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Revenue Management of Flexible Products

Guillermo Gallego

Department of Industrial Engineering and Operations Research, Columbia University, 326 Mudd Building,
New York, New York 10027, gmg2@columbia.edu

Robert Phillips

Nomis Solutions Inc., 1150 Bayhill Drive, Suite 305, San Bruno, California 94066,
robert.phillips@nomissolutions.com

A *flexible product* is a menu of two or more alternative, typically substitute, products offered by a constrained supplier using a sales or booking process. The supplier reserves the right to assign customers who purchase a flexible product to one of the alternatives at a time near the end of the booking process. An example would be an airline offering a morning flight consisting of specific flights serving the same market. Flexible products are currently offered by a number of industries including air cargo, tour operators, and Internet advertising. Flexible products have the advantage of increasing overall demand and enabling better capacity utilization at the cost of potentially cannibalizing high-fare demand for specific products. This paper introduces the concept of flexible products and derives conditions and algorithms for the management of a single flexible product consisting of two specific products. We use numerical simulation to illustrate the benefits from offering flexible products and discuss extension of the approach to more general settings.

Key words: revenue management; flexibility; stochastic models; real options
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1. Background and Introduction

We define a *flexible product* as a set of two or more alternatives serving the same market such that a purchaser of the flexible product will be assigned to one of the alternatives by the seller at a later date. This is in contrast to a *specific product* which, by definition, consists of a single alternative. For example, an airline passenger typically books a specific flight. If his first choice is not available at an acceptable price, he may book on a less preferred alternative or may choose not to travel. However, in addition to these specific products, an airline could also offer flexible products, each of which consists of a set of two or more flights serving the same market. A customer purchasing a flexible product would be guaranteed service by one of the alternatives, but the airline would not assign a specific flight until a later date.[1]

As an example of a flexible product, consider an airline with three morning flights from New York's

Kennedy airport (JFK) to San Francisco International airport (SFO). One flight departs at 8:00 A.M. and arrives at 11:00 A.M., the second departs at 9:00 A.M. and arrives at 12:30 P.M., and the third departs at 11:00 A.M. and arrives at 2:00 P.M. Customers can book seats on any one of the three flights as usual. However, in addition to these three specific products the airline might also offer a flexible product, say "JFK–SFO morning," at a discount. Customers purchasing the JFK–SFO morning product would be guaranteed a seat on one of the three flights, but they would not be informed which flight until later. The airline would have the luxury of observing specific demand for each of the individual flights before assigning the flexible passengers to the morning departures.

Although we have couched the definition in terms of an airline, flexible products are common in a number of other industries, such as Internet advertising, air cargo, tour operators, multiple property management, and opaque fares.

Internet Advertising. Internet service providers (ISPs) such as Yahoo, MSN, and Lycos sell capacity on different properties to advertisers, where the properties con-

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[1] The time at which the buyer of a flexible product is informed may vary from a few weeks to a day before departure, depending on airline policy.

sist of pages devoted to topics such as sports, finance, weather, maps, etc. Advertisers can purchase space on individual properties or they can buy capacity on a cheaper run-of-network basis. If they purchase run of network, the service provider can choose, effectively in real time, which property will host the advertisement. Some advertisers have a strong preference for a certain property: Nike might want its advertisements to appear on the sports page whereas American Airlines might want its ads to appear on the business page. These advertisers will tend to purchase specific capacity. Other advertisers who are more indifferent regarding placement (or are simply more price sensitive) can purchase the less expensive run of network and take their chances on where their ads will be placed. The total capacity of an ISP is fixed and equal to the sum of the capacities of its individual properties. Thus, run of network is a flexible product and individual properties are specific products. At least one ISP has set a booking limit on its sales of run of network in order to retain sufficient capacity to sell on a specific product basis.

Air Cargo. The majority of air cargo is sold on a reservation basis, similar to passenger sales. Shippers—primarily forwarders and consolidators—book capacity for their shipments on specific flights. This is known as a *flight-specific booking*. In addition to flight-specific bookings, however, some carriers offer *time-definite products*, in which the carrier specifies only the pick-up time and the delivery time. In this case, the carrier has the option to choose the flights it pleases to carry the shipment, subject only to the pick-up and delivery requirements. Here, the time-definite offering is a flexible product.

Tour Operators. European tour operators such as Airtours and Thompson sell tour packages that include both air transportation and lodging. For a popular destination such as Ibiza or the Costa del Sol, an operator will have space agreements with many different hotels. When she books her tour, a customer can specify a particular property within the resort. Or, for a discount, she can specify a desired quality level (say three stars) and the tour operator will choose the property for her. Under this flexible product alternative, the tour operator will assign the customer to whichever property will maximize profitability.

Multiple Property Management. Major hotel chains such as Marriott, Sheraton, or Hilton often operate several properties within a single metropolitan location such as Manhattan or San Francisco. These chains have the opportunity to sell a general location-based product to travellers who are largely indifferent among the specific properties within the general location. The Disney corporation faces a similar opportunity at Disney World, where it operates a number of different hotels, some with specific themes. There are travellers who strongly prefer a particular hotel, whereas others may be indifferent as long as the hotel is within Disney World.

Opaque Fares. Over the past decade, a number of Internet travel sellers such as Priceline and Hotwire have offered so-called *opaque fares* for hotels and airlines. Using an airline opaque fare, the purchaser buys a ticket (often at a discount) for a particular origin-destination and flight date without knowing the itinerary, airline, or exact flight-departure and flight-arrival times. He is informed of these details only after the purchase is consummated. Opaque fares were created specifically as an inferior product that could fill capacity without excessively cannibalizing full-fare demand.

Although opaque fares fit our definition of flexible products, it should be noted that currently they are primarily offered by distributors such as Priceline and Hotwire, rather than by the airlines themselves. In this situation, an airline makes inventory available to a distributor at a discount fare. The distributor can then combine that inventory with other inventory in the same market as an opaque product at a higher fare. For example, Delta might quote a fare of $250 for some Altanta–Denver round-trip seats to a distributor. The distributor could then combine these seats with inventory from another carrier (say United) and offer an Atlanta–Denver opaque product at a fare of (say) $270. The distributor allocates purchasers of the opaque product among the inventory based on a procedure pre-agreed with United and Delta. In this situation, the problem facing Delta and United is how many seats they should offer to the distributor at the discount fare, which is no different from the standard single-leg revenue management problem.

Flexible products could potentially be used in any situation in which a company offers several products

that some customers will consider as close substitutes, whereas others have strong preferences for a specific product. For example, in made-to-order manufacturing sellers could offer options under which a customer could either choose a specific slot or choose a less expensive flexible option under which delivery is guaranteed by some future date but the seller has the choice to choose the actual slot within that date.

Flexible products offer two advantages to sellers:

Risk Pooling. Flexible products can improve capacity utilization because customers can be assigned to products after demand uncertainty about specific products has been largely resolved, allowing the supplier to hedge against demand and capacity unbalances.

Demand Induction. Flexible products will be viewed as inferior to specific products by most consumers. This potentially allows them to be sold at a lower price than the specific products without excessively cannibalizing specific product demand. At sufficiently low fares, flexible products may induce demand from a segment of the population that would have not purchased a specific product.[2]

Although flexible products offer advantages, there is the risk is that poorly managed flexible products could lead to revenue deterioration through cannibalization of higher fare demands. Sellers of flexible products will need new approaches in order to maximize profitability and prevent excess cannibalization.

We investigate the case of a supplier with fixed perishable capacity offering a combination of flexible and specific products. In this case, the problem of managing and pricing flexible products belongs to the widely studied field of revenue management. A summary of the extensive revenue management literature can be found in Talluri and van Ryzin (2004). Because the majority of this literature relates to airlines, we will pose our analysis in terms of an airline. However, the reader should bear in mind that the concept is much more broadly applicable and, in fact, Internet service providers and tour operators, among others,

are currently utilizing flexible products more widely than the airlines do.

Airlines, hotels, and other service providers have, of course, long been aware of the risks presented by the combination of uncertain demand and immediately perishable capacity and a number of mechanisms have been proposed to help manage this risk. The most venerable of these mechanisms is overbooking—accepting more bookings on a flight than available capacity. Typically, overbooking has been a way for airlines to hedge against the risks of cancellations and no-shows. For this reason, overbooking models usually assume that the *denied boarding cost* of refusing a booked passenger is greater than the highest fare Rothstein (1971). If the denied boarding cost is less than the highest fare, an optimal policy allows overbooking—even in the absence of cancellations or no-shows. In this case, overbooking is done with the purpose of improving revenues by bumping lower-fare passengers in favor of higher-fare passengers. However, this *bumping strategy* is inconvenient for passengers, inflexible for airlines, and can result in high costs associated with involuntary denied boarding and reaccommodation if bumped passengers need to be rebooked on a competing flight.

Another relevant mechanism is the sale of deeply discounted *stand-by* tickets. Stand-by passengers are only accommodated if there are fewer shows from guaranteed bookings than the available capacity. If a stand-by passenger is not accommodated on the flight she booked, she will be accommodated on a future departure for the same destination that has available capacity. Stand-bys are similar to flexible products, with the difference that the stand-by ticket purchaser is not a priori guaranteed accommodation on one of a given set of flights. Thus, stand-bys are usually analyzed as a form of hedge against no-shows and overbookings, as in Rothstein (1971), rather than as a way to improve capacity utilization.

Talluri (2001) has proposed a flexible-booking approach for the case in which passengers are indifferent among a number of routing alternatives between an origin-destination pair. He calls a collection of such alternatives a *route set*. He proposes an immediate acceptance and routing heuristic for customers seeking to purchase a route set based on current leg bid prices. A booking request for a route set

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[2] The idea of offering an inferior product at a lower price to stimulate demand is not new: It is the motivation behind airline discount fares with advance-purchase and Saturday-night-stay restrictions. Moreover, airlines limit the availability of the discount fare inferior products to reserve capacity for full-fare demand.

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would be accepted if and only if its fare exceeded the minimum total bid price among alternatives within the route set. A customer who purchased from the route set would immediately be routed (and booked) on the alternative within the route set with the minimum total bid price. The flexible product approach we propose is considerably more general than the route set concept. Specifically, we do not assume that passengers are necessarily indifferent among the alternatives within a flexible product. In addition, we allow the airline the ability to route passengers purchasing flexible products near departure rather than at the time of booking.

Although flexible products appear to offer many advantages, it is not immediately clear how availability of these products should be managed by a seller offering both flexible and specific products. An airline offering only flexible products, each of which consists of a disjoint set of specific products, could manage each flexible product as a single flight with capacity equal to the total capacity of the constituent specific products. However, when a supplier offers a combination of flexible and specific products, it is not clear what kind of booking limits or nesting structures should be used. In this paper, we show that there can be significant differences in profitability, depending on the booking control mechanism used. We also show that, under reasonable assumptions, the benefits from offering flexible products can make it worthwhile to consider them as part of the overall market offering.

To our knowledge, the problem of managing flexible products has not been previously studied. The network management problem for specific products has been extensively studied in the airline by a number of authors including Phillips (1993), Talluri and van Ryzin (1998), Gallego and van Ryzin (1997), and Bertsimas and Popescu (2003), and in the hotel context by Bitran and Gilbert (1996), among others. However, these network management models do not incorporate the possibility of flexible bookings. In fact, flexible products can be considered an extension of the classic network management problem studied by these authors in a very specific fashion: Flexible products enable customers to purchase combinations of specific products linked by a Boolean "exclusive or" operation. Each specific product is a set of one or more resources linked by a Boolean "and" operation.[3] Thus, the concept of flexible products can be seen as extending the scope of revenue management to include combinations of resources linked by an "exclusive or" operation.

In this paper, we analyze in detail the two-period, two-flight case for an airline offering a flexible product in addition to specific products. We compare different control structures for flexible and specific bookings and derive algorithms for determining booking limits. We show that, under some conditions, it is optimal for a carrier to allow overbooking when managing flexible bookings, even in the absence of no-shows or cancellations. We present a consumer choice model that includes both the demand induction and cannibalization that would result from offering a flexible product. We use simulation to compare results under the various control structures and for different pricing scenarios, and provide insights into both the demand induction and risk-pooling benefits that an airline could achieve from offering flexible products. Finally, we discuss the extension of our analysis to full airline networks with arbitrary specifications of flexible products. Although we develop our model in the context of passenger airlines, the results extend directly to any industry that accepts bookings for multiple products or services using constrained capacity or inventory.

2. Two-Product Problem

Assume that an airline has two flights, say A and B, serving the same market, e.g., from the same origin to the same destination. Passengers book in two periods. In the first period, the airline sells *flexible* product (A, B) in addition to specific products at discounted fares. In the second period the airline sells specific products A and B, but not the flexible product (A, B). At some time after the end of the first period, the airline can allocate the customers who purchased (A, B) among flights A and B as it wishes. However, they must be accommodated on either flight A or flight B or the airline pays a denied boarding penalty

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[3] This is a very general characterization; in practice, specific products usually consist of a sequence of flights connecting to an origin and a destination, whereas each flexible product would include primarily flights (or flight sequences) serving a particular market.

to each passenger denied space. In addition, specific passengers must be accommodated on the flight that they choose or the airline must pay a denied boarding penalty to each unaccommodated passenger.

Notice that our model does not allow the sale of flexible products during the second period. This reflects the fact that customers who arrive in the second period are willing to pay a premium, in the form of higher fares, for the privilege of selecting the specific product they want late in the booking process. Offering flexible products to these customers might backfire because the discount needed to induce them to purchase a flexible product can be large, and flexible sales would then cannibalize full-fare demand. Another reason not to sell flexible products during the second period is that the time lag between the booking of a flexible product and the resolution of allocation is part of what makes a flexible product an inferior product.

We will study the problem of managing bookings for both flexible products and specific products in this simple setting under two scenarios: when the airline allows overbooking and when it does not.

Define

c^A, c^B = Capacity of flights.

g = Fare paid by flexible passengers during first period.

g^A, g^B = Fare paid by specific passengers during first period.

f^A, f^B = Fare paid by specific passengers during second period.

Y = Demand for flexible passengers during first period.

Y^A, Y^B = Demands for specific passengers during first period.

D^A, D^B = Demands for specific passengers during second period.

b = Maximum number of flexible bookings accepted during first period.

b_1^A, b_1^B = Maximum number of flight specific bookings accepted during first period.

b^A, b^B = Maximum number of flight specific bookings accepted during second period.

d = Gross penalty for each passenger denied boarding.

We assume that $0 < g < \min(g^A, g^B)$. Note that we have specified the overbooking penalty d as a gross

penalty, that is it is the sum of the ill-will cost, reaccommodation cost, and direct cost paid per denied boarding. We assume for simplicity that this cost is independent of the number and the mix of denied boardings. We further assume that $d > f^j > g^j$ for $j = A, B$. The assumption that the gross denied boarding cost exceeds all fares is standard, otherwise it is optimal for a revenue-maximizing airline to set no booking limit on a fare that exceeds the denied boarding cost.

In this model, the airline needs to decide how many units to make available for flexible and specific bookings during the first period and how to manage the remaining capacity during the second period. We assume that booking limits are set at the beginning of a period and cannot change during that period. The objective is to maximize the expected revenue net of denied boarding costs. We assume that the airline does not overbook in the first period; in other words, $b_1^j \leq c^j$, $j = A, B$, $b \geq 0$ and $b + b_1^A + b_1^B \leq c^A + c^B$. This assumption is realistic because airlines typically reserve at least some capacity to satisfy the demand for higher-fare products during the second period.

The expected revenue during the first period is given by

$$g^A E \min(Y^A, b_1^A) + g^B E \min(Y^B, b_1^B) + g E \min(Y, b).$$

Let $s^j = \min(Y^j, b_1^j)$ denote the number of seats booked by flight-specific passengers, $j = A, B$ during the first period, and let $s = \min(Y, b)$ denote the number of flexible seats booked during the first period. Notice that (s^A, s^B, s) is known at the beginning of the second period. Let $r^j \equiv c^j - s^j$, $j = A, B$ denote the residual capacity of the flights at the beginning of the second period and let

$$c(s^A, s^B, s) \equiv r^A + r^B - s \geq 0$$

denote the total residual capacity at the beginning of the second period. (For brevity, we will often write c for $c(s^A, s^B, s)$.) Given the vector $r = (r^A, r^B, c)$ of residual capacities, we want to determine how many seats, $b^j \geq 0$, to make available for sale for flight $j = A, B$ during the second period. Under a static control policy, the parameters b^j, $j = A, B$ are decided at the beginning of the second period before observing D^j, $j = A, B$. Later, we will consider the case

where bookings over the second period are managed dynamically.

Define

$$H(b^A, b^B) = f^A E \min(D^A, b^A) + f^B E \min(D^B, b^B) \quad (1)$$

$$\widehat{H}(b^A, b^B, c)$$
$$= H(b^A, b^B) - dE\big(\min(D^A, b^A) + \min(D^B, b^B) - c\big)^+. \quad (2)$$

Then, when overbooking is not allowed, the optimal expected profit during the second period is given by

$$h(r^A, r^B, c) \equiv \max_{b^A, b^B} H(b^A, b^B)$$

subject to integer b^j, $j = A, B$, satisfying $0 \leq b^j \leq r^j$ and $b^A + b^B \leq c$. When overbooking is allowed, the optimal expected profit is given by

$$\hat{h}(r^A, r^B, c) \equiv \max_{b^A, b^B} \widehat{H}(b^A, b^B, c)$$

subject to $0 \leq b^j \leq r^j$, $j = A, B$. The last term in Equation (2) represents expected denied boarding costs.

If overbooking is not allowed in the second period, the first-period problem is to select (b_1^A, b_1^B, b) to maximize the expected profit

$$\pi(b_1^A, b_1^B, b)$$
$$= g^A E \min(Y^A, b_1^A) + g^B E \min(Y^B, b_1^B) + g E \min(Y, b)$$
$$+ E\big[h(c^A - \min(Y^A, b_1^A), c^B - \min(Y^B, b_1^B), c^A + c^B$$
$$- \min(Y^A, b_1^A) - \min(Y^B, b_1^B) - \min(Y, b))\big], \quad (3)$$

where the maximization is over integers satisfying $0 \leq b_1^i \leq c^i$ for $i = A, B$ such that $b_1^A + b_1^B + b \leq c^A + c^B$. If overbooking is allowed in the second period, the first-period problem is to maximize

$$\hat{\pi}(b_1^A, b_1^B, b)$$
$$= g^A E \min(Y^A, b_1^A) + g^B E \min(Y^B, b_1^B) + g E \min(Y, b)$$
$$+ E\big[\hat{h}(c^A - \min(Y^A, b_1^A), c^B - \min(Y^B, b_1^B), c^A + c^B$$
$$- \min(Y^A, b_1^A) - \min(Y^B, b_1^B) - \min(Y, b))\big]. \quad (4)$$

It is evident that $\hat{\pi} \geq \pi$ and that

$$\lim_{d \to \infty} \hat{\pi}(b_1^A, b_1^B, b) = \pi(b_1^A, b_1^B, b),$$

which implies that, for sufficiently large d, it will not be optimal to overbook in the second period, and the optimal first-period allocations will be the same with and without overbooking.

We will concentrate on solving the second-period problem and later return to address the first-period problem. Throughout, we will assume that there are no cancellations and that all booked customers show up at the gate.

2.1. Second-Period Allocation Without Overbooking

The problem without overbooking may arise either as a policy choice by the airline or as a result of a very high denied boarding cost d. Because the second-period expected profit is increasing[4] in $b^j \leq c^j - s^j$, $j = A, B$ when $b^A + b^B < c$, it follows that any optimal solution must satisfy $b^A + b^B = c$. Notice that $b^A + b^B = c$ is equivalent to assigning $r^j - b^j$ of the s first-period flexible bookings to flight $j = A, B$ at the *beginning* of the second period. Under this policy, the airline could inform flexible customers of the allocation. Later, we will discuss dynamic policies where the assignment may be done later in the booking process.

The second-period problem without overbooking is

$$h(r^a, r^b, c) = \max_{b^A, b^B} \ H(b^A, b^B)$$
$$\text{s.t.} \ \ 0 \leq b^j \leq r^j \quad \text{for } j = A, B$$
$$b^A + b^B \leq c$$
$$b^A, b^B \quad \text{integer}.$$

Note that the objective function is concave. Let $F^j(x) \equiv \Pr(D^j \geq x)$ and consider the expected marginal seat revenues

$$\text{EMSR}^j(x) \equiv f^j E \min(D^j, x) - f^j E \min(D^j, x - 1)$$
$$= f^j F^j(x) \quad \text{for } x \in \{1, \ldots, r^j\}, \ j = A, B.$$

Let $\text{EMSR}(k; r^A, r^B)$ be the kth largest value in the set $\{\text{EMSR}^j(x), 1 \leq x \leq r^j, j = A, B\}$. We will follow the convention that sums over empty sets are zero.

THEOREM 1.

$$h(r^A, r^B, c) = \sum_{k=1}^{c} \text{EMSR}(k; r^A, r^B). \quad (5)$$

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[4] We use the terms increasing and decreasing in the weak sense.

PROOF. Suppose that b^A, b^B is an optimal allocation. Then

$$h(r^A, r^B, c) = f^A E \min(D^A, b^A) + f^B E \min(D^B, b^B)$$

$$= f^A \sum_{k=1}^{b^A} F^A(k) + f^B \sum_{k=1}^{b^B} F^B(k)$$

$$= \sum_{k=1}^{b^A} \text{EMSR}^A(k) + \sum_{k=1}^{b^B} \text{EMSR}^B(k).$$

We now argue by contradiction that $\text{EMSR}^A(k); k = 1, \ldots, b^A$ and $\text{EMSR}^B(k); k = 1, \ldots, b^B$ represent, collectively, the c largest EMSR values in the set $\{\text{EMSR}^j(x), 1 \le x \le r^j, j = A, B\}$. If not, then either $\text{EMSR}^A(b^A + 1) > \text{EMSR}^B(b^B)$ or $\text{EMSR}^A(b^A) < \text{EMSR}^B(b^B + 1)$. In the former case, $(b^A + 1, b^B - 1)$ is a better allocation, and in the latter case $(b^A - 1, b^B + 1)$ is a better allocation, contradicting the optimality of (b^A, b^B). $\square$

Finding the optimal allocation is similar to a bin-packing problem that reserves c seats for the most valuable second-period bookings, as measured in terms of their EMSR values.

EXAMPLE. If $f^A = \$350$, $f^B = \$330$, D^A follows a Poisson distribution with parameter 50, D^B follows a Poisson distribution with parameter 40, $r^A = 60$, $r^B = 38$, and $c = 83$, then $(b^A, b^B) = (47, 36)$ is an optimal allocation and

$$h(60, 38, 83) = \$350 E \min(D^A, 47) + \$330 E \min(D^B, 36)$$

$$= \$27,470.09.$$

Moreover, $\text{EMSR}^A(47) = \$239.16$, $\text{EMSR}^B(36) = \$250.00$, $\text{EMSR}^A(48) = \$220.62$, and $\text{EMSR}^B(37) = 232.21$. Thus,

$$h(60 + i, 38 + j, 84) = h(60, 38, 83) + \text{EMSR}^B(37)$$

$$= \$27,702.30$$

for all nonnegative integers i and j. Finally,

$$h(60, 38, 82) = h(60, 38, 83) - \text{EMSR}^A(47) = \$27,230.93.$$

Theorem 1 suggests two simple algorithms to find an optimal allocation depending on the size of s relative to $c + s$. If $s = 0$, then $b^j = r^j$, $j = A, B$ is optimal. This is, of course, the trivial single-period solution when only specific products are offered. If s is small

relative to $c + s$, then it is efficient to seek the s smallest EMSR values and subtract those from the allocation r^j. If c is small, however, then it is more efficient to find the c largest EMSR values.

2.2. Second-Period Allocation with Overbooking

Here we consider the situation where overbooking is allowed and there is a finite value of the denied boarding penalty d. To compute $\hat{h}(r^A, r^B, c)$ in this situation we need to study the marginal expected revenue for $0 \le b^j \le r^j$, $j = A, B$. It is easy to see that an optimal solution will satisfy $b^A + b^B \ge c$ and that overbookings will occur only if $b^A + b^B > c$. The restriction to $b^j \le r^j$ is imposed without loss of generality because the expected marginal seat revenue net of overbooking costs is negative for $b^j > r^j$ because $f^j < d$.

We can write the second-period allocation problem with overbooking as

$$\hat{h}(r^A, r^B, c) = \max_{b^A, b^B} \ \widehat{H}(b^A, b^B, c)$$

$$\text{s.t.} \ \ 0 \le b^j \le r^j \ \ \text{for } j = A, B$$

$$b^A, b^B \ \ \text{integer.}$$

For $b^A \le r^A$, $b^B \le r^B$ such that $b^A + b^B > c$ we can write the different equations as

$$\Delta_A \widehat{H}(b^A, b^B, c) \equiv \widehat{H}(b^A, b^B, c) - \widehat{H}(b^A - 1, b^B, c)$$

$$= F^A(b^A)[f^A - dF^B(c + 1 - b^A)]$$

$$\text{for } b^A \le r^A \quad (6)$$

$$\Delta_B \widehat{H}(b^A, b^B, c) \equiv \widehat{H}(b^A, b^B, c) - \widehat{H}(b^A, b^B - 1, c)$$

$$= F^B(b^B)[f^B - dF^A(c + 1 - b^B)]$$

$$\text{for } b^B \le r^B. \quad (7)$$

To see why Equation (6) holds, notice that the b^Ath booking will generate marginal revenue f^A only if $D^A \ge b^A$, and will cause an additional denied boarding only if $D^A \ge b^A$ and $D^B \ge c + 1 - b^A$.

From Equations (6) and (7) we can see that the function $\widehat{H}(b^A, b^B, c)$ is unimodal with respect to b^A and with respect to b^B, although not necessarily concave in those variables. Notice further that Equation (6) is independent of b^B and Equation (7) is independent of b^A when $b^A + b^B > c$ and $b^j \le r^j$, $j = A, B$.

Because $\Delta_A \widehat{H}(b^A, b^B, c)$ is independent of b^B and is decreasing in $b^A \leq r^A$ we can find the largest $b^A \leq r^A$ such that $\Delta_A \widehat{H}(b^A, b^B, c) \geq 0$. Similarly, we can find the largest $b^B \leq r^B$ such that $\Delta_B \widehat{H}(b^A, b^B, c) \geq 0$.

We will present two ways of obtaining an optimal solution for the second-period allocation problem with overbooking. The first will start from an optimal solution without overbooking and sequentially increase the allocation (b^A, b^B) as long as the objective function $\widehat{H}(b^A, b^B, c)$ continues to increase. This approach has the advantage that it calculates the incremental value of allowing overbooking as well as the optimal allocation. The second approach is more direct because it does not require calculation of the optimal solution to the allocation problem without overbooking. However, it also does not directly compute the gain from allowing overbooking.

To develop the first algorithm, we require the following lemma:

LEMMA 1. *Let* (b^A, b^B) *be an optimal solution without overbooking. Then, for any* $d > 0$ *there is an optimal solution with overbooking,* $(\bar{b}^A, \bar{b}^B)$, *such that* $\bar{b}^j \geq b^j$ *for* $j = A, B$.

PROOF. Suppose this is not the case. Then, by interchanging the labels A and B if necessary, there exists an optimal solution of the form $(b^A - k, b^B + j + k)$ with $j \geq 0$ and $k > 0$. Notice that

$$\text{EMSR}^A(b^A - k + 1) \geq \text{EMSR}^A(b^A) \geq \text{EMSR}^B(b^B + j + k).$$

Moreover, the last quantity is strictly larger than the expected marginal revenue minus denied boarding cost of the $b^B + j + k$th booking. This shows that the allocation $(b^A - k + 1, b^B + j + k - 1)$ has higher expected net revenue than the allocation $(b^A - k, b^B + j + k)$. By repeating this argument, if necessary, we see that there exists an optimal solution of the stated form. $\square$

From Lemma 1, we know we can start our search for an optimal allocation with overbooking from an optimal allocation without overbooking. Starting at the optimal allocation without overbooking, we can increase the allocations as long as the expected net marginal revenues calculated by Equations (6) and (7) are positive. We can use this result to develop an algorithm to calculate the second-period allocations when overbooking is allowed.

Algorithm to Compute Optimal Second Period Allocations and Optimal Expected Profit with Overbooking

(1) Compute an optimal solution (b^A, b^B) to the problem without overbooking as in Theorem 1. Let $\pi = f^A E \min(b^A, D^A) + f^B E \min(b^B, D^B)$ be the corresponding expected profit.

(2) If $b^A = r^A$ and $b^B = r^b$, stop. Otherwise, calculate $\Delta_i \widehat{H}(b^A, b^B, c)$ for $i = A, B$ such that $b^i \leq r^i$ according to (6) and (7). If both are less than or equal to zero, then stop. Otherwise go to Step 3.

(3) Add the highest $\Delta_i \widehat{H}(b^A, b^B, c)$ to π and update the allocation to $(b^A, b^B) \leftarrow (b^A + 1, b^B)$ if the highest EMSR was from A, and $(b^A, b^B) \leftarrow (b^A, b^B + 1)$ if the highest EMSR was from B. Go to Step 2.

EXAMPLE (CONT'D). If $r^A = 60$, $r^B = 38$, $c = 83$, and $d = 450$, then $(\bar{b}^A, \bar{b}^B) = (48, 38)$. Note that the residual capacity at the beginning of the second period is $c = 83$ seats, whereas the optimal allocation will allow up to $48 + 38 = 86$ bookings, meaning that it is optimal for the airline to accept the possibility of up to three denied boardings. The calculation of $(\bar{b}^A, \bar{b}^B) = (48, 38)$ and the associated expected profit, $\hat{h}(60, 38, 83)$ can be illustrated in terms of the steps from the optimal solution without overbooking, $(b^A, b^B) = (47, 36)$, as shown in Table 1. In this example, the additional expected revenue net of expected denied boarding costs from allowing overbooking is equal to \$21.63 or about 0.08% of the expected revenue without overbooking.

There is another, more direct, characterization of the optimal second-period allocations with overbooking that can simplify the calculation of the optimal allocations.

THEOREM 2. *Let* x^A *be the smallest integer such that* $F^B(x) \leq f^A/d$. *Similarly, define* x^B *as the smallest integer such that* $F^A(x) \leq f^B/d$. *If* $x^A + x^B \leq c + 1$, *then an optimal*

Table 1   **Calculation of** $\hat{h}(60, 38, 83)$ **in Example**

Term	Expression	Value
"No overbooking" solution	$350E \min(D^A, 47)$ $+330E \min(D^B, 36)$	\$27,470.09
First additional seat	$(330 - 450F^A(47))F^B(37)$	\$15.84
Second additional seat	$(350 - 450F^B(36))F^A(48)$	\$5.73
Third additional seat	$(330 - 450F^A(46))F^B(38)$	\$0.06
Total		\$27,491.71

overbooking allocation is given by $\bar{b}^j = \min(r^j, c + 1 - x^j)$, $j = A, B$.

PROOF. Suppose that $x^A + x^B \leq c + 1$, then $\bar{b}^A + \bar{b}^B \geq c + 1$, so the allocation $(\bar{b}^A, \bar{b}^B)$ overbooks at least one seat. The expected marginal revenue net of overbooking costs of the $\bar{b}^j$th seat is nonnegative, and becomes negative for the $b^j + 1$st seat for $j = A, B$. Thus, it is optimal to include the $\bar{b}^j$th seat for $j = A, B$, but it is not optimal to allocate additional seats. $\square$

COROLLARY 1. *Suppose that $(\bar{b}^A, \bar{b}^B) = (c + 1 - x^A, c + 1 - x^B)$ is an optimal overbooking allocation and that $x^A + x^B \leq c$, so the allocation overbooks at least two seats. Then $(\bar{b}^A - 1, \bar{b}^B - 1)$ is an optimal overbooking allocation when we reduce the residual capacity from c to $c - 1$.*

PROOF. From Theorem 2, $(c - x^A, c - x^B) = (\bar{b}^A - 1, \bar{b}^B - 1)$ is an optimal allocation for $c - 1$. $\square$

This means that over a certain range the optimal overbooking allowance *decreases by two seats* for each additional flexible seat sold during the first period.

EXAMPLE (CONT'D). Here $x^A = 36$ and $x^B = 46$ and $x^A + x^B = 82 \leq 84$, so $(\bar{b}^A, \bar{b}^B) = (84 - 36 = 48, 84 - 46 = 38)$ is an optimal overbooking allocation that overbooks three seats. If we reduce c to 82, the new optimal allocation is $(47, 37)$. If we reduce c to 81 the new optimal allocation changes to $(46, 36)$, coinciding with the allocation without overbooking.

Theorem 2 simplifies the computation of an optimal policy, but it does not give us the expected optimal profit. However, the expected optimal profit can be computed as follows: Let (b^A, b^B) be an optimal solution when overbooking is not allowed and let $(\bar{b}^A, \bar{b}^B) \geq (b^A, b^B)$ be an optimal solution when overbooking is allowed. Then, the additional expected revenue from allowing overbooking can be calculated simply by adding up the expected marginal seat revenues, that is,

$$\sum_{k=b^A+1}^{\bar{b}^A} [f^A - dF^B(c + 1 - k)]F^A(k)$$

$$+ \sum_{k=b^B+1}^{\bar{b}^B} [f^B - dF^A(c + 1 - k)]F^B(k).$$

COROLLARY 2. *Suppose that it is optimal to allow for overbookings. Then there is an optimal allocation $(\bar{b}^A, \bar{b}^B)$ such that $\bar{b}^A$ is increasing in f^A, decreasing in d, and*

Figure 1   Feasible Policies With and Without Overbooking

without overbooking   with overbooking
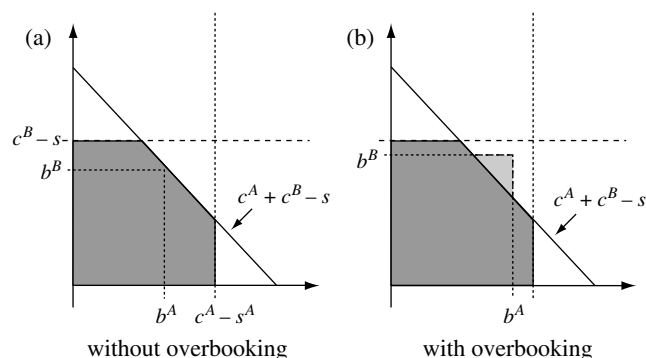
decreasing as the full fare demand D^B stochastically increases. A symmetric statement holds for $\bar{b}^B$.

Figure 1 shows why it may be optimal to set allocations that allow overbooking in the second period, even in the absence of no-shows and cancellations. The shaded region represents the available capacity for an airline that has flights with seating capacities of c^A and c^B and has already accepted s flexible-product bookings. On the one hand, if the airline decides not to allow overbooking, it is constrained to set its booking limits b^A and b^B such that $b^A + b^B = c^A + c^B - s$. On the other hand, if it allows overbooking, the number of possible booking policies is significantly expanded. Intuitively, allowing overbooking is optimal if the expected incremental gain from expanding the booking limits for the specific products outweighs the expected incremental risk of outcomes that would lead to bookings in the *overbooking region*.

2.3.   Second-Period Dynamic Allocation

In this section, we consider dynamically allocating seats in the second period. Suppose that the second period consists of T time intervals, and assume that at most one request for the high-fare product for either A or B (but not both) occurs during each time interval. This is similar to the booking model introduced by Lee and Hersh (1998). We will let

$$\tilde{h}(r^A, r^B, c) = V(0, r^A, r^B, c)$$

denote the optimal expected revenue from dynamically managing the residual capacity during the second period, where $V(t, r)$ can be calculated recursively according to

$$V(t, r) = V(t+1, r) + \sum_{j=A, B} p^j [f^j - \Delta^j V(t+1, r)]^+,$$

where $p^j \geq 0$ is the probability of a type j arrival, $p^A + p^B \leq 1$, $\Delta^j V(t, r) = V(t, r) - V(t, r - e^j)$, $e^A = (1, 0, 1)$, $e^B = (0, 1, 1)$, $V(T+1, r) = 0$, $V(t, r) = 0$ if the total residual capacity, i.e., the third component of r, is zero, and $V(t, r) = -\infty$ if either the first or the second component of r is negative. Note that this formulation assumes that a booking request of type j; $j = A, B$ will be accepted in period $t = 0, 1, \ldots, T$ only if $f^j \geq \Delta^j V(t+1, r)$.

We remark that it is easy to expand the dynamic formulation to time-varying arrival probabilities, time-varying fares, and to the case where flexible sales are allowed over the entire horizon, or a portion thereof. We also note that the second-period dynamic allocation approach will never result in overbooking.

2.4. First-Period Problem

We now turn to the problem of calculating the first-period allocations. At the beginning of the first period we need to determine the booking limits (b_1^A, b_1^B, b). Recall that the demands for discount flight-specific products for A and B and flexible products in the first period are denoted by Y^A, Y^B, and Y, respectively, with corresponding fares g^A, g^B, and g. Define $G^i(x) \equiv \Pr(Y^i \geq x)$ for $i = A, B$ and $G(x) \equiv \Pr(Y \geq x)$. Assume that overbooking is not allowed in the second period. For $r = (r^A, r^B, c) \geq (1, 1, 1)$, a simple consequence of Theorem 1 gives

$$\Delta^A h(r) \equiv h(r) - h(r - e^A)$$
$$= \max(\text{EMSR}^A(r^A), \text{EMSR}(c; r^A, r^B))$$
$$\Delta^B h(r) \equiv h(r) - h(r - e^B)$$
$$= \max(\text{EMSR}^B(r^B), \text{EMSR}(c; r^A, r^B))$$
$$\Delta^F h(r) \equiv h(r) - h(r - e^F) = \text{EMSR}(c; r^A, r^B).$$

Recall also that $\pi(b_1^A, b_1^B, b)$ is the expected total profit across both periods as specified in Equation (3). Define the marginal profits by

$$\Delta_A \pi(b_1^A, b_1^B, b) \equiv \pi(b_1^A, b_1^B, b) - \pi(b_1^A - 1, b_1^B, b)$$
$$\text{for } b_1^A \geq 1$$
$$\Delta_B \pi(b_1^A, b_1^B, b) \equiv \pi(b_1^A, b_1^B, b) - \pi(b_1^A, b_1^B - 1, b)$$
$$\text{for } b_1^B \geq 1$$
$$\Delta \pi(b_1^A, b_1^B, b) \equiv \pi(b_1^A, b_1^B, b) - \pi(b_1^A, b_1^B, b - 1)$$
$$\text{for } b \geq 1.$$

Then, we have, as an immediate corollary to Theorem 1

COROLLARY 3.

$$\Delta_A \pi(b_1^A, b_1^B, b) = \{g^A - E_{r^B, c}[\Delta^A h(c^A + 1 - b_1^A, r^B, c+1)]\}$$
$$\cdot G^A(b_1^A)$$
$$= \{g^A - E_{r^B, c}[\max(\text{EMSR}^A(c^A + 1 - b_1^A),$$
$$\text{EMSR}(c+1; c^A - b_1^A + 1, r^B))]\} G^A(b_1^A)$$

$$\Delta_B \pi(b_1^A, b_1^B, b) = \{g^B - E_{r^A, c}[\Delta^B h(r^A, c^B + 1 - b_1^B, c+1)]\}$$
$$\cdot G^B(b_1^B)$$
$$= \{g^B - E_{r^A, c}[\max(\text{EMSR}^B(c^B + 1 - b_1^B),$$
$$\text{EMSR}(c+1; r^A, c^B + 1 - b_1^B))]\} G^B(b_1^B)$$

$$\Delta \pi(b_1^A, b_1^B, b) = \{g - E_{r^A, r^B}[\Delta^F h(r^A, r^B, r^A + r^B + 1 - b)]\}$$
$$\cdot G(b)$$
$$= \{g - E_{r^A, r^B}$$
$$\cdot [\text{EMSR}(r^A + r^B + 1 - b; r^A, r^B)]\} G(b).$$

Let us start by analyzing the case of high EMSR values. If $\text{EMSR}^j(c^j) > g^j > g$ for $j = A, B$, then it is easy to verify that $(b_1^A, b_1^B, b) = (0, 0, 0)$ satisfies the first-order conditions. Consequently, it is only when the EMSR values fall below g^A and g^B that we start seeing action during the first period. To gain some intuition, we will now study two special cases. The first is the traditional case where we do not offer flexible products. In this case, $b = 0$, so

$$\text{EMSR}(r^A + r^B; r^A, r^B) = \min(\text{EMSR}^A(r^A), \text{EMSR}^B(r^B))$$
$$\leq f^j F^j(r^j).$$

This implies that

$$\Delta^j \pi(b_1^A, b_1^B, 0) = [g^j - f^j F^j(c^j + 1 - b_1^j)] G^j(b_1^j),$$

so the traditional EMSR rule applies; namely to make $b_1^j = (c^j - x^j)^+$ seats available for sale at fare g^j, where x^j is the largest integer such that $f^j F^j(x^j) \geq g^j$ for $j = A, B$.

Consider now, the case where only flexible products are sold during the first period. In this case $b_1^j = 0$ for $j = A, B$, then

$$\Delta \pi(0, 0, b) = [g - \text{EMSR}(c^A + c^B + 1 - b, c^A, c^B)] G(b).$$

This suggests that we set $b = (c^A + c^B - x)^+$ where x is the largest number such that $\text{EMSR}(x; c^A, c^B) \geq g$.

The challenge is to extend the analysis to the case when we allow both the forward selling of specific and flexible products. One reasonable point of departure is the traditional EMSR solution that reserves x^j seats for specific demands for flight j during the second period. This makes $b_1^j = (c^j - x^j)^+$ seats available for sale at g^j. Notice that if the demands Y^j, $j = A, B$ were such that $G^j(b_1^j) = 1$, then we can sell all of the b_1^j seats at g^j and we would not want to offer the flexible product. When the demands at g^j are weaker, we may end up selling $s^j \ll b_1^j$ seats and we would regret not allowing the sales of these seats at g. This suggests that selling flexible seats is only interesting when the demands Y^j, $j = A, B$ are not very strong.

The following proposition gives an upper bound on b_1^j, $j = A, B$:

PROPOSITION 1. $b_1^j \leq (c^j - x^j)^+$ where x^j is the largest integer such that $f^j F^f(x^j) \geq g^j$ for $j = A, B$.

PROOF. Consider the gradient $\Delta_A h$ at $(c^A - x^A)^+ + 1$. Then

$$\Delta^A \pi\big((c^A - x^A)^+ + 1, (c^B - x^B)^+, b\big)$$
$$\leq \big[g^A - f^A F^A(\min(c^A, x^A))\big] G^A(b_1^A)$$
$$\leq \big[g^A - f^A F^A(x^A)\big] G^A(b_1^A)$$
$$\leq 0.$$

A similar argument holds for flight B. □

Proposition 1 suggests a simple heuristic. Start with $b_1^j = (c^j - x^j)^+$, $j = A, B$ and search for the optimal number of flexible bookings to allow during the first period. The answer is the largest b such that

$$E_{r^A, r^B} \text{EMSR}(r^A + r^B + 1 - b; r^A, r^B) \leq g.$$

From this point, we can iterate between reducing the values b_1^j, $j = A, B$ for a fixed b and increasing b for fixed b_1^j, $j = A, B$ until total expected profit is no longer increasing. Our simulations indicate that this heuristic converges very quickly, usually after two or three iterations, to what appears to be a globally optimal solution.[5] We use a similar heuristic for the case where overbookings are allowed in the second period, and when second-period bookings are managed dynamically.

[5] Our simulations also show that expected revenue is very flat around this locally optimal point.

3. Numerical Results

In this section, we present numerical results for various settings of the problem parameters with and without overbooking. The purpose of these simulations is to gain insight into the benefits that an airline might achieve from flexible products, the relative magnitude of the risk pooling and demand induction benefits of flexible products, and the benefits of allowing overbooking.

3.1. Risk-Pooling Benefits

In the first set of simulations we focus on estimating the risk-pooling benefits from flexible products. We consider two flights A and B with identical capacities, $c^A = c^B = 100$ and two booking periods. In the second period, only full-fare specific bookings are received. The full fare is $200 and full-fare demand for each flight follows independent Poisson distributions with $\lambda_2^A = 75$ and $\lambda_2^B = 25$. We will compare the revenue obtained from the case when the airline offers *only* specific products in the first period to the case where the airline offers *only* flexible products in the first period. We will hold the expected total demand for the first constant in both cases so that the difference in revenue between the two cases can be considered a measure of the risk-pooling benefits from offering flexible products.

In the base case, we assume that only specific products are offered during the first period at a discount (i.e., no flexible products are offered). Each discount-specific passenger pays $150 and demands follow independent Poisson distributions with $\lambda_1^A = 80$ and $\lambda_1^B = 40$. In this case, the optimal booking limits for the two flights are $b^A = 31$ and $b^B = 78$ and the associated expected total revenue for both flights is $29,178.

We compare the base case to the situation in which only the flexible product is offered in the first period (i.e., no specific products are offered in the first period). We assume that the demand for the flexible product is the sum of the low-fare demands for the specific products. More precisely, we assume that the demand for the flexible product is Poisson with mean $\lambda_1^f = \lambda_1^A + \lambda_1^B = 120$. Because the flexible product is inferior to the specific products, we anticipate that we would have to sell it at a discounted price to keep the demand at the same level as it was. Suppose that to achieve this we need to sell the flexible product at

Table 2   Expected Revenue for Different Flexible Fare Levels, Assuming No Demand Induction

Alpha	Static control revenue ($)	Change from base (%)	Dynamic control revenue ($)	Change from base (%)
1	34,123	17.0	34,306	17.6
0.9	32,516	11.4	32,774	12.3
0.8	30,944	6.1	31,224	7.0
0.7	29,410	0.8	29,701	1.8
0.6	27,914	−4.3	28,205	−3.3

Note. The base case is offering only discount flight-specific products in the first period with a corresponding total revenue of $29,178.

fare $f^f = \$150\alpha$ where $\alpha \leq 1$. We will compute the expected profit for different αs and compare this to the expected profit of offering only specific products during the first period. The difference can be considered a measure of the benefit of risk pooling net of the discount needed to keep the aggregate demand constant.

The *Static Control Revenue* column in Table 2 shows the expected maximum revenue that could be gained from both flights assuming that only flexible products are offered in the first period and that static control without overbooking is applied to full-fare bookings in the second period. In other words, booking limits b^A and b^B are set optimally at the beginning of the second period with $b^A + b^B = c^A + c^B - s$ where s is the number of flexible bookings accepted in the first period. Full-fare bookings for each flight in the second period are then given by $\min[b^i, D^i]$ for $i = A, B$, where D^i is unconstrained demand. The *Dynamic Control Revenue* column shows the expected maximum revenue that could be achieved from full dynamic control of second-period full-fare bookings using the dynamic program described in §2.3. Revenue under both control mechanisms is compared against the base case under which specific products are offered at a discount in the first period. As expected, the expected total revenue from dynamic control of full-fare bookings is greater than that achieved from static control. It is interesting that in these cases, however, the vast majority of the benefits from flexible products can be achieved through static control.

Table 2 shows that the risk-pooling benefits provided by flexible products can be significant, even in the absence of any induced demand. Under static control, offering flexible products in the first period provides higher revenue than offering specific products, as long as the fare for the flexible products is greater than 70% of the specific fare—assuming that total expected demand remains the same.

3.2. Demand Induction and Cannibalization

For a more realistic estimate of the potential benefits of offering flexible products, we simulated the case when flexible products stimulated higher demand but also cannibalized demand from discount-specific products. To simulate the effect of offering a flexible product, we use a simple consumer-choice model that estimates both demand induction and cannibalization in a consistent fashion. Specifically, we assume that the fraction of buyers with a maximum willingness-to-pay (w.t.p.) for specific products has a joint distribution over $\Re_+^2$ of $g(w^A, w^B)$. We further assume that the total number of buyers is a Poisson random variable with parameter λ and that the w.t.p. distribution is independent of the total number of buyers. Finally, we assume that, for each customer, the maximum w.t.p. for the flexible product is a function of his w.t.p.s for the specific products according to

$$w(w^A, w^B) = pw^A + (1-p)w^B - \rho,$$

where p is the customer's probability that he will be assigned to flight A and $\rho \geq 0$ is his reduction in w.t.p. for the flexible product. Conceptually, ρ is the value of information that the buyer of the flexible product would pay to know to which flight she would be assigned at the time of booking. In a fully general model, both p and ρ would be random variables, possibly correlated with w^A and w^B. However, for simplicity we assume that $p = 1/2$ (the maximum-entropy assumption), and that both p and ρ are constant across the population.

For this model, we have assumed no recapture among products. That is, if a customer does not find his first choice available, he does not purchase. Although this is not fully realistic, it simplifies calculations and is fairly standard in revenue management analysis. It is also a conservative assumption—it tends to reduce the benefits of offering the flexible product because we have assumed that the customers who seek to buy the flexible product but cannot because of the booking limit are lost, whereas in reality some of them would be willing to buy the specific products.

For these simulations, we set the flight capacities at $c^A = c^B = 100$. As before, we consider two periods.

Table 3   Simulation Results for $\rho = 10$

Flexible fare ($)	λ_1^A	λ_1^B	λ_1^f	b^A	b^B	b	Total revenue ($)	Change (%)
60.00	4.82	0.42	299.23	14	4	88	25,495	−12.73
70.00	14.36	4.75	244.09	26	12	75	27,450	−6.03
80.00	24.61	9.78	191.81	31	18	65	29,195	−0.06
90.00	34.86	14.82	145.21	31	19	60	30,335	3.84
100.00	45.12	19.85	104.30	31	22	56	31,187	6.76
110.00	55.37	24.89	69.09	31	26	52	31,925	9.29
117.54	63.10	28.69	46.31	31	36	48	32,257	10.42
120.00	65.62	29.93	39.57	31	42	48	32,126	9.97
130.00	75.88	34.96	15.74	31	56	36	30,506	4.43
Base	83.74	40.00	0	31	78	0	29,212	0.00

In the first period, both flexible products and discount specific products are offered. The total population of buyers in the first period has mean $\lambda = 444$. The w.t.p. of buyers for flights A and B are given by independent uniform distributions on $(0, W^A)$ and $(0, W^B)$, respectively. The appendix describes how total demand for the flexible product and the two specific products, including both induction and cannibalization effects, can be calculated for this model. For our simulation, we set $W^A = 186$ and $W^B = 168$. In the second period, only full-fare specific products can book. The full fares for each flight are $200 and the discount fares $150. The full-fare demands were assumed Poisson with parameters $\lambda_2^A = 75$ and $\lambda_2^B = 25$.

Tables 3 and 4 show the results for $\rho = \$10$ and $\rho = \$30$, respectively. λ^i for $i = A, B, f$ are the mean demands for each product including induction and cannibalization. b^A, b^B, and b are the optimal booking first-period booking limits for A, B, and the flexible product respectively. In each case, offering flexible products at a very low fare leads to a loss in total
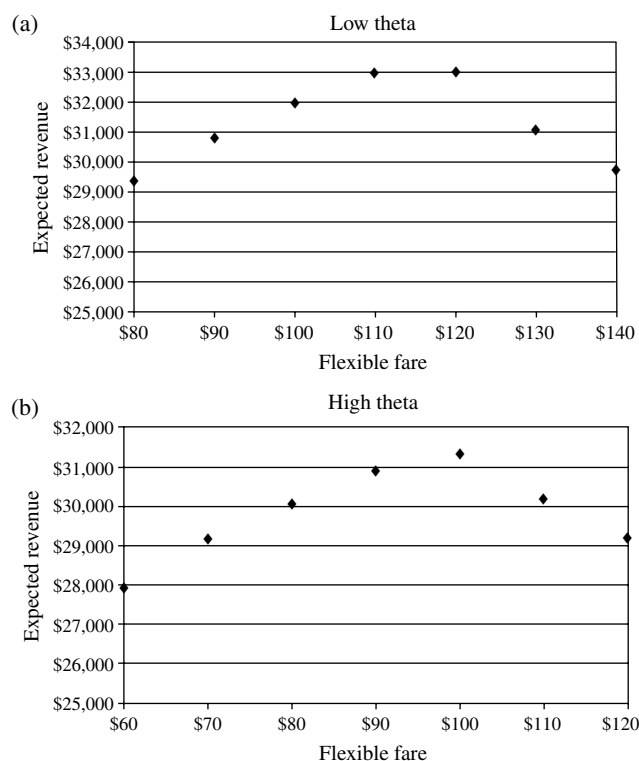
revenue because the loss from cannibalization exceeds the gain from demand induction. However, at a sufficiently high fare, offering flexibles begins to show positive benefits. These benefits begin to decline when the flexible fare becomes high enough that it is no longer inducing enough new demand to outweigh cannibalization. When the flexible fare is equal to $150 − \rho$, the benefit from offering flexibles drops to zero, because they no longer induce any additional demand. Total expected revenue as a function of flexible fare for both cases is shown in Figure 2.

We note that, for $\rho = 10$, the expected demand for all three products at any flexible fare g is the same as the expected demands for $\rho = 30$ at a fare $g - \$20.00$. However, for $\rho = 10$ the value of b is higher at g then for $\rho = 30$ at $g - \$20.00$, because flexibles are relatively more valuable at the higher fare. Consequently, the maximum achievable expected revenue is higher with the lower value of ρ. Similar reasoning shows that, for this choice model, maximum achievable expected revenue from offering flexible products is a decreasing function of ρ.

Figure 2   Expected Revenue as a Function of Flexible Fares

(a) Low theta

(b) High theta

Table 4   Simulation Results for $\rho = 30$

Flexible fare ($)	λ_1^A	λ_1^B	λ_1^f	b^A	b^B	b	Total revenue ($)	Change (%)
50.00	14.36	4.75	244.09	27	13	71	25,986	−11.05
60.00	24.61	9.78	191.81	31	19	64	27,917	−4.43
70.00	34.86	14.82	145.21	31	21	58	29,156	−0.19
80.00	45.12	19.85	104.30	31	23	54	30,089	3.00
90.00	55.37	24.89	69.09	31	25	51	30,878	5.70
97.54	63.10	28.69	46.31	31	37	47	31,368	7.38
100.00	65.62	29.93	39.57	31	42	46	31,344	7.30
110.00	75.88	34.96	15.74	31	56	36	30,191	3.35
Base	83.74	40.00	0	31	78	0	29,212	0.00

4. Summary and Extensions

In this paper we introduced the concept of flexible products and derived some revenue-management approaches for a supplier offering both flexible and specific products. We showed that, under reasonable assumptions, optimal booking limits can exceed the total seating capacity available, even in the absence of no-shows or cancellations. We derived simple algorithms for computing booking limits on both specific and flexible products in the two-flight, two-period case, both when overbooking is allowed and when it is not. We also formulated a dynamic program for optimal acceptance of flexible and specific bookings on a booking-by-booking basis. We used simulation to illustrate the potential benefits of flexible products both in terms of pure risk pooling, and under a consumer-choice model that represented both demand induction and cannibalization. Although managing flexible products adds complexity to the revenue management problem, our simulation indicates that offering flexible products can significantly improve profitability. The increased profitability comes from the following two sources.

• The lower price of the flexible product attracts additional customers who would otherwise not choose to purchase.

• Flexible products enable companies to wait for uncertainty on specific product demand to be resolved before the flexibles are assigned to products. This enables better usage of capacity.

Our simulations suggest that, under reasonable assumptions, the benefits of offering flexible bookings could be considerable—even when cannibalization from discount-specific products is included. The benefits vary widely depending on the unconstrained demand for specific products and how specific demand is distributed between flight alternatives. The results indicate that flexible products would have the highest benefit when the constituent-specific products have total demand that is low relative to capacity and when specific demand is unevenly distributed among flights (again, relative to capacity). For the parameters we chose, the value of overbooking was quite small relative to total revenue. Further research is needed to determine if this a general result.

The most immediate area for research is to extend the approach to a full network consisting of many flexible and specific products. Gallego et al. (2004) consider a fluid model over a network for both the independent demand model and for a class of consumer choice model. This model considers the choice of which flexible products should be offered before departure, as well as the booking limits that should be set in order to maximize expected revenue. The problem reduces to a linear program that can be efficiently solved by column generation for an important class of choice models. Gallego et al. (2004) also shows how the benefits of offering flexible products changes as a function of total demand for a network with fixed capacity.

Further issues for future research include the following.

(1) *Incorporation of General Consumer Choice Models.* A flexible product is not only an economic substitute for each of the specific products it contains—those specific products are also substitutes for each other. Thus, we could expect that closing any of the availabilities of the products might increase demand seen for the others. The implications of such substitutability for optimal revenue management of single flights has been studied to some extent (see Belobaba and Weatherford 1996, Andersson 1998, Talluri and van Ryzin 2004) but much less is known about the impacts of cross-flight substitution of specific products (see Belobaba and Hopperstadt 1999). An alternative model has been proposed by Gale and Holmes (1992, 1993), who consider a model under which consumers are initially uncertain about which flight they would prefer, but whose uncertainty resolves at the beginning of the second period.

A shortcoming of our consumer choice model is the implication that customers whose first choice would be to purchase the flexible product but who are not able to do so do not purchase at all. In reality, some fraction of those customers (in fact, all of those who had been cannibalized from the discount specifics) would be willing to purchase one of the specific products. Our model is consistent with the assumption of the majority of previous revenue management studies that demand for all products is independent of which other products and fare classes are open and closed. A more-realistic model would explicitly represent the increased demand for specific products once the flexible product was closed. Since this can only

result in higher revenue, the benefits of offering flexibles under the more-realistic model will only increase. The benefits of offering flexible products under a general model of consumer demand on a network is addressed in Gallego et al. (2004).

(2) *Competitive Models.* In this paper, we have assumed that offering low-price flexible products can induce additional demand. In reality, much of this induced demand would be drawn from competitors who are not offering flexible products. In this case, a competitor who is unable to offer flexible products (due to a limitation in his booking system, say), might retaliate by lowering his own specific fares. We could also consider the case of two competing carriers: Both can offer flexible products in a market but one has more flight frequencies. Does the ability to offer flexible products favor the carrier with greater flight frequency? If so, by how much and to what extent can the other carrier overcome this advantage by lower prices?

(3) *Alternative Control Structures.* We have investigated the benefits of offering flexible bookings when they are managed by setting a limit on their total availability. We believe this is the control structure that would be easiest for suppliers to implement, given existing reservation systems. Furthermore, we would expect that different fare classes for the same flexible product would be nested.

However, a simple limit on flexible bookings is not necessarily the best approach in all cases. In some situations more revenue might be generated through an alternative nesting approach. For example, in *nesting by fare* all products, both specifics and flexibles, could be nested on each leg according to total fare with a flexible product closing entirely if it is closed on any of its constituent legs. Although this approach might generate additional revenue in some cases, it easy to construct realistic cases in which it is inferior to the booking limit approach. For example, if there is a high level of anticipated full-fare demand for one specific product (say A) and low level for another (say B) it might be optimal to keep selling the flexible product consisting of A and B even if all discount products on A are closed—including those with higher fares. Comparison of alternative control structures for flexible products and determination of which one is "best" is an open issue, with the caveat that all such control structures are always inferior to the ideal of dynamically managing booking requests.

(4) *Variations.* In this model, we have assumed that the airline has complete discretion on which specific product to assign to a flexible customer. One variation would be to allow the flexible passenger to rank her alternatives. The airline could use one of a variety of schemes to match flexible customers with available flights in a way that best accommodates the customer's stated preferences. Alternatively, airlines could book the flexible passengers to a flight at the time of booking, or any time thereafter. This is the approach investigated by Talluri (2001) and actually used by the discount sales brokers Hotwire and Priceline. This approach unquestionably makes the flexible product more appealing to consumers, but this is a two-edged sword—anything that makes the flexible product more appealing increases cannibalization from the higher-fare specific products. Understanding how these tradeoffs would work in order to design the right portfolio of flexible products for an airline to offer is an open research topic.

Finally, although we have presented our analysis of flexible products entirely in a passenger airline context, we believe that the concept is applicable to any seller of products or services who uses different elements of capacity and faces uncertain demand. For any such manufacturer or service provider, offering flexible products has the potential to both increase demand and better balance demand with capacity. We are currently investigating the application of the flexible product concept to contract manufacturing, which shares these characteristics.

Acknowledgments

Appendix. Consumer Choice Model

The consumer-choice model assumes that fraction of buyers with maximum w.t.p. for specific products A and B has a joint distribution over $\Re_+^2$ of $g(w^A, w^B)$. We further assume that the total number of buyers is a Poisson random variable with parameter λ, and that the w.t.p. distribution is independent of the total number of buyers. Let f^i be the

discount fare for flight $i = A, B$. Then, when only specific products A and B are offered (i.e., no flexibles):

$$\lambda_1^A(f^A, f^B) = \lambda_1 \int_{f^A}^{\infty} \int_0^{w^A - f^A + f^B} g(w^A, w^B) \, dw^A \, dw^B,$$

where $\lambda_1^i(f^A, f^B)$ is the Poisson parameter for demand for flight $i = A, B$. A similar expression holds for $\lambda_1^B(f^A, f^B)$. We assume that w.t.p. follows a joint uniform distribution between 0 and W^i for $i = A, B$. Let $S^i = W^i - f^i$ be the maximum surplus for $i = A, B$. We will assume that $S^i > 0$. Then

$$\lambda_1^A(f^A, f^B) = \frac{\lambda_1\left[(S^A + f^B)S^A - 0.5(S^A)^2 - 0.5(S^A - S^B)_+^2\right]}{W^A W^B}$$

$$\lambda_1^B(f^A, f^B) = \frac{\lambda_1\left[(S^B + f^A)S^B - 0.5(S^B)^2 - 0.5(S^B - S^A)_+^2\right]}{W^A W^B}.$$

Assume now that we add a flexible product with fare $f < \min(f^A, f^B)$. We assume that the w.t.p. for the flexible product for a customer with specific w.t.p.s w^A, w^B is $w(w^A, w^B) = pw^A + (1-p)w^B - \rho$, where p is the customer's probability that he will be assigned to flight A. ρ is the reduction in w.t.p. for the flexible product. It is the *value of information* that the buyer of the flexible product would pay to know which flight he would be assigned to at the time of booking rather than later. In a fully general model, both p and ρ would be random variables, possibly correlated with w^A and w^B. For this simple model, however, we will assume that $p = 1/2$ (the maximum entropy assumption) and that ρ is constant across the population.

Now we can estimate the magnitude of the demand induced by the flexibles as well as the amount of demand cannibalized from each of the specific products. The induced portion of demand for the flexibles is given by the w.t.p.s that simultaneously satisfy the inequalities

$$w^A + w^B > 2(f + \rho), \quad w^A < f^A, \quad w^B < f^B.$$

The mean induced demand is given by the integral of g over this region times λ_1. Define $\Omega = f^A + f^B - 2(f + \rho)$. Note that, for nonzero induction, we must have $\Omega > 0$. In what follows we will assume that this condition holds. Let $\hat{\lambda}_1$ be the mean of *induced* flexible demand. Then, in the uniform case (recalling that $S^i > 0$ for $i = 1, 2$),

$$\hat{\lambda}_1 W^A W^B = \begin{cases} \lambda_1[f^A f^B - 2(f + \rho)^2] \\ \quad \text{if } 2(f + \rho) \leq \min(f^A, f^B) \\ \lambda_1[\min(f^A, f^B)(2\Omega - \min(f^A, f^B))]/2 \\ \quad \text{if } \min(f^A, f^B) < 2(f + \rho) \leq \max(f^A, f^B) \\ \lambda_1 \Omega^2 / 2 \quad \text{otherwise.} \end{cases}$$

We now turn to cannibalization. Given that we are offering a flexible product, the customers cannibalized from specific product A are those whose w.t.p.s satisfy the following inequalities:

$$w^A - f^A + f^B \geq w^B \geq w^A - 2(f^A - f - \rho), \quad w^A \geq f^A.$$

Similarly, cannibalization from B is given by

$$w^A + 2(f^B - f - \rho) \geq w^B \geq w^A - f^A + f^B, \quad w^B \geq f^B.$$

We now determine the amount of cannibalization that would occur assuming a uniform distribution of w.t.p.s. Let $\hat{\lambda}_1^i$ be the mean cannibalization from specific product $i = A, B$. Then,

$$\hat{\lambda}_1^A = \lambda_1\big[\Omega S^A - 0.5 \min[(S^A)^2, (\Omega - f^B)_+^2] - 0.5(S^A - S^B)_+^2$$
$$+ 0.5(S^A - \Omega - S^B)_+^2\big]/(W^A W^B),$$

$$\hat{\lambda}_1^B = \lambda_1\big[\Omega S^B - 0.5 \min[(S^B)^2, (\Omega - f^A)_+^2] - 0.5(S^B - S^A)_+^2$$
$$+ 0.5(S^B - \Omega - S^A)_+^2\big]/(W^A W^B).$$

Then, for any value of f, we can write

$$\lambda_1(f) = \hat{\lambda}_1 + \hat{\lambda}_1^A + \hat{\lambda}_1^B$$
$$\lambda_1^A(f) = \lambda_1^A(f^A, f^B) - \hat{\lambda}_1^A$$
$$\lambda_1^B(f) = \lambda_1^B(f^A, f^B) - \hat{\lambda}_1^B,$$

where $\lambda_1(f)$ is the mean flexible demand as a function of f and $\lambda_1^i(f)$ for $i = A, B$ are the specific demands after cannibalization, given f.

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