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# Pooling and Dependence of Demand and Yield in Multiple-Location Inventory Systems

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The benefits of inventory risk pooling are well known and documented. It has been proven in the literature that the expected costs of a centralized system are increasing in the degree of (positive) dependence of demand in an idealized newsvendor setting. Using the supermodular stochastic order to characterize dependence, we study a general two-tiered supply chain structure, in which both demand and supply yields are random, and prove that the expected costs are increasing in the degrees of positive dependence between demand and supply yield loss factors. Furthermore, using a distributionally robust optimization framework, we prove an analogous result for the case where demand and yield distributions are not precisely known.

**Keywords:** supply chain design; inventory sharing; stochastic orders; robust optimization

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## 1. Introduction

Demand uncertainty has long been the focus of research in supply chain management. A well-known strategy to guard against demand uncertainty is inventory risk pooling, first studied by Eppen (1979). In the newsvendor problem, pooling results in a smaller optimal total stocking quantity, while maintaining the same probability of meeting demand that balances underage and overage costs, and leads to a reduction in the overall expected costs. Following this work, the benefits of inventory pooling have since been well studied in the literature. One focus of the line of research is on how the benefits of pooling depend on demand variability. Eppen (1979) proves that, if demand follows a multivariate Gaussian distribution, then the benefits of pooling are increasing in demand variability. However, this relationship does not necessarily hold under general demand distributions (see, e.g., Benjaafar et al. 2005, Berman et al. 2011, Gerchak and He 2003).

There have been relatively fewer known results on the effect of demand correlation (or more generally, dependence) compared with those on demand variability. Because inventory sharing is only possible for products of closely substitutable nature, it is natural that candidate demand sources for pooling, e.g., for the same product at different outlets of a retail chain, are (positively) dependent to some degree. Eppen's (1979) result also implies that the value of

pooling is decreasing in pairwise correlations between components of Gaussian demand vectors. This result was generalized by Corbett and Rajaram (2006), who prove that the expected costs of the pooled system are increasing in the degree of positive dependence between demands at different locations. In these works, pooling is achieved by holding inventory at a central location that serves all demand sources. However, in the information-rich era, inventory sharing is often enabled by informational pooling. For example, IBM's service parts system flexibly allocates orders to stocking locations within a certain radius of the customer's location (Gresh and Kelton 2003). Similar arrangements are common for online retailers (Xu et al. 2009). Supply routing (Foreman et al. 2010) and the well-studied lateral transshipment strategy (see, e.g., Robinson 1990, Zhang 2005) are also common informational pooling arrangements.

Besides demand uncertainty, supply uncertainty has been a recent research focus, because of its significance as illustrated by supply chain disruptions following events such as Hurricane Katrina and the 2011 Japanese earthquake. It has been shown that the optimal supply chain strategies can be different under demand and supply uncertainties (e.g., Snyder and Shen 2006, Tomlin 2006). Where supply or disruption risks are present, the capability to share inventory at different locations via informational pooling helps

strike a balance between risk pooling and risk diversification (Mak and Shen 2012).

Despite a significant stream of literature on the operations and control of informational pooling, only a handful of formal results regarding the impact of demand dependence on these strategies are available. Zhang (2005) proves that the expected costs of a transshipment problem with identical retailers are increasing in demand dependence, by establishing a connection with the newsvendor problem. Van Mieghem and Rudi (2002) consider a general class of newsvendor network problems that generalize multidimensional newsvendor problems to incorporate resource capacity investment considerations and discretionary activities. They provide a result that, assuming demand follows a Gaussian distribution and the optimal profit is a submodular function of realized demand, the expected profit is decreasing in pairwise correlation of demand. These works motivate our quest for analytical results that are more general in terms of the form of inventory sharing and/or the demand (and in our case, yield) distributions. In this paper, we utilize the concepts of stochastic ordering and distributionally robust optimization in developing our results. The idea behind our analysis is to show that the cost function of the supply chain problem is supermodular, which suggests that demand and yield loss realizations have complementary contributions to total costs. Then, by using concepts of supermodular stochastic orders and distributionally robust optimization, we show that the expected costs of the system increase with demand (and yield) dependence under different settings, and that the largest (worst) possible expected costs arise from a joint distribution with perfect dependence. We begin by discussing the technical concepts that will be used later.

## 2. Preliminaries

A supermodular function can be defined as follows (Topkis 1998).

**DEFINITION 1.** A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is supermodular if  $f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y})$ , where  $\mathbf{x} \vee \mathbf{y}$  and  $\mathbf{x} \wedge \mathbf{y}$  are the componentwise maximum and minimum, respectively, of  $\mathbf{x}$  and  $\mathbf{y}$ .

For a supermodular function, the marginal impact of an input parameter on the function value increases in the other input parameters. Therefore, the input parameters are complements for increasing the function value. Next, the supermodular stochastic order is defined as follows (Shaked and Shanthikumar 2007).

**DEFINITION 2.** Let  $\tilde{\mathbf{X}}$  and  $\tilde{\mathbf{Y}}$  be two  $n$ -dimensional random vectors. If  $E[\psi(\tilde{\mathbf{X}})] \geq E[\psi(\tilde{\mathbf{Y}})]$  for all supermodular functions  $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$ , provided the expectations exist, then,  $\tilde{\mathbf{X}}$  is said to be larger than  $\tilde{\mathbf{Y}}$  under the *supermodular order* (denoted by  $\tilde{\mathbf{X}} \succeq_{sm} \tilde{\mathbf{Y}}$ ).

The interpretation of the supermodular order is that, as the input arguments of supermodular functions carry complementary effects, the function value tends to be more extreme as the input arguments vary in the same direction. Therefore, when a random vector has positively dependent components, the expected value of any supermodular function that takes its components as arguments tends to increase. In the literature, various properties of the supermodular order have been studied. For example, Joe (1990) studies its relationship with the weaker dependence measure of concordance stochastic order. Shaked and Shanthikumar (1997) study in depth the properties of the supermodular order, including its closure under various operations. Müller and Scarsini (2000) prove that the supermodular order satisfies the nine axioms for a multivariate positive dependence order (Joe 1997). Furthermore, it is also known that  $\tilde{\mathbf{X}} \succeq_{sm} \tilde{\mathbf{Y}}$  implies  $\tau_{\tilde{\mathbf{X}}} \geq \tau_{\tilde{\mathbf{Y}}}$ ,  $\rho_{\tilde{\mathbf{X}}} \geq \rho_{\tilde{\mathbf{Y}}}$ , and  $q_{\tilde{\mathbf{X}}} \geq q_{\tilde{\mathbf{Y}}}$  (Joe 1990, Shaked and Shanthikumar 2007), where  $\tau_{(\cdot)}$ ,  $\rho_{(\cdot)}$ , and  $q_{(\cdot)}$  denote Kendall's  $\tau$ , Spearman's  $\rho$ , and Blomqvist's  $q$ , respectively, all of which are common correlation measures of the corresponding random variables. The results discussed in the above references suggest that the supermodular order is a desirable characterization of positive dependence. For the commonly used Gaussian distributions, the following holds:

**EXAMPLE 1.** Let  $\tilde{\mathbf{X}}$  and  $\tilde{\mathbf{Y}}$  be two Gaussian random variables with the same marginal means and marginal variances. Let the correlation matrices of  $\tilde{\mathbf{X}}$  and  $\tilde{\mathbf{Y}}$  be denoted by  $\Sigma_{\tilde{\mathbf{X}}}$  and  $\Sigma_{\tilde{\mathbf{Y}}}$ , respectively. If  $\Sigma_{\tilde{\mathbf{X}}} \geq \Sigma_{\tilde{\mathbf{Y}}}$  component-wise, then  $\tilde{\mathbf{X}}$  is larger than  $\tilde{\mathbf{Y}}$  under the supermodular order.

We also note that, for random vectors with equal dimensions, the supermodular order also implies the sum-convex order, which is proposed by Corbett and Rajaram (2006) to characterize dependence of demand in multidimensional newsvendor settings.

**DEFINITION 3.** Let  $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n)$  and  $\tilde{\mathbf{Y}} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_m)$  be two random vectors. If  $E[\psi(\sum_{k=1}^n \tilde{X}_k)] \geq E[\psi(\sum_{k=1}^m \tilde{Y}_k)]$  for all univariate convex functions  $\psi(\cdot)$ , provided the expectations exist, then,  $\tilde{\mathbf{X}}$  is said to be larger than  $\tilde{\mathbf{Y}}$  under the sum-convex order (denoted by  $\tilde{\mathbf{X}} \succeq_{scx} \tilde{\mathbf{Y}}$ ).

**EXAMPLE 2.** Let  $\tilde{\mathbf{X}}$  and  $\tilde{\mathbf{Y}}$  be two  $n$ -dimensional random vectors such that  $\tilde{\mathbf{X}} \succeq_{sm} \tilde{\mathbf{Y}}$ . Then,  $\tilde{\mathbf{X}} \succeq_{scx} \tilde{\mathbf{Y}}$ .

One can check that many of the properties implying the sum-convex order discussed in Corbett and Rajaram (2006), in fact, imply the supermodular order as well. For example, following the proof of Proposition 5 in Corbett and Rajaram (2006), Example 1 can be generalized to cases where the random vectors

have general marginal distributions and dependence structures that can be completely characterized by the correlation matrices of some monotonic transformations of  $\tilde{X}$  and  $\tilde{Y}$ , i.e., with the normal copula (see, e.g., Clemen and Reilly 1999 for more in-depth discussion of copulas). Finally, we note that two random vectors that can be ranked using the supermodular order must belong to the same Fréchet class, defined as follows:

**DEFINITION 4.** Let  $\{F_1, \dots, F_n\}$  be a set of given marginal distributions. Its *Fréchet class* is given by the set of all  $n$ -dimensional joint distributions  $\mathbf{F}$  with the given marginal distributions.

Within a Fréchet class, there exists a largest element, defined as follows:

**DEFINITION 5.** For any distribution  $\mathbf{F}$  in the Fréchet class of  $\{F_1, \dots, F_n\}$ , it holds that  $\mathbf{F}(\mathbf{u}) \leq \bar{\mathbf{F}}(\mathbf{u}) = \min\{F_1(u_1), \dots, F_n(u_n)\}$ , for any  $\mathbf{u} = (u_1, \dots, u_n)$  in the domain of  $\mathbf{F}$ . The distribution function  $\bar{\mathbf{F}}(\mathbf{u})$  is an element of the same Fréchet class and is known as the *Fréchet upper bound*.

The Fréchet upper bound specifies a dependence structure that is known as *comonotonic* in the literature (e.g., Dhaene and Denuit 1999). Under this dependence structure, all components of the random vector varies in the same direction, i.e., they are perfectly positively dependent. More precisely, there exists a common univariate uniform  $[0, 1]$  random variable  $\tilde{U}$  such that  $\tilde{U}_i = F_i^{-1}(\tilde{U})$  for all  $i = 1, \dots, n$ .

### 3. Supply Chain Network with Inventory Sharing

Consider a two-tiered supply chain consisting of two layers of facilities. Let  $I = \{1, \dots, M\}$  denote a set of upstream facilities (e.g., distribution centers),  $J = \{1, \dots, N\}$  denote a set of downstream facilities (e.g., retailers), and  $E \subseteq I \times J$  denote the set of edges (e.g., shipping routes) between the two types of facilities. Each upstream facility  $i \in I$  orders an inventory of  $y_i$ . We assume that the order quantity vector  $\mathbf{y} = (y_1, y_2, \dots, y_M)$  is selected from a set  $\mathbf{Y} \subseteq \mathbb{R}_+^M$  before uncertainty in demand and supply is realized. We impose no restrictions on the set  $\mathbf{Y}$ , such that, for example, there can be minimum and maximum order sizes, and the inventory levels can be optimized or fixed. Because of imperfect reliability of suppliers, we assume that the facility  $i$  receives a quantity of  $y_i(1 - \tilde{\Delta}_i)$ , where  $\tilde{\Delta}_i$  ( $\in [0, 1]$ ) is a random variable referred to as the yield loss factor. Each downstream facility  $j \in J$  faces stochastic external demand of  $\tilde{D}_j$ . Let  $\tilde{\mathbf{U}}$  denote the vector  $\{\tilde{U}_1, \dots, \tilde{U}_M, \tilde{U}_{M+1}, \dots, \tilde{U}_{M+N}\} =$

$\{\tilde{\Delta}_1, \dots, \tilde{\Delta}_M, \tilde{D}_1, \dots, \tilde{D}_N\}$ , which follows some multivariate distribution  $\mathbf{F}$ . After demand and supply yield loss factors are realized (we use  $\mathbf{u}$  to denote a realization of  $\tilde{\mathbf{U}}$ ), the firm decides the quantity  $w_{ij}$  of inventory at upstream facility  $i$  to be used to satisfy demand at any  $j$  to which it is connected, i.e.,  $(i, j) \in E$ , incurring a shipping cost of  $c_{ij}$ . Any unmet demand at  $j$ , denoted by  $s_j$ , incurs an underage cost of  $p_j$  per unit; any leftover inventory at  $i$ , denoted by  $x_i$ , incurs an overage cost of  $h_i$  per unit. The total shipping, overage, and underage costs of the supply chain can be written as  $H(\mathbf{y}, \mathbf{u})$ , defined as the optimal objective value of the following linear program:

$$\min_{\mathbf{s}, \mathbf{x}, \mathbf{w} \geq 0} \sum_{i, j: (i, j) \in E} c_{ij} w_{ij} + \sum_{j \in J} p_j s_j + \sum_{i \in I} h_i x_i \quad (1)$$

$$\text{subject to } x_i + \sum_{j \in J: (i, j) \in E} w_{ij} = (1 - u_i) y_i \quad \text{for each } i = 1, \dots, M, \quad (2)$$

$$\sum_{i \in I: (i, j) \in E} w_{ij} + s_j = u_{M+j} \quad \text{for each } j = 1, \dots, N. \quad (3)$$

In the above, constraint (2) requires that inventory at any upstream facility  $i$  is either shipped to some downstream facility or held as leftovers. Similarly, constraint (3) requires that demand at any downstream facility be either met by shipments from some upstream facilities or unmet (incurring shortage). As the above formulation gives the total costs given a realization of demand, the expected total cost, under a joint distribution  $\mathbf{F}$ , can be obtained as  $E_{\mathbf{F}}[H(\mathbf{y}, \tilde{\mathbf{U}})]$ .

We note that the above model setting is quite general and includes many possible inventory (and capacity) sharing settings as special cases. We provide a few examples below.

**EXAMPLE 3.** The following classical settings can be obtained as special cases of our model:

1. To obtain the classical risk-pooling setting studied by Eppen (1979), we may set  $I = \{1\}$ ,  $J = \{1, \dots, N\}$ ,  $E = I \times J$ ,  $c_{ij} = 0$  for all  $(i, j) \in E$  and  $p_j = p$  for all  $j \in J$ .

2. To obtain the transshipment model studied by Robinson (1990), we may set  $I = J = \{1, \dots, N\}$ ,  $E = I \times I$  and  $c_{ii} = 0$  for all  $i = 1, \dots, M$ .

3. To obtain the chaining model of process flexibility studied by Jordan and Graves (1995), we may set  $I = J = \{1, \dots, N\}$ ,  $E = \{(i, i+1)\}_{i=1, \dots, N-1} \cup \{(N, 1)\}$ ,  $c_{ij} = 0$  for all  $(i, j) \in E$ ,  $h_i = 0$  for all  $i = 1, \dots, M$  and  $p_j = 1$  for all  $j = 1, \dots, N$ .

4. To obtain a two-tiered supply chain with a profit-maximizing objective, in which the marginal revenue of meeting each unit of demand at  $j$  is  $\hat{\pi}_j$ , the purchase cost of each unit of delivered inventory at  $i$  is



$\hat{\rho}_i$ , the unit shipping cost from  $i$  to  $j$  is  $\hat{c}_{ij}$ , and the unit salvage value for unsold leftover inventory at  $i$  the end of horizon is  $\hat{v}_i$  ( $< \hat{\rho}_i$ ), we may set  $p_j = \hat{\pi}_j$ ,  $h_i = \hat{\rho}_i - \hat{v}_i$ , and  $c_{ij} = \hat{\rho}_i + \hat{c}_{ij}$ . Then, by substituting constraint (2), one can see that the objective of the profit-maximizing problem is given by a constant minus the objective of the cost-minimization problem:

$$\begin{aligned} & \sum_{j \in J} \hat{\pi}_j (u_{M+j} - s_j) - \sum_{i \in I} \hat{\rho}_i (1 - u_i) y_i + \sum_{i \in I} \hat{v}_i x_i - \sum_{i, j: (i, j) \in E} \hat{c}_{ij} w_{ij} \\ &= \sum_{j \in J} \hat{\pi}_j u_{M+j} - \left[ \sum_{i, j: (i, j) \in E} c_{ij} w_{ij} + \sum_{j \in J} p_j s_j + \sum_{i \in I} h_i x_i \right]. \end{aligned}$$

5. To obtain a supply chain in which inventory is held at both upstream and downstream facilities, we can define, for each downstream facility  $j$ , an ancillary upstream facility  $i$  such that shipping cost  $c_{ij} = 0$ , and set  $h_i$  equal to the overage cost at  $j$ . This is a model for hybrid strategies where the inventory is partially localized (at downstream facilities), partially centralized (at upstream facilities), and can be flexibly allocated in response to realization of uncertainty.

## 4. The Cost of Pooling

### 4.1. When Marginal Distributions Are Known

First, we consider the case where the marginal distributions of  $\tilde{U}_k$ , for  $k = 1, \dots, M + N$ , are known and are denoted by  $F_k$ . That is, we consider the Fréchet class of  $\{F_k, k = 1, \dots, M + N\}$ . First, we will generalize the results of Corbett and Rajaram (2006) using the supermodular stochastic order. To begin, we first prove that the cost function of the two-tiered supply chain is supermodular. The proofs of all analytical results are provided in the online supplement (available at <http://dx.doi.org/10.1287/msom.2013.0469>).

**LEMMA 1.** *The function  $H(\mathbf{y}, \mathbf{u})$  is supermodular in  $\mathbf{u}$ .*

The supermodular property implies that the marginal cost of yield loss at one upstream facility is increasing in the yield loss at any other upstream facility and the realized demand at any downstream facility. The marginal cost of observing extra demand at any downstream facility is increasing in the realized demand at any other downstream facility and the yield loss at any upstream facility. We note that a recent paper by Simchi-Levi and Wei (2012) also utilizes a supermodular property to derive analytical results for the process flexibility network. By establishing supermodularity of flexible production edges, they answer important open questions in the literature, including providing a formal proof that the chaining structure is indeed optimal among all two-flexibility designs. In contrast, we develop understanding on the effect of the demand and supply yield dependence on expected costs. Both applications

help illustrate the power of supermodularity theory in developing understanding on supply chain (and process flexibility) structures.

Lemma 1 provides the technical condition needed to prove that the expected costs of the system increase in positive dependence of demand and yield loss factors. Intuitively, with a higher degree of demand dependence, it becomes more likely that different facilities observe high demand (yield loss) realizations at the same time, or low demand (yield loss) realizations at the same time. Because the marginal cost of extra demand (yield loss) at one facility is increasing in the demand (yield loss) values of others due to supermodularity, the expected costs is higher as demand (yield loss) values vary in the same direction. To prove this formally, we use the supermodular order to compare the degrees of positive dependence of random demand vectors. Our main result follows:

**PROPOSITION 1.** *Consider two multivariate random variables  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{U}}'$ , where  $\tilde{\mathbf{U}} \succeq_{sm} \tilde{\mathbf{U}}'$ . Then,*

$$\min_{\mathbf{y} \in \mathbf{Y}} E[H(\mathbf{y}, \tilde{\mathbf{U}})] \geq \min_{\mathbf{y} \in \mathbf{Y}} E[H(\mathbf{y}, \tilde{\mathbf{U}}')].$$

Proposition 1 is a generalization of the results of Eppen (1979), Corbett and Rajaram (2006), and Zhang (2005) to the general two-tiered supply chain setting in the presence of demand and supply uncertainty. Therefore, the benefits from informational and physical pooling diminish as demands at downstream facilities and yield losses at upstream facilities become more positively dependent. This holds regardless of the form of pooling, e.g., whether inventory is centralized at the same location, or transshipped among stocking locations, or dynamically rerouted from upstream to downstream facilities, and whether or not efficiency losses (transportation costs) are incurred.

Besides generalizing the classical result regarding the relationship between pooling benefits and demand dependence, Proposition 1 supports the insight that risk diversification in the presence of supply uncertainty ought to be carried out in a way that minimizes the positive dependence among yield loss factors, e.g., by contracting with suppliers that do not share common supply sources further upstream. Furthermore, suppliers should be selected in a way that minimizes dependence between demand and yield loss factors. Such dependence may arise, for example, from using a supplier with limited capacity. Under a high demand scenario, other firms selling the same product may scale up order quantities simultaneously, which causes the supplier to ration the limited supply and reduce the resulting supply yield to each individual firm. This result provides justification that the trade-off between the benefits of inventory sharing (informational or physical) and the cost to enable

it (e.g., information technology infrastructure investments) can be altered by both the structure of supply base as well as the nature of demand substitution.

The supermodular order is a general characterization of dependence that can be used under many distributions. For example, in problems where the marginal demand distributions at individual locations are nonnormal, and yet the dependence structure can be characterized by the correlation matrix of a monotonic transformation (i.e., the normal copula), our result implies that the expected costs are increasing in each element of the correlation matrix (following the proof of Proposition 5 of Corbett and Rajaram 2006). Next, we provide a complementary result that the largest possible expected costs are achieved by a perfectly positively dependent joint distribution.

**PROPOSITION 2.** *For a supermodular function  $h(\cdot)$ :  $\mathbb{R}^{M+N} \rightarrow \mathbb{R}$ , the joint distribution  $\tilde{\mathbf{F}}$  in the Fréchet class of  $\{F_k, k = 1, \dots, M+N\}$  that maximizes  $E_{\tilde{\mathbf{F}}}[h(\tilde{\mathbf{U}})]$  is given by the Fréchet upper bound, given by  $\tilde{\mathbf{F}}(\mathbf{u}) = \min\{F_1(u_1), \dots, F_{M+N}(u_{M+N})\}$ .*

By considering  $h(\cdot) = H(\mathbf{y}, \cdot)$ , Proposition 2 states that, when the marginal distributions of demand and yield loss factors are given, the comonotonic dependence structure leads to the largest expected cost. In the next section, we further show that these insights hold even when the marginal distributions are unknown.

#### 4.2. When Partial Distributional Information Is Known

Supply chain planners often have to work with new markets, products, and suppliers for which demand and yield data are limited. In such cases, fitting the precise (even marginal) distributions and establishing supermodular ordering relationships can be difficult. To tackle such problems, it is often assumed in the distributionally robust optimization literature that only descriptive statistics (e.g., supports, means, and variances) of random variables, which are relatively easier to estimate compared with fitting the complete distribution, are available. Then, the planner is assumed to be ambiguity averse and consider the worst expected objective value within the set of possible distributions (the ambiguity set) with the given descriptive statistics (e.g., Goh and Sim 2010).

Following this framework, we consider a model of uncertainty in which the exact marginal distributions of  $\tilde{U}_i$  are unknown, and instead, partial distributional information is given. For any positive integer  $k$ , let  $Q_{ik}$  denote the  $k$ th marginal moment of  $\tilde{U}_i$ . We also denote the support of  $\tilde{U}_i$  by  $S_i$ . Let  $\mathbb{S} = S_1 \times \dots \times S_{M+N} \subset \mathbb{R}_+^{M+N}$ . We assume knowledge of the support  $\mathbb{S}$  and some marginal moments  $Q_{ik}$  for  $i = 1, \dots, M+N$  and for  $k \in K$ , where  $K$  is a finite set of positive integers. For

example, if  $K = \{1, 2\}$ , the marginal means and second moments (and thus variances) are known.

For this alternative setting, we prove a result analogous to Proposition 2 that the worst-case distribution is given by a Fréchet upper bound. To begin, we first note that the worst-case expected value of any function  $h(\tilde{\mathbf{U}})$  can be obtained by solving the following moment problem:

$$\sup_{\mathbf{F}: d\mathbf{F}(\cdot) \geq 0} \int_{\mathbb{S}} h(\mathbf{u}) d\mathbf{F}(\mathbf{u}) \quad (4)$$

$$\text{subject to } \int_{\mathbb{S}} u_i^k d\mathbf{F}(\mathbf{u}) = Q_{ik}, \quad \text{for } i = 1, \dots, M+N, k \in K, \quad (5)$$

$$\int_{\mathbb{S}} d\mathbf{F}(\mathbf{u}) = 1. \quad (6)$$

In the above, the probability measure  $\mathbf{F}(\cdot)$  is selected to maximize the expected value of  $h(\cdot)$ . Constraints (5)–(6) stipulate that the moments of this chosen measure are equal to the given values, and the measure is valid so that the probability that the associated random variable lies in the support  $\mathbb{S}$  is equal to one. For any joint distribution satisfying these constraints, the corresponding marginal distributions  $F_i(\cdot)$  satisfy

$$\int_{S_i} u_i^k dF_i(u_i) = Q_{ik}, \quad \text{for } i = 1, \dots, M+N, k \in K, \quad (7)$$

$$\int_{S_i} dF_i(u_i) = 1. \quad (8)$$

**PROPOSITION 3.** *For a supermodular function  $h(\cdot)$ , the extremal distribution (i.e., optimal solution) to (4) is a Fréchet upper bound. That is, it satisfies  $\mathbf{F}(\mathbf{u}) = \min\{F_1(u_1), F_2(u_2), \dots, F_{M+N}(u_{M+N})\}$ , for some univariate distributions  $F_1, \dots, F_{M+N}$  satisfying the moment constraints (7)–(8).*

Substituting  $H(\mathbf{y}, \tilde{\mathbf{U}})$  in place of  $h(\tilde{\mathbf{U}})$ , Proposition 3 states that the counterpart of Proposition 2 holds, i.e., the worst expected cost is achieved by a comonotonic distribution, even under ambiguity on the marginal distributions. Furthermore, besides analyzing the worst dependence structure under distributional ambiguity, we are interested in showing the monotonicity relationship, in line with Proposition 1, between expected cost and dependence of demand and supply yield under the ambiguity setting. To this end, we consider the case where, in addition to the marginal distributional information specified in (5) and (6), the joint distribution  $\mathbf{F}$  of  $\tilde{\mathbf{U}}$  also satisfies a constraint that  $E_{\mathbf{F}}[g(\tilde{\mathbf{U}})]$  is equal to a known value, for a given supermodular function  $g(\cdot)$ :

$$\int_{\mathbb{S}} g(\mathbf{u}) d\mathbf{F}(\mathbf{u}) = \eta. \quad (9)$$

As discussed previously, the expected values of supermodular functions of a random variable are closely related to the dependence structure of the components of the random variable. For example, consider the function  $g(\tilde{\mathbf{U}}) = \prod_{i=1, \dots, M+N} g_i(\tilde{U}_i)$ , where  $g_i(\cdot)$  are nonnegative increasing functions (supermodular by Corollary 2.6.3 of Topkis 1998). A special case is  $g(\tilde{\mathbf{U}}) = \tilde{U}_i \tilde{U}_j$ , whose expected value  $E[\tilde{U}_i \tilde{U}_j]$  is the cross second moment (which gives the correlation coefficient, given fixed means and variances) typically used to measure positive dependence. It is also possible to consider  $g(\tilde{\mathbf{U}})$  to be any conic combination of  $\tilde{U}_i \tilde{U}_j$  for distinct  $i, j$  pairs, which is also a supermodular function. This is equivalent to specifying a conic combination of the correlation coefficients, given fixed means and variances. Similarly,  $E[g(\cdot)]$  can also represent other correlation measures such as Kendall's  $\tau$  or Spearman's  $\rho$  (or conic combinations thereof) by selecting an appropriate supermodular function. Therefore, imposing the additional constraint (9) can be interpreted as specifying the value of a generalized cross moment of  $\mathbf{F}$  that measures dependence. We also make the following mild technical assumptions:

**ASSUMPTION 1.** The value of  $\eta$  is bounded above by the value corresponding to any Fréchet upper bound, i.e.,  $\eta \leq \int_{\mathbb{S}} g(\mathbf{u}) d\tilde{\mathbf{F}}(\mathbf{u})$  for any  $\tilde{\mathbf{F}}$  satisfying  $\tilde{\mathbf{F}}(\mathbf{u}) = \min\{F_1(u_1), F_2(u_2), \dots, F_P(u_{M+N})\}$ , for some  $F_1, \dots, F_P$  satisfying the moment constraints (7)–(8) and  $dF_i(u_i) \geq 0$  for  $i = 1, \dots, M+N$ .

**ASSUMPTION 2.** The values of  $Q_{ik}$  for  $i = 1, \dots, M+N$ ,  $k \in K$  and  $\eta$  lie in the interior of the feasible set, i.e., the set for which there exists valid distributions with corresponding moment values.

Assumption 1 requires that the specified value of generalized moment to be used for measuring dependence cannot be even larger than what can be achieved under a comonotonic distribution with the given means and variances. This is a reasonable and mild assumption, considering that comonotonic distributions are already perfectly dependent. One may note that there exist comonotonic distributions under which  $E[g(\tilde{\mathbf{U}})] = 0$  for  $g(\tilde{\mathbf{U}}) = \tilde{U}_i \tilde{U}_j$ , such as the case where  $\tilde{U}_j = (\tilde{U}_i)^2$  and  $\tilde{U}_i$  follows a normal marginal distribution with zero mean. However, because the support  $\mathbb{S}$  is in the nonnegative orthant, such extreme examples can be ruled out. Assumption 2 is a mild Slater's-type condition needed for ensuring strong duality, and is commonly made in the analysis of moment problems (Bertsimas and Popescu 2004). For this alternative setting, we shall prove a result analogous to Proposition 1 that the worst-case expected costs are increasing in  $\eta$ , which can be interpreted as a measure of dependence. Similar to the previous

case, the ambiguity averse (worst-case) evaluation of the expected costs can be obtained by solving

$$\sup_{\mathbf{F}: d\mathbf{F}(\cdot) \geq 0} \int h(\mathbf{u}) d\mathbf{F}(\mathbf{u}) \quad (10)$$

subject to (5)–(6) and (9). By analyzing this distributionally robust formulation, we prove the following monotonicity result regarding its optimal objective value.

**PROPOSITION 4.** The optimal objective value to problem (10) is increasing in the value of  $\eta$ .

Substituting the supply chain cost function  $H(\cdot)$  in place of  $h(\cdot)$ , Proposition 4 suggests that, for an ambiguity-averse planner given only support and moment information of demand and yield loss, the worst-case expected costs of the supply chain are increasing in a measure of positive dependence of the random factors. The significance is that, in settings such as entering new markets and contracting with new suppliers, the same qualitative insight holds even if one cannot confirm whether the supermodular stochastic order holds. Instead, one simply needs to estimate some cross (generalized) moment  $E[g(\cdot)]$  and consider the ambiguity-averse objective. It is also interesting to consider different possible functions  $g(\cdot)$  to use. Besides the cross second moments (or conic combinations thereof), one may also substitute the cost function of another supply chain network structure, which is supermodular following Lemma 1, in place of  $g(\cdot)$ . Then, Proposition 4 suggests that the dependence structure exerts the same directional effect on the worst-case expected costs of different supply chain network configurations, such that increasing the expected costs of one configuration leads to an increase in any other.

## 5. Conclusion

We generalize the classical insights on the relationship between the risk-pooling benefits and the dependence of demand, due to Eppen (1979) and Corbett and Rajaram (2006), to general two-tiered supply chains with physical or informational pooling of inventory, in the presence of both demand and supply uncertainty, when demand and yield distributions can be known or unknown. We extend the traditional model by accounting for practical concerns such as the form of inventory sharing, i.e., physical or informational pooling, and the possible efficiency loss (e.g., extra transportation cost) incurred. Besides, our result covers the case with random supply yields and points out the importance of considering demand and supply dependence in supply chain network design. As a direction for future study, it will be interesting to develop further models to investigate conditions

under which cost functions in inventory problems are supermodular in demand and yield factors in general (e.g., when the network structure is not a bipartite graph), and the resulting implications.

### Supplemental Material

Supplemental material to this paper is available at <http://dx.doi.org/10.1287/msom.2013.0469>.

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### References

- Benjaafar S, Cooper WL, Kim JS (2005) On the benefits of pooling in production-inventory systems. *Management Sci.* 51(4):548–565.
- Berman O, Krass D, Tajbakhsh MM (2011) On the benefits of risk pooling in inventory management. *Production Oper. Management* 20(1):57–71.
- Bertsimas D, Popescu I (2004) Optimal inequalities in probability theory: A convex optimization approach. *SIAM J. Optim.* 15(3):780–804.
- Clemen RT, Reilly T (1999) Correlations and copulas for decision and risk analysis. *Management Sci.* 45(2):208–224.
- Corbett CJ, Rajaram K (2006) A generalization of the inventory pooling effect to nonnormal dependent demand. *Manufacturing Service Oper. Management* 8(4):351–358.
- Dhaene J, Denuit M (1999) The safest dependence structure among risks. *Insurance: Math. Econom.* 25(1):11–21.
- Eppen GD (1979) Effects of centralization on expected costs on the multi-location newsboy problem. *Management Sci.* 25(5):498–501.
- Foreman J, Gallien J, Alspaugh J, Lopez F, Bhatnagar R, Teo CC, Dubois C (2010) Implementing supply-routing optimization in a make-to-order manufacturing network. *Manufacturing Service Oper. Management* 12(4):547–568.
- Gerchak Y, He QM (2003) On the relation between the benefits of risk pooling and the variability of demand. *IIE Trans.* 35(11):1027–1031.
- Goh J, Sim M (2010) Distributionally robust optimization and its tractable approximations. *Oper. Res.* 58(4):902–917.
- Gresh DL, Kelton EI (2003) Visualization, optimization, and business strategy: A case study. Turk G, van Wijk JJ, Moorhead R, eds. *Proc. 14th IEEE Visualization 2003* (IEEE Computer Society, Los Alamitos, CA), 531–538.
- Joe H (1990) Multivariate concordance. *J. Multivariate Anal.* 35(1):12–30.
- Joe H (1997) *Multivariate Models and Dependence Concepts* (Chapman and Hall, London).
- Jordan WC, Graves SC (1995) Principles and benefits of manufacturing process flexibility. *Management Sci.* 41(4):577–594.
- Mak HY, Shen ZJ (2012) Risk pooling and risk diversification in supply chain design. *IIE Trans.* 44(8):603–621.
- Müller A (1997) Stop-loss order for portfolios of dependent risks. *Insurance: Math. Econom.* 21(3):219–233.
- Müller A, Scarsini M (2000) Some remarks on the supermodular order. *J. Multivariate Anal.* 73(1):107–119.
- Robinson LW (1990) Optimal and approximate policies in multi-period, multilocation inventory models with transshipments. *Oper. Res.* 38(2):278–295.
- Shaked M, Shanthikumar JG (2007) *Stochastic Orders* (Springer, New York).
- Shaked M, Shanthikumar JG (1997) Supermodular stochastic orders and positive dependence of random vectors. *J. Multivariate Anal.* 61(1):86–101.
- Simchi-Levi D, Wei Y (2012) Understanding the performance of the long chain and sparse designs in process flexibility. *Oper. Res.* 60(5):1125–1141.
- Snyder LV, Shen ZJM (2006) Supply and demand uncertainty in multi-echelon supply chains. Working paper, Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA.
- Tomlin B (2006) On the value of mitigation and contingency strategies for managing supply chain disruption risks. *Management Sci.* 52(5):639–657.
- Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press, Princeton, NJ).
- Van Mieghem JA, Rudi N (2002) Newsvendor networks: Inventory management and capacity investments with discretionary activities. *Manufacturing Service Oper. Management* 4(4):313–335.
- Xu PJ, Allgor R, Graves SC (2009) Benefits of reevaluating real-time order fulfillment decisions. *Manufacturing Service Oper. Management* 11(2):340–355.
- Zhang J (2005) Transshipment and its impact on supply chain members' performance. *Management Sci.* 51(10):1534–1539.