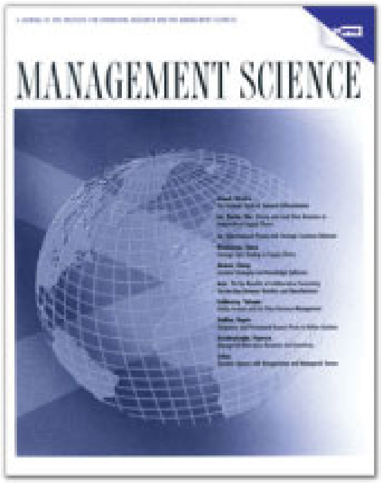

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Robust Comparative Statics of Risk Changes

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The standard method for establishing the comparative statics of risk changes in optimization problems has been confined to comparing unique interior solutions, relying on strong assumptions about payoff functions and decision variables. We propose a simple and intuitive approach that hinges on considerably weaker assumptions. Merging insights from the monotone comparative statics literature with insights from the risk apportionment literature, we show that the ranking of simple lottery pairs is all that is needed for establishing the comparative statics of risk changes. We use this approach to analyze the comparative statics of Nth-degree stochastic dominance shifts in a general setting with one and with multiple decision variables, and we show how these results can be applied to generalize the classical theories of precautionary saving, self-protection, and others.

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1. Introduction

Given the ubiquitous nature of decision making under uncertainty, it is not surprising that a substantial amount of research has been devoted to analyzing how changes in risk affect economic behavior. In this line of research, the most common method for characterizing the comparative statics of risk changes in optimization problems typically consists of two parts. First, it is established that the problem under consideration has a unique interior solution by relying on a number of "subordinate" assumptions (e.g., smooth and strictly concave payoff function, Inada conditions, convex choice sets). Second, stochastic dominance theorems are applied to first-order conditions to establish the set of payoff functions that always respond to the change in risk in a particular manner. As Milgrom and Shannon (1994) point out, the only role that the first set of assumptions plays in the comparative statics exercise is as "servants to a method," frequently hindering the economic significance and the generality of the results. For example, in the classical problem of precautionary saving (e.g., Leland 1968, Kimball 1990), it is typically assumed that the saving technology is continuous and linear, that consumers are risk averse, and that there is a unique interior solution. Given these assumptions, it is well known that a prudent consumer (i.e., one with convex marginal utility) will save more when faced with a higher risk in the sense of Rothschild and Stiglitz (1970). Yet, in a recent article Crainich et al. (2013) showed that risk lovers will also exhibit precautionary saving behavior if they are prudent. Therefore, at least in the precautionary saving problem, risk aversion is not necessary for the comparative statics result. In that problem, we may also wonder if a prudent consumer will also save more if the asset under consideration can only be purchased in discrete monetary amounts, or if saving is in the form of a take-it-or-leave-it project, or if capital has increasing returns. We show that, under certain regularity conditions, this is indeed the case.

More generally, we consider the following problem. Let $EU(\mathbf{x}, \tilde{\epsilon})$ denote the payoff function (expectation of the utility function) for a given agent, where $\mathbf{x}$ is a choice vector and $\tilde{\epsilon}$ is a random variable. We ask: When, and in what sense, can we say that the optimal choice of $\mathbf{x}$ increases when $\tilde{\epsilon}$ undergoes an increase in risk (within a given class of risk increases)? Rothschild and Stiglitz (1971), and a multitude of authors after them, have evaluated different applications of this problem when the risk change is a mean preserving spread as defined by Rothschild and Stiglitz (1970). More recently, the effect of higher-order risk changes has received a substantial amount of attention.[1] As mentioned above, the standard approach has been confined to comparing unique interior solutions, relying on strong assumptions about payoff functions and decision variables. In this paper, we present a simple and intuitive approach for comparative statics of risk that hinges on considerably weaker assumptions.

Our method builds upon two research strands. First, following Eeckhoudt and Schlesinger (2006) we study the link between optimal choice and preferences over the apportionment of risks in simple lotteries (see also Eeckhoudt et al. 2007, 2009; Tsetlin and Winkler 2009; Chiu et al. 2012; Jouini et al. 2013). For the most part, the purpose of this line of research has been to use simple

[1] For a unifying framework of many applications, see Jouini et al. (2013) and references therein.

lotteries to interpret the comparative statics results of optimization problems. We show that establishing the set of decision makers that unambiguously rank properly designed lottery pairs is all that is needed to order the set of maximizers (i.e., without any additional subordinate assumption).

The machinery required to establish such a result comes from the literature on monotone comparative statics (see, e.g., Topkis 1978, Milgrom and Roberts 1990, and Milgrom and Shannon 1994 for early developments of the theory).[2] With a few exceptions, the objective in this literature has been to determine the comparative statics of deterministic changes in one or more exogenous variables of the model without relying on the assumptions of the standard approach based on the implicit function theorem (e.g., small changes in the parameters, unique interior solutions). Athey (2002) extended this line of inquiry by establishing the class of distributional changes that have unambiguous predictions on optimal choice for a given class of decision makers. Broadly stated, Athey (2002) considered the following problem: Suppose that there is a single decision variable x, and let S denote the maximal set of agents whose choice of x under certainty is monotone increasing in an exogenous variable ϵ. Now suppose that the exogenous variable is random and undergoes a shift in its distribution. What is the weakest class of distributional shifts that guarantees monotone choices for all agents in S? The answer, as shown by Athey (2002), is monotone likelihood ratio (MLR) shifts, a subset of first-degree stochastic dominance.[3] Ormiston and Schlee (1993) had previously analyzed this problem under the assumption that a unique interior solution always exists. More recently, Quah and Strulovici (2009) extended Athey's result to a class of functions larger than S.[4] In this paper, we are similarly interested in obtaining robust conclusions about the comparative statics of distributional shifts. However, we focus on a larger class of shifts than those ordered by the MLR order (i.e., Nth-degree stochastic dominance shifts), and our objective is to establish the set of decision makers for which this given class of risk changes induces a particular behavior, placing special emphasis on how this set relates to preferences over simple lotteries.

[2] Topkis (1998) provides a textbook treatment of the methods as well as many applications. Amir (2005) provides a succinct review of the literature.

[3] Athey (2002) also studied a number of special cases of the mentioned result as well as problems with multidimensional risks and multiple decision variables. See Gollier (2001, Chap. 7) for a textbook treatment of some of Athey's results as well as several related problems and applications.

[4] See Quah and Strulovici (2012) for precise, general conditions under which the comparative statics under certainty are preserved under integration.

The paper proceeds as follows. In §2 we revisit the canonical precautionary saving model as a motivating example. After providing some basic background and reviewing some existing results from the theory of monotone comparative statics in §3, we use Ekern's (1980) definition of Nth-degree risk increases to analyze the mentioned comparative statics problem for the case of a single decision variable (§4). In §5 we extend the analysis to multiple decision variables, §6 shows how the main results apply to stochastic dominance orders other than Nth-degree risk increases, and §7 provides concluding remarks. The appendix collects the proofs of the main results, and the web appendix (available as supplemental material at http://dx.doi.org/10.1287/mnsc.2015.2202) provides supplementary proofs of the examples of applications given in the paper.

2. Motivating Example

Consider the standard two-date model of precautionary saving. In the absence of risk, the consumer has an income flow (y_0, y_1) and selects a consumption plan (c_0, c_1) by means of a savings technology that transforms date-0 income into date-1 income, one to one. Thus, given a level of savings x, the agent consumes $c_0 = y_0 - x$ at date-0 and $c_1 = x + y_1$ at date-1. The agent's objective is to maximize utility of lifetime consumption $U(x, y_1) = u(y_0 - x) + v(x + y_1)$. Naturally, we assume that the period utility functions u and v are increasing and, as usual, we will use numerical superscripts to denote the derivatives of these and other functions.[5] To simplify notation, and without any loss of generality, we will also assume that $y_1 = 0$. The consumer's problem under certainty can then be depicted as follows:

$$X_1 = \arg\max_{x \in B \subseteq \mathbb{R}} U(x, 0) = \arg\max_{x \in B \subseteq \mathbb{R}} \{u(y_0 - x) + v(x)\}. \quad (1)$$

Now suppose that a mean zero risk $\tilde{\epsilon}$ is added to the deterministic level of date-1 income $y_1 = 0$. In this case the consumer has

$$\begin{aligned} X_2 &= \arg\max_{x \in B \subseteq \mathbb{R}} EU(x, \tilde{\epsilon}) \\ &= \arg\max_{x \in B \subseteq \mathbb{R}} \{u(y_0 - x) + Ev(x + \tilde{\epsilon})\}. \quad (2) \end{aligned}$$

The textbook method for establishing comparative statics results in this context (e.g., Gollier 2001) is as follows.

Assumption 1. Suppose that the following conditions are true.

1. The solutions are interior,

[5] Specifically, given a unidimensional function $f(x)$, we denote by $f^{(k)}(x)$ its kth derivative. For a bidimensional function $f(x_1, x_2)$, we denote by $f^{(k_1, k_2)}(x_1, x_2)$ its (k_1, k_2)th cross derivative.

2. B is an interval of $\mathbb{R}$, and
3. U is smooth and strictly concave in x.

Then, there is a unique solution $X_1 = \{x_1^*\}$ that satisfies the first-order condition $u^{(1)}(y_0 - x_1^*) + v^{(1)}(x_1^*) = 0$, and there is a unique solution $X_2 = \{x_2^*\}$ that satisfies $u^{(1)}(y_0 - x_2^*) + Ev^{(1)}(x_2^* + \tilde{\epsilon}) = 0$. Jensen's inequality implies that $Ev^{(1)}(x_1^* + \tilde{\epsilon}) \geq v^{(1)}(x_1^*)$ if and only if $v^{(3)}(x) \geq 0$ for all x. Given concavity of U (i.e., $u^{(2)} < 0$ and $v^{(2)} < 0$), the condition $v^{(3)}(x) \geq 0$ then implies that, to restore the first-order condition, we must have $x_2^* \geq x_1^*$.

Using this model, and assuming that $u(x) = v(x)$ and $B = [0, y_0]$, Crainich et al. (2013) recently showed that risk lovers (i.e., $v^{(2)}(x) \geq 0$) also exhibit precautionary saving behavior if $v^{(3)}(x) \geq 0$. We will now show that their result is an example of a more general phenomenon.

Consider the following pair of 50:50 lotteries defined over date-1 consumption levels:

$$[x^l; x^h + \tilde{\epsilon}] \quad \text{and} \quad [x^h; x^l + \tilde{\epsilon}],$$

where $x^h \geq x^l$ and, as before, $\tilde{\epsilon}$ is a mean zero risk. According to Eeckhoudt and Schlesinger (2006), an individual who always ranks lottery $[x^l; x^h + \tilde{\epsilon}]$ over lottery $[x^h; x^l + \tilde{\epsilon}]$ is prudent. Such an individual prefers to allocate a high level of wealth x^h to the outcome of the lottery where the risk $\tilde{\epsilon}$ is present and a low level of wealth x^l to the outcome in which the risk is not present. This may occur because the individual perceives the risk as a harm and prefers to combine "bad" with "good," or because the individual is risk seeking and prefers to combine "good" with "good" and "bad" with "bad" (Crainich et al. 2013). Intuition then tells us that this individual will exhibit precautionary saving behavior as long as a solution to the saving problem exists (e.g., if $B = [0, y_0]$). On the other hand, we expect an imprudent individual (i.e., one who ranks lottery $[x^h; x^l + \tilde{\epsilon}]$ over lottery $[x^l; x^h + \tilde{\epsilon}]$) to reduce saving in response to income risk.

To see that this is indeed the case, let us return to optimization problems (1) and (2) and let us denote by $\bar{x}_1$ and $\bar{x}_2$ the greatest elements of X_1 and X_2, respectively. Since $\bar{x}_1$ and $\bar{x}_2$ are maximizers (not necessarily interior or unique) of (1) and (2), respectively, we have $U(\bar{x}_1, 0) \geq U(\bar{x}_2, 0)$ and $EU(\bar{x}_2, \tilde{\epsilon}) \geq EU(\bar{x}_1, \tilde{\epsilon})$. Using the period utility functions, we can write these conditions as $u(y_0 - \bar{x}_1) + v(\bar{x}_1) \geq u(y_0 - \bar{x}_2) + v(\bar{x}_2)$ and $u(y_0 - \bar{x}_2) + Ev(\bar{x}_2 + \tilde{\epsilon}) \geq u(y_0 - \bar{x}_1) + Ev(\bar{x}_1 + \tilde{\epsilon})$. Putting these two inequalities together, and multiplying throughout by $\frac{1}{2}$, we obtain

$$\tfrac{1}{2}v(\bar{x}_1) + \tfrac{1}{2}Ev(\bar{x}_2 + \tilde{\epsilon}) \geq \tfrac{1}{2}v(\bar{x}_2) + \tfrac{1}{2}Ev(\bar{x}_1 + \tilde{\epsilon}).$$

This condition shows that the optima of problems (1) and (2) is chosen to be in line with a preference over simple 50:50 unidimensional lotteries, i.e., with the ranking of $[\bar{x}_1; \bar{x}_2 + \tilde{\epsilon}]$ over $[\bar{x}_2; \bar{x}_1 + \tilde{\epsilon}]$. If $\bar{x}_2 \geq \bar{x}_1$ ($\bar{x}_1 \geq \bar{x}_2$), such a ranking is equivalent to Eeckhoudt and Schlesinger's (2006) characterization of prudence (imprudence). We can then conclude that a prudent consumer, in the precise sense defined by Eeckhoudt and Schlesinger (2006), always saves more in the presence of risk than in the absence of it, whereas an imprudent consumer always saves less.

Following Eeckhoudt and Schlesinger (2006, 2009), an alternative intuitive characterization of prudence is via the utility premium of Friedman and Savage (1948). Define the utility premium as follows:

$$\omega(x) = EU(x, \tilde{\epsilon}) - U(x, 0) = Ev(x + \tilde{\epsilon}) - v(x).$$

For a risk-averse individual, the utility premium captures the "pain" from adding the risk $\tilde{\epsilon}$ (i.e., $\omega(x)$ is negative), whereas for a risk-loving individual, the utility premium represents the "joy" from such an addition (i.e., $\omega(x)$ is positive). Now note that, regardless of the individual preferences for the presence of risk, the utility premium is increasing in wealth if and only if the decision maker is prudent in the sense of Eeckhoudt and Schlesinger (2006).[6] From the previous discussion we can conclude that a precautionary saving motive will arise whenever the utility premium is increasing in wealth.

The final step of this exercise is to establish the set of date-1 utility functions that rank lottery $[x^l; x^h + \tilde{\epsilon}]$ over lottery $[x^h; x^l + \tilde{\epsilon}]$ for all $x^h \geq x^l$ or, equivalently, that always have an increasing utility premium. Assuming that v is twice continuously differentiable and using Jensen's inequality, we obtain, as Eeckhoudt and Schlesinger (2006) did, that $[x^l; x^h + \tilde{\epsilon}]$ is preferred to $[x^h; x^l + \tilde{\epsilon}]$ for all $x^h \geq x^l$ if, and only if, $v^{(2)}(x)$ is increasing in x (i.e., $v^{(3)}(x) \geq 0$ if $v(x)$ is trice differentiable).

To summarize, a prudent consumer, characterized equivalently by a lottery preference à la Eeckhoudt and Schlesinger, a utility premium increasing in wealth, or a date-1 utility function with the property that $v^{(2)}(x)$ is increasing in x, will always exhibit precautionary saving. Although this result is not too surprising, what is remarkable is that we have dispensed with most assumptions of the standard method. In particular, if a solution exists this result holds for (1) both concave and convex date-1 utility functions v, as shown by Crainich et al. (2013);[7] (2) any date-0 utility function u (e.g.,

[6] Specifically, the expected utility of lottery $[x^l; x^h + \tilde{\epsilon}]$ is greater than the expected utility of lottery $[x^h; x^l + \tilde{\epsilon}]$ if $Ev(x^h + \tilde{\epsilon}) - v(x^h) \geq Ev(x^l + \tilde{\epsilon}) - v(x^l)$, or equivalently if $\omega(x^h) - \omega(x^l) \geq 0$.

[7] Of course, our result also implies that both risk-averse and risk-loving individuals who are imprudent in the Eeckhoudt–Schlesinger sense (equivalently, those decision makers whose utility function satifies $v^{(3)}(x) \leq 0$ when $v(x)$ is trice differentiable) will never display

convex, nondifferentiable); (3) both convex and nonconvex choice sets (e.g., indivisible saving technologies); and (4) any increasing monotone transformation of the saving technology (e.g., taxation of capital income, increasing marginal product of capital).

Our next objective is to extend this line of reasoning to more general contexts.

3. Preliminaries

This section introduces the basic background for the comparative statics analyses that follow. In §3.1. we present the class of risk changes under consideration. Then, in §3.2, we present some terminology from lattice theory and two important results from the theory of monotone comparative statics.

3.1. Changes in Risk

Let $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ denote two random variables with values in $[0, Q]$. For $i = 1, 2$, we denote by $F^{[1]}_{\tilde{\epsilon}_i}$ the distribution functions and for $k = 1, 2, \ldots$ we define the functions $F^{[k+1]}_{\tilde{\epsilon}_i}$ on $\mathbb{R}_+$ by $F^{[k+1]}_{\tilde{\epsilon}_i}(x) = \int_0^x F^{[k]}_{\tilde{\epsilon}_i}(t)\,dt$ for $x \in \mathbb{R}_+$. The class of risk changes that we analyze is based on the following characterization provided by Ekern (1980).

Definition 1. We say that $\tilde{\epsilon}_2$ is an increase in Nth-degree risk over $\tilde{\epsilon}_1$, and we denote it by $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$, if $F^{[N]}_{\tilde{\epsilon}_2}(x) \geq F^{[N]}_{\tilde{\epsilon}_1}(x)$ for all $x \in [0, Q]$ where the inequality is strict for some x and $F^{[k]}_{\tilde{\epsilon}_2}(Q) = F^{[k]}_{\tilde{\epsilon}_1}(Q)$ for $k = 1, \ldots, N$.

In words, an increase in Nth-degree risk represents an Nth-degree stochastic dominance shift in which the first $N-1$ moments of the distributions coincide.[8] For example, an increase in second-degree risk coincides with Rothschild and Stiglitz's (1970) mean preserving increase in risk, whereas an increase in third-degree risk coincides with a mean and variance preserving increase in risk that Menezes et al. (1980) labeled increase in downside risk.

The following result will be essential for the analysis that follows.

Lemma 1. *Let q be a given real valued function that is N times continuously differentiable. The following conditions are equivalent.*[9]

1. *For all pair $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$ such that $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$, we have $E[q(\tilde{\epsilon}_1)] \geq E[q(\tilde{\epsilon}_2)]$.*
2. *For all $x \geq 0$, we have $(-1)^N q^{(N)}(x) \leq 0$.*

As an illustration, for $N = 2$, we retrieve the classical result that an agent who dislikes mean preserving spreads is an agent whose utility function is concave.

3.2. Ordering Relations

Let X be a set with the partial order $\succcurlyeq$. For $\mathbf{x}'$ and $\mathbf{x}''$ elements of X, let $\mathbf{x} \vee \mathbf{x}'$ denote the least upper bound, or *join*, of these elements, if it exists. Similarly, let $\mathbf{x}' \wedge \mathbf{x}''$ denote the greatest lower bound, or *meet*, of these elements. The set X is a *lattice* if, for every pair of elements $\mathbf{x}'$ and $\mathbf{x}''$ in X, the join $\mathbf{x}' \vee \mathbf{x}''$ and the meet $\mathbf{x}' \wedge \mathbf{x}''$ exist. The real line $\mathbb{R}$ with the usual ordering $\geq$ is a lattice, with $x' \vee x'' = \max\{x', x''\}$ and $x \wedge x' = \min\{x', x''\}$ for x', x'' in $\mathbb{R}$. The Euclidean space $\mathbb{R}^n$ with the componentwise order is also a lattice, with $\mathbf{x}' \vee \mathbf{x}'' = \max(\mathbf{x}', \mathbf{x}'') = (x'_1 \vee x''_1, \ldots, x'_n \vee x''_n)$ and $\mathbf{x}' \wedge \mathbf{x}'' = \min(\mathbf{x}', \mathbf{x}'') = (x'_1 \wedge x''_1, \ldots, x'_n \wedge x''_n)$ for $\mathbf{x}', \mathbf{x}''$ in $\mathbb{R}^n$. Similarly, a subset B of X is a *sublattice* of X if it contains the join and the meet. For example, any subset of $\mathbb{R}$ is a sublattice, whereas B is a sublattice of $\mathbb{R}^n$ if it is closed under the max and min operators.

In comparing optimal solutions, it will be essential to have an ordering relation over sets. As is standard in the literature on monotone comparative statics, we use the strong set order.

Definition 2. For two sets B' and B'', we say that B'' is higher than B' in the strong set order, and we write it $B'' \geq_S B'$ if for any $\mathbf{x}' \in B'$ and $\mathbf{x}'' \in B''$ we have $\mathbf{x}' \vee \mathbf{x}'' \in B''$ and $\mathbf{x}' \wedge \mathbf{x}'' \in B'$.

For example, for two subsets B' and B'' of $\mathbb{R}$, we have $B'' \geq_S B'$ if for any $x' \in B'$ and $x'' \in B''$ we have $\max\{x', x''\} \in B''$ and $\min\{x', x''\} \in B'$.

The following definitions present important properties of functions defined on partially ordered sets.

Definition 3. Given a partially ordered set T, the function $f\colon T \to \mathbb{R}$ is increasing (decreasing) if $[f(\mathbf{t}') - f(\mathbf{t}'')] \geq 0$ ($[f(\mathbf{t}') - f(\mathbf{t}'')] \leq 0$) for all $\mathbf{t}' \succcurlyeq \mathbf{t}''$ in T.

Definition 4. Given a lattice X and a partially ordered set T, the function $f\colon X \times T \to \mathbb{R}$ has increasing (decreasing) differences in $(\mathbf{x}, \mathbf{t})$ if, for all $\mathbf{t}' \succcurlyeq \mathbf{t}''$ in T, the difference $f(\mathbf{x}, \mathbf{t}') - f(\mathbf{x}, \mathbf{t}'')$ is increasing (decreasing) in $\mathbf{x}$.

Definition 5. Given a lattice X, the function $f\colon X \to \mathbb{R}$ is supermodular if, for all $\mathbf{x}', \mathbf{x}''$ in X, we have $f(\mathbf{x}' \vee \mathbf{x}'') + f(\mathbf{x}' \wedge \mathbf{x}'') \geq f(\mathbf{x}') + f(\mathbf{x}'')$.[10]

When the ordering relations in Definitions 3–5 are strict, we will explicitly point it out. For example, we

precautionary saving behavior. In the analysis of Crainich et al. (2013), imprudent behavior does not arise because, by assumption, decision makers have consistent preferences for combining either "good with good" (implying risk seeking and prudence) or "good with bad" (implying risk aversion and prudence) (see Ebert 2013).

[8] In §6 we present results for the more general Nth-degree stochastic dominance order.

[9] Ekern (1980) established the fact that condition 2 implies condition 1. Specifically, Ekern proved that ϵ_2 is an increase in Nth-degree risk over ϵ_1 (as in Definition 1) if and only if $E[q(\tilde{\epsilon}_2)] < E[q(\tilde{\epsilon}_1)]$ for all N times continuously differentiable real valued function q such that $(-1)^N q^{(N)} < 0$. Lemma 1, established by Jouini et al. (2013) following along the lines of Denuit et al. (1999), extends this result by also showing the reverse implication.

[10] Note that a function of a single variable is supermodular.

will say that the function $f(\mathbf{x},\mathbf{t})$ has strictly increasing differences in $(\mathbf{x},\mathbf{t})$ if, for all $\mathbf{t}'\succ\mathbf{t}''$, the difference $f(\mathbf{x},\mathbf{t}')-f(\mathbf{x},\mathbf{t}'')$ is strictly increasing in $\mathbf{x}$; i.e., $f(\mathbf{x}',\mathbf{t}')-f(\mathbf{x}',\mathbf{t}'')>f(\mathbf{x}'',\mathbf{t}')-f(\mathbf{x}'',\mathbf{t}'')$ for all $\mathbf{x}'\succ\mathbf{x}''$.

When X is a subset of $\mathbb{R}^n$, T is a subset of $\mathbb{R}^m$, and $f\colon \mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is twice continuously differentiable on a given interval, we obtain the well-known characterization result, due to Topkis (1978), that $f(\mathbf{x},\mathbf{t})$ is supermodular in $\mathbf{x}$ for fixed $\mathbf{t}$ if and only if $\partial^2 f/\partial x_i\partial x_j\ge 0$ for all $i\ne j$, and $f(\mathbf{x},\mathbf{t})$ has increasing differences in $(\mathbf{x},\mathbf{t})$ if and only if $\partial^2 f/\partial x_i\partial t_j\ge 0$ for all $i=1,\dots,n$ and $j=1,\dots,m$. The following simple result will be important in our analysis.

Remark 1. If the real valued function $f(\mathbf{x},\mathbf{t})$ has increasing (decreasing) differences in $(\mathbf{x},\mathbf{t})$, then $v(\mathbf{x},\mathbf{t})=h_1(\mathbf{x})+h_2(\mathbf{t})+f(\mathbf{x},\mathbf{t})$ also has increasing (decreasing) differences in $(\mathbf{x},\mathbf{t})$.

Given these definitions, we are ready to present two fundamental comparative statics results due to Topkis (1978).[11]

Lemma 2 (Topkis's Monotonicity Theorem). *Let X be a lattice, B a sublattice of X, T a partially ordered set, and $f\colon X\times T\to\mathbb{R}$.*[12] *Suppose that $f(\mathbf{x},\mathbf{t})$ is supermodular in $\mathbf{x}$ on X for fixed $\mathbf{t}$ and has increasing differences (decreasing differences) in $(\mathbf{x},\mathbf{t})$ on $X\times T$. Then, for all $\mathbf{t}'\succcurlyeq\mathbf{t}''$, we have*

$$\arg\max_{\mathbf{x}\in B} f(\mathbf{x},\mathbf{t}')\ge_S \arg\max_{\mathbf{x}\in B} f(\mathbf{x},\mathbf{t}''),$$

$$\Big(\textit{resp.}\ \arg\max_{\mathbf{x}\in B} f(\mathbf{x},\mathbf{t}'')\ge_S \arg\max_{\mathbf{x}\in B} f(\mathbf{x},\mathbf{t}')\Big)$$

provided the sets of solutions are nonempty.

Lemma 3 (Topkis's Monotonicity Theorem (′)). *Let X be a lattice, B a sublattice of X, T a partially ordered set, and $f\colon X\times T\to\mathbb{R}$. Suppose that $f(\mathbf{x},\mathbf{t})$ is supermodular in $\mathbf{x}$ on X for fixed $\mathbf{t}$ and has strictly increasing differences (strictly decreasing differences) in $(\mathbf{x},\mathbf{t})$ on $X\times T$. Then, if $\mathbf{x}'\in\arg\max_{\mathbf{x}\in B} f(\mathbf{x},\mathbf{t}')$ and $\mathbf{x}''\in\arg\max_{\mathbf{x}\in B} f(\mathbf{x},\mathbf{t}'')$, with $\mathbf{t}'\succ\mathbf{t}''$, we have $\mathbf{x}'\succcurlyeq\mathbf{x}''$ ($\mathbf{x}'\preccurlyeq\mathbf{x}''$), provided the sets of solutions are nonempty.*

Note that Lemma 2 is more general than Lemma 3, in the sense that Lemma 3 invokes strictly increasing or decreasing differences. With such a stronger condition, Lemma 3 shows that the sets of optimal solutions can be ordered by a stronger relation than the strong set order. For example, if there is a single decision variable, $X=\mathbb{R}$, and $f(x,t)$ has strictly increasing differences, Lemma 3 implies that $x'\ge x''$ for every selection $x'\in\arg\max_{x\in B} f(x,t')$ and $x''\in\arg\max_{x\in B} f(x,t'')$ with $t'\succ t''$.

In the analysis that follows, we will demonstrate the usefulness of Topkis's theorems in the context of the comparative statics of risk changes (specifically, when T is ordered by $\succ_N$ as defined in §3.1) and we will illustrate the close link that exists between the concepts of increasing and decreasing differences with the idea of risk apportionment that has been so useful in providing intuition for the comparative statics of risk.

4. Single Decision Variable

We start by analyzing the case of a single decision variable, which is the context of many well-known applications. Given a payoff function $EU(x,\tilde\epsilon)$, where x is a unidimensional decision variable and $\tilde\epsilon$ is a random variable, the question we analyze is the following: How does $\arg\max_{x\in B} EU(x,\tilde\epsilon)$ vary with changes in the random variable that are ordered by a given stochastic dominance relation? Let us define the following problems.

Problem 1. $X_1=\arg\max_{x\in B} EU(x,\tilde\epsilon_1)$.

Problem 2. $X_2=\arg\max_{x\in B} EU(x,\tilde\epsilon_2)$, where $\tilde\epsilon_1\succ_N\tilde\epsilon_2$ and, for expositional clarity, we assume that the constraint set B is a subset of $\mathbb{R}$.[13] Our objective is to compare X_1 and X_2.

In this context, it is standard to obtain the comparative statics of risk changes by making use of conditions 1–3 of Assumption 1 (interior solutions, strict concavity, etc.). Given these assumptions, there is a unique solution $X_1=\{x_1^*\}$ that satisfies $EU^{(1,0)}(x_1^*,\tilde\epsilon_1)=0$, and similarly there is a unique solution $X_2=\{x_2^*\}$ that satisfies $EU^{(1,0)}(x_2^*,\tilde\epsilon_2)=0$. By Lemma 1, if $\tilde\epsilon_2$ is an increase in Nth-degree risk over $\tilde\epsilon_1$ we have $EU^{(1,0)}(x_1^*,\tilde\epsilon_2)\ge 0$ if and only if $(-1)^N U^{(1,N)}(x,\epsilon)\ge 0$ for all x,ϵ. Given the concavity of U, $EU^{(1,0)}(x_1^*,\tilde\epsilon_2)$ decreases with x, and so, to restore the first-order condition, we must have $x_2^*\ge x_1^*$.

To show that the mentioned assumptions are superfluous for the comparative statics result, we begin by appealing to the idea of risk apportionment. Clearly, preferences over the type of lotteries presented in the motivating example (i.e., those studied by Eeckhoudt and Schlesinger 2006) will not be sufficient to characterize the comparative statics in the present setting with stochastic dominance shifts. Eeckhoudt et al. (2009) presented a more general framework of

[11] We remark that Topkis's theorems provide sufficient conditions for monotone comparative statics. The interested reader can consult Milgrom and Shannon (1994) for necessary and sufficient conditions for monotone comparative statics. These latter conditions form the basis for Athey's (2002) results described in §1.

[12] Topkis's theorems also apply when the constraint set is increasing in $\mathbf{t}$ in the strong set order. See Quah (2007) for additional comparative statics results of changes in the constraint set and applications to problems involving risk.

[13] The assumption $B\subseteq\mathbb{R}$ will simplify the notation, but it is not consequential for our results, which would hold for arbitrary lattices.

risk apportionment via stochastic dominance. For the current purposes, we will need the more general, multidimensional characterization of risk apportionment via stochastic dominance developed by Tsetlin and Winkler (2009).

Let us consider the following pair of 50:50 bivariate lotteries:

$$\begin{aligned} L_G &= [x^l, \tilde{\epsilon}_1; x^h, \tilde{\epsilon}_2], \\ L_B &= [x^h, \tilde{\epsilon}_1; x^l, \tilde{\epsilon}_2], \end{aligned} \tag{3}$$

where $x^h \geq x^l$ and $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$. For a decision maker that always ranks L_G over L_B, we have

$$\begin{aligned} &\tfrac{1}{2} EU(x^l, \tilde{\epsilon}_1) + \tfrac{1}{2} EU(x^h, \tilde{\epsilon}_2) \\ &\quad \geq \tfrac{1}{2} EU(x^h, \tilde{\epsilon}_1) + \tfrac{1}{2} EU(x^l, \tilde{\epsilon}_2). \end{aligned} \tag{4}$$

Following Tsetlin and Winkler (2009), we can interpret this condition as saying that this individual prefers to combine a "bad" lottery with a "good" lotteryrather than combining two "bad" lotteries and two "good" lotteries, where "good" and "bad" are defined relative to increases in Nth-degree risk in one attribute ($\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$) and increases in first-degree risk in the other attribute ($x^h \geq x^l$). Using the terminology of §2, for this individual the "pain" caused by an Nth-degree risk increase can be ameliorated by higher levels of the attribute x. Clearly, an equally valid interpretation is that this condition reflects the preferences of an Nth-degree risk-loving individual for whom the "joy" caused by a risk increase can be strengthened by higher levels of the attribute x, as in our example in §2 of risk-loving individuals displaying prudence. In either case, the difference in the expected payoff $EU(x, \tilde{\epsilon}_2) - EU(x, \tilde{\epsilon}_1)$ (i.e., the "utility premium" for Nth-degree risk increases) is increasing in x. Intuition then suggests that, when the attribute x plays the role of a decision variable as in (1) and (2), an individual that always ranks L_G over L_B would select a higher level of this variable when faced with an Nth-degree risk increase.

In effect, note that saying that the utility premium is increasing in x (or equivalently, that the decision maker ranks L_G over L_B) can be characterized as follows:

$$EU(x, \tilde{\epsilon}_1) - EU(x, \tilde{\epsilon}_2) \text{ is decreasing in } x, \quad \text{for all } \tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2. \tag{5}$$

Condition (5) corresponds precisely with the property defined in the previous section as decreasing differences (with X ordered by $\geq$ and T ordered by $\succ_N$). This equivalency should not be too surprising: a function with decreasing differences characterizes substitutability between its elements, which is precisely what the idea of preferences for risk apportionment captures.[14] By Topkis monotonicity theorem, this condition is all that is needed to obtain the desired comparative statics result. Proposition 1 formalizes this link and it also establishes the set of decision makers for which the comparative statics result always holds.

Proposition 1. *Consider Problems 1 and 2, with $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and $B \subseteq \mathbb{R}$. Suppose that U is N times continuously differentiable in ϵ and that X_1, X_2 are nonempty. Then, $X_2 \geq_S X_1$ if, equivalently, one of the following conditions holds.*

1. *L_G is preferred to L_B for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $x^h \geq x^l$.*
2. *$(-1)^N U^{(0,N)}(x, \epsilon)$ is increasing in x for all ϵ.*

A dual result of Proposition 1 is that, if L_B is preferred to L_G for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $x^h \geq x^l$, so the decision maker prefers to combine higher risks with lower levels of the deterministic attribute (i.e., prefers to combine good-with-good and bad-with-bad lotteries, as perceived by a risk averter), and the payoff function obeys increasing differences, then $X_1 \geq_S X_2$.

We emphasize that the proposition is not restricted to unique or interior solutions and that it does not impose concavity of the payoff function, convexity of the choice set, or differentiability in the choice variable. Proposition 1 does assume that the sets of solutions are nonempty. A simple sufficient condition for this to be the case is that B is compact and U is continuous in x, in which case X_1 and X_2 are compact subsets of B and have least and greatest elements. By Proposition 1, the least and greatest elements of X_2 are at least as high as those of X_1. If, in addition, X_1 and X_2 are singletons: say $X_1 = \{x_1^*\}$ and $X_2 = \{x_2^*\}$, we have that $x_2^* \geq x_1^*$ under any of the enumerated conditions. The next result makes use of Lemma 3 to show that such an ordering also arises, without assuming a unique solution, if the decision maker has a strict ranking over the lotteries (3).

Proposition 2. *Consider Problems 1 and 2, with $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and $B \subseteq \mathbb{R}$. Suppose that L_G is strictly preferred to L_B for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $x^h > x^l$ (equivalently, $EU(x, \tilde{\epsilon})$ has strictly decreasing differences in $(x, \tilde{\epsilon})$). Then, if X_1, X_2 are nonempty and $x_1^* \in X_1$ and $x_2^* \in X_2$, we have $x_2^* \geq x_1^*$.*

We demonstrate these results in the context of a few well-known applications.

4.1. The Precautionary Demand for Saving Revisited

We begin by illustrating Proposition 1 in the context of precautionary saving. Maintaining most of the notation from §2, the decision maker's problem is now the following

$$\begin{aligned} X_i &= \arg\max_{x \in B \subset \mathbb{R}} EU(x, \tilde{\epsilon}_i) \\ &= \arg\max_{x \in B \subset \mathbb{R}} \{u(y_0 - x) + Ev(x + \tilde{\epsilon}_i)\} \end{aligned} \tag{6}$$

[14] In a different context, Schlesinger (2015) makes a similar point and explores the link between risk apportionment and lattices.

where $i = 1, 2$ and $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$. The motivating example of §2 is a special case of this framework with $\tilde{\epsilon}_1 = 0$ and $\tilde{\epsilon}_2 = \tilde{\epsilon}$, an example of $\succ_2$.

Eeckhoudt and Schlesinger (2008) analyzed problem (6) under the additional conditions 1–3 of Assumption 1. They showed that, given unique interior solutions $X_1 = \{x_1^*\}$ and $X_2 = \{x_2^*\}$, any decision maker who exhibits $(N+1)$th degree risk aversion, $(-1)^N v^{(N+1)}(x) \geq 0$, increases savings in response to an increase in Nth-degree risk, $x_2^* \geq x_1^*$. Next, we analyze problem (6) under the proposed approach.

Consider the following pair of lotteries defined over date-1 consumption:

$$[x^l + \tilde{\epsilon}_1; x^h + \tilde{\epsilon}_2], \tag{7}$$

$$[x^h + \tilde{\epsilon}_1; x^l + \tilde{\epsilon}_2]. \tag{8}$$

Suppose that the decision maker ranks (7) over (8) for all $x^h \geq x^l$ and all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$, which is an example of the idea of risk apportionment via stochastic dominance found in Eeckhoudt et al. (2009) (again, with the risks to be apportioned characterized via the partial orders $\succ_N$ and $\geq$). Since such a ranking is equivalent to the property of decreasing differences of $Ev(x + \tilde{\epsilon})$ and, by Remark 1, of $EU(x, \tilde{\epsilon})$, Proposition 1 readily implies the comparative statics result: $X_2 \geq_S X_1$ (i.e., for any $x_1 \in X_1$ and $x_2 \in X_2$, $\max\{x_1, x_2\} \in X_2$ and $\min\{x_1, x_2\} \in X_1$; see Definition 2). We repeat the argument to highlight the power and simplicity of the approach. Let $x_1 \in X_1$ and $x_2 \in X_2$. We have

$$\begin{aligned} 0 &\geq EU(\max\{x_1, x_2\}, \tilde{\epsilon}_2) - EU(x_2, \tilde{\epsilon}_2) \\ &= [u(y_0 - \max\{x_1, x_2\}) + Ev(\max\{x_1, x_2\} + \tilde{\epsilon}_2)] \\ &\quad - [u(y_0 - x_2) + Ev(x_2 + \tilde{\epsilon}_2)] \\ &\geq [u(y_0 - \max\{x_1, x_2\}) + Ev(\max\{x_1, x_2\} + \tilde{\epsilon}_1)] \\ &\quad - [u(y_0 - x_2) + Ev(x_2 + \tilde{\epsilon}_2)] \\ &= EU(\max\{x_1, x_2\}, \tilde{\epsilon}_1) - EU(x_2, \tilde{\epsilon}_1) \\ &\geq 0. \end{aligned}$$

The first inequality follows from the definition of X_2 and $x_2 \in X_2$. The second inequality follows from the assumption that lottery (7) is preferred to (8). Finally, the third inequality follows from the definition of X_1 and $x_1 \in X_1$. Since the inequalities are enclosed by zeroes, it follows that $\max\{x_1, x_2\} \in X_2$. Similarly, it is simple to show that $\min\{x_1, x_2\} \in X_1$, so we conclude that $X_2 \geq_S X_1$.

Last, note that, in line with Eeckhoudt and Schlesinger's (2008) analysis, lottery (7) is preferred to lottery (8) if and only if $(-1)^N v^{(N)}(x)$ is increasing in x (i.e., $(-1)^N v^{(N+1)}(x) \geq 0$ when $v(x)$ is smooth).[15] No additional assumptions are necessary to establish the comparative statics result. We summarize the preceding discussion as follows.

Example 1. Let X_i, $i = 1, 2$, be defined as in (6), with $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$. Suppose that v is N times differentiable and that B is compact. No restrictions are imposed on u. Then, $X_2 \geq_S X_1$ if, equivalently, one of the following conditions holds.

- Lottery (7) is preferred to lottery (8) for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $x^h \geq x^l$.
- $(-1)^N v^{(N)}(x)$ is increasing in x.

The following examples extend this line of reasoning in two directions. Example 2 looks at a more general problem of precautionary saving behavior with nonseparable utility analyzed recently by Jouini et al. (2013). Example 3 revisits Eeckhoudt and Schlesinger's (2008) analysis of precautionary saving with interest rate risk. In both examples we assume that the choice set is convex and that the payoff function is differentiable to make the link between our results and the results of the mentioned papers more transparent.

Example 2. Define $X_i = \arg\max_{x \in B} EU(x, \tilde{\epsilon}_i) = \arg\max_{x \in B} Eh(y_0 - x, x + \tilde{\epsilon}_i)$, $i = 1, 2$, where h is the (possibly nonseparable) utility of date-0 and date-1 consumption and all the variables are as defined in problem (6). Suppose that $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$, that h is $N+1$ times differentiable in the second argument and once in the first argument, and that B is a compact interval of $\mathbb{R}$. Then, $X_2 \geq_S X_1$ if, equivalently, one of the following conditions holds.

- The bivariate lottery $[y_0 - x^l, x^l + \tilde{\epsilon}_1; y_0 - x^h, x^h + \tilde{\epsilon}_2]$ is preferred to the bivariate lottery $[y_0 - x^h, x^h + \tilde{\epsilon}_1; y_0 - x^l, x^l + \tilde{\epsilon}_2]$ for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $x^h \geq x^l$.
- $(-1)^N (h^{(0, N+1)} - h^{(1, N)}) \geq 0$.

Example 3. Define $X_i = \arg\max_{x \in B} EU(x, \tilde{\epsilon}_i) = \arg\max_{x \in B} u(y_0 - x) + Ev(x\tilde{\epsilon}_i)$, $i = 1, 2$, where $\tilde{\epsilon}_i$ now represents the gross rate of interest. Suppose that $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$, that v is $N+1$ times differentiable with its successive derivatives alternating in sign, and that B is a compact interval of $\mathbb{R}$. No restrictions are imposed on u. Then, $X_2 \geq_S X_1$ if, equivalently, one of the following conditions holds.

- Lottery $[x^l \tilde{\epsilon}_1; x^h \tilde{\epsilon}_2]$ is preferred to lottery $[x^h \tilde{\epsilon}_1; x^l \tilde{\epsilon}_2]$ for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $x^h \geq x^l$ (both lotteries defined over date-1 consumption).[16]
- The measure of Nth-degree relative risk aversion (Eeckhoudt and Schlesinger 2008), $-v^{(N+1)}(x)x/v^{(N)}(x)$, is no smaller than N.

We remark that the equivalencies of the bulleted items in Examples 1–3 are not new results. Rather, it is the robustness of the comparative statics results that we want to emphasize.

[15] This follows from Lemma 1 or, more directly, from Proposition 1 by noting that $U^{(0, N)} = v^{(N)}$.

[16] This sort of lottery preference belongs to the class of risk apportionment with multiplicative attributes studied by Chiu et al. (2012).

4.2. Self-Protection and Background Risks

To illustrate Proposition 2 we present a version of the classical problem of self-protection that was recently analyzed by Wang and Li (2015) using an approach similar to ours. In this problem there are two dates and the decision maker exerts effort x at date-0 in order to reduce the probability of facing an adverse outcome $-L$ at date-1. The decision maker also faces a background risk at date-1 and the comparative statics of interest is whether an increase in this background risk increases or decreases effort. Let us define

$$\begin{aligned} X_i &= \arg\max_{x\in B} EU(x,\tilde{\epsilon}_i) \\ &= \arg\max_{x\in B}\{u(y_0-x)+P(x)Ev(y_1+\tilde{\epsilon}_i) \\ &\qquad +[1-P(x)]Ev(y_1-L+\tilde{\epsilon}_i)\}, \quad i=1,2, \end{aligned} \tag{9}$$

where y_0 and y_1 represent the nonstochastic endowments at date-0 and date-1, respectively, and we assume that $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$. Wang and Li (2015) established the following result.

Example 4. Let X_i, $i=1,2$, be defined as in (9), with $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$. Suppose that B is a compact interval of $\mathbb{R}$ and that u, v, and P are strictly increasing and smooth. Then, if the decision maker is strictly prudent, in the sense that lottery $[y_1-L+\tilde{\epsilon}_1; y_1+\tilde{\epsilon}_2]$ is strictly preferred to lottery $[y_1+\tilde{\epsilon}_1; y_1-L+\tilde{\epsilon}_2]$ for all $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$, and if $x_1^* \in X_1$ and $x_2^* \in X_2$, then $x_2^* \geq x_1^*$.

Thus, under the stated assumptions, any strictly prudent individual (weakly) increases effort when the background risk undergoes a mean preserving spread.

To demonstrate that this result is an example of Proposition 2, let us rewrite lifetime utility as follows: $u(y_0-x)+Ev(y_1-L+\tilde{\epsilon}_i)+P(x)Eg(\tilde{\epsilon}_i)$, where $g(\tilde{\epsilon}_i)=[v(y_1+\tilde{\epsilon}_i)-v(y_1-L+\tilde{\epsilon}_i)]$. Since $P(x)$ is assumed to be strictly increasing and, by strict prudence, $Eg(\tilde{\epsilon})$ is strictly decreasing in the order $\succ_2$ (i.e., $Eg(\tilde{\epsilon}_1)<Eg(\tilde{\epsilon}_2)$ for all $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$), the product $P(x)Eg(\tilde{\epsilon})$ has strictly decreasing differences in $(x,\tilde{\epsilon})$. Using Remark 1 we can then conclude that lifetime utility also has strictly decreasing differences in $(x,\tilde{\epsilon})$, from which the result follows. Of course, according to Proposition 2 the result applies more broadly (e.g., without any restrictions on the function $u(x)$), whereas Proposition 1 can be used to establish the comparative statics of risk for a broader set of decision makers.

5. Multiple Decision Variables

Let us now suppose that the choice vector is m-dimensional. Define the following problems where the constraint set B is assumed to be a sublattice of $\mathbb{R}^m$.

Problem 3. $X_1 = \arg\max_{\mathbf{x}\in B} EU(\mathbf{x},\tilde{\epsilon}_1)$.

Problem 4. $X_2 = \arg\max_{\mathbf{x}\in B} EU(\mathbf{x},\tilde{\epsilon}_2)$.

Proceeding as in the previous section, consider the following pair of 50:50 lotteries:

$$\begin{aligned} L_{\mathbf{G}} &= [\mathbf{x}^l, \tilde{\epsilon}_1; \mathbf{x}^h, \tilde{\epsilon}_2], \\ L_{\mathbf{B}} &= [\mathbf{x}^h, \tilde{\epsilon}_1; \mathbf{x}^l, \tilde{\epsilon}_2] \end{aligned} \tag{10}$$

where $\mathbf{x}^h \geq \mathbf{x}^l$ and $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$. A decision maker who always prefers $L_{\mathbf{G}}$ over $L_{\mathbf{B}}$ has

$$\begin{aligned} &\tfrac{1}{2}EU(\mathbf{x}^l,\tilde{\epsilon}_1)+\tfrac{1}{2}EU(\mathbf{x}^h,\tilde{\epsilon}_2) \\ &\quad \geq \tfrac{1}{2}EU(\mathbf{x}^h,\tilde{\epsilon}_1)+\tfrac{1}{2}EU(\mathbf{x}^l,\tilde{\epsilon}_2). \end{aligned} \tag{11}$$

Again, such a preference characterizes the property that the premium for Nth-degree risk changes, $EU(\mathbf{x},\tilde{\epsilon}_2)-EU(\mathbf{x},\tilde{\epsilon}_1)$, is increasing in $\mathbf{x}$. Equivalently, (11) corresponds to the notion of decreasing differences in Definition 4. Contrary to the case of a single decision variable, condition (11) is not sufficient to order the sets of maximizers X_1 and X_2. Intuitively, this condition implies that the decision maker prefers to combine higher risks with higher levels of all of the other variables. However, perceiving these latter variables to be substitutes for each other, the decision maker may choose to respond to the change in risk in Problems 3 and 4 by increasing some decision variables and decreasing others. Therefore, following most of the literature on monotone comparative statics (and Topkis's monotonicity theorems in particular), we will restrict the analysis to the case of complementary decision variables, where complementarity is captured by supermodularity of the payoff function.[17] Clearly, this assumption is not made without loss of generality since there are important multidimensional decision problems in which the controls are not complementary. We remark, however, that for the case with $m=2$ this assumption is not as restrictive since a simple change of variable can in most cases transform a problem in which the decision variables are substitutes into one in which the decision variables are complements.[18] We will illustrate this ordering-reversing "trick" in an example below.

Proposition 3. *Consider Problems 3 and 4 with $\tilde{\epsilon}_2 \succ_N \tilde{\epsilon}_1$. Suppose that B is a sublattice of $\mathbb{R}^m$, that U is N times continuously differentiable in ϵ and supermodular in $\mathbf{x}$ for fixed ϵ, and that X_1, X_2 are nonempty.*[19] *Then, $X_2 \geq_S X_1$ if, equivalently, one of the following conditions holds.*

[17] Note that supermodularity can be interpreted in terms of preference for lotteries that combine "good" with "good" and "bad" with "bad" rather than lotteries that combine "good" with "bad." In particular, given a payoff function with two attributes, supermodularity implies correlation-loving behavior in the sense of Eeckhoudt et al. (2007).

[18] Even in this case, the assumption restricts the decision variables to be always substitutes or always complements.

[19] As mentioned for the case of a single decision variable, a sufficient condition for the sets of solutions to be nonempty is that U is continuous and the constraint set B is compact.

1. *$L_{\mathbf{G}}$ is preferred to $L_{\mathbf{B}}$ for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $\mathbf{x}^h \geq \mathbf{x}^l$.*
2. *$(-1)^N U^{(\mathbf{0},N)}(\mathbf{x}, \epsilon)$ is increasing in $\mathbf{x}$ for all ϵ.*[20]

Moreover, if $L_{\mathbf{G}}$ is strictly preferred to $L_{\mathbf{B}}$ for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $\mathbf{x}^h > \mathbf{x}^l$, then, for $\mathbf{x}_1^* \in X_1$ and $\mathbf{x}_2^* \in X_2$, we have $\mathbf{x}_2^* \geq \mathbf{x}_1^*$.

Conversely, if U is supermodular in $\mathbf{x}$ for fixed ϵ but $L_{\mathbf{B}}$ is preferred to $L_{\mathbf{G}}$ for all $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ and all $\mathbf{x}^h \geq \mathbf{x}^l$ (i.e., $(-1)^N U^{(\mathbf{0},N)}(\mathbf{x}, \epsilon)$ is decreasing in $\mathbf{x}$, and the payoff function has increasing differences in $(\mathbf{x}, \epsilon)$), we have $X_1 \geq_S X_2$. Therefore, in either case (when $L_{\mathbf{B}}$ is preferred to $L_{\mathbf{G}}$ or $L_{\mathbf{B}}$ is preferred to $L_{\mathbf{G}}$) if all decision variables are complementary and they all respond in the same direction to the risk change when considered in isolation, they will still change in the same direction in the multiple control problem. If U is smooth, these conditions are equivalent to (1) supermodularity, where $(\partial^2/\partial x_i \partial x_j)U \geq 0$ for all $i \neq j$; and (2) decreasing differences, where $(-1)^N(\partial^{N+1}/\partial x_i \partial \epsilon^N)U \geq 0$ for all i (increasing differences, where $(-1)^N(\partial^{N+1}/\partial x_i \partial \epsilon^N)U \leq 0$ for all i).[21] We provide two applications of Proposition 3.

5.1. Borrowing and Human Capital Investments

Consider a stochastic version of Lochner and Monge-Naranjo's (2011) 2-date model of borrowing and human capital investments. Given an endowment of wealth y, in the first period the decision maker selects how much to borrow d and the level of human capital accumulation h. The monetary cost of human capital is $c(h)$. In the second period the consumer repays the debt, which has a cost of R, and receives wages $f(h)\tilde{\epsilon}$, where f is increasing and $\tilde{\epsilon} \geq 0$ is a random variable. Following Lochner and Monge-Naranjo (2011), we can interpret the multiplicative term $\tilde{\epsilon}$ as the worker's ability, which is uncertain in our setting. Define

$$\begin{aligned} X_i &= \arg\max_{\mathbf{x} \in B} EU(\mathbf{x}, \tilde{\epsilon}_i) \\ &= \arg\max_{(h,d) \in B} \{u(y - c(h) + d) + Ev(f(h)\tilde{\epsilon}_i - dR)\}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2. \end{aligned} \tag{12}$$

Note that (12) is an extension of the traditional model of precautionary saving. That is, given a fixed level of human capital investment and setting $d = -x$, problem (12) is formally equivalent to precautionary saving problem (6). The reason we use borrowing rather than saving as a decision variable is that, if u and v are concave as traditionally assumed, the payoff function is supermodular in (h, d) (i.e., borrowing and human capital investments are complementary), as required by Proposition 3. Of course, borrowing is not restricted to be positive, so any conclusion about changes in borrowing can be characterized as a change in saving in the opposite direction. It is in reference to this kind of change of variable that we said that supermodularity is not as restrictive in problems with two decision variables.

Now, according to our previous results, if borrowing and human capital investments respond in the same direction to a change in risk when considered in isolation, the consumer will still respond in the same direction when evaluating the joint decisions. In effect, intuition tells us that greater uncertainty surrounding the return from human capital will reduce borrowing and the accumulation of human capital (i.e., the payoff function has increasing differences). The following result provides conditions for this to be the case when $\tilde{\epsilon}_2$ is a mean preserving spread over $\tilde{\epsilon}_1$.

Example 5. Let X_i, $i = 1, 2$, be defined as in (12) and $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$. Suppose that B is a compact sublattice of $\mathbb{R}^2$, that v is concave and twice differentiable, and that u is concave. Without any loss of generality, also suppose that $R = 1$ and $f(h) = h$.[22] Then, $X_1 \geq_S X_2$ if, equivalently, one of the following conditions holds.

- Date-1 lottery $[h'\tilde{\epsilon}_1 - d', h''\tilde{\epsilon}_2 - d'']$ is preferred to date-1 lottery $[h''\tilde{\epsilon}_1 - d'', h'\tilde{\epsilon}_2 - d']$ for all $h' \geq h''$, $d' \geq d''$, and all $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$.
- $v^{(2)}(h\epsilon - d)h^2$ is decreasing in (h, d).

For example, if $v(x)$ is smooth, a mean preserving spread in the return to human capital will decrease the least and greatest selections of borrowing and human capital investment if the decision maker is risk averse, $v^{(2)} \leq 0$, or prudent, $v^{(3)} \geq 0$, and the measure of second-degree partial risk aversion (Chiu et al. 2012), $-v^{(3)}(h\epsilon - d)h\epsilon / v^{(2)}(h\epsilon - d)$, is no larger than 2.

5.2. Risky R&D Investments

Consider a monopolist who selects the production level q and the level of investment r (in units) of research and development (R&D). This investment allows the company to improve the methods of production, reducing per unit costs. The effectiveness of the R&D investment, however, is uncertain. In particular, suppose that the unit cost of production is given by $c(r^e)$, where $r^e = r\tilde{\epsilon}$ and the random variable $\tilde{\epsilon} \geq 0$ captures the effectiveness of R&D. The monetary cost

[20] That is, $(-1)^N(\partial^N U(\mathbf{x}, \epsilon)/\partial \epsilon^N)$ is monotonically increasing in x_i, $i = 1, \ldots, m$. See below for the equivalent differentiability conditions.

[21] These conditions include as a special case Topkis's (1978) characterization result, mentioned in §3, which arises when $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are scalars and $\tilde{\epsilon}_1 > \tilde{\epsilon}_2$ is a special case of $\succ_1$. As mentioned in the introduction, MLR shifts are also a special case of $\succ_1$. In effect, with such a stronger class of shifts, Athey (2002) and Quah and Strulovici (2009) show that the comparative statics result applies to a larger set of agents than those that rank $L_{\mathbf{G}}$ over $L_{\mathbf{B}}$ for all $\tilde{\epsilon}_1 \succ_1 \tilde{\epsilon}_2$ and all $\mathbf{x}^h \geq \mathbf{x}^l$.

[22] The assumption $f(h) = h$ would not be trivial if we were to restrict attention to unique interior solutions as Lochner and Monge-Naranjo (2011) do. In effect, Lochner and Monge-Naranjo assumed that each of the functions u, v, and f satisfies the Inada conditions (which, in turn, imply conditions 1–3 of our Assumption 1).

of the R&D investment is $g(r)$. The problem for the monopolist facing demand function $p(q)$ is to maximize expected profits. Define

$$X_i = \arg\max_{\mathbf{x}\in B} EU(\mathbf{x}, \tilde{\epsilon}_i) = \arg\max_{(q,r)\in B} E[p(q)q - c(r\tilde{\epsilon}_i)q - g(r)], \quad i = 1, 2. \quad (13)$$

Note that the profit function is supermodular in (q, r) as long as the per unit cost is decreasing. Using this fact, the following result characterizes the effect of a mean preserving spread on output and R&D investment.

Example 6. Let X_i, $i = 1, 2$, be defined as in (13) and $\tilde{\epsilon}_1 \succ_2 \tilde{\epsilon}_2$. Suppose that B is a compact sublattice of $\mathbb{R}^2$ and that $c(r\epsilon)$ is twice differentiable. If $c^{(1)}(r\epsilon) < 0$ and $c^{(2)}(r\epsilon)qr^2$ is increasing in (q, r) (e.g., if $c^{(2)}(r\epsilon)$ is positive and nondecreasing), then the profit function is supermodular in (q, r) and has increasing differences in $[(q, r), \tilde{\epsilon}]$; thus, $X_1 \geq_S X_2$.

Therefore, greater uncertainty about the effectiveness of R&D makes the firm more cautious, in the sense that the least and greatest selections of output and R&D spending decrease. Importantly, this result is independent of the properties of the demand function $p(q)$ and of the R&D cost function $g(r)$.

6. Other Stochastic Dominance Orders

Our results can be readily extended to stochastic dominance orders other than Nth-degree risk increases. We present two examples. First, consider Nth-degree stochastic dominance shifts as characterized by Jean (1980). Using the notation in Definition 1, $\tilde{\epsilon}_2$ is dominated by $\tilde{\epsilon}_1$ in the sense of Nth-degree stochastic dominance, and we denote it by $\tilde{\epsilon}_1 \succ_{NSD} \tilde{\epsilon}_2$, if $F_{\tilde{\epsilon}_2}^{[N]}(x) \geq F_{\tilde{\epsilon}_1}^{[N]}(x)$ for all $x \in [0, Q]$, where the inequality is strict for some x, and $F_{\tilde{\epsilon}_2}^{[k]}(Q) \geq F_{\tilde{\epsilon}_1}^{[k]}(Q)$ for $k = 1, \ldots, N-1$. Lemma 1 then applies to Nth-degree stochastic dominance shifts when (1) $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ is replaced by $\tilde{\epsilon}_1 \succ_{NSD} \tilde{\epsilon}_2$ and (2) the condition $(-1)^N q^{(N)}(x) \leq 0$ is replaced by the condition $(-1)^k q^{(k)}(x) \leq 0$ for $k = 1, \ldots, N$. It is therefore straightforward to extend Proposition 3 by introducing the following additional condition: $(-1)^k U^{(\mathbf{0},k)}(\mathbf{x}, \epsilon)$ is increasing in $\mathbf{x}$ for $k = 1, \ldots, N$ (equivalently, $L_{\mathbf{G}}$ is preferred to $L_{\mathbf{B}}$ for all $\tilde{\epsilon}_1 \succ_{NSD} \tilde{\epsilon}_2$ and all $\mathbf{x}^h \geq \mathbf{x}^l$).

Next, suppose that uncertainty is multidimensional. In particular, consider two random variables, $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$, and a dependence structure in the form of positive quandrant dependence (PQD) (see, e.g., Lehmann 1966).[23] Recall that $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$ have PQD if $\Pr(\tilde{\epsilon}_1 > t_1, \tilde{\epsilon}_2 > t_2) \geq \Pr(\tilde{\epsilon}_1 > t_1)\Pr(\tilde{\epsilon}_2 > t_2)$ for all t_1 and t_2. That is, compared to the baseline of independence, $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are more likely to take either high or low values at the same time. Given two independent random variables $(\tilde{\epsilon}_1^{\perp}, \tilde{\epsilon}_2^{\perp})$, where $\tilde{\epsilon}_i^{\perp}$ and $\tilde{\epsilon}_i$ are identically distributed, and a bivariate real valued function q that is twice differentiable, we have $q^{(1,1)} \geq 0 \Leftrightarrow Eq(\tilde{\epsilon}_1, \tilde{\epsilon}_2) \geq Eq(\tilde{\epsilon}_1^{\perp}, \tilde{\epsilon}_2^{\perp})$. Proposition 3 then applies as follows: if $U(\mathbf{x}, \epsilon_1, \epsilon_2)$ is supermodular in $\mathbf{x}$ and $U^{(\mathbf{0},1,1)}(\mathbf{x}, \epsilon_1, \epsilon_2)$ is increasing in $\mathbf{x}$ for all (ϵ_1, ϵ_2), we have $\arg\max_{\mathbf{x}\in B} EU(\mathbf{x}, \tilde{\epsilon}_1, \tilde{\epsilon}_2) \geq_S \arg\max_{\mathbf{x}\in B} EU(\mathbf{x}, \tilde{\epsilon}_1^{\perp}, \tilde{\epsilon}_2^{\perp})$.[24]

7. Concluding Remarks

The standard approach for establishing the comparative statics of risk changes in optimization problems has typically been confined to comparing unique interior solutions. Even within such a limited framework, it is often difficult to interpret the conditions involved in the comparative statics results, which frequently require signing high-order derivatives and cross derivatives. The idea of using preferences over simple lottery pairs to interpret and test those conditions, developed by Eeckhoudt and Schlesinger (2006), Eeckhoudt et al. (2009), and others, has been fundamental in advancing our understanding of decision making under risk. In this paper, we built upon the results of Topkis (1978) to show that such an idea can do even more: the ranking of properly designed lottery pairs is all that is needed to order the set of maximizers. A decision maker who ranks lottery pairs in a way that "more of x" is preferable whenever there is "less of $\tilde{\epsilon}$" (in our setting, an increase in Nth-degree risk) will accordingly, in the context of an optimization problem, increasingly choose x whenever $\tilde{\epsilon}$ "decreases." The assumptions frequently made to ensure unique and interior solutions do not change this basic intuition and, thus, are inessential for the comparative statics result. Moreover, this basic intuition still holds in problems with multiple decision variables that are complementary.

Although we assumed throughout that the decision maker behaves as an expected utility maximizer, our characterization of the comparative statics exercise via preferences over lotteries should permit us to extend the analysis to nonexpected utility frameworks. It would also be interesting to apply the methods studied in this paper to study distributional shifts other than stochastic dominance orders. In particular, we conjecture that these methods can be readily applied

[23] For related concepts of dependence where our analysis applies as well, see Yanagimoto and Okamoto (1969) and more recently Epstein and Tanny (1980).

[24] An application of this result is the 2-date model of precautionary saving with financial risks and background health risks of Denuit et al. (2011). Given an uncertain level of income $\tilde{\epsilon}_1$ and an uncertain level of health $\tilde{\epsilon}_2$, the consumer selects savings x to maximize $u(y-x) + Ev(x+\tilde{\epsilon}_1, \tilde{\epsilon}_2)$. We can then conclude that $\arg\max_{x\in B} u(y-x) + Ev(x+\tilde{\epsilon}_1, \tilde{\epsilon}_2) \geq_S \arg\max_{x\in B} u(y-x) + Ev(x+\tilde{\epsilon}_1^{\perp}, \tilde{\epsilon}_2^{\perp})$ whenever $v^{(1,1)}(x+\epsilon_1, \epsilon_2)$ is increasing in x.

to study the economic consequences of distributional shifts in the form of almost stochastic dominance (Leshno and Levy 2002).[25] Our results should also have ramifications beyond optimization problems. For example, in the context of deterministic parametric changes, monotonicity results have proved useful for the development of the theory of supermodular games (e.g., Topkis 1979, Vives 1990, Milgrom and Roberts 1994). In the same vein, the present results may be extended to analyze the comparative statics of risk changes in equilibrium models.[26] Finally, we only touched upon a few applications of the theory, and we expect that many more will follow.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2015.2202.

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Appendix. Proofs of Main Results

Propositions 1–3 are direct applications of Topkis's monotonicity theorems combined with Lemma 1. Next, we provide succinct proofs of these results.

Proof of Proposition 1

Let $x_1 \in X_1$ and $x_2 \in X_2$. We need to show that $\max\{x_1, x_2\} \in X_2$ and $\min\{x_1, x_2\} \in X_1$. We have

$$\begin{aligned} 0 &\geq EU(\max\{x_1, x_2\}, \tilde{\epsilon}_2) - EU(x_2, \tilde{\epsilon}_2) \\ &\geq EU(\max\{x_1, x_2\}, \tilde{\epsilon}_1) - EU(x_2, \tilde{\epsilon}_1) \geq 0. \end{aligned}$$

From left to right, the first inequality follows from the definition of X_2 and $x_2 \in X_2$. The second inequality follows from the assumption that L_G is always preferred to L_B, so condition (4) holds and the payoff function has decreasing differences in $(x, \tilde{\epsilon})$. The third inequality follows from the definition of X_1 and $x_1 \in X_1$. Since the inequalities are enclosed by zeroes, it follows that $\max\{x_1, x_2\} \in X_2$. Similarly, note that

$$\begin{aligned} 0 &\leq EU(x_1, \tilde{\epsilon}_1) - EU(\min\{x_1, x_2\}, \tilde{\epsilon}_1) \\ &\leq EU(x_1, \tilde{\epsilon}_2) - EU(\min\{x_1, x_2\}, \tilde{\epsilon}_2) \leq 0 \end{aligned}$$

where, from left to right, the first inequality follows from the definition of X_1 and $x_1 \in X_1$, the second inequality follows from the assumption that L_G is always preferred to L_B, and the third inequality follows from the definition of X_2 and $x_2 \in X_2$. Therefore, $\min\{x_1, x_2\} \in X_1$.

It remains to show that condition 2 is equivalent to condition 1 in the proposition. By Lemma 1 the condition $E[U(x^h, \tilde{\epsilon}_2) - U(x^l, \tilde{\epsilon}_2)] \geq E[U(x^h, \tilde{\epsilon}_1) - U(x^l, \tilde{\epsilon}_1)] \geq 0$ holds for all $\tilde{\epsilon}_2$ and $\tilde{\epsilon}_1$ such that $\tilde{\epsilon}_1 \succ_N \tilde{\epsilon}_2$ if and only if $(-1)^N[U^{(0,N)}(x^h, \epsilon) - U^{(0,N)}(x^l, \epsilon)] \geq 0$ for all ϵ, from which the result follows. □

Proof of Proposition 2

Let $x_1^* \in X_1$ and $x_2^* \in X_2$. Suppose that $x_1^* > x_2^*$. We have

$$0 \geq EU(x_1^*, \tilde{\epsilon}_2) - EU(x_2^*, \tilde{\epsilon}_2) > EU(x_1^*, \tilde{\epsilon}_1) - EU(x_2, \tilde{\epsilon}_1) \geq 0$$

where the first inequality follows from $x_2^* \in X_2$, the second strict inequality follows from the assumption that L_G is strictly preferred to L_B (i.e., strictly decreasing differences), and the third inequality follows from $x_1^* \in X_1$. Since the inequalities are enclosed by zeroes, there is a contradiction. □

Proof of Proposition 3

Following the line of reasoning in the proof of Proposition 1, let $\mathbf{x}_1 \in X_1$ and $\mathbf{x}_2 \in X_2$. We need to show that $\max(\mathbf{x}_1, \mathbf{x}_2) \in X_2$ and $\min(\mathbf{x}_1, \mathbf{x}_2) \in X_1$. We have

$$\begin{aligned} 0 &\geq EU(\max(\mathbf{x}_1, \mathbf{x}_2), \tilde{\epsilon}_2) - EU(\mathbf{x}_2, \tilde{\epsilon}_2) \\ &\geq EU(\max(\mathbf{x}_1, \mathbf{x}_2), \tilde{\epsilon}_1) - EU(\mathbf{x}_2, \tilde{\epsilon}_1) \\ &\geq EU(\mathbf{x}_1, \tilde{\epsilon}_1) - EU(\min(\mathbf{x}_1, \mathbf{x}_2), \tilde{\epsilon}_1) \\ &\geq 0 \end{aligned}$$

where the first inequality follows from the definition of X_2 and $\mathbf{x}_2 \in X_2$, the second inequality follows from the assumption that $L_\mathbf{G}$ is preferred to $L_\mathbf{B}$ (i.e., condition (11) holds, so the payoff function has decreasing differences in $(\mathbf{x}, \tilde{\epsilon})$), the third inequality follows from supermodularity, and the fourth inequality follows from the definition of X_1 and $\mathbf{x}_1 \in X_1$. The conclusion then follows. Furthermore, by Lemma 1, property (11) holds for all $\tilde{\epsilon}_2$ and $\tilde{\epsilon}_1$ such that $\tilde{\epsilon}_2 \succ_N \tilde{\epsilon}_1$ if and only if $(-1)^N[U^{(0,N)}(\mathbf{x}^h, \epsilon) - U^{(0,N)}(\mathbf{x}^l, \epsilon)] \geq 0$ for all ϵ. To prove the second part of the proposition, suppose that $\max(\mathbf{x}_1^*, \mathbf{x}_2^*) > \mathbf{x}_2^*$, let (11) be a strict inequality, and use the same argument as above to find a contradiction. □

[25] The recent extensions of the theory of almost stochastic dominance to high-order risk changes using the idea of risk apportionment (Tsetlin et al. 2015) and to settings with multiple variables (Denuit et al. 2014) should prove helpful in this endeavor.

[26] Nocetti and Smith (2015) present some results along these lines.

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