studied in the past; see, e.g., Diamond and Dybvig (1983), Rochet et al. (1996), Allen and Gale (2000), among other papers and the surveys by DeBandt and Hartmann (2000) and Staum (2013).

Once bankruptcy of one or several financial institutions has occurred or is imminent, the natural question is whether these institutions should be bailed out, and if yes, by whom.

Aghion et al. (1999) discuss exactly such a bailout problem and show how and when a bailout should be done. Cordella and Yeyati (2003) show that bailing out banks under certain conditions can outweigh possible moral hazard effects. The main problem generally is that if a lender of last resort is present, this might increase the risk appetite of financial institutions. If there is not such a lender of last resort, the resulting contagion effects and loss of confidence could seriously affect the financial system. Perotti and Suarez (2002) consider the possibility of solvent banks taking over distressed banks and show how this can stabilize the financial system. More recent approaches to optimal resolution of bank failures are provided in Acharya and Yorulmazer (2007) and Acharya and Yorulmazer (2008).

Acharya et al. (2011) focus on the related problem, namely the banks' choice of liquidity and how this depends on the resolution mechanism. They particularly provide evidence for the fact that providing liquidity support to failing banks can make them more likely to hold less liquidity in the first place.

Besides theoretical results on bank failure and contagion, there is also a wide range of empirical studies, such as Iyer and Peydro (2011) for Indian banks and Cont et al. (2013) for the Brazilian financial system. Many authors have also considered stress testing in financial networks (e.g., Amini et al. 2012) or general simulation studies (Upper and Worms 2004, Upper 2007, Elsinger et al. 2006). Many of those are built on the modelling paradigm of Eisenberg and Noe (2001).

Our paper complements the literature on dealing with financial networks under stress and bank merger decisions. Its spirit is closest to the work of Leitner (2005), who particularly investigated the role of linkage in interbank networks as a reason for private sector bailouts. Leitner (2005) also considers *optimal* networks in the sense that they provide an optimal trade-off between risk sharing and the collapse of the system. The main difference between our approach and the one by Leitner (2005) is that we consider a network as exogenously given, whereas Leitner (2005) takes an endogenous view on network formation. Leitner (2005) develops a network by considering agents who invest in several projects, and the success of these projects depends on the investments of the agents and of those who are directly linked to them. Hence, there is some incentive for agents to form linkages to increase their chances of success. We take the network as given and by doing so can in principle use our results to analyze every real-world network (assuming that the liabilities structure can be observed). Furthermore, we include costs for mergers and therefore introduce a different incentive structure for bailouts. We show that banks in such a network often have incentives to rescue other banks, and hence a lender of last resort is not required for the bailout itself but for insuring an appropriate coordination mechanism. The role of the regulator in our paper is therefore very similar to the one considered by Leitner (2005).

2. Definition of the Financial System

We use a model similar to Eisenberg and Noe (2001), though we modify their notation slightly. We consider a market with n banks with indices in $\mathcal{N} := \{1, \ldots, n\}$, which we represent by nodes in a network. Each bank has liabilities to other banks in the system. We represent these liabilities in terms of a matrix.

Definition 2.1 (Liabilities Matrix). The *liabilities matrix* is given by $\mathbf{L} \in \mathbb{R}^{n\times n}$, where the ijth entry L_{ij} represents the nominal liability of bank i to bank j. We assume that $L_{ij} \geq 0\ \forall\, i, j$ and $L_{ii} = 0\ \forall\, i$.

Those nominal liabilities that are strictly positive represent the directed edges of the network.

Definition 2.2 (Obligations). The *total nominal obligations* of bank i to all other banks in the system are given by $\bar{L}_i = \sum_{j=1}^n L_{ij}$, and $\bar{L}$ is the corresponding vector of the total nominal obligations.

Definition 2.3 (Relative Liabilities Matrix). Let $\mathbf{L}$ be a liabilities matrix and $\bar{L}$ the corresponding vector of total nominal obligations. The *relative liabilities matrix* $\Pi \in \mathbb{R}^{n\times n}$ is defined by

$$\pi_{ij} := \begin{cases} L_{ij}/\bar{L}_i & \text{if } \bar{L}_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the rows of Π all sum up to 1, if $\bar{L} > 0$. If for some $i \in \mathcal{N}$ the total liabilities $\bar{L}_i = 0$, then the corresponding row of Π sums up to 0.

Definition 2.4 (Net Assets). We denote by $e_i \geq 0$ the *net assets* of bank i from sources outside the banking system.[2] The corresponding vector of net assets is denoted by e.

Definition 2.5 (Financial System). Let e be a vector of net assets. We consider two constants $\alpha, \beta \in (0, 1]$, where the constant α is the fraction of the face value of net assets realized on liquidation, and the constant β is the fraction of the face value of interbank assets realized on liquidation. We define a *financial system* as a quadruple $(\mathbf{L}, e, \alpha, \beta)$, where $\mathbf{L}$ is a liabilities matrix.

Once a financial system $(\mathbf{L}, e, \alpha, \beta)$ is specified, one can immediately derive the corresponding relative liabilities matrix Π and the vector of total obligations $\bar{L}$. Similarly, if we start with a vector of total obligations $\bar{L}$ and a relative liabilities matrix Π, we can derive the corresponding liabilities matrix $\mathbf{L}$. This leads to equivalent definitions of a the financial system, which we will use interchangeably, as it is convenient.

Next, we introduce the clearing vector. A clearing vector specifies payments between the banks in the system that are consistent with some rules. The rules considered are the three rules proposed by (Eisenberg and Noe 2001, p. 239):

1. *Limited liabilities*: Each node never pays more than its available cash flow.
2. *Priority of debt claims over equity*: Paying off the liabilities L_{ij} has priority, even if the net assets e_i have to be used for that.
3. *Proportionality*: If default occurs, the defaulting bank pays all claimant banks in proportion to the size of their nominal claims on the assets of the defaulting bank.

This leads to the following definition.

Definition 2.6 (Clearing Vector). A *clearing vector* for the financial system $(\mathbf{L}, e, \alpha, \beta)$ is a vector $L^\star \in [0, \bar{L}]$ such that

$$L^* = \Phi(L^*),$$

where Φ is the function defined by

$$\Phi(L)_i \equiv \begin{cases} \bar{L}_i & \text{if } \bar{L}_i \leq e_i + \sum_{j=1}^n L_j \pi_{ji}, \\ \alpha e_i + \beta \sum_{j=1}^n L_j \pi_{ji} & \text{else.} \end{cases} \tag{1}$$

Simple but very important properties of the mapping Φ are given by the following result.

Lemma 2.7. *The mapping Φ has the following properties*:

(i) *Φ is bounded above by $\bar{L}$: for any L we have $\Phi(L) \leq \bar{L}$;*

(ii) *Φ is monotone: if $\tilde{L} \leq L$, then $\Phi(\tilde{L}) \leq \Phi(L)$.*

A proof of these results is provided in Appendix A.

The interpretation of the clearing vector L^* is that L_i^* represents the cash that bank i has available to pay out to other banks. The value of the assets (net and interbank) available to bank i will be the sum $e_i + \sum_j L_j^* \pi_{ji}$, and if this is at least $\bar{L}_i$, then bank i is able to meet its obligations. If this inequality does not hold, then bank i is in default and must call in its assets; it recovers only a fraction α of the net assets and a fraction β of the interbank assets. The two fractions may conceivably be different; we would typically expect that α would be low, because the bank would be having to sell off its loan portfolio, probably at a knock-down price. In contrast, β might be much closer to 1, because an obligation from a solvent bank would probably be paid in full (though perhaps with some negotiated discount to compensate for the inconvenience of early repayment). An obligation from a liquidated bank would also probably be paid in full, because L_j^* represents realized cash, but there might also be some deduction for the costs of receivership.

[2] This would be equity plus loans less deposits.

Thus, on default of bank i, there are actual losses of at least

$$(1-\alpha)e_i + (1-\beta)\sum_{j=1}^{n} L_j^* \pi_{ji}.$$

For $0<\alpha,\beta<1$, this is a positive amount—bank failure really costs something. This assumption is new and was not considered in Eisenberg and Noe (2001), who in effect have assumed $\alpha=\beta=1$ throughout. We will see how this extension leads to more realistic behavior and more interesting properties.

Remark 2.8. This model can be easily extended by using different fractions α_i,β_i corresponding to the different banks $1\le i\le n$ in the model.

We conclude this section by defining the value of a bank.

Definition 2.9 (Value of Banks). The *value* $\mathcal{V}$ of the banks corresponding to a clearing vector L in a financial system $(\mathbf{L}, e, \alpha, \beta)$ is defined as

$$\mathcal{V}(L,e)_i := (\Pi^\top L + e - L)_i \mathbb{1}_{\{L_i \ge \bar{L}_i\}}. \tag{2}$$

Remark 2.10. From Definition 2.9 it is clear that as soon as a bank's clearing payment is strictly less than its total liabilities, its value is zero. Moreover, $\Pi^\top L + e$ can be interpreted as cash flow into the bank and L as cash flow out of the bank. The value of the bank is then the net cash position after clearing, which is assumed to be 0 if the bank defaults. A similar concept was considered in Eisenberg and Noe (2001, p. 239).

3. Existence and Construction of Clearing Vectors

In the following we show that a clearing vector exists for all $0<\alpha,\beta\le 1$. We also propose an algorithm to compute a clearing vector. Eisenberg and Noe (2001) only considered the case $\alpha=\beta=1$.

3.1. Existence of Clearing Vectors

Theorem 3.1 (Existence of Clearing Vectors). *For every financial system* $(\mathbf{L}, e, \alpha, \beta)$ *there exist clearing vectors* L^* *and* L_* *such that for any clearing vector* L, *we have*

$$L_* \le L \le L^*. \tag{3}$$

Proof of Theorem 3.1. Consider a sequence of vectors $L^{(n)}$, $n=0,1,\ldots,$ defined recursively by $L^{(0)} := \bar{L}$,

$$L^{(n+1)} = \Phi(L^{(n)}) \tag{4}$$

for $n=0,1,\ldots,$ where Φ is as at (1). From Lemma 2.7 we have $L^{(1)} \le L^{(0)} = \bar{L}$, and hence for all n,

$$L^{(n+1)} \le L^{(n)}$$

by induction. Because all the $L^{(n)}$ are nonnegative, there is a monotone limit $L^* :=\downarrow \lim_{n\to\infty} L^{(n)}$. Notice that the set $A_n := \{i\colon L_i^{(n)} < \bar{L}_i\}$ is nondecreasing in n and therefore is eventually constant. Note that Φ is continuous from above. Hence, it is clear that L^* satisfies

$$L^* = \Phi(L^*), \tag{5}$$

that is, L^* is a clearing vector.

We may similarly start the recursion from the zero vector $L^{(0)} := 0$, in which case we obtain an increasing sequence of vectors, with limit $\hat{L}$. In contrast to the first situation, the limit $\hat{L}$ does not have to be a clearing vector unless $\alpha=\beta=1$ (see Example 3.3). This is because the function Φ is not continuous from below. If a bank just becomes solvent with the limit payment vector $\hat{L}$, one needs to restart the iteration from $\hat{L}$ and continue until one reaches the next limit (monotone convergence). Because there are at most n banks where that could happen, one has to restart the iteration at most $n-1$ times. We denote the vector that we obtain from this iteration by L_*. This vector is then the least clearing vector and a fixed point of Φ.

Because L^* is nonnegative and a fixed point of Φ, monotonicity of Φ implies that $L^* \ge L_*$. Moreover, because any clearing vector is bounded below by 0 and above by $\bar{L}$, we have that any clearing vector must be bounded between L_* and L^*. □

Remark 3.2. (i) It is clear from (3) that the greatest and least clearing vectors are unique.

(ii) The proof of Theorem 3.1 can be used directly to compute a clearing vector. However, it is not guaranteed that it will terminate after a finite number of steps. We will therefore present an algorithm in Definition 3.6 that returns a clearing vector after at most $n=|\mathcal{N}|$ steps.

(iii) An alternative proof of the existence results uses Tarski's fixpoint theorem (see Tarski 1955) and proceeds similarly, as in Eisenberg and Noe (2001). To apply Tarski's fixed-point theorem, however, we need a monotone map on a complete lattice. The set of clearing vectors is a complete lattice, but not by the usual pointwise monotonicity, as Example 3.3 shows.

Example 3.3. We consider the financial system

$$(\mathbf{L}, e, \alpha, \beta) = \left(\begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha, \beta \right)$$

with $\alpha=\beta=\frac{1}{2}$, $K_1 = 1/(1-\alpha) = 2$, and $K_2 = 2.2$. If we start from $L_i^{(0)} := 0$, $i\in\{1,2\}$, one can show by induction that $L_i^{(n)} = (2^n-1)/2^n$ and therefore $\lim_{n\to\infty} L_i^{(n)} = 1$ for $i=1,2$. But $(1,1)^\top$ is not a clearing vector because

$$\Phi\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

However, if we start the iteration again with $L^{(0),\text{new}} := \lim_{n\to\infty} L^{(n)} = (1,1)^\top$, $L^{(n+1),\text{new}} = \Phi(L^{(n),\text{new}})$, then we obtain

$$\lim_{n\to\infty} L^{(n),\text{new}} = \begin{pmatrix} 2 \\ 2.2 \end{pmatrix}.$$

Therefore, we have found the clearing vector $(2, 2.2)^\top$.

If we change the financial system by setting $K_1 = 2.2$, then $\binom{1}{1}$ and $\binom{2.2}{2.2}$ are both clearing vectors. If additionally, $\alpha = \beta = 1$, then only $\binom{2.2}{2.2}$ is a clearing vector.

Remark 3.4. The construction of the upper solution L^* is achieved by assuming initially that all banks are sound, then knocking out banks that are insolvent or that become insolvent as the result of the failures of others. Thus, we are building a solution that *spreads insolvency across the network.* The lower solution L_* is achieved by supposing initially that banks can only rely on their own assets e_i; any that can cover their interbank obligations with their own assets are certain to be secure, and thus their obligations will be paid in full. This in turn may make other insolvent banks secure; this solution *spreads solvency across the network.* From a practical point of view, it is likely to be the upper solution L^* that most concerns us; we would envisage a situation where initially all banks were (or were thought to be) solvent, and insolvency is seen spreading across the banking network.

Remark 3.5. Notice that in the case $\alpha = \beta = 1$ the function Φ defined in (1) takes the simpler form

$$\Phi(L)_i = \bar{L}_i \wedge \left(e_i + \sum_{j=1}^{n} L_j \pi_{ji} \right), \tag{6}$$

from which it is obvious that Φ is continuous and increasing.

It was shown in Eisenberg and Noe (2001) that if the banking network satisfied a technical condition (called *regularity*), and if $\alpha = \beta = 1$, then there is only one clearing vector: $L^* = L_*$.

3.2. Construction of Clearing Vectors

Theorem 3.1 gives a recursive method for calculating approximations to the greatest/least clearing vector which in practice converges very rapidly and stably. However, this algorithm is not guaranteed to converge in any fixed finite number of steps, and this is certainly a drawback, at least at the theoretical level.

In this subsection, we propose an algorithm that is a modification of the *fictitious default algorithm* introduced by Eisenberg and Noe (2001). Eisenberg and Noe (2001) derive their algorithm under the assumption that they are in the situation in which the clearing vector is unique. We do not make this assumption, because, in general, it does not hold for the situation studied here. Our algorithm will find the *greatest* clearing vector in at most n steps; moreover, on the way we find an interesting understanding of the way default spreads through the network, which we return to later.

Definition 3.6 (Greatest Clearing Vector Algorithm, (GA)). For a financial system $(\mathbf{L}, e, \alpha, \beta)$ the GA algorithm constructs a sequence $(\Lambda^{(\mu)})$ as follows. Again, $\Pi = (\pi_{ij})$ and $\bar{L}$ are defined as before.

1. Set $\mu = 0$, $\Lambda^{(0)} := \bar{L}$, and $\mathcal{I}_{-1} := \varnothing$.
2. For all nodes i, compute $v_i^{(\mu)} := \sum_{j=1}^{n} \Lambda_j^{(\mu)} \pi_{ji} + e_i - \bar{L}_i$.
3. Define

$$\mathcal{I}_\mu := \{1 \le i \le n \colon v_i^{(\mu)} < 0\},$$

the set containing all indices of insolvent banks, and

$$\mathcal{S}_\mu := \{1 \le i \le n \colon v_i^{(\mu)} \ge 0\},$$

the set containing all indices of solvent banks.

4. If $\mathcal{I}_\mu \equiv \mathcal{I}_{\mu-1}$, terminate the algorithm.
5. Otherwise, set

$$\Lambda_j^{(\mu+1)} := \bar{L}_j \quad \forall\, j \in \mathcal{S}_\mu$$

and determine the remaining clearing payments by finding the unique solution to the system of linear equations

$$x_i = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\mu} x_j \pi_{ji} \right\} \quad \forall\, i \in \mathcal{I}_\mu \tag{7}$$

and setting $\Lambda_i^{(\mu+1)} := x_i \ \forall\, i \in \mathcal{I}_\mu$.

6. Set $\mu \to \mu + 1$ and go back to step 2.

When the algorithm has terminated, the vector $\Lambda^{(\mu)}$ is a clearing vector.

Theorem 3.7. *The GA algorithm stated in Definition 3.6 produces a sequence of vectors $(\Lambda^{(\mu)})$ decreasing in at most n iterations to the greatest clearing vector.*

Hence, we have shown how we can find the greatest clearing vector. The proof is given in Appendix B.

There is an analogue of the GA algorithm that starts from $\Lambda^{(0)} = 0$, $\mathcal{I}_{-1} = \{1, \ldots, n\}$ and proceeds exactly as the GA algorithm. This algorithm produces a sequence $(\Lambda^{(\mu)})$ of vectors increasing to L_* in at most n steps. The method of proof is a direct translation of the proof of Theorem 3.7; because this result is of less interest, we omit it completely.

The construction of Theorem 3.7 leads to an increasing sequence $\mathcal{I}_\mu$ of insolvency sets, which have an important and natural interpretation.

Definition 3.8. We call the set $\mathcal{I}_\mu$ the *level-μ* insolvency set.

Notice that the level-0 insolvency set is the set of those banks that would default *even if all other banks paid their obligations in full*. The level-μ insolvency set is the set of all those banks that would not be able to meet their obligations *if all the level-*$(\mu-1)$ *insolvent banks were to default*. Thus, the insolvency sets $\mathcal{I}_\mu$ trace the spread of default through the financial system. We shall comment further on this later in the context of possible rescue schemes and policy for limiting the damage of bank failures.

4. Merged Banks as a Rescue Consortium

The model we have introduced already allows us to study how the default of one bank affects other banks in the system. However, it also allows us to study situations where a bank failure might be avoided by some solvent bank(s) stepping forward to rescue the failing bank. By so doing, the rescuer takes on the loss incurred by the distressed bank, but this may work out cheaper than the loss suffered by the rescuer once the losses of the defaulting bank have been spread and *amplified* by the network of interbank obligations. To model this amplification effect, the inclusion of default costs is essential. Note that even if such a rescue can be mounted, one has to address the question of how this can be organized, bearing in mind free-rider problems. We will come back to this problem in §6, once we have determined under which conditions a rescue is in principle possible.

Recall that the banks are indexed by $\mathcal{N}=\{1,\dots,n\}$. We define a bank merger in the following way. We assume that a merger is associated with costs. We model costs from the merger in terms of a vector $\kappa\in\mathbb{R}^n_+$. If bank i is involved in a merger, costs of size κ_i occur. We can now provide a formal definition of a bank merger.

Definition 4.1 (Merger). In the financial system $(\mathbf{L},e,\alpha,\beta)$ consisting of n banks, the *merger* of all banks in $B\subset\mathcal{N}$ is a new financial system $(\tilde{\mathbf{L}},\tilde{e},\alpha,\beta)$ indexed by $\tilde{\mathcal{N}}:=\{0\}\cup B^c$, where $B^c=\mathcal{N}\backslash B$. It is defined by

$$\tilde{e}:=Me-\tilde{\kappa},$$

where $\tilde{\kappa}\in\mathbb{R}^{|B^c|+1}$, $\tilde{\kappa}_0:=\sum_{i\in B}\kappa_i$, $\tilde{\kappa}_i=0\ \forall\, i\in B^c$, and

$$\tilde{\mathbf{L}}_{ij}:=\begin{cases}0 & \text{if } i=j=0,\\ \mathbf{L}'_{ij} & \text{otherwise},\end{cases}$$

where

$$\mathbf{L}':=M\mathbf{L}M^\top,$$

and M is the $|\tilde{\mathcal{N}}|\times|\mathcal{N}|$ matrix,

$$M_{0i}=\begin{cases}1 & \text{if } i\in B,\\ 0 & \text{else},\end{cases}$$

$$M_{ij}=\delta_{ij}\quad i,j\notin B.$$

From this definition, we see that for those banks that merge the corresponding liabilities to the other (nonmerging) banks in the network are just added up. The liabilities to those banks that merge are cancelled. Also the net assets of the merged bank are the sum of the net assets of the banks that merged. In addition, there are costs $\sum_{i\in B}\kappa_i$ associated with the merger. Hence, this is a very natural definition of a merger.

There is the possibility that a rescue consortium is formed that provides the necessary assets such that no bank defaults. This works as follows. Suppose the system $(\mathbf{L},e,\alpha,\beta)$ is subjected to stress by reducing the banks' initial net asset vector e to $\tilde{e}$. Here we assume that initially, with net assets e, all banks are sound. But under reasonable model assumptions there comes a point where one or more banks become insolvent if we reduce the net assets to $\tilde{e}$.

Definition 4.2 (Bailout Costs). Given a financial system $(\mathbf{L},\tilde{e},\alpha,\beta)$, we define

$$\delta:=\max\{0,-(\Pi^\top\bar{L}+\tilde{e}-\bar{L})\}$$

and refer to $\sum_{j=1}^n\delta_j=\sum_{j\in\mathcal{I}_0}\delta_j$ as the *bailout costs*.

In the following we define a rescue consortium assuming that the level-0 insolvency set is nonempty.

Definition 4.3 (Rescue Consortium). Let $(\mathbf{L},\tilde{e},\alpha,\beta)$ be a financial system where the level-0 insolvency set $\mathcal{I}_0$ is nonempty. The corresponding greatest clearing vector is denoted by L^*. We define

$$\tilde{V}:=\max\{0,\Pi^\top\bar{L}+\tilde{e}-\bar{L}\},\tag{8}$$

$$\Delta V:=\tilde{V}-\mathcal{V}(L^*,\tilde{e}).\tag{9}$$

A rescue consortium is a set $A\subseteq\mathcal{N}\backslash\mathcal{I}_0$ such that the following two conditions hold:

1. Rescue incentive:

$$\sum_{i\in A}\Delta V_i>\sum_{j\in\mathcal{I}_0}\delta_j+\sum_{k\in A\cup\mathcal{I}_0}\kappa_k.\tag{10}$$

2. Rescue ability:

$$\sum_{i\in A}\tilde{V}_i>\sum_{j\in\mathcal{I}_0}\delta_j+\sum_{k\in A\cup\mathcal{I}_0}\kappa_k.\tag{11}$$

Conditions (10) and (11) distinguish between situations in which banks *want* to rescue other banks and those in which they *can* rescue. We assume that both conditions need to be satisfied. We establish a relationship between these two conditions:

Theorem 4.4. (i) *Every rescue consortium that has an incentive to rescue the failing banks also has the ability to rescue the failing banks.*

(ii) *Suppose that the set of banks at risk of contagious default* $\mathcal{R}:=\bigcup_\mu\mathcal{I}_\mu\backslash\mathcal{I}_0$ *is nonempty, and suppose further that some subset* $A\subseteq\mathcal{R}$ *is able to rescue the failing banks. Then* A *also has an incentive to rescue the failing banks.*

A proof is provided in Appendix C.

Remark 4.5. We assume that a rescue consortium will always try to rescue all banks that are technically insolvent at once. One could change this definition and consider individual rescue consortia for different defaulting banks as well.

In the following we describe the new financial system once a rescue consortium was formed.

Definition 4.6 (Rescued Financial System). Let $(\mathbf{L}, \tilde{e}, \alpha, \beta)$ be a financial system in which the level-0 insolvency set $\mathcal{I}_0$ is nonempty, and suppose that a rescue consortium defined by a set of indices A exists. Then the *rescued financial system* is the financial system obtained by a merger of all banks in $\mathcal{I}_0 \cup A$.

From this definition, we immediately obtain the following results using the notation of Definition 4.6:

Lemma 4.7. *In the situation of Definition* 4.6, *the value of the new bank obtained as a merger of all banks in* $\mathcal{I}_0 \cup A$ *is* $\sum_{i\in A} \tilde{V}_i - \sum_{j\in \mathcal{I}_0} \delta_j - \sum_{k\in A\cup \mathcal{I}_0} \kappa_k > 0$.

The following result will be very useful to establish more general results about the existence of a rescue consortium.

Lemma 4.8. *Consider a financial system* $(\mathbf{L}, e, \alpha, \beta)$ *in which all banks are initially solvent. Suppose that the assets* e *are reduced to* $\tilde{e}$, *with* $\tilde{e}_i \le e_i\ \forall\, i$ *such that some banks have become insolvent*: $\mathcal{I}_0 \ne \varnothing$. *Let* L^* *be the greatest clearing vector in* $(\mathbf{L}, \tilde{e}, \alpha, \beta)$. *Then*

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} \big(\mathcal{V}(\bar{L}, e)_i - \mathcal{V}(L^*, \tilde{e})_i\big) \\ &= \sum_{i=1}^{n} (e_i - \tilde{e}_i) + \sum_{i=1}^{n} \bigg((1-\alpha)\tilde{e}_i + (1-\beta) \sum_{j=1}^{n} L_j^* \pi_{ji} \bigg) \mathbb{1}_{\{L_i^* < \bar{L}_i\}}. \end{aligned}$$

A proof of Lemma 4.8 is provided in Appendix C.

Remark 4.9. Lemma 4.8 shows how much money is lost in the financial system when the net assets are reduced from e to $\tilde{e}$. We see that for $\alpha, \beta < 1$, this loss is usually larger than $\sum_{i=1}^{n} (e_i - \tilde{e}_i)$.

As we will see in the following, accounting for default costs is necessary for the existence of a rescue consortium. Indeed, as we prove, for $\alpha = \beta = 1$ banks have no incentive to form rescue consortia.

Theorem 4.10 (Absence of Rescue Consortium). *Consider a financial system* $(\mathbf{L}, e, \alpha, \beta)$ *in which all banks are initially solvent. Suppose the assets* e *are reduced to* $\tilde{e}$, *with* $\tilde{e}_i \le e_i\ \forall\, i$, *with the result that at least one bank becomes level-*0 *insolvent. Suppose that* $\alpha = \beta = 1$. *Then no group of banks in the network has an incentive to rescue the insolvent bank*(*s*).

Again, a proof can be found in Appendix C. Therefore, we see that for $\alpha = \beta = 1$ it is never beneficial for a solvent bank to take over an insolvent bank. Note that this statement is true even if the costs for merger are zero. But as we will see, for α, $\beta < 1$ it can be beneficial for some bank(s) to take over an insolvent bank, and in many realistic situations it will be beneficial. We provide sufficient conditions for the existence of a rescue consortium in the following.

Theorem 4.11 (Presence of Rescue Consortium). *Let* $(\mathbf{L}, \tilde{e}, \alpha, \beta)$ *be a financial system with* $\alpha, \beta \in [0, 1)$. *Suppose that* $\mathcal{I}_0$ *is a proper subset of* $\mathcal{N}$: $\varnothing \subsetneq \mathcal{I}_0 \subsetneq \mathcal{N}$. *Let* L^* *be the corresponding greatest clearing vector. Let* κ *be the vector describing the costs for merger. Suppose*

$$\sum_{i=1}^{n} \bigg((1-\alpha)\tilde{e}_i + (1-\beta) \sum_{j=1}^{n} L_j^* \pi_{ji} \bigg) \mathbb{1}_{\{L_i^* < \bar{L}_i\}} > \sum_{k=1}^{n} \kappa_k.$$

Then there exists a rescue consortium.

A proof is given in Appendix C. We immediately get the following result in the absence of costs for merger.

Corollary 4.12. *Let* $(\mathbf{L}, \tilde{e}, \alpha, \beta)$ *be a financial system with* $\alpha, \beta \in [0, 1)$. *Suppose that* $\mathcal{I}_0$ *is a proper subset of* $\mathcal{N}$: $\varnothing \subsetneq \mathcal{I}_0 \subsetneq \mathcal{N}$. *Suppose that the costs for the merger are* $\kappa = 0$. *Let* L^* *be the corresponding greatest clearing vector and suppose that there exists a bank* k *such that* $L_k^* < \bar{L}_k$, *which satisfies at least one of the following two conditions*:

1. $\tilde{e}_k > 0$.
2. *There exists* $j \ne k$ *such that* $L_j^* \pi_{jk} > 0$.

Then there exists a rescue consortium.

Remark 4.13. Theorem 4.11 essentially tells us that as long as the losses caused by defaults of banks are greater than the costs for mergers, rescue consortia exist.

We see that in the presence of default costs a rescue consortium exists under very mild regularity conditions. The two conditions stated in Corollary 4.12 have a clear interpretation. The first condition requires that one defaulting bank with strictly positive net assets exist. This is a very natural assumption and is very often satisfied. The second condition requires the existence of at least one defaulting bank and another bank in the network that after clearing still has strictly positive liabilities to the defaulting bank. This condition is also often satisfied.

The statement of Corollary 4.12 can also be motivated by considering a *megamerger* in which all banks in the financial network merge into. The value of such a bank is $\sum_{i=1}^{n} e_i$ and hence strictly positive if at least one bank in the system has strictly positive net assets. Moreover, by definition such a megabank cannot go bankrupt because it does not have any liabilities to satisfy.

It is important to note that in the presence of costs for a merger, however, such megamergers will usually not be optimal for the network.

Key in all considerations regarding rescue consortia is that we assume that once the net assets are reduced, banks have time to actually form a rescue consortium. Hence, it is assumed that the default of a bank does not automatically trigger the clearing procedure, but banks may merge before clearing takes place. This is an important assumption. In a different setup, Leitner (2005) discusses the importance of a coordinating device, once default or possible contagion is unavoidable. We suppose that this coordination mechanism is available here too; provided the conditions of Theorem 4.11 are satisfied, a rescue consortium will be able to form.

We have seen so far that the existence of default costs is necessary for the existence of rescue consortia. One could take this further and assume that the parameters α, β could be chosen to some extent by regulators if they confiscate or tax assets of failing banks. Then there would be a much higher chance that the system would sort out its problem on its own without relying on bailouts using external money.

5. Examples

We consider now two symmetric examples and one asymmetric example to illustrate our results.

5.1. Circular Network

Definition 5.1. Let $n = 2N$, $N \in \mathbb{N}$, the (even) number of banks in the network. We refer to a financial system $(\mathbf{L}, e, \alpha, \beta)$ as *circular* if the net assets are $e = (1-\varepsilon, 1+\varepsilon, \ldots)^\top \in \mathbb{R}^n$ and the liabilities matrix $\mathbf{L} \in \mathbb{R}^{n\times n}$ is given by

$$\mathbf{L} = \begin{pmatrix} 0 & a & \cdots & 0 & 0 \\ 0 & 0 & a+\varepsilon & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & a \\ a+\varepsilon & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $a > 0$, $0 < \varepsilon \le 1$.

Figure 1 shows such a circular financial system. We see that the value of all banks is 1 and there are two types of banks. We investigate the effects of changing the net asset vector e to γe where $0 < \gamma \le 1$. By construction all banks are solvent if $\gamma = 1$.

We apply the GA algorithm from Definition 3.6 in the following. We set $\mu = 0$ and $\Lambda^{(0)} = \bar{L}$, where $\bar{L}_i = a$ if i is odd and $\bar{L}_i = a + \varepsilon$ if i is even. Then,

$$\begin{aligned} v_i^{(0)} &= \sum_{j=1}^n \Lambda_j^{(0)} \pi_{ji} + e_i - \bar{L}_i \\ &= \begin{cases} \varepsilon + \gamma(1-\varepsilon) & \text{if } i \text{ is odd}, \\ \gamma(1+\varepsilon) - \varepsilon & \text{if } i \text{ is even}. \end{cases} \end{aligned}$$

From there we see immediately that banks with odd indices cannot be level-0 insolvent, and all banks with an even index are level-0 insolvent if and only if

$$\gamma(1+\varepsilon) < \varepsilon \quad \Leftrightarrow \quad \gamma < \frac{\varepsilon}{1+\varepsilon}.$$

In the following we will always assume that

$$0 < \gamma < \frac{\varepsilon}{1+\varepsilon}, \tag{12}$$

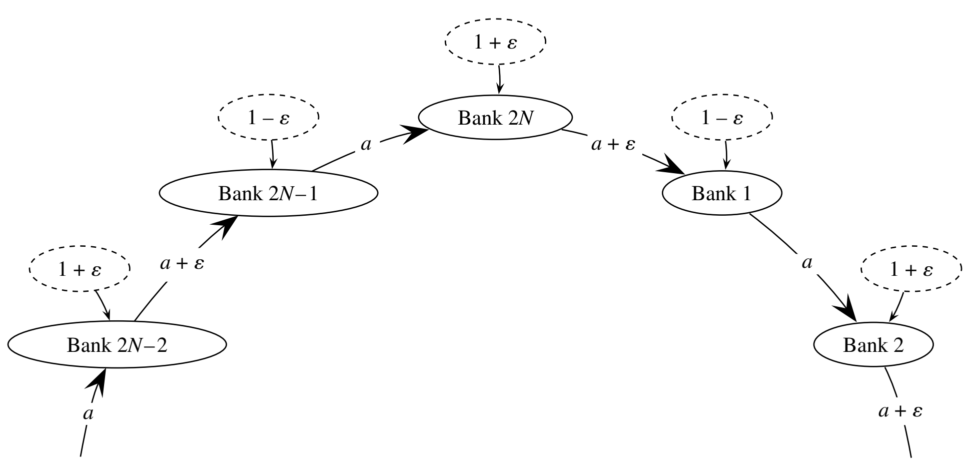
and hence default is triggered. Then $\mathcal{I}_0 = \{2, 4, \ldots, 2N\}$ and $\mathcal{S}_0 = \{1, 3, \ldots, 2N-1\}$. From the definition of the GA algorithm, we obtain

$$\Lambda_i^{(1)} = \begin{cases} a & \text{if } i \text{ is odd}, \\ \alpha\gamma(1+\varepsilon) + \beta a & \text{if } i \text{ is even}. \end{cases}$$

We now set $\mu = 1$ and investigate whether any level-1 insolvencies occur. We compute

$$\begin{aligned} v_i^{(1)} &= \sum_{j=1}^n \Lambda_j^{(1)} \pi_{ji} + e_i - \bar{L}_i \\ &= \begin{cases} \alpha\gamma(1+\varepsilon) + \beta a + \gamma(1-\varepsilon) - a & \text{if } i \text{ is odd}, \\ \gamma(1+\varepsilon) - \varepsilon & \text{if } i \text{ is even}. \end{cases} \end{aligned}$$

Figure 1 **Circular Arrangement**



We find that the banks with an odd index are level-1 insolvent if and only if

$$\alpha\gamma(1+\varepsilon)+a(\beta-1)+\gamma(1-\varepsilon)<0$$
$$\Leftrightarrow \quad \gamma < \frac{a(1-\beta)}{\alpha(1+\varepsilon)+(1-\varepsilon)}. \tag{13}$$

Note that this condition cannot be satisfied for $\beta = 1$. If this condition is not satisfied, $\Lambda^{(1)}$ is the clearing vector.

Suppose now that (13) holds. Then $\mathcal{I}_1 = \{1, 2, \ldots, 2N\}$, $\mathcal{S}_1 = \varnothing$. Furthermore,

$$\begin{aligned} x_1 &= \alpha\gamma(1-\varepsilon)+\beta x_{2N}, \\ x_2 &= \alpha\gamma(1+\varepsilon)+\beta x_1, \\ &\cdots \\ x_{2N-1} &= \alpha\gamma(1-\varepsilon)+\beta x_{2N-2}, \\ x_{2N} &= \alpha\gamma(1+\varepsilon)+\beta x_{2N-1}. \end{aligned}$$

One can check that

$$\Lambda_i^{(2)} = \begin{cases} \dfrac{\alpha\gamma}{1-\beta^2}((1-\varepsilon)+\beta(1+\varepsilon)) & \text{if } i \text{ is odd,} \\ \dfrac{\alpha\gamma}{1-\beta^2}(\beta(1-\varepsilon)+(1+\varepsilon)) & \text{if } i \text{ is even.} \end{cases}$$

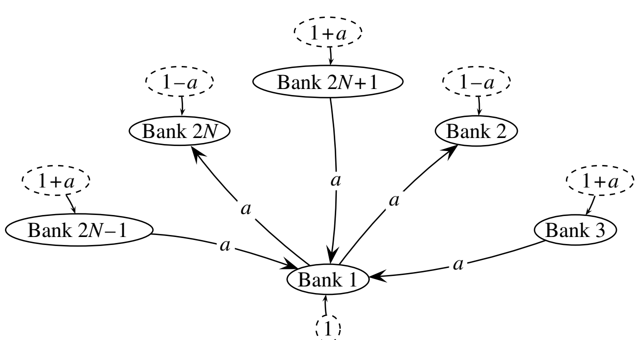
Because $\mathcal{I}_1 = \{1, 2, \ldots, 2N\} = \mathcal{I}_2$, the algorithm terminates and $\Lambda^{(2)}$ is the greatest clearing vector.

Now we investigate whether a rescue consortium exists. Suppose that (12) holds and suppose the costs for a merger in this network are $\kappa = 0 \in \mathbb{R}^{2N}$. Let $\alpha \in [0,1), \beta \in [0,1]$. Because $\gamma(1+\varepsilon) > \gamma(1-\varepsilon) > 0$, we obtain from Corollary 4.12 the existence of a rescue consortium, and the existence does not depend on whether (13) holds. In this example, the rescuing procedure leads to a megamerger of all banks in the network.

5.2. Star Network

We now consider a star arrangement, as shown in Figure 2.

Figure 2 **Star Arrangement**



Definition 5.2. Let $n = 2N+1$, $N \in \mathbb{N}$, the (odd) number of banks in the network. We refer to a network $(\mathbf{L}, e, \alpha, \beta)$ as *star shaped* if the net assets are $e = (1, 1-a, 1+a, 1-a, 1+a, \ldots\ldots)^\top$ and the liabilities matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{pmatrix} 0 & a & 0 & a & \cdots & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $0 < a < 1$.

Note that $\bar{L}_1 = Na$, $\bar{L}_i = a$ if $i > 1$ and i is odd, and $\bar{L}_i = 0$ for all even i.

Again we investigate the effects of changing the net asset vector e to γe where $0 < \gamma \le 1$. Again we use the GA algorithm from Definition 3.6. Let $\mu = 0$ and $\Lambda^{(0)} = \bar{L}$, where $\bar{L}_i = a$ if i is odd and $\bar{L}_i = 0$ if i is even. Then,

$$\begin{aligned} v_i^{(0)} &= \sum_{j=1}^n \Lambda_j^{(0)} \pi_{ji} + e_i - \bar{L}_i \\ &= \begin{cases} \gamma & \text{if } i = 1, \\ \gamma(1+a)-a & \text{if } i > 1, i \text{ is odd,} \\ \gamma(1-a)+a & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Hence, only banks with odd indices > 1 can become level-0 insolvent, and this happens if and only if $\gamma(1+a) < a$. Therefore, we will assume in the following that

$$0 < \gamma < \frac{a}{1+a}. \tag{14}$$

Then $\mathcal{I}_0 = \{3, 5, \ldots, 2N+1\}$, $\mathcal{S}_0 = \{1, 2, 4, \ldots, 2N\}$. Furthermore, $\Lambda_1^{(1)} = \bar{L}_1 = aN$, $\Lambda_i^{(1)} = \bar{L}_i = 0 \; \forall \, i \in \mathcal{S}_0 \backslash \{1\}$, and $\Lambda_i^{(1)} = \alpha\gamma(1+a) \; \forall \, i \in \mathcal{I}_0$. Now we set $\mu = 1$ and compute

$$\begin{aligned} v_i^{(1)} &= \sum_{j=1}^n \Lambda_j^{(1)} \pi_{ji} + e_i - \bar{L}_i \\ &= \begin{cases} \gamma + N\alpha\gamma(1+a) - Na & \text{if } i = 1, \\ \gamma(1+a)-a & \text{if } i > 1, i \text{ is odd,} \\ \gamma(1-a)+a & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

We find that bank 1 is level-1 insolvent if and only if

$$\gamma + N\alpha\gamma(1+a) - Na < 0$$
$$\Leftrightarrow \gamma < \frac{Na}{1+N\alpha(1+a)}. \tag{15}$$

If (15) is not satisfied, then $\Lambda^{(1)}$ is the clearing vector. If (15) is true, then $\mathcal{I}_1 = \{1, 3, 5, \ldots, 2N+1\}$, $\mathcal{S}_1 = \{2, 4, \ldots, 2N\}$. Furthermore, $\Lambda_i^{(2)} = 0 \ \forall i \in \mathcal{S}_1$ and

$$\begin{aligned} x_1 &= \alpha\gamma + \beta(x_3 + \cdots + x_{2N+1}) \\ x_3 &= \alpha\gamma(1+a), \\ &\ldots \\ x_{2N+1} &= \alpha\gamma(1+a). \end{aligned}$$

Hence, $\Lambda_1^{(2)} = \alpha\gamma + \beta N\alpha\gamma(1+a)$ and $\Lambda_i^{(2)} = \alpha\gamma(1+a)$ for all $i \in \mathcal{I}_1 \backslash \{1\}$. Then $\mathcal{I}_2 = \mathcal{I}_3$, and hence $\Lambda^{(2)}$ is the greatest clearing vector.

Again, we are interested in the existence of a rescue consortium. Suppose that (14) holds and the costs for merger in this network are $\kappa = 0 \in \mathbb{R}^{2N+1}$. Let $\alpha \in [0,1)$, $\beta \in [0,1]$. Because $\gamma > 0$ and $\gamma(1+a) > 0$, we obtain from Corollary 4.12 the existence of a rescue consortium, and the existence does not depend on whether (15) holds.

5.3. Asymmetric Network

Finally, we consider an asymmetric network consisting of six banks. Figure 3 shows the liabilities structure in this network and in (16) the liabilities matrix **L** and net assets e are given.

$$L = \begin{pmatrix} 0 & 4.94 & 2.47 & 5.59 & 0 & 0 \\ 6 & 0 & 0 & 2 & 0 & 0 \\ 0 & 13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 12 & 0 & 0 & 0 & 0 & 0 \\ 2.79 & 6.21 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 1 \\ 11.51 \\ 1.4 \\ 12.5 \\ 2 \end{pmatrix}. \tag{16}$$

To avoid level-0 insolvencies the net assets have to exceed

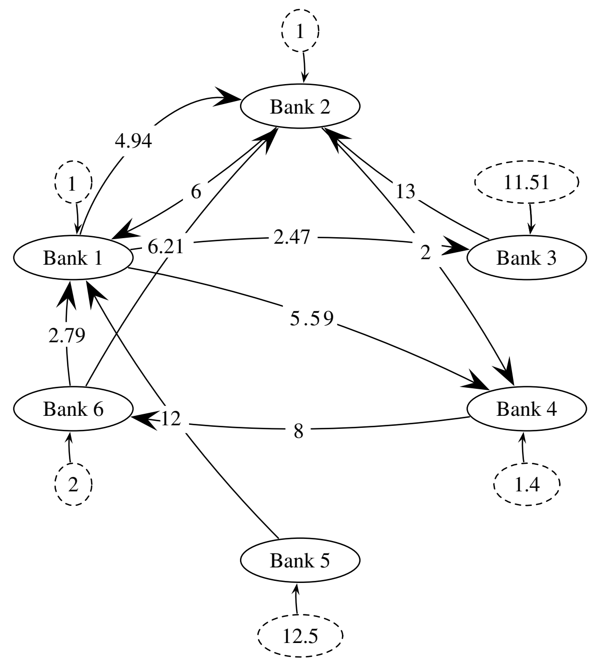
$$\xi = \max\{0, -(\Pi^\top \bar{L} - \bar{L})\} = (0, 0, 10.53, 0.41, 12, 1)^\top.$$

From this we see immediately that banks 3, 4, 5, and 6 can become level-0 insolvent. We see that if the net assets e are reduced to γe with $0 < \gamma < 1$, the first condition of Corollary 4.12 is still satisfied and that therefore a rescue consortium also exists in this asymmetric example if we assume $\kappa = 0 \in \mathbb{R}^6$.

6. Assessing and Controlling Contagion Risks

We have seen how default may spread through a financial system. A financial system that is initially solvent may get into difficulties because the net assets e fall to (or are revalued to) some lower value $\tilde{e}$, whereupon the set $\mathcal{I}_0$ of level-0 insolvent banks becomes nonempty; one or more banks are in difficulties. Section 3 explains what happens if the problems are left unchecked: without intervention, the recoverable amounts of the interbank loans fall from the nominal $\bar{L}$ to the greatest clearing vector L^*, and insolvency spreads from the initial level-0 set to level-1 insolvencies, level-2 insolvencies, etc. Section 4 explains what *can* happen if a group of solvent banks gets together and rescues the level-0 insolvent banks, and the circumstances in which there can be a rescue consortium with both the resources and the incentive to save the level-0 insolvents. However, what *can* happen is not necessarily what *does* happen, and, in general, a member of a rescue consortium would prefer *not* to take part in the rescue, because doing so will incur costs for the rescuer. It is far better to let someone else stand up and take the bullets! It may be that a potential rescuer may see benefits in the rescue that make the cost worthwhile; for example, the rescued banks may have strengths in geographical or business areas where the rescuer is weak, so the combination would strengthen the rescuer. But if no such synergy exists, the regulator must either have power to compel the banks that it regulates to participate in a rescue for the good of the system as a whole or else must have available sufficient cash to bail out the failing banks, which is the solution applied in the crisis of 2007–2008. This solution has generated huge resentment in the democracies that resorted to the emergency bailouts, as it was seen quite correctly as the state being left with an enormous bill, while the banks are able to continue much as before. The alternative, which involves regulators being equipped

Figure 3 Asymmetric Arrangement



with powers to compel banks to take part in rescues, would prove much more palatable to the electorates, though how it could be made to function in a landscape of overlapping regulatory responsibilities and transnational banks is far from clear. Nevertheless, we shall attempt in this section to make some suggestions in this direction.

But before we do this, we offer some possible diagnostics for the fragility of a financial system. We propose to base these on a thought experiment in which the eternal assets e of the individual banks are reduced by a common fraction;[3] as the losses mount, the pressure on the system increases until the set $\mathscr{I}_0$ of level-0 insolvent banks becomes nonempty. So we consider the situation where the net assets e have been reduced to $\tilde{e} = \gamma e$ for some $0 < \gamma < 1$ and the set $\mathscr{I}_0$ is nonempty. First we calculate the value $\tilde{V}$ of the banks with net assets $\tilde{e}$ on the assumption that all interbank liabilities are paid in full (8), and then we calculate the losses ΔV that are suffered if defaults occur, as given by (9). We then attempt to assemble a rescue consortium, starting with all banks in $\mathscr{I}_1$, recruiting banks in decreasing order of their potential losses ΔV_i, then moving on to $\mathscr{I}_2$ again adjoining banks in decreasing order of ΔV_i, continuing in this fashion through $\mathscr{I}_3, \mathscr{I}_4 \ldots$ until if necessary we add in banks that are in $\mathcal{N} \setminus \bigcup_\mu \mathscr{I}_\mu$, which would in any case survive the wave of bankruptcies, but nonetheless might suffer losses in the process. If at some stage we achieve a rescue consortium, then this rescue happens; otherwise, no rescue is possible, and the failure spreads through the network to bring down all the banks in $\bigcup_\mu \mathscr{I}_\mu$. Either way, we record the overall remaining value $v(\gamma)$ of the financial system. If a rescue consortium A exists, then we obtain from Lemma 4.7 that

$$\begin{aligned} v(\gamma) &= \sum_{i\in A} \tilde{V}_i - \sum_{j\in\mathscr{I}_0} \delta_j - \sum_{k\in A\cup\mathscr{I}_0} \kappa_k + \sum_{i\in\mathcal{N}\setminus(A\cup\mathscr{I}_0)} \tilde{V}_i \\ &= \sum_{i\in\mathcal{N}} (\Pi^\top \bar{L} + \gamma e - \bar{L})_i - \sum_{k\in A\cup\mathscr{I}_0} \kappa_k \\ &= \gamma\bar{e} - \sum_{k\in A\cup\mathscr{I}_0} \kappa_k, \end{aligned}$$

where $\bar{e} := \sum_{i\in\mathcal{N}} e_i$ is the initial value of the financial system before it was subjected to stress. We used the notation $L^*(\gamma)$ to indicate that the greatest clearing vector does depend on γ. If a rescue consortium does not exist, then we obtain from Lemma 4.8 that

$$\begin{aligned} v(\gamma) &= \sum_{i\in\mathcal{N}} \mathscr{V}(L^*(\gamma), \gamma e)_i \\ &= \gamma\bar{e} - \sum_{i=1}^{n} \left((1-\alpha)\gamma e_i + (1-\beta) \sum_{j=1}^{n} L_j^*(\gamma)\pi_{ji} \right) \mathbb{1}_{\{L_i^*(\gamma) < \bar{L}_i\}}. \end{aligned}$$

[3] This could represent a global decline in the value of assets used as collateral or the effects of a recession. The assumption is probably an oversimplification, but is the simplest story that could be told.

We then plot the function

$$\begin{aligned} \gamma \mapsto \lambda(\gamma) &:= \frac{\bar{e} - v(\gamma)}{\bar{e}} \\ &= \begin{cases} 1 - \gamma + \dfrac{\sum_{i=1}^{n} ((1-\alpha)\gamma e_i + (1-\beta)\sum_{j=1}^{n} L_j^*(\gamma)\pi_{ji}) \mathbb{1}_{\{L_i^*(\gamma) < \bar{L}_i\}}}{\bar{e}} \\ \qquad\qquad \text{without rescuing,} \\ 1 - \gamma + \dfrac{\sum_{k\in A\cup\mathscr{I}_0} \kappa_k}{\bar{e}} \quad \text{with rescuing.} \end{cases} \end{aligned}$$

The function λ measures the relative losses due to default. It measures the difference between the initial value of the system and the value of the stressed system divided by the initial value of the system. Note that the conditions in Corollary 4.12 on the existence of a rescue consortium are either satisfied for all $0 < \gamma < 1$ or not at all.

Another informative plot we can present will display the function

$$\gamma \mapsto \eta(\gamma) := \frac{|\{i \in \mathcal{N} \mid L_i^*(\gamma) < \bar{L}_i\}|}{|\mathcal{N}|}, \tag{17}$$

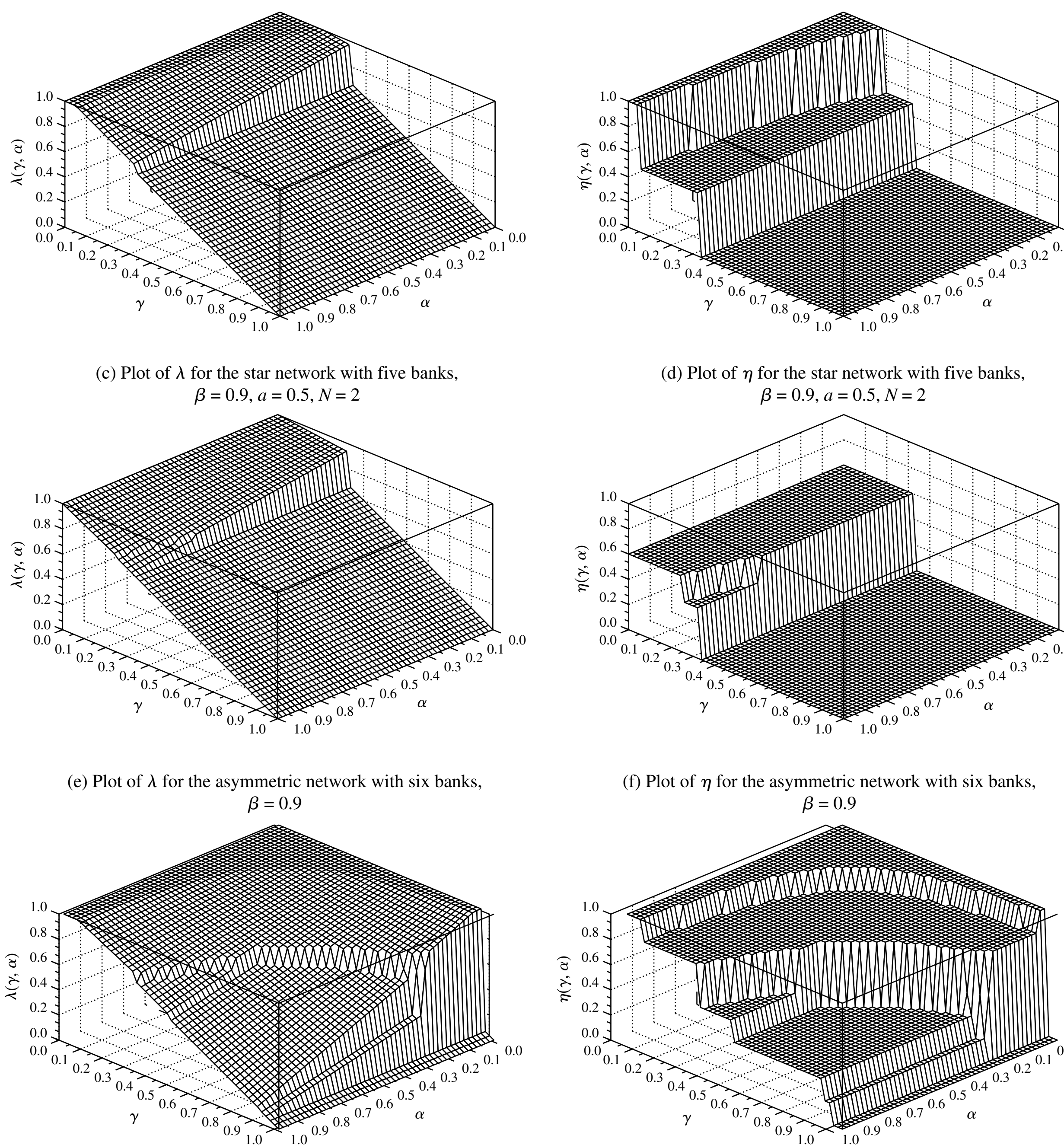
where $L^*(\gamma)$ is the greatest clearing vector if the net assets are given by $\tilde{e} = \gamma e$. This shows how the proportion of banks defaulting grows as γ decreases.

In the following we present these plots in Figures 4 and 5 for our examples from §5, assuming that no rescue consortium forms. This is for illustration only, as we know already from §5 that rescue consortia exist in all three examples. We immediately see a very different default behavior in the three networks.

We have chosen the parameters in the symmetric networks such that the initial default conditions (12) and (14) are satisfied at the same time. In all three examples we observe that if we decrease γ, i.e., if we decrease the net assets, both the losses due to default and the proportion of defaulting banks increase. This is an obvious result; what is more interesting is the exact behavior of this surface, which we can show in full detail. The sensitivity with respect to α is also very natural. The larger α is, the sounder the system is. In the asymmetric example we can see the effect of α particularly clearly. For large values of α we can observe almost individual defaults if we decrease γ. For small values of α we see many banks defaulting at once.

Furthermore, note the difference between Figure 4 and Figure 5. Figure 4 assumes $\beta = 0.9$, whereas Figure 5 presents the same examples for $\beta = 1$. We know already from our theoretical analysis that we will only find a total collapse of the circular network if $\beta < 1$. In our examples, the star-shaped network is the only example where some banks will never default no matter how we reduce γ, α, or β. This is because their liabilities are 0, so they can always satisfy them.

Figure 4 Plot of the Functions λ and η Considered as Functions in γ and α Without Rescuing for $\beta = 0.9$



(a) Plot of λ for the circular network with six banks, $\beta = 0.9$, $a = 1$, $\epsilon = 0.5$, $N = 3$

(b) Plot of η for the circular network with six banks, $\beta = 0.9$, $a = 1$, $\epsilon = 0.5$, $N = 3$

(c) Plot of λ for the star network with five banks, $\beta = 0.9$, $a = 0.5$, $N = 2$

(d) Plot of η for the star network with five banks, $\beta = 0.9$, $a = 0.5$, $N = 2$

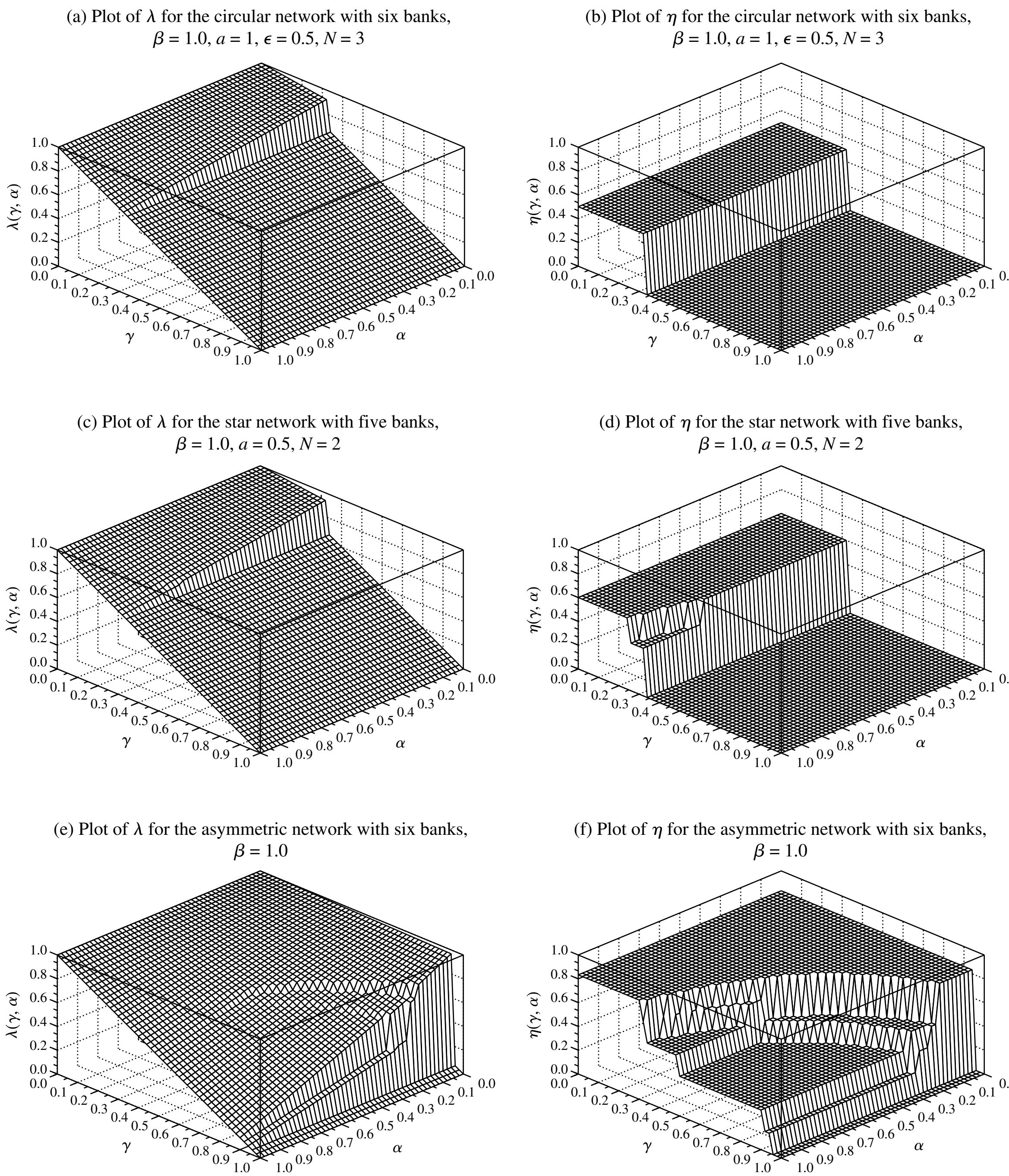
(e) Plot of λ for the asymmetric network with six banks, $\beta = 0.9$

(f) Plot of η for the asymmetric network with six banks, $\beta = 0.9$

These diagnostics give a readily understood picture of how fragile the given banking network is and are more informative than any attempt to define a single scalar "fragility index" for a financial system. They could be used to warn of looming dangers in the financial system, allowing time for corrective measures to be put in place.

As we have noted, left to their own devices, banks would be unwilling to step forward to take on the losses of the failing banks, and without the possibility of compulsion and some kind of coordination, a chaotic collapse would be the likely outcome. Here there is a role for a bank regulator. We shall suppose that the regulator is able to observe the entire financial system; in particular, he knows whether the banks in the level-1 insolvency set $\mathcal{I}_1$ are capable of rescuing the level-0 insolvent banks. If this is the case, then we propose that the rescue consortium will be made up of level-1 banks, which are, after all, those most perilously exposed to the level-0 banks. In the

Figure 5 **Plot of the Functions λ and η Considered as Functions in γ and α Without Rescuing for $\beta = 1$**



interests of containing the spread of contagion, we propose that the regulator should act as far as possible to leave banks that are distant from the level-0 banks completely unimpaired. If the banks in $\mathcal{I}_1$ cannot mount a rescue, then the regulator will widen the net and try to assemble a rescue from the banks in $\mathcal{I}_1 \cup \mathcal{I}_2$, and so on. For the sake of discussion,[4] let us suppose that the banks in $\mathcal{I}_1$ are capable of mounting a rescue. Then the regulator should be empowered to compel that group of banks to rescue the banks in $\mathcal{I}_0$. Different possible mechanisms could be proposed; here are a few that might be considered.

(i) The simplest possibility would be that each bank in $\mathcal{I}_1$ contributes to the bailout costs in proportion to the losses (see (9)) ΔV that they would experience if default were to occur, and each bank receives shares in the rescued banks in proportion to their contribution.

[4] There is no loss of generality here: it is just notationally more compact.

(ii) An alternative would be for the regulator to receive from each bank i in $\mathcal{I}_1$ a sealed bid for the fraction α_i of the defaulting banks that it was willing to take on, which would of course imply that bank i would assume responsibility for a fraction α_i of the bailout costs. This would allow banks to bid higher if they thought that taking over the failing banks might be advantageous to their own business. If $\sum_{i\in\mathcal{I}_1}\alpha_i \geq 1$, then the bids received are sufficient to cover the bailout costs, and banks in $\mathcal{I}_1$ are allocated fractions of the defaulting banks' assets and liabilities proportional to their bids α_i. If the total fractions bid fall short of 1, then each bank in $\mathcal{I}_1$ contributes to the bailout proportionally to its potential losses ΔV as in mechanism (i) above, but receives a fraction of the defaulting banks proportional to its bid. Thus, a bank that bids zero would pay nothing toward the bailout if the total fraction bid by the other banks was at least 1, but it would then run the risk that the total was less than 1, in which case it would end up contributing to the rescue, but not receiving any part of the assets of the rescued bank. This threat would hopefully induce banks to make a realistic offer toward the bailout.

(iii) Another somewhat riskier resolution mechanism is to allow the regulator to seize the assets of any failing bank, which would then pay out nothing to any bank to which it owed money. Those assets could now be used to compensate depositors, with any not used in this way being held by the government. The main reason for allowing the regulator to seize the assets of a failing bank would be to give other banks in the system a very strong incentive to mount a rescue. This mechanism would also give failing banks a stronger position when bargaining with other more solvent banks. However, this mechanism would probably not be preferred because as it stands it does not guarantee that a catastrophic meltdown would be avoided, even were that possible.

The mechanisms proposed here all have the property that the regulatory authority is not required to inject any cash to rescue the failing banks, which would presumably be preferred by a rational democratic government; the first two mechanisms would imply that the *total* cost to the financial system would be just the bailout costs, which is as low as it could possibly be.

Notice that although these proposals can arguably be effective in containing the damage caused by failure in an interconnected financial network, they would not work in the similar context of a failing sovereign nation, which could not be liquidated and shared among a consortium of rescuers, as a bank could.

7. Conclusion

This paper has extended the modelling framework of Eisenberg and Noe (2001) to allow for the fact that when a bank defaults and has to call in its loans, it never realizes the face value of those loans but instead suffers a real loss. We used this to show that without intervention a failure of one or more banks could spread through the financial system, destroying value and taking down more banks as a result. For a given network, we are able to find what the network would look like when the spread of default is finished; we can work out which banks—though initially solvent—will fail if the banks initially in difficulties fail—and then we can work out what other banks will fail as a consequence. Dominoes of default sweep through the network until eventually only solvent banks remain.

We are then able to analyze how failing banks might be rescued by consortia of other banks, establishing the important results that any consortium that has an incentive to rescue the failing banks also has the means and that any consortium of banks that would fail if default were allowed to spread would have an incentive to rescue if it had the means. These are hopeful conclusions, but not enough to ensure that failing banks will be rescued. We could have a situation where a group of banks might have the means to effect a rescue, but no incentive, as could happen if the banking network was in two geographical locations with weak linkages between them. Moreover, it does not deal with the moral hazard issue; a bank would prefer to let another bank do the rescuing and would indeed have to have a good reason to act apparently against shareholder interests by bailing out a failing bank. We therefore see no alternative to some regulatory backstop compulsion if the banking sector is to collectively make good the losses of some of its members that would threaten further losses and overall stability. Otherwise, governments will again be the unconscious underwriters of risky but profitable banking activities. Any framework of effective regulatory legislation runs the risk of driving financial services into the least regulated jurisdiction, so some effective international coordination would be needed here. We offer some thoughts on how a rescue mechanism might look, but do so tentatively, well aware of the complexities lurking around these issues.

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Appendix A. Proof of §2

PROOF OF LEMMA 2.7. (i) The first property follows immediately from the definition of Φ and the fact that $\alpha, \beta \in (0, 1]$.

(ii) Let $\mathcal{I} = \{i: \bar{L}_i > e_i + \sum_{j=1}^n L_j \pi_{ji}\}$ and $\tilde{\mathcal{I}} = \{i: \bar{L}_i > e_i + \sum_{j=1}^n \tilde{L}_j \pi_{ji}\}$. Clearly, $\mathcal{I} \subseteq \tilde{\mathcal{I}}$. For any $i \in \mathcal{I}$, the second alternative of the definition (1) obtains for both L and $\tilde{L}$, so it follows that $\Phi(\tilde{L})_i \le \Phi(L)_i$ for such i. For any $i \in \tilde{\mathcal{I}} \backslash \mathcal{I}$, because $\alpha, \beta \in (0, 1]$, we have

$$\Phi(\tilde{L})_i = \alpha e_i + \beta \sum_{j=1}^n \tilde{L}_j \pi_{ji} \le e_i + \sum_{j=1}^n \tilde{L}_j \pi_{ji} < \bar{L}_i = \Phi(L)_i. \quad \square$$

Appendix B. Proof of §3

Proof of Theorem 3.7. We shall first prove that

$$\Lambda^{(\mu+1)} \le \Lambda^{(\mu)} \quad \forall \mu = 0, \dots, n-1.$$

The proof proceeds by induction. To start the induction, we prove that $\Lambda^{(1)} \le \Lambda^{(0)} = \bar{L}$. According to step 5 of the GA algorithm, we have $\Lambda_i^{(1)} = \bar{L}_i = \Lambda_i^{(0)}$ for $i \in \mathcal{S}_0$, so now we just identify $\Lambda^{(1)}$ on the insolvency set $\mathcal{I}_0$, as specified by (7). We construct the unique solution x to (7) by a recursive method, starting from $x^{(0)} = \Lambda^{(0)}$ and defining recursively the $x^{(k)}$ by

$$x_i^{(k+1)} = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_0} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_0} x_j^{(k)} \pi_{ji} \right\}, \quad i \in \mathcal{I}_0. \tag{B1}$$

Now for $i \in \mathcal{I}_0$, we have

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_0} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_0} \Lambda_j^{(0)} \pi_{ji} \right\} \\ &\le e_i + \sum_{j=1}^n \Lambda_j^{(0)} \pi_{ji} < \bar{L}_i = \Lambda_i^{(0)} = x_i^{(0)}, \end{aligned} \tag{B2}$$

where the first inequality holds because $\alpha, \beta \in (0, 1]$ and $\Lambda^{(0)} = \bar{L}$ on $\mathcal{S}_0$, and the second inequality holds because of the definition of the insolvency set $\mathcal{I}_0$. Thus, we see that $x^{(1)} \le x^{(0)}$, so the sequence $x^{(k)}$ decreases to begin with, and hence because of the recursive definition decreases thereafter. The limit $x := \downarrow \lim_{k\to\infty} x^{(k)}$ solves (7). Hence, we find that $\Lambda^{(1)} \le \Lambda^{(0)}$. We will prove the uniqueness of the solution to (7) later.

To do the induction step from μ to $\mu+1$, we observe that the induction hypothesis is that $\Lambda^{(\mu)} \le \Lambda^{(\mu-1)}$; hence, $\mathcal{S}_\mu \subseteq \mathcal{S}_{\mu-1}$. So for all $j \in \mathcal{S}_\mu$, we have $\Lambda_j^{(\mu+1)} = \Lambda_j^{(\mu)} = \bar{L}_j$. Now we again construct the unique solution x to (7) by the obvious modification of the recursive recipe (B1):

$$x_i^{(k+1)} = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\mu} x_j^{(k)} \pi_{ji} \right\}, \quad i \in \mathcal{I}_\mu, \tag{B3}$$

this time starting with $x_i^{(0)} = \Lambda_i^{(\mu)}$ for $i \in \mathcal{I}_\mu$. The inequality (B2) evolves to

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\mu} \Lambda_j^{(\mu)} \pi_{ji} \right\} \\ &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{S}_{\mu-1} \backslash \mathcal{S}_\mu} \Lambda_j^{(\mu)} \pi_{ji} + \sum_{j \in \mathcal{I}_{\mu-1}} \Lambda_j^{(\mu)} \pi_{ji} \right\} \\ &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{S}_{\mu-1} \backslash \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\mu-1}} \Lambda_j^{(\mu)} \pi_{ji} \right\} \\ &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_{\mu-1}} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\mu-1}} \Lambda_j^{(\mu)} \pi_{ji} \right\}, \end{aligned} \tag{B4}$$

because $\Lambda^{(\mu)} = \bar{L}$ on $\mathcal{S}_{\mu-1}$. Now from (B4) we see that for $i \in \mathcal{I}_{\mu-1}$, we have $x_i^{(1)} = \Lambda_i^{(\mu)} = x_i^{(0)}$, and for $i \in \mathcal{I}_\mu \backslash \mathcal{I}_{\mu-1}$, the expression (B4) is

$$\begin{aligned} &\alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_{\mu-1}} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\mu-1}} \Lambda_j^{(\mu)} \pi_{ji} \right\} \\ &\quad \le e_i + \sum_{j \in \mathcal{S}_{\mu-1}} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\mu-1}} \Lambda_j^{(\mu)} \pi_{ji} \\ &\quad = e_i + \sum_{j=1}^n \Lambda_j^{(\mu)} \pi_{ji} \\ &\quad < \bar{L}_i = \Lambda_i^{(\mu)} = x_i^{(0)}. \end{aligned}$$

This ensures that the sequence $x^{(k)}$ starts decreasing and therefore will always be decreasing. Whatever the limit is, it cannot be bigger than $x^{(0)} = \Lambda^{(\mu)}$, so we learn that $\Lambda^{(\mu+1)} \le \Lambda^{(\mu)}$, as required.

The next task is to prove that $\Lambda^{(\mu)} \ge L^*$ for all μ, again by induction. It is clearly true when $\mu = 0$, so we now assume it is true up to μ and try to extend it to $\mu+1$. If we make the natural notation

$$\mathcal{I}_* = \left\{ i: e_i + \sum_j L_j^* \pi_{ji} < \bar{L}_i \right\},$$

then by the inductive hypothesis, we have $\mathcal{I}_\mu \subseteq \mathcal{I}_*$. Observe that for $i \in \mathcal{S}_\mu$ we have $\Lambda_i^{(\mu+1)} = \bar{L}_i \ge L_i^*$, so we just have to confirm that $\Lambda^{(\mu+1)} \ge L^*$ also on $\mathcal{I}_\mu$. For this, we return to the recursive construction (B3), and notice that when we start with $x_i^{(0)} = \Lambda_i^{(\mu)}$ on $\mathcal{I}_\mu$, we find

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\mu} \Lambda_j^{(\mu)} \pi_{ji} \right\} \\ &\ge \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\mu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\mu} L_j^* \pi_{ji} \right\} \\ &\ge \alpha e_i + \beta \sum_j L_j^* \pi_{ji} = L_i^*, \end{aligned}$$

exploiting the inductive hypothesis for the first inequality and the fact that $\bar{L} \ge L^*$ for the second, and finally using the defining property of L^* for the last equality, bearing in mind that $i \in \mathcal{I}_\mu \subseteq \mathcal{I}_*$. Once we have that $x^{(1)} \ge L^*$ on $\mathcal{I}_\mu$, the recursive recipe (B3) guarantees that $x^{(k)} \ge L^*$ on $\mathcal{I}_\mu$ for all k and hence that the limit $\Lambda^{(\mu+1)}$ is at least L^* on $\mathcal{I}_\mu$.

Next, observe that when the algorithm terminates, as it must, the final vector $\Lambda^{(\mu)}$ is a clearing vector. But we also know that $\Lambda^{(\mu)} \ge L^*$, as has just been proved; because L^* is the greatest clearing vector (Theorem 3.1), the only possibility is $L^* = \Lambda^{(\mu)}$, as required.

To finish the proof we show that the system of linear Equations (7) has a unique solution. We are solving

$$x_i = a_i + \beta \sum_{j \in \mathcal{I}_q} x_j \pi_{ji} \quad \forall i \in \mathcal{I}_q, \tag{B5}$$

where a is some known nonnegative vector. So the question is whether $I - \beta \Pi|_{\mathcal{I}_q \times \mathcal{I}_q}$ has an inverse. In the case where $\beta \in (0, 1)$, there is of course no problem—we have

$$(I - \beta \Pi|_{\mathcal{I}_q \times \mathcal{I}_q})^{-1} = \sum_{n \ge 0} \beta^n (\Pi|_{\mathcal{I}_q \times \mathcal{I}_q})^n, \tag{B6}$$

the sum being absolutely convergent. The case $\beta = 1$ is the only case that requires any thought.

If the largest Perron-Frobenius eigenvalue of $\Pi|_{\mathcal{I}_q\times\mathcal{I}_q}$ is less than 1, then the sum (B6) is convergent and defines the inverse. If not, then the Markov chain on $\mathcal{I}_q$ with transition matrix $\Pi|_{\mathcal{I}_q\times\mathcal{I}_q}$ has a recurrent closed class $C\subseteq\mathcal{I}_q$: $\sum_{l\in C}\pi_{il} = 1$ for all $i\in C$. Let us suppose that q is the first index for which $C\subseteq\mathcal{I}_q$; thus, we have that $C\cap\mathcal{S}_{q-1}\neq\varnothing$. We have for all $i\in C\cap\mathcal{S}_{q-1}$ that

$$\Lambda_i^{(q)} = \bar{L}_i > e_i + \sum_j \Lambda_j^{(q)}\pi_{ji} \geq \sum_{j\in C}\Lambda_j^{(q)}\pi_{ji};$$

and for $i\in C\cap\mathcal{I}_{q-1}$ we have

$$\Lambda_i^{(q)} = \alpha e_i + \sum_j \Lambda_j^{(q)}\pi_{ji} \geq \sum_{j\in C}\Lambda_j^{(q)}\pi_{ji};$$

summing over $i\in C$ leads to the contradiction $\sum_{i\in C}\Lambda_i^{(q)} > \sum_{i\in C}\Lambda_i^{(q)}$, so the hypothesized existence of a closed class $C\subseteq\mathcal{I}_q$ cannot hold, and the noninvertibility of the linear system does not arise. □

Appendix C. Proofs of §4

Proof of Theorem 4.4. (i) This result follows from (10) and (11) and the fact that $\mathcal{V}(L^*, \tilde{e})\geq 0$.

(ii) For $i\in\mathcal{R}$, we have $\mathcal{V}(L^*,\tilde{e}) = 0$, and hence $\Delta V_i = \tilde{V}_i$. □

Proof of Lemma 4.8. Note that in the original financial system $(\mathbf{L}, e, \alpha, \beta)$ the vector $\bar{L}$ is a clearing vector, because $\mathcal{I}_0 = \varnothing$. Hence,

$$\mathcal{V}(\bar{L}, e)_i = (\Pi^\top\bar{L} + e - \bar{L})_i = \sum_{j=1}^n \bar{L}_j\pi_{ji} + e_i - \bar{L}_i \geq 0 \quad \forall\, i\in\mathcal{N}.$$

Summing on i and using the fact that Π is a stochastic matrix[5] leads to the conclusion

$$\sum_{i=1}^n \mathcal{V}(\bar{L}, e)_i = \sum_{i=1}^n e_i. \tag{C1}$$

In the distressed financial system, for a bank i with $L_i^* < \bar{L}_i$ we have

$$L_i^* = \alpha\tilde{e}_i + \beta\sum_{j=1}^n L_j^*\pi_{ji},$$

and hence

$$\begin{aligned}\sum_{j=1}^n L_j^*\pi_{ji} + \tilde{e}_i - L_i^* &= \sum_{j=1}^n L_j^*\pi_{ji} + \tilde{e}_i - \left(\alpha\tilde{e}_i + \beta\sum_{j=1}^n L_j^*\pi_{ji}\right)\\ &= (1-\alpha)\tilde{e}_i + (1-\beta)\sum_{j=1}^n L_j^*\pi_{ji}.\end{aligned}$$

Thus, we can rewrite the value of a bank:

$$\begin{aligned}\mathcal{V}(L^*, \tilde{e})_i &= (\Pi^\top L^* + \tilde{e} - L^*)_i \mathbb{1}_{\{L_i^*\geq\bar{L}_i\}}\\ &= \sum_{j=1}^n L_j^*\pi_{ji} + \tilde{e}_i - L_i^*\\ &\quad - \left((1-\beta)\sum_{j=1}^n L_j^*\pi_{ji} + (1-\alpha)\tilde{e}_i\right)\mathbb{1}_{\{L_i^*<\bar{L}_i\}}.\end{aligned}$$

[5] In fact, there may be rows of Π that are identically zero, but these are rows corresponding to banks that owe nothing to any other bank, and these contribute nothing to the sum.

Again summing on i and using the fact that Π is stochastic gives us

$$\sum_{i=1}^n \mathcal{V}(L^*, \tilde{e})_i = \sum_{i=1}^n \tilde{e}_i - \sum_{i=1}^n\left((1-\beta)\sum_{j=1}^n L_j^*\pi_{ji} + (1-\alpha)\tilde{e}_i\right)\mathbb{1}_{\{L_i^*<\bar{L}_i\}}. \tag{C2}$$

Taking the difference of (C1) and (C2) proves the lemma. □

Proof of Theorem 4.10. Let L^* be the greatest clearing vector in $(\mathbf{L}, \tilde{e}, \alpha, \beta)$. Using the notation from Definitions 4.2 and 4.3, we see that

$$\tilde{V} = \max\{0, \Pi^\top\bar{L} + \tilde{e} - \bar{L}\},$$

$$\delta = \max\{0, -(\Pi^\top\bar{L} + \tilde{e} - \bar{L})\},$$

and hence

$$\begin{aligned}\tilde{V} - \delta &= \Pi^\top\bar{L} + \tilde{e} - \bar{L} = \Pi^\top\bar{L} + e - \bar{L} + (\tilde{e} - e)\\ &= \mathcal{V}(\bar{L}, e) + (\tilde{e} - e),\\ \Delta V &= \tilde{V} - \mathcal{V}(L^*, \tilde{e}) = \mathcal{V}(\bar{L}, e) - \mathcal{V}(L^*, \tilde{e}) + (\tilde{e} - e) + \delta.\end{aligned}$$

By summing over all components of ΔV and applying Lemma 4.8, we obtain

$$\begin{aligned}\sum_{i=1}^n \Delta V_i &= \sum_{i=1}^n (\mathcal{V}(\bar{L}, e)_i - \mathcal{V}(L^*, \tilde{e}))_i + \sum_{i=1}^n(\tilde{e}_i - e_i) + \sum_{i=1}^n \delta_i\\ &= \sum_{i=1}^n (e_i - \tilde{e}_i) + \sum_{i=1}^n\left((1-\alpha)\tilde{e}_i + (1-\beta)\sum_{j=1}^n L_j^*\pi_{ji}\right)\mathbb{1}_{\{L_i^*<\bar{L}_i\}}\\ &\quad - \sum_{i=1}^n(e_i - \tilde{e}_i) + \sum_{i=1}^n\delta_i\\ &= \sum_{i=1}^n\left((1-\alpha)\tilde{e}_i + (1-\beta)\sum_{j=1}^n L_j^*\pi_{ji}\right)\mathbb{1}_{\{L_i^*<\bar{L}_i\}} + \sum_{i=1}^n\delta_i.\end{aligned}$$

Because $\alpha = \beta = 1$, we obtain for all $A\subset\mathcal{N}\setminus\mathcal{I}_0$

$$\sum_{i=1}^n \Delta V_i = \sum_{i=1}^n \delta_i \geq \sum_{i\in A}\Delta V_i,$$

$$\sum_{i=1}^n \delta_i \not> \sum_{i=1}^n \delta_i + \sum_{k\in A\cup\mathcal{I}_0}\kappa_k$$

for all $\kappa\geq 0$. Hence, there is no incentive for any group of banks to form a rescue consortium. □

Proof of Theorem 4.11. Let e be a vector such that $e_i\geq\tilde{e}_i$ for all $i\in\mathcal{N}$ and such that the financial system $(\mathbf{L}, e, \alpha, \beta)$ does not contain a level-0 insolvent bank. (Note that such a vector always exists because we could choose $e = \tilde{e} + \delta$.)

As in the proof of Theorem 4.10, we obtain

$$\sum_{i=1}^n \Delta V_i = \sum_{i=1}^n\left((1-\alpha)\tilde{e}_i + (1-\beta)\sum_{j=1}^n L_j^*\pi_{ji}\right)\mathbb{1}_{\{L_i^*<\bar{L}_i\}} + \sum_{i=1}^n\delta_i.$$

Hence, we find that

$$\begin{aligned}&\sum_{i=1}^n \Delta V_i > \sum_{i=1}^n\delta_i + \sum_{k=1}^n\kappa_k\\ &\quad\Leftrightarrow \sum_{i=1}^n\left((1-\alpha)\tilde{e}_i + (1-\beta)\sum_{j=1}^n L_j^*\pi_{ji}\right)\mathbb{1}_{\{L_i^*<\bar{L}_i\}} > \sum_{k=1}^n\kappa_k.\end{aligned}$$

Hence, one can find a set of indices $A \subset \mathcal{N} \backslash \mathcal{I}_0$ such that $\sum_{i\in A} \Delta V_i > \sum_{k\in A\cup\mathcal{I}_0} \kappa_k$. With Theorem 4.4, the result follows. □

Proof of Corollary 4.12. With $\mu = 0$, one needs to prove that

$$\sum_{i=1}^{n}\left((1-\alpha)\tilde{e}_i + (1-\beta)\sum_{j=1}^{n}\pi_{ji}L_j^*\right)\mathbb{1}_{\{L_i^* < \bar{L}_i\}} > 0$$

and this inequality is true by our assumption that there exists a bank k that satisfies at least one of two properties, and we see that each property guarantees the strict positivity of the expression. □

References

Acharya VV, Yorulmazer T (2007) Too many to fail—An analysis of time-inconsistency in bank closure policies. *J. Financial Intermediation* 16(1):1–31.
Acharya VV, Yorulmazer T (2008) Cash-in-the-market pricing and optimal resolution of bank failures. *Rev. Financial Stud.* 21(6):2705–2742.
Acharya VV, Shin HS, Yorulmazer T (2011) Crisis resolution and bank liquidity. *Rev. Financial Stud.* 24(6):1773–1781.
Aghion P, Bolton P, Fries S (1999) Optimal design of bank bailouts: The case of transition economies. *J. Institutional Theoret. Econom.* 155(1):51–79.
Allen F, Gale D (2000) Financial contagion. *J. Political Econom.* 108(1):1–33.
Amini H, Cont R, Minca A (2012) Stress testing the resilience of financial networks. *Internat. J. Theoret. Appl. Finance* 15(1):1250006-1–1250006-20.
Beyeler WE, Glass RJ, Bech ML, Soramäki K (2007) Congestion and cascades in payment systems. *Physica A: Statist. Mech. Its Appl.* 384(2):693–718.
Cifuentes R, Shin HS, Ferrucci G (2005) Liquidity risk and contagion. *J. Eur. Econom. Assoc.* 3(2–3):556–566.
Cont R, Moussa A, Santos EB (2013) Network structure and systemic risk in banking systems. Fouque JP, Langsam J, eds. *Handbook of Systemic Risk* (Cambridge University Press, Cambridge, UK). Forthcoming.
Cordella T, Yeyati EL (2003) Bank bailouts: Moral hazard vs. value effect. *J. Financial Intermediation* 12(4):300–330.
DeBandt P, Hartmann O (2000) Systemic risk: A survey. Working Paper 35, European Central Bank, Frankfurt, Germany.
Diamond DW, Dybvig PH (1983) Bank runs, deposit insurance, and liquidity. *J. Political Econom.* 91(3):401–419.
Eisenberg L, Noe TH (2001) Systemic risk in financial systems. *Management Sci.* 47(2):236–249.
Elsinger H, Lehar A, Summer M (2006) Risk assessment for banking systems. *Management Sci.* 52(9):1301–1314.
Iyer R, Peydro JL (2011) Interbank contagion at work: Evidence from a natural experiment. *Rev. Financial Stud.* 24(4): 1337–1377.
Leitner Y (2005) Financial networks: Contagion, commitment, and private sector bailouts. *J. Finance* 60(6):2925–2953.
Nier E, Yang J, Yorulmazer T, Alentorn A (2007) Network models and financial stability. *J. Econom. Dynam. Control* 31(6): 2033–2060.
Perotti EC, Suarez J (2002) Last bank standing: What do I gain if you fail? *Eur. Econom. Rev.* 46(9):1599–1622.
Rochet JC, Tirole J, Rajan RG (1996) Interbank lending and systemic risk. *J. Money, Credit Banking* 28(4):733–762.
Staum JC (2013) Counterparty contagion in context: Contributions to systemic risk. Fouque JP, Langsam J, eds. *Handbook of Systemic Risk* (Cambridge University Press, Cambridge, UK). Forthcoming.
Tarski A (1955) A lattice-theoretical fixpoint theorem and its applications. *Pacific J. Math.* 5(2):285–309.
Upper C (2007) Using counterfactual simulations to assess the danger of contagion in interbank markets. BIS Working Paper 234, Bank for International Settlements, Basel, Switzerland.
Upper C, Worms A (2004) Estimating bilateral exposures in the German interbank market: Is there a danger of contagion? *Eur. Econom. Rev.* 48(4):827–849.