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Pricing and hedging American and hybrid strangles with finite maturity ☆

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ABSTRACT

This paper introduces variants of strangles, called Euro-American or hybrid strangles, and it promotes a new numerical pricing technique. We highlight and compare the properties of European, American, and hybrid strangles with pricing and hedging in mind. The new quadrature approach we propose can account for systems of coupled integral equations that locate the early exercise boundaries of finite-lived contracts. We show that this method is efficient, accurate, and fast for pricing all types of early exercisable strangles. Other advantages of this technique are that it avoids the non-monotonic gradient problem faced by others and it allows users to control for errors. We then investigate the hedging of all strangles, we derive analytical expressions for some Greek parameters, and we stress how these parameters can differ (or not) from each other.

1. Introduction

According to Chaput and Ederington (2005), strangles and straddles represent about 80% of option strategies. These strategies are used for risk management, volatility trading, and volatility speculation (see §11.4 of Hull (2012)). The classical strangle is a European-style strangle and it is comprises a long position in a European put option and a long position in a European call option, with both options being written on the same underlying asset and maturing at the same time. The call option strike is typically greater than the put option one, but in the case where they are equal, the position is termed a straddle. Strangle positions have recently been studied in terms of the American style, meaning that the holder can decide to exercise earlier than maturity (see Gerber and Shiu (1994) and Chiarella and Ziogas (2005), among others). An American strangle may be roughly viewed as a long position in an American put and an American call option equipped with a non-standard early exercise boundary (EEB) and a *self-closing* mechanism. This latter mechanism ensures that the right to sell (or to buy) disappears as soon as the holder decides to exercise the right to buy (or to sell).[1] As such, American strangles are not simple portfolios of standard American puts and calls.[2]

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[1] The self-closing mechanism has a number of interesting features for managers. Unlike sellers of portfolios made up of individual American options, sellers of American strangles avoid the risk of successive exercises because American strangles disappear once the early exercise is decided upon. For buyers, these contracts are cheaper than their rough synthetic portfolios.

[2] Studying contracts with no maturity, Moraux (2009) found that holders of individual American options can exercise earlier than holders of American strangles, can receive a greater value at the exercise time, and if they manage their position through the synthetic portfolio characteristics may exercise their right to sell or to buy rather suboptimally.

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In this study, we extend the family of strangles by introducing a new variant called Euro-American or hybrid strangles. These contracts can be exercised earlier than maturity given that only one side, the call side or the put side, is exercisable before expiration. Due to this possible early extinction, Euro-American contracts greatly differ from simple portfolios made up of independent American and European options. Euro-American strangles have several clear goals. From a practical viewpoint, these contracts can fit investors' and speculators' needs better than other strangles. Hybrids of American and European strangles can indeed offer the best of each of these variants. From a theoretical perspective, they can be used to understand and model real financial decisions. With real options in mind, strangle positions can model a firm decision to expand (exercise the call side) or to transfer (exercise the put side). In many cases, not all of these decisions can occur earlier than maturity, and if one side is delayed to maturity we have a hybrid strangle contract. From an option theory point of view, they contribute to an understanding of how American strangles work and how the opportunity for early exercise makes them different from standard European strangles. Hybrid contracts involve only a "one side" early exercise so we can assess the relative importance of it. Because a Euro-American strangle lies contractually between an American strangle and a European strangle, we can expect its price to be bounded by these two strangles. However, it is not *a priori* straightforward to predict how much cheaper or more expensive Euro-American strangles are compared to other strangles.

Pricing American-style contracts is known to be a challenge because the holder can exercise at any time before maturity. Two interconnected questions must be answered simultaneously: what is the best time to exercise and what is the resulting payoff? Mathematically, this free boundary problem can be addressed by solving a partial differential equation subject to some (boundary) conditions (see McKean (1965) for an early treatment).[3] Pricing American strangles can appear even more challenging because they depend on two interdependent, self-closing, and time-varying early exercise boundaries. For their part, Euro-American strangles depend on a single time-varying early exercise boundary only, but this is subtly influenced by the existing one-sided European feature of the contract, and furthermore, the opportunity to exercise at maturity disappears if an early exercise is decided beforehand. Such contracts have really special features to consider.

Past studies on the pricing of American strangles with finite maturity advocated the use of various advanced numerical techniques. Alobaidi and Mallier (2002) used Laplace transforms and derived analytical formulas for pricing American *straddles* as well as integral equations to locate early exercise boundaries. Unfortunately, the expressions they provide cannot be inverted analytically and they give no recommendation for numerical inversion. Chiarella and Ziogas (2005) (CZ) used Fourier transforms and derived analytical formulas for finite-lived American strangles as well as a system of coupled integral equations to locate early exercise boundaries. Finally, they employed a two-step algorithm for pricing, mixing a quadrature approach and an interpolation technique. This two-step approach seemed necessary to deal with the non-monotonic gradient problem they faced.[4,5] To avoid the above advanced methods, one may be tempted to consider simpler portfolios made up of two individual American call and put options. In some cases the pricing bias is indeed rather limited,[6] but in general, this way to proceed is hazardous because it can lead to suboptimal early exercise decisions (see Moraux (2009) for examples in the perpetual case).

In this study, we derive analytical formulas for all finite-lived strangles (among which are the new variant contracts) and promote a new numerical approach able to deal with the various systems of integral equations to locate early exercise boundaries. The quadrature method we introduce combines some Newton–Cotes weights and some fourth-order Gregory weights, as presented in Linz (1985, p.98) and Press et al. (2007, pp.159–160). This numerical approach in turn has several interesting features. First, it is a rather simple (one-step) numerical approach. Second, it is accurate, efficient, and fast. Third, the approach allows users to control for errors, and we show that numerical estimates tend to the true price from above as the number of discrete points increases.[7] Finally, the quadrature scheme faces no non-monotonic gradient problem and consequently requires neither repeated computations of early exercise boundaries nor extrapolation techniques.

We expect the prices of Euro-American strangles to lie between those of European and American comparable strangles. Indeed, our simulations reveal that Euro-American strangles can effectively be more expensive than comparable European strangles and cheaper than comparable American strangles, but we also provide scenarios where Euro-American strangles are as expensive as their European or American counterparts. Hence, the opportunities to buy or to sell the underlying asset earlier than the expiration date may be, in some contexts, effectively worthless. This information should be useful for potential users of existing early exercisable strangles because they should not pay for such opportunities.

Beyond introducing new variants of early exercisable strangles, our (financial) analysis differs from that of CZ. We split the price of strangles following the early exercise premium (EEP) representation advocated by Kim (1990).[8] Hence, in the present study, the American strangle is essentially the sum of a European strangle plus a premium to exercise it earlier than maturity. By contrast, CZ price the American strangle as a whole and then represent the contract as a portfolio made up of an American call and an American put with adjusted early exercise boundaries[9] (see their Proposition 7). Logically, most of their simulations compare early exercise boundaries of strangles to those of standard American options. Instead, we think it is important to consider European strangles as relevant benchmarks because we know their price analytically. We can then

[3] Various approaches have been developed for pricing American-style options. MacMillan (1986) provided an analytic approximation for valuing the early exercise premium, which is the price difference between an American option and its European equivalent. Kim (1990) later offered an intuitive representation of the early exercise premium (see also Jacka (1991) and Carr et al. (1992)). However, this representation requires the knowledge of the early exercise boundary at any point in time and consequently relies on a computational technique to solve the integral equation locating the boundary. Another way to proceed is to use numerical methods for solving the free boundary problem. These methods are now well known in finance. Finite difference-based methods are the most common (see, for instance, §28 of Wilmott (2007)) and Monte Carlo simulations are also used (see Broadie et al. (1997) and Longstaff and Schwartz (2001)).

[4] CZ wrote, "*the numerical scheme is firstly carried out using a time-step size ofhand is then repeated using $h/2$. In each case, since it is necessary to alternate between two different numerical integration schemes (for odd and even values), it turns out that the free boundaries have non-monotonic gradients. This is rectified by combining the two estimates using Richardson's extrapolation. Pricing the American strangle is then achieved via numerical integration using Simpson's rule, combined with the estimates*".

[5] Perpetual (American) strangles have also been studied in the literature. Gerber and Shiu (1994) provided the first pricing formula, but they did not solve the system of equations locating early exercise boundaries, nor did they provide any simulations. Moraux (2009) analysed these contracts as asymmetric rebates of double knock-out barrier options with special payoffs and found early exercise boundaries numerically. He then provided simulations and insights on price properties and on the optimal exercise policy as well as discussing hedging issues.

[6] E.g., CZ report a moderate 6% for the largest bias in their simulations.

[7] By way of comparison, the existing two-step quadrature approach over/underestimates the true price unexpectedly.

[8] see MacMillan (1986) and Barone-Adesi and Whaley (1987) for other approaches using the early exercise premium.

[9] Standard early exercise boundaries are not suitable anymore due to the self-closing mechanism.

explore the hedging parameters of every strangle and highlight how they differ from each other.

The rest of the paper is organized as follows. Section 2 presents the framework and problem statement. Section 3 presents the early exercise premium representation of American-style strangles. Section 4 presents and discusses the new numerical approach. Section 5 investigates the prices of American-style strangles with finite maturity. Section 6 considers the hedging issues and Greek parameters. The last section (Section 7) presents our conclusions.

2. The framework and problem statement

This section introduces contract specifications, notations, and hypotheses. The financial markets in our setting are perfect, efficient, and complete; trading takes place continuously and information is free. There are neither taxes nor transaction costs. There is a risk-free asset paying a known and constant interest rate denoted by r. There is also a risky asset, say a stock, paying a continuous dividend rate δ that underlies the different contracts. When the stock pays dividends, we know from standard option theory that it may be optimal to exercise standard American call options earlier than at expiration. The risk neutral price process of the underlying asset is denoted by $S = (S_t)_{t \geqslant 0}$ and it is described by

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \quad S_0 = x$$

where $W = (W_t)_{t \geqslant 0}$ is a standard Brownian motion, σ is the volatility, and $x \geqslant 0$. It follows that $S_t = x \exp\left[(r - \delta - \sigma^2/2)t + \sigma W_t\right]$. Throughout the paper, the normal cumulative density function is denoted by N and the normal probability density function is denoted by ϕ.

A strangle (K_1, K_2, T) is a contract expiring at time T that gives the holder the right to sell the underlying asset S at price K_1 and the right to buy at price K_2 (we omit S in previous parentheses because there is no ambiguity). Hereafter, we use the terms "call side" and "upper side" interchangeably for describing the right to buy embedded in the strangle. Similarly, the terms "put side" and "lower side" are used interchangeably for describing the right to sell. K_1 may therefore be termed the "put side" strike of the strangle and K_2 the "call side" strike of the strangle. If ever $K_1 = K_2$, the contract is a straddle. It should be noted that in this paper we do not discuss "American guts," for which $K_1 > K_2$. European strangles allow the holder(s) to sell the underlying asset at K_1 or to buy it at K_2 at the expiration of the contract only. American strangles allow the holder(s) to act at or before the expiration; that is, at any time between inception and termination.[10] The Euro-American or hybrid strangles that we introduce in this paper allow the holder(s) to sell or buy at the expiration of the contract and provide an opportunity to act earlier than at expiration on one side only. Such contracts can, for instance, allow the holder(s) to buy at any time before T and to buy or sell at time T, or the converse (i.e., to sell at any time before T and to buy or sell at time T), but not both.

It is well known (see for instance Merton (1973)) that the price $V(S, t)$ of every contract written on S satisfies the following fundamental partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + (r - \delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad t \in [0, T). \tag{1}$$

Of course, the contract considered here is assumed to be *alive*. For early exercisable contracts, this means that the time t-price of the underlying asset lies in the continuation region. Fig. 1 gives a typical representation of the price of a classical American strangle

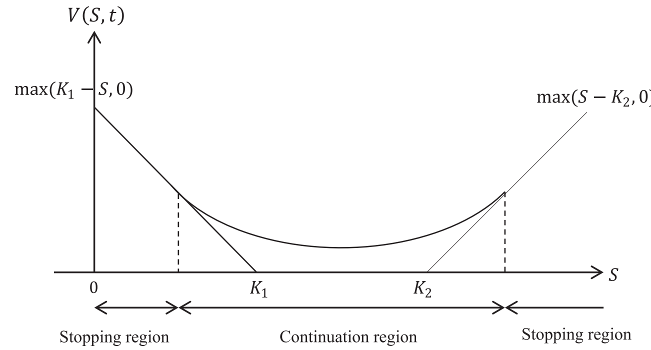

Fig. 1. American strangles continuation & stopping regions.

that highlights the intrinsic value (which is $\max(K_1 - S; 0) + \max(S - K_2; 0)$) and corresponding continuation and stopping regions. These regions are characterized by the early exercise boundaries that relate to the threshold values of the underlying asset where the decision to exercise early should intervene (because it is optimal for holders to do so). In Fig. 1, we emphasize these threshold values with dotted lines. For Euro-American strangles with an early exercisable call side feature (or, respectively, an early exercisable put side feature), the stopping region on the left (on the right) does not exist.

Let us now denote by $\tau = T - t$ the time to expiration, by $A_{c_1c_2}(S, \tau)$ the price of the American strangle, by $c_1(\tau)$ the lower early exercise boundary associated with the right to sell, and by $c_2(\tau)$ the upper early exercise boundary associated with the right to buy. Due to the exercise policy, finite-lived American-style contract boundaries are time dependent (see Merton (1973, pp.170–171) and depend on the remaining time to expiration. As long as the underlying stock price lies between these two thresholds, it is not optimal to exercise the American strangle (cf. the continuation region in Fig. 1). Consequently, the price of the American strangle satisfies the fundamental Eq. (1) with a final condition

$$\begin{aligned} A_{c_1c_2}(S_T, 0) = \max\left[0, (K_1 - S_T)\psi(S_T - c_1(0))\right] \\ + \max\left[0, (S_T - K_2)\psi(c_2(0) - S_T)\right], \quad S_T \geqslant 0, \end{aligned} \tag{2a}$$

and some specific boundary conditions

$$\lim_{S \downarrow c_1(\tau)} A_{c_1c_2}(S, \tau) = K_1 - c_1(\tau), \quad 0 \leqslant \tau \leqslant T, \tag{2b}$$

$$\lim_{S \uparrow c_2(\tau)} A_{c_1c_2}(S, \tau) = c_2(\tau) - K_2, \quad 0 \leqslant \tau \leqslant T, \tag{2c}$$

$$\lim_{S \downarrow c_1(\tau)} \frac{\partial A_{c_1c_2}(S, \tau)}{\partial S} = -1, \quad \lim_{S \uparrow c_2(\tau)} \frac{\partial A_{c_1c_2}(S, \tau)}{\partial S} = 1, \quad 0 \leqslant \tau \leqslant T. \tag{2d}$$

where $\psi(x)$ is the Heaviside step function defined by $\psi(x) = 0$ if $x \leq 0$ and $\psi(x) = 1$ if $x > 0$.

Eq. (2a) depicts the value of American strangles at expiration in the case where no early exercise occurs. At expiration, a rational investor exercises the in-the-money side of the American strangle. If ever the price of the underlying asset first reaches the early exercise boundary $c_1(\tau)$ at a given time τ before expiration, then the holder of the American strangle immediately exercises their right to sell the stock and the received payoff is Eq. (2b). If ever the underlying asset first reaches the early exercise boundary $c_2(\tau)$ at a given time τ before expiration, then the holder of the American strangle immediately exercises their right to buy the stock and the received payoff is as shown by Eq. (2c). Eqs. (2b) and (2c) represent value-matching conditions and Eqs. (2d) represent smooth-pasting conditions. They can be deduced from

[10] We know that holding an American-style strangle is not equivalent to being long in a portfolio made up of an American call option (K_2, T) and an American put option (K_1, T). Both rights (to sell/to buy) disappear when the owner decides to exercise one.

Merton (1973, pp.170–171) or Chiarella and Ziogas (2005, p.35). Notice that including step functions in the terminal payoff above is important to limit cash flows at maturity to $K_1 - c_1(0)$ or $c_2(0) - K_2$, where $c_1(0)$ and $c_2(0)$ stand for the limit of early exercise boundaries as τ tends to zero. It is impossible for the underlying asset price to end beyond the boundaries without crossing them beforehand and hence provoking an early exercise.

Euro-American strangles are new variants introduced in this study. Let us first consider Euro-American strangles with an early exercisable call side. These contracts offer the right to sell or buy the underlying asset at maturity and the right to buy it earlier. Similar to standard American call options, these contracts imply, in the presence of dividends, an upper early exercise boundary (placed on the price of the underlying asset) whose value at time t can be denoted by $h(\tau)$ with $\tau = T - t$. Holders of these contracts should exercise early if and when the price of the underlying asset reaches the boundary h. As long as the underlying stock price lies below the threshold h, it is not optimal for holders to exercise the right (to buy) earlier than at expiration. Consequently, the price of Euro-American strangles with an early exercisable call side, denoted by $EA_h(S,\tau)$, satisfies the fundamental Eq. (1) with the following final time and boundary conditions:

$$EA_h(S_T,0) = \max[0, K_1 - S_T] + \max[0, (S_T - K_2)\psi(h(0) - S_T)], \quad S_T \geqslant 0, \tag{3a}$$

$$\lim_{S \uparrow c_2(\tau)} EA_h(S,\tau) = h(\tau) - K_2, \quad 0 \leqslant \tau \leqslant T, \tag{3b}$$

$$\lim_{S \downarrow 0} EA_h(S,\tau) = 0, \quad 0 \leqslant \tau \leqslant T, \tag{3c}$$

$$\lim_{S \uparrow c_2(\tau)} \frac{\partial EA_h(S,\tau)}{\partial S} = 1, \quad 0 \leqslant \tau \leqslant T. \tag{3d}$$

Let us now consider Euro-American strangles with an early exercisable put side. These contracts offer the right to sell or buy the underlying asset at maturity and the right to sell it earlier. This hybrid contract implies a lower early exercise boundary (placed on the price of the underlying asset) whose value at time t is denoted by $l(\tau)$ with $\tau = T - t$. Holders of these contracts should exercise early if and when the price of the underlying asset reaches the boundary l. As long as the underlying stock price lies above the threshold l it is not optimal for holders to exercise the right (to sell) earlier than expiration. In view of this, the price of Euro-American strangles with an early exercisable put side, denoted by $EA_l(S,\tau)$, satisfies the fundamental Eq. (1) with the following final time and boundary conditions:

$$EA_l(S_T,0) = \max[0, (K_1 - S_T)\psi(S_T - l(0))] + \max[0, (S_T - K_2)], \quad S_T \geqslant 0, \tag{4a}$$

$$\lim_{S \downarrow l(\tau)} EA_l(S,\tau) = K_1 - l(\tau), \quad 0 \leqslant \tau \leqslant T, \tag{4b}$$

$$\lim_{S \uparrow \infty} EA_l(S,\tau) = 0, \quad 0 \leqslant \tau \leqslant T, \tag{4c}$$

$$\lim_{S \downarrow c_1(\tau)} \frac{\partial EA_l(S,\tau)}{\partial S} = -1, \quad 0 \leqslant \tau \leqslant T. \tag{4d}$$

3. Early exercise premium representation

Kim (1990), Jacka (1991), and Carr et al. (1992) showed that the price of standard American options can be decomposed into the price of a European option plus an early exercise premium having a special but intuitive integral representation. These authors demonstrated the EEP representation by different means. Hereafter, we provide EEP representations for all American and hybrid strangles. Such a representation strategy appears especially useful for early exercisable strangles for at least two reasons. First, we can price European strangles analytically. Second, European strangles can then play the role of common benchmarks for all contracts.[11]

Proposition 1 splits the price of an American strangle ($A_{c_1c_2}(S,\tau)$) into the price of a European strangle ($E(S,\tau)$) plus an early exercise premium ($EEP_{c_1c_2}(S,\tau)$) with an integral representation.

Proposition 1. *The price of an American strangle is given by:*

$$A_{c_1c_2}(S,\tau) = E(S,\tau) + EEP_{c_1c_2}(S,\tau) \tag{5}$$

where

$$\begin{aligned} E(S,\tau) &= K_1 e^{-r\tau} N(-d_2(S,\tau;K_1)) - Se^{-\delta\tau} N(-d_1(S,\tau;K_1)) \\ &\quad + Se^{-\delta\tau} N(d_1(S,\tau;K_2)) - K_2 e^{-r\tau} N(d_2(S,\tau;K_2)) \end{aligned}$$

and

$$\begin{aligned} EEP_{c_1c_2}(S,\tau) = \int_0^\tau &[K_1 r e^{-r(\tau-\eta)} N(-d_2(S,\tau-\eta;c_1(\eta))) \\ &- S\delta e^{-\delta(\tau-\eta)} N(-d_1(S,\tau-\eta;c_1(\eta))) \\ &+ S\delta e^{-\delta(\tau-\eta)} N(d_1(S,\tau-\eta;c_2(\eta))) \\ &- K_2 r e^{-r(\tau-\eta)} N(d_2(S,\tau-\eta;c_2(\eta)))] d\eta \end{aligned}$$

with early exercise boundaries $c_1(.)$ and $c_2(.)$ defined by

$$c_2(\tau) - K_2 = A_{c_1c_2}(c_2(\tau),\tau) \tag{6}$$

$$K_1 - c_1(\tau) = A_{c_1c_2}(c_1(\tau),\tau) \tag{7}$$

and $d_1(S,\tau;\beta) = \frac{\ln\left(\frac{S}{\beta}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2(S,\tau;\beta) = d_1(S,\tau;\beta) - \sigma\sqrt{\tau}.$

Proof. To prove Proposition 1, we follow Kim (1990) and divide the pricing problem into two parts. We can then solve these two sub-problems independently using the results of Kolodner (1956) and McKean, 1965 directly. □

Hence, the main difference with CZ is that we have recourse to the techniques introduced by Kim (1990) to deal with the standard American options. We split the pricing of strangles into two problems that we solve by using the results of McKean (1965) and Kolodner, 1956. We end up with the European component of the considered strangle and its EEP. By contrast, CZ consider the problem as a whole and apply a Fourier transform approach on the whole. They transform the PDE and apply the incomplete Fourier transform approach used by McKean to solve the free boundary problem. They decide to reorganize, to highlight the sum of an American-style put option and an American-style call option (with non-standard and adjusted EEPs). Of course our Proposition 1 conforms to their results.

The expression for the early exercise premium is semi-analytical only because it depends on a couple of early exercise boundaries to be determined. To this end, Eqs. (6) and (7) form a coupled integral equation system that we need to solve numerically by some techniques (see next section for details).

Proposition 2 details our core result on Euro-American strangles. The lengthy proof (relegated to the Appendix) uses the results of Kolodner (1956) and McKean, 1965 on free boundary problems.

Proposition 2.

(i) *The price of a Euro-American strangle with an American call side is given by:*

[11] This can be compared with CZ, who essentially favor another decomposition; that is, $A_{c_1c_2}(S,\tau) = \mathcal{C}(S,\tau,c_2(.)) + \mathcal{P}(S,\tau,c_1(.))$, where $\mathcal{C}(S,\tau,c_2(.))$, and $\mathcal{P}(S,\tau,c_1(.))$ stand for the prices of standard American calls and puts that use $c_2(.)$ and $c_1(.)$ for the respective early exercise boundaries.

$$EA_h(S,\tau) = E(S,\tau) + EEP_h(S,\tau) \quad (8)$$

where

$$EEP_h(S,\tau) = \int_0^\tau \left[S\delta e^{-\delta(\tau-\eta)} N(d_1(S,\tau-\eta;h(\eta))) \right. \\ \left. -K_2 r e^{-r(\tau-\eta)} N(d_2(S,\tau-\eta;h(\eta)))\right] d\eta$$

the early exercise boundary h is given by

$$h(\tau) - K_2 = EA_h(h(\tau),\tau). \quad (9)$$

(ii) *The Price of a Euro-American strangle with an American put side is given by:*

$$EA_l(S,\tau) = E(S,\tau) + EEP_l(S,\tau) \quad (10)$$

where

$$EEP_l(S,\tau) = \int_0^\tau \left[K_1 r e^{-r(\tau-\eta)} N(-d_2(S,\tau-\eta;l(\eta))) \right. \\ \left. - S\delta e^{-\delta(\tau-\eta)} N(-d_1(S,\tau-\eta;l(\eta)))\right.$$

and the early exercise boundary is given by

$$K_1 - l(\tau) = EA_l(l(\tau),\tau). \quad (11)$$

Proof. See Appendix A. □

The next step is to compute the Volterra integrals involved in Eqs. (5), (8), and (10).

4. Numerical implementation and algorithmic issues

We will use a quadrature approach to evaluate the integrals of Eq. (5) and to transform Eqs. (6) and (7). A quadrature is a way to approximate an integral in general. If f is a real function of, say, one variable defined on $[a,b]$, then the integral of f over $[a,b]$ may be approximated by

$$\int_a^b f(x)dx \approx h\sum_{k=0}^{n} w_k f(x_k)$$

where $(x_k)_{k=1,\dots,n}$ are equally spaced sample points such that $x_0 = a, x_k = x_0 + kh$, and $x_n = b$. The interval $[a,b]$ has been divided here into n subintervals of equal width (with $h = (b-a)/n$). Quadrature methods differ essentially in terms of weights (w_k); for instance, CZ use Cavalieri-Simpson weights (see Table 1).

The Cavalieri-Simpson approach has significant drawbacks when applied to American strangles. It requires an even number of intervals, as highlighted by Table 1, leads to non-monotonic gradients, requires a double computation of each early boundary, and then needs a technique to average both estimates. CZ suggest the use of Richardson's *extrapolation* technique to rectify this and to combine the two boundaries (found for h and $h/2$). Nevertheless, there is no possibility here to control for errors. To address this issue, we follow another way and propose a new quadrature method that is able to compute in the one-step integrals involved in the American and Euro-American strangles. We will see that this alternative way is accurate, efficient, and faster.

The new one-step quadrature method that we propose is as follows. Consider first a number of points n to approximate the EEB along the time line and i an index position taking values in $\{1,2,\dots,n\}$. The points of the EEB(s) are then obtained by computing integral(s) iteratively at i for i equal to 1 up to n. Each i is associated with a point of the EEB. $i = 1$ points to the first position before expiration (where we want to compute the EEB(s)). At each position i, integral equations are computed using i intervals and $(i+1)$ weighted endpoints. For weights, we use the closed-form formulas of Newton–Cotes when i is smaller than or equal to six

Table 1
Simpson's weights used by CZ.

n	w_0	w_1	w_2	w_3	w_4	·	w_{2k}
1	–	–	–	–	–	·	–
2	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	0	0	·	0
3	–	–	–	–	–	·	–
4	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	·	0
·	·	·	·	·	·	·	·
$2k-1$	–	–	–	–	–	·	–
$2k$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	·	$\frac{1}{3}$

With $k \in \{1,2,3,\dots\}$.
For odd values of n, the quadrature does not exist theoretically. Consequently, inserting an arbitrarily additional coefficient (4/3 or 2/3) for odd values of n, as CZ did, may lead to an additional bias in the integral approximation.

Table 2
Hybrid quadrature weights.

n	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	·	w_i
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	·	0
2	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	0	·	0
3	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{3}{8}$	0	0	0	0	·	0
4	$\frac{14}{45}$	$\frac{64}{45}$	$\frac{24}{45}$	$\frac{64}{45}$	$\frac{14}{45}$	0	0	0	·	0
5	$\frac{95}{288}$	$\frac{375}{288}$	$\frac{250}{288}$	$\frac{250}{288}$	$\frac{375}{288}$	$\frac{95}{288}$	0	0	·	0
6	$\frac{41}{140}$	$\frac{216}{140}$	$\frac{27}{140}$	$\frac{272}{140}$	$\frac{27}{140}$	$\frac{216}{140}$	$\frac{41}{140}$	0	·	0
7	$\frac{3}{8}$	$\frac{7}{6}$	$\frac{23}{24}$	1	1	$\frac{23}{24}$	$\frac{7}{6}$	$\frac{3}{8}$	·	0
·	·	·	·	·	·	·	·	·	·	·
i	$\frac{3}{8}$	$\frac{7}{6}$	$\frac{23}{24}$	1	1	1	1	1	·	$\frac{3}{8}$

Table 3
Degree of Exactness (DE) and error associated with Newton–Cotes.

i	DE	$\frac{M_i}{(i+2)!}$	$\frac{K_i}{(i+1)!}$
1	1	–	$\frac{1}{12}$
2	3	$\frac{1}{90}$	–
3	3	–	$\frac{3}{80}$
4	5	$\frac{8}{945}$	–
5	5	–	$\frac{275}{12096}$
6	7	$\frac{9}{1400}$	–

(listed in Table 2). For i greater than six, we use the fourth-order Gregory quadrature formulas[12] (derived by fitting cubic polynomials through successive groups of four points; see Linz (1985) and Press et al., 2007). Unlike Simpson's quadrature, this quadrature method can deal with odd and even values of intervals. The general form of Gregory weights is

$$\int_a^b f(x)dx \approx h\left[\begin{array}{c} \frac{3}{8}f(x_0) + \frac{7}{6}f(x_1) + \frac{23}{24}f(x_2) + f(x_3) + f(x_4)\dots \\ +f(x_{n-4}) + f(x_{n-3}) + \frac{23}{24}f(x_{n-2}) + \frac{7}{6}f(x_{n-1}) + \frac{3}{8}f(x_n) \end{array}\right].$$

In terms of convergence, a key advantage of this quadrature is that we have analytical expressions for computing errors for each value of i. Consequently, we can control for errors. For the Newton–Cotes part, the analytical expressions are as follows. For even values of i and any function $f \in C^{(i+2)}([a,b])$, the quadrature error is

$$E_i(f) = \frac{M_i}{(i+2)!} h^{i+3} f^{(i+2)}(\xi)$$

where $\xi \in [a,b]$ and $M_i = \int_0^i t\left(\prod_{k=0}^{i}(t-k)\right)dt < 0$. For odd values of i and any function $f \in C^{(i+1)}([a,b])$, the quadrature error is

$$E_i(f) = \frac{K_i}{(i+1)!} h^{i+2} f^{(i+1)}(\eta)$$

[12] Note that the Newton–Cotes weights may become negative for i greater than six and this causes numerical instability.

Table 4
Values of American strangles: convergence and speed.

Method	CZ				LAM			
S	n							
	100	200	400	800	100	200	400	800
0.75	0.275647	0.275647	0.275647	0.275647	0.275647	0.275647	0.275647	0.275647
1.00	0.100332	0.100332	0.100332	0.100332	0.100332	0.100332	0.100332	0.100332
1.25	0.038563	0.038562	0.038561	0.038561	0.038562	0.038561	0.038561	0.038561
1.50	0.092344	0.092341	0.092341	0.092340	0.092342	0.092341	0.092340	0.092340
1.75	0.255631	0.255632	0.255633	0.255633	0.255634	0.255634	0.255633	0.255633
Total subint.	300	600	1200	2400	200	400	800	1600
Time (sec)	64	244	1032	3920	51	193	758	3148

"Total Subint." represents the total number of subintervals needed for implementing the approach. "time" is the computational time needed to compute American strangle prices. The parameter values are $K_1 = 1, K_2 = 1.5, r = 5\%, \delta = 10\%, \sigma = 20\%$, and $\tau = 1$. Notice that for $n = 100, (100 + 2 \times 100) = 300$ subintervals have been used by CZ whereas LAM requires 2×100 subintervals only.

where $\eta \in [a, b]$ and $K_i = \int_0^i \prod_{k=0}^i (t-k)dt < 0$. The degree of exactness (DE hereafter) is equal to $i+1$ for even values of i and i for other values. For the Gregory part, the analytical expressions are quite lengthy and we refer to Linz (1985, pp.100–107), where a detailed and complete error analysis is provided.

Table 3 summarizes the numerical values of $M_i/(i+2)!$ and $K_i/(i+1)!$ and the DEs for the Newton–Cotes part.[13] It should be noted that from $i = 2$, the integration order is higher or equal to that of the extended Simpson's quadrature.[14] The DE of the fourth-order Gregory formulas is three. As emphasized earlier, our quadrature approach is suitable for even and odd values of i and only one step is necessary whatever the value of i. By comparison, the usual interpolation techniques implement the same algorithm twice (a first step for odd values of i and a second step for even values) and they then compute a weighted average of both. Consequently, our approach leads to less calculus and so should be less time consuming (this is verified below in Table 4). A pseudo-code for the hybrid quadrature is presented in Appendix C (see Algorithm 1).

Discretizing the time interval is the next step in a numerical quadrature strategy. Following Linz (1985) and Kim (1990), we divide the time interval into n parts of length h and define $\tau_i = ih$ for $i = 1, 2, 3, \ldots, n$ and $h = T/n$.

Before computing the price of an American or Euro-American strangle we must determine early exercise boundaries by iteratively solving integral equations for each value τ_i. The initial values for c_1, c_2, h and l are $c_1(0) = l(0) = \min(K_1, \frac{r}{\delta}K_1)$ and $c_2(0) = h(0) = \max(K_2, \frac{r}{\delta}K_2)$.[15] Then, for each step i, starting from $i = 1$, the only unknown variables in the system of integral equations are $c_1(\tau_i), c_2(\tau_i), h(\tau_i)$, and $l(\tau_i)$. Given that the values of $c_1(\tau_j), c_2(\tau_j), h(\tau_j)$, and $l(\tau_j)$ for $j < i$ are known from previous steps, we have:

– *For American strangles*

$$K_1 - c_1(ih) = E(c_1(ih), ih) + \widehat{EEP}_{c_1c_2}(c_1(ih), ih)$$
$$c_2(ih) - K_2 = E(c_2(ih), ih) + \widehat{EEP}_{c_1c_2}(c_2(ih), ih)$$

where

$$\begin{aligned} E(c_k(ih), ih) = {} & K_1 e^{-rih} N(-d_2(c_k(ih), ih; K_1)) \\ & - c_k(ih) e^{-\delta ih} N(-d_1(c_k(ih), ih; K_1)) \\ & + c_k(ih) e^{-\delta ih} N(d_1(c_k(ih), ih; K_2)) \\ & - K_2 e^{-rih} N(d_2(c_k(ih), ih; K_2)) \end{aligned}$$

and

$$\widehat{EEP}_{c_1c_2}(c_k(ih), ih) = h \sum_{j=0}^{i} w_j \begin{bmatrix} K_1 r e^{-rh(i-j)} N(-d_2(c_k(ih), (i-j)h; c_1(jh))) \\ -c_k(ih)\delta e^{-\delta h(i-j)} N(-d_1(c_k(ih), (i-j)h; c_1(jh))) \\ +c_k(ih)\delta e^{-\delta h(i-j)} N(d_1(c_k(ih), (i-j)h; c_2(jh))) \\ -K_2 r e^{-rh(i-j)} N(d_2(c_k(ih), (i-j)h; c_2(jh))) \end{bmatrix}$$

for $k = 1, 2$, given that the weights $(w_j)_j$ depend on the above hybrid scheme.

– *For Euro-American strangles with an American call side*

$$h(ih) - K_2 = E(h(ih), ih) + \widehat{EEP}_h(h(ih), ih)$$

where

$$\begin{aligned} E(h(ih), ih) = {} & K_1 e^{-rih} N(-d_2(h(ih), ih; K_1)) \\ & - h(ih) e^{-\delta ih} N(-d_1(h(ih), ih; K_1)) \\ & + h(ih) e^{-\delta ih} N(d_1(h(ih), ih; K_2)) \\ & - K_2 e^{-rih} N(d_2(h(ih), ih; K_2)) \end{aligned}$$

and

$$\widehat{EEP}_h(h(ih), ih) = h \sum_{j=0}^{i} w_j \begin{bmatrix} h(ih)\delta e^{-\delta h(i-j)} N(d_1(h(ih), (i-j)h; h(jh))) \\ -K_2 r e^{-rh(i-j)} N(d_2(h(ih), (i-j)h; h(jh))) \end{bmatrix}$$

– *For Euro-American strangles with an American put side*

$$K_1 - l(ih) = E(l(ih), ih) + \widehat{EEP}_l(l(ih), ih)$$

where

$$\begin{aligned} E(l(ih), ih) = {} & K_1 e^{-rih} N(-d_2(l(ih), ih; K_1)) \\ & - l(ih) e^{-\delta ih} N(-d_1(l(ih), ih; K_1)) \\ & + l(ih) e^{-\delta ih} N(d_1(l(ih), ih; K_2)) \\ & - K_2 e^{-rih} N(d_2(l(ih), ih; K_2)) \end{aligned}$$

and

$$\widehat{EEP}_l(l(ih), ih) = h \sum_{j=0}^{i} w_j \begin{bmatrix} K_1 r e^{-rh(i-j)} N(-d_2(l(ih), (i-j)h; l(jh))) \\ -l(ih)\delta e^{-\delta h(i-j)} N(-d_1(l(ih), (i-j)h; l(jh))) \end{bmatrix}$$

For d_1 we have the following limits:

$$\begin{aligned} \lim_{\xi \to \tau} N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) &= \lim_{\xi \to \tau} N(-d_1(c_2(\tau), \tau - \xi; c_1(\xi))) = 0 \\ \lim_{\xi \to \tau} N(d_1(c_2(\tau), \tau - \xi; c_2(\xi))) &= \lim_{\xi \to \tau} N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi))) = 0.5 \\ \lim_{\xi \to \tau} N(d_1(h(\tau), \tau - \xi; h(\xi))) &= \lim_{\xi \to \tau} N(-d_1(l(\tau), \tau - \xi; l(\xi))) = 0.5 \end{aligned}$$

[13] For more details on the error analysis, readers may consult (Quarteroni et al., 2007).
[14] For more efficiency, Linz (1985, p.98) recommends the use of the high-order quadrature when computing a Volterra integral equation type with lower values of i.
[15] Following Kim (1990), CZ give the proof for $c_1(0)$ and $c_2(0)$. We can obtain the results of $h(0)$ and $l(0)$ similarly.

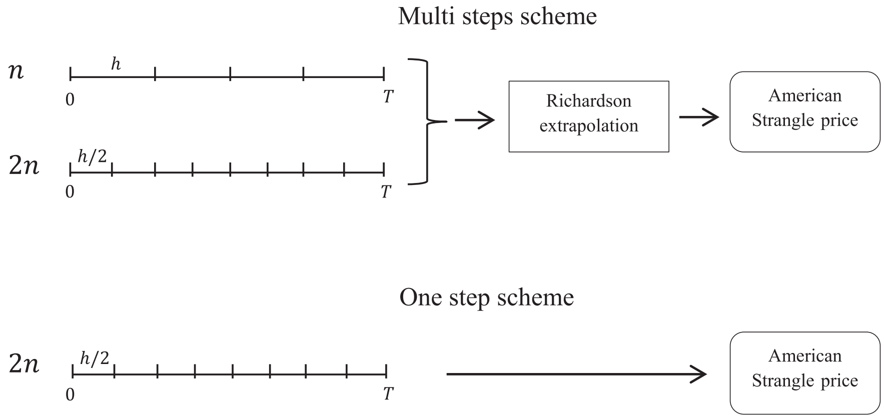

Fig. 2. Comparing numerical strategies.

These expressions are derived using the limits of the function $d_1(c_u(\tau), \tau - \xi; c_v(\xi))$ given by:

$$\lim_{\xi\to\tau} \frac{\ln\left(\frac{c_u(\tau)}{c_v(\xi)}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(\tau - \xi)}{\sigma\sqrt{\tau - \xi}} = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } u > v \\ -\infty & \text{if } u < v \end{cases}$$

for $u, v \in \{1, 2\}$.

From a financial point of view, it is notable that $\lim_{\xi\to\tau} N(d_1(c_1(\tau), \tau - \xi; c_2(\xi)))$ and $\lim_{\xi\to\tau} N(-d_1(c_2(\tau), \tau - \xi; c_1(\xi)))$ represent the probabilities that $c_1(\tau) > c_2(\xi)$ and $c_2(\tau) > c_1(\xi)$, respectively, within a short period of time $\tau - \xi$. By definition, at any time until maturity, we have $c_1(.) < c_2(.)$. So both probabilities are zero. Along similar lines, $\lim_{\xi\to\tau} N(d_1(c_2(\tau), \tau - \xi; c_2(\xi)))$, $\lim_{\xi\to\tau} N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)))$, $\lim_{\xi\to\tau} N(d_1(h(\tau), \tau - \xi; h(\xi)))$, and $\lim_{\xi\to\tau} N(-d_1(l(\tau), \tau - \xi; l(\xi)))$ represent the respective probabilities that $c_2(\tau) > c_2(\xi), c_1(\tau) < c_1(\xi), h(\tau) > h(\xi)$, and $l(\tau) < l(\xi)$ within a short period of time $\tau - \xi$. When ξ tends to τ, due to the property of the cumulative normal distribution function, the probabilities are equal to 0.5. In discrete time, the equations yield to:

$$\lim_{j\to i} N(d_1(c_1(ih), (i-j)h; c_2(jh))) = \lim_{j\to i} N(-d_1(c_2(ih), (i-j)h; c_1(jh))) = 0$$
$$\lim_{j\to i} N(d_1(c_2(ih), (i-j)h; c_2(jh))) = \lim_{j\to i} N(-d_1(c_1(ih), (i-j)h; c_1(jh))) = 0.5$$
$$\lim_{j\to i} N(d_1(h(ih), (i-j)h; h(jh))) = \lim_{j\to i} N(-d_1(l(ih), (i-j)h; l(jh))) = 0.5$$

Similar limits hold also for d_2. As a result, when j reaches the value of i, the numerical implementation does not involve the boundaries values at jh and ensures the existence of d_1 and d_2. To determine the unknown value of the boundaries at step i, we introduce two different functions:

$$f_1(x) = -K_1 + x + A(x, ih)$$
$$f_2(x) = -x + K_2 + A(x, ih)$$

where A is the price of the considered strangle contract, either a hybrid American strangle or a standard one, at time to maturity ih. The roots of f_1 (or f_2) computed at i correspond exactly to the lower (or upper) early exercise boundaries at that time.[16]

[16] The bisection method is used with a six-digit precision for the root searching result. The Newton–Raphson method, the bisection method, and other methods are available for root finding. By the nature of the contract, the holder of the option will choose the best of a put with strike K_1 and a call with strike K_2 (with a self-closing mechanism). Accordingly, the function $A(x, ih)$ is a decreasing function of x within $[0, K_1]$ and an increasing function of x within $[K_2, \infty)$ (see Fig. 1). This monotonic property is extended to functions f_1 and f_2 within the root searching intervals and ensures that the bisection method succeeds at any arbitrary level of confidence.

To analyze the complexity of our approach, assume that there are m operations in the bisection method (m depends on the targeted level of accuracy). To find each boundary at step i, we have $i+1$ evaluations of the integral equation. For n evaluations we have $((n+1)(n+2))/2$ operations. The total number of operations (considering an evaluation of the integrand as one operation) is $m[((n+1)(n+2))/2]$; its order is mn^2. This number highlights how the runtime of the code is influenced by the choice of n. This is the same as in Chiarella and Ziogas (2005) only when the Richardson's extrapolation step is ignored.

Appendix C shows the explicit pseudo-codes for the hybrid quadrature, the determination of the EEB, and the price of a Euro-American strangle with an early exercisable call side.[17] Algorithms 2 and 3 are iteratively and recursively linked. Algorithm 2 involves the index i, whereas Algorithm 3 refers to n. We can see that at each step i the function HybridQuad(i) gives the quadrature weights for $i+1$ endpoints.

5. Simulations

First, it is worth schematically comparing our method to that of CZ. Fig. 2 shows how the numerical scheme of CZ and ours differ in their principles. n and $2n$ are the numbers of the discretization length of the integrals in the price formula.

Next, Fig. 3 plots the call side early exercise boundary computed with the CZ numerical scheme before (checkered gray line) and after (black line) Richardson's extrapolation. The checkered gray line in this figure clearly shows the non-monotonic gradient problem. Interestingly, we can also notice a) that there is a need for an extrapolation technique and b) that the Richardson's extrapolation technique is far from being a simple averaging of external points.

Fig. 4 compares the call side early exercise boundary of CZ after the Richardson's *extrapolation* (gray line) with the one we obtain from our own procedure (black line). Both EEBs are associated with the same American strangle but both boundaries differ from each other. Our boundary appears to be lower than the one provided by the CZ algorithm. Consequently, our method suggests to exercise the call side of the strangle earlier and this increases the probability of exercising the call side. Due to the smoothness of the EEB pricing the accuracy should be improved in our context. It becomes clear that the choice of the quadrature method deserves particular attention and significantly affects the pricing and the management of American-Style strangles. Table 3, introduced below, will confirm these statements.

[17] Parameter *sigma*, the volatility, is implicitly used in functions d_1 and d_2.

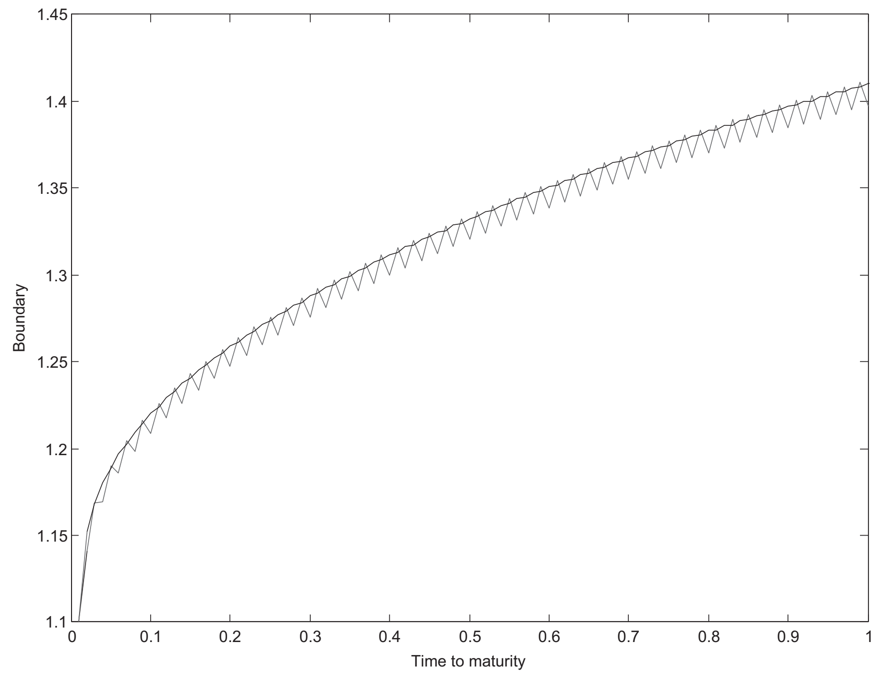

Fig. 3. Call side early exercise boundaries: CZ's algorithm. These two "call side" early exercise boundaries are provided by the Chiarella and Ziogas algorithm. The boundaries are considered before (gray checkered line) and after (black line) the Richardson's extrapolation technique to illustrate the non-monotonic gradient problem. Parameter values are $K_1 = 1, K_2 = 1.1, r = 5\%, \delta = 10\%, T = 1, \sigma = 20\%$ and $n = 100$.

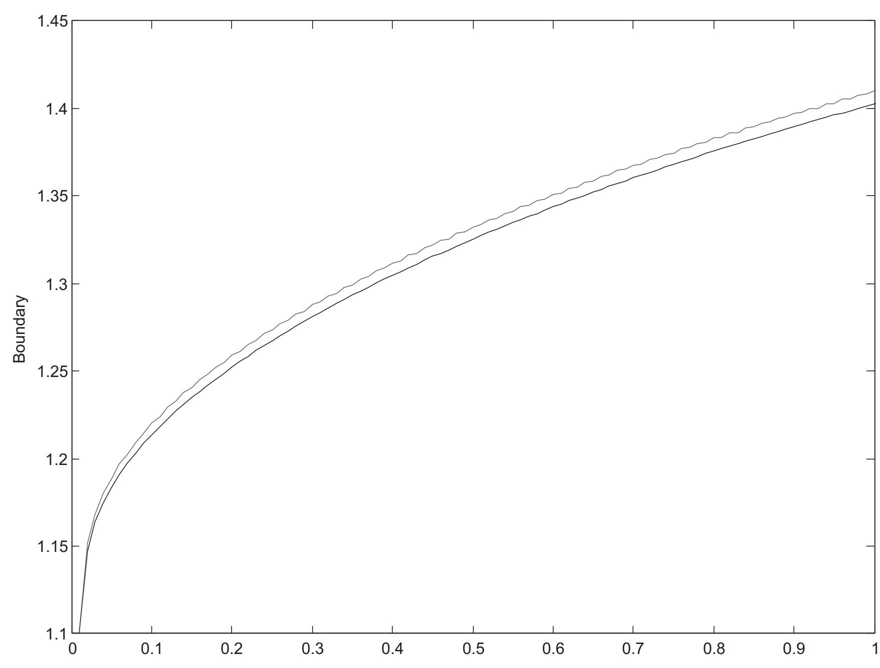

Fig. 4. Call side early exercise boundaries: a comparison. These two "call side" early exercise boundaries are provided by the CZ procedure (upper line) and our own procedure (lower line). The boundary of CZ is considered after Richardson's *extrapolation*. Our procedure does not need such a device. The parameter values are $K_1 = 1, K_2 = 1.1, r = 5\%, \delta = 10\%, T = 1, \sigma = 20\%$, and $n = 100$.

Using the one-step procedure saves computational time because it is run with an interval length of $h/2$ only, whereas the two-step algorithm uses both h and $h/2$. So, by construction, CZ use 50% more subintervals than we do. To assess the gain in time, we proceed as follows. We implement both of the codes for the different values of n that are the number of $h/2$ long subintervals. The computational times are then displayed on the bottom line of Table 4. The table reveals that for a given n, the convergence

Table 5
Prices of American strangles vs. Euro-American strangles.

S	American strangle	Euro-American strangle with American call side	Euro-American strangle with American put side	European strangle
Panel A: $r < \delta$ ($r = 5\%, \delta = 10\%$)				
1.0	0.156485	0.156485	0.152081	0.152081
1.1	0.169608	0.169608	0.158479	0.158479
1.2	0.219351	0.219351	0.195680	0.195680
1.3	0.299809	0.299809	0.255461	0.255461
Panel B: $r > \delta$ ($r = 10\%, \delta = 5\%$)				
0.7	0.299959	0.247329	0.299959	0.247329
0.8	0.205527	0.179077	0.205527	0.179077
0.9	0.156331	0.145270	0.156331	0.145270
1.0	0.156364	0.151920	0.156364	0.151920
Panel C: $r = \delta$ ($r = \delta = 5\%$)				
0.9	0.165262	0.163381	0.165033	0.163152
1.0	0.152594	0.151846	0.151853	0.151104
1.1	0.178339	0.178052	0.176449	0.176162
1.2	0.234506	0.234400	0.230451	0.230345

The parameter values are $K_1 = 1, K_2 = 1.001, T = 1$, and $\sigma = 20\%$. Both r, δ, and S vary.

towards the common long-run value is faster with our method. The computational time for the two-step procedure is at least about 25% longer. The maximum difference is obtained for $n = 400$, with a difference of 36%.

Table 4 compares the prices computed with the new quadrature method to those provided by CZ for $n = 100, 200, 400$, and 800. We consider different values of the stock price S to represent five different situations: in-the-money (on the call and put sides), at-the-money (on the call and put sides), and out-of-the-money (i.e., in between K_2 and K_2). The prices obtained by both methods share four decimal digits. When n increases, the fifth decimal converges to the same value. We can see that our algorithm converges faster than that of CZ in the sense that for an equivalent computational time our prices are closer to the exact solution than CZ's. Table 4 also shows that, as predicted, the price limit is reached from above when using our approach. By contrast, the CZ numerical approximation can converge from above or below (see for instance $S = 1.5$ and $S = 1.75$). The bottom line is that we can avoid the non-monotonic gradient problem, save computational time, improve the convergence rate, and control for errors.

Table 5 compares the prices of American and Euro-American strangles for three different environments (Panels A, B, and C) and for a range of asset prices chosen to highlight the price differences in all cases. We can check that the prices of the European and American strangles bound all other values. Panel A displays a situation where the risk-free rate is far smaller than the dividend rate. Here, the Euro-American strangles with American put sides are cheaper than their American strangle counterparts, whereas the Euro-American strangles with American call sides are as expensive. This panel highlights a situation where the lower early exercise boundary does not matter. In other words, the American strangle contract offers an opportunity to sell earlier than maturity that is worth nothing. Panel B provides simulations in an environment where the risk-free rate is far greater than the dividend rate. In that situation, the upper early exercise boundary does not matter, reflecting that an American strangle contract offers, in such a context, a worthless opportunity to buy earlier than maturity. Panel C highlights a situation where the risk-free rate is equal to the dividend rate. Here, the prices of both the Euro-American strangles are different from those of the standard American strangles.

6. Hedging management issues

It is important for traders and writers to hedge option positions. Dynamic hedging strategies suggest the need to consider price sensibilities. Delta (Δ), gamma (Γ), and theta (Θ) are in these respects key parameters. Kim-style representations of American and Euro-American strangles within the Black–Scholes model are useful to find analytical formulas. Proofs for the following hedging parameters are in Appendix B.

Delta, which is the first derivative of the position with respect to the stock price, measures the sensitivity of the contract price to the underlying stock price. This hedging ratio indicates the number of stocks to hold in a dynamic hedging strategy. From our previous representations of strangles, we have

$$\Delta_V = \frac{\partial V(S,\tau)}{\partial S} = \Delta_E + \Delta_{EEP} \tag{12}$$

where Δ_E is the delta of a European strangle ($\Delta_E = e^{-\delta\tau}(N(d_1(S,\tau;K_2)) - N(-d_1(S,\tau;K_1))))$ and Δ_{EEP} is the delta of the early exercise premium. For the American strangle, we have

$$\begin{aligned}\Delta^{AS}_{EEP} = \int_0^\tau \delta e^{-\delta(\tau-\eta)}\Big\{&N(d_1(S,\tau-\eta;c_2(\eta))) - N(-d_1(S,\tau-\eta;c_1(\eta)))\\ &+\frac{1}{\sigma\sqrt{\tau-\eta}}\Big[\phi(d_1(S,\tau-\eta;c_2(\eta)))\Big(1-\frac{K_2}{c_2(\eta)}\frac{r}{\delta}\Big)\\ &+\phi(-d_1(S,\tau-\eta;c_1(\eta)))\Big(1-\frac{K_1}{c_1(\eta)}\frac{r}{\delta}\Big)\Big]\Big\}d\eta.\end{aligned}$$

For a Euro-American strangle with an American upper side we have

$$\begin{aligned}\Delta^{EcAS}_{EEP} = \int_0^\tau \delta e^{-\delta(\tau-\eta)}\Big\{&N(d_1(S,\tau-\eta;h(\eta)))\\ &+\frac{1}{\sigma\sqrt{\tau-\eta}}\phi(d_1(S,\tau-\eta;h(\eta)))\Big(1-\frac{K_2}{h(\eta)}\frac{r}{\delta}\Big)\Big\}d\eta.\end{aligned}$$

For a Euro-American strangle with an American lower side, we have

$$\begin{aligned}\Delta^{EpAS}_{EEP} = \int_0^\tau \delta e^{-\delta(\tau-\eta)}\Big\{&-N(-d_1(S,\tau-\eta;c_1(\eta)))\\ &+\frac{1}{\sigma\sqrt{\tau-\eta}}\phi(-d_1(S,\tau-\eta;l(\eta)))\Big(1-\frac{K_1}{l(\eta)}\frac{r}{\delta}\Big)\Big\}d\eta.\end{aligned}$$

Gamma is the second derivative of the position with respect to the stock price. Viewed as the first derivative of the delta, this Greek parameter provides information on the rebalancing frequency of the self-financing replicating portfolio. It is a very important parameter for risk management because it informs option sellers about their exposure to an abrupt movement in the underlying asset price. Gamma is

$$\Gamma_V = \frac{\partial^2 V(S,\tau)}{\partial S^2} = \Gamma_E + \Gamma_{EEP} \tag{13}$$

where Γ_E is the gamma of a European strangle given by

$$\Gamma_E = \frac{e^{-\delta\tau}}{S\sigma\sqrt{\tau}}(\phi(d_1(S,\tau;K_1)) + \phi(d_1(S,\tau;K_2))).$$

For an American strangle, we have

$$\begin{aligned}\Gamma^{AS}_{EEP} = \int_0^\tau \frac{e^{-\delta(\tau-\eta)}}{S\sigma\sqrt{\tau-\eta}}\Big[&\phi(d_1(S,\tau-\eta;c_2(\eta)))\\ &\times\Big\{1-\frac{d_1(S,\tau-\eta;c_2(\eta))}{\sigma\sqrt{\tau-\eta}}\Big(1-\frac{K_2}{c_2(\eta)}\frac{r}{\delta}\Big)\Big\} + \phi(d_1(S,\tau-\eta;c_1(\eta)))\\ &\times\Big\{1-\frac{d_1(S,\tau-\eta;c_1(\eta))}{\sigma\sqrt{\tau-\eta}}\Big(1-\frac{K_1}{c_1(\eta)}\frac{r}{\delta}\Big)\Big\}\Big]d\eta.\end{aligned}$$

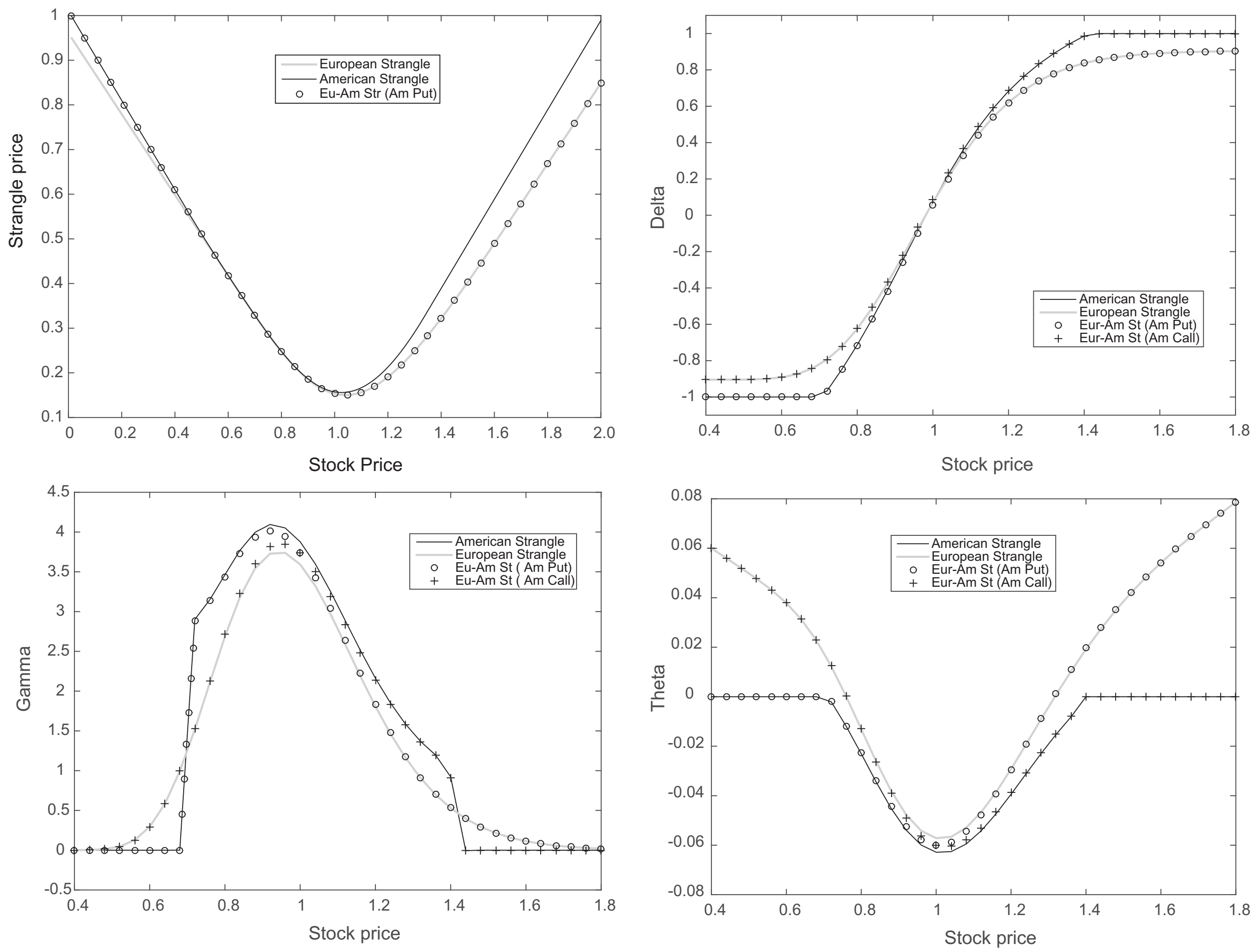

Fig. 5. Simulations. The graphs of these figures represent the prices and Greeks of the following strangles: European (gray line), American (black line), Euro-American with American call side (marker "+"), and Euro-American with American put side (marker "o"). The parameter values are $K_1 = 1, K_2 = 1.001, r = 10\%$ ($r = 0.5$ for prices graph), $\delta = 10\%, T = 1, \sigma = 20\%$.

For a Euro-American strangle with an American upper side, we have

$$\Gamma_{EEP}^{EcAS} = \int_0^{\tau} \frac{e^{-\delta(\tau-\eta)}}{S\sigma\sqrt{\tau-\eta}} \phi(d_1(S,\tau-\eta;h(\eta))) \left\{ 1 - \frac{d_1(S,\tau-\eta;h(\eta))}{\sigma\sqrt{\tau-\eta}} \left(1 - \frac{K_2}{h(\eta)} \frac{r}{\delta} \right) \right\} d\eta.$$

For a Euro-American strangle with an American lower side, we have

$$\Gamma_{EEP}^{EpAS} = \int_0^{\tau} \frac{e^{-\delta(\tau-\eta)}}{S\sigma\sqrt{\tau-\eta}} \phi(d_1(S,\tau-\eta;l(\eta))) \left\{ 1 - \frac{d_1(S,\tau-\eta;l(\eta))}{\sigma\sqrt{\tau-\eta}} \left(1 - \frac{K_1}{l(\eta)} \frac{r}{\delta} \right) \right\} d\eta.$$

The time decay theta is an important Greek parameter to consider (see Taleb (1997, pp.167–170) for a discussion). Because strangles satisfy the fundamental Black–Scholes Merton partial differential equation (Eq. (1)), this Greek parameter can be deduced from the delta and gamma. We have

$$\Theta_V = \frac{\partial V}{\partial \tau} = -(r-\delta)S\Delta_V - \frac{1}{2}\sigma^2 S^2 \Gamma_V + rV. \quad (14)$$

Theta mixes delta and gamma in a quite subtle way (see Eq. (14)). For American and Euro-American strangles, it is (of course) zero in stopping regions, because the time effect disappears once the contracts are exercised. Theta reaches its minimum when the option is approximately at the money (see Fig. 5). Actually, the value of an option decreases when it nears expiry without being in the money because the probability that the option expires out of the money increases.

From a computational perspective, the hedging parameters can be easily evaluated once we know the early exercise boundaries. These boundaries are computed with the recursive method introduced earlier. Clearly, this method is quite likely to be more efficient than the alternative methods relaying on perturbation schemes. In perturbation analysis, repeating the price computations is necessary and this compounds numerical errors. By contrast, the direct computation of Greek parameters, as permitted by our formulas, restricts the errors to those associated with the determination of the boundaries (see Huang et al. (1996) for an investigation of the recursive method).

Fig. 5 portrays the prices and Greek parameters of all the strangles on the same graph as a function of the underlying asset (contemporaneous) price. The upper-left graph illustrates the price differences between the strangles (for $r < \delta$).[18] The upper-right graph deals with the deltas. As expected, the deltas lie between −1 and 1. When the stock price is larger than K_2, the call side of the American strangle is in the money and the right to buy becomes influential. When the stock price is lower than K_1, the put side is in the money and the right to sell takes the lead. The gammas highlight the extinction of the early exercise strangles very clearly. The values simply drop to zero in such a case. All the gammas reach their maximum near strikes. The large gammas indicate that the delta

[18] In this case, the Euro-American strangle price with an American call side is similar to that of an American strangle.

hedging may be more demanding for these contracts compared to the European equivalent contracts.

7. Conclusions

This study reconsiders the pricing and hedging of strangles with finite maturity. It also introduces a new variant of these contracts called Euro-American strangles, and it provides a new (efficient and accurate) one-step numerical algorithm. The quadrature approach we propose combines Newton–Cotes and Gregory weighting schemes. Our algorithm avoids the non-monotonic gradient problem faced by the Chiarella and Ziogas' approach and has no recourse to any extrapolation scheme. We show that the computational time is significantly lowered and that the convergence rate is also improved. In addition, the new scheme allows us to control for errors. Our approach relies extensively on the Kim-style representation of American-style strangles. As a result of this, we can derive analytical expressions for prices and for hedging parameters. Finally, we make simulations to compare the parameters with each other. Our results suggest that the hedging of strangles deserves specific attention.

Appendix A. Solving PDE (1) with Kim's decomposition

The goal here is to solve the PDE (1) equipped with boundary conditions (3a)–(3d). Along the lines of Kim (1990), we transform the problem so we can solve it using the results of Kolodner (1956) and McKean, 1965. Due to the linear property of differentiation, we can divide the Euro-American strangle with an American upper side into two components

$$EA_h(S,\tau) = F(S,\tau) + G(S,\tau)$$

where both F and G satisfy PDE (1). The function F is precisely characterized by

$$F(S,0) = 0, S \geqslant 0, \tag{15}$$

$$F(S,\tau) = h(\tau) - K_2, \quad 0 \leqslant \tau \leqslant T, \tag{16}$$

$$\lim_{S\to h(\tau)} \frac{\partial F}{\partial S} = 1, \quad 0 \leqslant \tau \leqslant T. \tag{17}$$

while

$$G(S,0) = \max[0, K_1 - S] + \max[0, (S-K_2)\psi(h(0)-S)], \quad S \geqslant 0, \tag{18}$$

The solution of the second problem (i.e., the partial differential Eq. (1) with boundary condition (18)) is equivalent to the price of a portfolio made up of two independent standard options (a call and a put), given that the price of the underlying asset at maturity is at most equal to $h(0)$. In fact, in the case where there is no early exercise (of the American side), it is impossible for the underlying asset to be above $h(0)$ (due to continuity of the $h(.)$). The linear property of differentiation allows us to divide the problem into two subproblems and solve them separately. The first challenge is to solve the PDE(1) under the boundary condition

$$G_1(S,0) = \max[0, K_1 - S], S \geqslant 0.$$

The solution is available from the Black–Scholes framework and we have:

$$\begin{aligned}
G_1(S,\tau) &= E\left[e^{-r\tau}(K_1 - S_\tau)1_{S_\tau<K_1}|S_0 = S\right] \\
&= \int_{-\infty}^{\infty} e^{-r\tau}\left(K_1 - Se^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}u}\right)1_{Se^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}u}<K_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&= \int_{-\infty}^{-\frac{\ln(S/K_1)+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{-r\tau}\left(K_1 - Se^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}u}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du
\end{aligned}$$

where $E[.]$ is the mathematical expectation under the risk-neutral probability.

Separating the integral above yields:

$$\begin{aligned}
G_1(S,\tau) = K_1e^{r\tau}N\left(-\frac{\ln(S/K_1) + (r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right) \\
- Se^{-\delta\tau}N\left(-\frac{\ln(S/K_1) + (r-\delta+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).
\end{aligned}$$

The second challenge is to solve the PDE(1) under boundary condition

$$\max[0, (S-K_2)\psi(h(0)-S)], \quad S \geqslant 0.$$

Similarly, its solution is given by:

$$\begin{aligned}
G_2(S,\tau) &= E\left[e^{r\tau}(S_\tau - K_2)1_{K_2<S_\tau<h(0)}|S_0 = S\right] \\
&= \int_{-\infty}^{\infty} e^{r\tau}\left(Se^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}u} - K_2\right)1_{K_2<Se^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}u}<h(0)}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&= \int_{-\frac{\ln(S/K_2)+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{-\frac{\ln(S/h(0))+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{r\tau}\left(Se^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}u} - K_2\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&= \int_{\frac{\ln(S/h(0))+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\frac{\ln(S/K_2)+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{r\tau}\left(Se^{(r-\delta-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}u} - K_2\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&= \int_{-\infty}^{\frac{\ln(S/K_2)+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{r\tau}\left(Se^{(r-\delta-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}u} - K_2\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&\quad - \int_{-\infty}^{\frac{\ln(S/h(0))+(r-\delta-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{r\tau}\left(Se^{(r-\delta-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}u} - K_2\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du \\
&= Se^{-\delta\tau}N\left(\frac{\ln(S/K_2)+\left(r-\delta+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) - K_2e^{-r\tau}N\left(\frac{\ln(S/K_2)+\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&\quad - Se^{-\delta\tau}N\left(\frac{\ln(S/h(0))+\left(r-\delta+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) + K_2e^{-r\tau}N\left(\frac{\ln(S/h(0))+\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right).
\end{aligned}$$

The overall solution of the first problem is therefore given by:

$$\begin{aligned}
G(S,\tau) = G_1(S,\tau) + G_2(S,\tau) = K_1e^{-r\tau}N\left(-\frac{\ln(S/K_1) + \left(r-\delta-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
- Se^{-\delta\tau}N\left(-\frac{\ln(S/K_1) + \left(r-\delta+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
+ Se^{-\delta\tau}N\left(\frac{\ln(S/K_2) + \left(r-\delta+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
- K_2e^{-r\tau}N\left(\frac{\ln(S/K_2) + \left(r-\delta-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
- Se^{-\delta\tau}N\left(\frac{\ln(S/h(0)) + \left(r-\delta+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
+ K_2e^{-r\tau}N\left(\frac{\ln(S/h(0)) + \left(r-\delta-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right).
\end{aligned} \tag{19}$$

Kolodner (1956) and McKean (1965) analyzed the heat equation that is a free boundary problem similar to the system-formed PDE (1) under conditions (15)–(17). They solved this by means of Fourier transforms. Because it is long and tedious, we directly apply their results to our problem. We obtain:

$$F(S,\tau)=\int_0^\tau e^{-r(\tau-\xi)}\frac{e^{-[\alpha-\beta(\xi)]^2/2(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}}\times\left[\frac{h(\xi)\sigma}{2}+\left(\beta'(\xi)-\frac{\alpha-\beta(\xi)}{2(\tau-\xi)}\right)(h(\xi)-K_2)\right]d\xi$$

where

$$\beta(\xi)=\left[\ln(h(\xi))+\left(r-\delta-\frac{1}{2}\sigma^2\right)\xi\right]\Big/\sigma,$$
$$\alpha=\left[\ln(S)+\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau\right]\Big/\sigma.$$

By rearranging the terms of $F(S,\tau)$ we have

$$F(S,\tau)=\int_0^\tau e^{-r(\tau-\xi)}h(\xi)\frac{e^{-[\alpha-\beta(\xi)]^2/2(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}}\times\left[\frac{\sigma}{2}+\beta'(\xi)-\frac{\alpha-\beta(\xi)}{2(\tau-\xi)}\right]d\xi - K_2\int_0^\tau e^{-r(\tau-\xi)}\frac{e^{-[\alpha-\beta(\xi)]^2/2(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}}\times\left[\beta'(\xi)-\frac{\alpha-\beta(\xi)}{2(\tau-\xi)}\right]d\xi. \quad (20)$$

And, by noticing that

$$e^{-r(\tau-\xi)}h(\xi)\frac{e^{-[\alpha-\beta(\xi)]^2/2(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}}\times\left[\frac{\sigma}{2}+\beta'(\xi)-\frac{\alpha-\beta(\xi)}{2(\tau-\xi)}\right]=-e^{-\delta(\tau-\xi)}S\frac{\partial}{\partial\xi}N\left(\frac{\alpha-\beta(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)$$

and

$$e^{-r(\tau-\xi)}\frac{e^{-[\alpha-\beta(\xi)]^2/2(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}}\times\left[\beta'(\xi)-\frac{\alpha-\beta(\xi)}{2(\tau-\xi)}\right]=e^{-r(\tau-\xi)}\frac{\partial}{\partial\xi}N\left(\frac{\alpha-\beta(\xi)}{\sqrt{\tau-\xi}}\right)$$

as well as using integration by parts, Eq. (20) then gives

$$\begin{aligned}F(S,\tau)=&\ \delta S\int_0^\tau e^{-\delta(\tau-\xi)}N\left(\frac{\alpha-\beta(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi\\&-rK_2\int_0^\tau e^{-r(\tau-\xi)}N\left(\frac{\alpha-\beta(\xi)}{\sqrt{\tau-\xi}}\right)d\xi\\&+Se^{-\delta\tau}N\left(\frac{\ln(S/h(0))+(r-\delta+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\\&-K_2e^{-r\tau}N\left(\frac{\ln(S/h(0))+(r-\delta-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).\end{aligned}\quad (21)$$

Finally, by adding Eqs. (21) and (19) we obtain Eq. (8). The proof of Proposition (3) is similarly obtained.

Appendix B. Derivation of Greek parameters.

In this appendix we derive the deltas of American strangles. The deltas for European and Euro-American strangles, as well as gammas, may be found similarly. Once we get deltas and gammas, thetas can be deduced by using the fundamental Black–Scholes PDE (Eq. (14)). Let us consider an American strangle option

$$A_{c_1c_2}(S,\tau)=E(S,\tau)+EEP_{c_1c_2}(S,\tau)$$

According to the linearity of differentiation, we have

$$\Delta_A=\frac{\partial A_{c_1c_2}(S,\tau)}{\partial S}=\Delta_E+\Delta_{EEP}$$

where $\Delta_E=\frac{\partial E(S,\tau)}{\partial S}$ and $\Delta_{EEP}=\frac{\partial EEP_{c_1c_2}(S,\tau)}{\partial S}$. To simplify the proof let us divide Δ_E and Δ_{EEP} into two components. For Δ_E, we have $\Delta_E=\Delta_{E_1}+\Delta_{E_2}$ where

$$\Delta_{E_1}=\frac{\partial}{\partial S}\left[K_1e^{-r\tau}N(-d_2(S,\tau;K_1))-Se^{-\delta\tau}N(-d_1(S,\tau;K_1))\right]$$

and

$$\Delta_{E_2}=\frac{\partial}{\partial S}\left[Se^{-\delta\tau}N(d_1(S,\tau;K_2))-K_2e^{-r\tau}N(d_2(S,\tau;K_2))\right].$$

For Δ_{EEP}, we have $\Delta_{EEP}=\Delta_{EEP_1}+\Delta_{EEP_2}$ where

$$\Delta_{EEP_1}=\frac{\partial}{\partial S}\left[\int_0^\tau\left[K_1re^{-r(\tau-\eta)}N(-d_2(S,\tau-\eta;c_1(\eta)))-S\delta e^{-\delta(\tau-\eta)}N(-d_1(S,\tau-\eta;c_1(\eta)))\right]d\eta\right]$$

and

$$\Delta_{EEP_2}=\frac{\partial}{\partial S}\left[\int_0^\tau\left[S\delta e^{-\delta(\tau-\eta)}N(d_1(S,\tau-\eta;c_2(\eta)))-K_2re^{-r(\tau-\eta)}N(d_2(S,\tau-\eta;c_2(\eta)))\right]d\eta\right].$$

For Δ_E, things are straightforward and well known. We have: $\Delta_{E_2}=e^{-\delta\tau}N(d_1(S,\tau;K_2))+Se^{-\delta\tau}\frac{1}{S\sigma\sqrt{\tau}}\phi(d_1(S,\tau;K_2))-K_2e^{-r\tau}\frac{1}{S\sigma\sqrt{\tau}}\phi(d_2(S,\tau;K_2))$where the two last terms collapse. Δ_{E_1} is obtained with a similar calculation and we have $\Delta_{E_1}=-e^{-\delta\tau}N(d_1(S,\tau;K_1))$. The next step is to investigate Δ_{EEP_1} and Δ_{EEP_2}.

$$\Delta_{EEP_2}=\int_0^\tau\frac{\partial}{\partial S}\left[S\delta e^{-\delta(\tau-\eta)}N(d_1(S,\tau-\eta;c_2(\eta)))-K_2re^{-r(\tau-\eta)}N(d_2(S,\tau-\eta;c_2(\eta)))\right]d\eta$$

where the derivatives are

$$\delta e^{-\delta(\tau-\eta)}N(d_1(S,\tau-\eta;c_2(\eta)))+S\delta e^{-\delta(\tau-\eta)}\frac{\phi(d_1(S,\tau-\eta;c_2(\eta)))}{S\sigma\sqrt{\tau-\eta}}-K_2re^{-r(\tau-\eta)}\frac{\phi(d_2(S,\tau-\eta;c_2(\eta)))}{S\sigma\sqrt{\tau-\eta}}$$

Given that $Se^{-\delta\tau}\phi(d_1(S,\tau;K))=Ke^{-r\tau}\phi(d_2(S,\tau;K))$ is verified for all K, we obtain after simplification

$$\delta e^{-\delta(\tau-\eta)}N(d_1(S,\tau-\eta;c_2(\eta)))+\delta e^{-\delta(\tau-\eta)}\frac{\phi(d_1(S,\tau-\eta;c_2(\eta)))}{\sigma\sqrt{\tau-\eta}}\left(1-\frac{K_2}{c_2(\eta)}\frac{r}{\delta}\right)$$

With all in one we get

$$\Delta_{EEP_2}=\int_0^\tau\delta e^{-\delta(\tau-\eta)}\left[N(d_1(S,\tau-\eta;c_2(\eta)))+\frac{\phi(d_1(S,\tau-\eta;c_2(\eta)))}{\sigma\sqrt{\tau-\eta}}\times\left(1-\frac{K_2}{c_2(\eta)}\frac{r}{\delta}\right)\right]d\eta.$$

We can derive the expression for Δ_{EEP_1} with the same procedure. We get

$$\Delta_{EEP_1}=\int_0^\tau\delta e^{-\delta(\tau-\eta)}\left[-N(-d_1(S,\tau-\eta;c_1(\eta)))+\frac{\phi(-d_1(S,\tau-\eta;c_1(\eta)))}{\sigma\sqrt{\tau-\eta}}\left(1-\frac{K_1}{c_1(\eta)}\frac{r}{\delta}\right)\right]d\eta.$$

Appendix C. Pseudo-codes

Algorithm 1. Hybrid quadrature.

```
Function: HybridQuad
Input: n                          /* number of subinterval */
Output: w           /* vector of weights of n+1 endpoints */
begin
    Create vector w of length n+1 and full with default value 1;
    Switch n do
        case n = 1                /* Closed Newton-Cotes weights) */
            w(0)←—0.5; w(1)←—0.5;
        case n = 2
            w(0)←—1/3; w(1)←—4/3; w(2)←—w(0);
        case n = 3
            w(0)←—3/8; w(1)←—9/8;
            w(2)←—w(1); w(3)←—w(0);
        case n = 4
            w(0)←—14/45; w(1)←—64/45; w(2)←—24/45;
            w(3)←—w(1); w(4)←—w(0);
        case n = 5
            w(0)←—95/288; w(1)←—375/288; w(2)←—250/288;
            w(3)←—w(2); w(4)←—w(1); w(5)←—w(0);
        case n = 6
            w(0)←—41/140; w(1)←—216/140; w(2)←—27/140; w(3)←—272/140
            w(4)←—w(2); w(5)←—w(1); w(6)←—w(0);
        otherwise                 /* fourth-order Gregory weights) */
            w(0)←—3/8; w(1)←—7/6; w(2)←—23/24;
            w(n−2)←—w(2); w(n−1)←—w(1); w(n)←—w(0);
    return w
```

Algorithm 2. Euro-American strangle with early exercisable call side.

```
Function: EAS
Input: S, K1, K2, r, q, sigma, i, h, c2        /* t = ih stands for time to maturity */
Output: A                                      /* price of the strangle */
begin
    t ← ih;
    E ← ( K1 e^{-rt} N(−d2(S,t;K1)) − S e^{-qt} N(−d1(S,t;K1)) +
          S e^{-qt} N(d1(S,t;K2)) − K2 e^{-rt} N(d2(S,t;K2)) );
    EEP(v) ← (  S q e^{-q(i−v)h} N(d1(S,(i−v)h; c2(vh)))
               −K2 r e^{-r(i−v)h} N(d2(S,(i−v)h; c2(vh))) );
                                               /* new function of v */
    w←— HybridQuad (i)                         /* weights of the hybrid quadrature */
    sum ← 0
    for j←—0 to i do
        sum ← sum + w(j) × EEP(j)
    A ← E + h × sum;
    return A
```

Algorithm 3. Early exercise boundary.

```
Function: EEBoundary
Input: S, T, K1, K2, r, q, sigma, n
output: c2          /* vector of call side early exercise boundary */
begin
    h ← T/n ;
    Create vector c2 of length n+1 and full with default value 0 ;
    c2(0) ←max(K2, K2 × (r/q));
                    /* recursive computation of c2 starting with c2(0) */
    for i ← 1 to n do
        f2(x) ← x − K2 − EAS(x, K1, K2, r, q, sigma, i, h, c2)     /* new function */
        c2(i) ←Bisection(f2(x));        /* Bisection(f) gives the root of f */
    return c2
```

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