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Estimation of Choice-Based Models Using Sales Data from a Single Firm

Jeffrey P. Newman
School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332, jpn@gatech.edu

Mark E. Ferguson
Moore School of Business, University of South Carolina, Columbia, South Carolina 29208, mark.ferguson@moore.sc.edu

Laurie A. Garrow
School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332, laurie.garrow@ce.gatech.edu

Timothy L. Jacobs
US Airways, Phoenix, Arizona 85034, tim.jacobs@usairways.com

We develop a parameter estimation routine for multinomial logit discrete choice models in which one alternative is completely censored, i.e., when one alternative is never observed to have been chosen in the estimation data set. Our method is based on decomposing the log-likelihood function into marginal and conditional components. Our method is computationally efficient, provides consistent parameter estimates, and can easily incorporate price and other product attributes. Simulations based on industry hotel data demonstrate the superior computational performance of our method over alternative estimation methods that are capable of estimating price effects. Because most existing revenue management choice-based optimization algorithms do not include price as a decision variable, our estimation procedure provides the inputs needed for more advanced product portfolio availability and price optimization models.

Keywords: choice-based revenue management; discrete choice modeling; censored alternatives; sampling of alternatives
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1. Introduction

Understanding demand for products and services is an integral part of many fields. Within the revenue management (RM) community, there has been increased interest in modeling demand as the collection of individuals' decisions using discrete choice models. This interest is being driven by the fact that the majority of travel reservations are made online, which makes it easy for consumers to compare prices and product offerings across multiple competitors. Today's market conditions are different than those that existed when the first-generation RM systems were developed. Traditional RM systems, which assume independence among customer segment demand distributions, have struggled to adapt to these new market conditions, leading to calls for fundamentally new "choice-based" RM systems that use discrete choice models to forecast demand. Discrete choice models are appropriate for environments in which consumers can easily identify and compare product offerings.

The seminal work in choice-based RM was published by Talluri and van Ryzin (2004; hereafter referred to as TvR). They incorporate a discrete choice model of demand into their objective function to provide an exact analysis of the capacity allocation problem for a single-leg airline route. Subsequent research has focused on developing methods to more efficiently solve choice-based RM optimization problems or on developing theoretical extensions to network-level models or models that incorporate multiple customer segments. However, with a few notable exceptions, the majority of choice-based RM work has assumed that choice parameters were given; only a few papers have investigated the *estimation* of choice-based RM parameters.

Solving the underlying estimation problem for the demand model is crucial before any of the choice-based RM work can be implemented in practice. This problem is difficult because choice-based RM systems must be able to forecast the number of customers who purchase one of the firm's products as well as the number of customers who purchase competitors' products or who decide not to purchase. However, the data available in RM settings for the estimation of discrete choice parameters are based solely on observed purchases (or bookings) of the firm's products. This leads to a unique estimation problem: parameters must be estimated for a "no-purchase" alternative that was never observed to have been chosen in the estimation data. Consequently, existing estimation methods in the marketing and economics fields, such as those based on sampling of alternatives, are not viable because they require each alternative to have been observed as having been chosen at least once in the estimation data set.

To address this problem, TvR developed an estimation method that uses observed variations in a firm's bookings and product offerings over time to obtain an estimate for the constant of the censored no-purchase alternative. However, this estimation method, which is based on expectation-maximization routines, exhibits prohibitively long estimation times. To reduce estimation times, researchers have typically used a constants-only utility formulation, where the only parameters to be estimated are a set of identifiable constants for each of the firm's products and the no-purchase alternative. Although this makes the problem more tractable, it means that it is not possible to estimate parameters for product-level attributes, such as price.

Our paper addresses an important gap in the literature by applying a two-step estimation method for multinomial discrete choice models based on decomposing the log-likelihood function into particular marginal and conditional components. Our method is computationally efficient, provides consistent parameter estimates, and can easily incorporate price and other product attributes, which provide the inputs needed for more advanced product portfolio availability and price optimization models.

2. Problem Formulation and Literature Review

The single-leg choice-based RM problem proposed by TvR can be described by a few key features. A firm offers a variety of products for sale that vary in availability and/or attributes (e.g., price) over time. Customers are assumed to arrive at a constant but unknown rate over time, and each arriving customer can choose whether or not to purchase one of the firm's available products. The firm can observe sales (or booking) events but cannot observe customers who arrive and then choose not to purchase any of its products. The goal is to use the past sales observations to discover both the unknown customer arrival rate and the parameters of the process that governs customers' choices. Estimates for the customer arrival rate and underlying choice model can then be directly integrated into the objective function of a dynamic program (DP).

In TvR, the customer arrival process is modeled by dividing the booking horizon into a series of small discrete time slices, during which a customer might or might not arrive. The slices are assumed to be small enough that the probability of two customers arriving in the same time period is negligible. Arrivals are modeled as a Bernoulli process, and the arrival probability (denoted by λ) is interpreted as the probability that a customer arrives to the system in a given discrete time period within the booking horizon. Importantly, the population of customers includes those who arrive and purchase a product from the firm as well as those who arrive and either decide to purchase a product from a competitor *or* not to purchase at all. However, the data used for estimation contains only observations of customers who arrive and purchase a product from the firm. As a result, it is impossible to distinguish between periods in which a customer arrived and did not purchase from the firm (i.e., purchased from a competitor *or* did not purchase at all) and periods in which a customer did not arrive.

Given an arrival, TvR model a customer's product choice with a multinomial logit (MNL) discrete choice model. Such models are based on the concept of utility, which represents the "value" customers place on different attributes. Although we cannot observe utility directly, we can observe some of the factors that influence it, allowing us to model utility by expressing it as the sum of an observed component, V, and an unobserved random component, ϵ. The component V is typically assumed to be a linear-in-parameters function of attributes that vary across individuals and alternatives. Formally, $V_{ti} = \boldsymbol{\beta}'\mathbf{X}_{ti}$, where $\boldsymbol{\beta}$ is an unknown vector of k parameters and $\mathbf{X}_{ti}$ is a vector of k attributes of alternative i for a customer at time t. Those attributes often include a set of indicator variables referred to as alternative specific constants (ASCs), which indicate the identity of each product (i.e., ASC_i takes on a value of 1 when the alternative is product i and 0 otherwise). It is possible to specify a model where $\mathbf{X}_{ti}$ consists only of ASCs, which is referred to as a constants-only model. In the TvR formulation, the set of alternatives includes a no-purchase alternative, which we denote with $\oslash$, representing customers who decide to purchase products from a competitor or decide not to purchase at all.

Different discrete choice models can be derived via different assumptions on the unobservable component of utility, ϵ. The MNL model (McFadden 1974), which is one of the most common discrete choice models (and the one that has been used in almost all choice-based RM problems, including TvR), is derived by assuming the ϵ_{ti}'s are independent and identically distributed Gumbel (type I extreme value). Under this assumption, the choice probability for alternative i, and the probability for the no-purchase alternative for a customer arriving at time t are given, respectively, as

$$P_{ti}(\boldsymbol{\beta}, S_t, \mathbf{X}_t) = \frac{\exp(\boldsymbol{\beta}'\mathbf{X}_{ti})}{\sum_{j\in S_t}\exp(\boldsymbol{\beta}'\mathbf{X}_{tj}) + \exp(V_{t\oslash})}, \quad i \in S_t, \quad (1)$$

and

$$P_{t\oslash}(\boldsymbol{\beta}, S_t, \mathbf{X}_t) = \frac{\exp(V_{t\oslash})}{\sum_{j\in S_t}\exp(\boldsymbol{\beta}'\mathbf{X}_{tj}) + \exp(V_{t\oslash})}. \tag{2}$$

These probabilities are a function of the available choice set S_t available at time t, the matrix of attributes $\mathbf{X}_t$ of the available alternatives in that choice set and attributes of the choice itself (e.g., the day of the week when the purchase occurs), and a vector of parameters $\boldsymbol{\beta}$ that needs to be estimated. For convenience, we denote the complement of (2) as $P_{t\circledast}$ (i.e, $P_{t\circledast}(\boldsymbol{\beta}, S_t, \mathbf{X}_t) = 1 - P_{t\oslash}(\boldsymbol{\beta}, S_t, \mathbf{X}_t)$), which represents the probability of purchasing some product from the firm. We will also define a set of choice probabilities for the alternatives conditioned on the fact that some purchase is made:

$$P_{ti\mid\circledast}(\boldsymbol{\beta}\mid t, S_t, \mathbf{X}_t) = \frac{\exp(\boldsymbol{\beta}'\mathbf{X}_{ti})}{\sum_{j\in S_t}\exp(\boldsymbol{\beta}'\mathbf{X}_{tj})}. \tag{3}$$

Note that (3) specifically excludes the no-purchase alternative.

It is well known that utility maximization models are only unique up to the scale of utility (Ben-Akiva and Lerman 1985). Adding a constant to every utility value will result in an identical set of probabilities. Thus, it is not possible to generate unique parameter estimates for ASCs for each alternative, and at least one alternative in the model needs to be set as the reference and given a fixed constant utility value to ensure these parameter estimates are unique. TvR choose the reference alternative to be the no-purchase alternative by setting $V_{t\oslash} = 0$, but as will become evident in the next section, that choice will become restrictive. Instead, we parameterize the utility of the no-purchase alternative, denoting it as γ, and assume that the modeler imposes other suitable restrictions (see Ferguson et al. 2012) within the $\boldsymbol{\beta}$ vector to ensure the parameter estimates are unique.

In the application of the MNL model in a RM dynamic program, the customer behavioral parameters $\boldsymbol{\beta}$ and γ are assumed to be given, and S_t is used as the decision variable. However, finding the behavioral parameters requires observations of past behavior, where $\mathbf{X}_t$ and S_t are observed and $\boldsymbol{\beta}$ and γ are estimated. Our focus is on the estimation of the parameters, so for notational clarity the probabilities indicated by (1) are denoted as $P_{ti}(\boldsymbol{\beta}, \gamma)$; the choice probability of the no-purchase alternative and its complement are denoted $P_{t\oslash}(\boldsymbol{\beta}, \gamma)$ and $P_{t\circledast}(\boldsymbol{\beta}, \gamma)$, respectively; and the choice probabilities conditional on some purchase given in (3) are denoted as $P_{ti\mid\circledast}(\boldsymbol{\beta})$. Importantly, $P_{ti\mid\circledast}$ is a function of $\boldsymbol{\beta}$ but not γ. In each case, the conditionality upon the observed data and available choice set is implied.

A direct estimation of the likelihood-maximizing parameters of the arrival and discrete choice models is complicated by missing data. Formally, when there are N possible products for sale, the customer arrival and purchase processes can result in one of $N+2$ mutually exclusive and collectively exhaustive outcomes: N discrete outcomes associated with a customer arriving and choosing to purchase one of the N products offered by the firm, one discrete outcome associated with a customer arriving who chooses not to purchase any product from the firm, and one discrete outcome associated with no customer arriving.

An uncensored data observation would uniquely observe which of these $N+2$ outcomes occurred in each discrete time slice. A censored data observation differs in that the no-purchase and no-arrival outcomes are indistinguishable, jointly representing the same observable outcome. The complexity of the log-likelihood function for censored data arises because the censoring requires the summation of the no-purchase and no-arrival probabilities in calculating the likelihood, before taking the logarithm.

At an abstract level, the log-likelihood function is given as

$$LL(\boldsymbol{\beta}, \gamma, \lambda) = \sum_{t\in\mathcal{T}} \log(\Pr_t(\delta_t \mid \boldsymbol{\beta}, \gamma, \lambda)), \tag{4}$$

where $\mathcal{T}$ is the set of all discrete time slices in the estimation data, δ_t is the observed outcome in time slice t (e.g., a Y class ticket on a particular flight), and $\Pr_t(i\mid \boldsymbol{\beta}, \gamma, \lambda)$ is the probability of outcome i occurring in time slice t, given the model and parameters $\boldsymbol{\beta}$, γ, and λ. For the time slices in which a sale is observed (denote this set of time slices as $\mathcal{P}$), the value $\Pr_t(\delta_t \mid \boldsymbol{\beta}, \gamma, \lambda)$ is given by the joint probability that a customer arrives and that the arriving customer chooses to purchase product δ_t from the firm, giving

$$\Pr_t(\delta_t \mid \boldsymbol{\beta}, \gamma, \lambda) = \lambda P_{t\delta_t}(\boldsymbol{\beta}, \gamma), \quad \forall\, t\in\mathcal{P}.$$

For all other time slices (denote this set of time slices as $\bar{\mathcal{P}}$), the outcome is that a sale is not observed. The probability of that outcome is the sum of the probabilities of the two possible reasons: no purchase and no arrival, giving

$$\Pr_t(\delta_t \mid \boldsymbol{\beta}, \gamma, \lambda) = \underbrace{\lambda P_{t\oslash}(\boldsymbol{\beta}, \gamma)}_{\text{no purchase}} + \underbrace{(1-\lambda)}_{\text{no arrival}}, \quad \forall\, t\in\bar{\mathcal{P}}.$$

Thus, the resulting log-likelihood function for the model with censored data can be written as

$$LL(\boldsymbol{\beta}, \gamma, \lambda) = \sum_{t\in\mathcal{P}}[\log(\lambda P_{t\delta_t}(\boldsymbol{\beta}, \gamma))] + \sum_{t\in\bar{\mathcal{P}}}[\log(\lambda P_{t\oslash}(\boldsymbol{\beta}, \gamma) + (1-\lambda))]. \tag{5}$$

Maximum-likelihood estimators for $\boldsymbol{\beta}$, γ, and λ can be found by maximizing this log-likelihood function.

However, this function is not generally concave, and TvR suggest it may be hard to maximize. Nevertheless, the size and scope of some RM problems may allow for the direct maximization of this log-likelihood function. If a particular problem is found to be computationally tractable using regular maximum-likelihood estimators, then that approach should be preferentially employed. However, if directly maximizing (5) is found to be difficult or slow, or if there is concern about converging to local optima, then other approaches can be considered.

Instead of directly maximizing (5), TvR use the expectation-maximization (EM) method. (See Dempster et al. 1977 for a description of the EM method and Vulcano et al. 2010 for a description of how it is applied in the TvR formulation.) The EM method estimates the market size (i.e., the total number of "potential" customers) without having actual observations of no-purchase customers. The EM method avoids calculating the actual censored data log likelihood given in (5) and instead focuses on calculating the expected value of that log likelihood, which incorporates observed data as well as the expected value of unobserved data. In their Equation (15), TvR write an expected log-likelihood function for the sales model as

$$\begin{aligned} E[LL(\boldsymbol{\beta},\lambda)] = \sum_{t\in\mathcal{P}}[\log(\lambda)+\log(P_{t\delta_t}(\boldsymbol{\beta},0))] \\ +\sum_{t\in\bar{\mathcal{P}}}[\hat{a}(t)(\log(\lambda)+\log(P_{t\varnothing}(\boldsymbol{\beta},0))) \\ +(1-\hat{a}(t))\log(1-\lambda)], \end{aligned} \quad (6)$$

where $\hat{a}(t)$ is the expected value of $a(t)$, an indicator variable that takes on a value of 1 if an arrival occurred at time t and 0 otherwise. Where $t\in\bar{\mathcal{P}}$, the value of $a(t)$ cannot be observed because either a customer arrives but chooses to purchase nothing or no customer arrives—as noted above, these two outcomes are indistinguishable in the data. The EM algorithm iterates between updating the expected log-likelihood function given a distribution for the missing data (essentially, replacing $a(t)$ in (6) with its expected value) and maximizing that function, a process which is widely accepted as effective but slow. Wu (1983) shows that when the EM algorithm converges, it converges to a local stable point of the likelihood function. Often, this represents a local maximum for the underlying censored data log likelihood, but it could also be a saddle point or a local minimum of that function.

Maximizing the function given in (6) is appealing because $\boldsymbol{\beta}$ and λ are fully separable, and both components are easy to maximize, globally concave subproblems. However, this approach requires multiple iterations between the expectation and maximization steps. As we demonstrate in later sections, in practical applications, the number of iterations required to obtain accurate parameter estimates can be prohibitively large.

Problems that require estimating a discrete choice model using censored data, as is examined in this work, are found in multiple fields. Choice models are frequently used to estimate product substitution patterns in retailing applications and are used to support decisions related to pricing, stocking levels, and assortment planning. Vulcano et al. (2012) provide a general overview of methods (including non-choice-based methods) for estimating lost sales and substitution effects using retail transaction data, and Weatherford and Ratliff (2010) provide a review for the RM field. Within this body of literature, several papers have sought to apply or improve TvR's basic algorithm. A notable example is found in Vulcano et al. (2010), where the authors use simulations to examine some of the empirical properties of the TvR algorithm and illustrate an application using a real airline data set.

Within the RM field, a handful of other papers have examined problems closely related to ours—the estimation of choice parameters when one alternative is completely censored. The majority of these papers have used a constants-only approach (i.e., they do not estimate a price coefficient). Vulcano et al. (2012), van Ryzin and Vulcano (2012), Haensel and Koole (2011), and Farias et al. (2013) fall into this category. If the firm has not historically changed their prices (such as a roadside budget hotel or a vending machine), then this approach should work well. It may also be a preferred approach when the firm has historically changed their prices for their portfolio of products but all of the prices have been linked such that the marginal price of a particular product switch is always the same. Indeed, our estimation technique does not work in these two cases because we need some nonperfectly correlated variability in the historical prices to be able to estimate our price coefficients.

There have also been a few papers that have proposed methods that allow for product attribute variation over time. Although they initially focus on a constants-only model, Vulcano et al. (2012) note that it is possible to replace their preference weights formulation with a linear-in-parameters structure. However, the linear-in-parameters structure no longer has a closed-form solution in the maximization step of the EM algorithm, which eliminates the majority of the computational advantages associated with their algorithm. Another example is Ratliff et al. (2008), who propose a two-step approach that is similar in spirit to ours, but with an important difference (also shared by Vulcano et al. 2012): the utility of the no-purchase option is determined exogenously (using the historical market shares), whereas it is endogenous in TvR and our method.

The work closest to ours is Talluri (2009), which considers the parameter estimation problem and proposes a non-EM algorithm to estimate the model's parameter values. Talluri changes the problem by replacing the arrival rate of customers with an estimate of the total overall number of customers, thereby ignoring the order of the customer arrivals. This change allows Talluri to reformulate the problem into a two-step algorithm. In the first step, the parameters of the customer choice model are estimated. In the second step, the overall number of customers is estimated using a risk-ratio approach. Our algorithm mirrors this two-step concept, although we retain the formulation of arrivals happening over time from the TvR work and derive a second step directly from the log likelihood of that process. Although the method in Talluri (2009) appears theoretically appealing, we are not aware of any numerical results that demonstrate its effectiveness or computational efficiency relative to other methods. Thus, we add to this literature stream by providing some potential benchmarking results for evaluating other estimation methodologies.

3. Two-Step Parameter Estimation

As recognized in several of the papers described above, the TvR algorithm has two principal drawbacks. First, it utilizes the EM algorithm, which is notoriously slow. Second, it discretizes time into small slices of a size selected exogenously by the modeler. This size should be large enough to ensure stability of the model calculations but small enough so that the probability of two or more arrivals in the same time period is negligible (Vulcano et al. 2010). Selecting an appropriate size can be challenging for a particular application because there is generally no guidance on what is a reasonable size and different sizes can lead to different parameter estimates (Talluri 2009, and see §4.2 below). In this section, we propose a new parameter estimation methodology that addresses each of these drawbacks while maintaining the ability to estimate a rich utility specification with measurable product attribute elasticities in addition to ASCs.

We begin by recasting the estimation problem using a continuous time representation. (For clarity, we term discrete time intervals as time slices and continuous time intervals as windows.) Within the TvR discrete time formulation, customer arrivals are modeled as a Bernoulli process, with λ representing the probability of an arrival in any time slice. In the continuous time formulation, customer arrivals are modeled as a Poisson process, with λ representing the arrival rate. If the discrete slices and continuous windows are the same size and suitably small, the models are effectively interchangeable.

The discretization of time slices offered by TvR is convenient when using a discrete choice model of demand to solve a DP for calculating an optimal RM strategy because sales and inventory changes are assumed to occur one at a time, thereby limiting the number of possible state transitions. However, when estimating parameters for the discrete choice model and the arrival process, there is no need to assume time is discrete (and aggregating into larger but fewer time intervals can decrease computational times). Converting the estimation problem to a continuous time representation still allows for the application of the choice and arrival models in a discrete time DP because the $\boldsymbol{\beta}$ and γ parameters of the discrete choice model function independently from the time representation and converting from a continuous Poisson arrival process to a discrete Bernoulli arrival process is straightforward when the time slices are appropriately sized.

Specifying the log likelihood of the observed data in the continuous time formulation is more complicated than the discrete time log likelihood given in (5) because more than one sale can be observed in each time period. The set of (unordered) product sales is represented by the number of sales of each product (e.g., one Y class and one Q class ticket) and the number of combinations that can result in a particular sale distribution (e.g., customer one purchases Y and customer two purchases Q, or vice versa). As we will show below, this added combinatorial complexity can be effectively ignored in applications. To write the log likelihood, we define z_{tj} as the observed number of purchases of product j in time window t, $\mathbf{z}_t$ as the vector of such terms for all products and m_t as the total number of observed purchases in time window t such that $m_t = \sum_{i\in S_t} z_{ti}$. We also define η_t as the (unobserved) number of no-purchase arrivals in time window t and $\mathbf{P}_t(\boldsymbol{\beta}, \gamma)$ as a vector of the choice probabilities $P_{ti}(\boldsymbol{\beta}, \gamma)$ for all products i as well as the no-purchase alternative.

We can write the likelihood function as the product of the likelihood of the number of arrivals and the likelihood of the choices made by those arriving customers,

$$L(\boldsymbol{\beta}, \gamma, \lambda) = \prod_{t\in T}\Bigg[\sum_{\eta_t=0}^{+\infty} \mathbb{P}((\eta_t + m_t) \mid \lambda) \cdot \mathbb{M}(\eta_t, \mathbf{z}_t \mid \eta_t + m_t, \mathbf{P}_t(\boldsymbol{\beta}, \gamma))\Bigg], \tag{7}$$

where T is the set of all continuous time windows, $\mathbb{P}$ is the Poisson distribution probability mass function (PMF) for the arrivals given by

$$\mathbb{P}((\eta_t + m_t) \mid \lambda) = \frac{\lambda^{(\eta_t+m_t)} \exp(-\lambda)}{(\eta_t + m_t)!}, \tag{8}$$

and $\mathbb{M}$ is the multinomial distribution PMF for the choices given by

$$\mathbb{M}(\eta_t, \mathbf{z}_t \mid \eta_t + m_t, \mathbf{P}_t(\boldsymbol{\beta}, \gamma)) = \frac{(\eta_t + m_t)!}{\eta_t! \prod_{j\in S_t} z_{tj}!} (P_{t\varnothing}(\boldsymbol{\beta}, \gamma))^{\eta_t} \prod_{j\in S_t} (P_{tj}(\boldsymbol{\beta}, \gamma))^{z_{tj}}.$$

To simplify the notation, we assume that all time windows are of the same duration and that λ is the expected rate of arrivals per time window. This is easily generalized to time windows of varying lengths by scaling λ for each time window linearly with its duration.

If η_t were observed, the arrival and choice models would be fully separable, with a closed-form solution to maximize for λ and a globally concave problem in $\boldsymbol{\beta}$ and γ. However, because η_t is unobserved, we must find the marginal likelihood of the parameters unconditional on the values of η_t. To achieve this, we first decompose the multinomial component into a binomial purchase or no-purchase PMF and a multinomial PMF conditional on some purchase:

$$\begin{aligned}\mathbb{M}(\eta_t, \mathbf{z}_t \mid \eta_t + m_t, \mathbf{P}_t(\boldsymbol{\beta}, \gamma))\\ = \mathbb{B}(\eta_t \mid \eta_t + m_t, P_{t\oslash}(\boldsymbol{\beta}, \gamma))\mathbb{M}(\mathbf{z}_t \mid m_t, \mathbf{P}_{t\mid\circledast}(\boldsymbol{\beta})),\end{aligned}$$

where $\mathbf{P}_{t\mid\circledast}(\boldsymbol{\beta})$ is a vector of the conditional probabilities $P_{ti\mid\circledast}(\boldsymbol{\beta})$ given in (3) for all alternatives i excluding the no-purchase alternative, $\mathbb{B}$ is the binomial PMF given by

$$\begin{aligned}\mathbb{B}(\eta_t \mid \eta_t + m_t, P_{t\oslash}(\boldsymbol{\beta}, \gamma))\\ = \frac{(\eta_t + m_t)!}{\eta_t! m_t!}(P_{t\oslash}(\boldsymbol{\beta}, \gamma))^{\eta_t}(P_{t\circledast}(\boldsymbol{\beta}, \gamma))^{m_t}, \quad (9)\end{aligned}$$

and $\mathbb{M}$ is the multinomial PMF given by

$$\mathbb{M}(\mathbf{z}_t \mid m_t, \mathbf{P}_{t\mid\circledast}(\boldsymbol{\beta})) = \frac{m_t!}{\prod_{j\in S_t} z_{tj}!}\prod_{j\in S_t}(P_{tj\mid\circledast}(\boldsymbol{\beta}))^{z_{tj}}.$$

Because the time windows exhibit homogeneity in alternative attributes and availability ($\mathbf{X}_t$ and S_t remain fixed throughout time period t), the probability of a customer choosing the no-purchase alternative remains constant inside time window t, and the product of the binomial PMF given in (9) and the Poisson PMF given in (8) can be rewritten as the product of two independent Poisson PMF's for purchasing and nonpurchasing customers, with the arrival rates adjusted by the probability of purchase or no-purchase, respectively. Thus, the likelihood can now be written as

$$\begin{aligned}L(\boldsymbol{\beta}, \gamma, \lambda) = \prod_{t\in T}\Bigg[\mathbb{P}(m_t \mid \lambda P_{t\circledast}(\boldsymbol{\beta}, \gamma))\mathbb{M}(\mathbf{z}_t \mid m_t, \mathbf{P}_{t\mid\circledast}(\boldsymbol{\beta}))\\ \cdot \sum_{\eta_t=0}^{+\infty}\mathbb{P}(\eta_t \mid \lambda P_{t\oslash}(\boldsymbol{\beta}, \gamma))\Bigg]. \quad (10)\end{aligned}$$

The value of the summation in (10) is 1 by definition, leaving the likelihood of only the observed data, for which we take the logarithm to write the log likelihood as

$$\begin{aligned}LL(\boldsymbol{\beta}, \gamma, \lambda) = \sum_{t\in T}\Bigg[m_t\log[\lambda P_{t\circledast}(\boldsymbol{\beta}, \gamma)] - \lambda P_{t\circledast}(\boldsymbol{\beta}, \gamma)\\ - \log\Bigg[\prod_{j\in S_t} z_{tj}!\Bigg] + \sum_{j\in S_t} z_{tj}\log[P_{tj\mid\circledast}(\boldsymbol{\beta})]\Bigg]. \quad (11)\end{aligned}$$

Directly maximizing the log likelihood given in (11) can be complicated, but we propose simplifying the process with a two-step decomposition approach, drawing from well-known two-step approaches employed in economics (e.g., Pagan 1986). The first step considers only the last term inside the summation in (11), given as

$$LL_1(\boldsymbol{\beta}) = \sum_{t\in T}\sum_{j\in S_t} z_{tj}\log[P_{tj\mid\circledast}(\boldsymbol{\beta})]. \quad (12)$$

This part of the log likelihood is simply a MNL model of the choice among products offered by the firm, conditional on an observable purchase being made. Importantly, it is a function only of $\boldsymbol{\beta}$ and not of other parameters. The no-purchase option, as well as all no-purchase customers, are excluded from this model so that there is no unobserved data in the sample used to estimate the parameters. McFadden (1978) demonstrates that maximizing the log likelihood of a MNL model that includes only a subset of alternatives generates consistent parameter estimates. That is, the parameter estimates obtained from a MNL model that excludes the no-purchase alternative are consistent, even when the actual choice set does include the no-purchase alternative. The log likelihood of the MNL model in (12) is globally concave and has a unique maximum, so finding the maximum-likelihood estimators, denoted $\hat{\boldsymbol{\beta}}$, is straightforward.

The maximization of $LL_1(\boldsymbol{\beta})$ gives rise to the parameter γ, the no-purchase utility, described previously. TvR set the no-purchase alternative as the reference point for utility, but that alternative does not appear in this more limited conditional choice model, so it cannot be used as the utility reference point. Instead, we set one observable alternative as the reference, and later we estimate the utility of the no-purchase alternative against that reference.

Having found consistent estimators $\hat{\boldsymbol{\beta}}$, the second step in our approach is to maximize the log likelihood with respect to the other parameters, but holding $\boldsymbol{\beta}$ constant at the values of $\hat{\boldsymbol{\beta}}$. Instead of maximizing the log likelihood with respect to a large number of parameters, we only need to maximize the log likelihood with respect to two parameters: γ and λ. Because $\hat{\boldsymbol{\beta}}$ converges in probability to the same $\boldsymbol{\beta}^\star$ as would be found from simultaneously maximizing (11) across all its parameters, this second step will generate consistent parameter estimates for γ and λ as well.

The second step can be further simplified in several ways. First, several additive terms in (11) contain neither γ nor λ, so they can be ignored during maximization, reducing the problem to be solved in the second step to maximizing

$$LL_2(\gamma, \lambda) = \sum_{t\in T}[m_t\log(\lambda P_{t\circledast}(\hat{\boldsymbol{\beta}}, \gamma)) - \lambda P_{t\circledast}(\hat{\boldsymbol{\beta}}, \gamma)] \quad (13)$$

with respect to γ and λ. Second, the entire term $\sum_{j\in S_t}\exp(\hat{\boldsymbol{\beta}}'\mathbf{X}_{tj})$ in (2), which features prominently in calculating $P_{t\circledast}(\boldsymbol{\beta},\gamma)$, is now a constant and can be precalculated before the maximization process, greatly speeding up the subsequent computations. Lastly, maximizing (13) with respect to λ is easy for any value of γ. The first derivative of (13) with respect to λ is

$$\frac{\partial LL_2(\gamma,\lambda)}{\partial\lambda}=\sum_{t\in T}\left[\frac{m_t}{\lambda}-P_{t\circledast}(\hat{\boldsymbol{\beta}},\gamma)\right], \qquad (14)$$

and the second derivative is

$$\frac{\partial^2 LL_2(\gamma,\lambda)}{\partial\lambda^2}=\sum_{t\in T}\left[-\frac{m_t}{\lambda^2}\right], \qquad (15)$$

where m_t is nonnegative for all $t\in T$ and should be positive for at least some t; otherwise, the data would imply that the firm has never made a successful sale (thus, estimating any parameters at all would be impossible). Therefore, the second derivative is always negative and thus the function is globally concave, at least in this one dimension. Setting (14) equal to zero makes it convenient to solve for the estimator $\hat{\lambda}$ as a closed-form function of γ, with

$$\hat{\lambda}(\gamma)=\arg\max_{\lambda} LL_2(\gamma,\lambda)=\frac{\sum_{t\in T}[m_t]}{\sum_{t\in T}[P_{t\circledast}(\hat{\boldsymbol{\beta}},\gamma)]}. \qquad (16)$$

Thus, the function to be maximized in the second step can be reduced from two dimensions to one:

$$\widetilde{LL}_2(\gamma)=\sum_{t\in T}\big[m_t\log(\hat{\lambda}(\gamma)P_{t\circledast}(\hat{\boldsymbol{\beta}},\gamma)) - \hat{\lambda}(\gamma)P_{t\circledast}(\hat{\boldsymbol{\beta}},\gamma)\big]. \qquad (17)$$

Although (17) is not generally concave, the calculations in the maximization of $\widetilde{LL}_2$ are relatively simple for each time unit, and they need to be calculated for fewer time units than the discrete time model proposed by TvR (because each continuous time window can aggregate a large number of tiny discrete slices from $\mathcal{P}\cup\bar{\mathcal{P}}$). Moreover, the analytic gradient for this function is easy to derive and compute,

$$\frac{\partial\widetilde{LL}_2(\gamma)}{\partial\gamma}=\hat{\lambda}(\gamma)\sum_{t\in T}[P_{t\circledast}(\hat{\boldsymbol{\beta}},\gamma)P_{t\oslash}(\hat{\boldsymbol{\beta}},\gamma)] - \sum_{t\in T}[m_t P_{t\oslash}(\hat{\boldsymbol{\beta}},\gamma)], \qquad (18)$$

so that this second step can be solved efficiently.

We summarize our proposed two-step parameter estimation process to solve for the choice-based RM parameters as follows:

Step 1. Find $\hat{\boldsymbol{\beta}}=\arg\max_{\boldsymbol{\beta}}\{LL_1(\boldsymbol{\beta})\}$.

Step 2. Find $\hat{\gamma}=\arg\max_{\gamma}\{\widetilde{LL}_2(\gamma)\}$ and the corresponding value of $\hat{\lambda}(\hat{\gamma})$.

One drawback that our algorithm shares with the EM approach is that the inverse Hessian matrix of the functions that are calculated cannot be used alone to calculate the standard error of the estimators. Various methods have been proposed in the literature to calculate the standard errors for such problems. The simplest one, at least from a theoretical standpoint, is bootstrapping (Efron and Tibshirani 1994). The standard errors for two-step results reported in this paper are calculated in this manner.

Moreover, when parameters for a choice model are estimated using our two-step approach, there is a theoretical loss of efficiency (i.e., the standard errors of the estimates increase) because not all of the information in the observed data is being utilized for parameter estimation. However, there are multiple factors that suggest the efficiency loss will be small when using the two-step algorithm. First, the excluded alternative is a no-purchase alternative that does not have measurable attributes (price, quality, etc.). Second, we only infer and do not actually observe the no-purchase customers. Third, when the subset of observed choices represents a large fraction of the total (in this case, just one fewer than the original set), the loss in efficiency is typically small (Nerella and Bhat 2004).

As is common in two-step estimation approaches, it is possible that parameter estimates will be biased for small sample sizes. As the next section will show, our empirical tests suggest that any potential bias in this particular application is small, and the statistical reliability of the two-step algorithm is as good as, or better than, the TvR approach.

4. Simulation Testing

The principal benefits of implementing our two-step algorithm instead of EM are concrete: a reduction in computational time and an avoidance of any discretization problems, while still preserving the ability to estimate meaningful utility coefficients for price and other product attributes. To understand the magnitude of these benefits, we benchmarked the performance of our algorithm against the one proposed by TvR for a realistic industry application. We focus on a comparison with TvR because it provides the closest match to ours in terms of capabilities, i.e., both approaches allow for the estimation of product attribute coefficients (especially price) without requiring exogenous data or assumptions about market preferences. We compare the two approaches by simulating a purchase process that is modeled after an actual industry data set.

Using simulated data instead of the actual booking data allows us to concretely evaluate the performance of our estimation procedure in a controlled setting. With simulated data, we know the true underlying model parameters, whereas with real booking data, we do not. Using simulated data also allows us to test our procedure on different market environments using multiple replications.

We generated the simulations based on a single large urban hotel, with product attributes and customer parameters derived from "Hotel 1" in publicly available data taken from Bodea et al. (2009). We selected this hotel because it offered a variety of products (up to eight physically different room types could be available for booking at any given time). We then created a reasonable choice model specification, containing a complete set of ASCs, along with three parameters for price: one to represent a baseline price elasticity, one to represent the change in elasticity between same-day bookings and bookings that were made anytime earlier, and one final parameter to differentiate elasticity in bookings made at least two weeks in advance.

We set the customer arrival rate at 40 per day and the hotel's market share at approximately 10% (which is similar to the actual market share of the hotel upon which the simulated data is based). We used these parameters to generate a simulated data set containing 28-day booking curves for one year's worth of check-in days at a hotel with the same product types, average prices, and price variability as the real data. We incorporated changes in product availability by assuming a 6% probability of randomly closing each product type in the last 21 days of the booking curve; once closed, a product type remained closed for the remainder of the booking period. This resulted in each product being available about 65%–75% of the time. Each simulated data set contained approximately 50,000 purchase observations across 10,000 booking curve days.

The generation of the synthetic data, along with the subsequent estimation of model parameters, was conducted using Python v2.7.3 (van Rossum et al. 2001–2013) and SciPy v0.11 (Jones et al. 2001–2013). Components of the algorithm that involved maximizing a multinomial logit model (i.e., in the first step of the two-step process and for the MNL portion of the maximization step within EM) were solved using ELM v2 (Newman 2010–2013).

4.1. Comparing Two-Step vs. Expectation-Maximization Models

To apply the EM model, it is necessary to first select a time slice size. The EM models that have been estimated in the literature have used time slices of 10 to 15 minutes, and time slice arrival probabilities between 0.01 and 0.7. In this simulation, we adopt a time slice of 15 minutes. This corresponds to the probability of a customer arrival in each time slice of approximately 0.42; given a 10% market share, approximately 4% of time slices have an observed sale. The application of our two-step algorithm does not require any a priori decisions regarding the time slice size.

Table 1 Results for Hotel-Based Simulations

		EM (15-minute slices)		Continuous two-step	
Parameter	True values	Estimate	Deviation from true (%)	Estimate	Deviation from true (%)
β_{King1}	0	—		—	
β_{King3}	−0.9535	−1.073	12.6	−0.939	1.5
β_{King4}	0.0488	0.030	37.8	0.051	5.5
β_{Queen1}	−1.3131	−1.456	10.9	−1.323	0.7
β_{Special}	−1.0926	−1.192	9.1	−1.086	0.6
β_{Suite1}	2.3141	2.616	13.0	2.324	0.4
β_{Suite2}	−0.124	−0.056	54.6	−0.130	4.6
β_{TwoDbl}	−1.0738	−1.195	11.3	−1.089	1.5
β_{price}	−0.01719	−0.01822	6.0	−0.01783	3.7
$\beta_{\text{price, day}\geq 1}$	−0.00361	−0.00493	36.7	−0.00319	11.6
$\beta_{\text{price, day}\geq 14}$	−0.00193	−0.00216	11.9	−0.00150	22.1
γ	−5.3	−8.276	56.2	−5.283	0.3
λ (per day)	40	10.138	74.7	41.444	3.6
Total estimation time (40 runs)		12.4 hours		11.8 minutes	
Equivalent LL at best solution		−945,266.35		−943,330.12	

The results of this simulation are shown in Table 1. In this and subsequent tables, the arrival parameter λ is translated into an expected effective rate for the EM method and scaled to represent the expected number of arrivals per day for ease of interpretation across methodologies. Also, because the underlying frameworks are different, the final log likelihood of each algorithm is not directly comparable to that of the other algorithm. Instead, we recast each model as a discrete time model with extremely small time slices (one millisecond per slice) and the log likelihood of that (nearly) equivalent model is reported so as to be able to generate directly comparable log-likelihood measures.

Each algorithm was run 40 different times, each time using a different vector of starting values. However, the first step of the proposed two-step algorithm is a globally concave maximization problem, so any vector of initial values for those parameters will always converge to the same result. Therefore, the first step of the algorithm needs to be estimated just once using a single set of starting values; only the one-dimensional γ (no-purchase constant) parameter space needs to be well explored with multiple starting values, so the restart procedure only requires returning to the beginning of step 2. This alone provides a major advantage over the TvR algorithm, which requires each run to use new starting values for the entire parameter space.

For each run, the algorithm was allowed to converge to a solution, and the model with the best (i.e., highest) log-likelihood value was used to determine the parameter estimates. Because this is a simulation, we do not expect the parameter estimates to match the known underlying true values exactly. However, with one exception, the deviations of the parameter estimates from their true values are larger for the EM algorithm than for the two-step algorithm. In the case of the arrival rate (λ) and the no-purchase constant (γ), the deviations from the true values are substantially larger, so much so that it is clear that the EM algorithm has converged to the wrong solution. Moreover, no other EM solution (i.e., with an inferior log likelihood) converged to parameter estimates that were close to the true values for these parameters. Although it is reasonable to expect some additional error in these parameters because they are most closely tied to the unobserved data, the estimates derived from our proposed two-step algorithm are good, with less than 4% error for the arrival rate and less than 0.5% error for the no-purchase constant.

4.2. Understanding the Effects of Time Discretization

One other interesting outcome that can be seen in the results reported in Table 1 is that the log likelihood associated with the solution from the EM algorithm is actually inferior to the log likelihood associated with the solution from the two-step algorithm. Given the derivation of the algorithms, this is asymptotically impossible because the EM model is an unconstrained version of the two-step model. It is not entirely clear whether the problem is a result of a violation of the assumption that the probability of two arrivals in the same time slice is negligible or if it is simply a case of EM failing to converge to the global optimum, even when tested from numerous starting points.

To investigate this problem, another simulation experiment was conducted in which the EM algorithm was run on three simulated data sets representing 15-minute, 1-minute, and 1-second time slices. The results are shown in Table 2. Several trends emerge when the size of the time slices is decreased: the converged estimates of the parameters for the model move generally closer to the known true underlying values (especially for the λ and γ parameters), and the overall log likelihood of the results generated by the EM algorithm improves. Indeed, for the smallest time slices (i.e., the model that most closely represents the underlying continuous process), the log likelihood found by the EM algorithm is technically superior to that found by our proposed two-step algorithm, which is the asymptotic result predicted by the derivations of the models.

One nefarious trend also appears as time slice sizes are decreased: the computational time required to solve for those parameters increases dramatically. Changing from 15-minute to 1-minute time slices triples the computation time from approximately 15 to 45 minutes per run; changing to 1-second time slices increases the computation time to approximately six hours per run. Because the log likelihood is not globally concave, it is necessary to test several different vectors of initial values to achieve a high degree of confidence that a globally optimal solution has been found, so the long computation time associated with a single run is compounded. This contrasts against the solution times for our two-step algorithm, which requires only a few seconds per run and achieves nearly identical results to the best (slowest) EM formulation.

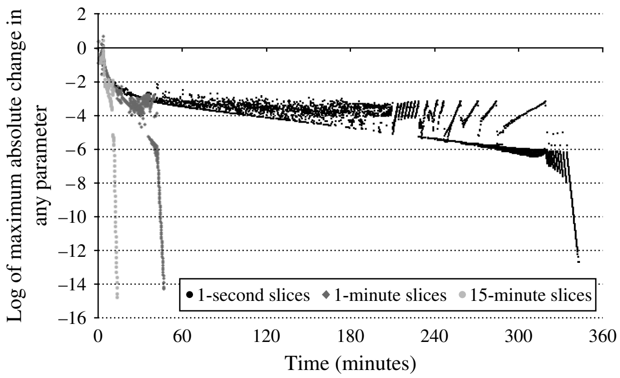
Moreover, if the EM algorithm is employed, our simulations demonstrate that great care must be used in setting the convergence criteria because improvements in parameter estimates can occur slowly. Figure 1 shows the convergence of three typical runs of the EM algorithm when using time slices of 15 minutes, 1 minute, and 1 second. In each case, the algorithm begins initially with some substantial parametric movement, and then improvement slows substantially. For the smallest time slices, during hours 2 to 5, the algorithm generally made improvement steps where the largest change in any model parameter was on the

Table 2 **Results for Hotel-Based Simulations with Various Small Time Slices**

Parameter	True values	EM (15 minutes)	EM (1 minute)	EM (1 second)	Continuous two-step
β_{King1}	0	—	—	—	—
β_{King3}	−0.954	−1.095	−1.033	−0.969	−0.960
β_{King4}	0.049	0.041	0.051	0.063	0.061
β_{Queen1}	−1.313	−1.444	−1.378	−1.309	−1.307
β_{Special}	−1.093	−1.156	−1.112	−1.065	−1.062
β_{Suite1}	2.314	2.722	2.563	2.377	2.325
β_{Suite2}	−0.124	0.004	−0.033	−0.082	−0.080
β_{TwoDbl}	−1.074	−1.179	−1.132	−1.080	−1.074
β_{price}	−0.01719	−0.01855	−0.01828	−0.01751	−0.01700
$\beta_{\text{price, day}\geq 1}$	−0.00361	−0.00489	−0.00400	−0.00354	−0.00375
$\beta_{\text{price, day}\geq 14}$	−0.00193	−0.00214	−0.00206	−0.00194	−0.00212
γ	−5.300	−8.433	−6.869	−5.485	−5.341
λ (per day)	40	9.800	18.775	37.018	38.854
Average estimation time		9.9 hours	32.6 hours	9.72 days[a]	12.2 minutes
Equivalent LL at best solution		−919,524.80	−917,722.10	−917,363.60	−917,395.50

[a]Extrapolated to represent computational time for 40 runs (the number of runs used in the other columns) from the 23.53 hours needed to estimate the model from four different vectors of initial values.

Figure 1 EM Convergence Measure by Computation Time



order of 0.1% or less. Figure 2 shows the progression of the estimate of the λ parameter in these runs. It is clear that the slow convergence is not merely a matter of refining the estimators to numerous significant figures, and stopping the slowest (but most refined) model after an hour or two of computation will give notably biased parameter estimates. During the hours when the algorithm was making stepwise changes of less than 0.1% in all of the parameters, the total movement in the arrival rate parameter exceeded 10%. Vulcano et al. (2010, p. 376) similarly report finding "bad behavior in the estimation procedure" when the time slices were too small. This slow convergence also reduces the potential of the obvious extension of our algorithm to a three-step process—using the two-step algorithm to generate a vector of starting values for the EM algorithm to finish. In our testing, applying such a three-step process reduced the average six-hour computational time of the regular EM algorithm by only about an hour. The final bit of convergence is the slowest part.

4.3. Empirical Unbiasedness

To further evaluate the two-step algorithm, we applied it to 50 distinct simulated data sets and compiled the results in Table 3. The average of the estimates in each case is close to the true value, and the standard

Figure 2 EM Daily Arrival Rate Estimate by Computation Time

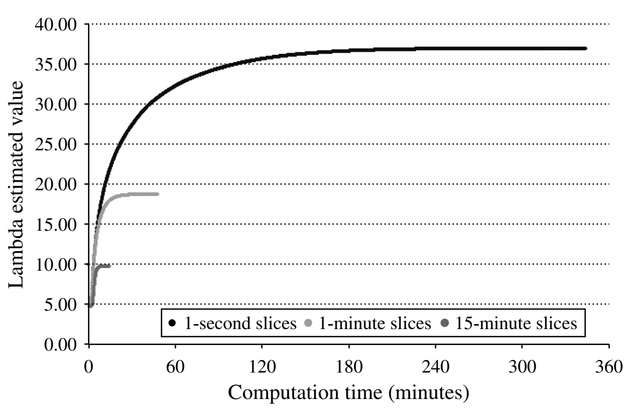


Table 3 Average Results of Two-Step Algorithm over Numerous Simulations

Parameter	True values	Mean estimates	Std. dev. estimates	Error in mean (%)	Coeff. of variation
β_{King1}	0	—	—	—	
β_{King3}	−0.954	−0.9550	0.0203	0.2	0.021
β_{King4}	0.049	0.0487	0.0199	0.2	0.409
β_{Queen1}	−1.313	−1.3124	0.0242	0.1	0.018
β_{Special}	−1.093	−1.0944	0.0239	0.2	0.022
β_{Suite1}	2.314	2.3217	0.0384	0.3	0.017
β_{Suite2}	−0.124	−0.1207	0.0317	2.7	0.262
β_{TwoDbl}	−1.074	−1.0771	0.0247	0.3	0.023
β_{price}	−0.01719	−0.017190	0.000410	0.0	0.024
$\beta_{\text{price, day}\geq 1}$	−0.00361	−0.003622	0.000419	0.3	0.116
$\beta_{\text{price, day}\geq 14}$	−0.00193	−0.001908	0.000209	1.1	0.109
γ	−5.3	−5.3175	0.1060	0.3	0.020
λ (per day)	40	39.5224	2.6443	1.2	0.067

errors of the estimates are all reasonably small. The coefficients of variation for the parameter estimates are typically less than 0.1, and the mean estimate for each parameter is less than 0.2 standard errors from the true value. Because the EM algorithm is so slow, it was not feasible to benchmark these results against EM results. Comparisons against the faster EM models seem pointless because those models exhibit obvious flaws, and to compare against the 1-second time slices model would require months of dedicated computational time on a modern desktop PC versus just a few hours to generate the results seen in Table 3 for the two-step algorithm.

4.4. Robustness

The simulations examined above employ synthetic data that we constructed to be as close as possible to the real hotel data set. We also ran simulations estimating parameters under a number of other scenarios (varying arrival rates, product availability rules, and the popularity of various product types) to evaluate the robustness of our approach (and its advantage over the TvR method) across a variety of environments. Ten simulated data sets, each consisting of 90 check-in days, were generated for each scenario, and five vectors of random starting values were used for the parameter estimation algorithms for each simulated data set. Because these simulations are focused primarily on the robustness of the two-step approach, we follow Vulcano et al. (2010) in terminating the EM algorithm after 300 iterations if it has not already converged. The reported average solution times for the TvR algorithm are therefore significantly less than they would be if that algorithm were allowed to run to the more tightly converged solutions that are necessary to achieve their statistically efficient solutions.

We first tested the estimation algorithms with larger and smaller arrival rates, with rates set at four, 40, or 400 arrivals per day. For the TvR algorithm, time was

Table 4 Mean Parameter Estimates for Simulations with Varying Arrival Rates

		4/day		40/day		400/day	
Parameter	True values	EM	Two-step	EM	Two-step	EM	Two-step
β_{King1}	0	—	—	—	—	—	—
β_{King3}	−0.954	−0.924	−0.876	−1.016	−0.953	−1.036	−0.952
β_{King4}	0.049	0.099	0.142	0.037	0.047	0.028	0.047
β_{Queen1}	−1.313	−1.291	−1.230	−1.360	−1.301	−1.402	−1.315
β_{Special}	−1.093	−1.033	−0.993	−1.160	−1.107	−1.174	−1.097
β_{Suite1}	2.314	2.523	2.469	2.420	2.306	2.470	2.301
β_{Suite2}	−0.124	−0.127	−0.123	−0.117	−0.140	−0.101	−0.132
β_{TwoDbl}	−1.074	−1.052	−0.996	−1.114	−1.072	−1.136	−1.070
β_{price}	−0.01719	−0.01744	−0.01781	−0.01759	−0.01736	−0.01782	−0.01715
$\beta_{\text{price, day}\geq 1}$	−0.00361	−0.00409	−0.00300	−0.00413	−0.00334	−0.00438	−0.00367
$\beta_{\text{price, day}\geq 14}$	−0.00193	−0.00205	−0.00217	−0.00200	−0.00207	−0.00206	−0.00191
γ	−5.300	−6.033	−5.577	−6.517	−5.378	−7.060	−5.321
λ (per day)	{4, 40, 400}	2.812	3.447	23.046	37.794	192.477	394.661
Mean computation time		18.6 minutes	56.7 seconds	26.0 minutes	57.4 seconds	39.0 minutes	62.7 seconds

discretized into 15-minute periods for the four and 40 per day simulations, and 1-minute periods for the 400 per day simulations (with 15-minute periods, there could be at most only 96 arrivals per day). The results, summarized in Table 4, clearly show that the two-step algorithm performs well, completes calculations much faster, and generates estimates that are on average closer to true parameters than the iteration-limited TvR method. Moreover, in numerous individual trials, especially in the 400 per day arrival rate scenario, the best fitting TvR parameter estimates were far from the true estimates for the γ and λ parameters. It is possible that every tested starting point for the parameters in those trials may have been converging to a nonoptimal stationary point representing an "autocratic monopolist" scenario, where the probability of the no-purchase option is zero. In no case did the two-step algorithm ever result in those boundary condition parameters.

We also tested the estimation algorithms under different product availability scenarios. In this set of experiments, instead of employing the availability criteria described earlier, we implemented randomized product availability for each booking day. We set the most expensive premium product as always available and divided the remaining products in a 50/50 split of higher and lower priced sets. In the first scenario, higher priced products were made unavailable 10% of the time, and lower priced products were made unavailable 20% of the time. In the second scenario, the unavailability ratios were set to 20% and 80%, respectively, and in the final scenario all but the highest priced product were unavailable 80% of the time. A summary of the results from these scenarios is shown in Table 5.

Finally, we tested the estimation algorithms using different sets of values for the ASCs. These values control the relative overall popularity of each of the products available for sale, after accounting for price. In each set of experiments, all the ASCs used to generate the synthetic data were perturbed by adding a random

Table 5 Mean Parameter Estimates for Simulations with Varying Product Availability

		10%–20% closure		20%–80% closure		80% closure	
Parameter	True values	EM	Two-step	EM	Two-step	EM	Two-step
β_{King1}	0	—	—	—	—	—	—
β_{King3}	−0.954	−1.007	−0.963	−0.940	−0.956	−1.093	−0.959
β_{King4}	0.049	0.018	0.034	0.066	0.016	0.016	0.034
β_{Queen1}	−1.313	−1.346	−1.306	−1.261	−1.285	−1.502	−1.381
β_{Special}	−1.093	−1.129	−1.094	−1.066	−1.110	−1.232	−1.067
β_{Suite1}	2.314	2.393	2.333	2.442	2.329	2.550	2.300
β_{Suite2}	−0.124	−0.121	−0.129	−0.048	−0.119	−0.075	−0.086
β_{TwoDbl}	−1.074	−1.121	−1.084	−1.071	−1.105	−1.254	−1.108
β_{price}	−0.01719	−0.01710	−0.01731	−0.01712	−0.01744	−0.01850	−0.01716
$\beta_{\text{price, day}\geq 1}$	−0.00361	−0.00442	−0.00358	−0.00341	−0.00355	−0.00462	−0.00368
$\beta_{\text{price, day}\geq 14}$	−0.00193	−0.00224	−0.00198	−0.00162	−0.00210	−0.00233	−0.00154
γ	−5.300	−6.320	−5.343	−5.059	−5.440	−7.813	−5.280
λ (per day)	40	26.552	41.128	37.389	40.891	14.184	41.263
Mean computation time		52.3 minutes	67.6 seconds	42.2 minutes	67.6 seconds	34.6 minutes	68.1 seconds

value drawn from a uniform distribution bounded by −2 and 2. The results generally mirror those seen above, namely, solution times around 60–70 seconds for the two-step approach and 45–60 minutes for the EM algorithm, along with much superior estimates (closer to the known true values) for the arrival rates and γ parameters and generally better estimates for the other parameters as well.

5. Parameter Application

A principle advantage of the two-step algorithm, as compared with the majority of other algorithms proposed for the censored data problem, is that it allows for the estimation of utility coefficients on product attributes, not just product-specific constants. This is desirable for applications that involve joint pricing and assortment decisions. To illustrate this, we conducted a simplified simulation using the same hotel considered in §4. The results in Table 6 are based on synthetic data, using the real prices and assortments offered by the hotel (including the relative frequency that each combination was offered), synthetic choices generated from a MNL choice model, and a Poisson arrival model with parameters derived from the actual data. The synthetic data contains two-month booking curves preceding each of 100 simulated weekday booking dates. To simplify the example, arrival rates and price elasticities are simulated and modeled as constant across the length of the booking curve, although they need not be.

Table 6 shows expected choice probabilities for each arriving customer facing the indicated price and assortment offering. Because the actual offerings by this hotel varied widely in both pricing and assortment (there are 77 distinct price and assortment variations present in the weekday booking data for this hotel), four examples are shown in the table: the first two examples correspond to a set of high and low prices when all products are available, and the last two examples correspond to a set of high and low prices when a more limited subset of products is available. In each case, the true simulation probabilities are shown, along with the probabilities derived from the two-step algorithm and the probabilities derived from the algorithm given in Vulcano et al. (2012), denoted as VvRR, which generates parameters quickly but is not able to estimate price coefficients. It is important to note that the VvRR probability estimates respond only to changes in assortment and not to changes in price, whereas the two-step probability estimates respond to both changes in assortment and changes in price. By responding to both, the proposed two-step approach is better able to match the underlying purchase probabilities for any particular offering.

5.1. The Efficient Frontier

The results of this simulation were also used to construct an efficient frontier (EF) of possible offerings that should be considered by a revenue maximizing DP like that proposed in TvR. For the two-step algorithm, the expected revenue is calculated based on the estimated probabilities and the associated product prices for each individual portfolio. For VvRR, pricing is not considered in probability generation, so expected revenue is calculated using the average prices (when offered) for each product.

The graph in Figure 3 shows that, on average, where it exists the VvRR-based EF roughly mirrors the two-step frontier, suggesting that when desired sales rates are reasonably low (e.g., when the booking day is distant or inventory is quite limited) DPs based on inputs from VvRR and the two-step algorithm will perform similarly, even though the two-step algorithm offers substantially more detail and a better behavioral representation. However, using the two-step algorithm

Table 6 Selected Product Assortment and Pricing Portfolios with True and Modeled Probabilities

Product	NoPurch	Suite1	Suite2	King1	Queen1	TwoDbl	Special	King4	King3
Prices	N/P	529	429	399	359	359	359	359	329
True probability (%)	37.65	8.59	4.18	7.93	4.24	5.39	5.29	16.56	10.18
Two-step probability (%)	39.98	8.19	3.99	7.72	4.14	5.10	5.13	16.03	9.71
VvRR probability (%)	56.65	4.60	2.25	6.22	3.15	3.79	3.91	12.14	7.29
Prices	N/P	629	529	469	429	439	429	429	399
True probability (%)	68.99	2.82	1.37	4.36	2.33	2.50	2.91	9.11	5.60
Two-step probability (%)	70.02	2.74	1.34	4.25	2.28	2.38	2.82	8.82	5.34
VvRR probability (%)	56.65	4.60	2.25	6.22	3.15	3.79	3.91	12.14	7.29
Prices	N/P	599	499	439	None	None	None	None	None
True probability (%)	82.80	5.67	2.76	8.77	0.00	0.00	0.00	0.00	0.00
Two-step probability (%)	83.52	5.44	2.65	8.40	0.00	0.00	0.00	0.00	0.00
VvRR probability (%)	81.26	6.59	3.22	8.92	0.00	0.00	0.00	0.00	0.00
Prices	N/P	669	569	499	None	None	None	None	None
True probability (%)	93.60	1.93	0.94	3.54	0.00	0.00	0.00	0.00	0.00
Two-step probability (%)	93.65	1.92	0.93	3.50	0.00	0.00	0.00	0.00	0.00
VvRR probability (%)	81.26	6.59	3.22	8.92	0.00	0.00	0.00	0.00	0.00

Figure 3 Simulated Efficient Frontiers Generated by Two-Step and VvRR Using Real Price and Assortment Offerings

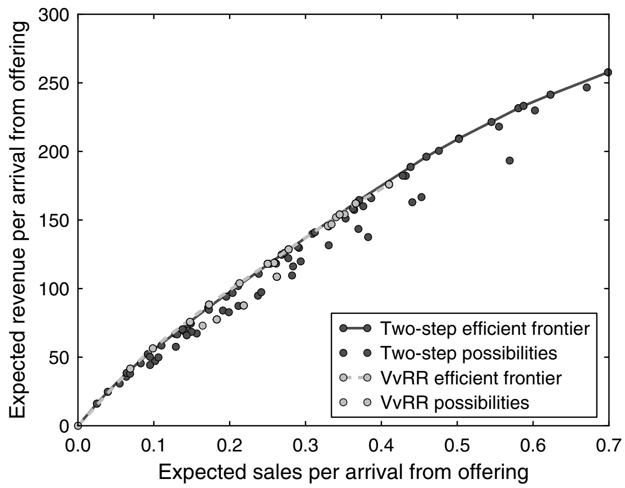
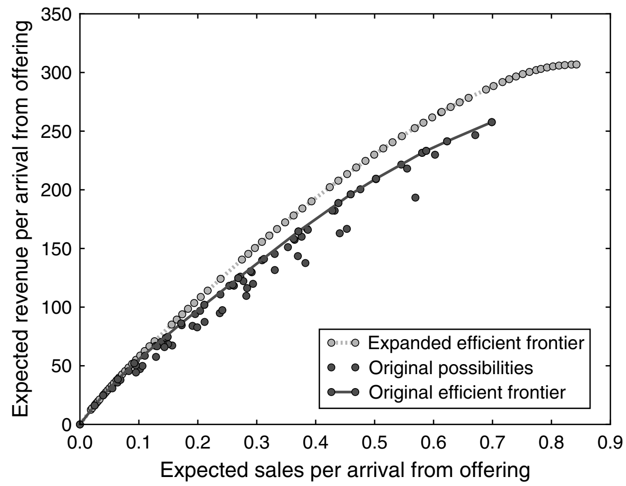


Figure 4 Simulated Efficient Frontiers Generated Using Price and Assortment Offerings from Actual Observations and an Expanded Set of Potential Offerings



to find price elasticity can discover more profitable product assortment combinations in the upper part of the EF, where the total number of sales is more heavily influenced by price discounting.

5.2. Expanding the Efficient Frontier

A principal advantage of employing the two-step algorithm introduced here instead of the simpler EM methodology introduced in VvRR is that this method preserves the ability to estimate meaningful coefficients on the attributes of the alternatives, particularly for price. This can be useful in RM because it provides opportunities to adjust prices and product assortments, which can result in portfolios of offerings that lie above the EF as described in TvR.

As described in TvR, for any given assortment S along the EF, there is an associated selling probability $Q(S) = 1 - P_{\oslash}(S)$ and expected revenue $R(S) = \sum_{j\in S} P_j(S) r_j$. If in that assortment there is some product j where the price r_j is such that $(\partial R(S)/\partial r_j)/(\partial Q(S)/\partial r_j) < 0$, then a superior set of prices exists for that assortment, which would lie clearly above the EF because it would dominate the current assortment with the current pricing. (For typical utility maximizing choice models, increasing the price of any product will cause an increase in the probability of the no-purchase alternative, so generally $\partial Q(S)/\partial r_j$ will be negative, meaning a positive value of $\partial R(S)/\partial r_j$ would be sufficient for this condition to be true.) Moreover, if we define S^+ as the adjacent assortment along the EF (where $R(S^+) > R(S)$ and $Q(S^+) > Q(S)$), then if there is some price r_j where $(\partial R(S)/\partial r_j)/(\partial Q(S)/\partial r_j) > (R(S^+) - R(S))/(Q(S^+) - Q(S))$, there is the possibility of expanding the EF with some set of prices for assortment S that does not dominate the current prices for S but introduces a new point along the EF between S and S^+. An analogous logic also applies for expanding the EF between S and the adjacent assortment along the EF with lower R and Q.

For the hotel simulated above, most of the 77 pricing and assortment points offered by the hotel are not actually on the true EF, as shown in Figure 4. The points along the expanded EF were discovered by systematically varying the prices of each individual product in each assortment. Although the EF shown in Figure 4 is not optimized (discovering the true optimal EF is beyond the scope of this work), it is generally 10% better than the limited EF generated by actual pricing observations.

Naturally, this results in a continuum of pricing and assortment portfolios that would need to be incorporated into a DP to achieve maximum revenue, which is cumbersome and undesirable. Still, selecting a few pricing and assortment portfolios that lie above the fixed-prices EF will provide the opportunity to achieve greater revenue than a model that does not incorporate the ability to set prices. A limited number of discrete portfolios along the theoretical EF defined by varying pricing and assortment should suffice for achieving a result reasonably close to the theoretical maximum revenue, although how many such points are sufficient and an analysis of the revenue benefits over a fixed price scenario is beyond the scope of this work.

6. Conclusions

To date, there have been many theoretical papers that have examined the potential of using choice-based models for RM applications. However, only a few papers have examined how the underlying choice and arrival rate parameters needed for these RM applications can be estimated. In part, this is because estimation of parameters for choice-based RM applications presents a new estimation challenge: namely, a parameter needs to be estimated for an alternative that is completely censored and is never observed to have been chosen in the estimation data set. Thus, existing estimation methods for choice-based samples, which can be used for partially censored (but not completely censored) alternatives, are of limited viability in the RM context.

Our formulation of the choice-based estimation problem represents a streamlined approach for estimating discrete choice modeling parameters for data sets in which one of the alternatives is completely censored and is never observed to have been chosen. We decompose the estimation problem into two steps: the first step uses conditional likelihood that estimates the discrete choice model parameters for the firm's products, and the second step is based on the full likelihood that estimates (as a one-dimensional optimization problem) the arrival rate and overall market share. These types of two-step algorithms are common in discrete choice modeling applications and other econometric applications. As in the case of our paper, these two-stage approaches can often provide theoretically consistent, but inefficient estimates. For example, in the late 1970s and early 1980s, the limited-information maximum-likelihood (LIML) estimator was used by transportation modelers for nested logit models. The LIML represented a two-stage approach, in which parameters specific to one or more subsets of alternatives were computed first, and then aggregated into a composite utility that was used as an input to solve for parameters common across all alternatives. It was known that the full-information maximum-likelihood (FIML) estimator, which solved for all parameters simultaneously, was more efficient and thus theoretically superior (Hensher 1986). However, during this time period, LIML was attractive because it could solve problems that were not computationally tractable using FIML.

A similar relationship can be drawn between our method and the TvR method. The TvR method represents a theoretically consistent and efficient estimation approach, but it requires a tremendous amount of computational resources, which makes its practical application challenging using current technology. Our method represents a theoretically consistent and (slightly) less efficient method, but it dominates in computational performance and can easily handle estimation of large models with a rich set of product attributes. This provides researchers and practitioners with the ability to estimate a broad set of models for which one alternative is never observed to have been chosen. Within the RM community, this includes models that incorporate price, which can subsequently be used as a decision variable for more advanced product portfolio availability and price optimization applications.

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References

Ben-Akiva M, Lerman SR (1985) *Discrete Choice Analysis: Theory and Application to Travel Demand*, MIT Press Series in Transportation Studies (MIT Press, Cambridge, MA).

Bodea T, Ferguson M, Garrow L (2009) Choice-based revenue management: Data from a major hotel chain. *Manufacturing Service Oper. Management* 11(2):356–361.

Dempster A, Laird N, Rubin D (1977) Maximum likelihood from incomplete data via the EM algorithm. *J. Royal Statist. Soc. B* 39(1):1–38.

Efron B, Tibshirani RJ (1994) *An Introduction to the Bootstrap* (Chapman & Hall/CRC Press, Boca Raton, FL).

Farias VF, Jagabathula S, Shah D (2013) A nonparametric approach to modeling choice with limited data. *Management Sci.* 59(2): 305–322.

Ferguson M, Garrow L, Newman J (2012) Application of discrete choice models to choice-based revenue management problems: A cautionary note. *J. Revenue Pricing Management* 11(5):536–547.

Haensel A, Koole G (2011) Estimating unconstrained demand rate functions using customer choice sets. *J. Revenue Pricing Management* 10(5):438–454.

Hensher DA (1986) Sequential and full information maximum likelihood estimation of a nested logit model. *Rev. Econom. Statist.* 68(4):657–667.

Jones E, Oliphant T, Peterson P, et al. (2001–2013) SciPy: Open source scientific tools for Python. http://www.scipy.org/.

McFadden D (1974) Conditional logit analysis of qualitative choice behavior. Zarembka P, ed. *Frontiers in Econometrics*, Chap. 4 (Academic Press, New York), 105–142.

McFadden D (1978) Modelling the choice of residential location. Karlquist A, Lundqvist L, Snickers F, Weibull JW, eds. *Spatial Interaction Theory and Residential Location* (North-Holland, Amsterdam), 75–96.

Nerella S, Bhat CR (2004) Numerical analysis of effect of sampling of alternatives in discrete choice models. *Transportation Res. Record* 1894:11–19.

Newman J (2010–2013) ELM (Easy Logit Modeler), Version 2. http://elm.newman.me.

Pagan A (1986) Two stage and related estimators and their applications. *Rev. Econom. Stud.* 53(4):517–538.

Ratliff RM, Rao BV, Narayan CP, Yellepeddi K (2008) A multi–flight recapture heuristic for estimating unconstrained demand from airline bookings. *J. Revenue and Pricing Management* 7(2):153–171.

Talluri K (2009) A finite population revenue management model and a risk-ratio procedure for the joint estimation of population size and parameters. Technical report, Universitat Pompeu Fabra, Barcelona, Spain.

Talluri K, van Ryzin G (2004) Revenue management under a general discrete choice model of consumer behavior. *Management Sci.* 50(1):15–33.

van Rossum G, et al. (2001–2013) Python 2.7.3. www.python.org.

van Ryzin G, Vulcano G (2012) An expectation-maximization algorithm to estimate a general class of non-parametric choice models. Working paper, Columbia Business School, Columbia University, New York.

Vulcano G, van Ryzin G, Chaar W (2010) Choice-based revenue management: An empirical study of estimation and optimization. *Manufacturing Service Oper. Management* 12(3):371–392.

Vulcano G, van Ryzin G, Ratliff R (2012) Estimating primary demand for substitutable products from sales transaction data. *Oper. Res.* 60(2):313–334.

Weatherford LR, Ratliff RM (2010) Review of revenue management methods with dependent demands. *J. Revenue and Pricing Management* 9(4):326–340.

Wu J (1983) On the convergence properties of the EM algorithm. *Ann. Statist.* 11(1):95–103.