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# On the Equilibrium Behavior of a Supply Chain Market for Capacity

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A capacity market is a business-to-business exchange in which equally capable suppliers compete with one another to satisfy generic orders from diverse buyers. The market is asymmetric because the buyers can carry inventory of the products ordered but the suppliers cannot store their capacity. In such a market, we might expect to see something like a price for capacity emerge to equilibrate demand and supply. The financial risk of participating in such a market will be driven by the volatility of the capacity price. In this paper we develop a model to explore the behavior of such a market and demonstrate, for example, that volatility of the price for capacity increases, to a point, when inflexibility of the capacity increases. We can also make statements about how the resolution of price uncertainty in the capacity market is related to the resolution of demand uncertainty faced by the buyers. Another contribution of the paper is to explain the role of market characteristics in how the market acts to minimize shortages caused by consumer demand uncertainty. We use continuous time stochastic optimal control techniques and numerical experiments to demonstrate these insights.

*Key words*: incentives and contracting; supply chain management; stochastic methods
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## 1. Introduction

Supply chain economics is the study of how firms engaged in production and distribution can coordinate their behavior to provide goods and services to consumers in a timely and cost-effective manner. As the cost of information decreases, there are increasingly more opportunities for firms engaged in assembly and distribution to seek out alternative sources of supply and rapidly establish contracts for the delivery of complex goods. Likewise, suppliers are able to rapidly adapt their manufacturing processes to meet demands for diverse specialized products. They are capable of serving many different clients from a single base of installed capacity. Market mechanisms emerge to solve the extensive coordination and allocation problems that exist in decentralized, deregulated supply chains. The form of these market mechanisms is a topic of active research (Cachon 2003).

We define a capacity market to be a collection of firms that exchange orders for diverse goods whose manufacture requires very similar equipment and processes and for which many firms in the collection possess the basic production capability but no one firm possesses a unique ability that discriminates their product. Textiles, machined parts, and semiconductor chips are categories of products for which such markets might exist. Jaikumar and Upton (1993) provided technical and commercial conditions that are required for the emergence of capacity markets. The primary requirement is the flexibility of the capacity to produce a diverse product set. Computer-controlled flexible machining, for example, can be used to produce metal parts of consistently high quality. Any part is defined by a computer code, which is fed into a machine to produce it. Numerous firms can accept the code and translate raw materials into a finished part. As a result of this development, the capacity is indistinguishable, and the actual source of the capacity is less relevant. Thus, the capacity itself can be treated as a commodity.

Jaikumar and Upton (1993) gave an example of a thriving capacity market in the textile industry. This market exists in Prato, Italy, with thousands of suppliers engaged in the different operations of weaving, spinning, finishing, etc. The trade of the capacity is brokered by hundreds of agents, called *impannatori*, who have information about the suppliers' capacities and loading. The brokers perform three functions: they find spare capacity to source a production order, they find buyers to consume spare capacity, and they mediate in the negotiations. The market mechanism has helped the firms to take full advantage of the associated flexibility.

A capacity market may also be useful for capital-intensive industries. For such industries, Kleindorfer and Wu (2003) argued that a capacity market can play an important role in improving capacity utilization and demand fulfillment through trading of residual capacity and excess demand by the sellers and buyers, respectively. The residual capacity arises from incomplete utilization of capacity by long-term contracts. Similarly, excess demand arises for buyers because long-term contracts cannot fulfill their dynamic needs. These potential benefits should lead to the creation of several such markets in the future.

Intuitively, we would expect a capacity market in a capital-intensive industry to behave differently than a capacity market for a more labor-intensive industry. In particular, because of the inability to rapidly adjust production volume to demand, we would predict price in capital-intensive markets to be more volatile. There can be high financial risks to forming and speculating in such markets. We would also expect that if buyers in a capacity market have the ability to store the capacity they purchase, in the form of goods inventory, for example, then this could significantly dampen the price volatility.

In this paper, we create a mathematical framework for testing such intuition and developing new insights on the resolution of uncertainty in price over time. We imagine an economy of capacity owners who accept orders from buyers who sell the resulting goods to consumers. Demand and supply are equated in a market for capacity by means of a market-clearing, or equilibrium, price. By examining the mathematical form of the equilibrium price and its standard deviation, and by calculating their characteristics, we can make predictive statements about the behavior of this economy.

The economy is represented as a continuous time stochastic economic equilibrium model. We formalize a continuous time extension of the martingale model of forecast evolution (CTMMFE) to model the evolution of consumer demand uncertainty. Using the mechanics of stochastic control, we derive necessary and sufficient conditions that the equilibrium price of capacity must satisfy. For the special case in which the cost functions are symmetric quadratic, we are able to derive a closed-form expression for the equilibrium price process and its forecast.

Whereas analysis of the model is conducted for general convex cost functions, for numerical experiments we focus on asymmetric quadratic cost functions. This permits us to describe different supply chains using a small number of parameters. In particular, a single parameter, the cost of volume flexibility, where volume flexibility is defined as the ease of modulating production with respect to capacity, describes the cost of varying production. Because production in excess of nominal capacity is likely to be more expensive than penalties for underutilization, we introduce a second parameter, the cost of overproduction. We use these two parameters to capture the difference between capital-intensive and labor-intensive capacity markets: the cost of volume flexibility and over production are likely much higher for capital-intensive industries (semiconductor chips) than for labor-intensive industries (upholstery).

Similarly, on the buyers' side, we use two parameters, the cost of inventory flexibility, where inventory flexibility is defined as the ease of varying inventory from the ideal level of zero, and the cost of backorders, to capture the differences between markets with regard to how well inventory and backorders can be used to buffer the capacity market from demand fluctuations.

Also, for numerical purposes, our parametrization of the demand process allows us to control both the instantaneous variance of demand and the resolution of demand uncertainty through forecasting process. With this small set of parameters, there are thus three interesting dimensions of supply chain characteristics that can be explored: volume flexibility, inventory flexibility, and demand predictability.

The rest of this paper is structured as follows. In §2, we discuss the relevant literature. In §3, we introduce our model, and in §4 we formalize the CTMMFE. In §5, we solve for the equilibrium price in the market for capacity. In §6, we obtain insights using our model and conclude in §7.

## 2. Literature Review

Given our focus on the structure of price in a business-to-business (B2B) market for capacity, this paper contributes to the recent literature on B2B exchanges. Kleindorfer and Wu (2003) provided a comprehensive review of the then-extant related literature. This literature primarily examines the following two questions: How does the presence of a short-term source/spot market influence sourcing and other operational decisions? Under what conditions will a spot market become viable? Papers that analyze the second question also identify conditions under which spot markets will dominate long-term contracts. It should be noted that the analysis of these two questions is not entirely disparate; several papers consider both of these questions simultaneously. Our classification below is based on our perception of the primary focus of each paper.

The influence of B2B exchanges on operational decisions is driven by the use of such exchanges for short-term coordination of supply and demand. This coordination could provide significant economic benefits in capital-intensive industries whose technology does not scale easily. In a supply chain consisting of a single seller and multiple buyers, Lee and Whang (2002) examined the effect of a spot market on the total quantity sold by the seller and on the allocative efficiency. For the same setup, Milner and Kouvelis (2007) analyzed the impact of a spot market on sourcing and production decisions and determined the price in the spot market endogenously. Serel et al. (2001) looked at the effect of the spot market on the amount of capacity reserved in a capacity reservation contract. Related issues were also analyzed by Akella et al. (2002), Spinler et al. (2003), and Wu and Kleindorfer (2005). For a summary of these papers, see Kleindorfer and Wu (2003).

Papers in the second category examine the viability of spot markets and associated contract design issues. Mendelson and Tunca (2000) determined necessary and sufficient conditions for the existence of a spot market. Mendelson and Tunca (2007) examined the effect of private information in procurement allocation between a spot market and a long-term contract. Tunca and Zenios (2006) identified conditions under which long-term contracts coexist with a B2B exchange, where the exchange utilizes procurement auctions for price discovery. Peleg et al. (2002) also identified conditions for the use of an auction-based exchange and analyzed price in the exchange. There also exists a large body of literature that has looked at supply chain coordination through contracting; see Cachon (2003) for a comprehensive review.

We position this paper in the body of literature that has derived price endogenously (e.g., Mendelson and Tunca 2007, Milner and Kouvelis 2007) in a spot market. The formulation of prices as the endogenous output of a mathematical model of economic equilibrium is a classical contribution of the economics literature as formalized, for example, by Debreu (1959) and Benavie (1972). What distinguishes our work is our exclusive focus on the structure of the price process in a spot market for capacity. Our goal is to investigate how operational factors such as the cost of backorders affect the resolution of price uncertainty over time.

Finally, we briefly describe two areas of research in finance with which this work shares a few similarities. The area of market microstructure examines the process by which demand translates into price in the financial markets. These papers also analyze how the market price incorporates information over time. For extensive reviews of literature in this area, see O'Hara (1995) and Madhavan (2000). Clearly, the broad goals of this paper are similar to those of the literature in this area. Our work is differentiated through explicit consideration of several supply chain characteristics.

On the other hand, the area of financial economics is relevant because of its focus on the valuation of assets such as shares, futures, and options (Huang and Litzenberger 1988, Duffie 2001); we too are interested in deriving insights on the valuation of production capacity. Another dimension of commonality is the extensive use of the principles of general equilibrium theory and the rational expectations hypothesis. One more, though minor, similarity is the continuous time nature of our model and the use of Brownian motion to model uncertainty. This also leads to similarity in the optimization tools that we employ.

In spite of these broad similarities, this work is substantially different from the financial economics area simply because the characterization of a capacity market is markedly different from a financial market. As an example, the utility functions of agents in the two types of markets are different. The participants in a capacity market face inventory/backlog costs as well as production-related costs, which agents in a financial market do not face. As a result, our model has a different structure than models used in financial economics. Another dimension along which the two markets differ is that whereas several financial assets such as shares can be held over time, capacity is perishable. One way in which this point is significant is that the methodology used in the pricing of financial derivative instruments (e.g., futures) cannot be used directly in a capacity market. We discuss this issue in §7.

## 3. The Market Model

We consider a continuous time market for the homogeneous capacity used for the production of a class of products. The market has numerous agents of two types: owners of capacity (sellers) and customers of capacity (buyers). The products themselves may be heterogeneous (such as logic chips or machined parts), but because each of the sellers is equally capable, the demand for products can be translated simply into demand for capacity using a measure such as machine time. Because of the heterogeneity of the underlying products and the multiplicity of buyers, the sellers opt not to hold inventory in finished goods; they operate strictly on a produce-to-order basis. Buyers, on the other hand, are willing to hold inventory of these products, and they do so to buffer themselves from the uncertainties of the capacity market and consumer demand. From the modeling perspective, therefore, we consider a market for a single homogeneous product (machine hours of capacity) and interpret all transactions in the real economy by translating them in terms of this capacity product.

We assume that there is no lead time between order placement and order delivery in the capacity market. If the consumer demand exceeds the available inventory at the buyer, the excess demand is backlogged. We ignore competition among buyers for shares of the end-consumer demand. This is applicable if either the buyers are geographically dispersed or the buyers' products are sufficiently differentiated. Finally, all agents in this market are assumed to be price takers.

We next develop the cost models for the buyers and sellers. There are $B$ buyers in the market who place production orders and satisfy demand from end consumers. Let $I_j(t)$ denote the net inventory (on-hand inventory less backorders) of buyer $j$ at time $t$. For simplicity, we assume that $I_j(0) = 0$ for all $j$. Let $x_j(t)$ be the instantaneous rate of order placement by buyer $j$ at time $t$.

Let $F_j(t)$ denote the cumulative demand from consumers for the sales of buyer $j$ through time $t$. (We develop an expression for $F_j(t)$ in §4.) This demand process is exogenous to the model. Using definitions of $I_j$, $F_j$ and $x_j$,

$$I_j(t) = \int_0^t x_j(u)\,du - F_j(t).$$

Let $P(\cdot)$ be the equilibrium price process for capacity. Given a price process $P(\cdot)$, buyer $j$'s problem is to choose a production order policy $x_j(\cdot)$ to minimize the total expected cost of production orders and inventory/shortfall costs. Because the consumer demand is exogenous, we do not include the revenue earned by the buyer in the consumer market in the objective function. The finite horizon version of this problem is

$$\begin{aligned} \min_{x_j \in \mathcal{U}_j} \quad & E\int_0^T \{G(I_j(t)) + P(t)x_j(t)\}\,dt \\ \text{s.t.} \quad & dI_j(t) = x_j(t)\,dt - dF_j(t), \quad t \in [0, T], \\ & x_j(t) \geq 0, \quad t \in [0, T], \\ & I_j(0) = 0, \end{aligned} \tag{1}$$

where $G(\cdot)$ is the inventory cost function, and $\mathcal{U}_j$ is the set of admissible controls, assumed to be a convex set. (Admissibility is a technical condition defined in §1.1 of the online supplement (available at http://dx.doi.org/10.1287/msom.1120.0409); see Assumption 3.) We ignore any end-of-horizon costs to simplify the analysis.

As noted before, we will embed the buyer model within an equilibrium model. To make the equilibrium problem analytically tractable, we make two simplifying assumptions. First, we assume that the inventory cost function $G(\cdot) \in \mathscr{C}^1$ is strictly convex and identical for all buyers. The assumption of convexity and continuous differentiability of the inventory cost function are standard in the supply chain literature. The assumption of commonality of $G(\cdot)$ across buyers means that the only difference among buyers lies in their individual demand processes. This assumption can be relaxed in the special case in which the inventory cost function is symmetric quadratic. (The details are available in the first author's Ph.D. thesis (Sapra 2004).) Second, we do not require an admissible order placement process $x_j$ to be nonnegative. When conducting numerical experiments, we are careful to choose parameter values so that the probability of negative order rates is kept quite small.

Observe that we assume that each buyer plans for the future using the same price process for the capacity market. This is an application of the rational expectation hypothesis, which is common in market equilibrium analysis (Muth 1961). The argument is somewhat circular: If a unique equilibrium price process exists, then every rational agent would choose to use that process in making decisions, and the equilibrium price process is the unique process that makes those decisions consistent by clearing the market at each point in time.

Even though few papers in supply chain management have utilized the theory of rational expectations in their models, this theory has been used to analyze several other economic situations, e.g., the efficient markets theory of asset prices, the permanent income theory of consumption, and the price evolution of storable commodities (Sargent 2008).

We next develop a cost model for sellers. There are $S$ sellers in the market. These sellers own production facilities and accept production orders. Let $C_k$ denote the capacity of seller $k$, and let $y_k(t)$ be the instantaneous rate of production of seller $k$ at time $t$.

Each seller faces the following stochastic control problem. Given an equilibrium price process $P(\cdot)$, seller $k$'s problem is to choose a production rate policy, $y_k(\cdot)$, to minimize the total expected cost of volume flexibility and production less the revenue derived from production. The finite horizon version of this problem is

$$\begin{aligned} \min_{y_k \in \mathcal{U}_k} \quad & E\int_0^T \{K(y_k - C_k) - (P(t) - c_k)y_k(t)\}\,dt \\ \text{s.t.} \quad & dY_k(t) = y_k(t)\,dt, \quad t \in [0, T], \\ & y_k(t) \geq 0, \quad t \in [0, T], \\ & Y_k(0) = 0, \end{aligned} \tag{2}$$

where $Y_k(t)$ is the cumulative production by time $t$, $K(y_k - C_k)$ is the cost of volume flexibility, and $c_k$ is the marginal cost of production and transportation.

Some comments on the seller model are as follows. First, as is common in the supply chain literature, we assume that the production and transportation cost is linear in production volume. Second, we assume that $K(\cdot) \in \mathscr{C}^1$ and that $K(y_k - C_k)$ is a strictly convex function of $y_k$ with a minimum at $C_k$. Thus, $C_k$ may be interpreted as the ideal rate of production; any deviation of the rate of production from $C_k$ incurs a penalty. Third, for the sake of simplicity, we assume that $C_k$ is deterministic. This assumption is not critical, and most of our results would still hold (Proposition 5.1.2 and Corollary 5.2.1) with suitable modifications if we take $C_k$ to be an exogenous continuous time stochastic process.

Fourth, we define $\mathcal{U}_k$ as the set of admissible controls, and we assume that $\mathcal{U}_k$ is a convex set. Fifth, as in the buyer model, for analytical purposes, we relax the nonnegativity restriction on the values that an admissible $y_k$ may take. However, in our numerical studies, we are careful to choose parameter values such that the production rate is nonnegative with high probability. Sixth, similar to the buyer model, we ignore the end-of-horizon costs to simplify the analysis. Seventh and finally, unlike the buyers, the sellers need not have the capability to rationally forecast the equilibrium prices. The reason is that the sellers do not require future price information. Their optimal decisions are myopic because they cannot store capacity and do not store inventory (we will see this in §5).

The equilibrium price at any instant is determined by the balance of supply and demand for capacity. Each buyer determines the instantaneous rate of order placement as a function of capacity price. Similarly, each seller chooses the instantaneous rate of production as a function of capacity price. The equilibrium price is determined such that

$$\sum_k y_k(t) = \sum_j x_j(t) \quad \text{for all } 0 \le t \le T. \tag{3}$$

The combination of buyer and seller models for each buyer and seller together with the above market-clearing equation is referred to as the *market model*. Our goal with this model is to characterize the equilibrium price that clears the market. As mentioned, this is a classical approach in the economics literature (e.g., Debreu 1959). While solving the model, similar to Mendelson and Tunca (2007), we do not impose an explicit restriction that the spot price be nonnegative. We do assume that the model parameters are such that the price remains nonnegative with a high probability.

So far we have developed cost models for the buyers and sellers without specifying a model for the consumer demand. We derive the foundational results for a consumer demand model in the following section.

## 4. Continuous Time Martingale Model of Forecast Evolution

In this section, we develop a continuous time model for the evolution of forecasts for the demand of a good. This model is a continuous time analogue of the martingale model of forecast evolution (MMFE) developed by Graves et al. (1986) and Heath and Jackson (1994), and we refer to it as CTMMFE. (In the MMFE model, forecast updates are assumed to occur after discrete intervals of time.) Both MMFE and CTMMFE are not forecasting techniques; they simply provide a framework to represent the evolution of forecasts over time. The actual forecasts could be made using any forecasting technique such as time-series methods or a combination of human judgement and statistical techniques. The MMFE and CTMMFE models treat the forecasts over time as realizations of a stochastic process, namely, a martingale process; that is, the forecasts are the conditional expectation of the future demand, given all the available information. Because the forecasts, by definition, are estimates of the future demand, this assumption is natural. For a detailed discussion on the merits of MMFE models, see Heath and Jackson (1994).

The CTMMFE model is inspired by the model for price of interest rate futures contracts developed by Heath and Jara (2003). They assumed that the futures contract price evolves as a continuous time martingale between time 0 and the realization of the spot price; we make this same assumption for the rate of demand. We extend their model by adding details for application in a production-inventory context and by introducing new parameters.

To ensure readability, the technically complete derivation is deferred to the online supplement. Let $\mathbf{W}(\cdot) \equiv (W_1(\cdot), W_2(\cdot), \ldots, W_n(\cdot))$ be an $n$-dimensional Weiner process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $f(s,t)$ denote the forecast at time $s$ of the rate of demand at time $t$. Thus, $f(t,t)$ represents the actual rate of demand at time $t$. We assume that the forecasts, $f(\cdot, t)$, evolve as a martingale process for any given $t \in [0, \infty)$. Thus, $f(0,t)$ represents the initial unbiased forecast of $f(t,t)$, and $f(s,t)$, the forecast at $s$ of the rate of demand at $t$, is the conditional expectation of $f(t,t)$ given the information known by time $s$. This assumption allows us to represent the forecasting process as a stochastic integral:

$$\begin{aligned} f(t,t) &= f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t)\, dW_i(s) \\ &= f(0,t) + \int_0^t \boldsymbol{\sigma}(s,t)\, \mathbf{dW}(s)^T, \end{aligned} \tag{4}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Hence,

$$f(s,t) = f(0,t) + \int_0^s \boldsymbol{\sigma}(u,t)\, \mathbf{dW}(u)^T, \quad \text{and}$$

$$d_s f(s,t) = \boldsymbol{\sigma}(s,t)\, \mathbf{dW}(s)^T = \sum_{i=1}^{n} \sigma_i(s,t)\, dW_i(s), \tag{5}$$

where $d_s f(s,t)$ can be interpreted as the forecast update at time $s$. Equation (4) implies that for a given $f(0,t)$, the use of any unbiased forecasting technique corresponds to a suitable form of $\boldsymbol{\sigma}(s,t)$, where

$\boldsymbol{\sigma}(s, t)$ is not necessarily deterministic. For simplicity, however, we consider only deterministic choices for $\boldsymbol{\sigma}(s, t)$. An example of a class of models that this assumption excludes is the class of multiplicative models. In multiplicative models, forecast updates are proportional to the magnitude of the current forecast. Extending the results of this paper to include multiplicative models is a significant mathematical challenge, and we leave it for future research.

Assuming $\boldsymbol{\sigma}(s, t)$ to be deterministic, the variance of forecast update between $s$ and $t$, $f(t, t) - f(s, t)$, is equal to $\int_s^t (\sum_{i=1}^n \sigma_i^2(u, t))\, du$, which decreases in $s$; that is, the amount of demand uncertainty to be resolved, which is measured by the variance of forecast update over time to go, decreases as the time of realization of demand approaches. This is the meaning of resolution of demand uncertainty through forecasts. Likewise, in §6, we will interpret the reduction in volatility of price forecast update over time to go as the resolution of price uncertainty.

Because $f(t, t)$ is the rate of demand at $t$, cumulative demand by $t$, $F(t)$, is equal to $\int_0^t f(s, s)\, ds$, which implies that

$$dF(t) = f(t, t)\, dt. \tag{6}$$

A more detailed version of the CTMMFE model is available in Sapra and Jackson (2012) as well as in Sapra (2004).

In the following section, we use the CTMMFE in the consumer demand model and solve the market model.

## 5. Solution to the Market Model for Capacity

In this section, we present and discuss the solution to the market model for capacity. We begin by sketching the outline of our approach.

### 5.1. Solution Approach

Our solution approach consists of four major steps. In the first step, we determine the necessary and sufficient conditions for the optimality of the buyer and seller models. Whereas the derivation of such conditions for the buyer model requires tools from convex analysis, the conditions for the seller model follow directly. In the second step, we aggregate these conditions over all the buyers and sellers and combine them using the equilibrium condition, Equation (3). In the third step, we develop an auxiliary model called the *integrated model*. The integrated model corresponds to a centralized supply chain in which both the production and inventory-related decisions are controlled by a single agent. The integrated model is defined in such a manner so that the necessary and sufficient conditions for its optimality coincide with the optimality conditions for the market model. In the fourth and final step, we solve the integrated model using tools from stochastic control theory and use this solution to obtain the equilibrium solution to the market model.

**5.1.1. Optimal Control of the Buyer Model.** We assume that the instantaneous demand for buyer $j$ at time $t$ is equal to

$$dF_j(t) \equiv \left(D_j + \int_0^t \boldsymbol{\sigma}_j(u, t)\, \mathbf{dW}(u)^T\right) dt, \tag{7}$$

where $\mathbf{W}(\cdot)$ is defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. This model is obtained by setting $f(0, t) = D_j$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_j$ in Equation (4). The parameter $D_j$ is the estimate at time 0 for the rate of demand at all future instants including time $t$; it is also the expected rate of demand for all the instants. One way to interpret the above model is as follows: For any $t$, buyer $j$ updates its forecast between time 0 and $t$, with $D_j$ being the initial forecast. The cumulative forecast update through $t$ is equal to $\int_0^t \boldsymbol{\sigma}_j(u, t)\, \mathbf{dW}(u)^T$.

A few remarks on the demand model for the buyers are as follows. First, the embedding of the CTMMFE model within the demand model allows us to capture the notion that consumer demand uncertainty is not resolved suddenly but over time. At the same time, the model serves another purpose: it facilitates an analysis of how equilibrium price uncertainty is resolved over time. This point will become clearer in §6 when we discuss our computational experiments. Second, the assumption that $f(0, t)$ is a constant ($D_j$) is not critical for most of the results that we derive; all the analytical results in this paper can be extended even when $f(0, t)$ is time dependent. Third, the demand process for any buyer is exogenous to the market.

Fourth, the same Weiner process drives demand for all the buyers. The $n$-dimensional Weiner process represents $n$ sources of uncertainty (for example, the state of the economy, the rate of inflation, interest rates, etc.). To keep the notation simple, we set $n = 1$ in the subsequent analysis; all of our results can be extended easily to the case of $n > 1$.

Fifth and finally, because of the presence of a stochastic integral on the right-hand side (RHS) in (7), it is possible for the demand rate to become negative. More precisely, the distribution of the rate of demand at any instant as seen at time 0 is normally distributed, which is a common modeling choice in the operations literature. Negative demand could be interpreted as customer returns, but we assume that the mean $D_j$ is large enough so that the probability of negative demand is negligible.

Using Equation (7), the rate of change of net inventory of buyer $j$ is given by

$$dI_j(t) = \left(x_j(t) - D_j - \int_0^t \sigma_j(u, t)\, dW(u)\right) dt. \tag{8}$$

In the following proposition, we discuss the optimal solution for the buyer model. Before stating the result, we define the adjoint variable pair $\bar{p}_j$: $[0, T] \times \Omega \to \Re$ and $\bar{q}_j$: $[0, T] \times \Omega \to \Re$ by the following stochastic differential equation:

$$d\bar{p}_j(t) = G'(\bar{I}_j(t))\,dt + \bar{q}_j(t)\,dW(t),$$
$$\bar{p}_j(T) = 0.$$

The adjoint variable $\bar{p}_j$ can be interpreted as the shadow price corresponding to the net inventory resource. The second adjoint variable $\bar{q}_j$ is not constrained to satisfy any differential equation. There exists a unique pair $\bar{p}_j$, $\bar{q}_j$ that solves the above equation if $\int_0^T I_j^2(t)\,dt < \infty$ (Cadenillas and Karatzas 1995). The technical assumptions needed for all the results of this section are deferred to §1.1 in the online supplement.

Observe that the shadow price $\bar{p}_j$ is the per unit benefit obtained in terms of reduction of cost when buyer $j$ applies the control $\bar{x}_j$. On the other hand, $P$ is the price paid per unit control applied. A buyer faced with a situation in which the shadow price exceeds the market price will increase his or her order rate without limit. Such excess demand will surely result in a rise in the market price of capacity. Similarly, if the shadow price is less than the market price, the buyer will continually reduce his or her orders. This puts downward pressure on the market price. Thus, the market price cannot be said to be in equilibrium if it differs from $\bar{p}_j(t)$; that is, in equilibrium, $P(t) = \bar{p}_j(t)$ for all buyers (and so the marginal price paid by them for one unit is equal to the marginal reduction in expected future cost due to the availability of that unit). This, indeed, is the equation that is necessary (and sufficient) for the buyer model to have a bounded solution, and we state it formally in the following proposition.

Proposition 5.1.1. *A necessary condition for the buyer model to have a bounded solution is $P(t) = \bar{p}_j(t)$. Furthermore, if $P(t) = \bar{p}_j(t)$, then any $x_j$ that results in $P(t) = \bar{p}_j(t)$ is optimal.*

The proof of the above proposition as well as other results in the paper are available in the online supplement.

The above proposition shows that the condition $P(t) = \bar{p}_j(t)$ is necessary for the buyer model to have a solution. The controls $x_j$ that ensure $\bar{p}_j(t) = P(t)$, assuming at least one such control exists, should thus be thought of as *stable* solutions. Because these are the only solutions for which the buyer model has a bounded solution, without which the capacity market cannot have an equilibrium, we use them in computing an equilibrium solution to the market model.

Note also that the equation $P(t) = \bar{p}_j(t)$ implies that all the buyers have the same shadow price for inventory at any instant. Summing this equation over all the buyers gives

$$P(t) = \frac{\bar{p}(t)}{B}, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega, \tag{9}$$

where $\bar{p}(t) = \sum_j \bar{p}_j(t)$.

**5.1.2. Optimal Control of the Seller Model.** Because the objective function in the seller model (2) does not have any state variable, the integrand can be optimized directly for every $t$. Thus, a control $\bar{y}_k$ is optimal if and only if

$$P = c_k + K'(\bar{y}_k - C_k), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega.$$

The above result implies that the optimal rate of production for seller $k$ at time $t$ depends only on the price at time $t$ and does not require information regarding the future price. The intuition behind this result is as follows. The seller's choice of rate of production has no bearing on the future costs because nothing is carried over from $t$ to the future. By definition, capacity cannot be stored or backlogged. The quantity produced is also not carried because the seller makes to order. As a result, the optimal rate of production depends only on the current price and is independent of the forecasts of the future prices.

Because our focus is on the second-order behavior of $P$, we set the constant $c_k$ equal to 0 without loss of generality in the subsequent analysis. Rearranging terms in the above equation,

$$\bar{y}_k = C_k + K'^{-1}(P), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega.$$

Let $y = \sum_k y_k$, and let $C = \sum_k C_k$. Summing the last equation over all sellers results in

$$SK'^{-1}(P) + (C - \bar{y}) = SK'^{-1}\left(\frac{\bar{p}}{B}\right) + (C - \bar{y}) = 0, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega, \tag{10}$$

where the RHS is obtained by substituting for $P$ from (9).

To obtain the optimal solution to the market model, we use the equilibrium condition (3) to link the optimal solutions to the buyer and seller models. In the subsequent analysis, we assume that the optimal solution to the market model exists. In the following proposition, we provide both necessary and sufficient conditions for the optimality of market variables when $G'(\cdot)$ is a linear function. Otherwise, we provide only sufficient conditions.

Proposition 5.1.2. *Let $\bar{I} = \sum_j \bar{I}_j$, $\bar{q} = \sum_j \bar{q}_j$, $\sigma = \sum_j \sigma_j$, and $I_j(0) = 0$, $\forall j$. If the vector of market variables,*

*$(\bar{I}, \bar{y}(=\bar{x}), \bar{p}, \bar{q})$ satisfies the following system of equations in equilibrium at time $t \in [0, T]$, then it is optimal:*

$$\begin{gathered} d\bar{I}(t) = \left(\bar{x}(t) - D - \int_0^t \sigma(s,t)\, dW(s)\right) dt, \\ d\bar{p}(t) = BG'\left(\frac{\bar{I}(t)}{B}\right) dt + \bar{q}(t)\, dW(t), \\ \bar{p}(T) = 0, \\ \bar{x}(t) = \bar{y}(t) = C + SK'^{-1}\left(\frac{\bar{p}(t)}{B}\right), \quad \bar{x}(t) \in \Re. \end{gathered} \tag{11}$$

*Furthermore, if $G'(I_j)$ is a linear function, then the above conditions are necessary as well.*

To obtain the optimal solution to the market model, we define another model, which we refer to as the *integrated model*. This model corresponds to an integrated supply chain (hence the name) in which the whole supply chain is owned by a single agent. Our approach involves obtaining the necessary and sufficient conditions for optimality in the integrated model and showing them to be equivalent to the necessary and sufficient conditions for optimality in the market model. Thus, a solution to the integrated model would automatically lead to a solution to the market model.

### 5.2. Integrated Model

In this subsection, we state the integrated model. The mathematical objective of this model is to choose a production policy $y_\iota$ to minimize the sum of volume flexibility and inventory/shortfall costs over a finite horizon. The model is as follows:

$$\begin{aligned} \min_{y_\iota \in \mathcal{U}_\iota} \ & E\int_0^T \{K_\iota(y_\iota - C_\iota(t)) + G_\iota(I_\iota(t))\}\, dt \\ \text{s.t. } & dI_\iota(t) = \left(y_\iota(t) - D_\iota - \int_0^t \sigma_\iota(u,t)\, dW(u)\right) dt, \\ & \qquad t \in [0, T], \\ & I_\iota(0) = 0, \end{aligned} \tag{12}$$

where all the notation has the same interpretation as for the market model. We add a distinguishing mark, subscript $\iota$, to indicate the integrated model. As for the buyer and seller models, the technical assumptions necessary to define and solve the integrated model are provided in §1.5 in the online supplement. Also, similar to the buyer and seller models, we require only that $y_\iota$ take real values but not necessarily nonnegative values.

To establish equivalence between the integrated and market models, we employ the following calibration among the volume flexibility and inventory cost functions for these models:

$$G'_\iota(I) = G'\left(\frac{I}{B}\right), \tag{13}$$

$$K'_\iota(y - C) = K'\left(\frac{y - C}{S}\right). \tag{14}$$

In the following subsection, we derive the necessary and the sufficient conditions for the optimality of the integrated model.

**5.2.1. Optimal Control of the Integrated Model.** We use the stochastic maximum principle for convex cost functions derived by Cadenillas and Karatzas (1995) to solve the integrated model. To apply the maximum principle, we first define the Hamiltonian function as

$$\begin{aligned} &H_\iota(y_\iota, I_\iota, p_\iota, q_\iota) \\ &\quad = -G_\iota(I_\iota) - K_\iota(y_\iota - C_\iota) \\ &\qquad + p_\iota\left(y_\iota - D_\iota - \int_0^t \sigma_\iota(u,t)\, dW(u)\right). \end{aligned} \tag{15}$$

The adjoint variable pair $\bar{p}_\iota$: $[0, T] \times \Omega \to \Re$ and $\bar{q}_\iota$: $[0, T] \times \Omega \to \Re$ is defined in the same manner as for the buyer model by the following stochastic differential equation:

$$\begin{gathered} d\bar{p}_\iota(t) = G'_\iota(\bar{I}_\iota(t))\, dt + \bar{q}_\iota(t)\, dW(t), \\ \bar{p}_\iota(T) = 0. \end{gathered}$$

The variable $\bar{p}_\iota$ could be interpreted as the shadow price for the net inventory resource $\bar{I}_\iota$.

According to Theorem 3.2 of Cadenillas and Karatzas (1995), if the objective function is convex in the state and control variables, then $\bar{y}_\iota$ is an optimal control variable if and only if

$$\max_{y_\iota \in \mathcal{U}_\iota} H_\iota(y_\iota, \bar{I}_\iota, \bar{p}_\iota, \bar{q}_\iota) = H_\iota(\bar{y}_\iota, \bar{I}_\iota, \bar{p}_\iota, \bar{q}_\iota), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega.$$

Optimizing the Hamiltonian yields

$$\bar{y}_\iota = C_\iota + K_\iota'^{-1}(\bar{p}_\iota), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega.$$

The necessary and the sufficient conditions for optimality of the integrated model are similar to the sufficient conditions for optimality of the market model. As a result, if an optimal solution to the integrated model is known, the corresponding optimal solution to the market model can be easily obtained. We formally state this observation in the following corollary.

Corollary 5.2.1. *Let $C_\iota = C$, $D_\iota = D$, and $\sigma_\iota = \sigma$. If $(\bar{I}_\iota, \bar{y}_\iota, \bar{p}_\iota, \bar{q}_\iota)$ is an optimal solution of the integrated model, then $(\bar{I}, \bar{y}(=\bar{x}), \bar{p}, \bar{q})$ is an equilibrium solution to the market model, where*

$$\bar{p} = B\bar{p}_\iota, \quad \bar{q} = B\bar{q}_\iota, \quad \bar{y} = \bar{y}_\iota, \quad \bar{I} = \bar{I}_\iota, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega.$$

**5.2.2. Symmetric Quadratic Cost Model.** In this section, we briefly discuss a special case in which the volume flexibility and inventory cost functions are symmetric quadratic. In this case, we are able to obtain the optimal solution to the integrated model in closed form. Given the relationship between the integrated and market models, this means that we can solve the market model as well and obtain optimal market variables, including the equilibrium price process, in closed form. (Details may be found in the first author's Ph.D. thesis (Sapra 2004).)

In particular, our analysis shows that when both cost functions are symmetric quadratic, the equilibrium price at time $t$ can be written in the following form:

$$P(t) = a(t) + \int_0^t b(s,t)\,dW(s), \tag{16}$$

where $a$ and $b$ are deterministic functions of market parameters. One advantage of the above functional form is that it allows us to illustrate the notion of price forecasts easily. To see this, let the forecast at $s$ of price at $t$ be $P(s,t)$. Using the same idea as in the CTMMFE model, we define $P(s,t)$ to be the conditional expectation at $s$ of price at $t$ given all the information available at $s$. Using the above expression for the equilibrium market price, this leads to

$$P(s,t) = a(t) + \int_0^s b(u,t)\,dW(u), \tag{17}$$

and the forecast update at $s$ of the market price at $t$, $d_sP(s,t)$, is given by

$$d_sP(s,t) = b(s,t)\,dW(s).$$

Observe that if we compare the above equation with (5), then it becomes clear that $b(s,t)$ plays the same role in the price forecasting process that $\sigma(s,t)$ does in the forecasting process for the rate of demand. This means that the function $b(\cdot,t)$ is a measure of the variance of forecast updates for the price at $t$.

## 6. Numerical Experiments

In this section, we use numerical experiments with the model to derive insights into the equilibrium behavior of the capacity market. Depending upon the objective, the experiments can be classified into three categories. In the first set of experiments, which are reported in §6.2, we study the effect of the model parameters on inventory and backorder levels and production volatility in the market. Because inventory is held and production is varied to minimize shortages caused by uncertainty in demand, this set of experiments illustrates how the relative use of inventory and production variation changes with model parameters. In the second set of experiments, which are reported in §6.3, we investigate how model parameters influence price volatility and the time distribution of price learning. In the final set of experiments, which are reported in §6.4, we test whether the results reported in §§6.2 and 6.3 are robust to the functional form of the cost functions employed.

We begin by describing the setup for the numerical experiments.

### 6.1. Experimental Setup

**6.1.1. Cost Functions and Parameters.** The market is characterized by $G(\cdot)$, the buyers' inventory-backorder cost function; $K(\cdot)$, the volume flexibility cost function; and $\sigma^2(\cdot,\cdot)$, which characterizes the evolution of demand forecasts. For the experiments, we assume the following forms for these functions:

$$\begin{gathered} G(I) = \pi\big((I^+)^2 + \alpha(I^-)^2\big), \\ K(y-C) = \kappa\big(((y-C)^-)^2 + \beta((y-C)^+)^2\big), \\ \sigma(s,t) = \xi\exp(\lambda(s-t)), \end{gathered} \tag{18}$$

where $\pi$, $\kappa$, $\beta$, $\xi$, $\lambda > 0$, and $\alpha \geq 1$. Note that both $G$ and $K$, in general, are asymmetric, and we refer to them as asymmetric quadratic cost functions throughout the rest of this paper. The asymmetry allows us to vary the relative costs of on-hand inventories, backorders, overproduction, and underproduction using the parameters $\pi$, $\alpha$, $\kappa$, and $\beta$.

Two remarks about the model parameters are as follows. First, the parameter $\lambda$ can be interpreted as the inverse rate of demand learning: the greater the value of $\lambda$, the smaller the fraction of variance of rate of demand at time $t$ resolved by time $s$; that is, as $\lambda$ increases, the ratio

$$\frac{\int_0^s \sigma^2(u,t)\,du}{\int_0^t \sigma^2(u,t)\,du} = \frac{\exp(-2\lambda(t-s)) - \exp(-2\lambda t)}{1-\exp(-2\lambda t)}$$

decreases for each value of $s$. Put another way, as $\lambda$ increases, more of the variance in forecast change is concentrated in the period of time immediately before the time of the actual demand. However, changes in $\lambda$ also affect the variance of rate of demand (as computed at time 0), given by

$$\int_0^t \sigma^2(u,t)\,du = \frac{\xi^2}{\lambda}\big(1-\exp(-2\lambda t)\big).$$

Consequently, when investigating changes in the rate of demand learning, we vary $\lambda$ but choose values of $\xi$ to keep the variance of the rate of demand constant.

Second, we consider only those values of $\kappa$ that are at least five times the value of $\pi$ in the experiments that we report on in §§6.2 and 6.3. The reason for this condition is as follows. While conducting the experiments, we search for the optimal rate of production

**Table 1** Data for Numerical Experiments

| $t$ | $T$ | $S/B$ | $I_0$ | $\alpha$ | $\beta$ | $C$ | $D$ | $\xi$ | $\lambda$ | $\kappa$ | $\pi$ | $\delta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 110 | 200 | 1 | 0 | 10 | 5 | 5 | 5 | 0.1 | 1 | 100 | 1 | 0.05 |

within a compact interval. To prevent distortion of results due to the optimal rate of production being at a boundary point, we require that the maximum probability of the optimal rate of production at a boundary point over all time and state combinations be no more than $10^{-5}$. In our experiments, this requirement is satisfied only when $\kappa \geq 5\pi$. For more details, see §1.7 in the online supplement.

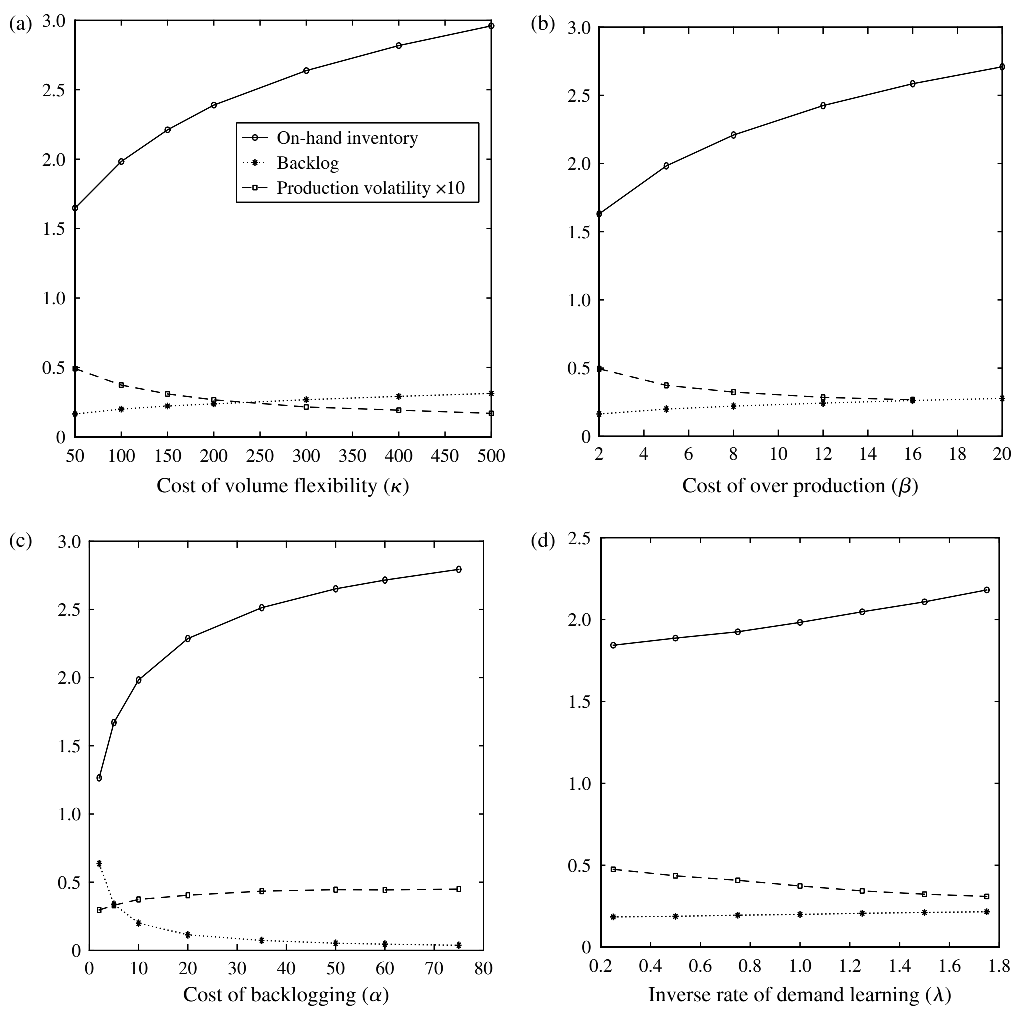
**6.1.2. Base Case Data.** For the experiments reported on in §§6.2 and 6.3, the base case data are given in Table 1. We discretize the planning horizon into intervals of length $\delta$. (Recall that $D$ is the expected rate of demand aggregated over all the buyers, $S$ is the number of sellers, and $B$ is the number of buyers.)

Observe that we have chosen the value of $t$ to be 110, which is near the midpoint of the time horizon. This is to focus on long-term or steady-state behavior and to minimize the effects of initialization and end of horizon.

**6.1.3. Trajectory Stabilization and Steady State.** For each numerical experiment, we solve a discrete-time approximation to the integrated model (12). From this solution we derive trajectories of expected values and variances for state variables of interest, such as on-hand inventory and production, over the planning horizon. Sample plots of these trajectories are included in the online supplement (Figure 1). What we observe in these plots is a common pattern in which the trajectories rise rapidly and then stabilize, remaining relatively flat for a long period of time. Near the end of the horizon, the trajectories depart from the stable path and then rapidly converge to a value very different from the stable value.

By studying the impact of structural parameters on trajectory midpoints, we are presumably developing

**Figure 1** Expected On-Hand Inventory and Backlog and Variance of Rate of Production as a Function of Market Parameters at $t = 110$



insights into their effect on steady-state behavior. A note of caution is in order, however. In some experiments, we did not observe trajectory stabilization, for example, when $\kappa$ or $\beta$ was large. Provided $\kappa \leq 500$ and $\beta \leq 20$, however, the trajectory midpoints do appear to be stable. Though not precise, we refer to these trajectory midpoint values as steady-state values for the purpose of developing intuition.

Further details of the numerical solution approach may be found in §1.7 of the online supplement.

### 6.2. Role of Supply Chain Parameters in Managing Consumer Demand Uncertainty

It is easy to see that when demand is deterministic the optimal rate of production should be equal to the rate of demand for the set of parameters used in our experiments, because we have taken $C = D$. Furthermore, no on-hand inventory would be held, and no backlog would occur. In the presence of uncertainty, however, the buyers must hold on-hand inventory, and sellers must vary the rate of production to minimize shortages. (Any demand still remaining unsatisfied will be backlogged.) Therefore, on-hand inventory and volume flexibility may be interpreted as instruments that can be used to reduce shortages in the presence of demand uncertainty. In this subsection, we discuss the role of market parameters in the relative use of these instruments.

To understand the effect of these parameters, we plot expected on-hand inventory, expected backlog, and variance of rate of production at time $t = 110$ for multiple values of $\kappa$, $\lambda$, $\alpha$, and $\beta$ in Figure 1. (As note before, we have considered only those values of $\kappa$ and $\beta$ for which the initialization and end-of-horizon effects have vanished by $t$.) A summary of interesting observations from the figure are as follows.

1. As $\kappa$ increases, the variance of the rate of production decreases, and both expected on-hand inventory and backlog increase, with on-hand inventory increasing at a greater rate than backlog (Figure 1(a)). The increase in expected on-hand inventory could be explained by two factors. The first factor is the market's holding greater amounts of inventory in a bid to minimize shortages, because using volume flexibility is now more expensive. (Assuming this is true, an increase in expected backlog implies that the effect of reduction in volume flexibility is not completely covered by an increase in on-hand inventory.)

The other factor is demand randomness; that is, the increase in expected on-hand inventory may be occurring due to inventory buildup caused by demand randomness, because the variation in rate of production decreases as $\kappa$ increases. However, we believe that if demand randomness were completely explaining the increase in on-hand inventory, the expected backlog should have increased by the same amount as expected on-hand inventory, as $\kappa$ increases, because randomness should not favor on-hand inventory over backlog. To illustrate this idea, consider an example in which volatility in net inventory is completely determined by demand randomness. Consider the extreme case in which $\kappa$ is arbitrarily large. In this case, the rate of production will be equal to $C$ at time $t$. Recall that the net inventory at any time is equal to cumulative production less cumulative demand by that time. Because $C = D$ in our experiments, the net inventory will be equal to $\int_0^t \int_0^v \sigma(u, v)\, dW(u)\, dv$, which has an expectation equal to 0. This means that the expectation of on-hand inventory is equal to that of backlog.

Because expected on-hand inventory increases at a rate greater than expected backlog as $\kappa$ increases, at least some of the increase in on-hand inventory appears to be because it is preferred over backlogs as a production compensating strategy.

2. As $\beta$ increases, production beyond capacity incurs greater penalty. As a result, production-capacity mismatch decreases (in general), and the variance of the rate of production decreases (Figure 1(b)). Similar to the discussion in Observation 1 above, it appears that the market increases the use of on-hand inventory to minimize shortages, though, as before, some of the increase could be driven by uncertainty itself. Also, as in Observation 1, the expected backlog increases, implying that the increased use of on-hand inventory is not able to completely make up for the loss of volume flexibility.

3. As $\alpha$ increases, Figure 1(c) implies that demand uncertainty is managed by holding greater amounts of on-hand inventory and through volume flexibility. The net effect is a reduction in expected backlog, as one would expect.

4. Finally, as $\lambda$ increases, on-hand inventory and backlog increase, but production variance decreases (albeit weakly), implying that delayed learning of demand results in increased use of on-hand inventory and a reduced use of volume flexibility to minimize shortages (Figure 1(d)).

### 6.3. Impact of Supply Chain Parameters on Price Volatility and Temporal Distribution of Price Learning

To measure price volatility and temporal distribution of price learning, we define a fundamental unit of uncertainty, the variance of price forecast updates. Specifically, we define the variance of the price forecast update at time $s$ for the spot price at time $t$, $v(s, t)$, by

$$v(s, t) = \mathrm{Var}\big(P(s + \delta, t) - P(s, t)\big), \quad s + \delta \leq t,$$

where $P(s, t)$ is the forecast at $s$ of the spot price at $t$. From these fundamental units, we compute the variance of the cumulative price forecast update by time $s$

for the price at time $t$, $V(s,t)$, by summing the variance of price forecast updates for all intervals in $[0,s]$:

$$V(s,t) = \sum_{n=0}^{\lfloor s/\delta \rfloor} v(n\delta, t), \quad s+\delta \le t. \tag{19}$$

Note that all the variance computations (e.g., the variance of price forecast updates) are done with respect to the information known at time 0.

We use the standard deviation of the forecast at $s$ of price at time $t$, $\text{SDP}(s,t) := \sqrt{V(s,t)}$, to measure the volatility of price, and the degree of advance price resolution at time $s$ for spot price at time $t$, $\text{DAPR}(s,t) := V(s,t)/V(t,t)$, $s \le t$, to measure the temporal distribution of price learning.

If $\text{DAPR}(s,t) > (s/t)$ $(<s/t)$ for all $s \in (0,t)$, we say the price process exhibits early (delayed) learning. If, as a result of a change in a supply chain parameter, the standard deviation of the forecast of spot price, $\text{SDP}(s,t)$, increases (respectively, decreases), we say that the change has made the forecasts of the price more (respectively, less) volatile. If the change results in the degree of advance price resolution decreasing (respectively, increasing) for all values of $s$ in $(0,t)$, we say that the change has made the market slower (respectively, faster) to learn. We assume that such a trend in either dimension (more volatile or slower to learn) makes the market for capacity less desirable for all participants who must make plans based on price forecasts (i.e., the buyers in our model).

Although it is not as common to consider the timing of the resolution of uncertainty as it is to consider the magnitude of uncertainty, it is reasonable to assume that early resolution of uncertainty is preferable to later resolution. For example, even if a lottery ticket does not pay off for several years, the ticket purchaser would prefer to know whether the ticket is a winner earlier rather than later. The knowledge has value for all sorts of contingent activities, such as life insurance purchases. Likewise, in a capacity market, if forecast accuracy is known to improve rapidly, as would happen with early learning, then contingent activities such as workforce planning can be made with greater confidence if these activities can be delayed to exploit the improved forecast accuracy. For example, if 90% or more of the variance of spot price is resolved a month in advance, then there will be much higher confidence in the month-out workforce plan than if only, say, 10% of the variance were resolved by that time.

We plot both metrics, SDP and DAPR, with respect to $\kappa$ in Figure 2. Because for $\kappa > 500$ the market may not have stabilized by $t$, we have split the plots depending upon whether $\kappa$ is less than or greater than 500. In Figures 2(a) and 2(b), we plot SDP and DAPR for $\kappa \le 500$. On the other hand, in Figures 2(c) and 2(d), we plot the same metrics for $\kappa > 500$. For such values of $\kappa$, the market may still be in an initialization stage, and we are not really observing relationships that might hold in steady state. Nevertheless, the dynamics are interesting.

We also plot the SDP and DAPR with respect to $\alpha$; see Figure 3. A summary of insights from these experiments is as follows:

1. For low values of $\kappa$ ($\kappa < 500$), the standard deviation of spot price and its forecasts increase with $\kappa$ (Figure 2(a)). The DAPR also increases with $\kappa$.

On the other hand, for higher values of $\kappa$ ($\kappa > 500$), the standard deviation of the spot price and its forecasts first increase and then decrease with $\kappa$ (Figure 2(c)). Likewise, the degree of advance price resolution first increases and then decreases. This may not reflect steady-state behavior, but it suggests competing phenomenon are at work.

To understand these observations, substitute (18) into the first-order condition (10). (We discuss the case of spot price here; the explanation for price forecasts is similar.) After simplifying, we obtain

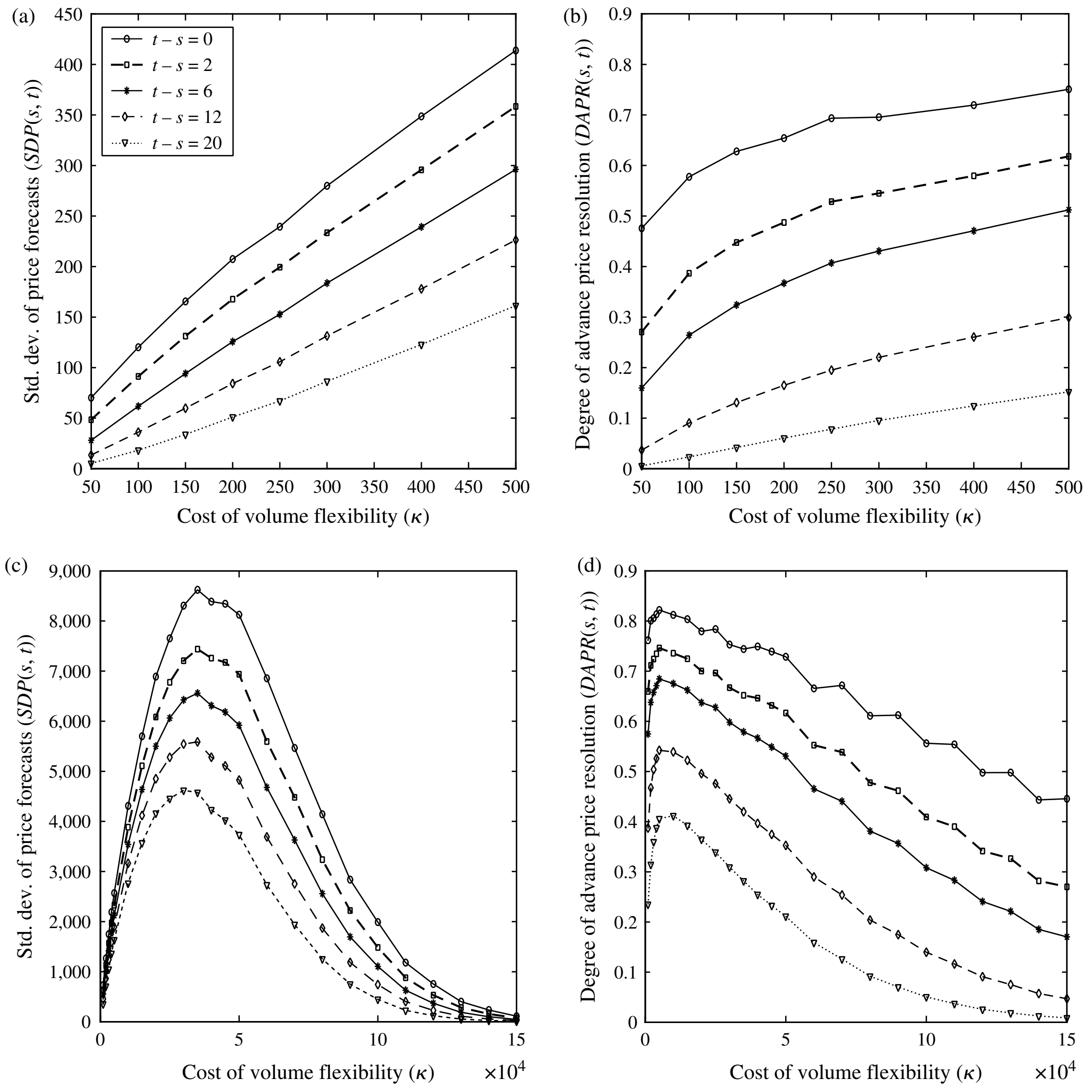
$$P(t) = 2\kappa\big(-(y-C)^- + \beta(y-C)^+\big).$$

From this, the parameter $\kappa$ can be seen to play two roles in determining the price of capacity: In the first place, it amplifies the cost of a capacity mismatch. For low values of $\kappa$ we would anticipate considerable variance in production-capacity mismatch. This variance is then magnified as a variance in price. However, for large values of $\kappa$, the difference between $y(t)$ and $C$ will approach 0. Assuming this difference approaches 0 faster than the rate of increase of $\kappa$, the price of capacity will approach 0 for large values of $\kappa$. With this interpretation it would be natural to observe an increase followed by a decrease in price volatility as $\kappa$ increases.

A similar result is obtained when the value of $\beta$, which is a measure of the relative cost of overproduction, is adjusted; the details are omitted.

2. As the cost of backorders, $\alpha$, increases, the degree of advance price resolution decreases (Figure 3(b)). On the other hand, the standard deviation of price forecasts as a function of $\alpha$ depends on time to go $(t-s)$ (Figure 3(a)). For large values of time to go, the standard deviation of price forecasts decreases with $\alpha$. However, the trend reverses as time to go decreases. Another way to look at this observation is that for low values of backlogging cost, the standard deviation of price forecasts remains relatively stable as the time of realization of demand approaches. However, for high cost of backorders, the standard deviation of price forecasts changes more significantly as the time of realization of demand approaches.

**Figure 2** Standard Deviation of Price Forecasts and Temporal Distribution of Price Learning as a Function of $\kappa$




3. As the rate of demand learning increases (i.e., $\lambda$ decreases), the standard deviation of the spot price and its forecasts increase. Also, as the rate of demand learning increases, the degree of advance price resolution increases as well. (Figures are omitted.) This is a satisfying result. It suggests, for example, that improvements in demand forecasting systems that improve the timing of the resolution of demand uncertainty can also improve the timing of the resolution of price uncertainty.

### 6.4. Robustness of Computational Results

Most of the managerial insights reported so far in this section are based on asymmetric quadratic cost functions for both the volume flexibility cost as well as inventory-related costs. Because these functional forms are hypothesized, a natural question to ask is, will these insights continue to hold if other functional forms are used? To answer this question, we repeated the numerical experiments in §§6.2 and 6.3 for five other functional forms, all of which are convex in $y$ and $I$. The functional forms are as follows:

$$\begin{gathered} G(I)=\pi\{(I^+)^{1.01}+\alpha(I^-)^{1.01}\}, \\ K(y-C)=\kappa\{((y-C)^-)^2+\beta((y-C)^+)^2\}; \end{gathered} \tag{20}$$

$$\begin{gathered} G(I)=\pi\{(I^+)^{1.01}+\alpha(I^-)^{1.01}\}, \\ K(y-C)=\kappa\{((y-C)^-)^{1.5}+\beta((y-C)^+)^{1.5}\}; \end{gathered} \tag{21}$$

$$\begin{gathered} G(I)=\pi\{(I^+)^{1.5}+\alpha(I^-)^{1.5}\}, \\ K(y-C)=\kappa\{((y-C)^-)^{1.5}+\beta((y-C)^+)^{1.5}\}; \end{gathered} \tag{22}$$

$$\begin{gathered} G(I)=\pi\{(I^+)^{1.01}+\alpha(I^-)^{1.01}\}, \\ K(y-C)=\kappa\{((y-C)^-)^{1.01}+\beta((y-C)^+)^{1.01}\}; \end{gathered} \tag{23}$$

$$G(I)=\pi\{I^2\}, \quad K(y-C)=\kappa\{(y-C)^2\}. \tag{24}$$

Note that because asymmetric linear functions of the form $a(\cdot)^+ + b(\cdot)^-$ are not $\mathscr{C}^1$, which is a

**Figure 3** Standard Deviation of Price Forecasts and Temporal Distribution of Price Learning as a Function of $\alpha$

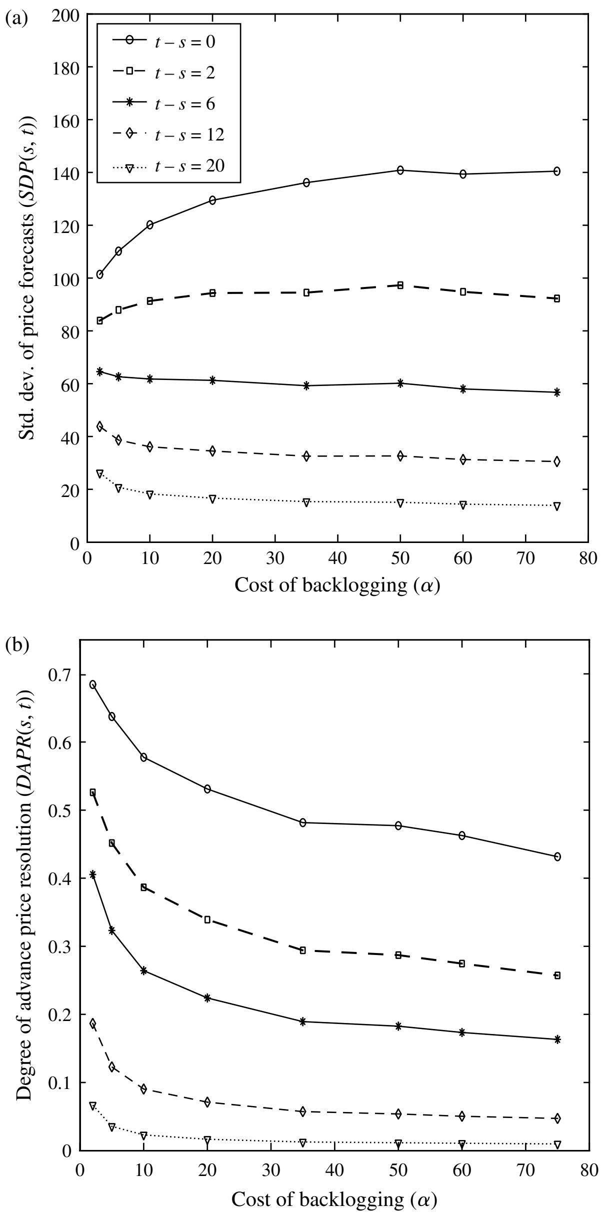
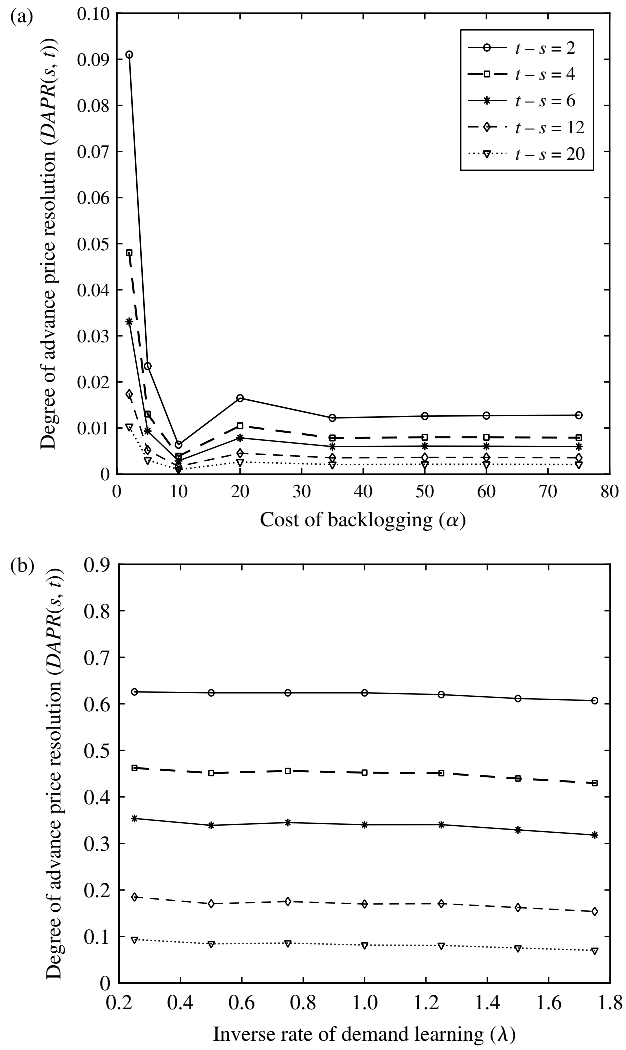



**Figure 4** Sample Figures to Illustrate Analysis on Robustness




*Notes.* For (a), $\kappa = 15$, and the cost functions are as given in Equation (23). For (b), $\kappa = 20$, and the cost functions are as given in Equation (21). The rest of the parameter values are same as in Table 1.

requirement to apply the stochastic maximum principle, we instead have considered asymmetric polynomial functions with a power of 1.01. We refer to this class of functions as *near-linear* functions.

A summary of observations from this set of experiments is as follows:

1. In general, the insights derived using the asymmetric quadratic cost functions remain valid for all the other cost functional forms except when both inventory and volume flexibility cost functions are near linear. As an example, the relationship between $\text{DAPR}(s,t)$ and $\alpha$ when both cost functions are near linear does not match that of the case in which both cost functions are asymmetric quadratic (Figure 4(a)).

At the same time, we note that the validity of results in the case in which both cost functions are near-linear cost functions is doubtful. The reason is that for most of these experiments, there is a significant probability that the optimal control lies at one of the boundary points. (See our discussion earlier in this section.) For example, for the experiments summarized in Figure 4(a), this probability varied from 3.2% to 4.1%, which is significantly higher compared to other cost functions, for which the maximum value of this probability is $10^{-5}$. Because of this observation, we conjecture that the differences in trends when cost functions are near linear are not structural.

2. The plots of $\text{SDP}(s,t)$ and $\text{DAPR}(s,t)$ with respect to $\lambda$ when both cost functions are asymmetric quadratic do not always have the same trends as other cost functions. (Both metrics show a mildly

decreasing trend with respect to $\lambda$ for the asymmetric quadratic cost function. However, in Figure 4(b), which corresponds to the cost functions defined in Equation (21), for $t-s=6$, DAPR$(s,t)$ for $\lambda=0.5$ is less than $\lambda=0.75$.) Because the trends are weak in general, it is difficult to say whether these differences are caused by structural differences between different cost functions or by numerical approximations.

That the insights broadly remain similar across different cost functional forms implies that our results are robust to the choice of cost functional form.

## 7. Conclusion

In this paper, we analyze a dynamic supply chain market for capacity and obtain necessary and sufficient conditions that the equilibrium price, if it exists, should satisfy. According to the conditions, the equilibrium price must be equal to the shadow price of net inventory for all the buyers and to the marginal cost of production for all the sellers. The equilibrium price can be obtained in closed form when the cost functions are symmetric quadratic. For general convex cost functions, we show how to compute the standard deviation of price forecasts. This permits us to conduct a sensitivity analysis of price volatility with respect to model parameters. Our experiments also shed light on how the market minimizes shortages that occur because of consumer demand uncertainty under various settings.

In general, we have derived several insights on the equilibrium behavior of a capacity market, and these insights appear to hold for a broad range of cost functions. These are new results in the literature of supply chain economics.

One interesting extension of our work will be consideration of derivatives such as forwards and futures for capacity in the model. The standard approach used to compute prices of financial derivatives employs the *arbitrage* principle. The use of this principle involves a hypothetical series of trades in which the underlying security is purchased and held that result in the same payoff as the derivative. Unfortunately, this approach cannot be replicated for capacity derivatives because capacity, being perishable, cannot be held. One possibility is an integrated approach that computes the spot price as well as derivative prices simultaneously. Although this is a challenging approach, we believe that it is possible to implement it under a suitable set of assumptions.

Finally, this work illustrates application of stochastic control tools in a supply chain problem. Such tools could be quite useful in modeling dynamic systems, but their application in operations management is rare. We hope that this work will inspire other researchers to identify applications where this set of tools could be applied to advantage.

### Electronic Companion

An electronic companion to this paper is available as part of the online version at http://dx.doi.org/10.1287/msom.1120.0409.

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