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Measuring the Bullwhip Effect: Discrepancy and Alignment Between Information and Material Flows

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Abstract. The bullwhip effect is a phenomenon commonly observed in supply chains. It describes how demand variance amplifies from a downstream site to an upstream site due to demand information distortion. Two different bullwhip effect measures have been used in the literature. Theorists analyze the bullwhip effect based on the *information* flow (i.e., order and demand information), whereas most empiricists measure it according to the *material* flow (i.e., shipment and sales data). It is unclear how much the discrepancy between these two measures is, and, if significant, how to reconcile the discrepancy. In this paper, we illustrate and quantify the discrepancy under three inventory systems. For the system with stationary demand and ample supply, we show that the bullwhip effect measure based on the material-flow data is always greater than that based on the information flow. For the system with correlated demand and for the system with supply shortages, we derive conditions under which the material flow measure is either greater or less than the information flow measure. We find that the discrepancy is driven by four factors: stocking level, lead time, demand correlation, and supply service level. We further propose a method to reduce the discrepancy by using the sample variances of aggregated sales data. Our method works for common demand processes with short-range dependence, and it does not require the knowledge of the underlying base-stock levels.

Supplemental Material: The online appendix is available at <https://doi.org/10.1287/msom.2016.0590>.

Keywords: the bullwhip effect • information and material flows • measurement

1. Introduction

The bullwhip effect is a well-known phenomenon of demand information distortion in supply chains; namely, demand information tends to be more volatile as it propagates upstream (Lee et al. 1997). The bullwhip effect has a direct impact on supply chain performance. The amplification of demand variability going upstream requires each firm along the chain to carry more safety stock; it also affects order flow which, in turn, affects the material flow going downstream. It is crucial for supply chain managers to understand the causes of the bullwhip effect and to develop mitigation strategies. Since the seminal work of Lee et al. (1997), two research streams have emerged: modeling and empirical. In the modeling stream, researchers have studied the bullwhip effect in systems with various demand processes and ordering policies, whereas in the empirical stream, researchers have measured the extent of the bullwhip effect in many industry cases. These two research streams have reinforced each other in deepening our understanding of the bullwhip effect. The modeling research generates insights and forms hypotheses for the empirical research. The empirical research identifies where and how the effect occurs and

discovers new phenomena to motivate the modeling research.

There have been two primary bullwhip effect measures used in the modeling and empirical research. Lee et al. (1997) described the bullwhip effect as a form of demand information distortion of a single-item supply chain (i.e., diapers in Procter & Gamble's supply chain). The amplification of demand variances is measured according to the demand and order information. The subsequent modeling research adopted this *information flow* definition for analysis (e.g., Cachon 1999, Chen et al. 2000b, Aviv 2003, Chen and Lee 2012). On the other hand, when measuring the bullwhip effect with empirical data, researchers often had to resort to *material flow* data (either item-, firm-, or industry-level shipments and sales) as proxies for the order and demand information, because the information flow data (especially the demand data) were hard to obtain (for a discussion, see Cachon et al. 2007, p. 460). Intuitively, at the individual item level, the material flow measure should be consistent with the information flow measure if a supply firm can always fulfill the orders placed by a downstream firm. In reality, however, *backorders* do occur due to inventory shortages and/or limited production/shipping capacity at the

supply firm. Consequently, these two measures may diverge, leading to inconsistent measurements of the bullwhip effect.

In this paper, we investigate this measurement discrepancy issue. Specifically, we seek to answer the following questions: How much is the discrepancy between these two measures? What are the factors driving the extent of the discrepancy? When these two measures are significantly different, how to reconcile them based on the sales data? We investigate these issues in the following three inventory models.

We first consider a base model with stationary demand and ample supply from an outside vendor. A stationary base-stock policy is optimal for this model (see Zipkin 2000, p. 378). Under such a policy, the order quantity in a period is equal to the demand realized in the previous period, and there is no information bullwhip effect. In this case, the *information bullwhip measure*, defined as the ratio between the variances of order and demand, is equal to one. On the other hand, the material flow comprises the shipments from an outside vendor and the sales to the customer. Because of the ample supply from the outside vendor, the variance of the shipments from the vendor is the same as that of demand. Interestingly, we show that the variance of shipments is always greater than that of sales, and that their difference can be expressed as a simple product of on-hand inventory and backorders. This result implies that backlogging under the base-stock policy has a smoothing effect on the material flow going downstream. It also indicates that the *material bullwhip measure*, defined as the ratio between the variances of shipment and sales, is always greater than the information bullwhip measure. A numerical study suggests that the discrepancy between the two measures can be as high as 60%. Cachon et al. (2007) also observed that the variance of demand is more than twice of that of sales in certain industries. Thus, our model prediction is supported by their empirical observation. Our analysis further shows that these two measures are close when the base-stock level is high (i.e., the service level is high) or when the lead time is long. Indeed, several empirical studies have examined the inventory/service level to ensure that these two measures are good approximation to each other. For example, Bray and Mendelson (2012) investigated how demand information updating affects the bullwhip effect. They verified that the sales data were a good proxy for the demand because the firms under study had high inventory levels. Similarly, Osadchiy et al. (2016) studied the relationships of system risks (i.e., the covariance between sales and market returns) between supply chain firms. They also confirmed in their data that the service levels were high and thus the sales data served as a good proxy for the demand.

We next extend the base model by considering an autoregressive (AR(1)) demand process. A state-dependent base-stock policy is optimal for this model, and the information bullwhip effect exists under the optimal policy even when the replenishment lead time is zero (Lee et al. 1997). Under the zero lead time assumption, we show that there exists a threshold—when the autocorrelation coefficient of the demand process is above (below) the threshold, the material bullwhip measure underestimates (overestimates) the information bullwhip measure. This discrepancy result suggests that when the demand is highly correlated between periods, sales variability can be greater than the demand variability (due to backlogging and the nonstationary base-stock policy). We further show that this insight holds for the positive lead time case.

In the third inventory model, we relax the ample supply assumption. Specifically, we assume that orders from the downstream firm can be backlogged at the vendor site due to supply shortages. Both the downstream firm and the vendor implement a stationary base-stock policy. As a result, there is no information bullwhip effect in the supply chain. We show that the material bullwhip measure at the downstream firm can overestimate (underestimate) the information bullwhip measure when the base-stock level at the vendor site is either high or low (within an intermediate range). This result suggests that the vendor service level can affect the discrepancy between the two measures. It shows that backlogging at the vendor site has an inherent smoothing effect on the shipment variability to the downstream firm. Such smoothing effect vanishes when the vendor has either sufficiently high or sufficiently low service levels. Furthermore, we show that in a two-stage system with independent and identically distributed (i.i.d.) demand, the material bullwhip measure for the entire system is always greater than the information bullwhip measure for the entire system. This insight can be further generalized to a serial system with an arbitrary number of stages. It is worth noting that our analysis is based on the exact characterization of the material flow in a serial inventory system, which has not been explored before in the literature (see a discussion in Section 2).

To resolve the discrepancy between the information and material bullwhip measures, we need a method to estimate demand variance based on the observed sales data (in the presence of backlogging). When one has full access of the firm's backorder data, it is straightforward to recover the demand information from the sales and backorders data. However, for most outside parties (such as the firm's supply chain partners, competitors, or academic researchers), the backorder data are usually not observed. To address this challenge, we develop an estimation method that does not depend on the knowledge of the backorder data. We first note that the sales in a period consist of the current-period

demand plus backorders (if any) from the previous period minus the new backorders (if any) of the current period. Moreover, the aggregated sales over multiple periods consist of the aggregated demand plus backorders (if any) from the period before aggregation minus the backorders (if any) of the final period of aggregation. With the above observation, we show that if the two backorder terms in the aggregated sales are (nearly) independent, such as when the demands cross different periods are independent or when demands far apart in time are (nearly) independent, then the demand variance can be recovered by a simple expression of the sample variances of aggregated sales. This estimation method works for common demand processes with short-range dependence, and it does not require the knowledge of the underlying base-stock levels. Thus, it can be applied even if systems are operated under nonoptimal policies.

In summary, the contribution of our paper is threefold. First, we quantify the extent of discrepancy between the information and material bullwhip measures and identify four factors that drive the discrepancy: stocking level, lead time, demand correlation, and supply service level. One can examine the conditions of these factors to determine whether or not there is a significant discrepancy between the two measures. Second, when the discrepancy is significant, we propose a method to align the two measures based on the sales data. Our method works for demand processes with short-range dependence, such as the AR(1) process and other common time-series processes. It can also be extended to the time aggregation and product aggregation models of Chen and Lee (2012). Finally, our comparison of demand variability and shipment/sales variability reveals interesting managerial insights. We show that the material flow can be very different from the information flow. Managers, who make operational decisions and financial arrangements according to the demand information (e.g., inventory stocking, labor and transportation scheduling, shipment contracts, payment terms, etc.), may also want to take into account the material flow variability, as the latter has a direct impact on the logistics-related costs and the firm's financial flow.

The rest of this paper is organized as follows. Section 2 provides a literature review. Section 3 contains the analysis of the material flow for a single-stage system with i.i.d. demand. Section 4 extends the analysis to the system with an AR(1) demand process. Section 5 extends the analysis to the system with supply shortages. Section 6 provides a discrepancy reduction method to align the information and material bullwhip measures. Section 7 concludes the paper. A few selected proofs are included in the appendix; the remaining proofs can be found in the online appendix.

2. Literature Review

There is an extensive literature on the bullwhip effect. The existing analytical models in the literature have mostly focused on the information flow distortion between orders and demand. In a single-stage model, Lee et al. (1997) identified four causes for order variability amplification in supply chains. Cachon (1999) studied the effect of batch orders on order variabilities. Chen et al. (2000a) showed that certain demand forecast methods (such as moving average and exponential smoothing) can increase order variability. Aviv (2001, 2002, 2003, 2007) studied the information sharing and collaborative forecasting in supply chains and discussed the different implications between order uncertainty and variability. Chen and Lee (2009) proposed a generalized order-up-to policy for studying the information sharing and order variability control in supply chains. Chen and Lee (2012) studied the impact of batch size, seasonality, and product aggregation on the bullwhip effect. They pointed out that the bullwhip effect may be masked under these effects. Our paper differs from this literature in that we study the material flow variability rather than the information flow variability.

To our knowledge, Kahn (1987) is the only paper that studied the material flow variability in single-stage systems with zero lead time. He showed that the variability of shipments is higher than that of sales under a lost-sales system with time-correlated demand and a backorder system with i.i.d. demand. Our contributions are as follows: First, we extend these single-stage models with positive lead time and show that the difference of the shipment and sales variabilities is a function of the system's on-hand inventory and backorders. This result connects the system's inventory states to the sales variability, which was not known before in the literature. Second, we extend Kahn's (1987) model to the two-stage system and the time-correlated system with backlogging and find that the variability of shipments may not be greater than that of the sales. Third, we provide a method based on the historical sales data to estimate the information bullwhip measure.

Most prior research on the bullwhip effect involving multistage models often made simplifying assumptions to decouple the stages. For example, Graves (1999) assumed a high internal service level between the two stages, so that the two stages could be analyzed separately. Lee et al. (2000) assumed that the backordered inventory at the upstream stage can be "borrowed" from an alternative source and is required to be returned to the source after usage (the same assumption was also used by Gaur et al. 2005, Gilbert 2005, and Chen and Lee 2009). Aviv (2003) cautioned that the decoupling assumption may lead to a poor system performance and proposed using a coordinated inventory policy for the two stages (similar policies were also

used in Aviv (2001, 2002, 2007). De Kok (2012) studied a one warehouse–multiretailer system in which the retailers forecast demand with an exponential smoothing method. He showed that reducing the inventory order frequency helps mitigate the bullwhip effect. The two-stage model in our paper is similar to the coordinated two-level inventory system of Aviv (2003). However, our focus is to analyze the material flow dynamics in the system and to analytically compare the information and material bullwhip measures.

There is also a large empirical literature on the bullwhip effect. Economists hypothesized that a *production smoothing* effect exists (i.e., production is less volatile than sales), as inventory can serve as a buffer to cope against the demand uncertainty. However, the majority of the empirical evidence finds that production (or the amount of shipment in our term) is more volatile than sales in many industries, e.g., the TV set industry (Holt et al. 1968), the retail industry (Blinder 1981, Mosser 1991), the automobile industry (Blanchard 1983, Kahn 1992), the cement industry (Ghali 1987), and many others (Miron and Zeldes 1988, Fair 1989). We refer the reader to Cachon et al. (2007) and Chen and Lee (2016) for comprehensive reviews of this literature. Rong et al. (2009) studied the forward and reverse bullwhip effects in both experimental and simulation studies. More recently, Udenio et al. (2015) studied the impact of the 2008 financial crisis on the bullwhip effect, where the liquidity reduction during the crisis led to inventory destocking in firms along a supply chain. Bray and Mendelson (2015) further investigated the bullwhip effect and production smoothing in an automotive manufacturing sample. Duan et al. (2015) investigated the bullwhip effect in a daily product-level data set and found that the bullwhip effect at the product level is more significant than that measured at the firm or industry level. Our study complements this literature. We show that under certain conditions, backlogging under the base-stock policy actually plays a role in smoothing the material flow going downstream, which provides an alternative explanation for these empirical findings. Note that the material flow smoothing effect is different from the order smoothing effect observed in van Donselaar et al. (2010), where the order quantity was constrained due to system capacity. In fact, these two smoothing effects work in opposite directions: the hard capacity constraint has a smoothing effect on the order information flow propagating upstream, because the order quantity is truncated by the capacity constraint (Chen and Lee 2012); backlogging under the base-stock policy, on the other hand, has a smoothing effect on the material flow going downstream.

3. Stationary Demand

Consider a periodic-review, single-stage inventory system. Time is divided into periods of length one, and

we count the time forward (i.e., $t = 0, 1, 2, \dots$). Let D_t denote the customer demand in period t . In this section, we shall assume D_t is i.i.d. between periods. Let $\mu (> 0)$ denote the mean of the demand in a period. Let $D_t^{t+\tau}$ denote the sum of the demand from period t to $t + \tau$, i.e., $D_t^{t+\tau} = \sum_{i=0}^{\tau} D_{t+i}$, with $D_t^t = D_t$. The system operates under a stationary base-stock policy; that is, the system reviews its inventory order position (equal to outstanding orders plus on-hand inventory minus backorders) and orders up to the base-stock level s . We assume that the system replenishes from an outside vendor with ample supply. The replenishment lead time is a constant L . Customer demand will be fulfilled immediately if the system has enough on-hand inventory; unmet demand is fully backlogged. Let $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ denote the expectation and the variance of a random variable, respectively.

The sequence of events is as follows: At the beginning of a period, (1) a shipment sent from the outside vendor L periods ago is received, and (2) an order is placed. During the period, customer demand occurs and is fulfilled immediately if there is on-hand inventory. To facilitate the subsequent discussion, we separate this event into two: (3) demand is realized and (4) a shipment is sent to the customer at the end of the period.

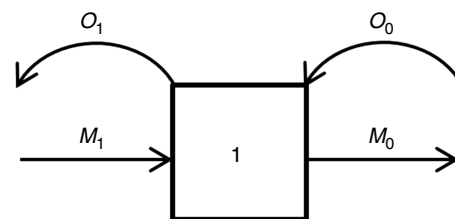
Figure 1 illustrates the information and material flows in a single-stage system. Define the following variables in period t :

- $O_1(t)$ = order quantity placed to the outside vendor at Event (2);
- $M_1(t)$ = shipment released to the system from the outside vendor after Event (2);
- $O_0(t)$ = customer demand occurred in period t at Event (3) = D_t ;
- $M_0(t)$ = realized sales to the customer at Event (4);
- $IL(t)$ = inventory level after Event (4)
- $I(t)$ = on-hand inventory = $(IL(t))^+$;
- $B(t)$ = backordered quantity = $(IL(t))^- = (-IL(t))^+$,

where $(\cdot)^+ = \max\{\cdot, 0\}$ and $(\cdot)^- = -\min\{\cdot, 0\}$.

Since our main goal is to study the discrepancy between the information and material bullwhip measures, it is useful to define these measures. Specifically,

Figure 1. Information and Material Flows in a Single-Stage System



we define the information bullwhip effect as the ratio between the variances of order and demand, i.e.,

$$r^O = \frac{\mathbb{V}[O_1(t)]}{\mathbb{V}[O_0(t)]} = \frac{\mathbb{V}[O_1(t)]}{\mathbb{V}[D_t]}. \quad (1)$$

We also refer to r^O as the information bullwhip ratio. This ratio indicates how significant the demand information is distorted by the system. The bullwhip effect exists if $r^O > 1$, meaning that the order variability is amplified. We define the material bullwhip effect as the ratio between the variances of shipments and sales, i.e.,

$$r^M = \frac{\mathbb{V}[M_1(t)]}{\mathbb{V}[M_0(t)]}. \quad (2)$$

We also refer to r^M as the material bullwhip ratio. When $r^M > 1$, the material flow is smoothed going downstream. In our subsequent analysis, we will focus on comparing the difference between the two measures r^M and r^O .

Under the base-stock policy, the system orders the previous period's demand, i.e., $O_1(t) = D_{t-1}$. Consequently, we have $\mathbb{V}[O_1(t)] = \mathbb{V}[D_{t-1}] = \mathbb{V}[D_t]$, and thus $r^O = 1$. There is no bullwhip effect for the information flow.

Now, let us examine the material flow, i.e., the shipment $M_1(t)$ and the sales $M_0(t)$. Since the outside vendor has ample supply, we have $M_1(t) = O_1(t) = D_{t-1}$. For $M_0(t)$, from the flow conservation property, we have

$$M_0(t) = O_0(t) + B(t-1) - B(t). \quad (3)$$

This equation states that the shipment in period t is equal to the total order to fill, i.e., $O_0(t) + B(t-1)$, minus the new backorders $B(t)$ after shipping. Thus, to characterize $M_0(t)$, we only need to characterize $B(t)$ for each t .

We first consider an order cycle starting at the beginning of period t . The ordering decision made in period t will affect the inventory variable in period $t+L$; that is,

$$IL(t+L) = s - \sum_{i=0}^L O_0(t+i) = s - D_t^{t+L}.$$

Thus, the on-hand inventory is given by $I(t+L) = (s - D_t^{t+L})^+$, and the backorder is given by $B(t+L) = (-IL(t+L))^+ = (D_t^{t+L} - s)^+$. Substituting $B(t+L)$ into (3), we obtain

$$M_0(t+L) = D_{t+L} + (D_{t-1}^{t+L-1} - s)^+ - (D_t^{t+L} - s)^+. \quad (4)$$

Since our goal is to compare the variability of $M_0(t)$ and $M_1(t)$, we shift the time index in (4) for this purpose; that is,

$$M_1(t) = D_{t-1}, \quad (5)$$

$$M_0(t) = D_t + (D_{t-L-1}^{t-1} - s)^+ - (D_{t-L}^t - s)^+. \quad (6)$$

Based on the above equations, we can establish the following result.

Proposition 1. *In a single-stage system with i.i.d. demand, for any given lead time $L \geq 0$,*

$$\begin{aligned} \mathbb{V}[M_1(t)] - \mathbb{V}[M_0(t)] \\ = 2\mathbb{E}[I(t-1)B(t)] = 2\mathbb{E}[(s - D_{t-L-1}^{t-1})^+(D_{t-L}^t - s)^+] \geq 0, \end{aligned}$$

which implies $r^M \geq r^O$.

Proposition 1 provides an explicit expression of the difference between the shipment variance and the sales variance. Specifically, we show that this difference is two times the expectation of the product of the on-hand inventory and backorders. To our knowledge, this is the first expression in the literature that makes such a connection. This expression enables us to study the impact of the base-stock level s as well as the lead time L on the gap between the material bullwhip ratio r^M and the actual information bullwhip ratio r^O . It is clear from the expression that the shipment variance equals the sales variance only when $s = 0$ or $s = \infty$. Thus, the material bullwhip ratio r^M is a good approximation to the actual information bullwhip ratio r^O only when the base-stock level is sufficiently high or sufficiently low. The magnitude of the discrepancy can be fairly significant. To see this, consider a special case with $L = 0$. In this case,

$$\begin{aligned} \mathbb{V}[M_1(t)] - \mathbb{V}[M_0(t)] &= 2\mathbb{E}[(s - D_{t-1})^+ \cdot (D_t - s)^+] \\ &= 2\mathbb{E}[I(t-1)] \cdot \mathbb{E}[B(t)], \end{aligned}$$

where the difference of the variances is simply two times the average on-hand inventory and the average backorders in the system.

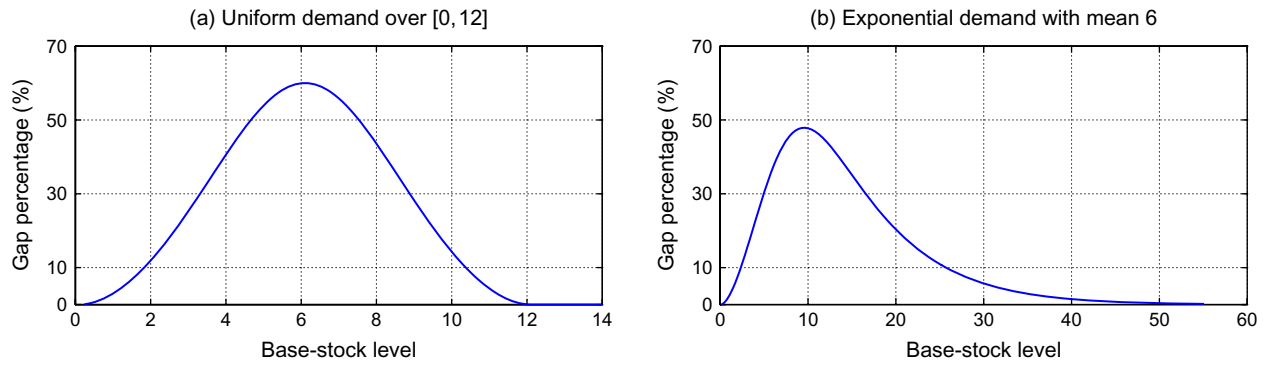
Figure 2(a) illustrates the percentage of the gap between r^M and r^O , i.e., $(r^M - r^O)/r^O \times 100\%$, under a uniform demand distribution with $L = 0$.¹ We observe that the gap percentage is unimodal in the base-stock level s , with the highest value around 60% attained at the base-stock level $s = d/2$. Figure 2(b) illustrates the percentage of the gap between r^M and r^O under an exponential demand distribution.² We again observe that the gap percentage is unimodal in s , with the highest value around 48%.

Proposition 2 below shows the impact of base-stock level s on the gap of the two bullwhip measures under a normal demand distribution.

Proposition 2. *When demand D_t follows an i.i.d. normal distribution $N(\mu, \sigma^2)$, the expression $2\mathbb{E}[(s - D_{t-L-1}^{t-1})^+ \cdot (D_{t-L}^t - s)^+]$ is unimodal in s . The gap between r^M and r^O is increasing in s when $s \leq (L+1)\mu$ and is decreasing in s when $s > (L+1)\mu$.*

When demand follows a normal distribution, the above result shows that the gap between r^M and r^O is decreasing (increasing) in the base-stock level s if the

Figure 2. (Color online) Impact of Base-Stock Level on the Bullwhip Ratio Gap Percentage



system service level is greater (less) than 50%. The gap reaches its highest value when $s = (L + 1)\mu$. The implication is that the material bullwhip ratio r^M is a good approximation to the information bullwhip ratio r^O when the system maintains a relatively high service level. Indeed, several empirical studies have examined the inventory service level to ensure the two measures are good approximations to each other. For example, Bray and Mendelson (2012) investigated how demand information updating affects the bullwhip effect. They verified that the sales data were a good proxy for the demand because the firms under study had high inventory levels. Similarly, Osadchiy et al. (2016) studied the relationships of system risks (i.e., the covariance between sales and market returns) between supply chain firms. They also confirmed in their data that the service levels were high and thus the sales data served as a good proxy for the demand.

We can further establish the following result about the impact of the replenishment lead time on the difference of these two bullwhip measures.

Proposition 3. Suppose that the demand D_t follows an i.i.d. normal distribution $N(\mu, \sigma^2)$. Given the optimal base-stock level $s = (L + 1)\mu + z\sigma\sqrt{L + 1}$, where z is the safety factor determined by the system inventory holding and backorder cost parameters, $2\mathbb{E}[(s - D_{t-L-1}^+)(D_{t-L}^+ - s)^+]$ is decreasing in lead time L . Thus, the gap between r^M and r^O is also decreasing in L .

Proposition 3 shows that the gap between the two bullwhip measures is decreasing in lead time L when the service level is fixed. An intuition for this result is that the variability of D_t is an upper bound for the variability of sales $M_0(t)$. Thus, a longer lead time makes the backorder larger, which, in turn, makes $M_0(t)$ more variable and close to the variability of demand. It is interesting to note that while demand and sales behave more differently as the backorder becomes larger, their variance measures nevertheless become closer to each other.

4. Correlated Demand

In this section, we extend the base model to the correlated demand case. Specifically, we assume that the demand D_t follows an AR(1) demand model as studied in Lee et al. (1997):

$$D_t = \mu + \rho D_{t-1} + \varepsilon_t, \quad (7)$$

where $|\rho| < 1$ and ε_t is an i.i.d. random variable with normal distribution $N(0, \sigma^2)$. The lead time demand D_t^{t+L} follows a normal distribution with mean and variance given as

$$\begin{aligned} \mathbb{E}[D_t^{t+L} | D_{t-1}] &= \frac{\rho(1 - \rho^{L+1})}{1 - \rho} \cdot D_{t-1} + \mu \cdot \sum_{k=1}^{L+1} \frac{1 - \rho^k}{1 - \rho}, \\ \mathbb{V}[D_t^{t+L} | D_{t-1}] &= \sigma^2 \sum_{k=1}^{L+1} \left(\frac{1 - \rho^{L-k+2}}{1 - \rho} \right)^2. \end{aligned}$$

Thus, under the free-return assumption of Lee et al. (1997), we can show that the optimal base-stock level in period t is given by

$$\begin{aligned} s_t &= \frac{\rho(1 - \rho^{L+1})}{1 - \rho} \cdot D_{t-1} + \mu \cdot \sum_{k=1}^{L+1} \frac{1 - \rho^k}{1 - \rho} \\ &\quad + z \cdot \sigma \sqrt{\sum_{k=1}^{L+1} \left(\frac{1 - \rho^{L-k+2}}{1 - \rho} \right)^2}, \end{aligned}$$

where z is the safety factor determined by the system inventory holding and backorder cost parameters. Under the optimal base-stock policy s_t , the order in period t is given by

$$\begin{aligned} O_1(t) &= s_t - (s_{t-1} - D_{t-1}) \\ &= \rho^{L+2} D_{t-2} + \frac{1 - \rho^{L+2}}{1 - \rho} \mu + \frac{1 - \rho^{L+2}}{1 - \rho} \varepsilon_{t-1}. \end{aligned} \quad (8)$$

Because the outside vendor has ample supply, we have $M_1(t) = O_1(t)$. We can further generalize the sales Equation (4) under the AR(1) demand as follows:

$$\begin{aligned} M_0(t + L) &= D_{t+L} + (D_{t-1}^{t+L-1} - s_{t-1})^+ - (D_t^{t+L} - s_t)^+ \\ &= \rho^{L+2} D_{t-2} + \frac{1 - \rho^{L+2}}{1 - \rho} \mu + \sum_{k=0}^{L+1} \rho^{L-k+1} \varepsilon_{t+k-1} \end{aligned}$$

$$+ \left(\sum_{k=0}^L \frac{1-\rho^{L-k+1}}{1-\rho} \varepsilon_{t+k-1} - \bar{z} \right)^+ - \left(\sum_{k=0}^L \frac{1-\rho^{L-k+1}}{1-\rho} \varepsilon_{t+k} - \bar{z} \right)^+, \quad (9)$$

where $\bar{z} = z \cdot \sigma \sqrt{\sum_{k=1}^{L+1} ((1-\rho^{L-k+2})/(1-\rho))^2}$.

Because $\mathbb{V}[M_1(t)] = \mathbb{V}[O_1(t)]$, to compare the bullwhip ratio measures r^M and r^O , it suffices to compare $\mathbb{V}[M_0(t)]$ and $\mathbb{V}[D_t]$. Consider the case of $L = 0$. We have

$$M_0(t) = \rho^2 D_{t-2} + \mu(1+\rho) + \varepsilon_t + \rho \varepsilon_{t-1} + (\varepsilon_{t-1} - \bar{z})^+ - (\varepsilon_t - \bar{z})^+, \quad (10)$$

where $\bar{z} = z\sigma$. From (7), we know that D_{t-2} is independent of ε_t and ε_{t-1} . Therefore,

$$\mathbb{V}[M_0(t)] = \mathbb{V}[\varepsilon_t + \rho \varepsilon_{t-1} + (\varepsilon_{t-1} - \bar{z})^+ - (\varepsilon_t - \bar{z})^+] + \mathbb{V}[\rho^2 D_{t-2}].$$

Also, note that $\mathbb{V}[D_t] = \mathbb{V}[\varepsilon_t + \rho \varepsilon_{t-1}] + \mathbb{V}[\rho^2 D_{t-2}]$. Hence, to compare $\mathbb{V}[M_0(t)]$ and $\mathbb{V}[D_t]$, it suffices to compare the two terms $\mathbb{V}[\varepsilon_t + \rho \varepsilon_{t-1} + (\varepsilon_{t-1} - \bar{z})^+ - (\varepsilon_t - \bar{z})^+]$ and $\mathbb{V}[\varepsilon_t + \rho \varepsilon_{t-1}]$. We note that when $\rho = 0$, the comparison reduces to that of the i.i.d. demand case (see Proposition 1). We can establish the following result.

Proposition 4. In a single-stage system with AR(1) demand and zero lead time,

$$\mathbb{V}[D_t] - \mathbb{V}[M_0(t)] = 2\mathbb{E}[(\bar{z} - \varepsilon_t)^+] \cdot \mathbb{E}[(\varepsilon_t - \bar{z})^+] - 2\rho \mathbb{E}[\varepsilon_t(\varepsilon_t - \bar{z})^+].$$

There exists a threshold $0 < \bar{\rho} < 1$ such that if $\rho < \bar{\rho}$, $\mathbb{V}[M_0(t)] < \mathbb{V}[D_t]$, which implies $r^M > r^O$; otherwise, if $\rho \geq \bar{\rho}$, $\mathbb{V}[M_0(t)] \geq \mathbb{V}[D_t]$, which implies $r^M \leq r^O$.

The above result shows that, under the AR(1) demand, the autocorrelation coefficient ρ has a direct

impact on the gap between the two bullwhip measures r^M and r^O . When the demands are moderately correlated, the material bullwhip ratio r^M overestimates the information bullwhip ratio r^O . This result generalizes the insight from our base model with i.i.d. demand. However, when the demand is highly correlated, the result reverses; i.e., the material bullwhip ratio r^M underestimates the actual information bullwhip ratio r^O . In this case, the sales variability becomes greater than the demand variability. This is due to backlogging and the nonstationary base-stock policy in the system.

In the case of positive lead time $L > 0$, we can show that the result continues to hold in the neighborhoods of $\rho = 0$ and $\rho = 1$.

Proposition 5. In a single-stage system with AR(1) demand and lead time $L > 0$, there exist two thresholds $\underline{\rho} \leq \bar{\rho}$ such that if $0 \leq \rho \leq \underline{\rho}$, $\mathbb{V}[M_0(t)] < \mathbb{V}[D_t]$, which implies $r^M > r^O$, and if $\bar{\rho} \leq \rho < 1$, $\mathbb{V}[M_0(t)] \geq \mathbb{V}[D_t]$, which implies $r^M \leq r^O$.

The above two propositions indicate that when the demand is highly correlated between periods, managers should expect the variability of sales to be greater than the variability of demand (even under the optimal policy). Several numerical illustrations are presented below.

Figure 3 illustrates the demand variance $\mathbb{V}[D_t]$, the sales variance $\mathbb{V}[M_0(t)]$, and r^O and r^M for different values of the autocorrelation coefficient ρ , with $L = 2$, $\sigma = 1$, and $z = 1.3$ (90% service level). It shows that the result of Proposition 4 continues to hold for positive lead times. We experiment with different service levels and obtain similar insights. We also observe that the two bullwhip measures become close to each other as the service level increases.

Figure 4 illustrates the gap percentage between r^M and r^O for different lead times $L = 1, 2, 5$, with the same demand parameters as in Figure 3. It shows that, all else equal, a longer lead time leads to larger underes-

Figure 3. (Color online) Impact of the Demand Autocorrelation on Variances and the Bullwhip Ratios ($L = 2$)

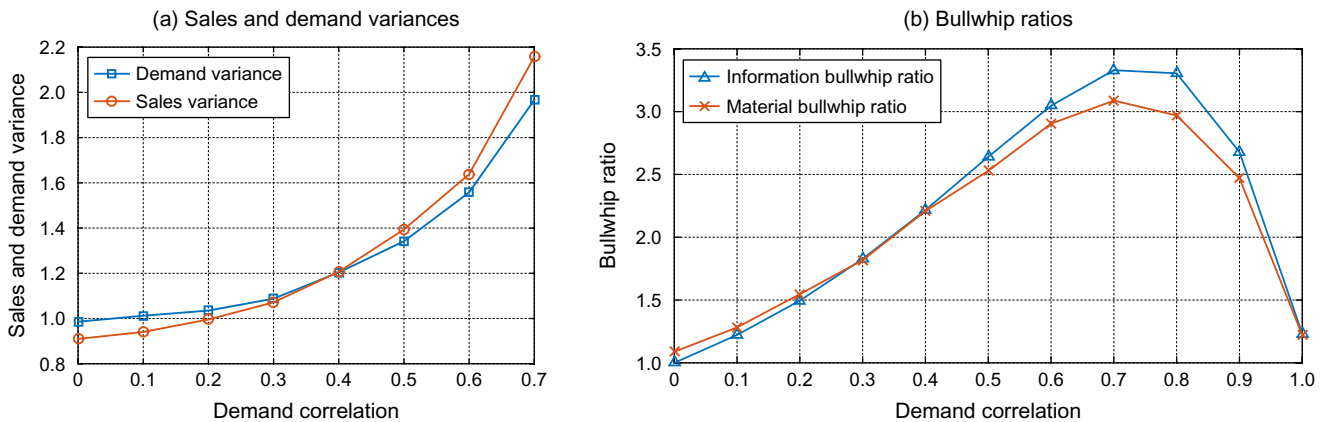
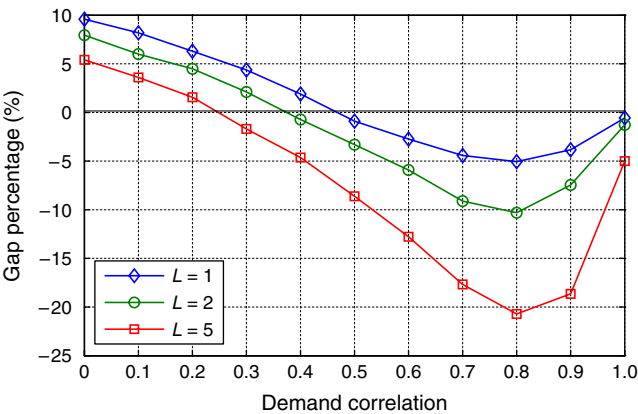


Figure 4. (Color online) Impact of the Demand Autocorrelation on the Bullwhip Ratio Gap Percentage



timination percentage gap when the demand is highly correlated.

5. Supply Shortages

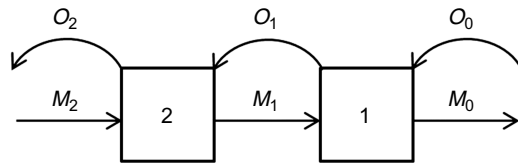
In this section, we relax the ample supply assumption of the base model. Specifically, we assume that orders can be backlogged at the vendor site due to supply shortages. For ease of exposition, we refer to the downstream firm as stage 1 and the outside vendor as stage 2. We also index the outside vendor's supplier as stage 3 and the end customer as stage 0. There is a positive lead time L_j for stage j . Stage 1 and stage 2 operate under a local base-stock policy.³ Under this policy, stage j ($=1, 2$) views stage $j-1$'s order as its local demand and places an order in each period. At the beginning of each period, stage j reviews its local inventory order position (equal to outstanding orders plus on-hand inventory minus stage j 's backorders) and orders up to the local base-stock level s_j .

The sequence of events for stage 2 is slightly different from that of the single-stage system in Section 3 as stage 2 receives and fills stage 1's order at the beginning of a period. Specifically, (1) a shipment sent from the stage 2's supplier L_2 periods ago is received, (2) an order from stage 1 is received, (3) an order is placed with the stage 2's supplier, according to the updated inventory order position in Event (2), and (4) a shipment is sent to stage 1. All these events occur at the beginning of a period. The sequence of events for stage 1 is the same as that of the single-stage system in Section 3. We assume that the stages perform the ordering event sequentially from downstream to upstream, whereas the shipping events occur sequentially from upstream to downstream in a period.

Figure 5 shows the two-stage model with the material and information flows in the opposite directions. Define the following state variables in period t for stage $j = 1, 2$:

$O_j(t)$ = order quantity from stage j to its upstream supplier;

Figure 5. Information and Material Flows in a Two-Stage System



$M_j(t)$ = shipment released to stage j from its upstream supplier at Event (4);

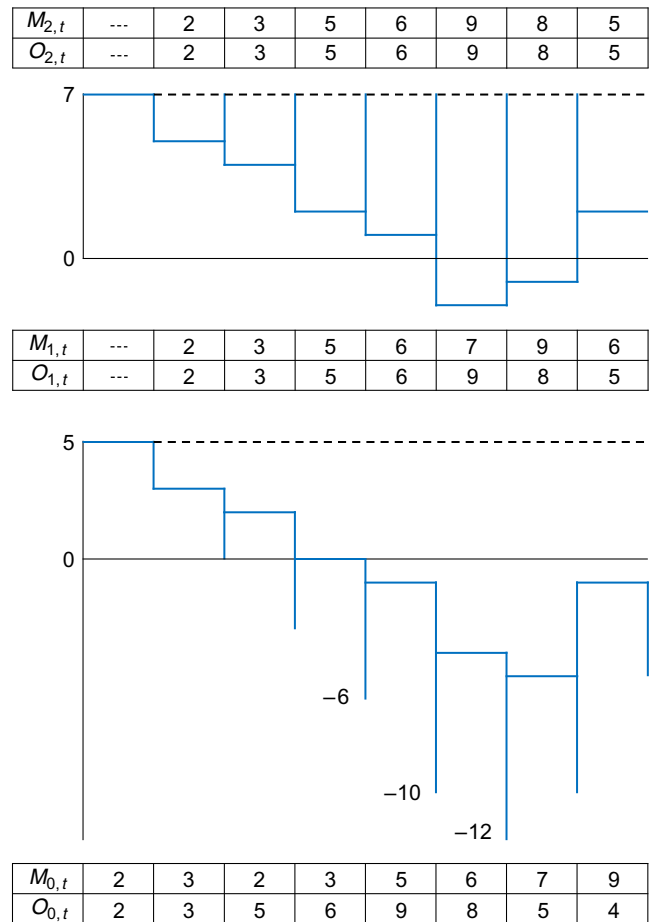
$IL_j(t)$ = local inventory level for stage j after Event (4);

$B_j(t)$ = local backorders $= (IL_j(t))^- = (-IL_j(t))^+$.

Similarly, we define $O_0(t)$ and $M_0(t)$ to represent the customer demand and sales that occurred in period t , respectively. Below we present an example to illustrate the sequence of events.

Figure 6 plots the inventory level curves $IL_1(t)$ and $IL_2(t)$ under a demand sample path, with $L_1 = L_2 = 1$, $s_1 = 5$, and $s_2 = 7$. Consider the period $t = 4$. At the end of period 3, stage 1's backorder is $B_1(3) = 6$. At the beginning of period 4, stage 1 first receives a shipment of five units, followed by placing an order of six units (because $D_3 = 6$). During the period, nine units of demand arrive. The total order to fill is $6 + 9 = 15$ units.

Figure 6. (Color online) A Two-Stage Example for Order and Shipment Dynamics



Thus, stage 1 ships all the available five units, making $B_1(4) = 10$. We turn to stage 2. The six units of order from stage 1 become stage 2's local demand in period 4. After receiving five units of inventory at the beginning of period 4, stage 2's inventory level goes back to seven, so it can ship six units to stage 1 to fulfill this order.

We extend the definition of the information bullwhip ratio and the material bullwhip ratio in Section 3 by adding a stage index j ; that is, the material bullwhip ratio for stage j is

$$r_j^M = \frac{\mathbb{V}[M_j(t)]}{\mathbb{V}[M_{j-1}(t)]},$$

and the information bullwhip ratio for stage j is

$$r_j^O = \frac{\mathbb{V}[O_j(t)]}{\mathbb{V}[O_{j-1}(t)]}.$$

Clearly, $O_0(t) = D_t$. Under the local base-stock policy, stage $j = 1, 2$ will order the previous period's local demand. Thus, $O_2(t) = O_1(t) = D_{t-1}$. Consequently, we have $r_1^O = r_2^O = 1$, implying that there is no bullwhip effect for the information flow in the two-stage system. Below we examine the material flow, i.e., the shipments $M_1(t)$ and $M_2(t)$, and the sales $M_0(t)$ in the system.

5.1. Material Flow Dynamics

We now derive the shipment variables. By applying the flow conservation property in (3) for $j = 0, 1, 2$, we obtain

$$M_j(t) = O_j(t) + B_{j+1}(t-1) - B_{j+1}(t), \quad (11)$$

where $B_3(t) \equiv 0$. To characterize $M_j(t)$, we only need to determine $B_{j+1}(t)$ in each period t .

Obtaining expressions for the backorder variables in the two-stage system is quite complicated, because stage 1 may not be able to receive what it orders from stage 2. Thus, we need to consider the impact of stage 2's order decision on the backorder variables.

We first consider an order cycle starting from stage 2 in period t . It is clear that the ordering decision made at stage 2 in period t will directly and indirectly affect the inventory variables from period t to $t + L_1 + L_2$. Specifically, stage 2's order in period t will arrive at stage 1 in period $t + L_2$, which affects the inventory availability for stage 1 in that period as well as the resulting net inventory level at stage 1 in period $t + L_1 + L_2$. We now specify the detailed dynamics.

Since stage 2's supplier has ample supply, the inventory order position is equal to s_2 in each period:

$$IL_2(t + L_2) = s_2 - \sum_{i=1}^{L_2} O_1(t + i) = s_2 - D_t^{t+L_2-1}.$$

Thus,

$$B_2(t + L_2) = (-IL_2(t + L_2))^+ = (D_t^{t+L_2-1} - s_2)^+. \quad (12)$$

Now, consider stage 1 in period $t + L_2$. stage 1 places an order to s_1 , but stage 2 may have backorders. Thus, stage 1 cannot get what it orders, so the inventory in-transit position is given by

$$ITP_1(t + L_2) = s_1 - B_2(t + L_2),$$

and the resulting inventory level in period $t + L_1 + L_2$ is

$$\begin{aligned} IL_1(t + L_1 + L_2) &= ITP_1(t + L_2) - \sum_{i=L_2}^{L_1+L_2} O_0(t + i) \\ &= s_1 - B_2(t + L_2) - D_{t+L_2}^{t+L_1+L_2}. \end{aligned}$$

Therefore,

$$\begin{aligned} B_1(t + L_1 + L_2) &= (-IL_1(t + L_1 + L_2))^+ \\ &= (D_{t+L_2}^{t+L_1+L_2} + B_2(t + L_2) - s_1)^+. \end{aligned} \quad (13)$$

With (12) and (13), we are able to characterize backorders $B_j(t)$ in the system for all t . Substituting (12) and (13) into (11), we can obtain the shipment variables as follows:

$$\begin{aligned} M_1(t + L_2) &= D_{t+L_2-1} + (D_{t-1}^{t+L_2-2} - s_2)^+ - (D_t^{t+L_2-1} - s_2)^+, \end{aligned} \quad (14)$$

$$\begin{aligned} M_0(t + L_1 + L_2) &= D_{t+L_1+L_2} + (D_{t+L_2-1}^{t+L_1+L_2-1} + (D_{t-1}^{t+L_2-2} - s_2)^+ - s_1)^+ \\ &\quad - (D_{t+L_2}^{t+L_1+L_2} + (D_t^{t+L_2-1} - s_2)^+ - s_1)^+. \end{aligned} \quad (15)$$

Since our goal is to compare the variability of $M_0(t)$, $M_1(t)$, and $M_2(t)$, we shift the time index in (14) to (15) for this purpose; that is,

$$M_2(t) = D_{t-1}, \quad (16)$$

$$M_1(t) = D_{t-1} + (D_{t-L_2-1}^{t-2} - s_2)^+ - (D_{t-L_2}^{t-1} - s_2)^+, \quad (17)$$

$$\begin{aligned} M_0(t) &= D_t + (D_{t-L_1-1}^{t-1} + (D_{t-L_1-L_2-1}^{t-L_1-2} - s_2)^+ - s_1)^+ \\ &\quad - (D_{t-L_1}^{t-1} + (D_{t-L_1-L_2}^{t-L_1-1} - s_2)^+ - s_1)^+. \end{aligned} \quad (18)$$

It is clear that (16) and (17) have the same structure as (5) and (6) in the single-stage system. Therefore, the results of Propositions 1–3 can be directly carried over to stage 2, i.e., the material bullwhip ratio $r_2^M \geq r_2^O = 1$. Moreover, we can show that the material bullwhip ratio for the entire system, defined as $\mathbb{V}[M_2(t)]/\mathbb{V}[M_0(t)]$, is always greater than one.

Proposition 6. In a two-stage system with i.i.d. demand and lead times $L_1 \geq 0$ and $L_2 \geq 1$, $\mathbb{V}[D_t] = \mathbb{V}[M_2(t)] \geq \mathbb{V}[M_0(t)]$, which implies $r_2^M r_1^M \geq r_2^O r_1^O$.

The above result holds for any s_1 and s_2 , suggesting that in a two-stage system with i.i.d. demand, the material bullwhip measure for the entire system is always greater than the information bullwhip measure for the entire system. This insight can be further generalized to a serial system with arbitrary number of stages.

It remains to compare the two bullwhip measures at stage 1. When $s_2 = \infty$, stage 2 has ample supply, and the two-stage system is effectively reduced to a single-stage system. In this case, the results of Propositions 1–3 can also be carried over to stage 1, i.e., the material bullwhip ratio $r_1^M \geq r_1^O = 1$.

Below we will focus on studying $r_1^M = \mathbb{V}[M_1(t)] / \mathbb{V}[M_0(t)]$ in the case of $s_2 < \infty$, i.e., the outside vendor (stage 2) does not have ample supply, and orders may be backlogged at the vendor site. In this case, the variability of the material flow at stage 1 depends on the lead times and the base-stock levels of both stages 1 and 2, which makes it difficult to obtain an explicit expression as in the single-stage system. We shall adopt an alternative approach based on the sample path analysis to compare the variances of shipment and sales for stage 1.

5.2. Comparison of Bullwhip Measures

Let us first consider the case with $s_2 \geq s_1$ and $L_2 = 1$. To illustrate, consider a special case with $L_1 = 0$ below. Under this system configuration, the shipment dynamics are given by

$$M_1(t) = D_{t-1} + (D_{t-2} - s_2)^+ - (D_{t-1} - s_2)^+, \quad (19)$$

$$M_0(t) = D_t + (D_{t-1} + (D_{t-2} - s_2)^+ - s_1)^+ - (D_t + (D_{t-1} - s_2)^+ - s_1)^+. \quad (20)$$

Because demand is random, there are periods with backorders along a demand sample path. Let us call the time interval that contains consecutive backorder periods the “backorder cycle” and the other time intervals the “nonbackorder” intervals. We first consider the nonbackorder interval. The shipment $M_0(t)$ in the nonbackorder interval may not necessarily be equal to the demand D_t because stage 2 may have backorders. However, under the condition $s_2 > s_1$, we can show that when stage 1 is in the nonbackorder period, stage 2 will naturally be in the nonbackorder period as well, because both face the same local demand D_t . Thus, we have $M_0(t) = D_t$ if t is in the nonbackorder interval.

We next consider a backorder cycle from period t to $t + \tau$ at stage 1, with $B_1(t - 1) = 0$, $B_1(t) > 0$, $B_1(t + 1) > 0, \dots, B_1(t + \tau - 1) > 0$, and $B_1(t + \tau) = 0$. In this cycle, the sequence of sales are given by

$$M_0(t + i) = \begin{cases} s_1, & \text{if } i = 0, \\ M_1(t + i), & \text{if } 1 \leq i \leq \tau - 1, \\ D_{t+\tau-1} + D_{t+\tau} + (D_{t+\tau-2} - s_2)^+ - s_1, & \text{if } i = \tau. \end{cases} \quad (21)$$

From the above expression, we can show that $\sum_{i=0}^{\tau} M_1(t + i + 1)^2 \geq \sum_{i=0}^{\tau} M_0(t + i)^2$. Thus, by the ergodic property of a stationary process, it follows that $\mathbb{V}[M_1(t)] \geq \mathbb{V}[M_0(t)]$. The above analysis can be further

extended to the general case with $L_1 \geq 0$. The result is summarized in the following proposition.

Proposition 7. *In a two-stage system with i.i.d. demand and stage 2 lead time $L_2 = 1$, for any given stage 1 lead time $L_1 \geq 0$, if $s_2 \geq s_1$, then $\mathbb{V}[M_1(t)] \geq \mathbb{V}[M_0(t)]$, which implies $r_1^M \geq r_1^O$.*

The above proposition can be viewed as a further generalization of Proposition 1. Here we show that the insight $r_1^M \geq r_1^O$ continues to hold if the vendor site is subject to inventory shortages. However, to ensure the result holds, the vendor stage needs to have a short replenishment lead time and a higher base-stock level than the downstream stage. This result can be generalized to the system with $L_2 > 1$, which is summarized in the next two propositions.

Proposition 8. *In a two-stage system with i.i.d. demand and lead times $L_1 \geq 0$ and $L_2 \geq 1$, for any s_1 , there exists a threshold $\bar{s}_2(s_1) \geq 0$ such that, for any $s_2 \geq \bar{s}_2(s_1)$, $\mathbb{V}[M_1(t)] \geq \mathbb{V}[M_0(t)]$, which implies $r_1^M \geq r_1^O$.*

The above result provides further support for the intuition that $r_1^M \geq r_1^O$ when the vendor stage keeps a relatively high base-stock level. We note that when $L_2 = 1$, we have $\bar{s}_2(s_1) = s_1$ from the result of Proposition 7.

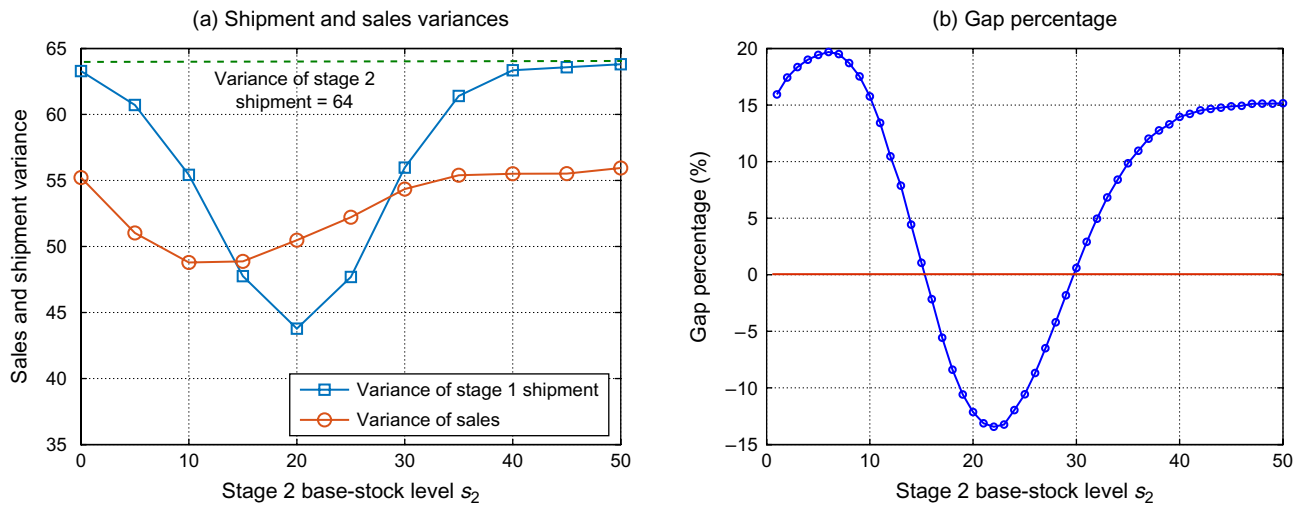
Now let us consider the other extreme case where stage 2 keeps zero inventory, i.e., $s_2 = 0$. In this case, the vendor site becomes a cross-docking facility (e.g., Eppen and Schrage 1981). The material flow dynamics equations (17) and (18) become the following:

$$M_1(t) = D_{t-L_2}, \\ M_0(t) = D_t + (D_{t-L_1-L_2-1} - s_1)^+ - (D_{t-L_1-L_2} - s_1)^+.$$

The above expressions have the same structure as the material flow equations (5) and (6) in the single-stage system. Thus, the result of Proposition 1 can be directly applied to this case, i.e., $r_1^M \geq r_1^O = 1$. In fact, this result can be generalized as follows.

Proposition 9. *In a two-stage system with i.i.d. demand and lead times $L_1 \geq 0$ and $L_2 \geq 1$, suppose that the demand has a finite bound, i.e., $D_t < d$ ($d = \infty$ if $D_t < \infty$). Then, for any given $s_1 < (L_1 + L_2 + 1)d$, there exists a threshold $\underline{s}_2(s_1) > 0$ such that, for any $s_2 \leq \underline{s}_2(s_1)$, $\mathbb{V}[M_1(t)] \geq \mathbb{V}[M_0(t)]$, which implies $r_1^M \geq r_1^O$.*

The above result shows that the result $r_1^M \geq r_1^O$ continues to hold if the vendor stage keeps low inventory. The intuition behind this result is that when the vendor stage becomes closely resembling a cross-docking facility, the combination of the vendor stage and its supplier can be viewed as one ample supply source with some extra lead time delay. Thus, the single-stage system result can be applied here. We note that in the above result we need the condition $s_1 < (L_1 + L_2 + 1)d$ to

Figure 7. (Color online) Impact of Stage 2 Base-Stock Level

rule out the unrealistic case in which stage 1 provides 100% service level to the end customer. Combining Propositions 8 and 9, we arrive at the conclusion that when stage 2 has either relatively high or low inventory, the material bullwhip ratio r_1^M is greater than or equal to the actual information flow bullwhip ratio r_1^O at stage 1.

Figure 7 provides an illustration of the shipment and sales variances and the gap percentage between r_1^M and $r_1^O = 1$ under the normal demand distribution $N(20, 8)$ (with lead times $L_2 = 1$ and $L_1 = 0$ and stage 1 base-stock level $s_1 = 30$). We observe from the figure that there exist three regions based on stage 2 base-stock level: When s_2 is either high or low, $r_1^M > r_1^O$, and when s_2 takes a value in an intermediate range, the result reverses to $r_1^M < r_1^O$. These results suggest that the vendor service level can affect the discrepancy between the information and material bullwhip measures. It also shows that backlogging at the vendor site has a smoothing effect on the shipment variability to the downstream firm.

6. Discrepancy Reduction Method

In the previous sections, we illustrated and quantified the discrepancy between the information and material bullwhip measures in three inventory models. To resolve the discrepancy, we need a method to estimate demand variance based on the available sales data (in the presence of backlogging). When one has full access to a firm's backorder data, it is straightforward to recover the demand information from the sales and backorders data. However, for most outside parties (such as the firm's supply chain partners, competitors, or academic researchers), the backorder data are usually not available. Thus, an estimation method that does not rely on the backorder data would present great value to these outside parties. In this section, we

provide an estimation method to achieve this under a fairly general demand process.

Consider that the demand process D_t follows a general, common stationary time series (which can be temporally correlated). Let N be the time aggregation period. Define the ratio between the variance of the aggregated demand and the variance of the single-period demand as

$$f(N) = \frac{\mathbb{V}[D_t^{t+N-1}]}{\mathbb{V}[D_t]}, \quad (22)$$

where $f(N)$ is a function depending on the aggregation period and the underlying demand model parameters. For example, for the i.i.d. demand process, we have $f(N) = N$. For the AR(1) demand process, it can be shown that

$$f(N) = \left(\frac{1 - \rho^N}{1 - \rho} \right)^2 + (1 - \rho^2) \sum_{i=1}^{N-1} \left(\frac{1 - \rho^{N-i}}{1 - \rho} \right)^2. \quad (23)$$

Let s_t be an arbitrary state-dependent base-stock level for period t . For example, s_t can be a nonstationary value that depends on the historical demand data. We know that

$$M_0(t+L) = D_{t+L} + (D_{t-1}^{t+L-1} - s_{t-1})^+ - (D_t^{t+L} - s_t)^+.$$

It follows that

$$\begin{aligned} \sum_{i=0}^N M_0(t+L+i) \\ = D_{t+L}^{t+N+L} + (D_{t-1}^{t+L-1} - s_{t-1})^+ - (D_{t+N}^{t+N+L} - s_{t+N})^+. \end{aligned} \quad (24)$$

Intuitively, the total sales in $[t+L, t+L+N]$ is equal to the total demand in the same time period (the first term on the right hand side) plus the backorder level in period $t+L-1$ (the second term) minus the backorder

level in period $t + N + L$ (the third term). If the backorder terms $(D_{t-1}^{t+L-1} - s_{t-1})^+$ and $(D_{t+N}^{t+N+L} - s_{t+N})^+$ are i.i.d. for any t , we can then obtain a simple expression for the demand variance based on the variances of the aggregated sales. The following theorem summarizes this result.

Theorem 1. Suppose that D_t is a stationary time series and there exists $T > 0$ such that D_t and D_{t+i} are independent for any t and $i \geq T$. Then, for any $N \geq T + L - 1$, the following holds:

$$\mathbb{V}[D_t] = \frac{\mathbb{V}[\sum_{i=0}^{2N+1} M_0(t+i)] - \mathbb{V}[\sum_{i=0}^N M_0(t+i)]}{f(2N+2) - f(N+1)},$$

where $f(\cdot)$ is defined in (22).

An important feature of the above theorem is that the demand variance can be recovered from the formula without knowing the underlying base-stock levels. Thus, the formula holds even if the system is operated under nonoptimal policies.

In general, the two backorder terms may be correlated. However, for demand processes with short-range dependence (i.e., α mixing with exponential decay rate; see Billingsley 1995), the two backorder terms are nearly independent as they become far apart in time when the aggregation period N increases. Thus, we can still apply the formula to estimate the demand variance as long as $N \gg 0$, i.e.,

$$\mathbb{V}[D_t] \approx \frac{\mathbb{V}[\sum_{i=0}^{2N+1} M_0(t+i)] - \mathbb{V}[\sum_{i=0}^N M_0(t+i)]}{f(2N+2) - f(N+1)}.$$

Many common time series possess the property of short-range dependence. Below we provide two illustrative examples.

6.1. i.i.d. Demand

For the i.i.d. demand model, it is clear that when $N \geq L$, the two backorder terms are independent. Recall that $f(N) = N$. Thus, the demand variance can be recovered from the variances of the aggregated sales as follows/

Corollary 1. Suppose that D_t follows an i.i.d. process. In a single-stage system, for any given lead time $L \geq 0$ and $N \geq L$,

$$\mathbb{V}[D_t] = \frac{\mathbb{V}[\sum_{i=0}^{2N+1} M_0(t+i)] - \mathbb{V}[\sum_{i=0}^N M_0(t+i)]}{N+1}.$$

We can further extend the above result to the case when the outside vendor does not have ample supply, i.e., the two-stage system studied in Section 5. In this system, from expression (18), we can show that

$$\begin{aligned} \sum_{i=0}^N M_0(t+i) &= D_t^{t+N} + (D_{t-L-1}^{t-1} + (D_{t-L-1-L_2-1}^{t-L_2-2} - s_2)^+ - s_1)^+ \\ &\quad - (D_{t+N-L_1}^{t+N} + (D_{t+N-L_1-L_2-1}^{t+N-L_1-1} - s_2)^+ - s_1)^+. \end{aligned}$$

When $N \geq L_1 + L_2$, the second and third terms in the above expression do not have overlapping demand terms and thus are i.i.d. Leveraging this observation, we obtain the following result.

Corollary 2. In a two-stage system with i.i.d. demand and lead times $L_1 \geq 0$ and $L_2 \geq 1$, for any $N \geq L_1 + L_2$, the following holds:

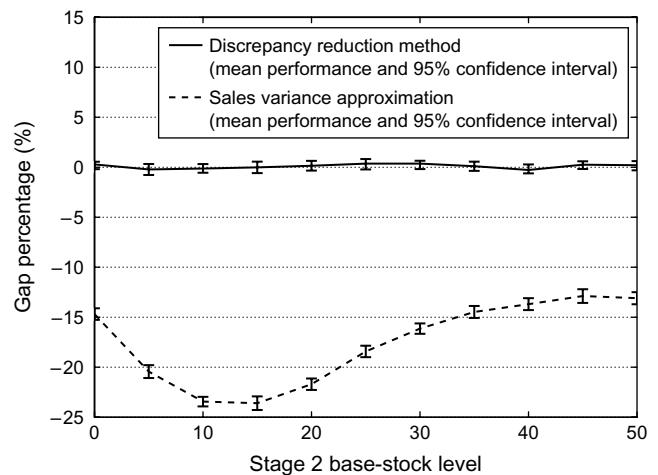
$$\mathbb{V}[D_t] = \frac{\mathbb{V}[\sum_{i=0}^{2N+1} M_0(t+i)] - \mathbb{V}[\sum_{i=0}^N M_0(t+i)]}{N+1}.$$

Thus, to estimate the demand variance in this system, one only needs a sufficiently long history of sales data. There is no need to know the underlying base-stock levels at either stage, even though the observed sales data depend on these base-stock levels. In fact, this formula can be generalized to a general J -stage serial system as long as $N \geq \sum_{j=1}^J L_j$, where L_j is the lead time at stage j .

To illustrate how our discrepancy reduction method can align between the information and material bullwhip measures, we compute the estimated demand variance $\mathbb{V}^e[D_t]$ from simulated data based on Corollary 2 and compare its performance with that of the commonly used sales variance approximation method. The performance is measured based on the gap percentage relative to the actual demand variance $\mathbb{V}[D_t]$, i.e., $(\mathbb{V}^e[D_t] - \mathbb{V}[D_t])/\mathbb{V}[D_t]$.

Figure 8 shows the performance comparison between our method and the sales variance approximation method under the normal demand distribution $N(20, 8)$, with lead times $L_2 = 1$, $L_1 = 0$, and stage 1 base-stock level $s_1 = 30$. To reduce estimation error, we compute $\mathbb{V}^e[D_t]$ for $N = 2, 3, 4$, and 5 based on the simulated data and then take an average of the four estimated values to obtain the demand variance estimation. We repeat the simulation for 30 times to generate 30 independent demand variance estimates based on which

Figure 8. Comparison of Estimation Methods Under i.i.d. Demand



we compute the mean and 95% confidence interval of the gap percentage. It is clear from the figure that our method yields a significantly better estimate of $\mathbb{V}[D_t]$ than does the sales variance approximation method in all cases.

6.2. AR(1) Demand

Now suppose that D_t follows an AR(1) model. It can be verified that the AR(1) process is α mixing with exponential decay rate (Billingsley 1995). Thus, D_t and D_{t+i} become nearly independent as i increases, and we can apply the formula given in Theorem 1 to estimate the demand variance as long as N is sufficiently large.

Specifically, we can choose K distinctive sufficiently large N values, denoted by N_k . We have, for $1 \leq k \leq K$,

$$\mathbb{V}[D_t] \approx \frac{\mathbb{V}[\sum_{i=0}^{2N_k+1} M_0(t+i)] - \mathbb{V}[\sum_{i=0}^{N_k} M_0(t+i)]}{f(2N_k+2) - f(N_k+1)},$$

where $f(\cdot)$ is given by (23). Take natural logarithm on both sides of the above equation, and define

$$\begin{aligned} y(N_k) &= \ln\left(\mathbb{V}\left[\sum_{i=0}^{2N_k+1} M_0(t+i)\right] - \mathbb{V}\left[\sum_{i=0}^{N_k} M_0(t+i)\right]\right), \\ g(N_k, \rho) &= \ln(f(2N_k+2) - f(N_k+1)), \\ a &= \ln(\mathbb{V}[D_t]). \end{aligned}$$

We arrive at, for $1 \leq k \leq K$,

$$y(N_k) = a + g(N_k, \rho) + e_k,$$

where $y(N_k)$ is the observation point based on the sales data, and e_k is the error term. Thus, given the K observation points, we can apply the least squares method to estimate the unknown parameters a and ρ ; that is, a and ρ can be estimated by solving the following optimization problem:

$$\min_{\hat{a}, 0 \leq \hat{\rho} \leq 1} \sum_{k=1}^K [y(N_k) - \hat{a} - g(N_k, \hat{\rho})]^2. \quad (25)$$

After obtaining the estimate \hat{a} , we can convert it to the estimated demand variance as

$$\mathbb{V}^e[D_t] = e^{\hat{a}}.$$

Note that the above estimation procedure is not limited to the AR(1) process. One can apply the same approach to estimate the demand variance for any demand processes with short-range dependence. All one needs to do is to determine the function $f(N)$ based on the underlying demand process parameters and then ensure that the number of observation points is greater than the number of parameters to be estimated, such that the least squares method can work properly.

As a numerical illustration, we jointly estimate the demand variance $\mathbb{V}^e[D_t]$ and demand autocorrelation coefficient ρ based on the above least squares method (25) and then compare its performance with that of the sales variance and autocorrelation approximation method.

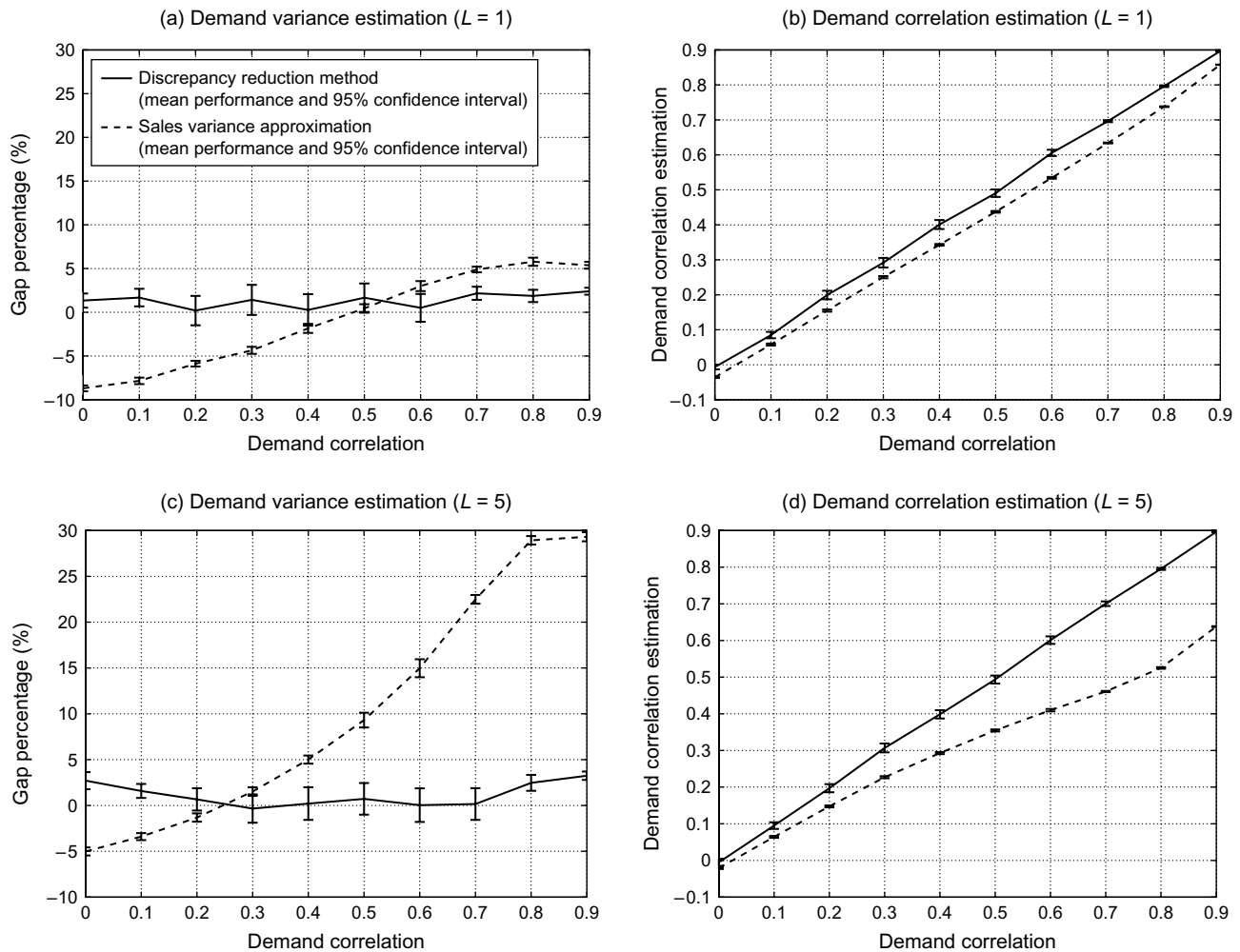
Figure 9 shows the performance comparison between our estimation method and the commonly used sales approximation method. As in Section 6.1, we measure the performance based on the gap percentage relative to the actual demand variance. In each simulation run, we fix $\sigma = 1$ and $z = 1.3$ (90% service level) and compute the gap percentage for the variance estimation and actual value for the correlation estimation under lead times $L = 1$ and $L = 5$. In our least squares estimation method, we choose $N_k = 2, 5, 8$, and 11 when $L = 1$, and $N_k = 6, 8, 10$, and 12 when $L = 5$. We repeat the simulation 30 times to generate 30 independent demand variance estimates, based on which we compute the mean and 95% confidence interval of the gap percentage. It is clear from the figure that our method yields significantly better estimates of $\mathbb{V}[D_t]$ and ρ than does the sales approximation method in almost all cases, except for two variance estimation cases ($\rho = 0.5$ for $L = 1$ and $\rho = 0.3$ for $L = 5$) in which the estimates are not statistically different.

7. Concluding Remarks

In this paper, we investigate the discrepancy issues between the information and material bullwhip measures. We illustrate and quantify the extent of the discrepancy in three inventory models. In the model with i.i.d. demand, we show that the material bullwhip measure always overestimates the information bullwhip measure. The magnitude of the overestimation is driven by the base-stock level and the replenishment lead time. For the model with AR(1) demand, we find that the material bullwhip measure can either overestimate or underestimate the information bullwhip measure, depending on the level of demand correlation. For the model with supply shortages, we find that the material bullwhip measure at the downstream firm can either overestimate or underestimate the information bullwhip measure, depending on the supply service level. Furthermore, we show that in a serial system with i.i.d. demand, the material bullwhip measure for the entire system is always greater than the information bullwhip measure for the entire system. Thus, we have identified four factors that affect the discrepancy: stocking level, lead time, demand correlation, and supply service level. We have also derived conditions for these factors, to examine whether or not there is a significant discrepancy between the two measures.

When the discrepancy is significant, we propose an estimation method to align the two measures. Our method is based on a simple relationship between the

Figure 9. Comparison of Estimation Methods Under AR(1) Demand



demand variance and the variances of the aggregated sales we have discovered in this study. It works for demand processes with short-range dependence, such as the AR(1) process and other common time-series processes. Because the estimation method involves the aggregated sales over time, it can be extended to estimate aggregated demand variance over time (i.e., measurement under time aggregation). The method can also be extended to estimate aggregated demand variance under the product aggregation model of Chen and Lee (2012), where the replenishment lead times and decision periods of products under aggregation are assumed to be identical. For products with different replenishment intervals, it remains an open problem whether the method would work effectively. We hope that our study sets the stage for future research in this direction. Moreover, we have tested and validated our estimation method based on simulated data. It would be interesting to further validate our method with real-world empirical data.

Finally, our work presents a first step to understanding the difference between demand variability and

shipment/sales variability. We show that the material flow can be very different from the information flow. Strategies designed to counter demand variability amplification could inadvertently increase material flow variability. For example, Chen and Lee (2009) studied an optimal order variability control strategy to minimize the total supply chain inventory costs under a forecast evolution model. The optimal strategy reduces the upstream supplier's demand uncertainty by postponing a portion of the order update quantity to future periods, but it also increases the variability of the final order, which leads to a higher variability of shipments sent from the supplier. From a perspective of the total supply chain costs, the demand information, transmitted from downstream to upstream, affects inventory-related costs as well as labor and warehousing costs. On the other hand, the material flow, triggered by the demand and order process and delivered from upstream to downstream, affects logistics-related costs. Thus, when making operational decisions and financial arrangements according to the demand information, managers should also take into account the material flow variability, as the latter has a direct

impact on the shipping capacity utilization and the firm's financial flow. A model that accounts for total supply chain costs (e.g., inventory, shipping, capacity, and labor scheduling costs) would be an interesting direction for future research.

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Appendix. Selected Proofs

Proof of Proposition 1. First consider the special case of $L = 0$. By the expression (6), we have

$$M_0(t) = D_t + (D_{t-1} - s)^+ - (D_t - s)^+ = \min\{D_t, s\} + (D_{t-1} - s)^+.$$

Because D_t and D_{t-1} are i.i.d., it follows that

$$\mathbb{E}[M_0(t)] = \mathbb{E}[\min\{D_t, s\}] + \mathbb{E}[(D_t - s)^+] = \mathbb{E}[D_t].$$

With this result, we have

$$\begin{aligned} \mathbb{V}[M_0(t)] &= \mathbb{E}[(\min\{D_t, s\} + (D_{t-1} - s)^+)^2] - (\mathbb{E}[D_t])^2 \\ &= \mathbb{E}[(\min\{D_t, s\})^2] + 2\mathbb{E}[\min\{D_t, s\} \cdot (D_{t-1} - s)^+] \\ &\quad + \mathbb{E}[(D_{t-1} - s)^+)^2] - (\mathbb{E}[D_t])^2 \\ &= \mathbb{E}[(\min\{D_t, s\})^2] + 2\mathbb{E}[\min\{D_t, s\} \cdot (D_t - s)^+] \\ &\quad + \mathbb{E}[(D_t - s)^+)^2] - (\mathbb{E}[D_t])^2, \end{aligned}$$

where the last equality follows from the fact that D_t and D_{t-1} are i.i.d. Also, note that

$$\begin{aligned} \mathbb{V}[D_t] &= \mathbb{E}[(D_t)^2] - (\mathbb{E}[D_t])^2 \\ &= \mathbb{E}[(\min\{D_t, s\} + (D_t - s)^+)^2] - (\mathbb{E}[D_t])^2 \\ &= \mathbb{E}[(\min\{D_t, s\})^2] + 2\mathbb{E}[\min\{D_t, s\} \cdot (D_t - s)^+] \\ &\quad + \mathbb{E}[(D_t - s)^+)^2] - (\mathbb{E}[D_t])^2 \\ &= \mathbb{E}[(\min\{D_t, s\})^2] + 2s \cdot \mathbb{E}[(D_t - s)^+] \\ &\quad + \mathbb{E}[(D_t - s)^+)^2] - (\mathbb{E}[D_t])^2, \end{aligned}$$

where the last equality follows from $\mathbb{E}[\min\{D_t, s\} \cdot (D_t - s)^+] = s \cdot \mathbb{E}[(D_t - s)^+]$. Thus,

$$\begin{aligned} \mathbb{V}[M_1(t)] - \mathbb{V}[M_0(t)] &= \mathbb{V}[D_t] - \mathbb{V}[M_0(t)] \\ &= 2(s - \mathbb{E}[\min\{D_t, s\}]) \cdot \mathbb{E}[(D_t - s)^+] \\ &= 2\mathbb{E}[(s - D_t)^+] \cdot \mathbb{E}[(D_t - s)^+]. \end{aligned}$$

For the general case with $L \geq 0$, by the expression (6), we have

$$M_0(t) = D_t + (D_{t-L-1} + D_{t-L}^{t-1} - s)^+ - (D_{t-L}^{t-1} + D_t - s)^+.$$

Note that D_{t-L}^{t-1} is independent of D_t and D_{t-L-1} . Conditional on $D_{t-L}^{t-1} = x$, by applying the result of the case of $L = 0$, we have

$$\begin{aligned} \mathbb{V}[M_1(t) \mid D_{t-L}^{t-1} = x] - \mathbb{V}[M_0(t) \mid D_{t-L}^{t-1} = x] \\ &= 2\mathbb{E}[(s - x - D_t)^+] \cdot \mathbb{E}[(D_t - s + x)^+] \\ &= 2\mathbb{E}[(s - x - D_{t-L-1})^+] \cdot \mathbb{E}[(D_t - s + x)^+]. \end{aligned}$$

Thus, it follows that

$$\mathbb{V}[M_1(t)] - \mathbb{V}[M_0(t)] = 2\mathbb{E}[(s - D_{t-L-1}^{t-1})^+] \cdot \mathbb{E}[(D_{t-L}^{t-1} - s)^+] \geq 0,$$

which implies $r^M = \mathbb{V}[M_1(t)]/\mathbb{V}[M_0(t)] \geq 1 = r^O$. \square

Proof of Theorem 1. Define $A_{t-1} = (D_{t-L-1}^{t-1} - s_{t-L-1})^+$, $A_{t+N} = (D_{t+N-L}^{t+N} - s_{t+N-L})^+$, and $A_{t+2N+1} = (D_{t+2N+1-L}^{t+2N+1} - s_{t+2N+1-L})^+$. Then (24) becomes $\sum_{i=0}^N M_0(t+i) = D_t^{t+N} + A_{t-1} - A_{t+N}$. Thus,

$$\begin{aligned} \mathbb{V}\left[\sum_{i=0}^N M_0(t+i)\right] &= \mathbb{V}[D_t^{t+N}] + 2\mathbb{E}[D_t^{t+N} A_{t-1}] - 2\mathbb{E}[D_t^{t+N} A_{t+N}] \\ &\quad + \mathbb{E}[A_{t-1}^2] + \mathbb{E}[A_{t+N}^2] - 2\mathbb{E}[A_{t-1} A_{t+N}]. \end{aligned}$$

Similarly, we have $\sum_{i=0}^{2N+1} M_0(t+i) = D_t^{t+2N+1} + A_{t-1} - A_{t+2N+1}$. Thus,

$$\begin{aligned} \mathbb{V}\left[\sum_{i=0}^{2N+1} M_0(t+i)\right] &= \mathbb{V}[D_t^{t+2N+1}] + 2\mathbb{E}[D_t^{t+2N+1} A_{t-1}] \\ &\quad - 2\mathbb{E}[D_t^{t+2N+1} A_{t+2N+1}] + \mathbb{E}[A_{t-1}^2] \\ &\quad + \mathbb{E}[A_{t+2N+1}^2] - 2\mathbb{E}[A_{t-1} A_{t+2N+1}]. \end{aligned}$$

When D_t is stationary and $N \geq T + L - 1$, it can be shown that A_{t-1} , A_{t+N} , and A_{t+2N+1} are i.i.d. Also note that $D_t^{t+2N+1} A_{t+2N+1} = D_t^{t+N} A_{t+2N+1} + D_{t+N+1}^{t+2N+1} A_{t+2N+1}$, where the second term and $D_t^{t+N} A_{t+N}$ can also be shown as i.i.d. Therefore,

$$\begin{aligned} \mathbb{V}\left[\sum_{i=0}^{2N+1} M_0(t+i)\right] - \mathbb{V}\left[\sum_{i=0}^N M_0(t+i)\right] \\ &= \mathbb{V}[D_t^{t+2N+1}] - \mathbb{V}[D_t^{t+N}] + 2\mathbb{E}[D_{t+N+1}^{t+2N+1} A_{t-1}] \\ &\quad - 2\mathbb{E}[D_t^{t+N} A_{t+2N+1}] \\ &= \mathbb{V}[D_t^{t+2N+1}] - \mathbb{V}[D_t^{t+N}] \\ &= [f(2N+1) - f(N)]\mathbb{V}[D_t]. \quad \square \end{aligned}$$

Proof of Corollary 1. The result follows from Theorem 1 by letting $T = 1$ and $f(N) = N$. \square

Proof of Corollary 2. Let $L = L_1 + L_2$. Because $N \geq L$, by repeatedly using Equation (18), we have

$$\sum_{i=0}^N M_0(t+i) = D_t^{t+N} + (A_{t-L-1}^{t-1} - s_1)^+ - (A_{t+N-L}^{t+N} - s_1)^+,$$

where $A_{t-L-1}^{t-1} = D_{t-L-1}^{t-1} + (D_{t-L-1}^{t-2} - s_2)^+$ and $A_{t+N-L}^{t+N} = D_{t+N-L}^{t+N} + (D_{t+N-L}^{t+N-1} - s_2)^+$. Then

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{i=0}^N M_0(t+i)\right)^2\right] \\ &= \mathbb{E}[(D_t^{t+N})^2] + 2\mathbb{E}[(A_{t+N-L}^{t+N} - s_1)^+)^2] \\ &\quad - 2\mathbb{E}[D_t^{t+N} (A_{t+N-L}^{t+N} - s_1)^+] + 2\mathbb{E}(D_t^{t+N}) \mathbb{E}[(A_{t-L-1}^{t-1} - s_1)^+] \\ &\quad - 2\mathbb{E}[(A_{t-L-1}^{t-1} - s_1)^+] \mathbb{E}[(A_{t+N-L}^{t+N} - s_1)^+], \end{aligned} \quad (\text{A.1})$$

where (A.1) follows because $N \geq L$, and thus A_{t-L-1}^{t-1} and A_{t+N-L}^{t+N} are i.i.d., and the former is independent from D_t^{t+N} . Recall that $\mathbb{E}[D_t] = \mu$. With the same logic as above, we have

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{i=0}^{2N+1} M_0(t+i)\right)^2\right] \\ &= \mathbb{E}[(D_t^{t+2N+1} + D_{t+N+1}^{t+2N+1} + (A_{t-L-1}^{t-1} - s_1)^+ - (A_{t+2N+1-L}^{t+2N+1} - s_1)^+)^2] \\ &= \mathbb{E}[(D_t^{t+2N+1})^2] + 2\mathbb{E}[(A_{t+2N+1-L}^{t+2N+1} - s_1)^+)^2] \\ &\quad - 2\mathbb{E}[D_{t+N+1}^{t+2N+1} (A_{t+2N+1-L}^{t+2N+1} - s_1)^+] \\ &\quad + 2\mathbb{E}(D_{t+N+1}^{t+2N+1}) \mathbb{E}[(A_{t-L-1}^{t-1} - s_1)^+] \\ &\quad - 2\mathbb{E}[(A_{t-L-1}^{t-1} - s_1)^+] \mathbb{E}[(A_{t+2N+1-L}^{t+2N+1} - s_1)^+], \end{aligned} \quad (\text{A.2})$$

where we use that A_{t-L-1}^{t-1} and $A_{t+2N-L+1}^{t+2N+1}$ are i.i.d. and both are independent from D_t^{t+N} . Plugging (A.1) and (A.2) into the variance expressions, we have

$$\begin{aligned} & \mathbb{V}\left[\sum_{i=0}^{2N+1} M_0(t+i)\right] - \mathbb{V}\left[\sum_{i=0}^N M_0(t+i)\right] \\ &= \mathbb{E}\left[\left(\sum_{i=0}^{2N+1} M_0(t+i)\right)^2\right] - 4(N+1)^2\mu^2 \\ &\quad - \mathbb{E}\left[\left(\sum_{i=0}^N M_0(t+i)\right)^2\right] + (N+1)\mu^2 \\ &= \mathbb{E}[(D_t^{t+2N+1})^2] - 4(N+1)^2\mu^2 - \mathbb{E}[(D_t^{t+N})^2] + (N+1)\mu^2 \\ &= \mathbb{V}[D_t^{t+2N+1}] - \mathbb{V}[D_t^{t+N}] = (N+1)\mathbb{V}[D_t], \end{aligned}$$

which completes the proof. \square

Endnotes

¹ $\mathbb{V}[M_1(t)] - \mathbb{V}[M_0(t)] = (s^2/(2d^2))((d-s)^+)^2$, where the demand density is uniform over $[0, d]$.

² $\mathbb{V}[M_1(t)] - \mathbb{V}[M_0(t)] = (2e^{-\lambda s}/\lambda^2)(\lambda s + e^{-\lambda s} - 1)$, where the demand density is exponential $\lambda e^{-\lambda t}$.

³ There exists a one-to-one equivalence between a local base-stock policy and an echelon base-stock policy (see Axsäter and Rosling 1993).

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