



Management Science

Publication details, including instructions for authors and subscription information:
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To cite this article:

Arash Asadpour, Hamid Nazerzadeh (2016) Maximizing Stochastic Monotone Submodular Functions. *Management Science* 62(8):2374-2391. <http://dx.doi.org/10.1287/mnsc.2015.2254>

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Maximizing Stochastic Monotone Submodular Functions

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We study the problem of maximizing a stochastic monotone submodular function with respect to a matroid constraint. Because of the presence of diminishing marginal values in real-world problems, our model can capture the effect of stochasticity in a wide range of applications. We show that the adaptivity gap—the ratio between the values of optimal adaptive and optimal nonadaptive policies—is bounded and is equal to $e/(e-1)$. We propose a polynomial-time nonadaptive policy that achieves this bound. We also present an adaptive myopic policy that obtains at least half of the optimal value. Furthermore, when the matroid is uniform, the myopic policy achieves the optimal approximation ratio of $1 - 1/e$.

Keywords: submodular maximization; stochastic optimization; adaptivity gap; influence spread in social networks

History: Received May 6, 2014; accepted April 11, 2015, by Dimitris Bertsimas and Yinyu Ye, optimization. Published online in *Articles in Advance* December 18, 2015.

1. Introduction

The problem of maximizing submodular functions has been extensively studied in operations research and computer science. For a ground set \mathcal{A} , the set function $f: 2^{\mathcal{A}} \rightarrow \mathbb{R}$ is called *submodular* if for any two subsets $S, T \subseteq \mathcal{A}$ we have $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$. An equivalent definition is that the marginal value of adding any element is diminishing. In other words, for any $S \subseteq T \subseteq \mathcal{A}$ and $j \in \mathcal{A}$, we have $f(T \cup \{j\}) - f(T) \leq f(S \cup \{j\}) - f(S)$. Also, a set function f is called *monotone* if for any two subsets $S \subseteq T \subseteq \mathcal{A}$ we have $f(S) \leq f(T)$.

Because of the presence of diminishing marginal values, a wide range of optimization problems that arise in the real world can be modeled as maximizing a monotone submodular function with respect to some feasibility constraints. One instance is the welfare maximization problem (see, e.g., Dobzinski and Schapira 2006, Vondrák 2008, Feige 2009), which is to find an optimal allocation of resources to agents where the utilities of the agents are submodular. Here, submodularity corresponds to the law of diminishing returns in the economy (see Samuelson and Nordhaus 2009).

Another application of this problem is capital budgeting, in which a risk-averse investor with a limited budget is interested in finding the optimal investment portfolio over different projects (see, e.g., Weingartner 1963, Ahmed and Atamtürk 2011). Risk aversion can be modeled by concave utility functions where, informally speaking, the utility of a certain expectation is higher than the expectation of the utilities

of the corresponding uncertain outcomes. As argued by Ahmed and Atamtürk (2011), the utility function of a risk-averse investor over a set of investment possibilities can be modeled by submodular functions. Such functions are also nonnegative and monotone by nature.

Another example is the problem of viral marketing and maximizing influence through the network (see, e.g., Kempe et al. 2003, Mossel and Roch 2010), where the goal is to choose an initial “active” set of people and to provide them with free coupons or promotions so as to maximize the spread of a technology or behavior in a social network. It is well known that under a wide variety of models for influence propagation in networks (e.g., cascade model of Kempe et al. 2003) the expected size of the final cascade is a submodular function of the set of initially activated individuals. Also, as a result of budget limitations, the number of people that we can directly target is bounded. Hence, the problem of maximizing influence can be seen as a maximizing submodular function problem subject to cardinality constraints.

Yet another example is the problem of optimal placement of sensors for environmental monitoring (see, e.g., Krause and Guestrin 2005, 2007), where the objective is to place sensors in the environment in order to most effectively reduce uncertainty in observations. This problem can be modeled by entropy minimization, and because of concavity of the entropy function, it is a special case of submodular optimization.

For these problems and many others, the constraints can be modeled by a matroid. A finite *matroid* \mathcal{M} is defined by a pair $(\mathcal{A}, \mathcal{I})$, where \mathcal{A} is a ground set of elements and \mathcal{I} is a collection of subsets of \mathcal{A} (called the *independent sets*) with the following properties:

1. Every subset of an independent set is independent; i.e., if $T \in \mathcal{I}$ and $S \subseteq T$, then $S \in \mathcal{I}$.
2. If S and T are two independent sets and T has more elements than S , then there exists an element in T that is not in S and when added to S still gives an independent set.

Two important special cases are *uniform matroid* and *partition matroid*. In a uniform matroid, all the subsets of \mathcal{A} of size at most k , for a given k , are independent. Uniform matroids represent cardinality constraints. A partition matroid is defined over a partition of set \mathcal{A} , where every independent set includes at most one element from each set in the partition.

The celebrated result of Nemhauser et al. (1978) shows that for maximizing nonnegative monotone submodular functions over uniform matroids, the greedy algorithm gives a $(1 - 1/e \approx 0.632)$ -approximation of the optimal solution. Later, Fisher et al. (1978) showed that for optimizing over matroids, the approximation ratio of the greedy algorithm is $\frac{1}{2}$. More recently, Călinescu et al. (2011) proposed a better approximation algorithm with a ratio of $1 - 1/e$. It also has been shown that this factor is optimal (in the value oracle model, where we only have access to the values of $f(S)$ for all the subsets S of the ground set) if only a polynomial number of queries is allowed (see Nemhauser and Wolsey 1978, Feige 1998).

However, all these algorithms are designed for deterministic environments. In practice, one must deal with the stochasticity caused by the uncertain nature of the problem's parameters, the incomplete information about the environment, and so on. For instance, in welfare maximization, the quality of the resources may be unknown in advance, or in the capital budgeting problem, some projects taken by an investor may fail as a result of unexpected events in the market. Yet another example is from viral marketing, where some people whom we target might not adopt the behavior (e.g., receive a coupon or a promotion but not purchase the product). Also, in the environmental monitoring example, it is expected that a nonnegligible fraction of sensors might not be functioning properly for various reasons.

All these possibilities motivate the study of submodular maximization in the stochastic setting. In such settings, the outcome of the elements in the selected set is not known in advance and will only be discovered after they are chosen.

1.1. Setting

Here, we provide a brief overview of our setting. Later in §2, we will discuss the model in detail. We

study the problem of maximizing a monotone submodular function f over a set of n random variables, namely, $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$. A policy π picks the elements one by one (perhaps, based on the realized value of the previous elements) until it stops. Once π stops, the state of the world is a random vector $\Theta^\pi = (\theta_1, \theta_2, \dots, \theta_n)$, where θ_i denotes the realization of X_i if i is picked by the policy and is equal to 0 otherwise. The objective of stochastic submodular maximization is to optimize the expected value of a policy, i.e., maximize $\mathbb{E}[f(\Theta^\pi)]$, subject to feasibility. We model the feasibility constraint using a matroid. For a given matroid \mathcal{M} defined on the ground set of the aforementioned random variable set \mathcal{A} , a policy π is called feasible if the subset of random variables it picks is always an independent set of \mathcal{M} .

We consider both *adaptive* (general) and *nonadaptive* policies. In adaptive policies, at each point in time all the information regarding the previous actions of the policy is known. In other words, the policy has access to the actual realized value of all the elements it has picked so far. By contrast, nonadaptive policies do not have access to such information and should make their decisions (about which random variables to pick) before observing the outcome of any of them.

The main reason behind considering the notion of adaptivity is that they could be very hard, sometimes even impossible, to implement in practice for various reasons. We will discuss such reasons in more detail later, but to name a couple, it could be very costly to wait long enough to observe the outcome of the previous actions, or sometimes it is not possible to measure such outcomes accurately. On the other hand, nonadaptive policies are highly restricted and hence may perform poorly in terms of value compared with adaptive ones. To study this trade-off between simplicity and value, we study the notion of an *adaptivity gap*, which is defined as the ratio of the expected value of the optimal adaptive policy versus the expected value of the optimal nonadaptive one.

1.2. Contributions

We show that the adaptivity gap of the stochastic monotone submodular maximization (SMSM) problem is equal to $e/(e-1) \approx 1.59$. In other words, we prove that there exists a nonadaptive policy that achieves at least an $(e-1)/e$ fraction of the value of the optimal adaptive policy. We also provide an example to show that our analysis of the adaptivity gap is tight. For that, we use a simple instance of the stochastic max k -coverage problem. Furthermore, we show that the adaptivity gap can be unbounded if the objective function is submodular but nonmonotone (see §6.4).

We will then present a generalization of the continuous greedy method of Călinescu et al. (2011) and construct a nonadaptive policy that approximates the

optimal adaptive policy within a factor of $(e-1)/e - \epsilon$ for any arbitrary ϵ . Our algorithm runs in polynomial time in terms of the size of the problem and $1/\epsilon$. This policy does not necessarily coincide with the optimal nonadaptive policy; however, because of the tightness of our adaptivity gap result, this is essentially the best approximation ratio that one could hope for with respect to the optimal adaptive policy.

In §5, we focus on myopic policies. We study the natural extension of the myopic policy studied in Fisher et al. (1978) to a stochastic environment. This policy iteratively chooses an element with the maximum *expected* marginal value, *conditioned* on the outcome of the previous elements. We show that the approximation ratio of this policy with respect to the optimal adaptive policy is $\frac{1}{2}$ for *general matroids*. In addition, we show that the approximation ratio of this adaptive myopic policy is $1/(\kappa+1)$ if the feasible domain is given by the intersection of κ matroids. We also prove that over a uniform matroid (i.e., subject to a cardinality constraint) the approximation ratio of this policy is $1 - 1/e$. We will discuss the results of Nemhauser and Wolsey (1978) and Feige (1998) to show that the approximation ratio of $1 - 1/e$ is optimal if only polynomial algorithms are allowed.

This paper settles the status of SMSM problem as presented in Table 1. The values of the optimal nonadaptive and the optimal adaptive policies are denoted by OPT_{NA} and OPT_{A} , respectively. The “ \oplus ” sign in the table means that the bound is exact; i.e., the lower bound and the upper bound coincide. Also, “ \dagger ” means that the algorithm is optimum; i.e., it is computationally intractable to improve the approximation bound. Finally, “ \ddagger ” means that our analysis of the algorithm is tight.

1.3. Related Work

Goemans and Vondrák (2006) proposed adaptive and nonadaptive policies for the stochastic covering problem where the goal is to cover all elements of a target set using stochastic subsets with the minimum (expected) number of subsets. Guillory and Bilmes (2010) studied a similar set cover problem but took a worst-case analysis approach. They showed that in the worst case, the adaptivity gap can be very large,

and they provide optimal (up to a constant factor) algorithms for their problem.

Since the publication of our preliminary results in Asadpour et al. (2008), adaptive policies for other interesting notions of stochastic submodular maximization have been studied. Golovin and Krause (2011b) introduced the notion of *adaptive submodularity*. Adaptive submodularity is a generalization of our setting in which the random variables of interest (namely, X_1, X_2, \dots, X_n) are not necessarily independent. Instead, it suffices that the function f satisfies diminishing conditional expected marginal returns over partial realizations. They demonstrated the effectiveness and applicability of this notion over a wide range of important optimization problems by providing approximately optimal adaptive algorithms; see §6.3. Also, Golovin and Krause (2011a) provided examples to show that the adaptivity gap for certain covering problems (where the objective is to *minimize* the cost) may not be constant.

Very recently, Adamczyk et al. (2014) studied the problem of stochastic submodular probing with applications in online dating, kidney exchange, and Bayesian mechanism design. This problem investigates a generalized notion of adaptive policies. In the model, a ground set of elements is given where every element is *active* independently at random with some preknown probability. A policy can probe elements sequentially, but if an element turns out to be active, it has to be included in the solution. The goal is, naturally, to find a feasible subset of active elements that maximizes the value of a submodular set function. The feasibility of the eventual subset of elements is determined by k^{in} matroid constraints. Also, the subset of elements being probed should be feasible and obey k^{out} matroid constraints. They provide $(1 - 1/e)/(k^{\text{in}} + k^{\text{out}} + 1)$ -approximation policies for this problem for $k^{\text{in}} \geq 0$ and $k^{\text{out}} \geq 1$. We note that as a complementary result to that paper, when $k^{\text{in}} = 1$ and $k^{\text{out}} = 0$, the nonadaptive myopic policy provided by our Proposition 2 gives a $(1 - 1/e)/(k^{\text{in}} + k^{\text{out}} + 1) = (e - 1)/2e$ -approximation for the stochastic submodular probing problem. Also, the nonadaptive policy resulting from Theorem 2 provides a $(1 - 1/e - \epsilon)$ -approximation in polynomial time in terms of the size of the problem and $1/\epsilon$.

Table 1 Bounds for the Algorithms and the Adaptivity Gap

Constraint	Nonadaptive	Myopic adaptive	Adaptivity gap
Uniform matroid	Myopic policy $\frac{e-1}{2e} \text{OPT}_{\text{A}}$ [Proposition 2]	$\frac{e-1}{e} \text{OPT}_{\text{A}}^{\dagger}$ [Theorem 4]	$\frac{e}{e-1}^{\oplus}$ [Theorem 1]
General matroid	Stochastic continuous greedy $\frac{e-1}{e} \text{OPT}_{\text{A}}^{\dagger}$ [Theorem 2]	$\frac{1}{2} \text{OPT}_{\text{A}}^{\dagger}$ $(\frac{1}{\kappa+1} \text{OPT}_{\text{A}} \text{ for the intersection of } \kappa \text{ matroids})^{\ddagger}$ [Theorem 3]	

Another closely related work to ours is by Chan and Farias (2009), who studied a rather general stochastic depletion problem. Their optimization problem is defined over a time horizon where at each step a policy decides on an action. Each action stochastically generates a reward and depletes some of the resources. The goal is to maximize the expected reward subject to the resource constraints. They show that under certain “submodularity” constraints, the myopic policy obtains at least half the value of the optimal (off-line) solution—a benchmark stronger than the optimal online policy. An important difference between the framework in Chan and Farias (2009) and ours is the ordering imposed in their model on the sequence of actions a policy can take, whereas in our model, to prove our results, we need to exploit the matroid properties. Although our proposed adaptive policies are not applicable to their setting, our nonadaptive policy can be applied to their setting using a partition matroid to obtain a $(1 - 1/e)$ -approximation with respect to the optimal adaptive (online) algorithms.

1.3.1. Organization. The rest of the paper is organized in the following way. In §2 we formally introduce our problem and define adaptive and nonadaptive policies and the concept of adaptivity gap. In §3 we study the adaptivity gap and show that it is equal to $e/(e - 1)$. In §4 we provide a polynomial-time algorithm to find a nonadaptive policy that matches the adaptivity gap ratio. The simple myopic policies are investigated in §5, and worst-case performance guarantees have been provided for them. We discuss some natural variations of our model in §6. Finally, we conclude in §7 by mentioning some interesting directions for future research. Some of the proofs in the paper are relegated to the online appendix (available as supplemental material at <http://dx.doi.org/10.1287/mnsc.2015.2254>).

2. Model

In this section, we define our framework for stochastic submodular optimization. Consider a ground set $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$ of n independent random variables. Let \mathbb{R}_+ denote the set of nonnegative real numbers (including 0). Each random variable X_i is defined over a domain $\Omega_i \subseteq \mathbb{R}_+$, and its probability distribution function over Ω_i has density g_i .¹ Let $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be a monotone submodular value function. In other words,

$$\forall x, y \in \mathbb{R}_+^n: f(x \vee y) + f(x \wedge y) \leq f(x) + f(y),$$

¹ Throughout this paper, we work with continuous probability distributions. All our theorems and their corresponding proofs directly apply to the probability distributions over discrete domains by using Dirac delta functions.

where $x \vee y$ denotes the component-wise maximum and $x \wedge y$ the component-wise minimum of x and y . Note that submodular set functions are special cases of the above definition. This generalization to the continuous domain allows us to capture wider aspects of the problem. For instance, we can model *partial* contributions from the elements. This could be useful in many of the applications that we have mentioned before. For example, in viral marketing, targeted people could have different levels of enthusiasm about the new product or technology. Or in the sensor placement problem, some sensors in the network might not become fully functional and cover only a part of the area they were supposed to cover.

A policy, formally defined below, consists of a sequence of actions (π_1, π_2, \dots) . Each *action* corresponds to the selection of an element from \mathcal{A} (or perhaps none, denoted by \emptyset , corresponding to the empty set). After an element is selected, its actual value is realized. The state of the problem Θ is represented by an n -tuple $(\theta_1, \theta_2, \dots, \theta_n)$. Here, $\theta_i = x_i$ if X_i has been selected by the policy in any of the previous steps, and its actual value is realized to be x_i ; otherwise, $\theta_i = \circ$. The symbol \circ , different from value zero, stands for null, representing that the corresponding element has not been selected yet.

Let $\Omega := \prod_{i=1}^n (\Omega_i \cup \{\circ\})$ be the set of all possible states. Consider a policy π where the state of the problem before taking the j th action is given by $\Theta_{j-1}^\pi \in \Omega$. The j th action, denoted by π_j , is a mapping from the current state to an element from set $\mathcal{A} \cup \{\emptyset\}$. Before taking any action, the state of the problem is $\Theta_0^\pi = (\circ, \circ, \dots, \circ)$. For instance, if we choose to pick X_3 in our first action (i.e., $\pi_1 = X_3$) and its value is realized to be x_3 , then the state will be $\Theta_1^\pi = (\circ, \circ, x_3, \circ, \dots, \circ)$. If at some step j the action \emptyset is selected, then the state of the problem will remain unchanged; i.e., $\Theta_j^\pi = \Theta_{j-1}^\pi$. We can now formally define the notion of policy.

DEFINITION 1 (POLICY). A policy $\pi: \Omega \rightarrow [0, 1]^n$ is a mapping from the state space Ω to a distribution over the elements in $\mathcal{A} \cup \{\emptyset\}$. Let $\Theta_{j-1}^\pi = (\theta_1, \theta_2, \dots, \theta_n)$ represent the state of the problem before taking the j th action. Then, $\pi(\Theta_{j-1}^\pi)$ is a distribution over $\{X_k \mid \theta_k = \circ\} \cup \{\emptyset\}$. If the policy is deterministic, $\pi(\Theta_{j-1}^\pi)$ will correspond to exactly one element in $\{X_k \mid \theta_k = \circ\} \cup \{\emptyset\}$. The action of the policy, denoted by π_j , corresponds to choosing an element of $\{X_k \mid \theta_k = \circ\} \cup \{\emptyset\}$ according to distribution $\pi(\Theta_{j-1}^\pi)$. Subsequently, if the policy chooses element $X_i \neq \emptyset$, the state of the problem after taking the j th action will be $\Theta_j^\pi = (\theta_1, \theta_2, \dots, \theta_{i-1}, x_i, \theta_{i+1}, \dots, \theta_n)$, where x_i is the realized value of X_i (drawn from distribution g_i). Finally, if at any point $\pi_j(\Theta_{j-1}^\pi) = \emptyset$, then the selection of the elements stops. We denote by Θ^π the random n -tuple corresponding to the last state reached by the policy.

In the following example, we discuss the evolution of a simple policy defined on a ground set consisting of three elements.

EXAMPLE 1. Suppose that the ground set of elements is $\{X_1, X_2, X_3\}$. Also, suppose that each element, once picked, will have a realized value of either 10 or 100 with probabilities 0.4 and 0.6, respectively. Consider a simple policy π that in the first step picks element X_3 . In the next step, π will pick X_1 if the realized value of X_3 is realized to be 10. Otherwise, it picks either X_1 or X_2 , each of which with probability 0.5. After that, π does not pick anything. Note that as we have explained before, the initial state of the problem is by definition $\Theta_0^\pi = (\circ, \circ, \circ)$. The first action of this policy is $\pi_1 = X_3$. After that, the state of the problem, i.e., Θ_1^π , will be either $(\circ, \circ, 10)$ or $(\circ, \circ, 100)$ with probabilities 0.4 and 0.6, respectively. Now, if $\Theta_1^\pi = (\circ, \circ, 10)$, then the second action $\pi_2(\circ, \circ, 10)$ will be to choose X_1 . Consequently, the state of the problem afterwards, i.e., Θ_2^π , will be either $(10, \circ, 10)$ or $(100, \circ, 10)$ with probabilities 0.4 and 0.6, respectively. On the other hand, if $\Theta_1^\pi = (\circ, \circ, 100)$, then the second action will be a random one that picks either X_1 or X_2 , each with probability 0.5. In the first case, the state of the problem, i.e., Θ_2^π , will be either $(10, \circ, 10)$ or $(100, \circ, 10)$ with probabilities 0.4 and 0.6, respectively. Similarly, in the second case, Θ_2^π will be either $(\circ, 10, 10)$ or $(\circ, 100, 10)$ with probabilities 0.4 and 0.6, respectively.

From the discussion above, we conclude that for this specific policy π we have

$$\Theta_2^\pi = \begin{cases} (10, \circ, 10) & \text{w.p. } 0.4 \times 0.4 = 0.16, \\ (100, \circ, 10) & \text{w.p. } 0.4 \times 0.6 = 0.24, \\ (10, \circ, 100) & \text{w.p. } 0.6 \times 0.5 \times 0.4 = 0.12, \\ (100, \circ, 100) & \text{w.p. } 0.6 \times 0.5 \times 0.6 = 0.18, \\ (\circ, 10, 10) & \text{w.p. } 0.6 \times 0.5 \times 0.4 = 0.12, \\ (\circ, 100, 10) & \text{w.p. } 0.6 \times 0.5 \times 0.6 = 0.18. \end{cases} \quad (1)$$

After that, the policy stops. Hence, Θ^π will be equal to Θ_2^π . \square

Note that a policy sequentially picks new elements from the ground set \mathcal{A} (perhaps based on the outcomes of the previous selections). We emphasize that we only consider policies *without* replacements (i.e., only elements whose corresponding entries in the current state are null can be selected). As a consequence, once the value of an element is realized, it will be fixed and cannot be changed by picking that element again. As we discuss in §6.2, this assumption is only made to simplify the presentation, and all of our results hold without it.

The value of any state $\Theta = (\theta_1, \theta_2, \dots, \theta_n)$, denoted by $f(\Theta)$, is equal to the value of function f over

those elements of \mathcal{A} that already have been chosen. Formally,

$$f(\Theta) := f(\zeta_1, \zeta_2, \dots, \zeta_n), \quad \text{where } \zeta_i = \begin{cases} \theta_i & \theta_i \neq \circ, \\ 0 & \theta_i = \circ. \end{cases} \quad (2)$$

For instance, for a state $\Theta = (10, \circ, 100)$, the value of $f(\Theta)$ is equal to $f(10, 0, 100)$. Finally, the expected value of a policy π after the j th action is equal to $\mathbb{E}[f(\Theta_j^\pi)]$. Here, the expectation is taken over all possible realizations of the state Θ_j^π as an outcome of the first j actions taken by π . Throughout this paper the parameter for the expectation will be clear from the context. We will make this parameter explicit wherever some confusion may arise. For instance, if needed, we can represent the aforementioned expectation by $\mathbb{E}_\Theta[f(\Theta_j^\pi)]$.

DEFINITION 2 (VALUE OF A POLICY). The value of a policy π is defined as the expected value of its final state, i.e., $\mathbb{E}[f(\Theta^\pi)]$.

In the above definition, the expectation is taken over all realization of Θ^π . We refer the reader to the end of Definition 1 for Θ^π . Thus, for the example we discussed earlier, by following Equation (1) we have

$$\begin{aligned} \mathbb{E}[f(\Theta^\pi)] &= \mathbb{E}[f(\Theta_2^\pi)] \\ &= 0.16f(10, 0, 10) + 0.24f(100, 0, 10) \\ &\quad + 0.12f(10, 0, 100) + 0.18f(100, 0, 100) \\ &\quad + 0.12f(0, 10, 100) + 0.18f(0, 100, 0). \end{aligned}$$

Throughout this paper, we assume that all such expected values are finite. Because of the monotonicity of f , it is enough to assume that $\mathbb{E}[f(\Theta^\pi)]$ is finite for a policy π that picks all the elements in \mathcal{A} .

Finally, we take into account the constraints that might appear in our optimization problems.

DEFINITION 3 (A FEASIBLE POLICY). Consider a matroid $\mathcal{M}(\mathcal{A}, \mathcal{F})$ defined on a ground set $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$ of n random variables. A policy π is feasible with respect to \mathcal{M} if and only if with probability 1 the set of elements chosen by π belongs to \mathcal{F} . We denote by $\Pi(\mathcal{A}, \mathcal{F})$ the set of all such feasible policies.

Now, we are ready to formally define the problem of maximizing a stochastic monotone submodular function in the presence of a matroid feasibility constraint.

DEFINITION 4 (SMSM: MAXIMIZING A STOCHASTIC MONOTONE SUBMODULAR FUNCTION WITH MATROID CONSTRAINT). An instance of the SMSM problem is defined with tuple $(f, \mathcal{A}, \mathcal{F})$, where $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$ is a set of n nonnegative independent

random variables, $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is a monotone submodular value function, and matroid $M(\mathcal{A}, \mathcal{F})$ represents the constraints. The goal is to find a feasible policy (with respect to \mathcal{M}) that obtains the maximum (expected) value, i.e., $\max_{\pi \in \Pi(\mathcal{A}, \mathcal{F})} \mathbb{E}[f(\Theta^\pi)]$.

In addition to “adaptive” policies, we also consider nonadaptive policies that do not rely on the outcomes of the previous actions to make a decision. When a policy progresses, more information about the actual realization of the state will be revealed. Nonadaptive policies do not use such information and choose all the elements in advance.

DEFINITION 5 (NONADAPTIVE POLICIES). A policy π is called *nonadaptive* if, for each state Θ , $\pi(\Theta)$ depends only on which elements were previously chosen but *not* their realizations. A nonadaptive policy is defined by the set (and not the order) of the chosen elements.

Since nonadaptive policies choose their set of actions in advance (not sequentially), they are usually simpler to implement. Furthermore, sometimes it is not possible to implement an adaptive policy or it might be difficult to learn the realizations of the random variables. For instance, in the capital budgeting problem, it might be costly for the investor to wait for outcomes of the projects to be realized before making another decision. In the case of viral marketing, it is difficult to get feedback from each individual and find out about the outcome of the previous actions.

However, one drawback of nonadaptive policies is that they may not perform as well as the adaptive policies. The optimal policy for a given instance of the SMSM problem might be a complicated adaptive policy, far from being nonadaptive. Hence, there exists a trade-off between the value obtained by the policy on one side and its practicality and convenience (in terms of both computation and implementation) on the other side. However, contemplating between these two options is reasonable for strategists only if the performance of the nonadaptive policies is not too far from that of the adaptive ones. This observation motivates the comparison between the value of adaptive and nonadaptive policies in SMSM problems. This can be measured by studying the adaptivity gap as defined by Dean et al. (2005, 2008).

DEFINITION 6 (ADAPTIVITY GAP). The adaptivity gap is defined as the upper bound on the ratio of the value of the optimal adaptive policy to the value of the optimal nonadaptive policy. More precisely,

$$\text{Adaptivity Gap} = \sup_{(f, \mathcal{A}, \mathcal{F})} \frac{\sup_{\pi \in \Pi(\mathcal{A}, \mathcal{F})} \mathbb{E}_\Theta[f(\Theta^\pi)]}{\sup_{S \in \mathcal{I}} \mathbb{E}_\Theta[f(\Theta_S)]},$$

where the first supremum is taken over all instances $(f, \mathcal{A}, \mathcal{F})$ of the SMSM problem, and for $S \in \mathcal{I}$, the vector Θ_S corresponds to the outcome of the nonadaptive policy that picks set S of the elements.

3. The Adaptivity Gap of SMSM

It is easy to observe from Definition 6 that the adaptivity gap is at least equal to 1 simply because nonadaptive policies are a special case of adaptive (general) policies. However, a high adaptivity gap for a problem suggests that good adaptive policies are far superior than the nonadaptive ones. Similarly, a low (close to 1) adaptivity gap is a certificate that one can get the benefits of nonadaptive policies (such as the relative simplicity in their optimization and also implementation) without losing much in the obtained value compared with that of the adaptive policies. In this section, we show that for the SMSM problem, the adaptivity gap is bounded. The main result of this section is as follows.

THEOREM 1 (ADAPTIVITY GAP). *The adaptivity gap of SMSM is equal to $e/(e-1)$.*

In the rest of this section, we prove the above theorem by analyzing the optimal nonadaptive and adaptive policies and compare their performances. We start with an example that shows that the above ratio is tight.

3.1. A Tight Example: Stochastic Maximum k -Coverage

Given a collection \mathcal{A} of the subsets of $\{1, 2, \dots, m\}$, the goal of the *maximum k -coverage* problem is to find k subsets from \mathcal{A} such that their union has the maximum cardinality (see Feige 1998). In the stochastic version, each random variable X_i with some probability covers a subset of $\{1, 2, \dots, m\}$ and with the remaining probability covers no element (i.e., the realization of X_i would be the empty set). The subset that an element of \mathcal{A} would cover is revealed only after choosing the element according to a given probability distribution. It is easy to see that this problem is a special case of the SMSM with respect to a uniform matroid $\mathcal{M} = (\mathcal{A}, \{S \subseteq \mathcal{A} : |S| \leq k\})$ constraint.

To give a lower bound on the adaptivity gap, consider the following instance: a ground set $\{1, 2, \dots, m\}$ and a collection $\mathcal{A} = \{X_j^{(i)} \mid 1 \leq i \leq m, 1 \leq j \leq m^2\}$ of its subsets are given. Note that, here, n , the total number of elements in the ground set, is equal to m^3 . For every i, j , define $X_j^{(i)}$ to be the one-element subset $\{i\}$ with probability $1/m$ and the empty set with probability $1 - 1/m$. The goal is to cover the maximum number of the elements of the ground set by selecting at most $k := m^2$ subsets from \mathcal{A} . We denote this instance by SMSM1(m).

LEMMA 1. *The optimal nonadaptive policy for the problem SMSM1(m) is to pick m subsets from each of the collections $\mathcal{A}^{(i)} = \{X_j^{(i)} \mid 1 \leq j \leq m^2\}$ for every i . The expected value of this policy is $(1 - 1/e - o(1))m$.*

Therefore, for large enough values of m , the expected value of the nonadaptive policy above is arbitrarily close to $(1 - 1/e)m$. The proof is given in the online appendix and is based on a convexity argument.

We now consider the following myopic adaptive policy π : start with $i = 1$ and pick the elements of $\mathcal{A}^{(i)}$ one by one until one of them is realized as $\{i\}$ or all of elements in $\mathcal{A}^{(i)}$ are chosen. Then increase i by 1. Continue the iteration until either all the elements are covered or m^2 subsets from \mathcal{A} are selected. The following lemma gives a lower bound on the number of elements in the ground set covered by the adaptive policy. The proof is given in the online appendix.

LEMMA 2. *The expected number of elements in $\{1, 2, \dots, m\}$ covered by π described above is $(1 - o(1))m$.*

By combining the results of Lemmas 1 and 2, we have that the adaptivity gap of SMSM1(m) is at least $e/(e - 1)$.

3.2. Bounds on the Optimal Nonadaptive Policy

In this section, we study the optimal nonadaptive policy in more detail. We start with some definitions. Let $S \subseteq \mathcal{A}$ be a subset of variables. Also, let vector $\Theta_S = (\theta_1, \dots, \theta_n)$ denote the realization of set S , where $\theta_i = x_i$ for $X_i \in S$ and $\theta_i = \circ$ for $X_i \notin S$. The value obtained by choosing set S after the realization is equal to $\mathbf{E}[f(\Theta_S)]$. Let g_i and G_i be the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of random variable X_i , respectively. We introduce $g_S: \mathbb{R}_+^n \rightarrow \mathbb{R}$ to represent the p.d.f. of observing $\Theta = (\theta_1, \dots, \theta_n)$ while selecting S :

$$g_S(\Theta) := \Pr[\Theta_S = \Theta] = \prod_{i: X_i \in S} g_i(\theta_i),$$

where $g_S(\theta)$ is defined to be 0 if there exists $j \notin S$ such that $\theta_j \neq \circ$. Note that the definition above relies on the assumption that the realizations of elements chosen by the policy are independent from each other. As we show in §6.3, the adaptivity gap can be arbitrarily large if this assumption does not hold.

Now, for the sake of simplicity in notation, we define function $F: 2^{\mathcal{A}} \rightarrow \mathbb{R}^+$ as the expected value obtained by choosing set S ; i.e.,

$$F(S) = \mathbf{E}[f(\Theta_S)] = \int_{\Theta \in \Omega} f(\Theta) g_S(\Theta) d\Theta, \quad (3)$$

where, as defined before, $\Omega = \prod_{i=1}^n (\Omega_i \cup \{\circ\})$ is the set of all possible states. Therefore, the optimal nonadaptive policy for SMSM is equivalent to choosing a set $S \in \mathcal{J}$ that maximizes $F(S)$ with respect to the desired matroid constraint. Hence, the following proposition holds.

PROPOSITION 1 (OPTIMAL NONADAPTIVE POLICY). *For any instance of SMSM, each set $S^* \in \arg \max_{S \in \mathcal{J}} F(S)$ corresponds to an optimal nonadaptive policy.*

In the online appendix, we prove the following lemma that shows F is monotone and submodular.

LEMMA 3 (PROPERTIES OF FUNCTION F). *If $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is monotone and submodular, then function $F: 2^{\mathcal{A}} \rightarrow \mathbb{R}^+$, defined in Equation (3), is a monotone and submodular set function.*

We now define the notion of *fractional nonadaptive policies*. A fractional nonadaptive policy \mathcal{S}_y is defined with a vector $y \in [0, 1]^n$. Policy \mathcal{S}_y chooses elements in the (random) set Y , a set that includes each $X_i \in \mathcal{A}$ with probability y_i , independently for every i .

We call fractional nonadaptive policy \mathcal{S}_y *feasible in expectation* if vector y lies in the base polytope of \mathcal{M} denoted by $\mathcal{B}(\mathcal{M})$. The base polytope of \mathcal{M} is the convex hull of the characteristic vectors of all bases of \mathcal{M} ; i.e.,

$$\mathcal{B}(\mathcal{M}) = \text{conv}\{1_S \mid S \in \mathcal{J}, \text{ and } S \text{ is a basis}\}. \quad (4)$$

For instance, suppose \mathcal{M} is a uniform matroid of rank d , and for $y = (y_1, y_2, \dots, y_n)$, we have $\sum_{i=1}^n y_i \leq d$. Observe that in this case, the fractional nonadaptive policy \mathcal{S}_y is feasible in expectation, and the expected number of elements chosen by this policy is at most d .

With slight abuse of notation, we denote the expected value obtained by fractional nonadaptive policy \mathcal{S}_y by $F(y)$. Namely,

$$F(y) := \sum_{Y \subseteq \{1, \dots, n\}} \left[\left(\prod_{i \in Y} y_i \prod_{i \notin Y} (1 - y_i) \right) F(Y) \right], \quad (5)$$

where, as defined in Equation (3), we have $F(Y) = \mathbf{E}_\Theta[f(\Theta_Y)]$.

Fractional nonadaptive policies extend the space of (integral) nonadaptive policies. Therefore, it is easy to see that $\max_{S \in \mathcal{J}} F(S) \leq \max_{y \in \mathcal{B}(\mathcal{M})} F(y)$. We show that this inequality is in fact tight, and any vector $y \in \mathcal{B}(\mathcal{M})$ can be rounded to an integral corner point $Y \in \mathcal{B}(\mathcal{M})$ corresponding to a subset $S \in \mathcal{J}$ such that $F(S) \geq F(y)$. Note that Lemma 3 shows that function $F(Y)$ used in Equation (5) is a *submodular set function*. Hence, $F(y)$ is the same as the “multilinear extension” that has been defined by Călinescu et al. (2011) to study the problem of maximizing (general deterministic) submodular functions. Therefore, by applying the pipage rounding procedure of Călinescu et al. (2011) to function $F(y)$, we immediately obtain the following lemma.

LEMMA 4 (FRACTIONAL VS. INTEGRAL NONADAPTIVE POLICIES). *For any $y \in \mathcal{B}(\mathcal{M})$, there exists a set $S \in \mathcal{J}$ such that $F(S) \geq F(y)$. Moreover, such a set can be found in polynomial time.*

The above lemma implies that $\max_{y \in \mathcal{B}(\mathcal{M})} F(y)$ is a (tight) *lower bound* on the performance of the optimal nonadaptive policy.

3.3. Bounds on the Optimal Adaptive Policy

We start this section by making a few observations about adaptive policies. Consider an arbitrary adaptive policy ADAPT. Any such policy decides to choose a sequence of elements where the decision on which element to choose at any step might depend on the realized values of the previously chosen elements. Therefore, any specific state of the world will result in a distribution over the sequence of elements that will be chosen by ADAPT.² Any adaptive policy can be described by a (possibly randomized) decision tree in which at each step an element is being added to the current selection. Each path from the root to a leaf of this tree corresponds to a subset of elements and occurs with some certain probability. These probabilities, covering all the possible scenarios for ADAPT, sum to 1. Let y_i be the probability that element $X_i \in \mathcal{A}$ is eventually chosen by ADAPT. Also, let β_Θ be the probability that the final state Θ is reached by ADAPT. Then we have the following properties:

1. $\int_{\Theta \in \Omega} \beta_\Theta d\Theta = 1$,
2. $\forall \Theta \in \Omega: \beta_\Theta \geq 0$, and
3. $\forall i, x_i \in \Omega_i: \int_{\Theta: \theta_i = x_i} \beta_\Theta d\Theta^{-i} = y_i g_i(x_i)$,

where $d\Theta^{-i}$ represents $\prod_{j \neq i} d\theta_j$. The first two properties hold because β defines a probability measure on the space of all feasible outcomes. The third property implies that the probability that we observe an outcome x_i as a realized value of X_i among all possible states Θ reached by the policy is equal to the probability that X_i is chosen (i.e., y_i) multiplied by the probability that the realization is equal to x_i . We emphasize that we use the independence among the random variables to ensure that this property holds. We also emphasize that, as we will see in §6.3, the adaptivity gap could be unbounded if there are dependencies among the variables.

Since every policy satisfies the above properties, we can establish an upper bound on the value of any adaptive policy. Hence, we define the function $f^+: [0, 1]^n \rightarrow \mathbb{R}$ as follows:

$$f^+(y) := \sup_{\alpha} \left\{ \int_{\Theta} \alpha_\Theta f(\Theta) d\Theta: \int_{\Theta} \alpha_\Theta d\Theta = 1, \alpha_\Theta \geq 0, \forall i, \right. \\ \left. x_i \in \Omega_i: \int_{\Theta: \theta_i = x_i} \alpha_\Theta d\Theta^{-i} = y_i g_i(x_i) \right\}. \quad (6)$$

The supremum is taken over all probability measures α that satisfy the three properties above.

² We refer the reader to Definition 1 for policy and emphasize that we allow our policies to be *randomized*. This is the reason that we mention a distribution over different sequences of elements (and not just a specific sequence) here.

Another observation is that for an optimal adaptive policy, vector y lies in the base polytope of \mathcal{M} ; see Equation (4). Using these observations, we obtain the following lemma, which is proved in the appendix.

LEMMA 5 (AN UPPER BOUND ON ADAPTIVE POLICIES). *The expected value of the optimal adaptive policy is at most $\sup_{y \in \mathcal{B}(\mathcal{M})} \{f^+(y)\}$.*

We are now ready to prove Theorem 1.

3.4. Proof of Theorem 1

Lemma 5 shows that $\sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$ is an upper bound on the performance of the optimal adaptive policy. Now consider any $y \in \mathcal{B}(\mathcal{M})$. Below, we present Lemma 6, which shows $(1 - 1/e)f^+(y) \leq F(y)$. On the other hand, Lemma 4 implies that there exists a set $S \in \mathcal{F}(\mathcal{M})$ such that $F(y) \leq F(S)$. Recall that $F(S)$ is in fact the expected value gained by a nonadaptive policy that selects set S . Hence, for every $y \in \mathcal{B}(\mathcal{M})$, there exists a nonadaptive policy that achieves a value of at least $(1 - 1/e)f^+(y)$. Therefore, the optimal nonadaptive policy corresponding to selecting set $S^* \in \arg \max_S \{F(S)\}$ has a value of at least $(1 - 1/e) \sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$ or, equivalently, at least a $(1 - 1/e)$ fraction of the value of the optimal adaptive policy. We remind the reader that the example in §3.1 shows that the factor is tight.

LEMMA 6 (UPPER BOUND ON f^+). *For any monotone submodular function f and any vector $y \in \mathcal{B}(\mathcal{M})$, we have $f^+(y) \leq (e/(e - 1))F(y)$.*

The main technical ingredient in the proof of the lemma above is a generalization of Lemmas 3.7 and 3.8 in Vondrák (2007) to continuous submodular functions for which we provide a new stochastic dominance result.

PROOF OF LEMMA 6. We start with introducing the following notation. For a vector $\Theta = (\theta_1, \theta_2, \dots, \theta_n)$, define $\Theta^{\dagger j} = (\theta_1^{\dagger j}, \theta_2^{\dagger j}, \dots, \theta_n^{\dagger j})$ as a random vector whose entries are defined as below:

$$\theta_i^{\dagger j} := \begin{cases} \theta_i & i \neq j, \\ \max\{\theta_j, X\} & i = j, \text{ where } X \text{ is independently} \\ & \text{drawn from } g_j. \end{cases} \quad (7)$$

One can think of $\Theta^{\dagger j}$ as the same as Θ , except for the j th entry for which we draw a number X from the distribution g_j and override the entry with X if and only if it increases the value of the entry.

Now, for any j we define an independent Poisson clock \mathcal{C}_j that sends signals with rate y_j throughout the time. We start with $\Theta = (0, 0, \dots, 0)$ at $t = 0$. Once a clock \mathcal{C}_j sends a signal, we replace Θ by $\Theta^{\dagger j}$. By abuse of notation, we denote the value of vector Θ at time t

by $\Theta(t)$. We first show that $\mathbb{E}[f(\Theta(1))] \leq F(y)$. We will do so by proving a stochastic dominance result.

Consider $F(y)$, defined in Equation (5). Note that each entry j in Θ_Y is zero (null) if $j \notin Y$. Otherwise, it is drawn from the probability distribution g_j , independent from other variables. Hence, $F(y)$ can be written as $\mathbb{E}[f(\theta_1, \theta_2, \dots, \theta_n)]$, where the entries are all independent and the c.d.f. of θ_j is given by a function $\eta_j: \mathbb{R}_+ \mapsto [0, 1]$ and $\eta_j(x) = (1 - y_j) + y_j G_j(x)$.

On the other hand, as a result of the independence of the Poisson clocks, the entries of $\Theta(1)$ are independent random variables. In particular, the clock \mathcal{C}_j signals k times between $t = 0$ and $t = 1$, where k is a Poisson random variable with rate y_j . If this clock signals k times, by the construction of $\Theta(t)$ for $0 \leq t \leq 1$, the j th entry of $\Theta(1)$ will be the maximum of k random variables, each drawn independently from the probability distribution g_j . In this case, the c.d.f. of the j th entry will be simply given by the function $G_j(x)^k$. Summing over all k and incorporating the Poisson distribution function, we can summarize that the c.d.f. of the j th entry of $\Theta(1)$ is given by a function $\gamma_j: \mathbb{R}_+ \mapsto [0, 1]$, where $\gamma_j(x) = \sum_{k=0}^{\infty} (y_j^k / k!) e^{-y_j} G_j(x)^k$.

By the properties of Poisson distribution, we have

$$\begin{aligned} \gamma_j(x) &= \sum_{k=0}^{\infty} \frac{y_j^k}{k!} e^{-y_j} G_j(x)^k \\ &= \frac{e^{-y_j}}{e^{-y_j G_j(x)}} \sum_{k=0}^{\infty} \left(\frac{(y_j G_j(x))^k}{k!} e^{-y_j G_j(x)} \right) \\ &= \frac{e^{-y_j}}{e^{-y_j G_j(x)}} = e^{-y_j(1-G_j(x))}. \end{aligned}$$

However, comparing the two cumulative distribution functions discussed above, we have

$$\gamma_j(x) = e^{-y_j(1-G_j(x))} \geq 1 - y_j(1 - G_j(x)) = \eta_j(x).$$

This means that for every j , the random variable drawn from γ_j is stochastically dominated by that from η_j . Therefore,

$$\mathbb{E}[f(\Theta(1))] \leq F(y). \quad (8)$$

Now, we compare the value $\mathbb{E}[f(\Theta(1))]$ with $f^+(y)$. Let $t \in [0, 1)$ be fixed. For each j , the chance that the clock \mathcal{C}_j sends a signal during the interval $[t, t + dt)$ is simply $y_j dt$ for a sufficiently small dt . Hence,

$$\begin{aligned} \mathbb{E}[f(\Theta(t + dt)) - f(\Theta(t)) \mid \Theta(t) = \Theta] \\ = \sum_{j=1}^n y_j dt (\mathbb{E}[f(\Theta^{\uparrow j})] - f(\Theta)). \end{aligned} \quad (9)$$

Note that the expectation is taken over the random draw that appears in the definition of $\Theta^{\uparrow j}$, i.e., the random variable X in Definition (7).

We define an auxiliary function $f^*: [0, 1]^n \rightarrow \mathbb{R}$ as the following:

$$f^*(y) := \inf_{\Theta} \left\{ f(\Theta) + \sum_{j=1}^n y_j (\mathbb{E}[f(\Theta^{\uparrow j})] - f(\Theta)) \right\}, \quad (10)$$

where the infimum is taken over all possible states Θ . Therefore, the right-hand side of Equation (9) is at least $(f^*(y) - \mathbb{E}[f(\Theta)]) dt$. As a result, the following bound can be derived on the derivative of $\mathbb{E}[f(\Theta(t))]$:

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[f(\Theta(t))] &= \frac{1}{dt} \mathbb{E}[f(\Theta(t + dt)) - f(\Theta(t)) \mid \Theta(t) = \Theta] \\ &\geq f^*(y) - \mathbb{E}[f(\Theta(t))]. \end{aligned} \quad (11)$$

Solving the differential equation above, we obtain $\mathbb{E}[f(\Theta(t))] \geq (1 - e^{-t}) f^*(y)$. In particular,

$$\left(1 - \frac{1}{e}\right) f^*(y) \leq \mathbb{E}[f(\Theta(1))]. \quad (12)$$

In Appendix A.1, Lemma 16, we show that $f^+(y) \leq f^*(y)$, $y \in \mathcal{B}(\mathcal{M})$. Combining with inequalities (8) and (12), we have $f^+(y) \leq (e/(e-1)) F(y)$, which concludes the proof of the lemma. \square

4. A Polynomial-Time $(1 - 1/e - \epsilon)$ -Approximate Nonadaptive Policy

In this section, we present polynomial-time nonadaptive policies for the SMSM problem. We assume that the values of function F are accessible via an oracle. We discuss the computation of function F in §6.1. We show that under standard assumptions (when g_j 's are bounded or constant Lipschitz continuous), the value of function F can be computed within any polynomially small error term using sampling.

Let us first consider the following myopic nonadaptive policy formally defined in Figure 1. This policy, at each step, adds a feasible element with the highest marginal contribution to the set of selected elements. Recall that using Lemma 3, we showed that function F is monotone submodular.

The classic result of Fisher et al. (1978) implies that the greedy algorithm obtains an approximation ratio of $\frac{1}{2}$ for maximizing monotone submodular functions, i.e., at the end of the algorithm $F(S_i) \geq \frac{1}{2} \max_{S \in \mathcal{F}} \{F(S)\}$. Therefore, using Theorem 1, we have the following.

PROPOSITION 2 (MYOPIC NONADAPTIVE POLICY). *The myopic nonadaptive policy for any instance of SMSM obtains an expected value of at least $\frac{1}{2} \times (1 - 1/e) \approx 0.316$ times the expected value of the optimal adaptive policy.*

Figure 1 The Nonadaptive Myopic Policy

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Initialize  $i = 0, S_0 = \emptyset, U_0 = \emptyset$ .
Repeat
   $i \leftarrow i + 1$ .
  Find  $X_i \in \arg \max_{X \in \mathcal{A} \setminus (U_{i-1} \cup S_{i-1})} \mathbb{E}[F(S_{i-1} \cup \{X_i\})]$ .
  If  $S_{i-1} \cup \{X_i\} \in \mathcal{F}$ , then
     $S_i \leftarrow S_{i-1} \cup \{X_i\}; U_i \leftarrow U_{i-1}$ .
  else
     $U_i \leftarrow U_{i-1} \cup \{X_i\}; S_i \leftarrow S_{i-1}$ .
Until  $(U_i \cup S_i = \mathcal{A})$ .
The nonadaptive policy selects set  $S_i$ .

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The above result provides a performance guarantee for a simple policy; however, it does not match the adaptivity gap. Theorem 1 shows that there exists a nonadaptive policy corresponding to the selection of set $S^* \in \arg \max_{S \in \mathcal{F}(\mathcal{M})} F(S)$ that achieves at least a $(1 - 1/e)$ fraction of the value of the optimal adaptive policy. However, this does not immediately provide an efficient algorithm to find such a nonadaptive policy. The reason is that finding $\arg \max_{S \in \mathcal{F}(\mathcal{M})} F(S)$, i.e., maximizing a monotone submodular function with respect to a matroid constraint, is not computationally tractable. As mentioned before, the maximum k -cover, described in §3.1, is a special case of this problem. Feige (1998) shows that it is not possible to find an approximation ratio better than $1 - 1/e$ for the maximum k -cover problem unless $\text{NP} \subset \text{TIME}(n^{O(\log \log n)})$. Hence, $1 - 1/e$ is the best possible approximation ratio achievable for any policy that can be implemented within polynomial time.

In the proof of Theorem 1, we used a dynamic process $\Theta(t): t \in [0, 1]$ to find a vector y^* such that $F(y^*) \geq (1 - 1/e) \sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$. This process takes an arbitrary $y \in \mathcal{B}(\mathcal{M})$ and starts with $\Theta(0) = (0, 0, \dots, 0)$. It keeps track of n independent Poisson clocks $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$, where the rate of \mathcal{C}_j is y_j , and whenever a clock \mathcal{C}_j sends a signal throughout the time $0 \leq t \leq 1$, the process substitutes Θ with $\Theta^{\uparrow j}$ (see Equation (7) for the definition of the latter). If we were given an appropriate vector y (for instance, the y that corresponds to the probability of each element being picked by the optimum adaptive policy), then we could simulate this process by discretizing the time (we will discuss discretizing later in detail). However, we do not know any such vector a priori. The idea of our algorithm is to use a greedy approach to select the entries that are going to be updated at any given point in time. See Figure 2 for the description of our algorithm.

Our method is a generalization of the continuous greedy approach of Călinescu et al. (2011) performed on function F as defined in Equation (5). In fact, for the deterministic problem (i.e., when each X_j , if picked, will always have a realized value of 1) our method becomes the equivalent of the continuous greedy method of Călinescu et al. (2011).

Figure 2 The Stochastic Continuous Greedy Algorithm

- (i) Let $\delta = 1/(\lceil 3d/\epsilon \rceil)$, where d denotes the rank of the matroid $\mathcal{M} = (\mathcal{A}, \mathcal{F})$; $n = |\mathcal{A}|$, and ϵ is the discretization parameter. Start from $t = 0$ and $y(0) = 0$.
- (ii) Let $R(t)$ be a set containing each element $X_j \in \mathcal{A}$ independently with probability $y_j(t)$. Let $\Theta_{R(t)}$ be the realization of set $R(t)$, i.e., a vector whose j th entry is independently drawn from the probability distribution function g_j if $X_j \in R(t)$ and is 0 if $X_j \notin R(t)$. Let $w_j(t)$ be an estimate of $\mathbb{E}_{\Theta_{R(t)}}[f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})]$, obtained by averaging over $(4/\delta^2)(1 + \ln n - 0.5 \ln \delta)$ samples of $\Theta_{R(t)}$.
- (iii) Let $I(t)$ be a maximum-weight independent set in \mathcal{M} according to the weights $w_j(t)$. Let $y(t + \delta) = y(t) + \delta \cdot \mathbb{1}_{I(t)}$.
- (iv) Increment $t := t + \delta$; if $t < 1$, go back to (ii). Otherwise, return $y(1)$.

4.1. The High-Level Idea

Our algorithm (whose description can be seen below) keeps track of a vector $y(t)$, starting from $y(0) = (0, 0, \dots, 0)$ (see step (i)). It then updates $y(t)$ by taking steps of size δ (for some carefully chosen δ , as we will discuss later) throughout the time, until $t = 1$. To find out how to update $y(t)$, our algorithm first calculates the potential marginal contribution of each y_j to the current $F(y(t))$. To mimic the function F as defined in Equation (5), our algorithm produces a randomly selected set $R(t)$ that contains each element X_j with probability $y_j(t)$ independently at random. Then it simulates a realization of $R(t)$ —namely, $\Theta_{R(t)}$ —using the probability distributions g_j . The marginal contribution of the element y_j to $F(y(t))$ for this specific realization can be measured simply by $f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})$. To estimate the expected marginal contribution of y_j , our algorithm takes the average of the mentioned subtraction over a large enough number of samples (see step (ii)). Finally, our algorithm finds a maximum-weight independent set according to the weights defined as the expected marginal contribution of each element. This can be done via the classic greedy algorithm for finding the maximum-weight independent sets in matroids (see Rado 1957, Gale 1968, Edmonds 1971). Finally, $y(t + \delta)$ is obtained by adding δ to the entries of $y(t)$ corresponding to the elements of the maximum-weight independent set (see step (iii)).

4.2. The Analysis

To analyze the stochastic continuous greedy algorithm, we follow a similar path to that of the proof of Theorem 1 in the previous section. We show that our sampling in step (ii) provides a good approximation of the weights, and in particular, the weight of the independent set selected at step (iii) in each iteration is close to the weight of the actual maximum-weight independent set. For the rest of this section, “with high probability” means with probability of at least

$(1 - 1/\text{poly}(n))$. The proofs of the following lemmas can be found in the online appendix.

LEMMA 7. *If the set $I(t)$ is chosen by step (iii) of the stochastic continuous greedy algorithm at time t , then with high probability, for any t we have*

$$\begin{aligned} & \sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})] \\ & \geq \left(\max_{I \in \mathcal{F}} \sum_{j: X_j \in I} \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})] \right) - 2d\delta \cdot \text{OPT}, \end{aligned}$$

where OPT is the value of the optimum adaptive policy.

Now, we focus on the change in the value of $F(y(t + \delta))$ compared to that of $F(y(t))$. We show that the actual difference between $F(y(t + \delta))$ and $F(y(t))$ is bounded from below by $\delta(1 - d\delta)$ times the weight of the maximum independent set selected by step (iii). We note that this lemma could be seen as analogous to Equation (9) in our continuous Poisson process.³

LEMMA 8. *At each t we have*

$$\begin{aligned} & F(y(t + \delta)) - F(y(t)) \\ & \geq \delta(1 - d\delta) \sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})], \end{aligned}$$

where d is the rank of the matroid \mathcal{M} .

The final step in the analysis of our algorithm is to provide an analogous statement to the differential equation of Equation (11). This is done through the following lemma, whose proof is presented in the appendix.

LEMMA 9. *Throughout the algorithm, with high probability, for every t we have the following:*

$$\begin{aligned} & F(y(t + \delta)) - F(y(t)) \\ & \geq \delta(1 - 3d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\} - F(y(t)). \end{aligned}$$

Now we are ready to finish the analysis of the algorithm. Let us first denote the value of $(1 - 3d\delta) \cdot \sup_{y \in B(\mathcal{M})} \{f^*(y)\}$ by \mathcal{X} . From Lemma 9 we have $\mathcal{X} - F(y(t + \delta)) \leq (1 - \delta)(\mathcal{X} - F(y(t)))$ with high probability throughout the algorithm. By induction and considering the fact that $F(y(0)) = 0$, we get $\mathcal{X} - F(y(k\delta)) \leq (1 - \delta)^k \mathcal{X}$ for every k such that $k\delta \leq 1$. In particular, for $k = 1/\delta$ we will have

$$\mathcal{X} - F(y(1)) \leq (1 - \delta)^{1/\delta} \mathcal{X} \leq \frac{1}{e} \mathcal{X}.$$

³ We remark that the proof of Lemma 8 does not depend on the accuracy of the estimation and only uses a stochastic dominance result.

Thus, $F(y(1)) \geq (1 - 1/e)\mathcal{X} \geq (1 - 1/e - 3d\delta) \cdot \sup_{y \in B(\mathcal{M})} \{f^*(y)\}$. Note that $\delta = 1/(\lceil 3d/\epsilon \rceil)$. Hence, the point $y(1) \in B(\mathcal{M})$ found by the stochastic continuous greedy algorithm with high probability has the value of at least $(1 - 1/e - \epsilon) \sup_{y \in B(\mathcal{M})} \{f^*(y)\}$. Therefore, we obtain the following result.

LEMMA 10. *For any $\epsilon > 0$, the stochastic continuous greedy algorithm finds a vector $y^* \in \mathcal{B}(\mathcal{M})$ in polynomial time in $|\mathcal{A}|$ and $1/\epsilon$ such that*

$$F(y^*) \geq \sup_{y \in \mathcal{B}(\mathcal{M})} (1 - 1/e - \epsilon) f^*(y).$$

By Lemma 4 and using vector y^* mentioned in the above lemma, we can find in polynomial time a subset $S^* \in \mathcal{F}$ such that $F(S^*) \geq F(y^*) \geq \sup_{y \in \mathcal{B}(\mathcal{M})} (1 - 1/e - \epsilon) f^*(y)$. Recall that by Lemma 16, we have $f^*(y) \geq f^+(y)$ for every $y \in \mathcal{B}(\mathcal{M})$. Hence, $F(S^*) \geq (1 - 1/e - \epsilon) \sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$. On the other hand, by Lemma 5, $\sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$ is an upper bound on the value of the optimal adaptive policy. Therefore, we can conclude the following.

THEOREM 2 (APPROXIMATELY OPTIMAL POLYNOMIAL-TIME ADAPTIVE POLICY). *For any $\epsilon > 0$ and any instance of an SMSM problem, a nonadaptive policy that obtains a $(1 - 1/e - \epsilon)$ -fraction of the value of the optimal adaptive policy can be found in polynomial time (with respect to $|\mathcal{A}|$ and $1/\epsilon$).*

5. Approximation Ratio of Simple Myopic Policies

In this section, we present an adaptive myopic policy for the SMSM problem with a general matroid constraint. The set of feasible solutions is the intersection of κ matroids. Let $\mathcal{M}_1 = (\mathcal{A}, \mathcal{F}_1)$, $\mathcal{M}_2 = (\mathcal{A}, \mathcal{F}_2)$, \dots , $\mathcal{M}_\kappa = (\mathcal{A}, \mathcal{F}_\kappa)$ be κ matroids all defined on the ground set of elements \mathcal{A} . The feasible set for the SMSM problem that we study in this section is $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \cap \dots \cap \mathcal{F}_\kappa$. We will present a myopic policy whose approximation ratio with respect to the optimal adaptive policy is $1/(\kappa + 1)$. At the end of this section, we explain that the myopic policy achieves the approximation ratio of $1 - 1/e$ if $\kappa = 1$ and the matroid is uniform.

The policy is given in Figure 3. Remember that the feasible set of elements is given by \mathcal{F} . At each iteration, from the elements in \mathcal{A} that have not yet been picked, the policy chooses an element with the maximum expected marginal value. We denote by S_t the set of elements chosen by the adaptive policy up to iteration t . By abuse of notation, let s_t denote the realizations of all of the elements in S_t . Also, U_t is the set of elements probed but not chosen by the policy as a result of the matroid constraint.

Figure 3 The Adaptive Myopic Policy

```

Initialize  $t = 0, S_0 = \emptyset, U_0 = \emptyset, s_0 = \emptyset$ .
While  $(U_t \cup S_t \neq \mathcal{A})$  do
   $t \leftarrow t + 1; S_t \leftarrow S_{t-1}; U_t \leftarrow U_{t-1}; s_t \leftarrow s_{t-1}$ .
  Repeat
    Find  $X_i \in \arg \max_{X_i \in \mathcal{A} \setminus (U_t \cup S_t)} \mathbb{E}[F(S_{t-1} \cup \{X_i\}) | s_{t-1}]$ .
    If  $S_{t-1} \cup \{X_i\} \notin \mathcal{F}$ , then
       $U_t \leftarrow U_t \cup \{X_i\}$ ,
    else
       $S_t \leftarrow S_t \cup \{X_i\}$ .
      Observe  $x_i$  and update the state  $s_t$ ,
      i.e.,  $s_t \leftarrow s_{t-1} \cup \{x_i\}$ .
  Until  $(S_t \neq S_{t-1})$  or  $(U_t \cup S_t = \mathcal{A})$ .
End (while)

```

Note the exit condition for the repeat loop. The repeat loop will continue until either a new element X_i is added to the set of selected elements (i.e., $S_t = S_{t-1} \cup \{X_i\} \neq S_{t-1}$) or all elements in the ground set \mathcal{A} are probed (i.e., $U_t \cup S_t = \mathcal{A}$). In the former, the policy will continue probing more elements to possibly add new elements to S_t . In the latter, however, after exiting the repeat loop, the policy immediately jumps out of the while loop, too. Consequently, the policy ends.

THEOREM 3 (ADAPTIVE MYOPIC POLICY). *If feasible domain \mathcal{F} is the intersection of κ matroids, then the approximation ratio of the adaptive myopic policy with respect to any optimal adaptive policy is $1/(\kappa + 1)$.*

The proof is given in Appendix A.2. To prove the above bound, we will rely on the following lemma proved by Fisher et al. (1978). Note that in each iteration of the while loop, at least one new element of \mathcal{A} is probed. Hence, $|\mathcal{A}|$ is an upper bound on the number of iterations of the while loop in the myopic policy. For clarity of presentation, if the policy ends at some iteration t^* where $t^* < |\mathcal{A}|$, we define $U_i := U_{t^*}$ and $S_i := S_{t^*}$ for all $i > t^*$. (In other words, we assume that the policy continues until the $|\mathcal{A}|$ th iteration but does not do anything after iteration t^* .)

LEMMA 11 (FISHER ET AL. 1978). *Let $P \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \dots \cap \mathcal{F}_\kappa$ be any arbitrary subset in the intersection of κ matroids, $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_\kappa$. For $0 \leq i \leq |\mathcal{A}|$, let $C_i = P \cap (U_{i+1} \setminus U_i)$, where U_i 's are coming from the adaptive myopic policy. We will have $\sum_{i=0}^t |C_i| \leq \kappa t$ for every $1 \leq t \leq |\mathcal{A}|$.*

Note that C_i 's, U_i 's, and S_i 's are random sets that depend on the decisions made by the policy and possibly based on the outcomes of the previous realizations of the chosen elements. However, the lemma holds for every realization path because the above property is a consequence of the matroid constraint, not the realizations of the element chosen by the policy.

Define $\Delta_t := F(S_t) - F(S_{t-1})$ for $1 \leq t \leq |\mathcal{A}|$. The main difficulty in proving the theorem compared with the nonstochastic case is that the realized values of Δ_t are not always decreasing—note that

$\mathbb{E}[\Delta_t | s_t] \geq \mathbb{E}[\Delta_{t+1} | s_t]$ does not necessarily hold as a result of the stochastic nature of the problem. (This is because at time t , the actual value of Δ_t has been realized, and it may have turned out to be very small. On the contrary, $\mathbb{E}[\Delta_t | s_{t-1}] \geq \mathbb{E}[\Delta_{t+1} | s_{t-1}]$ always holds, and this is what we are going to use in our proof.) In addition, the sequence of elements chosen by the optimal adaptive policy is random; see the proof in the appendix for more details. We point out that $1/(1 + \kappa)$ is a tight bound on the approximation ratio of the myopic policy (Fisher et al. 1978). In Appendix A.2, we show that the myopic policy obtains a stronger approximation ratio if the constraint matroid is uniform.

THEOREM 4 (UNIFORM MATROID). *Over a uniform matroid, the approximation ratio of the adaptive myopic policy with respect to the optimal adaptive policy is equal to $1 - 1/e$.*

We note that achieving any approximation ratio better than the above is computationally intractable. In fact, as mentioned earlier, even in deterministic settings the problem of maximizing a submodular function with respect to cardinality constraints cannot be approximated with a ratio better than $1 - 1/e$ within polynomial time; see Feige (1998).

6. Discussions

In this section, we discuss some of our previous assumptions and explain how our results would change in their absence.

6.1. Computation of F

The problem of calculating F and functions similar to it has appeared in several related works and is usually solved via sampling. For instance, in the framework of Vondrák (2007, 2008), where the author deals with a similar notion of F but only with Bernoulli distribution for all distributions g_i -s, it is shown that with repeated sampling one can get an estimation of F within a multiplicative error of $1 \pm 1/\text{poly}(n)$. The following proposition generalizes their results to Lipschitz continuous functions.

PROPOSITION 3 (ADDITIVE ERROR). *Suppose f is K -Lipschitz continuous and for all i , $\text{Var}[X_i] \leq \mathcal{V}$. Then, for any arbitrary values of $0 < \epsilon < 1$ and $\delta > 0$, the average value of $f(\Theta_s)$ for $t = \lceil K^2 n^5 \mathcal{V} \epsilon^{-1} \delta^{-0.5} \rceil$ independent samples is within the interval $(\mathbb{E}[f(\Theta_s)] - \epsilon, \mathbb{E}[f(\Theta_s)] + \epsilon)$, with probability at least $1 - \delta$.*

The proof is given in the online appendix. The above lemma could be used to achieve any desirable polynomially small additive error. For instance, $\lceil K^2 n^5 \mathcal{V} \rceil$ samples is enough to ensure that with probability $1 - n^{-2}$, the average sampled value of $f(\Theta_s)$ is within the interval $(\mathbb{E}[f(\Theta_s)] - n^{-2}, \mathbb{E}[f(\Theta_s)] + n^{-2})$.

Note that if we have an upper bound on the values of f (for instance, where X 's can only take binary values), then we can use Chernoff inequality in the proof. That allows us to get the same bounds by only $O(\log K + \log n + \log \frac{1}{\epsilon})$ samples. Also, since in this case the values F can take would be bounded, one can easily come up with multiplicative error terms. This is formalized via the following proposition (proved in the online appendix).

PROPOSITION 4 (MULTIPLICATIVE ERROR). *Suppose for any realization Θ_S with $\Pr[\Theta_S] > 0$, we have $1 \leq f(\Theta_S) \leq \mathcal{F}$. Then, for any arbitrary values of $0 < \epsilon < 1$ and $0 < \delta$, the average value of $f(\Theta_S)$ for $t = \lceil -2\mathcal{F}^2 \ln(\delta/2) \epsilon^{-1} \rceil$ independent samples is within the interval $(\mathbb{E}[f(\Theta_S)](1 - \epsilon), \mathbb{E}[f(\Theta_S)](1 + \epsilon))$, with probability at least $1 - \delta$.*

Throughout this paper we have assumed that the values of the form $\mathbb{E}[f(\Theta_S)]$, or equivalently $F(S)$, for every subset S are accessible via an oracle. However, if such an oracle is not available, then one can incorporate the above lemmas to achieve our results with the desirable degree of precision. To observe this, we note that our algorithms have at most n steps, and at each step there are at most n values of the form $\mathbb{E}[f(\Theta_S)]$ that should be estimated. Hence, at most n^2 estimations of function $\mathbb{E}[f(\Theta_S)]$ will be needed. Because of Proposition 3, each of these estimations will have at most ϵ additive error with probability at least $1 - \delta$ if the average is taken over $\lceil K^2 n^2 \mathcal{V} \epsilon^{-1} \delta^{-0.5} \rceil$ samples. Therefore, union bound implies that with a probability at least $1 - n^2 \delta$, all these samples will be within ϵ error from their true value. Since all the inequalities in our proofs have at most n terms, an error of at most ϵn will appear in our final results. Hence, if we set $\delta = n^{-3}$, we need to average over $\lceil K^2 n^2 \mathcal{V} (\epsilon/n)^{-1} (n^{-3})^{-0.5} \rceil$ (or simply, $\lceil K^2 n^{4.5} \mathcal{V} \epsilon^{-1} \rceil$) samples throughout our algorithms to ensure that with probability at most $1 - n^2 \cdot n^{-3}$ (or simply, $1 - 1/n$) our results will stay valid with an additional error of ϵ .

6.2. Choosing Elements with Replacement

So far, in our model we did not allow for multiple draws of any element. In other words, once an element is picked and its value is realized, it cannot be picked again later. However, this assumption is made without loss of generality and to simplify the presentation. This follows from the observation that one can reduce the problem in which replacements are allowed to our original setting by creating identical and independent copies of each random variable to simulate multiplicity.

Consider the problem of maximizing a stochastic monotone submodular function defined on a function $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$. Also, suppose that we allow for multiple selections of the entries, and if an entry corresponding to any X_i is selected multiple times, then

its final value will be the *maximum* of all the realized values in the draws.⁴ More formally, we allow for m copies of each X_i , where m is an upper bound on the maximum number of elements that could be picked. (For instance, with matroid constraints, m could be the rank of the matroid.) Then, we construct m copies of each X_i —namely, $X_1^{(1)}, X_2^{(1)}, \dots, X_m^{(1)}$. Let a_i denote the number of times element X_i is selected, and let $x_1^{(1)}, x_2^{(1)}, \dots, x_{a_i}^{(1)}$ represent the corresponding realizations. As before, the value of all unrealized elements $x_k^{(i)}$, $a_i + 1 \leq k \leq m$, is defined to be zero. The final value of the objective function will be denoted by function $\phi: \mathbb{R}_+^{mn} \rightarrow \mathbb{R}_+$, which is defined as follows:

$$\begin{aligned} \phi(x_1^{(1)}, \dots, x_m^{(1)}, x_1^{(2)}, \dots, x_m^{(2)}, \dots, x_1^{(n)}, \dots, x_m^{(n)}) \\ := f(\max_k \{x_k^{(1)}\}, \max_k \{x_k^{(2)}\}, \dots, \max_k \{x_k^{(n)}\}). \end{aligned} \quad (13)$$

The next lemma shows that ϕ is monotone and submodular. The proof is presented in the appendix.

LEMMA 12. *If function $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is monotone submodular, then function $\phi: \mathbb{R}_+^{mn} \rightarrow \mathbb{R}_+$, as defined in Equation (13), will be monotone submodular.*

The above lemma shows that stochastic monotone submodular optimization problems with replacement, i.e., when multiple selection of an element might change its value, can be reduced to our original setting in which replacement is not allowed. Hence, our approximation guarantees and the adaptivity gap will still hold if we allow for multiple draws. Our tightness results for approximation guarantees come from deterministic examples. Therefore, for the same reason, they hold as well when selection with replacement is allowed.

We note that the above observations are in sharp contrast to minimization problems (e.g., minimum set cover) and threshold problems in which the goal is to reach a value of function above a certain hard-constrained threshold. For instance, in the problem of stochastic set cover (where the objective is to cover all the elements of a ground set using the minimum number of a given collection of its subsets), allowing a policy to choose multiple copies of an element significantly reduces the adaptivity gap, as shown by Goemans and Vondrák (2006).

6.3. Large Adaptivity Gap Without the Independence Assumption

In this section, we present an example that shows that the adaptivity gap without the independence assumption can be at least $n/2$. Consider an instance

⁴ This is in line with the current literature on selection problems with “multiplicity.” For instance, see Goemans and Vondrák (2006), where the multiple selection of a set in the stochastic set cover problem is considered.

of SMSM2(n, M) with the ground set $\mathcal{A} = \{X_0, X_1, \dots, X_n\}$, where X_0 , with equal probability of $1/n$, takes one of the values in $\{1, 2, \dots, n\}$. Also, for $i, 1 \leq i \leq n$, we have $X_i = M \times \mathbf{I}\{X_0 = i\}$ for some $M \gg n$, i.e., $X_j = M$ if $X_0 = j$; otherwise, $X_j = 0$. The goal is to maximize a linear function f that corresponds to the sum of the realized values of the chosen elements. The constraint is to select at most two elements, i.e., an independent set from the matroid $\mathcal{M} = (\mathcal{A}, \{S \subseteq \mathcal{A}, |S| \leq 2\})$.

It is easy to see that any nonadaptive policy would yield a value of at most $2M/n$ because each element selected will contribute at most M/n in expectation. However, an adaptive policy can select X_0 first and then select $j = x_0$ and, therefore, obtain a value of at least M (exactly $M + j$ to be more precise). Hence, the adaptivity gap is at least $n/2$.

Recently, Golovin and Krause (2011a, b) show that the myopic adaptive policy provides an $(e - 1)/e$ -approximation to the optimal adaptive policy under a cardinality constraint when the adaptive submodularity assumption holds. Under adaptive submodularity, the random variables of interest (namely, X_1, X_2, \dots, X_n) are not necessarily independent. Instead, it suffices that the function f satisfies diminishing conditional expected marginal returns over partial realizations. In addition, they show an approximation factor of $1/(\kappa + 1)$ for myopic adaptive policies under the intersection of κ matroids. Their results show that our Theorems 3 and 4 will also hold in an adaptive submodular setting. In addition, see Agrawal et al. (2012), who introduce the notion of *price of correlations* and study the effect of ignoring correlations in optimization problems.

6.4. Large Adaptivity Gap Without the Monotonicity Assumption

Another important assumption in our model is the monotonicity of the objective function. This assumption is necessary to obtain our adaptivity gap results. Here, we provide an example of nonmonotone stochastic submodular optimization with an arbitrarily large adaptivity gap. Our example is an instance of the stochastic maximum k -coverage (similar to §3.1), with an additional probing cost appearing in the objective. Formally, we are given $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$, a collection of finite sets, where a cost c_i is associated with any X_i . The goal is as follows:

$$\max_{S \subseteq \{1, 2, \dots, n\}, |S| \leq k} \left\{ \left| \bigcup_{i \in S} X_i \right| - \sum_{i \in S} c_i \right\}.$$

We note that $|\bigcup_{i \in S} X_i|$ is a submodular function of S , and $\sum_{i \in S} c_i$ is an additive function of S . Hence, the objective is submodular (but not necessarily monotone) in S .

In the stochastic version of the problem, each X_i —if chosen—will cover a set A_i with probability p independently at random or will turn out to be empty with probability $1 - p$. We now formally show that this is a special case of our setting. We work with the state space $(\theta_1, \theta_2, \dots, \theta_n)$ by starting from $(\circ, \circ, \dots, \circ)$. Once an element X_i is chosen, if it turns out to be nonempty, then $\theta_i := 1$; otherwise, $\theta_i := -1$. We define the value function $f: \{-1, \circ, 1\}^n \rightarrow \mathbb{R}$ as follows:

$$f(\theta_1, \theta_2, \dots, \theta_n) := \left| \bigcup_{\theta_i=1} A_i \right| - \sum_{i: \theta_i \neq \circ} c_i. \quad (14)$$

Similar to the definition of the value function (see Equation (2)), we replace \circ with 0 whenever we calculate the value of function f . The proofs of Lemmas 13–15 are presented in the appendix.

LEMMA 13. *The function f defined by formula (14) is submodular.*

Now, we consider the problem of SMSM3(n, c, p) in which all A_i 's are equal to $\{1\}$, all c_i 's are equal to a fixed $c := -(1 - p) \ln(1 - p)$, and also $k = n$ (meaning any subset $S \subseteq \{1, 2, \dots, n\}$ can be chosen).

LEMMA 14. *The optimal nonadaptive policy for the problem SMSM3(n, c, p) is to pick a subset S consisting of only one element. The expected value of this policy is $p - c$.*

Now, consider an adaptive policy π that picks the elements in $\{1, 2, \dots, n\}$ one by one, until the first time that an X_i turns out to be nonempty and, hence, covers $\{1\}$. Then the policy stops.

LEMMA 15. *The expected value of the adaptive policy π is at least $(1 - (1 - p)^n) - c/p$.*

By combining the lemmas above, we observe that the adaptivity gap of SMSM3(n, c, p) is at least as large as

$$\frac{(1 - (1 - p)^n) - c/p}{p - c} = \frac{p - c - p(1 - p)^n}{p(p - c)} = \frac{1}{p} - \frac{(1 - p)^n}{p - c}.$$

Note that fixing p , by choosing large enough values of n , the bound gets arbitrarily close to $1/p$. Hence, taking the value of p sufficiently close to 0 makes the adaptivity gap arbitrarily large.

Note that in our example, the value function f takes negative values at some points. Our previous results were derived under the assumption that f is defined on R_+^n and only takes nonnegative values. This is without loss of generality for monotone functions if we assume the value of taking “no action” is zero (i.e., $f(\circ, \dots, \circ) = 0$). We leave it as an open question to find the adaptivity gap for nonmonotone nonnegative stochastic submodular maximization.

7. Concluding Remarks

In this paper we studied the problem of maximizing monotone submodular functions with respect to matroid constraints in a stochastic setting. Our model can be applied to various problems that involve both diminishing marginal returns and a stochastic environment. To capture the effect of partial contributions, we considered real-valued submodular functions instead of submodular set functions. We showed that a myopic adaptive policy is guaranteed to achieve a $(1 - 1/e \approx 0.63)$ -approximation of the optimal (adaptive) policy. Also, we studied the concept of the adaptivity gap to compare the performance of nonadaptive policies (which are very easy to implement) to that of adaptive policies. We showed that a very straightforward myopic nonadaptive policy achieves a $(\frac{1}{2} \times (1 - 1/e) \approx 0.316)$ -approximation of the value of the optimal policy. Moreover, we showed that the adaptivity gap for these problems is at most $e/(e - 1)$, which implies the existence of a nonadaptive policy that achieves a 0.63-approximation of the optimal policy. Finally, we provided a polynomial algorithm to find a nonadaptive policy that achieves a $(0.63 - \epsilon)$ -approximation of the optimal policy for any positive ϵ . In our proofs, we generalized the techniques from the previous works in the literature (especially those in Vondrák 2008 and Călinescu et al. 2011) to the stochastic and continuous setting by proving new stochastic dominance results that could be of independent interest.

There are many interesting questions that we leave open for future research. One such question is whether our algorithms perform better for the specific submodular functions that appear in practice. For instance, the submodular functions that arise from the context of viral marketing have a very specific structure as a result of the properties of the underlying graph of relationships in social networks (such as being self-similar and scale-free; see Barabási and Réka 2009). Exploiting these properties may lead to improved analysis of our policies or to new policies with better performance guarantees. Also, studying similar classes of functions for which the adaptivity gap is bounded would be an interesting extension. We believe that one promising class of such functions could be the almost-concave functions, which are the functions whose values can be approximated closely by a concave function.

Another interesting direction would be to study the problem of maximizing *nonmonotone* submodular functions in a stochastic setting; applications include facility location problem and maximum entropy sampling (see Lee et al. 2010). The example provided in §6.4 showed that the adaptivity gap of nonmonotone stochastic submodular maximization can

be unbounded if we do not assume the nonnegativity of the objective function. We leave it as an open question to find the adaptivity gap for *nonmonotone nonnegative* stochastic submodular maximization. We emphasize that finding provably good approximate solutions for this problem has turned out to be more difficult than that of the monotone case even in the deterministic setting. Various methodologies have been used to approach the problem, including local search (Lee et al. 2010, Vondrák 2013) and simulated annealing (Gharan and Vondrák 2011). Feldman et al. (2011) were the first to extend the *continuous greedy* approach of Călinescu et al. (2011) to the nonmonotone setting and provided a $1/e$ -approximation for the problem under cardinality constraint. Recently, Buchbinder et al. (2014) built on the *double greedy* approach⁵ and used randomization to deal with nonmonotonicity of the objective function. A promising approach to extend their results to the stochastic setting would be to combine it with our stochastic continuous greedy and develop a unified *stochastic continuous double greedy* approach that could handle both the stochasticity of the environment and the nonmonotonicity of the objective function.

Supplemental Material

Supplemental material to this paper is available at <http://dx.doi.org/10.1287/mnsc.2015.2254>.

Acknowledgments

The authors thank Amin Saberi, Jan Vondrák, Marek Adamczyk, Chandra Chekuri, and the anonymous referees whose comments and suggestions significantly improved this work. The authors acknowledge that the example in §3.1 was brought to their attention by Jan Vondrák.

Appendix. Proofs

A.1. Appendix to §3

PROOF OF LEMMA 5. An optimal adaptive policy only chooses independent sets of \mathcal{M} . Because of monotonicity, all of these are also the bases of the matroid. Hence, an optimal adaptive policy ends up selecting one of the basis of the matroid. Thus, vector y , whose i th entry represents the probability that element $X_i \in \mathcal{A}$ is ultimately chosen by this policy, is a convex combination of the characteristic vectors of the basis of \mathcal{M} . Moreover, the expected value of the adaptive policy is bounded by $f^+(y)$ because the policy has to satisfy the three properties mentioned earlier. \square

⁵ The double greedy approach was introduced by Buchbinder et al. (2012) for unconstrained nonmonotone submodular optimizations. Their technique generalizes on the continuous greedy approach by keeping track of two solutions (starting from $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$, instead of only one solution starting from $(0, 0, \dots, 0)$ as in the continuous greedy) and continuously moving them toward each other by a greedy rule until they coincide at time $t = 1$.

LEMMA 16. For $y \in \mathcal{B}(\mathcal{M})$, we have $f^+(y) \leq f^*(y)$.

PROOF. Consider the function f^+ . Following the notation of $\Theta^{\dagger j}$, for every $x \in \mathbb{R}$, we define $\Theta^{\dagger j:x}$ as a vector that has all entries except the j th one as the same as Θ , and its j th entry is defined as $\max\{\theta_j, x\}$. Fix any $y \in \mathcal{B}(\mathcal{M})$, any feasible probability measure α with respect to y (see Definition (6) for f^+), and any given vector $\underline{\Theta} \in \mathbb{R}_+^n$. We have

$$\begin{aligned} & \int_{\Theta} \alpha_{\Theta} f(\Theta) d\Theta \\ & \leq \int_{\Theta} \alpha_{\Theta} \left[f(\Theta) + \sum_{j=1}^n (f(\Theta^{\dagger j, \theta_j}) - f(\Theta)) \right] d\Theta \\ & = f(\Theta) + \sum_{j=1}^n y_j \left[\int_{\theta_j > \theta_j} (f(\Theta^{\dagger j, \theta_j}) - f(\Theta)) g_j(\theta_j) d\theta_j \right]. \end{aligned}$$

The inequality above holds because of the submodularity of f , and the equality is a consequence of the properties of the feasible probability measure α . However, $\int_{\theta_j > \theta_j} f(\Theta^{\dagger j, \theta_j}) g_j(\theta_j) d\theta_j$ is simply $\mathbb{E}[f(\Theta^{\dagger j})]$. Hence,

$$\int_{\Theta} \alpha_{\Theta} f(\Theta) \leq f(\underline{\Theta}) + \sum_{j=1}^n y_j (\mathbb{E}[f(\Theta^{\dagger j})] - f(\underline{\Theta})).$$

Note that the inequality above holds for any arbitrary $y \in \mathcal{B}(\mathcal{M})$, any feasible probability measure α , and any vector $\underline{\Theta} \in \mathbb{R}_+^n$. Thus, we have

$$\begin{aligned} f^+(y) &= \sup_{\alpha} \left\{ \int_{\Theta} \alpha_{\Theta} f(\Theta) d\Theta \right\} \\ &\leq \inf_{\underline{\Theta}} \left\{ f(\underline{\Theta}) + \sum_{j=1}^n y_j (\mathbb{E}[f(\Theta^{\dagger j})] - f(\underline{\Theta})) \right\} \\ &= f^*(y). \quad \square \end{aligned} \quad (15)$$

A.2. Appendix to §5

PROOF OF THEOREM 3. Let P be the (random) set of elements chosen by the optimal adaptive policy. Let S denote the final selection of the elements by our myopic policy; i.e., $S = S_{|\mathcal{A}|}$. Moreover, let t be any arbitrary number between 1 and $|\mathcal{A}|$. Consider a realization s_t of S_t . By submodularity and monotonicity of F we have

$$\begin{aligned} \mathbb{E}[F(P) - F(S) | s_t] &\leq \mathbb{E}[F(P \cup S) - F(S) | s_t] \\ &\leq \mathbb{E} \left[\sum_{l \in P \setminus S} (F(S + l) - F(S)) | s_t \right]. \end{aligned}$$

The expectations in the above inequality are taken over the probability distribution of all possible realizations, conditioned on the realized values of elements in S_t being equal to s_t .

Since the above inequality holds for all s_t , we have

$$\mathbb{E}[F(P)] - \mathbb{E}[F(S)] \leq \mathbb{E} \left[\sum_{l \in P \setminus S} (F(S + l) - F(S)) \right]. \quad (16)$$

Now, define $C_t = P \cap (U_{t+1} \setminus U_t)$ for all $0 \leq t \leq |\mathcal{A}|$. In other words, C_t represents the elements of the optimum solution P that our myopic policy has probed (but not picked) after

picking its t th element and before picking its $(t+1)$ st element. By the construction of the policy, we have $U_0 \subseteq U_1 \subseteq \dots \subseteq U_{|\mathcal{A}|}$. Hence, $(U_{t+1} \setminus U_t)$ and $(U_{t'+1} \setminus U_{t'})$ are disjoint for any $t \neq t'$. Therefore, C_t 's are all disjoint. On the other hand, Lemma 11 implies that $|C_0| = 0$, and hence $C_0 = \emptyset$. Therefore, $\bigcup_{t=1}^{|\mathcal{A}|} C_t$ represents all the elements in P that were not selected by our policy; i.e., $\bigcup_{t=1}^{|\mathcal{A}|} C_t = P \setminus S$. As a result, we can rewrite inequality (16) as follows:

$$\mathbb{E}[F(P)] - \mathbb{E}[F(S)] \leq \sum_{t=1}^{|\mathcal{A}|} \mathbb{E} \left[\sum_{l \in C_t} (F(S + l) - F(S)) \right].$$

By expanding the expectation, we have

$$\mathbb{E}[F(P)] - \mathbb{E}[F(S)] \leq \sum_{t=1}^{|\mathcal{A}|} \int_{s_{t-1}: S_{t-1} \in \mathcal{F}} \mathbb{E} \left[\sum_{l \in C_t} F(S + l) - F(S) | s_{t-1} \right] \Pr[s_{t-1}] ds_{t-1}. \quad (17)$$

Fix any s_{t-1} and $l \in C_t$. Note that the myopic policy probes the elements in decreasing order of their expected marginal value, and it has picked the t th element of S (i.e., the only element in $S_t \setminus S_{t-1}$) before $l \in C_t$. Hence, we have $\mathbb{E}[F(S_{t-1} + l) - F(S_{t-1}) | s_{t-1}] \leq \mathbb{E}[F(S_t) - F(S_{t-1}) | s_{t-1}] = \mathbb{E}[\Delta_t | s_{t-1}]$. Now, since F is submodular and $S_{t-1} \subseteq S$, for any $l \in C_t$ we have $\mathbb{E}[\sum_{l \in C_t} F(S + l) - F(S) | s_{t-1}] \leq \mathbb{E}[\sum_{l \in C_t} \Delta_t | s_{t-1}]$. By plugging the above inequality into (17), we get

$$\mathbb{E}[F(P)] - \mathbb{E}[F(S)] \leq \sum_{t=1}^{|\mathcal{A}|} \int_{s_{t-1}: S_{t-1} \in \mathcal{F}} \mathbb{E} \left[\sum_{l \in C_t} \Delta_t | s_{t-1} \right] \Pr[s_{t-1}] ds_{t-1}.$$

Using telescopic sums and the linearity of expectation, we derive the following; here, $\Delta_{|\mathcal{A}|+1}$ is defined as 0:

$$\begin{aligned} \mathbb{E}[F(P)] - \mathbb{E}[F(S)] &\leq \sum_{t=1}^{|\mathcal{A}|} \int_{s_{t-1}: S_{t-1} \in \mathcal{F}} \mathbb{E} \left[\sum_{l \in C_t} \sum_{j=t}^{|\mathcal{A}|} (\Delta_j - \Delta_{j+1}) | s_{t-1} \right] \Pr[s_{t-1}] ds_{t-1} \\ &= \sum_{j=1}^{|\mathcal{A}|} \sum_{t=1}^j \int_{s_{t-1}: S_{t-1} \in \mathcal{F}} \mathbb{E} \left[\sum_{l \in C_t} (\Delta_j - \Delta_{j+1}) | s_{t-1} \right] \Pr[s_{t-1}] ds_{t-1}. \end{aligned}$$

Note that by applying the law of total probability, for every t and j the integral term in the above is in fact equal to $\mathbb{E}[\sum_{l \in C_t} (\Delta_j - \Delta_{j+1})]$. Again, we use the law of total probability but this time by conditioning on s_{j-1} instead of s_{t-1} . We will have

$$\begin{aligned} \mathbb{E}[F(P)] - \mathbb{E}[F(S)] &\leq \sum_{j=1}^{|\mathcal{A}|} \sum_{t=1}^j \int_{s_{j-1}: S_{j-1} \in \mathcal{F}} \mathbb{E} \left[\sum_{l \in C_t} (\Delta_j - \Delta_{j+1}) | s_{j-1} \right] \Pr[s_{j-1}] ds_{j-1}. \end{aligned}$$

Note that the term $(\Delta_j - \Delta_{j+1})$ in the innermost summation does not depend on the index of the sum, i.e., $l \in C_t$. Hence, the right-hand side can be written as follows:

$$= \sum_{j=1}^{|\mathcal{A}|} \sum_{t=1}^j \int_{s_{j-1}: S_{j-1} \in \mathcal{F}} (E[|C_t| \mathbb{E}[\Delta_j - \Delta_{j+1} | s_{j-1}]] \Pr[s_{j-1}]) ds_{j-1}.$$

Note that conditioned on s_{j-1} , the term $\mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}]$ is by definition a constant, and we can take it out from the outer expectation. Hence,

$$\begin{aligned} \mathbf{E}[F(P)] - \mathbf{E}[F(S)] \\ \leq \sum_{j=1}^{|S|} \int_{s_{j-1}: s_{j-1} \in \mathcal{F}} \left(\mathbf{E} \left[\sum_{t=1}^j |C_t| | s_{j-1} \right] \mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}] \right) \Pr[s_{j-1}] ds_{j-1}. \end{aligned} \quad (18)$$

We now use Lemma 11, which implies that in every realization $\sum_{t=1}^j |C_t| \leq \kappa j$. We also use the fact that because of the submodularity and the rule of the policy, we have $\mathbf{E}[(\Delta_j - \Delta_{j+1}) | s_{j-1}] \geq 0$. We conclude that

$$\begin{aligned} \mathbf{E}[F(P)] - \mathbf{E}[F(S)] \\ \leq \sum_{j=1}^{|S|} \int_{s_{j-1}: s_{j-1} \in \mathcal{F}} \left(\mathbf{E} \left[\sum_{t=1}^j |C_t| | s_{j-1} \right] \mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}] \right) \\ \cdot \Pr[s_{j-1}] ds_{j-1} \\ \leq \sum_{j=1}^{|S|} \int_{s_{j-1}: s_{j-1} \in \mathcal{F}} \kappa j \mathbf{E}[(\Delta_j - \Delta_{j+1}) | s_{j-1}] \Pr[s_{j-1}] ds_{j-1} \\ = \sum_{j=1}^{|S|} \int_{s_{j-1}: s_{j-1} \in \mathcal{F}} \kappa \mathbf{E}[\Delta_j | s_{j-1}] \Pr[s_{j-1}] ds_{j-1} \\ = \kappa \sum_{j=1}^{|S|} \mathbf{E}[\Delta_j] = \kappa \mathbf{E}[F(S)]. \end{aligned}$$

Therefore, $\mathbf{E}[F(P)] \leq (\kappa + 1)\mathbf{E}[F(S)]$, as desired. Finally, Fisher et al. (1978) have shown that even in the nonstochastic setting, in the worst case, the approximation ratio of the greedy algorithm (hence the myopic policy) is equal to $1/(\kappa + 1)$. Therefore, this bound is tight. \square

PROOF OF THEOREM 4. This proof is similar to the proof of Kleinberg et al. (2004) for submodular set functions. The main technical difficulty in proving our claim is that the optimal adaptive policy here is a random set whose distribution depends on the realized values of the elements of \mathcal{S} .

Let P denote the (random) set chosen by an optimal adaptive policy. Also, denote the marginal value of the t th element chosen by the myopic policy by Δ_t ; i.e., $\Delta_t = F(S_t) - F(S_{t-1})$. Now consider a realization s_t of S_t . Because F is stochastic monotone submodular, we have

$$\begin{aligned} \mathbf{E}[F(P) | s_t] &\leq \mathbf{E}[F(P \cup S_t) | s_t] \\ &\leq \mathbf{E} \left[F(S_t) + \sum_{l \in P} (F(S_t + l) - F(S_t)) | s_t \right]. \end{aligned} \quad (19)$$

The above expectations are taken over all realization of P such that the realized values of elements in S_t are according to s_t . Because the myopic policy chooses the element with maximum marginal value, for every $l \in P$, we have $\mathbf{E}[\Delta_{t+1} | s_t] \geq \mathbf{E}[F(S_t + l) - F(S_t) | s_t]$. Therefore, we get $\mathbf{E}[F(P) | s_t] \leq \mathbf{E}[F(S_t) + k\Delta_{t+1} | s_t]$. Since this inequality holds for every possible path in the history, by adding up all such inequalities for all t , $0 \leq t \leq k-1$, we have

$$\mathbf{E}[F(P)] \leq \mathbf{E}[F(S_k)] + k\mathbf{E}[\Delta_{k+1}] = \mathbf{E}[\Delta_1 + \dots + \Delta_k] + k\mathbf{E}[\Delta_{k+1}].$$

We multiply the t th inequality, $0 \leq t \leq k-1$, by $(1 - 1/k)^{k-1-t}$, and we add them all. The sum of the coefficients of $\mathbf{E}[F(P)]$ is equal to

$$\begin{aligned} \sum_{t=0}^{k-1} \left(1 - \frac{1}{k}\right)^{k-1-t} &= \sum_{t=0}^{k-1} \left(1 - \frac{1}{k}\right)^t = \frac{1 - (1 - 1/k)^k}{1 - (1 - 1/k)} \\ &= k \left(1 - \left(1 - \frac{1}{k}\right)^k\right). \end{aligned} \quad (20)$$

On the right-hand side, the sum of the coefficients corresponding to the term $\mathbf{E}[\Delta_t]$, $1 \leq t \leq k$, is equal to

$$\begin{aligned} k \left(1 - \frac{1}{k}\right)^{k-t} + \sum_{j=t}^{k-1} \left(1 - \frac{1}{k}\right)^{k-1-j} \\ = \left(1 - \frac{1}{k}\right)^{k-t} + k \left(1 - \left(1 - \frac{1}{k}\right)^{k-t}\right) = k. \end{aligned} \quad (21)$$

Thus, by inequalities (20) and (21), we conclude that $(1 - (1 - 1/k)^k)\mathbf{E}[F(P)] \leq \sum_{t=1}^k \mathbf{E}[\Delta_t] = \mathbf{E}[F(S_k)]$. Hence, the approximation ratio of the myopic policy is at least $1 - 1/e$. \square

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