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Split-case sorting systems are used in many retail supply chains where items must be distributed in less-than-case quantities, such as orders shipped directly to customers by catalogers and e-tailers or shipments made in less-than-case quantities from a distribution center to a retail store. These systems are particularly effective in order-packing systems where the same item is needed for multiple orders. Items are inducted into a circular sorting conveyor system one unit at a time and then delivered to an order-packing bin designated for a particular customer or retail store. We develop analytical performance models that incorporate the stochastic operating conditions faced by these systems. Our model allows system designers to predict the sorting capacity for different system configurations. More importantly, we use the model to develop insight into the system design and operation.
(*Distribution; Retail; Sortation; Material Handling; Throughput Models; Approximate Queueing Models*)

1. Introduction

Retail distribution is changing faster today than any time in recent memory. From large discount chains such as Wal-Mart to online retailers such as Amazon.com, retailers have been fundamentally changing their supply chains to reduce inventory and become more agile (Fisher et al. 2000). Distribution centers are no longer large inventory storage areas simply holding goods for future shipments. They are now used to add value to the final product through last-minute customization, to perform predistribution functions such as price ticketing or packaging and to serve as a transit point to sort incoming bulk shipments into small customized shipments with no storage. Moreover, shipment sizes are plunging as retailers seek to reduce inventory and facilitate micro-marketing at their stores (Vitzthum 2001). To enable these just-in-time distribution strategies, cost-effective sortation systems are required. For direct retailers like J. Crew and Amazon.com, efficient sorting and shipping of small orders is one key to survival in the hypercompetitive environment of Internet retailing. Success stories linking automated warehousing and B2C e-commerce have littered the business and trade press (Anders 1999, Maloney 2000) and have even made the pages of general publications like *The New Yorker* (Gladwell 1999).

Sortation is one of the primary functions of a distribution center. Over the course of a day, customer orders (e.g., a single household for a direct retailer, a particular retail store for a discount merchandiser or specialty retailer) are communicated to the distribution center. One simple approach to fulfill these orders is to assign each order to an order-picker and send the picker out to the warehouse to gather each item. For such manually oriented distribution systems, Coyle et al. (1996) found that order-picking costs average 65% of the total distribution center operating costs. To increase picker efficiency, many distribution centers batch orders together in what is

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called a wave. Batching orders in a wave allows an order-picker to pick an entire case of a popular item that will be used to fulfill many orders. However, the cases must then be split into items and those items sorted into customer orders. The process of sorting cases of product into orders with individual items is referred to as a split-case sorting.

The sorting process itself is often accomplished via an automated sorting system. A typical automated sorter uses a circular conveyor system constructed above bins that hold the shipping carton for each order. Upon reaching the appropriate bin, items are automatically diverted into a shipping carton. When the order is complete, packers inspect the carton and move it onto a conveyor that carries the carton out to the appropriate truck dock.

Very little research has been conducted on the design and operation of split-case sorting systems. Past research on general conveyor systems has concentrated on production systems serviced by conveyor systems (for early models, see Muth 1972, 1977; El Sayed et al. 1976; and Proctor et al. 1977; see Bastani 1990 for a survey). Likewise, past research on sortation systems has concentrated on general accumulation/sortation systems used to sort cartons of product into the appropriate shipping lanes. Recirculation or conveyor blocking is fundamental in most accumulation sorting systems because the number of orders is greater than the number of shipping lanes and there can also be blocking at the shipping lanes (see Bozer and Sharp 1985, Bozer et al. 1988, Meller 1997, and Johnson 1998 for research on such systems). In split-case sorting systems, the number of orders is always less than or equal to the number of packing bins; thus, there is no blocking or recirculation of items. Other areas of loosely related research are open-face order picking (Bartholdi and Eisenstein 1995 and Bartholdi et al. 2001), warehouse order picking (Ratliff and Rosenthal 1983 and Yoon and Sharp 1996), freight terminal layout and operations (Gue 1999 and Bartholdi and Gue 2000), worker-machine interference (Niebel 1982), and queueing systems with blocking (Gross and Harris 1998).

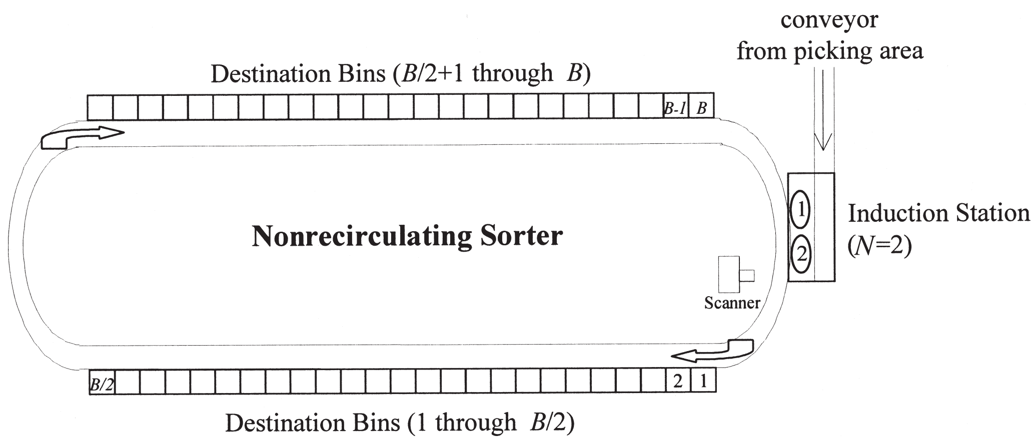
In this paper, we describe our research in developing analytical models of system performance that incorporate the stochastic elements of demand, product mix, and human interaction. This research was motivated by sorting problems we originally observed at Russell Athletic, where product is often shipped directly to retail stores in less-than-case quantities. Throughout our research project, we visited and interviewed many different companies that applied sorting technologies, including Sears, Federal Express, Ingram Book, J. Crew, Goodies Clothing, and Hanover Direct (a direct merchant with multiple catalog titles and a provider of e-fulfillment services for major e-tailers). During our visits we observed a wide range of systems, some of which were very successful, while others vastly underperformed their original specification. In this paper, we will uncover reasons why some implementations were successful while others failed.

We continue our paper in §2 by describing the context for split-case sorting operations and the physics of an automated sorting system. One of the most important aspects of system operation is the induction of items onto the sorter. In §3 we develop a stochastic model of the induction process and show how throughput is affected by the system design and the stochastic elements of system operation. In §4 we develop an approximate performance model to consider general distributions for the induction process and the impact of process variance on system throughput. In §5 we examine the robustness of our approximate model. Finally, in §6 we summarize our results and discuss their implication for success and failure of automated sorting systems.

2. Description of Automated Sortation System

In an automated sortation system, each customer order is assigned its own bin and the sorter is able to accommodate B customer orders at a time. Inductors located at induction stations place individual items onto a circular conveyor made up of discrete trays. For most systems, induction is performed manually. A worker removes items from a case and tosses them onto the moving trays as they pass the station. A barcode scanner reads the items as they leave the station,

Figure 1 Nonrecirculating Sorter (Top View)



passing the information to the computer that is controlling the sorter. Later, the computer will signal the system to divert the item into the appropriate bin for packing into a shipping carton. Figure 1 illustrates a top-view of a typical nonrecirculating sorter with B bins and $N = 2$ inductors.

Several different technologies are available for automated sortation systems (e.g., bomb-bay, tilt-tray, cross-belt sorters). The technologies are similar in basic sorting functionality but are distinguished by discharge mechanism, with bomb-bay systems as the most simple (a tray opens above the bin) and cross-belt systems as the most sophisticated (a conveyor on the tray moves the item onto the tray and then off of it into the bin). The choice between the different discharge mechanisms is based on the physical dimensions of the items to be sorted and is influenced by such factors as the weight, size, and delicacy of the item. For example, apparel can be easily dropped into a bin, while more fragile goods like consumer electronics require the gentle treatment of a cross-belt sorter. The cost of these systems ranges from \$350,000–\$750,000 for small–medium systems ($30 \leq B \leq 80$), with large sophisticated systems exceeding \$1M.

2.1. Assumptions and Notation

To model the performance of the sorter, we assume that the induction process governs the throughput of the system. This assumption is true in most cases because the output of the sorting system is dependent on the output of three subsystems: picking, induction, and packing. And although it is true that the throughput of each subsystem is dependent on the labor assigned to that activity, only induction is also limited by the investment and configuration of the hardware. Thus, it is rare to find a system where the induction process is not the bottleneck of the sorting system. In practice, this assumption is supported by noting that the picking process usually works ahead of the induction area by staging cases in front of the sorter, and the packing area includes a mechanism to ensure that packing never blocks the sorter (e.g., a two-level system where each bin serves two orders, allowing one order to wait for a ''buttoning up'' process (dunnage is added to the carton, the lid is taped shut, and shipping labels are attached) while the other order is accumulating).

Because induction is typically performed manually, induction is a stochastic process governed by the speed of the worker and his/her ability to place a stream of items onto the trays. In most cases, the trays are moving much faster than the worker can induct, and thus the worker must time the toss carefully to ensure the item is well centered in the tray to allow accurate reading by the scanner for proper diversion later. Workers develop their own pace, but hesitation and other natural variation occurs when tossing the items onto the trays. (We will examine automated induction in §4.)

The inductor's nominal induction rate, λ, is defined as the unconstrained induction rate of an inductor working in isolation. Of course, factors such as the conveyor speed and inductor conflict may limit the inductor's effective induction rate, which we denote by λ'. First, the conveyor speed (s) defines the rate at which potential empty trays arrive at an induction station—limiting the effective induction rate (i.e., $\lambda' \leq s$). Additionally, when multiple inductors are operating on the sorter, trays passing an induction station may already be occupied by an item. This forces the inductor to wait until an open tray arrives. Thus, the effective induction rate must be lower than the nominal rate (i.e., $\lambda' \leq \lambda$) and may be much lower in some cases.

The throughput of the induction system may seem to be an increasing function in s. However, this is not the case because conveyor speeds can be set too fast for the human inductors. When this occurs, throughput actually decreases as s increases because the inductors are not able to ''time'' their induction attempts and miss inducting into open trays. Moreover, the physics of the item (size, weight, friction, etc.) determine how quickly the item can be discharged, therefore limiting the speed of the sorting system. In this paper, we assume s has been set at a level that is appropriate for the induction process and the items being sorted.

Once inducted, items travel to the bin assigned to the corresponding order. We assume that the destination bin for each inducted item is *iid* uniformly distributed amongst all bins. For example, if an inductor is inducting large, blue sweatshirts into the system, each sweatshirt is equally likely to be destined for any bin. This assumption is justified in many realistic cases where there is significant commonality among items in a wave of orders and orders are randomly assigned to bins. We will explore the impact of this assumption later in the paper.

3. Analytical Models for the Induction Process

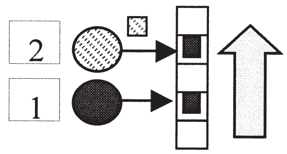
In this section we model cases where multiple inductors may be employed at single or multiple induction stations. When multiple inductors are working at a single station, we say they are working *side-by-side.* Multiple induction stations (each with one or more inductors) are referred to as *splitting* the inductors. When induction stations are split, they are typically placed at equal increments around the sorter. For example, two stations would be placed at 3 and 9 o'clock, while three stations would be placed at 4, 8, and 12 o'clock.

3.1. Induction Throughput Model for Side-by-Side Inductors

Consider a system where each inductor is characterized by his/her ability to induct an item onto the conveyor as it passes the induction station. In a side-by-side system, we number inductors 1, 2, 3, . . . , N, with 1 being upstream of 2, and 2 upstream of 3, and so on. Denote p_i as the probability that inductor i—working in isolation—is able to induct an item onto a tray passing the induction station ($p_i < 1$). This probability, p_i, which can be directly estimated by observation of the inductor, is a function of both the time to gather the next item to be inducted as well as the worker's skill in placing an item onto a moving conveyor. Therefore, this probability is determined by his/her nominal (mean) induction rate, λ_i, and the speed of the conveyor ($p_i = \lambda_i/s$, $\lambda_i < s$). Using this approach, we are implicitly assuming that the time for an inductor to pick up the next item is small enough in relationship to conveyor speed to ensure finite p. For this Bernoulli process, the number of empty trays between successful inductions for an inductor working in isolation follows a geometric distribution. We will examine other stochastic processes to describe induction later in the paper.

While it is easy to characterize the induction process of a single inductor, multiple inductors create ''queueing-like'' interference with each other that degrades system performance. Because multiple inductors are almost always employed, the steady state throughput of the system is the mean total effective induction rate of all inductors. Throughout this paper we will concentrate on steady state, mean induction rates (both nominal and effective), but we will drop the word ''mean'' from this point forward. We denote λ_i' as the effective induction rate of inductor i ($\lambda_i' <$

Figure 2 Induction Station with Two Side-by-Side Inductors



s). For a single inductor, the effective induction rate is equal to the nominal induction rate; that is $\lambda_1' = \lambda_1 = sp_1$.

Now consider a second inductor working next to the first inductor at the same station. Without any upstream interference, the second inductor has probability p_2 of loading a tray. However, after retrieving an item from the case, the inductor may find that the tray passing the induction station is occupied by an item placed by the first inductor. Thus, the second inductor must wait until an empty tray appears (Figure 2). If the inductor must wait for an empty tray, we have observed that inductors look ahead to see the first available tray and have more time to synchronize their toss, thus ensuring a successful load. Therefore, we assume they will successfully load the item in the next empty tray. We also assume that the inductors act independently, so the arrival process of full trays to the second inductor is a Bernoulli process. Therefore, the effective induction rate of Inductor 2 is the inverse of the sum of the expected number of trays between nominal load attempts $(1/p_2 - 1)$ plus the expected number of trays until an empty tray appears $(1/(1 - p_1))$:

$$\lambda_2' = \left[\left(\frac{1}{p_2} - 1\right) + \frac{1}{1 - p_1}\right]^{-1} s.$$

Redefining this expression in terms of the nominal induction rates, combining terms, and inverting the left-hand side yields

$$\lambda_2' = \left[\frac{1}{\lambda_2/s} - 1 + \frac{1}{1 - \lambda_1/s}\right]^{-1} s = \left[\frac{(\lambda_2/s)(1 - \lambda_1/s)}{1 - \lambda_1/s + \lambda_1\lambda_2/s^2}\right] s. \quad (1)$$

Therefore, the total effective induction rate for two side-by-side inductors is

$$\Lambda_{SBS}' = \lambda_1' + \lambda_2', \quad (2)$$

where $\lambda_1' = sp_1$ and λ_2' is found by (1).

With this exact model of two side-by-side inductors, we can explore important design questions, such as the benefit of having a second inductor under different scenarios of conveyor speed and nominal induction rates (we will present approximate models for systems with more than two inductors and non-Bernoulli induction processes in §4). Figure 3 shows the total effective induction rate (relative to the conveyor speed) for two identical inductors over a range of nominal induction rates (relative to the conveyor speed). Also included is an upper bound on the in-

Figure 3 Total Effective Induction Rate for Two Identical, Side-by-Side Inductors

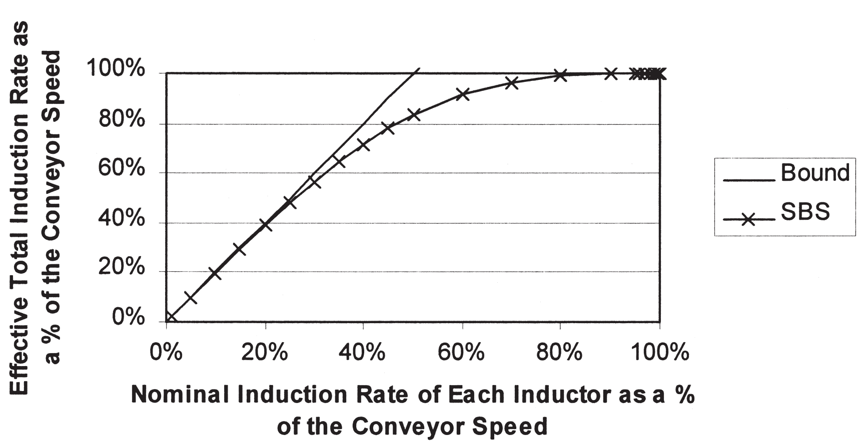
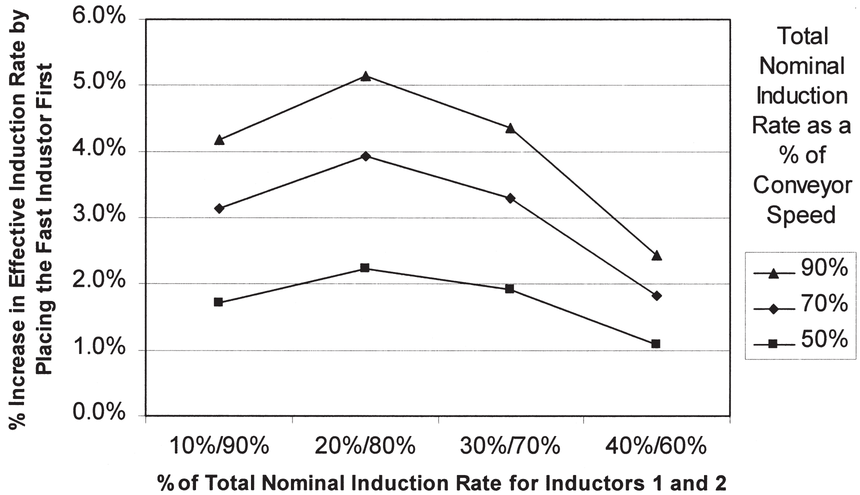


Figure 4 Impact of Placing the Fast Inductor First in a Side-by-Side Configuration



duction rate if no interference was present. It is easy to see that as the induction rate of the first inductor approaches the conveyor speed, the value of the second inductor vanishes.

Another interesting operating question is the configuration of two inductors who operate at different speeds. The following result shows that for two inductors operating side-by-side, the effective throughput is maximized by placing the fast inductor first. This result implies that highly skilled workers should be placed at the first inductor position, while those in training should be placed downstream.

Result 1. For two side-by-side inductors with nominal induction probabilities p_1 and p_2, and constants a and b, where $1 > a > b > 0$, the total effective induction rate is larger for a system with $p_1 = a$ and $p_2 = b$ than a system with $p_1 = b$ and $p_2 = a$. (Proof provided in the Appendix.)

Figure 4 shows the impact of placing the fast inductor first for various levels of the two inductors' total induction rate (the three curves) and various levels of imbalance between the two inductors (the four points on each curve). As can be seen from the three curves, the improvement increases with total induction rate because downstream interference increases with total induction rate. And for a given total induction rate, the improvement is the greatest when the imbalance is significant but not extreme. In such a case with significant (but not extreme) imbalance, the first inductor creates significant downstream interference, eroding the second inductor's (relatively large) nominal rate. For example, in the 20/80% and the 30/70% cases, if the slower inductor is placed first, the slower inductor's rate is high enough to create significant interference for the faster inductor placed downstream. However, for the 10/90% case, the slower inductor's rate is so low as to create hardly any interference for the faster inductor. As one would expect, when the inductors are relatively balanced (40/60%) the improvement is also less significant.

3.2. Induction Throughput Model for Split Inductors

Because upstream inductors working side-by-side significantly interfere with the induction rate of downstream workers, many equipment vendors recommend splitting induction stations into two or more stations spread equally around the sorter. Such an approach is more expensive because it requires additional bar-code readers, conveyors, and accumulation space for cases of items waiting to be sorted. Thus, a key question is to determine the magnitude of possible throughput increases from splitting induction stations.

Consider a sorter with two inductors working at opposite ends (3 and 9 o'clock) of the sorting system with an equal number of order bins between them. Recall that, without any special presorting, the items inducted by both stations are equally likely to be destined for any bin around the sorter. Thus, one half of the items inducted at Station 1 are delivered to a bin before reaching Station 2, and likewise for those items inducted at Station 2. In this case, *both* inductors face some full trays that interfere with their induction rate.

Because items inducted at Station 1 are equally likely to be delivered to any bin around the sorter, the exit of full trays before Station 2 acts like a thinning process with probability ½ that any item inducted at Station 1 exits into a bin before Station 2. Thus, we will assume the arrival process of full trays to Station 2 is still a Bernoulli process with rate $\lambda_1'/2$. While not exact, this assumption is reasonable because the thinning process reduces the interdependent interference between the two inductors. The effective induction rate at Station 2 may be determined with (1) as

$$\lambda_2' = \left[\frac{1}{\lambda_2/s} - 1 + \frac{1}{1 - \lambda_1'/(2s)}\right]^{-1} s.$$

However, notice that λ_2' is dependent on λ_1' and vice versa because they both interfere with each other. Thus,

$$\lambda_1' = \left[\frac{1}{\lambda_1/s} - 1 + \frac{1}{1 - \lambda_2'/(2s)}\right]^{-1} s.$$

With two equations and two unknowns, we can readily solve this system of equations to find λ_1' and λ_2'. However, to simplify our analysis and draw insight into our system design, we assume (without loss of generality) that both inductors have the same nominal induction rate (i.e., $p_1 = p_2 = p$ and $\lambda_1 = \lambda_2 = \lambda = ps$) and, thus, the same effective induction rate. Reorganizing in terms of λ', we obtain the quadratic function, $(\lambda - s)\lambda'^2 + (2s^2 + s\lambda)\lambda' - 2s^2\lambda = 0$, which has one feasible (positive) root:

$$\lambda' = \frac{-(2s^2 + s\lambda) + \sqrt{(2s^2 + s\lambda)^2 + 8s^2\lambda(\lambda - s)}}{2(\lambda - s)}. \tag{3}$$

Therefore, the total effective induction rate for a split system with two equivalent inductors is

$$\Lambda_{SPL}' = 2\lambda', \tag{4}$$

where λ' is found in (3).

We now can make a direct comparison of the split vs. side-by-side systems. For example, consider a system where $s = 70$ and $p_1 = p_2 = 35/70$. From (2) we see that $\Lambda_{SBS}' = 35.00 + 23.33 = 58.33$, and from (4) we see that $\Lambda_{SPL}' = 2(30.69) = 61.38$. The result below indicates that, all other factors equal, a split system will always outperform a side-by-side system.

Result 2. For two inductors each with nominal induction rate λ ($\lambda < s$), the total effective induction rate of a split system is larger than that of a side-by-side configuration ($\Lambda_{SPL}' > \Lambda_{SBS}'$). Proof provided in the Appendix.

Figure 5 compares the performance of the split and side-by-side systems (with a theoretical upper bound on performance with no interference) over a range of nominal induction rates. Notice that for low induction rates (relative to the conveyor speed), the two systems are nearly identical. However, the difference between the two systems grows as the nominal induction rate approaches the conveyor speed.

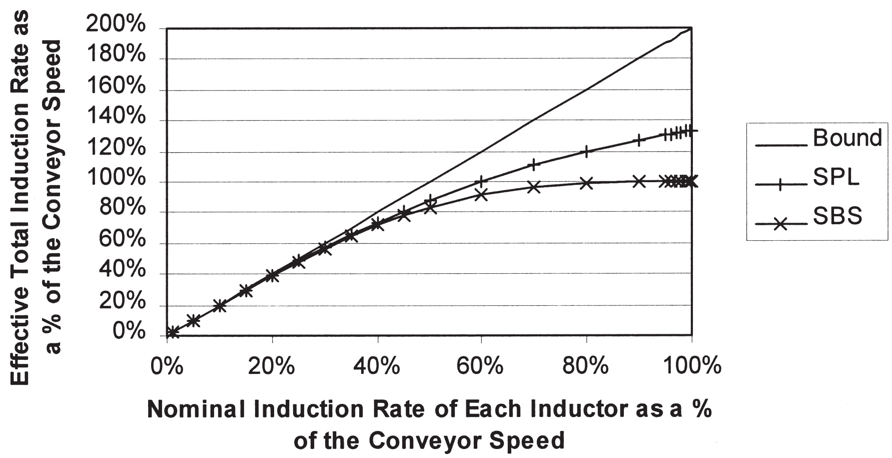
Up to this point we have considered only cases where the inductors' nominal induction rates are less than the conveyor speed ($\lambda_i < s$, and thus $p_i < 1\ \forall i$). For the side-by-side case, if the first inductor can keep up with the conveyor, then the second inductor never sees an empty tray and the total effective induction rate is s. However, for the split system with inductors who induct items at the speed of the conveyor, the total effective rate exceeds s as shown in the following result.

Result 3. For two inductors working in a split configuration with nominal induction rates limited by the conveyor speed (i.e., $\lambda_i = s$, $i = 1, 2$), the total effective induction rate is expressed as $\Lambda_{SPL}' = (\frac{4}{3})s$.

Proof. Note that $\lambda_1' = s - \frac{1}{2}\lambda_2'$ because Inductor 1 can keep up with the conveyor, filling every empty tray, and one half of the items inducted at Station 2 are delivered to a bin before reaching Station 1 (and likewise for those items inducted at Station 1). We also note that both inductors are equivalent, which means that $\lambda_1' = \lambda_2'$, thus $\lambda_1' = s - \frac{1}{2}\lambda_1'$. Simplifying, $\lambda_1' = \frac{2}{3}s$ and $\Lambda_{SPL}' = \lambda_1' + \lambda_2' = \frac{4}{3}s$. □

This observation leads to a more general result.

Figure 5 Comparison of the Total Effective Induction Rate for Two Identical Inductors in Either a Side-by-Side or Split Configuration



Consider a side-by-side system. As inductors are added to the station, the total effective rate rises until it reaches the speed of the conveyor (s). Thus, the system capacity is limited by the conveyor speed. However, as we saw for two split inductors working at the speed of the conveyor, the total effective induction rate is 133% of the conveyor speed. What would happen if we added a third, fourth, or Nth inductor—each equally placed around the conveyor?

RESULT 4. For N inductors working in an equally spaced split configuration (with $B > N$ evenly distributed between the inductors) with nominal induction rates limited by the conveyor speed (i.e., $\lambda_i = s$, $i = 1, \ldots, N$), the total effective induction rate is expressed as $\Lambda'_{SPL} = (2N/(N + 1))\ s$. Moreover, $\lim_{N\to\infty} \Lambda'_{SPL} = 2s$.

PROOF. For each inductor, $1/N$ of the items inducted at that station exit the trays into a bin before arriving at the next, and all subsequent induction stations. Also, the effective induction rate at each station is limited by the number of empty trays arriving at that station. Thus, $\lambda'_1 = s - [((N - 1)/N)\lambda'_2 + ((N - 2)/N)\lambda'_3 + \cdots + (1/N)\lambda'_N]$. Utilizing the fact that the effective induction rates will be equivalent at each station, we find that $\Lambda'_{SPL} = \Sigma_{i=1}^{N} \lambda'_i = (2N/(N + 1))s$. Taking the limit of this expression as N increases yields the final result. □

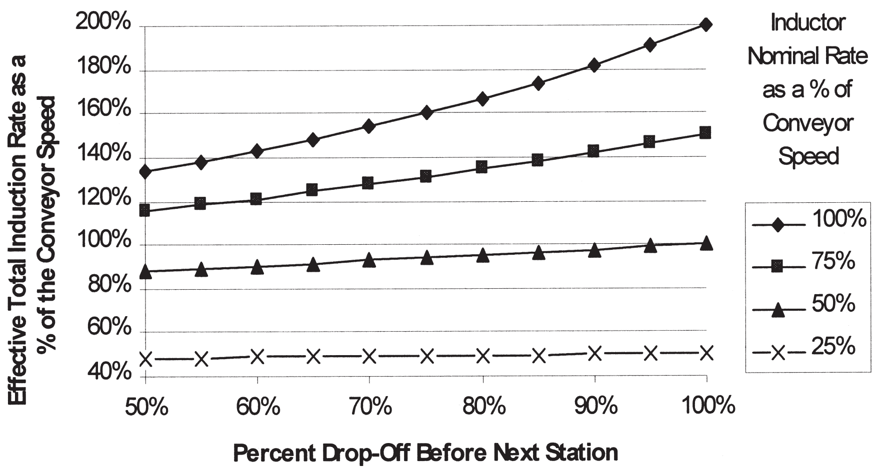
Intuitively, this result makes sense because as $N \to \infty$, every tray will be filled the moment it outputs its item, and on average a tray goes half way around before dumping again. Therefore, every tray outputs an average of twice per revolution, or $\Lambda'_{SPL} = 2s$.

So we have seen that splitting induction stations provides small benefits for a few slow inductors, but that it can double the capacity of the sorter for many inductors who are limited only by the conveyor speed. Another way to further expand the capacity of a sorter in a split configuration is by presorting.

3.3. Presorting and Its Impact on Sorter Throughput

One key driver of sorting performance that we have not yet addressed is item commonality. Without item commonality, automated sorting systems as described in this paper would have less value because the sorter would only be used to assemble items picked by different order-pickers in the warehouse. However, as commonality among orders increases, automated sorting becomes more attractive because it is easy to find a group (or wave) of orders with like contents—thus facilitating case-quantity picking (e.g., 50 orders that all require a large, blue sweatshirt). Typically, for picking efficiency, distributors will try to group orders that have like contents and sort them in a single wave. A further refinement of this approach is to find two smaller subgroups of orders that

Figure 6 **Impact of Presorting on a Split Configuration with Two Inductors**



have high commonality within the subgroup, but limited commonality across subgroups. Using a split approach with two stations, half of the bins (from 3 to 9 o'clock) could be assigned to the first group, with the second half of the sorter dedicated to the other subgroup. The first induction station would induct items that are largely destined for the first subgroup and would thus drop off before reaching the second induction station (and likewise for the second induction station). Such an approach reduces the interference between the two induction stations.

Examining our model for the split system, we can see that such presorting changes the probability of an induction station observing a full tray. Rather than 50% of the orders dropping off before the next station, we may be able to achieve a higher percentage (e.g., 70 or 80%). Denote d as the probability that an item drops off before the next induction station. For two induction stations with the same nominal induction rate and drop-off rate, we express the effective induction rate for each inductor to be

$$\lambda' = \left[\frac{1}{\lambda/s} - 1 + \frac{1}{1-(1-d)\lambda'/s}\right]^{-1} s.$$

Solving for the appropriate root of the quadratic expression yields

$$\lambda' = \Big\{-[s^2/(1-d) + s\lambda] + \sqrt{[s^2/(1-d)+s\lambda]^2 + 4s^2\lambda(\lambda-s)/(1-d)}\Big\} \div [2(\lambda - s)], \tag{5}$$

and the total effective rate $\Lambda'_{SPL} = 2\lambda'$. Note that (3), the previously presented expression for the effective throughput of an inductor in a split two-inductor system, is a special case of (5) with $d = 0.50$. Using $d = 0.75$ in our previous example ($s = 70$ and $p_1 = p_2 = 35/70$), Λ'_{SPL} increases from 61.38 to 65.64.

Figure 6 shows the impact of presorting for different drop-off rates (d) and relative induction rates. For low (25%) induction rates (relative to the conveyor speed), presorting has very limited impact because interference between the induction stations is low. However, as the relative induction rate rises, interference increases and so does the value of presorting. For example, in the case where the induction rate is limited by the conveyor speed (nominal rate is 100% of conveyor speed), presorting has a significant impact. With no presorting (represented by a 50% drop-off rate), we saw that two split inductors could achieve total effective induction rates of 133% of the conveyor speed. However, with a 75% drop-off rate they can achieve 160% of the conveyor speed, and with a drop-off rate near 100% they can double the conveyor speed (200%). At its extreme (≈ 100% drop-off), presorting allows each new inductor added to the system to contribute its full nominal rate to the total effective rate. While this result seems great, recall that commonality among orders is the key to reducing picking time in the warehouse. So while a small amount of presorting can increase sorter capacity with little warehouse picking impact, massive presorting may eliminate some of the value of automated sorting systems.

4. Induction Process Variance and Throughput

While the Bernoulli process does a good job representing many real induction processes, automated induction, worker training, and process improvements can lower the variance of the number of trays between induction attempts. During our on-site study of operating facilities, we noticed that some system users had developed job aids, such as ergonomic racking systems, or added electronic pacing devices, such as lights. Coupled with worker training and experience, such process improvements reduced the variance between induction attempts. For example, in one three-inductor system, we noticed that the third inductor very rarely saw more than three full trays in a row, and the second inductor very rarely saw more than two full trays in a row. In fact, this makes sense because the third inductor could not be justified if the probability that he/she saw more than three full trays in a row was high (because that would indicate lots of blocking and, therefore, poor utilization of the third inductor). However, given the relative nominal induction rates of the workers and speed of the conveyor, our Bernoulli-based model from §3 would predict more blocking than we observed in some real systems. To examine the impact of induction variance, we developed an approximate model for general induction processes. A key parameter for this model is the variance of the number of trays between induction attempts, which could be estimated from direct observation or from past experience.

4.1. Approximate Model for Low Induction Variance

To develop an approximation for low variance induction, we used an approach analogous to the development of an approximate waiting-time expression from a queueing model. Many approximations for waiting times in $G/G/1$ queueing systems are based on a weighted combination of upper and lower bounds (Gross and Harris 1998). For example, Marchal (1978) developed an approximation on waiting time based on bounds that reflected the observation that an upper bound was tight for high traffic intensity. Using a weighting quotient, he constructed an approximation that performed well in many circumstances and was also binding for known extreme cases ($M/G/1$ and $D/D/1$).

Because we are interested in cases with lower variance of trays between induction attempts than that of the geometric distribution created by the Bernoulli process, we consider the geometric case to be a lower bound on system throughput (referred to as Λ^G). Of course, it is not a true lower bound for all systems because those with higher variance than the Bernoulli or correlation structures that increased blocking would be worse. For cases with more than two side-by-side inductors, we propose the approximate approach of sequentially solving for the effective induction rates using (1), starting with the second inductor as follows:

$$\lambda_i^G = \left[\frac{1}{\lambda_i/s} - 1 + \frac{1}{1 - \sum_{j=1}^{i-1} \lambda_j^G/s} \right]^{-1} s, \qquad (6)$$

for $i = 2, 3, 4, \ldots,$ where $\lambda_1^G = \lambda_1$. The geometric estimate of total system induction rate is then $\Lambda^G = \Sigma_{i=1}^N \lambda_i^G$. This is an approximation because the output process of the second and subsequent inductors is no longer a Bernoulli process. To create a simple optimistic estimate, we modeled the likelihood of a downstream inductor observing a full tray as a *single* Bernoulli trial—which we refer to as the finite estimate. If the inductor sees a full tray, we assume that the next tray will be open with probability 1. For example, if the effective total rate of the upstream inductors is λ', we define the distribution of full trays seen by the downstream inductor simply as $p = \lambda'/s$ for one full tray and zero otherwise. Thus, our optimistic finite estimate of the effective induction rate for inductor i in a side-by-side configuration is

$$\lambda_i^F = \left[\frac{1}{\lambda_i/s} + \sum_{j=1}^{i-1} \frac{\lambda_j^F}{s}\right]^{-1} s, \tag{7}$$

for $i = 2, 3, 4, \ldots, \lambda_i = s$, and $\lambda_1^F = \lambda_1$. Again, for multiple inductors, the optimistic finite estimate of each inductor is solved sequentially starting with the second inductor and the total rate is $\Lambda^F = \Sigma_{i=1}^N \lambda_i^F$, with an upper limit of s (that is, if $\Lambda^F > s$, simply set it to s). Note this is not a true upper bound on system performance because deterministic (or near-deterministic) systems could be balanced to achieve even higher rates. For example, if two inductors often worked together, they may develop a rhythm to make empty spaces even more predictable. Nevertheless, with the constant change of items sorted and warehouse staff, (7) provides a very realistic upper bound in many real cases. For cases with split inductors with varying drop-off rates, both the geometric and finite estimates are addressed, as in §3.2, by solving the system of equations to find the effective rates.

With optimistic and pessimistic estimates of effective total induction, we developed a two-moment approximate expression. Our expression strives to capture the effect of induction variance and to reflect the effective induction rates relative to the conveyor speed. First, for each inductor, let s/λ_i and Var_i be the mean and variance of trays between an induction attempt. Let Var_i^G be the variance of the trays between induction attempts for inductor i if it followed a geometric distribution (driven by the Bernoulli process). Finally, define

$$\rho_i = \begin{cases} \displaystyle\sum_{j=1}^{i-1} \frac{\lambda_j}{s} & \text{if } \displaystyle\sum_{j=1}^{i-1} \frac{\lambda_j}{s} < 1, \\ 1 & \text{otherwise,} \end{cases} \tag{8}$$

as the upstream induction intensity. Employing the observation that high variance or high induction intensity leads to more interference and blocking between inductors, we developed a weighting quotient for each inductor, w_i, that ranges from zero to one and tends towards one for high variance and/or induction intensity:

$$w_i = \rho_i^{(1-\Sigma_{j=1}^{i} \mathrm{var}_j / \Sigma_{j=1}^{i} \mathrm{var}_j^G)}. \tag{9}$$

Using that quotient, our approximate expression for an effective induction rate (for both side-by-side or split) is

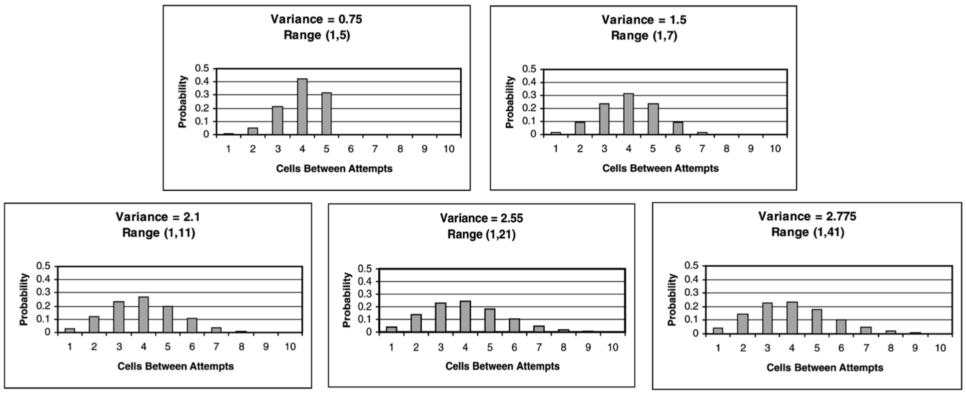
$$\lambda_i^A = w_i \lambda_i^G + (1 - w_i)\lambda_i^F. \tag{10}$$

Notice that this expression is tight for the case of our Bernoulli process because when $\mathrm{Var}_j = \mathrm{Var}_j^G \ \forall j$, our expression yields the geometric estimate. Likewise, when ρ_i becomes large, our estimate tends towards the geometric estimate. For example, when the *nominal* upstream intensity is 100% (note that the effective rate of the upstream servers would be less than 100% due to interference), our approximate expression becomes the geometric estimate, which fully captures the high interference and limits the remaining possible inductions to few open trays. On the other hand, for very low variance and low induction intensity the chance of seeing multiple full trays is also low and our approximate expression is more heavily influenced by the optimistic finite model that assumes no sequential full trays. In the next section we examine the robustness of our approximate expression.

5. Robustness of Approximate Expression

To better understand the performance of our approximate expression, we conducted an extensive set of simulation experiments over a wide range of configurations and induction processes. The base set of scenarios included seven levels of total induction intensity (ranging from 40 to 100%). Within each level of induction intensity, we examined five levels of inductor variance by using five different distributions of trays between induction attempts (TBA $= s/\lambda$), resulting in a total of 35 different scenarios. Figure 7 shows a sample of the different discrete distributions of TBA used in the scenarios. These distributions included a range of variance, range, and skewness values. Also, within the base set of scenarios, we examined cases where both inductors followed the

Figure 7 **Five Different Discrete Distributions of the Nominal Induction Process with Same Mean (4.0) but Different Range (Only 1–10 Shown) and Variance**



same stochastic inductor process and cases where the inductors worked at different rates following different distributions of cells between induction attempts. For comparison purposes, the conveyor speed was fixed at $s = 100$ trays per minute. For each scenario, we replicated the simulation 20 times for 1,100 minutes, truncating the first 100 minutes to eliminate initial transient bias (Law and Kelton 1991). Tables 1–4 (which are included in the Appendix) report the system parameters for each simulation experiment, our estimate of system throughput (in items per minute), which is based on the presented geometric and finite estimates, and the simulation result with its associated 95% confidence interval half-width.

Table 1 shows the results for side-by-side induction, while Table 2 shows the results for a split configuration. As we can see from the tables, our approximate expression performed very well under both configurations. For each case we report two error statistics—the average percent error to measure bias and the average absolute percent error to measure dispersion. For the side-by-side case, the average percent error was 0.13%, with the average absolute percent error at 0.18%. For the split configuration, the results were even better, with the 0.02 and 0.05% for the average and absolute percent error, respectively. While our approximate expression is unable to capture the full interaction effects of the two inductors, it still performs well over a wide range of induction intensities and variance.

To further examine the impact of multiple side-by-side inductors, we ran two more groups of scenarios with three and four inductors (another 35 scenarios). Tables 3 and 4 show that our expression continues to perform very well in these more complex configurations with the worst errors within about 2%. Particularly in low-variance scenarios, the interaction of multiple inductors can lead to brief periods of synchronization, allowing the inductors to achieve higher throughput. However, as can be seen from the tables, our expression remains very robust with the average and absolute percent error for three inductors at -0.20 and 0.46%, respectively; with four inductors it was -0.02 and 0.58%, respectively. In total, the results from over 100 different scenarios show that our expression is very robust with little bias and average dispersion well within 0.5%.

6. Conclusions

In this paper we presented the results of our research on modeling the performance of split-case sorting systems. By modeling the induction attempts as a Bernoulli process, we have developed throughput models for different systems' configurations. We have also established many interesting insights into system design, such as the negative impact of interference for multiple inductors, that split configurations can significantly outperform side-by-side systems, that side-by-side inductors should be placed from fastest to slowest, and that presorting can significantly improve throughput in split configurations. For cases where the induction process is less variable (due to automation, worker training, or process improvements), we have developed an approximate model of system throughput that we showed via simulation is very robust.

Possibly the most valuable aspect of any model is its ability to explain behavior in real applications. In the process of conducting this research, we visited many different installations of sorting systems to better understand the important issues. During those visits, we saw some sorters that performed well, while others never met the expectations of the users. In one case, an application at Russell Athletic was so unsuccessful that the whole system was scrapped after the first year because it never met throughput targets within designed manpower requirements. After a review of the system design, we quickly realized that initial throughput estimates were based solely on nominal induction rates—ignoring inductor interaction effects! At another unsuccessful application we observed, it was clear that item commonality was so low that inductors were inducting a small number of many items—driving up variability and reducing throughput. In a successful application at J. Crew, we observed that the users had learned, through trial-and-error, that presorting could significantly improve throughput during holiday crunch periods. In every case, the insights from our model allowed us to understand and explain both the good and bad applications.

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Appendix

RESULT 1. For two side-by-side inductors with nominal induction probabilities p_1 and p_2 and constants a and b, where $1 > a > b > 0$, the total effective induction rate is larger for a system with $p_1 = a$ and $p_2 = b$ than a system with $p_1 = b$ and $p_2 = a$.

PROOF. Define Λ'_F as the total effective induction rate for the case in which the fast inductor is placed first ($p_1 = a$, $p_2 = b$) and Λ'_S as the case in which the slow inductor is placed first ($p_1 = b$, $p_2 = a$). Thus,

$$\Lambda'_F = as + s\left[\frac{1}{b} - 1 + \frac{1}{1-a}\right]^{-1},$$

$$\Lambda'_S = bs + s\left[\frac{1}{a} - 1 + \frac{1}{1-b}\right]^{-1},$$

and we wish to show that $\Lambda'_F > \Lambda'_S$,

$$as + s\left[\frac{1}{b} - 1 + \frac{1}{1-a}\right]^{-1} > bs + s\left[\frac{1}{a} - 1 + \frac{1}{1-b}\right]^{-1}.$$

Canceling like terms and simplifying we have

$$a - b > \left[\frac{1}{a} - 1 + \frac{1}{1-b}\right]^{-1} - \left[\frac{1}{b} - 1 + \frac{1}{1-a}\right]^{-1},$$

$$a - b > \left[\frac{a - ab}{1 - b + ab}\right] - \left[\frac{b - ab}{1 - a + ab}\right],$$

$$a - b > \frac{a - a^2 + 2a^2b - b + b^2 - 2ab^2}{(1 - b + ab)(1 - a - ab)}.$$

Because $1 > a > 0$ and $1 > b > 0$, the denominator is clearly positive, thus (multiplying through and simplifying),

$$a^2b - a^3b + a^3b^2 - ab^2 + ab^3 - a^2b^3 > 0,$$

$$ab(a - b)(1 - (a + b) + ab) > 0,$$

$$ab(a - b)(1 - a)(1 - b) > 0,$$

which is clearly true because, by assumption, $ab > 0$, $a - b > 0$, $(1 - a) > 0$, and $(1 - b) > 0$. □

RESULT 2. For two inductors each with nominal induction rate λ ($\lambda < s$), the total effective induction rate of a split system is larger than that of a side-by-side configuration ($\Lambda'_{SPL} > \Lambda'_{SBS}$).

PROOF. First recall,

$$\Lambda'_{SBS} = sp + s\left[\frac{1}{p} - 1 + \frac{1}{1-p}\right]^{-1},$$

and $\Lambda'_{SPL} = 2\lambda'$, where the effective induction rate for split inductors is

$$\lambda' = \left[\frac{1}{\lambda/s} - 1 + \frac{1}{1 - \lambda'/(2s)}\right]^{-1} s.$$

Note that we can obtain a lower bound on the effective induction rate for split systems (Λ'_{SPL}) by replacing the probability that an inductor sees a full tote, $\lambda'/(2s)$, with $\lambda/(2s)$. This gives a lower bound on effective induction because it assumes a higher level of interference between inductors ($\lambda > \lambda'$):

$$\lambda' = \left[\frac{1}{\lambda/s} - 1 + \frac{1}{1 - \lambda'/(2s)}\right]^{-1} s$$

$$> \left[\frac{1}{\lambda/s} - 1 + \frac{1}{1 - \lambda/(2s)}\right]^{-1} s.$$

Table 1 Results for Two Side-by-Side Inductors

Simulation Experiment	AVG TBA Inductor 1	AVG TBA Inductor 2	Total Intensity	VAR TBA Inductor 1	VAR TBA Inductor 2	Total Induction: Geometric Estimate (Λ^G)	Total Induction: Finite Estimate (Λ^F)	Total Induction: Approx. Estimate (Λ^A)	Total Induction: Simulation Estimate	Simulated Half-Width 95%	% Error
1	5	5	40.0%	0.80	0.80	39.05	39.23	39.19	39.30	0.01	0.27%
2	5	5	40.0%	1.30	1.30	39.05	39.23	39.19	39.23	0.02	0.10%
3	5	5	40.0%	2.40	2.40	39.05	39.23	39.19	39.21	0.03	0.07%
4	5	5	40.0%	3.20	3.20	39.05	39.23	39.18	39.21	0.03	0.07%
5	5	5	40.0%	3.60	3.60	39.05	39.23	39.18	39.21	0.03	0.06%
6	4	4	50.0%	0.75	0.75	48.08	48.53	48.41	48.60	0.02	0.40%
7	4	4	50.0%	1.50	1.50	48.08	48.53	48.40	48.50	0.04	0.21%
8	4	4	50.0%	2.10	2.10	48.08	48.53	48.39	48.46	0.03	0.15%
9	4	4	50.0%	2.55	2.55	48.08	48.53	48.38	48.45	0.04	0.14%
10	4	4	50.0%	2.78	2.78	48.08	48.53	48.37	48.42	0.04	0.08%
11	3	4	58.3%	1.00	0.75	55.55	56.41	56.09	56.33	0.03	0.41%
12	3	4	58.3%	1.33	1.50	55.55	56.41	56.07	56.24	0.03	0.30%
13	3	4	58.3%	1.60	2.10	55.55	56.41	56.05	56.19	0.04	0.25%
14	3	4	58.3%	1.80	2.55	55.55	56.41	56.04	56.15	0.04	0.20%
15	3	4	58.3%	1.90	2.78	55.55	56.41	56.03	56.08	0.05	0.08%
16	3	3	66.7%	1.00	1.00	61.90	63.33	62.76	63.08	0.03	0.50%
17	3	3	66.7%	1.33	1.33	61.90	63.33	62.73	62.94	0.04	0.34%
18	3	3	66.7%	1.60	1.60	61.90	63.33	62.69	62.86	0.04	0.27%
19	3	3	66.7%	1.80	1.80	61.90	63.33	62.67	62.81	0.05	0.22%
20	3	3	66.7%	1.90	1.90	61.90	63.33	62.66	62.75	0.05	0.15%
21	2	4	75.0%	0.75	0.75	70.00	72.22	71.03	71.14	0.03	0.16%
22	2	4	75.0%	0.83	1.50	70.00	72.22	70.98	71.06	0.05	0.12%
23	2	4	75.0%	0.90	2.10	70.00	72.22	70.93	70.94	0.04	0.01%
24	2	4	75.0%	0.95	2.55	70.00	72.22	70.90	70.87	0.06	−0.05%
25	2	4	75.0%	0.98	2.78	70.00	72.22	70.88	70.86	0.06	−0.03%
26	2	3	83.3%	0.75	1.00	75.00	78.57	76.49	76.76	0.04	0.35%
27	2	3	83.3%	0.83	1.33	75.00	78.57	76.42	76.56	0.05	0.19%
28	2	3	83.3%	0.90	1.60	75.00	78.57	76.35	76.44	0.04	0.11%
29	2	3	83.3%	0.95	1.80	75.00	78.57	76.31	76.30	0.05	−0.01%
30	2	3	83.3%	0.98	1.90	75.00	78.57	76.28	76.26	0.06	−0.02%
31	2	2	100.0%	0.75	0.75	83.33	90.00	85.68	85.79	0.03	0.13%
32	2	2	100.0%	0.83	0.83	83.33	90.00	85.55	85.51	0.05	−0.05%
33	2	2	100.0%	0.90	0.90	83.33	90.00	85.45	85.31	0.05	−0.16%
34	2	2	100.0%	0.95	0.95	83.33	90.00	85.37	85.12	0.05	−0.29%
35	2	2	100.0%	0.98	0.98	83.33	90.00	85.32	85.06	0.05	−0.31%

TBA = Trays Between Attempts. Average % Error = 0.13%.
Average (Absolute) % Error = 0.18%.

Table 2 Results for Two Split Inductors

Simulation Experiment	AVG TBA Inductor 1	AVG TBA Inductor 2	Total Intensity	VAR TBA Inductor 1	VAR TBA Inductor 2	Total Induction: Geometric Estimate (Λ^G)	Total Induction: Finite Estimate (Λ^F)	Total Induction: Approx. Estimate (Λ^A)	Simulation Estimate	Simulated Half-Width 95%	% Error
1	5	5	40.0%	0.80	0.80	39.15	39.23	39.21	39.24	0.01	0.06%
2	5	5	40.0%	1.30	1.30	39.15	39.23	39.21	39.21	0.02	0.00%
3	5	5	40.0%	2.40	2.40	39.15	39.23	39.21	39.23	0.03	0.04%
4	5	5	40.0%	3.20	3.20	39.15	39.23	39.21	39.22	0.04	0.02%
5	5	5	40.0%	3.60	3.60	39.15	39.23	39.21	39.21	0.03	0.01%
6	4	4	50.0%	0.75	0.75	48.34	48.53	48.48	48.54	0.02	0.14%
7	4	4	50.0%	1.50	1.50	48.34	48.53	48.47	48.52	0.03	0.09%
8	4	4	50.0%	2.10	2.10	48.34	48.53	48.47	48.50	0.03	0.06%
9	4	4	50.0%	2.55	2.55	48.34	48.53	48.46	48.49	0.03	0.06%
10	4	4	50.0%	2.78	2.78	48.34	48.53	48.46	48.48	0.04	0.03%
11	3	4	58.3%	1.00	0.75	55.76	56.09	55.98	56.02	0.03	0.07%
12	3	4	58.3%	1.33	1.50	55.76	56.09	55.97	56.00	0.03	0.05%
13	3	4	58.3%	1.60	2.10	55.76	56.09	55.96	56.01	0.04	0.08%
14	3	4	58.3%	1.80	2.55	55.76	56.09	55.96	56.00	0.04	0.07%
15	3	4	58.3%	1.90	2.78	55.76	56.09	55.96	55.96	0.05	0.01%
16	3	3	66.7%	1.00	1.00	62.77	63.32	63.10	63.20	0.03	0.16%
17	3	3	66.7%	1.33	1.33	62.77	63.32	36.09	63.17	0.04	0.12%
18	3	3	66.7%	1.60	1.60	62.77	63.32	63.08	63.12	0.04	0.07%
19	3	3	66.7%	1.80	1.80	62.77	63.32	63.07	63.12	0.05	0.08%
20	3	3	66.7%	1.90	1.90	62.77	63.32	63.06	63.09	0.05	0.04%
21	2	4	75.0%	0.75	0.75	70.14	70.82	70.53	70.51	0.05	−0.02%
22	2	4	75.0%	0.83	1.50	70.14	70.82	70.51	70.49	0.04	−0.02%
23	2	4	75.0%	0.90	2.10	70.14	70.82	70.50	70.45	0.04	−0.06%
24	2	4	75.0%	0.95	2.55	70.14	70.82	70.49	70.44	0.06	−0.07%
25	2	4	75.0%	0.98	2.78	70.14	70.82	70.48	70.41	0.04	−0.10%
26	2	3	83.3%	0.75	1.00	76.22	77.35	76.77	76.80	0.04	0.04%
27	2	3	83.3%	0.83	1.33	76.22	77.35	76.75	76.74	0.05	−0.01%
28	2	3	83.3%	0.90	1.60	76.22	77.35	76.73	76.72	0.05	−0.01%
29	2	3	83.3%	0.95	1.80	76.22	77.35	76.71	76.67	0.06	−0.05%
30	2	3	83.3%	0.98	1.90	76.22	77.35	76.70	76.67	0.07	−0.04%
31	2	2	100.0%	0.75	0.75	87.69	89.90	88.47	88.49	0.04	0.02%
32	2	2	100.0%	0.83	0.83	87.69	89.90	88.42	88.41	0.06	−0.02%
33	2	2	100.0%	0.90	0.90	87.69	89.90	88.39	88.36	0.06	−0.03%
34	2	2	100.0%	0.95	0.95	87.69	89.90	88.36	88.30	0.06	−0.07%
35	2	2	100.0%	0.98	0.98	87.69	89.90	88.35	88.29	0.07	−0.07%

TBA = Trays Between Attempts. Average % Error = 0.02%.
Average (Absolute) % Error = 0.05%.

Table 3 Results for Three Side-by-Side Inductors

Simulation Experiment	AVG TBA Inductor 1	AVG TBA Inductor 2	AVG TBA Inductor 3	Total Intensity	VAR TBA Inductor 1	VAR TBA Inductor 2	VAR TBA Inductor 3	Total Induction: Geometric Estimate (Λ^G)	Total Induction: Finite Estimate (Λ^F)	Total Induction: Approx. Estimate (Λ^A)	Total Induction: Simulation Estimate	Simulated Half-Width 95%	% Error
1	5	5	5	60%	0.80	0.80	0.80	56.78	57.77	57.39	57.68	0.02	0.50%
2	5	5	5	60%	1.30	1.30	1.30	56.78	57.77	57.38	57.44	0.02	0.09%
3	5	5	5	60%	2.40	2.40	2.40	56.78	57.77	57.36	57.32	0.02	−0.08%
4	5	5	5	60%	3.20	3.20	3.20	56.78	57.77	57.35	57.27	0.03	−0.13%
5	5	5	5	60%	3.60	3.60	3.60	56.78	57.77	57.34	57.24	0.04	−0.18%
6	4	4	4	75%	0.75	0.75	0.75	68.38	70.83	69.66	70.28	0.02	0.88%
7	4	4	4	75%	1.50	1.50	1.50	68.38	70.83	69.61	69.74	0.02	0.20%
8	4	4	4	75%	2.10	2.10	2.10	68.38	70.83	69.56	69.54	0.03	−0.03%
9	4	4	4	75%	2.55	2.55	2.55	68.38	70.83	69.52	69.45	0.04	−0.11%
10	4	4	4	75%	2.78	2.78	2.78	68.38	70.83	69.50	69.35	0.04	−0.22%
11	3	3	3	100%	1.00	1.00	1.00	83.52	90.85	86.07	86.76	0.04	0.79%
12	3	3	3	100%	1.33	1.33	1.33	83.53	90.85	85.94	86.13	0.04	0.22%
13	3	3	3	100%	1.60	1.60	1.60	83.52	90.85	85.83	85.74	0.05	−0.11%
14	3	3	3	100%	1.80	1.80	1.80	83.52	90.85	85.75	85.49	0.04	−0.30%
15	3	3	3	100%	1.90	1.90	1.90	83.52	90.85	85.71	85.38	0.06	−0.39%
16	2	2	2	150%	0.75	0.75	0.75	97.62	100.00	99.96	99.10	0.05	−0.87%
17	2	2	2	150%	0.83	0.83	0.83	97.62	100.00	99.84	98.87	0.04	−0.98%
18	2	2	2	150%	0.90	0.90	0.90	97.62	100.00	99.73	98.69	0.04	−1.06%
19	2	2	2	150%	0.95	0.95	0.95	97.62	100.00	99.65	98.57	0.05	−1.10%
20	2	2	2	150%	0.98	0.98	0.98	97.62	100.00	99.60	98.57	0.05	−1.05%

TBA = Trays Between Attempts. Average % Error = −0.20%.
Average (Absolute) % Error = 0.46%.

Table 4 Results for Four Side-by-Side Inductors

Simulation Experiment	AVG TBA Inductors 1–4	Total Intensity	VAR TBA Inductors 1–4	Total Induction: Geometric Estimate (Λ^G)	Total Induction: Finite Estimate (Λ^F)	Total Induction: Approx. Estimate (Λ^A)	Total Induction: Simulation Estimate	Simulated Half-Width 95%	% Error
1	5	80%	0.80	72.62	75.70	74.05	74.82	0.02	1.03%
2	5	80%	1.30	72.62	75.70	74.02	74.22	0.03	0.27%
3	5	80%	2.40	72.62	75.70	73.96	73.80	0.03	−0.22%
4	5	80%	3.20	72.62	75.70	73.92	73.63	0.04	−0.39%
5	5	80%	3.60	72.62	75.70	73.90	73.56	0.04	−0.45%
6	4	100%	0.75	84.61	92.03	87.07	88.90	0.03	2.06%
7	4	100%	1.50	84.61	93.06	86.95	87.36	0.03	0.47%
8	4	100%	2.10	84.61	92.06	86.84	86.74	0.03	−0.11%
9	4	100%	2.55	84.61	92.06	86.76	86.42	0.04	−0.39%
10	4	100%	2.78	84.61	92.06	86.72	86.29	0.05	−0.50%
11	3	133%	1.00	95.92	100.00	98.47	98.87	0.01	0.41%
12	3	133%	1.33	95.92	100.00	98.33	98.12	0.02	−0.22%
13	3	133%	1.60	95.92	100.00	98.23	97.65	0.03	−0.59%
14	3	133%	1.80	95.92	100.00	98.14	97.39	0.02	−0.77%
15	3	133%	1.90	95.52	100.00	98.10	97.25	0.03	−0.88%

TBA = Trays Between Attempts. Average % Error = −0.02%.
Average (Absolute) % Error = 0.58%.

Now we will show that our lower bound on Λ'_{SPL} is greater than Λ'_{SBS}. Substituting $p = \lambda/s$, we have

$$2\left[\frac{1}{p} - 1 + \frac{1}{1 - p/2}\right]^{-1} s > sp + s\left[\frac{1}{p} - 1 + \frac{1}{1 - p}\right]^{-1}.$$

Simplifying both sides yields

$$\frac{p(2 - p)}{1 - p/2 + p^2/2} > \frac{p^3 - 2p^2 + 2p}{1 - p + p^2}.$$

Because the denominators are both clearly positive, further simplification yields $1 > p$, which is true by definition. $\square$

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