



## Management Science

Publication details, including instructions for authors and subscription information:  
<http://pubsonline.informs.org>

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To cite this article:

Hong Liu, Mark Loewenstein, (2013) Market Crashes, Correlated Illiquidity, and Portfolio Choice. Management Science 59(3):715-732. <http://dx.doi.org/10.1287/mnsc.1120.1561>

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# Market Crashes, Correlated Illiquidity, and Portfolio Choice

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The recent financial crisis highlights the importance of market crashes and the subsequent market illiquidity for optimal portfolio selection. We propose a tractable and flexible portfolio choice model where market crashes can trigger switching into another regime with a different investment opportunity set. We characterize the optimal trading strategy in terms of coupled integro-differential equations and develop a quite general iterative numerical solution procedure. We conduct an extensive analysis of the optimal trading strategy. In contrast to standard portfolio choice models, changes in the investment opportunity set in one regime can affect the optimal trading strategy in another regime even in the absence of transaction costs. In addition, an increase in the expected jump size can increase stock investment even when the expected return remains the same and the volatility increases. Moreover, we show that misestimating the correlation between market crashes and market illiquidity can be costly to investors.

*Key words:* market crashes; portfolio choice; correlated illiquidity

*History:* Received October 27, 2010; accepted March 15, 2012, by Wei Xiong, finance. Published online in *Articles in Advance* September 4, 2012.

## 1. Introduction

The recent financial crisis highlights several potentially important fundamental elements for optimal portfolio choice. First, event risks such as a market crash may be significant; second, market liquidity may dry up after a crash; third, the probability of another crash may increase after a crash; and fourth, other investment opportunity set parameters (e.g., market volatility) may also change after a crash. However, the optimal trading strategy in the presence of market crashes that can trigger changes in the investment opportunity set has not been studied in the existing literature.

In this paper, we develop a flexible portfolio choice model for a small investor that incorporates correlated market crashes and changes in the investment opportunity set. For example, both liquidity and volatility may change after a crash and crashes themselves may be correlated in our model. This model captures the essence of all the above-mentioned important features but still remains tractable. More specifically, we consider the optimal trading strategy of a constant relative risk averse (CRRA) investor who derives utility from terminal wealth and can trade a riskless asset and a risky stock continuously. Stock price crashes in a liquid regime can trigger switching into an illiquid regime where other parameters such as crash intensity, expected return, and volatility can

also change. Similarly, large upward price jumps in the illiquid regime can trigger regime switching into the liquid regime.<sup>1</sup>

Because of the possibility of price jumps, the coupled Hamilton–Jacobi–Bellman (HJB) equations become integro-differential variational equations, which makes our problem much more difficult to solve, even numerically than that of Jang et al. (2007), who do not consider event risks. Remarkably, we are able to develop an iterative procedure that solves for the value function as a sequence of solutions to ordinary differential equations, which significantly reduces computation intensity. This iterative procedure can be readily applied to many other optimal portfolio choice problems and significantly simplifies computation, especially for those involving coupled nonlinear HJB equations. As in the pure diffusion case, the no-transaction region is characterized by two regime-dependent boundaries within which the investor maintains the ratio of the dollar amount in the riskless asset to the dollar amount in the risky

<sup>1</sup> We take these changes after a market crash as exogenously given. There is a large literature on why liquidity and other parameters may change after a crash (see, e.g., Geanakoplos 2003, Diamond and Rajan 2011). Our model can also be consistent with a model where investors learn from crashes and update their beliefs about the investment opportunity set after a crash, although we do not explicitly model this learning process to keep tractability.

asset whenever possible. In contrast to the pure diffusion case, however, this ratio can jump outside these boundaries, which requires an immediate discrete transaction back to the closest boundary. We characterize the value function and provide some analytical comparative statics and an extensive numerical analysis to illustrate how various elements of our model affect the optimal trading strategy.

In contrast to standard portfolio choice models, changes in the investment opportunity set in one regime can affect the optimal trading strategy in another regime even in the absence of transaction costs. This is because the correlation between a market crash (or an upward jump) and regime switching makes the impact of stock investment in one regime dependent on the investment opportunity set in the other regime. In absence of this correlation and transaction costs, the portfolio choice is independent of changes in the investment opportunity set of a different regime. Thus, our model differs from those with investment opportunity set changes that are independent of the stock price risk (e.g., Merton 1971, Jang et al. 2007).<sup>2</sup>

We illustrate quantitative conditions under which an investor should sell stock after a crash even when the market becomes less liquid. Not surprisingly, this sale typically occurs when the investment opportunity set significantly worsens after a market crash (e.g., much higher volatility or much greater further crash intensity) and the worsened environment may persist for a period of time. Intuitively, this is because a significantly worsened investment opportunity set and the expected long duration of the illiquid regime make the marginal benefit of selling the stock outweigh the marginal cost of incurring the necessary transaction cost. This finding is consistent with “flight to quality” after a crash, but in sharp contrast to the contrarian style prediction of the standard portfolio selection models with independent and identically distributed (i.i.d.) returns (e.g., Merton 1971). We show that even a small increase in the after-crash volatility (e.g., from 12% to 20%) may trigger a shift from stock to the risk-free asset. On the other hand, it may also be optimal to buy more stock or not to trade at all upon a crash. In general, to determine the optimal trading strategy after a crash, the investor trades off the benefit of rebalancing due to the change in the investment opportunity set and the cost of transaction. Loosely speaking, the greater the change in the investment opportunity set and the greater the

expected duration of the illiquid regime, the greater the benefit of rebalancing. Depending on the relative magnitude of the benefit and cost, the investor may choose to sell, to buy, or to wait out the illiquid regime after a crash.

We show that an increase in the expected jump size may *increase* the optimal stockholding even if the expected return remains the same and the return volatility increases. Intuitively, increasing the expected jump size (but keeping the expected return constant) may help an investor by making returns less negatively skewed and price jumps can help *reduce* rebalancing costs across regimes. To understand the latter effect, suppose a large price drop triggers the illiquid regime and the optimal fraction of wealth that should be invested in stock decreases. With a large price drop, the fraction of wealth invested in stock is already lower, so a transaction may be no longer necessary. Therefore, this transaction cost reduction effect may make the investor hold more stock in the liquid regime.

In addition, we show that misestimating the correlation between market crashes and market illiquidity can be costly to investors. For example, if an investor underestimates the correlation between market crashes and market illiquidity and adopts the corresponding “optimal” trading strategy under the wrong estimation, the certainty equivalent wealth loss from this trading strategy can be as high as 3.5% of the investor’s initial wealth in some reasonable scenarios.

Closely related works include the literature on portfolio selection with transaction costs but without event risks (e.g., Constantinides 1986, Davis and Norman 1990, Dumas and Luciano 1991, Shreve and Soner 1994, Liu and Loewenstein 2002), and the literature on portfolio selection with event risks but without transaction costs (Liu et al. 2003). The closest works to ours are Liu et al. (2003), Jang et al. (2007), Framstad et al. (2001), and Øksendal and Sulem (2005). Liu et al. (2003) examine the optimal trading strategy when the stock price follows a jump diffusion process with stochastic volatility. However, they do not consider the joint impact of the correlated market crashes and market illiquidity. Jang et al. (2007) use a regime switching model to show that transaction costs can have a first-order effect when an investment opportunity set varies through time. In contrast to our model, they do not consider the effect of market crashes on trading strategies. Framstad et al. (2001) study the optimal consumption/investment problem with an infinite horizon in a jump diffusion setting with constant proportional transaction costs. However, they do not examine the effect of crash-triggered investment opportunity set changes, which are important features for understanding the optimal trading strategy in a financial crisis. In addition, they

<sup>2</sup> Although we present the model with two regimes, our methodology extends to more regimes in which all the parameters as well as jump distributions can change across regimes. Therefore, our methodology can be used to solve optimal portfolio problems with transaction costs where the stock price risk is correlated with a wide variety of changes in the investment opportunity set.

do not offer a numerical procedure to solve for the optimal strategy. Øksendal and Sulem (2005) provide theoretical results on some types of optimal control problems with jump diffusions and offer some examples to illustrate the application of their theory. However, they do not provide theoretical or numerical analysis on the portfolio choice problem in the presence of market crashes and correlated changes in the investment opportunity set. The correlation significantly complicates the theoretical and numerical analysis and the theoretical methods provided by Øksendal and Sulem (2005) no longer apply without significant changes, because the optimal trading strategy in one regime can depend on the investment opportunity set after a crash even in the absence of transaction costs.

The rest of this paper is organized as follows. In §2 we describe our portfolio choice model in a two-regime framework. We provide characterization of the value function and the no-transaction region. Section 3 describes an iterative procedure to compute the optimal trading strategy. Section 4 provides some analytical comparative statics on the optimal trading boundaries. We conduct an extensive numerical analysis, on the optimal trading strategy in §5. We conclude in §6 and provide proofs in the appendix.

## 2. The Basic Model

### 2.1. The Asset Market

An investor can trade two assets in the financial market: one risk free (“the bond”) and one risky (“the stock”). There are two regimes with different liquidity: regime 0 (liquid, lower transaction costs) and regime 1 (illiquid), across which other parameter values may also change. We use  $\iota_t \in \{0, 1\}$  as a state variable to indicate the regime at time  $t$ . The time  $t$  interest rate is  $r(\iota_t)$ . The investor can buy the stock at the ask price  $(1 + \theta(\iota_t))S_t$  and sell it at the bid price  $(1 - \alpha(\iota_t))S_t$ , where  $\theta(\iota) \geq 0$  and  $0 \leq \alpha(\iota) < 1$  represent the proportional transaction cost rates in regime  $\iota$ , and  $S_t$  denotes the stock price without transaction costs. We assume that the stock price  $S_t$  may jump. To capture the idea that a downward jump may have a different impact compared to an upward jump, we sort a stock price jump into an up jump (“U”) and a down jump (“D”), occurring at the jump times of independent Poisson processes  $N^j$  with intensities  $\eta^j(\iota)$  for  $j \in \{U, D\}$ , respectively, and random jump sizes  $J^D - 1 \in (-1, 0)$ ,  $J^U - 1 \in [0, \infty)$ . The stock price process then evolves as

$$dS_t = (\mu(\iota_t) - r(\iota_t))S_{t-}dt + \sigma(\iota_t)S_{t-}dw_t + (J_t^U - 1)S_{t-}dN_t^U + (J_t^D - 1)S_{t-}dN_t^D, \quad (1)$$

where

$$\nu(\iota) = \eta^U(\iota)E[J^U - 1] + \eta^D(\iota)E[J^D - 1] \quad (2)$$

represents the expected return compensation for the presence of jumps so that the instantaneous stock expected return is  $\mu(\iota)$  with  $\mu(\iota) > r(\iota)$ ,  $w$  is a one-dimensional Brownian motion,  $\sigma(\iota)$  is the stock return volatility, and  $J_t^{U,D}$  are the time  $t$  realizations of  $J^{U,D}$ . We assume for simplicity that the jump sizes are drawn from identical independent distributions at each time.<sup>3</sup> Let  $J$  be the greatest lower bound that satisfies  $\text{Prob}\{J^{U,D} \geq J\} = 1$ .

To capture the idea that liquidity changes may be correlated with price jumps (e.g., downward jumps may be positively correlated with switching into an illiquid regime), we decompose each of the jump processes into two independent components. Specifically, let

$$N_t^U = N_{1t}^U + N_{2t}^U, \quad N_t^D = N_{1t}^D + N_{2t}^D,$$

where if  $N_{1t}^U$  or  $N_{1t}^D$  jumps then the current regime switches into the other regime, and  $N_{2t}^U$  and  $N_{2t}^D$  are independent of regime switching. In addition,  $N_{1t}^j$  has an intensity of  $\eta_1^j(\iota_t)$  with  $\eta_1^j(\iota) + \eta_2^j(\iota) = \eta^j(\iota)$  for  $i = 1, 2$ ,  $\iota \in \{0, 1\}$  and  $j \in \{U, D\}$ . To model the possibility that regimes can also change because of other factors such as general macroeconomic conditions, we assume that regime also switches into a different regime at the jump times of another independent Poisson jump process  $N^R$  with intensity  $\xi(\iota)$ . With these assumptions, we have that the state variable  $\iota_t$  evolves (almost surely) as

$$d\iota_t = \begin{cases} dN_{1t}^D + dN_t^R & \text{if } \iota_{t-} = 0, \\ -(dN_{1t}^U + dN_t^R) & \text{if } \iota_{t-} = 1. \end{cases} \quad (3)$$

To understand Equation (3), suppose the current regime is liquid (i.e.,  $\iota_{t-} = 0$ ). Equation (3) then indicates that if  $N_{1t}^D$  jumps then we have a downward jump in the stock price and the regime switches into the illiquid regime ( $\iota_t = 1$ ). On the other hand, if  $N^R$  jumps, then the regime also shifts but there is no jump in the stock price. Finally, if  $N_{2t}^D$  jumps then there is a downward jump in the stock price but the market stays in the liquid regime. Similarly, suppose the current regime is illiquid (i.e.,  $\iota_{t-} = 1$ ). Equation (3) implies that if  $N_{1t}^U$  jumps then we have an upward jump in the stock price and the regime switches into the liquid regime ( $\iota_t = 0$ ). On the other hand, if  $N^R$  jumps, then the regime also shifts but the stock price does not jump. Finally, if  $N_{2t}^U$  jumps then there is an upward jump in the stock price but the market stays in the illiquid regime.

Thus, we have a fairly parsimonious model that nests many possible submodels and allows the investment opportunity set including liquidity to be

<sup>3</sup> Alternatively, maintaining the independence of jump sizes through time, we could let the jump distribution vary with the state variable  $\iota_t$ .



correlated with stock prices in many interesting ways. For example, a pure jump diffusion model with constant proportional transaction costs is obtained by setting  $\eta_1^D = \eta_1^U = \xi = 0$ . Our model also allows regime switching and changes in the other parameter values to be correlated with stock price jumps. For example, after a downward price jump, the regime may switch, and the volatility, expected return, or further crash intensity may become higher.

When  $\alpha(\iota) + \theta(\iota) > 0$ , the above model gives rise to equations governing the evolution of the dollar amount invested in the bond,  $x_t$ , and the dollar amount invested in the stock,  $y_t$ :

$$dx_t = r(\iota_t)x_{t-}dt - (1 + \theta(\iota_t))dI_t + (1 - \alpha(\iota_t))dD_t, \quad (4)$$

$$dy_t = (\mu(\iota_t) - \nu(\iota_t))y_{t-}dt + \sigma(\iota_t)y_{t-}dw_t + (J_t^U - 1)y_{t-}dN_t^U + (J_t^D - 1)y_{t-}dN_t^D + dI_t - dD_t, \quad (5)$$

where the processes  $D$  and  $I$  represent the cumulative dollar amount of sales and purchases of the stock, respectively. These processes are nondecreasing, right continuous adapted processes with  $D(0) = I(0) = 0$ .

Let  $x_0$  and  $y_0$  be the given initial dollar amounts in the bond and the stock, respectively. We let  $\Theta(x_0, y_0)$  denote the set of admissible trading strategies  $(D, I)$  such that (4) and (5) are satisfied given (3) and the investor is always solvent, i.e.,<sup>4</sup>

$$x_t + (1 - \max(\alpha(0), \alpha(1)))y_t \geq 0, \quad \forall t \geq 0, \quad (6)$$

which, as in Liu et al. (2003), restricts the ratio  $x_t/y_t$ .

## 2.2. The Investor's Problem

The investor's problem is to choose admissible trading strategies  $D$  and  $I$  so as to maximize  $E[u(x_\tau + (1 - \alpha(\iota_\tau))y_\tau)]$  for an event that occurs at the first jump time  $\tau$  of a standard, independent Poisson process with intensity  $\lambda$ . Thus,  $\tau$  is exponentially distributed with parameter  $\lambda$ , i.e.,

$$P\{\tau \in dt\} = \lambda e^{-\lambda t} dt.$$

This formulation can capture bequest, accidents, retirement, and many other events that happen on uncertain dates.<sup>5</sup> If  $\tau$  is interpreted to represent the investor's uncertain lifetime (as in Merton 1971), the investor's average lifetime is then  $1/\lambda$  and the variance of the investor's lifetime is accordingly  $1/\lambda^2$ .

<sup>4</sup> Because the jump size is not bounded above, to maintain solvency, the investor cannot short and so  $y \geq 0$ .

<sup>5</sup> We also used the method proposed by Liu and Loewenstein (2002) to solve the case with a deterministic horizon. The solution shows that the exponentially distributed horizon case is a close approximation to the case with a long horizon (about 20 years). Because this finding is similar to that in Liu and Loewenstein (2002), we do not report it in the paper to save space.

Assuming a CRRA preference, we can then write the value function as

$$v(x, y, \iota) = \sup_{(D, I) \in \Theta(x, y)} E \left[ \frac{(x_\tau + (1 - \alpha(\iota_\tau))y_\tau)^{1-\gamma}}{1-\gamma} \middle| \iota_0 = \iota \right]. \quad (7)$$

As shown in the appendix, similar to Merton (1971), Liu and Loewenstein (2002), and Jang et al. (2007), Equation (7) can be rewritten as the following recursive form:

$$v(x, y, \iota) = \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^\infty e^{-\delta(\iota)t} \left( \eta_1^D(\iota)v(x_t, y_t, J_t^D, 1 - \iota) + \eta_2^D(\iota)v(x_t, y_t, J_t^D, \iota) + \eta_1^U(\iota)v(x_t, y_t, J_t^U, 1 - \iota) + \eta_2^U(\iota)v(x_t, y_t, J_t^U, \iota) + \xi(\iota)v(x_t, y_t, 1 - \iota) + \lambda \frac{(x_t + (1 - \alpha(\iota))y_t)^{1-\gamma}}{1-\gamma} \right) dt \right], \quad (8)$$

where

$$\delta(\iota) = \lambda + \xi(\iota) + \eta^U(\iota) + \eta^D(\iota). \quad (9)$$

## 2.3. Optimal Policies with No Transaction Costs

For the purpose of comparison, we first consider the case without transaction costs (i.e.,  $\alpha(\iota) = \theta(\iota) = 0$ ,  $\iota \in \{0, 1\}$ ). Define the total wealth  $W_t = x_t + y_t$  and let  $\pi_t$  be the fraction of wealth invested in the stock. The investor's problem becomes

$$v(W, \iota) = \sup_{\{\pi_t: t \geq 0\}} \lambda E \left[ \int_0^\infty e^{-\lambda t} \frac{W_t^{1-\gamma}}{1-\gamma} dt \middle| \iota_0 = \iota, W_0 = W \right], \quad (10)$$

subject to (3), the dynamic budget constraint

$$dW_t = (r(\iota_t) + \pi_{t-}(\mu(\iota_t) - \nu(\iota_t) - r(\iota_t)))W_{t-}dt + \pi_{t-}\sigma(\iota_t)W_{t-}dw_t + \pi_{t-}W_{t-}((J_t^U - 1)dN_t^U + (J_t^D - 1)dN_t^D), \quad (11)$$

and the solvency constraint  $W_t \geq 0$ . Without transaction costs, the optimal trading strategy is to invest a constant fraction  $\pi(\iota)$  of wealth in stock in regime  $\iota$  and the value function in regime  $\iota$  is of the form

$$v(W, \iota) = M(\iota) \frac{W^{1-\gamma}}{1-\gamma}.$$

From the HJB partial differential equation, it is straightforward to show that for  $\iota = 0, 1$ ,  $(M(\iota), \pi(\iota))$  solves

$$a(\pi(\iota), \iota)M(\iota) + h(\pi(\iota), \iota)M(1 - \iota) + \lambda = 0 \quad (12)$$

and

$$\pi(\iota) = \arg \max_{\pi} \left( \frac{a(\pi, \iota)M(\iota) + h(\pi, \iota)M(1 - \iota) + \lambda}{1 - \gamma} \right), \quad (13)$$

where

$$a(\pi, \iota) = (r(\iota) + \pi(\mu(\iota) - r(\iota) - \nu(\iota)) - \frac{1}{2}\gamma\pi^2\sigma(\iota)^2) \cdot (1 - \gamma) - \delta(\iota) + \eta_2^U(\iota)E[(1 + \pi(J^U - 1))^{1-\gamma}] + \eta_2^D(\iota)E[(1 + \pi(J^D - 1))^{1-\gamma}], \quad (14)$$

and

$$h(\pi, \iota) = \eta_1^U(\iota)E[(1 + \pi(J^U - 1))^{1-\gamma}] + \eta_1^D(\iota)E[(1 + \pi(J^D - 1))^{1-\gamma}] + \xi(\iota). \quad (15)$$

Equations (12) and (13) yield four equations for four unknowns  $(M(\iota), \pi(\iota))$  ( $\iota = 0, 1$ ) that can be easily solved numerically. As in Merton (1971) and Liu et al. (2003), conditions on the parameters and the jump distribution are required for the existence of the optimal solution.

**ASSUMPTION 1.** The solution  $(M(\iota), \pi(\iota))$  to (12) and (13) is such that  $M(\iota) > 0$  for  $\iota = 0, 1$ .

The positivity of  $M(\iota)$  rules out the case where the investor can achieve bliss levels of utility and ensures the existence of an optimal portfolio.<sup>6</sup> We summarize the main result for this no-transaction-cost case in the following theorem.

**THEOREM 1.** Suppose that  $\alpha(0) = \theta(0) = \alpha(1) = \theta(1) = 0$ . Then under Assumption 1, for  $0 \leq t < \tau$  the optimal stock investment policy  $\pi_t^*$  in regime  $\iota_t$  is equal to  $\pi(\iota_t)$  as defined in (13) and the lifetime expected utility is

$$v(W, \iota) = M(\iota) \frac{W^{1-\gamma}}{1-\gamma}, \quad (16)$$

where  $(M(\iota), \pi(\iota))$  solves (12) and (13) for  $\iota = 0, 1$ .

In addition to the standard trade-off between the excess return and variance, in determining the optimal trading strategy the investor also takes into account the impact of stock price jumps in choosing the optimal portfolio. More interestingly, in contrast to Jang et al. (2007), the optimal trading strategy in one regime can depend on the investment opportunity set in the other regime even in the absence of transaction costs. This cross-regime dependence comes from the dependence of  $h(\pi, \iota)$  in Equation (13) on  $\pi$ , which is due to the key feature of our model: the correlation between price jumps and regime switching. Without this correlation (e.g.,  $h(\pi, \iota) = \xi(\iota)$ , or  $h(\pi, \iota) = 0$ ),  $h(\pi, \iota)$  would not depend on  $\pi$  and the portfolio choice would be unaffected by the investment opportunity set in the other regime. Intuitively, the impact of price jumps is regime dependent and with the correlation between jumps and regime switching,

<sup>6</sup> It can be easily verified that Assumption 1 reduces to the well-known Merton condition in the absence of jumps and regime shifts.

this impact becomes dependent on the investment opportunity sets in both regimes. Because of the cross-regime dependence, the optimal portfolio consists of an extra regime hedging component compared to the existing literature (e.g., Merton 1971, Jang et al. 2007).

**REMARK 1.** If  $J = 0$ , then the investor never leverages (i.e.,  $\pi_t^* \leq 1$ ). In general, when  $J < 1$ , leverage is limited because solvency requires  $x + yJ \geq 0$  or  $\pi \leq 1/(1 - J)$ . This is why Equation (13) is not written in terms of the first-order conditions.

#### 2.4. Optimal Policies with Transaction Costs

Suppose now that  $\alpha(\iota) + \theta(\iota) > 0$  for  $\iota = 0, 1$ . As in Liu and Loewenstein (2002), the value functions are homogeneous of degree  $1 - \gamma$  in  $(x, y)$ . This implies that for  $\iota = 0, 1$ ,

$$v(x, y, \iota) = y^{1-\gamma} \psi(z, \iota), \quad \text{where } z \equiv \frac{x}{y}, \quad (17)$$

for some concave function  $\psi: (\alpha(\iota) - 1, \infty) \times \{0, 1\} \rightarrow \mathbb{R}$ .

In the presence of transaction cost, the solvency region in each regime splits into three regions: buy region, sell region, and no-transaction (NT) region. Because of the time homogeneity of the value function, these regions can be identified by two critical numbers (instead of functions of time)  $\underline{z}(\iota)$  and  $\bar{z}(\iota)$  in regime  $\iota$ . The buy region corresponds to  $z \geq \bar{z}(\iota)$ , the sell region to  $z \leq \underline{z}(\iota)$ , and the NT region to  $\underline{z}(\iota) < z < \bar{z}(\iota)$ . We illustrate these three regions in regime  $\iota$  in Figure 1. As in the pure diffusion case, the investor does not trade as long as the ratio  $z$  remains inside the NT region. However, as soon as the ratio  $z$  moves out of the NT region, the investor immediately trades a minimum amount to get back to the NT region. Thus, the investor only trades on a measure zero set of times and follows a singular control strategy in stock trading.<sup>7</sup> However, in contrast to the pure diffusion cases previously studied, the ratio  $z$  can jump out of the NT region, which is followed by an immediate lump-sum transaction to the closest boundary of the NT region. Moreover, when the regime shifts, an investor might also need to make a lump-sum trade to the new boundary in the new regime.

Following Jang et al. (2007), we have the following coupled HJB equation:<sup>8</sup>

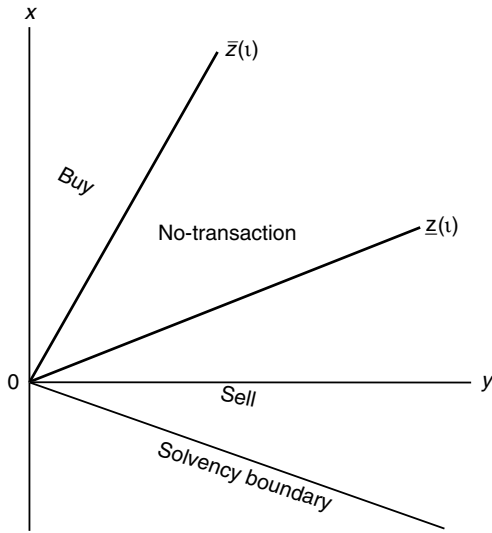
$$\max\{\mathcal{L}v, (1 - \alpha(\iota))v_x - v_y, -(1 + \theta(\iota))v_x + v_y\} = 0, \quad \iota = 0, 1, \quad (18)$$

<sup>7</sup> Although singular controls are a tractable way to solve for optimal portfolio strategies with transaction cost, they require trading at an infinite rate on a measure zero set of time points.

<sup>8</sup> The HJB equation follows from the fact that

$$e^{-\delta(\iota)t} v(x_t, y_t, \iota) + \int_0^t e^{-\delta(\iota)s} \left( f(x_s, y_s, \iota) + \lambda \frac{(x_s + (1 - \alpha(\iota))y_s)^{1-\gamma}}{1 - \gamma} \right) ds$$

is a martingale for the optimal policy. The expression of  $f(x, y, \iota)$  follows directly from (8).

Figure 1 The Solvency Region in Regime  $\iota$ 

where

$$\begin{aligned} \mathcal{L}v = & \frac{1}{2}\sigma(\iota)^2 y^2 v_{yy} + r(\iota)xv_x + (\mu(\iota) - \nu(\iota))yv_y - \delta(\iota)v \\ & + f(x, y, \iota) + \lambda \frac{(x + (1 - \alpha(\iota))y)^{1-\gamma}}{1 - \gamma}, \end{aligned} \quad (19)$$

$$\begin{aligned} f(x, y, \iota) = & \eta_1^U(\iota)E[v(x, yJ^U, 1 - \iota)] + \eta_2^U(\iota)E[v(x, yJ^U, \iota)] \\ & + \eta_1^D(\iota)E[v(x, yJ^D, 1 - \iota)] \\ & + \eta_2^D(\iota)E[v(x, yJ^D, \iota)] \\ & + \xi(\iota)v(x, y, 1 - \iota). \end{aligned} \quad (20)$$

Using (17), we can simplify (18) to get the following integro-differential variational equations:

$$\begin{aligned} \max\{\mathcal{L}_1\psi, (z + 1 - \alpha(\iota))\psi_z(z, \iota) - (1 - \gamma)\psi(z, \iota), \\ -(z + 1 + \theta(\iota))\psi_z(z, \iota) + (1 - \gamma)\psi(z, \iota)\} = 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \mathcal{L}_1\psi = & \frac{1}{2}\sigma(\iota)^2 z^2 \psi_{zz}(z, \iota) + \beta_2(\iota)z\psi_z(z, \iota) \\ & + \beta_1(\iota)\psi(z, \iota) + g(z, \iota), \\ g(z, \iota) = & \eta_1^U(\iota)E[\psi(z/J^U, 1 - \iota)(J^U)^{1-\gamma}] \\ & + \eta_2^U(\iota)E[\psi(z/J^U, \iota)(J^U)^{1-\gamma}] \\ & + \eta_1^D(\iota)E[\psi(z/J^D, 1 - \iota)(J^D)^{1-\gamma}] \\ & + \eta_2^D(\iota)E[\psi(z/J^D, \iota)(J^D)^{1-\gamma}] \\ & + \xi(\iota)\psi(z, 1 - \iota) + \lambda \frac{(z + 1 - \alpha(\iota))^{1-\gamma}}{1 - \gamma}, \\ \beta_1(\iota) = & -\delta(\iota) - (1 - \gamma)(\gamma\sigma(\iota)^2/2 - \mu(\iota) + \nu(\iota)), \\ \beta_2(\iota) = & \gamma\sigma(\iota)^2 - \mu(\iota) + r(\iota) + \nu(\iota). \end{aligned} \quad (22)$$

### 3. An Iterative Procedure to Find Optimal Trading Strategy

The fact that the ratio  $z$  can jump out of the NT region (reflected by the presence of the term  $g(z, \iota)$  in Equation (22)) and the regime can shift complicates the problem significantly. We now develop an iterative technique that solves the investor's problem using a sequence of closed-form expressions.

First, we choose an initial function  $v^0(x, y, \iota)$  that is finite, concave, increasing, and homogeneous such that  $v^0(x, y, \iota) \geq v(x, y, \iota)$  for  $\iota = 0, 1$ . For concreteness and ease of boundary conditions, we assume that

$$v^0(x, y, \iota) = M(\iota) \frac{(x + y)^{1-\gamma}}{1 - \gamma}, \quad (23)$$

where  $M(\iota)$  are the coefficients for the no-transaction-cost case.

Then to compute  $v^{i+1}(x, y, \iota)$ , for  $i = 0, 1, 2, \dots$ , we can solve the following recursive structure:

$$\begin{aligned} v^{i+1}(x, y, \iota) = & \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^\infty e^{-\delta(\iota)t} \left( f^i(x_t, y_t, \iota) \right. \right. \\ & \left. \left. + \lambda \frac{(x_t + (1 - \alpha(\iota))y_t)^{1-\gamma}}{1 - \gamma} \right) dt \right], \end{aligned} \quad (24)$$

where

$$\begin{aligned} f^i(x_t, y_t, \iota) = & \eta_1^U(\iota)v^i(x_t, y_t J_t^U, 1 - \iota) + \eta_2^U(\iota)v^i(x_t, y_t J_t^U, \iota) \\ & + \eta_1^D(\iota)v^i(x_t, y_t J_t^D, 1 - \iota) \\ & + \eta_2^D(\iota)v^i(x_t, y_t J_t^D, \iota) \\ & + \xi(\iota)v^i(x_t, y_t, 1 - \iota). \end{aligned} \quad (25)$$

Lemma 1 in the appendix guarantees the convergence of this iterative procedure and the concavity of the limit function  $\hat{v} (\equiv \lim_{i \rightarrow \infty} v^i)$ . To facilitate the proof that  $\hat{v}$  is indeed the value function, note that as before, for  $\iota = 0, 1$ , because of the homogeneity of  $v^i(x, y, \iota)$ , there exists a function  $\psi^i$  such that

$$v^i(x, y, \iota) = y^{1-\gamma} \psi^i\left(\frac{x}{y}, \iota\right).$$

Solving (24) reduces to finding functions  $\psi^i(z, \iota)$  for  $\iota = 0, 1$  such that

$$\begin{aligned} \frac{1}{2}\sigma(\iota)^2 z^2 \psi_{zz}^i + \beta_2(\iota)z\psi_z^i + \beta_1(\iota)\psi^i + g^{i-1}(z, \iota) = 0, \\ i = 1, \dots, n, \end{aligned} \quad (26)$$

where

$$\begin{aligned} g^{i-1}(z, \iota) = & \eta_1^U(\iota)E[\psi^{i-1}(z/J^U, 1 - \iota)(J^U)^{1-\gamma}] \\ & + \eta_2^U(\iota)E[\psi^{i-1}(z/J^U, \iota)(J^U)^{1-\gamma}] \\ & + \eta_1^D(\iota)E[\psi^{i-1}(z/J^D, 1 - \iota)(J^D)^{1-\gamma}] \\ & + \eta_2^D(\iota)E[\psi^{i-1}(z/J^D, \iota)(J^D)^{1-\gamma}] \\ & + \xi(\iota)\psi^{i-1}(z, 1 - \iota) + \lambda \frac{(z + 1 - \alpha(\iota))^{1-\gamma}}{1 - \gamma}, \end{aligned}$$

and  $\beta_1(\iota)$  and  $\beta_2(\iota)$  are the same as in (21). We then have the following result:

**THEOREM 2.** As  $i \rightarrow \infty$ , the functions  $v^i(x, y, \iota) = y^{1-\gamma} \psi^i(x/y, \iota)$  converge to the value function  $v(x, y, \iota)$  for  $\iota = 0, 1$ .

**PROOF.** See the appendix.

Theorem 2 shows that the iterative procedure can indeed closely approximate the value function and the corresponding optimal trading strategy. The basic intuition for the convergence of this iterative procedure is the monotonicity of the value function:  $v^i(x, y, \iota) \leq v^{i-1}(x, y, \iota)$ , as directly implied by the optimization structure (24) and the fact that  $v^i(x, y, \iota)$  is bounded by the value function  $v^0(x, y, \iota)$  for the no-transaction-cost case and the value function for investing only in the risk-free asset. The iterative procedure is formally similar to that used to solve a discrete time infinite horizon dynamic programming problem, where a time period in our model corresponds to the time between adjacent Poisson jumps. The proof of Theorem 2 implies that this iterative procedure can be readily applied to many other optimal portfolio choice problems and can significantly simplify computation especially for those problems that involve coupled nonlinear HJB equations.<sup>9</sup>

#### 4. Analytical Comparative Statics

The optimal trading strategy in regime  $\iota$  is no trading if  $\underline{z}(\iota) \leq z \leq \bar{z}(\iota)$ , selling stock to the boundary  $\underline{z}(\iota)$  if  $z < \underline{z}(\iota)$ , and buying stock to the boundary  $\bar{z}(\iota)$  if  $z > \bar{z}(\iota)$ . In contrast to a diffusion model, it is possible that  $z$  jumps out of the NT region, which would be followed by an immediate transaction back to the closest boundary. When the regime shifts, the optimal strategy may or may not involve an immediate transaction depending on how the boundaries change across regimes. It is helpful to discuss three possible types of NT regions across regimes we will encounter in our numerical work later.

**Case 1: Separated.** For example,  $\underline{z}(0) < \bar{z}(0) < \underline{z}(1) < \bar{z}(1)$ . In this case investors may sell some stock and buy more of the risk-free asset right after a stock price crash. This is consistent with the so-called flight-to-quality phenomenon, but in sharp contrast with the contrarian strategy predicted by a model with i.i.d. returns. This case occurs if the regime shifts from the liquid regime ( $\iota = 0$ ) to the illiquid regime ( $\iota = 1$ ) after a downward price jump and the new ratio  $z$  right

after the crash stays in the sell region of the illiquid regime. This case will typically obtain when the shift in the investment opportunity set across regimes is large and the expected time spent in the new regime is long so that the required transaction cost is justified.

**Case 2: Nested.** For example,  $\underline{z}(1) < \underline{z}(0) < \bar{z}(0) < \bar{z}(1)$ . In this case, if the regime shifts from the liquid regime and the jump magnitude at this time is not too large, then the investor will optimally not rebalance. However, if the regime shifts from the illiquid regime to the liquid regime, then the investor may buy or sell stock even without a price jump. In this case an investor optimally reduces transaction frequency until market conditions improve. Intuitively, this case will occur when the difference in the investment opportunity set is relatively small, the time spent in the illiquid regime is relatively short, and the transaction costs are relatively large.

**Case 3: Overlapping but nonnested.** For example,  $\underline{z}(0) < \underline{z}(1) < \bar{z}(0) < \bar{z}(1)$ . In contrast to Case 2, in the absence of upward jumps, the investor never sells the stock when the regime shifts from the illiquid regime to the liquid regime. This case lies between Cases 1 and 2 and typically occurs when the difference in the investment opportunity set is moderate and the transaction costs are relatively small.

We now present an upper bound on the lowest sell boundary of the two regimes in terms of the Merton lines (i.e., the optimal portfolio in the no-transaction-cost case).

**PROPOSITION 1.** Let  $z^*(\iota) = 1/\pi^*(\iota) - 1$ , where  $\pi^*(\iota)$  is the optimal portfolio in the no-transaction-cost case in regime  $\iota$ . Then

1. for an i.i.d. returns case, if  $h(\pi, \iota) = 0$ , then  $\underline{z}(\iota) \leq (1 - \alpha(\iota))z^*(\iota)$ ;
2. for a general case, if  $h(\pi, \iota) \neq 0$ , then

$$\min\{\underline{z}(\iota), \underline{z}(1 - \iota)\} \leq \max\{(1 - \alpha(\iota))z^*(\iota), (1 - \alpha(1 - \iota))z^*(1 - \iota)\}. \quad (27)$$

Our next result provides lower bounds on the transaction boundaries in the regime with the highest utility.

**PROPOSITION 2.** Suppose either  $h(\pi, \iota) = 0$  (i.i.d. case) or  $v(x, y, \iota) \geq v(x, y, 1 - \iota)$  with  $r(\iota) = r(1 - \iota)$ .<sup>10</sup> Then the buy boundary satisfies

$$\bar{z}(\iota) \geq (1 + \theta(\iota)) \left( \frac{\gamma \sigma(\iota)^2}{2(\mu(\iota) - r(\iota))} - 1 \right), \quad (28)$$

<sup>9</sup> For example, one could have a model with  $n$  regimes in which the coefficients and the jump distributions vary across regimes. This procedure can then solve an optimal portfolio problem with transaction costs with correlated stock price risk and investment opportunity set.

<sup>10</sup> Sufficient conditions for  $v(x, y, \iota) \geq v(x, y, 1 - \iota)$  are given in Proposition 3 in the appendix. The method of proof uses the iterative construction of the value function and can be used to provide a variety of comparative statics.



and the sell boundary satisfies

$$\bar{z}(\iota) \geq (1 - \alpha(\iota)) \left( \frac{\gamma \sigma(\iota)^2}{2(\mu(\iota) - r(\iota))} - 1 \right). \quad (29)$$

Propositions 1 and 2 give useful bounds on the transaction boundaries. In the i.i.d. case or in the regime with the highest utility, if  $\pi^*(\iota) > 0$ , then the sell boundary is always below the Merton line, the optimal ratio of bond to stock with no transaction costs. However, the sell boundary is not always a decreasing function of the transaction cost rate. For example, if  $\pi^*(\iota) > 1$  (i.e., with leverage), then the sell boundary can be above the Merton line for a large enough transaction cost rate. This can be seen from the extreme case where  $\alpha(\iota) = 1$  and thus the sell boundary has to be on or above zero by (29). The bounds obtained in Propositions 1 and 2 can be also useful for validating numerically computed boundaries.

## 5. Numerical Results

To gain some understanding of how the various elements of our model are optimally traded off, we now present a baseline case and perform comparative statics to see how the optimal boundaries behave.

To capture the idea that large price jumps likely have greater correlations with changes in the investment opportunity set than small jumps, for our numerical analysis we consider a slightly more general setting where large jumps can have different correlations from small jumps. More specifically, we divide jumps into large downward jumps, large upward jumps and moderate (upward or downward) jumps and allow large jumps to be correlated with regime switching. In other words, we assume

$$\begin{aligned} dS_t = & (\mu(\iota_t) - \nu(\iota_t))S_{t-}dt + \sigma(\iota_t)S_{t-}dw_t \\ & + (J_t^U - 1)S_{t-}dN_t^U + (J_t^D - 1)S_{t-}dN_t^D \\ & + (J_t^M - 1)S_{t-}dN_t^M, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \nu(\iota) = & \eta^U(\iota)E[J^U - 1] + \eta^D(\iota)E[J^D - 1] \\ & + \eta^M(\iota)E[J^M - 1], \end{aligned} \quad (31)$$

and the moderate jump Poisson process  $N_t^M$  is independent of all other jumps. All the previous results are easily extended to this case.<sup>11</sup>

For our numerical analysis, we use as our default parameters  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\theta(0) = \theta(1) = 0$ ,  $\mu(0) = \mu(1) = 7\%$ ,  $r(0) = r(1) = 1\%$ ,  $\gamma = 5$ , and  $\lambda = 0.04$ . These parameters represent an equity premium

of 6% in both regimes and an expected horizon of 25 years. The round-trip transaction cost is 0.5% in the liquid regime and 2.5% in the illiquid regime. We also assume in the baseline case volatilities are the same in both regimes, i.e.,  $\sigma(0) = \sigma(1)$ .

In our baseline case we assume that a jump arrives on average once every two years and jump intensities do not change across regimes, that is,  $\eta^U + \eta^M + \eta^D = 0.5$  with  $\eta^U(0) = \eta^U(1) = \eta^U$ ,  $\eta^M(0) = \eta^M(1) = \eta^M$ , and  $\eta^D(0) = \eta^D(1) = \eta^D$ .<sup>12</sup>

For  $i \in \{U, M, D\}$ , log jump size  $\log(J^i)$  is assumed to be truncated normal with parameters  $\mu_J$  and  $\sigma_J$  and support interval  $[a^i, b^i]$ , where  $a^U = \bar{R} > 0$ ,  $b^U = \infty$ ,  $a^M = \bar{R} < 0$ ,  $b^M = \bar{R}$ ,  $a^D = -\infty$ , and  $b^D = \bar{R}$ .

To determine the remaining baseline parameters, we calibrate the model to match the variance (0.0082), skewness (−1.33), and excess kurtosis (34.92) reported in Campbell et al. (1996, p. 21) for daily log returns. This procedure leads to  $\sigma(0) = \sigma(1) = 0.1190$ ,  $\mu_J = -0.0259$ , and  $\sigma_J = 0.0666$ . As default parameter values, we set  $\bar{R} = -0.03$  and  $\bar{R} = 0.03$ , which implies the average large up jump size is 7.0%, the average large down jump size is −7.8%, and the average moderate jump size is 0.0%. Using these parameter values, we obtain that  $\eta^U = 0.1003$ ,  $\eta^D = 0.2377$ , and  $\eta^M = 0.1620$ . These parameters indicate that the probability of a large down jump is greater than the probability of a large up jump, consistent with the negative skewness of the stock returns.

We assume regime 0 switches to regime 1 if and only if a large down jump occurs. To accomplish this, we set  $\xi(0) = 0$ ,  $\eta_1^U(0) = 0$ ,  $\eta_2^U(0) = \eta^U$ ,  $\eta_1^D(0) = \eta^D$ , and  $\eta_2^D(0) = 0$ . These choices capture the idea that worsened liquidity conditions are usually accompanied by large downward jumps in the stock price. Thus, the ratio  $x/y$  will jump up whenever there is a shift from the liquid to the illiquid regime.

Our baseline assumption is that the expected duration of the illiquid regime is one year. To accomplish this, we set  $\xi(1) = 0.9367$  and  $\eta_1^U(1) = 0.0633$ , which implies that the correlation between a large upward jump and switching into the liquid regime is 20%. Our remaining parameters are set to be consistent with our assumptions that the jump intensities, do not vary across regimes. The relation that  $\eta_1^U(1) + \eta_2^U(1) = \eta^U = 0.1003$  dictates that  $\eta_2^U(1) = 0.0370$ . The moderate jump intensity remains fixed at  $\eta^M = 0.1620$ . For the large down jump intensities, we set  $\eta_1^D(1) = 0$  so that the regime does not shift back to the liquid regime coincident with a large down jump. Thus,  $\eta^D = \eta_2^D(1) = 0.2377$ .

The default parameter values are summarized in Table 1.

<sup>11</sup> The analytical results for this generalized case are presented in an earlier version of the paper that is available from the authors.

<sup>12</sup> We also conducted analysis on different baseline cases with lower jump frequencies, which implies larger jump sizes on average. The qualitative results are the same.

**Table 1** Default Parameter Values

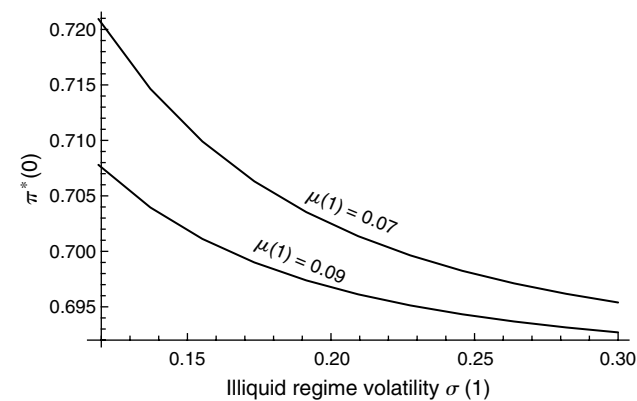
Parameters	$\mu_J$	$\sigma_J$	$\sigma(0)$	$\sigma(1)$	$\mu(0)$	$\mu(1)$	$r$
Values	-0.0259	0.0666	0.1190	0.1190	0.07	0.07	0.01
Parameters	$\lambda$	$\gamma$	$\xi(0)$	$\xi(1)$	$\theta(0)$	$\theta(1)$	$\alpha(0)$
Values	0.04	5	0	0.9367	0	0	0.005
Parameters	$\alpha(1)$	$\eta_1^U(0)$	$\eta_1^D(1)$	$\eta_2^D(0)$	$\eta_2^U(0)$	$\eta^U$	$\eta_2^U(1)$
Values	0.025	0	0	0	0.1003	0.1003	0.0370
Parameters	$\eta_1^U(1)$	$\eta^M$	$\eta_1^D(0)$	$\eta_2^D(1)$	$\eta^D$	$\bar{R}$	$\bar{R}$
Values	0.0633	0.1620	0.2377	0.2377	0.2377	-0.03	0.03

We now examine the optimal portfolio policies for our model. For ease of comparison with the no-transaction-cost case, we will present the transaction boundaries in terms of the fraction of wealth invested in the risky asset:  $\pi = y/(x + y)$ . Let  $\underline{\pi}(t) = 1/(\bar{z}(t) + 1)$  and  $\bar{\pi}(t) = 1/(z(t) + 1)$ . The optimal policy is equivalent to maintaining the fraction  $\pi$  between  $\underline{\pi}(t)$  and  $\bar{\pi}(t)$ .

### 5.1. Without Transaction Costs

For our baseline case with no transaction costs it is optimal to hold 70.8% of wealth in the risky asset in both regimes. In the absence of jumps (the i.i.d. Merton case), the optimal fraction of wealth invested in stock is 84.7%. The reduction in the stock investment compared to the i.i.d. case is caused by the jumps that result in higher volatility and negative skewness. We now graphically illustrate the cross-regime hedging effect in the absence of transaction costs in both regimes, as discussed in §2. In Figure 2, we plot the regime 0 optimal fraction of wealth  $\pi^*(0)$  against the volatility in regime 1 for two cases:  $\mu(1) = 7\%$  and  $\mu(1) = 9\%$ , in the absence of transaction costs in both regimes. Figure 2 shows that

**Figure 2** Optimal Portfolio in Regime 0,  $\pi^*(0)$ , as a Function of  $\sigma(1)$  Without Transaction Costs



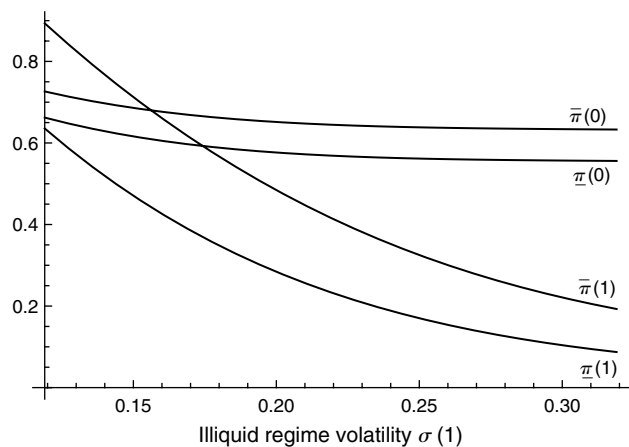
*Note.* This figure shows how the optimal fractions vary with the volatility in the illiquid regime  $\sigma(1)$  for parameters  $\mu_J = -0.0259$ ,  $\sigma_J = 0.0666$ ,  $\sigma(0) = 0.1190$ ,  $\mu(0) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = \alpha(1) = 0$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ ,  $\eta_2^U(0) = \eta^U = 0.1003$ ,  $\eta_2^U(1) = 0.0370$ ,  $\eta_1^U(1) = 0.0633$ ,  $\eta^M = 0.1620$ , and  $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.2377$ .

in contrast to standard models (e.g., Merton 1971, Jang et al. 2007), the optimal trading strategy in regime 0 depends on the investment opportunity set in regime 1 even when both regimes are perfectly liquid. For example, the optimal fraction of wealth invested in stock in the liquid regime decreases from 70.8% to 69.6% when the volatility in the illiquid regime increases from 11.9% to 20%. On the other hand, if the expected return in regime 1 is also higher (e.g., 0.09), then the net hedging demand may be positive or negative depending on whether the expected return increase or the volatility increase dominates.

### 5.2. Changes in the Volatility in the Illiquid Regime

Next, we examine the effect of the post-crash changes in market volatility and liquidity. It is well documented that both volatility and illiquidity tend to be greater after a crash. Accordingly, in Figure 3 we show how the optimal transaction boundaries vary when volatility rises and a market becomes less liquid after a crash. In the absence of transaction costs, it is optimal to always keep 70.8% (in both regimes) of the wealth in stock in our baseline case. With positive transaction costs, this policy is no longer optimal. Figure 3 implies that it is optimal to keep the fraction of wealth invested in stock between 66.1% and 72.5% in the liquid regime and between 63.5% and 87.5% in the illiquid regime. The transaction boundaries of the two regimes are thus nested. In other words, the investor does not transact when the regime shifts from the liquid regime to the illiquid regime unless the price drop is too large in magnitude. The trading frequency in

**Figure 3** Optimal Trading Boundaries as a Function of  $\sigma(1)$



*Note.* This figure shows how the optimal trading boundaries vary with the volatility in the illiquid regime  $\sigma(1)$  for parameters  $\mu_J = -0.0259$ ,  $\sigma_J = 0.0666$ ,  $\sigma(0) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ ,  $\eta_2^U(0) = \eta^U = 0.1003$ ,  $\eta_2^U(1) = 0.0370$ ,  $\eta_1^U(1) = 0.0633$ ,  $\eta^M = 0.1620$ , and  $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.2377$ .

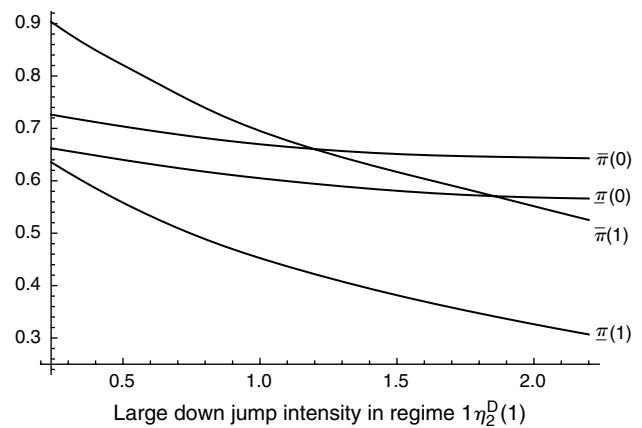
the illiquid regime is lower, as implied by the wider no-transaction region due to greater transaction costs.

As the volatility in the illiquid regime increases, all transaction boundaries move downward, which implies that the investor decreases stock investment not only in the illiquid regime but also in the liquid regime, because of the cross-regime hedging. As expected, the transaction boundaries in the illiquid regime are much more sensitive to the illiquid regime volatility increase than those in the liquid regime. Figure 3 shows that even for modest increases in volatility in the illiquid regime, the NT region in the illiquid regime can move well below that in the liquid regime. This implies that even small increases in the after-crash volatility can make investors sell stock and buy more of the risk-free asset right after the price crash, behaving like flight to quality, that was commonly observed after a market crash. For example, after a 5% price crash, if the price crash reflects significantly worsened fundamentals (e.g., much greater uncertainty of the stock payoff) and as a result the after-crash stock volatility increases from 11.9% to 20%, then the investor will keep the fraction of wealth invested in stock between 56.0% and 63.7% in the liquid regime, and between 23.5% and 43.7% in the illiquid regime. Therefore, upon the crash, the investor will sell stock so that the stock fraction becomes 43.7% after rebalancing. In contrast, standard portfolio choice models with i.i.d. returns (e.g., Merton 1971) predict the opposite: After a price drop, the investor should buy more to rebalance. Thus, the deterioration of investment opportunity set after a crash may contribute to the flight to quality behavior.

### 5.3. Changes in the Intensity of a Large Downward Jump in the Illiquid Regime

Our baseline case assumes that only liquidity changes after a market crash. However, after a market crash, the probability of another crash might also change. Figure 4 shows how the optimal trading boundaries vary as we vary the intensity of a large downward jump in the illiquid regime ( $\eta_2^D(1)$ ). For higher values of the intensity of another market crash in the illiquid regime, the investor is more likely to sell right after a market crash in the liquid regime, all else equal. Recall that varying the jump parameters does not affect the expected return but affects variance, skewness, and kurtosis. A large negative jump accompanied by the transition into the illiquid regime where large downward jumps occur more frequently represents a significant deterioration of the investment opportunity set. Thus, the investor might optimally incur the transaction cost to sell some stock upon a crash, similar to the effect of increased volatility. For example, if the intensity of further crash increases to 2.0 after a crash (i.e., on average one crash every six

Figure 4 Optimal Trading Boundaries as a Function of  $\eta_2^D(1)$



Note. This figure shows how the optimal trading boundaries vary with the intensity of the downward jump  $\eta_2^D(1)$  for parameters  $\mu_J = -0.0259$ ,  $\sigma_J = 0.0666$ ,  $\sigma(0) = \sigma(1) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ ,  $\eta_2^U(0) = \eta_2^U = 0.1003$ ,  $\eta_2^U(1) = 0.0370$ ,  $\eta_1^U(1) = 0.0633$ ,  $\eta^M = 0.1620$ , and  $\eta_1^D(0) = \eta^D = 0.2377$ .

months), then the investor will keep the fraction of wealth invested in stock between 56.8% and 64.5% in the liquid regime, and between 32.6% and 55.1% in the illiquid regime. Therefore, upon a crash, the investor may sell some stock to reach 55.1%.

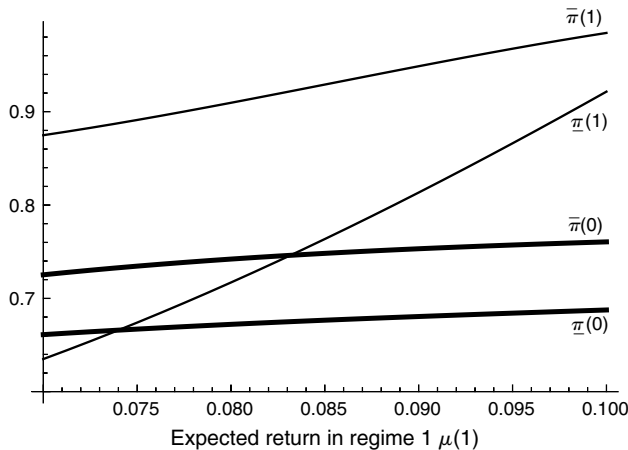
In addition, Figure 4 suggests that in anticipation of this increase in the crash intensity, the investment in the liquid regime is also reduced.

### 5.4. Changes in the Expected Return in the Illiquid Regime

Figure 5 shows how the transaction boundaries vary as a function of the illiquid-regime expected return  $\mu(1)$ . If the after-crash expected return goes up (as found in empirical studies; e.g., Fama and French 1989, Ferson and Harvey 1991), the NT region can be higher in the illiquid regime than in the liquid regime, which implies that the investor would hold more stock in the illiquid regime. In this case the investor will always buy more of the risky asset to take advantage of the higher expected return when the regime shifts from the liquid to the illiquid regime and liquidate the position when the market becomes more liquid. For example, if the after-crash expected return increases to 9%, then it is optimal for the investor to keep the fraction of wealth in stock between 68.1% and 75.4% in the liquid regime and between 80.1% and 95.3% in the illiquid regime. Figures 3–5 suggest that the effect of the increased expected return on the optimal trading strategy counteracts the effect of the increased volatility and the increased crash intensity. The flight to quality behavior ensues when the volatility and the crash intensity effects dominate.

As the expected return in the illiquid regime increases, the NT region shrinks. Intuitively, when

Figure 5 Optimal Trading Boundaries as a Function of  $\mu(1)$



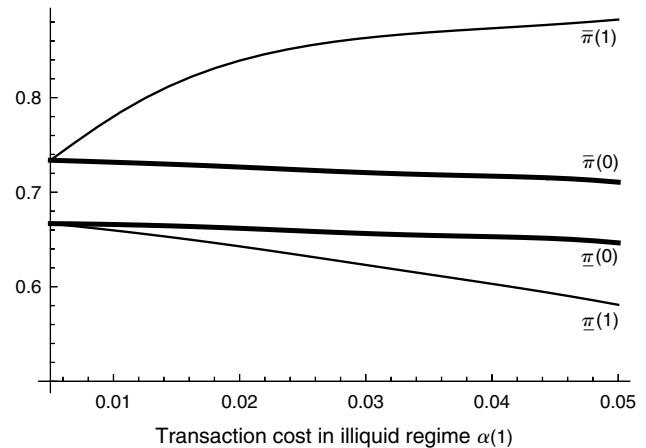
Note. This figure shows how the optimal trading boundaries vary with the expected return in the illiquid regime  $\mu(1)$  for parameters  $\mu_J = -0.0259$ ,  $\sigma_J = 0.0666$ ,  $\sigma(0) = \sigma(1) = 0.1190$ ,  $\mu(0) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ ,  $\eta_2^U(0) = \eta^U = 0.1003$ ,  $\eta_2^U(1) = 0.0370$ ,  $\eta_1^U(1) = 0.0633$ ,  $\eta^M = 0.1620$ , and  $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.2377$ .

an investor determines the no-transaction region size, he trades off the transaction cost payment and the risk exposure variation (i.e., the variation in the fraction of wealth invested in stock). As the expected return increases, the stock price tends to go up faster and therefore the dollar amount invested in the stock tends to increase faster. As a result, the probability of hitting the lower boundary and incurring transaction costs at the lower boundary becomes lower. In addition, because the investor sells at the upper boundary when the price and thus his wealth goes up and buys at the lower boundary when the price and thus his wealth goes down, the marginal utility cost of transaction cost payment at the upper boundary is lower than that at the lower boundary. Therefore, to avoid too much risk exposure variation, the investor can increase the lower boundary more than the upper boundary without significantly increasing the marginal utility cost of transaction cost payment. This causes the no-transaction region to decrease with the expected return in the illiquid regime, as also shown in Liu and Loewenstein (2002, Figure 6). When the expected return in the illiquid regime is much higher than that in the liquid regime, the probability of hitting the lower boundary becomes much lower in the illiquid regime, and thus the no-transaction region can become even smaller than that in the liquid regime even though the transaction cost rate is much higher in the illiquid regime.

### 5.5. Changes in the Illiquidity in the Illiquid Regime

Figure 6 shows how the transaction boundaries change as the transaction costs vary in the illiquid

Figure 6 Optimal Trading Boundaries as a Function of  $\alpha(1)$



Note. This figure shows how the optimal trading boundaries vary with the transactions cost in the illiquid regime  $\alpha(1)$  for parameters  $\mu_J = -0.0259$ ,  $\sigma_J = 0.0666$ ,  $\sigma(0) = \sigma(1) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ ,  $\eta_2^U(0) = \eta^U = 0.1003$ ,  $\eta_2^U(1) = 0.0370$ ,  $\eta_1^U(1) = 0.0633$ ,  $\eta^M = 0.1620$ , and  $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.2377$ .

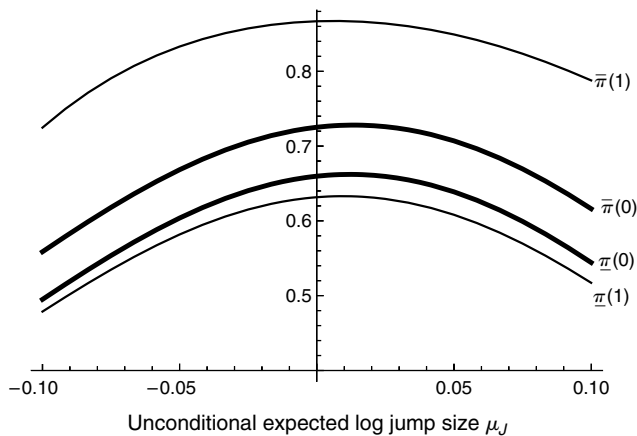
regime. The illiquid regime NT region nests the liquid regime NT region. For large transaction costs in the illiquid regime, the investor significantly widens the NT region in the illiquid regime to reduce trading frequency. Thus, for large transaction costs, it is optimal to try to wait out the illiquid regime. As transaction costs in the illiquid regime increase, the investor also optimally holds less stock in the liquid regime.

### 5.6. Changes in Mean of the Log Jump Size

The next set of results addresses the sensitivity to the jump size distribution. For this we return to our baseline model and maintain the assumption that the unconditional log jump size  $\log(J)$  is normally distributed with mean  $\mu_J$  and volatility  $\sigma_J$ . As we vary  $\mu_J$  or  $\sigma_J$ , we change the values of  $\eta^U$ ,  $\eta^M$ , and  $\eta^D$  so that as before large up jumps correspond to a greater than 3% jump size, moderate jumps between -3% and 3%, and large down jumps to less than -3%. We maintain all other assumptions. Note that as  $\mu_J$  goes down, the jumps tend to be more negatively skewed and, in addition, the possibility of a large down jump goes up while the possibility of a large up jump decreases. Thus, as we decrease  $\mu_J$ , in our baseline model, it becomes more likely that the liquid regime shifts to the illiquid regime.

Figure 7 shows how the optimal transaction boundaries vary against  $\mu_J$  when the expected return remains the same at 7% in both the liquid and the illiquid regimes. This figure reveals, similar to the findings in Liu et al. (2003), that the optimal trading boundaries are nonmonotonic with some asymmetry across positive and negative jumps. It also implies that an increase in the expected jump size may increase the

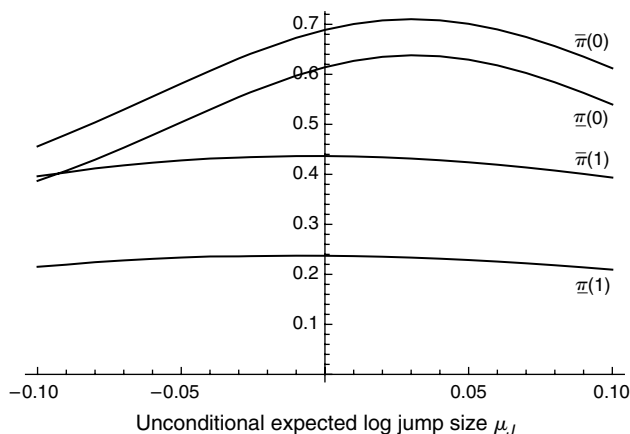


**Figure 7** Optimal Trading Boundaries as a Function of  $\mu_J$ 

*Note.* This figure shows how the optimal trading boundaries vary with  $\mu_J$  for parameters  $\sigma_J = 0.0666$ ,  $\sigma(0) = \sigma(1) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ , and  $\eta_1^U(1) = 0.0633$ .

optimal stockholding even if the expected returns are held constant and the total return volatility increases. Intuitively, when the expected value of a jump is positive, the jump helps the investor by introducing a positive skew to returns. However, the jump also increases return volatility. Whether the optimal stock investment increases or decreases depends on whether the skewness effect or the volatility effect dominates. The asymmetry occurs because downward jumps tend to introduce a negative skew, which tends to bring the investor closer to the solvency boundary and the associated higher marginal utility.

Figure 8 shows how the optimal trading boundaries vary as we change  $\mu_J$  as before, but instead

**Figure 8** Optimal Trading Boundaries as a Function of  $\mu_J$  with  $\sigma(1) = 20\%$ 

*Note.* This figure shows how the optimal trading boundaries vary with  $\mu_J$  for parameters  $\sigma_J = 0.0666$ ,  $\sigma(0) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ , and  $\eta_1^U(1) = 0.0633$ .

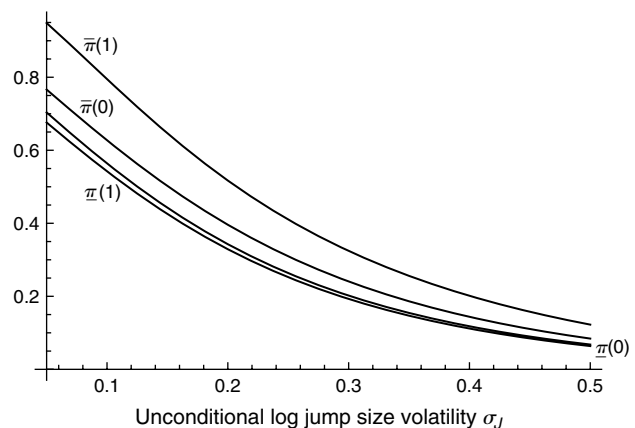
we assume that the after-crash volatility is 20% in the illiquid regime. Again we see the nonmonotonic transaction boundaries, but now they are almost always separated and the NT region is lower in the illiquid regime than in Figure 7, which is driven by the worsened investment opportunity set in the illiquid regime. Interestingly, in the liquid regime the NT region in Figure 8 is quite similar to that in Figure 7 except for large negative jump sizes. This occurs because even though the investor knows he will hold less of the risky asset in the illiquid regime following a market crash, he also knows that a market crash will already make him hold less of the risky asset even without any trading, and thus the required transaction cost payment may be small when the regime switches. Therefore, it is less costly to hold more of the risky asset in the liquid regime with a larger expected jump size.

### 5.7. Changes in Volatility of the Log Jump Size

Figure 9 shows how the optimal transaction region varies with the unconditional log jump size volatility  $\sigma_J$  (with the same expected return in both regimes). When  $\sigma_J$  gets large, the transaction boundaries generally go down because the increase in volatility makes the stock less attractive.

### 5.8. How Other Parameters Affect the Optimal Trading Boundaries and Hedging Demands

We show how other parameters affect the optimal trading boundaries in the top section of Table 2. The “baseline” row corresponds to the baseline case and the other rows correspond to a change in the stated parameter alone from the baseline case. Consistent with Liu and Loewenstein (2002), as the expected investment horizon  $1/\lambda$  decreases, the investor invests

**Figure 9** Optimal Trading Boundaries as a Function of  $\sigma_J$ 

*Note.* This figure shows how the optimal trading boundaries vary with  $\sigma_J$  for parameters  $\mu_J = -0.0259$ ,  $\sigma(0) = \sigma(1) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ , and  $\eta_1^U(1) = 0.0633$ .

**Table 2** Boundaries and Hedging Demands as Other Parameter Values Change

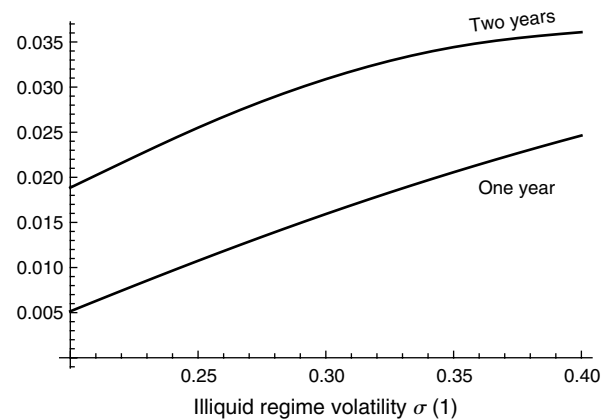
Parameters	With TC				Without TC	
	$\pi(0)$	$\bar{\pi}(0)$	$\pi(1)$	$\bar{\pi}(1)$	$\pi^*(0)$	$\pi^*(1)$
Baseline	0.661	0.725	0.635	0.875	0.708	0.708
$\lambda = 0.05$	0.657	0.722	0.630	0.877	0.708	0.708
$r = 0.005$	0.719	0.778	0.698	0.907	0.765	0.765
$\gamma = 4$	0.839	0.888	0.812	0.952	0.883	0.883
$\mu(0) = 0.08$	0.743	0.800	0.631	0.883	0.815	0.708
$\sigma(0) = 0.1$	0.813	0.861	0.633	0.893	0.917	0.708
$\alpha(0) = 0.1\%$ , $\alpha(1) = 0.25\%$	0.688	0.724	0.683	0.742	0.708	0.708
$\rho(1) = 0.1$	0.658	0.723	0.633	0.870	0.708	0.708
$\rho(0) = 0.6$	0.666	0.732	0.636	0.884	0.708	0.708
Percentage hedging demand						
Baseline	-1.2	-1.5	4.0	16.9	0	0
$\lambda = 0.05$	-1.7	-1.9	3.6	17.3	0	0
$r = 0.005$	-1.5	-1.6	3.2	13.3	0	0
$\gamma = 4$	-1.6	-1.5	3.3	5.3	0	0
$\mu(0) = 0.08$	-5.6	-5.1	3.4	18.1	-0.8	0.0
$\sigma(0) = 0.1$	-9.3	-8.0	3.8	19.4	-0.9	0.0
$\alpha(0) = 0.1\%$ , $\alpha(1) = 0.25\%$	-0.1	-0.2	0.6	1.6	0	0

Note. TC, transaction costs.

less in the stock to reduce the impact of transaction costs. As expected, with a decrease in interest rate, risk aversion, volatility, the correlation between the large downward jump and switching into the illiquid regime ( $\rho(0)$ ), or an increase in expected return, stock investment increases. With a decrease in the correlation between the large upward jump and switching into the liquid regime ( $\rho(1)$ ), stock investment decreases. Similar to the effect of increasing the transaction cost rate in the illiquid regime, the NT regions widen in both regimes when the liquidity in the liquid regime increases.

The bottom section of Table 2 examines how hedging demands vary with parameter values. It is difficult to come up with a precise measure of hedging demands in a model with transaction costs and jumps. However, a reasonable way to measure hedging demands is to compare the optimal portfolio policy when changes in the investment opportunity set are correlated with the jumps to that when the correlation is zero (i.e.,  $\eta_1^D(0) = \eta_1^U(1) = \xi(0) = \xi(1) = 0$ ). The bottom part of Table 2 reports the percentage difference between these portfolios. As expected, to hedge against changes in the investment opportunity set from regime shifts, the investors increase (decrease) stock investment when regime switching into a better (worse) investment opportunity set is possible. It is interesting to note that the magnitude of the changes in the transaction boundaries is much higher than those in the no-transaction-cost case.

**Figure 10** Certainty Equivalent Wealth Loss as a Function of  $\sigma(1)$



Notes. This figure shows how the certainty equivalent wealth loss from using wrong estimates as a fraction of the initial wealth varies with  $\sigma(1)$  for true parameters  $\mu_L = -0.0259$ ,  $\sigma_L = 0.0666$ ,  $\sigma(0) = 0.1190$ ,  $\mu(0) = \mu(1) = 0.07$ ,  $r = 0.01$ ,  $\lambda = 0.04$ ,  $\gamma = 5$ ,  $\xi(0) = 0$ ,  $\xi(1) = 0.9367$ ,  $\theta(0) = \theta(1) = 0$ ,  $\alpha(0) = 0.5\%$ ,  $\alpha(1) = 2.5\%$ ,  $\bar{R} = -0.03$ ,  $\bar{R} = 0.03$ ,  $\eta_1^U(0) = \eta_1^D(1) = \eta_2^D(0) = 0$ ,  $\eta_2^U(0) = \eta_1^U = 0.1003$ ,  $\eta_2^U(1) = 0.0370$ ,  $\eta_1^U(1) = 0.0633$ ,  $\eta_1^M = 0.1620$ , and  $\eta_1^D(0) = \eta_2^D(1) = \eta^D = 0.2377$ . The wrong estimates are  $\eta_1^D(0) = 0.1^2 \times \eta^D$ , and  $\eta_2^D(0) = (1 - 0.1^2) \times \eta^D$ .

### 5.9. Certainty Equivalent Wealth Loss from Misestimation of the Correlation Between Market Crash and Market Illiquidity

So far we have been focusing on the analysis of the optimal trading strategies. Next, as an example, we show the economic significance of correctly taking into account the correlation between market crashes and market illiquidity. Specifically, suppose an investor underestimates the correlation between market crashes and market illiquidity to be 0.1, although the true correlation is 1, and adopts the optimal trading strategy that is based on the wrong estimate. We compute the certainty equivalent wealth loss as a fraction of his initial wealth from this misestimation. Figure 10 plots this loss against the volatility in the illiquid regime for two cases, one with expected illiquid regime duration of one year and the other two years. Figure 10 shows that misestimation of the correlation is costly to the investor. For example, the equivalent wealth loss can be as high as 2% for the one-year case and about 3.6% for the two-year case. This finding indicates the economic importance of correctly taking into account the correlation between market crashes and market illiquidity.

## 6. Conclusions

In this paper, we develop a tractable workhorse model of optimal portfolio choice with market crashes and correlated changes in the investment opportunity set (e.g., higher illiquidity, greater volatility). Our analysis demonstrates that the presence of these risks can be an important factor in determining an optimal

portfolio. We also provide an efficient iterative solution procedure that can be applied to a wide class of models with coupled integro-differential equations with free boundaries. Given its incorporation of many of the important determinants of portfolio selection and its tractability, our model provides an attractive framework for studying the joint qualitative and quantitative impact of event risks, liquidity risks, and time-varying return dynamics.

Several extensions to our analysis are immediate. For example, one can examine the effects of a deterministic horizon by using the methodology proposed in Liu and Loewenstein (2002). Although our model is formulated with two regimes, the extension to  $n$  regimes with differing jumps distributions is conceptually straightforward.

### Acknowledgments

The authors thank Dilip Madan, George Constantinides, Phil Dybvig, Jeongmin Lee, Yingzi Zhu, and participants at the 2007 China International Conference in Finance and 2008 American Finance Association (AFA) meetings for their helpful comments. The authors are grateful to Francis Longstaff (AFA discussant) for his helpful suggestions. The comments of two anonymous referees, an anonymous associate editor, and Wei Xiong (the department editor) improved the paper significantly.

### Appendix. Proofs

**PROOF OF EQUATION (8).** Consider the special case where only  $\eta_1^D(0)$  and  $\lambda$  are nonzero and the current regime is regime 0. Let  $\hat{\tau}$  denote the random downward jump time. Upon the downward jump, the stock amount becomes  $y_{\hat{\tau}} J_{\hat{\tau}}^D$  and the regime becomes regime 1. Therefore, the value function becomes  $v(x_{\hat{\tau}}, y_{\hat{\tau}} J_{\hat{\tau}}^D, 1)$  right after the jump. Then the value function in regime 0 can be rewritten as

$$\sup_{(D, I) \in \Theta(x, y)} E \left[ 1_{\hat{\tau} < \tau} v(x_{\hat{\tau}}, y_{\hat{\tau}} J_{\hat{\tau}}^D, 1) + 1_{\tau < \hat{\tau}} \frac{(x_{\tau} + (1 - \alpha(\tau))y_{\tau})^{1-\gamma}}{1 - \gamma} \right]. \quad (32)$$

Because  $\hat{\tau}$  and  $\tau$  follow independent exponential distributions with parameters  $\eta_1^D(0)$  and  $\lambda$ , (32) becomes

$$\begin{aligned} & \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^{\tau} e^{-\eta_1^D(0)t} \eta_1^D(0) v(x_t, y_t J_t^D, 1) dt \right. \\ & \quad \left. + \int_{\tau}^{\infty} e^{-\eta_1^D(0)t} \eta_1^D(0) \frac{(x_t + (1 - \alpha(t))y_t)^{1-\gamma}}{1 - \gamma} dt \right] \\ &= \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^{\infty} e^{-\lambda s} \lambda \left( \int_0^s e^{-\eta_1^D(0)t} \eta_1^D(0) v(x_t, y_t J_t^D, 1) dt \right. \right. \\ & \quad \left. \left. + e^{-\eta_1^D(0)s} \frac{(x_s + (1 - \alpha(s))y_s)^{1-\gamma}}{1 - \gamma} \right) ds \right] \\ &= \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^{\infty} e^{-(\lambda + \eta_1^D(0))t} \left( \eta_1^D(0) v(x_t, y_t J_t^D, 1) \right. \right. \\ & \quad \left. \left. + \lambda \frac{(x_t + (1 - \alpha(t))y_t)^{1-\gamma}}{1 - \gamma} \right) dt \right], \quad (33) \end{aligned}$$

after integrating out  $s$  by interchanging  $s$  and  $t$ .

The same procedure leads to Equation (8) in the paper when other intensities are also nonzero. Q.E.D.

**PROOF OF THEOREM 1.** To save space, we only provide the main steps for the proof because the details are standard. Let  $v(W, \iota)$  be as defined in (16) and define

$$M_t = \frac{W_t^{1-\gamma}}{1-\gamma} 1_{\{t=\tau\}} + v(W_t, \iota_t) 1_{\{t<\tau\}}. \quad (34)$$

Then using (12), (13), and the generalized Itô's lemma (see, e.g., Harrison 1985, §4.7), one can show that  $M_t$  is a supermartingale for any admissible policy  $\pi$  that is bounded away from the solvency constraint and a martingale for the claimed optimal policy  $\pi_t^*$ , which implies that  $M_0 \geq E[M_{\tau}]$  by the optional sampling theorem, i.e.,

$$v(W, \iota) \geq E \left[ \frac{W_{\tau}^{1-\gamma}}{1-\gamma} \mid W_0 = W, \iota_0 = \iota \right], \quad (35)$$

and with equality for the claimed optimal policy. Q.E.D.

The following result is useful for proving the validity of the iterative approach.

**LEMMA 1.** For  $\iota = 0, 1$  and  $i = 0, 1, \dots, n$ ,  $v^i$  is increasing, concave, and satisfies

$$v^i(x, y, \iota) \geq v^{i+1}(x, y, \iota) \geq \frac{\lambda}{\lambda - (1 - \gamma) \min(r(\iota), r(1 - \iota))} \cdot \frac{(x + (1 - \alpha(\iota))y)^{1-\gamma}}{1 - \gamma}. \quad (36)$$

In addition,  $v^i$  satisfies  $v^i(p x, p y) = p^{1-\gamma} v^i(x, y)$  for any  $p > 0$ . Thus, these functions converge to concave functions  $\hat{v}(x, y, \iota)$ . The convergence is uniform on compact subsets of the interior of the solvency region.

**PROOF.** Monotonicity, concavity, and homogeneity are fairly obvious for  $v^1$ , which inherits these properties from  $v^0$  and subsequently  $v^{i+1}$  inherits these properties from  $v^i$  (see, e.g., Shreve and Soner 1994). The inequalities

$$v^i(x, y, \iota) \geq \frac{\lambda}{\lambda - (1 - \gamma) (\min(r(\iota), r(1 - \iota)))} \cdot \frac{(x + (1 - \alpha(\iota))y)^{1-\gamma}}{1 - \gamma} \quad (37)$$

follow from the fact that an investor must have greater utility than that obtained from liquidating the risky asset investment and investing all wealth in the riskless asset until time  $\tau$ . The other inequalities  $v^i(x, y, \iota) \geq v^{i+1}(x, y, \iota)$  are deduced as follows. Observe that for  $\iota = 0, 1$ , we have

$$\begin{aligned} v^1(x, y, \iota) &= \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^{\infty} e^{-\delta(\iota)t} \left( f^0(x_t, y_t, \iota) \right. \right. \\ & \quad \left. \left. + \lambda \frac{(x_t + (1 - \alpha(\iota))y_t)^{1-\gamma}}{1 - \gamma} \right) dt \right] \\ &\leq \sup_{\pi} E \left[ \int_0^{\infty} e^{-\delta(\iota)t} \left( f^0(x_t, y_t, \iota) + \lambda \frac{(x_t + y_t)^{1-\gamma}}{1 - \gamma} \right) dt \right] \\ &= v^0(x, y, \iota), \quad (38) \end{aligned}$$

where the first equality follows from the dynamic programming principle for the no-transaction-cost case and the

inequality holds because admissible trading policies with transaction costs are also admissible in the no-transaction-cost case. Now assume  $v^i(x, y, \iota) \leq v^{i-1}(x, y, \iota)$ . Then this implies that

$$\begin{aligned} v^{i+1}(x, y, \iota) &= \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^\infty e^{-\delta(\iota)t} \left( f^i(x_t, y_t, \iota) \right. \right. \\ &\quad \left. \left. + \lambda \frac{(x_t + (1 - \alpha(\iota))y_t)^{1-\gamma}}{1 - \gamma} \right) dt \right] \\ &\leq \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^\infty e^{-\delta(\iota)t} \left( f^{i-1}(x_t, y_t, \iota) \right. \right. \\ &\quad \left. \left. + \lambda \frac{(x_t + (1 - \alpha(\iota))y_t)^{1-\gamma}}{1 - \gamma} \right) dt \right] \\ &= v^i(x, y, \iota). \end{aligned} \quad (39)$$

The last statements of the lemma follow from (Rockafellar 1970, Theorem 10.8). Q.E.D.

**PROOF OF THEOREM 2.** Assumption 1 implies that  $\beta_1(\iota) < 0$ . Thus, the homogeneous solution to (26) is given by  $\psi_1(z, \iota) = |z|^{n_1(\iota)}$  and  $\psi_2(z, \iota) = |z|^{n_2(\iota)}$ , where

$$\begin{aligned} n_{1,2}(\iota) &= \frac{(\sigma(\iota)^2/2 - \beta_2(\iota)) \pm \sqrt{(\sigma(\iota)^2/2 - \beta_2(\iota))^2 - 4\beta_1(\iota)}}{\sigma(\iota)^2} \end{aligned} \quad (40)$$

with  $n_1(\iota) > 0$  and  $n_2(\iota) < 0$ . This leads to the general solution to (26) (see Boyce and DiPrima 2001):

$$\psi^i(z, \iota) = C_1^i(\iota)\psi_1(z, \iota) + C_2^i(\iota)\psi_2(z, \iota) + \psi_p^i(z, \iota), \quad (41)$$

where  $C_1^i(\iota)$  and  $C_2^i(\iota)$  are integration constants and  $\psi_p^i(z, \iota)$  is the particular solution

$$\psi_p^i(z, \iota) = u_1^i(z, \iota)\psi_1(z, \iota) + u_2^i(z, \iota)\psi_2(z, \iota), \quad (42)$$

where

$$\begin{aligned} u_1^i(z, \iota) &= \int^z \frac{\psi_2(s, \iota)}{\psi_1'(s, \iota)\psi_2(s, \iota) - \psi_1(s, \iota)\psi_2'(s, \iota)} \frac{2g^{i-1}(s, \iota)}{\sigma(\iota)^2 s^2} ds, \\ u_2^i(z, \iota) &= - \int^z \frac{\psi_1(s, \iota)}{\psi_1'(s, \iota)\psi_2(s, \iota) - \psi_1(s, \iota)\psi_2'(s, \iota)} \frac{2g^{i-1}(s, \iota)}{\sigma(\iota)^2 s^2} ds; \end{aligned}$$

equivalently,

$$\begin{aligned} \psi_p^i(z, \iota) &= \psi_p^{i-1}(z, \iota) + \int^z \frac{\psi_1(s, \iota)\psi_2(z, \iota) - \psi_1(z, \iota)\psi_2(s, \iota)}{\psi_1'(s, \iota)\psi_2(s, \iota) - \psi_1(s, \iota)\psi_2'(s, \iota)} \\ &\quad \cdot \frac{2(g^{i-1}(s, \iota) - g^{i-2}(s, \iota))}{\sigma(\iota)^2 s^2} ds. \end{aligned}$$

For  $\iota = 0, 1$ , the HJB equations imply that

$$\psi^i(z, \iota) = \begin{cases} A^i(\iota) \frac{(z + 1 + \theta(\iota))^{1-\gamma}}{1 - \gamma} & \text{if } z \geq \bar{z}^i(\iota), \\ C_1^i(\iota)\psi_1(z, \iota) + C_2^i(\iota)\psi_2(z, \iota) + \psi_p^i(z, \iota) & \text{if } \max(\bar{z}^i(\iota), 0) \leq z < \bar{z}^i(\iota), \\ \hat{C}_1^i(\iota)\psi_1(z, \iota) + \hat{C}_2^i(\iota)\psi_2(z, \iota) + \psi_p^i(z, \iota) & \text{if } \min(\bar{z}^i(\iota), 0) \leq z < 0, \\ B^i(\iota) \frac{(z + 1 - \alpha(\iota))^{1-\gamma}}{1 - \gamma} & \text{if } \alpha(\iota) - 1 < z \leq \bar{z}^i(\iota), \end{cases}$$

for some constants  $A^i(\iota)$ ,  $B^i(\iota)$ ,  $C_1^i(\iota)$ ,  $C_2^i(\iota)$ ,  $\hat{C}_1^i(\iota)$ , and  $\hat{C}_2^i(\iota)$  and the boundaries  $\bar{z}^i(\iota)$  and  $\bar{z}^i(\iota)$ .

For  $\iota = 0, 1$ , because  $\mu(\iota) > r(\iota)$ , the buy boundary must lie in the region  $y > 0$ . If the buy and sell boundaries  $\bar{z}^i(\iota)$  and  $\bar{z}^i(\iota)$  are positive, then the third branch is vacuous and the value function is  $C^2$  in the entire solvency region. However, the sell boundary  $\bar{z}^i(\iota)$  can be nonpositive whereas the buy boundary  $\bar{z}^i(\iota)$  is positive. In this case, the homogeneous solution suggests that  $\hat{C}_2^i(\iota)$  must take the same value as  $C_2^i(\iota)$ , which must be equal to  $-\lim_{z \rightarrow 0} u_2^i(z, \iota)$  to keep the value function finite. In addition, one can show by L'Hôpital's rule that  $\lim_{z \rightarrow 0} \psi^i(z, \iota) = -g^{i-1}(0, \iota)/\beta_1(\iota)$ .

The case where  $\bar{z}^i(\iota) = \infty$  only arises when it is optimal to never buy stock. Intuitively, this can happen when the transaction cost is large and the investor's expected lifetime is short, as shown in Liu and Loewenstein (2002). A similar, albeit more complex, set of conditions will arise in our model. In this case, to keep the value function finite, we must have  $C_1^i(\iota) = -\lim_{z \rightarrow \infty} u_1^i(z, \iota)$ . One can show that

$$\lim_{y \rightarrow 0} y^{1-\gamma} \psi^i\left(\frac{x}{y}, \iota\right) = \lim_{y \rightarrow 0} \frac{y^{1-\gamma} g^{i-1}(x/y, \iota)}{\delta(\iota) - (1 - \gamma)r(\iota)},$$

which agrees with the direct computation in (24) if it is optimal to never buy stock given an initial position 100% in cash.

Using a similar approach to those in Shreve and Soner (1994) or Framstad et al. (2001), one can show that there exist constants  $A^i(\iota)$ ,  $B^i(\iota)$ ,  $C_1^i(\iota)$ ,  $C_2^i(\iota)$ ,  $\hat{C}_1^i(\iota)$ , and  $\hat{C}_2^i(\iota)$  and the boundaries  $\bar{z}^i(\iota)$  and  $\bar{z}^i(\iota)$ , which make  $\psi^i(z, \iota)$  a  $C^2$  function in the solvency region except at  $z = 0$  or  $z = \infty$ . We can thus iteratively compute the optimal boundaries and value functions for each  $i$  by following the approach described in Liu and Loewenstein (2002).

Lemma 1 implies that by passing to a subsequence if necessary we must have as  $i \rightarrow \infty$ ,  $A^i(\iota) \rightarrow A(\iota)$ ,  $B^i(\iota) \rightarrow B(\iota)$ ,  $C_1^i(\iota) \rightarrow C_1(\iota)$ ,  $C_2^i(\iota) \rightarrow C_2(\iota)$ ,  $\hat{C}_1^i(\iota) \rightarrow \hat{C}_1(\iota)$ ,  $\bar{z}^i(\iota) \rightarrow \bar{z}(\iota)$ , and  $\bar{z}^i(\iota) \rightarrow \bar{z}(\iota)$ , for some constants  $A(\iota)$ ,  $B(\iota)$ ,  $C_1(\iota)$ ,  $C_2(\iota)$ ,  $\hat{C}_1(\iota)$ ,  $\bar{z}(\iota)$ , and  $\bar{z}(\iota)$ . Note that  $\bar{z}(\iota) > \alpha(\iota) - 1$  and  $\bar{z}(\iota) > \bar{z}(\iota)$ . For a complete proof one would need to provide verification theorems for the functions obtained in each iteration as well as for the limiting value function. Because this part is fairly long, involved, and very similar to those in Jang et al. (2007), Shreve and Soner (1994), and Framstad et al. (2001), we omit it to minimize repetition. We proceed to show that in all possible cases the limiting value function in Lemma 1 is a solution to the HJB equation with boundary conditions for the investor's problem and thus satisfy the conditions in the verification theorem for the limiting value function.<sup>13</sup>

First for a fixed  $\iota$ , suppose  $0 < \bar{z}(\iota) < \bar{z}(\iota) < \infty$ . Define

$$\psi(z, \iota) = \begin{cases} A(\iota) \frac{(z + 1 + \theta(\iota))^{1-\gamma}}{1 - \gamma} & \text{if } z \geq \bar{z}(\iota), \\ C_1(\iota)\psi_1(z, \iota) + C_2(\iota)\psi_2(z, \iota) + \psi_p(z, \iota) & \text{if } \bar{z}(\iota) \leq z \leq \bar{z}(\iota), \\ B(\iota) \frac{(z + 1 - \alpha(\iota))^{1-\gamma}}{1 - \gamma} & \text{if } \alpha(\iota) - 1 < z \leq \bar{z}(\iota). \end{cases}$$

<sup>13</sup> By construction, corresponding conditions are satisfied for each iteration.



Then by the convergence of the constants, we have that  $\psi^i$  converges uniformly to  $\psi$  on any compact set of the solvency region, in particular in  $[\underline{z}(\iota), \bar{z}(\iota)]$ . The functions  $g^i$  are concave and converge uniformly on compact sets to a limiting concave function  $g$  as defined in (23), Rockafellar (1970, Theorem 10.8), and Lemma 1. Observe that  $\psi_p^i$  and its first and second derivatives also converge uniformly on compact sets. Thus, we see that for  $z \in (\underline{z}(\iota), \bar{z}(\iota))$ , the function  $C_1(\iota)\psi_1(z, \iota) + C_2(\iota)\psi_2(z, \iota) + \psi_p(z, \iota)$  solves (21) in the NT region.

Observe from the  $C^2$  property of the  $\psi^i$  we have (suppressing the fixed  $\iota$  dependence)

$$A^i(\bar{z}^i + 1 + \theta)^{-\gamma} = C_1^i\psi_1'(\bar{z}^i) + C_2^i\psi_2'(\bar{z}^i) + \psi_p'(\bar{z}^i), \quad (43)$$

$$B^i(\underline{z}^i + 1 - \alpha)^{-\gamma} = C_1^i\psi_1'(\underline{z}^i) + C_2^i\psi_2'(\underline{z}^i) + \psi_p'(\underline{z}^i), \quad (44)$$

$$-\gamma A^i(\bar{z}^i + 1 + \theta)^{-\gamma-1} = C_1^i\psi_1''(\bar{z}^i) + C_2^i\psi_2''(\bar{z}^i) + \psi_p''(\bar{z}^i), \quad (45)$$

$$-\gamma B^i(\underline{z}^i + 1 - \alpha)^{-\gamma-1} = C_1^i\psi_1''(\underline{z}^i) + C_2^i\psi_2''(\underline{z}^i) + \psi_p''(\underline{z}^i). \quad (46)$$

So thanks to the uniform convergence of  $\psi_p^i$ , in the limit, we have

$$A(\bar{z} + 1 + \theta)^{-\gamma} = C_1\psi_1'(\bar{z}) + C_2\psi_2'(\bar{z}) + \psi_p'(\bar{z}), \quad (47)$$

$$B(\underline{z} + 1 - \alpha)^{-\gamma} = C_1\psi_1'(\underline{z}) + C_2\psi_2'(\underline{z}) + \psi_p'(\underline{z}), \quad (48)$$

$$-\gamma A(\bar{z} + 1 + \theta)^{-\gamma-1} = C_1\psi_1''(\bar{z}) + C_2\psi_2''(\bar{z}) + \psi_p''(\bar{z}), \quad (49)$$

$$-\gamma B(\underline{z} + 1 - \alpha)^{-\gamma-1} = C_1\psi_1''(\underline{z}) + C_2\psi_2''(\underline{z}) + \psi_p''(\underline{z}). \quad (50)$$

So  $\psi$  is a solution to the HJB Equation (21) with the boundary conditions.

Next, assume for a fixed  $\iota$  that  $0 = \underline{z}(\iota) < \bar{z}(\iota) < \infty$ . The basic approach above still works. However, we must recognize that because  $\psi^{i,j}(z)$  converge to a finite valued concave function, we must have  $C_2(\iota) = -\lim_{z \rightarrow 0} u_2(z, \iota)$  to keep the value function finite. Otherwise, the situation above still applies.

The case  $\underline{z}(\iota) < 0 < \bar{z}(\iota) < \infty$  is also similar to the above. In this case, we can write the limiting function as

$$\psi(z, \iota) = \begin{cases} A(\iota) \frac{(z + 1 + \theta(\iota))^{1-\gamma}}{1-\gamma} & \text{if } z \geq \bar{z}(\iota), \\ C_1(\iota)\psi_1(z, \iota) + C_2(\iota)\psi_2(z, \iota) + \psi_p(z, \iota) & \text{if } 0 \leq z \leq \bar{z}(\iota), \\ \hat{C}_1(\iota)\psi_1(z, \iota) + C_2(\iota)\psi_2(z, \iota) + \psi_p(z, \iota) & \text{if } \underline{z}(\iota) \leq z \leq 0, \\ B(\iota) \frac{(z + 1 - \alpha(\iota))^{1-\gamma}}{1-\gamma} & \text{if } \alpha(\iota) - 1 < z \leq \underline{z}(\iota), \end{cases}$$

where we must recognize that  $C_2(\iota) = -\lim_{z \rightarrow 0} u_2(z, \iota)$  to keep the value function finite.

Finally, we must also consider the possibility that  $\bar{z}(\iota) = \infty$ . Again the proof is similar to the above arguments once we recognize this requires restrictions on  $C_1(\iota)$ . We leave the details to the determined reader. Q.E.D.

**PROOF OF PROPOSITION 2.** We will prove the proposition for the case  $h(\pi, \iota) \neq 0$ . The case where  $h(\pi, \iota) = 0$  follows

from identical arguments. Evaluating Equation (18) at  $\underline{z}(\iota)$  and some algebra gives

$$\begin{aligned} & -\frac{\gamma}{2}\sigma(\iota)^2(1-\alpha(\iota))^2B(\iota) + (\mu(\iota) - r(\iota))(1-\alpha(\iota))(\underline{z}(\iota) \\ & + 1 - \alpha(\iota))B(\iota) + \left(\frac{\lambda}{1-\gamma} + r(\iota)B(\iota) - \frac{\lambda B(\iota)}{1-\gamma}\right)(\underline{z}(\iota) + 1 - \alpha(\iota))^2 \\ & + \frac{f(x, y, \iota) - (\xi(\iota) + \eta^U(\iota) + \eta^D(\iota))v(x, y, \iota) - yv_y(x, y, \iota)v(\iota)}{y^{1-\gamma}(\underline{z}(\iota) + 1 - \alpha(\iota))^{-\gamma-1}} = 0 \end{aligned} \quad (51)$$

which, defining  $\pi(\iota) = (1 - \alpha(\iota))/(\underline{z}(\iota) + 1 - \alpha(\iota))$  becomes

$$\begin{aligned} & -\frac{\gamma}{2}\sigma(\iota)^2\pi(\iota)^2B(\iota) + (\mu(\iota) - r(\iota))B(\iota)\pi(\iota) \\ & + \frac{\lambda}{1-\gamma} + r(\iota)B(\iota) - \frac{\lambda B(\iota)}{1-\gamma} \\ & + \frac{f(x, y, \iota) - (\xi(\iota) + \eta^U(\iota) + \eta^D(\iota))v(x, y, \iota) - yv_y(x, y, \iota)v(\iota)}{y^{1-\gamma}(\underline{z}(\iota) + 1 - \alpha(\iota))^{1-\gamma}} = 0. \end{aligned} \quad (52)$$

Because  $v(x, y, 1 - \iota) \leq v(x, y, \iota)$ ,

$$\begin{aligned} f(x, y, \iota) & \leq \xi(\iota)v(x, y, \iota) + \eta^U(\iota)E[v(x, y)^U, \iota] \\ & + \eta^D(\iota)E[v(x, y)^D, \iota], \end{aligned} \quad (53)$$

and the inequality  $v(x, yJ, \iota) \leq v(x, y, \iota) + yv_y(x, y, \iota)(J - 1)$  gives us

$$\frac{f(x, y, \iota) - (\xi(\iota) + \eta^U(\iota) + \eta^D(\iota))v(x, y, \iota) - yv_y(x, y, \iota)v(\iota)}{y^{1-\gamma}(\underline{z}(\iota) + 1 - \alpha(\iota))^{1-\gamma}} \leq 0. \quad (54)$$

Therefore,

$$\begin{aligned} & -\frac{\gamma}{2}\sigma(\iota)^2\pi(\iota)^2B(\iota) + (\mu(\iota) - r(\iota))B(\iota)\pi(\iota) + \frac{\lambda}{1-\gamma} \\ & + r(\iota)B(\iota) - \frac{\lambda B(\iota)}{1-\gamma} \geq 0. \end{aligned} \quad (55)$$

Because  $\lambda/(1 - \gamma) + r(\iota)B(\iota) - \lambda B(\iota)/(1 - \gamma) \leq 0$  ( $r$  is constant), we have

$$-\frac{\gamma}{2}\sigma(\iota)^2\pi(\iota)^2B(\iota) + (\mu(\iota) - r(\iota))B(\iota)\pi(\iota) \geq 0, \quad (56)$$

and the bound follows. The bound on the buy boundary follows from similar arguments. Q.E.D.

The following proposition provides some sufficient conditions for  $v(x, y, \iota) \geq v(x, y, 1 - \iota)$ .

**PROPOSITION 3.** Suppose  $\xi(\iota) = \xi(1 - \iota) = 0$ ,  $r(\iota) = r(1 - \iota)$ ,  $\delta(\iota) \leq \delta(1 - \iota)$ ,  $\eta_1^U(\iota) \leq \eta_2^U(1 - \iota)$ ,  $\eta_2^U(\iota) \geq \eta_1^U(1 - \iota)$ ,  $\eta_1^D(\iota) \leq \eta_2^D(1 - \iota)$ ,  $\eta_2^D(\iota) \geq \eta_1^D(1 - \iota)$ ,  $\mu(\iota) \geq \mu(1 - \iota)$ ,  $\sigma(\iota) \leq \sigma(1 - \iota)$ ,  $\alpha(\iota) \leq \alpha(1 - \iota)$ , and  $\theta(\iota) \leq \theta(1 - \iota)$ . Then  $v(x, y, \iota) \geq v(x, y, 1 - \iota)$ .

**PROOF.** Under the conditions stated in the proposition, one can show that  $M(\iota) \geq M(1 - \iota)$ , so recalling notation

from Equation (25), we have  $f^0(x, y, \iota) \geq f^0(x, y, 1 - \iota)$ . By construction, on  $\text{NT}_\iota$  we have

$$\begin{aligned} & \frac{1}{2} \sigma(\iota)^2 y^2 v_{yy}^1(\iota) + r(\iota) x v_x^1(\iota) + (\mu(\iota) - \nu(\iota)) y v_y^1(\iota) - \delta(\iota) v^1(\iota) \\ & + f^0(x, y, \iota) + \lambda \frac{(x + (1 - \alpha(\iota))y)^{1-\gamma}}{1 - \gamma} = 0. \end{aligned}$$

Therefore, using Ito's Lemma, for any trading strategy

$$\begin{aligned} v^1(x, y, \iota) & \geq E \left[ \int_0^\infty e^{-\delta(\iota)t} \left( f^0(x_t, y_t, \iota) + \lambda \frac{x_t + (1 - \alpha(\iota))y_t}{1 - \gamma} \right) dt \right] \\ & \geq E \left[ \int_0^\infty e^{-\delta(1-\iota)t} \left( f^0(x_t, y_t, 1 - \iota) \right. \right. \\ & \quad \left. \left. + \lambda \frac{x_t + (1 - \alpha(1-\iota))y_t}{1 - \gamma} \right) dt \right] \quad (57) \end{aligned}$$

$$\begin{aligned} v^1(x, y, \iota) & \geq \sup_{(D, I) \in \Theta(x, y)} E \left[ \int_0^\infty e^{-\delta(1-\iota)t} \left( f^0(x_t, y_t, 1 - \iota) \right. \right. \\ & \quad \left. \left. + \lambda \frac{x_t + (1 - \alpha(1-\iota))y_t}{1 - \gamma} \right) dt \right] \\ & = v^1(x, y, 1 - \iota). \end{aligned}$$

We now observe that this implies that  $f^1(x, y, \iota) \geq f^1(x, y, 1 - \iota)$ , which using the above arguments gives  $v^2(x, y, \iota) \geq v^2(x, y, 1 - \iota)$ . Iterating the above arguments, we find that  $v^i(x, y, \iota) \geq v^i(x, y, 1 - \iota)$  for all  $i$ , which implies  $v(x, y, \iota) \geq v(x, y, 1 - \iota)$  from Theorem 2. Q.E.D.

The following lemma is used to prove Proposition 1.

LEMMA 2. For all  $x, y$  in the solvency region, we have

$$1. \quad v(x, y, \iota) \geq \frac{B(\iota)}{1 - \gamma} (x + (1 - \alpha(\iota))y)^{1-\gamma}. \quad (58)$$

$$2. \quad \frac{M(\iota)}{1 - \gamma} \geq \frac{B(\iota)}{1 - \gamma}. \quad (59)$$

PROOF OF LEMMA 2.

Statement 1. It is always feasible to trade to the sell region. Indeed, the quantity to sell to reach the sell boundary is  $\Delta y$ , which is the solution to

$$\frac{x + (1 - \alpha(\iota))\Delta y}{y - \Delta y} = \underline{z}(\iota), \quad (60)$$

so

$$\Delta y = \frac{\underline{z}(\iota)y - x}{\underline{z}(\iota) + 1 - \alpha(\iota)}. \quad (61)$$

Therefore,

$$\begin{aligned} v(x, y, \iota) & \geq v(x + (1 - \alpha(\iota))\Delta y, y - \Delta y, \iota) \\ & = \frac{B(\iota)}{1 - \gamma} (x + (1 - \alpha(\iota))\Delta y + (1 - \alpha(\iota))(y - \Delta y))^{1-\gamma} \\ & = \frac{B(\iota)}{1 - \gamma} (x + (1 - \alpha(\iota))y)^{1-\gamma}. \quad (62) \end{aligned}$$

Statement 2. Suppose, to the contrary,  $M(\iota)/(1 - \gamma) < B(\iota)/(1 - \gamma)$ . Then we have

$$\begin{aligned} & \frac{M(\iota)}{1 - \gamma} (x + (1 - \alpha(\iota))y)^{1-\gamma} < \frac{B(\iota)}{1 - \gamma} (x + (1 - \alpha(\iota))y)^{1-\gamma} \\ & \leq v(x, y, \iota), \quad (63) \end{aligned}$$

where the last inequality follows from Statement 1. However, for  $y = 0$ , this gives

$$\frac{M(\iota)}{1 - \gamma} (x)^{1-\gamma} < v(x, 0, \iota), \quad (64)$$

which cannot hold because the value function with no transaction costs must be at least as big as the value function with transaction costs. Q.E.D.

PROOF OF PROPOSITION 1. Evaluate the HJB equation (18) at  $\underline{z}(\iota)$ , define  $\pi(z) = (1 - \alpha(\iota))/(z + 1 - \alpha(\iota))$ , and use (2) to get for all  $z \leq \underline{z}(\iota)$

$$\frac{a(\pi(z), \iota)B(\iota) + h(\pi(z), \iota)B(1 - \iota) + \lambda}{1 - \gamma} \leq 0 \quad (65)$$

with equality when  $z = \underline{z}$  and  $a, h$  are the same as the no-transaction-cost case. Recall

$$\frac{a(\pi(z), \iota)M(\iota) + h(\pi(z), \iota)M(1 - \iota) + \lambda}{1 - \gamma} \leq 0 \quad (66)$$

with equality when  $\pi(z) = \pi^*(\iota)$ .

If  $h = 0$  as in the pure jump diffusion model in Statement 2 of Lemma 2, and  $a(\pi, \iota) < 0$  imply

$$\frac{a(\pi^*(\iota), \iota)M(\iota) + \lambda}{1 - \gamma} = 0 \leq \frac{a(\pi^*(\iota), \iota)B(\iota) + \lambda}{1 - \gamma}. \quad (67)$$

If the inequality is strict, the right-hand side must be less than or equal to zero from Equation (65), and this implies that  $\pi(\underline{z}(\iota)) > \pi^*(\iota)$ . If the inequality is an equality, then  $\pi(\underline{z}(\iota)) = \pi^*(\iota)$ . In either case this implies that  $\underline{z}(\iota) \leq (1 - \alpha(\iota))z^*(\iota)$ .

If  $h$  is not zero, then because  $a < 0$ , we have for all  $z$  in the sell region,

$$\frac{B(\iota)}{1 - \gamma} \geq \frac{-h(\pi(z), \iota)B(1 - \iota) - \lambda}{(1 - \gamma)a(\pi(z), \iota)}. \quad (68)$$

In regime  $1 - \iota$  we have

$$\frac{B(1 - \iota)}{1 - \gamma} \geq \frac{-h(\pi(z), 1 - \iota)B(\iota) - \lambda}{(1 - \gamma)a(\pi(z), 1 - \iota)}. \quad (69)$$

The two inequalities (68) and (69) then imply for all  $z_1$  and  $z_2$  in the sell regions,

$$\begin{aligned} & \frac{B(\iota)}{1 - \gamma} \\ & \geq \frac{\lambda(h(\pi(z_1), \iota) - a(\pi(z_2), 1 - \iota))}{(1 - \gamma)(a(\pi(z_1), \iota)a(\pi(z_2), 1 - \iota) - h(\pi(z_1), \iota)h(\pi(z_2), 1 - \iota))}, \quad (70) \end{aligned}$$

$$\begin{aligned} & \frac{B(1 - \iota)}{1 - \gamma} \\ & \geq \frac{\lambda(h(\pi(z_2), 1 - \iota) - a(\pi(z_1), \iota))}{(1 - \gamma)(a(\pi(z_1), \iota)a(\pi(z_2), 1 - \iota) - h(\pi(z_1), \iota)h(\pi(z_2), 1 - \iota))}, \quad (71) \end{aligned}$$

where the numerator and denominator are positive from Assumption 1. These must hold for every choice of  $z_1$  in the

sell region of regime  $\iota$  and  $z_2$  in the sell region in regime  $1 - \iota$ . We also have from Statement 2 of Lemma 2,

$$\frac{B(\iota)}{1 - \gamma} \leq \frac{M(\iota)}{1 - \gamma}$$

$$= \frac{\lambda(h(\pi^*(\iota), \iota) - a(\pi^*(1 - \iota), 1 - \iota))}{(1 - \gamma)(a(\pi^*(\iota), \iota)a(\pi^*(1 - \iota), 1 - \iota) - h(\pi^*(\iota), \iota)h(\pi^*(1 - \iota), 1 - \iota))}, \quad (72)$$

$$\frac{B(1 - \iota)}{1 - \gamma} \leq \frac{M(1 - \iota)}{1 - \gamma}$$

$$= \frac{\lambda(h(\pi^*(1 - \iota), 1 - \iota) - a(\pi^*(\iota), \iota))}{(1 - \gamma)(a(\pi^*(\iota), \iota)a(\pi^*(1 - \iota), 1 - \iota) - h(\pi^*(\iota), \iota)h(\pi^*(1 - \iota), 1 - \iota))}. \quad (73)$$

Therefore, for all choices of  $z_1$  and  $z_2$  in the sell regions, we have

$$\frac{a(\pi(z_1), \iota)M(\iota) + \lambda}{1 - \gamma}$$

$$+ \frac{h(\pi(z_1), \iota)\lambda(h(\pi(z_2), 1 - \iota) - a(\pi(z_1), \iota))}{(1 - \gamma)(a(\pi(z_1), \iota)a(\pi(z_2), 1 - \iota) - h(\pi(z_1), \iota)h(\pi(z_2), 1 - \iota))}$$

$$\leq \frac{a(\pi(z_1), \iota)B(\iota) + h(\pi(z_1), \iota)B(1 - \iota) + \lambda}{1 - \gamma} \leq 0. \quad (74)$$

If  $z^*(1 - \iota)$  is in the sell region for regime  $1 - \iota$ , then evaluating the above expression at  $z_2 = z^*(1 - \iota)$  and  $z_1 = z^*(\iota)$  gives

$$\frac{a(\pi^*(\iota), \iota)M(\iota) + \lambda}{1 - \gamma}$$

$$+ \frac{h(\pi^*(\iota), \iota)\lambda(h(\pi^*(1 - \iota), 1 - \iota) - a(\pi^*(\iota), \iota))}{(1 - \gamma)(a(\pi^*(\iota), \iota)a(\pi^*(1 - \iota), 1 - \iota) - h(\pi^*(\iota), \iota)h(\pi^*(1 - \iota), 1 - \iota))}$$

$$= 0 \leq \frac{a(\pi^*(\iota), \iota)B(\iota) + h(\pi^*(\iota), \iota)B(1 - \iota) + \lambda}{1 - \gamma} \leq 0.$$

Therefore, the inequality cannot be strict. If it is an equality, then  $\pi(z(\iota)) = \pi(z^*(\iota))$ , otherwise we must have  $\pi(z(\iota)) > \pi(z^*(\iota))$ ; in other words  $z^*(\iota)$  cannot be in the sell region for regime  $\iota$ . A symmetric argument in regime  $1 - \iota$  says that if  $z^*(\iota)$  is in the sell region for regime  $\iota$ , then  $z(1 - \iota)$  cannot be in the sell region for regime  $1 - \iota$ . This then gives the bounds. Q.E.D.

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