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# Price Competition Under Mixed Multinomial Logit Demand Functions

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In this paper, we postulate a general class of price competition models with mixed multinomial logit demand functions under affine cost functions. In these models, the market is partitioned into a finite set of market segments. We characterize the equilibrium behavior of this class of models in the case where each product in the market is sold by a separate, independent firm. We identify a simple and very broadly satisfied condition under which a pure Nash equilibrium exists and the set of Nash equilibria coincides with the solutions of the system of first-order-condition equations, a property of essential importance to empirical studies. This condition specifies that in every market segment, each firm captures less than 50% of the *potential* customer population when pricing at a specific level that, under the condition, is an upper bound for a rational price choice for the firm irrespective of the competitors' prices. We show that under a somewhat stronger, but still broadly satisfied, version of the above condition, a *unique* equilibrium exists. We complete the picture by establishing the existence of a Nash equilibrium, indeed a unique Nash equilibrium, for markets with an *arbitrary* degree of concentration, under sufficiently tight price bounds. We discuss how our results extend to a continuum of customer types. A discussion of the multiproduct case is included. The paper concludes with a discussion of implications for structural estimation methods.

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## 1. Introduction and Summary

Our primary goal in this paper is to characterize the equilibrium behavior of price competition models with mixed multinomial logit (MMNL) demand functions under affine cost structures. In such models, the market is partitioned into a finite set or a continuous spectrum of customer segments, differentiated by, for example, demographic attributes, income level, and/or geographic location. In each market segment, the firms' sales volumes are given by a multinomial logit (MNL) model. In spite of the huge popularity of MMNL models in both the theoretical and empirical literature, it is not known, in general, whether a Nash equilibrium[1] of prices exists and whether the equilibria can be uniquely characterized as the solutions to the system of first-order-condition (FOC) equations. (This system of equations is obtained by specifying that all firms' marginal profit values equal 0.) Indeed, as the next section elaborates, there are many elementary price competition models in which either *no* or a *multiplicity* of Nash equilibria exist.

Characterization of the equilibrium behavior in price competition models with MMNL demand functions has remained a formidable challenge because the firms' profit functions fail, in general, to have any of the standard structural properties under which the existence of an equilibrium can be established. For example, the profit functions fail to be quasi-concave. (When firms offer multiple products, this quasi-concavity property is absent, even in a *pure* rather than a *mixed* MNL model; this was shown by Hanson and Martin 1996, with a counterexample in a three-product monopoly model.)

Consider, for example, the seminal paper by Berry et al. (1995), which studies market shares in the U.S. automobile industry that introduced, at least in the empirical industrial organization literature, a new estimation methodology to circumvent the problem that prices, as explanatory variables of sales volumes, are typically endogenously determined. The paper postulates an MNL model with random coefficients

[1] Henceforth, "equilibrium" will refer to pure strategy equilibrium unless otherwise stated.

for the industry. One of the empirical methods developed in the paper is based on estimating the model parameters as those under which the observed price vector satisfies the FOC equations. Berry et al. (1995, Footnote 12) acknowledge that it is unclear whether their model possesses an equilibrium, let alone a unique equilibrium. Even if these questions can be answered in the affirmative, so that the observed price vector can be viewed as the unique price equilibrium, it is unclear whether it is necessarily identified by the FOC equations on which the estimation method relies.[2]

In a more recent example, Thomadsen (2005a) points out that in many empirical studies the distance between the consumer and each of the competing product outlets or service providers is naturally and essentially added to the specification of the utility value. (Examples following this practice include Manuszak 2000, Dubé et al. 2002, Bradlow et al. 2005, Thomadsen 2005b, Davis 2006, and Allon et al. 2011.) Distance attributes depend jointly on the firm and the consumer. Such geography-dependent utility functions can be cast as special cases of the general model in Caplin and Nalebuff (1991), the most frequently employed foundation for the existence of an equilibrium. However, Thomadsen (2005a) points out that the conditions in Caplin and Nalebuff (1991) that guarantee the existence of an equilibrium do not apply to such specifications, except for very restrictive geographical distributions of the (potential) consumer base. Similar difficulties in the application of the Caplin–Nalebuff existence conditions arise when the utility functions involve other attributes that depend jointly on the firm/customer type combination, for example, brand loyalty characteristics as in Dubé et al. (2009, 2011); see §3.

We identify a simple and very broadly satisfied condition under which a Nash equilibrium exists and the set of Nash equilibria coincides with the solutions of the system of FOC equations, a property of essential importance to empirical studies. Our existence condition merely requires that any single product captures less than a majority of the *potential* customer population in any of the market segments; moreover, this market share restriction only needs to hold when the product is priced at a level that under the condition is shown to be an *upper bound* for a rational price choice, irrespective of the prices charged for the competing products. To guarantee *uniqueness* of the Nash equilibrium, a second condition is needed, restricting any given product's market share to a third of the potential market. No restrictions whatsoever are required with respect to the distribution of population sizes across the different market segments.

We develop our theory, first, under the assumption that the market is partitioned into a finite set of segments, such that in each segment market shares are determined on the basis of a pure MNL model. Many empirical studies follow this approach, segmenting the market geographically and/or on the basis of demographic attributes (e.g., gender, race, age, and income bracket). Other empirical models consider a *continuum* of customer types, by treating some of the parameters in the consumer choice model as continuous random variables. Although, in §6.2, we show that all of our results carry over to such settings, it is more difficult to verify our existence and uniqueness conditions because the market share restriction has to apply for each of the market segments or customer types. Indeed, the condition may sometimes fail to hold when the modeler assumes that some of the coefficients in the utility functions have distributions with infinite support, thus allowing for rare customer types with arbitrary relative weights for different attributes. See §4 for a more complete discussion.

Our results differ from those in the seminal paper by Caplin and Nalebuff (1991) in three ways: (i) the model specification, (ii) the conditions guaranteeing existence of a Nash equilibrium, and (iii) the analytical approach.

In terms of the model specification, our class of MMNL models generalizes that of Caplin and Nalebuff (1991), itself a generalization of many existing models in the industrial organization literature. In particular, along with a similar utility measure for the outside option, our MMNL model is based on postulating a utility function for each product and market segment that consists of three parts: the first component is an arbitrary function of the product's nonprice attributes and the non-income-related customer characteristics in the given market segment. The second term captures the impact of the customer's income level and the third term captures the impact of the product's price. The income sensitivity functions are fully general, whereas the price sensitivity functions are concave and decreasing. The fourth and final term denotes a random utility component with an extreme value distribution as in standard multinomial logit models. Our structure generalizes

[2] Berry et al. (1995, Footnote 12, p. 853) write, "We assume that a Nash equilibrium to this pricing game exists, and that the equilibrium prices are in the interior of the firms' strategy sets (the positive orthant). While Caplin and Nalebuff (1991) provide a set of conditions for the existence of equilibrium for related models of single-product firms, their theorems do not easily generalize to the multiproduct case. However, we are able to check numerically whether our final estimates are consistent with the existence of an equilibrium. Note that none of the properties of the estimates require uniqueness of equilibrium, although without uniqueness it is not clear how to use our estimates to examine the effects of policy and environmental changes." We explain several reasons why the conditions in Caplin and Nalebuff (1991) fail to apply to the Berry et al. (1995) model, beyond its multiproduct feature.

that in Caplin and Nalebuff (1991) in two ways: First, Caplin and Nalebuff specify the first term in the utility functions as a weighted average of the non-price-related product attributes, with each customer type or market segment characterized by a unique vector of weights. Our model specification allows for an arbitrary structural dependence on the customer types. Second, their price–income sensitivity function is specified as a concave function of the difference of the customer's income and the product's price, as opposed to our general function of income and price.

In terms of the existence conditions, beyond operating within a narrower class of consumer choice models, Caplin and Nalebuff (1991) require that the distribution of population sizes across the different customer types satisfies specific ($\rho$-concavity) properties that, as mentioned above, are violated in many applications. We impose *no* restrictions on this distribution. We do require the above market share condition, whereas Caplin and Nalebuff (1991) do not.

Finally, in terms of the analysis approach, Caplin and Nalebuff (1991) identify a set of conditions under which each firm's profit function is quasi-concave in its price variable over the complete price space; this represents the standard approach in equilibrium analyses of competition models, establishing desired structural properties on the full strategy space. Our existence condition does not imply that the firms' profit functions have any of these structural properties on the *full* price space. Instead, our approach is to (i) identify a compact region in the feasible price space on which the profit functions are quasi-concave in the firm's own price(s) or one in which they possess the so-called single-point-crossing property, discussed in §4; this guarantees the existence of an equilibrium in the restricted price region. We then establish that (ii) the equilibria identified with respect to the restricted region continue to be equilibria in the full price region and (iii) that no equilibria exist outside the identified restricted price region.

Although the above results characterize the equilibrium behavior for all but heavily concentrated markets we complete the picture, giving a condition for the existence of a Nash equilibrium, indeed a *unique* Nash equilibrium, for markets with an *arbitrary* degree of concentration: The condition specifies that the maximum feasible price vector falls below a given upper bound. In other words, to guarantee that a market with an arbitrary degree of concentration has a (unique) Nash equilibrium, sufficiently tight exogenous price limits must prevail, whereas no such limits are needed when (one of) the above market concentration condition(s) applies. Another important distinction is that under the price limit condition, the equilibrium may reside at the boundary of the feasible price region and therefore fails to satisfy the FOC equations. A counterexample shows that if neither a very high level of market concentration can be excluded nor the feasible price region sufficiently confined, there may be *no* Nash equilibrium for the price competition model. We also discuss the implications of these results for econometricians both in settings where a specific price vector is *observed* and *assumed* to be the (or an) equilibrium and those where neither the model parameters nor a price equilibrium is observed.

As in Caplin and Nalebuff (1991), we assume that each firm sells a single product. Settings where the firms offer an arbitrary number of products are discussed in §10 of the online appendix. The remainder of this paper is organized as follows. Section 2 provides a review of the relevant literature. Section 3 introduces the consumer choice model. Section 4 presents our equilibrium existence and uniqueness results. Section 5 develops the example showing that a Nash equilibrium may fail to exist in the absence of any conditions precluding highly concentrated markets or, alternatively, enforcing sufficiently tight price limits. Section 6 discusses extensions of our base model that allow for a continuous specification of customer types. Section 7 describes the implications of these results for the econometrician attempting to estimate the model parameters.

## 2. Literature Review

There has been a plethora of price competition models for industries with differentiated products or services, beginning with the seminal paper by Bertrand (1883). One important class of such competition models employs demand functions based on an MNL discrete choice model. This model was proposed by McFadden (1976), a contribution later awarded with the 2000 Nobel Price in Economics. As explained in the introduction, the model may be derived from an underlying random utility model (see (1) in §3), with homogeneous coefficients, i.e., the special case where the customer population does not need to be segmented. Luce and Suppes (1965) attribute this derivation to an unpublished manuscript by Holman and Marley. The MNL model has been widely used in the economics, marketing, and operations management literature, among many other fields (see, e.g., Ben-Akiva and Lerman 1993, Anderson et al. 2001, Talluri and Van Ryzin 2005). The MNL model satisfies the so-called independence of irrelevant alternatives (IIA) axiom, according to which the *ratio* of any pair of firms' market shares is independent of the set of other alternatives that are offered to the consumers. This axiom was first postulated by Luce (1959), but Debreu (1960) pointed out that the IIA property is highly restrictive, as illustrated by his famous red bus–blue bus example: the relative market share of an alternative is, in general, significantly affected if a close substitute to this alternative is added to the choice set.

To remedy this problem, Ben-Akiva (1973) introduced the so-called nested logit model, where the choice process is modeled as a two-stage nested process: the consumer first selects among broad classes of alternatives (e.g., air versus ground transportation) and subsequently a specific variant among the selected class of alternatives (e.g., a specific flight). This approach still ignores systematic differences in the way different customer segments trade off relevant attributes of the various products or services. To address the issue of systematic customer heterogeneity, the MMNL model was introduced, apparently first by Boyd and Mellman (1980) and Cardell and Dunbar (1980); earlier papers in the 1970s (e.g., Westin 1974) had derived a similar model by treating, in a single-segment model, the attribute vector as random with a given distribution. The properties of the MMNL model have been extensively studied in the economics and marketing literature (see, e.g., Train et al. 1987, Steckel and Vanhonacker 1988, Gonul and Srinivasan 1993, Berry 1994, Jain et al. 1994). More recently, McFadden and Train (2000) showed that, under mild conditions, any discrete choice model derived from random utility maximization generates choice probabilities that can be approximated, arbitrarily closely, by an MMNL model. Moreover, these authors showed that MMNL models enjoy numerical and estimation advantages beyond other discrete choice models. (It would be of considerable interest to extend our results to the general class of choice models considered by McFadden and Train 2000.)

Whether or not a Nash equilibrium exists in a Bertrand price competition model depends fundamentally on the structure of the demand functions as well as the cost structure. The same applies to the uniqueness of the equilibrium. Milgrom and Roberts (1990) and Topkis (1998) identified broad classes of demand functions under which the resulting price competition model is supermodular, a property guaranteeing the existence of a Nash equilibrium.

More specifically, for the pure MNL model with a cost structure that is affine in the sales volume, Anderson et al. (2001) established the existence of a (unique) Nash equilibrium in the special case where all firms are symmetric, i.e., have identical characteristics. Bernstein and Federgruen (2004) extended this result for the case of general asymmetric firms and a generalization of MNL models referred to as attraction models. For the same model, Gallego et al. (2006) provide sufficient conditions for the existence of a unique equilibrium, under cost structures that depend on the firm's sales volume according to an increasing convex function. Konovalov and Sándor (2009) recently showed that the existence of a unique equilibrium can be guaranteed in the multiproduct generalization of a pure MNL-price competition model.

Seemingly minor variants of the pure MNL model may result in a fundamentally different equilibrium behavior of the associated price competition model. For example, Cachon and Harker (2003) report that under a simple piecewise linear transformation of the MNL demand functions, and a cost function that is proportional to the square root of the sales volume, the model may have no, one, or multiple equilibria as a single parameter is varied. (This is demonstrated with an example involving two symmetric firms.) Similar erratic behavior was demonstrated by Chen and Wan (2003) for what is, arguably, the seminal price competition model for service competition, presented in Luski (1976) and Levhari and Luski (1978).

For price competition models with nested logit demand functions, Liu (2006) recently established the existence of a unique Nash equilibrium. As mentioned in the introduction, the seminal paper by Caplin and Nalebuff (1991) established sufficient conditions for the existence of a price equilibrium when the demand functions are based on a broad class of MMNL models. Caplin and Nalebuff (1991) show that, under these conditions, a unique price equilibrium exists in the case of a duopoly *or* when products are characterized by their price and a single, one-dimensional attribute, whereas the density of the customer type distribution is log-concave (see Dierker 1991 for an alternative treatment). As mentioned by many authors, for example, Berry et al. (1995) and Thomadsen (2005a), these sufficient conditions are often not satisfied in many industry-based models.

The papers by Peitz (2000, 2002) show that a price equilibrium exists in certain variants of the Caplin and Nalebuff (1991) model, allowing for settings where customers maximize their utility functions subject to a budget constraint or when they may purchase an arbitrary amount of each of the products in the market, as opposed to a single unit. Unfortunately, the utility functions in Peitz (2000, 2002) do not depend on the product prices, so the firms' incentive to mitigate price levels arises purely from the customers' budget constraints. Mizuno (2003) established the existence of a unique price equilibrium for certain classes of models (e.g., logit, nested logit) in which the demand functions *are* log-supermodular. As we show at the end of §4, this property fails to apply in general MMNL models.

As explained in the introduction, our model assumptions generalize those made by Caplin and Nalebuff (1991). Our paper also builds on results in Thomadsen (2005a) that provide a sufficient condition for the existence of a price equilibrium—but not its uniqueness—when the demand functions arise from a general MMNL model; his condition relates the firm's variable cost rate to the value of the nonprice related variables in the utility measures (see (1) in §3). It is difficult to assess how widely applicable the condition is.

## 3. The Price Competition Model

Consider an industry with $J$ competing single-product firms, each selling a specific good or service. The firms differentiate themselves via an arbitrary series of observable product characteristics as well as their price. Each firm faces a cost structure that is affine in the expected sales volume. Customers are assumed to purchase only one unit and can be segmented into $K$ distinct groups, each with a known population size. (In §6, we discuss models with a continuum of customer types or market segments. All of the results obtained in §4, for the case of a finite set of market segments, continue to apply there.) If the potential buyers in the model represent consumers, the different segments may, for example, represent different geographical areas, in combination with socioeconomic attributes such as age, gender, race, income level, number of years of formal education, occupational and marital status, etc. In the case of business-to-business (B2B) markets, the different segments may again represent different geographical regions; industry subsectors (government agencies, educational institutions, for-profit companies); and firm size levels.[3] When modeling, for example, an industry of automobile part suppliers, each automobile manufacturer may represent a segment by itself. The chosen segmentation should reflect the various observable factors that may impact how different product attributes are traded off by the potential buyers. We use the following notation for all firms $j = 1, \ldots, J$ and customer segments $k = 1, \ldots, K$:

$x_j$ = an $L$-dimensional vector of observable *nonprice* attributes for firm $j$;
$c_j$ = the variable cost rate for firm $j$;
$p_j$ = the price selected by firm $j$; $p_j \in [p_j^{\min}, p_j^{\max}]$ with $0 \le p_j^{\min} \le c_j \le p_j^{\max}$;
$h_k$ = the population size of customer segment $k$;
$S_{jk}$ = expected sales volume for firm $j$ among customers in segment $k$;
$S_j$ = expected aggregate sales volume for firm $j$ across all customer segments;
$\pi_{jk}$ = expected profit for firm $j$ derived from sales to customers in segment $k$;
$\pi_j$ = expected aggregate profits for firm $j$.

We thus assume that each firm selects its price from a given closed interval of feasible prices. To our knowledge, compact feasible price ranges are required for any of the known approaches to establish the existence of a Nash equilibrium.[4] At the same time, the restriction is without loss of essential generality. Consider first $p_j^{\min}$. In the absence of other considerations, we may set $p_j^{\min} = 0$.[5] As for $p_j^{\max}$, price limits may result from a variety of sources, for example, government regulation, maximum price levels specified by suppliers or franchisers, limits set by industry organizations, or branding considerations. In other settings, where no such exogenous price limits prevail, one can always select unrestrictive upper bounds for $p_j^{\max}$ that are well above reasonable price choices. (For example, no fast food meal will be priced beyond \$100, say, and no subcompact car beyond the \$40,000 level.) Moreover, we will show that under a widely applicable condition and $p^{\max}$ sufficiently large, the choice of $p^{\max}$ has no impact on the price equilibrium.

Market shares within each customer segment may be derived from a standard random utility model as follows. First, let

$$u_{ijk} = U_{jk}(x_j) + F_{jk}(Y_i) + G_j(p_j) + \epsilon_{ijk}, \quad j = 1, \ldots, J; \\ k = 1, \ldots, K; \text{ and } i = 1, 2, \ldots \tag{1}$$

denote the utility attributed to product $j$ by the $i$th customer in segment $k$, with income or firm size $Y_i$. Similarly, the utility associated with the no-purchase option is given by

$$u_{i0k} = U_{0k}(x_1, \ldots, x_J) + F_{0k}(Y_i) + \epsilon_{i0k}, \\ k = 1, \ldots, K; \ i = 1, 2, \ldots. \tag{2}$$

Recall that $x_j$ is a vector of *observable* product attributes. Conversely, $\epsilon_{ijk}$ denotes a random *unobserved* component of customer utility. The functions $\{U_{jk}, j = 0, \ldots, J\}$ are completely general as are the *income sensitivity functions* $F_{jk}(\cdot)$. Because we exclude Veblen goods, $G_j(p_j)$ is decreasing in the price level $p_j$. Throughout the paper we use the terms "increasing"

[3] Firm size may, for example, be defined as the firm's annual revenues or its capital value.

[4] Caplin and Nalebuff (1991), for example, assume that prices are selected from a closed interval $[p^{\min}, p^{\max}]$ with $p_j^{\min} = c_j$ and $p_j^{\max} = Y$, the consumer's income level. We make no up-front specification for these limits, allowing $0 \le p_j^{\min} < c_j$ and $p_j^{\max} \ne Y$. Indeed, for certain durable or investment goods and certain income levels, $p_j$ may be in excess of $Y$.

[5] We assume $p^{\min} < c$ to ensure, under our existence conditions, that any Nash equilibrium $p^* > p^{\min}$; see Lemma 4.1 in §4.

and "decreasing" to mean "nondecreasing" and "nonincreasing" respectively. The $G_j$ functions in (1) are twice differentiable and concave; i.e.,

$$g_j(p_j) \equiv |G'_j(p_j)| > 0 \quad \text{and} \quad g_j(p_j) \text{ increasing in } p_j, \quad j = 1, \ldots, J, \tag{3}$$

where $g_j(p_j)$ denotes the (absolute value of the) marginal change in the utility value of product $j$ due to a marginal change in its price. (Below, we discuss an alternative interpretation of the $g_j$ functions.) Many model specifications in the literature employ a $G(\cdot)$ function common to all products $j = 1, \ldots, J$; see, for example, the various models listed in §3 of Caplin and Nalebuff (1991) or the recent literature on willingness-to-pay distributions (e.g., Meijer and Rouwendal 2006, Sonnier et al. 2007). However, in some applications, even the marginal utility shift due to a price increase may differ among the different competing products.[6]

We also assume, without loss of practical generality, that

$$\lim_{p_j \uparrow \infty} G_j(p_j) = -\infty. \tag{4}$$

In other words, a product's market share, in any market segment, falls below any given threshold if the price is sufficiently high.

The separable dependence of the utility functions on income level and price is, of course, less than fully general.[7] Some MMNL models or MNL models with random coefficients have, for example, modeled the joint dependence on income and price by adding a term $\Gamma(Y_i - p_j)$ to the utility functions, with $\Gamma(\cdot)$ an increasing, concave function; see, for example, the general model in Caplin and Nalebuff (1991), discussed in more detail below, as well as Berry et al. (1995). Caplin and Nalebuff (1991) showed that many of the earlier consumer choice models arise as a special case of their model, including the classical models by Hotelling (1929) and Lancaster (1966), Perloff and Salop (1985), Jaskold and Thisse (1979), Shaked and Sutton (1982), Economides (1989), Christensen et al. (1975), and Anderson et al. (2001). All of these models specify $\Gamma(Y_i - p_j) = \beta(Y_i - p_j)$ or $\Gamma(Y_i - p_j) = \beta \log(Y_i - p_j)$. This includes the consumer choice model in the later, seminal paper by Berry et al. (1995).

[6] Caplin and Nalebuff (1991) already recognized the value of allowing for product-dependent price–income sensitivity functions. As explained below, see (14), they confine themselves to the case when these functions differ by a proportionality constant only, thus assuming that for any pair of products, the ratio of the marginal utility changes due to a \$1 price increase remains constant, irrespective of the products' price levels.

[7] One would like to generalize our results to settings where the utility functions' dependence on the income level and price are given by general functions $G_{jk}(Y_k, p_j)$, $j = 1, \ldots, J$ and $k = 1, \ldots, K$.

However, the equilibrium results in Caplin and Nalebuff (1991) are derived either assuming that (i) all consumers have an equal income level $Y$ or (ii) that the function $\Gamma(\cdot)$ is affine. Under either assumption, the Caplin–Nalebuff framework can be represented as a special case of the specification in (1), with either (i) $F_{jk}(\cdot) \equiv 0$ or (ii) $F_{jk}(\cdot)$and $G_j(\cdot)$ linear functions.[8]

To complete the specification of utility functions (1) and (2), $\{\epsilon_{ijk}, j = 0, 1, \ldots, N\}$ is an independent and identically distributed (i.i.d.) sequence of random variables, for all $i = 1, 2, \ldots$ and $k = 1, \ldots, K$. We further assume that the random components $\epsilon_{ijk}$ follow a type 1 extreme value or Gumbel distribution:

$$\Pr[\epsilon_{ijk} \le z] = \exp\left[-\exp\left(-\left(\frac{x}{\delta} + \gamma\right)\right)\right], \quad j = 0, \ldots, J;\ k = 1, \ldots, K;\ i = 1, 2, \ldots, \tag{5}$$

where $\gamma$ is Euler's constant (0.5772) and $\delta$ is a scale parameter. The mean and variance of the random terms $\{\epsilon_{ijk}\}$ are $E[\epsilon_{ijk}] = 0$ and $\mathrm{var}[\epsilon_{ijk}] = \delta^2\pi^2/6$, respectively. Without loss of generality, we scale, for each customer segment $k = 1, \ldots, K$, the units in which the utility values are measured such that $\delta = 1$. This random utility model results in the well-known MNL model for demand for product $j$ among customers of segment $k$:

$$S_{jk} = h_k \frac{e^{U_{jk}(x_j) + F_{jk}(Y_k) + G_j(p_j)}}{e^{U_{0k}(x_1, \ldots, x_J) + F_{0k}(Y_k)} + \sum_{m=1}^{J} e^{U_{mk}(x_m) + F_{mk}(Y_k) + G_m(p_m)}}; \quad j = 1, \ldots, J;\ k = 1, \ldots, K. \tag{6}$$

Aggregating the sales volumes over all segments, we get the following expected sales functions:

$$S_j = \sum_{k=1}^{K} S_{jk} = \sum_{k=1}^{K} h_k \frac{e^{[U_{jk}(x_j) + F_{jk}(Y_k) + G_j(p_j)]}}{e^{U_{0k}(x_1, \ldots, x_J) + F_{0k}(Y_k)} + \sum_{m=1}^{J} e^{U_{mk}(x_m) + F_{mk}(Y_k) + G_m(p_m)}}, \quad j = 1, \ldots, J. \tag{7}$$

An alternative foundation for the sales volume formula (6) is to assume that among potential customers in segment $k$, each firm $j$ and the no-purchase option have a so-called attraction value given by

$$a_{jk} = e^{U_{jk}(x_j) + F_{jk}(Y_k) + G_j(p_j)}, \quad j = 1, \ldots, J,\ k = 1, \ldots, K, \tag{8}$$

$$a_{0k} = e^{U_{jk}(x_j) + F_{0k}(Y_k)}, \quad k = 1, \ldots, K. \tag{9}$$

[8] Berry et al. (1995) appear, in the presence of income heterogeneity, to allow for a price–income sensitivity function that is nonseparable; i.e., $g_j(Y_i, p_j) = \alpha \log(Y_i - p_j)$ (see Equation (2-7a) in Berry et al. 1995). As mentioned in the introduction, Footnote 12 of Berry et al. (1995) suggests that only the multiproduct feature of their model precludes reliance on Caplin and Nalebuff (1991). In actuality, the choice of a nonseparable price–income sensitivity function provides a second reason why the existence results in Caplin and Nalebuff (1991) do not apply to their model.

Under the four intuitive axioms specified in Bell et al. (1975), this uniquely gives rise to the demand volumes specified in (6).

The above consumer choice model thus distinguishes between two types of customer heterogeneity: (i) heterogeneity that is attributable to observable customer attributes such as their geographical location or socioeconomic profile and (ii) intrinsic heterogeneity not explained by any systematic or observable customer attributes. This model specification covers most random utility models in the literature. As an example, consider the following general specification used in Berry (1994):

$$u_{ij} = x_j\beta_i + \xi_j - \alpha p_j + \epsilon_{ij}, \quad j = 1, \ldots, J;\ i = 1, 2, \ldots, \tag{10}$$

$$\beta_{il} = \beta_l + \sigma_l \zeta_{il}, \quad l = 1, \ldots, L \text{ and } i = 1, 2, \ldots. \tag{11}$$

Here, $\{\epsilon_{ij}\}$ is again a sequence of unobservable random noise terms that is i.i.d., and $\beta_i$ is a customer-specific $L$-vector of parameters, specified by (11). The vector $[\alpha, \beta, \sigma]$ is a $2L+1$ dimensional string of parameters. Finally, the sequences $\{\zeta_{il}\}$ are random sequences with zero mean, which may or may not be observable, where $\xi_j$ is used to represent an unobservable utility component that reflects attributes of firm $j$ unobserved by the modeler but with common value among the customers.

To verify that the general structure in Berry (1994) can be treated as a special case of (1)–(5), assume the $\{\zeta_{il}\}$ distributions are discrete and segment the customer population such that all customers in any segment $k$ share the same $\zeta_{il}$ value for each of the $L$ observable product attributes; i.e., $\zeta_{il} = \hat{\zeta}_{kl}$ for all customers $i$ in segment $k$. Specifying $U_{jk}(x_j) = \xi_j + \sum_{l=1}^{L} x_{jl}[\beta_l + \theta\hat{\zeta}_{kl}]$ and $G_j(Y_i, p_j) = -\alpha p_j$, we note that the general Berry model arises as a special linear specification of our structure. A restriction inherent in the Berry model is the assumption that $\alpha$, the marginal disutility for firm $j$'s product due to a marginal price increase, is uniform across all products and all price and income levels. In many practical applications, price sensitivity may vary significantly along any one of these dimensions.

Other MMNL consumer choice models employ one or more measurable attributes that depend on the specific firm and customer segment combination. For example, if the customer segmentation is in part based on the customer's geographic location, a measure $d_{jk}$ for the distance between customer segment $k$ and firm $j$ may be added to the specification in (1) as follows:

$$U_{jk}(x_j) = \Delta_j(x_j, d_{jk}) + \xi_{jk}, \tag{12}$$

with $\xi_{jk}$, again, an unobservable component in firm $j$'s utility measure that is common among all customers of segment $k$.

In other applications, the distance measure $d_{jk}$ refers to a measure of a priori affinity. If, for example, on the basis of nationalistic sentiments, customers have a propensity to buy from a domestic provider, this may be modeled by basing the segmentation in part on the consumer's nationality and defining the distance $d_{jk} = 0$ if segment $k$ represents the same nationality as firm $j$ and $d_{jk} > 0$ otherwise. Alternatively, the a priori affinity may be based on past purchasing behavior. Both the economics and the marketing literature have addressed that customers tend to be inert or firm/brand loyal; i.e., because of explicit or psychological switching costs, customers tend to stay with their current provider or brand, even if they would otherwise be more attracted by a competitor. Dubé et al. (2009), for example, model this as an MMNL model, segmenting customers, in part, on the basis of the firm most recently patronized; a distance measure $d_{jk}$ is added to the utility measure where $d_{jk} = 0$ if customers of segment $k$ used to buy from firm $j$ and $d_{jk} = 1$ otherwise.

Another general model was introduced in the seminal paper by Caplin and Nalebuff (1991) with the specific objective of establishing the existence of a price equilibrium for a broad class of consumer choice models. This general model assumes that each potential customer $i$ is characterized by a weight vector $\alpha_i \in \Re^L$ as well as an income level $Y_i$, such that

$$u_{ij} = \sum_{l=1}^{L} \alpha_{il} H_l(x_{jl}) + \beta_j \Gamma(Y_i - p_j), \quad j = 1, \ldots, J \text{ and } i = 1, \ldots, N, \tag{13}$$

for given functions $\Gamma(\cdot)$ and $H_l(\cdot)$, with $\Gamma(\cdot)$ concave and increasing, and for given constants $\beta_j > 0$, $j = 1, \ldots, J$.[9] In other words, the Caplin–Nalebuff model assumes that customers characterize each product $j$ in terms of a transformed attribute vector $x'_j$, the $l$th component of which is given by $x'_{jl} \equiv H_l(x_{jl})$, $j = 1, \ldots, J$ and $l = 1, \ldots, L$. Customers then aggregate the (transformed) attribute values via a linear aggregate measure, with different customers applying a different weight vector $\alpha$ to the attribute values. Assuming the distribution of $\alpha$ is discrete,[10] we obtain

[9] Caplin and Nalebuff (1991) consider, in addition, a generalization of (13) in which the $L$-dimensional vector of product attributes $x$ is first transformed into a $L'$-dimensional vector of utility benefits $t(x)$. Instead of (13), the utility value of firm $j$ for customer $i$ is then specified as $u_{ij} = \sum_{l=1}^{L'} \alpha_{il} t_l(x_j) + \beta_j g(Y_i - p_j)$. This specification can also be shown to be a special case of our model. The authors state, however, that in most applications, preferences take the simpler form of (13).

[10] Caplin and Nalebuff (1991) allow for continuous distributions of $\alpha$ as well.

the Caplin–Nalebuff structure as a special case of our random utility model (1)–(5) as follows: segment the customer population into segments such that all customers in a segment share the same $\alpha$ values. In other words, for all customers in segment $k$, $\alpha_{il} = \alpha_l^{(k)}$. The Caplin–Nalebuff model (13) thus arises as a special case of our model with

$$U_{kj}(x_j) = \sum_{l=1}^{L} \alpha_l^{(k)} H_l(x_{jl}), \quad \forall\, j=1,\ldots,J \text{ and } k=1,\ldots,K; \tag{14}$$

the scale parameter $\delta$ of the $\{\epsilon_{jk}\}$ variables are chosen such that $\delta = 0$.[11] Alternatively, the $L$-dimensional attribute vector $x_j$ may be partitioned into a part that is observable and one that is unobservable by the econometrician: $x = [x', x'']$ with $x'$ an $L'$-dimensional vector of observable attributes and $x''$ a $J$-dimensional vector of product indicator variables; i.e., $x_{j,L'+j} = 1$ and $x_{j,L'+m} = 0$ $\forall\, m \neq j$. If the weights $\{\alpha_l$: $l = L'+1,\ldots,L'+J\}$ follow independent Gumbel distributions, denoting (unobserved) utility components, and each point $(\alpha_1,\ldots,\alpha_{L'})$ represents one of finitely many market segments, we retrieve a (specific type of) MMNL model where the mixture is over the given distribution of $(\alpha_1,\ldots,\alpha_{L'})$ only. As mentioned above, the price and income sensitivity functions $F_j(\cdot)$ and $G_j(\cdot)$ can be specified to fit the second term in (13) in both of the two cases considered by Caplin and Nalebuff (1991): (i) a constant income level $Y_i$ and (ii) a general income distribution but linear $\Gamma(\cdot)$ function.

To obtain the existence of a Nash equilibrium in this price competition model, the authors assume, further, that the probability density function $f(\alpha)$ of the consumer attribute vector $\alpha$ is $\rho$-concave for a specific value of $\rho$, i.e., for any pair of points $\alpha^{(0)}$ and $\alpha^{(1)}$ in the convex support of the distributions and any scalar $0 < \lambda < 1$:

$$f(\lambda\alpha^{(0)} + (1-\lambda)\alpha^{(1)}) \geq [\lambda f(\alpha^{(0)})^{\rho} + (1-\lambda) f(\alpha^{(1)})^{\rho}]^{1/\rho} \quad \text{and} \quad \rho = -1/(L+1). \tag{15}$$

Thomadsen (2005b) shows that geographic distance measures can be incorporated in this specification by appending an indicator vector for each of the $J$ firm locations. However, Thomadsen (2005a) also shows that the requirement of a $\rho$-concave probability density function for the customer attribute vector $\alpha$ precludes all but the most restrictive geographic customer distributions. In addition, under the Caplin–Nalebuff model, the price and income sensitivity functions for the different products $j = 1,\ldots,J$ differ from each other only in the proportionality constant $\beta_j$. Moreover, the customer's income and the product's price impact the product's utility value only via their difference. This represents a significant restriction, in particular, when dealing with items or services, the unit price of which constitutes a negligible fraction of a typical customer's income.

[11] Caplin and Nalebuff (1991) represent the proportionality constant $\beta_j$ as the $(n+1)$st utility benefit measure associated with the product; i.e., $\beta_j = t_{n+1}(x_j)$.

We conclude this section with a few preliminary results related to our model. It is easily verified that, in each market segment, the price sensitivity of each firm's demand with respect to its own price is given by

$$\frac{\partial S_{jk}}{\partial p_j} = -g_j(p_j) S_{jk}\left(1 - \frac{S_{jk}}{h_k}\right), \quad j=1,\ldots,J; \quad k=1,\ldots,K, \tag{16}$$

so that

$$g_j(p_j) = -\left(\frac{1}{S_{jk}}\frac{\partial S_{jk}}{\partial p_j}\right) \Big/ \left(1 - \frac{S_{jk}}{h_k}\right) = \frac{-\partial \log S_{jk}}{\partial p_j} \Big/ \left(1 - \frac{S_{jk}}{h_k}\right), \quad j=1,\ldots,J;\ k=1,\ldots,K. \tag{17}$$

In other words, $g_j(p_j)$ may be interpreted as the percentage increase in firm $j$'s market share, due to a unit price decrease, expressed as a fraction of the percentage of market segment $k$ not yet captured by the firm. We, therefore, refer to $g_j(\cdot,\cdot)$ as the *price penetration rate*. Similarly, the price sensitivity of firm $j$'s demand with respect to the competitor's price is given by

$$\frac{\partial S_{jk}}{\partial p_m} = g_m(p_m) S_{mk} S_{jk}/h_k, \quad m \neq j. \tag{18}$$

We assume, without loss of essential generality, that for all market segments $k = 1,\ldots,K$:

$$\left|\frac{\partial S_{jk}}{\partial p_j}\right| \geq \sum_{m\neq j} \frac{\partial S_{jk}}{\partial p_m}, \quad j=1,\ldots,J. \tag{19}$$

This condition is a classical dominant-diagonal condition (see, e.g., Vives 2001) and merely precludes that a uniform price increase by all $J$ firms would result in an increase of any of the firms' expected sales volume.

## 4. The Equilibrium Behavior in the Price Competition Model

In this section, we provide a sufficient condition under which the price competition model permits a Nash equilibrium and a second, somewhat stronger, condition under which this equilibrium is unique. These conditions merely preclude a very high degree of market concentration and are easily verified on the basis of the model primitives only. We conclude the section with a sufficient condition for a (unique) Nash equilibrium that applies to markets with an arbitrary degree of market concentration. Unlike, for example, the existence conditions in Caplin and Nalebuff (1991), our conditions allow for arbitrary distributions of the population sizes $\{h_k: k = 1, \ldots, K\}$ in the various customer segments.

Recall that for any of the $K$ market segments, $g_j(p_j)$ may be interpreted as the percentage increase in firm $j$'s market share—expressed as a function of the percentage of the market segment not yet captured by the firm—due to a unit decrease in the firm's prices. Similarly, let

$$\omega_j(p_j) = (p_j - c_j)g_j(p_j) \tag{20}$$

denote a dimensionless elasticity, i.e., for any of the $K$ market segments, the percentage increase in firm $j$'s market share—expressed as a function of the percentage of the market not yet captured by the firm—due to a one percent decrease in the variable profit margin. As the product of two continuous functions $\omega_j(p_j)$ is continuous, with $\omega_j(c_j) = 0$ and $\lim_{p_j \uparrow \infty} \omega_j(p_j) = \infty$. By the intermediate value theorem, we conclude that for any critical elasticity level $\eta > 0$ there exists a price level $\bar{p}_j(\eta) > c_j$, with $\omega_j(\bar{p}_j(\eta)) = \eta$. Moreover, $\omega_j$ is strictly increasing as the product of an increasing and a strictly increasing function, implying the existence of a unique price level $\bar{p}_j(\eta)$ such that for all $(p_j^1, p_j^2)$ with $p_j^1 \le \bar{p}_j(\eta) \le p_j^2$,

$$\begin{aligned}\omega_j(p_j^1) \le \omega_j(\bar{p}_j(\eta)) &= (\bar{p}_j(\eta) - c_j)g_j(\bar{p}_j(\eta)) \\ &= \eta \le \omega_j(p_j^2).\end{aligned} \tag{21}$$

Moreover, because $\omega_j(\cdot)$ is strictly increasing, the larger one chooses $\eta$, the desired elasticity value $\omega_j(\cdot)$, the larger the uniquely corresponding price level $\bar{p}_j(\eta)$:

$$\text{For all } \eta^1 < \eta^2\text{: } \bar{p}_j(\eta^1) < \bar{p}_j(\eta^2), \quad j = 1, \ldots, J. \tag{22}$$

Our main condition for the existence of a Nash equilibrium in the interior of the price region, or even a unique such equilibrium, consists of excluding the possibility of excessive market concentration. In particular, existence of a Nash equilibrium can be guaranteed if any single firm captures less than 50% of the potential market in any customer segment when it prices at a level that under the condition, will be shown to be an upper bound for the firm's equilibrium price choice. Similarly, if every single firm captures less than one third of the potential market in each segment (again when pricing at a level that, under the condition, is shown to be an upper bound for its price choice), a unique Nash equilibrium can be guaranteed. Frequently, the market share bounds for the various firms arise because the no-purchase option, itself, has a dominant share of the market (under the above price settings). Marketing scientists (e.g., Villas-Boas 2012) have conjectured that a unique equilibrium is likely to exist in this case. Thus, in a given market, the following condition may apply for some maximum market share $0 < \mu < 1$.

$C(\mu)$: In each market segment $k = 1, \ldots, K$, each firm $j$ captures less than $\mu$ of the market among all potential customers when pricing at the level $\bar{p}_j((1-\mu)^{-1})(j = 1, \ldots, J)$ (irrespective of what prices the competitors choose within the feasible price range).

Clearly, if condition $C(\mu_1)$ applies, then $C(\mu_2)$ applies for all $\mu_2 \ge \mu_1$. Below, we describe various model examples where condition $C(1/2)$ or $C(1/3)$ applies. Note that each firm $j$'s market share, in each market segment $k$, can be evaluated in closed form using (6). As mentioned, the critical maximal market shares $\mu$ of importance in the results below are $\mu = 1/2$ and $\mu = 1/3$.

The following lemma shows that, under $C(\mu)$, any firm $j$'s relevant price region may be restricted to $[c_j, \bar{p}_j((1-\mu)^{-1})]$. (The proofs of all lemmas are relegated to §9 of the online appendix.)

Lemma 4.1. *Fix $\mu > 0$. Under condition $C(\mu)$, the best response of any firm $j$ to any given feasible price vector $p_{-j}$ is a price $c_j < p_j^*(p_{-j}) < \bar{p}_j((1-\mu)^{-1})$.*

Thus, the market concentration test $C(\mu)$ is conducted while setting each firm's price level above what (under the condition) is rational. Therefore, because rational firms will price below $\bar{p}((1-\mu)^{-1})$, condition $C(\mu)$ does not preclude that, in equilibrium, a firm captures a share above $\mu$ in some or all market segments.

There are different ways in which condition $C(\mu)$ may be verified efficiently. Because a firm's market share is maximized when all competitors adopt maximal prices, employing the closed-form market share expression given by (6), condition $C(\mu)$ is easily verified as follows:

$$\begin{aligned}&\big(e^{[U_{jk}(x_j)+F_{jk}(Y_k)+G_j(\bar{p}_j)]}\big) \\ &\quad\cdot \Bigg(e^{U_{0k}(x_1,\ldots,x_J)+F_{0k}(Y_k)} + e^{[U_{jk}(x_j)+F_{jk}(Y_k)+G_j(\bar{p}_j)]} \\ &\qquad + \sum_{m\neq j} e^{[U_{mk}(x_m)+F_{mk}(Y_k)+G_m(p_m^{\max})]}\Bigg)^{-1} \le \mu, \\ &\qquad\qquad \forall\, j = 1, \ldots, J,\ k = 1, \ldots, K,\end{aligned} \tag{23}$$

where $\bar{p}_j$ is shorthand notation for $\bar{p}_j((1-\mu)^{-1})$. Thus, verification of $C(\mu)$ reduces to the evaluation of $JK$ closed-form market shares. (Often, it is possible to identify which firm, market segment, or firm/segment combination achieves the maximal market share, further reducing the computational effort; see the examples below.) Clearly, the larger the value chosen for $p^{\max}$, the stronger condition $C(\mu)$ becomes. Therefore, if one is unwilling to specify $p^{\max}$ up front, there are two alternative ways to proceed. First, one may determine, $\hat{p}(\mu)$ as the smallest of the unique roots of the following $JK$ equations in the single variable $p$:

$$\begin{aligned}&(e^{[U_{jk}(x_j)+F_{jk}(Y_k)+G_j(\bar{p}_j)]})\\&\cdot\Bigg(e^{U_{0k}(x_1,\ldots,x_J)+F_{0k}(Y_k)}+e^{[U_{jk}(x_j)+F_{jk}(Y_k)+G_j(\bar{p}_j)]}\\&\qquad+\sum_{m\neq j}e^{[U_{mk}(x_m)+F_{mk}(Y_k)+G_m(p_m)]}\Bigg)^{-1}=\mu,\\&\qquad\qquad j=1,\ldots,J \text{ and } k=1,\ldots,K. \qquad (24)\end{aligned}$$

Condition $C(\mu)$ is satisfied for any $p^{\max}\le\hat{p}(\mu)$. If $\hat{p}(\mu)$ is in excess of a reasonable upper bound for the products' prices, $p^{\max}$ may be set to $\hat{p}(\mu)$ without loss of generality and $C(\mu)$ may be assumed up front. A second, much stronger version of $C(\mu)$ is obtained by letting $p^{\max}\to\infty$. It is, therefore, independent of the boundary of the feasible region and only dependent on the vector $\bar{p}$.

$C'(\mu)$: Each individual firm $j$ has, in each of the market segments, an expected utility measure that falls below that of the no-purchase option by at least $\log(\mu^{-1}-1)$, assuming the firm's product is priced at the level $\bar{p}_j((1-\mu)^{-1})$; i.e.,

$$\begin{aligned}&U_{jk}(x_j)+F_{jk}(Y_k)+G_j(\bar{p}_j)((1-\mu)^{-1})+\log(\mu^{-1}-1)\\&\quad\le U_{0k}(x_1,\ldots,x_J)+F_{0k}(Y_k),\\&\qquad\qquad\forall\, j=1,\ldots,J,\ k=1,\ldots,K. \qquad (25)\end{aligned}$$

This much stronger condition $C'(\mu)$ has the additional advantage of emphasizing the importance of the value of the outside option. As mentioned, existence of a Nash equilibrium is tested with respect to the critical market share $\mu=0.5$. Lemma 4.2 shows that an even stronger, yet still widely applicable condition can be identified.

$C''(1/2,p)$: Fix a price vector $p^{\min}\le p\le\bar{p}(2)$. The no-purchase option is adopted by the majority of each market segment when the firms adopt the price vector $p$.

Lemma 4.2. (a) *For all* $0<\mu<1$: $C'(\mu)\Rightarrow C(\mu)$.
(b) *Fix* $p^{\min}\le p\le\bar{p}(2)$. $C''(1/2,p)\Rightarrow C'(1/2)\Rightarrow C(1/2)$.

For $\mu=1/2$, condition $C(1/2)$ is easily satisfied in many of the applications we are familiar with, as are its stronger versions $C'(1/2)$ and $C''(1/2,p)$. In these industrial organization studies, no single firm captures the majority of the potential market in any market segment (in particular, when pricing at a most unfavorable price level).

Consider, for example, the drive-thru fast food industry studied by Allon et al. (2011). In their consumer choice model, the conditional indirect utility of consumer $i$ from fast food outlet $j$ is specified as follows:

$$u_{ij}=X'_{k(j)}\zeta+\delta D_{ij}-\gamma P_j-\alpha W_{k(j)}+\eta_{ij}, \qquad (26)$$

where $k(j)$ denotes the chain $k$ to which outlet $j$ belongs; $X_{k(j)}$ is a column vector of observed properties of the chain to which outlet $j$ belongs; $D_{ij}$ is the distance between consumer $i$ and outlet $j$; $P_j$ is the price of a (standard) meal at outlet $j$; $W_{k(j)}$ is the waiting time standard of chain $k$; $\eta_{ij}$ is the portion of the utility of individual $i$ at outlet $j$ that is unobserved by the modeler; and $(\alpha,\beta,\gamma,\delta,\zeta)$ represents a parameter string with $\zeta$ an array of the same dimension as $X$.

The indirect utility associated with the no-purchase option is specified as

$$u_{i0}=\beta_0+M_i\pi+\eta_{i0}. \qquad (27)$$

Here, $M_i$ is a row vector specifying the consumer's demographic attributes, with binary entries, and $\eta_{i0}$ denotes the unobserved portion of the utility measure; $(\beta_i,\pi)$ is another string of parameter values. Allon et al. (2011) computed 95% confidence intervals for all parameters. We have tested conditions C(1/2) and C(1/3) under the most adversarial possible parameter values within these confidence intervals.

To maximize the market share of any given outlet in any one of the market segments (combinations of demographic groups and geographic regions) all of the parameters should be set at their lower bound values in the 95% confidence intervals, with the exception of the parameters in the $\zeta$-string—which should be set at their upper bound values; see (23). (Note that all of the explanatory variables in the utility functions are nonnegative.) The conditions in (25) for $\mu=1/2$ and $\mu=1/3$ also require the values $\bar{p}_j(2)$ and $\bar{p}_j(1.5)$. In this model $g_j(p_j)=\gamma$, so by (22),

$$\bar{p}_j(2)=c_j+2\gamma^{-1}\ge 2\gamma^{-1}\ge 2\bar{\gamma}^{-1} \quad\text{and}\quad \bar{p}_j(1.5)\ge 1.5\bar{\gamma}^{-1}, \qquad (28)$$

with $\bar{\gamma}$ the lower bound of the 95% confidence interval for $\gamma$. Thus, a robust verification of condition C′(1/2) can be obtained by checking that (25) holds with $\bar{p}_j(2)$ replaced by $2\bar{\gamma}^{-1}$. Similarly, C′(1/3) may be verified to hold by checking that (25) holds, with $\bar{p}_j(1.5)$ replaced by $1.5\bar{\gamma}^{-1}$. Following this bounding procedure, we have computed that the maximum upper bound for any outlet's market share in any of the market segments is 14%, implying that condition $C(1/3)$ and, a fortiori, C(1/2) are easily satisfied, thus guaranteeing that the model has a unique equilibrium that satisfies the system of FOC equations.

Thomadsen (2005b) studies the drive-thru fast food industry in Santa Clara county, focusing on the outlets of the two largest chains, McDonalds and Burger King. The specification of the utility functions in his model is different from that in Allon et al. (2011). However, as in the latter model, the market is partitioned into a finite number of segments, based on demographic and geographical attributes of the customers. Verification of (25) is easily performed when the values of all explanatory variables are known. (These are, understandably, not reported in the paper.) In the absence of these values, all that can be confirmed with certainty is that condition C(1/2), but not C(1/3), is satisfied, assuming the minimum distance between a census tract centroid and an outlet is at least one mile.[12] The likelihood that conditions C(1/2) and C(1/3) can be confirmed in the presence of the above data is, further, enhanced by the fact that in the study year 1999, only 6% of the Santa Clara county population consumed a meal from a McDonalds outlet on an average day, and a smaller percentage a Burger King meal (see Thomadsen 2005b, p. 909).[13]

As a third example, we consider the Davis (2001) empirical study of the movie theater industry. The model in Davis (2001), again, employs a finite segmentation of the customer population[14] based on geographical regions (census tracts), five age brackets, four income levels, and three race/ethnic groupings: Caucasians, African Americans, and others. From the results in the Davis (2001, Table 4) full model, one infers that any given movie/theater combination achieves the highest market share for any market segment that is located at a negligible distance to the theater and consists of individuals below the age of 25, with a mean income above \$50,000 and belonging to the "other" racial/ethnic group. In addition, for such a market segment, any given movie/theater combination maximizes its market share if the theater has adopted a customer service line, is a seven-screen theater, has a digital theater sound system and an auditorium quality certification provided by Lucas Arts, and features a movie during its first week in its theater but not during the first national release week. Because the utility functions in the model are, again, linear in the price variable, lower bounds for the ticket price $\bar{p}_j(2)$ and $\bar{p}_j(1.5)$ are again given by $\bar{p}_j(2) \geq 2/\gamma$ and $\bar{p}_j(1.5) \geq 1.5/\gamma$, with $\gamma$ the price sensitivity coefficient. Thus, based on the estimated value of $\gamma = 0.09$, $\bar{p}_j(2) \geq \$20.83$ and $\bar{p}_j(1.5) \geq \$15.63$. One easily verifies that condition C(1/3) and, a fortiori, C(1/2) generously apply.[15]

As a last example, consider the ready-to-eat cereal industry, which is widely characterized as one "with high concentration, high price-cost margins" (Nevo 2001, p. 307); see Schmalensee (1978) and Scherer (1982) for similar characterizations. In this industry, each of the competing manufacturers offers a series of cereals, so an adequate representation of this industry requires a multiproduct competition model as in §10 of the online appendix. (Indeed, Nevo 2001 has estimated such a multiproduct MMNL model for the industry.) In spite of this industry being viewed as one of high concentration, the *aggregate* market share of the Kellogg Company, the largest competitor, varied between 41.2% in the first quarter of 1988 and 32.6% in the last quarter of 1992, with market shares calculated among all cereal consumers as opposed to the potential consumer population. Because the value of the explanatory variables in the utility functions are unreported, the reader is unable to verify the conditions $C(\mu)$, but for the modeler the verification is easy.

We now establish that, under condition $C(1/2)$, a Nash equilibrium exists and that the set of Nash equilibria coincides exactly with the solutions to the system of FOC equations.

Theorem 4.3. *Assume condition* $C(1/2)$ *applies. Define* $\hat{p}(2)$ *by* $\hat{p}(2)_j = \min\{\bar{p}_j(2), p_j^{\max}\}$.

(a) *The price competition model has a Nash equilibrium, and every Nash equilibrium* $p^* \leq \hat{p}(2)$.

[12] The test is based on the estimated parameter values in the second estimated model; see column (2) in Table 4 of Thomadsen (2005b). Thomadsen (2005b) does not report confidence intervals around these values, so that the much more conservative test reported above for the Allon et al. (2011) model cannot be carried out. The parameters from the two alternative model specifications are very similar, having similar implications for conditions $C(1/2)$ and $C(1/3)$.

[13] Because McDonalds has the largest market share in the county, this means that at most 12% of the population consumed a meal at any outlet of the two chains considered in the Thomadsen (2005b) model.

[14] Davis (2006) employs a different specification of the utility function, one in which the utility of the no-purchase option is no longer a function of these demographic attributes. Instead, this part of the customer heterogeneity is modeled by adding Normally distributed noise terms to a constant. See §6 for a discussion of existence conditions under a continuous spectrum of consumer types.

[15] Davis (2001) concludes that on an average night, only 1 in 150 people attend any movie, explaining why in this industry, condition $C(1/3)$ is so easily established. Once again, Davis (2001) does not report any confidence intervals precluding the above, much more conservative verification test of conditions C(1/3) and C(1/2). The values of $\bar{p}_j(2)$ and $\bar{p}_j(1.5)$ are to be compared with a maximum observed ticket price of \$7.50 across 36 markets included in the study, based on 1996 data.

(b) *If $\bar{p}(2) \le p^{\max}$, every Nash equilibrium $p^*$ is a solution to the FOC*:

$$\frac{\partial \pi_j}{\partial p_j} = \sum_{k=1}^{K} S_{jk}\left[1 - (p_j - c_j)g_j(p_j)\left(1 - \frac{S_{jk}}{h_k}\right)\right] = 0, \quad \forall\, j = 1, \ldots, J, \tag{29}$$

*and has $c < p^* < \bar{p}(2)$.*

(c) *Every solution, $p^{\min} \le p^* \le p^{\max}$ to the FOC is a Nash equilibrium.*

Proof. To simplify the notation, we write $\hat{p}$ as shorthand for $\hat{p}(2)$ and $\bar{p}$ as shorthand for $\bar{p}(2)$.

(a) To prove the result on the full price cube, we first establish the existence of a Nash equilibrium $p^*$ in the interior of the restricted price cube $X_{j=1}^{J}[p_j^{\min}, \hat{p}_j]$. Existence follows from the Nash–Debreu theorem because each firm's restricted action set $[p_j^{\min}, \hat{p}_j]$ is a compact, convex set and as the profit function $\pi_j(p)$ is concave in $p_j$ on the complete price cube $X_{j=1}^{J}[p_j^{\min}, \hat{p}_j]$. Concavity follows by differentiating (29) with respect to $p_j$ as follows:

$$\begin{aligned}
&\frac{\partial^2 \pi_j}{\partial p_j^2} \\
&= \sum_{k=1}^{K}\left\{-S_{jk}g_j(p_j)\left(1 - \frac{S_{jk}}{h_k}\right)\left[1 - (p_j - c_j)g_j(p_j)\left(1 - \frac{S_{jk}}{h_k}\right)\right]\right. \\
&\qquad - S_{jk}g_j(p_j)\left(1 - \frac{S_{jk}}{h_k}\right) \\
&\qquad \left. + \frac{S_{jk}}{h_k}(p_j - c_j)g_j(p_j)S_{jk}(-g_j(p_j))\left(1 - \frac{S_{jk}}{h_k}\right)\right\} \\
&\quad - g_j'(p_j)(p_j - c_j)\sum_{k=1}^{K} S_{jk}\left(1 - \frac{S_{jk}}{h_k}\right), \\
&= \sum_{k=1}^{K} h_k g(p_j)\left(\frac{S_{jk}}{h_k}\right)\left(1 - \frac{S_{jk}}{h_k}\right) \\
&\quad \cdot \left[-2 + g_j(p_j)(p_j - c_j)\left(1 - 2\frac{S_{jk}}{h_k}\right)\right] \\
&\quad - g_j'(p_j)(p_j - c_j)\sum_{k=1}^{K} S_{jk}\left(1 - \frac{S_{jk}}{h_k}\right) < 0.
\end{aligned} \tag{30}$$

To verify the inequality, note that the second term on the right-hand side of (30) is negative because $g_j(p_j)$ is increasing in $p_j$ (see (3)). As to the first term, it follows from (21) that $g_j(p_j)(p_j - c_j) \le 2$ for all $p_j \le \bar{p}_j$ and, in particular, for all $p_j \le \hat{p}_j$. Thus, because $S_{jk}/h_k \ge 0$,

$$\left[-2 + g_j(p_j)(p_j - c_j)\left(1 - 2\frac{S_{jk}}{h_k}\right)\right] \le 0, \quad k = 1, \ldots, K. \tag{31}$$

We have shown that a price vector $p^*$ exists that is a Nash equilibrium on the restricted price cube $X_{j=1}^{J}[p_j^{\min}, \hat{p}_j]$. To show that $p^*$ is a Nash equilibrium on the full price range $X_{j=1}^{J}[p_j^{\min}, p_j^{\max}]$ as well, it suffices to show that $\pi_j(p_j, p_{-j}^*) < \pi_j(\hat{p}_j, p_{-j}^*) \le \pi_j(p_j^*, p_{-j}^*)$ $\forall\, p_j \in (\hat{p}_j, p_j^{\max})$.

This interval is nonempty only if $\bar{p}(2)_j < p_j^{\max}$. In this case, the first inequality follows from Lemma 4.1, whereas the second inequality follows from $p^*$ being a Nash equilibrium on the price cube $X_{j=1}^{J}[p_j^{\min}, \hat{p}_j]$. In view of Lemma 4.1 and because $p^{\min} \le c$, any price equilibrium $p^* \in X_{j=1}^{J}[p_j^{\min}, \hat{p}_j]$.

(b) In this case, $\bar{p}(2) = \hat{p}(2)$. To show that any Nash equilibrium $p^*$ is, in fact, an interior point of $X_{j=1}^{J}[c_j, \bar{p}_j]$, and hence a solution of the FOC (29), note that $\partial \pi_j(c_j, p_{-j}^*)/\partial p_j = \sum_{k=1}^{K} S_{jk} = S_j > 0$, whereas $\partial \pi_j(\bar{p}_j, p_{-j}^*)/\partial p_j < 0$, by Lemma 4.1.

(c) Assume $p^* \le p^{\max}$ solves the FOC (29). By (22), $p_j^* < \bar{p}_j$, and hence $p_j^* \le \hat{p}_j$, $\forall\, j = 1, \ldots, J$. In view of the concavity of $\pi_j(p_j, p_{-j})$ in $p_j$ on the price cube $X_{j=1}^{J}[p_j^{\min}, \hat{p}_j]$, $p^*$ is a Nash equilibrium on this price cube, and by the proof of part (a) on the full price range $X_{j=1}^{J}[p_j^{\min}, p_j^{\max}]$ as well. $\square$

The above proof technique builds on that in Thomadsen (2005a) for the specific class of models and existence condition considered there. (The latter may be viewed as a stronger version of $C'(1/2)$, itself a sufficient condition for $C(1/2)$; see Lemma 4.2.)

The following theorem establishes that a unique Nash equilibrium can be guaranteed under the slightly stronger condition $C(1/3)$. Analogous to $\hat{p}(2)$, define $\hat{p}(1.5)$ by $\hat{p}(1.5)_j = \min\{\bar{p}(1.5)_j, p_j^{\max}\}$.

Theorem 4.4. *Assume condition $C(1/3)$ applies.*

(a) *The price competition model has a unique Nash equilibrium $p^* \le \tilde{p}(3/2)$.*

(b) *If $\tilde{p}(1.5) \le p^{\max}$, the unique Nash equilibrium satisfies the FOC equation* (29) *and, vice versa,* (29) *has a unique solution.*

Proof. (a) Because $C(1/3) \Rightarrow C(1/2)$, it follows from Theorem 4.3 that a Nash equilibrium $p^*$ exists. Moreover, following the proof of that theorem, replacing $\mu = 1/2$ by $\mu = 1/3$, we obtain that $c \le p^* < \bar{p}(1.5)$; hence, $c \le p^* \le \hat{p}(1.5)$. Also, no Nash equilibrium may exist outside the price cube $\mathcal{P} = X_{j=1}^{J}[c_j, \hat{p}_j(3/2)]$. It thus suffices to show that even when the feasible price space is restricted to $\mathcal{P}$, no alternative equilibria may arise. (If an additional equilibrium $p^{**} \in \mathcal{P}$ were to exist, it would, a fortiori, be an equilibrium on the restricted price space $\mathcal{P}$.) We establish this by showing that on the price region $\mathcal{P}$:

$$\left|\frac{\partial \pi_j^2}{\partial p_j^2}\right| > \sum_{m \ne j}\left|\frac{\partial^2 \pi_j}{\partial p_j \partial p_m}\right|, \quad j = 1, \ldots, J. \tag{32}$$

This inequality is a sufficient condition for the best response function to be a contraction mapping (see Vives 2001) and for the equilibrium to be unique. Fix $j=1,\ldots,J$. By the definition of $\bar{p}_j(3/2)=\bar{p}_j((1-1/3)^{-1})$ and (21), we have

$$g_j(p_j)(p_j-c_j)<3/2 \quad \forall p_j<\bar{p}_j(3/2). \tag{33}$$

For all $j=1,\ldots,J$ and $k=1,\ldots,K$, let $R_{jk}\equiv 1-g_j(p_j)(p_j-c_j)(1-2(S_{jk}/h_k))$. We first prove that for all $j=1,\ldots,J$ and $k=1,\ldots,K$ and all $c_j<p_j<\bar{p}_j(3/2)$:

$$2-g_j(p_j)(p_j-c_j)\left(1-2\frac{S_{jk}}{h_k}\right)=1+R_{jk}>|R_{jk}|. \tag{34}$$

The inequality is immediate when $R_{jk}>0$. When $R_{jk}<0$, we must have $0<1-2(S_{jk}/h_k)<1$. (The upper bound always holds because $S_{jk}/h_k>0$; the lower bound holds because if $(1-2(S_{jk}/h_k))<0$, $R_{jk}>0$, as $g_j(p_j)>0$ and $p_j>c_j$.) By (31), this implies that $g_j(p_j)(p_j-c_j)\cdot(1-2(S_{jk}/h_k))<3/2$, so that $0>R_{jk}>-0.5$ and $2-g_j(p_j)(p_j-c_j)(1-2(S_{jk}/h_k))=1+R_{jk}>0.5\geq|R_{jk}|$ in case $R_{jk}<0$ as well.

Differentiating the right-hand side of (29) with respect to $p_m$, for $m\neq j$, we obtain the following:

$$\begin{aligned}\frac{\partial^2\pi_j}{\partial p_j\partial p_m}&=\sum_{k=1}^{K}\frac{\partial S_{jk}}{\partial p_m}\left[1-(p_j-c_j)g_j(p_j)\left(1-2\frac{S_{jk}}{h_k}\right)\right]\\&=\sum_{k=1}^{K}\frac{\partial S_{jk}}{\partial p_m}R_{jk}.\end{aligned}\tag{35}$$

It follows from (32) and (17) that

$$\begin{aligned}\frac{\partial^2\pi_j}{\partial p_j^2}&=-\sum_{k=1}^{K}\left|\frac{\partial S_{jk}}{\partial p_k}\right|(1+R_{jk})\\&\quad-g_j'(p_j)(p_j-c_j)\sum_{k=1}^{K}S_{jk}\left(1-\frac{S_{jk}}{h_k}\right),\end{aligned}\tag{36}$$

so that

$$\begin{aligned}\left|\frac{\partial^2\pi_j}{\partial p_j^2}\right|&=\sum_{k=1}^{K}\left|\frac{\partial S_{jk}}{\partial p_k}\right|(1+R_{jk})+g_j'(p_j)(p_j-c_j)\sum_{k=1}^{K}S_{jk}\left(1-\frac{S_{jk}}{h_k}\right)\\&\geq\sum_{k=1}^{K}\left|\frac{\partial S_{jk}}{\partial p_k}\right|(1+R_{jk})>\sum_{k=1}^{K}\left|\frac{\partial S_{jk}}{\partial p_k}\right||R_{jk}|\\&\geq\sum_{k=1}^{K}\left(\sum_{m\neq j}\frac{\partial S_{jk}}{\partial p_m}\right)|R_{jk}|=\sum_{m\neq j}\sum_{k=1}^{K}\left|\frac{\partial S_{jk}}{\partial p_m}\right||R_{jk}|\\&\geq\sum_{m\neq j}\left|\sum_{k=1}^{K}\frac{\partial S_{jk}}{\partial p_m}R_{jk}\right|=\sum_{m\neq j}\left|\frac{\partial^2\pi_j}{\partial p_j\partial p_m}\right|.\end{aligned}\tag{37}$$

To verify the first equality in (37), note that both terms on the right-hand side are negative. (Negativity of the first term follows from $1+R_{jk}\geq 0$; see (34). Negativity of the second term follows from $g_j'(p_j)>0$; see (3), $p_j>c_j$ and $S_jk/h_k\leq 1$.) The first inequality follows because the second term on the left-hand side is nonnegative, as just verified. The second inequality employs (34). The third inequality is based on (19). The second equality in (37) follows from $\partial S_{jk}/\partial p_m>0$; see (19) and the fact that $g_m(p_m)\geq 0$. The last inequality in (37) follows from the triangular inequality, and the last equality uses (35).

(b) If $\bar{p}(1.5)\leq p^{\max}$, it follows from the proof of Theorem 4.3(b) that the unique Nash equilibrium $p^*$ satisfies the FOC (29). Conversely, again by the proof of Theorem 4.3(b), any solution $p^*$ to the FOC (29) must have $p^*\leq\bar{p}(1.5)$ and hence must be a Nash equilibrium; because the Nash equilibrium is unique, the system of equations must have a unique solution. □

The conditions needed for existence and uniqueness, $C(\mu)$, bear a remarkable relation to standard policy criteria used to define "moderately" or "highly concentrated" markets. The Department of Justice (DOJ) and the Federal Trade Commission (FTC) measure the degree of concentration in a market via the Herfindahl-Hirschman Index (HHI), defined as the sum of the squares of the market shares represented as percentages. (This index has a maximum value of 10,000 in case of a monopoly and approaches 0 if the market is divided among a very large number of competitors with an equal market share.) The DOJ–FTC 1992 Horizontal Merger Guidelines define a market with an HHI below 1,000 as "unconcentrated," those between 1,000 and 1,800 as "moderately concentrated," and those with an HHI above 1,800 as "highly concentrated."[16] Interestingly, when the market share restriction in C(1/3) is violated for the aggregate market under the equilibrium prices, the minimum possible HHI equals 1,111, and 2,500 when C(1/2) is violated.[17] (Thus, although it is unclear what the cutoff value of 1,800 was based on, it corresponds with the average of the minimum HHI values when C(1/2) and C(1/3) are violated.) These 1992 DOJ–FTC guidelines were updated in April 2010, and the new HHI cutoff level for a "highly concentrated market" has been increased from 1,800 to 2,500, the minimal value corresponding with $C(1/2)$ in the sense explained above. Of course, the Horizontal Merger Guidelines test aggregate market shares, whereas in conditions $C(1/2)$ and $C(1/3)$ a test is applied to the market shares in every market segment separately.

[16] The FTC calculates the HHI based on the anticipated postmerger equilibrium, measuring market shares as a percentage of aggregate sales in the industry. Our $C(\mu)$ conditions put "market concentration" in a favorable light, measuring each firm $j$'s market share as a percentage of the total potential customer population and under the assumption that the firm selects as a price level $\bar{p}_j((1-\mu)^{-1})$, which by Theorem 4.3 is an upper bound for the firm's equilibrium price, under the condition. (At the same time, the market share is assessed, assuming the competitors choose their maximum price.)

[17] These minima arise when a single firm captures one third or half of the market, respectively, with the remainder of the market being divided equally among infinitely many competitors.

We have not yet addressed whether and under what conditions a price equilibrium exists in the few industries where a very high level of concentration does arise, in some of the market segments, and a single firm captures the majority of the potential customer population. Aksoy-Pierson et al. (2010) show that such an existence guarantee can indeed be given, but only in the presence of potentially restrictive exogenous price limits. (See §3 for a discussion of a variety of settings where firms operate with exogenous price limits.) Indeed, the following theorem, proven in Aksoy-Pierson et al. (2010), shows that the price competition model is supermodular and has, in fact, a unique equilibrium when the feasible price range is such that any firm $j$'s variable profit margin is at a level where the above defined elasticity $\omega(p_j)$ is no larger than one (i.e., $p^{\max} \le \bar{p}(1)$).

Theorem 4.5. *Assume $p_j^{\max} \le \bar{p}(1)$, $j = 1, \ldots, N$. The price competition model has a unique Nash equilibrium.*

Recall from (22) that $\bar{p}(0) < \bar{p}(0.5) < \bar{p}(2)$. Thus, Theorem 4.5 is more restrictive than Theorems 4.3 and 4.4 in that it requires the feasible price range to be more confined. On the other hand, no maximum market share limits are required by this theorem. The proof of Theorem 4.5, contained in Aksoy-Pierson et al. (2010), is based on the price competition game being supermodular under the condition $p^{\max} \le \bar{p}(1)$; in particular, the profit functions have the so-called "single-crossing property." Indeed, the single-crossing property can be shown to hold in general for the segment-by-segment profit functions $\{\pi_{jk}\colon j = 1, \ldots, J,\ k = 1, \ldots, K\}$ without any restrictions on the model parameters; see Lemma 9.1 in the online appendix. Unfortunately, the *aggregate* profit functions $\{\pi_j\}$ may fail to have this single-crossing property on arbitrarily large price regions. However, in practice, the single-crossing property often carries over to the aggregate profit functions, even on very large price regions, so that existence of a (unique) Nash equilibrium is guaranteed even in markets that are highly concentrated, i.e., where condition $C(1/2)$ fails.

## 5. Counterexample

The following counterexample demonstrates that a condition like $C(1/2)$, broadly applicable as it is, is required to guarantee the existence of a Nash equilibrium; i.e., a Nash equilibrium cannot be expected to exist in the fully general model. Our counterexample was inspired by Dubé et al. (2008), who exhibit that multiple equilibria may arise in a price competition model with two firms (no outside good), three customer segments, and a combination of linear and constant elasticity of substitution (CES) demand functions. As is often the case with such counterexamples, this one is stylized.

Consider a market with two firms and three consumer segments (i.e., $J = 2, K = 3$) whose consumer utility functions are defined as follows:

$$\text{Firm 1:}\quad U_{i11} = 11 - p_1 + \epsilon_{i11};\quad U_{i12} = 4 - p_1 + \epsilon_{i12};$$
$$U_{i13} = 2 - p_1 + \epsilon_{i13};$$
$$\text{Firm 2:}\quad U_{i21} = 4 - p_2 + \epsilon_{i21};\quad U_{i22} = 11 - p_2 + \epsilon_{i22};$$
$$U_{i23} = 2 - p_2 + \epsilon_{i23}.$$

(As in the general model, the $\epsilon$ terms are random with a type 1 extreme value distribution.) In this example, potential consumers in segment 1 (2) have a major preference for firm 1 (2). In contrast, consumers in market segment 3 attribute the same expected utility to both products when equally priced. Following the derivation of (6) and (7), the demand functions for firms 1 and 2 are, therefore, given by

$$D_1 = h_1 \frac{e^{11-p_1}}{e^{11-p_1} + e^{4-p_2}} + h_2 \frac{e^{4-p_1}}{e^{4-p_1} + e^{11-p_2}} + h_3 \frac{e^{2-p_1}}{e^{2-p_1} + e^{2-p_2}}, \tag{38}$$

$$D_2 = h_1 \frac{e^{4-p_2}}{e^{11-p_1} + e^{4-p_2}} + h_2 \frac{e^{11-p_2}}{e^{4-p_1} + e^{11-p_2}} + h_3 \frac{e^{2-p_2}}{e^{2-p_1} + e^{2-p_2}}. \tag{39}$$

The profit for each firm is given by $\pi_j = (p_j - c_j)D_j$. The following set of parameters specifies a game without a Nash equilibrium: $c_1 = c_2 = 1$, $h_1 = 2$, $h_2 = 2$, $h_3 = 4$, and $p_{j,\max} = 10$. The following defines a cycle of best responses that is reached from any starting point in the feasible price region $[1, 10] \times [1, 10]$, where $br_1(p_2)$ denotes the best response of firm 1 to firm 2's price choice, $p_2$, and vice versa for $br_2(p_1)$:

$$br_1(6.01) = 5.59,\quad br_2(5.59) = 5.32;$$
$$br_1(5.32) = 5.16,\quad br_2(5.16) = 5.08;$$
$$br_1(5.08) = 5.03,\quad br_2(5.03) = 9.80;$$
$$br_1(9.80) = 8.51,\quad br_2(8.51) = 7.45;$$
$$br_1(7.45) = 6.62,\quad br_2(6.62) = 6.01.$$

Note that the parameters specified above violate condition C(1/2): $\bar{p}_j(2) = c_j + 2 = 3$, $j = 1, 2$, and the market share of firm 1 in segment 1 is $e^8/(e^8 + e^{-6}) > 0.5$ when $p_1 = 3$, $p_2 = 10$. (Counterexamples with a no-purchase option may be created by specifying that $U_{i0k} = -M + \epsilon_{i0k}$ for $M$ sufficiently large.)

The counterexample not only demonstrates the necessity of a condition like C(1/2) but also reinforces that the existence of a (unique) equilibrium cannot be taken for granted. With many structural estimation models relying on the existence of a (unique) equilibrium when estimating market parameters and evaluating policies, it is important to note that without an existence guarantee for an equilibrium, these methods may result in flawed estimates.

## 6. A Continuum of Customer Types

In some applications, a continuum of customer types needs to be considered in the consumer choice model. Our model is easily respecified to allow for a continuum of customer types $\theta \in \Theta$, with a density function $h(\theta)$. Let

$$u_{ij}(\theta) = U_j(x_j \mid \theta) + F_j(Y \mid \theta) + G_j(p_j) + \epsilon_{ij}(\theta), \quad j = 1, \ldots, J \text{ and } i = 1, 2, \ldots, \tag{40}$$

$$u_{i0}(\theta) = U_0(x_1, \ldots, x_J \mid \theta) + F_0(Y \mid \theta) + \epsilon_{i0}(\theta), \quad i = 1, 2, \ldots. \tag{41}$$

Here, $u_{ij}(\theta)$ denotes the utility value attributed by the $i$th customer of type $\theta \in \Theta$. For all $j = 0, \ldots, J$, $\{\epsilon_{ij}(\theta)\}$ represents a sequence of independent random variables with Gumbel distributions. It is easily verified that the demand functions in (7) need to be replaced by

$$\begin{aligned} S_j &= \int_{\theta \in \Theta} h(\theta) S_{j\theta}\, d\theta \\ &= \int_{\theta \in \Theta} h(\theta)(e^{[U_j(x_j|\theta)+F_j(Y|\theta)+G_j(p_j)]}) \\ &\quad \cdot \left( e^{[u_0(x_1,\ldots,x_J|\theta)]+F_0(Y|\theta)} + \sum_{m=1}^{J} e^{[U_m(x_m|\theta)+F_m(Y|\theta)+G_m(p_m)]} \right)^{-1} d\theta. \end{aligned} \tag{42}$$

All of the results in §4 continue to apply, with the conditions $C(\mu)$ now specified as follows.

$C(\mu)$: For each customer type $\theta \in \Theta$, each firm $j$ captures less than a fraction $\mu$ of the market among all potential customers when pricing at the level $\bar{p}_j((1-\mu)^{-1})$ $(j = 1, \ldots, J)$ (irrespective of which prices the competitors choose from the feasible price range).

As mentioned in the introduction, verification of condition $C(\mu)$ may be more involved in the case of a continuum of customer types. Starting with the Berry et al. (1995) paper, many empirical models add random shock terms to some of the parameters in the utility functions $U_{jk}(x_j)$ to add an additional level of heterogeneity because of unobservable factors, beyond the additive unobservable heterogeneity included by the noise terms $\epsilon_{ijk}$, and either instead of or beyond heterogeneity due to observable customer characteristics; see (1). Often, these random shock terms are assumed to follow continuous distributions of a numerically convenient type, for example, a multivariate Normal distribution. Such specifications imply the existence of customer types who attribute an arbitrarily large weight to one of the product attributes and are hence in vast majority attracted to a single product, irrespective of other product attributes or the magnitude of price differences. The presumed existence of such extreme customer types, beyond arguably being unrealistic, causes the market concentration restriction in condition $C(\mu)$ to be violated; the latter needs to hold for every customer type, even those that are very rare and therefore hardly impact the structure of the firms' aggregate profit functions. If random shock terms with a parsimonious distributional description are deemed to be necessary in the model specification, this problem can be avoided by specifying distributions with a bounded support, for example, uniform, triangular or noncentral beta or truncated Normal distributions: Feenstra and Levinsohn (1995), for example, model the consumer's ideal attribute vector as uniformly distributed on a finite cube in attribute space. In this case, it suffices to check the market share condition $C(\mu)$ for the corner points of the cube. In state-of-the-art estimation procedures, the integrals in the sales volume functions (42) are evaluated by drawing samples from the distributions of customer types, a process that is as easily carried out for the above bounded support distributions as it is for Normals, say.

Other modelers avoid random shocks in the parameters of the utility functions altogether confining themselves to a *discrete* distribution of heterogeneity, i.e., a finite segmentation of the market based on observable customer characteristics alone. This is referred to as the "latent class" model; see, for example, Kamakura and Russell (1989), Davis (2001) Thomadsen (2005b), Allon et al. (2011), and the textbook on market segmentation by Wedel and Kamakura (2000).

As with other specification challenges, the natural desire for broader structures of customer heterogeneity must be traded off against the risks of overspecification, for example, the difficulty of estimating additional sets of parameters, and the ability to interpret the resulting market segments. The above observations indicate that an additional consideration is that overly refined segmentations, allowing for extreme customer types, may result in a violation of condition $C(\mu)$ that leaves the modeler without a foundation to estimate parameters based on equilibrium conditions.

## 7. Structural Estimation Methods

In this section, we discuss the implications of our results for the econometrician desiring to estimate the parameters of a model with MMNL demand function. Empiricists implicitly or explicitly "assume" the following:

*Empiricist assumption* (*EA*): The model possesses an equilibrium.[18]

The problems arising as a result of the potential existence of multiple equilibria or no equilibrium have been featured prominently in recent papers as well as the fact that a solution to the system of FOC equations may fail to be an equilibrium and vice versa (see, e.g., Tamer 2003, Schmedders and Judd 2005, Aguirregabiria and Mira 2007,[19] Ferris et al. 2008, Ciliberto and Tamer 2009).

We distinguish between two types of estimation settings: estimation under an observed price vector and estimation absent price observations.

### 7.1. Structural Estimation with a Given Observed Price Vector

In this subsection, we consider settings where the price vector $p^*$ is observed. Sometimes, the market shares of the different firms are observable as well (see, e.g., Berry et al. 1995); often, sales volumes are unobservable (see, e.g., Thomadsen 2005b, Nevo 2000, Allon et al. 2011). Either way, we will show that conditions $C(1/2)$ and $C(1/3)$ are useful to ensure that the estimated model satisfies the basic assumption (EA), i.e., that it has a Nash equilibrium or a unique Nash equilibrium, respectively. Exact verification of these conditions $C(\cdot)$ requires knowledge of the model's parameter values and is, therefore, to be carried out *after* these parameters have been estimated. Indeed, even though the $C(\mu)$ conditions verify that no firm has a market share in excess of $\mu$, in any one of the market segments, these market shares need to be evaluated under price conditions different from the observed price vector $p^*$; thus, even if the equilibrium market shares corresponding with $p^*$ are observable, they fail to be the basis for verification of the $C(\cdot)$ condition. (See below for a discussion of how the condition $C(\cdot)$ can sometimes be used *prior* to estimating the model, whether equilibrium sales volumes are observable or not.)

Typically, the observed price vector $p^*$ is an interior point of the feasible price region; i.e., $p^{\min} < p^* < p^{\max}$. Thus, under the basic model assumption (EA), it is necessary that $p^*$ satisfies the system of FOC equations. This permits the use of estimation methods (e.g., GMM) that compute the string of parameter combinations, under which $p^*$ satisfies this system of equations as closely as possible. However, because under the estimated parameter vector, $p^*$ satisfies the FOC equations (29) closely does not guarantee that $p^*$ is a Nash equilibrium, let alone the unique Nash equilibrium, in the competition model that arises under the estimated parameters. Verification of condition $C(1/2)[C(1/3)]$ establishes these equilibrium results, thus establishing that the estimation results are consistent with the fundamental assumption (EA).

Direct verification that $p^*$ is a Nash equilibrium under the estimated parameters may, in principle, be accomplished without resorting to conditions $C(1/2)$ or $C(1/3)$. Instead, one may test whether

$$p_j^* = \arg\max_{p_j} \pi_j(\cdot, p_{-j}^*), \quad j = 1, \ldots, J. \tag{43}$$

However, solving the $J$ global maximization problems in (43) may be considerably more challenging than the simple evaluations of the $JK$ market choices in (23), the verification test of $C(\mu)$. (Recall that the profit functions have many local optima.) Moreover, because any parameter string, including the generated "best fit" parameter string, achieves a less than perfect fit of the system of FOC equations (29), the maximizing prices to the right of (43) are likely to be somewhat distinct from the $\{p_j^*\}$ prices, raising the question when the equilibrium test (43) can be considered to be "satisfied." Finally, as discussed in §4, one ideally wants to verify that the model has a Nash equilibrium under any combination of parameters selected from the estimated confidence intervals, not just the single vector of point estimates. In §4, we have demonstrated how condition $C(\mu)$ can be verified to hold, robustly, for all such parameter conditions by testing specific combinations of boundary values of the confidence intervals. In contrast, we are not aware of a parametric extension of the equilibrium test in (43).

To verify that $p^*$ is the unique Nash equilibrium, condition $C(1/3)$ provides a broadly applicable sufficient condition that, again, requires no more than the evaluation of $JK$ market shares; see (23).

Finally, beyond their use as a *post* estimation test to verify whether the estimated model has a Nash equilibrium, or, better yet, a unique Nash equilibrium, conditions $C(1/2)$ and $C(1/3)$ may be useful ex ante (i.e., prior to engaging in a challenging estimation project) to assess whether they are satisfied over the likely range of parameter conditions, thus providing a level of confidence that the model, after estimation, can be guaranteed to possess a (unique) equilibrium. In most applications, very high degrees of market concentration can be ruled out on a priori grounds, and condition $C(1/2)$ may be assumed to

[18] See, for example, the quote in the introduction of Berry et al. (1995).

[19] Aguirregabiria and Mira (2007, p. 2) note, for example, "The existence of multiple equilibria is a prevalent feature in most empirical games where best response functions are nonlinear in other players' actions. Models with multiple equilibria do not have a unique reduced form, and this incompleteness may pose practical and theoretical problems in the estimation of structural parameters."

hold up front. Two examples are the aforementioned drive-thru fast food industry, an industry modeled with MMNL demand functions in both Thomadsen (2005b) and Allon et al. (2011), as well as the movie industry modeled by Davis (2001, 2006). Elementary statistical studies reveal that even when aggregating across all chains, the fast food industry captures a minority of the potential market in any relevant demographic segment. In §11 of the online appendix, we discuss, briefly, how and to what extent condition $C(1/2)$ may be used to narrow the search for "best fitting" parameters.

After the model parameters are estimated, most studies proceed to conduct counterfactual investigations. To predict changes in the price equilibrium and corresponding sales volumes resulting from a given change in one or several of the model's parameters, it is important to know whether a unique equilibrium exists. The uniqueness conditions in Theorems 4.4 and 4.5 can again be used for this purpose: as mentioned, the former reduces to making the $JK$ comparisons in (23) with $\mu = 1/3$ using the estimated parameters, whereas the latter reduces to the vector comparison $p^{\max} \le \bar{p}(1)$.

If condition $C(1/2)$ applies but condition $C(1/3)$ fails, one may still be able to establish that $p^*$ is the unique equilibrium, based on an ex post numerical test. After all, under $C(1/2)$, in view of Theorem 4.3, it suffices to verify that the system of FOC (29) has the observed price vector $p^*$ as its unique solution on the cube $X_{j=1}^{J}[p^{\min}, p^{\max}]$ under the parameter estimates by employing any of the known algorithms that identify all solutions to a system of equations. Thus, the characterization in parts (b) and (c) of Theorem 4.3 of the set of Nash equilibria as the solutions to (29) may be of great value in empirical studies.

An alternative ex post uniqueness test, under $C(1/2)$, is to verify that the single nonlinear function given by the determinant of the Jacobian matrix associated with (29) has no root; i.e.,

$$\det J(p) \neq 0, \quad \forall\, p \in X_{j=1}^{J}[\hat{c}_j, \bar{p}_j], \tag{44}$$

where $J(p)$ is an $J \times J$ matrix with $J(p_{mj}) = \partial^2\pi_m/\partial p_m \partial p_j$. The validity of (44) follows from Kellogg (1976). (Recall that Theorem 4.3(b) excludes the existence of equilibria on the boundary of the price region.)

### 7.2. Structural Estimation of the Game in the Absence of an Observed Price Vector

In other studies, the parameters of the price competition game need to be estimated in the absence of an observed price vector. This happens, for example, when estimating dynamic multistage games (see, e.g., Doraszelski and Pakes 2007). Most estimation methods consist of optimizing some objective $L(\theta, p(\theta))$ over all possible parameters vectors $\theta$ and all price vectors $p(\theta)$ that arise as a Nash equilibrium under $\theta$. The objective may be a maximum likelihood function or pseudo-maximum likelihood function (see Aguirregabiria and Mira 2002, 2007). Alternatively, it may be a (generalized) method-of-moments norm (see, e.g., Pakes et al. 2007). The characterization of the equilibria $p(\theta)$ as the solutions to the FOC equations (29) helps, once again, enormously for any of these estimation methods: Traditional estimation methods, starting with Rust's (1987) (nested) fixed point algorithmic approach, have projected the associated optimization problems onto the parameter space $\Theta$, solving an optimization problem of the type

$$\min\bigl\{L(\theta, p(\theta)) \mid \theta \in \Theta \text{ and } p(\theta) \text{ is an equilibrium under } \theta\bigr\}. \tag{45}$$

This means that a search is conducted through the parameter space, and whenever a specific trial parameter vector $\hat{\theta} \in \Theta$ is evaluated, all associated price equilibria $p(\hat{\theta})$ are computed. As pointed out, for example, by Aguirregabiria and Mira (2007), this approach may be infeasible even for simple models. A further complication is that even the computation of the equilibria $p(\theta)$ for any single-parameter vector $\theta$ may be very difficult. Many have concluded that games in which multiple equilibria may exist cannot be estimated and have restricted themselves to highly stylized model specifications in which uniqueness of the equilibrium can be guaranteed. Fortunately, no such model restrictions are necessary. The prevalence of multiple equilibria can comfortably be dealt with as long as the set of equilibria can be characterized as the solutions to a (closed-form) set of equations like the FOC equations (29). Within the context of our class of price competition models, this characterization is obtained by Theorem 4.3. Instead of optimizing the projected unconstrained problem (45), Theorem 4.3 permits us to estimate the parameters by solving the *constrained* optimization problem:

$$\min\{L(p, \theta)\colon \theta \in \Theta \text{ and } (29)\}. \tag{46}$$

As explained above, if $C(1/2)$ can be assumed on a priori grounds, in view of Lemma 4.1, constraints (23) could be added to (46) because these represent necessary conditions under $C(1/2)$,

$$\min\{L(p, \theta)\colon \theta \in \Theta,\ (29) \text{ and } (23)\}. \tag{47}$$

We refer to §7.1 for a discussion of how uniqueness of an equilibrium can be guaranteed ex post.

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