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Semi-analytical valuation for discrete barrier options under time-dependent Lévy processes

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Simple analytical solutions for the prices of discretely monitored barrier options do not yet exist in the literature. This paper presents a semi-analytical and fully explicit solution for pricing discretely monitored barrier options when the underlying asset is driven by a general Lévy process. The explicit formula only involves elementary functions, and the Greeks are also explicitly available with little additional computation. By performing a $\mathcal{Z}$-transform, we reduce the valuation problem to an integral equation. This equation is solved analytically with the solution expressed in terms of a Fourier cosine series. We then manage to analytically carry out the $\mathcal{Z}$-transform inversion, and obtain a semi-analytical formula for pricing discrete barrier options. We establish the theoretical error bound and analyze the convergence order of our method. Numerical implementation demonstrates that our numerical results are accurate and efficient, and match up with the results from the benchmark methods in the literature.

1. Introduction

Barrier options are among the most actively-traded exotic options, and yet closed-form analytical pricing formulae are not available, except for simple cases where the breach of the barrier condition is continuously monitored. On the one hand, in market practice, almost all the barriers of options written on many asset classes (such as equity, commodity, interest rates) are discretely monitored at a regular frequency which is usually daily time intervals. The survey article of Kou in the handbook (Birge and Linetsky, 2007) reports that "due to regulatory and practical issues, most of path-dependent options traded in markets are discrete path-dependent options.... In practice most, if not all, barrier options traded in markets are discretely monitored. In other words, they specify fixed time points for the monitoring of the barrier (typically daily closings)." Besides practical implementation issues, there are some legal and financial reasons why discretely monitored barrier options are preferred to continuously monitored barrier options. For example, some discussions in trader's literature ("Derivatives Week", May 29th, 1995) concerned that, when the monitoring is continuous, extraneous barrier breach may occur in less liquid markets while the major western markets are closed, and may lead to certain arbitrage opportunities.

For the evaluation of these discretely monitored barrier options, the continuously-monitoring-based approximation can sometimes introduce significant pricing errors because of overestimation of the probability to cross the barrier. On the other hand, discrete monitoring policies result in significant computational complexities in determining the prices of discrete barrier options. As a result, as argued by Kou in the handbook (Birge and Linetsky, 2007), "there are essentially no closed-form solutions for discrete barrier options, except those obtained using an integration form of N-dimensional normal distribution functions (N is the number of monitoring points)." Indeed, even under the Black and Scholes (1973) model, almost no simple analytical solutions are available so far. This paper contributes a simple, explicit and semi-analytical solution to efficiently determine the prices of discrete barrier options.

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In the literature, pricing discrete barrier options has been an actively pursued topic in the last decade. There are a range of techniques for pricing discrete barrier options numerically or analytically. Asymptotic approximations, Monte Carlo simulations, binomial and trinomial trees and fast numerical transforms are popular choices. Birge and Linetsky (2007) (Chap. 8) provide a review of the research methods on discrete barrier and lookback options. Interested readers are referred to their paper and the references therein. Here we briefly summarize the methods on discrete barrier options relevant to our approach.

In the numerical methods stream, lattice methods are among the most popular. It is known that the straightforward binomial tree is not efficient in pricing discrete barrier options, due to the inefficiencies in computing discrete extreme values of the sample paths involved in the payoffs. Broadie et al. (1999) proposed an enhanced trinomial tree method, explicitly using the continuity correction in Broadie et al. (1997) and a shifted node. Other lattice methods include adjusting the position of nodes (Tian, 1999; Boyle and Tian, 1999), refining branching near the barrier, or combining a lattice, simulation, and quadrature method (Duan et al., 2003). Recently Dai and Lyuu (2010) introduce a new approach by combining binomial and trinomial trees to achieve improvements in efficiency. Another popularly adopted general method is Monte Carlo simulations. Because barrier options may involve events (e.g. barrier crossing) with very small probabilities, straightforward simulations may have large variances. Variance reduction techniques, notably importance sampling and conditional sampling methods using a Brownian bridge, can be used to achieve significant variance reduction. Instead of giving a long list of related papers, we refer the readers to the book on Monte Carlo simulation by Glasserman (2003) and the recent publication by Joshi and Tang (2010) and references therein. Since the price of a discrete barrier option can be formulated as a solution to a partial differential equation (PDE), one can also use various finite difference methods; see Boyle and Tian (1998), Zvan et al. (2000), and Zhu and De Hoog (2010). Golbabai et al. (2013) develop an accurate numerical method for pricing discrete double barrier options by solving the PDE with a parabolic finite element method. Alternatively, the prices of discrete barrier options can also be written in terms of N-dimensional integrals, so numerical integration methods can be used, see Fusai and Recchioni (2007) etc. The main problem of N-dimensional integrations is that they are computationally inefficient.

Of the analytical methods so far there are basically the following approaches for discrete barrier options: (1) the continuity correction approximation (Broadie et al., 1997); (2) the Wiener–Hopf method (Fusai et al., 2006); (3) the Hilbert transform-based semi-analytical method (Feng and Linetsky, 2008); and (4) other methods (the Broadie and Yamamoto, 2005 method based on fast Gaussian transforms, the Petrella and Kou, 2004 Laplace transform method, the Howison and Steinberg, 2007's asymptotic expansions method, etc.).

The continuity correction, initiated by Broadie et al. (1997) and based on the Black–Scholes model, is to approximate a discrete barrier option by the price of a theoretical counterpart that assumes continuous monitoring and has new adjusted barriers. The main idea of this approach is to relate the discrete barrier option price to the continuous one by using classical results on the overshoot asymptotics of the Gaussian random walk. Recently, Jun (2013) extends the idea to barrier options with two barriers. Fuh et al. (2013) and Dia and Lamberton (2011) extend the continuity correction to jump-diffusion models. In these extensions to models with jumps, they have imposed an assumption that "the correction factor does not depend on the jump part of the process, so that the continuity correction for jump-diffusion models is the same as that for the Black–Scholes model." This argument is questionable because they provided neither a theoretical foundation nor particular numerical experiments supporting this assumption. In this paper we investigate this issue and present numerical comparisons in Section 6.1 and Table 1. We find that, for a pure jump-diffusion model (with no diffusion component), the price of a discretely monitored barrier option can be significantly different from the continuously monitored counterpart. This implies that, for better accuracy, the continuity correction should not be independent of the jump part and a continuity correction term is necessary for a pure jump model.[3]

Feng and Linetsky (2008)'s method involves a sequential evaluation of Hilbert transforms of the product of the Fourier transform of the value function at the previous barrier monitoring date and the characteristic function of the Lévy process. Due to the difficulty of analytically working out these Hilbert transforms and the final inverse Fourier transform, numerically computing these transforms would incur heavy computational burdens. In order to speed up the efficiency, the authors adopt the fast Hilbert transform numerical method to make the approach practically applicable. Therefore, this approach can be viewed as a semi-analytical method.

The Wiener–Hopf method and the Fourier cosine series method, which are more closely relevant to this paper, have also been applied to pricing discretely monitored barrier options, see Fang and Oosterlee (2009) and Levendorskiĭ (2004). Fusai et al. (2006) derive an exact solution for pricing discrete barrier options in the Black–Scholes framework. They perform a $\mathcal{Z}$-transform to translate the partial differential equation into an integral equation, and then solve the half-spaced integral equation with a Wiener–Hopf method. However, they do not find an analytical inversion of the $\mathcal{Z}$-transform as a result of the complicated Wiener–Hopf method and thus have to resort to numerically performing the inverse $\mathcal{Z}$-transform at the end. Green et al. (2010) extend the approach to price discrete barrier options by taking into account more general Lévy processes and double-barrier options. For the latest development of this method applied to discretely monitored exotic options, interested readers are referred to Fusai et al. (2016) and the references therein. On the other hand, the Fourier cosine series method in Fang and Oosterlee (2009) involves a recursive backward numerical induction procedure, which is still time consuming.[4]

Our approach in this paper follows Fusai et al. (2006)'s method to obtain an integral equation, but we solve the equation using a Fourier cosine series method. Our approach in this paper differs from the previous work in the following ways. Firstly, although it follows Fusai et al. (2006) in deriving the integral equation, it solves the integral equation by using a Fourier cosine series expansion rather than the involved Wiener–Hopf method which applies only to the Black–Scholes model. Our method works for more general classes of underlying processes including in particular the Lévy process. Secondly, the significant advantage of our Fourier-cosine-based solution is that the inverse $\mathcal{Z}$-transform can be analytically carried out and explicitly expressed using matrices. Different from the recursive backward numerical induction procedure of Fang and Oosterlee (2009), we analytically solve the equations and no recursive backward induction procedure is needed any more. As a result, the final formula is of a simple form including only elementary functions with no integrations at all. The Greeks are also immediately available in explicit forms with little additional computation. Thirdly, the explicit solution is computationally efficient. For example, our code takes 0.02 s to price a one-year daily mon-

[3] Very recently, Fuh et al. (2015) published a working paper and raised the same concern as us. They also proposed a new continuity correction approximation to take into account jump components.

[4] Skabelin (2014) recently gave a presentation with a new analytical solution at Cornell University. We are keen to study the methodology when the paper is publicly available.

itored down-and-out put with 10^{-8} accuracy in the Black–Scholes model. Finally, our approach is presented in a general Lévy process, (see, for example, Cont and Tankov, 2004 for detailed discussions about the applications of Lévy processes in modeling stock processes and pricing derivatives). As such, our approach only requires the knowledge of the characteristic function of the underlying asset price process, and is applicable to complex variations of barrier options, such as discrete double-barrier options, barrier options when the processes have time-dependent coefficients, barrier options with time-dependent barriers, or barrier options with non-equally spaced monitoring time interval, etc.

In general, pricing these complex variations of barrier options is an even harder task. In the literature, there are significant interests in studying the valuation of double barrier options. Under the Black–Scholes model, Kunitomo and Ikeda (1992) first provide a pricing formula for double barrier options expressed as the sum of an infinite series. For transform-based methods, Geman and Yor (1996) and Pelsser (2000) obtain Laplace transforms of prices of double barrier options, and apply numerical inversion procedures to obtain prices. Sbuelz (2005) proposes a static hedging strategy for double barrier options using a portfolio of single barrier options. All these mentioned papers consider the problems in the Black–Scholes setting. Fusai et al. (2006) comment that the valuation of double barrier options using their Wiener–Hopf factorization method "is sometimes involved and requires suitable numerical approximation", and see also Boyarchenko and Levendorskiĭ (2012). Our approach is also a transform-based method, but can be extended with simple modifications to the valuation of double barrier options and other variations in the general time-dependent Lévy processes (see Section 7).

The rest of this paper is organized as follows. In Section 2, we first introduce notations of Lévy processes and barrier options, and then in a theorem we present our semi-analytical explicit pricing formula for discrete barrier options. We remark that this theorem is the main result of the current paper. Section 3 gives the details of our derivation of the pricing formula in the general Lévy process framework. Section 4 presents two particular cases, the Black–Scholes model and the CGMY model, as examples to further illustrate our approach. In Section 5, we rigorously analyze the total error and establish the theoretical error bounds. The convergence analysis is also provided. In Section 6, we apply our formula to price discrete barrier options in different scenarios and compare with the results from the numerical methods, the analytical approximation, and the semi-analytical methods in the literature. Greeks are also computed with little additional effort. Section 7 demonstrates with examples that our approach can deal with more general scenarios. Our conclusion is given in Section 8.

2. Pricing discrete barrier options in Lévy processes - the main results

2.1. Lévy processes for asset price dynamics

Suppose that $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{Q})$ is a filtered probability space, and here $\mathcal{F}_t$ is a filtration satisfying the usual conditions. Let $T \in [0, \infty]$ denote the time horizon which, in general, can be infinite. The financial market is assumed to be frictionless and absence of arbitrage opportunities, and the risk-neutral probability measure $\mathbb{Q}$ exists. The dynamics of an underlying asset price $(S_t)_{t\,\ge\,0}$ under $\mathbb{Q}$ is described as

$$S_t = e^{X_t}, \quad t > 0. \tag{1}$$

Here $X = (X_t)_{0<t<T}$ is assumed to be a Lévy process. That is, it has independent and stationary increments and possesses the property of continuity in probability, i.e. for any $\epsilon > 0$, $\lim_{h\to0}\mathbb{P}(|X_{t+h} - X_t| > \epsilon) \to 0$. Note that we set $X_0 = \log(S_0)$.

According to the Lévy-Itô decomposition theorem, a general Lévy process admits the following representation

$$X_t = X_0 + \gamma t + \sigma W_t + \int_0^t \int_{\mathbb{R}\setminus 0} z\widetilde{N}(ds, dz). \tag{2}$$

Here $\gamma \in \mathbb{R}$ is a constant and will be determined to ensure that $e^{X_t}\exp(-(r-\delta)t)$ is a $\mathbb{Q}$ martingale. r and δ denote the continuous instantaneous interest rate and dividend rate, respectively. $W \equiv \{W_t\colon t \ge 0\}$ is a standard Brownian motion and $\int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}(ds, dz)$ is a jump process with (possibly) infinitely many jumps. It is this jump process which essentially is responsible for the huge diversity of Lévy processes and also for the discontinuities or jumps in the paths of X_t. $\widetilde{N}(ds, dz)$ denotes the number of jumps of size dz in the infinitesimal time interval dt. All sources of randomness are *mutually independent*. From the martingale property of $\widetilde{N}(t, \mathcal{B})$, we can define the Lévy measure ν as

$$\nu(\mathcal{B}) = \mathbb{E}[\widetilde{N}(1, \mathcal{B})], \quad \text{where } \mathcal{B} \text{ is a Borel set on } \mathbb{R}\setminus\{0\}. \tag{3}$$

Lévy processes provide a generalization of the sum of independent and identically distributed (i.i.d.) random variables. Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths[5]. Other examples of Lévy processes are the Poisson and compound Poisson processes, and they have finite numbers of jumps happening in a given time interval. These are often called "jump-diffusion" processes, and the paths can be described by

$$X_t = X_0 + \gamma t + \sigma W_t + \left(\sum_{k=1}^{N_t} J_k - t\lambda\kappa\right), \tag{4}$$

where $N = (N_t)_{0<t<T}$ is a Poisson process with parameter λ (i.e. $\mathbb{E}[N_t] = \lambda t$) and $J = (J_k)_{k\ge1}$ is an i.i.d. sequence of random variables with probability distribution J and $\mathbb{E}[J] = \kappa < \infty$. Hence, J describes the distribution of the jumps, which arrives according to the Poisson process.

By the Lévy–Khintchine Theorem (see, for example, Bertoin 1998), a Lévy process can be uniquely determined by the Lévy or characteristic triplet (γ, σ, ν). $\gamma \in \mathbb{R}$ is called the drift term, $\sigma \in \mathbb{R} > 0$ the Gaussian or diffusion coefficient and ν the Lévy measure. Moreover, the characteristic function of a Lévy process X_t has the following Lévy–Khinchin representation (Cont and Voltchkova, 2005),

$$\phi(\omega; t) \triangleq \mathbb{E}[e^{i\omega X_t}|\mathcal{F}_0] = e^{i\omega X_0}\exp(t\psi(\omega)), \ \ i = \sqrt{-1}, \tag{5}$$

with

$$\psi(\omega) = -\frac{\omega^2\sigma^2}{2} + i\omega\gamma + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x 1_{\{|x|<1\}})\nu(dx). \tag{6}$$

In Table 2 we list some well-known parameterized Lévy processes and their associated characteristic exponents $\psi(\omega)$. In order to ensure that $e^{X_t}\exp(-(r-\delta)t)$ is a $\mathbb{Q}$ martingale, the following conditions on the triplet (γ, σ, ν) need to be satisfied

$$\int_{|y|>1} \nu(dy)e^y < \infty, \tag{7}$$

and

$$\gamma = \gamma(\sigma, \nu) = r - \delta - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x1_{\{|x|<1\}})\nu(dx). \tag{8}$$

Here r and δ are both assumed to be constant over time for simplicity at the moment. Section 7 relaxes this assumption to incorporate time-dependent parameters, such as $r(t)$ or $\sigma(t)$.

[5] The simplest Lévy process is the linear drift, which has a continuous and deterministic sample paths.

2.2. Barrier options and notations

Barrier options are a type of popularly-traded path-dependent exotic options. If the price of the underlying asset reaches a pre-specified level (called a barrier), a knock-in barrier option would be triggered to be able to receive a payoff, while a knock-out option would become invalidated to receive a payoff. For example, the up-and-out call and put options with strike K and barrier B have payoffs $(S(T)-K)^{+}1_{\{M(0,T)<B\}}$ and $(K-S(T))^{+}1_{\{M(0,T)<B\}}$ at maturity T. Here $M(0,T)=\max_{0\le t\le T}(S_t)$ and $1_{\{\,\cdot\,\}}$ is the indicator function of the event $\{\,\cdot\,\}$. Similarly, up-and-in call and put options have payoffs $(S(T)-K)^{+}1_{\{M(0,T)>B\}}$ and $(K-S(T))^{+}1_{\{M(0,T)>B\}}$. Down-and-in and down-and-out options are similar with $M(0, T)$ replaced by $m(0, T)$ which is defined as $m(0,T)=\min_{0\le t\le T}(S_t)$. Sometimes, an up-and-out option can be coupled with a down-and-out option to form a hybridized option, called a double-barrier-out option. For example, a double-barrier-out call option has a payoff $(S(T)-K)^{+}1_{\{m(0,T)>B_L\}}1_{\{M(0,T)<B_H\}}$. Here B_L and B_H, satisfying $B_L<S_0<B_H$, denote the down barrier and the up barrier, respectively.

In the discussion above, the minimum $m(0, T)$ or the maximum $M(0, T)$ is the global extremum along the entire time domain $[0, T]$. In a discrete time setting, the minimum (maximum) of the asset price is determined by comparing a finite number of discretely monitored instants. We assume that the monitoring instants are equally spaced in time. More precisely, consider the asset value $S(t)$, monitored in the interval $[0, T]$ at a sequence of equally spaced monitoring points, $0=t_0<t_1<\ldots<t_n<\ldots<t_N=T$, then $M(0,N,T)=\max_{0\le j\le N}(S_{t_j})$[6]. Obviously, the total number of monitoring times, denoted by N, (or the frequency of monitoring), will affect the value of a barrier option. The more frequently the barrier is checked, the more likely the in- (or out-) condition of a barrier option is triggered.

2.3. Main results: explicit and semi-analytical pricing formula for barrier options

Theorem 1 (Pricing Formula)**.** *If the underlying stock follows the exponential Lévy process (1), then the price of a double-barrier-out call option expiring at time T with N monitoring instants, denoted by $C(S_0,t=0,T,N,B_L,B_H)$, can be explicitly valuated by the following function*

$$C(S_0,t=0,T,N,B_L,B_H) = e^{-rT}\left[A_j\left(\log S_0,\frac{T}{N}\right)\right]_{1\times M}\times\mathbf{A}^{N-1}\times\mathbf{f}, \tag{9}$$

where

$$\mathbf{f}=[f_k]_{M\times 1},\ \text{with } f_k=\int_{D_1\cap D_2}b_k(x)f(x,0)dx, \tag{10}$$

$$A_j(x)=\frac{2}{b-a}\mathrm{Re}\left[\phi\left(\frac{j\pi}{b-a};x,\frac{T}{N}\right)\times\exp\left(-i\frac{ja\pi}{b-a}\right)\right], \tag{11}$$

and

$$\mathbf{A}=[A_{kj}]_{M\times M},\ \text{with } A_{kj}=\int_{D_1\cap D_2}b_k(x)A_j(x)dx. \tag{12}$$

Here f(x, 0) is the payoff function of the option with its expression given in Eq. (21), $\phi(\omega;x,\frac{T}{N})$ is the characteristic function of the Lévy process, and $b_k(x)=\cos\left(\frac{k\pi\,(x-a)}{b-a}\right)$. The detailed expressions for carrying out the integrations above are presented in Appendix A and Appendix B with these notations. The controlling variables M, a, b, D_1 and D_2 are introduced in Section 3.3.

It is also straightforward to extend the above formula to value a barrier option that starts at an initial time $t_{n-1}<t<t_n$ where $n=1,2,\ldots,N$. This is motivated from hedging purposes where we need the Greeks not only at the monitoring dates, but also at intermediate time points between monitoring dates. Define

$$\tilde{A}_j(x,s)=\frac{2}{b-a}\mathrm{Re}\left[\phi\left(\frac{j\pi}{b-a};x,s\right)\times\exp\left(-i\frac{ja\pi}{b-a}\right)\right], \tag{13}$$

and

$$\tilde{\mathbf{A}}(s)=[\tilde{A}_{kj}(s)]_{M\times M},\ \text{with } \tilde{A}_{kj}(s)=\int_{D_1\cap D_2}b_k(x)\tilde{A}_j(x,s)dx. \tag{14}$$

We have the following corollary.

Corollary 1. *If the underlying stock follows the exponential Lévy process (1), then the price of a double-barrier-out call option expiring at time T with initial time point t satisfying $t_{n-1}<t<t_n$, $n=1,2,\ldots,N$, denoted by $C(S_t, t, T, N, n, B_L, B_H)$, can be explicitly valuated by the following function*

$$C(S_t,t,T,N,n,B_L,B_H) = e^{-r(T-t)}\left[\tilde{A}_j\left(\log S_t,n\frac{T}{N}-t\right)\right]_{1\times M}\times\left[\tilde{\mathbf{A}}\left(\frac{T}{N}\right)\right]^{N-n}\times\mathbf{f}, \tag{15}$$

where the intermediate functions are defined in Eqs. (13) and (14).

Note that when $n=1$ and $t=0$, the formula given in (15) degenerates to Formula (9) as expected.

3. Derivation of the main results

3.1. Notations and problem formulation

In the rest of this paper, we shall use g to denote a general contingent claim, so g can denote the value of a call option, a put option, or even other complex derivatives. Based on the Lévy process (1), Cont and Voltchkova (2005) derive the governing partial integro-differential equation for the un-discounted price of a contingent claim g as:

$$\frac{\partial g}{\partial\tau}=\mathcal{L}[g(x)]+(r-\delta)\frac{\partial g}{\partial x}. \tag{16}$$

Here, the infinitesimal generator $\mathcal{L}$ is

$$\mathcal{L}g(x)=\frac{\sigma^2}{2}\left[\frac{\partial^2 g}{\partial x^2}-\frac{\partial g}{\partial x}\right](x)+\int_{-\infty}^{\infty}\left[g(x+y)-g(x)-(e^y-1)\frac{\partial g}{\partial x}(x)\right]\nu(dy). \tag{17}$$

However, solving this equation to obtain the value function g for the general process (1) is extremely difficult, even for a vanilla option. We follow Fusai et al. (2006)'s procedure to frame the pricing problem of a barrier option as an integral equation which is then solved.

Let $0=t_0<t_1<\ldots<t_n<\ldots<t_N=T$ be the monitoring dates, T the option maturity. Write $\tau=T-t$ to denote the time to maturity, and $\tau_n=T-t_{N-n}$ $(n=0,\ldots,N)$. In the rest of this paper, we shall use τ (and τ_n) as the variable in the time space, and let $\Delta\tau$ (or Δt exchangeably) be the fixed equally-spaced time period between two successive monitoring dates ($\Delta\tau=\Delta t=\tau_{n+1}-\tau_n$). The time interval between the nth and $(n+1)$th barrier checking is $[\tau_{n-1},\tau_n]$.[7] We also define $l=\log B_L$ and $h=\log B_H$ to denote the

[6] In the rest of the paper, we will use S_k (or X_k) to denote S_{t_k} (or X_{t_k}) for notation simplicity when no confusion will be caused.

[7] In our notation settings for the barrier options, the first barrier checking, indexed by $n=0$, takes place at the maturity date, T or τ_0, and the last barrier checking, which is the Nth checking and indexed by $n=N-1$, takes place at t_1 or τ_{N-1}. The current time is $t=0$ or τ_N.

down and up barriers in the X-space. They are all constants active at all times t_n.

In the following, we consider a double-barrier-out call option with a payoff $(S_T-K)^+$ at the maturity if not knocked out. For generality we denote the price of this barrier option in the interval $\tau_{n-1}\le\tau<\tau_n$ as $g(x, \tau, n; l, h)$ (or simply $g(x, \tau, n)$). It is the price when the nth check of the barrier boundaries has already been completed, but the $(n+1)$th check has not commenced yet. As it can be understood, $g(x, \tau_n, n)$ denotes the value of an option at time τ_n^- before checking the barrier conditions of the option, while $g(x,\tau_n,n+1)$ means the value at time τ_n^+ immediately after checking the conditions of the barrier boundaries. As a result, after the barrier checking routine is completed at time τ_n, we can record the outcome of the checking and update $g(x,\tau_n,n+1)$. That is,

$$\begin{cases} g(x,\tau_n,n+1)=g(x,\tau_n,n)\mathbf{1}_{\{l<x<h\}}, \\ g(x,\tau_0,1)=(e^x-K)^+\mathbf{1}_{\{l<x<h\}}. \end{cases} \tag{18}$$

To further simplify the notations, let us consider the function $g(x, \tau, n)$ only at the monitoring times τ_n, and define

$$f(x,n)=g(x,\tau_n^-,n). \tag{19}$$

That is, the value of $g(x, \tau, n)$ at the left extremity, τ_n, of the time interval just before the $(n+1)$th checking of the barriers. In Section 7 we shall discuss the price of barrier options at any arbitrary time rather than the monitoring time τ_n. The current value of a barrier call option with N total monitoring instants, denoted by $C(x_0, N)$ is,

$$C(x_0,N)=f(x_0,n)=g(x,\tau_n^-,n). \tag{20}$$

The initial value of the series $\{f(x,n)\}_{(n=0,\dots,N,\dots)}$ is set to be

$$f(x,0)=(e^x-K)^+1_{\{l<x<h\}}. \tag{21}$$

In addition, according to the risk-neutral expectation theory, the value of $f(x,n+1)$ can be linked with $f(x, n)$, (which is equivalent to the solution of the partial differential equation (16) with the condition (18)) by (see also Fusai et al., 2006)

$$f(x,n+1)=e^{-r\Delta t}\int_{-\infty}^{\infty}p(\xi;x,\Delta\tau)f(\xi,n)1_{\{l<\xi<h\}}d\xi, \quad n=0,1,\dots, \tag{22}$$

where the kernel $p(\xi;\, x,\, \Delta\tau)$ is the conditional probability density function of the random variable ξ at time τ_n, conditional on the information filtration $\mathcal{F}_{\tau_{n+1}}$ (such as the value of x at time τ_{n+1}). Note that an iterative application of (22) provides an evaluation of discrete barrier options in terms of recursive integrations, but from a numerical point of view it is hardly feasible when the number of observation points becomes large, say more than 10.

3.2. $\mathcal{Z}$-transform

We now perform a $\mathcal{Z}$-transform, as defined by Poularikas (2000), with reference to the sequence $\{f(x, n)\}$ in the above difference equation,

$$F(x,z)\triangleq\mathcal{Z}[f(x,n)]=\sum_{n=0}^{\infty}z^{-n}f(x,n). \tag{23}$$

The $\mathcal{Z}$-transform $\sum_{n=0}^{\infty}z^{-n}f(x,n)$ is in fact a Laurent series in z with coefficients $f(x, n)$ and its radius of convergence can be computed. This series converge uniformly for all points in the z-plane that lie outside the circle of the radius of convergence. We assume the following:

Assumption 1. $z\in\mathbb{C}$ is the transform variable and it satisfies the condition $|z|>1$.

The condition in Assumption 1 will ensure the existence and well-definedness of the $\mathcal{Z}-$transform in (23). Note that by definition the coefficients $f(x, n)$ are positive and bounded in n for any x. In particular, from Eqs. (21) and (22), we have that $0<f(x, n)<e^x$ holds. We can check that $\liminf_{n\to\infty}f(x,n)^{-1/n}\ge\lim_{n\to\infty}e^{-x/n}=1$ holds, then by the Cauchy–Hadamard theorem (page 55–56 of Lang, 2002), the power series in (23) is convergent with radius greater than 1. Thus the $\mathcal{Z}-$transform exists for $|z^{-1}|<1$ or equivalently for $|z|>1$.

Following this transform definition and multiplying both sides of (22) by z^{-n}, we obtain

$$z^{-n}f(x,n)=\frac{e^{-r\Delta t}}{z}\int_{-\infty}^{\infty}p(\xi;x,\Delta t)z^{-(n-1)}f(\xi,n-1)1_{\{l<\xi<h\}}d\xi, \quad n=1,\dots. \tag{24}$$

Then sum over all $n\ge 1$. With Assumption 1, we have that $\sum_{n=1}^{\infty}z^{-n}f(x,n)$ converges uniformly for all points satisfying $|z|>1$, From Fubini Theorem, we can interchange the order of integration and summation, and obtain

$$\begin{aligned}\sum_{n=1}^{\infty}z^{-n}f(x,n) &=\frac{e^{-r\Delta t}}{z}\int_{-\infty}^{\infty}p(\xi;x,\Delta t)\sum_{n=1}^{\infty}z^{-(n-1)}f(\xi,n-1)1_{\{l<\xi<h\}}d\xi \\ &=\frac{e^{-r\Delta t}}{z}\int_{-\infty}^{\infty}p(\xi,x,\Delta t)\sum_{n=0}^{\infty}z^{-n}f(\xi,n)1_{\{l<\xi<h\}}d\xi.\end{aligned} \tag{25}$$

Adding $f(x, 0)$ to both sides of Eq. (25), we derive the following integral equation for $F(x, z)$,

$$F(x,z)=\frac{e^{-r\Delta t}}{z}\int_{-\infty}^{\infty}p(\xi;x,\Delta t)F(\xi,z)1_{\{l<\xi<h\}}d\xi+f(x,0). \tag{26}$$

These procedures have transformed the discrete, iterative integral expression (or an n-dimensional integration) into a continuous integral equation. The advantage is that we do not need to solve the original integral equation system recursively, which leads to savings in computational time. However, the difficulty is to obtain the inverse $\mathcal{Z}$-transform. Like many other complex-valued integral transforms, obtaining the inverse $\mathcal{Z}$-transform analytically is usually difficult. For example, Fusai et al. (2006) derive the above integral equation in the Black–Scholes framework and solve the integral equation analytically, but they have to numerically evaluate the inverse $\mathcal{Z}$-transform. Numerically computing complex-valued integral transforms requires analysis of the singularities and sometimes leads to severe oscillation or instability because of the singularities and multi-branch cuts (see, for example, Albrecher et al., 2007 and Lord and Kahl, 2010).

3.3. Solutions in Fourier-cosine series

In this paper, rather than solving the integral equation using the Wiener–Hopf method, we solve the integral equation using a Fourier-cosine series. The advantage of our Fourier-cosine-series solution is that the inverse $\mathcal{Z}$-transform can be worked out analytically. Also, the Wiener–Hopf method is very involved and as a result it is difficult to use. Our approach significantly simplifies the whole procedure, and the solution from the integral equation is of a simple form. The main idea is to reconstruct the whole integral (not just the integrand) from its Fourier cosine series expansion, extracting the series coefficients directly from the integrand.

Although the density $p(\xi,\, x,\, \Delta t)$ is explicitly known for the Black–Scholes model, that is usually not the case for other Lévy models. Computing $p(\xi,\, x,\, \Delta t)$ requires the inverse Fourier transform of characteristic functions. We assume that on a finite interval, $[a,b]\in\mathbb{R}$, the density $p(\xi,\, x,\, \Delta t)$ can be expressed in terms of a Fourier-cosine series expansion

$$p(\xi, x, \Delta t) = \sum_{j=0}^{\infty} A_j(x) b_j(\xi), \tag{27}$$

with

$$b_j(\xi) = \cos\left(\frac{j\pi\,(\xi - a)}{b - a}\right), \tag{28}$$

and

$$A_j = \begin{cases} \dfrac{1}{b-a}, & \text{if } j = 0, \\ \dfrac{2}{b-a}\displaystyle\int_a^b p(\xi, x, \Delta t)\cos\left(j\pi\frac{\xi - a}{b - a}\right)d\xi, & \text{if } j > 0. \end{cases} \tag{29}$$

Since any real function has a cosine expansion when it is finitely supported, the derivation starts with a truncation of the infinite integration range in (26). Due to the conditions for the existence of a Fourier transform, the integrands in (26) have to decay to zero at $\pm\,\infty$ and we can truncate the integration range in a proper way without losing much accuracy. A detailed convergence and error analysis is given in Section 5.

Following Fang and Oosterlee (2008), the coefficient A_j in the expansion can be recovered from the characteristic function. Specifically, suppose that $[a, b] \in \mathbb{R}$ is chosen such that the truncated integral approximates the infinite counterpart very well, then

$$A_j = \frac{2}{b-a}\int_a^b p(\xi, x, \Delta t)\cos\left(j\pi\frac{\xi - a}{b - a}\right)d\xi \tag{30}$$

$$\begin{aligned} &= \frac{2}{b-a}\mathrm{Re}\left[\phi_1\left(\frac{j\pi}{b-a}\right) \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right] \\ &\simeq \frac{2}{b-a}\mathrm{Re}\left[\phi\left(\frac{j\pi}{b-a}\right) \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right], \end{aligned} \tag{31}$$

where $\phi_1(\omega; x, \Delta t) = \int_a^b e^{i\omega\xi} p(\xi, x, \Delta t)d\xi$ is the characteristic function defined on the truncated region $[a, b]$ and $\phi(\omega; x, \Delta t) = \int_{\infty}^{\infty} e^{i\omega\xi} p(\xi, x, \Delta t)d\xi$ is the characteristic function defined on $\mathbb{R}$. The approximation error of using $\phi(\,\cdot\,)$ to replace $\phi_1(\,\cdot\,)$ will be analyzed in Section 5.

Furthermore, because the modulus of A_j decays with the speed of e^{-j^2} in the Black–Scholes model when $j \to \infty$ and for other Lévy processes it will decay either exponentially or algebraically (see Section 5 and Fang and Oosterlee, 2008 for more discussions), the infinite sum in Eq. (27) can be approximated by a finite number of terms as

$$p(\xi, x, \Delta t) \approx \sum_{j=0}^{M} A_j(x) b_j(\xi), \tag{32}$$

where M is the total number of terms of the Fourier-cosine series used to approximate the density $p(\xi,\ x,\ \Delta t)$. From the convergence analysis in Section 5, we can always increase M to achieve any desired accuracy.

Introduce the notation $D_1 = \{x : x \in \mathbb{R}, a \le x \le b\}$ for the interval $[a,\ b]$, and later use it to truncate the probability density function $p(\xi,\ x,\ \Delta t)$. Due to the conditions for the existence of a Fourier transform, the density function decays rapidly to zero as $x \to \pm\infty$, and truncating the infinite integration range should not lose significant accuracy. Also, introduce the notation D_2 to denote the domain in X-space[8] where the barrier option remains alive. For example, $D_2 = \{x \le h\}$ for an up-and-out option with barrier h, $D_2 = \{h \le x\}$ for an up-and-in option with up-barrier h, and $D_2 = \{x \le l\}$ for a down-and-in option with down-barrier l. For a double-barrier-out option with barriers l and h, $D_2 = \{l \le x \le h\}$.

Substituting the expression (27) in the integral Eq. (26), we have

$$\begin{aligned} F(x, z) &\simeq \frac{e^{-r\Delta t}}{z}\int_{D_1\cap D_2} p(\xi, x, \Delta t)F(\xi, z)d\xi + f(x, 0) \\ &= \frac{e^{-r\Delta t}}{z}\int_{D_1\cap D_2}\sum_{j=0}^{M} A_j(x)b_j(\xi)F(\xi, z)d\xi + f(x, 0) \\ &= \frac{e^{-r\Delta t}}{z}\sum_{j=0}^{M} A_j(x)\int_{D_1\cap D_2} b_j(\xi)F(\xi, z)d\xi + f(x, 0). \end{aligned} \tag{33}$$

Integrating both sides of the above equation, we have

$$\begin{aligned} &\int_{D_1\cap D_2} b_k(x)F(x, z)dx \\ &= \frac{e^{-r\Delta t}}{z}\sum_{j=0}^{M}\int_{D_1\cap D_2} b_k(x)A_j(x)dx\int_{D_1\cap D_2} b_j(\xi)F(\xi, z)d\xi \\ &\quad + \int_{D_1\cap D_2} b_k(x)f(x, 0)dx. \end{aligned} \tag{34}$$

For notational simplicity, we introduce the following vectors and matrices:

$$\mathbf{F} := [F_k(z)]_{M\times 1}, \text{ with } F_k(z) = \int_{D_1\cap D_2} b_k(x)F(x, z)dx, \tag{35}$$

$$\mathbf{f} := [f_k]_{M\times 1}, \text{ with } f_k = \int_{D_1\cap D_2} b_k(x)f(x, 0)dx, \tag{36}$$

and

$$\mathbf{A} := [A_{kj}]_{M\times M}, \text{ with } A_{kj} = \int_{D_1\cap D_2} b_k(x)A_j(x)dx. \tag{37}$$

The detailed expressions for carrying out the integrations above are presented in Appendices A and B. With these notations, Eq. (34) can be written as the matrix equation:

$$\mathbf{F} = \frac{e^{-r\Delta t}}{z}\mathbf{A}\mathbf{F} + \mathbf{f}, \tag{38}$$

which can be rewritten as

$$\left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right] \times \mathbf{F} = \mathbf{f}.$$

This matrix equation is solvable if the following assumption holds:

Assumption 2. for a suitable matrix norm[9] $||\cdot||$, we assume that the transform variable $z \in \mathbb{C}$ satisfies $|z| > R := e^{-r\Delta t}||\mathbf{A}||$.

[8] The domain of the barrier option should be presented in the original S space, but as X has a one-to-one mapping relationship with S, we shall present all the barrier conditions in the X space.

[9] In particular, it satisfies the sub-multiplicative property $||\mathbf{AB}|| \le ||\mathbf{A}|| \cdot ||\mathbf{B}||$, which is Axiom (4) in Section 5.6 on page 290 of Horn and Johnson (1985).

Corollary 2. *If Assumption 2 holds, then the matrix* $\left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right]$ *is invertible and its inverse has the following power series representation*

$$\left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right]^{-1} = \mathbf{I} + \frac{e^{-r\Delta t}}{z}\mathbf{A} + \frac{e^{-2r\Delta t}}{z^2}\mathbf{A}^2 + \ldots + \frac{e^{-nr\Delta t}}{z^n}\mathbf{A}^n + \ldots$$

for $|z| > e^{-r\Delta t}||\mathbf{A}||$.

Proof. We first recall the following Lemma: □

Lemma 1 (Corollary 5.6.16 of Horn and Johnson, 1985)**.** *A matrix* $\mathbf{B} \in \mathbf{M_n}$ *is invertible if there is a matrix norm* $||\cdot||$ *such that* $||\mathbf{I} - \mathbf{B}|| < 1$. *If this condition is satisfied,*

$$\mathbf{B}^{-1} = \sum_{k=0}^{\infty}(\mathbf{I} - \mathbf{B})^k.$$

In our setting, we just need to choose $\mathbf{B} = \mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}$ in Lemma 1, and we then can verify that the condition $||\mathbf{I} - \mathbf{B}|| < 1$ is equivalent to Assumption 2.

So we have

$$\mathbf{F} = \left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right]^{-1} \times \mathbf{f}, \tag{39}$$

where $\mathbf{I}$ is an $M \times M$ identity matrix.

Finally, inserting the expression of $\mathbf{F}$ into Eq. (33), we have

$$F(x,z) = \frac{e^{-r\Delta t}}{z}[A_j(x)]_{1\times M} \times \left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right]^{-1} \times \mathbf{f} + f(x,0). \tag{40}$$

3.4. Inverse $\mathcal{Z}$-transform

Throughout this section, we assume Assumptions 1 and 2, which are equivalent to $|z| > \max(R, 1)$. In general, the inverse $\mathcal{Z}$-transform requires the computation of the following complex-valued integral:

$$f(x,n) = \mathcal{Z}^{-1}[F(x,z)] = \frac{1}{2\pi i}\oint_C F(x,z)z^{n-1}dz. \tag{41}$$

However, recall that for the series $\{f(x,n), n = 0, \ldots, \infty\}$, the $\mathcal{Z}$-transform is defined as

$$F(x,z) = \mathcal{Z}[f(x,n)] = \sum_{n=0}^{\infty} z^{-n} f(x,n). \tag{42}$$

This suggests that the value of $f(x, n)$ is obtained as the coefficient of z^{-n} in the power series expansion of $F(x, z)$ in terms of z^{-1}, when $F(x, z)$ is analytic for $|z| > \max(R, 1)$.

Recall from Corollary 2 that the power series expansion of the matrix is

$$\left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right]^{-1} = \mathbf{I} + \frac{e^{-r\Delta t}}{z}\mathbf{A} + \frac{e^{-2r\Delta t}}{z^2}\mathbf{A}^2 + \ldots + \frac{e^{-nr\Delta t}}{z^n}\mathbf{A}^n + \ldots$$

$$\text{for } |z| > \max(R, 1). \tag{43}$$

Then, $F(x, z)$ can also be expressed as a power series expansion in terms of z^{-1}

$$\begin{aligned} F(x,z) &= \frac{e^{-r\Delta t}}{z}[A_j(x)]_{1\times M} \times \left[\mathbf{I} - \frac{e^{-r\Delta t}}{z}\mathbf{A}\right]^{-1} \times \mathbf{f} + f(x,0) \\ &= [A_j(x)]_{1\times M} \times \left(\frac{e^{-r\Delta t}}{z}\mathbf{I} + \frac{e^{-2r\Delta t}}{z^2}\mathbf{A} + \frac{e^{-3r\Delta t}}{z^3}\mathbf{A}^2 \ldots \right. \\ &\quad \left. + \frac{e^{-(n+1)r\Delta t}}{z^{(n+1)}}\mathbf{A}^n + \ldots\right) \times \mathbf{f} + f(x,0). \end{aligned} \tag{44}$$

As mentioned, $f(x, n)$ is the coefficient of z^{-n} in the expansion, so

$$\begin{aligned} f(x,n) &= \mathcal{Z}^{-1}[F(x,z)] \\ &= e^{-rn\Delta t}[A_j(x)]_{1\times M} \times \mathbf{A}^{n-1} \times \mathbf{f}. \end{aligned} \tag{45}$$

In summary, the price of a barrier option at time $t = 0$, can be computed by

$$C(S_0, t=0, T, N, B_L, B_H) = e^{-rT}[A_j(\log S_0)]_{1\times M} \times \mathbf{A}^{N-1} \times \mathbf{f}, \tag{46}$$

and this completes the proof of Theorem 1.

One may argue that our solution is an approximation rather than an exact solution because of the errors incurred by the truncation $[a, b]$ and the use of finitely many terms M to approximate the Fourier cosine expansion which shall have infinite number of terms. From the error and convergence analysis in Section 5, it is shown that one can choose the truncation interval $[a, b]$ as wide as possible and choose M as large as possible so as to achieve any desired accuracy, e.g. up to the precision limit of a computer system. In addition, for the numerical implementation of any exact solution involving an integration over an infinite interval, such as the Black–Scholes formula or the Fourier inversion formula for the Heston stochastic volatility model, one always has to truncate the infinite interval into a finite interval, like our $[a, b]$. Similarly, programs that implement any exact solutions expressed by a summation of infinite terms, such as Merton (1976)'s option pricing formula or the exact solutions of Fusai et al. (2006), will have to terminate at a finite total number of terms. Finally, although we demonstrate our procedure above through a double-barrier-out call option, we remark that the approach is applicable to other more general settings as discussed in Section 7.

4. Special cases: a demonstration

4.1. Double barrier-out options in Black–Scholes model

The probability density function in the Black–Scholes model is

$$p(\xi, x, \Delta t) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{\frac{-[\xi - x - (r - \delta - \frac{1}{2}\sigma^2)\Delta t]^2}{2\sigma^2\Delta t}}, \tag{47}$$

where $\mu = (r - \delta - \frac{1}{2}\sigma^2)\Delta t$. This function can be presented in terms of the Fourier-cosine series as

$$p(\xi, x, \Delta t) = \sum_{j=0}^{M} A_j(x) b_j(\xi), \tag{48}$$

with

$$A_j(x) = \begin{cases} \dfrac{1}{b-a}, & j = 0, \\ \dfrac{2}{b-a} e^{-\frac{j^2\pi^2\sigma^2\Delta t}{2(b-a)^2}} \cos\left(\dfrac{j\pi\,(a - x - \mu)}{b-a}\right), & j \geq 1. \end{cases} \tag{49}$$

Let $c = \max(a, l)$ and $d = \min(b, h)$, then $[c, d] = D_1 \cap D_2$. So, we can compute

$$\begin{aligned} A_{kj} &= \int_c^d b_k(x) A_j(x) dx \\ &= \frac{2}{b-a} e^{-\frac{j^2\pi^2\sigma^2\Delta t}{2(b-a)^2}} \int_c^d \cos\left(\frac{j\pi\,(x + \mu - a)}{b-a}\right) \cos\left(\frac{k\pi\,(x-a)}{b-a}\right) dx \\ &= \frac{2e^{-\frac{j^2\pi^2\sigma^2\Delta t}{2(b-a)^2}}}{\pi\left(j^2 - k^2\right)} \left[\cos\left(\frac{\pi\,((c + d - 2a)(k - j) - 2j\mu)}{2(b-a)}\right)\right. \\ &\quad \times \sin\left(\frac{\pi\,((c-d)(k-j))}{2(b-a)}\right)(j+k) \end{aligned}$$

$$+\cos\left(\frac{\pi\ ((c+d-2a)(k+j)+2j\mu)}{2(b-a)}\right)$$
$$\sin\left(\frac{\pi\ ((d-c)(k+j))}{2(b-a)}\right)(j-k)\Bigg]. \tag{50}$$

4.2. Up-and-out options in CGMY model

To the best of the authors' knowledge, there exists no analytical explicit probability density function for the CGMY model. However, the characteristic function is explicitly known as:

$$\phi(\omega; x_0, t) = \mathbb{E}[e^{i\omega X_t}|\mathcal{F}_0] = \exp\,(i\omega x_0 + t \times \psi(\omega)), \tag{51}$$

with

$$\psi(\omega) = C\Gamma(-Y)\big[(M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y\big]. \tag{52}$$

Therefore, the probability density function can be expressed in terms of the Fourier-cosine series as

$$p(\xi, x, \Delta t) = \sum_{j=0}^{M} A_j(x) b_j(\xi), \tag{53}$$

with

$$A_j(x) = \begin{cases} \dfrac{1}{b-a}, & j=0, \\ \dfrac{2}{b-a}\mathrm{Re}\left[\phi\left(\dfrac{j\pi}{b-a}\right)\times \exp\left(-i\dfrac{ja\pi}{b-a}\right)\right] & j\geq 1. \end{cases} \tag{54}$$

For an up-and-out option, we have $c = a$, $d = \min(b, h)$, and

$$A_{kj} = \int_c^d b_k(x) A_j(x) dx$$
$$= \frac{2}{b-a}\mathrm{Re}\left[\exp\left(\Delta t \times \psi\left(\frac{j\pi}{b-a}\right)\right)\right.$$
$$\left.\times \int_c^d b_k(x)\exp\left(i\frac{j\pi(x-a)}{b-a}\right)dx\right]. \tag{55}$$

5. Convergence and error analysis

In this section, we conduct a theoretical analysis of the total approximation error and the convergence by utilizing the Fourier-cosine series to solve the integral equation (26). Recall that there are three approximation steps in our derivation. The total error of the approximation contains three parts:

(1) The integration range truncation error from (33) is

$$\epsilon_1 := \frac{e^{-r\Delta t}}{z}\int_{D_2/(D_1\cap D_2)} p(\xi; x, \Delta t) F(\xi, z) d\xi. \tag{56}$$

(2) The series truncation error on $[a, b]$ in (32) is

$$\epsilon_2 := \frac{e^{-r\Delta t}}{z}\sum_{j=M+1}^{\infty} A_j(x)\int_{D_1\cap D_2} b_j(\xi) F(\xi, z) d\xi. \tag{57}$$

(3) The error of approximating $\phi_1(\cdot)$ by $\phi(\cdot)$ in (31) is

$$\epsilon_3 := \frac{2e^{-r\Delta t}}{z(b-a)}\sum_{j=0}^{M}\mathrm{Re}\left(\int_{D_2/(D_1\cap D_2)} e^{ij\pi\frac{\xi-a}{b-a}} p(\xi; x, \Delta t) d\xi\right)$$
$$\times \int_{D_1\cap D_2} b_j(\xi) F(\xi, z) d\xi. \tag{58}$$

Although there are three parts of errors, we show that the key to bounding the approximation errors lies in the decay rate of the Fourier cosine series coefficient. We remark that our error analysis is analogous to that of Fang and Oosterlee (2008), who also use the Fourier cosine series method. In particular, we follow their definitions of algebraic (or exponential) index of convergence and accept their Propositions 4.1 and 4.2 as the foundation of our analysis.

Lemma 2. *The error ϵ_3 can be bounded by*

$$|\epsilon_3| < \frac{2}{b-a}|\epsilon_1| + Q|\epsilon_4|,$$

where Q is some constant independent of M and

$$\epsilon_4 := \int_{D_2/(D_1\cap D_2)} p(\xi; x, \Delta t) d\xi.$$

Proof. Assume that $p(\xi;\ x,\ \Delta t)$ is a real function. Denote $V_j := \int_{D_1\cap D_2} b_j(\xi) F(\xi, z) d\xi$, we shall rewrite (58) as

$$\epsilon_3 := \frac{2e^{-r\Delta t}}{z(b-a)}\sum_{j=0}^{M} V_j \int_{D_2/(D_1\cap D_2)} \cos\left(j\pi\frac{\xi-a}{b-a}\right) p(\xi; x, \Delta t) d\xi$$
$$= \frac{2e^{-r\Delta t}}{z(b-a)}\left(\sum_{j=0}^{\infty} V_j \int_{D_2/(D_1\cap D_2)} \cos\left(j\pi\frac{\xi-a}{b-a}\right) p(\xi; x, \Delta t) d\xi\right.$$
$$\left. - \sum_{j=M+1}^{\infty} V_j \int_{D_2/(D_1\cap D_2)} \cos\left(j\pi\frac{\xi-a}{b-a}\right) p(\xi; x, \Delta t) d\xi\right)$$
$$= \frac{2}{b-a}\epsilon_1 - \frac{2e^{-r\Delta t}}{z(b-a)}\sum_{j=M+1}^{\infty} V_j \int_{D_2/(D_1\cap D_2)} \cos\left(j\pi\frac{\xi-a}{b-a}\right)$$
$$p(\xi; x, \Delta t) d\xi. \tag{59}$$

Applying Propositions 4.1 and 4.2 of Fang and Oosterlee (2008), we know that the series V_j show at least algebraic convergence with the algebraic convergence index n, then we have

$$\left|\sum_{j=M+1}^{\infty} V_j \cos\left(j\pi\frac{\xi-a}{b-a}\right)\right| \leq \sum_{j=M+1}^{\infty}|V_j| \leq \frac{Q^*}{M^{n-1}} \leq Q^*,$$
$$M \gg 1, n \geq 1, \tag{60}$$

for some positive constant Q^*. Then from (59), we have

$$|\epsilon_3| < \frac{2}{b-a}|\epsilon_1| + Q|\epsilon_4|, \tag{61}$$

with $Q = Q^*\frac{2e^{-r\Delta t}}{z(b-a)}$ and $\epsilon_4 := \int_{D_2/(D_1\cap D_2)} p(\xi; x, \Delta t) d\xi$. □

Thus both ϵ_1 and ϵ_3 are related to the truncation range and as the range $[a, b]$ becomes sufficiently wide (e.g. when $a < l$ and $b > h$ in the double barrier case), the errors ϵ_1, ϵ_3 and ϵ_4 converge to 0, and the overall approximation error will be dominated by ϵ_2. Thus in the following we shall analyze the convergence behavior of ϵ_2.

Note that

$$\epsilon_2 = \frac{e^{-r\Delta t}}{z}\sum_{j=M+1}^{\infty} A_j(x) V_j,$$

We assume that the density is typically smoother than the payoff functions in finance and that the coefficients $A_j(x)$ decay faster than V_j. Consequently, the product of $A_j(x)$ and V_j converges faster than either one, and we can bound this product

$$\left|\sum_{j=M+1}^{\infty} A_j(x) V_j\right| \leq \sum_{j=M+1}^{\infty}|A_j(x)|.$$

This means that the error ϵ_2 is dominated by the series truncation error of the density function. For densities in the class $\mathbb{C}^\infty([a, b])$, Fang and Oosterlee (2008) already show (in their Lemma 4.2) that the truncation error can be bounded by

$$|\epsilon_2| < P\exp(-M\nu),$$

where $\nu > 0$ is a constant and P is a term that varies less than exponentially with M. For densities having discontinuous derivatives, their Lemma 4.3 states that the truncation error can be bounded by

$$|\epsilon_2| < \frac{\overline{P}}{M^{\beta-1}},$$

where $\overline{P}$ is a constant, and $\beta \geq n \geq 1$, and here n is the algebraic index of convergence for V_j.

To summarize, the overall approximation error of using the Fourier-cosine series to solve the integral equation (26) is given by the following statement.

Lemma 3. *With a properly chosen truncation $[a, b]$ of the integration range, the overall approximation error for approximating the true density function $p(\xi, x, \Delta t)$ in (26) by its Fourier-cosine series either converges exponentially for density functions that belong to $\mathbb{C}^\infty([a, b]) \subset \mathbb{R}$, i.e.*

$$|\epsilon| < \left(1 + \frac{2}{b-a}\right)|\epsilon_1| + Q|\epsilon_4| + Pe^{-M\nu}, \tag{62}$$

or algebraically for density functions with a discontinuity in one of its derivatives, i.e.

$$|\epsilon| < \left(1 + \frac{2}{b-a}\right)|\epsilon_1| + Q|\epsilon_4| + \frac{\overline{P}}{M^{\beta-1}}. \tag{63}$$

The overall error will decay to 0 when $[a, b] \to D_2$ and $M \to \infty$, and our approximate solution (40) will converge to the exact solution of the integral equation (26).

The above Lemma 3 establishes the desired property that our solution (40) to the integral equation (26) has an overall approximation error converging to 0 at either an exponential or algebraic convergence order depending on the smoothness of the density function being approximated. Thus the accuracy of our method can always be improved by choosing proper truncation regions $[a, b]$ and increasing M, i.e. the number of terms we use in the Fourier cosine series. This theoretical property is later confirmed by numerical evidence in Section 6.5.

6. Numerical examples and greeks

In this section, we shall present numerical examples to show the accuracy and efficiency of our semi-analytical and explicit formula. We compare our results with well-known methods in the literature, including analytical approximate or exact solutions, and several numerical methods.

6.1. Comparison with the continuity correction approximation in Fuh et al. (2013)

Fuh et al. (2013) extend the continuity correction approximation of Broadie et al. (1997) from the Black–Scholes model to the Kou and Wang (2004) model with a double exponential jump diffusion. They show that the discrete barrier or lookback options can be approximately priced by their continuously-monitored counterpart's price with a simple continuity correction.

Table 1 gives the comparisons between our results, the Monte Carlo simulation (MC), and Fuh et al. (2013)'s results which are retrieved from their Table 2. Although their approach is an analytical approximation, it requires the numerical calculation of a two-dimensional inverse Laplace transform. As a result, the numerical computation of their approximation is quite inefficient in comparison with our approach.[10] For example, the calculation of the price of an option in Table 1 takes about 8 min for their code in contrast to 0.3 s consumed by our code in the same computer system. This is understandable because inverting a two-dimensional Laplace transform is a heavy computational burden and sometimes is even unstable due to possible singularities.

More importantly, Fuh et al. (2013) (and Dia and Lamberton, 2011) derive their continuity correction based on the assumption that the continuity correction depends only on the volatility of diffusion (i.e., σ) but not the jump diffusion part. When σ is

[10] We acknowledge our appreciation to the authors of Fuh et al. (2013), in particular, Sheng-Feng Luo, for gratefully sharing their codes and so enabling us to compare the computational efficiencies and accuracies.

Table 1
The results of up-and-in put options in the Fuh et al. (2013) model. This compares our results with those at Table 2 in Fuh et al. (2013) which presents a continuity correction approach. The up-and-in put options have $S_0 = 90$, $K = 96$, $B_H = 96$, $r = 0.1$, $T = 0.2$, $p = 50\%$, and $\eta_1 = \eta_2 = \eta$. The numerical results of our formula are obtained by setting $M = 5000$ in the Fourier-cosine series. The MC estimates with standard deviation in brackets were obtained by running 100,000,000 sample paths without variance reductions.

λ	Monitoring	$\eta = 10$			$\eta = 5$		
	Totals	MC	Fuh's CC	Ours	MC	Fuh's CC	Ours
Panel A: $\sigma/\sigma_{total} = 0.9$							
12	$N = 5$	5.7831 (0.0012)	6.97	5.7840	11.7885 (0.0022)	15.00	11.7867
	$N = 50$	11.494 (0.0017)	11.58	11.4930	24.5466 (0.0030)	25.07	24.5473
48	$N = 5$	11.8535 (0.0024)	15.41	11.8540	19.6611 (0.0031)	27.17	19.6631
	$N = 50$	24.6575 (0.0031)	25.29	24.6589	44.3251 (0.0041)	46.43	44.3214
Panel B: $\sigma/\sigma_{total} = 0.5$							
12	$N = 5$	1.8862 (0.0009)	2.20	1.8855	4.8253 (0.0013)	5.88	4.8243
	$N = 50$	3.1817 (0.0008)	3.23	3.1812	8.8570 (0.0016)	8.96	8.8564
48	$N = 5$	5.4956 (0.0013)	7.46	5.4966	11.2267 (0.0023)	16.07	11.2277
	$N = 50$	10.0178 (0.0018)	10.26	10.0195	22.3824 (0.0031)	22.94	22.3836
Panel C: $\sigma/\sigma_{total} = 0.1$							
12	$N = 5$	0.7674 (0.0004)	1.01	0.7676	1.8410 (0.0008)	2.42	1.8407
	$N = 50$	1.0041 (0.0005)	1.07	1.0404	2.5075 (0.0009)	2.58	2.5084
48	$N = 5$	3.9890 (0.0010)	5.91	3.9895	8.4252 (0.0019)	12.87	8.4247
	$N = 50$	6.0339 (0.0013)	6.26	6.0333	13.1241 (0.0021)	13.63	13.1244
Panel D: $\sigma/\sigma_{total} = 0$ (a pure jump diffusion model)							
12	$N = 5$	0.7375 (0.0004)	0.97	0.7372	1.8009 (0.0008)	2.43	1.8020
	$N = 50$	0.9451 (0.0004)	0.97	0.9449	2.3620 (0.0009)	2.43	2.3618
48	$N = 5$	3.9340 (0.0011)	6.05	3.9337	8.32522 (0.0019)	13.04	8.3258
	$N = 50$	5.8260 (0.0013)	6.05	5.8257	12.8721 (0.0026)	13.04	12.8719

1. σ/σ_{total} is computed as $\sqrt{\frac{\sigma^2}{\sigma^2+2\lambda\left(\frac{p}{\eta_1^2}+\frac{1-p}{\eta_2^2}\right)}}$, as defined in Fuh et al. (2013).

Table 2
The characteristic functions of some parameterized Lévy processes. The characteristic functions is defined as

$$\phi(\omega; x_0, t) = \exp\left(i\omega x_0 + t \times \psi(\omega)\right). \quad \text{(C1)}$$

The exponent of the characteristic functions, $\psi(\omega)$, is computed as

$$\psi(\omega) = -\frac{\omega^2\sigma^2}{2} + i\omega(r - \delta - \frac{\sigma^2}{2}) + h(i\omega) - i\omega h(1). \quad \text{(C2)}$$

The function $h(x)$ is tabulated below.

Models	Functions $h(x)$
Panel A: finite-activity models	
Geometric Brownian model	0
Merton model	$\lambda(e^{x\mu_J + \frac{\sigma_J^2}{2}x^2} - 1)$
Kou model	$\lambda(p_u \frac{\mu_u}{\mu_u - x} + (1 - p_d)\frac{\mu_d}{\mu_d + x} - 1)$
Panel B: infinite-activity models	
Variance Gamma	$-\log(1 - x\upsilon\theta - \frac{1}{2}\sigma^2\upsilon x^2)/\upsilon$
CGMY	$C\gamma(-Y)[(M - x)^Y - M^Y + (G + x)^Y - G^Y]$
Normal Inverse Gaussian	$\delta[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + x)^2}]$

small or zero, this approach makes no correction at all, implying that the discretely monitored options have the same value as the continuously monitored counterparts. To examine this argument, we set σ to be small or even 0 and compute the prices of both discrete and continuous barrier options. As it can be seen from Panels C and D of Table 1, our exact results match up extremely well with the Monte Carlo simulation results. However, "the higher jump volatility (in Panel C of Table 1) worsens the performance of Fuh et al. (2013)'s approach, especially for the case with high jump intensity and large jump mean size", as reported by Fuh et al. (2013) themselves. In Panel D when it is a pure jump diffusion model, the errors of the approximation are extremely large (in particular when $\lambda = 12$). The approximation of Fuh et al. (2013) is independent with the monitoring times, because the approximation makes no continuity correction and so is equal to the continuously-monitoring case. We therefore disagree with the claim of Fuh et al. (2013) and Dia and Lamberton (2011) that there is "no necessity for a continuity correction in a pure jump process".

In summary, our formula, as a semi-analytical and explicit solution for a general Lévy process framework including the double exponential jump-diffusion model as a special case, greatly improves the efficiency and accuracy over Fuh et al. (2013)'s approach for pricing discrete barrier options.

6.2. Comparison with the Wiener–Hopf technique in Fusai et al. (2006) and Green et al. (2010)

Fusai et al. (2006) solve the integral equation using the Wiener–Hopf technique and numerically compute the inverse $\mathcal{Z}$-transform, and initially confine their analysis to down-and-out options in the Black–Scholes model. Later, Green et al. (2010) extend the approach to take into account more general Lévy processes and double-barrier options, but only present numerical results for the Black–Scholes model. Green et al. (2010) also comment that the inverse $\mathcal{Z}$-transform "for large N becomes extremely slow" and that numerical integration or truncated summation, "remain too slow to be effective for practical purposes". As a result, they have to apply a Padé approximation to approximate their Wiener–Hopf kernel $K(z)$ and speed up the computational efficiency, but it still takes about 1.5 min to compute one data point with accuracy $O(10^{-5})$ when the barrier is close to the spot value of the underlying stock.[11] Readers interested in the Padé approximation are referred to Green et al. (2010) and Abrahams (2000).

Table 3 presents the prices of down-and-out call options in the Black–Scholes model, comparing our results with Fusai et al. (2006) and Green et al. (2010), as well as other numerical methods. As expected, our results match up with theirs perfectly well. We also note that the computational time of our approach, which is coded in MATLAB 2012, takes 0.03 s for the price of a barrier option in the Black–Scholes model. Finally, the accuracy of the continuity correction approximation becomes worse when the down barrier is closer to the strike of the option.

We remark that it may not be fair to directly compare the computational times quoted above, simply because our method was implemented in Matlab while Fusai et al. (2006)'s method was coded in Mathematica. Although the two computer programs were executed on the same computer, the difference in computational times can be large due to different programming languages. The same comment also applies to other computational times of existing methods quoted from the literature. Since the main focus of this paper is to present our new approach to the valuation of the barrier option, the fair and comprehensive numerical comparisons are left to future research.

As it can be seen in Eq. (46), the computation of a barrier option consists of three components: $\mathbf{f}$, which is determined by the payoff structure of the options (call options, or put options, or binary options), $\mathbf{A}^{N-1}$, which is determined by the exponentials of

[11] We appreciate the authors of Fusai et al. (2006) for sending us their Mathematica codes.

Table 3
The prices of down-and-out call options in the Black–Scholes model. The table compares our results with the results obtained from the Wiener–Hopf method in Fusai et al. (2006) and Green et al. (2010). The model parameters of the Black–Scholes model are chosen to be the same as Table 1 in Fusai et al. (2006) and Table 5.1 in Green et al. (2010). $S_0 = K = 100$, $r = 10\%$, $\sigma = 0.3$ and $T = 0.2$. The numerical results of our formula are obtained by setting $M = 5000$ in the Fourier series. The MC estimates with standard deviation in brackets were obtained by running 100,000,000 sample paths without variance reductions. All the other results of the numerical methods were retrieved from Fusai et al. (2006) and Green et al. (2010).

B_L	Fusai et al. (2006)	Green et al. (2010)	Ours	MC	Trinomial tree	Continuity correction
Panel A: $N = 5$						
89	6.28076	–	6.2807551	6.2808 (0.0012)	6.281	6.284
95	5.67111	5.67111	5.6711051	5.6712 (0.0015)	5.671	5.64562
99	4.48917	4.48917	4.4891724	4.4886 (0.0018)	4.489	4.04952
99.9	–	4.13824	4.1382432	4.1374 (0.0019)	–	3.46577
Panel B: $N = 25$						
89	6.20995	–	6.2099516	6.2090 (0.0013)	6.210	6.210
95	5.08142	–	5.0814152	5.0827(0.0017)	5.081	5.084
99	2.81244	2.81244	2.8124393	2.8115 (0.0019)	2.812	2.673
99.9	–	–	2.1902492	2.1912 (0.0018)	–	–

the Lévy processes, and $[A_j(x)]_{1\times M}$, which includes the variable $\log S_0$ and the exponentials. As a result, the formula can be very efficient to compute option prices for a range of initial asset prices by just re-computing $[A_j(x)]_{1\times M}$ and keeping $\mathbf{f}$ and $\mathbf{A}^{N-1}$ the same. Similarly, it can efficiently compute option prices for a range of strike prices by only re-computing $\mathbf{f}$.

6.3. Comparison with the fast-Hilbert transform of Feng and Linetsky (2008)

The above numerical comparisons are confined to the Black–Scholes model. In this section we compare our results for other Lévy processes with those reported by Feng and Linetsky (2008) who develop a Hilbert transform method to price discretely-monitored single-barrier and double-barrier options for Lévy processes. To obtain the price of a barrier option with N total monitoring instants, $C(S_0, t=0, N)$, their algorithm requires the calculation of $N-1$ Hilbert transforms and an inverse Fourier transform. In order to speed up the computation, Feng and Linetsky (2008) apply numerical methods (the fast Hilbert transforms and the fast Fourier transform) to implement these transforms. Table 4 presents the numerical comparisons between ours and those in Table 7.1 of Feng and Linetsky (2008), showing that all the results match up exactly.

In terms of computational time, Table 4 presents our computational time used in generating the numerical results. Feng and Linetsky (2008) stated that “we attain accuracy of 10^{-8} in the pure jump NIG model in between 1.38 and 3.94 s, and just 0.016 s for Black–Scholes model.” So, the speed of our method shown in Table 4 is comparable to that of Feng and Linetsky (2008). As remarked before, the comparison of computational times may not be fair because of the fast development of computing technologies and the improved computational efficiency it brings.

The error convergence of the fast Hilbert transform is exponential for models with rapidly decaying characteristic functions, similar to the Fourier COS method as shown in our convergence and error analysis in Section 5. However, the Fourier COS method is particularly suitable for our setting because we can explicitly invert the $\mathcal{Z}$-transform and express the final option price formula as in (46) after we recover the coefficients $b_k(\cdot)$ and $A_j(\cdot)$. In addition, their method requires care in determining the analytical strips when numerically computing the Hilbert and inverse Fourier transforms, but our formula has no need for this.

6.4. Greeks

As a semi-analytical and explicit formula, Eq. (46) can be further utilized to derive all the Greeks (Delta and Gamma etc.) of barrier options. One of the advantages of our method is that the Greeks can be computed efficiently and explicitly with little additional computational effort. For example, we can differentiate Eq. (46) with respect to $S(=\exp(x))$, which is contained only in the vector $[A_j(x)]_{1\times N}$.

$$A_j(x) = \frac{2}{b-a}\mathrm{Re}\left[e^{ix\frac{j\pi}{b-a}} \times \exp\left(\Delta t \times \psi\left(\frac{j\pi}{b-a}\right)\right) \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right]. \tag{64}$$

This gives

$$\frac{\partial A_j(x)}{\partial x} = \frac{2}{b-a}\mathrm{Re}\left[i\frac{j\pi}{b-a}e^{ix\frac{j\pi}{b-a}} \times \exp\left(\Delta t \times \psi\left(\frac{j\pi}{b-a}\right)\right) \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right], \tag{65}$$

and

$$\frac{\partial^2 A_j(x)}{\partial x^2} = \frac{2}{b-a}\mathrm{Re}\left[-\left(\frac{j\pi}{b-a}\right)^2 e^{ix\frac{j\pi}{b-a}} \times \exp\left(\Delta t \times \psi\left(\frac{j\pi}{b-a}\right)\right) \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right] = -\left(\frac{j\pi}{b-a}\right)^2 A_j(x). \tag{66}$$

Thus, to compute Delta (Δ) and Gamma (Γ), all we need to do is to modify the last step (45) by replacing the vector $[A_j(x)]_{1\times N}$.

Theorem 2 (Hedging Ratio)**.** *If the underlying stock follows an exponential Lévy process (1), then the Delta (Δ) and Gamma (Γ) of a double-barrier-out call option, can be computed as follows*

$$\Delta = \frac{\partial C}{\partial S} = \frac{e^{-rT}}{S}\left[\frac{\partial A_j(x)}{\partial x}\right]_{1\times N} \times \mathbf{A}^{N-1} \times \mathbf{f}, \tag{67}$$

and

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-rT}}{S^2}\left[\frac{\partial^2 A_j(x)}{\partial x^2} - \frac{\partial A_j(x)}{\partial x}\right]_{1\times N} \mathbf{A}^{N-1} \times \mathbf{f}, \tag{68}$$

where the full expressions of $[A_j(x)]_{1\times M}$, $\mathbf{A}^{N-1}$ and $\mathbf{f}$ are given in Appendix A.

As it can be seen, to compute Delta (Δ) and Gamma (Γ) in addition to the option price, the terms $\mathbf{A}^{N-1}$ and the $\mathbf{f}$ are kept the same, so there is no additional computational cost. One only needs to compute one additional $1\times N$ vector to obtain the first and second derivatives of $[A_j(x)]_{1\times N}$ with respect to x. We will present the Greeks under the Black–Scholes model and the CGMY model. For other models, the approach can be applied in a similar manner. Fig. 1 demonstrates the prices of a double-barrier put option under the Black–Scholes model and the CGMY model with different monitoring frequencies and different initial underlying stock prices. Figs. 2 and 3 present the Greeks for a double-barrier put option under the Black–Scholes model or the CGMY model, respectively. Table 5 compares the prices and the Greeks from our formulae with the results of Petrella and Kou (2004). As it can be observed, very good agreements are achieved in the Black–Scholes model. However, we alert that their numerical method tends to overestimate the prices and the Greeks of up-and-out put options in the Merton model, in comparison with the results from either our semi-analytical formulae or Monte Carlo simulations. This bias may be caused by the numerical inversion of Laplace transforms.

6.5. Convergence of the Fourier-cosine series

Following Fang and Oosterlee (2008) we select the interval of integration $[a, b]$ as

$$[a, b] = [c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}}]. \tag{69}$$

Here c_n denotes the n-th cumulant of $\log S_T$ and is presented in Appendix A of Fang and Oosterlee (2008). We have chosen $L = 15$ in our numerical examples. Larger values of parameter L would require larger number of terms M in the Fourier-cosine series to reach the same level of accuracy. Fig. 4 shows the convergence and relative errors of down-and-out call options in the Black–Scholes model and double-barrier-out put options in the CGMY model. As we have tested in our numerical examples, after 100 terms have been included in the Fourier-cosine series ($M > 100$),

Table 4
The prices of vanilla options and one-year daily-monitored barrier options in various Lévy models. The table compares our results in Lévy processes with those in Table 7.1 of Feng and Linetsky (2008), for one-year vanilla call and put options ($N = 1$) and daily-monitored single or double knock-out barrier options ($N = 252$). The options are at-the-money $S = K = 100$, and the barriers (when they come into effect) are set to be $B_L = 80$ and $B_H = 120$. The prices are given with 10^{-9} accuracy. It takes 0.030 s for our code to complete the computation of an option under the Black–Scholes model, and 2.50 s under the CGMY model, because the series in the CGMY model converge slower than in the Black–Scholes model. Feng and Linetsky (2008) stated that “we attain accuracy of 10^{-8} in the pure jump NIG model in between 1.38 and 3.94 s, and just 0.016 s for Black–Scholes model” .

Panel A:	model parameters (the same as Feng and Linetsky, 2008's Table 7.1)					
	BS: $\sigma = 0.2$					
	Merton: $\sigma = 0.1,\ \lambda = 3,\ \mu_J = -5\%$ and $\sigma_J = 8.6\%$					
	Kou: $\sigma = 0.1,\ \lambda = 3,\ p = 0.3,\ \eta_1 = 40$ and $\eta_2 = 12$					
	CGMY: $C = 4,\ G = 50,\ M = 60$ and $Y = 0.7$					
Panel B:	the results					
Models	Methods	V. Call	UOP	DOC	DBC	DBP
BS	FL	9.22700551	6.13865136	9.15141382	1.22420234	1.72868009
	Ours	9.227005508	6.138651363	9.151413819	1.224202343	1.728680086
Merton	FL	9.01731154	5.93687139	8.97945779	2.07502090	1.60065569
	Ours	9.017311537	5.936871388	8.979457788	2.075020902	1.600655689
Kou	FL	8.87700487	5.77759181	8.86025111	2.49384291	1.43836344
	Ours	8.877004873	5.777591808	8.860251109	2.493842912	1.438363440
CGMY	FL	9.18819989	6.10938803	9.11932528	1.30878441	1.77036472
	Ours	9.188199886	6.109388030	9.119325279	1.308784414	1.770364724
Panel C:	the required computational time in seconds of our method for the double-barrier-out put option (DBP)					
Models			Accuracy			
	$O(10^{-1})$	$O(10^{-3})$	$O(10^{-5})$	$O(10^{-7})$	$O(10^{-9})$	
BS	0.0037	0.0081	0.0133	0.0190	0.0376	
Merton	0.0109	0.0824	0.1203	0.3877	0.5598	
Kou	0.0198	0.1183	0.3890	0.5257	1.7040	
CGMY	0.0126	0.18143	0.7034	1.3962	2.3389	

Table 5
The prices, Deltas and Gammas of up-and-out put options in the Black–Scholes (1973) model and the Merton (1977) model. This table compares the prices and Greeks from our formulae with the results in Tables 4 and 9 of Petrella and Kou (2004). The numerical results of our formula are obtained by setting $M = 1000$ in the Fourier-cosine series. The MC estimates with standard deviation in brackets were obtained by running 10,000,000 sample paths without variance reductions. Although the results of Petrella and Kou (2004) match up well with ours in the Black–Scholes model, their results consistently overestimate ours or the results from Monte Carlo (MC) simulations. The columns entitled by “Time” refer to the required computational time in seconds for getting a price data. The “Time” column of Petrella and Kou (2004) method was taken from their Tables 4 and 9. The MC method requires in general 5–7 min to get one data point for the Matlab code running in a 2-year-old laptop.

Monitoring	Petrella and Kou (2004)				Our approach				MC
Totals	Price	Delta	Gamma	Time	Price	Delta	Gamma	Time	Price
Panel A:	Black–Scholes (1973) model with different monitoring times[a]								
	$S_0 = 100,\ K = 100,\ r = 0.05,\ \delta = 0.0,\ \sigma = 0.3,\ T = 1$								
	$B_H = 101.$[b]								
$N = 5$	6.010	−0.4541	0.0213	1.01	6.009507	-0.454080	0.021289	0.011	6.009(0.0038)
$N = 20$	3.611	−0.5202	0.0391	4.25	3.610719	-0.520174	0.039141	0.012	3.610(0.0032)
$N = 80$	2.180	−0.5798	0.0677	25.24	2.179642	-0.579743	0.067669	0.012	2.178(0.0026)
$N = 160$	1.738	−0.6120	0.0832	73.89	1.738253	-0.612043	0.083201	0.013	1.738(0.0023)
Panel B:	Merton (1977) model with different up barrier levels								
	$S_0 = 100,\ K = 100,\ r = 0.05,\ \delta = 0.0,\ T = 0.2,\ \sigma_D = 0.353,\ N = 50$								
	$\mu_J = -0.01,\ \sigma_J = 0.141,\ \lambda = 6.22\ (\sigma_{Total} = \sqrt{\sigma_D^2 + \lambda \cdot (\sigma_J^2 + \mu_J^2)} = 0.5).$								
$B_H = 101$	2.528	−0.8547	0.0960	46	2.523282	−0.854249	0.096067	0.105	2.523(0.0026)
$B_H = 105$	5.130	−0.7457	−0.0116	46	5.122815	−0.745039	−0.011484	0.108	5.124(0.0032)
$B_H = 109$	6.604	−0.6004	0.0025	46	6.593279	−0.599912	0.002542	0.104	6.592(0.0034)
$B_H = 113$	7.368	−0.5210	0.0102	46	7.354876	−0.520650	0.010270	0.104	7.356(0.0035)

[a] Petrella and Kou (2004)'s results in Panel A are retrieved from their Table 4, and their results in Panel B are from Table 9 of their paper.
[b] We remark there was a typo in Petrella and Kou (2004)'s Table 4. The up-barrier parameter there should be $B_H = 101$ rather than 110.

the relative errors are smaller than 0.5% already in the Black–Scholes model. In the CGMY model, it converges slower, requiring 200 terms for the relative errors to be smaller than 0.8% when $N = 252$.

7. Further extensions

Note that our approach can be applied to other more general settings with little additional effort. For example, it can take into account barrier options under Lévy processes with time-dependent-coefficients, options with time-dependent barriers, non-equally spaced monitoring dates, and the valuation of barrier options at any time $t \in [0, T]$ which is not necessarily a barrier monitoring date. Finally, our approach can be applied to cash-or-nothing options. The only required modification is to substitute the expression of the cash-or-nothing payoff for the vanilla call or put payoffs. We demonstrate these through the following two cases based on the Black-Scholes model.

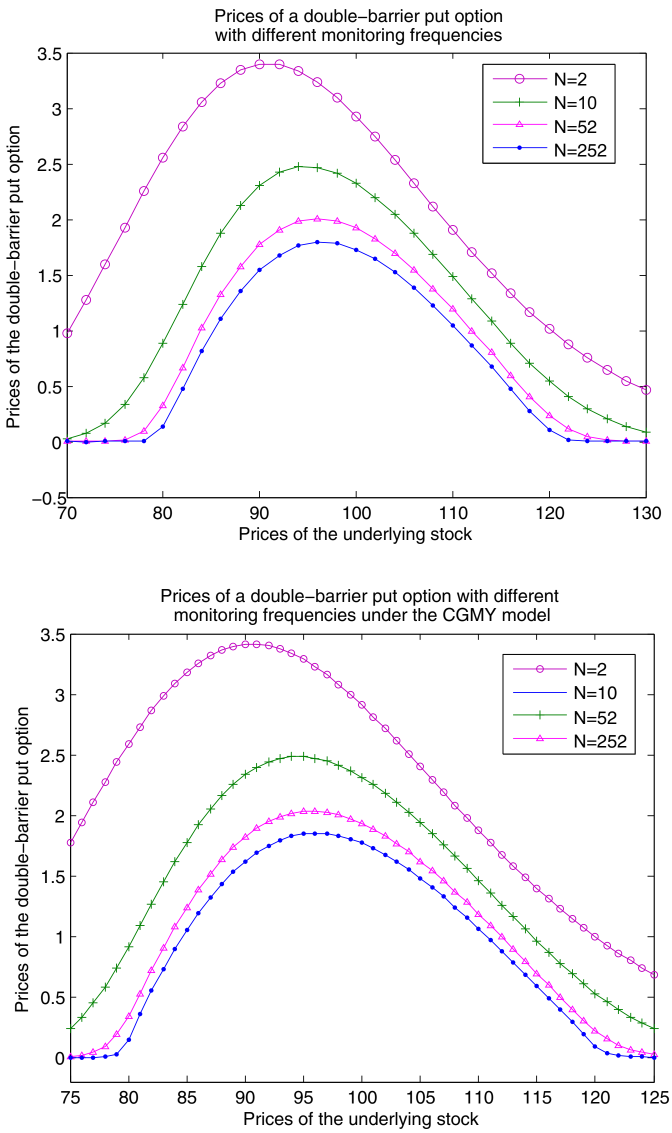

Fig. 1. The values of a double-barrier put option under the Black–Scholes model and the CGMY model with different monitoring frequencies. Model parameters are presented in Panel A of Table 4.

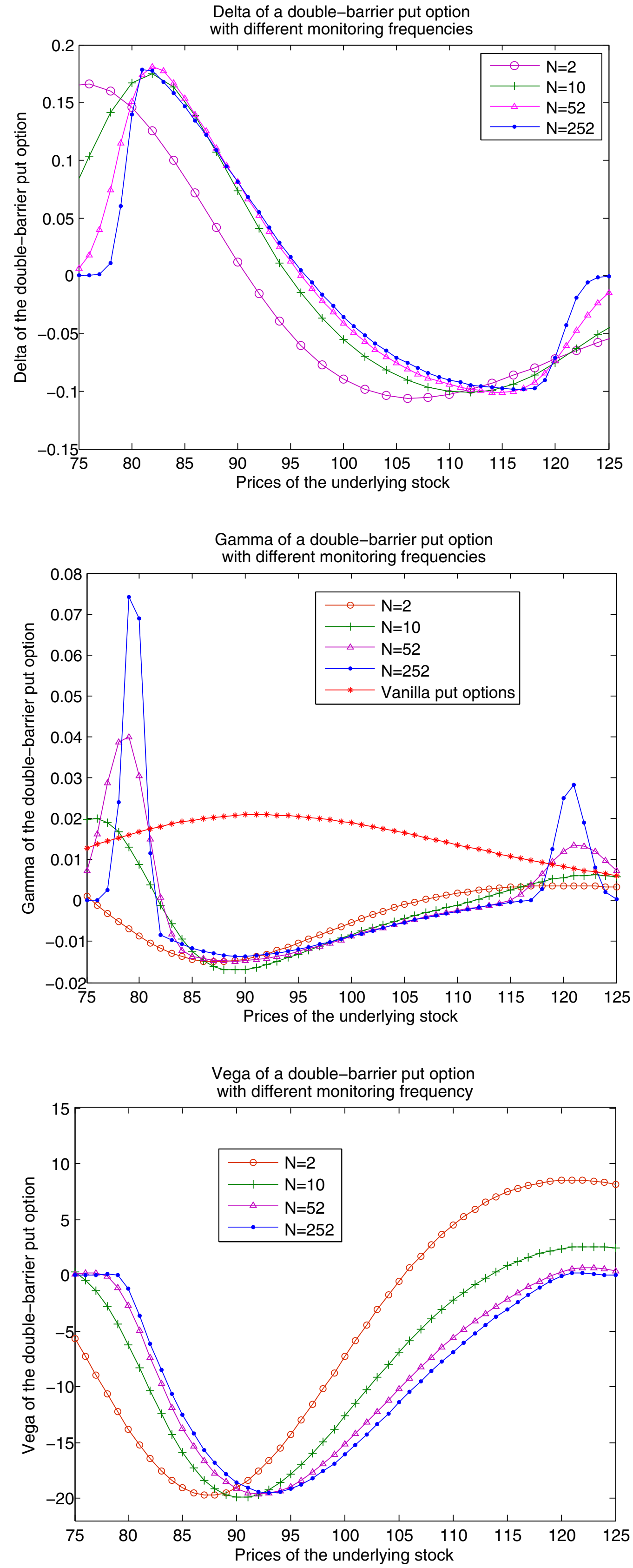

Fig. 2. The Delta, Gamma and Vega of a double-barrier put option under the Black–Scholes model with different monitoring frequencies. Model parameters are presented in Panel A of Table 4.

7.1. *Lévy processes with time-dependent coefficients*

We first show how to extend the approach to solve discretely-monitored barrier options under time-dependent Lévy processes. In fact, from the partial differential equations (PDE) perspective, the problem of pricing a barrier option with time-dependent parameters is not a trivial extension of time invariant models. That is because the corresponding PDEs for the value function of a barrier option have time-varying coefficients in the time-dependent parameters framework and are usually hard to solve.

In the literature, several attempts have been made to attack the time-dependent problem. For example, Roberts and Shortland (1997) apply the hazard rate tangent approximation method to evaluate upper and lower bounds for price of barrier options in the time-dependent-coefficient Black–Scholes model. Lo et al. (2003) present a simple approach for computing upper and lower bounds (in closed-form) for price of a barrier option. This research considers continuously-monitored barrier options. Pricing discretely-monitored barrier options under time-dependent

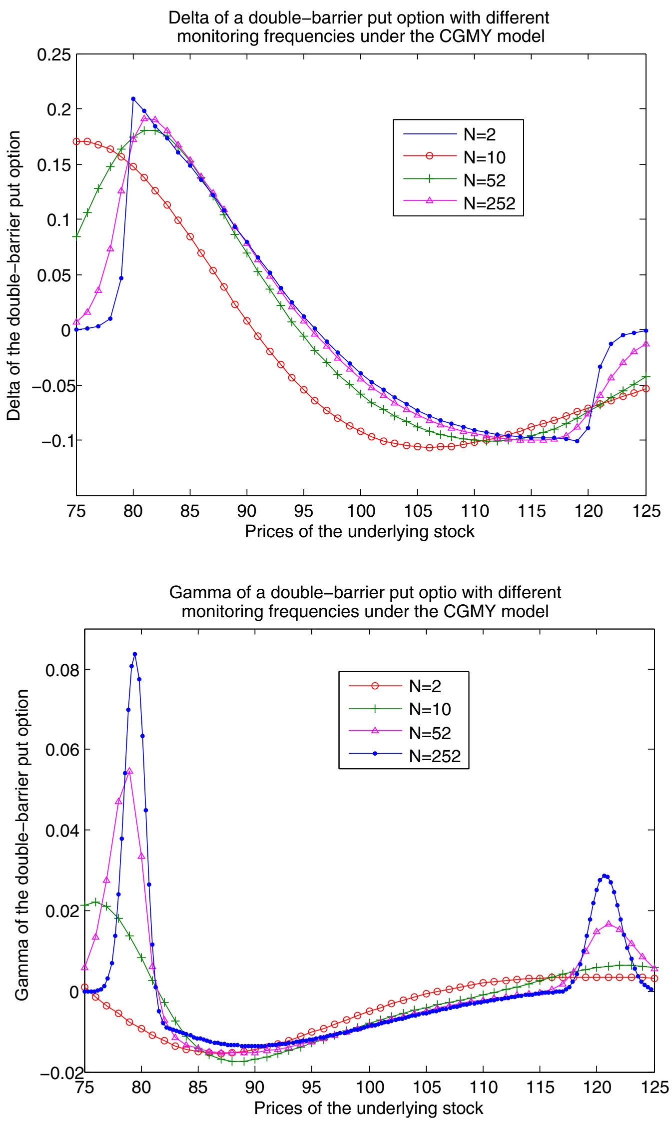

Fig. 3. The Delta and Gamma of a double-barrier put option under the CGMY model with different monitoring frequencies. Model parameters are presented in Panel A of Table 4.

models has not yet been studied. We modified our approach to deal with this challenge. By analogy with the derivation in Section 3, we are able to derive a pricing formula in the following theorem.

Theorem 3 (Time-Dependent-Coefficient Pricing Formula)**.** *If the underlying stock follows the stochastic process*

$$dS_t = r(t)S_t dt + \sigma(t)S_t dW_t, \tag{70}$$

then the price of a double-barrier-out call option can be evaluated by the following function

$$C(S_0, t{=}0, N, B_L, B_H) = e^{-rT}[A_j(\log S_0)]_{1\times M}\times\left(\prod_{n=1}^{n=N-1}\mathbf{A}_n\right)\times\mathbf{f}. \tag{71}$$

Here, the $\mathbf{A}_n$ $(n = 1, \ldots, N)$ *are obtained by substituting* $\frac{\int_{\tau_{n-1}}^{\tau_n} r(s)ds}{\tau_n-\tau_{n-1}}$ *and* $\sqrt{\frac{\int_{\tau_{n-1}}^{\tau_n}\sigma^2(s)ds}{\tau_n-\tau_{n-1}}}$ *for* r *and* σ *in* $\mathbf{A}$*.*

Table 6 presents the values of up-and-out call options from both our formula and Lo et al. (2003) when the interest rate is assumed to have the term structure $r(t) = r_0[1 + c_0e^{-t}]$ with $r_0 = 10\%$ and $c_0 = 5\%$. As can be observed from the table, when the total monitoring number increases from $N = 252$ to 50,000, the values of an up-and-out call option are decreasing and approaching to the price of the continuously-monitored barrier option, which is computed by Lo et al. (2003)'s or the finite-difference method. It is intuitive, because when a continuously monitored barrier is touched the option is immediately knocked in or out, while in discretely monitored conditions, barriers only come into effect at discretely monitored time instants.

7.2. Time-dependent-barrier options

We now consider barrier options which allow the (up or down) barriers to keep changing as time passes. One type of such options allows option purchasers choose a barrier period covering a segment of time either at the beginning (front end) or the end (rear end) of the option life. This feature makes them more flexible than the regular barrier options for an investor having a particular view on an underlying asset in a certain period of time. Hui (1997) discusses this type of time-dependent barrier options and derives analytical formulae for pricing continuously monitored time-dependent barrier of this type. Zhu and De Hoog (2010) present an improved finite difference numerical method to price barrier options with complex barrier structures. We show that our approach can be modified to deal with the discretely monitored barrier options with complex barrier structures.

Theorem 4 (Time-Dependent-Barrier Options Pricing Formula)**.** *If the underlying stock follows a stochastic process*

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{72}$$

Table 6

The prices of up-and-out call options in the Black–Scholes (1973) model with time-dependent coefficients. This table presents the prices of discretely-monitored barrier options from our formulae and the continuously-monitored barrier options which are shown in Tables 1 and 2 of Lo et al. (2003). The numerical results of our formula are obtained by setting $M = 5000$ in the Fourier-cosine series.

T	Lo et al. (2003)	Finite Difference	Monitoring totals of our approach (N)			
			252	5000	20,000	50,000
Panel A:	$S_0 = 10,\ K = 11,\ B_H = 15,\ \sigma = 0.2,\ \delta = 0$ $r(t) = 0.1 + 0.05e^{-t}$					
0.25	0.175093	0.175094	0.177117	0.176054	0.175639	0.175236
1	0.509009	0.511674	0.5416708	0.523853	0.514789	0.512207
Panel B:	$S_0 = 50,\ K = 50,\ B_H = 70,\ r = 10\%,\ \delta = 5\%$ $\sigma^2(t) = 0.03 + 0.02t$					
0.25	2.082428	2.082437	2.082929	2.082566	2.082501	2.082485
1	2.723146	2.736401	2.812427	2.763162	2.747060	2.738031

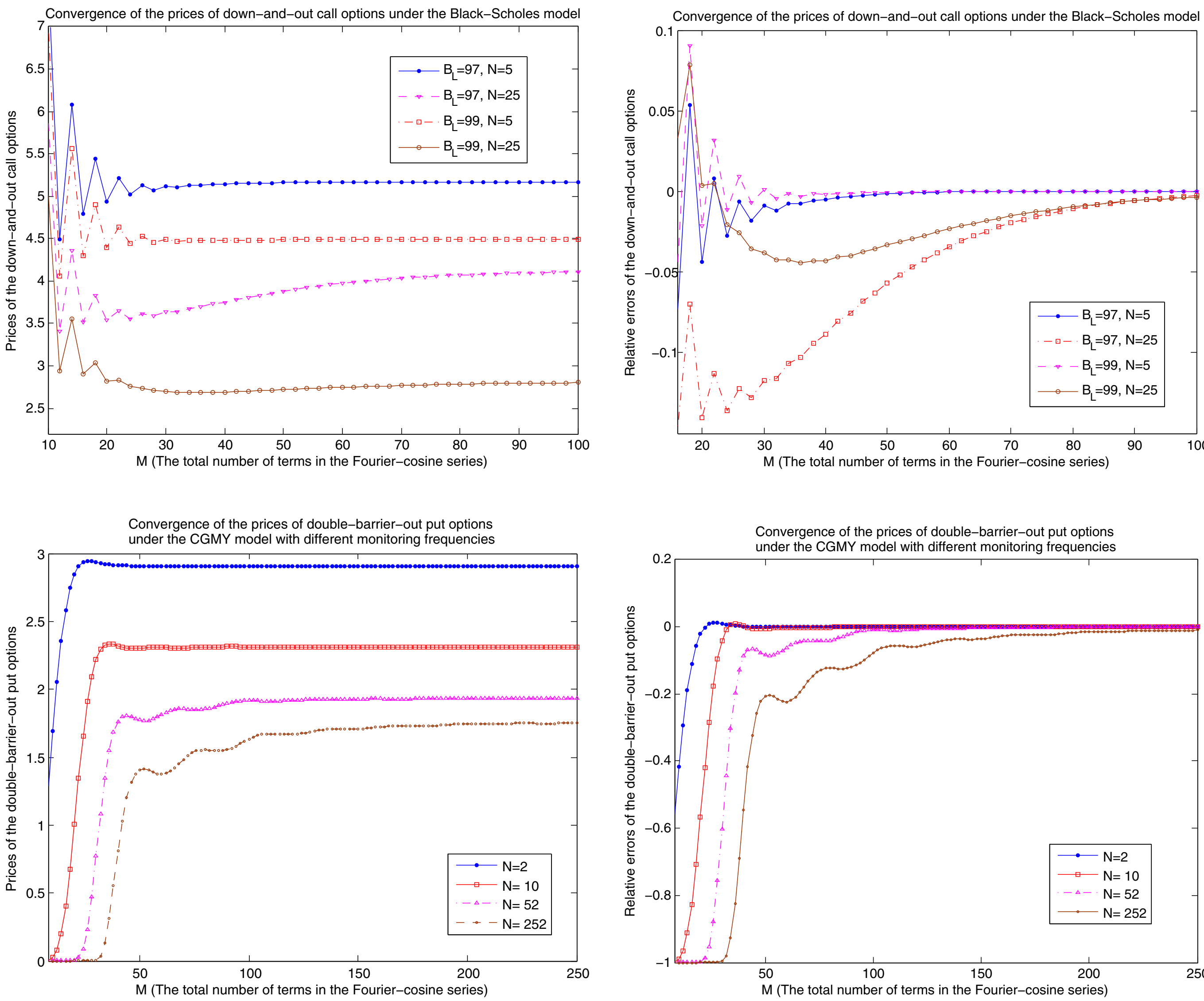

Fig. 4. The convergence of our formulae in terms of Fourier-cosine series for pricing discrete barrier options under the Black–Scholes model and the CGMY model . Model parameters for the down-and-out call options in the Black–Scholes model are presented in Table 3, and for the double-barrier-out put options in the CGMY model are presented in Table 4.

then the price of a double-barrier-out call option with barriers $B_L(t)$ and $B_H(t)$ can be evaluated by the following function

$$C(S_0, t=0, N, B_L, B_H) = e^{-rT}[A_j(\log S_0)]_{1\times M} \times \left(\prod_{n=1}^{n=N-1} \mathbf{A}_n\right) \times \mathbf{f}. \tag{73}$$

Here, $\mathbf{A}_n$ $(n = 1, \dots, N)$ *are obtained by substituting* $\log(B_L(\tau_n))$ *and* $\log(B_H(\tau_n))$ *for* l *and* h *in* $\mathbf{A}$*.*

We present in Table 7 the values of up-and-out call options with a beginning (front end) barrier, although our formula above can be applied to the case with general continuous functions for $B_L(t)$ and $B_H(t)$ such as soft-barrier options (see Haug, 1998). Table 7 shows the values of up-and-out call options from our discrete formula converge from above to the continuously-monitored limit which was computed by Zhu and De Hoog (2010)'s finite difference method, or Haug (1998)'s analytical formula.

Table 7
The prices of an early-finish-barrier (front end) up-and-out call option in the Black–Scholes (1973) model. The table presents the prices of a discretely-monitored barrier call option with the up-and-out barrier existing only in the first half of the tenor. The results are obtained from our formula and Table 8 of Zhu and De Hoog (2010) which presents an improved finite difference method for continuously-monitored barrier options. The numerical results of our formula are obtained by setting $M = 5000$ in the Fourier-cosine series. Haug (1998) provides an analytical and exact solution for continuously-monitored early-finish barrier options.

Parameters:	$S_0 = 1.0,\ K = 1.0,\ T = 2,\ \sigma = 0.1305,\ r = 0,\ \delta = 0$ $B_H(t) = 1.25$ if $t \le 1$; otherwise $B_H(t) = \infty$.							
Haug (1998)	Zhu & De Hoog		Crank–Nicolson		Our approach (N)			
Analytic	$\delta t = 0.5$	$\delta t = 0.02$	$\delta t = 0.5$	$\delta t = 0.02$	504	5000	20,000	50,000
0.0536	0.054003	0.053636	0.0629731	0.0540093	0.054753	0.054011	0.053824	0.053739

8. Conclusions

This paper presents a closed-form and semi-analytical solution for the evaluation of discretely-monitored single or double barrier options for asset prices modeled using a general Lévy process. The solution is derived by casting the pricing problem into an integral equation using a $\mathcal{Z}$-transform and by solving the equation with a Fourier-cosine series. The analyticity and explicitness of our pricing formula are due to the success of carrying out the inverse $\mathcal{Z}$-transform. This is a great advantage over previous pricing formulae in the literature. As a result, a closed-form, semi-analytical solution is derived to price discretely-monitored barrier options. The efficiency and accuracy of our formula is validated and demonstrated by comparing with other major approaches in the literature, either numerical methods (e.g., Monte Carlo simulation, finite difference method, trinomial tree method, Laplace transform) or analytical methods (e.g., Fusai et al., 2006, Green et al., 2010, Fuh et al., 2013).

Another main point to be included in our concluding remarks is that we disagree with the argument in Fuh et al. (2013) that "there is no need of correction when the underlying is a pure jump model" when extending Broadie et al. (1997)'s continuity correction approximation. We show that, even when no diffusion component appears in the Merton (1976) model, the values of discretely-monitored barrier options are still different from those continuously-monitored limits, implying the necessity of a correction in a pure jump model.

We show that the Greeks for a barrier option can be explicitly computed with little additional effort. We demonstrate that our approach is generic in dealing with barrier options under Lévy processes with time-dependent coefficients, options with time-dependent barriers, non-equally spaced monitoring dates, or for the evaluation of barrier options at any intermediate time not necessarily coinciding with any barrier monitoring date.

Appendix A. Integration of $\int_{D_1\cap D_2} b_k(x) f(x,0)dx$

Here, $D_1 = \{x : x \in \mathbb{R}, a \le x \le b\}$ denotes the interval $[a, b]$ used to truncate the probability density function $p(\xi, x, \Delta t)$. The notation D_2 denotes the domain in X-space where the barrier option remains alive. For example, for a double-barrier-out option with barriers l and h, there is $D_2 = \{l \le x \le h\}$.

As the stock price process S_t is the exponential of the Lévy process X_t,

$$S_t = e^{X_t}, \quad t > 0. \tag{A1}$$

The payoff, $f(x, 0)$, of a call option at expiration is

$$f(x,0) = (S_T - K)^+ = (e^{x_T} - K)^+ = \left(e^{x_T} - e^{\log(K)}\right)^+. \tag{A2}$$

Given that

$$b_k(x) = \cos\left(\frac{k\pi\,(x-a)}{b-a}\right), \tag{A3}$$

the integration, denoted by $f_k = \int_{x\in D_1\cap D_2} b_k(x) f(x,0)dx$, can be analytically carried out with the help of the Maple software package. For example, for a double-barrier-out call options with payoff

$$f(x,0) = (e^x - K)^+ \mathbf{1}_{x\in D_2}, \tag{A4}$$

f_k can be computed as

$$\begin{aligned} f_k &= \int_{D_1\cap D_2} b_k(x)(e^x - K)^+ dx \\ &= \int_c^d b_k(x)(e^x - K)dx. \end{aligned} \tag{A5}$$

Here we have defined $c = \max(a, l, \log(K))$ and $d = \min(b, h)$ with $l = \log(B_L)$ and $h = \log(B_H)$. l and h are the translations of the down-barrier B_L and the up-barrier B_H in the stock price S-space to the Lévy process X-space. Then

$$f_k = I, \tag{A6}$$

with

$$I = \begin{cases} e^d - e^c + K(c-d), & \text{if} \quad k = 0, \\ \mathrm{Re}\Big[e^{-\frac{ik\pi a}{b-a}} \times (b-a) \times \Big(\frac{K}{ik\pi}\Big(e^{\frac{i\pi ck}{b-a}} - e^{\frac{i\pi dk}{b-a}}\Big) & \\ \quad - \frac{1}{i\pi k - a + b}\Big(e^{\frac{(ik\pi+b-a)c}{b-a}} - e^{\frac{(ik\pi+b-a)d}{b-a}}\Big)\Big)\Big], & \text{if} \quad k > 0. \end{cases} \tag{A7}$$

For a put option with payoff

$$f(x,0) = (K - e^x)^+ \mathbf{1}_{\{x\in D_2\}}, \tag{A8}$$

we have the similar result

$$\begin{aligned} f_k &= \int_{D_1\cap D_2} b_k(x)(K - e^x)^+ dx \\ &= -I, \end{aligned} \tag{A9}$$

with $c = \max(a, l)$ and $d = \min(b, h, \log(K))$.

Appendix B. Expression of $A_{kj} = \int_{D_1\cap D_2} b_k(x) A_j(x) dx$

From the definition, we have

$$\begin{aligned} A_{kj} &= \int_{D_1\cap D_2} b_k(x) A_j(x) dx \\ &= \frac{2}{b-a}\mathrm{Re}\left[\int_{D_1\cap D_2} b_k(x)\phi\left(\frac{j\pi}{b-a}, x\right)dx \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right] \\ &= \frac{2}{b-a}\mathrm{Re}\left[\int_{D_1\cap D_2} b_k(x) e^{ix\frac{j\pi}{b-a}} dx \times \exp\left(\Delta t \times \psi\left(\frac{j\pi}{b-a}\right)\right)\right. \\ &\quad \left. \times \exp\left(-i\frac{ja\pi}{b-a}\right)\right] \\ &= \frac{2}{b-a}\mathrm{Re}\left[\int_{D_1\cap D_2} b_k(x) e^{i\frac{j\pi(x-a)}{b-a}} dx \times \exp\left(\Delta t \times \psi\left(\frac{j\pi}{b-a}\right)\right)\right]. \end{aligned} \tag{B1}$$

For a double-barrier (put or call) option with barriers B_L and B_H in S-space or $l = \log(B_L)$ and $h = \log(B_H)$ in the X-space, we have that $D_1 \cap D_2 = [c, d]$ with $c = \max(a, l)$ and $d = \min(b, h)$. As a result, the integration above is computed as

$$\int_c^d b_k(x) e^{i\frac{j\pi(x-a)}{b-a}} dx$$

$$= \begin{cases} d - c, & \text{if} \quad j = k = 0. \\ \frac{d-c}{2} + \frac{b-a}{2\pi k}\sin\left(\frac{\pi k(d-c)}{b-a}\right) e^{\frac{i\pi k(d-2a+c)}{b-a}}, & \text{if} \quad 0 < j = k, \\ \frac{b-a}{\pi(j^2-k^2)}\left[\left(ij\cos\left(\frac{k\pi(c-a)}{b-a}\right)\right.\right. & \\ \quad \left. + k\sin\left(\frac{k\pi(c-a)}{b-a}\right)\right) e^{\frac{i\pi j(c-a)}{b-a}} & \\ \quad - \left(ij\cos\left(\frac{k\pi(d-a)}{b-a}\right)\right. & \\ \quad \left.\left. + k\sin\left(\frac{k\pi(d-a)}{b-a}\right)\right) e^{\frac{i\pi j(d-a)}{b-a}}\right], & \text{if} \quad j \ne k. \end{cases}$$

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