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Joint Pricing and Production Decisions in an Assemble-to-Order System

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This paper studies coordinated pricing and production decisions in an assemble-to-order system. We first show that unlike in make-to-stock systems, a state-dependent base-stock list-price policy is optimal. The optimal state-dependent base-stock levels and list prices may increase or decrease as demand backlogs increase, whereas demand backlogs always improve the optimal expected profit. Because the problem easily becomes intractable under general system settings, we next develop a simple heuristic policy. The heuristic policy decouples inventory replenishment, pricing, and component allocation decisions in a coordinated way. We provide a sufficient condition that ensures the optimality of the heuristic policy, and present a numerical study to demonstrate its performance when the condition is not met. The numerical study also shows how the performance of the heuristic policy is affected by various market and operational conditions, and by the structure of the assemble-to-order system. By focusing on the simple W-model, we show how the heuristic pricing decisions are made in response to changes in inventory levels and various cost parameters.

Keywords: assemble-to-order system; dynamic pricing; inventory control; heuristic policy; marketing–operations interface
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1. Introduction

Manufacturing firms such as IBM, Dell, and HP consistently identify the imbalance between supply and demand as one of their biggest challenges. Whether it results in unmet demand or accumulation of unsold products, it impacts the bottom line. To address this problem, several operational strategies, such as dynamic pricing and assemble-to-order (ATO), have been applied in practice. In academia, there has also been growing research on dynamic pricing and ATO strategies (see Song and Zipkin 2003 and Chen and Simchi-Levi 2012 for reviews on these literatures). Although the two strategies are often applied simultaneously in practice, the academic literature has been considering them separately. In this paper, we study the two strategies in a single context.

Coordinating pricing and production decisions in ATO systems is a challenging problem. Most manufacturing firms that adopt ATO strategies, such as computer manufacturers, sell a family of substitutable products. In such cases, the price of a product affects not only the demand for the product but also the demands for other products. In ATO systems, different products are also connected through the common components they share. The two levels of connectivity make coordinating pricing and production decisions a challenging problem. For example, if an expensive component is overstocked, firms may want to discount the price of a product that uses this component. In ATO systems, there can be multiple such products, and thus firms need to determine for which product to offer a price discount. Because discounting the price of a product increases the demands for all components used by the product, the decision should also be made in consideration of the inventory levels of all related components.

In this paper, we study the problem of optimizing inventory replenishment, pricing, and component allocation decisions in an ATO system. We consider a finite-horizon periodic-review system consisting of multiple components and multiple products. We show that unlike in make-to-stock (MTS) systems, a state-dependent base-stock list-price policy is optimal. The optimal control parameters depend on demand backlogs. In ATO systems, common components create dependencies among products. When there are backlogged demands, it is optimal to order sufficient amounts of components to fulfill them. These components, however, can be used for any products. The inventories ordered for demand backlogs loosen the dependencies among products, which in turn affect the optimal state-dependent base-stock levels and list prices. We show that demand backlogs always improve the optimal expected profit, whereas their impacts on the optimal control parameters are not uniform. We also show that in ATO systems, one can control the expected shortage cost of a product via the prices of other products instead of safety stocks.

Except for very simple ATO systems, one cannot fully solve the problem because of the complex structure of the optimal policy and the curse of dimensionality of dynamic programming. Thus, we develop a heuristic policy, which is easy to implement in practice. The heuristic policy decouples inventory replenishment, pricing, and component allocation decisions in a coordinated way. The heuristic policy has a simple structure, and the control parameters of the heuristic policy can be obtained from tractable stochastic programs. We provide a sufficient condition that ensures the optimality of the heuristic policy, and demonstrate its performance when the condition is not met via a numerical study. The numerical study also shows how various market and operational conditions and the structure of the ATO system affect the performance of the heuristic policy.

By focusing on the simple W-model, we provide a complete characterization of the heuristic pricing policy. The W-model consists of two products and three components, and each product requires one unit of a dedicated component and one unit of a common component. We first show how the prices of the two products respond to the changes in the inventory levels of the three components. Bish and Wang (2004) also study a pricing problem in a system with two products and three resources. In our problem each dedicated component is a complement to the common component, whereas in Bish and Wang (2004) each dedicated resource is a substitute to the flexible resource. We discuss how this difference affects the properties of the optimal solution. We next show how the prices change as various cost parameters change. These results provide insights on how to make coordinated pricing and production decisions in complex ATO systems.

Our work falls within the literature on joint pricing and production decisions. This problem was first studied by Whitin (1955) in a single-period setting. The multiperiod versions of the problem have received great attention during the last two decades. For the case with a single product and a zero procurement lead time, the problem has been studied under several different model settings. For example, Federgruen and Heching (1999) consider the case of demand backlogs, Chen and Simchi-Levi (2004a, b) consider the case of a fixed ordering cost and demand backlogs, and Chen et al. (2006) and Song et al. (2009) consider the case of a fixed-ordering cost and lost sales. Huh and Janakiraman (2008) extend the results by considering multiple demand-related decisions (e.g., pricing and seller incentive decisions). When there are multiple products or when the procurement lead time is positive, the optimal policy no longer has a simple structure. For this reason, little has been discussed for these cases. Zhu and Thonemann (2009) fully characterize the structure of the optimal policy for a system with two products, and Song and Xue (2007) partially characterize the structure of the optimal policy for general multiproduct systems. For the case of a positive lead time, Pang et al. (2012) provide a partial characterization of the optimal policy. One of our contributions is to partially characterize the structure of the optimal policy in an ATO setting. To our knowledge, we are the first to show the dependencies between the optimal control parameters and demand backlogs, which only arise in ATO settings.

Despite the complexity of the optimal policy in general model settings, there have been limited efforts to develop heuristic solutions in this literature. One of the few exceptional works is done by Federgruen and Heching (2002), who proposed a heuristic policy and an approximate solution for a distribution system. Bernstein et al. (2012) recently developed a heuristic solution for a single product system with a positive procurement lead time. We also contribute to this stream of research by developing a heuristic policy for an ATO system. We refer the reader to Chen and Simchi-Levi (2012) for the latest review on this literature.

Our research is also related to the literature on ATO systems. Unlike our study, this literature has not considered dynamic pricing decisions. In addition, this literature has mostly considered stationary problems. In contrast, we consider a nonstationary problem, in which the value of coordinating pricing and production decisions is known to be significant (Federgruen and Heching 1999). By considering a nonstationary problem, we investigate the dynamic interaction between demand backlogs and optimal inventory replenishment decisions, which has not been discussed in the literature. Unlike this literature, however, we limit our focus to the case of a zero procurement lead time. The literature on periodic review ATO systems (e.g., Zhang 1997, Hausman et al. 1998, Agrawal and Cohen 2001, Akçay and Xu 2004) has grown in a way that the performance of a new heuristic policy is measured against existing ones. Because there is no prior study that considers joint pricing and production decisions in an ATO setting, we cannot compare our heuristic policy with an existing one. By limiting the scope to the case of a zero procurement lead time, we obtain a tractable upper-bound problem, and we measure the performance of the heuristic policy using this upper bound. We provide an extension of the heuristic policy to the positive lead-time case in the electronic companion (available as supplemental material at http://dx.doi.org/10.1287/msom.2014.0492). For recent advances in this literature, we refer the reader to Doğru et al. (2010), Bernstein et al. (2011), and the references therein.

The rest of this paper is organized as follows. In §2, we introduce the model and formulation. In §3, we discuss the properties of the optimal policy. In §4, we develop a heuristic policy and an upper-bound problem. In §5, we discuss the heuristic pricing policy in details. In §6, we present a numerical study and conclude in §7. We defer all proofs to the electronic companion.

2. Model

We consider a firm that makes pricing and production decisions in an ATO system over T time periods. The ATO system consists of I components indexed by $i \in \{1, 2, \ldots, I\}$ and J products indexed by $j \in \{1, 2, \ldots, J\}$. We use subscripts for time periods and superscripts to indicate each element of a vector. The bill of materials is given in an $I \times J$ dimensional matrix B. For example, product j requires $B^{i,j}$ units of component i. We provide a summary of notations in the appendix.

The sequence of events is as follows: At the beginning of each period t, the firm first observes the levels of *component* inventories $x_t \in \mathbb{R}^I$, and the amounts of backlogged *product* demands $s_t \in \mathbb{R}^J$. Next, the firm places component replenishment orders and determines the prices of the products $p_t \in \mathbb{R}^J$. During period t, the demands for the products $D_t(p_t) \in \mathbb{R}^J$ are realized. At the end of period t, the firm receives the components ordered at the beginning of period t. We denote the inventory levels after replenishment by $y_t \in \mathbb{R}^I$. Given y_t units of inventories, the firm determines how many units of products to produce, $z_t \in \mathbb{R}^J$. Unmet demands are backlogged, and the problem proceeds to period $t+1$. The firm earns revenues from all realized demands, and demand backlogs incur unit shortage costs $b_t \in \mathbb{R}^J$. Unit procurement costs of components are given as $c_t \in \mathbb{R}^I$, and unit assembly costs of products are given as $a_t \in \mathbb{R}^J$. Finally, component inventories incur unit holding costs $h_t \in \mathbb{R}^I$. At period $T+1$, the firm fulfills all backlogged demands and salvages all remaining components. The salvage values of components are the same as the procurement costs at period $T+1$. All revenues and costs are discounted at the rate α per period.

When making pricing decisions, the firm is uncertain about the demands for the products. We model the uncertain demands as $D_t(p_t) = \epsilon_t q_t(p_t) + \xi_t$. Random variable $\epsilon_t \in \mathbb{R}$ represents the multiplicative demand uncertainty, and random vector $\xi_t \in \mathbb{R}^J$ represents the additive demand uncertainty. We assume that $E[\epsilon_t] = 1$ and $E[\xi_t] = [0, \ldots, 0]'$, and thus the deterministic vector function $q_t(p_t)$, indicates the expected demands for the products. We also assume that the demands at different time periods are independent. When formulating the problem, we use the expected demands q_t as decision variables instead of p_t. This change of variables helps characterize the structure of the optimal policy, and we refer the reader to Chen and Simchi-Levi (2012) for related discussions. For given expected demands q_t, the prices can be obtained by an inverse demand function $p_t(q_t)$. We denote the lower and upper bounds on q_t by $\underline{q}_t$ and $\bar{q}_t$, respectively. We further assume that the inverse demand function meets the following regularity conditions.

Assumption 1. 1. *$p_t(q_t)$ is twice differentiable, bounded, and decreasing in q_t.*
2. *$q_t' p_t(q_t)$ is bounded and concave in q_t.*

Next we formulate the firm's problem as a dynamic program, which has two decision points in each period. Because the firm keeps inventories at the component level and demands are backlogged at the product level, both x_t and s_t are state variables. At period $T+1$, the firm fulfills all backlogged demands and salvages all remaining inventories. Thus, the value-to-go function is

$$v_{T+1}(x_{T+1}, s_{T+1}) = c_{T+1}' x_{T+1} - (c_{T+1}' B + \alpha a_{T+1}') s_{T+1}.$$

At the end of period $t \in \{1, \ldots, T\}$, the firm produces z_t units of products with y_t units of components to meet the demands $d_t = D_t(q_t) + s_t$. Thus, the value-to-go function at the end of period t is given as

$$\begin{aligned} g_t(y_t, d_t) = \max_{z_t} \{ & v_{t+1}(y_t - Bz_t, d_t - z_t) - a_t' z_t \\ & - h_t'(y_t - Bz_t) - b_t'(d_t - z_t) \} \end{aligned} \tag{1}$$
$$\text{s.t.} \quad Bz_t \le y_t, \quad 0 \le z_t \le d_t.$$

Finally, at the beginning of period $t \in \{1, \ldots, T\}$, the firm places component replenishment orders, and determines the sales prices. The inventory levels after replenishment, y_t, should satisfy that $y_t \ge x_t$. Thus, the value-to-go function at the beginning of period t is given as

$$\begin{aligned} v_t(x_t, s_t) = \max_{y_t \ge x_t,\, \underline{q}_t \le q_t \le \bar{q}_t} \{ & \alpha E[p_t(q_t)' D_t(q_t)] - c_t'(y_t - x_t) \\ & + \alpha E[g_t(y_t, D_t(q_t) + s_t)] \}. \end{aligned} \tag{2}$$

We define

$$\begin{aligned} J_t(y_t, s_t, q_t) = {} & \alpha E\big[p_t(q_t)' D_t(q_t)\big] - c_t' y_t \\ & + \alpha E\big[g_t(y_t, D_t(q_t) + s_t)\big]. \end{aligned}$$

Then, (2) can be simplified as

$$v_t(x_t, s_t) = c_t' x_t + \max_{y_t \ge x_t,\, \underline{q}_t \le q_t \le \bar{q}_t} J_t(y_t, s_t, q_t).$$

When B is an identity matrix, there is a one-to-one correspondence between components and products, and the solution of the component allocation problem (1) is $z_t = \min\{y_t, d_t\}$. In this case, the problem is reduced to the problem of joint pricing and production control in a make-to-stock (MTS) system. When the set of feasible price vectors consists of a singleton, the problem is reduced to the inventory control problem in a periodic-review ATO system with a zero procurement lead time.

3. Optimal Policy

In this section, we discuss structural properties of the optimal policy. The result of this section shows how the optimal decisions in ATO systems differ from those in MTS systems. We first prove the concavity of value-to-go functions, which establishes the structure of the optimal policy when the beginning inventory levels are low.

Proposition 1. *The following statements are true for every t:*

1. *$g_t(y_t, d_t)$ is concave in (y_t, d_t).*
2. *$J_t(y_t, s_t, q_t)$ is concave in (y_t, s_t, q_t).*
3. *$v_t(x_t, s_t)$ is concave in (x_t, s_t).*
4. *We define*

$$(y_t^*(s_t), q_t^*(s_t)) = \mathop{\arg\max}_{y_t,\, \underline{q}_t \le q_t \le \bar{q}_t} J_t(y_t, s_t, q_t).$$

If $x_t \le y_t^*(s_t)$, it is optimal to replenish inventories up to $y_t^*(s_t)$ and set the expected demands at $q_t^*(s_t)$.

In MTS systems studied in the literature (e.g., Federgruen and Heching 1999, Song and Xue 2007, Zhu and Thonemann 2009), the *base-stock list-price* policy is known to be optimal. Under this policy there exist constant base-stock levels, and if the net-inventory levels are below these levels, it is optimal to replenish net inventories up to these levels and offer the list prices, which are also constant. In the ATO setting, a *state-dependent base-stock list-price policy* is optimal. Note that $y_t^*(s_t)$ defined in Proposition 1 indicates the optimal order-up-to levels for on-hand inventories. The optimal state-dependent base-stock levels, which are the order-up-to levels for net inventories, are thus given as $y_t^*(s_t) - Bs_t$. In the ATO setting, $y_t^*(s_t) - Bs_t$ and $q_t^*(s_t)$ are dependent on demand backlogs s_t.

To explain why the optimal control parameters depend on demand backlogs, we first consider a simple example. The example also shows how common components affect the prices of the products that use them. Consider a system with two products and one common component. Suppose that $b_t^2 \gg b_t^1$. In this case, product 2 always gets a higher priority than product 1 in component allocation. Thus, the total expected shortage and holding costs at period t are given as

$$E\big[h_t^1(y_t - D_t^1(q_t) - s_t^1 - D_t^2(q_t) - s_t^2)^+\big] + E\big[b_t^2(D_t^2(q_t) + s_t^2 - y_t)^+ + b_t^1(D_t^1(q_t) + s_t^1 - (y_t - D_t^2(q_t) - s_t^2)^+)^+\big].$$

We define $ss_t = y_t - q_t^1 - s_t^1 - q_t^2 - s_t^2$, which indicates the safety-stock level at period t, and we assume that $\epsilon_t = 1$ for simplicity. Then, the total operating costs at period t can be simplified as

$$E\big[h_t^1(ss_t - \xi_t^1 - \xi_t^2)^+ + b_t^2(\xi_t^2 - ss_t - q_t^1 - s_t^1)^+ + b_t^1(q_t^1 + \xi_t^1 + s_t^1 - (q_t^1 + ss_t + s_t^1 - \xi_t^2)^+)^+\big]. \quad (3)$$

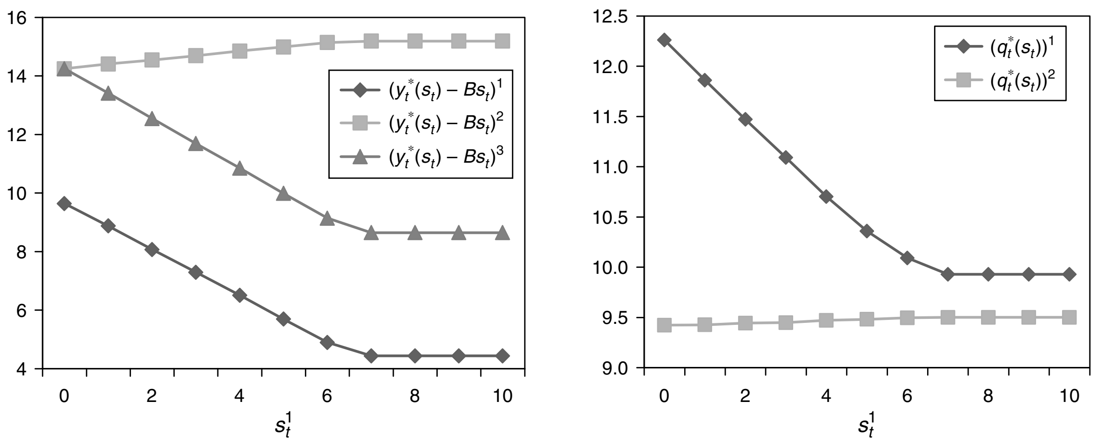
First note that (3) depends on ss_t, q_t^1, and s_t^1, but not on q_t^2 and s_t^2. For given s_t, the expected shortage cost of product 2 (i.e., the second term of (3)) can be reduced by increasing either ss_t or q_t^1. In other words, both safety stocks and the expected demands for product 1 can reduce the shortage cost of product 2. They, however, incur different types of costs. The increase in ss_t increases the expected holding cost (i.e., the first term), whereas the increase in q_t^1 increases the expected shortage cost of product 1 (i.e., the third term). When $h_t^1 \gg b_t^1$, increasing the expected demand for product 1 is a cheaper method to reduce the shortage cost of product 2 than keeping more safety stocks.

In (3) q_t^1 and s_t^1 have the exact same impact on the total cost. In contrast, the increase in q_t^1 reduces the profit margin of product 1, whereas s_t^1 has no impact on the expected revenue at period t. Hence, when there is a large amount of backlogged demands for product 1 (i.e., when s_t^1 is large), q_t^1 should not be increased beyond the most profitable level. For this reason, the optimal expected demand for product 1 decreases as s_t^1 increases. This change in turn affects the optimal state-dependent base-stock level of the component.

The impacts of demand backlogs on the optimal control parameters are not uniform. We next extend the previous example to show the mixed impacts of backlogs. Now suppose that the system consists of two products and three components. Each product $j \in \{1, 2\}$ requires one unit of component j and one unit of component 3, i.e., each product has an additional dedicated component. We use the following cost parameters and demand functions for each j: $q_t^j(p_t) = 100 - 1.6p_t^j$, $\text{Prob}(\epsilon_t = 0.9) = \text{Prob}(\epsilon_t = 1.1) = 0.5$, $\xi_t^j \sim \mathcal{N}(0, 10)$, $c_t = [5, 5, 45]'$, $a_t = [0, 0]'$, $h_t = [4, 4, 36]'$, and $b_t = [1, 50]'$. The two products are symmetric except for the shortage costs. As in the previous example, the shortage cost of product 1 is substantially smaller than that of product 2 and is also much smaller than the holding cost of the common component. Such a cost setting may arise when components are expensive and customers are willing to wait for backlogged orders, such as in the market of new high-technology products. We compute the optimal state-dependent base-stock levels and expected demands for period $t = T$, which are shown in Figure 1.

The optimal expected demand that maximizes the profit without operating costs is 10 for each product. When $s_t = 0$, the optimal expected demand for product 1 is much higher than 10, and that for product 2 is lower

Figure 1 Optimal Base-Stock Levels and Expected Demands When $s_t^2 = 0$



than this level, as shown in Figure 1. Because the shortage cost of product 1 is much smaller than the holding cost of the common component, the expected demand for product 1 is set at a high level to reduce the expected shortage costs for product 2, as in the previous example. As s_t^1 increases, the optimal expected demands for product 1 and the optimal state-dependent base-stock levels of components 1 and 3 all decrease. Because backlogged demands for product 1 can also reduce the expected shortage costs for product 2, the optimal expected demands for product 1 converge to the optimal risk-free level as s_t^1 increases. The optimal state-dependent base-stock levels of components 1 and 3, which are used by product 1, also decrease as their expected demands decrease.

As s_t^1 increases, the optimal expected demand for product 2 and the optimal state-dependent base-stock level of its dedicated component increase. Note that the holding cost of the common component and the shortage cost of product 2 are both large, i.e., both the underage and overage costs of product 2 are large. Hence, to reduce the degree of demand uncertainty, the optimal expected demand for this product is set at a lower level than the optimal risk-free level. When the firm has backlogged demands for product 1, it can use the components ordered for the backlogs to meet the demand for product 2 without the risk of incurring holding costs. Hence, the overall cost of the demand uncertainty decreases as s_t^1 increases, and the optimal expected demands for product 2 converge to the optimal risk-free level. The optimal state-dependent base-stock level of component 2, which is used only by product 2, also changes in the same direction.

In ATO systems, the price of a product affects other products in two different ways. First, the price of a product directly affects the demands for other products. Second, the price of a product affects the demands for multiple components, which in turn affect the prices of other products that also use these components. In this example, we used a demand model in which the direct impact does not exist. Thus, the sharp difference in the optimal expected demands for the two products and their responsiveness to demand backlogs show the significance of the indirect impact.

Although demand backlogs can either increase or decrease the optimal control parameters, their impact on the optimal expected profit is always consistent. Note that $v_t(x_t, s_t) - c_t' x_t + (c_t' B + \alpha a_t') s_t$ indicates the optimal value-to-go function after adjusting the values of on-hand inventories and the costs of fulfilling demand backlogs. The following proposition shows that the adjusted value-to-go function always increases in s_t.

Proposition 2. *For every* x_t,

$$v_t(x_t, s_t) - c_t' x_t + (c_t' B + \alpha a_t') s_t$$

increases in s_t.

Demand backlogs from the previous period can be seen as certain demands at the current period. If there are backlogged demands, it is optimal to order enough components to fulfil them. Later, these components can be used for any products. In other words, demand backlogs provide the firm with additional flexibilities when it makes component allocation decisions. Thus, the firm's adjusted value-to-go function always increases as demand backlogs increase. From this result, one may conjecture that the optimal safety-stock levels decrease as demand backlogs increase, and so do the optimal state-dependent base-stock levels. However, as shown in the previous example, demand backlogs can also increase the optimal state-dependent base-stock levels.

Doğru et al. (2010) show that in a one-period ATO model with no pricing decisions, the expected operating cost can be reduced if the initial backlog levels are optimized along with the inventory levels. Proposition 2 does not imply that the optimal policy intentionally backlogs certain amounts of demands, i.e., backlog levels are optimized. It implies only that after shortage costs are incurred, demand backlogs can help the decisions in the future. In practice, shortage costs outweigh the benefit of having backlogs, and thus intentionally backlogging demands is a costly strategy. However, because backlogged demands enable flexible component allocation decisions in the future, the true underage cost of a product in ATO systems is smaller than its direct shortage cost.

The dependencies of the adjusted value-to-go function, state-dependent base-stock levels, and list prices on demand backlogs disappear if a certain condition is met. We denote the optimal solution of (1) for given (y_t, d_t) by $z_t^*(y_t, d_t)$.

Proposition 3. *Suppose that* $z_t^*(y_t^*(0), D_t(q_t^*(0))) > 0$ *holds with probability one. Then,*

1. $y_t^*(s_t) - Bs_t = y_t^*(0)$,
2. $q_t^*(s_t) = q_t^*(0)$, *and*
3. $v_t(x_t, s_t) - c_t' x_t + (c_t' B + \alpha a_t')s_t = v_t(0, 0)$ *for every* $x_t - Bs_t \le y_t^*(0)$.

The condition in Proposition 3 implies that the firm always produces at least one unit of each product when there are no backlogged demands. Recall the example that we discussed earlier. In this example, the shortage cost of product 1 is much smaller than the holding cost of the common component. In this case, it is cheaper to backlog the demand for product 1 than to incur inventory holding costs. Thus, the condition in Proposition 3 is not met in this case. However, in practice, shortage costs are usually assessed at much higher levels than inventory holding costs. Thus, backlogging the entire demand for a product is a costly decision and is unlikely to be executed under the optimal policy.

When the beginning inventory level of any component is above the optimal order-up-to level, the optimal policy no longer has a simple structure. For example, suppose that $x_t^i > (y_t^*(s_t))^i$ for some i and $x_t^k < (y_t^*(s_t))^k$ for every $k \ne i$, i.e., only component i is overstocked. In this case, the optimal replenishment decision for each component $k \ne i$ can be either ordering a positive amount or not ordering. Because component i is overstocked, it may be optimal to discount the price of a product that uses this component. This change also affects the demands for other components that are used by the same product, and thus their optimal order-up-to levels also change. In addition, the price discount may decrease the demands for other products, which in turn reduces the optimal order-up-to levels of certain components. Because of this complexity, to fully characterize the structure of the optimal policy and compute the optimal expected profit, one needs to obtain different optimal order-up-to levels and prices for every state that does not satisfy $x_t \le y_t^*(s_t)$. Due to the curse of dimensionality of dynamic programming, this problem is not tractable except for very simple systems. Hence, in the rest of the paper, we focus on a heuristic solution and its properties.

4. Heuristic Policy

4.1. Decoupling Heuristic Policy

In this section, we first develop a heuristic policy. The main idea of the heuristic policy is to decouple inventory replenishment, pricing, and component allocation decisions while ensuring their coordination. The heuristic policy has a simple structure, and we derive the control parameters of the heuristic policy from tractable stochastic programs. Thus, the policy is easy to implement in practice.

We first introduce a myopic problem to obtain the control parameters of the heuristic policy. If the firm myopically assesses the value and the cost of demand backlogs and component inventories, it makes pricing and production decisions as in the following two-stage stochastic program:

$$\begin{aligned} \hat{v}_t &= \max_{y_t,\, \underline{q}_t \le q_t \le \bar{q}_t} \big\{ \alpha E[p_t(q_t)' D_t(q_t)] - c_t' y_t \\ &\qquad + \alpha E[\hat{g}_t(y_t, D_t(q_t))] \big\}, \\ \hat{g}_t(y_t, d_t) &= \max_{z_t} \big\{ c_{t+1}'(y_t - Bd_t) - a_t' z_t - h_t'(y_t - Bz_t) \\ &\qquad - (b_t + \alpha a_{t+1})'(d_t - z_t) \big\}, \\ &\quad \text{s.t. } Bz_t \le y_t, \quad 0 \le z_t \le d_t. \end{aligned} \tag{4}$$

We denote the optimal solutions of (4) by $\hat{y}_t$ and $\hat{q}_t$. The difference between the original dynamic program and the myopic problem is that the myopic problem approximates the value of $v_{t+1}(x_{t+1}, s_{t+1})$ at $c_{t+1}' x_{t+1} - (c_{t+1}' B - \alpha a_{t+1}')s_{t+1}$ without considering the impact of x_{t+1} and s_{t+1} in the future. By considering pricing decisions, this problem extends the two-stage stochastic programs studied by Song and Zipkin (2003) and Harrison and Van Mieghem (1999), which in turn are special cases of the newsvendor network model introduced by Van Mieghem and Rudi (2002). In contrast to these problems, which are two-stage stochastic linear programs, our problem is not linear. However, the problem also falls within the category of two-stage convex stochastic optimization problems, which one can efficiently solve via various numerical methods (see, e.g., Birge and Louveaux 1997, Shapiro et al. 2009).

Using $\hat{y}_t$ and $\hat{q}_t$, we next define the *decoupling heuristic policy*. Under the base-stock list-price policy, which is optimal for single product MTS systems (Federgruen and Heching 1999), a price discount is offered if the product is overstocked, and otherwise inventories are replenished up to the base-stock level and the list price is offered. We design the heuristic policy in a similar way. We sequentially describe how the heuristic policy makes inventory replenishment, pricing, and component allocation decisions.

Inventory Replenishment Decisions. The inventory replenishment policy is an independent base-stock policy with the base-stock levels $\hat{y}_t$. Independent base-stock policies have been widely adopted in the ATO literature (e.g., Hausman et al. 1998, Akçay and Xu 2004, Lu and Song 2005, Lu et al. 2010). Under this policy, the firm orders $(\hat{y}_t - x_t + Bs_t)^i$ units of component i if $(x_t - Bs_t)^i \le \hat{y}_t^i$, and otherwise orders zero units of component i. On-hand inventory levels after replenishment are thus given as $y_t = \max\{x_t, \hat{y}_t + Bs_t\}$. Recall that the myopic base-stock levels are obtained under the assumption that remaining inventories at a given time period do not affect the decisions at future time periods. In our problem, the firm can dynamically adjust the prices, and thus it can quickly sell excess inventories. Hence, leftover inventories have a limited impact on future decisions, and $\hat{y}_t$ are reasonable base-stock levels.

Pricing Decisions. As in the base-stock list-price policy, if there are no excess inventories, i.e., if $y_t = \hat{y}_t + Bs_t$, the expected demands are set at $\hat{q}_t$. Otherwise, the firm adjusts the prices to reduce the expected holding costs. We define $\widehat{ss}_t = \hat{y}_t - B\hat{q}_t$, which indicates the safety-stock levels for period t. Then the pricing decisions are made by the following problem:

$$\max_{\underline{q}_t \le q_t \le \bar{q}_t} \Big\{ \underbrace{\alpha p_t(q_t)' q_t - (c_t' B + \alpha a_t')(q_t + s_t)}_{\text{(a)}} - \underbrace{(\alpha h_t + c_t - \alpha c_{t+1})'[y_t - B(q_t + s_t)]}_{\text{(b)}} \Big\}$$
$$\text{s.t. } B(q_t + s_t) \le y_t - \widehat{ss}_t. \tag{5}$$

Without inventory consideration, the firm would maximize the difference between expected revenue and production costs, i.e., part (a) of the objective function. However, leftover inventories incur holding costs, i.e., part (b). The solution for (5) maximizes the expected profit in consideration of inventory holding costs. Note that this deterministic problem ignores demand uncertainty. The ignored demand uncertainty is managed by safety stocks. By meeting the resource constraint $B(q_t + s_t) \le y_t - \widehat{ss}_t$, the firm always reserves $\widehat{ss}_t$ units of inventories for demand uncertainty.

Component Allocation Decisions. For given y_t and d_t, the component allocation decisions are made by the following optimization problem:

$$\max_{z_t} \big\{ c_{t+1}'(y_t - Bd_t) - a_t' z_t - h_t'(y_t - Bz_t) - (b_t + \alpha a_{t+1})'(d_t - z_t) \big\} \tag{6}$$
$$\text{s.t. } Bz_t \le y_t, \quad 0 \le z_t \le d_t.$$

This problem minimizes the instantaneous holding and shortage costs as in the myopic problem (4). Note that the net-inventory levels at the beginning of period $t+1$, i.e., $x_{t+1} - Bs_{t+1}$, are independent of z_t. In other words, how to allocate components at period t does not affect the inventory replenishment and pricing decisions at period $t+1$. Thus, the component allocation decisions are made such that the total operating cost at the current period is minimized. Plambeck and Ward (2006) show that a similar myopic component allocation policy is asymptotically optimal in an ATO system.

The decoupling heuristic policy has a simple structure, and one can efficiently compute the decision variables of this policy. To obtain $\hat{y}_t$ and $\hat{q}_t$, one needs to solve T two-stage stochastic programs. The pricing problem (5) is a deterministic convex optimization problem, and the component allocation problem (4) is a deterministic linear program. One can solve such deterministic convex optimization problems in a short amount of time even when the number of decision variables is large (Boyd and Vandenberghe 2004). Hence, the heuristic policy is implementable in practice.

Although simple, the heuristic policy takes into account all determinants of the firm's profit: revenue, production costs, and inventory costs. In addition, the heuristic policy makes pricing and inventory decisions in a coupled manner. Thus, the heuristic policy may achieve a near-optimal expected profit. Next we briefly discuss when the heuristic policy is indeed optimal.

The base-stock levels and list prices of the heuristic policy are from the myopic problem, looking only one period ahead when assessing the value and the cost of remaining inventories and demand backlogs. These base-stock levels and list prices are optimal control parameters if the net-inventory levels after replenishment are independent of the beginning inventory levels and demand backlogs.

PROPOSITION 4. *If $z_t^*(y_t^*(0), D_t(q_t^*(0))) > 0$ and $BD_t(\hat{q}_t) \ge \hat{y}_t - \hat{y}_{t+1}$ hold with probability one for every t, the decoupling heuristic policy is optimal.*

The first condition is the same as the condition in Proposition 3, which ensures that the optimal base-stock levels are state-independent. The second condition implies that the beginning net-inventory levels at each period, i.e., $\hat{y}_t - BD_t(\hat{q}_t)$, are always equal to or less than the order-up-to levels, i.e., $\hat{y}_{t+1}$. Because $BD_t(\hat{q}_t) \ge 0$, this condition holds if $\hat{y}_t$ does not decrease over time. Roughly speaking, the heuristic policy is optimal if demands do not decrease over time, e.g., when demands are stationary. In §6, we demonstrate the performance of the heuristic policy when demands are decreasing and fluctuating over time.

4.2. Upper-Bound Problem

In this section, we construct an upper bound of the optimal expected profit. The difference between the optimal expected profit and the expected profit under the heuristic policy is the most proper performance measure. However, because one cannot compute the optimal expected profit in general ATO systems, we compare the expected profit under the heuristic policy with a tractable upper bound of the optimal expected profit.

We construct the upper bound problem by decoupling the dynamic program into two-stage stochastic programs. To do so, we make two assumptions: The first assumption is that the firm can return inventories at the same price as the procurement cost at every period. This assumption removes the constraint $y_t \geq x_t$ in (2), and thus the optimal decisions at period t no longer depend on x_t. The second assumption is that the firm can produce negative units of products. Producing a negative one unit of product j results in positive $B^{i,j}$ units of component i. This assumption removes the constraint $z_t \geq 0$ in (1), and thus the optimal decisions at period t do not depend on s_t. Under these two assumptions, the problem for period t is given as:

$$\begin{aligned}\bar{v}_t &= \max_{y_t,\, \underline{q}_t \leq q_t \leq \bar{q}_t} \{\alpha E[p_t(q_t)' D_t(q_t)] - c_t' y_t \\ &\qquad + \alpha E[\bar{g}_t(y_t, D_t(q_t))]\},\\ \bar{g}_t(y_t, d_t) &= \max_{z_t} \{c_{t+1}'(y_t - Bd_t) - a_t' z_t - h_t'(y_t - Bz_t) \\ &\qquad - (b_t + \alpha a_{t+1})'(d_t - z_t)\},\\ &\text{s.t. } Bz_t \leq y_t, \quad z_t \leq d_t. \end{aligned} \tag{7}$$

Then, the following property holds for every t.

Proposition 5. *For every x_t and s_t, $v_t(x_t, s_t) \leq c_t' x_t - (c_t' B + \alpha a_t') s_t + \sum_{k=t}^{T} \alpha^{k-1} \bar{v}_k$.*

The two-stage stochastic program (7) is identical to the myopic problem (4) except that the constraint $z_t \geq 0$ is further relaxed.

5. Pricing Decisions Under Decoupling Heuristic Policy

In this section, we discuss the heuristic pricing policy in detail. By focusing on the W-model, we provide a complete characterization of the heuristic pricing policy. In particular, we show how the prices respond to changes in the inventory levels and various cost parameters. The objective of this section is to generate managerial insights on how to make coordinated pricing decisions in complex ATO systems. When a component with a large holding cost is overstocked, the firm may want to discount the price of a product that uses this component. In ATO systems, there can be multiple such products, and thus the firm needs to determine how to adjust the prices of all the products that use the overstocked component. In doing so, the firm should take into account the inventory levels of other components used by those products.

The W-model refers to the system that consists of two products and three components with the following bill of materials. Each product $j \in \{1, 2\}$ requires one unit of component j and one unit of component 3. In other words, components 1 and 2 are dedicated components, and component 3 is a common component. The W-model has been widely used in the literature because of its simplicity (e.g., Doğru et al. 2010). In this section, we relax the upper and lower bounds on q_t, and further assume that the inverse demand function $p_t(q_t)$ meets the following condition.

Assumption 2. *For each $j \in \{1, 2\}$ and $k \neq j$,*

$$\left|\frac{\partial^2 p_t(q_t)' q_t}{\partial (q_t^j)^2}\right| > \left|\frac{\partial^2 p_t(q_t)' q_t}{\partial q_t^j \partial q_t^k}\right|.$$

This condition implies that the marginal revenue change in the demand for a product, i.e., $\partial p_t(q_t)' q_t / \partial q_t^j$ is more sensitive to the demand for that product than to the demand for the other product. The assumption holds for widely used demand models such as the linear demand model (Vives 1999, Chap. 6.2) and the multinomial logit demand model (Song and Xue 2007).

For notational convenience, we define one additional variable. Under the heuristic policy, the inventory levels after replenishment are $y_t = \max\{x_t, \hat{y}_t + Bs_t\}$. Recall that $\widehat{ss}_t = \hat{y}_t - B\hat{q}_t$ are the safety-stock levels for period t. Thus, $e_t = y_t - Bs_t - B\widehat{ss}_t$ indicates the amounts of available inventories at period t. With e_t, we can simplify the pricing problem (5) as

$$\begin{aligned}\max_{q_t} \quad & \Big\{\overbrace{p_t(q_t)' q_t - (a_t' + c_{t+1}' B - h_t' B) q_t}^{H_t(q_t)}\Big\} \\ \text{s.t.} \quad & Bq_t \leq Be_t.\end{aligned} \tag{8}$$

We denote the optimal solution by $\tilde{q}_t(e_t)$.

To help the characterization of $\tilde{q}_t(e_t)$, we define four special solutions of (8). Because the problem is a single-period problem, we omit the time index and use subscripts to indicate these four special solutions. We define

$$\begin{aligned} q_A &= \arg\max_{q} H_t(q),\\ q_B^1(e^2) &= \arg\max_{q^1 \mid q^2 = e^2} H_t(q),\\ q_B^2(e^1) &= \arg\max_{q^2 \mid q^1 = e^1} H_t(q),\\ q_C(e^3) &= \arg\max_{q \mid q^1 + q^2 = e^3} H_t(q). \end{aligned}$$

Note that q_A is the optimal solution for (8) when the resource constraint $Bq \leq Be$ is relaxed. Similarly, $q_B^j(e^k)$ is the optimal solution when the resource constraint is relaxed and additionally q^k is fixed at e^k for $k \neq j$. Finally, $q_C(e^3)$ is the optimal solution when the resource constraint is relaxed and $q^1 + q^2$ is fixed at e^3.

Depending on e, $\tilde{q}_t(e)$ can take one of seven different forms. We divide the space of e into the following seven sets:

$$\Omega_1 = \{e\colon e^1 \geq q_A^1 \text{ and } e^2 \geq q_A^2 \text{ and } e^3 \geq q_A^1 + q_A^2\},$$
$$\Omega_2 = \{e\colon e^1 \leq q_A^1 \text{ and } e^2 \geq q_B^2(e^1) \text{ and } e^3 \geq e^1 + q_B^2(e^1)\},$$
$$\Omega_3 = \{e\colon e^1 \geq q_B^1(e^2) \text{ and } e^2 \leq q_A^2 \text{ and } e^3 \geq q_B^1(e^2) + e^2\},$$

$$\Omega_4 = \{e\colon e^1 \le q^1_{\mathrm{B}}(e^2) \text{ and } e^2 \le q^2_{\mathrm{B}}(e^1) \text{ and } e^3 \ge e^1 + e^2\},$$
$$\Omega_5 = \{e\colon e^1 \ge q^1_{\mathrm{C}}(e^3) \text{ and } e^2 \ge q^2_{\mathrm{C}}(e^3) \text{ and } e^3 \le q^1_{\mathrm{A}} + q^2_{\mathrm{A}}\},$$
$$\Omega_6 = \{e\colon e^1 \ge e^3 - e^2 \text{ and } e^2 \le q^2_{\mathrm{C}}(e^3) \text{ and } e^3 \le q^1_{\mathrm{B}}(e^2) + e^2\},$$
$$\Omega_7 = \{e\colon e^1 \le q^1_{\mathrm{C}}(e^3) \text{ and } e^2 \ge e^3 - e^1 \text{ and } e^3 \le e^1 + q^2_{\mathrm{B}}(e^1)\}.$$

Then, the following properties hold:

Proposition 6. *For every e, $e \in \Omega_k$ for some $k \in \{1, 2, \ldots, 7\}$, and*

1. *If $e \in \Omega_1$, $\tilde{q}_t(e) = q_{\mathrm{A}}$.*
2. *If $e \in \Omega_2$, $\tilde{q}_t(e) = [e^1, q^2_{\mathrm{B}}(e^1)]'$.*
3. *If $e \in \Omega_3$, $\tilde{q}_t(e) = [q^1_{\mathrm{B}}(e^2), e^2]'$.*
4. *If $e \in \Omega_4$, $\tilde{q}_t(e) = [e^1, e^2]'$.*
5. *If $e \in \Omega_5$, $\tilde{q}_t(e) = q_{\mathrm{C}}(e^3)$.*
6. *If $e \in \Omega_6$, $\tilde{q}_t(e) = [e^3 - e^2, e^2]$.*
7. *If $e \in \Omega_7$, $\tilde{q}_t(e) = [e^1, e^3 - e^1]$.*

We can divide the seven cases of Proposition 6 into two groups. The first four cases arise when the inventory level of the common component is high, and the last three cases arise when the inventory level of the common component is low, and thus it becomes the bottleneck of the optimal solution. Figure 2(a) shows the space of (e^1, e^2) and $\tilde{q}_t(e)$ of the first four cases. In all these cases, the inventory level of component 3 is sufficiently high so that $\tilde{q}_t(e)$ does not depend on e^3. When the firm also has sufficient inventories of components 1 and 2, i.e., when $e \in \Omega_1$, the firm sets the expected demands at q_{A}, which is the optimal solution for $H_t(q)$ without any resource constraint. If the firm increases the expected demands above these levels, the loss in the profit margin outweighs the savings in the inventory holding costs. When the inventory level of component 2 is high but that of component 1 is low, i.e., when $e \in \Omega_2$, the firm increases the expected demand for product 1 up to e^1, which is the maximum amount of product 1 that can be produced. At the same time, the firm increases the expected demand for product 2 only up to $q^2_{\mathrm{B}}(e^1)$, which is the interior solution for $H_t(q)$ when $q^1 = e^1$. Above this level, the loss in the profit margin of product 2 outweighs the savings in the inventory holding costs of components 2 and 3. Because the demand for each product may depend on the price of the other product, q^2_{A} and $q^2_{\mathrm{B}}(e^1)$ may not be the same unless $e^1 = q^1_{\mathrm{A}}$. Depending on the demand model, $q^2_{\mathrm{B}}(e^1)$ can be either increasing or decreasing in e^1. Part 3 is the symmetric result to part 2. Finally, when the inventory levels of both dedicated components are low, i.e., when $e \in \Omega_4$, the firm increases the expected demands for the two products such that it uses all inventories of components 1 and 2.

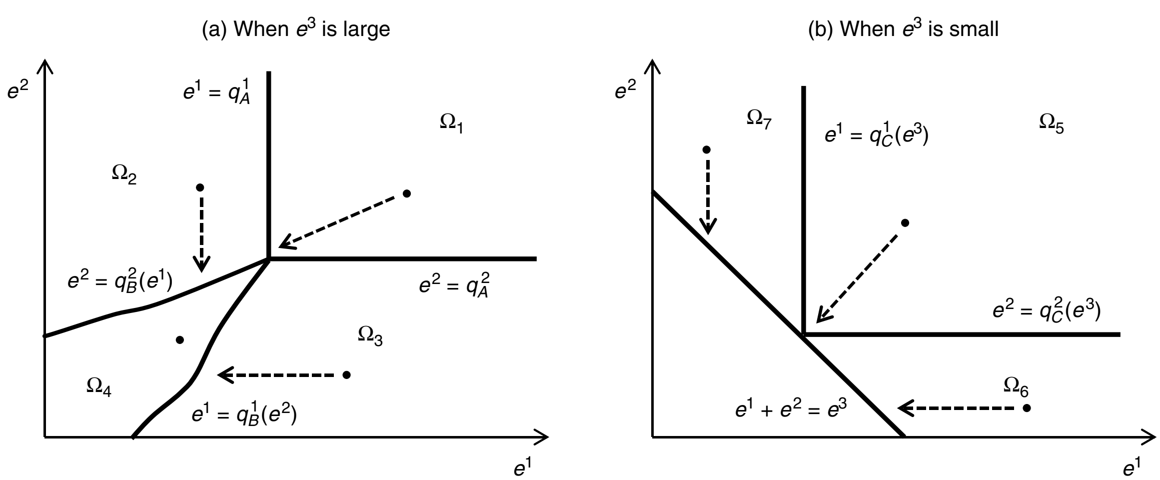
Parts 5–7 of Proposition 6 arise when the common component is scarce. Figure 2(b) shows the space of (e^1, e^2) and $\tilde{q}_t(e)$ in these cases. Because the common component is scarce, the firm sets the expected demands such that it uses all inventories of the common component. Hence, the pricing decision is essentially about how to allocate the inventories of component 3 to the two products. When distributing e^3 units of the common component, $q_{\mathrm{C}}(e^3)$ are the levels of expected demands at which the marginal profit increase is the same for the two products, i.e., $\partial H_t(q)/\partial q^1 = \partial H_t(q)/\partial q^2$. If the firm has enough inventories of components 1 and 2 to meet $q_{\mathrm{C}}(e^3)$, i.e., if $e \in \Omega_5$, then it sets the expected demands at these levels. If the inventory level of either component 1 or 2 is not sufficient for $q_{\mathrm{C}}(e^3)$, i.e., if $e \in \Omega_6$ or $e \in \Omega_7$, then the firm sets the expected demands such that it depletes all inventories of the scarce dedicated component and component 3.

With this characterization, we next discuss how the prices change as the inventory levels change.

Lemma 1. 1. *If $e \in \Omega_5 \cup \Omega_6 \cup \Omega_7$, then for every $\hat{e}$ such that $\hat{e} \ge e$ and $\hat{e}^3 = e^3$, $\hat{e} \in \Omega_5 \cup \Omega_6 \cup \Omega_7$.*

2. *If $e \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, then for every $\hat{e}$ such that $\hat{e}^1 = e^1$, $\hat{e}^2 = e^2$, and $\hat{e}^3 \ge e^3$, $\hat{e} \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$.*

Figure 2 **Space of (e^1, e^2) and Optimal Solutions**



As discussed, the first four cases of Proposition 6 arise when e^3 is sufficiently large compared to e^1 and e^2, whereas the last three cases arise when e^3 is small. Part 1 of the lemma implies that as e^1 or e^2 increases, e moves from $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ to $\Omega_5 \cup \Omega_6 \cup \Omega_7$ at most once. Part 2 of the lemma implies that as e^3 increases, e crosses the two sets at most once in the opposite direction. Using the result, we prove the following properties.

Proposition 7. *The following properties hold for each $j \in \{1, 2\}$ and $k \neq j$:*

1. *$\tilde{q}_t^j(e)$ is increasing in e^j and e^3.*
2. *If $q_B^j(e^k)$ is decreasing in e^k, $\tilde{q}_t^j(e)$ is decreasing in e^k.*
3. *If $q_B^j(e^k)$ is increasing in e^k, $\tilde{q}_t^j(e)$ is quasiconcave in e^k.*
4. *$\tilde{q}_t^1(e) + \tilde{q}_t^2(e)$ is increasing in e^j.*

As shown in part 1, as the inventory level of the common component increases, the expected demands for the two products both increase. In contrast, the inventory level of a dedicated component has different impacts on the two products. As shown in part 1, the expected demand for each product always increases as the inventory level of its dedicated component increases. When $e \in \Omega_5 \cup \Omega_6 \cup \Omega_7$, i.e., when the common component is scarce, the total amount of products that can be produced is fixed at e^3, i.e., $\tilde{q}_t^1(e) + \tilde{q}_t^2(e) = e^3$. Hence, the positive correlation between e^j and $\tilde{q}_t^j(e)$ implies a negative correlation between e^j and $\tilde{q}_t^k(e)$. When the common component is sufficient, i.e., when $e \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, the cross-price elasticity of demands determines the relationship between $\tilde{q}_t^j(e)$ and e^k. Depending on whether $q_B^j(e^k)$ is increasing or decreasing in e^k, $\tilde{q}_t^j(e)$ is either quasiconcave or uniformly decreasing in e^k. Although the increase in the inventory level of a dedicated component has mixed impacts on the two products, it always increases the total expected demands for the two as shown in part 4.

Bish and Wang (2004) also studied a pricing problem in a system with two products and three resources. In their problem, each product $j \in \{1, 2\}$ requires either one unit of a dedicated resource *or* one unit of a flexible resource. In contrast, in our problem each product requires one unit of the dedicated component *and* one unit of the common component. Our characterization of the optimal solution and that of Bish and Wang (2004) show the differences between ATO strategies and flexible-resource strategies. In Bish and Wang (2004), when the inventory level of the flexible resource is very large, the optimal solution is always the interior solution, which corresponds to q_A of our problem because both products can be produced only with the flexible resource. In our problem, even when e^3 is very large, the optimal solution can take four different forms, shown in parts 1–4 of Proposition 6, because the amounts of products that can be produced are always limited by the inventory levels of the dedicated components. When the common component is scarce, the optimal solution of our problem can take one of the three different forms shown in parts 5–7. In contrast, in Bish and Wang (2004) there are five different forms of the optimal solution that fully consumes the flexible resource.

The difference between our problem and that of Bish and Wang (2004) becomes more apparent when we consider the impact of the inventory level of a dedicated component (resource) on the allocation of the common component (flexible resource). When the common component is scarce, i.e., when $e \in \Omega_5 \cup \Omega_6 \cup \Omega_7$, the amount of the common component allocated to product j, i.e., $\tilde{q}_t^j(e)$ decreases as the inventory level of the other product's dedicated component, i.e., e^k, increases. This result is caused by the complementarity between the dedicated components and the common component in the ATO setting. In contrast, in Bish and Wang (2004) the amount of the flexible resource that can be allocated to a product increases as the inventory level of the other product's dedicated resource increases. The inventory level of a dedicated resource indicates the amount of the corresponding product that can be produced without the flexible resource. Hence, as this amount increases, more units of the flexible resource can be used to produce the other product. In other words, the impact of e^k on $\tilde{q}_t^j(e)$ is the opposite to our case. In Bish and Wang (2004) the flexible resource is a substitute to the dedicated resources.

Proposition 6 shows that the firm would not increase the expected demands beyond q_A even when all components are heavily overstocked. Similarly, when the demand for product j is fixed at e^j, the firm would not increase the expected demand for product $k \neq j$ above $q_B^k(e^j)$. In other words, q_A and $q_B^j(e^k)$ set the bounds on price discounts. When allocating the inventories of component 3 to the two products, the firm follows $q_C(e^3)$ to the extent possible. Hence, $q_C(e^3)$ characterizes the relative depths of price discounts to offer for the two products.

Proposition 8. *The following properties hold for each $j \in \{1, 2\}$ and $k \neq j$:*

1. *q_A^j, $q_B^j(e^k)$, and $q_C^j(e^3)$ are increasing in h_t^j, and decreasing in a_t^j and c_{t+1}^j.*
2. *q_A^j, $q_B^j(e^k)$, and $q_C^j(e^3)$ are independent of c_t.*
3. *q_A^j, $q_B^j(e^k)$ are increasing in h_t^3 and decreasing in c_{t+1}^3, whereas $q_C^j(e^3)$ is independent of them.*

Part 1 implies that the depth of a price discount to offer for a product increases in the inventory holding costs of its dedicated components. By increasing the expected demand for a product, the firm can reduce the potential holding costs that it may otherwise incur, and thus a deeper discount should be offered when the holding costs are larger. As shown in parts 1 and 2, the price discount is independent of the procurement costs at the current period, but it depends on the procurement costs at the next period. The costs of the components that are already ordered are sunk costs, and thus their values should be assessed at the procurement costs at the next period. If the cost of a certain component is expected to drop sharply in the upcoming period, then the firm should deeply discount the prices of the products that consume this component. Part 3 implies that the costs of the common component affect the bounds on the price discounts, i.e., q_{A} and $q^j_{\mathrm{B}}(e^k)$, but they do not affect the decisions on how to distribute the common component, i.e., $q_{\mathrm{C}}(e^3)$. The same amount of holding cost savings can be achieved by increasing the expected demand for any products that use the common component. Thus, the decisions on how to allocate inventories depend only on the costs of dedicated components.

6. Numerical Study

This section presents a numerical study. The purpose of the numerical study is twofold. First, we demonstrate the overall performance of the heuristic policy. Second, we investigate how the performance of the heuristic policy is affected by various market and operational conditions and by the structure of the ATO system. The analytical result in §4 shows that the heuristic policy is optimal when demands are either increasing or stationary over time. Thus, in this section we consider only the cases in which demands are decreasing and fluctuating over time.

6.1. Setup and Overall Performance

We first define a performance measure. Let v^h be the total expected profit under the heuristic policy. We compare v^h with $\sum_{t=1}^{T}\alpha^{t-1}\bar{v}_t$, which is the upper bound of the optimal expected profit given in Proposition 5. We define $\overline{\text{gap}} = ((\sum_{t=1}^{T}\alpha^{t-1}\bar{v}_t) - v^h)/(\sum_{t=1}^{T}\alpha^{t-1}\bar{v}_t)$, which is an upper bound of the true optimality gap.

To examine the overall performance of the heuristic policy, we first test it under an ATO system used by Zhang (1997) and Akçay and Xu (2004). The ATO system consists of four products and five components, and the bill of materials is

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The demand function has the following form: $D_t(p_t) = a_t(L - Ap_t) + \xi_t$, where

$$L = \begin{bmatrix} 50 \\ 40 \\ 55 \\ 30 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 1 & -0.1 & 0 \\ 0 & -0.1 & 1.2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We define the scalar a_t as $a_t = (1-\delta(t-1)) \times (1 + \eta\sin((\pi/2)t))$, where δ defines how demands rapidly decrease over time, and η defines the degree of demand seasonality. To prohibit a_t from taking a negative value, we define $\Delta = 1 - \delta(T-1)$, and directly determine the value of Δ instead of δ. We set $\underline{q}_t$ and $\bar{q}_t$ at 70% and 200%, respectively, of the optimal $\overline{q}_t$ in the deterministic setting. We assume that ξ^j_t is normally distributed and is independent of ξ^k_t for $k \neq j$. We set the variances of ξ^j_t such that the coefficients of variation of demands are cv for every t and j, and we truncate the values of ξ^j_t symmetrically so that $D_t(\underline{q}_t) \geq 0$ is ensured. Unit costs are given as $c_t = [15, 15, 10, 20, 15]'$, $a_t = [3,3,3,3]'$, $h_t = \lambda \times c_t$, and $b_t = \rho \times \lambda \times [560, 410, 460, 260]'$ for every t. The value of λ defines the relative size of operating costs compared to production costs, and the value of ρ defines the relative size of shortage costs compared to holding costs.

With this system setting, we test the performance of the heuristic policy under a total of 147 problem instances. As the base parameter set, we use $T = 10$, $\Delta = 0.4$, $\eta = 0.6$, $cv = 1.5$, $\lambda = 0.15$, and $\rho = 0.6$. Then we construct the first 49 problem instances with each combination of $T \in \{4, 6, \ldots, 16\}$ and $\Delta \in \{0.2, 0.3, \ldots, 0.8\}$, and the second 49 instances with each combination of $\eta \in \{0.2, 0.3, \ldots, 0.8\}$ and $cv \in \{0.5, 0.75, \ldots, 2\}$. We construct the last 49 problem instances with each combination of $\lambda \in \{0.09, 0.11, \ldots, 0.21\}$ and $\rho \in \{0.3, 0.4, \ldots, 0.9\}$. In the 147 problem instances, the average value of $\overline{\text{gap}}$ is 3.2%, the median value is 2.8%, and the maximum value is 13.6%. We report further details of these results in the next section. We also test the heuristic policy under other ATO systems and present the results in §§6.2 and 6.3. Under other ATO systems, the heuristic policy shows similar performances.

Given that $\overline{\text{gap}}$ is an upper bound of the true optimality gap, this result shows that the heuristic policy performs well in most problem instances tested. The result also implies that the difference between the optimal expected profit and the upper bound is also small. This is a surprising result because in every problem instance we tested, demands decrease and fluctuate over time and the degree of demand uncertainty is significant. Recall from §4.2 that we construct the upper bound by making two assumptions on the model. The first assumption is that the firm can produce negative units of products to obtain positive amounts of components. Producing a negative unit of a product implies that the firm backlogs the entire demand for this product. Unless the holding cost of a certain component is substantially larger than the shortage cost of a product that uses it, the firm is unlikely to backlog the entire demand for a product under the optimal policy. Hence, relaxing this constraint hardly affects the optimal decision. The second assumption is that the firm can always return the inventories from the previous period. Under this assumption, unused inventories do not affect the decisions at future time periods. In inventory control problems without pricing decisions, this assumption may yield a significant gap between the optimal expected profit and the upper bound. However, when the firm jointly makes pricing and inventory decisions, it can quickly consume excess inventories by discounting prices. Thus, unused inventories have a limited impact on the decisions to be made in the future. Hence, the second assumption is also not a strong assumption. The heuristic policy, which is designed under similar modeling assumptions, performs well for the same reason.

6.2. Impact of Market and Operational Conditions
In this section, we discuss how various market and operational conditions affect the performance of the heuristic policy. We conduct four sets of numerical experiments. In the first three sets, we use the same ATO system as in the previous section. In the last set, we use an ATO system with two products that have the same bill of materials but different demand functions. Otherwise noted, we use the base parameter set described in the previous section.

The first set of experiments investigates how the degree of demand seasonality and the degree of demand uncertainty affect the performance of the heuristic policy. Figure 3 reports $\overline{\text{gap}}$ under various values of η and cv. For each cv, the optimality gap increases as the degree of demand seasonality η increases. When demands do not fluctuate over time, the optimality gap is negligible despite the decreasing trend of demands. When $\eta = 0.8$, the expected demands at period 1 are more than nine times of those at period 3, which is a rather extreme scenario. For each given η, the optimality gap also increases in the coefficient of variation of demands. When demands are highly uncertain and significantly fluctuate over time, component inventories are frequently overstocked.

Figure 3 Impact of Demand Seasonality and Uncertainty

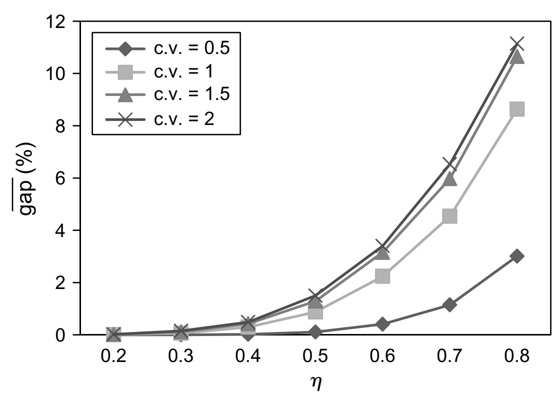
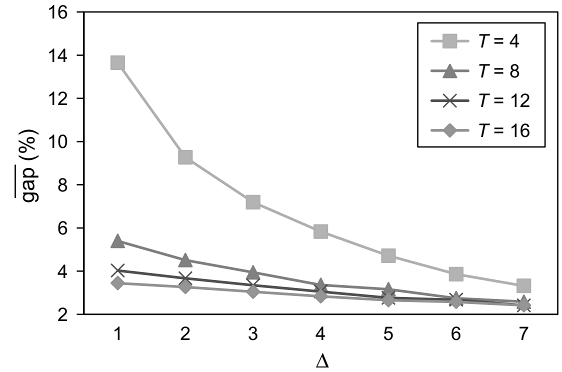


Figure 4 Impact of Planning Horizon Length and Demand Reduction Rate

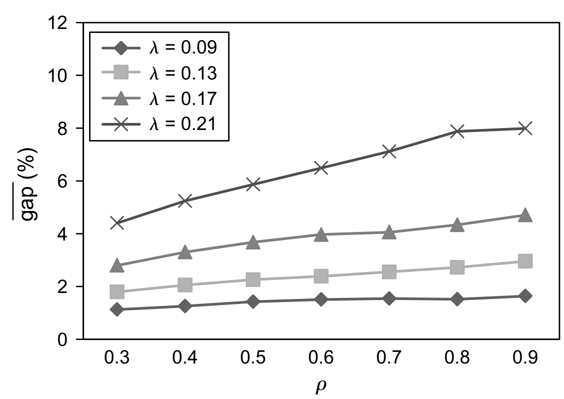


The cost of making suboptimal pricing and production decisions is greater when inventories are more frequently overstocked.

The second set of experiments examines the impacts of the length of the planning horizon and the demand reduction rate. In Figure 4, we report $\overline{\text{gap}}$ under various values of Δ and T. For each T, the optimality gap decreases as Δ increases. Recall that the value of Δ indicates the ratio between the expected demands at periods 1 and T after adjusting seasonality. Hence, demands decrease more rapidly over time when Δ is smaller. The heuristic policy performs better when demands decrease more gradually over time. For each Δ, the optimality gap gets smaller as T increases. When Δ is fixed, the demand reduction rate increases as T gets smaller. The performance of the heuristic policy depends on how rapidly demands decrease over time, not on the overall length of the planning horizon.

In Figure 5, we report the result of the third set of experiments, which shows the impact of holding and shortage costs. Recall that λ determines the ratio between operating and procurement costs, and ρ determines the ratio between shortage and holding costs. For each λ, the optimality gap increases in ρ but at a rather minor rate. The prices are adjusted from the list prices only when there are overstocked components,

Figure 5 Impact of Holding and Shortage Costs



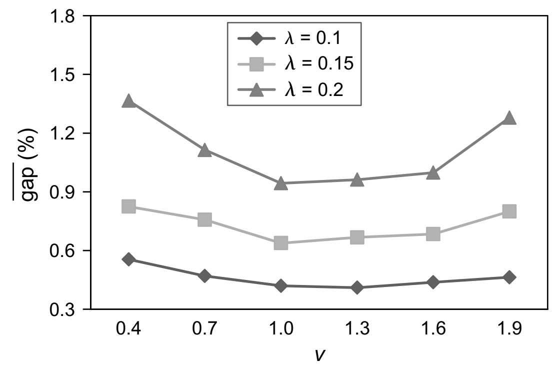


Figure 6 Impact of Price Sensitivity

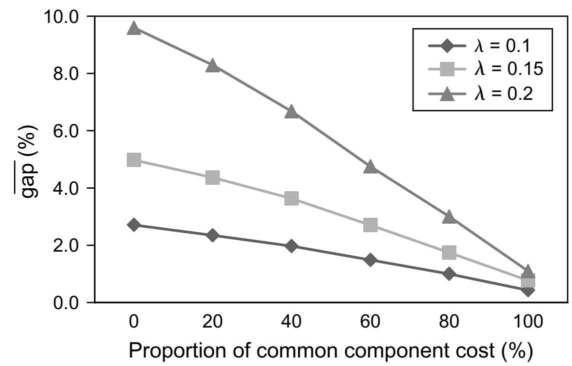


Figure 7 Impact of the Proportion of Common Component Cost

and doing so helps reduce the potential inventory holding costs. Hence, the performance of the heuristic policy is more sensitive to the holding costs than to the shortage costs. The optimality gap increases in λ, which implies that the cost of using the heuristic policy is greater when the operating costs are larger.

The last set of experiments investigates how the relative price sensitivities of the products that share common components affect the performance of the heuristic policy. For this experiment, we consider a system consisting of two products that have the same bill of materials. Product 1 of this system is the same as product 1 of the previous experiments. For product 2, we use the same setting as product 1 except that its expected demand function is $q_t^2(p_t^2) = \nu q_t^1(p_t^1)$. This demand model satisfies $dq_t^2(p_t^2)/dp_t^2 = \nu(dq_t^1(p_t^1)/dp_t^1)$, which implies that product 2 is more price-sensitive than product 1 if $\nu > 1$, and otherwise the opposite is true.

Figure 6 reports $\overline{\text{gap}}$ for various values of ν and λ. For each λ, the optimality gap is the smallest when $\nu = 1$, and it increases as ν diverges from 1. In other words, the heuristic policy performs the best when the two products have similar price sensitivities. If the two products have substantially different price sensitivities, only the price of the product with a large price sensitivity will be discounted when components are overstocked because doing so will increase the demands for the components with a minor loss in the profit margin. In contrast, if the price sensitivities of the two products are similar, similar levels of price discounts will be offered for the two products. In the latter case, the burden of excess inventories is equally distributed to the two products, whereas in the former case only one product takes all the burden of excess inventories.

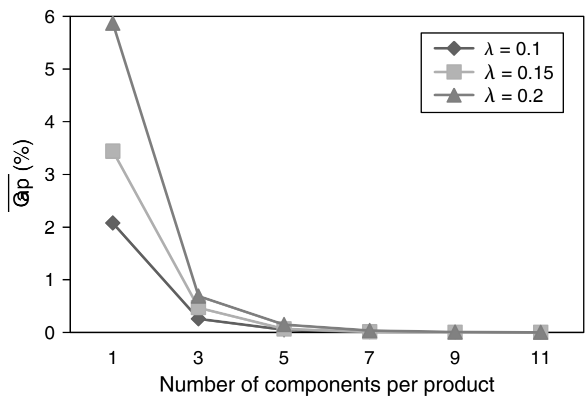
6.3. Impact of System Structure

In this section, we discuss how the structure of the ATO system affects the performance of the heuristic policy. We conduct three experiments. In all three experiments, we consider systems with symmetric products. All products have the same demand function and the shortage cost as product 1 used in previous experiments. The bill of materials and component costs are different in each of the three experiments. However, the total component procurement cost for one product is always set at 55, and the holding costs are given as $h_t = \lambda \times c_t$. We used the following parameters for the results of this section: $T = 10$, $\Delta = 0.5$, $\eta = 0.5$, $cv = 2$, $\lambda \in \{0.1, 0.15, 0.2\}$, and $\rho = 0.6$.

The first experiment investigates how the performance of the heuristic policy is affected by the cost of common components. For this experiment, we use the W-model discussed in §5. While maintaining the total component procurement cost of each product at 55, we change the procurement cost of the common component from 0 to 55. Figure 7 shows how the optimality gap changes as the proportion of the common component cost changes from 0% to 100%. When the proportion is 0%, the ATO system is essentially a MTS system, and when it is 100% the system is a distribution system. As shown in Figure 7, the optimality gap decreases as the proportion increases. Note that as the cost of the common component increases, the value of the ATO strategy, i.e., the value of postponement, increases. The heuristic policy performs better when the value of postponement is greater.

The second experiment examines how the interconnectivity of different products in the system affects the performance of the heuristic policy. For this test, we consider a system with 11 symmetric products and 11 symmetric components. While maintaining the total component cost for each product at 55, we change the number of components each product requires, and change the procurement cost of each component accordingly. If a product requires n different components, then the procurement cost of each component is $55/n$ and each component is used by n different products. The bill of materials is determined such that all products are symmetric. In this setting, as the number of components that each product uses increases, the number of different products that use the same component also increases. The number of

Figure 8 **Impact of Interconnectivity**

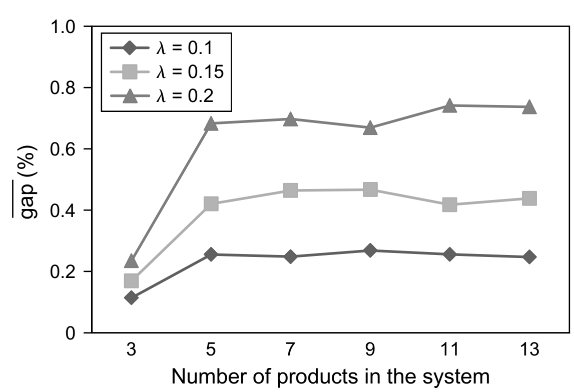


components for a product quantifies the degree of interconnectivity of different products in the system.

Figure 8 shows that the optimality gap decreases as the degree of interconnectivity increases. When the number of products that use a common component is large, the uncertainties residing in the demands for those products are pooled to a great degree, and thus the demand for the component has a small degree of uncertainty. The heuristic policy performs well when the risk-pooling effect in the ATO system is significant.

The last experiment investigates the impact of the size of the ATO system. For this experiment, we change the number of products in the system while maintaining the number of components that each product uses at 3. The number of products and components are always the same, and the procurement cost of each component is 55/3. Figure 9 shows how the optimality gap changes as the number of products increases. For each λ, the optimality gap is particularly small when there are three products, whereas there is no clear dependency between the number of products and the performance of the heuristic policy in other cases. When the number of products is three, all products in the system require the exact same set of components, and thus inventories of the three components are always perfectly synchronized. Except for this special case, the number of products, i.e., the size of the ATO system, does not affect the performance of the heuristic policy.

Figure 9 **Impact of the Number of Products in the System**



7. Conclusion

In this paper, we have studied the problem of jointly optimizing pricing and production decisions in an ATO system. Unlike in MTS systems, a state-dependent base-stock list-price policy is shown to be optimal. The optimal state-dependent base-stock levels and list prices may increase or decrease as demand backlogs increase, whereas demand backlogs always improve the optimal expected profit by providing additional flexibilities in component allocation. Due to the intractability of the optimal solution in general systems, we have developed a simple heuristic policy, which decouples inventory replenishment, pricing, and component allocation decisions in a coordinated way. We have shown that the heuristic policy is optimal when demands do not decrease over time, and via a numerical study we have shown that the heuristic policy also performs well when demands decrease and fluctuate over time.

We have investigated how the performance of the heuristic policy is affected by various factors. The heuristic policy performs well when components are shared by many products, when the common components are expensive, and when the products that share components have similar price sensitivities. The performance of the heuristic policy deteriorates as the frequency of inventory overstocking increases. The performance is not strongly correlated with the size of the ATO system, the length of planning horizon, or shortage costs.

For the W-model, we have provided a complete characterization of the heuristic pricing policy. The increase in the inventory level of a component increases the expected demands for the products that use this component, whereas it can reduce the expected demands for other products. When a component that is shared by multiple products is overstocked, pricing decisions on these products are primarily dependent on the inventory levels of other required components. When the firm can increase the demands for several such products, a deeper price discount should be offered for a product whose dedicated components have larger holding costs, and smaller procurement costs in the upcoming period.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/msom.2014.0492.

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Appendix. Summary of Notation

Demand and cost parameters

$c_t \in \mathbb{R}^I$: unit procurement costs
$a_t \in \mathbb{R}^J$: unit assembly costs
$h_t \in \mathbb{R}^I$: unit holding costs
$b_t \in \mathbb{R}^J$: unit shortage costs
$D_t(p_t) = \epsilon_t q_t(p_t) + \xi_t \in \mathbb{R}^J$: demands for products
$p_t(q_t)$: inverse demand function
$B = [B^{i,j}]$: bill of materials

State and decision variables

$x_t \in \mathbb{R}^I$: inventory levels before replenishment
$y_t \in \mathbb{R}^I$: inventory levels after replenishment
$s_t \in \mathbb{R}^J$: demand backlogs
$p_t \in \mathbb{R}^J$: sales prices
$q_t \in \mathbb{R}^J$: expected demands
$z_t \in \mathbb{R}^J$: production quantities

Optimal solution and value functions

$v_t(x_t, s_t)$: value-to-go function at the beginning of period t
$y_t^*(s_t)$: optimal order-up-to levels
$g_t(y_t, d_t)$: value-to-go function at the end of period t
$q_t^*(s_t)$: expected demands under list prices

Other notations

$\bar{v}_t, \bar{g}_t(y_t, d_t)$: value functions of upper-bound problem
$\hat{v}_t, \hat{g}_t(y_t, d_t)$: value functions of myopic problem
$\hat{y}_t$: order-up-to levels for heuristic policy
$\hat{q}_t$: expected demands under heuristic list prices
$H_t(q_t) = p_t(q_t)' q_t - (a_t' - h_t' B + c_{t+1}' B) q_t$
$\tilde{q}_t(e) = \arg\max_{Bq \le Be} H_t(q_t)$
$q_{\mathrm{A}} = \arg\max_q H_t(q_t)$
$q_{\mathrm{B}}^j(e^k) = \arg\max_{q^j \mid q^k = e^k} H_t(q_t)$
$q_{\mathrm{C}}(e^3) = \arg\max_{q \mid q^1 + q^2 = e^3} H_t(q_t)$

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