Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Heuristic Methods for Centralized Control of One-Warehouse, N-Retailer Inventory Systems

Sven Axsäter, Johan Marklund, Edward A. Silver,

Heuristic Methods for Centralized Control of One-Warehouse, *N*-Retailer Inventory Systems

Sven Axsäter • Johan Marklund • Edward A. Silver
Department of Industrial Management and Logistics, Lund University, P.O. Box 118, 221 00 Lund, Sweden
Leeds School of Business, University of Colorado at Boulder, 419 UBC Boulder, Colorado 80309-0419
Faculty of Management, University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada T2N 1N4
sven.axsater@iml.lth.se • johan.marklund@colorado.edu • silver@mgmt.ucalgary.ca

This paper considers a periodic-review, two-echelon inventory system with one central warehouse and several retailers facing stochastic demand. The retailers replenish their stock from the warehouse, which in turn places orders at an outside supplier with infinite capacity. Transportation times and costs are constant. No ordering costs are considered, but warehouse replenishments must be multiples of a given batch quantity. The objective is to find policies that minimize holding and backorder costs. The standard approach to approximately solve this problem is to use a ''balance'' assumption, meaning that negative stock allocations to the retailers are possible. This approach may lead to considerable errors for problems with large differences between the retailers in terms of service requirements and demand characteristics. To handle such situations we suggest and evaluate two computationally tractable heuristics: the Virtual Assignment ordering rule for warehouse replenishments and the Two-step Allocation rule for allocating stock from the warehouse to the retailers. Numerical evidence shows that, especially when combining these heuristics, we obtain considerable improvements for many problems over the standard approach. Savings of up to 50% have been recorded.
(*Supply Chain Inventories; Stochastic Demand; Ordering and Allocation*)

1. Introduction

In this paper we look at a two-echelon inventory system consisting of a central warehouse supplying a commodity to a number of local warehouses or retailers that face stochastic customer demand. Important questions for cost-efficient management of such supply chains are how much total inventory to carry within the system and where to deploy it so as to minimize the total inventory holding and backorder costs. This paper addresses these two questions by introducing new heuristics for improved centralized inventory control in situations where all installations inspect their inventories periodically. Specifically, we focus on situations where there are large differences between the retailers in terms of their perceived shortage costs and their demand characteristics.

Periodic-review multistage inventory control has attracted a lot of interest from the research community during the last decades, and there are a wide range of specialized models designed to handle different types of systems and conditions. For an overview see Federgruen (1993). Many of these models, including the present work, have to various extents been inspired by the seminal work of Clark and Scarf (1960). In their paper they introduce the echelon-stock concept and characterize optimal inventory policies in a serial system with finite time horizon and no set-up costs at downstream installations. Their optimal-

1523-4614/02/0401/0075$05.00
1526-5498 electronic ISSN

ity results for serial systems were later generalized to an infinite time horizon by Federgruen and Zipkin (1984b), to assembly systems by Rosling (1989), and to batch ordering by Chen (2000). Apart from their analysis of serial systems, Clark and Scarf also briefly discuss how their approach can be modified to deal with arborescent systems in an approximate manner, using the so-called ''balance'' assumption. The essence of this assumption is that the warehouse is allowed to make negative stock allocations to the retailers. Consequently, the risk-pooling effect of keeping stock at the central warehouse to guard against future stock imbalances between retailers disappears. It also means that only the sum of all the inventories in the system (the echelon stock) is of interest for the ordering decision at the warehouse. Furthermore, the optimal stock allocation is achieved by looking at the consequences only in the immediate future. This myopic perspective reduces the complex multiperiod allocation problem into a sequence of independent single-period (i.e., myopic) allocation problems.

Even though the ''balance'' assumption might seem somewhat unrealistic, it has since been used extensively in the inventory literature and has been shown to produce solutions of very good quality in many different situations, see, for example, Eppen and Schrage (1981), Federgruen and Zipkin (1984a, b, c), Federgruen (1993), van Houtum et al. (1996), Verrijdt and de Kok (1996), van der Heijden et al. (1997) and references therein. Still, as indicated in, for example, Zipkin (1984), Federgruen and Zipkin (1984a, c), McGavin et al. (1997), Kumar and Jacobson (1998), and explicitly illustrated in this paper, the ''balance'' assumption is less appropriate in situations with long order cycles and large differences between the retailers in terms of service requirements and demand characteristics.

For the PUSH type, periodic-review systems considered in this research, the inventory replenishment decisions can be divided into two categories:

(1) When and how much should the warehouse order from the outside supplier, i.e., *the warehouse ordering problem*;

(2) How should the available inventory be allocated among the different facilities, i.e., *the allocation problem.*

In this paper we look at two heuristics for the warehouse ordering problem: the classical Clark and Scarf approach, which boils down to finding an echelon-stock reorder point, and a new heuristic called the Virtual Assignment ordering rule. Similarly, for the allocation problem we consider the classical myopic allocation rule and a new heuristic called the Two-step Allocation rule. The performances of the different combinations of ordering and allocation rules are evaluated using simulation. The numerical study shows that in cases with large warehouse order quantities and large differences in retailer service requirements and/or demand characteristics, the new heuristics, especially when combined, offer significant savings in expected holding and backorder costs. Cost reductions as high as 50% have been recorded. Another interesting observation from the numerical study is that when the demand variability increases in relation to the average demand, the portion of the total system stock kept at the warehouse decreases.

The rest of this section is devoted to a brief discussion on related models. The specific assumptions defining the considered inventory system are discussed in §2. Detailed mathematical descriptions of the ordering and allocation rules are given in §3. Section 4 presents a numerical study, which illustrates the performance of the different heuristics. A short summary and some concluding remarks can be found in §5.

Related Literature

Apart from the already mentioned work by Clark and Scarf (1960), there exist a number of related publications dealing with periodic-review, two-echelon arborescent inventory systems. Eppen and Schrage (1981) consider a model with identical retailers and normally distributed demand where the central warehouse is not allowed to carry any stock. They consider base-stock policies and fixed-period ordering policies at the warehouse and a myopic-type allocation policy. Federgruen and Zipkin (1984a, c) extend the Eppen and Schrage model in a number of ways, including nonidentical retailers, nonstationary demand, and (s, S) ordering policies at the warehouse. Of particular relevance is the empirical evaluation of generalized myopic-allocation policies in Federgruen and Zipkin (1984c). Their allocation policies, however, are based on the assumption of no warehouse stock. Kumar and Jacobson (1998) also build on the Eppen and Schrage model, relaxing the simplifying ''balance'' assumption (referred to as the allocation assumption). They provide methods for evaluating system costs and the probability that inventory imbalances will occur under different procurement and allocation policies. Jönsson and Silver (1987) deal with a depotless system, analyzing the effect of allowing redistribution of the retailer inventories through lateral transshipments just before the last time period in the order cycle. In a sense, their division of the entire order cycle into two subperiods of different length is similar to the Two-step Allocation heuristic presented in this paper. With the model by Jönsson and Silver (1987) as a starting point, Schwarz (1989) analyzes the value of risk-pooling over the outside supplier lead time. He concludes that the value is especially high in situations with significant demand variability, long warehouse lead times, and short retailer lead times.

Jackson (1988) considers an extension of the Eppen and Schrage model where the warehouse is allowed to carry stock. The analysis is focused on a more elaborate allocation policy (a so-called ship-up-to-*S* policy), which, unlike the myopic methods, explicitly takes into account the risk-pooling benefits of keeping stock at the warehouse. Erkip (1984) considers the same problem setting as Jackson (1988), and proposes an alternative allocation scheme, in which a fraction of the warehouse inventory is allocated at the beginning of the order cycle and the rest is retained for an additional allocation opportunity later in the cycle. Jackson and Muckstadt (1989) consider retailers that are different but that still have identical cost parameters. Their focus is to analytically investigate the risk-pooling effect of keeping stock at the warehouse; they also present an allocation policy related to the Two-step Allocation policy and the ideas in Erkip (1984).

McGavin et al. (1993) consider a one-warehouse *N*-identical retailer system with lost sales. They show that to minimize lost sales, the optimal allocation policy is to balance the inventory at every allocation opportunity, i.e., to maximize the lowest retailer-inventory position. Furthermore, they design two allocation heuristics, one based on an infinite-retailer model and one simplified version called the 50/25 heuristic. In both cases, the order cycle is divided into two allocation intervals. From a numerical study, they conclude that a second allocation opportunity, late in the replenishment cycle, seems to capture most of the risk-pooling effect of keeping stock at the warehouse. Virtual assignment of stock to retailers, which is the basis for our warehouse ordering heuristic, was first suggested by Graves (1996).

2. Problem Formulation

Inventory positions are reviewed periodically. We assume that all events take place at the *beginning* of a period in the following sequence:

(1) The warehouse decides whether to order from the outside supplier.

(2) Deliveries from the outside supplier arrive at the warehouse.

(3) All or part of the stock on hand at the warehouse is allocated to the retailers.

(4) Deliveries from the warehouse arrive at the retailers.

(5) The stochastic demand takes place at the retailers.

All stockouts are backordered. There are holding costs incurred both at the central warehouse and at the retailers proportional to the inventory on hand. Each retailer is also charged a constant backorder cost per unit per period. There are no ordering costs or constraints associated with deliveries from the warehouse to the retailers. The deliveries from the outside supplier to the warehouse are restricted to be integer multiples of a *given* batch quantity (for example, determined by a deterministic EOQ model). No explicit ordering costs are considered. The stochastic demand at the retailers is stationary and independent across periods and retailers. We introduce the following notation:

N = number of retailers;

L_0 = lead time (integral number of periods) for an order generated by the warehouse;
L_j = transportation time (integral number of periods) for a delivery from the warehouse to retailer j;
Q_0 = warehouse batch size;
$f_j^n(v)$ = probability density function for demand at retailer j over n periods;
$F_j^n(v)$ = cumulative distribution function for demand at retailer j over n periods;
$f^n(v)$ = probability density function for total system demand over n periods (the convolution of $f_j^n(v)$ for $j = 1, 2, \ldots, N$);
μ_j = average demand per period at retailer j;
σ_j = standard deviation of the demand per period at retailer j;
h_j = holding cost per unit and period at installation j ($h_j \geq h_0$ for $j > 0$);
e_j = echelon-holding cost per unit and period at installation j, i.e., $e_0 = h_0$, and $e_j = h_j - h_0$ for $j > 0$;
p_j = backorder cost per unit per period at retailer j.

3. Methodology

This section provides detailed descriptions of the considered heuristics: the Classical Approach, the Virtual Assignment ordering rule, and the Two-step Allocation rule. The presentation is based on the assumption that the customer demand is continuous. The necessary modifications to handle discrete demand processes are straightforward.

3.1. The Classical Approach

Clark and Scarf proposed a heuristic based on the "balance" assumption, which from here on is denoted the Classical Approach. One consequence of the "balance" assumption is that it is optimal for the retailers to use order-up-to-S policies (Clark and Scarf 1960). We introduce the notation:

S_j = order-up-to level for retailer j;
B_j = expected backorder level at retailer j;
C_j' = expected total of echelon-holding costs and backorder costs at retailer j.

All considered costs are on a per period basis.

Assume that retailer j orders up to S_j in period t. The expected backorder level in period $t + L_j$ is then

$$B_j(S_j) = \int_{S_j}^{\infty} (v - S_j) f_j^{L_j+1}(v)\, dv. \quad (1)$$

The corresponding expected total echelon-holding and shortage costs are

$$C_j'(S_j) = e_j[S_j - (L_j + 1)\mu_j + B_j(S_j)] + p_j B_j(S_j). \quad (2)$$

Assume, now, that the warehouse applies an echelon-stock reorder point policy with

R_0 = echelon-stock reorder point at the warehouse,

and let

C_0' = expected echelon-holding costs at the warehouse.

Under very general conditions it can be shown that the warehouse echelon inventory position (after ordering, if necessary) is uniformly distributed on $[R_0, R_0 + Q_0]$ (see Hadley and Whitin 1963). The expected echelon-holding costs at the warehouse corresponding to the order-up-to-levels, S_j, are then

$$C_0'(R_0, S_1, \ldots, S_N) = e_0\left[R_0 + Q_0/2 - (L_0 + 1)\sum_{j=1}^{N} \mu_j + \sum_{j=1}^{N} B_j(S_j)\right]. \quad (3)$$

Without loss of generality, all costs that depend on S_j are assigned to retailer j. As a consequence, (2) and (3) are replaced by

$$C_j(S_j) = e_j[S_j - (\mathrm{L}_j + 1)\mu_j] + (p_j + h_j)B_j(S_j) \quad (4)$$

and

$$C_0(R_0) = e_0\left[R_0 + Q_0/2 - (L_0 + 1)\sum_{j=1}^{N} \mu_j\right]. \quad (5)$$

It is easy to show that $C_j(S_j)$ is convex in S_j. The optimal S_j can therefore be obtained from the first-order "Newsvendor" condition

$$1 - F_j^{L_j+1}(S_j^*) = \frac{e_j}{p_j + h_j}. \tag{6}$$

If the echelon-stock at the warehouse, u, is sufficient, then S_j^* is allocated to retailer j. If not, the available quantity is used to minimize the total retailer costs C^r, i.e., if $\Sigma_{j=1}^N S_j^* > u$, the restricted problem (7) is solved.

$$C^r(u) = \min_{\sum_{j=1}^N S_j \le u} \sum_{j=1}^N C_j(S_j). \tag{7}$$

If the constraint $\Sigma_{j=1}^N S_j \le u$ is relaxed by a Lagrange multiplier $\lambda \ge 0$, the solution of (7) must satisfy the following condition, which is a slight variation of (6):

$$1 - F_j^{L_j+1}(S_j) = \frac{e_j + \lambda}{p_j + h_j}. \tag{8}$$

The echelon-stock level u is obtained as the warehouse inventory position minus the system demand during L_0 periods. Let

$P(y)$ = expected additional retailer costs due to limited stock for a given warehouse echelon inventory position y;

$$P(y) = \int_{y-\sum_{j=1}^N S_j^*}^{\infty} f^{L_0}(v)\left[C^r(y - v) - \sum_{j=1}^N C_j(S_j^*)\right] dv. \tag{9}$$

Given the "balance" assumption, the total expected costs $C(R_0)$ can then be determined as

$$C(R_0) = C_0(R_0) + \sum_{j=1}^N C_j(S_j^*) + \frac{1}{Q_0}\int_{R_0}^{R_0+Q_0} P(y)\, dy. \tag{10}$$

By taking the derivative with respect to R_0, we get the optimality condition

$$[P(R_0) - P(R_0 + Q_0)]/Q_0 = e_0. \tag{11}$$

Under the Classical Approach, (11) is used to determine the warehouse reorder point, R_0, and (10) to get a lower bound for the total costs (because the "balance" assumption is a relaxation). The computational procedure is to first determine the order-up-to-levels S_j^* from (6). Next the function $C^r(u)$ in (7) is tabulated for $u \le \Sigma_{j=1}^N S_j^*$. The additional costs $P(y)$ in (9) and the costs $C(R_0)$ according to (10) are obtained by numerical integration.

The determination of R_0 is based on the "balance" assumption. If the echelon stock at the warehouse $u \le \Sigma_{j=1}^N S_j^*$, we apply (7). This means that negative allocations to some retailers might occur. When implementing the solution, this is not feasible. Therefore, the actual allocations are determined by solving a modified version of (7), the so-called myopic-allocation problem, see (12). Let x_j denote the inventory position of retailer j just before allocation (u still represents the warehouse echelon stock):

$$\min_{\substack{\sum_{j=1}^N S_j \le u \\ S_j \ge x_j}} \sum_{j=1}^N C_j(S_j). \tag{12}$$

Since $C_j(S_j)$ is convex, the optimal solution to (12) can be obtained using the following standard approach. First disregard the constraints $S_j \ge x_j$. The resulting problem is equivalent to (7), which can be solved using the optimality conditions (8) obtained from Lagrangian relaxation. If the solution for some k does not satisfy $S_k \ge x_k$, we add constraints by setting all such $S_k = x_k$ (remember that $C_j(S_j)$ is convex). In other words, the retailers that would get negative allocations based on the "balance" assumption are given nothing. The procedure is repeated until we get a feasible (and optimal) solution to (12).

3.2. The Virtual Assignment Ordering Rule

The Classical Approach for determining R_0, due to the "balance" assumption, underestimates the expected total of carrying and shortage costs at the retail level. The system stock can be used more "efficiently" by allowing complete redistribution of retailer inventory positions in the beginning of each period. The resulting solution will therefore underestimate the required system stock. We now consider a new decision rule, the Virtual Assignment ordering rule, which instead *overestimates* the required stock. The idea is to keep track of all the inventory positions in the system and, based on this information, assign stock to the retailers as soon as it is ordered from the external supplier. Clearly, this is a more restrictive assumption than delaying the allocation decision until the units are delivered to the warehouse. As a result, it will lead to higher stocking levels. It is important to emphasize that this virtual assignment of stock is only used in the decision rule for warehouse replenishments. The final allocation of stock to retailers is, as before, decided when the stock has reached the warehouse. Virtual assignment of stock to retailers is also used in a similar way by Axsäter (1993) and Graves (1996). The idea of using an ordering rule based on information about the individual stocking levels at all installations in the system is also used in Marklund (1997). However, apart from the general idea his approach is completely different from the present work.

Note that a system where the allocation decision is made at the same moment as the items are ordered from the external supplier can be replaced by an equivalent system with the warehouse lead time set to zero and the lead time for retailer j equal to $L_0 + L_j$. Let the corresponding order-up-to level for retailer j at time t be denoted by $\bar{S}_j$. The expected backorder level in period $t + L_0 + L_j$ is then, in complete analogy with (1),

$$\bar{B}_j(\bar{S}_j) = \int_{\bar{S}_j}^{\infty} (v - \bar{S}_j) f_j^{L_0+L_j+1}(v)\, dv. \qquad (13)$$

Similarly, as in (4),

$$\bar{C}_j(\bar{S}_j) = e_j[\bar{S}_j - (L_0 + L_j + 1)\mu_j] + (p_j + h_j)\bar{B}_j(\bar{S}_j). \qquad (14)$$

We denote the optimal order-up-to-level by $\bar{S}_j^*$, which is based on (6), except $F_j^{L_0+L_j+1}(S_j^*)$ is used instead of $F_j^{L_j+1}(\bar{S}_j^*)$.

Let

IP_0 = warehouse echelon inventory position before a possible order.

As before, the inventory position of retailer j is denoted x_j. Assume that after ordering the warehouse echelon inventory position is u. As in (12) the corresponding retailer costs can then be expressed as

$$\bar{C}(u) = \min_{\substack{\sum_{j=1}^N \bar{S}_j \le u \\ \bar{S}_j \ge x_j}} \sum_{j=1}^{N} \bar{C}_j(\bar{S}_j). \qquad (15)$$

Note that if $u \ge \Sigma_{j=1}^N \bar{S}_j^*$, we get $\bar{C}(u) = \Sigma_{j=1}^N \bar{C}_j(\bar{S}_j^*)$. Obviously, $\bar{C}(u)$ is nonincreasing with u. It can be shown that $\bar{C}(u)$ is convex.

The new decision rule, to be applied at the beginning of every time period, can be formally stated as follows.

The Virtual Assignment Ordering Rule. Order mQ_0 units, where m is the smallest nonnegative integer satisfying

$$e_0Q_0 \ge \bar{C}(IP_0 + mQ_0) - \bar{C}[IP_0 + (m + 1)Q_0]. \qquad (16)$$

In other words, order until the additional holding costs are higher than the savings in the retailer costs (compare with (11)). The decision rule is to be applied in every period and considers not only the warehouse echelon-stock inventory position but also the retailer inventory positions x_j.

3.3. The Two-Step Allocation Rule

Under the "balance" assumption the myopic allocation rule used in the Classical Approach (see (12)) is optimal. However, there is a risk-pooling benefit of retaining stock at the warehouse that is completely neglected in the myopic approach. To illustrate this, consider a system where $h_0 = h_i$ or equivalently $e_i = 0$ ($i = 1, 2, \ldots, N$). From (4) and (12) we can see that this means that the warehouse will never carry any inventory. Intuition tells us that this hardly can be the best course of action if the order cycle is long. In general, there must be a potential gain in saving some inventory at the warehouse to be allocated later in the order cycle. The Two-step Allocation rule is designed to recognize this fact by letting the warehouse reserve a suitable amount of stock for subsequent allocations. The rule is used at the beginning of *every* time period, thereby making use of the most recent information.

Ideally, one would like to determine the retailer allocations at the beginning of each period by solving a stochastic dynamic-programming problem covering at least all time periods until the next delivery arrives at the warehouse. (This statement is based on the assumption that each delivery to the warehouse is large enough to satisfy the retailer requirements at least in the first period in the order cycle. As a result, there are no risk-pooling benefits of retaining stock at the warehouse between order cycles.) Unfortunately, such a dynamic-programming approach is computationally unrealistic except for very few retailers, short order cycles, and small-size, discrete demand distributions (see, for example, Jackson and Muckstadt 1989). Some kind of approximation scheme is needed.

The Two-step Allocation method can be divided into two hierarchical levels:

(1) Determination of how much of the available inventory should be retained at the warehouse and how much is available for allocation in the current period.

(2) Determination of the quantity to be shipped to each retailer in the current period, given the amount available for immediate allocation.

Inspired by Jönsson and Silver (1987), Jackson and Muckstadt (1989), and McGavin et al. (1993), we divide the time until the next order arrival at the warehouse into two sub-periods, denoted sp1 and sp2. In every period prior to the last period in the order cycle, the two-step procedure makes an allocation as if there will be only one remaining allocation after the current period. Now, consider the allocation decision in an arbitrary time period. Let $t_r > 1$ denote the estimated number of periods remaining in the current order cycle (if $t_r = 1$, then (12) is applied). If there are outstanding orders at the warehouse, it is easy to determine t_r since it is known exactly when the first batch will arrive. If there are no orders in transit to the warehouse, we estimate $t_r > L_0$ from the expected length of an order cycle ($=Q_0/\Sigma_{j=1}^{N} \mu_j$, rounded to the closest integer), adjusted for the number of periods since the last delivery.

Additional notation:

D_j^n = demand at retailer j during n time periods, stochastic variable;

$\mathbf{D}^n$ = the vector $(D_1^n, D_2^n, \ldots, D_N^n)$, i.e., demand at retailer 1, 2, . . . , N during n time periods;

E_0 = echelon stock at the warehouse before allocation in the current period;

y_j = inventory position at retailer j after allocation in sp1;

z_j = inventory position at retailer j after allocation in sp2.

As above, define

$$B_j^k(S_j) = \int_{S_j}^{\infty} (v - S_j) f_j^{L_j+k}(v)\, dv; \tag{17}$$

$$C_j^k(S_j) = e_j[S_j - (L_j + k)\mu_j] + (p_j + h_j)B_j^k(S_j). \tag{18}$$

To determine how much of the available inventory should be allocated in sp1 and sp2, respectively, so as to minimize the total expected cost, we would like to solve

$$\min_{\substack{\sum_{j=1}^{N} y_j \le E_0 \\ y_j \ge x_j}} \left\{ \sum_{j=1}^{N} \sum_{k=1}^{\text{sp1}} C_j^k(y_j) \right.$$

$$\left. + \underset{\mathbf{D}^{\text{sp1}}}{E}\left[\min_{\substack{\sum_{j=1}^{N} (z_j + D_j^{\text{sp1}}) \le E_0 \\ z_j \ge y_j - D_j^{\text{sp1}}}} \sum_{j=1}^{N} \sum_{k=1}^{\text{sp2}} C_j^k(z_j) \right] \right\}. \tag{19}$$

Even though (19) considers only two periods, it is very time consuming to solve exactly even for small problems (see, for example, Jackson and Muckstadt 1989). To find an approximate solution we use the following heuristic approach based on a desire to treat the two periods sequentially and to solve the problem just once for certain values of t_r and E_0.

First consider the myopic type problem (20) for sp1, ignoring the effects in sp2 for some given value of u, $0 \le u \le E_0$. Note that the constraints $y_j \ge x_j$ are disregarded, which implies that the entire echelon stock is available for allocation at the warehouse. The reason for this assumption is to enable tabulation of the cost-minimizing amount of stock available for immediate allocation, u^*, as a function of the system stock E_0.

$$\text{TC}^{\text{sp1}}(u) = \min_{\substack{\sum_{j=1}^{N} y_j = u \\ y_j \ge 0}} \left\{ \sum_{j=1}^{N} \sum_{k=1}^{\text{sp1}} C_j^k(y_j) \right\}. \tag{20}$$

The solution to (20), $y_j(u)$ for $j = 1, 2, \ldots, N$, is the best allocation in sp1 given the allocated amount, u, when ignoring the cost effects in sp2 and the constraints $y_j \ge x_j$.

To solve the convex problem (20), we relax the condition $\Sigma_{j=1}^{N} y_j = u$ using a Lagrangian multiplier λ. For a given λ the relaxed problem decomposes into N newsvendor problems with solutions

$$y_j^*(\lambda) = \max(0, y_j'), \tag{21}$$

where y_j' must satisfy the following optimality condition

$$\sum_{k=1}^{\mathrm{sp1}} F_j^{L_j+k}\left(\frac{y_j' - (L_j + k)\mu_j}{\sigma_j(L_j + k)^{1/2}}\right) = \frac{\mathrm{sp1}(p_j + h_0) - \lambda}{p_j + h_j}. \tag{22}$$

Note that the left-hand side of (22) is monotonically increasing in y_j', which means that the solution is unique.

The optimal multiplier value, λ^*, corresponding to the optimal solution to (20), must satisfy the condition

$$\sum_{j=1}^{N} y_j^*(\lambda^*) = u. \tag{23}$$

Since $y_j^*(\lambda)$ is nonincreasing in λ, see (21) and (22), it is easy to find λ^* and $y_j(u) = y_j^*(\lambda^*)$ by applying a simple search procedure over λ, for example, with starting value $\lambda = 0$. Note that the optimality conditions above are sufficient due to the convexity of the original problem (20).

Second, apply (24) to determine the minimum expected cost in sp2 given the allocation $y_j(u)$ ($j = 1, 2, \ldots, N$) in sp1, obtained from (20),

$$\mathrm{TC}^{\mathrm{sp2}}(u) = \underset{\mathbf{D}^{\mathrm{sp1}}}{E}\left[\min_{\substack{\sum_{j=1}^{N}(z_j + D_j^{\mathrm{sp1}}) \le E_0 \\ z_j \ge y_j(u) - D_j^{\mathrm{sp1}}}} \sum_{j=1}^{N}\sum_{k=1}^{\mathrm{sp2}} C_j^k(z_j)\right]. \tag{24}$$

The total expected cost for a given u is $\mathrm{TC}(u) = \mathrm{TC}^{\mathrm{sp1}}(u) + \mathrm{TC}^{\mathrm{sp2}}(u)$. Searching over different values of u ($0 \le u \le E_0$) will render the value u^*, which gives the lowest total cost. When determining $\mathrm{TC}^{\mathrm{sp1}}(u)$ for increasing values of u, one can stop when $\mathrm{TC}^{\mathrm{sp1}}(u)$ starts to increase.

Depending on the demand distribution and the size of N, (24) might still be cumbersome to solve. The problem is the large number of possible demand realizations that have to be considered to determine the expectation in (24). To reduce the computational effort it is suggested that for each retailer j the actual distribution of D_j^{sp1} be replaced by a sparsely discrete probability function with correct mean and variance. Note that this approximation is only used to obtain the outer expectation in (24) and does not affect the evaluation of the cost functions C_j^k. In the numerical study presented in §4, we have fitted three-point distributions to describe D_j^{sp1} when computing (24). As a result we only have 3^N demand combinations to consider.

The approximate method described above is used to determine u^* for each t_r that might occur and a discrete number of values of E_0. For each t_r, u^*/E_0 is tabulated as a function of E_0. To obtain u^*/E_0 for some intermediate value of E_0, simple linear interpolation is used. This means that it is not necessary to solve the Two-step Allocation problem at the beginning of every period. Instead, we only have to perform a simple linear interpolation. Of course, initially the table need to be constructed, but that is done only once.

Although the approximation might seem rather rough, the numerical study in §4 indicates that it works quite well. Furthermore, the relative solutions u^*/E_0 seem to be stable for adjacent values of E_0, which implies that the generated tables need rather few values of E_0. In the numerical study we have, for example, used a grid of 5, i.e., $E_0 = 0, 5, \ldots, E_0^{\max}$ (with $E_0^{\max} = Q_0 + \Sigma_1^N S_j^*$). This meant 10–50 values (for each value of t_r). In constructing the table we must also decide on a maximum value of t_r, denoted $t_r^{\max}$. It is clear that $t_r^{\max}$ should cover a warehouse order cycle with a high probability. In the numerical study $t_r^{\max}$ was chosen as the average order cycle ($=Q_0/\Sigma_{j=1}^N \mu_j$) rounded to the next higher integer plus two periods. If for some reason the table is too small, i.e., $E_0 > E_0^{\max}$ or $t_r > t_r^{\max}$, the ratio u^*/E_0 corresponding to the largest tabulated value $E_0^{\max}$ or $t_r^{\max}$, respectively, is used.

Through the above approach, the maximum amount of stock available for immediate allocation u^* can be determined at the beginning of *each* period. The actual allocation is obtained from the myopic allocation problem (12) with $u = u^*$. Note especially that although t_r is always divided into two subperiods sp1 and sp2, the TA heuristic is applied in the same way in each period.

Summary of the Two-Step Allocation Heuristic.

(1) At the beginning of each time period, use the table to find the value of u^* for the current values of t_r and E_0.

(2) Determine the allocation in the current time period by solving the myopic allocation problem (12) with $u = u^*$.

To implement the Two-step Allocation heuristic it remains to choose the lengths of sp1 and sp2. In the numerical study (§4) we report results obtained using $\text{sp1} = t_r - 1$ and $\text{sp2} = 1$, which, in general, turned out to be a good choice, providing a high and stable performance compared to the Classical Approach. This means that the amount available for immediate allocation in the first period is chosen so as to cover the demand for the next $t_r - 1$ periods and the allocations in periods 2, 3, . . . , $t_r - 1$ will only be marginal adjustments of this initial allocation. Evidently, the allocation in period 1 then tends to be too large (although never larger than in the Classical Approach since the myopic problem (12) determines the actual allocations). The solution will be similar to a solution where only two allocations are allowed. Such a solution, with allocations only in the first and the last period, however, has been reported to be close to optimum (see, for example, Jönsson and Silver 1987 and McGavin et al. 1993). Since the Two-step Allocation heuristic is applied in every period, a natural alternative is to use $\text{sp1} = 1$ and $\text{sp2} = t_r - 1$. This means aggregating all periods beyond the upcoming period. In this case another potential drawback occurs. Since the model assumes just a single allocation for the remaining $t_r - 1$ periods, the amount needed to be saved for this allocation is overestimated. The result might be that some retailers experience shortage costs early in the order cycle while there is an abundance of stock at the warehouse simultaneously incurring holding costs. In the numerical study this drawback was in some instances quite serious, rendering costs significantly higher than the alternative with $\text{sp1} = t_r - 1$ and even higher than the Classical Approach. On the other hand, in some instances using $\text{sp1} = 1$ and thereby retaining more stock at the warehouse leads to considerable cost reductions. An intermediate alternative where the two intervals were chosen as equal as possible was also tested. As expected this leads to solutions with a performance between the two extreme alternatives. In conclusion, the best choice of sp1 and sp2 is contingent on the particular problem. However, choosing a long sp1 implies an overall lower risk of really bad performance. For a further discussion, we refer to §4.7.

Figure 1 Considered Control Heuristics Defined in Terms of Involved Ordering/Allocation Policies

		Allocation Policies	
		Classical Approach	Two-step Allocation
Warehouse Ordering Policies	Classical Approach	CA/CA	CA/TA
	Virtual Assignment	VA/CA	VA/TA

4. Numerical Results

We considered 68 test problems and four combinations of ordering/allocation policies (Figure 1). Most of the problems concern a system consisting of one central warehouse and three retailers. (We also evaluated eight problems with five retailers, see §4.6.) In 51 of the test problems, the customer demands are normally distributed. In the remaining 17 problems, the demand is discrete and has a negative-binomial distribution. The negative-binomial distribution is used to avoid occurrences of negative demand when studying distributions with high coefficients of variation. The first 40 problems, dealt with in §4.1 and §4.2, have the following parameter values in common: $L_1 = L_2 = L_3 = 1$, $h_0 = 0.9$, $h_1 = h_2 = h_3 = 1$, $\mu_1 = \mu_2 = \mu_3 = 2$.

What distinguishes one problem from the others is the order quantity used at the warehouse, Q_0, the warehouse lead time, L_0, the retailer backorder costs, p_j, and the standard deviations of demand for different retailers, σ_j. In §4.3 we evaluate four problems with considerably higher retailer holding costs $h_1 = h_2 = h_3 = 5$. The problems in §4.4 are the same as problems 1–8 in §4.1 except for the backorder costs, which are divided by a factor 10. Section 4.5 considers another variation of problems 1–8, where the

Table 1 Specific Parameter Values for Problems 1–32 with Corresponding Lower Bounds on Total Costs Obtained Through the Classical Approach, TC_{LB}

Problem	Demand	σ	L_0	p_1	p_2	p_3	Q_0	TC_{LB}
1	Normal	0.5	5	20	35	50	20	12.25
2	Normal	0.5	5	20	35	50	40	20.20
3	Normal	0.5	5	5	35	65	20	11.03
4	Normal	0.5	5	5	35	65	40	18.30
5	Normal	0.5	1	20	35	50	20	11.48
6	Normal	0.5	1	20	35	50	40	19.62
7	Normal	0.5	1	5	35	65	20	10.61
8	Normal	0.5	1	5	35	65	40	18.08
9	Normal	1.0	5	20	35	50	20	17.27
10	Normal	1.0	5	20	35	50	40	24.51
11	Normal	1.0	5	5	35	65	20	15.48
12	Normal	1.0	5	5	35	65	40	22.06
13	Normal	1.0	1	20	35	50	20	15.38
14	Normal	1.0	1	20	35	50	40	22.96
15	Normal	1.0	1	5	35	65	20	14.17
16	Normal	1.0	1	5	35	65	40	21.22
17	NegBin	2.0	5	20	35	50	20	32.25
18	NegBin	2.0	5	20	35	50	40	37.49
19	NegBin	2.0	5	5	35	65	20	29.21
20	NegBin	2.0	5	5	35	65	40	33.94
21	NegBin	2.0	1	20	35	50	20	28.33
22	NegBin	2.0	1	20	35	50	40	33.98
23	NegBin	2.0	1	5	35	65	20	25.96
24	NegBin	2.0	1	5	35	65	40	31.36
25	NegBin	4.0	5	20	35	50	20	73.52
26	NegBin	4.0	5	20	35	50	40	75.52
27	NegBin	4.0	5	5	35	65	20	66.47
28	NegBin	4.0	5	5	35	65	40	68.72
29	NegBin	4.0	1	20	35	50	20	66.89
30	NegBin	4.0	1	20	35	50	40	69.33
31	NegBin	4.0	1	5	35	65	20	60.47
32	NegBin	4.0	1	5	35	65	40	62.93

warehouse order quantities are larger. In §4.6 we consider eight problems with five retailers (otherwise the problems are again similar to problems 1–8 in §4.1).

All simulation results are for the exact demand distributions, i.e., no approximations are invoked in the simulation evaluation. The policies under consideration are computed according to the descriptions in §3. For constructing the TA table, see §3.3, TC^{sp2} needs to be computed according to (24). As mentioned before, this is done by replacing the correct distributions of D_j^{sp1} with three-point distributions fitted against the correct mean and variance. In case of normally distributed demand this involves replacing the standard $N(0, 1)$ distribution with a symmetric distribution with the possible outcomes $\{-3, 0, 3\}$ and the corresponding probabilities $\{1/18, 8/9, 1/18\}$. In case of negative-binomial demand we use asymmetric three-point distributions (for details see Appendix A). In §§4.1–4.6 we report detailed results for the Two-step Allocation heuristic with sp1 $= t_r - 1$ (see §3.3). In §4.7 we investigate the effect of other choices of sp1 and sp2. The main focus is on sp1 $= 1$, but we also comment on the results for sp1 $= \lceil t_r/2 \rceil$ ($\lceil x \rceil =$ the smallest integer $\geq x$). Note, sp2 $= t_r -$ sp1.

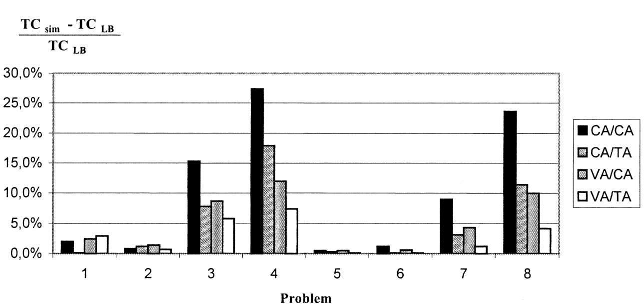
Finally it should be noted that all the presented cost figures *exclude* the holding costs for items in transit between the warehouse and the retailers ($= h_0 \sum_{j=1}^{N} L_j \mu_j$), since their expected values are not affected by the different policies used.

Table 2 Simulation Results for Problems 1–8

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
1	CA/CA	5.42	5.23	7.07	12.49 (0.13)	12.25	2.0%
	CA/TA	5.56	5.18	6.70	12.26 (0.08)		0.1%
	VA/CA	6.50	5.51	6.05	12.55 (0.06)		2.4%
	VA/TA	6.57	5.51	6.04	12.61 (0.06)		2.9%
2	CA/CA	13.09	5.42	7.28	20.37 (0.09)	20.20	0.8%
	CA/TA	13.19	5.35	7.25	20.44 (0.08)		1.2%
	VA/CA	13.92	5.53	6.57	20.49 (0.08)		1.4%
	VA/TA	13.94	5.46	6.41	20.35 (0.06)		0.7%
3	CA/CA	4.27	4.61	8.45	12.72 (0.21)	11.03	15.3%
	CA/TA	4.66	4.34	7.23	11.89 (0.12)		7.8%
	VA/CA	5.63	5.10	6.36	11.99 (0.09)		8.7%
	VA/TA	5.74	4.89	5.93	11.67 (0.06)		5.8%
4	CA/CA	10.48	4.69	12.84	23.32 (0.30)	18.30	27.4%
	CA/TA	10.97	4.48	10.62	21.59 (0.19)		17.9%
	VA/CA	11.95	4.98	9.45	20.50 (0.16)		12.0%
	VA/TA	12.14	4.77	7.52	19.66 (0.11)		7.4%
5	CA/CA	5.04	5.22	6.50	11.54 (0.05)	11.48	0.5%
	CA/TA	5.11	5.20	6.38	11.52 (0.05)		0.3%
	VA/CA	5.28	5.31	6.26	11.54 (0.05)		0.5%
	VA/TA	5.36	5.25	6.13	11.49 (0.05)		0.1%
6	CA/CA	12.75	5.39	7.11	19.86 (0.07)	19.62	1.2%
	CA/TA	12.84	5.38	6.88	19.72 (0.06)		0.1%
	VA/CA	12.98	5.39	6.76	19.74 (0.06)		0.6%
	VA/TA	12.98	5.38	6.80	19.78 (0.06)		0.1%
7	CA/CA	4.23	4.69	7.34	11.57 (0.09)	10.61	9.0%
	CA/TA	4.57	4.48	6.42	10.94 (0.06)		3.1%
	VA/CA	4.53	4.81	6.54	11.07 (0.07)		4.3%
	VA/TA	4.76	4.66	5.98	10.74 (0.04)		1.2%
8	CA/CA	10.54	4.73	11.81	22.35 (0.14)	18.08	23.6%
	CA/TA	10.78	4.56	9.36	20.14 (0.13)		11.4%
	VA/CA	11.89	5.01	8.02	19.90 (0.10)		10.0%
	VA/TA	11.96	4.82	6.87	18.83 (0.07)		4.1%

Note. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

Figure 2 Relative Cost Increase of Using Different Heuristics Compared with the Lower Bound Obtained Through the Classical Approach for Problems 1–8 (see also Table 2)



4.1. Main Problem Set

Our main problem set contains 32 problems with the same standard deviation of demand for all retailers, i.e., $\sigma_j = \sigma$. The values of Q_0, p_j ($j = 1, 2, 3$), L_0, and σ for the different problems are displayed in Table 1.

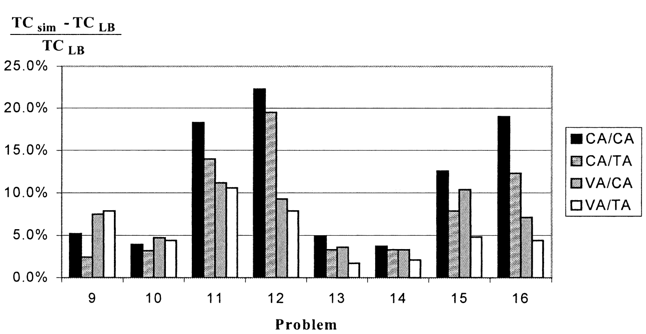
Tables 2–5 show the total expected costs obtained through simulation for the heuristics and relate them to the lower bound (10) for the total cost obtained from the Classical Approach. These tables also depict how much of the holding cost and the total cost is incurred at the retailers and at the central warehouse, respectively. From these figures we can further deduce how the total stock is divided between the central warehouse and the retailers as a group. The rel-

Table 3 Simulation Results for Problems 9–16

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
9	CA/CA	4.40	10.38	13.76	18.16 (0.16)	17.27	5.2%
	CA/TA	4.74	10.03	12.94	17.68 (0.18)		2.4%
	VA/CA	6.55	11.08	12.01	18.56 (0.11)		7.5%
	VA/TA	6.82	10.95	11.81	18.63 (0.12)		7.9%
10	CA/CA	11.03	10.58	14.43	25.46 (0.14)	24.51	3.9%
	CA/TA	11.28	10.43	14.01	25.29 (0.16)		3.2%
	VA/CA	13.05	11.04	12.60	25.65 (0.12)		4.7%
	VA/TA	13.19	10.91	12.39	25.58 (0.13)		4.4%
11	CA/CA	3.25	9.19	15.06	18.31 (0.28)	15.48	18.3%
	CA/TA	3.80	8.62	13.84	17.64 (0.24)		14.0%
	VA/CA	5.53	10.30	11.69	17.22 (0.12)		11.2%
	VA/TA	5.78	9.95	11.34	17.12 (0.13)		10.6%
12	CA/CA	8.72	9.36	18.27	26.99 (0.28)	22.06	22.3%
	CA/TA	9.08	9.04	17.28	26.36 (0.33)		19.5%
	VA/CA	11.23	10.15	12.90	24.12 (0.15)		9.3%
	VA/TA	11.48	9.87	12.32	23.80 (0.15)		7.9%
13	CA/CA	3.24	10.01	12.89	16.13 (0.12)	15.38	4.9%
	CA/TA	3.44	9.81	12.44	15.88 (0.11)		3.3%
	VA/CA	3.70	10.33	12.24	15.94 (0.10)		3.6%
	VA/TA	3.89	10.10	11.75	15.64 (0.09)		1.7%
14	CA/CA	10.11	10.42	13.69	23.80 (0.10)	22.96	3.7%
	CA/TA	10.15	10.45	13.57	23.72 (0.12)		3.3%
	VA/CA	10.70	10.64	13.07	23.72 (0.11)		3.3%
	VA/TA	10.72	10.76	11.75	23.44 (0.10)		2.1%
15	CA/CA	2.61	9.13	13.34	15.95 (0.13)	14.17	12.6%
	CA/TA	3.07	8.53	12.19	15.26 (0.14)		7.9%
	VA/CA	3.24	9.54	12.41	15.65 (0.15)		10.4%
	VA/TA	3.60	9.09	11.23	14.74 (0.10)		4.8%
16	CA/CA	8.44	9.30	16.81	25.25 (0.25)	21.22	19.0%
	CA/TA	8.81	9.15	15.01	23.82 (0.22)		12.3%
	VA/CA	10.14	9.93	12.59	22.73 (0.12)		7.1%
	VA/TA	10.16	9.70	12.00	22.16 (0.11)		4.4%

Note. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

Figure 3 Relative Cost Increase of Using Different Heuristics Compared with the Lower Bound Obtained Through the Classical Approach for Problems 9–16 (see also Table 3)



ative cost increase of using the different heuristics compared to the lower bound obtained by applying the "balance" assumption is also illustrated graphically in Figures 2–5 (note the scale differences).

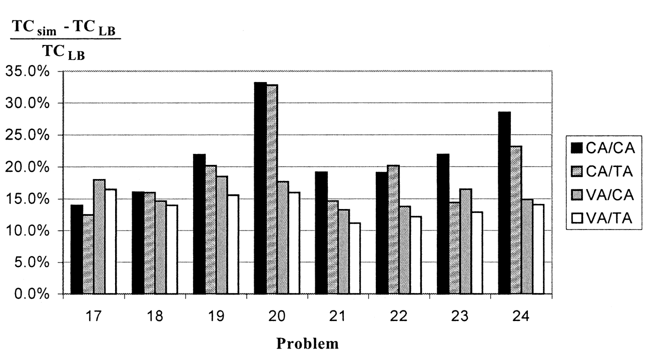
When considering the results for problems 1–32 we conclude that our heuristics will significantly reduce the costs for problems where the retailers have major differences in their perceived shortage costs. The cost reductions are higher for problems with larger warehouse batch quantities (see also §4.5). With large stochastic demand variations the savings are considerable also for some other problems, e.g., problems 26 and 30. In general, the best heuristic is VA/TA, i.e., a combined application of Virtual Assignment ordering

Table 4 Simulation Results for Problems 17–24

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
17	CA/CA	1.21	25.32	35.55	36.76 (0.52)	32.25	14.0%
	CA/TA	2.83	23.74	33.44	36.27 (0.51)		12.5%
	VA/CA	4.82	30.45	33.24	38.06 (0.25)		18.0%
	VA/TA	5.71	29.29	31.86	37.57 (0.23)		16.5%
18	CA/CA	4.55	26.85	38.96	43.51 (0.57)	37.49	16.1%
	CA/TA	5.43	25.94	38.06	43.49 (0.53)		16.0%
	VA/CA	8.48	29.95	34.52	43.00 (0.34)		14.7%
	VA/TA	9.32	28.99	33.41	42.73 (0.35)		14.0%
19	CA/CA	0.85	22.36	34.75	35.60 (0.61)	29.21	21.9%
	CA/TA	2.18	21.10	32.93	35.11 (0.63)		20.2%
	VA/CA	3.35	27.18	28.25	34.60 (0.38)		18.5%
	VA/TA	4.36	26.10	29.42	33.78 (0.27)		15.6%
20	CA/CA	3.27	23.36	41.94	45.21 (0.85)	33.94	33.2%
	CA/TA	4.02	22.80	41.07	45.09 (0.89)		32.8%
	VA/CA	6.97	27.21	32.99	39.96 (0.38)		17.7%
	VA/TA	7.61	26.32	31.76	39.37 (0.35)		16.0%
21	CA/CA	0.07	22.86	33.69	33.76 (0.46)	28.33	19.2%
	CA/TA	1.64	21.19	30.86	32.50 (0.44)		14.7%
	VA/CA	0.51	26.10	31.58	32.09 (0.29)		13.3%
	VA/TA	2.11	23.20	29.40	31.51 (0.33)		11.2%
22	CA/CA	2.89	25.71	37.58	40.47 (0.46)	33.98	19.1%
	CA/TA	2.98	25.54	37.86	40.84 (0.45)		20.2%
	VA/CA	4.47	27.81	34.19	38.66 (0.31)		13.8%
	VA/TA	4.73	27.21	33.42	38.15 (0.32)		12.2%
23	CA/CA	0.04	20.20	31.60	31.64 (0.47)	25.96	21.9%
	CA/TA	1.18	19.05	28.52	29.70 (0.40)		14.4%
	VA/CA	0.46	23.83	29.78	30.24 (0.33)		16.5%
	VA/TA	1.67	21.65	27.63	29.30 (0.32)		12.9%
24	CA/CA	2.19	22.78	38.10	40.29 (0.61)	31.36	28.5%
	CA/TA	2.28	22.56	36.36	38.64 (0.52)		23.2%
	VA/CA	4.25	25.82	31.79	36.04 (0.32)		14.9%
	VA/TA	4.42	25.06	31.37	35.79 (0.33)		14.1%

Note. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

Figure 4 Relative Cost Increase of Using Different Heuristics Compared with the Lower Bound Obtained Through the Classical Approach for Problems 17–24 (see also Table 4)



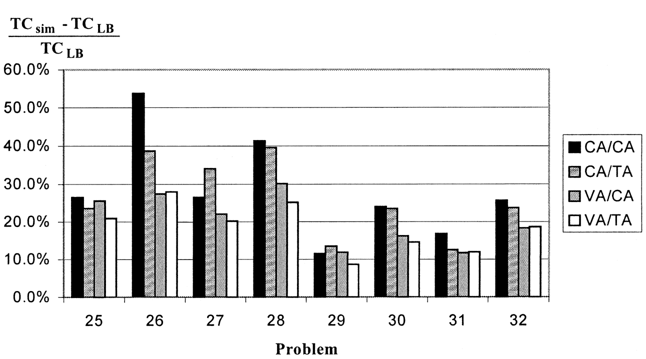
at the warehouse and Two-step Allocation to the retailers. The savings, compared to CA/CA, are often 10–15%. Also CA/TA (Two-step Allocation with classical warehouse ordering) and VA/CA (Virtual Assignment ordering with myopic classical allocation), which perform about as well, give considerable savings compared to the Classical Approach CA/CA for many problems. For other problems (similar shortage costs across retailers and/or small batch sizes) the classical CA/CA approach performs very well, rendering costs that are just above the lower bounds obtained by (10). Our heuristics also give costs close to the lower bounds but not necessarily lower than the CA/CA approach (the differences are not statistically

Table 5 Simulation Results for Problems 25–32 with $\sigma = 4$

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
25	CA/CA	0.00	54.90	93.02	93.02 (2.36)	73.52	26.5%
	CA/TA	1.92	52.88	88.92	90.84 (2.35)		23.6%
	VA/CA	1.05	75.68	91.31	92.36 (1.50)		25.6%
	VA/TA	2.78	73.74	86.14	88.92 (1.21)		20.9%
26	CA/CA	0.07	57.44	116.06	116.13 (3.57)	75.52	53.8%
	CA/TA	0.62	58.13	104.13	104.75 (2.59)		38.7%
	VA/CA	2.48	76.32	93.75	96.23 (1.51)		27.4%
	VA/TA	4.16	73.64	92.47	96.63 (1.70)		28.0%
27	CA/CA	0.00	48.89	84.06	84.06 (2.48)	66.47	26.5%
	CA/TA	1.45	47.07	87.66	89.11 (2.57)		34.1%
	VA/CA	0.54	66.88	80.63	81.17 (1.34)		22.1%
	VA/TA	1.99	65.82	77.89	79.88 (1.12)		20.2%
28	CA/CA	0.04	51.98	97.14	97.18 (2.60)	68.72	41.4%
	CA/TA	0.78	50.45	95.15	95.93 (2.83)		39.6%
	VA/CA	1.79	68.05	87.71	89.50 (1.68)		30.2%
	VA/TA	3.12	66.53	82.89	86.01 (1.31)		25.2%
29	CA/CA	0.00	48.39	74.61	74.61 (1.33)	66.89	11.5%
	CA/TA	1.91	46.06	74.02	75.93 (1.42)		13.5%
	VA/CA	0.00	56.92	74.84	74.84 (1.13)		11.9%
	VA/TA	1.92	53.31	70.82	72.74 (1.18)		8.7%
30	CA/CA	0.00	50.82	85.94	85.94 (1.64)	69.33	24.0%
	CA/TA	0.00	50.72	85.64	85.64 (1.55)		23.5%
	VA/CA	0.05	63.12	80.48	80.53 (1.08)		16.2%
	VA/TA	0.14	62.64	79.33	79.47 (1.08)		14.6%
31	CA/CA	0.00	42.13	70.60	70.60 (1.62)	60.47	16.8%
	CA/TA	1.41	40.34	66.62	68.03 (1.66)		12.5%
	VA/CA	0.00	50.66	67.53	67.53 (1.24)		11.7%
	VA/TA	1.25	48.24	66.49	67.74 (1.30)		12.0%
32	CA/CA	0.00	45.67	79.10	79.10 (1.59)	62.93	25.7%
	CA/TA	0.00	45.74	77.87	77.87 (1.60)		23.7%
	VA/CA	0.03	57.46	74.41	74.44 (1.27)		18.3%
	VA/TA	0.35	55.77	74.31	74.66 (1.20)		18.6%

Note. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

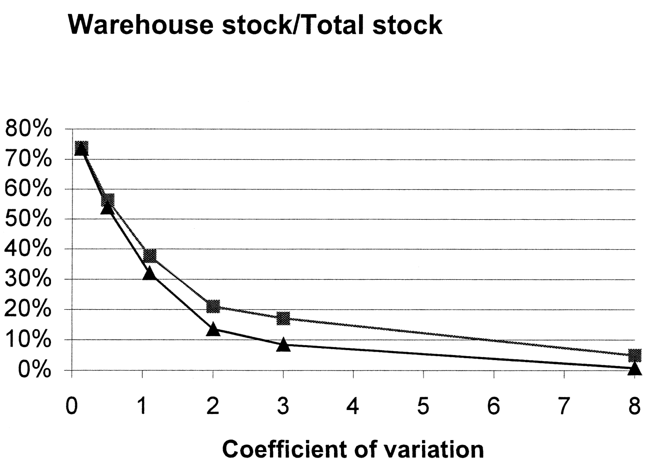
Figure 5 Relative Cost Increase of Using Different Heuristics Compared with the Lower Bound Obtained Through the Classical Approach for Problems 25–32 (see also Table 5)



significant). However, as expected, we can see that a longer warehouse lead time will reduce the efficiency of Virtual Assignment ordering. The explanation is quite intuitive. Due to the "balance" assumption, the Classical Approach tends to underestimate the need for system stock. Virtual Assignment ordering means, on the other hand, that the need for system stock is overestimated. However, for long warehouse lead times Virtual Assignment ordering tends to go a little too far in that direction, as a result the holding costs become relatively high.

The standard deviation σ of retailer demand does not seem to affect the relative performance of the heuristics in any major way. However, a larger variability

Figure 6 Fraction of Average Total Stock (Excluded Stock in Transit Between the Warehouse and Retailers) Kept at the Central Warehouse



Note. VA/TA, $Q_0 = 40$, $p_1 = 5$, $p_2 = 35$, $p_3 = 65$, ■ $L_0 = 5$, and ▲ $L_0 = 1$.

Table 6 Specific Parameter Values for Problems 33–40 with Corresponding Lower Bounds on Total Costs Obtained Through the Classical Approach, TC_{LB}

Problem	$(\sigma_1, \sigma_2, \sigma_3)$	L_0	p_1	p_2	p_3	Q_0	TC_{LB}
33	(1, 0.5, 0.1)	5	20	35	50	20	12.63
34	(1, 0.5, 0.1)	5	20	35	50	40	20.39
35	(1, 0.5, 0.1)	5	5	35	65	20	10.58
36	(1, 0.5, 0.1)	5	5	35	65	40	17.65
37	(0.1, 0.5, 1)	5	20	35	50	20	13.39
38	(0.1, 0.5, 1)	5	20	35	50	40	21.23
39	(0.1, 0.5, 1)	5	5	35	65	20	12.52
40	(0.1, 0.5, 1)	5	5	35	65	40	19.68

Note. Demand is normally distributed.

will increase the gap between the different heuristics and the lower bound. An interesting observation, which can be made from Tables 2–5, is that the fraction of the total stock kept at the central warehouse will decrease as the coefficient of variation increases. The behavior is explicitly illustrated for the VA/TA policy in Figure 6. (Figure 6 includes results for two additional coefficients of variation that are not given in Tables 2–5.)

The managerial implication is that in situations with high demand variability only a small part of the total stock should be kept at the warehouse. Apparently, when the demand variability increases, the relative importance of the difference in holding costs and the risk-pooling benefits decrease in relation to the potential backorder costs that can be avoided if more stock is available at the retailers. To investigate this further we experimented with identical holding costs at the warehouse and at the retailers, eliminating the holding cost incentive to keep stock at the warehouse. The curves corresponding to those in Figure 6 then initially decrease faster and then become much flatter but in essence demonstrate a similar behavior. To understand this, consider first a case with a very low coefficient of variation. If we initially allocate to each retailer what will most certainly be needed during the cycle, this will give enough protection against demand variations in all periods except for the last one. We can therefore save all the "safety stock" to the allocation in the beginning of the last period. When the coefficient of variation increases, we need more retailer "safety stock" earlier in the order cycle, which means that the average warehouse stock will decrease.

4.2. Problems with Different Coefficients of Variation Across Retailers

The second group of problems (33–40), have normally distributed demand but different standard deviations across retailers. The warehouse lead time is $L_0 = 5$ for all problems (see Table 6). Note that in problems 33–36, high standard deviations correspond to low shortage costs and vice versa. The opposite is true for problems 37–40.

The results are given in Tables 7 and 8 and illustrated in Figures 7 and 8 (note the scale differences). The results for problems 33–36 confirm the observations for problems 1–32. The cost reduction compared with the classical approach for problem 35 is as high as 40%. It is interesting to note that the gap between the Classical Approach and the lower bound is much higher for problems 33–36 than for any of the other problems. It seems that the classical myopic allocation will, due to the "balance" assumption, allocate far too much to the retailer with the highest demand variability and lowest shortage cost. On the other hand, the results in Table 8 indicate that the myopic allocation approach benefits from situations with positive

Table 7 Simulation Results for Problems 33–36

Problem	Control policy	C^W_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
33	CA/CA	5.51	9.63	15.14 (0.18)	12.63	19.9%
	CA/TA	5.78	7.57	13.35 (0.14)		5.7%
	VA/CA	6.20	7.70	13.90 (0.12)		10.1%
	VA/TA	6.41	6.71	13.12 (0.09)		3.9%
34	CA/CA	12.94	10.23	23.17 (0.12)	20.39	13.6%
	CA/TA	12.97	8.52	21.49 (0.15)		5.4%
	VA/CA	13.52	8.78	22.30 (0.12)		9.4%
	VA/TA	13.58	7.35	20.93 (0.11)		2.6%
35	CA/CA	4.00	18.88	22.88 (0.37)	10.58	116.3%
	CA/TA	4.58	12.01	16.59 (0.33)		56.8%
	VA/CA	4.58	13.85	18.43 (0.27)		74.2%
	VA/TA	5.06	8.63	13.69 (0.21)		29.4%
36	CA/CA	10.06	25.25	35.31 (0.30)	17.65	100.1%
	CA/TA	10.52	21.25	31.77 (0.36)		80.0%
	VA/CA	10.81	19.78	30.59 (0.28)		73.3%
	VA/TA	11.11	16.63	27.74 (0.34)		57.2%

Note. C^W_{sim} = warehouse costs; C^R_{sim} = retailer costs; TC_{sim} = total costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

Table 8 Simulation Results for Problems 37–40

Problem	Control policy	C^W_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
37	CA/CA	5.83	8.31	14.14 (0.12)	13.39	5.6%
	CA/TA	6.05	7.44	13.49 (0.09)		0.1%
	VA/CA	6.92	7.16	14.08 (0.09)		5.2%
	VA/TA	6.99	6.95	13.94 (0.09)		4.1%
38	CA/CA	13.33	8.76	22.09 (0.09)	21.23	4.1%
	CA/TA	13.37	8.47	21.51 (0.11)		1.3%
	VA/CA	14.23	6.78	21.71 (0.08)		2.3%
	VA/TA	14.52	6.98	21.50 (0.08)		1.3%
39	CA/CA	4.67	9.16	13.83 (0.16)	12.52	10.5%
	CA/TA	4.96	8.47	13.43 (0.15)		7.3%
	VA/CA	6.71	6.78	13.49 (0.08)		7.7%
	VA/TA	6.81	6.78	13.59 (0.08)		8.5%
40	CA/CA	11.05	11.19	22.24 (0.17)	19.68	13.0%
	CA/TA	11.11	10.77	21.88 (0.19)		11.1%
	VA/CA	13.12	8.06	21.18 (0.11)		7.6%
	VA/TA	13.27	7.41	20.68 (0.10)		5.1%

Note. C^W_{sim} = warehouse costs; C^R_{sim} = retailer costs; TC_{sim} = total costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

correlation between high demand variability and high shortage costs.

To give an intuitive explanation of these results, imagine a situation where there are several periods of the order cycle left and a myopic allocation rule is applied. If a retailer has a high backorder cost rate and experiences high demand variability, it will be optimal from a myopic perspective to allocate it a large portion of the available stock. With several periods left before the next warehouse replenishment arrives, shortages are bound to occur. Consequently, it seems logical from this *cycle* perspective, to put most of the stock at the retailer with the highest backorder cost rate. If, on the other hand, the retailer in question has a low demand variability the myopic approach will allocate it less stock. However, with sev-

Figure 7 Relative Cost Increase of Using Different Heuristics Compared with the Lower Bound Obtained Through the Classical Approach for Problems 33–36 (see also Table 7)

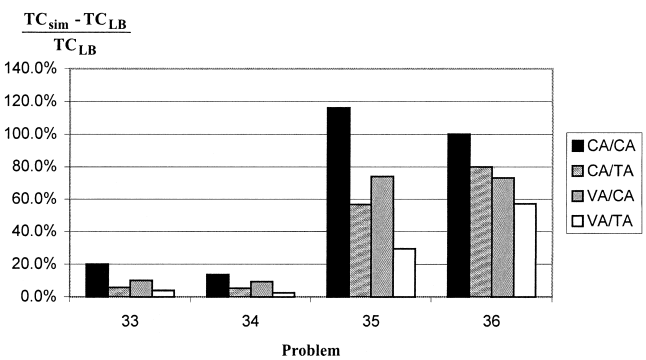
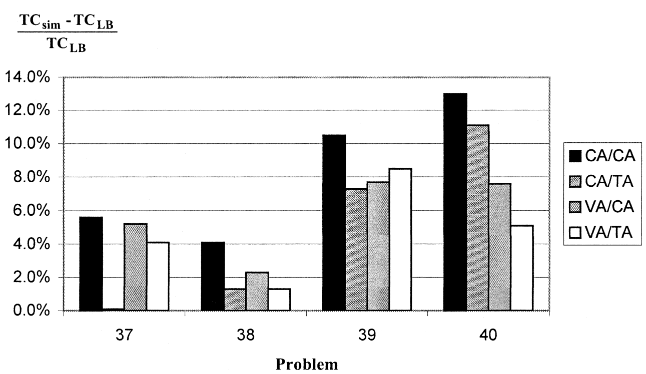


Figure 8 Relative Cost Increase of Using Different Heuristics Compared with the Lower Bound Obtained Through the Classical Approach for Problems 37–40 (see also Table 8)



eral periods of the order cycle left this might be very costly from a cycle perspective.

Although the new heuristics lead to significant improvements, there is still, in many cases, a large gap remaining to the lower bound. Since we do not know how close the true optimum is to the lower bound, the absolute performance of the heuristics for these problems remains unknown, still they certainly outperform the Classical Approach.

4.3. Problems with Higher Retailer Holding Costs

If the retailer holding costs are considerably higher than the warehouse holding cost, the Classical Approach has a stronger incentive to keep stock at the warehouse. We can therefore expect the differences between our new heuristics and the Classical Approach to be smaller. To check this we changed the holding costs to $h_1 = h_2 = h_3 = 5$ in four problems (see Table 9).

The problems in Table 9 (denoted 41–44) were chosen as the two previous problems with the lowest and the two previous problems with the highest relative simulated cost reduction when going from CA/CA to VA/TA, i.e., problems 9, 17, 35, and 36.

Although the savings are lower than for the original problems, the cost reductions when applying the Two-step Allocation heuristic are still substantial for problems 43 and 44. The Virtual Allocation heuristic, however, does not give any significant improvement compared to the Classical Approach for warehouse replenishments.

4.4. Problems with Lower Backorder Costs

The backorder costs in the previous problems are relatively high. Therefore we also analyzed eight problems with very low backorder costs. These problems were generated from problems 1–8 by reducing the backorder cost coefficients by a factor 10 ($p' = p/10$, i.e., $p' = 0.5$, 3.5, 6.5 and 2.0, 3.5, 5.0, respectively). The results are given in Table 10.

If we compare with the corresponding results for problems 1–8 in Table 2 it is obvious that our new heuristics give even larger cost reductions in the case of lower backorder costs. However, for some problems the gap to the lower bound is very large.

4.5. Problems with Larger Warehouse Order Quantities

The results in §4.1 indicated that the cost reductions when using our heuristics were higher for larger warehouse order quantities. This is not surprising, because the assumptions underlying the Classical Approach can be expected to cause larger errors when the order cycles are longer. To check this further, eight problems with considerably larger warehouse batches were evaluated. These problems were constructed by replacing the order quantities $Q_0 = 20$, and $Q_0 = 40$ in problems 1–8 by $Q_0 = 60$, and $Q_0 = 80$, respectively. The results are presented in Table 11.

If the results in Table 11 are compared to the cor-

Table 9 Simulation Results for Problems 41–44 with Holding Costs $h_1 = h_2 = h_3 = 5$

Problem	Control policy	C^W_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
41 (9′)	CA/CA	7.77	36.29	44.06 (0.26)	43.94	0.2%
	CA/TA	7.73	36.11	43.84 (0.26)		0.0%
	VA/CA	8.20	36.02	44.22 (0.27)		0.6%
	VA/TA	8.31	35.85	44.16 (0.27)		0.5%
42 (17′)	CA/CA	9.42	86.27	95.69 (0.76)	92.75	3.2%
	CA/TA	9.63	85.63	95.26 (0.86)		2.7%
	VA/CA	10.22	84.91	95.13 (0.83)		2.6%
	VA/TA	10.28	84.08	94.36 (0.70)		1.7%
43 (35′)	CA/CA	5.65	19.68	25.33 (0.29)	20.28	24.9%
	CA/TA	5.84	16.24	22.08 (0.18)		8.9%
	VA/CA	5.56	20.15	25.71 (0.30)		26.8%
	VA/TA	5.78	16.25	22.03 (0.17)		8.6%
44 (36′)	CA/CA	11.82	26.75	38.57 (0.31)	27.43	40.6%
	CA/TA	12.06	21.73	33.79 (0.29)		23.2%
	VA/CA	11.77	26.86	38.63 (0.32)		40.8%
	VA/TA	12.03	21.88	33.91 (0.29)		23.6%

Note. Original problems: 9, 17, 35, 36. C^W_{sim} = warehouse costs; C^R_{sim} = retailer costs; TC_{sim} = total costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

responding results for problems 1–8 in Table 2, we can again conclude that larger warehouse batches imply larger savings when using our heuristics. The average cost reductions in absolute terms when changing from CA/CA to VA/TA are 0.45, 1.82, 3.04, and 5.99 for $Q_0 = 20$, 40, 60, and 80, respectively.

4.6. Problems with Five Retailers

We have also evaluated eight problems with five retailers. The problems were again generated from problems 1–8. Retailers 1 and 2 are identical and have the same data as retailer 1 in problems 1–8. Retailer 3 has the same data as retailer 2 in problems 1–8. Finally, retailers 4 and 5 are identical with the same data as retailer 3 in the original problems 1–8. The results are presented in Table 12.

When comparing to the results in Table 2, we note that the Virtual Allocation heuristic performs less well (see, e.g., problems 1 and 61). This is natural since the approximation when applying this heuristic (i.e., a preliminary early assignment of stock to individual retailers) is more drastic for the case of five retailers. Furthermore, we note that CA/TA performs very well and may be an interesting alternative to VA/TA when the number of retailers is increasing. Also as expected, the classical approach has a better performance than in problems 1–8 since the retailers have increased in number and similarity.

4.7. Tests with Other Subperiods

As discussed in §3.3 and illustrated by the results in §4.1–4.6, using sp1 = $t_r - 1$ (and sp2 = 1) tends to produce robust results compared to the Classical Approach, but it is not necessarily the best choice of sp1 and sp2 in all problem instances. To investigate the effect of other choices of sp1 and sp2, we evaluated all problems in §4.1 and §4.2 with normally distributed demand, i.e., problems 1–16 and problems 33–40 with sp1 = 1 (and sp2 = $t_r - 1$). Intuitively this choice of sp1 makes sense since we apply TA at the beginning of every period. Table 13 shows the results for the overall best policy VA/TA. For comparison, we also give the results for sp1 = $t_r - 1$ and when applying the classical approach CA/CA (all figures given as the relative cost increase from the lower bound). These latter results can also be found in Tables 2, 3, 7, and 8. The results for sp1 = 1 are comparable to the results with sp1 = $t_r - 1$ for most problems. However, there are some notable differences. In problems 7, 8, 15, and 16 using sp1 = 1 leads to considerable cost increases compared to using sp1 = $t_r - 1$. For problem 8,

Table 10 Simulation Results for Problems 45–52 with Decimated Shortage Costs, i.e., $p_i' = p_i/10$ for All Retailers

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
45 (1′)	CA/CA	2.52	2.79	5.99	8.51 (0.03)	7.86	8.3%
	CA/TA	2.74	2.59	5.58	8.32 (0.03)		5.9%
	VA/CA	2.86	2.93	5.52	8.38 (0.03)		6.6%
	VA/TA	3.04	2.72	5.26	8.30 (0.03)		5.6%
46 (2′)	CA/CA	6.54	2.82	9.10	15.64 (0.04)	13.85	12.9%
	CA/TA	6.62	2.84	9.00	15.62 (0.04)		12.8%
	VA/CA	7.04	2.93	8.31	15.35 (0.04)		10.8%
	VA/TA	7.04	2.92	8.35	15.39 (0.04)		11.1%
47 (3′)	CA/CA	0.58	1.54	11.79	12.37 (0.10)	5.26	135.2%
	CA/TA	1.29	1.20	9.88	11.17 (0.09)		112.4%
	VA/CA	0.94	1.82	9.79	10.73 (0.08)		104.0%
	VA/TA	1.62	1.41	8.25	9.87 (0.07)		87.6%
48 (4′)	CA/CA	1.61	1.48	25.78	27.39 (0.14)	8.39	226.5%
	CA/TA	1.61	1.50	25.47	27.08 (0.14)		222.8%
	VA/CA	4.96	2.43	11.74	16.70 (0.07)		99.0%
	VA/TA	5.17	2.34	11.10	16.27 (0.07)		93.9%
49 (5′)	CA/CA	2.40	2.76	5.77	8.17 (0.02)	7.64	6.9%
	CA/TA	2.50	2.70	5.64	8.14 (0.02)		6.5%
	VA/CA	2.76	2.92	5.34	8.10 (0.02)		6.0%
	VA/TA	2.85	2.84	5.18	8.03 (0.02)		5.1%
50 (6′)	CA/CA	6.61	2.87	8.82	15.43 (0.03)	13.75	12.2%
	CA/TA	6.61	2.86	8.82	15.44 (0.04)		12.3%
	VA/CA	8.36	3.17	6.48	14.84 (0.03)		7.9%
	VA/TA	8.40	3.20	6.48	14.88 (0.03)		8.2%
51 (7′)	CA/CA	0.50	1.52	11.77	12.27 (0.07)	5.16	137.8%
	CA/TA	0.81	1.36	10.86	11.67 (0.08)		126.2%
	VA/CA	2.39	2.66	5.36	7.75 (0.03)		50.2%
	VA/TA	2.84	2.30	4.49	7.33 (0.02)		42.1%
52 (8′)	CA/CA	1.59	1.48	25.52	27.11 (0.11)	8.34	225.1%
	CA/TA	1.58	1.48	25.58	27.16 (0.11)		225.7%
	VA/CA	8.27	3.07	6.19	14.46 (0.03)		73.4%
	VA/TA	8.06	3.01	6.30	14.36 (0.03)		72.2%

Note. Original problems: 1–8. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

the relative gap to the lower bound increases from 4.1% to 30.2%, and VA/TA actually becomes significantly more costly than the CA/CA method. On the other hand, for problems 35 and 36 (problems with different coefficients of variation across retailers) using sp1 = 1 reduces the costs significantly. The relative gap to the lower bound decreases from 29.4% to 6.4% for problem 35 and from 57.2% to 10.1% for problem 36. For problem 35 this means that applying the VA/TA policy with sp1 = 1 instead of CA/CA reduces the costs by just over 50%. The conclusion to be drawn is that using sp1 = 1 is a riskier choice than sp1 = $t_r - 1$ for an arbitrary problem. It can lead to considerable cost savings but can also increase the costs drastically; it might even give higher costs than the Classical Approach.

From an operational point of view using sp1 = 1 means that more units are retained at the warehouse than if sp1 = $t_r - 1$ is used (see discussion in §3.3). As a compromise between the two extreme choices of sp1, a plausible alternative is of course sp1 = $\lceil t_r/2 \rceil$. If we focus on problems 8, 16, and 35 with the largest cost differences, the simulated costs TC_{sim} were 19.32, 22.58, and 13.38, i.e., always between the costs for sp1 = $t_r - 1$ and sp1 = 1.

A final remark is that also with the other subperi-

Table 11 **Simulation Results for Problems 53–60 with Larger Warehouse Order Quantities, i.e., $Q_0 = 20$ and $Q_0 = 40$ Are Replaced by $Q_0 = 60$ and $Q_0 = 80$, Respectively**

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
53 (1″)	CA/CA	21.06	5.45	7.66	28.72 (0.09)	28.47	0.9%
	CA/TA	21.08	5.37	7.49	28.57 (0.09)		0.4%
	VA/CA	21.81	5.52	7.01	28.82 (0.08)		1.2%
	VA/TA	21.86	5.46	6.74	28.60 (0.07)		0.5%
54 (2″)	CA/CA	30.43	5.48	8.08	38.51 (0.18)	36.87	4.4%
	CA/TA	30.66	5.37	8.18	38.84 (0.19)		5.3%
	VA/CA	29.50	5.50	7.49	36.99 (0.18)		0.3%
	VA/TA	29.90	5.48	7.27	37.17 (0.17)		0.8%
55 (3″)	CA/CA	17.06	4.73	18.04	35.10 (0.29)	25.80	36.0%
	CA/TA	17.31	4.56	15.77	33.08 (0.26)		28.2%
	VA/CA	18.43	4.89	12.58	31.01 (0.21)		20.2%
	VA/TA	18.67	4.70	11.17	29.84 (0.19)		15.7%
56 (4″)	CA/CA	23.41	4.76	24.48	47.89 (0.51)	33.36	43.6%
	CA/TA	22.85	4.64	20.21	43.06 (0.44)		29.1%
	VA/CA	23.35	4.86	18.20	42.55 (0.40)		27.5%
	VA/TA	24.20	4.79	14.79	38.99 (0.31)		16.9%
57 (5″)	CA/CA	20.82	5.44	7.44	28.27 (0.07)	28.01	0.9%
	CA/TA	20.84	5.38	7.27	28.11 (0.07)		0.4%
	VA/CA	21.04	5.47	7.21	28.25 (0.07)		0.9%
	VA/TA	21.02	5.43	7.26	28.28 (0.07)		1.0%
58 (6″)	CA/CA	29.89	5.43	8.01	37.90 (0.17)	36.49	3.9%
	CA/TA	29.81	5.41	7.71	37.52 (0.18)		2.8%
	VA/CA	29.51	5.52	7.43	36.94 (0.18)		1.2%
	VA/TA	30.02	5.51	7.27	37.29 (0.18)		2.2%
59 (7″)	CA/CA	16.94	4.72	17.02	33.96 (0.23)	25.65	32.4%
	CA/TA	17.20	4.60	14.97	32.16 (0.23)		25.4%
	VA/CA	20.00	5.10	8.29	28.29 (0.11)		10.3%
	VA/TA	20.21	4.98	6.96	27.18 (0.08)		6.0%
60 (8″)	CA/CA	24.30	4.67	25.66	49.96 (0.48)	33.25	50.3%
	CA/TA	23.75	4.62	21.84	45.58 (0.45)		37.1%
	VA/CA	28.67	5.16	8.41	37.08 (0.21)		11.5%
	VA/TA	29.44	5.05	7.42	36.86 (0.19)		10.9%

Note. Original problems: 1–8. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

ods tested, the warehouse stock decreases with the demand variability in basically the same way as shown in Figure 6.

5. Conclusions

In this paper, we have considered a well-known periodic-review two-echelon arborescent inventory problem. The Classical Approach to solve this problem is to use the so-called ''balance'' assumption, which essentially means that negative stock allocations to the retailers are assumed possible. This approximation is used both when determining the warehouse reorder point and the retailer order-up-to levels. Consequently, the needed system stock is underestimated and too much stock is allocated to the retailers. It is known that this approach works well in many situations, but also that considerable errors may occur when there are large differences between the retailers in terms of their perceived shortage costs and their demand characteristics; a common situation in practice. To handle such situations we have suggested and evaluated two new heuristics: the Virtual Assignment rule for warehouse replenishments, and the Two-step Allocation rule for allocating stock from the warehouse to the retailers. Both of these can be implemented with reasonable

Table 12 Simulation Results for Problems 61–68 with Five Retailers

Problem	Control policy	C^W_{sim}	H^R_{sim}	C^R_{sim}	TC_{sim}	TC_{LB}	$(TC_{sim} - TC_{LB})/TC_{LB}$
61 (1‴)	CA/CA	4.21	8.53	10.74	14.95 (0.13)	14.78	1.2%
	CA/TA	4.32	8.35	10.59	14.91 (0.12)		0.9%
	VA/CA	6.43	9.31	9.74	16.16 (0.08)		9.3%
	VA/TA	6.42	9.21	9.67	16.09 (0.08)		8.9%
62 (2‴)	CA/CA	11.29	8.81	11.42	22.71 (0.11)	22.38	1.5%
	CA/TA	11.38	8.68	10.99	22.37 (0.09)		0.0%
	VA/CA	13.11	9.21	10.08	23.19 (0.09)		3.6%
	VA/TA	13.19	9.19	10.02	23.21 (0.09)		3.7%
63 (3‴)	CA/CA	3.25	7.57	11.17	14.42 (0.17)	13.24	8.9%
	CA/TA	3.58	7.20	9.98	13.56 (0.10)		2.4%
	VA/CA	5.05	8.39	9.23	14.28 (0.09)		7.9%
	VA/TA	5.25	8.36	9.08	14.33 (0.08)		8.2%
64 (4‴)	CA/CA	9.12	7.75	14.72	23.84 (0.24)	20.21	18.0%
	CA/TA	9.57	7.37	11.78	21.35 (0.15)		5.6%
	VA/CA	11.16	8.30	10.56	21.72 (0.11)		7.5%
	VA/TA	11.33	8.12	9.83	21.16 (0.08)		4.7%
65 (5‴)	CA/CA	3.58	8.37	10.27	13.85 (0.09)	13.82	0.2%
	CA/TA	3.74	8.45	10.17	13.91 (0.08)		0.7%
	VA/CA	4.11	8.64	9.82	13.93 (0.07)		0.8%
	VA/TA	4.19	8.70	9.88	14.07 (0.07)		1.8%
66 (6‴)	CA/CA	10.81	8.73	10.80	21.61 (0.08)	21.61	0.0%
	CA/TA	10.87	8.71	10.87	21.74 (0.08)		0.6%
	VA/CA	11.26	8.91	10.37	21.63 (0.07)		0.1%
	VA/TA	11.27	8.81	10.44	21.71 (0.08)		0.5%
67 (7‴)	CA/CA	2.91	7.47	10.06	12.97 (0.10)	12.62	2.8%
	CA/TA	3.23	7.25	9.58	12.81 (0.08)		1.5%
	VA/CA	3.37	7.78	9.41	12.78 (0.07)		1.3%
	VA/TA	3.52	7.64	9.15	12.67 (0.06)		0.4%
68 (8‴)	CA/CA	9.00	7.74	13.39	22.39 (0.18)	19.85	12.8%
	CA/TA	9.46	7.35	10.73	20.19 (0.08)		1.7%
	VA/CA	9.53	7.91	11.99	21.52 (0.14)		8.4%
	VA/TA	9.85	7.57	10.19	20.04 (0.07)		1.0%

Note. The problems were obtained from problems 1–8 by choosing $p_1''' = p_2''' = p_1$, $p_3''' = p_4''' = p_5''' = p_3$. Otherwise all data are the same. C^W_{sim} = warehouse costs; H^R_{sim} = retailer holding costs; C^R_{sim} = total retailer costs; TC_{sim} = total system costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

computational effort, comparable to the requirements of the Classical Approach.

Especially when combining the two new heuristics we obtain considerable cost reductions for many problems. The improvements are most significant when the retailers have different service requirements and/or demand characteristics and when the warehouse uses a large batch quantity. From our numerical study, we can see that for such problems the costs are, in general, reduced by about 10–15%. However, the largest cost reduction was as high as 50%. For problems where the Classical Approach works well, our heuristics give comparable results, and we cannot see any significant cost differences. When the number of retailers becomes larger the performance of Virtual Assignment ordering is deteriorating, and it may be better to combine the Classical Approach for warehouse ordering with the Two-step Allocation rule. In general, we can conclude that the techniques that we have suggested seem to provide very interesting alternatives to the Classical Approach in most practical situations. Apart from providing the means to evaluate the performance of our heuristics, the numerical results also clearly indicate that the portion of the total stock kept at the warehouse should decrease when the demand variability increases.

Table 13 Simulation Results for Problems 1–16 and 33–40 with sp1 = 1 and Control Policy VA/TA

Problem	TC_{sim} VA/TA sp1 = 1	$(TC_{sim} - TC_{LB})/TC_{LB}$ VA/TA sp1 = 1	VA/TA sp1 = $t_r - 1$	CA/CA
1	12.56 (0.06)	2.5%	2.9%	2.0%
2	20.40 (0.06)	1.0%	0.7%	0.8%
3	11.64 (0.06)	5.5%	5.8%	15.3%
4	19.25 (0.07)	5.2%	7.4%	27.4%
5	11.51 (0.05)	0.3%	0.1%	0.5%
6	19.74 (0.07)	0.6%	0.1%	1.2%
7	11.59 (0.07)	9.2%	1.2%	9.0%
8	23.54 (0.12)	30.2%	4.1%	23.6%
9	18.34 (0.10)	6.2%	7.9%	5.2%
10	25.60 (0.11)	4.4%	4.4%	3.9%
11	16.77 (0.10)	8.3%	10.6%	18.3%
12	23.68 (0.14)	7.3%	7.9%	22.3%
13	15.64 (0.08)	1.7%	1.7%	4.9%
14	23.49 (0.11)	2.3%	2.1%	3.7%
15	15.34 (0.11)	8.3%	4.8%	12.6%
16	25.68 (0.19)	21.0%	4.4%	19.0%
33	12.84 (0.09)	1.7%	3.9%	19.9%
34	20.76 (0.10)	1.8%	2.6%	13.6%
35	11.26 (0.07)	6.4%	29.4%	116.3%
36	19.43 (0.10)	10.1%	57.2%	100.1%
37	13.75 (0.11)	2.7%	4.1%	5.6%
38	21.46 (0.08)	1.0%	1.3%	4.1%
39	13.49 (0.08)	7.7%	8.5%	10.5%
40	20.63 (0.08)	4.8%	5.1%	13.0%

Note. Previous results for VA/TA with sp1 = $t_r - 1$ and for CA/CA are given for comparison. TC_{sim} = total costs (standard deviation in parenthesis); TC_{LB} = lower bound obtained through the Classical Approach.

Acknowledgments

The research leading to this paper was partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant No. A1485, by the Carma Chair at the University of Calgary, and by the Swedish Transport and Communications Research Board.

Appendix

This appendix shows how we have fitted a three-point distribution to the mean and variance of $D_j^{sp1} \geq 0$ in the case of negative-binomial demand.

Let

μ = expected demand under sp1 = $E(D_j^{sp1})$;

σ = standard deviation of demand under sp1.

For the negative-binomial distribution the coefficient of variation $\sigma^2/\mu \geq 1$.

The integer outcomes $a < b < c$ were chosen in the following way:

$a = 0$;

b = smallest integer larger or equal to μ;

$c \geq \mu + \sigma^2/\mu$ as close as possible to $\mu + 3\sigma$.

Note that $b < \mu + 1 \leq \mu + \sigma^2/\mu \leq c$.

The corresponding probabilities are obtained in the following way:

$$P_0 = \frac{\sigma^2 + \mu^2 + bc - \mu(b + c)}{bc};$$

$$P_b = \frac{c\mu - \sigma^2 - \mu^2}{(c - b)b};$$

$$P_c = 1 - P_0 - P_b = \frac{\sigma^2 + \mu^2 - \mu b}{(c - b)b}.$$

It is easy to verify that these probabilities are nonnegative and that the distribution has mean μ and standard deviation σ.

References

Axsäter, S. 1993. Optimization of order-up-to-*S* policies in two-echelon inventory systems with periodic review. *Naval Res. Logist.* **40** 245–253.

Chen, F. 2000. Optimal policies for multi-echelon inventory problems with batch ordering. *Oper. Res.* **48** 376–389.

Clark, A.J., H. Scarf. 1960. Optimal policies for a multi-echelon inventory problem. *Management Sci.* **6** 475–490.

Eppen, G., L. Schrage. 1981. Centralized ordering policies in a multi-warehouse system with leadtimes and random demand. L. Schwarz, ed. *Multi-Level Production/Inventory Control Systems: Theory and Practice.* North Holland, Amsterdam, The Netherlands, 51–69.

Erkip, N. 1984. Approximate policies in multi-echelon inventory systems. Unpublished Ph.D. dissertation, Department of Industrial Engineering and Engineering Management, Stanford University.

Federgruen, A. 1993. Centralized planning models for multi-echelon inventory systems under uncertainty. S. C. Graves et al., eds. *Handbooks in OR & MS Vol. 4,* North Holland, Amsterdam, The Netherlands, 133–173.

——, P. Zipkin. 1984a. Approximations of dynamic multilocation production and inventory problems. *Management Sci.* **30** 69–84.

——, ——. 1984b. Computational issues in an infinite horizon, multi-echelon inventory model. *Oper. Res.* **32** 818–836.

——, ——. 1984c. Allocation policies and cost approximations for multilocation inventory systems. *Naval Res. Logist.* **31** 97–129.

Graves, S. C. 1996. A multiechelon inventory model with fixed replenishment intervals. *Management Sci.* **42** 1–18.

Hadley, G., T. M. Whitin. 1963. *Analysis of Inventory Systems.* Prentice Hall, Englewood Cliffs, NJ.

Jackson, P. L. 1988. Stock allocation in a two-echelon distribution system or what to do until your ship comes in. *Management Sci.* **34** 880–895.

——, J. A. Muckstadt. 1989. Risk pooling in a two-period, two-echelon inventory stocking and allocation problem. *Naval Res. Logist.* **36** 1–26.

Jönsson, H., E. A. Silver. 1987. Analysis of a two-echelon inventory control system with complete redistribution. *Management Sci.* **33** 215–277.

Kumar, A., S. H. Jacobson. 1998. Optimal and near-optimal decisions for procurement and allocation of a critical resource with a stochastic consumption rate. Working paper #98019, University of Michigan Business School.

Marklund, J. 1997. Centralized inventory control in a two-level distribution system with Poisson demand. Working paper, Lund University, Sweden.

McGavin, E. J., L. B. Schwarz, J. E. Ward. 1993. Two-interval inventory-allocation policies in a one-warehouse, *N*-identical-retailer distribution system. *Management Sci.* **39** 1092–1107.

——, J. E. Ward, L. B. Schwarz. 1997. Balancing retailer inventories. *Oper. Res.* **45** 820–830.

Rosling, K. 1989. Optimal inventory policies for assembly systems under random demands. *Oper. Res.* **37** 565–579.

Schwarz, L. B. 1989. A model for assessing the value of warehouse risk-pooling: Risk-pooling over the outside supplier leadtimes. *Management Sci.* **35** 828–842.

van der Heijden, M. C., E. B. Diks, A. G. de Kok. 1997. Stock allocation in general multi-echelon distribution systems with (*R, S*) order-up-to policies. *Internat. J. Production Econom.* **49** 157–174.

van Houtum, G. J., K. Inderfurth, W. H. M. Zijm. Materials coordination in stochastic multi-echelon systems. *Eur. J. Oper. Res.* **95** 1–23.

Verrijdt, J. H. C. M., A. G. de Kok. 1996. Distribution planning for divergent depotless two-echelon network under service constraints. *Eur. J. Oper. Res.* **89** 341–354.

Zipkin, P. 1984. On the imbalance of inventories in multi-echelon systems. *Math. Oper. Res.* **9** 402–423.

The consulting Senior Editor for this manuscript was Paul Zipkin. This manuscript was received on October 23, 1998, and was with the authors 706 days for 7 revisions. The average review cycle time was 59 days.