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Robust Portfolio Choice with Learning in the Framework of Regret: Single-Period Case

Andrew E. B. Lim
NUS Business School, National University of Singapore, Singapore 119245; and Department of Industrial Engineering and Operations Research, University of California, Berkeley, Berkeley, California 94720, lim@ieor.berkeley.edu

J. George Shanthikumar
Department of Operations Management, Krannert School of Management, Purdue University, West Lafayette, Indiana 47907, jshanthi@purdue.edu

Gah-Yi Vahn
London Business School, Regent's Park, London NW1 4SA, United Kingdom, gvahn@london.edu

In this paper, we formulate a single-period portfolio choice problem with parameter uncertainty in the framework of relative regret. Relative regret evaluates a portfolio by comparing its return to a family of benchmarks, where the benchmarks are the wealths of fictitious investors who invest optimally given knowledge of the model parameters, and is a natural objective when there is concern about parameter uncertainty or model ambiguity. The optimal relative regret portfolio is the one that performs well in relation to all the benchmarks over the family of possible parameter values. We analyze this problem using convex duality and show that it is equivalent to a Bayesian problem, where the Lagrange multipliers play the role of the prior distribution, and the learning model involves Bayesian updating of these Lagrange multipliers/prior. This Bayesian problem is unusual in that the prior distribution is endogenously chosen by solving the dual optimization problem for the Lagrange multipliers, and the objective function involves the family of benchmarks from the relative regret problem. These results show that regret is a natural means by which robust decision making and learning can be combined.

Key words: parameter uncertainty; ambiguity; model uncertainty; learning; regret; relative regret; competitive analysis; portfolio selection; Bayesian methods; objective-based loss functions; convex duality
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1. Introduction

Consider an investor living in a world where there are risky assets and a risk-free money market account. Log returns for the risky assets are independent and identically distributed (i.i.d.) normal, but the mean and variance are not known to the investor (though they are constant over the data window, which may be small). We consider a single-period portfolio choice problem where the agent is endowed with a finite sample of historical returns data and makes a one-shot allocation decision after observing the last data point. Parameter estimation is possible because data are generated i.i.d. from a known class of models with constant but unknown parameters, and concern for robustness is legitimate because the data window may be small and convergence of the learning model may be slow. We would like to understand how parameter estimation/learning and robust decision making can be combined in this setting.

Portfolio selection with parameter uncertainty is commonly formulated in a Bayesian framework where the investor makes an exogenous specification of the prior and maximizes expected utility of terminal wealth. One challenge with the Bayesian approach, which has substantial normative implications, is that the solution can be very sensitive to the specification of the prior.[1] In particular, "relatively small" changes in the mean of the prior (for the expected return of a log-normal distribution) can translate into a large deterioration in performance if the prior variance is small or even moderate, whereas a uniform prior (the "obvious" choice when the investor has no prior information) gives the solution of the classical Merton/Markowitz problem with the sample mean of realized returns plugged in for the true expected return, and it is well known that this strategy performs badly (Brandt 2010, Rogers 2001). More generally, the posterior distribution for the mean return

[1] Of course, this is a desirable feature if the investor has strong prior views that he wishes to incorporate in the optimization problem.

converges at the same rate as the sample mean, so a poor choice takes a long time to correct itself.

In this paper, we address the problem of portfolio choice with parameter uncertainty by adopting a relative regret objective. The essential feature of relative regret is that an investor's allocation is assessed by comparing his or her wealth to a family of benchmarks, where benchmarks are wealths of fictitious investors who behave optimally given knowledge of the parameter value, and the goal is to maximize the worst-case *relative* performance with respect to this family of benchmarks. Early results on this notion can be found in Blackwell (1956), Hannan (1957), and Savage (1951), whereas more recent analysis of problems with this objective include Bergemann and Schlag (2011), Lim et al. (2011), Lobel and Perakis (2010), and Perakis and Roels (2008). Also related is the work on universal portfolios (Cover 1991), though its focus is the asymptotic regret of certain online policies, as opposed to finite horizon results, which are the focus of this paper (see also Besbes and Zeevi 2009, Cesa-Bianchi and Lugosi 2006, Foster et al. 1999). More generally, though portfolio choice with parameter uncertainty and model ambiguity has attracted substantial attention in recent years (see, for example, Garlappi et al. 2007, Goldfarb and Iyengar 2003, Gundel 2005, Maenhout 2004, Schied 2005, Tutuncu and Koenig 2004), most of this literature adopts an absolute worst case, as opposed to a relative performance/regret, objective. One exception is Lim et al. (2011), which we discuss in more detail below.

A second feature of our model is that a finite sample of historical returns (assumed i.i.d. normal but with unknown mean and variance) is available to the investor when he/she makes a decision. Historical data have value because they can be used to learn model parameters, and our regret-optimizing investor endowed with a moderate amount of data lives somewhere between the highly nonstationary data-absent world, in which decisions need to be made with robustness as a primary concern, at the one extreme, and the stationary data-abundant world, in which parameters can be learned with little statistical uncertainty, at the other. The analysis of his or her problem is interesting to us because it gives insight into the trade-off between robust decision making, data-driven optimization, and learning for the world "in between." Our model puts no restrictions on how the investor uses the data, and one contribution of this paper is to characterize both the investment decision and the update rule for learning parameters that are optimal in this setting.

We analyze our regret problem using convex duality and show that the optimal learning model, *over all potential update rules*, involves Bayesian updating of the Lagrange multiplier that solves the dual problem, which plays the role of the prior. We also show that the optimal portfolio is the solution of an interesting (though nonstandard) Bayesian portfolio choice problem where the objective involves the family of benchmarks associated with the relative regret problem. In particular, the investor's decision is evaluated by comparing his or her wealth to that of each of the benchmark investors and averaging over the posterior. Roughly speaking, investors are rewarded for performing well relative to benchmarks that look plausible given the posterior; if the posterior is relatively flat (so all models are still plausible), then the investor seeks to do well relative to all the benchmarks. On the other hand, the investor will narrow his or her attention to a smaller set of benchmarks or models as the posterior becomes more concentrated. We mention here that probabilistic interpretations of Lagrange multipliers in the context of regret are also made in Perakis and Roels (2008) and Lim et al. (2011), though neither considers learning or updating because the decision maker has no data or has already incorporated it in the uncertainty set (Perakis and Roels 2008) or lives in a highly nonstationary world in which even recent data have no value (Lim et al. 2011). The problem of combining robustness with learning is also discussed in Hansen and Sargent (2005, 2007) and Knox (2002), though from a worst-case perspective.

Our relative regret objective can be interpreted as a loss function that evaluates an estimator in the context of the application and the associated decision-making goals. The idea that inference and decision making should go hand in hand goes back at least to Wald (1939, p. 302), who stated:

> The question as to how the form of the weight (i.e., loss) function $W(\theta, \omega)$ should be determined, is not a mathematical or statistical one. The statistician who wants to test certain hypotheses must first determine the relative importance of all possible errors, which will entirely depend on the special purpose of his investigation.

(See also McCloskey 1985.) Crudely stated, a lousy estimator (according to some loss function) may be perfectly acceptable if it can be combined with an investment decision that consistently delivers large profits, whereas an "optimal" estimator is of little value if decisions using the resulting estimates consistently perform poorly.

The outline of our paper is as follows. We formulate the single-period market model in §2 and introduce two relative regret objectives in §3. The first of these is more standard, whereas the second (which is the major focus of this paper) is original and can be interpreted as an objective-based loss function. We establish connections between our relative regret model and Bayesian problems in §4 using convex duality. In particular, we show that Lagrange multipliers in this duality relationship play the role of the prior in the Bayesian problem, and that the solution of the regret problem involves Bayesian updating of the Lagrange multiplier/prior characterized as the solution of the associated dual problem. Computational studies are provided in §5.

The single-period model in this paper can be extended to dynamic trading, which will be discussed elsewhere. An interesting feature of this extension is that the learning model involves a posterior that is tilted using the family of benchmarks.

2. Model

2.1. Market, Model, and Investor

2.1.1. Financial Market. Assume that there is a risk-free asset $S_0(t)$ and n risky assets $S_1(t), \ldots, S_n(t)$. The risk-free asset has a continuously compounded interest rate of r, and its price is given by $S_0(t) = e^{rt}$. We assume that the interest rate is known to the investor. The prices of the n risky assets evolve in continuous time according to

$$S_i(t) = S_i(0)\exp\{\mu_i t + \sigma_i W(t)\}, \quad i = 1, 2, \ldots, n, \quad (1)$$

where $W(t)$ is an n-dimensional standard Brownian motion, the scalar μ_i is the rate of return for stock i, and the n-dimensional row vector $\sigma_i = [\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in}] \in \mathbb{R}^{1\times n}$ is the volatility of this stock. We assume throughout this paper that μ_i and σ_i are constant. The column vector

$$\mu = [\mu_1, \mu_2, \ldots, \mu_n]' \in \mathbb{R}^{n\times 1}$$

is the vector of returns for all the risky assets, and the $n \times n$ matrix

$$\sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is the volatility. We assume, as is standard, that the nondegeneracy assumption holds: $Q = \sigma\sigma' \geq \delta I$ for some constant $\delta > 0$. The column vector of stock prices is denoted by

$$S(t) = [S_1(t), S_2(t), \ldots, S_n(t)]'.$$

2.1.2. Investor's Observations/Data. We assume in this paper that the parameters $H = (\sigma\sigma', \mu)$ for the stock price model (1) are constant, and that the investor knows the model family (1) but does not know the parameter values beyond the fact that they belong to some uncertainty set $\mathcal{H}$. The only assumption about the uncertainty set $\mathcal{H}$ is that it is compact. (For example, $\mathcal{H}$ might be a "confidence interval/region" associated with statistical point estimates of the parameters, subjective uncertainty regions specified by the investor around forecasted means, or a finite set of models that the investor wishes to consider, etc.) We also assume that the investor does not observe the stock prices continuously but samples the process at discrete times $t\delta$ for $t = 0, 1, 2, \ldots, T$ (i.e., t indexes the number of sample points that have been seen by the investor, whereas $\delta > 0$ is the time interval between each observation). Equivalently, the investor is seeing a sequence of log returns $\mathcal{R}(1), \mathcal{R}(2), \ldots, \mathcal{R}(T)$, where

$$\mathcal{R}(t+1) = \begin{bmatrix} \mathcal{R}_1(t+1) \\ \vdots \\ \mathcal{R}_n(t+1) \end{bmatrix}, \quad t = 0, 1, 2, \ldots, T,$$

is an n-dimensional random vector with entries being the log returns for the individual stocks over time period $[t\delta, (t+1)\delta)$:

$$\begin{aligned} \mathcal{R}_j(t+1) &\triangleq \ln \frac{S_j((t+1)\delta)}{S_j(t\delta)} \\ &= \mu_i\delta + \sigma_i[W((t+1)\delta) - W(t\delta)] \\ &= \mu_i\delta + \sqrt{\delta}\sigma_i Z(t+1), \end{aligned}$$

where

$$Z(t+1) \triangleq \frac{W((t+1)\delta) - W(t\delta)}{\sqrt{\delta}}.$$

Observe that $Z(1), Z(2), \ldots, Z(T)$ is a sequence of n-dimensional i.i.d. standard normal random variables. Clearly, $\mathcal{R}(t+1)$ is multivariate normal:

$$\mathcal{R}(t+1) \sim N(\delta\mu, \delta Q), \quad (2)$$

with mean $\mu\delta$ and covariance matrix δQ.

2.1.3. Investment Decision. Consider an investor with wealth x. The investor (correctly) assumes that prices evolve in continuous time according to a model of the form (1), but does not know the parameter values (σ, μ). Instead, the investor has observed T historical returns over time periods of size δ, $\mathcal{R}(1), \mathcal{R}(2), \ldots, \mathcal{R}(T)$ (or equivalently, has seen stock prices $S(0), S(\delta), \ldots, S(T\delta)$), and wishes to make an investment decision over the time interval $[T\delta, (T+1)\delta)$ following the realization of the last observation. The investor can use knowledge of the T historical returns to make his/her decision. More formally, let

$$\begin{aligned} \mathcal{G}_T &\triangleq \sigma\{\mathcal{R}(1), \mathcal{R}(2), \ldots, \mathcal{R}(T)\} \\ &= \sigma\{S(0), S(\delta), \ldots, S(T\delta)\} = \sigma\{\underline{S}(t\delta)\} \end{aligned}$$

denote the σ-algebra generated by relative returns $\mathcal{R}(1), \mathcal{R}(2), \dots, \mathcal{R}(T)$ (equivalently, stock prices $S(0)$, $S(\delta), S(2\delta), \dots, S(T\delta)$). The investor's decision $\pi = [\pi_1, \pi_2, \dots, \pi_n]'$ for the interval $[T\delta, (T+1)\delta)$ is a $\mathcal{G}_T$-measurable random vector. We assume that π_i is the proportion of wealth invested in stock i, whereas $1 - \pi'\mathbf{1}$ is the proportion invested in the bond. Under this assumption, it follows that the investor's wealth at time $(T+1)\delta$ (after the realization of return $\mathcal{R}(T+1)$) is given by

$$\begin{aligned} x_\pi^H(T+1) &= \sum_{i=1}^m x\pi_i e^{\mathcal{R}_i(T+1)} + x\left(1 - \sum_{i=1}^n \pi_i\right) e^{r\delta} \\ &= x\left\{\sum_{i=1}^n \pi_i e^{\delta\mu_i + \sqrt{\delta}\sigma_i Z(T+1)} + \left(1 - \sum_{i=1}^n \pi_i\right) e^{r\delta}\right\}. \end{aligned} \tag{3}$$

For the sake of mathematical convenience, let us assume that δ is small. It can then be shown that

$$x_\pi^H(T+1) \simeq x \exp\left\{\delta\left[r + b'\pi - \tfrac{1}{2}\pi' Q\pi\right] + \sqrt{\delta}\pi'\sigma Z(T+1)\right\},$$

where $b = [b_1, b_2, \dots, b_n]'$ is an n-dimensional vector of real numbers $b_i \triangleq \mu_i + (1/2)\sigma_i\sigma_i' - r$, and $Z(T+1)$ is a standard n-dimensional normal random variable that is independent of returns $\mathcal{R}(1), \mathcal{R}(2), \dots, \mathcal{R}(T)$ (or equivalently, of realized stock prices $S(0), S(\delta), \dots, S(T\delta)$).[2] With this in mind, we shall assume that the investor's wealth is defined by

$$x_\pi^H(T+1) = x \exp\left\{\delta\left[r + b'\pi - \tfrac{1}{2}\pi' Q\pi\right] + \sqrt{\delta}\pi'\sigma Z(T+1)\right\} \tag{4}$$

for the remainder of this paper. Alternatively, (4) is the wealth at $T+1$ if there is continuous rebalancing (by a computer, say) between times T and $T+1$ so as to maintain the proportions π of wealth in each stock, with the understanding that the investor does not change π between $T\delta$ and $(T+1)\delta$ once it has been specified at $T\delta$.

Finally, because we will be dealing directly with the wealth equation (4) rather than the log returns model (2), it is more convenient for us to talk about uncertainty in $H = (Q, b)$ instead of uncertainty in (Q, μ). In particular, we assume that the investor does not know (Q, b) beyond the fact that it lies in some compact uncertainty set $\mathcal{H}$.

2.2. Prior Distributions in Bayesian Models

When there is parameter uncertainty, it is common to adopt a Bayesian framework. In this section, we present an example that shows that the solution of the Bayesian problem is sensitive to the prior distribution. Sensitivity to the prior can be of concern if specification of the prior distribution is difficult (e.g., it is often difficult to translate a particular qualitative prior view into a joint distribution, particularly if there are many uncertain variables).

[2] This follows from the so-called log-linear approximation of the wealth equation (3), which becomes exact when $\delta \downarrow 0$ (see Campbell and Viciera 2002 for more details).

Table 1 **Summary of Priors for Bayesian Investors**

	Investor 1	Investor 2	Investor 3
Prior mean m	0.15	0.2	0.25
Prior precision τ	25	25	25

Suppose there is a single stock with i.i.d. log-normal returns described by (2) and parameters $\mu = 20\%$ and $\sigma = 20\%$. We assume that σ is known to all investors but μ is not. Consider three Bayesian investors. We assume that each investor knows $\sigma = 20\%$, but has a different (normal) prior on the unknown mean μ. These are summarized in Table 1. Observe that each of the priors has a different mean m but the same precision τ (recall that precision $\tau = (\text{variance})^{-1}$). A precision of 25 is the same as a standard deviation of 20%. Observe that the mean of Investor 2's prior $m = 0.2$ equals the mean μ of the distribution generating the returns.

We simulated data consisting of $n = 10$ years of annual returns $\mathcal{R}(1), \dots, \mathcal{R}(10)$ using the "true model" $(\mu, \sigma) = (0.2, 0.2)$ and updated the priors of each of the investors using Bayes' rule. It is well known that posteriors are normal with mean and precision

$$m' = \frac{\tau m + (n/\sigma^2)\bar{\mu}_n}{\tau + n/\sigma^2}, \qquad \tau' = \tau + \frac{n}{\sigma^2},$$

where $\bar{\mu}_n = [\mathcal{R}(1) + \cdots + \mathcal{R}(n)]/n$ is the sample mean of the historical returns. Our historical sample mean took the value $\bar{\mu}_n = 0.2126$ (which is relative close to the actual value $\mu = 0.2$). Each Bayesian investor then solved the single-period asset allocation problem

$$\begin{cases} \max_\pi (1/\gamma)\mathbb{E}[x(1)^\gamma] \\ \text{subject to} \\ x(1) = x(0)\exp\left\{\left[r + (\mu - r)\pi - \tfrac{1}{2}\pi^2\sigma^2\right] + \pi\sigma Z(T+1)\right\} \\ x(0) = 1, \quad \text{prior on } \mu \sim N(m', \tau') \end{cases}$$

using their updated parameters. It can be shown that

$$\pi = \left[1 - \gamma\left(1 + \frac{1}{\sigma^2\tau'}\right)\right]^{-1} \frac{m' - r}{\sigma^2}$$

is the optimal portfolio for the Bayesian investors (with (m', τ') as above). For the historical returns we generated, we obtained

$$\pi_1 = 22.954, \qquad \pi_2 = 23.62, \qquad \pi_3 = 24.3$$

for Investors 1, 2, and 3, respectively. It is interesting to compare this to the optimal portfolio $\psi = (1/(1-\gamma))(\mu - r)/\sigma^2$ of a fictitious investor with constant relative risk aversion (CRRA) utility who knows the model parameters (μ, σ) and who solves

$$\begin{cases} \max\limits_{\psi} (1/\gamma)\mathbb{E}[y(1)^\gamma] \\ \text{subject to} \\ y(1) = y(0)\exp\left\{\left[r + (\mu - r)\psi - \frac{1}{2}\psi^2\sigma^2\right] + \psi\sigma Z(T+1)\right\} \\ y(0) = 1. \end{cases}$$

For our model parameters, $\psi = 23.4$.

Observe that if prior precision τ is set to 0, the commonly accepted default when the investor has no information about μ,

$$\pi = \left[1 - \gamma\left(1 + \frac{1}{n}\right)\right]^{-1} \frac{\bar{\mu}_n - r}{\sigma^2},$$

which is essentially the portfolio ψ with the sample mean $\bar{\mu}_n$ substituted in place of the unknown mean μ and a small correction to the risk-aversion parameter. It is well known, however, that this plug-in approach does not perform well out of sample (Brandt 2010, Rogers 2001).

Consider now the following experiment. We generated 1,000,000 samples of annual returns using the model $(\mu, \sigma) = (20\%, 20\%)$. For each sample we recorded the end-of-year wealth $x_i(1)$ of each of the Bayesian investors π_i $(i = 1, 2, 3)$ as well as the wealth $y(1)$ of the "knowledgeable" investor who invests according to ψ. Figures 1–3 are histograms of log relative wealth, i.e., $\log[x_i(1)/y(1)]$, for each of the Bayesian investor's. The most striking observation is

Figure 1 **Histogram of Log Relative Wealth of Investor 1 Relative to the Knowledgeable Investor**

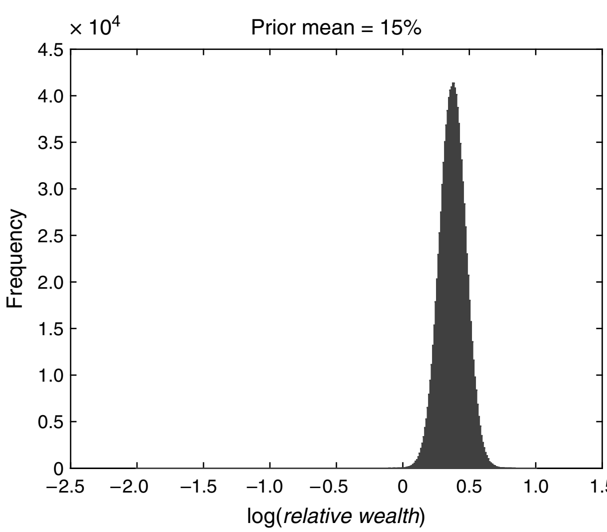
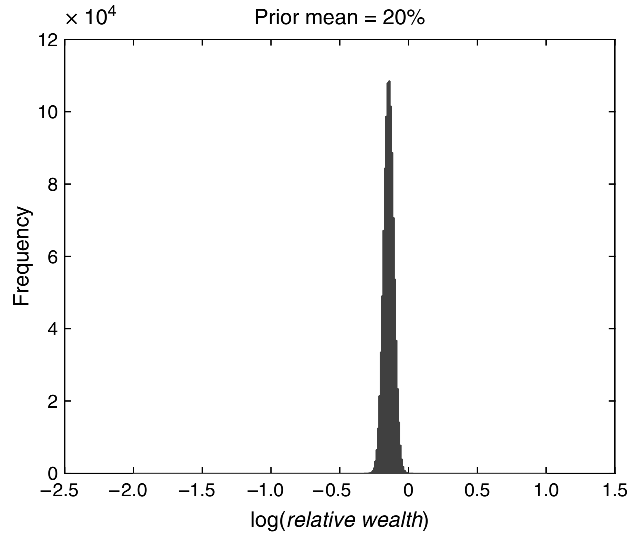


Figure 2 **Histogram of Log Relative Wealth of Investor 2 Relative to the Knowledgeable Investor**



the large difference between these three histograms given that the difference in prior specification is relatively small.[3]

2.3. Worst-Case Model

Worst-case models are commonly proposed when there is model uncertainty. A typical formulation of this problem is

$$\begin{cases} \max\limits_{\pi \in \mathcal{G}_T} \min\limits_{H \in \mathcal{H}} \mathbb{E}_H[(1/\gamma) x_\pi^H(\delta)^\gamma] \\ \text{subject to} \\ x_\pi^H(\delta) = x\exp\left\{\delta\left[r + b'\pi - \frac{1}{2}\pi' Q\pi\right] \right. \\ \qquad\qquad \left. + \sqrt{\delta}\pi'\sigma Z(T+1)\right\}, \end{cases} \tag{5}$$

for which the solution is[4]

$$\pi^* = \frac{1}{1-\gamma}[Q^*]^{-1} b^*, \tag{6}$$

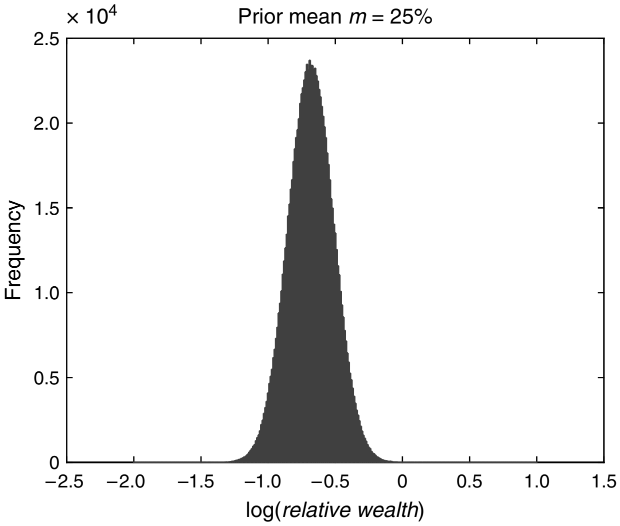
$$H^* = (Q^*, b^*) = \underset{\mu \geq 0,\, \mu(\mathcal{H}) = 1}{\arg\min}\ \mathbb{E}_\mu[b' Q^{-1} b]$$

$$= \underset{(Q,\, b) \in \mathcal{H}}{\arg\min}\, b' Q^{-1} b. \tag{7}$$

[3] Figures 1–3 may be puzzling to some in that Investor 1 outperforms Investors 2 and 3 relative to the benchmark, even though the mean of the prior chosen by Investor 2 coincides with the mean of the data-generating process (i.e., $\mu = 0.2$). This is because Figures 1–3 show relative performance *conditional on a particular realization of 10 years of data*, and by chance, this realization was such that Investor 1's portfolio outperformed those of the other two investors. For other data samples, the ordering will differ. The main point of this example is not that one particular investor outperforms the others, but rather that relatively small differences in the prior can substantially affect performance.

[4] In writing the equality in (7), we mean that the solution of the first optimization problem (over probability measures with support $\mathcal{H}$) is degenerate with mass 1 on the solution $(Q^*, b^*) \in \mathcal{H}$ of the second.

Figure 3 **Histogram of Log Relative Wealth for Investor 3 Relative to the Knowledgeable Investor**



The solution can be described as follows: (i) Find the model (Q^*, b^*) in $\mathscr{H}$ with the smallest Sharpe ratio (i.e., Equation (7)), and (ii) solve a standard Bayesian problem with a prior that puts all its mass on (Q^*, b^*) (i.e., Equation (6)).

The solution (6)–(7) is problematic on several grounds. First, investing according to a prior that puts all its mass on the model with the smallest Sharpe ratio seems overly pessimistic and is sensitive to the choice of uncertainty set. This feature is a consequence of the worst-case objective, which is only concerned about performance for the worst-case model (Q^*, b^*), but is unconcerned about underperforming when "better" models (e.g., the one with the largest Sharpe ratio) apply. Second, because the worst-case prior (7) is degenerate, then so too is the posterior; that is, the "worst-case" equilibrium portfolio (6) resolutely sticks to the "worst-case" model (Q^*, b^*) and ignores the data $\mathscr{R}(1), \mathscr{R}(2), \ldots, \mathscr{R}(T)$, even if they strongly suggest that returns are not being generated by (Q^*, b^*) but by some other model. In other words, *learning does not occur, even when it is possible.*

In the remainder of this paper, we consider relative regret as a framework for formulating portfolio selection problems with parameter uncertainty. In contrast to the standard worst-case approach, relative regret favors decisions that perform well in both pessimistic (low Sharpe ratio) and optimistic (large Sharpe ratio) scenarios and will use (rather than ignore) data to infer the model parameters so as to increase the likelihood that it performs reasonably well in all cases. That being said, one feature of our model is that there is no prior specification, and the update rule is not imposed. Rather, we are interested to understand the learning model that comes up as part of the optimal solution.

3. Relative Regret

In this section, we formulate two portfolio optimization problems within the setup described in §2 and analyze the solution of these problems. Both problems involve relative regret objectives. The first of these is the classical relative regret (see, for example, Terlizzese 2006), whereas the second is our own. A key feature of both problems is that the investor, though ignorant of the model parameters, has the opportunity to learn. As such, a major focus of our work in subsequent sections (particularly §4) is the characterization of the learning model associated with the optimal solution of the problems formulated in this section. We refer the reader to Lim et al. (2011) for a version of this paper where learning is not possible.

3.1. Relative Regret I: Standard Model

Consider the problem

$$\max_{\pi\in\mathscr{G}_T}\min_{H\in\mathscr{H}}\frac{\mathbb{E}_H U(x_\pi^H(\delta))}{\max_\psi \mathbb{E}_H U(y_\psi^H(\delta))}. \tag{8}$$

This objective can be understood as follows:

- The investor begins by proposing a decision rule $\pi\in\mathscr{G}_T$ or, equivalently, a measurable function $f\colon \mathbb{R}^{T+1}\to\mathbb{R}$, which maps the T observed returns $\mathscr{R}(1),\ldots,\mathscr{R}(T)$ to an investment position $\pi\equiv f(\mathscr{R}(1),\ldots,\mathscr{R}(T))$. Note that every $\mathscr{G}_T$-measurable random variable (r.v.) can be represented as a measurable function of $\mathscr{R}(1),\ldots,\mathscr{R}(T)$. We emphasize that π (equivalently, f) cannot depend explicitly on the parameters (Q, b) because they are not known to the investor.
- Once this decision rule $\pi\in\mathscr{G}_T$ has been revealed, nature chooses a parameter $H=(Q,b)$ from the set $\mathscr{H}$.
- For the chosen policy $\pi\in\mathscr{G}_T$ and model $H\in\mathscr{H}$, the investor's wealth at time $(T+1)\delta$ is given by (4), and the expected utility $\mathbb{E}_H U(x_\pi^H(\delta))$ in the numerator of (8) can be computed. In addition, the denominator is calculated by optimizing over ψ given knowledge of the model H chosen by nature:

$$\begin{cases} \mathbb{E}_H U(y^H(\delta)) = \max\limits_{\psi} \mathbb{E}_H U(y_\psi^H(\delta)) \\ \text{subject to} \\ y_\psi^H(\delta) = y(0)\exp\big\{\delta\big[r + b'\pi - \tfrac{1}{2}\psi' Q\psi\big] \\ \qquad\qquad + \sqrt{\delta}\psi'\sigma Z(T+1)\big\}. \end{cases} \tag{9}$$

The ratio of these two quantities is precisely the relative regret objective (8).

- The investor chooses the policy $\pi\equiv f(\cdot)$ and nature the model H to satisfy the equilibrium condition (8).

An axiomatic justification for this objective is given in Terlizzese (2006); see also Hayashi (2008).

3.1.1. CRRA Utility Function: $U(x) = (1/\eta)x^\eta$, $\eta<1$. A more explicit computation can be done for the model (8) if the utility function is assumed to be CRRA. More specifically, observing that $y(\delta)$ is a log-normal random variable given by (9), it follows that

$$\begin{aligned}
&\frac{1}{\eta}\mathbb{E}_H(y_\pi^H(\delta))^\eta \\
&= \frac{1}{\eta}y(0)^\eta \mathbb{E}_H \exp\left\{\eta\delta\left[r+b'\pi-\frac{1}{2}\psi'Q\psi\right]\right. \\
&\qquad\qquad \left. +\eta\sqrt{\delta}\psi'\sigma Z(T+1)\right\} \\
&= \frac{1}{\eta}y(0)^\eta \exp\left\{\eta\delta\left[r+b'\psi-\frac{1-\eta}{2}\psi'Q\psi\right]\right\} \\
&= \frac{1}{\eta}y(0)^\eta \exp\left\{\delta\eta\left[r-\frac{1-\eta}{2}\left(\psi-\frac{Q^{-1}b}{1-\eta}\right)'Q\left(\psi-\frac{Q^{-1}b}{1-\eta}\right)\right.\right. \\
&\qquad\qquad \left.\left. +\frac{1}{2(1-\eta)}b'Q^{-1}b\right]\right\} \qquad (10)
\end{aligned}$$

(where the first equality is just the moment-generating function of a normal r.v.). It now follows that the benchmark investor's optimal portfolio (the solution of (9) using (10)) is given by

$$\psi^H = \arg\max_\psi \frac{1}{\eta}\mathbb{E}_H U(y_\pi^H(\delta)) = \frac{1}{1-\eta}Q^{-1}b \qquad (11)$$

when the model is $H=(Q,b)$. Substituting ψ^H into the wealth equation in (9), it follows that the benchmark investor's optimal wealth is given by

$$\begin{aligned}
y^H(\delta) = y(0)\exp\left\{\delta\left[r+\frac{1-2\eta}{2(1-\eta)^2}b'Q^{-1}b\right]\right. \\
\left. +\frac{\sqrt{\delta}}{1-\eta}b'Q^{-1}\sigma Z(T+1)\right\}, \qquad (12)
\end{aligned}$$

and the denominator of (8) (from substituting (11) into (10)) is

$$\begin{aligned}
&\mathbb{E}_H U(y^H(\delta)) \\
&\quad = \frac{1}{\eta}y(0)^\eta \exp\left\{\delta\eta\left[r+\frac{1}{2(1-\eta)}b'Q^{-1}b\right]\right\}. \qquad (13)
\end{aligned}$$

On the other hand, for the portfolio $\pi \in \mathscr{G}_T$ and model $H=(Q,b)\in\mathscr{H}$, the investor's utility function (the numerator of (8)) satisfies

$$\mathbb{E}_H U(x_\pi^H(\delta)) = \mathbb{E}_H[\mathbb{E}_H\{U(x_\pi^H(\delta)) \mid \mathscr{G}_T\}].$$

Observing that (conditional on $\mathscr{G}_T$) the exponent of

$$U(x_\pi^H(\delta)) = \frac{x(0)^\eta}{\eta}e^{\delta\eta\left[r+b'\pi-\frac{1}{2}\pi'Q\pi\right]+\eta\sqrt{\delta}\pi'\sigma Z(T+1)}$$

is a standard normal r.v. with mean $\delta\eta(r+b'\pi-\frac{1}{2}\pi'Q\pi)$ and variance $\delta\eta^2\pi'Q\pi$, it follows from the formula for the moment-generating function of a normal r.v. that

$$\begin{aligned}
&\mathbb{E}_H[U(x_\pi^H(\delta)) \mid \mathscr{G}_T] \\
&\quad = \frac{1}{\eta}x(0)^\eta \exp\left\{\delta\eta\left[r+b'\pi-\frac{1-\eta}{2}\pi'Q\pi\right]\right\},
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbb{E}_H U(x_\pi^H(\delta)) \\
&\quad = \frac{1}{\eta}x(0)^\eta \mathbb{E}_H \exp\left\{\delta\eta\left[r+b'\pi-\frac{1-\eta}{2}\pi'Q\pi\right]\right\}. \qquad (14)
\end{aligned}$$

Substituting (13) and (14) into the relative regret objective in (8), we obtain

$$\begin{aligned}
\frac{\mathbb{E}_H U(x_\pi^H(\delta))}{\max_\psi \mathbb{E}_H U(y_\psi^H(\delta))} &= \frac{\mathbb{E}_H U(x_\pi^H(\delta))}{\mathbb{E}_H U(y^H(\delta))} \\
&= \mathbb{E}_H \exp\left\{\delta\eta\left[b'\pi-\frac{1-\eta}{2}\pi'Q\pi-\frac{1}{2}\frac{b'Q^{-1}b}{1-\eta}\right]\right\},
\end{aligned}$$

and it follows that (8) is equivalent to

$$\begin{aligned}
&\max_{\pi\in\mathscr{G}_T}\min_{H\in\mathscr{H}} \frac{\mathbb{E}_H U(x_\pi^H(\delta))}{\max_\psi \mathbb{E}_H U(y_\psi^H(\delta))} \\
&= \max_{\pi\in\mathscr{G}_T}\min_{H\in\mathscr{H}} \mathbb{E}_H \exp\left\{\delta\eta\left[b'\pi-\frac{1-\eta}{2}\pi'Q\pi-\frac{1}{2}\frac{b'Q^{-1}b}{1-\eta}\right]\right\}. \qquad (15)
\end{aligned}$$

It does not appear that an explicit expression for the equilibrium solution of (15) is possible. However, we show in §4 that an approximate solution, which becomes exact when $\delta \downarrow 0$, can be obtained.

3.2. Relative Regret II: Objective-Based Loss Function

In this section we introduce an alternative relative regret problem. We adopt the same family of models $\mathscr{H}$ as in §3.2, the same (approximate) wealth equation (4) for the investor, and the same definition (9) of the benchmark $y^H(\delta)$. The essential difference comes in the way that the investor's wealth $x_\pi^H(\delta)$ is compared to that of the benchmark investor, $y^H(\delta)$.

3.2.1. Benchmark Investor. As in (9), the benchmark investor solves a portfolio selection problem with full knowledge of the model parameters $H=(Q,b)$. More specifically, suppose that the benchmark investor has utility function $U^B(y)$ and that he/she solves the portfolio selection problem

$$\left\{\begin{aligned}
&\mathbb{E}_H U^B(y^H(\delta)) \equiv \max_\psi \mathbb{E}_H U^B(y_\psi^H(\delta)) \\
&\text{subject to} \\
&y_\psi^H(\delta) = y(0)\exp\{\delta[r+b'\psi-\tfrac{1}{2}\psi'Q\psi] \\
&\qquad\qquad +\sqrt{\delta}\psi'\sigma Z(T+1)\},
\end{aligned}\right. \qquad (16)$$

where as before, ψ^H denotes the optimal solution of this problem, and $y^H(\delta)$ (a random variable) is the associated optimal wealth.

3.2.2. Relative Regret Problem. Consider the following relative regret problem:

$$\begin{cases} \max_{\pi \in \mathcal{G}_T} \min_{H \in \mathcal{H}} \mathbb{E}_H \left[U\left(\dfrac{x_\pi^H(\delta)}{y^H(\delta)} \right) \right] \\ \text{subject to} \\ x_\pi^H(\delta) \text{ is given by (4)}, \\ y^H(\delta) \text{ is defined via (16).} \end{cases} \tag{17}$$

The key difference between (17) and the relative regret problem (8) is the way that $x_\pi^H(\delta)$ and $y^H(\delta)$ are compared. In this regard, it is worth noting that the "comparison function" $U(z)$ in the objective and the utility function $U^B(y)$ need not be the same.

The model (17) can be described as follows.

- The investor begins by declaring a policy $\pi \in \mathcal{G}_T$ (or equivalently, by specifying some measurable function $f: \mathbb{R}^{T+1} \to \mathbb{R}$). Nature, endowed with knowledge of this policy π (equivalently, function f), follows up by choosing a model $H \in \mathcal{H}$. Asset returns $\mathcal{R}(1), \ldots, \mathcal{R}(T)$ are then generated under nature's model H.

At time T, the investor adopts the position $\pi = f(\mathcal{R}(1), \ldots, \mathcal{R}(T))$ (after seeing all the returns), whereas nature invests according to ψ^H (the optimal solution of (16) corresponding to H).

- Once the positions π and ψ^H have been taken, one more return realization $\mathcal{R}(T+1)$ is generated under nature's model H and the wealths of the investor $x_\pi^H(\delta)$ (given by (4)) and the benchmark $y^H(\delta)$ (given by (16)) are realized. Conditional on $\pi = f(\mathcal{R}(1), \mathcal{R}(2), \ldots, \mathcal{R}(T))$ and H, the distribution of the ratio $x_\pi^H(\delta)/y^H(\delta)$ is fully characterized, and we can calculate

$$\mathbb{E}_H U\left(\frac{x_\pi^H(\delta)}{y^H(\delta)} \right).$$

We use this objective to compare $x_\pi^H(\delta)$ and $y^H(\delta)$.

- The investor and nature choose π and H, respectively, to satisfy the equilibrium condition associated with (17).

We reiterate that the essential difference between the models (8) and (17) is the way that $x_\pi^H(\delta)$ and $y^H(\delta)$ are compared. In (8) they are compared by evaluating the ratio of their expected utilities, whereas in (17) we compute the expectation of the comparison function $U(z)$ applied to the ratio $x_\pi^H(\delta)/y^H(\delta)$.

Several additional comments are worth making. First, unlike the objective (8), we are not aware of an axiomatic foundation for (17), though we believe that this is an issue worth pursuing. On the other hand, it will be shown that the solution of (8) is a limiting case of (17) when the utility/comparison functions are CRRA/power type. Another advantage of (17) is that it gives us some degree of control over the "risk aversion" of the benchmark investor (through the choice of $U^B(y)$) as well as the distance measure between $x_\pi^H(\delta)$ and $y^H(\delta)$ through the choice of $U(z)$. Finally, there is a natural extension of (17) to multiperiod problems that is relatively easy to analyze (this will be done elsewhere). The same cannot be said about (8).

Objective-Based Loss Function. Suppose we have a set of models $\mathcal{M}(\mathcal{H}) = \{M(H), H \in \mathcal{H}\}$ parameterized by $H \in \mathcal{H}$ and data $(\mathcal{R}_1, \ldots, \mathcal{R}_T)$ generated from one of the models $M(H^*)$ in this family. A classical problem in statistics is to estimate the unknown parameter H^*. An estimator is a function $\hat{H} = g(\mathcal{R}_1, \ldots, \mathcal{R}_T)$ of data to the parameter set $\mathcal{H}$, and the quality of the estimator (a random variable) is evaluated using a loss function $\mathcal{L}(g(\mathcal{R}_1, \ldots, \mathcal{R}_T), H)$ and some criterion; for example, the min-max criterion is

$$\min_g \max_{H \in \mathcal{H}} \mathbb{E}_H \left[\mathcal{L}(g(\mathcal{R}_1, \ldots, \mathcal{R}_T), H) \right]. \tag{18}$$

Among other things, the loss function $\mathcal{L}$ represents our attitudes concerning the distribution of the error $g(\mathcal{R}_1, \ldots, \mathcal{R}_T) - H$.

The objective in (17) can be interpreted as a min-max criterion for an *objective-based loss function* $U(x_\pi^H(\delta), y^H(\delta)) \equiv U(x_\pi^H(\delta)/y^H(\delta))$, which we now describe. A fundamental difference is that our primary concern is performance of a *decision* $\pi = f(\mathcal{R}_1, \ldots, \mathcal{R}_T)$, which may involve an estimate of the parameters somewhere in its definition, rather than the quality of the estimator itself (though of course, both issues are related).

The components of our model have analogs with (18) as follows:

- The optimal (random) wealth $y^H(\delta)$ for every given parameter value in (17) is analogous to the parameter value H in the loss function $\mathcal{L}(g(\mathcal{R}_1, \ldots, \mathcal{R}_T), H)$. The benchmark in the parameter estimation problem (18) is H, whereas our benchmark is the optimal wealth for the data-generating parameter.
- Investment decisions $\pi = f(\mathcal{R}_1, \ldots, \mathcal{R}_T)$ are analogous to the estimator $\hat{H} = g(\mathcal{R}_1, \ldots, \mathcal{R}_T)$ in (18).
- Whereas the objective in (18) compares closeness of an estimator $g(\mathcal{R}_1, \ldots, \mathcal{R}_T)$ to the parameter H that generates the data, we compare the performance $x_\pi^H(\delta)$ of the mapping $\pi = f(\mathcal{R}_1, \ldots, \mathcal{R}_T)$ to the optimal performance $y^H(\delta)$ for each parameter.
- We are concerned with the distribution of investor performance $x_\pi^H(\delta)$ relative to $y^H(\delta)$, represented by the comparison function $U(x, y) = (1/\gamma)(x/y)^\gamma$, which is analogous to concern about the distribution of $g(\mathcal{R}_1, \ldots, \mathcal{R}_T) - H$ as expressed through the choice of loss function in (18). In this regard, classical regret (8) is different because it compares average performance instead of performance for each realization.

3.2.3. Power Utility and Comparison Functions: $U^B(y) = (1/\eta)y^\eta$ and $U(z) = (1/\gamma)z^\gamma$. In this section we consider the relative regret problem (17) under the assumption that the benchmark investor's utility function as well as the comparison function are power-type functions: $U^B(y) = (1/\eta)y^\eta$ $(\eta < 1)$ and $U(z) = (1/\gamma)z^\gamma$ $(\gamma < 1)$. As shown in (11) and (12), the benchmark investor's problem (16) with a CRRA utility has an explicit solution

$$\psi^H = \frac{1}{1-\eta}Q^{-1}b, \tag{19}$$

and

$$y^H(\delta) = y(0)\exp\left\{\delta\left[r + \frac{1-2\eta}{2(1-\eta)^2}b'Q^{-1}b\right] + \frac{\sqrt{\delta}}{1-\eta}b'Q^{-1}\sigma Z(T+1)\right\} \tag{20}$$

is the associated benchmark investor's wealth. It now follows that the normalized wealth process $z_\pi^H(\delta) = x_\pi^H(\delta)/y^H(\delta)$ satisfies

$$\frac{x_\pi^H(\delta)}{y^H(\delta)} = \frac{x(0)}{y(0)}\exp\left\{\delta\left[b'\pi - \frac{1}{2}\pi'Q\pi - \frac{1-2\eta}{2(1-\eta)^2}b'Q^{-1}b\right] + \sqrt{\delta}\left[\pi - \frac{1}{1-\eta}Q^{-1}b\right]'\sigma Z(T+1)\right\},$$

so the relative regret problem (17) becomes

$$\begin{cases} \max\limits_{\pi\in\mathcal{G}_T}\min\limits_{H\in\mathcal{H}}\mathbb{E}_H[U(z_\pi^H(\delta))] \\ \text{subject to} \\ z_\pi^H(\delta) = z(0)\exp\left\{\delta\left[b'\pi - \frac{1}{2}\pi'Q\pi - \frac{1-2\eta}{2(1-\eta)^2}b'Q^{-1}b\right] + \sqrt{\delta}\left[\pi - \frac{1}{1-\eta}Q^{-1}b\right]'\sigma Z(T+1)\right\}, \end{cases} \tag{21}$$

where we have substituted $z_\pi^H(\delta)$ for $x_\pi^H(\delta)/y^H(\delta)$. Choosing $U(z) = (1/\gamma)z^\gamma$ as the comparison function in (21), it follows from the log normality of $z_\pi^H(\delta)$ (conditional on $\mathcal{G}_T$ and H) that the conditional expectation

$$\begin{aligned} &\mathbb{E}_H[U(z_\pi^H(\delta)) \mid \mathcal{G}_T] \\ &= \frac{1}{\gamma}\mathbb{E}_H[z_\pi^H(\delta)^\gamma \mid \mathcal{G}_T] \\ &= \frac{z(0)^\gamma}{\gamma}\mathbb{E}_H\left[\exp\left\{\delta\gamma\left[b'\pi - \frac{1}{2}\pi'Q\pi - \frac{1-2\eta}{2(1-\eta)^2}b'Q^{-1}b\right] + \gamma\sqrt{\delta}\left[\pi - \frac{1}{1-\eta}Q^{-1}b\right]'\sigma Z(T+1)\right\} \,\Big|\, \mathcal{G}_T\right] \\ &= \frac{z(0)^\gamma}{\gamma}\exp\left\{\delta\gamma\left[\frac{1-\eta-\gamma}{1-\eta}b'\pi - \frac{1-\gamma}{2}\pi'Q\pi - \frac{1-2\eta-\gamma}{2(1-\eta)^2}b'Q^{-1}b\right]\right\}. \end{aligned}$$

It now follows that (21) is equivalent to

$$\max_{\pi\in\mathcal{G}_T}\min_{H\in\mathcal{H}}\frac{z(0)^\gamma}{\gamma}\mathbb{E}_H\left[\exp\left\{\delta\gamma\left[\frac{1-\eta-\gamma}{1-\eta}b'\pi - \frac{1-\gamma}{2}\pi'Q\pi - \frac{1-2\eta-\gamma}{2(1-\eta)^2}b'Q^{-1}b\right]\right\}\right]. \tag{22}$$

It is interesting to note the similarity between (15) and (22). We shall expand further on this in later sections.

4. Optimal Portfolio and Learning Model

We characterize the learning model and optimal portfolio for the relative regret problem with power-type utility and comparison functions. We also derive an approximate solution that is exact in the limit $\delta \downarrow 0$ and shows the explicit dependence of the optimal portfolio on data.

4.1. Convex Duality and Robust Learning

We begin by characterizing the learning model and optimal portfolio using results from convex analysis. For the purposes of accessibility, we derive our results for the case that the uncertainty set $\mathcal{H} = \{H_1, \ldots, H_m\} = \{(b_1, Q_1), \ldots, (b_m, Q_m)\}$ is finite and, for simplicity, assume that the comparison and utility functions are power-type functions (i.e., $U(z) = (1/\gamma)z^\gamma$ and $U^B(y) = (1/\eta)y^\eta$). The case of compact and uncountable uncertainty set $\mathcal{H}$ is handled using similar ideas but requires some heavier machinery and is proven in the appendix.

Observe first that when $\mathcal{H} = \{H_1, \ldots, H_m\} = \{(b_1, Q_1), \ldots, (b_m, Q_m)\}$ and utility and comparison functions are power type, that (17) is equivalent to

$$\begin{cases} \nu^* = \max\limits_{\pi,\kappa}\kappa \\ \text{subject to} \\ \mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H_i}(\delta)}{y^{H_i}(\delta)}\right)^\gamma\right] \ge \kappa, \quad i = 1, \ldots, m. \end{cases} \tag{23}$$

For every model, (22) gives

$$\mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H_i}(\delta)}{y^{H_i}(\delta)}\right)^\gamma\right] = \frac{z(0)^\gamma}{\gamma}\mathbb{E}_{H_i}\left[\exp\left\{\delta\gamma\left[\frac{1-\eta-\gamma}{1-\eta}b_i'\pi - \frac{1-\gamma}{2}\pi'Q_i\pi - \frac{1-2\eta-\gamma}{2(1-\eta)^2}b_i'Q_i^{-1}b_i\right]\right\}\right]. \tag{24}$$

Observe that $\mathbb{E}_{H_i}[(1/\gamma)(x_\pi^{H_i}(\delta)/y^{H_i}(\delta))^\gamma]$ is concave in π whenever $\gamma < 0$, and (23) is a convex optimization problem in (π, κ) when this condition holds.

Let $\mu_i \ge 0$ denote the Lagrange multiplier for the ith constraint in (23). Clearly, if (π, κ) is feasible for (23), and $\mu_i \geq 0$ for $i = 1, \ldots, m$, then

$$\begin{aligned} L(\pi, \kappa, \mu) &= \kappa + \sum_{i=1}^{m} \mu_i \left\{ \mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H_i}(\delta)}{y^{H_i}(\delta)} \right)^{\gamma} \right] - \kappa \right\} \\ &= \kappa\left(1 - \sum_{i=1}^{m} \mu_i\right) + \sum_{i=1}^{m} \mu_i \mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H_i}(\delta)}{y^{H_i}(\delta)} \right)^{\gamma} \right] \\ &\geq \nu^*. \end{aligned}$$

Define the dual objective function

$$\begin{aligned} \psi(\mu) &= \max_{\pi \in \mathcal{G}_T, \kappa} L(\pi, \kappa, \mu) \\ &= \max_{\pi \in \mathcal{G}_T, \kappa} \left\{ \kappa\left(1 - \sum_{i=1}^{m} \mu_i\right) + \sum_{i=1}^{m} \mu_i \mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H_i}(\delta)}{y^{H_i}(\delta)} \right)^{\gamma} \right] \right\}. \end{aligned}$$

Clearly, $\psi(\mu)$ is finite if and only if $\sum_{i=1}^{m} \mu_i = 1$, under which it follows that

$$\psi(\mu) = \max_{\pi \in \mathcal{G}_T} \sum_{i=1}^{m} \mu_i \mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H_i}(\delta)}{y^{H_i}(\delta)} \right)^{\gamma} \right].$$

Observing that the Lagrange multipliers $\mu = [\mu_1, \ldots, \mu_m]'$ are all nonnegative and sum to 1, it follows that μ can be interpreted as a probability measure on the class of models $\mathcal{H} = \{H_1, \ldots, H_m\}$, and that the summation in the dual function is nothing but an expectation where the Lagrange multiplier μ plays the role of a prior distribution. Particularly,

$$\psi(\mu) = \max_{\pi \in \mathcal{G}_T} \mathbb{E}_{\mu}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right], \tag{25}$$

which is nothing but a Bayesian problem with prior μ.

All that remains is to relate the dual function $\psi(\mu)$ to the original optimization problem (23). The following result is an immediate consequence of Lagrangian duality (see Theorem 1 of Luenberger 1968, p. 224). The general proof of this result (which extends the analysis to the case when $\mathcal{H}$ is possibly uncountable though compact) can be found in the appendix.

Proposition 1. *Suppose that $\mathcal{H}$ is compact, $\gamma < 0$, and $\eta < 1$. Let ν^* denote the optimal value of the relative regret problem* (17) *(or* (23)*), and let $\psi(\mu)$ be the value function (dual function) for the Bayesian problem* (25) *when the prior (Lagrange multiplier) is μ. Then dual optimization problem*

$$\psi(\mu^*) = \min_{\mu \geq 0, \, \mu(\mathcal{H})=1} \psi(\mu) \tag{26}$$

has a solution μ^*, and the optimal regret objective value satisfies $\nu^* = \psi(\mu^*)$. The optimal portfolio for the relative regret problem is the maximizer in (25) *under μ^*, namely,*

$$\pi^* = \arg\max_{\pi \in \mathcal{G}_T} \mathbb{E}_{\mu^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right]. \tag{27}$$

Proposition 1 tells us that the solution of the relative regret problem is also the solution of a Bayesian problem (27), where the optimal Lagrange multiplier μ^* characterized by (26) also plays the role of a prior distribution on the set of models $\mathcal{H}$.

This result allows us to see the dependence of the optimal portfolio π^* on the observations $\underline{\mathcal{R}}_T = \{\mathcal{R}(1), \ldots, \mathcal{R}(T)\}$, and hence to characterize the optimal learning model. Specifically, if μ^* is the optimal prior/Lagrange multiplier from (26), and $\mu_T^* = [\mu_T^*(1), \ldots, \mu_T^*(m)]$ is the posterior obtained from Bayesian updating conditional on data $\underline{\mathcal{R}}_T$, then the objective function can be written as

$$\begin{aligned} \mathbb{E}_{\mu^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right] &= \mathbb{E}_{\mu^*}\left[\mathbb{E}_{\mu^*}\left\{ \left(\frac{1}{\gamma} \frac{x(\delta)}{y(\delta)} \right)^{\gamma} \,\middle|\, \underline{\mathcal{R}}_T \right\} \right] \\ &= \mathbb{E}_{\mu^*}\left\{ \mathbb{E}_{\mu_T^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right] \right\}, \end{aligned}$$

where

$$\mathbb{E}_{\mu_T^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right] = \begin{cases} \displaystyle\sum_{i=1}^{m} \mu_T^*(i) \mathbb{E}_{H_i}\left[\frac{1}{\gamma}\left(\frac{x^{H_i}(\delta)}{y^{H_i}(\delta)} \right)^{\gamma} \right] & \mathcal{H} \text{ is finite}, \\ \displaystyle\int_{\mathcal{H}} \mathbb{E}_H\left[\frac{1}{\gamma}\left(\frac{x_\pi^{H}(\delta)}{y^{H}(\delta)} \right)^{\gamma} \right] \mu_T^*(dH) & \\ \qquad\qquad \mathcal{H} \text{ is uncountable and compact}. \end{cases}$$

This means that

$$\begin{aligned} \psi(\mu^*) &= \max_{\pi \in \mathcal{G}_T} \mathbb{E}_{\mu^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right] \\ &= \mathbb{E}_{\mu^*}\left\{ \max_{\pi \in \mathcal{G}_T} \mathbb{E}_{\mu_T^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right] \right\}, \end{aligned}$$

where the second equality follows from the $\mathcal{G}_T$-measurability of π. The dependence of the optimal portfolio π^* on the data can now be summarized as follows.

Proposition 2. *Suppose that $\mathcal{H}$ is compact, that $\gamma < 0$ and $\eta < 1$, and that the Lagrange multiplier/prior distribution μ^* on $\mathcal{H}$ is the solution of the dual problem* (26). *Let $\mu_T^* = [\mu_T^*(1), \ldots, \mu_T^*(m)]$ denote the posterior distribution on $\mathcal{H}$ obtained from Bayesian updating of μ^* given the observations $\underline{\mathcal{R}}_T = \{\mathcal{R}(1), \ldots, \mathcal{R}(T)\}$. Then*

$$\pi^* = \arg\max_{\pi \in \mathcal{G}_T} \mathbb{E}_{\mu_T^*}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)} \right)^{\gamma} \right] \tag{28}$$

is optimal for the relative regret problem (17).

We mention here that probabilistic interpretations of Lagrange multipliers in the context of regret are also made in Perakis and Roels (2008) and Lim et al. (2011), though neither considers learning or parameter estimation because the decision maker has no data or has already incorporated them in the uncertainty set (Perakis and Roels 2008) or lives in a highly nonstationary world in which even recent data have no value (Lim et al. 2011).

4.2. Optimal Portfolio: Approximate Solution

Although we have characterized the optimal portfolio (28) and optimal dual variable/prior (26), both appear difficult to compute. In this section we derive an approximate characterization of the optimal portfolio π^* and the associated Lagrange multiplier/prior μ^*, which becomes exact as $\delta \to 0$. The basic idea is to approximate the right-hand side of (28) so that the maximization over π can be solved explicitly and a closed-form expression for the dual function $\psi(\mu)$ can be obtained. The resulting dual problem is easier to solve than (26), and the expression for π^* shows us the explicit dependence of the (approximately) optimal solution on the posterior μ_T^*.

As a start, recall that

$$\mathbb{E}_H\left[\frac{1}{\gamma}\left(\frac{x_\pi^H(\delta)}{y^H(\delta)}\right)^\gamma\right] = \frac{1}{\gamma}\mathbb{E}_H\left[\exp\left\{\delta\gamma\left[\frac{1-\eta-\gamma}{1-\eta}b'\pi - \frac{1-\gamma}{2}\pi' Q\pi - \frac{1-2\eta-\gamma}{2(1-\eta)^2}b'Q^{-1}b\right]\right\}\right].$$

A Taylor expansion of $\exp\{\cdots\}$ in orders of δ gives

$$\begin{aligned}\frac{1}{\gamma}\mathbb{E}_H \exp\{\cdots\} &= \frac{1}{\gamma}\mathbb{E}_H\left[1+\delta\gamma\left\{\frac{1-\eta-\gamma}{1-\eta}b'\pi - \frac{1-\gamma}{2}\pi'Q\pi - \frac{1-2\eta-\gamma}{2(1-\eta)^2}b'Q^{-1}b\right\} + o(\delta)\right] \\ &= \mathbb{E}_H\left[\frac{1}{\gamma} + \delta\frac{1-\gamma}{2}\left\{2\frac{1-\eta-\gamma}{(1-\eta)(1-\gamma)}b'\pi - \pi'Q\pi - \frac{1-2\eta-\gamma}{(1-\eta)^2(1-\gamma)}b'Q^{-1}b\right\} + o(\delta)\right].\end{aligned}$$

The dual function (25) can now be written as

$$\begin{aligned}\psi(\mu) &= \mathbb{E}_\mu\left\{\max_{\pi\in\mathcal{G}_T}\mathbb{E}_{\mu_T}\left[\frac{1}{\gamma}\left(\frac{x(\delta)}{y(\delta)}\right)^\gamma\right]\right\} \\ &= \frac{1}{\gamma} + \delta\frac{1-\gamma}{2}\mathbb{E}_\mu\left\{\max_{\pi\in\mathcal{G}_T}\mathbb{E}_{\mu_T}\left[2\frac{1-\eta-\gamma}{(1-\eta)(1-\gamma)}b'\pi - \pi'Q\pi - \frac{1-2\eta-\gamma}{(1-\eta)^2(1-\gamma)}b'Q^{-1}b\right]\right\} + o(\delta).\end{aligned}$$

With this in mind, define

$$\begin{aligned}\bar\psi(\mu) &= \mathbb{E}_\mu\left[\max_{\pi\in\mathcal{G}_T}\mathbb{E}_{\mu_T}\left\{2\frac{1-\eta-\gamma}{(1-\eta)(1-\gamma)}b'\pi - \pi'Q\pi - \frac{1-2\eta-\gamma}{(1-\eta)^2(1-\gamma)}b'Q^{-1}b\right\}\right] \\ &= \mathbb{E}_\mu\left\{\max_{\pi\in\mathcal{G}_T}\left[2\frac{1-\eta-\gamma}{(1-\eta)(1-\gamma)}\mathbb{E}_{\mu_T}(b)'\pi - \pi'\mathbb{E}_{\mu_T}(Q)\pi\right]\right\} - \frac{1-2\eta-\gamma}{(1-\eta)^2(1-\gamma)}\mathbb{E}_\mu(b'Q^{-1}b),\end{aligned}$$

where

$$\mathbb{E}_{\mu_T}(Q) \equiv \int_{\mathcal{H}} Q\mu_T(dH), \quad \mathbb{E}_{\mu_T}(b) \equiv \int_{\mathcal{H}} b\mu_T(dH)$$

and

$$\mathbb{E}_\mu(b'Q^{-1}b) = \mathbb{E}_\mu[\mathbb{E}_{\mu_T}(b'Q^{-1}b)].$$

By completing the square, the above is equivalent to

$$\begin{aligned}\bar\psi(\mu) = &-\mathbb{E}_\mu\left\{\min_{\pi\in\mathcal{G}_T}[\pi - C_1[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b)]' \cdot\mathbb{E}_{\mu_T}(Q)[\pi - C_1[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b)]\right\} \\ &+ C_1^2\,\mathbb{E}_\mu\{\mathbb{E}_{\mu_T}(b)'[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b)\} - C_2\mathbb{E}_\mu(b'Q^{-1}b),\end{aligned}$$

where

$$C_1 = \frac{1-\gamma-\eta}{(1-\gamma)(1-\eta)} \quad\text{and}\quad C_2 = \frac{1-2\eta-\gamma}{(1-\eta)^2(1-\gamma)}.$$

Clearly,

$$\bar\pi^\mu \triangleq C_1[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b) \tag{29}$$

is the optimal portfolio, from which it follows that

$$\bar\psi(\mu) = C_1^2\,\mathbb{E}_\mu\{\mathbb{E}_{\mu_T}(b)'[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b)\} - C_2\mathbb{E}_\mu(b'Q^{-1}b). \tag{30}$$

Recalling that

$$\psi(\mu) = \frac{1}{\gamma} + \delta\frac{1-\gamma}{2}\bar\psi(\mu) + o(\delta),$$

(and noting that the coefficient of $\bar\psi(\mu)$ is positive) it follows that an approximate solution $\bar\mu^*$ of the dual problem (26) can be obtained by solving

$$\bar\mu^* = \operatorname*{arg\,min}_{\mu\ge 0,\,\mu(\mathcal{H})=1} \bar\psi(\mu), \tag{31}$$

whereas

$$\bar\pi^* = C_1[\mathbb{E}_{\bar\mu_T^*}(Q)]^{-1}\mathbb{E}_{\bar\mu_T^*}(b) \tag{32}$$

is an approximate solution for the regret problem.

A similar calculation/approximation can be carried out for (15). In this case, the (approximate) dual problem is again given by (31), where

$$\bar\psi(\mu) = \mathbb{E}_\mu\{\mathbb{E}_{\mu_T}(b)'[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b)\} - \mathbb{E}_\mu(b'Q^{-1}b).$$

The (approximate) optimal policy is given by

$$\bar\pi^* = \frac{1}{1-\eta}[\mathbb{E}_{\mu_T^*}(Q)]^{-1}\mathbb{E}_{\mu_T^*}(b).$$

Interestingly, this is the extreme case of $\gamma \to -\infty$ in (30) and (32), which coincides with a large aversion to missing the benchmark.

5. Example

In this section, we plot the approximate dual function (30) and solve for the approximate optimal prior and portfolio (31)–(32) for a simple example. We illustrate the effect of the number of data points on the dual function and the optimal prior.

For simplicity, we consider two assets and three models, and assume the covariance matrix is known. Thus our uncertainty set is $\mathcal{H} = \{(b_1, Q), \ldots, (b_3, Q)\}$, and we choose the following models for annualized mean returns

$$b_1 = \begin{pmatrix} 0.055 \\ 0.020 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0.035 \\ 0.010 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0.015 \\ 0.035 \end{pmatrix}$$

and the following annualized covariance matrix:

$$Q = \begin{pmatrix} 0.0087 & 0.0037 \\ 0.0037 & 0.0063 \end{pmatrix}.$$

We also set $\eta = -3$, $\gamma = -5$, $r = 0.03$, and observation frequency $\delta = 1/12$, which corresponds to monthly observations; T denotes the number of monthly return samples in the investor's data set.

To begin, observe that the approximate dual function (30) can be written as

$$\begin{aligned}
\bar{\psi}(\mu) &= C_1^2 \mathbb{E}_\mu\{\mathbb{E}_{\mu_T}(b)'[\mathbb{E}_{\mu_T}(Q)]^{-1}\mathbb{E}_{\mu_T}(b)\} - C_2\mathbb{E}_\mu(b'Q^{-1}b) \\
&= C_1^2 \mathbb{E}_\mu\left\{\left(\sum_{i=1}^{3}\mu_T(i)b_i\right)' Q^{-1}\left(\sum_{i=1}^{3}\mu_T(i)b_i\right)\right\} \\
&\quad - C_2\sum_{i=1}^{3}\mu_i(b_i'Q^{-1}b_i) \\
&= C_1^2 \sum_{i=1}^{3}\sum_{j=1}^{3} b_i'Q^{-1}b_j\mathbb{E}_\mu[\mu_T(i)\mu_T(j)] \\
&\quad - C_2\sum_{i=1}^{3}\mu_i(b_i'Q^{-1}b_i) \\
&= C_1^2 \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} b_i'Q^{-1}b_j\mu_k\mathbb{E}_\mu[\mu_T(i)\mu_T(j) \mid H_k \text{ is true}] \\
&\quad - C_2\sum_{i=1}^{3}\mu_i(b_i'Q^{-1}b_i),
\end{aligned} \tag{33}$$

where $\mu_T(i)$ is the posterior probability that the model H_i is correct after T observations of log returns $\{\mathcal{R}(1), \ldots, \mathcal{R}(T)\}$:

$$\begin{aligned}
&\mu_T(i) \\
&\quad := p(H_i \text{ is true} \mid \mathcal{R}(1), \ldots, \mathcal{R}(T)) \\
&\quad = \frac{\mu_i p(\mathcal{R}(1), \ldots, \mathcal{R}(T) \mid H_i \text{ is true})}{\sum_{l=1}^{m}\mu_l p(\mathcal{R}(1), \ldots, \mathcal{R}(T) \mid H_l \text{ is true})} \\
&\quad = \frac{\mu_i \prod_{t=1}^{T} p(\mathcal{R}(t) \mid H_i \text{ is true})}{\sum_{l=1}^{m}\mu_l \prod_{t=1}^{T} p(\mathcal{R}(t) \mid H_l \text{ is true})} \\
&\quad = \frac{\mu_i \exp\{-\frac{1}{2}\sum_{t=1}^{T}(\mathcal{R}(t)-\delta\nu_i)'(\delta Q)^{-1}(\mathcal{R}(t)-\delta\nu_i)\}}{\sum_{l=1}^{m}\mu_l \exp\{-\frac{1}{2}\sum_{t=1}^{T}(\mathcal{R}(t)-\delta\nu_l)'(\delta Q)^{-1}(\mathcal{R}(t)-\delta\nu_l)\}}.
\end{aligned}$$

Now for a given μ, computing the second expression of $\bar{\psi}(\mu)$ in (33) is easy; for the first term we employ Monte Carlo integration (see Figure 4 for the algorithm). We find the optimal prior by initially computing $\bar{\psi}(\mu)$ over the whole domain $\sum_{i=1}^{3}\mu_i = 1$, $\mu_i \geq 0$, $i = 1, 2, 3$ coarsely discretized, and zooming in on the region of interest with higher degree of accuracy in both domain discretization and Monte Carlo integration.

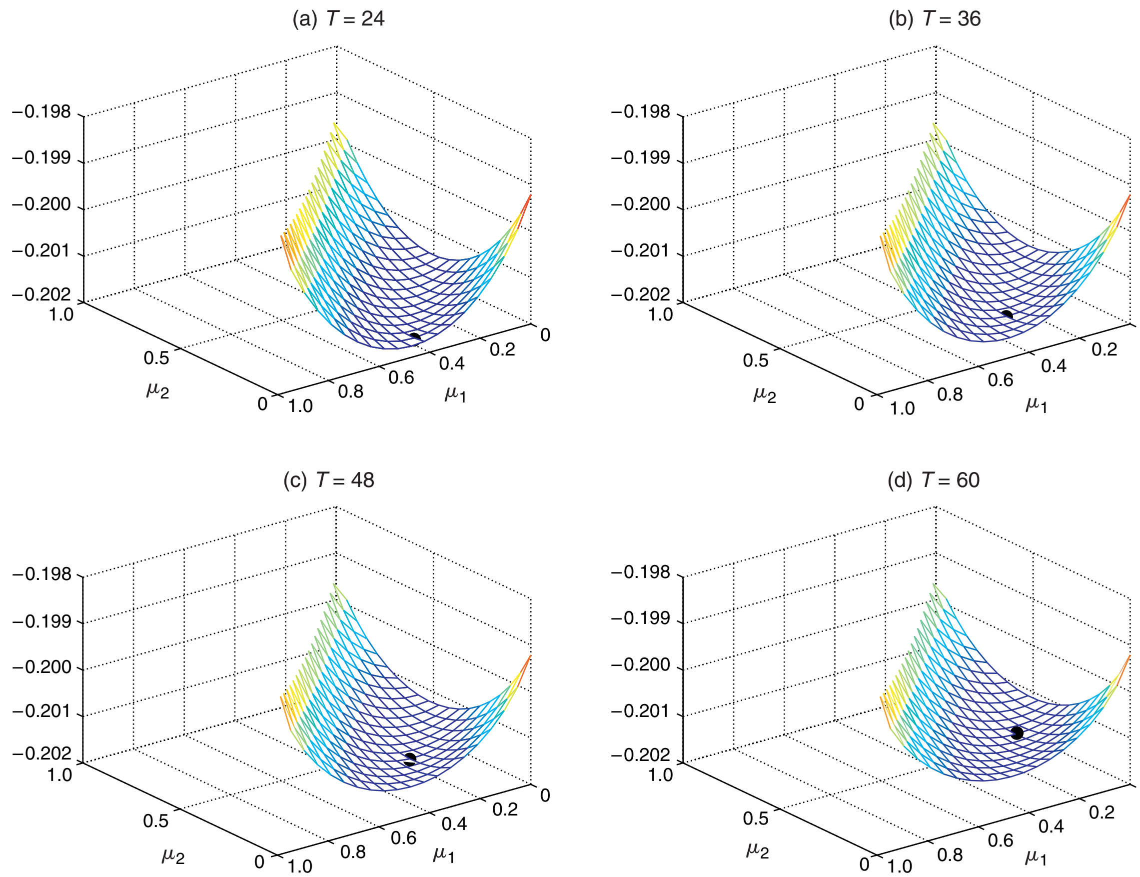
We plot the approximate dual function surfaces in Figure 5 for $T = 24, 36, 48$, and 60 observations, corresponding to two to five years of monthly observations. Observe that the dual function surface becomes flatter with increasing number of observations. This makes sense intuitively; when the number of observations is small, the impact of learning is not significant, and robustness concerns dominate. It may be worthwhile to focus on doing well relative to some models while ignoring others. However, with an increasing number of observations, learning takes over, and it eventually becomes worthwhile to consider all models; for a large number of data points, any prior that weighs all the models performs reasonably well, hence the flatter surface. For this example, robustness means focusing on Models 1 and 2 when there are only 24 data points (see Table 2). When there are ≥ 36 data points, learning becomes significant enough that the prior puts weight on all models. Hence, for a substantial number of data points, the optimal prior still weights all models but the flatness of the dual function means that any prior that weights all of the models performs reasonably well.

In this regard, we compare the performance of the approximate optimal portfolio (32) corresponding to the approximate optimal prior $\bar{\mu}^*$ with the portfolio that corresponds to using a uniform prior $\mu_{\text{unif}} = [1/3, 1/3, 1/3]$. For the comparison, we simulate annualized portfolio log return $12 \times \log(x_\pi^H(\delta)/x(0))$ using (4) under the three different models. We plot

Figure 4 **Algorithm for Computing $\mathbb{E}_\mu[\mu_T(i)\mu_T(j) \mid H_k \text{ is true}]$ for Given μ and i, j, k**

Initialize *Nsim*, *Nint*, *Value* = *Nsim* × 1 zero vector
for *csim* = 1 to *Nsim* **do**
  for *cint* = 1 to *Nint* **do**
    Generate $\mathbf{R} = \{\mathcal{R}(1), \ldots, \mathcal{R}(T)\}$ under model H_k
    $Value(csim) = Value(csim) + \mu_T(i; \mathbf{R})\mu_T(j; \mathbf{R})$
  end for
  $Value(csim) = \dfrac{1}{Nint} Value(csim)$
end for
return sample mean(*Value*), sample std(*Value*)

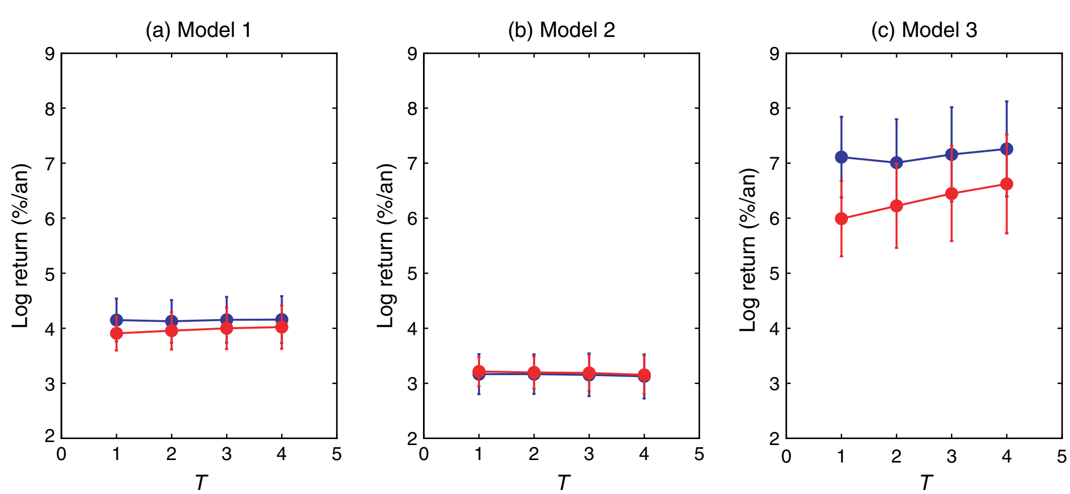
Figure 5 **Dual Function Surfaces to First Order δ for Different Values of T**



Note. The minimum value is shown by a filled black circle.

the mean log returns with one-standard-error bars in Figure 6. The mean log returns of the approximate solution performs better than or equal to the uniform prior across all three models and for all time periods. In addition, the mean log-return gap decreases with increasing learning periods T, which is consistent with the flattening of the dual function surface—that the performance of μ_{unif} increases over time.

Figure 6 **Annualized Portfolio Mean Log Returns of the Approximate Optimal Prior (Blue) and the Uniform Prior (Red) for the Three Different Models**



Notes. Error bars indicate one standard error. The mean log returns of the approximate solution performs no worse than the uniform prior for all time periods. In addition, the mean log-return gap decreases with increasing learning periods T, which is consistent with the observation of flattening dual function surface in the previous paragraph.

Table 2 Optimal Prior Computation for the Approximate Dual Problem

T	$\bar{\mu}^*$	$\psi(\bar{\mu}^*)$ mean (std.)	$\bar{\nu}^*$ mean (std.)	Duality gap at $\bar{\mu}^*$ (% of $\psi(\bar{\mu}^*)$)
24	$(0.421, 0.053, 0.526)'$	−0.2018 (0.0002)	−0.2024 (0.0003)	0.0006 (0.3%)
36	$(0.368, 0.158, 0.474)'$	−0.2016 (0.0002)	−0.2022 (0.0003)	0.0006 (0.3%)
48	$(0.316, 0.211, 0.474)'$	−0.2014 (0.0002)	−0.2020 (0.0002)	0.0006 (0.3%)
60	$(0.208, 0.313, 0.480)'$	−0.2012 (0.0002)	−0.2018 (0.0002)	0.0006 (0.3%)

For the final reported values, we discretize the domain by increments of 0.05 and use $Nsim = 100$ and $Nint = 5{,}000$ for Monte Carlo integration. To examine the validity of the approximate solution, we also report objective values $\bar{\nu}^*$ for the (primal) relative regret problem (23), where $\pi = \bar{\pi}^*$ is the approximately optimal prior given by the solution of (32). Clearly, $\bar{\nu}^*$ is a lower bound to the true optimal regret value of (23). We note that the gap between $\bar{\nu}^*$ and the approximate dual function $\bar{\psi}(\cdot)$ evaluated at the optimal solution $\bar{\mu}^*$ of the approximate dual problem is always within 0.4% of the value of $\bar{\nu}^*$, which suggests that (31)–(32) is a good approximation of the exact problem (23).

6. Conclusion

This paper was motivated by our interest in understanding how learning and robust decision making could be combined. One of its main contributions is a characterization of the optimal learning model and investment portfolio for a portfolio selection problem with parameter uncertainty where the objective is to minimize relative regret. Specifically, we show using convex duality that the optimal learning model is Bayesian where the prior is endogenously specified and corresponds to the Lagrange multipliers that solve the dual problem. The optimal investment portfolio is the solution of a nonstandard Bayesian problem where the posterior is obtained by Bayesian updating of the optimal dual variables/prior using Bayes' rule. The problem of minimizing relative regret can be interpreted as one of minimizing a loss function in which estimators are evaluated not by the associated statistical errors, but by the performance of investment policies that use these estimators relative to a benchmark. Our results can be interpreted as a characterization of the optimal estimator for this loss function.

The single-period model in this paper can be extended to a multiperiod investment model, which will be discussed elsewhere. An interesting feature of this extension is that the learning model involves a posterior that is tilted using the family of benchmarks. It is also possible to consider alternative benchmarks, such as those that weight recent returns more heavily than past returns, which is of interest when returns are only locally stationary.

Finally, this paper has focused primarily on theoretical questions associated with characterizing the optimal learning model and optimal investment for our relative regret model. We have neglected the important applied question as to whether it is possible to solve the dual problem or some approximation of it efficiently. We plan to address this in subsequent work.

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Appendix. Proof of Proposition 1

The problem (17) is equivalent to

$$\begin{cases} \max_{\pi\in\mathcal{G}_T,\,\kappa} \kappa \\ \text{subject to} \\ \mathbb{E}_H\left(\dfrac{1}{\gamma}\dfrac{x_\pi^H(\delta)}{y^H(\delta)}\right)^{\gamma} \geq \kappa, \quad \forall H \in \mathcal{H}, \end{cases} \tag{34}$$

which is a convex optimization problem in (π, κ) for all values of $\gamma < 0$ and $\eta < 1$. We will analyze this problem using convex duality, for which the following definitions are required. For more details, the reader should consult Luenberger (1968).

Let $C(\mathcal{H})$ denote the space of real-valued continuous functionals on $\mathcal{H}$ with sup-norm

$$\|g\| \triangleq \sup_{(Q,b)\in\mathcal{H}} |g(Q, b)|, \quad \forall g \in C(\mathcal{H}).$$

The linear space $C(\mathcal{H})$ with this norm is a Banach space (Dunford and Schwartz 1988). Let

$$\mathcal{P} \triangleq \{g \in C(\mathcal{H}) \mid g(Q, b) \geq 0, \forall (Q, b) \in \mathcal{H}\}$$

define the positive cone in $C(\mathcal{H})$. It is easy to see that $\mathcal{P}$ has a nonempty interior.[5] We say that $f \geq g$ for $f, g \in C(\mathcal{H})$ if $f - g \in \mathcal{P}$ and $g \leq 0$ if $-g \in \mathcal{P}$. We write $g > 0$ if

[5] This is needed for certain strong duality results.

$g(Q, b) > 0$ for every $(Q, b) \in \mathscr{H}$. Next, let $\mathscr{B}(\mathscr{H})$ denote the set of Borel sets of $\mathscr{H}$. The dual (or conjugate) space $C^*(\mathscr{H})$ is (isomorphic to) the set of measures defined on $\mathscr{B}(\mathscr{H})$ with bounded total variation:

$$C^*(\mathscr{H}) = \left\{ \mu \;\Big|\; \int_{\mathscr{H}} |\mu(dH)| < \infty \right\};$$

see, for example, Section IV.6.3 of Dunford and Schwartz (1988). Observe that elements of $C^*(\mathscr{H})$ are signed measures. The dual cone of $\mathscr{P}$ is defined by $\mathscr{P}^* \triangleq \{\mu \in C^*(\mathscr{H}) \mid \int_{\mathscr{H}} f\, d\mu \geq 0,\ \forall f \in \mathscr{P}\}$ (see Luenberger 1968) and is equal to the subset of $\mathscr{C}^*(\mathscr{H})$ consisting of positive measures:

$$\mathscr{P}^* = \{\mu \in C^*(\mathscr{H}) \mid \mu(A) \geq 0,\ \forall A \in \mathscr{B}(\mathscr{H})\}. \tag{35}$$

We write $\mu \geq 0$ when $\mu \in \mathscr{P}^*$.

Let $\mu \in \mathscr{P}^*$ be arbitrary and (π, κ) be feasible for (34). It is clear that

$$\begin{aligned} L(\pi, \kappa, \mu) &\triangleq \kappa + \int_{H\in\mathscr{H}} \left[\mathbb{E}_H \left(\frac{1}{\gamma} \frac{x_\pi^H(\delta)}{y^H(\delta)} \right)^\gamma - \kappa \right] \mu(dH) \\ &= \kappa(1 - \mu(\mathscr{H})) + \int_{H\in\mathscr{H}} \mathbb{E}_H \left(\frac{1}{\gamma} \frac{x_\pi^H(\delta)}{y^H(\delta)} \right)^\gamma \mu(dH) \\ &\geq \nu^*. \end{aligned} \tag{36}$$

Define the dual function $\psi(\mu)$ as

$$\begin{aligned} \psi(\mu) &\triangleq \max_{\kappa\in\mathbb{R},\, \pi\in\mathscr{G}_T} L(\pi, \kappa, \mu) \\ &= \max_{\kappa\in\mathbb{R},\, \pi\in\mathscr{G}_T} \kappa(1 - \mu(\mathscr{H})) + \int_{H\in\mathscr{H}} \mathbb{E}_H \left(\frac{1}{\gamma} \frac{x_\pi^H(\delta)}{y^H(\delta)} \right)^\gamma \mu(dH). \end{aligned}$$

From our construction of $L(\pi, \kappa, \mu)$, $\psi(\mu)$ is an upper bound on ν^* for every $\mu \in \mathscr{P}^*$. This upper bound is finite if and only if $\mu(\mathscr{H}) = 1$ (i.e., a probability measure on $\mathscr{H}$), from which it follows that

$$\psi(\mu) = \max_{\pi\in\mathscr{G}_T} \int_{H\in\mathscr{H}} \mathbb{E}_H \left(\frac{1}{\gamma} \frac{x_\pi^H(\delta)}{y^H(\delta)} \right)^\gamma \mu(dH).$$

Observing that the integral is nothing but an expectation with respect to a probability distribution μ, we adopt the notation

$$\mathbb{E}_\mu \left[\frac{1}{\gamma} \left(\frac{x(\delta)}{y(\delta)} \right)^\gamma \right] \equiv \int_{H\in\mathscr{H}} \mathbb{E}_H \left[\frac{1}{\gamma} \left(\frac{x_\pi^H(\delta)}{y^H(\delta)} \right)^\gamma \right] \mu(dH),$$

and we can write

$$\psi(\mu) = \max_{\pi\in\mathscr{G}_T} \mathbb{E}_\mu \left\{ \frac{1}{\gamma} \left(\frac{x(\delta)}{y(\delta)} \right)^\gamma \right\},$$

which is precisely the dual function (25). It now follows from Theorem 1 of Luenberger (1968, p. 224) that the optimal relative regret objective value ν^* and the dual objective value $\psi(\mu)$ are related by

$$\nu^* = \psi(\mu^*) = \min_{\mu\geq 0,\, \mu(\mathscr{H})=1} \psi(\mu),$$

and that the optimal solution π^{μ^*} of (27) with prior μ^* is also the optimal solution of the relative regret problem (34).

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