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A Characterization of the SSD-Efficient Frontier of Portfolio Weights by Means of a Set of Mixed-Integer Linear Constraints

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In this paper, the set of all second-order stochastic dominance (SSD)-efficient portfolios is characterized by using a series of mixed-integer linear constraints. Our derivation employs a combination of the first-order conditions of the utility maximization problem together with a judicious use of binary variables. This result opens the door to the formulation of optimizations whose objective function is free to select a particular portfolio out of the entire SSD-efficient set.

Keywords: stochastic dominance; mixed-integer linear programming; portfolio theory
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1. Introduction

There has been a relatively recent explosion of research that tries to raise the status of the stochastic dominance methodology within the field of portfolio theory. The seminal work of Post (2003), Kuosmanen (2004), and Dentcheva and Ruszczynski (2003) has been, to a very large extent, the driving engine of this increased interest. Although these new contributions have developed theory whose application involves the whole spectrum of this field, its most interesting derivations have focused on the concept of second-order stochastic dominance (SSD) efficiency. In particular, it has become viable to establish whether a given asset (or portfolio of assets) can be found optimal for some investor with increasing and concave preferences when the feasible set of choices is any portfolio of the assets in a given investment universe. The merit of these advances is fully understood when one realizes that until their arrival, only pairwise comparison could be implemented; that is, it was only possible to establish whether a given asset or portfolio was optimal when the only alternative was another specific asset or fixed portfolio.

All the above authors make use of linear programming optimization to develop their arguments; however, Post (2003) chooses to focus on the first-order conditions of the utility maximization problem, whereas Kuosmanen (2004) and Dentcheva and Ruszczynski (2003) have as a departure point the cumulative distribution function of the assets' returns. Even though the approach proposed by Post (2003), as opposed to the other two, is computationally affordable when it comes to large samples, it has as its major drawback that it does not identify in general an SSD-efficient portfolio, a drawback that is also shared by the method proposed by Dentcheva and Ruszczynski (2003).

After this turning point, theoretical extensions of these results have proliferated. Post (2008) exploited further his own linear program to construct an SSD dominant (but generally not efficient) portfolio from the dual solution. This identification is also the goal pursued by Kopa and Chovanec (2008), who developed a linear programming approach based on conditional value at risk.

As it turns out, a more ambitious determination of the entire SSD-efficient *frontier* has remained elusive, and the bulk of the work has gravitated around the design of tests of the SSD efficiency of a single portfolio. The most notorious exception is probably the work of Dentcheva and Ruszczynski (2006), who exploit their earlier arguments and by resorting to a set of linear constraints characterize the set of portfolios dominating a given return. An important advantage of this approach is that an optimization with an arbitrary objective function is implementable. However, this approach creates a feasible set whose elements are not necessarily SSD efficient.

The goal of this paper is to present a methodology that fills this gap. Our main theorem presents a series of mixed-integer linear constraints that fully characterize the whole set of SSD-efficient portfolios, and hence it fully accomplishes Levy's (1992) invitation to determine such a set. As a result, an optimization whose arbitrary objective function is free to select

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any particular portfolio from the entire SSD-efficient set thus becomes possible. Our arguments rely on the first-order conditions of the utility maximization problem and the use of binary variables to impose the desired antimonotonicity between marginal utility and the portfolio's returns.

2. A Characterization of SSD Efficiency

Within the confines of static modeling, consider an investment universe containing N assets, where all uncertainty is captured by S states of nature with associated probabilities denoted by p_s with $s=1,\ldots,S$. In this simple setup, the random (gross) returns of the assets can thus be represented by vectors $x_s \in \Re_+^N$, $s=1,\ldots,S$ giving the returns of all assets at state of nature s. Hence, a typical element of these vectors, x_{is}, denotes the particular realization at state of nature s of the return of the ith asset, $i=1,\ldots,N$. It is further assumed that all these N returns are risky and that there is no state for which all their realizations are equal to zero. In addition, there exists a risk-free asset whose constant return is denoted by x_0.[1]

Investors may diversify, that is, they may allocate portions of their wealth to each individual asset. These allocations or *portfolio weights* will be constrained to be elements of a polytope whose form is given by

$$\underline{W} \equiv \{w \in \Re^{N+1}\colon Aw \le b\},$$

where A is an $M \times (N+1)$-dimensional matrix of coefficients with typical element a_{ji}, which premultiplies w to give the left-hand side of M inequalities. Also, w_i represents a typical element of the $(N+1)$-dimensional vector w giving the weight on the ith risky asset, where w_0 gives the weight on the risk-free asset. Furthermore, b_j is an element of the M-dimensional vector b giving the right-hand side of inequality j. The polytope $\underline{W}$ is assumed to be closed and bounded.[2]

Investors make their decisions by maximizing a von Neumann–Morgenstern utility function $u\colon \Re \to \Re$, whose role is to represent their particular preferences. This utility function is usually constrained to be continuous, nondecreasing, and concave to satisfy reasonable economic assumptions. Hence, denote by $\underline{U}$ the set of such functions.[3]

Thus, the main focus of our analysis will be the solutions of the problem

$$\max_{w\in \underline{W}} \sum_{s=1}^{S} p_s u(x_s^T w) \tag{1}$$

for any $u \in \underline{U}$. This optimization is the basis of our definition of efficiency.

Definition 1. Portfolio w is SSD-efficient if it solves (1) for some $u \in \underline{U}$.

We will denote by $\underline{W}_{SSD}$ the set of SSD-efficient portfolios. Our starting point will be the approach followed by Post (2003), which crucially relies on a clever insight whose extension to our general framework reads as follows.

Lemma 2. *Portfolio w is in $\underline{W}_{SSD}$ if and only if $w \in \underline{W}$ and there exist vectors $m=(m_1,\ldots,m_S)^T \in \Re_+^S$ and $\mu=(\mu_1,\ldots,\mu_M)^T \in \Re_+^M$ such that*

$$\sum_{s=1}^{S} p_s m_s x_{is} = \sum_{j=1}^{M} a_{ij}\mu_j, \quad i=1,\ldots,N, \tag{2}$$

$$\sum_{s=1}^{S} p_s m_s x_0 = 1, \tag{3}$$

$$\left(\sum_{i=0}^{N} a_{ij} w_i - b_j\right)\mu_j = 0, \quad j=1,\ldots,M, \tag{4}$$

and

$$(m_s - m_r)(x_s^T w - x_r^T w) \ge 0, \quad r,\, s=1,\ldots,S. \tag{5}$$

Proof. It follows from a straightforward application of the generalization of the Karush–Kuhn–Tucker (KKT) theorem for subdifferentiable functions and the arguments in the proof of Theorem 2 in Post (2003). □

Equations (2)–(4) result from the KKT conditions, and, in particular, equality (3) is obtained by normalizing the constraint associated with the risk-free asset so that its right-hand side expression, which initially involves the constraint multipliers, is set equal to one. This normalization is useful since in its absence, for any constant $a > 0$ and any SSD-efficient portfolio w, m, and μ satisfy (2) and (4) if and only if $a \times m$ and $a \times \mu$ satisfy those conditions too.[4] Its presence

[1] The existence of a riskless asset is by no means a necessary assumption, but it is included in our analysis to make it more general. The results in its absence can be adjusted in a straightforward manner as we will see later.

[2] Post (2003), Kuosmanen (2004), Post and van Vliet (2006), among others, use the less general case of the simplex as the admissible set of portfolios. Although such specification is harmless for efficiency tests of diversified long-only portfolios, an analysis of the efficient set is highly dependent on the correct specification of the admissible portfolios.

[3] Sometimes the definition of stochastic dominance efficiency requires the admissible utility functions to be strictly increasing (see, e.g., Post 2003). Our approach can also be adjusted without much trouble to fit such criteria.

[4] Note also that without this normalization, the conditions of the theorem hold for any w in $\underline{W}$ by simply choosing vectors m and μ containing only zeros.

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is inconsequential for the efficiency classification, but not for degree measures of inefficiency. Various alternative standardizations are discussed in Post (2003), Post and van Vliet (2006), and Kopa and Post (2014).

Each element of the vector m represents the slope of one of the segments of a piecewise utility function that supports a vector w in $\underline{W}_{SSD}$. As is shown in Post (2003), a complete such function can be constructed from these values. Since this utility function has to be an element of $\underline{U}$, (5) guarantees its concavity, whereas its nondecreasing character results from the nonnegativity of the elements of m. The conditions in (5) are also known as the antimonotonicity conditions, and they require the ranking of returns established by any feasible portfolio to be associated with an exactly reversed ranking in m.

Post (2003) uses a similar result to introduce a linear program that checks the efficiency of a *given* portfolio. In his framework, w is thus fixed, and this implies that all the conditions above are clearly linear in m. He also reduces the number of constraints associated with the antimonotonicity conditions by reordering the states at the data preprocessing stage. Things are quite different when w is also variable rather than exogenously given because a multiplicative structure is introduced as a result. However, we will show that in this case it is still possible to express (2)–(5) as a set of mixed-integer linear constraints. This is the key result of this paper, whose proof will be divided into two lemmas. The first one will show that the KKT conditions (2)–(4) accept such type of linear structure, whereas the second one will demonstrate that the same applies to the antimonotonicity conditions (5). All these results are then combined in a final theorem to characterize the set of portfolios in $\underline{W}_{SSD}$.

In what follows, we refer to those variables that can only take values 0 and 1 as binary.

Lemma 3. *Let w and μ be elements of $\underline{W}$ and $\Re^M_+$, respectively. Condition (4) holds if and only if there exists a vector of binary variables $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_M)^T$ such that*

$$b_j - \sum_{i=0}^{N} a_{ij} w_i \le (1-\varepsilon_j)B, \quad j = 1, \ldots, M, \tag{6}$$

$$\mu_j \le \varepsilon_j E, \quad j = 1, \ldots, M, \tag{7}$$

where B and E are large positive constants.

Proof. The proof of this result is straightforward by noting that since each ε_j is binary, whenever $b_j - \sum_{i=1}^{N} a_{ij} w_i$ is strictly positive, we must have $\varepsilon_j = 0$, something that in turn forces μ_j to be equal to zero. The large constants B and E guarantee that $b_j - \sum_{i=1}^{N} a_{ij} w_i$ and μ_j can attain any relevant value. These constants always exist given that $\underline{W}$ is bounded and that μ_j must be finite. □

Thus, (2)–(4) can be expressed in the way we anticipated. We turn now our attention to the antimonotonicity conditions. Given the variable nature of the portfolio weights, this is clearly the most difficult task. Kuosmanen (2004), Scaillet and Topaloglou (2010), and Kopa and Post (2014) faced similar challenges in different contexts, and they also produced endogenous ordering of the outcomes ranking by introducing additional model variables. Our arguments will again be summarized in a new lemma.

Consider a series $Q \equiv S(S+1)$ of binary variables, y_{rs}, with $r \ne s$, which take value one if state r is followed immediately by state s in such ranking, and zero otherwise. Since the ranking needs to go through all states once and only once, any admissible ranking vector y must satisfy

$$\sum_{s=0,\, s\ne r}^{S} y_{rs} = 1, \quad \forall\, r = 0, \ldots, S, \tag{8}$$

$$\sum_{r=0,\, r\ne s}^{S} y_{rs} = 1, \quad \forall\, s = 0, \ldots, S. \tag{9}$$

These conditions resemble the definition of a permutation matrix given by Kuosmanen (2004), although there are some important differences. Whereas a unitary element in a permutation matrix selects a state's position in a particular ranking, a value of one in our case captures a sequence of two consecutive states in such ranking. In addition, the reader should note that our conditions introduce a dummy state 0. Its role is to determine the first and last state in a given ranking (state i will be the first one whenever y_{0i} takes value one and it will be the one associated with the smallest portfolio's return if such value is taken by y_{i0}). Since our goal is to let the vector of weights of the portfolio be a variable, our formulation is more convenient.

However, the above set of conditions is not enough, since we do not want the ranking to split into *subrankings*. In other words, if we have four states 1, 2, 3, and 4, plus the dummy state 0, the above constraints do not rule out a ranking that, for example, starts at 3 and goes to 4, and back to 3, and then "jumps" to 0, and visits 1 and 2 afterward to return to 0 at the end; that is, a ranking vector with components $y_{01} = y_{12} = y_{20} = y_{34} = y_{43} = 1$ and 0 otherwise satisfies the constraints above, but it does not have associated an admissible ranking of the states for our purposes.

A well-known way to add a *no-split* condition goes as follows. Introduce S nonnegative real variables $v_1, \ldots, v_S$. Any admissible ranking vector must also satisfy the $(S-1) \times S$ constraints

$$v_r - v_s + (S+1) y_{rs} \le S, \quad r, s = 1, \ldots, S,\ r \ne s. \tag{10}$$

To understand these constraints, note first that we impose that the ranking must start at the dummy state 0 (that is why we do not define a variable v_0

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above). The above inequalities rule out any subranking of any length less than $S+1$. For example, with $S=4$, the values $y_{23}=y_{34}=y_{42}=1$ are not possible because we would have

$$v_2 - v_3 + 5 \le 4,$$
$$v_3 - v_4 + 5 \le 4,$$
$$v_4 - v_2 + 5 \le 4,$$

which is a set of constraints that if added up gives the wrong inequality:

$$15 \le 12.$$

Thus, (8)–(10) give the set of all possible rankings of the states. The job will be completed by guaranteeing that the returns of the portfolio and their associated vector of slope coefficients are increasing and decreasing, respectively, for one such ranking of states. This is accomplished by imposing

$$x_r^T w - x_s^T w \ge (y_{rs}-1)C, \quad r,s=1,\dots,S,\ r\neq s, \tag{11}$$

$$m_r - m_s \le (1-y_{rs})D, \quad r,s=1,\dots,S,\ r\neq s, \tag{12}$$

where C and D are large enough positive constants. Hence, whenever $y_{rs}=1$, the conditions above imply that for the two *subsequent* states r and s, we must have

$$x_r^T w - x_s^T w \ge 0,$$
$$m_r - m_s \le 0.$$

In addition, whenever $y_{rs}=0$, we obtain

$$x_r^T w - x_s^T w \ge -C,$$
$$m_r - m_s \le D,$$

conditions that are never binding provided that C and D are large enough.

With all the above arguments, we have proved the following result.

LEMMA 4. *Let w and m be elements of $\underline{W}$ and $\Re_+^S$, respectively. Condition (5) holds if and only if there exists a vector $v=(v_1,\dots,v_S)^T$ in $\Re_+^S$ and a series of Q binary variables, y_{rs}, with $r\neq s$ such that*

$$\sum_{s=0,\, s\neq r}^{S} y_{rs} = 1, \quad r=0,\dots,S, \tag{13}$$

$$\sum_{r=0,\, r\neq s}^{S} y_{rs} = 1, \quad \forall s=0,\dots,S, \tag{14}$$

$$v_r - v_s + (S+1)y_{rs} \le S, \quad r,s=1,\dots,S,\ r\neq s, \tag{15}$$

$$x_r^T w - x_s^T w \ge (y_{rs}-1)C, \quad r,s=1,\dots,S,\ r\neq s, \tag{16}$$

$$m_r - m_s \le (1-y_{rs})D, \quad r,s=1,\dots,S,\ r\neq s, \tag{17}$$

where C and D are large positive constants.

Finally, our three lemmas above easily lead to our main result below, which therefore requires no proof.

THEOREM 5. *w is in $\underline{W}_{SSD}$ if and only if $w\in\underline{W}$ and there exist two vectors of variables $m=(m_1,\dots,m_S)^T$ and $v=(v_1,\dots,v_S)^T$ in $\Re_+^S$, a vector of variables μ in $\Re_+^M$, a vector $\varepsilon=(\varepsilon_1,\dots,\varepsilon_M)^T$ of binary variables, and a series of Q binary variables, y_{rs}, with $r\neq s$ such that (2), (3), (6), (7), and (13)–(17) hold for some large fixed positive constants B, C, D, and E.*

The case with no risk-free asset is easily handled by setting $x_0=1$ and $w_0=a_{ij}=0$ for all $j=1,\dots,M$.

3. Computational Issues

3.1. Handling Ties

Two types of ties will be taken into consideration in this subsection. First, there is the possibility of the existence of at least two states for which the return of any given w in $\underline{W}_{SSD}$ is equal, that is, the existence of r and s, $r\neq s$, for which $x_r^T w = x_s^T w$. Second, there may be cases in which the entire return vector is equal for at least two states, that is, cases in which $x_r = x_s$ for some r and s, $r\neq s$. As it turns out, a reduction in the number of variables and constraints can be accomplished in the *absence* of the former and in the *presence* of the latter.

Let us begin with the first type of tie. As we pointed out in the general case, the introduction of the conditions (8) and (9) is not enough to fix a particular ranking because of the possibility of subsplits. In addition, we need to guarantee that the returns of the portfolio and their associated vector of slope coefficients are increasing and decreasing, respectively, for a given valid ranking of states. Remarkably enough, for those elements of $\underline{W}_{SSD}$ whose payoffs present no ties, both requirements can be handled in one shot by imposing

$$x_r^T w - x_s^T w \ge (y_{rs}-1)C + \epsilon, \quad r,s=1,\dots,S,\ r\neq s, \tag{18}$$

$$m_r - m_s \le (1-y_{rs})D, \quad r,s=1,\dots,S,\ r\neq s, \tag{19}$$

where ϵ is any strictly positive constant satisfying

$$\epsilon \le \min_{r,s}\{x_r^T w - x_s^T w\colon\ x_r^T w - x_s^T w > 0\},$$

a value that always exists in the absence of ties in the payoffs of w. If we leave aside ϵ, the arguments in explaining the role of these constraints is identical to what we presented above. The value ϵ eliminates subsplits in the ranking of states avoiding the need to resort to the conditions in (10). To see this, note that, for example, with $S>4$, the values $y_{23}=y_{34}=y_{42}=1$ are not possible because they lead to the inequalities

$$x_2^T w - x_3^T w \ge \epsilon,$$

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$$x_3^T w - x_4^T w \geq \epsilon,$$
$$x_4^T w - x_2^T w \geq \epsilon,$$

which, if added, yield the wrong inequality:

$$0 \geq 3\epsilon.$$

The second type of tie is a common feature when resorting to pseudosamples. The researcher in that case has at her disposal an initial number S of empirical observations (states) of the vector of returns. Our characterization in Theorem 5 can be used to establish the (in-sample) set of SSD-efficient portfolios by using $p_s = 1/S$ for all $s = 1, \ldots, S$. Pseudosamples can then be obtained by randomly drawing with replacement a number S of vectors of returns. Since some of these drawings will pick the same vector, let $K \leq S$ be the number of *unique* instances of these drawings and let them be indexed as $k = 1, \ldots, K$. Also, denote by l_k the number of repetitions in the pseudosample of each one of these unique vectors of returns. Our characterization of the (in-pseudosample) SSD-efficient set gives a reduced number of variables and constraints. The number of states is now K, and the probability associated with each one of them is set to l_k/S for all $k = 1, \ldots, K$.

3.2. Setting the Value of the Constants B, C, D, and E

One way to proceed is to simply assign a very large number to these constants. However, in many applications it may be more efficient in terms of computational time to tighten the value of these constants by assigning a specific value to each individual constraint. Indeed, if we begin, for example, with C, it is not difficult to see that for any two states r and s, with $r \neq s$, we can compute

$$C_{rs} \equiv \max_{w \in \underline{W}} x_s^T w - x_r^T w,$$

and hence we can rewrite the constraints (16) as

$$x_r^T w - x_s^T w \geq (y_{rs} - 1) C_{rs}, \quad r, s = 1, \ldots, S, \ r \neq s.$$

This approach has an additional advantage. Note that whenever we have $C_{rs} \leq 0$, it is clear that $x_r^T w - x_s^T w \geq 0$, which implies that y_{sr} must be zero. Thus, we can remove this variable from the problem and all its associated constraints.

Similarly, for every $j = 1, \ldots, M$ we can compute

$$B_j \equiv \max_{w \in \underline{W}} b_j - \sum_{i=0}^{N} a_{ji} w_i$$

and rewrite the constraints in (6) as

$$b_j - \sum_{i=0}^{N} a_{ji} w_i \leq (1 - \varepsilon_j) B_j, \quad j = 1, \ldots, M.$$

Constraint-specific values of the constant E can also be found. In this case, for every $j = 1, \ldots, M$ we can compute[5]

$$E_j \equiv \max_{m,\mu} \mu_j$$
$$\text{s.t.} \begin{cases} \displaystyle\sum_{s=1}^{S} p_s m_s x_{is} = \sum_{j=1}^{M} a_{ij} \mu_j, & i = 1, \ldots, N, \\ \displaystyle\sum_{s=1}^{S} p_s m_s x_0 = 1, \end{cases}$$

where m and μ must be further constrained to be in $\Re_+^S$ and $\Re_+^M$, respectively. We can then rewrite the inequalities (7) as

$$\mu_j \leq \varepsilon_j E_j, \quad j = 1, \ldots, M.$$

Finally, we can also follow a similar approach to obtain a specific value of D for each individual constraint in (17). This can be attained by first finding

$$D_{rs} \equiv \max_{m,\mu,w,\varepsilon} m_r - m_s$$
$$\text{s.t.} \begin{cases} \displaystyle\sum_{s=1}^{S} p_s m_s x_{is} = \sum_{j=1}^{M} a_{ij} \mu_j, & i = 1, \ldots, N, \\ \displaystyle\sum_{s=1}^{S} p_s m_s x_0 = 1, \\ \displaystyle b_j - \sum_{i=0}^{N} a_{ji} w_i \leq (1 - \varepsilon_j) B_j, & j = 1, \ldots, M, \\ \mu_j \leq \varepsilon_j E_j, & j = 1, \ldots, M, \end{cases}$$

where m, μ, and w must be further constrained to be in $\Re_+^S$, $\Re_+^M$, and $\underline{W}$, respectively, and ε is an M-dimensional vector of binary variables. The set of inequalities in (12) can be rewritten as

$$m_r - m_s \leq (1 - y_{rs}) D_{rs}, \quad r, s = 1, \ldots, S, \ r \neq s.$$

In addition, if D_{rs} happens to be less or equal than zero, this implies that y_{sr} must be zero, and we can again discard this variable and all its associated constraints.

4. Conclusions

Our main result opens the door to the formulation of optimizations whose objective function is free to select a particular portfolio out of the entire *SSD-efficient frontier*. This approach can deliver a whole range of applications within portfolio theory and asset pricing. For example, a direct application of our result

[5] The solution to this program is bounded as long as $a_{ij} \neq 0$ for at least one i.

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can clearly be exploited to develop even further the work on *SSD spanning* pioneered by Post (2001, 2002) along the lines suggested in Longarela (2014). Within this very same field of research, the mathematical programming section of Arvanitis et al. (2015) can also benefit from our contribution. This may lead to a faster pace of expansion of stochastic dominance methods in finance. The nonparametric character of these methods and their weak economic assumptions are appealing features that justify this ambition. Obviously, there is still a long way to go, since the applicability of these methods is still insufficient compared with other alternative approaches. In particular, within the field of portfolio theory, the gap is still large compared with the current applicability of mean-variance analysis.

Efforts should also be placed in developing additional conditions that rule out unreasonable preferences within those that satisfy monotonicity and concavity. A tempting way to go is to try to extend our main result to higher orders of stochastic dominance by using the arguments presented in Post and Kopa (2013). Unfortunately, such an approach is likely to leave us empty handed, since the linearity obtained in their results entirely relies on the fact that the portfolio weights are fixed.[6] An alternative way to Proceed, which may be more successful, is presented in Post et al. (2015).

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[6] For example, the system of inequalities presented in Corollary 2 of Post and Kopa (2013, p. 324), contains constraint (14.1), where the portfolio returns are raised to powers higher than 1 for any stochastic dominance efficiency of order higher than 2. This does not represent an obstacle when the portfolio weights are given since the portfolio returns are automatically given too. However, it becomes a problem when those weights are model variables. Linearity in this latter case does not hold anymore.

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