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# Optimal Dynamic Auctions for Revenue Management

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## 1. Introduction

Revenue management involves segmenting customers, setting prices, and controlling capacity to maximize the revenue generated from a fixed capacity. See McGill and van Ryzin (1999) for a review. However, with the rise of the Internet, many industries have lately begun experimenting with alternative pricing mechanisms, such as auctions, guaranteed purchase contracts, group purchasing, etc. (see van Ryzin 2000). Given the capabilities that online channels provide, such innovations are not surprising. But they do raise some important theoretical and practical questions. In particular, exactly which mechanisms should be used to maximize revenue in any given context? How should an optimal mechanism be designed and operated? And how much benefit (if any) can be obtained from a better mechanism?

We make some initial progress in addressing these questions. Specifically, we consider a natural variation of the traditional single-leg, multiperiod revenue management problem (e.g., see Brumelle and McGill 1993) in which buyers bid strategically in each period for a limited supply of capacity. Our model closely follows the assumptions of classical auction theory as described in the seminal work of Vickrey (1961), the influential paper of Milgrom and Weber (1982), and the recent survey by Klemperer (1999). As in this auction literature, we assume buyers have private valuations for a unit of capacity, and they act strategically to maximize their utility (i.e., their value minus the price they pay).

Recently, some papers have addressed the link between revenue management and auctions (e.g., Segev et al. 2001). What distinguishes our work, however, is that we model the dynamics of sales and inventory over time along with the strategic behavior of bidders. Specifically, we assume there are $T$ periods and in each period $t$, a new set of buyers bids for the remaining capacity. The seller must determine winners and award payments in period $t$ before observing the bids (or even the number of bidders) in future periods. Bidders act strategically in response to the seller's policy. We give a precise description of an optimal allocation of units in this dynamic setting and also show modifications of two traditional auction mechanisms (first and second price) to achieve the optimal allocation.

We then compare the optimal mechanism to a stylized version of the traditional revenue management settings, which we call a list price, capacity-controlled mechanism—or LPCC for short. The LPCC mechanism sets a fixed, take-it-or-leave-it price in each period together with a limit on the number of units of capacity that can be sold at the list price. We show theoretically that the LPCC mechanism is optimal when there is at most one buyer in each period—or asymptotically as the number of bidders and units to be sold grows large. Our numerical results show that LPCC revenues decrease relative to the optimal mechanism when the number of bidders per period increases (i.e., there is more ''aggregation'' of buyers) and when the product is relatively scarce. Moreover, this difference grows larger if the variability in buyers' valuations or in the number of bidders per period increases. These results suggest that some aggregation of buyers is necessary to achieve a benefit from using an auction mechanism for revenue management, and that scarcity and more variation in willingness to pay will help as well.

## 2. Model Formulation

### 2.1. Description of the Model

A seller has an initial inventory of $C$ units of a good that she wants to sell over a finite time horizon $T$. She does this by conducting a sequence of auctions, indexed by $t = T, T - 1, \ldots, 1$. The time index is assumed to run backwards, and smaller values of $t$ represent later points in time.

In period $t$, $N_t$ risk neutral potential buyers (bidders) arrive. $N_t$ is a nonnegative, discrete-valued random variable, distributed according to a known probability mass function $g(\cdot)$, with support $[0, M]$ for some $M > 0$, and strictly positive first moment. We will assume that buyers do not select their time of arrival, and that they participate in only one auction period.

Each buyer wishes to purchase at most one unit and has a reservation value $v_i^t$, $1 \leq i \leq N_t$, which represents the maximum amount buyer $i$ is willing to pay for the object. When the context is clear, we will drop the time index and write $v_i$. Reservation values are private information, independent and identically distributed draws from a distribution $F(\cdot)$, which is strictly increasing with a continuous density function $f(\cdot)$ on the support $[\underline{v}, \bar{v}]$, with $F(\underline{v}) = 0$ and $F(\bar{v}) = 1$. Without loss of generality, assume $\underline{v} = 0$ throughout. We will use $v$ both for the random vector of valuations (from the seller's perspective) and for its realization, where the meaning should be clear from the context. To simplify notation and subsequent analysis we restrict attention to distribution functions $g$ and $F$ that do not depend on the time $t$. The extension to time-dependent distributions is straightforward. The distributions $F$ and $g$ are assumed common knowledge to the seller and all potential buyers. Without loss of generality, we assume that the unit salvage value for the seller at time $t = 0$ is $v_0 = 0$.

The seller's problem is to design an auction mechanism that maximizes her expected revenue. The auctioneer will specify a set of rules (the mechanism) according to which the auction will be conducted. These rules may depend on the time $t$ and the remaining inventory at the beginning of each period, denoted by $x$. Each bidder (based on his private valuation, his knowledge of the distribution functions $F$, $g$, and the set of rules established by the auctioneer), chooses his bid (or strategy) to maximize his expected utility. Then, the auctioneer observes the set of submitted bids and applies the rules specified earlier to decide the number of units to award in period $t$ and the payments to be made by the various bidders. Before proceeding, we briefly review some results on optimal auction design that are used in our analysis.

### 2.2 Results from the Theory of Optimal Auctions

The basic results on optimal auctions that we use are from Myerson (1981) and Maskin and Riley (2000). Consider an auction in which we are selling one or more homogeneous objects to $n$ buyers. Each buyer $i$ wants at most one of the objects, which he values at $v_i$. As mentioned above, the values $v_i$ are private information, but it is common knowledge that $v_i$'s are iid with distribution $F$.

Maskin and Riley (2000) showed the rather remarkable fact that the expected seller's revenue can be expressed *only* in terms of the allocation rule $q_i(v_i, v_{-i})$—independent of the bidders' payments—where $q_i(v_i, v_{-i}) = 1$ if bidder $i$ is awarded an item, and 0 otherwise. Its requirements are rather general: The theorem holds provided $q_i(\cdot, v_{-i})$ are increasing in $v_i$ and the buyers with value equal to zero have zero expected surplus in equilibrium (see Maskin and Riley 2000, Proposition 2). Specifically, the expected revenue for the seller is given by

$$E_{v_i, v_{-i}}\left[\sum_{i=1}^{n} J(v_i) q_i(v_i, v_{-i})\right], \tag{1}$$

where $J(v) = v - 1/\rho(v)$, and $\rho(v) = f(v)/[1 - F(v)]$ is the hazard rate function associated with the distribution $F$. From this fact, it follows that all mechanisms that result in the same allocations $q$ for each realization of $v$ yield the same expected revenue. This is the so-called *revenue equivalence theorem.*

Moreover, Equation (1) can be used to design an optimal mechanism by simply choosing the allocation rule $q^*(v)$ that maximizes $\Sigma_{i=1}^{n} J(v_i) q_i(v_i, v_{-i})$, subject to any constraints one might have on the allocation (e.g., we have $k$ items to sell so we may require that the allocation $q$ satisfies $\Sigma_i q_i \le k$).

In optimal auction problems, it is useful to make the assumption that $J(v)$ is strictly increasing in $v$. If we define $v^* = \max\{v : J(v) = 0\}$ (and by convention, $v^* = \infty$ if $J(v) < 0$, $\forall v$), then from (1) it follows that it is never optimal to allocate a unit to a buyer with valuation $v_i < v^*$. This simple observation is the basis for determining optimal reserve prices.

The trick in applying this approach to auction design is to find an implementable mechanism that in fact produces the optimal allocation $q^*(v)$. This requires a separate analysis. Thus, the analysis proceeds in two steps: (1) Find an optimal allocation $q^*(v)$, then (2) Find an implementable mechanism that produces $q^*(v)$ for each realization $v$.

### 2.3. Dynamic Programming Formulation

In each period $t$, the seller must choose an auction mechanism that maximizes her total expected revenues. Define the value function $V_t(x)$ as the maximum expected revenue obtainable from periods $t, t-1, \dots, 1$ given that there are $x$ units remaining at time $t$. Using (1), the Bellman Equation (see Bertsekas 1995) for $V_t(x)$ in terms of $q(v)$ is

$$V_t(x) = E_{N_t,v}\left[\max_q \left\{\sum_{i=1}^{N_t} J(v_i)q_i + V_{t-1}(x-p) : q_i \in \{0,1\}, p = \sum_{i=1}^{N_t} q_i, p \le x\right\}\right], \quad (2)$$

with boundary conditions

$$V_t(0) = 0, \qquad t = 1, \dots, T \quad \text{and}$$
$$V_0(x) = 0, \qquad x = 1, \dots, C. \quad (3)$$

The problem is to choose for each period $t$ an allocation $q(\cdot)$ that maximizes $V_T(C)$.

## 3. Optimal Dynamic Allocations and Mechanisms

The solution of the dynamic program (2)–(3) crucially depends on the structural properties of the marginal value of capacity, defined by $\Delta V_t(x) \equiv V_t(x) - V_t(x-1)$. First, we begin by reformulating the dynamic program. Define:

$$R(p) = \begin{cases} 0 & \text{if } p = 0, \\ \sum_{i=1}^{\min\{p,N_t\}} J(v_{(i)}) & \text{if } p > 0. \end{cases}$$

Note that $\max\{\Sigma_{i=1}^{N_t} J(v_i)q_i : q_i \in \{0,1\}, \Sigma_i q_i = \min\{p, N_t\}\} = R(p)$, and that the integrality constraints can be relaxed to $0 \le q_i \le 1$. Formulation (2) can therefore be rewritten in terms of $p$ as follows:

$$V_t(x) = E_{N_t,v}\left[\max_{0\le p\le x} \{R(p) + V_{t-1}(x-p)\}\right],$$

subject to (3). Let $p^*(x)$ be the optimal number of bids to accept at time $t$ with inventory position $x$. Clearly, $p^*(x) \le N_t$. Let $\Delta R(i) \equiv R(i) - R(i-1)$, and rewrite $V_t(x)$ in the form

$$V_t(x) = E_{N_t,v}\left[\max_{0\le p\le x}\left\{\sum_{i=1}^{p} [\Delta R(i) - \Delta V_{t-1}(x-i+1)]\right\}\right] + V_{t-1}(x),$$

where the sum is defined to be 0 if $p = 0$.

Let $n_t$ denote any realization of the random variable $N_t$ and $v$ be a realization of bidders' types. The following lemma shows that the optimal allocation in each period is rather simple provided $\Delta V_{t-1}(x)$ is decreasing in $x$.

Lemma 1. *If $\Delta V_{t-1}(x)$ is decreasing in $x$, then for any realization $(v, n_t)$,*

$$p^*(x) = \begin{cases} \max\{1 \le p \le \min\{x, n_t\} : \Delta R(p) > \Delta V_{t-1}(x-p+1)\} & \textit{if } R(1) > \Delta V_{t-1}(x), \\ 0 & \textit{otherwise.} \end{cases}$$

The following theorem establishes:

Theorem 1. *$\Delta V_t(x)$ is decreasing in $x$.*

Theorem 1 and Lemma 1 show how the seller should allocate units. In particular, note that $\Delta R(i) = J(v_{(i)})$ for $i = 1, \dots, N_t$. So the decision rule in Lemma 1 about the optimal number of bids to accept is simply based on sorting the values $v_i$ and progressively awarding items to the highest value bidders until $J(v_{(i)})$ drops below the marginal opportunity cost $\Delta V_{t-1}(x - i + 1)$. Regarding the computation of the expected revenue $V_t(x)$, we have used Monte Carlo simulation, taking advantage of the previous results to reduce the complexity of the algorithm.

### 3.1. Mechanism Design

The next step is to construct auction mechanisms that implement the optimal allocation policy derived above. We show that appropriately modified versions of the first- and the second-price auctions achieve this objective.

**3.1.1. Second-Price Auction.** In the traditional single unit, second-price—or Vickrey—auction, the winner pays the price of the second highest bid, and it is a dominant strategy for the buyers to bid their own values. A similar result is true in a traditional, $k$-unit auction where all $k$ winners pay the $(k + 1)$th highest bid. However, if one uses a straightforward application of the second-price mechanism in our setting, it is no longer optimal for buyers to bid their valuation. But if we let

$$\hat{v}_i \equiv J^{-1}(\Delta V_{t-1}(x - i + 1)), \qquad i \geq 1,$$

then the following modified second-price mechanism avoids this pitfall: given the vector of submitted bids $b$, the seller will award $k$ items, where

$$k = \max\{i \geq 1 : b_{(i)} > \hat{v}_i\}, \tag{4}$$

and $k = 0$ if $b_{(1)} \leq \hat{v}_1$; and all winners will pay

$$v^{(2\text{nd})}_{(k+1)} = \max\{b_{(k+1)}, \hat{v}_k\}, \tag{5}$$

where $b_{(k+1)}$ is the $(k + 1)$th highest bid and $\hat{v}_k$ is the threshold to award the $k$th unit. Ties between bids are broken by randomization.

PROPOSITION 1. *In the modified second-price auction with allocation and payments given by (4)–(5), the buyers' dominant strategy is to bid their own values, and the mechanism is optimal.*

**3.1.2. First-Price Auction.** In a first-price auction, items are awarded to the highest bidders, and winners pay their bids. This type of mechanism may be more natural in certain applications. To establish that the first-price auction achieves the same expected revenue as the second-price mechanism described above, it suffices to show that there exists a symmetric equilibrium bidding strategy $B(\cdot)$ that is strictly increasing in the bidder's value.

Bidders are informed of the time $t$, the remaining inventory $x$, and the following allocation rule used by the seller: given a vector of bids $b$, the seller will award $k$ items, where $k = \max\{i \geq 1 : B^{-1}(b_{(i)}) > \hat{v}_i\}$ and $k = 0$ if $B^{-1}(b_{(1)}) \leq \hat{v}_1$. Here, $B(\cdot)$ is the equilibrium bid function, which we show can be computed by the seller. The items are awarded to the highest bidders, and winners pay their bids.

Our main result is the following:

PROPOSITION 2. *The first-price auction has a symmetric, equilibrium, strictly increasing bidding strategy $b_i = B(v_i)$, which can be computed in closed form, and the mechanism is optimal.*

## 4. A Suboptimal Revenue Management Mechanism

We next compare the optimal mechanism to a suboptimal one that approximates a traditional revenue management mechanism. We call this mechanism the list-price, capacity-controlled mechanism (LPCC). Under LPCC, the seller sets a take-it-or-leave-it price at the beginning of each period and calculates a threshold—or capacity control—on the number of units she is willing to award at the list price. Both prices and capacity controls are set optimally. A dominant strategy for the buyers is to accept the offered price if and only if their own value exceeds the seller's list price.

We use dynamic programming to solve LPCC. Moreover, we have proved that LPCC is in fact optimal when any of the following conditions hold: (a) there is at most one bidder per period, (b) there are more units to sell than potential buyers, and (c) asymptotically as the number of bidders and units to sell grows large ($C, N_t \uparrow \infty$).

We also performed some numerical experiments to understand when the optimal mechanism is most beneficial. Bidders' valuations were assumed to be uniformly distributed between zero and one. We only report on two such experiments here.

**Varying Concentration of Bidders Per Period.** We assume the seller starts with $C = 10$ units and the total number of bidders in all periods is constant at

**Table 1 Revenue for Different Concentration of Bidders**

| Bidders per Period | Number of Periods | Optimal Revenue Mean | Optimal Revenue 95% CI | LPCC Revenue Mean | LPCC Revenue Gap |
|---|---|---|---|---|---|
| 64 | 1 | 8.3037 | (8.2952, 8.3122) | 7.7594 | 6.65% |
| 32 | 2 | 8.2471 | (8.2395, 8.2547) | 7.8627 | 4.75% |
| 16 | 4 | 8.1713 | (8.1657, 8.1769) | 7.9314 | 3.00% |
| 8 | 8 | 8.1077 | (8.1011, 8.1143) | 7.9729 | 1.74% |
| 4 | 16 | 8.0523 | (8.0454, 8.0592) | 7.9963 | 0.78% |
| 2 | 32 | 8.0294 | (8.0232, 8.0356) | 8.0089 | 0.33% |
| 1 | 64 | 8.0173 | (8.0069, 8.0277) | 8.0157 | 0.15% |

**Table 2 Revenue and Variance in Bidders' Distributions**

| Range of Types Min | Range of Types Max | Optimal Revenue Mean | Optimal Revenue 95% CI | LPCC Revenue Mean | LPCC Revenue Gap |
|---|---|---|---|---|---|
| 9.5 | 10.5 | 102.6558 | (102.6404, 102.6712) | 102.1848 | 0.46% |
| 9 | 11 | 105.3121 | (105.2939, 105.3303) | 104.4558 | 0.81% |
| 8 | 12 | 110.5931 | (110.5473, 110.6389) | 109.1265 | 1.33% |
| 6 | 14 | 121.1814 | (121.1032, 121.2596) | 118.7877 | 1.98% |
| 4 | 16 | 131.7713 | (131.6502, 131.8924) | 128.7275 | 2.31% |
| 2 | 18 | 142.4494 | (142.2860, 142.6128) | 138.8575 | 2.52% |
| 0 | 20 | 153.1282 | (152.9471, 153.3093) | 149.1259 | 2.61% |

64. We then vary the number of periods from 1 to 64, so that the number of bidders per period varied. The results are summarized in Table 1.

The first observation is that the optimal revenue increases as the concentration of bidders increases. This is intuitive because as the seller observes more bidders' valuations per period, she is making allocation decisions with reduced uncertainty about future bid values. Moreover, an increase in concentration creates more direct bidding competition among buyers.

**Variability in Buyers' Valuations.** Bidders' valuations were assumed to be uniformly distributed with mean 10. The variance of the valuations was changed by adjusting the range of the distribution (the "Min" and "Max" values in Table 2). There were $T = 5$ periods, $N_t = 10$ bidders per period, and $C = 10$ units to sell.

Results are shown in Table 2. The main observation is that the seller benefits from increased variability in bidders' valuations.

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