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Supply Streams

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A supply stream is a continuous version of a supply chain. It is like a series inventory system, but stock can be held at any point along a continuum, not just at discrete stages. We assume stationary parameters and aim to minimize the long-run average total cost. We show that a stationary continuous-stage echelon base-stock policy is optimal. That is, at each geographic point along the supply stream, there is a target echelon inventory level, and the optimal policy at all times is to order and dispatch material so as to move the echelon inventory position as close as possible to this target. We establish this result by showing that the solutions to certain discrete-stage systems converge monotonically to a limit, as the distances between the stages become small, and this limit solves the continuous-stage system. With demand approximated by a Brownian motion, we show that, in the continuous-stage limit, the supply stream model is equivalent to one describing first-passage times. This linkage leads to some interesting and useful results. Specifically, we obtain a partial differential equation that characterizes the optimal cost function, and we find a closed-form expression for the optimal echelon base-stock levels in a certain special case, the first in the inventory literature. These expressions demonstrate that the well-known square-root law for safety stock does not apply in this context.

Key words: inventory/production; multiechelon; uncertainty; probability; diffusion; partial differential equations
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1. Introduction

A supply stream is a continuous version of a supply chain. It is like a series inventory system, but stock can be held at any point along a continuum, not just at discrete stages. Time too is continuous.

Imagine that a single product is carried on barges on some waterway. Demand and supply occur in whole barge loads. Demand occurs, randomly, only at one end of the waterway, and the source is at the other end. At any point in time, one can launch a barge at the source. The barge usually travels at a certain maximum speed toward the destination. At any point along the way, however, the barge can slow down or even stop and then start moving again. Once the barge reaches the destination, its goods are stored in inventory until they are demanded.

Why stop a barge in midstream or even slow it down? The answer is to save on holding costs. We assume a cost of holding inventory on the barge that increases as it moves toward its destination. This relation is sometimes called a cost-time profile (see Fooks 1993, Schraner and Hausman 1997, Chaudhari 2007). Suppose the destination is a port. The port docking fee may be larger than that on the waterway. The closer the barge is to the port, the more expensive it is to dock. In contrast, of course, as the barge moves, it gets closer to being available to meet demand and thus avoid shortage costs. Similar stories can be told about other modes of transportation.

As a model of traffic, this one is stylized in many ways, but a few deserve mention at the outset. First, those stops and starts and changes in speed are costless. Imagine that the crew can just cut the engines, drop anchor, and turn to unpaid leisure activities. Second, there are no capacity limits and therefore there is no congestion. Each barge has a maximum speed, but any number of barges can pass a given point at any time. Third, the speed of each barge is unaffected by exogenous influences, such as weather, or internal impediments, such as breakdowns. Fourth, there are no explicit scale economies, like fixed costs. Here, a barge load is the system's basic quantity unit. These assumptions are similar to those of many standard logistics models.

A supply stream can also describe a fluid that flows from a source to a destination. Think of a pipeline or a channel moving gasoline, natural gas, or water. The fluid moves only in one direction, with a maximum velocity, but otherwise we can costlessly control the flow rate at each point. (Again, as a description of fluid dynamics, this model omits some important real factors. Our pipes have infinite capacities and therefore zero pressures.)

Or imagine a production process with many stages. We approximate it by a continuum. A unit of product can move through the system at a maximum rate, but it can slow down or stop anywhere. Many production processes really are continuous in this sense. Oil refining and chemical processing are examples. (Our assumption that product can be stored at any point is a stretch in those contexts, however. Also, they too have finite capacities.) Likewise, consider a conveyor in a distribution center. One can speed up a certain stage by adding more workers or slow it down by shifting workers.

The model is a natural continuous limit of discrete-stage models. Such discrete-stage models are well understood. Their analysis was pioneered by Clark and Scarf (1960) and refined by Federgruen and Zipkin (1984) and Chen and Zheng (1994). Concepts and methods from this literature, such as echelon inventory, play prominent roles in our discussion. These models too make simplifying assumptions like those above—unlimited capacity, no explicit scale economies, etc.

We formulate the continuous-stage model in §2. Demand is a compound Poisson process, and all parameters are stationary. There are costs for holding inventory, which vary continuously over space, and a penalty cost for backorders at the point of customer demand. The objective is to minimize the long-run average total cost.

To establish the optimal control policy for the continuous-stage model, we first (in §3) construct a discrete-stage model by allowing the system to retain inventory only at fixed, discrete points. We know that a policy of simple form, called a stationary echelon base-stock policy, is optimal for this discrete model, and we have a simple algorithm to compute one. We then (in §4) recover analogous results for the continuous model by taking limits of the discrete one, as the distances between the stocking points become small. We demonstrate that the optimal costs and optimal policies of the discrete-stage models converge monotonically, and these limits solve the original continuous-stage model. In particular, a stationary continuous-stage echelon base-stock policy is optimal. That is, at each geographic point along the supply stream, there is a target echelon inventory level, and the optimal policy at all times is to order and dispatch material so as to move the echelon inventory position as close as possible to this target. The limiting result also provides simple bounds on the optimal policy and cost for the original model.

In §5 we approximate demand by a Brownian motion. (More precisely, we approximate the optimal cost in equilibrium by treating demand as a Brownian motion.) We show that, in the continuous-stage limit, the model is equivalent to one that describes the first-passage time of standard Brownian motion to a moving boundary. (A similar equivalence holds for certain discrete-stage models, as pointed out in §3.) This linkage is interesting and fruitful. That first-passage time model has been studied intensively, and the results immediately apply to the supply-stream model. In particular, we obtain a partial differential equation that characterizes the optimal-cost function. This leads to closed-form expressions for the optimal echelon base-stock levels in certain special cases. These are the only closed-form solutions of inventory network models we know of. We point out and explain some interesting qualitative features of these solutions. Their behavior is different from what we might expect by extrapolation from single-stage models. In particular, the well-known square-root law for safety stock does not apply in this context. We also discuss various numerical solution methods. All proofs are in Online Appendix A (available as supplemental material at http://dx.doi.org/10.1287/msom.2013.0431). In Online Appendix B, we show how to evaluate any given policy and to find the best one with tractable algorithms for Poisson demand.

How should one utilize these results to analyze a real supply stream? We believe it is worthwhile to *formulate* a continuous-stage model along the lines suggested here. To obtain a solution, however, one reasonable approach is to discretize the stages and solve the resulting model using the standard algorithm for discrete-stage systems (namely, (8) below). We do recommend trying several discretizations, to study the effects of changing the number and placement of stocking points. The limiting result mentioned above assures that, if done carefully, this approach yields a near-optimal solution to the original model.

This interplay between discrete and continuous formulations has a long history. For a short and entertaining survey covering the work of such luminaries as Laplace, Fourier, and Maxwell, see Narasimhan (2010). One of his conclusions is, "Thus, remarkably, the discrete world represented by the difference equation and the abstract world represented by the differential equation together help us comprehend the world around us in ways that neither can individually achieve" (p. 1005).

Concerning the assumption that inventory can be held anywhere, in many cases this is literally true. Barges, trucks and ships do in fact stop en route, though sometimes for reasons other than those modeled here, such as work rules, fatigue, and congestion downstream. In other cases, however, storage requires an expensive special facility. For example, even an oil refinery uses storage tanks to hold intermediate products at various stages in the process. But the location of such facilities is an important decision, and the possibilities are typically continuous. Our model provides a useful tool to guide such design decisions. The solution of the supply-stream model may suggest where to locate stocking facilities. This same idea is elaborated by Gallego and Zipkin (1999) in the context of discrete stages.

There have been a few similar models in the literature. Axsäter and Lundell (1984) pose the problem and analyze a few cases numerically. Recently, Berling and Martínez-de-Albéniz (2011a) show that, under the discounted-cost criterion, for Poisson demand and piecewise-constant or linear holding costs, an echelon base-stock policy is optimal. Berling and Martínez-de-Albéniz (2011b) extend the model to allow certain forms of expediting. As previously discussed, all these models are stylized in some ways and therefore limited in depicting the real world. But together they comprise valuable first steps in modeling continuous-stage supply chains.

Continuous-time, continuous-state models have been extensively used to study queueing network models; see, for example, Harrison (1985) and Chen and Yao (2001). Their applications to inventory models have been relatively scant. Examples include Bather (1966), Reiman et al. (1999), and Plambeck and Ward (2006, 2008). We hope this paper can help generate more interest in this domain.

2. Formulation

We use the continuous variable $t \geq 0$ to indicate time. In space, the system covers a finite, semiopen interval, denoted $[0, U)$. Customer demand occurs at point $u = 0$, and point $u = U$ represents an outside supplier with ample stock. When we order a unit from the supplier, it starts at U and travels toward 0. We scale space so that the time it would take a unit traveling at maximum speed to reach point 0 from $u \in [0, U)$ is precisely u. Once a unit enters the system at U, we can choose to hold it at any point u along the way or to make it move more slowly than the maximum speed. Echelon u means the subsystem comprising point u and all points downstream, that is, the interval $[0, u]$, $0 \leq u < U$. When a demand occurs, if there is enough stock at point 0, the demand is satisfied. Otherwise, the demand is satisfied as much as possible, and the unsatisfied portion is backlogged, to be filled later.

Demand is represented by a compound Poisson process $\mathbf{D} = \{D(t)\colon t \geq 0\}$ with positive jumps. Here, $D(0) = 0$, and $D(t)$ is the cumulative demand in $(0, t]$, $t > 0$. Also, let $D(s, t) = D(t) - D(s)$, $s \leq t$. Thus, the sample paths of $\mathbf{D}$ are nondecreasing and càdlàg (right continuous with left limits, or RCLL). Also, $\mathbf{D}$ has stationary, independent increments. The underlying Poisson process has rate λ, and the jumps are distributed as a positive random variable X. The average demand rate is μ, with $0 < \mu \equiv E[D(1)] = \lambda E[X] < \infty$. Let $\Psi_0(t, x) = \Pr\{D(t) > x\}$ denote the complementary cumulative distribution function (cdf) of $D(t)$ and $\Psi_1(t, x) = E[[x - D(t)]^-]$ its loss function. For the range of $\mathbf{D}$, we consider two cases, demand (X has a positive density on the interval $(0, \bar{X})$ for some $\bar{X} > 0$). The results apply to both cases unless indicated otherwise.

The control is described by the supply process $\mathbf{S} = \{S(t, u)\colon t \geq 0,\ u \in [0, U)\}$, where $S(t, u)$ is the cumulative supply to echelon $u \in [0, U)$ in time $[0, t]$, $t > 0$. The set of *admissible* controls, denoted $\mathcal{S}$, are those satisfying the following conditions:

- $\mathbf{S}$ is adapted to the filtration generated by $\mathbf{D}$. In particular, $\mathbf{S}$ is nonanticipating. Thus, supply decisions can use information about past demand but not future demand.
- The sample paths of $\mathbf{S}$ are nondecreasing and càdlàg in both t and u.
- $\mathbf{S}$ satisfies

$$S(s, u) \leq S(t, u + s - t), \quad t < s < t + U - u. \tag{1}$$

This constraint expresses the condition that goods can move toward 0 at a rate no more than 1. When $\mathbf{D}$ is integer valued, so is $\mathbf{S}$, because only whole units can be used to meet demand; when $\mathbf{D}$ is continuous, $\mathbf{S}$ is real valued. The initial supply $S(0, u)$ is a given constant. It is nonnegative, nondecreasing, and càdlàg in u.

These properties are consistent with our usual notions of orders and shipments. Suppose $S(0, u) = 0$, and we order one unit at time 0. We subsequently move the unit so that its position describes a continuous, nonincreasing curve $\bar{u}(t)$, where $\bar{u}(0) = U$, $\bar{u}(t) < U$, $t > 0$, and, according to (1), $\bar{u}(s) - \bar{u}(t) \geq -(s - t)$ for $s > t$. Then, for $t > 0$, $S(t, u) = 1$ for $u \geq \bar{u}(t)$ and 0 otherwise. This function is indeed càdlàg as well as nondecreasing in both t and u.

In general, an order *at* time t does not affect $S(t, u)$ itself but rather $S(s, u)$ for $s > t$. The same is true of a decision to ship a unit from its current position to a lower one. Letting $t-$ or $u-$ denote a limit from below and $t+$ or $u+$ a limit from above, we can say that an order at time t increments $S(t+, U-)$. (This is a concise way of saying that, for any $s > t$, there exists $v < U$, such that $S(s, v) > S(t, v)$.) Likewise, a shipment from point u increments $S(t+, u-)$.

There is a positive linear holding cost for inventory at each point in $[0, U)$, whether the unit is moving or not, and a positive linear penalty cost for backorders at point 0.

$$b = \text{unit backorder penalty cost rate},$$

$$\hat{h}(u) = \text{unit holding cost rate at space point } u.$$

Also, let

$$h(u, v) = -[\hat{h}(v) - \hat{h}(u)], \quad u \leq v;$$
$$h(u) = h(0, u);$$
$$H(u) = \int_0^u h(v)\, dv;$$
$$H(u, v) = H(v) - H(u), \quad u \leq v.$$

Assume that $\hat{h}(u)$ is continuous on $[0, \bar{U}]$ and continuously differentiable (C^1) on $(0, \bar{U})$, for some $\bar{U} > U$, with $\hat{h}(U) = 0$. The function $h(u)$ measures how much cheaper it is to hold inventory at u instead of 0. Thus, $h(0)=0$, and $h(U)=\hat{h}(0)$. Denote the derivative of h by $h'(u)$ and assume $h'(u)>0$, $u\in(0,\bar{U})$. Also, assume the limit $h'(0+)$ exists and is finite. (Some of these assumptions can be relaxed, as mentioned in §4.2.) Rescale the costs so that $\hat{h}(0)+b=1$.

The objective is to minimize the long-run average total cost of the system. As usual in steady-state analysis of inventory systems, we can and do ignore linear purchase and shipment costs. The average total of such costs is a constant, independent of the control, for any control with finite average cost.

Denote

$$\begin{aligned} I(t,u) &= \text{net inventory in echelon } u \text{ at time } t \\ &= S(t,u)-D(t). \end{aligned}$$

This is the (on-hand, non-net) inventory in echelon u, minus backorders at point 0. Note that the backorder is $B(t)\equiv[I(t,0)]^-$. Thus, the inventory in echelon u is $I(t,u)+B(t)$. The total cost rate at time t can be written as

$$\begin{aligned} C(t\mid \mathbf{S}) &= \int_{0+}^{U-}\hat{h}(u)\,dI(t,u)+\hat{h}(0)[I(t,0)]^+ + bB(t) \\ &= \{\hat{h}(u)[I(t,u)+B(t)]\}_{u=0+}^{U-} \\ &\quad -\int_0^U \hat{h}'(u)\big[I(t,u)+B(t)\big]\,du \\ &\quad +\hat{h}(0)\big[I(t,0)+B(t)\big]+bB(t) \\ &= \int_0^U h'(u)\big[I(t,u)+B(t)\big]\,du+bB(t) \\ &= \int_0^U h'(u)I(t,u)\,du+(\hat{h}(0)+b)B(t) \\ &= \int_0^U h'(u)I(t,u)\,du+[I(t,0)]^-. \end{aligned}$$

The average cost is

$$C(\mathbf{S})=\limsup_{T\to\infty}\left\{\frac{1}{T}E\left[\int_0^T C(t\mid \mathbf{S})\,dt\right]\right\}.$$

Suppose we follow a stationary policy, such that for each u the random variables $I(t,u)$ have a stationary distribution over t. Let $I(u)$ denote a random variable with this same distribution. The mean cost rate in steady state is then

$$E\left[\int_0^U h'(u)I(u)\,du+[I(0)]^-\right]. \tag{2}$$

Under standard ergodic assumptions, $C(\mathbf{S})$ equals this quantity with probability 1.

3. Preliminaries: Discrete-Stage Model

To help understand and solve the supply stream model, we first consider policies where stock can be held only at discrete stages. This leads to a discrete-stage model. As mentioned in the introduction, such models are well understood. We present a brief review. Our treatment follows Zipkin (2000, §8.3), but with some different notation. Here, however, time is still continuous, and costs accumulate continuously, as above. This fact leads to some minor adjustments to the model.

3.1. Definition

The discrete-stage system contains N stages, with $N=2^k$ for a nonnegative integer k. The stages are spaced equally at distance $\varepsilon=U/N$ apart, specifically at $u=0,\varepsilon,2\varepsilon,\ldots,U-\varepsilon$. Stage 0 obtains its resupply from stage ε, stage ε from stage 2ε, etc. Stage $U-\varepsilon$ obtains stock from the outside source at U. This is a restriction of the model above. Goods between stages must keep moving at rate 1. Thus, ε is also the travel time from one stage to the previous one. We can express this condition as another constraint,

$$S(s,n\varepsilon)=S(t,n\varepsilon+s-t),\quad t<s<t+\varepsilon,\quad 0\le n<N, \tag{3}$$

in addition to (1). Let us refer to this as system k. Let $\mathcal{S}_k\subset\mathcal{S}$ denote the set of admissible controls for system k, namely, those that satisfy (3) as well as the three conditions above.

3.2. Echelon Base-Stock Policies

For $u=\varepsilon,2\varepsilon,\ldots,U$, $I(t,u-)$ is called the *echelon inventory transit position* at $u-\varepsilon$. This is $I(t,u-\varepsilon)$ plus inventory in transit from stage u, or in other words, $I(t,u)$ minus the inventory at u itself. It is something we partly control. As explained above, when we order a unit at time t, we increment $S(t+,U-)$ and thereby $I(t+,U-)$. When we pull a unit from point u and send it toward $u-\varepsilon$, we increment $I(t+,u-)$.

It turns out that a stationary policy based on monitoring and controlling the $I(t,u-)$, called an *echelon base-stock policy*, is optimal for this system. Such a policy is specified by a càdlàg (left-continuous with right limits, or LCRL) step function $y_k=y_k(u)$, $u\in(0,U]$, with steps at $u=n\varepsilon$. When $\mathbf{D}$ and hence $I(t,u)$ are integer valued, so is y_k. When $\mathbf{D}$ is continuous, y_k is real valued.

The idea is to try to keep $I(t,u-)$ as close as possible to $y_k(u)$. The value $y_k(U)$ governs external orders. Do not order while $I(t,U-)>y_k(U)$. When $I(t,U-)$ falls below $y_k(U)$, immediately order the difference. Thus, perhaps after a finite (with probability 1) initial period when $I(t+,U-)>y_k(U)$, the policy keeps $I(t+,U-)=y_k(U)$, and external orders precisely equal demands. The rest of y_k controls flow within the system. For $u=n\varepsilon$, while $I(t,u-)>y_k(u)$, make no shipments from u. When $I(t,u-)$ falls below $y_k(u)$, immediately dispatch the difference from u, provided there is enough stock there. If not, dispatch what is there. So, again perhaps after an initial period, $I(t+, u-) \le y_k(u)$. It is not hard to see that this policy describes an admissible control.

One can restrict attention to nonnegative y_k. Given any y_k with $y_k(u) < 0$ for some u, consider the revised policy $[y_k(u)]^+$. This policy has lower backorder costs than y_k and no greater holding costs.

Here is some notation: When x is an integer variable, for a function $a(x)$, denote

$$\Delta_x a(x) = a(x+1) - a(x),$$

as usual. Also, when ε is understood, for a function $a(u)$ of the real variable u, denote

$$\Delta_u a(u) = a(u+\varepsilon) - a(u).$$

An echelon base-stock policy does lead to a stationary distribution for the $I(t, u)$. (In fact, the system arrives at this steady state after a finite initial period, with probability 1.) This distribution can be described recursively as follows: Let $\bar{\mathbf{D}} = \{\bar{D}(u)\colon u \ge 0\}$ denote a process identical in law to $\mathbf{D}$, but with parameter u instead of t. We use $\bar{\mathbf{D}}$ to describe equilibrium behavior, whereas $\mathbf{D}$ describes real, finite time. Then, the steady-state random variables $I(u)$ satisfy

$$\begin{aligned} I(U-) &= y_k(U);\\ I(u) &= I((u+\varepsilon)-) - \Delta_u \bar{D}(u), \quad u = n\varepsilon,\ 0 \le n < N;\\ I(u-) &= \min\{y_k(u), I(u)\}, \quad u = n\varepsilon,\ 0 < n < N. \end{aligned} \tag{4}$$

Here, $\Delta_u \bar{D}(u)$ is the random variable $\bar{D}(u, u+\varepsilon)$, the demand during the interstage travel time. It follows that

$$I(u) = \min_v \{y_k(v) - \bar{D}(u, v)\colon v = m\varepsilon,\ n < m \le N\}, \quad u = n\varepsilon,\ 0 \le n < N. \tag{5}$$

Also, by (3), for $u \in (n\varepsilon, (n+1)\varepsilon)$, $I(u) = I((n+1)\varepsilon-) - \bar{D}(u, (n+1)\varepsilon)$.

We can use this steady-state distribution to compute the average cost (2). Let

$$c_k(u, x \mid y_k) = E\left[\int_0^u h'(v) I(v)\, dv + [I(0)]^- \mid I(u-) = x, y_k\right], \quad u = n\varepsilon,\ n > 0;$$

$$\bar{c}_k(u, x \mid y_k) = E\left[\int_0^u h'(v) I(v)\, dv + [I(0)]^- \mid I(u) = x, y_k\right], \quad u = n\varepsilon,\ n \ge 0.$$

(The conditioning event for c_k means start (4) at $u = n\varepsilon$ instead of U with $I(u-) = x$, and continue the recursion from that point using y_k. Interpret $\bar{c}_k$ similarly.) The variable x is integer valued for integer demand and real for continuous demand. We have $\bar{c}_k(0, x \mid y_k) = [x]^-$, and for $u = n\varepsilon$, $n > 0$,

$$\bar{c}_k(u, x \mid y_k) = c_k(u, y_k(u) \wedge x \mid y_k).$$

For $u = n\varepsilon$, $n \ge 0$,

$$\begin{aligned} &c_k(u+\varepsilon, x \mid y_k)\\ &= E\left[\int_0^{u+\varepsilon} h'(v) I(v)\, dv + [I(0)]^- \mid I((u+\varepsilon)-) = x, y_k\right]\\ &= E\left[\int_u^{u+\varepsilon} h'(v)(x - \bar{D}(v, u+\varepsilon))\, dv + \bar{c}_k(u, x - \Delta_u \bar{D}(u) \mid y_k)\right]\\ &= \int_u^{u+\varepsilon} h'(v)\big[x - \mu(u+\varepsilon-v)\big]\, dv\\ &\quad + E\big[\bar{c}_k(u, x - \Delta_u \bar{D}(u) \mid y_k)\big]. \end{aligned}$$

The integral in this expression equals

$$\begin{aligned} &\big\{h(v)[x - \mu(u+\varepsilon)] - \mu[H(v) - vh(v)]\big\}_{v=u}^{u+\varepsilon}\\ &\quad = \Delta_u h(u)\big[x - \mu(u+\varepsilon)\big] - \mu\big[\Delta_u H(u) - \Delta_u(uh(u))\big]\\ &\quad = \Delta_u h(u) x - \mu\big[\Delta_u H(u) - \varepsilon h(u)\big]. \end{aligned}$$

So the following algorithm evaluates c_k and $\bar{c}_k$ recursively:

$$\begin{aligned} \bar{c}_k(0, x \mid y_k) &= [x]^-;\\ c_k(u+\varepsilon, x \mid y_k) &= \Delta_u h(u) x - \mu[\Delta_u H(u) - \varepsilon h(u)]\\ &\quad + E[\bar{c}_k(u, x - \Delta_u \bar{D}(u) \mid y_k)], \quad u \ge 0;\\ \bar{c}_k(u, x \mid y_k) &= c_k(u, y_k(u) \wedge x \mid y_k), \quad u \ge \varepsilon. \end{aligned} \tag{6}$$

The average cost (2) is

$$\bar{c}_k(U, y_k(U) \mid y_k) = c_k(U, y_k(U) \mid y_k).$$

This has the same form as a standard discrete-stage model, except for the term $\mu[\Delta_u H(u) - \varepsilon h(u)]$. The standard model has a term $\mu\varepsilon\Delta_u h(u)$ instead. This is because in the standard model, stock in transit between stages is charged the holding cost of the upstream stage. Here, the holding cost varies continuously. This change just adds a constant to each of the functions above, independent of the policy y_k and x. (It does depend on u, of course.) Also, in the standard model, the expected cost rate we call $c_k(u+\varepsilon, x \mid y_k)$ is normally assigned to point u. This is merely a difference in labeling. Our labeling turns out to be convenient.

For $y_k \ge 0$ and $x \le 0$, an induction shows that

$$\begin{aligned} \bar{c}_k(u, x \mid y_k) = c_k(u, x \mid y_k) &= h(u)x - \mu H(u) + E[\bar{D}(u) - x]\\ &= -[1 - h(u)]x + \mu[u - H(u)] \equiv J(u, x). \end{aligned} \tag{7}$$

This is independent of y_k and k.

A similar algorithm determines an optimal policy y_k^* and its cost c_k^*.

$$\bar{c}_k^*(0,x) = [x]^-;$$
$$c_k^*(u+\varepsilon,x) = \Delta_u h(u)x - \mu[\Delta_u H(u) - \varepsilon h(u)] + E[\bar{c}_k^*(u, x-\Delta_u \bar{D}(u))], \quad u \ge 0; \tag{8}$$
$$y_k^*(u) = \arg\min_x \{c_k^*(u,x)\}$$
$$\bar{c}_k^*(u,x) = c_k^*(u, y_k^*(u) \wedge x), \quad u \ge \varepsilon.$$

The main results are, for each k,

- for all (u,x), $0 \le \bar{c}_k^*(u,x) \le c_k^*(u,x)$;
- for each u, $c_k^*(u,x)$ and $\bar{c}_k^*(u,x)$ are convex and $\bar{c}_k^*(u,x)$ is nonincreasing in x;
- $0 \le y_k^* < \infty$, and $\bar{c}_k^*(u,x) = c_k^*(u,x) = J(u,x)$, $x \le 0$;
- $c_k^*(u,x)$ is in fact strictly convex in x for $x \ge 0$, so the minimal $y_k^*(u)$ in (8) is unique;
- for integer demand, $\Delta_x \bar{c}_k^* \le \Delta_x c_k^*$;
- for continuous demand, c_k^* and $\bar{c}_k^*$ are C^1 in x for $x > 0$, with $\partial \bar{c}_k^*/\partial x \le \partial c_k^*/\partial x$; and
- y_k^* is indeed the best echelon base-stock policy, that is, $c_k^*(\cdot,\cdot) = c_k(\cdot,\cdot \mid y_k^*) \le c_k(\cdot,\cdot \mid y_k)$ for any other y_k.

As previously mentioned, this last fact implies that y_k^* is optimal over all policies for system k. Let $c_k^{**} = c_k^*(U, y_k^*(U))$ denote the optimal average cost.

We remark that, in the usual presentation of the discrete-stage (and discrete-time) model with continuous demand, it is assumed that the distribution of the analogue of $\Delta_u \bar{D}(u)$ is absolutely continuous everywhere. In that case, c_k^* is C^1 in x for all x. Here, there is a kink at $x = 0$, because of the mass of Ψ_0 there. This is why we say only that c_k^* is C^1 for $x > 0$. (Of course, it is also C^1 for $x < 0$.) By convexity, however, the limit $\partial c_k^*(u, 0+)/\partial x$ exists, with $\partial c_k^*(u,0+)/\partial x \ge -[1-h(u)]$.

3.3. First-Passage Times and Stockout Probabilities

Next, we review an interesting characterization of y_k^* (van Houtum and Zijm 1991). (Later, in §§4 and 5, we shall show that there is an analogous characterization in the continuous-stage model.) For the case of continuous demand and real x, assume $y_k^*(u) > 0$ for all $u = n\varepsilon$, $n > 0$. Then,

$$\frac{\partial}{\partial x} c_k^*(u,x) = h(u) - \Pr\{I(0) \le 0 \mid I(u-) = x, y_k^*\}, \quad x > 0. \tag{9}$$

In particular, since $y_k^*(u)$ solves $\partial c_k^*/\partial x = 0$,

$$\Pr\{I(0) \le 0 \mid I(u-) = y_k^*(u), y_k^*\} = h(u). \tag{10}$$

Moreover, as in (5),

$$[I(0) \mid I(u-) = y_k^*(u), y_k^*] = \min_v \{y_k^*(v) - \bar{D}(v)\colon v = m\varepsilon, 0 < m \le n\}.$$

So the event $\{I(0) > 0 \mid I(u-) = y_k^*(u), y_k^*\}$ is equivalent to $\{\bar{D}(v) < y_k^*(v)\colon v = m\varepsilon, 0 < m \le n\}$. As pointed out by de Kok and Fransoo (2003), this can be interpreted as a statement about a first-passage time. Define the random variable

$$\tau_k = \inf_n \{n\varepsilon\colon n > 0, \bar{D}(n\varepsilon) \ge y_k^*(n\varepsilon)\}.$$

This is the first-passage time of the process $\bar{\mathbf{D}}$, observed at the discrete points $n\varepsilon$, to the function y_k^*. So for $u = n\varepsilon$,

$$\{I(0) > 0 \mid I(u-) = y_k^*(u), y_k^*\} \iff \{\tau_k > u\}.$$

This and (10) imply

$$\Pr\{\tau_k \le u\} = h(u).$$

In other words, the $y_k^*(u)$ are those values, such that the first-passage time τ_k has cdf h. (A similar but more complex relation holds for integer demand; see Doğru et al. 2008.) This connection will be useful later.

Notice that (10) specifies the stockout probability under the optimal policy, namely, $h(u)$, for a system of length u. This relation is independent of the demand distribution. Once the holding-cost structure is known, this performance measure is immediate. This finding generalizes a well-known property of the single-stage model (see, e.g., Zipkin 2000, §6.4). Because $h(u)$ is increasing, the longer the system is, the higher is the stockout probability. This is perhaps not obvious; larger u means lower holding cost, so one might expect to find a *lower* stockout probability. But this logic is flawed. The result refers to the stockout probability at 0, not u, and between u and 0 lie all the points $v < u$ with their larger holding costs.

Sobel (2004) derives a related performance measure—the fill rate—for discrete-time systems. This quantity does involve the demand distribution. Consistent with the finding here, however, longer supply chains have lower fill rates.

4. Optimal Policy

In this section, we characterize the optimal policy and cost for the continuous-stage model. We show that they are the limits of the optimal policy and cost of the discrete-stage model, respectively, as the distances between the stages go to zero. In particular, we show that a stationary echelon base-stock policy is optimal.

In the continuous-stage system, a stationary echelon base-stock policy can be specified by a càdlàg function $y = y(u)$. Its range is determined by that of $\mathbf{D}$. The value $y(U)$ governs external orders. When $I(t, U-)$ falls below $y(U)$, order the difference. In steady state, $I(U-) = y(U)$, and external orders equal demands. The rest of the policy tries to keep $I(t, u-)$ as close as possible to $y(u)$. When $I(t, u-)$ falls below $y(u)$, immediately dispatch the difference from u, to the extent possible given $I(t, u)$. As in the discrete-stage model, perhaps after a finite initial period, $I(t+, u-) \le y(u)$. Clearly, the corresponding $\mathbf{S} \in \mathcal{S}$. It is not hard to show that, in equilibrium, as in (5),

$$I(u) = \inf_v \{y(v) - \bar{D}(u, v)\colon u < v \le U\}. \tag{11}$$

4.1. Monotonicity and Limits of Discrete-Stage Solutions

We now relate the discrete-stage solution to the continuous-stage model. Note that each y_k defined in §3.2 counts as a policy for the continuous system. In fact, any policy $y(u)$ that is constant over an interval $u \in (v, w)$ retains no inventory in the interval in steady state. (Of course, all inventory must *pass through* the interval, but none of it stops or slows down there.) So a step-function y retains inventory only at its jump points.

The functions $c_k(u, x \mid y_k)$ are defined in §3.2 for discrete $u = n\varepsilon$. We can interpolate them for all u as follows: for $n\varepsilon < u \le (n+1)\varepsilon$, set

$$\begin{aligned} c_k(u, x \mid y_k) &= h(n\varepsilon, u)x - \mu\big[H(n\varepsilon, u) - (u - n\varepsilon)h(n\varepsilon)\big] \\ &\quad + E\big[\bar{c}_k\big(n\varepsilon, x - \bar{D}(n\varepsilon, u) \,\big|\, y_k\big)\big]. \end{aligned} \tag{12}$$

Also, define $c_k(0, x \mid y_k) = [x]^-$. Similarly, we can interpolate the functions c_k^*. Because each $\bar{c}_k^*(n\varepsilon, x)$ is convex in x, so is $c_k^*(u, x)$ for all u. Also, set $y_k^-(u) = \arg\min_x\{c_k^*(u, x)\}$, $u > 0$. (Note that this is not the same as the step function y_k^*, although they agree at the points $u = n\varepsilon$.) Also, interpolate $\bar{c}_k^*(u, x) = c_k^*(u, y_k^-(u) \wedge x)$, $n\varepsilon < u < (n+1)\varepsilon$.

The recursion (6) evaluates any step-function policy y_k. Now, consider a policy y for the continuous-stage problem. Construct a sequence of step-function policies y_k that approximate y in a sensible way, e.g., $y_k(u) = y(\lceil u/\varepsilon \rceil \varepsilon)$. Then the functions $c_k(u, x \mid y_k)$ converge to the limit $c(u, x \mid y)$, where

$$\begin{aligned} &c(u, x \mid y) \\ &\quad = E\left[\int_0^u h'(v)I(v)\,dv + [I(0)]^- \mid I(u-) = x, y\right], \end{aligned} \tag{13}$$

and, as in (11),

$$\begin{aligned} &\big[I(v) \,|\, I(u-) = x, y\big] \\ &\quad = \min\Big\{x - \bar{D}(v, u), \\ &\qquad\qquad \inf_w\big\{y(w) - \bar{D}(v, w)\colon v < w < u\big\}\Big\}. \end{aligned} \tag{14}$$

(The sequence $c_k(u, x \mid y_k)$ just expresses the definition of this integral.) The average cost (2) of the policy is thus $c(U, y(U) \mid y)$. Also, by (7), for $y \ge 0$ and $x \le 0$,

$$c(u, x \mid y) = J(u, x), \tag{15}$$

independent of y.

Demonstrating the convergence of the c_k^* is a bit harder.

Proposition 1. (a) *For each (u, x), $c_k^*(u, x)$ and $\bar{c}_k^*(u, x)$ are nonincreasing in k. These functions converge pointwise to a common limit c^* as $k \to \infty$. This function $c^*(u, x)$ is convex and nonincreasing in x for each u.*

(b) *For integer (continuous) demand, $\Delta_x c_k^*$ and $\Delta_x \bar{c}_k^*$ ($\partial c_k^*/\partial x$ and $\partial \bar{c}_k^*/\partial x$) are nonincreasing in k. These functions too converge pointwise to limits as $k \to \infty$. For each $u > 0$, $y_k^*(u)$ and $y_k^-(u)$ are nondecreasing in k, and they converge to a limit $y^*(u)$ (for now possibly ∞). This $y^*(u)$ minimizes $c^*(u, x)$ over x.*

(c) *For integer (continuous) demand, $\Delta_x \bar{c}_k^*$ ($\partial \bar{c}_k^*/\partial x$) converges to its limit uniformly in (u, x). (Thus, c^* is C^1 in x, $x > 0$.) For both cases, c^* is continuous in u, and y^* is càdlàg.*

(d) *The policy y^* is optimal among echelon base-stock policies. Specifically, $c^*(\cdot\,, \cdot) = c(\cdot, \cdot \mid y^*) \le c(\cdot, \cdot \mid y)$ for all y.*

(e) *For $u > 0$ and $x < y^*(u)$, c^* is C^1 in u, and*

$$\frac{\partial}{\partial u} c^*(u, x) = h'(u)x + \lambda\big[E[c^*(u, x - X)] - c^*(u, x)\big]. \tag{16}$$

For integer demand, presuming $y^*(u) < \infty$, this holds also for $x = y^*(u)$.

(f) *For integer demand,*

$$y^*(u) = \inf\big\{x\colon \Delta_x c^*(u-, x') < 0, x' < x\big\}, \tag{17}$$

$$\begin{aligned} y^*(u) &= \inf\big\{x\colon x \ge 0, h'(u) \\ &\qquad + \lambda E\big[\Delta_x c^*(u, x - X)\big] \ge 0\big\}, \end{aligned} \tag{18}$$

and $y^*(u)$ is finite for all $u > 0$. Analogous results hold for continuous demand.

Let $c^{**} = c^*(U, y^*(U))$ denote the limiting optimal average cost.

Proposition 2. *The policy y^* is optimal for the continuous-stage system.*

Note that, by the proof of Proposition 1(c), the first-passage-time interpretation for continuous demand also carries over to the limit. Specifically,

$$\Pr\{\tau \le u\} = \Pr\big\{I(0) \le 0 \mid I(u-) = y^*(u), y^*\big\} = h(u).$$

Again, under the optimal policy, $h(u)$ is the cdf of τ, as well as the stockout probability of a system of length u.

Proposition 1 also implies that $c_0^*(u, x)$ is an upper bound on $c^*(u, x)$ and that $y_0^-(u)$ is a lower bound on $y^*(u)$. These bounds are easy to compute. By (12),

$$c_0^*(u, x) = h(u)x - \mu H(u) + \Psi_1(u, x) \tag{19}$$

and

$$y_0^-(u) = \arg\min_x\{c_0^*(u, x)\}. \tag{20}$$

In the case of continuous demand, $y_0^-(u)$ solves

$$0 = \frac{\partial}{\partial x} c_0^*(u, x) = h(u) - \Psi_0(u, x),$$

provided this equation has a positive solution; otherwise, $y_0^-(u) = 0$. For integer demand, the corresponding first-order condition is $y_0^-(u) = \min\{x: h(u) - \Psi_0(u, x) \geq 0\}$. This calculation for each u is just the solution of a single-stage system, which has the same form as a newsvendor problem.

This result is analogous to the bounds for discrete-stage systems obtained by Gallego and Zipkin (1999), Shang and Song (2003), Dong and Lee (2003), and Gallego and Özer (2005). (They also derive lower bounds on c_k^* and upper bounds on y_k^*. We have not been able to extend those results to the continuous-stage model.)

4.2. Discussion and Extensions

Notice that $y_0^-(u)$ need not be increasing in u. It is true that $\bar{D}(u)$ is stochastically increasing in u, so $\Psi_0(u, x)$ is increasing. So the solution to the equation $\Psi_0(u, x) = \bar{h}$ for a *fixed constant* $\bar{h}$ is increasing in u. But $h(u)$ is not fixed. It is increasing, and this tends to *reduce* the solution $y_0^-(u)$. The net effect of these opposing influences depends on the particulars of h and Ψ_0. The same contention appears in (8) for larger k. In $c_k^*(u + \varepsilon, x)$, the expectation over $\Delta_u \bar{D}(u)$ tends to push $y_k^*(u + \varepsilon)$ above $y_k^*(u)$, but the positive linear term $\Delta_u h(u)x$ tends to pull it down. Thus, we should not be surprised to find a y_k^* that is decreasing in some places. The same logic applies to the limiting y^*.

However, for both the discrete- and continuous-stage systems, given a policy y that is *not* nondecreasing, one can construct another policy $\acute{y}$ that *is* nondecreasing and that is equivalent to y. Specifically, $\acute{y}(u) = \min\{y(v): u \leq v \leq U\}$. (For example, consider system k with $y_k(u) > y_k(u + \varepsilon)$ for some u. After a finite time, the system reaches and remains in states with $I(t, (u + \varepsilon)-) \leq y_k(u + \varepsilon)$. So $I(t, u-) \leq I(t, (u + \varepsilon)-) \leq y_k(u + \varepsilon)$. The policy *tries* to pull more stock from $u + \varepsilon$, to raise $I(t, u-)$ closer to $y_k(u)$, but it can not; there is no more stock there to pull. One can prove the equivalence formally using (5).) Having computed an optimal policy y_k^* (or y^*), one can construct another policy $\acute{y}_k^*$ (or $\acute{y}^*$), also optimal, that is nondecreasing. (The first-passage-time interpretations, however, apply only to y_k^* and y^*, not $\acute{y}_k^*$ and $\acute{y}^*$.)

This notion allows us to relax the assumption that $h'(u) > 0$. For system k, in (8), if $\Delta_u h(u - \varepsilon) \leq 0$, then $c_k^*(u, x)$ is decreasing in x, so $y_k^*(u) = \infty$, and $\bar{c}_k^*(u, x) = c_k^*(u, x)$. The recursion can continue any way. The resulting policy y_k^* includes some infinite values, which are awkward to interpret or implement. However, provided $y_k^*(U) < \infty$, the policy $\acute{y}_k^*$ has only finite values. The same considerations apply to y^*. As in Proposition 1(f), to ensure $y^*(U) < \infty$, we must still assume $h'(U) > 0$. Otherwise, the model is ill posed, like a single-stage model with nonpositive holding cost.

Also, we can allow discontinuous (but still càdlàg) h and $\hat{h}(U) > 0$. These extensions complicate the notation and arguments, but not the basic results.

Incidentally, the restriction to equally spaced points is purely for convenience. One can define the recursions (6) and (8) for any sequence of points u. (Online Appendix B provides an example.) The monotonicity results in Proposition 1 go through for any refinement of a given sequence (i.e., a new sequence obtained by adding more points to an old one). The convergence results also remain valid, provided the sequence becomes dense.

To further understand the model and its solution, the next section examines a specific (approximate) demand process. Online Appendix B discusses another one—Poisson demand.

5. Brownian-Motion Approximation

In this section we focus on a specific demand process and provide a detailed characterization of the optimal cost and policy.

5.1. Discussion

Assume the equilibrium demand process $\bar{\mathbf{D}}$ has positive, finite variance rate $\sigma^2 = V[\bar{D}(1)]$. We shall approximate $\bar{\mathbf{D}}$ by a Brownian motion $\check{\mathbf{D}}$ with the same first two moment parameters, μ and σ^2. The state variable x is now continuous.

For a single-stage system, the normal approximation of lead-time demand (which corresponds to $\bar{D}(U)$ here) is a well-established technique. It is often accurate. It not only makes computation easier, but it also reveals the primary effects of the system's parameters on the optimal policy and its cost. It is thus of great practical and pedagogical value. The aim of this section is to investigate whether analogous results can be found for supply streams. As we shall see, the answer is yes, partly. We show that the effects of μ and σ^2 on the solution are quite transparent, as for single-stage systems. (See (21) below.) Also, for certain cost functions h, we obtain the entire solution in closed form; for others we obtain nearly explicit solutions. But in general, the computation remains challenging. We explain the challenges and present methods for dealing with them.

Recall that we assumed that the demand increments are nonnegative. The theory underlying the discrete-stage model relies on this assumption. A Brownian motion, of course, violates it. So, we cannot allow $\mathbf{D}$ to be a Brownian motion. However, it is possible to perform the calculations indicated in (6) and (8) using normal demand increments, treating the results as approximations. (Federgruen and Zipkin 1984 do just that for the case of two stages.) This is exactly analogous to the use of normal demand in a single-stage model.

Specifically, consider the discrete model of §3.1 with one stage; that is, $k = 0$ and $N = 1$. A base-stock policy has only one parameter, $y = y(U)$. After a finite time, $I(t, U-) = y$, and $I(t+U, 0) = y - D(t, t+U)$. In equilibrium, therefore, $I(U-) = y$, and $I(0) = y - \bar{D}(U)$. The average cost as a function of y is $c_0(U, y \mid y)$. We can approximate this *function* by another one, replacing $\bar{D}(U)$ with $\breve{D}(U)$. We expect this approximation to be accurate, precisely when the central limit theorem applies, namely, when $\bar{D}(U)$ is the sum of many (say M) independent components. If so, optimization over the approximate function will produce good results. Abundant numerical evidence supports this notion (see, e.g., Zipkin 2000, §6.4).

We apply the same idea for each k. The equilibrium distribution is described by (4), and we approximate the increments $\Delta_u \bar{D}(u)$ there and in the equivalent cost functions of (6) and (8) by $\Delta_u \breve{D}(u)$. Again, we expect the approximation to be accurate under the conditions of the central limit theorem. However, as N increases, we divide $\bar{D}(U)$ into smaller and smaller pieces. To continue to invoke the central limit theorem at each step, therefore, we must assume that M increases along with N.

Most of the development of §4 remains valid with this approximation. For instance, the functions c_k^* are convex in x, and they converge to a limit c^*. There are some differences, however. It is no longer necessarily true that $y_k^* \geq 0$ and $y^* \geq 0$. Also, neither $c_k^*(u, x)$ nor $c^*(u, x)$ equals $J(u, x)$ for $x \leq 0$. (However, as $x \to -\infty$, the difference goes to 0.) The smoothness properties of c_k^* and c^* are also different; we shall consider those below. And, of course, Proposition 2 does not apply; we are not discussing the original control problem, only the class of echelon base-stock policies in equilibrium.

Several authors, starting with Bather (1966), have studied single-stage models, assuming Brownian-motion demand $\mathbf{D}$ at the outset. These models are different from ours, and they lead to different results. For example, in the special case of Bather's model with no fixed order cost, his policy reduces to a base-stock policy. The analogue of $I(U-)$ is *not* the constant y, but instead y plus an exponential random variable, representing the negative excursions of $\mathbf{D}$. The average cost of any such policy—and the optimal one—is also different.

DeCroix et al. (2005) consider a discrete-stage system with returns as well as demands. This leads to net demand increments that can be negative. They show how to do an exact analysis. Suffice it to say that this requires far more intricate calculations than the model here.

Our approach differs also from that of Harrison (1985) and others to the approximation of control problems. There, the starting point is an intractable problem. One considers a sequence of such problems that, when properly rescaled, converges to a limiting control problem, whose dynamics are driven by a Brownian motion. This problem often is much easier to analyze than the original. Our problem, in contrast, is quite tractable, at least qualitatively. Here, Brownian motion enters the analysis only in equilibrium and only for a class of policies known to be optimal. The purpose is mainly to simplify the calculations and sensitivity analysis. (Recall, however, that our system is uncapacitated. The approach of Harrison (1985) is most powerful in dealing with capacitated systems. We suspect that, to extend our model to finite capacities, something like that approach will prove essential.)

5.2. Transformations

We now apply a sequence of transformations to the optimal cost function c_k^*. The aim is to obtain a direct characterization of the first-passage times τ_k (in this subsection) and their limit as $k \to \infty$ (in §5.3). The characterization turns out to be closely related to the well-studied first-passage time problem in the Brownian motion literature (§5.3). This connection, in turn, leads to closed-form solutions and bounds for the optimal policy (§5.4) and efficient solution approaches for computing the optimal policy (§5.5).

Denote by ϕ the standard normal density function and Φ_0 its complementary cdf. The interpolation formula (12), with the approximate increments $\breve{D}(n\varepsilon, u)$ in place of $\bar{D}(n\varepsilon, u)$, implies that, for all $u > 0$ and $x \in \mathbb{R}$, c_k^* is twice continuously differentiable (C^2) in x. This (and more; see Cannon 1984) follows from the smoothness of ϕ.

First, define

$$p_k^*(u, x) = c_k^*(u, x) - J(u, x),$$

where $J(u, x) = h(u)x - \lambda H(u)$. In these terms, the recursion (8) with the approximate demand increments $\Delta_u \breve{D}(u)$ becomes

$$\begin{aligned}
\bar{p}_k^*(0, x) &= [x]^+;\\
p_k^*(u+\varepsilon, x) &= E\big[\bar{p}_k^*(u, x - \Delta_u \breve{D}(u))\big], \quad u \geq 0;\\
\frac{\partial p_k^*}{\partial x}(u, y_k^*(u)) &= 1 - h(u);\\
\bar{p}_k^*(u, x) &= p_k^*(u, y_k^*(u) \wedge x);\\
&\quad + [1 - h(u)][x - y_k^*(u)]^+, \quad u \geq \varepsilon.
\end{aligned}$$

Next, standardize the variable x, the policy y_k^*; and $\breve{\mathbf{D}}$. Denote

$$w=\frac{x-\mu u}{\sigma},\quad z_k^*(u)=\frac{y_k^*(u)-\mu u}{\sigma},\quad W(u)=\frac{\breve{D}(u)-\mu u}{\sigma},$$

and define

$$q_k^*(u,w)=p_k^*(u,\mu u+\sigma w)/\sigma.$$

Now, (8) becomes

$$\begin{aligned}\bar{q}_k^*(0,w)&=[w]^+;\\ q_k^*(u+\varepsilon,w)&=E[\bar{q}_k^*(u,w-\Delta_u W(u))],\quad u\ge 0;\\ \frac{\partial q_k^*}{\partial w}(u,z_k^*(u))&=1-h(u);\\ \bar{q}_k^*(u,w)&=q_k^*(u,z_k^*(u)\wedge w)\\ &\quad+[1-h(u)][w-z_k^*(u)]^+,\quad u\ge\varepsilon.\end{aligned}$$

This construction shows that, in the original recursion (8), up to the normal approximation, the parameters μ and σ affect the solution as linear translation and scale factors. Specifically, $y_k^*(u)=\mu u+\sigma z_k^*(u)$, and

$$\begin{aligned}c_k^{**}&=c_k^*(U,y_k^*(U))\\ &=J\big(U,\mu U+\sigma z_k^*(U)\big)+\sigma q_k^*\big(U,z_k^*(U)\big).\end{aligned}\tag{21}$$

The same is true of the limits y^* and c^{**}. Both can be expressed in terms of z^*, the standardized version of y^*. This effect is precisely analogous to the familiar result for single-stage systems mentioned above. Note that the effect of σ^2 is correctly described as a square-root law. However, recalling the discussion in §4.2, we are comparing two entire systems, not the solutions for different u within one system. Thus, the problem of determining y_k^* reduces to finding z_k^*.

Now, since c_k^* is C^2 in x, q_k^* is C^2 in w. So we can define

$$G_k^*(u,w)=\frac{\partial q_k^*(u,w)}{\partial w},$$

$$g_k^*(u,w)=\frac{\partial G_k^*(u,w)}{\partial w}=\frac{\partial^2 q_k^*(u,w)}{\partial w^2}.$$

In terms of G_k^*, (8) becomes

$$\begin{aligned}\bar{G}_k^*(0,w)&=\mathbf{1}\{w\ge 0\};\\ G_k^*(u+\varepsilon,w)&=E[\bar{G}_k^*(u,w-\Delta_u W(u))],\quad u\ge 0;\\ G_k^*(u,z_k^*(u))&=1-h(u);\\ \bar{G}_k^*(u,w)&=G_k^*(u,z_k^*(u)\wedge w),\quad u\ge\varepsilon.\end{aligned}\tag{22}$$

And in terms of g_k^*, denoting by δ the Dirac delta centered at 0,

$$\begin{aligned}\bar{g}_k^*(0,w)&=\delta(w);\\ g_k^*(u+\varepsilon,w)&=E\big[\bar{g}_k^*(u,w-\Delta_u W(u))\big],\quad u\ge 0;\\ \int_{-\infty}^{z_k^*(u)} g_k^*(u,v)\,dv&=1-h(u);\\ \bar{g}_k^*(u,w)&=\begin{Bmatrix} g_k^*(u,w) & w<z_k^*(u). \\ 0 & w\ge z_k^*(u)\end{Bmatrix},\quad u\ge\varepsilon.\end{aligned}\tag{23}$$

A careful look at (22) and (23) reveals that they describe the first-passage time τ_k of the process $\breve{\mathbf{D}}$ observed at discrete points $u=n\varepsilon$ to the boundary y_k^*. (We do not need the condition $y_k^*(u)>0$ here to get (10), because of the smoothness of c_k^*.) First, note that τ_k is also the first-passage time of the standardized process $\mathbf{W}$ to the standardized boundary z_k^*.

Proposition 3.

$$\begin{aligned}G_k^*(u,w)&=\Pr\{W(u)\le w,\ \tau_k>u-\varepsilon\},\quad u=n\varepsilon>0;\\ \bar{G}_k^*(u,w)&=\Pr\{W(u)\le w,\ \tau_k>u\},\quad u=n\varepsilon\ge 0.\end{aligned}$$

Consequently, for $u>0$, $\bar{g}_k^*(u,w)$ is the probability density of $[W(u)=w,\ \tau_k>u]$ (i.e., the conditional density of $[W(u)=w\mid\tau_k>u]$, times $\Pr\{\tau_k>u\}$).

5.3. Limiting First-Passage Time

Let us now consider the limit as $k\to\infty$. Let τ denote the first-passage time of $\mathbf{W}$, observed at all $u>0$, to the limiting z^*, the standardized version of y^*; that is, $\tau=\inf_u\{u>0\colon W(u)\ge z^*(u)\}$. By Proposition 1, we know that G_k^* converges to a limit, say G^*, and $\bar{G}_k^*$ converges to the same limit. Indeed, the limit of (22) as $k\to\infty$ is one way to construct τ, using a standard construction of $\mathbf{W}$. We therefore have the following result:

Corollary 4. *As $k\to\infty$, τ_k converges to τ in distribution. That is,*

$$G^*(u,w)=\Pr\{W(u)\le w,\ \tau>u\}.$$

The first-passage time τ of a standard Brownian motion $\mathbf{W}$ to a boundary z^* and its distribution h have received much scrutiny. The *original* first-passage time problem takes the boundary z^* as given and aims to determine h. Our problem (given h, determine z^*) is called the *inverse* first-passage time problem.

For the original problem, clearly, any càdlàg boundary z^* leads to a well-defined distribution h. (On certain intervals, $z^*(u)$ can even be infinite. If so, h is constant on those intervals.) However, many boundaries lead to the trivial distribution $h(0)=\Pr\{\tau=0\}=1$. To avoid this, the boundary must be an *upper* function. The precise condition is rather technical (see, e.g., Peskir 2002a). A sufficient condition is $z^*(0+)>0$. For an upper z^*, $h(0)=\Pr\{\tau=0\}=0$. If an upper z^* is continuous on $[0,\infty]$, then so is h. And, if z^* is C^1 on $(0,\infty)$, then so is h, with $h'(u)>0$, $u>0$ (see Peskir 2002b, Theorem 4.1, and the references therein).

Cheng et al. (2007) consider the inverse problem. (They consider the first passage of $\mathbf{W}$ *down* to a lower boundary, instead of *up* to an upper boundary. We translate their results to the latter scenario.) Their setting is quite general. Their $\mathbf{W}$ is a diffusion process. (Because we assume Brownian motion, however, we state only their results for this case.) They assume

h is any càdlàg cdf with $h(0)=0$. Their approach begins with a variational inequality. They prove that this inequality has a unique solution, namely, G^*, in the viscosity sense. (Under our stronger conditions on h, our Propositions 1 and 3 comprise an alternate proof that G^* is indeed the solution.) The boundary z^* is then defined as $z^*(u)=\sup\{w\colon G^*(u,w)<1-h(u)\}$. This is an upper function. When h is continuous, then so is G^*. When h is also C^1 on $(0,\infty)$, then G^* is C^1 in (u,w) for $u>0$ and $w\in\mathbb{R}$. Also, if $h'(u)>0$, then $z^*(u)$ is finite.

Most importantly for our case (where h is C^1), for $u>0$,

$$g^*(u,w)=\lim_{k\to\infty}\{g_k^*(u,w)\}=\lim_{k\to\infty}\{\bar{g}_k^*(u,w)\}=\frac{\partial G^*(u,w)}{\partial w}.$$

This limit is continuous, and it is the density of $[W(u)=w,\tau>u]$. This implies that the limiting c^* is C^1 in (u,w) and C^2 in w. Moreover, G^* itself is C^2 in w for $w<z^*(u)$ and $u>0$, and G^* and z^* jointly satisfy a partial differential equation (pde):

$$G^*(0+,w)=\mathbf{1}\{w\ge 0\},$$

and for $u>0$,

$$\frac{\partial G^*}{\partial u}=\frac{1}{2}\frac{\partial^2 G^*}{\partial w^2},\quad w<z^*(u);$$
$$G^*(u,z^*(u))=1-h(u);$$
$$G^*(u,w)=G^*(u,z^*(u)),\ \ w\ge z^*(u).$$

(One can also derive this pde from (22) by a limiting argument, *assuming* G^* is smooth enough.) This in turn implies (see, e.g., Cannon 1984) that g^* is C^1 in (u,w) and C^2 in w for $w<z^*(u)$, $u>0$, and g^* and z^* jointly satisfy the pde

$$g^*(0+,w)=\delta(w),$$

and for $u>0$,

$$\frac{\partial g^*}{\partial u}=\frac{1}{2}\frac{\partial^2 g^*}{\partial w^2},\quad w<z^*(u);$$
$$\frac{1}{2}\frac{\partial g^*}{\partial w}(u,z^*(u))=-h'(u);$$
$$g^*(u,z^*(u))=0;$$
$$g^*(u,w)=0,\quad w\ge z^*(u).$$

The equation involving $\partial g^*/\partial w$ follows from

$$\frac{\partial G^*}{\partial u}=\frac{1}{2}\frac{\partial^2 G^*}{\partial w^2}=\frac{1}{2}\frac{\partial g^*}{\partial w}.$$

This pde is essentially the same one used to describe the original first-passage time problem (see, e.g., Daniels 1982, Lerche 1986, Durbin 1988).

Let us restate this last pde in a slightly different form.

$$g^*(0+,w)=\delta(w), \tag{24}$$

and for $u>0$,

$$\frac{\partial \tilde{g}^*}{\partial u}=\frac{1}{2}\frac{\partial^2 \tilde{g}^*}{\partial w^2},\quad w\in\mathbb{R}; \tag{25}$$
$$\frac{1}{2}\frac{\partial \tilde{g}^*}{\partial w}(u,z^*(u))=-h'(u); \tag{26}$$
$$\tilde{g}^*(u,z^*(u))=0; \tag{27}$$
$$g^*(u,w)=\tilde{g}^*(u,z^*(u)\wedge w). \tag{28}$$

In this formulation we seek *two* functions, g^* and an auxiliary function $\tilde{g}^*$, linked by Equation (28). The auxiliary function extends the pde past the boundary. In this sense it is simpler than g^* itself. The key point is, that we do not care at all about $\tilde{g}^*(u,w)$ for $w>z^*(u)$. This flexibility will prove useful later.

Let us briefly summarize some other results on first-passage times of Brownian motion:

- The exact solution is known in a few special cases, discussed below. But the general problem remains challenging. There are several numerical methods available, also discussed below.
- A good deal is known about the asymptotic behavior (for small and large u) of h and z^* (see, e.g., Peskir 2002a, Redner 2001, Zipkin 2013).
- The exact relation between h and z^* can be expressed by certain integral equations (see, e.g., Peskir 2002b, Jaimungal et al. 2009).
- The latter paper also shows the effects on z^* of certain transformations of h and vice versa. The simplest of these is a change of scale: For any positive γ, if we change $h(u)$ to $h^\gamma(u)=h(u/\gamma)$, then the corresponding boundary is $z^{\gamma*}(u)=\sqrt{\gamma}z^*(u/\gamma)$. The reader may recognize this as a diffusion scaling, which rescales both the u and the w dimensions in a way that preserves the law of $\mathbf{W}$. The effects on the other parts of the supply-stream model are equally simple:

$$U^\gamma=\gamma U,\qquad H^\gamma(u)=\gamma H(u/\gamma),$$
$$J^\gamma(u,x)=\gamma J(u/\gamma,x/\gamma),$$
$$q^{\gamma*}(u,w)=\sqrt{\gamma}q^*(u/\gamma,w/\sqrt{\gamma}),$$
$$y^{\gamma*}(u)=\gamma\mu(u/\gamma)+\sqrt{\gamma}\sigma z^*(u/\gamma),$$

and, as in (21),

$$c^{\gamma**}=-[1-h(U)]\big[\gamma\mu U+\sqrt{\gamma}\sigma z^*(U)\big]$$
$$+\gamma\mu[U-H(U)]+\sqrt{\gamma}\sigma q^*(U,z^*(U)).$$

The effect of γ, then, is similar to that of changing μ and σ^2, keeping σ^2/μ constant.

5.4. The Method of Images and Closed-Form Solutions

The method of images, a venerable technique in physics, was introduced to first-passage problems by

Daniels (1982) and elaborated by Lerche (1986). It is a synthetic method—it enables one to construct both a problem and its solution by fairly simple means. In terms of the system (24)–(28) above, the idea is as follows: Pick *any* initial condition for $\tilde{g}^*$ that does not interfere with the condition (24) for g^*. That is, freely choose $\tilde{g}^*(0+, w)$ for $w > 0$, leaving $\tilde{g}^*(0+, w) = g^*(0+, w) = \delta(w)$ for $w \le 0$. Solve the pde (25) with no additional conditions. Given the solution $\tilde{g}^*$, use Equation (27) to *define* $z^*(u)$. Then, use Equation (26) to *determine* $h(u)$. In this way we find explicit solutions to certain cases, albeit artfully constructed ones.

One fairly general form for $\tilde{g}^*(0+, w)$ is $-A(dw)$, where A is a positive, σ-finite measure on the positive real numbers with

$$\int_0^\infty \phi(w) A(dw) < \infty.$$

The solution to (25) can then be written as

$$\tilde{g}^*(u, w) = \frac{1}{\sqrt{u}}\phi\left(\frac{w}{\sqrt{u}}\right) - \int_0^\infty \frac{1}{\sqrt{u}}\phi\left(\frac{\xi - w}{\sqrt{u}}\right) A(d\xi).$$

For each $u > 0$ separately, $z^*(u)$ solves

$$\tilde{g}^*(u, w) = 0.$$

It turns out (see Lerche 1986) that there is a unique solution, and the resulting $z^*(u)$ is concave and smooth (infinitely differentiable). Also, if the support of A starts at $\eta \ge 0$ (i.e., $\eta = \inf\{w\colon A(0, w) > 0\}$), then $z^*(0+) = \frac{1}{2}\eta$. So the solution is consistent with (24). Then,

$$h(u) = \Phi_0\left(\frac{z^*(u)}{\sqrt{u}}\right) + \int_0^\infty \Phi_0\left(\frac{\xi - z^*(u)}{\sqrt{u}}\right) A(d\xi)$$

and

$$h'(u) = \frac{1}{2u^{3/2}} \int_0^\infty \xi\phi\left(\frac{\xi - z^*(u)}{\sqrt{u}}\right) A(d\xi).$$

At worst, these integrals can be computed numerically. In some cases, they are even easier.

5.4.1. One Point. Suppose A is concentrated at a single point, $\xi > 0$. Specifically, $A = a\delta(\xi - w)$, where $a > 0$. For this case $z^*(u)$ solves

$$0 = \tilde{g}^*(u, w) = \frac{1}{\sqrt{u}}\phi\left(\frac{w}{\sqrt{u}}\right) - a\frac{1}{\sqrt{u}}\phi\left(\frac{\xi - w}{\sqrt{u}}\right),$$

or

$$\ln(a)u = \tfrac{1}{2}[(\xi - w)^2 - w^2] = \tfrac{1}{2}\xi^2 - \xi w.$$

Thus, setting $\rho = -\ln(a)/\xi$,

$$z^*(u) = \tfrac{1}{2}\xi + \rho u.$$

Also,

$$\begin{aligned} h(u) &= \Phi_0\left(\frac{\frac{1}{2}\xi + \rho u}{\sqrt{u}}\right) + a\Phi_0\left(\frac{\frac{1}{2}\xi - \rho u}{\sqrt{u}}\right), \\ h'(u) &= \frac{\xi}{2u^{3/2}}\phi\left(\frac{\frac{1}{2}\xi + \rho u}{\sqrt{u}}\right). \end{aligned} \tag{29}$$

This is an inverse Gaussian (or Wald or Bachelier–Lévy) distribution. It is indeed the distribution of the first-passage time to the affine boundary $z^*(u)$.

The following result summarizes this discussion:

Proposition 5. *For the h in* (29), *the optimal policy is the affine function*

$$y^*(u) = \mu u + \sigma z^*(u) = \tfrac{1}{2}\xi\sigma + (\mu + \rho\sigma)u.$$

Clearly, by selecting the parameters correctly, we can recover any affine function with positive intercept.

From an inventory-theoretic viewpoint, it is surprising to learn that the cost functions h of this form are the most basic, canonical ones, in the sense of leading to the simplest possible policies y^*. We cannot think of an intuitive reason why this should be so. We don't mean that these h are odd or implausible, only that we can discern nothing obviously special about them. In contrast, for linear h, there appears to be no closed-form solution. These facts suggest that supply streams are indeed complicated systems. The interplay between their dynamics and economics leads to fairly subtle behavior. Apart from the transformation discussed above and a few examples presented in Gallego and Zipkin (1999), we lack a solid understanding of the impacts of different forms of h.

This case certainly indicates the limits of the rule that safety stock should be proportional to the standard deviation of lead-time demand, namely, $\sigma\sqrt{u}$. That rule, up to the normal approximation, correctly compares several single-stage systems. A single multistage system is different. The distribution of stock within it depends on the holding-cost structure as well as the demand parameters. For the particular structure (29), the safety stock grows linearly. (In the case $a = 1$, it is constant.) This echoes the point in §4.2 about monotonicity. Here, y^* may increase (when $\mu + \rho\sigma > 0$), but not as $\sqrt{u}$.

5.4.2. Several Points. Next, suppose A is concentrated at several equally spaced points, specifically, $A = \sum_{j=1}^J a_j \delta(j\xi - w)$, where the $a_j > 0$. Here, $z^*(u)$ solves

$$0 = \tilde{g}^*(u, w) = \frac{1}{\sqrt{u}}\phi\left(\frac{w}{\sqrt{u}}\right) - \sum_j a_j \frac{1}{\sqrt{u}}\phi\left(\frac{j\xi - w}{\sqrt{u}}\right)$$

or

$$\sum_{k=1}^J a_k \exp\left(-\frac{(k\xi)^2/2 - k\xi w}{u}\right) = 1. \tag{30}$$

Denoting $\theta = \exp(\xi w/t)$, this is equivalent to

$$\sum_{k=1}^J a_k \exp\left(-\frac{(k\xi)^2}{2u}\right)\theta^k - 1 = 0.$$

Again, this polynomial has exactly one positive root for all $u > 0$. Set

$$h(u) = \Phi_0\left(\frac{z^*(u)}{\sqrt{u}}\right) + \sum_j a_j \Phi_0\left(\frac{j\xi - z^*(u)}{\sqrt{u}}\right),$$

$$h'(u) = \frac{1}{2u^{3/2}} \sum_j a_j (j\xi) \phi\left(\frac{j\xi - z^*(u)}{\sqrt{u}}\right).$$

This looks something like a mixture of inverse Gaussian distributions, but not quite, because $z^*(u)$ is not linear, and it depends on all the a_j.

The solution $z^*(u)$ is not as simple as in the one-point model. However, we can obtain a fairly simple upper bound on it. For each j, let θ_j denote the positive solution of the polynomial equation

$$\sum_{k=1}^{j} a_k \theta^k = 1. \tag{31}$$

Set $\rho_j = \ln(\theta_j)/\xi$ and

$$\pi_j = \left(\frac{\xi}{2}\right) \frac{\sum_{k=1}^{j} k^2 a_k \theta_j^k}{\sum_{k=1}^{j} k a_k \theta_j^k},$$

and define

$$z^+(u) = \min_j \{\pi_j + \rho_j u\}.$$

This is a piecewise-linear, concave function. Clearly, θ_j and ρ_j are decreasing in j, and π_j is increasing.

Proposition 6. $z^*(u) \le z^+(u)$.

There are reasons to expect that $z^+(u)$ approximates $z^*(u)$ fairly well. For $J = 1$, they are identical. For $J \ge 1$, both are concave, and they are asymptotically equal for both small and large u (see Zipkin 2013).

5.4.3. Other Solutions. Finally, suppose A has a density, that is, $A(d\xi) = a(\xi)d\xi$. As discussed in Lerche (1986) and Zipkin (2013), there are several cases where the required calculations can be performed in closed form.

Besides the method of images, there are a few other explicit (but not simple) solutions to first-passage time problems. One such result is the distribution of τ for a square-root boundary. The formula is an infinite series involving parabolic cylinder functions (see Sato 1977, Novikov et al. 1999). So, notwithstanding the warnings above, there do exist cost functions h that lead to square-root optimal policies.

5.5. Numerical Methods

Apart from special cases like those above, one must use numerical methods to solve both original and inverse first-passage time problems. Approximations for the original problem are discussed by Daniels (1996), and Song and Zipkin (2011) develop one for the inverse problem.

Several exact methods are available for the original problem. Daniels (1996) and Lo et al. (2002) use the method of images within an approximation scheme. Durbin (1992) develops a series representation. Wang and Pötzelberger (1997) discretize u and approximate the boundary by a piecewise-linear function. This approach requires the calculation of certain expectations of a multivariate normal, which they perform by Monte Carlo simulation. Novikov et al. (1999) point out that these multivariate expectations can be expressed recursively as sequences of univariate expectations and performed by direct numerical integration. They also provide a detailed error analysis.

Zucca and Sacerdote (2009) adapt the approach of Wang and Pötzelberger to the inverse problem. (They also develop an alternative method, based on one of the integral equations mentioned above.) With the recursive formulation of Novikov et al. (1999), this algorithm is similar in broad outline to (8). Of course, the details are different. Our (8) optimizes a cost function c^*, whereas their algorithm works with the density g^*. Also, they use a piecewise-linear approximation of the boundary, whereas we use a step function.

These observations suggest that (8) may be an effective method for the inverse first-passage time problem. (It is noteworthy that this algorithm, versions of which date to 1960, does solve the problem.) The main effort lies in the convolutions required to update $c_k^*(u+\varepsilon, x)$ from $\bar{c}_k^*(u, x)$. Such calculations are well suited for the fast Gauss transform, a technique that can speed them up considerably (see Greengard and Strain 1991, Broadie and Yamamoto 2003). Also, the cost-minimization framework enjoys the advantage of monotone convergence, as in Proposition 1. Equivalently, one could solve (22), which also converges monotonically. (We have not attempted to compare this approach to the other available techniques, however.)

We should also mention that there are various methods for solving more general moving-boundary and stochastic-control problems (see, e.g., Crank 1987, Kushner and Dupuis 1992, Fleming and Soner 1993, Kumar and Muthuraman 2004).

6. Concluding Remarks

This paper develops a continuous-stage serial supply chain model, called a supply stream. In this model, stock can be held at any point along a continuum from the origin to the destination. The model is a natural continuous limit of the discrete-stage serial inventory system, which has been studied intensively in the literature, starting from the seminal work of Clark and Scarf (1960). We show that the optimal policy for the continuous-stage system is a stationary echelon base-stock policy, namely, the limit of the optimal policies for the discrete-stage systems, as the distances between the stages go to zero.

We further show that, when the demand process is approximated by a Brownian motion, the computation of the optimal policy is related to the inverse first-passage time problem studied in the Brownian motion literature. Using this connection, we obtain the first closed-form expression for the optimal echelon base-stock levels in inventory networks. This expression demonstrates that the well-known square-root law for safety stock that is prevalent in the supply chain literature no longer holds for supply streams.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/msom.2013.0431.

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