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# Improved Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover

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Order crossovers occur when replenishment orders arrive in a sequence that is different than the one in which they were placed. Order crossovers require that optimal reorder levels be set with regard to the inventory shortfall distribution rather than the lead-time demand distribution. Assuming periodic review and independent lead times, this paper suggests simple approximations of the shortfall distribution by showing that the variance of the number of orders outstanding is bounded above by the standard deviation of lead time divided by $\sqrt{3}$. Using this bound in a normal approximation improves significantly upon the common practice of basing policies on the lead-time demand distribution. A negative binomial approximation of the shortfall, based on its exact variance, offers even greater improvement, at the cost of some additional informational and computational requirements.

*Key words*: inventory policies/management; base-stock policies; stochastic lead time; order crossover
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## 1. Introduction

In this paper we consider the standard periodic review inventory model, where at the start of each period, an order is placed that returns the inventory position to some base-stock level $S$. The probability distribution of lead-time demand, which is the quantity of demand that occurs between the placement and receipt of an order, is commonly used in calculating $S$. The lead-time demand distribution is a mixture that depends on the probability distribution of the lead time and the convolution of demand over multiple periods. Although the base-stock level can be computed from this mixture, this computation is quite cumbersome (albeit straightforward), and requires information about the entire lead-time and demand distributions that may not always be available. For tractability, a normal approximation of the lead-time demand distribution is commonly used to set the base-stock level.

Using the lead-time demand distribution to set the base-stock level $S$ is appropriate when orders always arrive in the same sequence in which they were placed. However, when stochastic lead times allow for the possibility of out-of-sequence "order crossovers," then this distribution is no longer correct. Instead, the distribution for the inventory shortfall (the outstanding inventory still on order, plus the current period's demand) should be used. Robinson et al. (2001) show that the cost of incorrectly setting the base-stock level $S$ from the lead-time demand distribution, rather than from the shortfall distribution, can be quite high even if crossovers are infrequent.

To visualize circumstances where order crossovers might occur, we divide the replenishment lead time into two components: (1) the time required for the supplier to produce the order and (2) the time required for transportation. The production component includes the time to transmit the order to the supplier, any delay before the start of production, and the actual production time. Order crossover cannot occur during production when there is a single supplier who processes each order sequentially on a firstcome, first-served (FCFS) basis. However, if any of these conditions are violated, then order crossover is possible. A non-FCFS scheduling discipline can clearly cause crossover. With multiple suppliers, an order that is delayed because of congestion at one supplier might be produced after an order that is subsequently placed with a less-congested supplier. Crossovers could also occur at a single supplier when orders are processed in parallel.

Order crossover in the transportation component of replenishment lead time can be caused by geography, transportation time variability, or the use of multiple transportation modes. For example, crossover in rail transportation is sometimes due to transportation time variability resulting from the scheduling practices in that industry. Even with a single supplier, the use of multiple transportation modes or providers could cause order crossover if one could sometimes be slower than another.

Order crossover can be correctly accounted for by using the shortfall distribution, which is based on the number of orders outstanding. Like the lead-time demand, the shortfall is a mixture of demand over multiple periods. Because the mixing probabilities reflect the number of outstanding orders rather than the lead time, they may be cumbersome to compute. Even a normal approximation of the shortfall distribution is more difficult to construct than a normal approximation of the lead-time demand distribution, because its variance depends on the variance of the number of orders outstanding, the exact computation of which, in turn, depends on the entire distribution of the lead time.

In order to easily characterize the shortfall distribution, we develop a simple and tractable upper bound on the variance of the number of orders outstanding. Specifically, we prove that the *variance* of the number of orders outstanding is bounded above by the *standard deviation* of the lead time divided by $\sqrt{3}$. This bound facilitates base-stock approximations that are computed by assuming that the shortfall is distributed according to either the normal or negative binomial distributions.

We conduct a comprehensive numerical study of the cost performance of six heuristic policies based on either the normal or negative binomial distributions and one of three estimates of the variance of the number of orders outstanding. We show that a heuristic base-stock level based on the normal distribution with an approximation of the variance of the shortfall, as estimated by our new upper bound, generally does quite well except for very high service levels. It is clearly superior to the commonly used heuristic based on the variance of the lead-time demand (which would be appropriate in the absence of order crossover), while requiring neither additional information nor additional computational effort. We also show that the best performance comes from a heuristic based on a negative binomial approximation of the shortfall distribution using the true variance of the shortfall. Although this heuristic requires some additional information, its computation is still relatively simple.

One consequence of our upper bound on the variance of the number of orders outstanding is that this variance will be much less than the variance of the lead time when the lead-time variance is large. In turn, the shortfall will be less variable than the lead-time demand, so that inventory policies based on the lead-time demand will suggest too much safety stock.

In the remainder of this paper, we first summarize the relevant literature in §2. We then describe our inventory replenishment model and highlight the importance of the variance of the number of orders outstanding in §3. We compute bounds on, and approximations of, the variance of the number of orders outstanding in §4. We evaluate the performance of our shortfall-based heuristic policies in §5, and conclude in §6.

## 2. Literature Review

Hadley and Whitin (1963, §§4–12) point out that the analysis of inventory models with independent and stochastic lead times is difficult because of order crossover. For convenience, the base-stock level is often based on the distribution of lead-time demand, which has sometimes been justified in the literature by the assumption that order crossovers are sufficiently rare that they can be ignored. In contradiction, Robinson et al. (2001) provide a counterexample where inventory control costs can be 60% above the optimal, even when the probability of crossover is less than 10%.

Some models preclude order crossover. For example, in contrast to the assumption of independent lead times in this paper, Kaplan (1970), Nahmias (1979), and Ehrhardt (1984) all assume that the lead times of outstanding orders are dependent. In particular, they assume that the receipt of an order $l$ periods after it was placed implies that all orders placed more than $l$ periods ago have also been received. Song and Zipkin (1996b) generalize this model by using a Markov process to describe how the supply system state can evolve over time, which allows orders to progress stochastically through the replenishment pipeline while precluding them from crossing over one another.

One model that allows for order crossover is the continuous review $(S-1, S)$ base-stock model with Poisson demand and independent lead times. Using Palm's Theorem (1938), Hadley and Whitin (1963, §§4–13) show that the inventory shortfall in this case has a Poisson distribution with a mean equal to the average lead time multiplied by the average demand rate. Surprisingly, the shortfall distribution does not depend on the variance of either the lead time or the number of orders outstanding. Feeney and Sherbrooke (1966) extend their result by allowing demand to follow a compound Poisson distribution.

A few other authors have analyzed the effect of order crossover in continuous-review $(Q, r)$ models, including Hayya et al. (1995), Song and Zipkin (1996a), He et al. (1998), and Hayya et al. (2001). In particular, Song and Zipkin (1996a) focus on Poisson demand and approximate the variance of the shortfall by the minimum of two quantities. The first quantity is derived from a heavy traffic model that allows crossover, and the second quantity is the variance when no crossover occurs.

Our paper is most closely related to that of Zalkind (1976, 1978) and Robinson et al. (2001), who examine the effects of order crossover in a periodic review base-stock model. They show that the optimal order-up-to point $S^*$ should be based on the distribution of the shortfall rather than the lead-time demand. Under a base-stock policy, the shortfall distribution depends on the distribution of the number of orders outstanding. Robinson et al. (2001) provide an iterative algorithm for calculating this distribution based on the distribution of the lead time. Both Robinson et al. (2001) and Zalkind (1976) show that the number of orders outstanding has the same mean as the lead time, but an equal or smaller variance. A limited numerical study by Robinson et al. (2001) analyzes normal and negative binomial approximations of the shortfall distribution; both yield base-stock levels that are superior to the common normal approximation of lead-time demand. We extend Robinson et al. (2001) by developing an upper bound on the variance of the number of orders outstanding, which facilitates the computation of both the normal and negative binomial approximations. We also conduct a comprehensive numerical study of base-stock heuristics based on various normal and negative binomial approximations of the shortfall distribution.

## 3. The Model and Base-Stock Approximations

Consider a periodic review model where $D_t$ is the stochastic and stationary demand in any period $t$, and $L_t$ is the stochastic and stationary lead time for an order that is placed at the start of period $t$, independent of its size $Q_t$. We assume that $L_t$ is measured in periods of fixed length, and is nonnegative and integer valued. We define $f_l \doteq \Pr\{L = l\}$, and $F_l$ to be its corresponding cumulative distribution function (c.d.f.). We assume that $D_t$ and $L_t$ are jointly independent over time, and that all unsatisfied demand is backordered. Also, we let $z$ denote a realization of any random variable $Z$, and we let $\mu_Z = \mathrm{E}(Z)$ denote its expected value and $\sigma_Z^2 = \mathrm{Var}(Z)$ denote its variance. Furthermore, we define $Z^{(L)}$ to be the $L$-fold convolution of $Z$.

Following a single opportunity to place an order at the beginning of each period $t$, some of the outstanding orders may arrive, including the current order if $L_t = 0$. Under a base-stock policy, the order quantity $Q_t$ will equal the demand in the previous period, $D_{t-1}$, provided that the initial inventory is not too high. Following the realization of demand $D_t$ over the course of the period, per-unit holding and penalty costs ($h$ and $p$, respectively) are finally assessed against the ending inventory level (on-hand inventory minus backorders).

Define $SF_t$ to be the inventory shortfall at the end of period $t$; this random variable equals the total amount of inventory that has been ordered in or prior to period $t$, but that has not arrived, plus the demand in period $t$, $D_t$. Note that $SF_t$ is invariant with the base-stock level $S$, and that the ending inventory level is $S - SF_t$. We define the random variable $N$ to be the number of orders that remain outstanding after the beginning of a period when a new order has been placed and any earlier orders might have arrived, and define $g_n \doteq \Pr\{N = n\}$, so that the inventory shortfall $SF$ can be represented as a mixture of the demands of $N+1$ periods with mixing probabilities $\{g_n\}$:

$$\Pr\{SF = sf\} = \sum_{n=0}^{\infty} g_n \cdot \Pr\{D^{(n+1)} = sf\}. \tag{1}$$

Note that the shortfall is equal to $N+1$ periods' worth of demand at the end of every period, when inventory costs are assessed: the $N$ orders that remain outstanding, plus the current period's demand. For technical completeness, we note in passing that orders of size zero are included in the definition of $N$.

The expected cost per period can be written as a function of the base stock $S$ as

$$C(S) = h \sum_{sf \le S} (S - sf) \Pr\{SF = sf\} + p \sum_{sf \ge S+1} (sf - S) \Pr\{SF = sf\}.$$

The well-known optimal base stock that minimizes $C(S)$ is

$$S^* \doteq \min\{S \mid \Pr(SF \le S) \ge r\},$$

where $r$ is the target service level

$$r \doteq \frac{p}{p+h}.$$

The actual service level may, of course, vary from this target level under heuristic policies.

Both Robinson et al. (2001) and Zalkind (1976, 1978) show that if orders cannot strictly cross (i.e., if $L$ is either deterministic or can assume one of two adjacent values), then $N$ is identical in distribution to $L$ and, moreover, the shortfall $SF$ is identical in distribution to the demand over the lead time, $LTD$:

$$\Pr\{LTD = ltd\} = \sum_{l=0}^{\infty} f_l \cdot \Pr\{D^{(l+1)} = ltd\}. \tag{2}$$

In this case, base-stock levels set according to the target service level $r$ with respect to $LTD$ and $SF$ are, of course, identical and optimal. In contrast, when orders can cross over one another, Robinson et al. (2001) show that incorrectly using the distribution of $LTD$ to set the base-stock level $S$ will typically result in unnecessarily high inventory levels (and costs) because $\sigma_L^2 > \sigma_N^2$, and so $\sigma_{LTD}^2 > \sigma_{SF}^2$.

To avoid the mixture calculations, the base-stock level is often approximated under the assumption that the $LTD$ distribution is normal as

$$S_{LTD}^N \doteq \mu_{LTD} + \sigma_{LTD} \cdot \Psi^{-1}(r), \tag{3}$$

rounded to the nearest integer, where $\Psi(\cdot)$ is the unit normal c.d.f., and the mean and variance are computed using the well-known result for the sum of a random number $L+1$ of independent and identically distributed (i.i.d.) random variables $D$:

$$\mu_{LTD} = (\mu_L + 1) \cdot \mu_D, \tag{4}$$

$$\sigma_{LTD}^2 = (\mu_L + 1) \cdot \sigma_D^2 + \mu_D^2 \cdot \sigma_L^2. \tag{5}$$

Despite the dangers of this approach outlined by Eppen and Martin (1988), this heuristic offers two distinct advantages. First, the awkward mixture calculations in (2) are avoided. Second, there is no longer any need to know the entire distribution of $L$; only its first two moments are needed.

One possible improvement to this approach when order crossover is possible is to approximate the base stock using a normal approximation of the shortfall $SF$ rather than of $LTD$,

$$S_{SF}^N \doteq \mu_{SF} + \sigma_{SF} \cdot \Psi^{-1}(r) \tag{6}$$

(again rounded to the nearest integer), where the first two moments of the shortfall are computed as the sum of a random number $N+1$ of orders outstanding, where each order is the previous period's i.i.d. demand $D$:

$$\mu_{SF} = (\mu_N + 1) \cdot \mu_D, \tag{7}$$

$$\sigma_{SF}^2 = (\mu_N + 1) \cdot \sigma_D^2 + \mu_D^2 \cdot \sigma_N^2. \tag{8}$$

These computations require the first two moments of the distribution of $N$, which Robinson et al. (2001) implicitly show to be

$$\mu_N = \mu_L, \tag{9}$$

$$\sigma_N^2 = \sum_{l=0}^{\infty} F_l \cdot (1 - F_l). \tag{10}$$

Note that $\mu_{SF} = \mu_{LTD}$, which is easily calculated, and that the only difference between $\sigma^2_{LTD}$ and $\sigma^2_{SF}$ is that $\sigma^2_N$ in (8) replaces $\sigma^2_L$ in (5). Computing $\sigma^2_N$ in (10), however, requires knowledge of the entire distribution of the lead time $L$. Thus, the two major advantages of using a normal approximation—easier calculations and reduced data requirements—are both compromised when the shortfall distribution is used in place of the lead-time demand distribution.

In the following section, we develop a tractable and robust approximation for $\sigma^2_N$ that resolves this issue and makes computing a base-stock policy from the normal shortfall approximation just as easy as from the commonly used normal lead-time demand approximation. Because the shortfall distribution is unimodal (Zalkind 1976), albeit not normal, much of Eppen and Martin's (1988) concern with the normal approximation becomes moot. We also evaluate heuristic policies based on approximating the shortfall distribution by a negative binomial distribution.

## 4. Bounds on the Variance of the Number of Orders Outstanding

Although a wide variety of distributions $\{f_l\}$ can have the same first two moments, the distributions $\{g_n\}$ that each generates are remarkably similar. For example, consider the three distinctly different lead-time probability distributions shown in Figure 1 that all have $\mu_L = 2$ and $\sigma^2_L = 2$. Figure 2 shows that the three corresponding distributions of the number of orders outstanding $\{g_n\}$ are remarkably similar to one another

**Figure 1** **Distributions of the Lead Times $\{f_l\}$ with $\mu_L = \sigma^2_L = 2$**

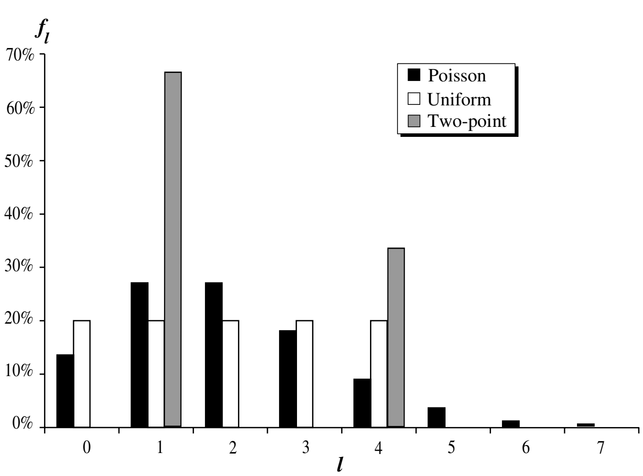
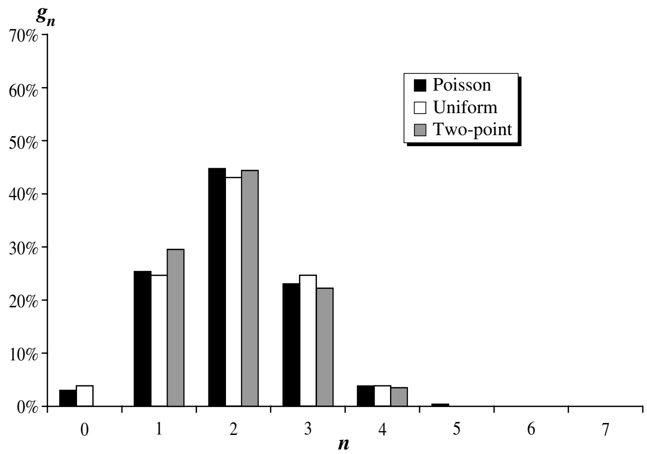



**Figure 2** **Distributions of the Number of Orders Outstanding $\{g_n\}$ with $\mu_L = \sigma^2_L = 2$**




given their disparate origins, with variances that vary from one another by at most 20% (between $\sigma^2_N = 0.667$ for the two-point distribution and $\sigma^2_N = 0.800$ for the uniform distribution). This relatively narrow range for $\sigma^2_N$ suggests that a reasonable approximation of it could yield good base-stock approximations, particularly given that $\sigma^2_N$ contributes to only one of two terms in $\sigma^2_{SF}$, which in turn contributes to only one of two terms in the normal base-stock approximation. We approximate $\sigma^2_N$ by constructing an upper bound based on the first two moments of the lead-time distribution, $\mu_L$ and $\sigma^2_L$.

The problem of finding the distribution of the lead time $\{f_l\}$ that maximizes $\sigma^2_N$ given the first two moments $\mu_L$ and $\sigma^2_L$ can be formulated as the following quadratic program:

$$\text{(P1)} \quad \max \; \sigma^2_N = \sum_{l=0}^{\infty} F_l \cdot (1 - F_l)$$

$$\text{s.t.} \; \sum_{l=0}^{\infty} f_l = 1,$$

$$\sum_{l=0}^{\infty} l \cdot f_l = \mu_L,$$

$$\sum_{l=0}^{\infty} l^2 \cdot f_l = \sigma^2_L + \mu^2_L,$$

$$f_l \geq 0, \quad l = 0, 1, \ldots.$$

Lemma 1. *The unique solution to* (P1) *is a "quasi-uniform" distribution.*

Although the theorems presented in this section can be stated concisely, their proofs are long and tedious. They have therefore all been relegated to an accompanying document "Appendix to Improved Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover," by Bradley and Robinson (2005), which is available at http://informs.org/Pubs/Supplements/MSOM/1526-5498-2005-07-04-0319-app.pdf.

By "quasi-uniform," we refer to a discrete probability distribution with equivalent mass on all points, with the possible exception of the endpoints. The probability mass on the endpoints is no greater than the mass on the intermediate points unless the lowermost endpoint is zero. In that case, the mass on zero can be greater than the mass on the intermediate points. Lemma 1 makes it possible to derive the following upper bound on the number of orders outstanding.

Proposition 1. *For any lead-time distribution with variance* $\sigma_L^2$,

$$\sigma_N^2 \le \frac{\sigma_L}{\sqrt{3}}.$$

One of the most striking aspects of this proposition, besides its simplicity, is that this upper bound on the *variance* of the number of orders outstanding is proportional to the *standard deviation* of the lead time. Thus, whenever the lead time is highly variable, the variance of the number of orders outstanding will be significantly less than the variance of the lead time and, in turn, the variance of the shortfall will be less than the variance of the lead-time demand. The following lower bound on the variance of the number of orders outstanding is also a linear function of the standard deviation of the lead time, $\sigma_L$.

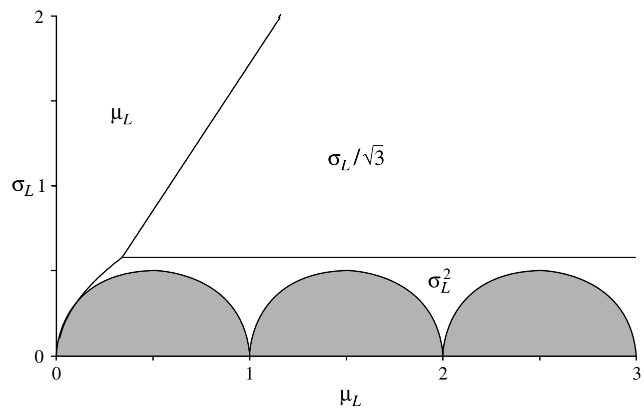
Corollary 1. *For any* $\mu_L$ *and* $\sigma_L^2$, *if the replenishment lead time* $L$ *is restricted to some range* $\mu_L \pm k\sigma_L$, *then*

$$\sigma_N^2 \ge \left(\frac{k}{k^2+1}\right)\sigma_L. \tag{11}$$

The upper bound of $\sigma_L/\sqrt{3}$ can be further tightened by incorporating two other simple upper bounds. First, both Robinson et al. (2001) and Zalkind (1976, 1978) have shown that $\sigma_N^2 \le \sigma_L^2$ for all distributions of lead time $L$. Second, it is trivial to show that $\sigma_N^2 \le \mu_L$:

$$\sigma_N^2 = \sum_{l=0}^{\infty} F_l(1-F_l) \le \sum_{l=0}^{\infty}(1-F_l) = \mu_L.$$

**Figure 3** **Tightest Bounds on $\sigma_N^2$**




Combining these three bounds yields the upper bound

$$\hat{\sigma}_N^2 \doteq \min\left\{\sigma_L^2, \mu_L, \frac{\sigma_L}{\sqrt{3}}\right\}. \tag{12}$$

Figure 3 shows which of these three bounds is the tightest for various $\mu_L$ and $\sigma_L$. Our newly developed bound of $\sigma_L/\sqrt{3}$ is often the tightest, while $\mu_L$ is the tightest bound for distributions with large coefficients of variation $c_v \doteq \sigma_L/\mu_L > \sqrt{3}$, and $\sigma_L^2$ is the tightest when the lead-time variability is small; i.e., when $\sigma_L < 1/\sqrt{3}$. (Note that the scalloped, shaded region in Figure 3 represents those $\sigma_L$ that are infeasible because of noninteger means $\mu_L$ of the integer-valued random variable $L$.)

We can approximate $\sigma_{SF}^2$ in (8) by replacing $\sigma_N^2$ with its upper bound $\hat{\sigma}_N^2$:

$$\hat{\sigma}_{SF}^2 \doteq (\mu_N+1)\cdot\sigma_D^2 + \mu_D^2\cdot\hat{\sigma}_N^2, \tag{13}$$

which can then, in turn, be used to compute a new heuristic base-stock level

$$S_{\widehat{SF}}^N \doteq \mu_{SF} + \hat{\sigma}_{SF}\Psi^{-1}(r), \tag{14}$$

rounded to the nearest integer.

Of course, these are not the only possible bounds on $\sigma_N^2$. Lower and upper bounds can be computed using partial summations, where the parameter $m$ is used to control for accuracy:

$$\sigma^2_{N:LB} = \sum_{l=0}^{m} F_l(1-F_l). \tag{15}$$

A corresponding upper bound is

$$\sigma^2_{N:UB} = \sigma^2_{N:LB} + \sum_{l=m+1}^{\infty}(1-F_l) = \sigma^2_{N:LB} + \mu_L - \sum_{l=0}^{m}(1-F_l)$$
$$= \mu_L - \sum_{l=0}^{m}(1-F_l)^2. \tag{16}$$

Both bounds in (12) and (16) are simple to compute. The decision of which bound to use rests in part on what information is available. If only the first two moments of the lead-time distribution are known, then (12) must be used. However, if the first $m+1$ values of the lead-time distribution are known, then (15) and (16) could be calculated.

Finally, we note that this entire estimation problem could be avoided if information were collected about $N$ rather than $L$. Although corporate information systems are typically set up to collect information about each order, including its lead time $L$, there is no theoretical barrier to collecting information about $N$ directly. When crossover is possible, it might be advisable to reprogram information systems so that $\sigma^2_N$ can be estimated directly.

## 5. Performance Evaluation

We can construct base-stock approximations of either the lead-time demand distribution (as would be appropriate in the absence of order crossover) or the shortfall distribution. With the shortfall distribution, we can use either the true variance $\sigma^2_N$ or our upper bound $\widehat{\sigma}^2_N$. We designate these three options by the subscripts $LTD$, $SF$, or $\widehat{SF}$, depending on whether (5), (8), or (13) is used, respectively. In each of these three cases, we can use either a normal or a negative binomial distribution, which we characterize by the superscripts $N$ and $NB$, respectively. We undertake a numerical study to evaluate these six heuristic base-stock levels—$S^N_{LTD}$, $S^{NB}_{LTD}$, $S^N_{SF}$, $S^{NB}_{SF}$, $S^N_{\widehat{SF}}$, and $S^{NB}_{\widehat{SF}}$—relative to the optimal base stock $S^*$. The three normal heuristics $S^N_{\bullet}$ are all rounded to the nearest integer value. We will eventually focus on three of these alternatives: $S^N_{LTD}$, $S^N_{\widehat{SF}}$, and $S^{NB}_{SF}$. $S^N_{LTD}$ is commonly used in practice, and so it provides a benchmark that we hope to improve upon. Among the other five heuristics, $S^N_{\widehat{SF}}$ and $S^{NB}_{SF}$ provide the best accuracy.

Heuristics based on the normal distribution require the inverse cumulative normal function $\Psi^{-1}(r)$, which is available in virtually all computational packages. This is not the case for the inverse cumulative negative binomial function, whose probability distribution needs to be iteratively calculated and summed.

We evaluated a broad range of probability distributions for the demand $D$ and the lead time $L$. To avoid the tedious calculations of convoluted demand, we assume that $D$ follows a Poisson distribution. (As a direct result of this assumption, the variance exceeds the mean for both the lead-time demand $LTD$ and shortfall $SF$ distributions, so that either can be easily approximated by a negative binomial distribution, which shares this property. However, for demand distributions with $\sigma^2_D < \mu_D$, this will not always be true.) We also assume that the lead time $L$ belongs to the broad family of power series distributions (described below) with mean $\mu_L$ and variance $\sigma^2_L$. Thus, the state space for evaluating these heuristics depends on four parameters: $\mu_D$, $\mu_L$, $\sigma^2_L$, and $r$. We evaluated each heuristic policy by computing the incremental cost $\Delta(S)$ that would be incurred if that heuristic was used rather than the optimal base stock $S^*$:

$$\Delta(S) \doteq \frac{C(S)-C(S^*)}{C(S^*)}.$$

For each heuristic base-stock policy, we calculated and graphed $\Delta(S)$ for 200 values of $r \in [0.800, 0.999]$ and 81 values of $\sigma_L \in [0.0, 8.0]$, for each of $3^2$ values of $\mu_D$ and $\mu_L \in \{2, 6, 10\}$, for a total of 145,800 parameter combinations. In comparing the following graphs of $\Delta(S)$, please recognize that the vertical scales differ considerably from one another.

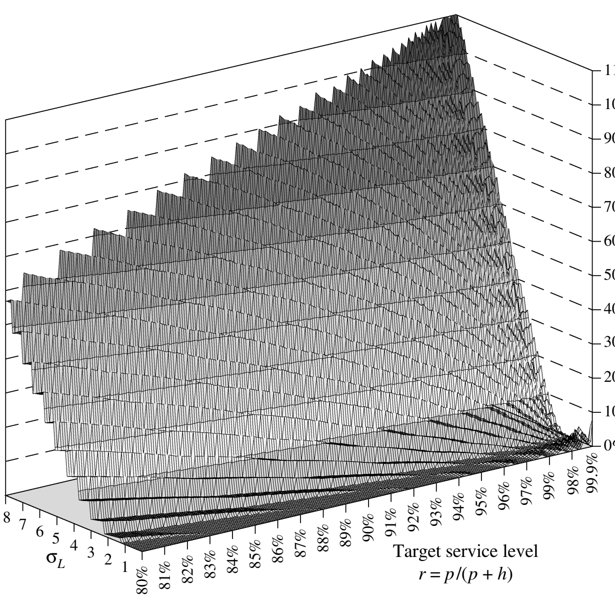
In testing the various heuristic policies, we will need to construct discrete lead-time distributions that match a specified mean $\mu_L$ and variance $\sigma^2_L$. We have chosen the family of power series distributions (PSDs), which includes the binomial, Poisson, and negative binomial distributions, depending on whether the variance is less than, equal to, or greater than the mean. (We exclude the logarithmic distribution, which is also a member of this family.) Because the binomial and negative binomial distributions both converge to the Poisson distribution (from different sides) as $\sigma_L^2 \to \mu_L$, the PSDs encompass a broad family of unimodal distributions.

For technical completeness when $\sigma_L^2 < \mu_L$, we note that because the number of trials must be integer, most values of $(\mu_L, \sigma_L^2)$ cannot be represented by a single binomial $(n, p)$ distribution. In such cases, we instead construct a mixture of two binomial distributions with adjacent values of $n$: $n_1 \doteq \lfloor \mu_L^2/(\mu_L - \sigma_L^2) \rfloor$ and $n_2 \doteq n_1 + 1$. By defining $p_1 \doteq \min\{\mu_L/n_1, 1\}$, then $p_2$ and the mixing probability are uniquely chosen so that the mean and variance of this mixture match $\mu_L$ and $\sigma_L^2$.

### 5.1. The Lead-Time Demand Heuristics $S_{LTD}^N$ and $S_{LTD}^{NB}$

Although it is the most commonly used, the heuristic based on the normal approximation of the lead-time demand $S_{LTD}^N$ was, on average, far worse than any of the heuristics based on the shortfall distribution. Its cost increment $\Delta(S_{LTD}^N)$ increases with $\mu_D$, $\sigma_L$, and $r$, and decreases with $\mu_L$. Even with the most favorable values of the means ($\mu_L = 10$, $\mu_D = 2$), Figure 4 shows that the cost of using $S_{LTD}^N$ can be more than twice as great as the optimal policy $S^*$. Because its negative binomial sibling $S_{LTD}^{NB}$ performed even worse while being more difficult to calculate, it was quickly dropped from further consideration.

**Figure 4** Relative Cost Performance $\Delta(S_{LTD}^N)$ of the Normal Heuristic with Lead-Time Variance $\sigma_{LTD}^2$ ($\mu_L = 10$, $\mu_D = 2$)




### 5.2. The Normal Shortfall Heuristics $S_{SF}^N$ and $S_{\widehat{SF}}^N$

We observed that $S_{SF}^N$ uniformly underperforms $S_{\widehat{SF}}^N$, perhaps because the larger variance of the distribution underlying $S_{\widehat{SF}}^N$ offsets the normal distribution's lack of positive skewness, which we observed throughout our entire numerical test bed. In consideration also of its higher computational requirements, we do not consider $S_{SF}^N$ further. The cost increment $\Delta(S_{\widehat{SF}}^N)$ decreases with both $\mu_D$ and $\mu_L$, does not clearly change with $\sigma_L$, and is significantly positive only for high values of $r$. The worst case ($\mu_D = 2$, $\mu_L = 2$) and best case ($\mu_D = 10$, $\mu_L = 10$) results are shown in Figures 5 and 6, respectively. In the worst case examined, $S_{\widehat{SF}}^N$ results in costs of 36.62% above optimum for very large values of $r$, although it performs within 5.16% of optimal for all $r \le 97.4\%$. The (relatively) poor performance for extremely high values of $r$ results from the skewness of the shortfall distribution, which is absent in the normal approximation. Because the tail of the shortfall distribution for high fractiles is thicker than the approximating normal distribution, $S_{\widehat{SF}}^N$ will increasingly underestimate $S^*$ as $r \to 100\%$. As shown in Figure 6, in the best case ($\mu_D = 10$, $\mu_L = 10$), $S_{\widehat{SF}}^N$ always performs within 2.02%

**Figure 5** Relative Cost Performance $\Delta(S_{\widehat{SF}}^N)$ of the Normal Heuristic with Approximate Variance $\widehat{\sigma}_{SF}^2$ ($\mu_L = 2$, $\mu_D = 2$)

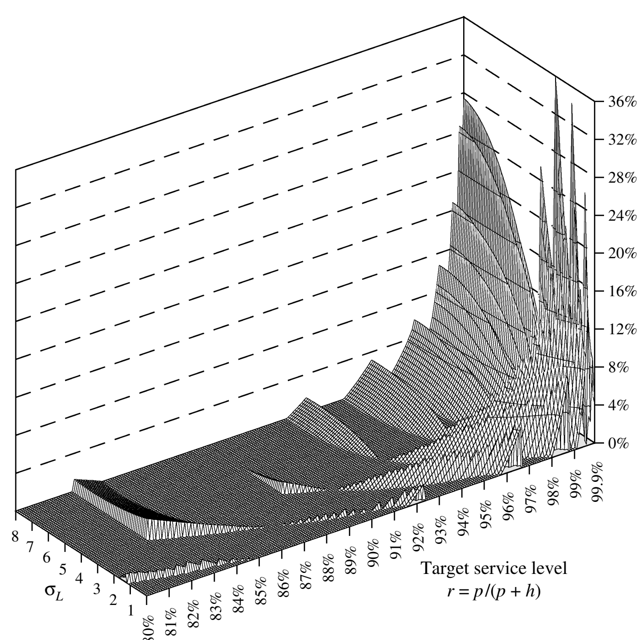


**Figure 6** Relative Cost Performance $\Delta(S^N_{\widehat{SF}})$ of the Normal Heuristic with Approximate Variance $\hat{\sigma}^2_{SF}$ ($\mu_L = 10$, $\mu_D = 10$)

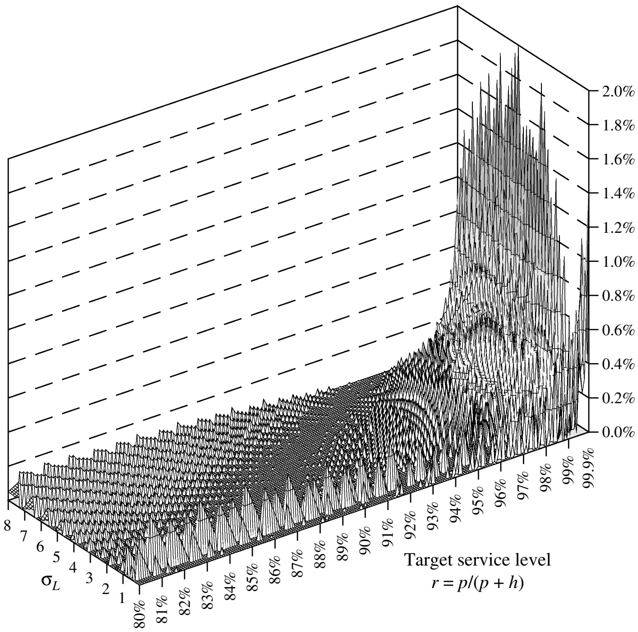
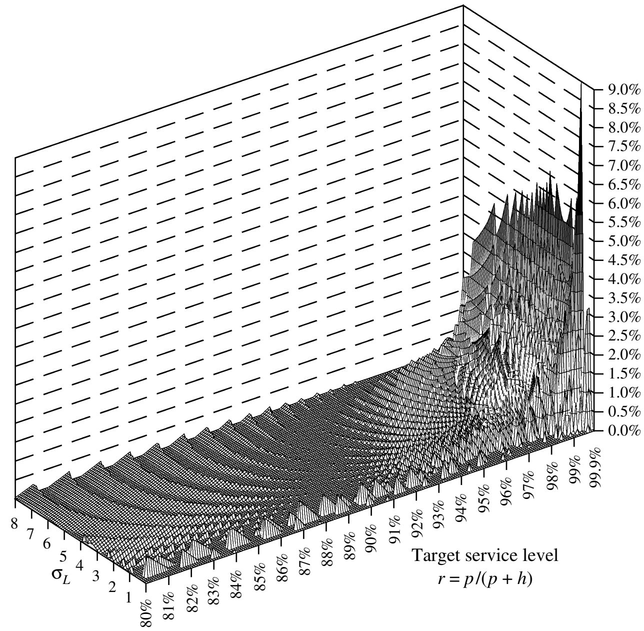



**Figure 7** Relative Cost Performance $\Delta(S^{NB}_{SF})$ of the Negative Binomial Heuristic with True Variance $\sigma^2_{SF}$ ($\mu_L = 2$, $\mu_D = 10$)




of optimal and is within 0.37% of optimal for all $r \le 98.3\%$.

The heuristic $S^N_{\widehat{SF}}$ outperformed $S^N_{LTD}$ in all but a small number of instances. Specifically, $\Delta(S^N_{LTD}) < \Delta(S^N_{\widehat{SF}})$ for only 0.95% of the parameter combinations, again, because the shortfall distribution is positively skewed. In these cases, the overestimation of the shortfall variance inherent in $S^N_{LTD}$ offset the lack of positive skewness of the normal distribution and provided a better fit for high target service levels $r$.

### 5.3. The Negative Binomial Shortfall Heuristics $S^{NB}_{SF}$ and $S^{NB}_{\widehat{SF}}$

Finally, we turn to the two negative binomial-based heuristic base-stock policies $S^{NB}_{\widehat{SF}}$ and $S^{NB}_{SF}$. Because the negative binomial distribution is positively skewed like the shortfall distribution, overestimating the variance of the shortfall no longer provides an advantage as it did with the normal distribution. Consistent with this, we observed that $S^{NB}_{SF}$ outperforms $S^{NB}_{\widehat{SF}}$ throughout. With the exception of $\mu_D = 2$, we found that $S^{NB}_{\widehat{SF}}$ is not clearly superior to its easily calculated normal equivalent $S^N_{\widehat{SF}}$; it is clearly worse when both $\mu_D$ and $\mu_L$ are either 6 or 10. Because there are so few situations where $S^{NB}_{\widehat{SF}}$ dominates both $S^{NB}_{SF}$ and $S^N_{\widehat{SF}}$, we do not consider it further. $\Delta(S^{NB}_{SF})$ increases with $\mu_D$, $\sigma_L$, and $r$, and decreases with $\mu_L$. As shown in Figure 7, the worst-case performance ($\mu_L = 2$, $\mu_D = 10$) of $S^{NB}_{SF}$ is always within 9.15% of optimal and is within 1.16% of optimal for $r \le 98.0\%$. For its best case ($\mu_L = 10$, $\mu_D = 2$), $S^{NB}_{SF}$ is optimal for 96.0% of the test bed and never exceeds the optimal cost by more than 1.10%.

### 5.4. Summary of Heuristic Performance

The overall performance of the six heuristics across the entire test bed of 145,800 parameter combinations is summarized in Table 1. It is clear from this table that when viewed in aggregate, the performances of the two heuristics based on lead-time demand, $S^N_{LTD}$ and $S^{NB}_{LTD}$, are dramatically worse than the other four heuristics. For example, they are on average at least 60% more expensive than the optimal policy, while the other heuristics average less than 0.6% above optimal. Moreover, $S^N_{LTD}$ and $S^{NB}_{LTD}$ are within 5% of the optimal cost less than 22% of the time, compared to at least 97% of the time for the other heuristics. The poor performance of the two heuristics based on the lead-time demand reinforces the previous findings of Robinson et al. (2001).

**Table 1** **Summary Cost Performance $\Delta(S)$ Across the Test Bed of 145,800 Parameter Combinations**

| | Mean | Std. dev. | 95th percentile | 99th percentile | Worst case | $\Pr\{\Delta = 0\}$ | $\Pr\{\Delta \le 1\%\}$ | $\Pr\{\Delta \le 5\%\}$ |
|---|---|---|---|---|---|---|---|---|
| $S^N_{LTD}$ | 64.02 | 60.18 | 180.06 | 237.85 | 290.11 | 9.97 | 14.38 | 20.85 |
| $S^N_{\widehat{SF}}$ | 0.32 | 1.30 | 1.42 | 5.54 | 36.62 | 61.00 | 93.27 | 98.85 |
| $S^N_{SF}$ | 0.59 | 2.29 | 2.85 | 9.73 | 58.18 | 59.16 | 87.58 | 97.44 |
| $S^{NB}_{LTD}$ | 69.14 | 86.89 | 231.71 | 403.47 | 1,089.11 | 10.02 | 14.23 | 21.49 |
| $S^{NB}_{\widehat{SF}}$ | 0.38 | 1.10 | 1.98 | 5.50 | 23.19 | 57.31 | 89.80 | 98.80 |
| $S^{NB}_{SF}$ | 0.07 | 0.29 | 0.40 | 1.41 | 9.15 | 77.43 | 98.25 | 99.98 |

*Note.* All numbers are in percent.

When cost performance, informational requirements, and computational complexity are all taken into account, two of the remaining four heuristic policies dominate the others. The first heuristic based on the normal distribution with the approximate variance of the shortfall, $S^N_{\widehat{SF}}$, is very easy to compute and requires computational effort virtually equivalent with that required for $S^N_{LTD}$. Its cost performance is an average of 0.32% more than the cost of the optimal policy and is within 5% of it 98.85% of the time. For very high target service levels $r$ and low values of the mean lead time and demand, however, its cost can exceed that of the optimal policy by up to 36.62%. The informational requirements for $S^N_{\widehat{SF}}$ are low: like $S^N_{LTD}$, it requires only the first two moments of the lead-time and demand distributions.

The second heuristic is based on the negative binomial distribution with the true variance of the shortfall $S^{NB}_{SF}$. Its cost performance is superb: it averages only 0.07% over that of the optimal policy and is within 5% virtually all (99.98%) of the time. Compared to the more easily computed normal heuristic $S^N_{\widehat{SF}}$, its worst-case performance is substantially better—9.15% versus 36.62%. Because both the variance of the number of orders outstanding and the inverse negative binomial distribution require simple iterative calculations, its implementation is somewhat more involved. Another disadvantage of $S^{NB}_{SF}$ relative to $S^N_{\widehat{SF}}$ is that it requires more information: unless information systems were reprogrammed to estimate $\sigma^2_N$ rather than $\sigma^2_L$, the complete distribution of the lead time must be known rather than only its first two moments.

## 6. Conclusions

Whenever stochastic lead times allow for the possibility of replenishment orders crossing over one another, the base-stock level should be calculated from the distribution of the shortfall rather than the distribution of the lead-time demand. Because both distributions are mixtures, their derivations depend on the complete distributions of both demand and the lead time, and are somewhat involved even under the assumption of joint independence. Thus, for expediency, a normal approximation of the lead-time demand distribution is often used in its stead. When order crossover is possible, the normal approximation of the shortfall distribution has heretofore been more difficult to use because its variance $\sigma^2_{SF}$ depends on $\sigma^2_N$, the exact computation of which requires knowledge of the entire lead-time distribution and a cumbersome summation.

The simple bounds that we develop in this paper serve two purposes: (1) they make the normal approximation of the shortfall distribution tractable and (2) they highlight the importance of setting base-stock levels with the shortfall rather than the lead-time demand distribution. Setting a base stock with a normal approximation of the shortfall distribution using the upper bound $\widehat{\sigma}^2_N$ depends on only the first two moments of the lead time, just as the popular *LTD* approximation does. The upper bound on $\sigma^2_N$ of $\sigma_L/\sqrt{3}$ indicates that the *variance* of $N$ can never exceed a limit that is proportional to the *standard deviation* of $L$. So when $\sigma^2_L$ is large, the difference between $\sigma^2_N$ and $\sigma^2_L$ can be dramatic, which demonstrates the importance of using the shortfall distribution rather than the lead-time demand distribution to set base-stock levels. Thus, using the shortfall distribution leads to less costly inventory policies than does the lead-time demand distribution.

Our numerical tests show that when order crossover is possible, distributions matching the mean $\mu_{LTD}$ and variance $\sigma^2_{LTD}$ of the lead-time demand are very poor substitutes for ones that use the variance of the shortfall. Further, the skewness of the shortfall distribution is well matched by that of the negative binomial distribution. The performance of the negative binomial base-stock level $S^{NB}_{SF}$ is generally excellent. Its drawbacks are both informational and computational: its implementation requires knowledge of the entire lead-time distribution, and iterative calculations for both the variance of the number of orders outstanding $\sigma^2_N$ as well as the inverse negative binomial function.

In situations where it is important to minimize computational effort and information requirements, the base-stock level $S^{N}_{\widehat{SF}}$ (derived from the normal distribution with the true mean of the shortfall distribution $\mu_{SF}$ and the approximate variance $\widehat{\sigma}^2_{SF}$) generally does almost as well as $S^{NB}_{SF}$. The larger variance $\widehat{\sigma}^2_{SF}$ effectively compensates for the lack of skewness of the normal distribution, except for high values of the target service level $r$.

## Acknowledgments

The authors are grateful for the suggestions of two anonymous referees, which substantially improved this paper. Of particular note, they suggested the possibility of collecting data on $N$ directly, and proposed the lower and upper bounds on $\sigma^2_N$ in (15) and (16), both of which were obvious to the authors only in hindsight.

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