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Neil Geismar, Milind Dawande, Divakar Rajamani, Chelliah Sriskandarajah,

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# Managing a Bank's Currency Inventory Under New Federal Reserve Guidelines

Neil Geismar
Department of Management and Marketing, Prairie View A&M University, P. O. Box 519, MS 2315, Prarie View, Texas 77446-0638, hngeismar@pvamu.edu

Milind Dawande, Divakar Rajamani, Chelliah Sriskandarajah
School of Management, University of Texas at Dallas, 2601 North Floyd Road, Richardson, Texas 75080
{milind@utdallas.edu, divakar@utdallas.edu, chelliah@utdallas.edu}

New currency recirculation guidelines implemented by the Federal Reserve System (Fed) of the United States are intended to reduce the overuse of its currency processing services by depository institutions (banks). These changes are expected to have a significant impact on operating policies at those depository institutions that handle large volumes of currency. We describe two business models that capture the flow of currency between a bank and the Fed; the first model captures the current operations of most banks, while the second is expected to be adapted by many banks in response to the new guidelines. Motivated by our work with Brink's, Inc., to assess the economic impact that banks will sustain from these guidelines, we present a detailed analysis that provides managers of banks with optimal strategies to manage the flow of currency to and from the Fed for a variety of cost structures and demand patterns. Given this insight into a bank's optimal behavior, the Fed can also use our analysis to fine tune its guidelines to achieve the desired goals.

*Key words*: currency supply chain; Federal Reserve System; depository institutions; cross-shipping; custodial inventory
*History*: Received: March 12, 2006; accepted: August 10, 2006.



## 1. Introduction

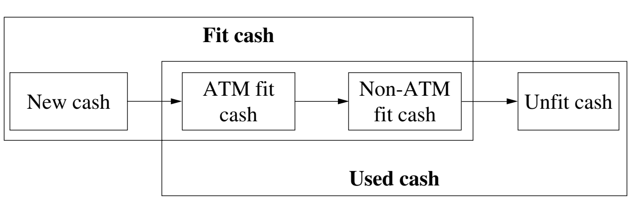
Despite the growth in debit cards, smart cards, and electronic transactions by consumers, and the prevalence of high-value commercial transactions that are made electronically, cash (i.e., currency and coins) is still the most widely used daily consumer payment mechanism. Cash is favored for its ease and anonymity: It ensures the user's privacy by leaving no record. Cash is the only means of transactions that requires no bank account; about 10% of consumers in the United States do not have bank accounts (Kennickell et al. 2000). In the UK, nearly three in four of all personal payments are made by cash, and the forecast is that 66% of all personal payments in 2011 will still be made by currency and coins (Association for Payment Clearing Services 2005). In the United States, there has been a 76% increase in the amount of currency in circulation since 1990 (Blacketer and Evetts 2004): The volume of U.S. currency at the end of 2004 was 24.2 billion notes (Gage and McCormick 2005).

The Federal Reserve Bank (Fed) of New York has estimated that the value of U.S. currency in circulation will exceed US$1,000 billion by 2010; currently it is $690 billion, up from $492 billion five years ago. While it is likely that more and more payrolls will be automated via direct deposit, there are indications of an increase in the amount of currency circulated through ATMs. Between 1996 and 2003, there have been a 200% increase in the number of ATMs in the United States (Blacketer and Evetts 2004) and a corresponding growth in the need for *fit cash*, i.e., currency whose condition makes it suitable for circulation to the public.

To fulfill its mission of providing currency services to depository institutions (banks) so they can meet the public's currency demand, the Fed spends a significant fraction of its annual budget (about 15% or, equivalently, $387 million in 2003) on currency management operations. One of the reasons for this increased spending is that 30%–50% of the currency deposited with the Fed is reordered by the same banks in the same denominations in less than five days. The Fed believes that if banks were more active in recycling currency, then fewer demands would be made on the Fed's currency processing services, and the economy would require less currency in circulation. Hence the Fed is following a trend that is common to several countries by attempting to move from a centralized business model toward a privatized one. In a centralized model, a country's central bank provides all currency services; current examples include the United States and France. Semiprivatized models include the features that the Fed is adopting: reduced circulation and currency processing services, and custodial inventory (described in §2). Such systems are used in Canada and the UK. In a privatized system (e.g., Australia and South Africa), the only services provided by the central bank are the introduction of new cash and the destruction of unfit cash (Blacketer and Evetts 2004).

To understand a bank's cash management policy, we must first understand how currency is classified (see Figure 1). In its physical form, the Fed divides currency into four categories. *New cash* is produced by the Bureau of Engraving and Printing and then introduced into the currency supply by the Fed. Once in this supply, the currency is called *used cash*. This classification contains the remaining three categories. The first of these is *ATM-fit cash*, which contains notes that are of sufficient quality to be dispensed via ATMs. Currency that is suitable for most transactions, but not for ATMs is called *non-ATM-fit cash*. New cash, ATM-fit cash, and non-ATM-fit cash are referred to collectively as *fit cash*. The final category is *unfit cash*, which is soiled, torn, or defaced and is therefore unacceptable for circulation and will be destroyed by the Fed. Because coins are significantly less perishable and have lesser value, the guidelines and challenges in dealing with them are entirely different and are not considered in this paper.

**Figure 1** **Cash Life Cycle**




Two papers related to ours use dynamic programming to derive note ordering policies for a country's central bank. The objective of each study is to minimize the central bank's total cost. For Israel's different denominations of currency, Ladany (1997) considers the lifetimes of banknotes; the effects of inflation and economic growth on demand, ordering and holding costs; and periodic changes in the designs of banknotes, to derive optimal policies. Massoud (2005) extends this work for a general central bank by also allowing for economic shocks, counterfeiting, lead times for the printing of currency, and production delays. There are also earlier empirical studies of the demand for banknotes in different countries; the interested reader may consult Fase and van Nieuwkerk (1976, 1977), Fase et al. (1979), Fase (1981), and Boeschoten and Fase (1992).

The flow of currency from the Fed to banks to customers and back constitutes a closed-loop supply chain. Applicable studies of closed-loop supply chains and reverse logistics include Dekker et al. (2004), Fleischmann et al. (1997), Guide and Van Wassenhove (2003), and Rajamani et al. (2006). Additionally, this paper has a tangential relation to studies of cash management and balancing by firms, including early qualitative work by Whistler (1967), Eppen and Fama (1968, 1969), Girgis (1968), Neave (1970), and the quantitative studies by Vial (1972), Constantinides (1976), Constantinides and Richard (1978), and Smith (1989).

Broadly speaking, our paper deals with supply chain issues in the service sector with the bank and the Fed as the two parties. The emphasis, however, is more on an operational level—to obtain a detailed, cost-minimizing schedule of a bank's deposits to and withdrawals from the Fed. Because of its focus on operational scheduling issues, the recent work in supply chain scheduling (Hall and Potts 2003, Chen and Vairaktarakis 2005) is relevant to this study.

Before we proceed with the description of the new currency recirculation guidelines, we summarize the following major contributions of this paper:

(1) We introduce two models that explain the flow of currency inventory, and the consequent cost implications, between a depository institution (bank) and the Fed. The first model (§3) captures the current operations of most banks, whereas the second

model (§4) is expected to be adopted by many banks in response to the Fed's revised guidelines.

(2) We present a detailed analysis that provides managers of depository institutions with optimal strategies to manage the flow of currency to and from the Fed. Because most banks do not currently have the necessary infrastructure to fully implement the advanced offerings in the Fed's new recirculation guidelines, our analysis of the basic model is intended to help managers in the short term. Our analysis of the second model should be helpful for long-term planning as banks decide whether or not to participate in the advanced offerings.

(3) Our structural analysis of a bank's response to the guidelines will also help the Fed in understanding the impact of some of the operational details in its new guidelines. As such, the Fed can use our analysis to fine tune these details.

Section 2 describes the Fed's new currency recirculation guidelines that motivate this study. Section 3 presents a detailed analysis of the basic model for analyzing a bank's currency handling policies. We examine the structure of optimal policies and show the dominance of a specific subclass of policies. The results of the analysis are then used to offer managerial insights for the basic model and its generalizations. Section 4 analyzes a more complex model that uses custodial inventory. Section 5 concludes this study and makes recommendations for future research.

## 2. Fed's New Currency Recirculation Guidelines

Under the Fed's current guidelines, it accepts deposits of used cash from depository institutions, *fit sorts* that currency (i.e., separates it into fit and unfit cash), removes unfit cash from circulation, and provides fit cash to depository institutions, who, in turn, use that fit cash to meet the demands from customers. The Fed believes that banks overuse these cash processing services (the banks' incentive for doing so is described in the next paragraph). This perceived overuse motivated the new guidelines, which are designed to encourage banks to use the currency deposited by customers to fill withdrawal orders. This would reduce the Fed's expenses in three ways. First, the Fed would handle fewer transactions, thereby lowering its labor costs. Second, if banks were to recycle currency, rather than shipping used cash to the Fed and then receiving shipments of fit cash from the Fed, less currency would be in transit. In the absence of such recycling, the Fed often requests that more notes be printed to compensate for this currency in transit. Thus, recycling would reduce the Fed's printing expenses. Third, to recycle deposited currency, banks must fit sort it, rather than passing this task onto the Fed.

There are two primary reasons that banks do not recycle currency, but instead prefer to fill their customers' needs with currency ordered directly from Reserve Banks and to deposit the notes received from their customers to Reserve Banks. First, fit sorting increases costs because it requires additional labor and expensive machines. Second, as with other types of inventory, holding currency represents a cost of lost opportunity to a bank: currency held in a bank's vault does not earn interest, but currency on account at the Fed can (by lending to another institution). Therefore, depository institutions wish to minimize their vault currency holdings, while still maintaining enough to meet customers' demands. This can be done most effectively by increasing the frequency of their deposits of currency to the Fed and the frequency of their orders of currency from the Fed.

Such actions by banks have led the Fed to define *cross-shipping* as depositing fit or nonfit sorted currency and ordering the same denomination during the same business week within a Federal Reserve zone (Federal Reserve 2003). The Fed wants to minimize or eliminate this practice, but currently its only tool to curtail cross-shipping is to deny service, which conflicts with its mission to provide currency services to depository institutions. Thus, the Fed plans to implement a recirculation fee that will be charged on cross-shipped currency. The fee would not be activated by deposits of unfit cash. Furthermore, the fee also would not apply to \$50 and \$100 notes because these notes are a relatively minor component of cross-shipped currency and, more importantly, because of the risk that depository institutions might recirculate high-denomination counterfeit notes.

The second component of the Fed's new guidelines is the *Custodial Inventory Program*, which would allow depository institutions to deposit into custodial inventories \$5, \$10, and \$20 notes without subjecting the withdrawals made in the same week to cross-shipping fees. A *custodial inventory* contains fit cash deposited to the Federal Reserve, hence it earns interest for the bank, but it is located within the bank's secured facility and is segregated from its operating cash. An additional benefit is that custodial inventories may allow depository institutions to avoid the costs of preparing and transporting their temporarily surplus currency to and from Federal Reserve offices (Federal Reserve 2003). Because banks may use custodial inventory to meet demand, this program provides an incentive for banks to fit sort used cash at their expense and to keep the resulting fit cash in custodial inventory. The program further encourages fit sorting by the banks by forbidding cash withdrawn from the Fed from being deposited into custodial inventory.

These new guidelines are "intended to encourage private-sector behavioral changes that would lower the overall societal costs of cash processing and distribution by curtailing overuse of a free governmental service." The Fed also states that "any costs incurred by depository institutions are estimated to be significantly smaller than the costs that Reserve Banks will avoid if the institutions reduce or cease cross-shipping currency" (Federal Reserve 2003).

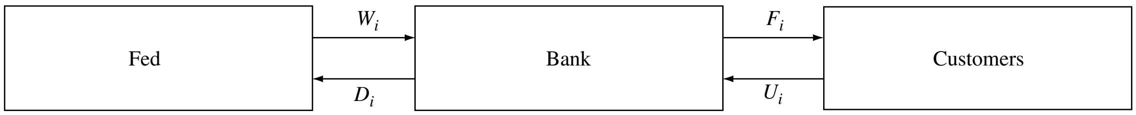
Our models study the impact of these guidelines on a bank's operations. They follow practice in that transactions with the Fed are performed at the depository institution level. This means that within a Federal Reserve zone, a bank and all of its branches must deal with the Fed as one unit, i.e., on each day, it can make at most one deposit and one withdrawal to cover the needs of all branches. For any combination of parameter values, we provide a bank with a policy that minimizes its expenses. Given this insight into a bank's optimal behavior, the Fed can adjust its cross-shipping fee or other aspects of its guidelines to achieve its goals.

## 3. The Basic Currency Management Model

The basic model captures the current operations of many banks: there is no custodial inventory, and no fit sorting is performed before deposits are made to the Fed. Hence, any withdrawal made during a week in which a deposit is made is subject to a cross-shipping fee. Additionally, cash is shipped directly between the bank and the Federal Reserve. Within this model, the bank's decision variables are the amount it deposits and the amount it withdraws in its transactions with the Fed each day. The aggregation of these decisions over $n$ weeks forms an *$n$-week policy*. We provide a formal definition of *policy* at the end of §3.2. When discussing policies, the demand for fit cash by customers, and their deposits of used cash, we consider only a single denomination (\$5, \$10, or \$20). The bank's policy for each denomination may be determined independently of the other denominations because a cross-shipping fee is charged only if currency of the same denomination is deposited and withdrawn during the same week. Furthermore, the transportation charges are calculated per bundle shipped, where a *bundle* is 1,000 notes of the same denomination, so there is no incentive to aggregate different denominations into a single shipment.

Customers' demand for fit notes of a particular denomination from the bank on day $i$ is $F_i$ bundles, and the deposits of used notes of that denomination that the bank receives from customers on day $i$ is $U_i$ bundles. In practice, an important characteristic that these demands are likely to satisfy is weekly periodicity, i.e., $F_i \approx F_{i+5}$ and $U_i \approx U_{i+5}$. Our discussions with Brink's, Inc. confirmed this. For example, within a week, the demand for fit cash is typically the highest on a Friday, whereas the value of deposited used notes is typically the highest on a Monday. These values do change from one week to the next, but the variation is negligible. Furthermore, the analysts at Brink's, Inc. indicated that a reasonably accurate estimate of these values for each weekday may be obtained by averaging the realizations over the past several weeks. Therefore we assume that $F_i$ and $U_i$ are known and constant weekly, i.e., $F_i = F_{i+5}$ and $U_i = U_{i+5}$, $\forall\, i$, where the bank's business operating time is five days per week (Monday to Friday). For our analysis of the basic model, we use this assumption until the end of §3.2. Then, §3.3 solves the special case in which the customer demands are the same each day and customer deposits are the same each day (i.e., $F_i = F$; $U_i = U$). Finally, we demonstrate (in §3.4) that the

**Figure 2** Cash Flows in Basic Model




solution of this special case provides a close approximation to the case when $U_i$ and $F_i$ vary within a week.

To assess a policy's cost, we must specify the bank's daily schedule. During business hours on day $i$, for a given denomination, the bank receives $U_i$ bundles of used cash and disperses $F_i$ bundles of fit cash via customer transactions. At some point during the day, the logistics provider arrives to collect the bank's deposit of $D_i$ bundles of used cash and to deliver $W_i$ bundles of fit cash withdrawn from the Fed (see Figure 2). Because of the processing required, a day's deposit is prepared before the bank opens, so the currency received from customers on day $i$ ($U_i$) cannot be part of day $i$'s deposit ($D_i$). Similarly, day $i$'s withdrawal ($W_i$) is not added to inventory until the bank closes, so it cannot be used to satisfy demand on day $i$ ($F_i$). After close of business, the used cash that was collected during the day is separated by denomination and counted to compute the inventory level ($I_i^u$) of the denomination in question. The fit currency inventory ($I_i^f$) is also measured then (see Figure 3). Therefore, the inventory balance equations are

$$I_i^u = I_{i-1}^u - D_i + U_i \tag{1}$$

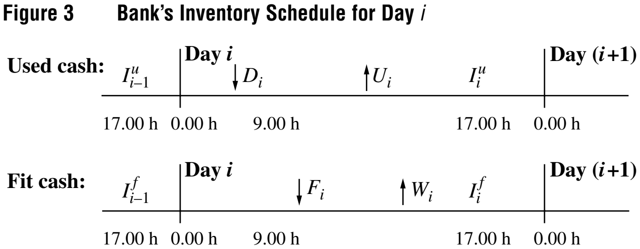
$$I_i^f = I_{i-1}^f - F_i + W_i. \tag{2}$$

The bank's daily schedule and the requirement that all demand must be satisfied imply

$$I_{i-1}^u \geq D_i, \quad \text{so } I_i^u \geq U_i \quad \text{and} \tag{3}$$

$$I_{i-1}^f \geq F_i, \quad \text{so } I_i^f \geq W_i. \tag{4}$$

**Figure 3** Bank's Inventory Schedule for Day $i$




These observations prove the following lemma, which states that the minimum inventory levels before opening on day $i$ are the previous day's received used cash ($U_{i-1}$) and exactly enough fit cash to meet the day's demand ($F_i$).

Lemma 1. *The inventory of used cash on day $i$ ($I_i^u$) is minimized by depositing that day's starting used cash: $I_{i-1}^u = D_i$, which implies $I_i^u = U_i$. The inventory of fit cash on day $i$ ($I_i^f$) is minimized by starting with just enough to meet the day's demand: $I_{i-1}^f = F_i$, which implies $I_i^f = W_i$.*

Note that when computing a policy's cost in this model, transportation costs are not considered because the bank receives (respectively, deposits) $n\sum_{i=1}^5 F_i$ ($n\sum_{i=1}^5 U_i$) in fit (used) cash from (to) the Fed during the course of each $n$-week policy. Thus the transportation cost is the same for all $n$-week cyclic policies because the third-party logistic providers (e.g., Brink's, Inc.; Loomis, Fargo & Co.; Dunbar Armored) that usually provide the transportation services charge a fixed rate for each bundle transported.

In an $n$-week cyclic policy for a bank, the schedule of deposits and withdrawals at the Fed is repeated every $n$ weeks: $W_i = W_{i+5n}$, $D_i = D_{i+5n}$, $n \in \mathbb{Z}^+$. Cyclic policies can be easily understood, implemented, and analyzed; therefore they are a preferred mechanism for specifying the schedule of deposits and withdrawals in practice. More importantly, there exists a cyclic policy that minimizes long-term average per-week costs; we show this result at the end of §3.2. Prior to the mathematical justification of cyclic policies, we introduce examples of cyclic policies in §3.1 to develop intuition. We then derive structural results for optimal cyclic policies in §3.2. We conclude this section with managerial insights (§3.3) and generalizations (§3.4).

### 3.1. Example One-Week and Two-Week Cyclic Policies

We now briefly describe two policies and compute their costs. When labeling specific policies, the subscript will indicate the number of weeks it covers,

**Table 1** Potentially Optimal One-Week and Two-Week Cycles and Their Costs if $U_i = U$ and $F_i = F$, $i = 1, \ldots, 5$

| Name | Week 1 | Week 2 | Cost |
|---|---|---|---|
| Policy $P_1 5$ | D D D D D<br>WWWWW | D D D D D<br>WWWWW | $10Uh + 10Fh + 10Fe$ |
| Policy $P_2 0$ | WWWWW | D D D D D | $25Uh + 25Fh$ |
| Policy $P_2 1$ | WWWWW | D D D D D<br>W | $25Uh + 20Fh + Fe$ |
| Policy $P_2 2$ | WWWWW | D D D D D<br>WW | $25Uh + 16Fh + 2Fe$ |
| Policy $P_2 3$ | WWWWW | D D D D D<br>WWW | $25Uh + 13Fh + 3Fe$ |
| Policy $P_2 4$ | WWWWW | D D D D D<br>WWWW | $25Uh + 11Fh + 4Fe$ |
| Policy $P_2 5$ | WWWWW | D D D D D<br>WWWWW | $25Uh + 10Fh + 5Fe$ |

and the normal-sized number indicates the number of days that withdrawals are made in the last week. For example, $P_2 1$ is a two-week cyclic policy in which a withdrawal is made on only one day during the second week (see Table 1). That these two numbers are sufficient to uniquely specify all potentially optimal one-week and two-week cyclic policies is proven by the results of §§3.2 and 3.3. They imply that there is only one possibly optimal one-week policy and that an optimal two-week cyclic policy $P_2 l$ begins with a week in which there is no deposit to the Fed, but a withdrawal is made each day. During week 2, a deposit is made each day, but there are withdrawals only on each of the last $l$ days of the cyclic policy. General $n$-week policies will be defined in §3.3.2.

A policy's cost depends on two parameters. The inventory holding charge per day is $h$. This cost is assessed on each bundle of currency (used or fit) that is held overnight. Because $h$ includes the cost of capital, its value depends on the denomination of the notes considered. The per bundle cross-shipping fee is $e$. The Fed is in the process of determining the value of $e$ for each denomination.

Policy $P_1 5$ is similar to the current operations of most banks: a deposit of $U_{i-1}$ and a withdrawal of $F_{i+1}$ are made each day $i$, so a bank faces inventory cost of $U_{i-1}h + F_{i+1}h$ each day and must pay a cross-shipping fee of $F_{i+1}e$ each day (see Table 1). The cost of this policy is $\sum_{i=1}^{5}(U_i h + F_i h + F_i e)$.

In policy $P_2 0$, the bank only makes withdrawals from the Fed each day of the first week, and only makes deposits to the Fed each day of the second week (see Table 1). Because the bank holds $U_5$ bundles of used cash over the weekend before the first week, the bank's inventory of used cash during the first week is (in bundles) $U_5 + U_1$ over Monday night, $U_5 + U_1 + U_2$ over Tuesday night, and so on, and $U_5 + \sum_{i=1}^{5} U_i$ over Friday night (and the weekend), so the total inventory charge for used cash for the first week is $5U_5 h + \sum_{i=1}^{5}(6-i)U_i h$. During each day of the second week, the bank's overnight holding charge on used cash applies only to that day's receipts, for a total of $\sum_{i=1}^{5} U_i h$. Therefore the total inventory charge for used cash for both weeks is $5U_5 h + \sum_{i=1}^{5}(7-i)U_i h$.

The overnight inventory for fit cash in the first week is $F_{i+1}$ for each night Monday through Thursday. Friday night's inventory is $2F_1 + F_2 + \cdots + F_5$, because the bank will receive no fit cash throughout the second week, so it must have enough fit cash to last the week and the following Monday. Therefore, in the second week, over Monday night, it holds $F_2 + \cdots + F_5 + F_1$, over Tuesday night, it holds $F_3 + F_4 + F_5 + F_1$, and so on; finally, over Friday night, it holds $F_1$, so the total fit cash inventory charge for the second week (Monday night through Friday night) is $(F_2 + 2F_3 + 3F_4 + 4F_5 + 5F_1)h$. Hence the two-week total charge for all inventory is $(5U_5 + \sum_{i=1}^{5}(7-i)U_i + \sum_{i=1}^{5}(i+1)F_i + 5F_1)h$, with no cross-shipping charge. The other two-week policies are hybrids of the previous two policies (see Table 1).

In Table 1, we list all potentially optimal one-week and two-week cyclic policies and their costs if $U_i = U$ and $F_i = F$, $i = 1, \ldots, 5$. The reasons for their selection become clear in the next subsection. We provide the two-week cost of Policy $P_1 5$ for easy comparison to the two-week policies.

### 3.2. Structural Results for Optimal Policies

Although a bank faces dynamic demand for currency, none of the classical dynamic lot-sizing algorithms (e.g., Wagner and Whitin 1958, De Matteis and Mendoza 1968, Silver and Meal 1973) are applicable because of the bank's cost structure: there is no setup fee for delivery of currency, and the cross-shipping fee is charged only if the bank deposits in the same week that it withdraws. Hence we now perform a structural decomposition on the set of all feasible policies to simplify the task of finding an optimal policy. We assume that all policies begin on Monday, so every Friday's index will be a multiple of five: $i = 5j$, for some $j \in \mathbb{Z}^+$. Note that the cost of an $n$-week policy is $C = \sum_{i=1}^{5n}(I_i^u h + I_i^f h + x_i W_i e)$, where $x_i = 1$ if a withdrawal on day $i$ is subject to a cross-shipping charge, $x_i = 0$ otherwise. Recall that the requirement that all demand must be met implies $I_{i-1}^f \geq F_i$, $\forall\, i$.

The following lemma is analogous to a result concerning dynamic lot sizing in classical inventory theory. Its proof is given in the appendix.

Lemma 2. *In an optimal policy, withdrawals of fit cash from the Fed will be made so that the current inventory of fit cash is exhausted on the day that the next order of fit cash arrives from the Fed, i.e.,* $W_i > 0 \Leftrightarrow I_{i-1}^f = F_i$.

The next lemma states that in an optimal policy, deposits are made every day or not at all in a given week. Furthermore, each deposit should exhaust the current inventory of used cash. For example, see policies $P_2 0$, $P_2 1$, $P_2 2$, $P_2 3$, $P_2 4$, $P_2 5$ (Table 1) in which deposits are made every day in week 2 and none in week 1.

Lemma 3. *If there is a set of optimal policies in which* $D_q > 0$ *for some* $q \in \{5j+1, \ldots, 5j+5\}$, $j \in \mathbb{Z}^+ \cup \{0\}$, *then there is at least one element of that set in which* $D_{5j+1} = I_{5j}^u, \ldots, D_q = I_{q-1}^u, \ldots, D_{5j+5} = I_{5j+4}^u$. *Moreover,* $D_i = U_{i-1}$, $i = 5j+2, \ldots, 5j+5$ *for this policy.*

Proof. $D_q > 0$ for some $q \in \{5j+1, \ldots, 5j+5\}$ implies that $x_i = 1$ for $i = 5j+1, \ldots, 5j+5$, so the cost for this week is $C = \sum_{i=5j+1}^{5j+5}(I_i^u h + I_i^f h + W_i e)$. Any additional deposits will effect $\sum_{i=5j+1}^{5j+5} I_i^u h$ but have no bearing on $\sum_{i=5j+1}^{5j+5}(I_i^f h + W_i e)$. According to Lemma 1, $\sum_{i=5j+1}^{5j+5} I_i^u h$ is minimized by $D_i = I_{i-1}^u$ for $i = 5j+1, \ldots, 5j+5$. This and Equation (1) imply $D_i = U_{i-1}$, $i = 5j+2, \ldots, 5j+5$. □

Note that if $D_i > 0$, $i = 1, \ldots, 5n$, in an $n$-week cyclic policy, then all withdrawals will be subject to cross-shipping fees. Hence, to minimize costs, withdrawals should be made each day: $W_i = F_{i+1}$, $i = 1, \ldots, 5n$. This implies that $P_1 5$ is the only possibly optimal one-week cyclic policy.

Lemma 4 states that if no deposits are made on a given day, then withdrawals should be made each day of that week. For example, see policies $P_2 0$, $P_2 1$, $P_2 2$, $P_2 3$, $P_2 4$, $P_2 5$ (Table 1). In those polices, withdrawals are made every day in week 1, during which no deposits are made.

Lemma 4. *If there is a set of optimal policies in which* $D_q = 0$ *for some* $q \in \{5j+1, \ldots, 5j+5\}$, $j \in \mathbb{Z}^+ \cup \{0\}$, *then there is at least one element of that set in which* $W_i = F_{i+1}$, $i = 5j+1, \ldots, 5j+4$, *and* $W_{5j+5} \geq F_{5j+6}$.

Proof. Lemma 3 implies that we can limit our study to only policies in which either $D_i = I_{i-1}^u$, $i = 5j+1, \ldots, 5j+5$, or $D_i = 0$, $i = 5j+1, \ldots, 5j+5$. Hence, during a week in which $D_q = 0$ for some $q \in \{5j+1, \ldots, 5j+5\}$, $D_i = 0$, for $i = 5j+1, \ldots, 5j+5$, so the holding cost for used cash is fixed, and withdrawals from the Fed will not be subject to cross-shipping fees. Therefore, according to Lemma 1, the week's cost is minimized by $I_{i-1}^f = F_i$, for $i = 5j+1, \ldots, 5j+4$. This and Equation (2) imply $W_i = F_{i+1}$, $i = 5j+2, \ldots, 5j+4$. $W_{5j+5}$ is not restricted to $F_{5j+6}$ because it may satisfy demand for more days than just Monday of the following week. □

Lemma 5 states that if a withdrawal is made on a particular day, then a withdrawal should be made on each of the remaining days of that week. For example, in policy $P_2 3$ (Table 1), a withdrawal is made on the third day of week 2. Thus, withdrawals are also made on the fourth and fifth days of week 2. In general, policy $P_n l$ has a withdrawal on each of the last $l$ days of week $n$.

Lemma 5. *If there is a set of optimal policies in which* $W_q > 0$ *for some* $q \in \{5j+1, \ldots, 5j+5\}$, $j \in \mathbb{Z}^+ \cup \{0\}$, *then there is at least one element of that set in which* $W_q = F_{q+1}, \ldots, W_{5j+4} = F_{5j+5}$, $W_{5j+5} \geq F_{5j+6}$.

Proof. $W_q > 0$ implies $I_{q-1}^f = F_q$ by Lemma 2. $I_{q-1}^f = F_q$ is disbursed on day $q$, so the amount that the bank withdraws during the rest of the week must satisfy $\sum_{i=q}^{5j+5} W_i \geq \sum_{i=q+1}^{5j+6} F_i$. Because the deposits are used cash, the demand for fit cash for the rest of the week, and the value of $x_i$, $i = q, \ldots, 5j+5$ are set, $C$ is minimized by minimizing $\sum_{i=q}^{5j+5} I_i^f$, which is done by $W_q = F_{q+1}, \ldots, W_{5j+4} = F_{5j+5}$, $W_{5j+5} \geq F_{5j+6}$. $W_{5j+5}$ is not restricted to $F_{5j+6}$ because it may satisfy demand for more days than just Monday of the following week. □

In §3.3, we show how this section's results limit the cyclic policies that need to be studied when searching for optimality. We end this section by proving sufficiency of cyclic policies for minimizing the long-term average per-week cost of the currency management system. This result is relatively straightforward, so we only provide a brief discussion.

To study a bank's operations, certain parameters must be known: $h$, $e$, and $F_i$, $U_i$, for $i=1,\ldots,5$. With this information, we can define the state of the system as follows.

Definition. The *state* of a bank's currency management system can be specified by its current inventory levels for used cash ($I_i^u$) and for fit cash ($I_i^f$).

Let $\mathcal{F}$ denote the set of all feasible states. An *operating sequence* for a bank's currency management operations is an infinite sequence of successive states resulting from feasible operations starting from an initial state. The *long-term average per-week cost* of an operating sequence is the limit of the average of the costs for week 1, week $2,\ldots,$ week $n$, as $n\to\infty$. Since the Fed ships and receives cash only in bundles of 1,000 notes, $I^u$ and $I^f$ must be integers. Furthermore, it is easy to see that both $I^u$ and $I^f$ must be bounded from below and above in an optimal operating sequence. Thus $\mathcal{F}$ is finite.

Definition. A *policy* for the bank is a function $d: \mathcal{F}\to\mathcal{F}$ such that there exists a state $S\in\mathcal{F}$ for which the infinite sequence $T(d,S)\equiv\{S, d(S), d^2(S),\ldots, d^n(S),\ldots\}$ is an operating sequence, where $d^2(S)=d(d(S))$, $d^3(S)=d(d^2(S)),\ldots, d^n(S)=d(d^{n-1}(S)),\ldots.$

The finiteness of $\mathcal{F}$ implies that the infinite sequence of states resulting from *any* policy is a repeating sequence. Every policy repeats a minimal sequence of deposits and withdrawals. The minimal sequence is a state-preserving sequence: the state of the currency management system at the beginning is identical to the state of the system at the end of the sequence. We therefore refer to a sequence resulting from a policy as a *cyclic sequence*. It is straightforward to establish that there exists a cost-minimizing operating sequence that can be generated by a policy. Thus we have the existence of a cyclic sequence that minimizes costs.

Regarding cyclic policies, we consider only those that have been reduced to their fewest number of weeks. For example, a policy that covers $jn$ weeks, $j\in\mathbb{Z}^+$, and is the same $n$-week policy performed $j$ times is only considered as an $n$-week policy. In general, if for an $m+n$ week cyclic policy $P_{m+n}$ we have $(I_{5m}^u, I_{5m}^f)=(I_{5(m+n)}^u, I_{5(m+n)}^f)$, then this policy can be decomposed into one $m$-week cyclic policy $P_m$ and one $n$-week cyclic policy $P_n$. Furthermore, either $C(P_m)/m\le C(P_{m+n})/(m+n)$ or $C(P_n)/n\le C(P_{m+n})/(m+n)$, i.e., $P_{m+n}$ cannot be a minimum length optimal policy.

The concepts of states and cyclic sequences also allow us to state and prove the following result.

Theorem 1. *Suppose policy $Pn$ is an $n$-week minimum length optimal policy. If $Pn$ contains $m$ weeks with no deposits* ($0\le m\le n-1$), *then those $m$ weeks with no deposit must be consecutive.*

Proof. Suppose weeks $i$ and $j$ in policy $Pn$ each have no deposits and are preceded by weeks with deposits. Because weeks $i$ and $j$ each have withdrawals on each day (Lemma 4), each week begins with $I^u=U_5$ and $I^f=F_1$. Hence, by the observations immediately preceding the statement of the theorem, policy $Pn$ cannot be a minimum length optimal policy. □

Without loss of generality, we will represent all potentially optimal $n$-week policies, except $P_15$, as beginning with the $m$ weeks in which no deposits are made, and concluding with $n$–$m$ weeks in which a deposit is made on each day. Lemma 4 thus implies that a withdrawal is made on each day of the first $m$ weeks. It follows that all potentially optimal cyclic policies begin with state $(I_0^u, I_0^f)=(U_5, F_1)$.

### 3.3. Managerial Insights

We now determine the circumstances under which particular cyclic policies are optimal. We begin with one-week and two-week policies and then generalize to $n$-week ($n\ge 3$) cyclic policies. To gain insight into which policy is optimal under specific circumstances, we consider the special case in which customer demands are the same each day and customer deposits are the same each day. These values are obtained from weekly averages: the demand for fit cash on each day is $F$ bundles, and the used cash received on each day is $U$ bundles, where $F=(1/5)\sum_{i=1}^5 F_i$, $U=(1/5)\sum_{i=1}^5 U_i$, $k=U/F$. The value of $k$ will be important for determining an optimal policy for a given instance. The following results will be generalized to $U_i$ and $F_i$, $i=1,\ldots,5$ in §3.4.

**3.3.1. Results for One-Week and Two-Week Cyclic Policies.** Lemmas 2–5 imply that there are only one possibly optimal one-week cyclic policy and six possibly optimal two-week cyclic policies (Table 1). We can now specify the optimal one-week or two-week cyclic policies for all possible values of $k$, $e$, and $h$.

Theorem 2. *Within the set of one-week and two-week cyclic policies, the following two-week policies are optimal for the stated values of $e/h$ and of $k$:*

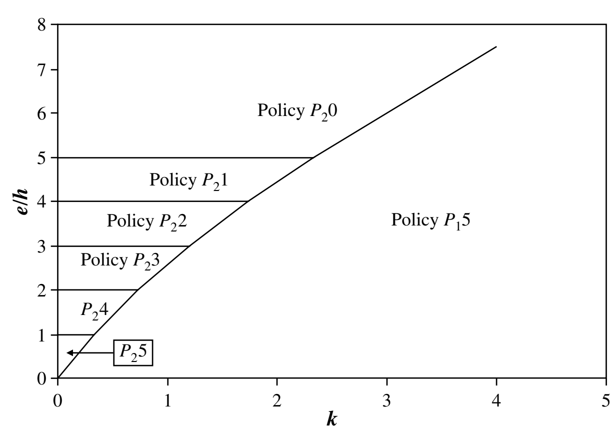
$P_2 5$: $e/h \le 1$, $15k \le 5e/h$
$P_2 4$: $1 \le e/h \le 2$, $15k \le 6e/h - 1$
$P_2 3$: $2 \le e/h \le 3$, $15k \le 7e/h - 3$
$P_2 2$: $3 \le e/h \le 4$, $15k \le 8e/h - 6$
$P_2 1$: $4 \le e/h \le 5$, $15k \le 9e/h - 10$
$P_2 0$: $5 \le e/h$, $15k \le 10e/h - 15$

*Policy $P_1 5$ is optimal over this set in all other cases.*

Proof. We find the values of $e/h$ for which $P_2 l$ is optimal over all two-week policies by noting that $C(P_2(l-1)) \ge C(P_2 l) \Leftrightarrow e \le (6-l)h$, which is true because $C(P_2(l-1)) - C(P_2 l) = (6-l)Fh - Fe \ge 0 \Leftrightarrow e \le (6-l)h$. We verify the values for $k$ by comparing each two-week policy to policy $P_1 5$: $C(P_2 l)/2 \le C(P_1 5) \Leftrightarrow e \ge (15k + \sum_{q=1}^{5-l} q)h/(10-l)$. The results follow. □

Figure 4 graphically presents the regions in which the different one-week and two-week policies are optimal. Note that Theorem 2 implies that it is sufficient to know only $e/h$, rather than the values of $e$ and $h$ individually.

**Figure 4** Regions of Optimality for One-Week and Two-Week Policies




**3.3.2. Results for $n$-Week Cyclic Policies.** We now generalize the results of the previous subsection to larger cyclic policies. To simplify matters, we will be interested only in minimum length optimal cyclic policies. Recall that without loss of generality, we consider only those potentially optimal policies (except $P_1 5$), which begin with the consecutive weeks in which no deposits are made. Furthermore, Lemma 4 implies that a withdrawal is made on each day of these first $m$ weeks.

To represent longer cyclic policies, we must augment our notation to $P_n^m l$. As before, the subscript $n$ will indicate the number of weeks the policy covers, and the normal-sized number $l$ indicates that withdrawals are made on each of the last $l$ days of the policy. The superscript $m$ signifies the number of weeks with no deposits; if the number of weeks with no deposits is less than two, we omit the superscript. For example, $P_4^2 3$ is a four-week cyclic policy in which the first two weeks have no deposits, but a withdrawal is made each day. Deposits are made each day of weeks 3 and 4. After week 2, no withdrawal is made until the last three days of the fourth week. To generalize, an optimal $n$-week cyclic policy $P_n^m l$ will begin with $m$ weeks in which there is no deposit to the Fed, but a withdrawal is made each day. Theorem 3 implies that during weeks $m+1$ through $n$, a deposit is made each day, but there is no withdrawal until the last $l$ days of the cyclic policy and $l \le 4$. If $m = n-1$, then $l \le 5$.

We first limit the number of potentially optimal $n$-week cyclic policies in which $m$ weeks have no deposits. Next, we prove that policy $P_n^m l$ with $m \ge 2$ cannot be optimal if $k > 3/(5m+3)$. Following that, optimality conditions for policy $P_n l$ are given. Next, we show that policy $P_n^m l$ with $m \ge 2$ can be optimal only if $n = m+1$, which leads to optimality conditions for policy $P_n^{n-1} l$. This subsection concludes with an $O(e/h)$ algorithm for an optimal cyclic policy, given $e/h$ and $k$.

Theorem 3. *In any optimal $n$-week ($n \ge 3$) cyclic policy with $m \ge 1$ weeks with no deposit, no withdrawals are made during days $5m+1$ through $5n-4$, inclusive, unless $m = n-1$, in which case a withdrawal may be made on day $5m+1 = 5n-4$.*

Proof. The proof of Lemma 5 can easily be extended to show that if $W_i > 0$ for some $i \ge 5m+1$,

**Table 2** Potentially Optimal Three-Week Policies and Their Costs if $U_i = U$ and $F_i = F$, $i = 1, \ldots, 5$

| Policy | M T W T F | M T W T F | M T W T F | Cost |
|---|---|---|---|---|
| $P_3 4$ | <br>WWWWW | D D D D D | D D D D D<br>WWWW | $30Uh + 36Fh + 4Fe$ |
| $P_3 3$ | <br>WWWWW | D D D D D | D D D D D<br>WWW | $30Uh + 43Fh + 3Fe$ |
| $P_3 2$ | <br>WWWWW | D D D D D | D D D D D<br>WW | $30Uh + 51Fh + 2Fe$ |
| $P_3 1$ | <br>WWWWW | D D D D D | D D D D D<br>W | $30Uh + 60Fh + Fe$ |
| $P_3 0$ | <br>WWWWW | D D D D D | D D D D D | $30Uh + 70Fh$ |
| $P_3^2 5$ | <br>WWWWW | <br>WWWWW | D D D D D<br>WWWWW | $70Uh + 15Fh + 5Fe$ |
| $P_3^2 4$ | <br>WWWWW | <br>WWWWW | D D D D D<br>WWWW | $70Uh + 16Fh + 4Fe$ |
| $P_3^2 3$ | <br>WWWWW | <br>WWWWW | D D D D D<br>WWW | $70Uh + 18Fh + 3Fe$ |
| $P_3^2 2$ | <br>WWWWW | <br>WWWWW | D D D D D<br>WW | $70Uh + 21Fh + 2Fe$ |
| $P_3^2 1$ | <br>WWWWW | <br>WWWWW | D D D D D<br>W | $70Uh + 25Fh + Fe$ |
| $P_3^2 0$ | <br>WWWWW | <br>WWWWW | D D D D D | $70Uh + 30Fh$ |

then $W_j = F_{j+1}$, for $j = i, \ldots, 5n-1$, and $W_{5n} = F_1$. Thus the theorem can be proven by showing that no withdrawal is made on day $5n-4$. If a withdrawal is made on day $5n-4$ in some $n$-week cyclic policy, then that policy is the concatenation of policy $P_{n-1}^m l$ and policy $P_1 5$, for some integer $l \in [0, 5]$. Hence the $n$-week cyclic policy cannot be an optimal minimum length cyclic policy. If $l = 5$ and $n-1 \geq m+2$, then the same argument can be applied to show that policy $P_{n-1}^m l$ is a concatenation of shorter policies. This could repeat until we have $P_{m+1}^m 5$, in which case a withdrawal is allowed on day $5m+1$, and $P_{m+1}^m 5$ is potentially optimal. □

To illustrate this result, Table 2 lists all potentially optimal three-week policies. In general, any optimal $n$-week ($n \geq 2$) cyclic policy $P_n^m l$ has the following structure:

$D_1 = \cdots = D_{5m} = 0$; $D_{5m+1} = (5m+1)U$; $D_{5m+2} = \cdots = D_{5n} = U$.

$W_1 = \cdots = W_{5m-1} = F$; $W_{5m} = (5(n-m) - l + 1)F$.

$W_i = 0$; $i = 5m+1, \ldots, 5n-l$; $0 \leq l \leq 4$ ($0 \leq l \leq 5$ if $m = n-1$).

$W_i = F$, $i = 5n+1-l, \ldots, 5n$; $0 \leq l \leq 4$ ($0 \leq l \leq 5$ if $m = n-1$).

The cost of such a cyclic policy is

$$C(P_n^m l) = \left(5n + \sum_{q=1}^{5m} q\right) Uh + \left(5n + \sum_{q=1}^{5(n-m)-l} q\right) Fh + lFe. \tag{5}$$

These values for $n = 3$ can be found in Table 2.

The following theorem limits the values of $k$ for which we need to consider policies in which $m \geq 2$. A proof is provided in the appendix.

Theorem 4. *If $k > 3/(5m+3)$, then policy $P_n^m l$ ($m \geq 2$) cannot be optimal.*

Note that $C(P_n^m l) \leq C(P_n^m (l-1))$ if and only if $(5(n-m) - l + 1)Fh - Fe \geq 0$. In turn, the latter inequality is satisfied if and only if $e \leq [5(n-m) - l + 1]h$. We therefore have the following result that limits the values of $e/h$ for which a given policy can be optimal.

Lemma 6. *If $[5(n-m) - l]h \leq e \leq [5(n-m) - l + 1]h$, for $l = 1, \ldots, 4$, then the $n$-week policy $P_n^m l$ is optimal over the set of all $n$-week policies that have $m$ weeks with no deposits. If $e \geq 5(n-m)h$, then $P_n^m 0$ is optimal over the set of all $n$-week policies that have $m$ weeks with no deposits. If $n = m+1$ and $e \leq h$, then $P_n^{n-1} 5$ is optimal over the set of all $n$-week policies that have $m$ weeks with no deposits.*

The following theorem shows how the value of $k$ determines which of these policies with $m = 1$ is optimal.

Theorem 5. *The $n$-week ($n \geq 3$) cyclic policy $P_n l$, $l = 0, \ldots, 4$ is optimal over all $j$-week ($1 \leq j \leq n$) cyclic policies if it satisfies Lemma 6 and if $k$ satisfies the following two inequalities:*

$$15k \leq [5n - l]\frac{e}{h} - \sum_{q=1}^{5(n-1)-l} q \tag{6}$$

$$15k \geq (n-1)l\frac{e}{h} + (n-1)\sum_{q=1}^{5(n-1)-l} q - n\sum_{q=1}^{5(n-2)} q. \tag{7}$$

Proof. Lemma 6 and (7) imply $k \geq 3$, so policy $P_n^m l$ in which $m \geq 2$ cannot be optimal. The two conditions (6) and (7) can be derived by comparing formula (5) to the per-week costs for policies $P_1 5$ and $P_j 0$, $2 \leq j \leq n-1$, respectively. That this is sufficient to prove the theorem follows from Lemma 6, which also implies

**Figure 5** **Regions of Optimality for One-Week, Two-Week, Three-Week, and Four-Week Policies with $m = 1$**

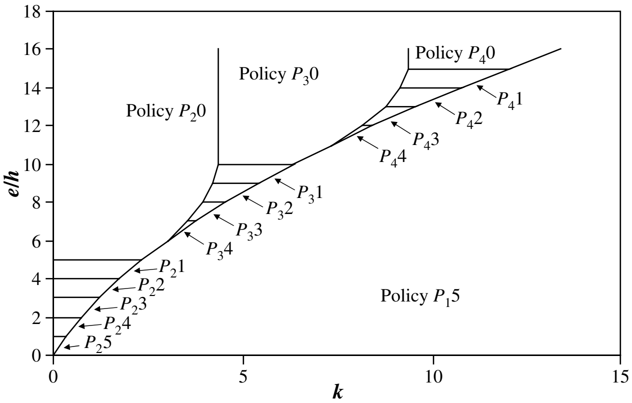
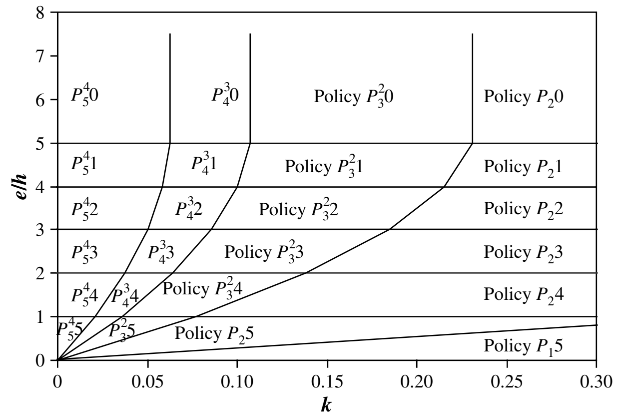



**Figure 6** **Regions of Optimality for Two-Week, Three-Week, Four-Week, and Five-Week Policies $P_n^{n-1} l$, $l = 0, \ldots, 5$**




that the only possibly optimal $j$-week policies, $2 \le j \le n-1$, are $P_j 0$. $\square$

For three-week and four-week policies with $m = 1$, inequalities (6) and (7) yield the following regions of optimality.

$$
\begin{array}{lll}
P_3 4: & 8e/h - 3 \le 15k \le 11e/h - 21, & 6h \le e \le 7h \\
P_3 3: & 6e/h + 11 \le 15k \le 12e/h - 28, & 7h \le e \le 8h \\
P_3 2: & 4e/h + 27 \le 15k \le 13e/h - 36, & 8h \le e \le 9h \\
P_3 1: & 2e/h + 45 \le 15k \le 14e/h - 45, & 9h \le e \le 10h \\
P_3 0: & 65 \le 15k \le 15e/h - 55, & 10h \le e \\
P_4 4: & 12e/h - 22 \le 15k \le 16e/h - 66, & 11h \le e \le 12h \\
P_4 3: & 9e/h + 14 \le 15k \le 17e/h - 78, & 12h \le e \le 13h \\
P_4 2: & 6e/h + 53 \le 15k \le 18e/h - 91, & 13h \le e \le 14h \\
P_4 1: & 3e/h + 95 \le 15k \le 19e/h - 105, & 14h \le e \le 15h \\
P_4 0: & 140 \le 15k \le 20e/h - 120, & 15h \le e
\end{array}
$$

Figure 5 graphically presents the regions in which the different one-week, two-week, three-week, and four-week policies with $m = 1$ are optimal.

We now shift our attention to policies with at least two weeks with no deposits. We first reduce the set of policies to be considered, then we provide optimality conditions. The proof of the following theorem is given in the appendix.

Theorem 6. *Policy $P_n^m l$, $m \ge 2$ can be optimal only if $n = m + 1$.*

It follows from Theorem 5 that the minimum value of $k$ for which a cyclic policy with $n \ge 3$ and one week with no deposits can be optimal is $k = 3$. Hence, policy $P_n^{n-1} l$ with $n \ge 3$, which can be optimal only if $k \le 3/(5n-2)$ (by Theorem 4), need only be compared to policy $P_2 l$ or to other policies $P_\nu^{\nu-1} l$ with $\nu \ge 3$.

Theorem 7. *Policy $P_n^{n-1} l$, $n \ge 3$, $l = 0, \ldots, 5$ is optimal over all $j$-week policies ($0 \le j \le n$) if Lemma 6 is satisfied and*

$$
k \le \frac{l(e/h) + \sum_{q=1}^{5-l} q}{(n-1)\sum_{q=1}^{5(n-1)} q - n \sum_{q=1}^{5(n-2)} q}.
$$

Proof. We first observe that the theorem holds for $n = 3$ as a direct result of (31) in the appendix. For general $n$, the condition of the theorem holds if and only if the per-week cost for $P_n^{n-1} l$ is less than that for $P_{n-1}^{n-2} l$. That this one comparison is sufficient follows from the theorem holding for $n = 3$, from that policy $P_n l$, $n \ge 3$, can be optimal only if $k \ge 3$, and from the right-hand side of the condition being decreasing in $n$. $\square$

Figure 6 shows the regions for which policies $P_3^2 l$, $P_4^3 l$, and $P_5^4 l$, $l = 0, \ldots, 5$ are optimal. Note that the scale of the horizontal axis for this figure is significantly different from that of previous figures.

Lemma 6, Theorem 5, and Theorem 7 imply that the following $O(e/h)$ algorithm finds an optimal policy for a particular bank.

Algorithm Find Policy.

**Input:** $e/h, k, N$

If $k \ge 7/3$, then ($P_2 l$, $l = 1, \ldots, 5$ cannot be optimal, nor can $P_n^{n-1} l$, $n \ge 3$, $l = 0, \ldots, 5$).

*Step* 1. Find $P_n l$, the longest potentially optimal policy:

(a) Choose $n$ so that $(e/h + 4)/5 \le n < (e/h + 4)/5 + 1$.

(b) Choose $l$ so that $5(n-1) - e/h < l \le 5(n-1) - e/h + 1$

*Step* 2. Choose the optimal policy from among $P_1 5$, $P_n l$, and $P_j 0$, $2 \le j \le n-1$:

If $15k \ge (5n-l)e/h - \sum_{q=1}^{5(n-1)-l} q$, then choose policy $P_1 5$,

Else if $15k > (n-1)le/h + (n-1)\sum_{q=1}^{5(n-1)-l} q - n\sum_{q=1}^{5(n-2)} q$, then choose policy $P_n l$,

Else if $15k > (n-2)\sum_{q=1}^{5(n-2)} q - (n-1)\sum_{q=1}^{5(n-3)} q$, then choose policy $P_{n-1} 0$,

$\vdots$

Else if $15k > (n-i-1)\sum_{q=1}^{5(n-i-1)} q - (n-i)\sum_{q=1}^{5(n-i-2)} q$, then choose policy $P_{n-i} 0$,

$\vdots$

Else if $15k > 2\sum_{q=1}^{10} q - 3\sum_{q=1}^{5} q$, then choose policy $P_3 0$,

Else choose policy $P_2 0$.

Else (optimal policy will be $P_1 5$ or $P_n^{n-1} l$, $n \ge 2$, $l = 0, \ldots, 5$)

*Step* 3. Choose $l$ so that $5 - e/h \le l \le 5 - e/h + 1$. If $5 - e/h + 1 < 0$, set $l = 0$.

*Step* 4. Choose the optimal policy from among $P_1 5$ and $P_\nu^{\nu-1} l$, $2 \le \nu \le N$:

If $15k \ge (10-l)e/h - \sum_{q=1}^{5-l} q$, then choose policy $P_1 5$.

Else define the function $K$ by

$$K\left(l, n, \frac{e}{h}\right) = \frac{l(e/h) + \sum_{q=1}^{5-l} q}{(n-1)\sum_{q=1}^{5(n-1)} q - n\sum_{q=1}^{5(n-2)} q}.$$

If $k \ge K(l, 3, e/h)$, then choose policy $P_2 l$.

Else if $k \ge K(l, 4, e/h)$, then choose policy $P_3^2 l$.

$\vdots$

Else if $k \ge K(l, \nu+1, e/h)$, then choose policy $P_\nu^{\nu-1} l$.

$\vdots$

Else if $k \ge K(l, N, e/h)$, then choose policy $P_{N-1}^{N-2} l$.

Else choose policy $P_N^{N-1} l$.

**Stop.**

We now use the results and insights gained in this section to analyze the scenario in which $U_i$ and $F_i$ are not constant.

### 3.4. Generalizations to Nonconstant $U_i$ and $F_i$

The results of §3.3 were derived with the assumption that the used cash received each day is a constant $U$ and the fit cash distributed each day is a constant $F$. The analysis, however, remains useful even if the used cash received and the fit cash distributed, $U_i$ and $F_i$, respectively, $i = 1, \ldots, 5$, vary from day to day within a week. To demonstrate this, we performed computational experiments to investigate how well Algorithm Find Policy operates on general data. We compared the solution returned by the algorithm to the optimal policy in the class of all one-week, two-week, and three-week policies. Of 100 instances, 15 data sets were generated, where an instance consists of $U_i$ and $F_i$, $i = 1, \ldots, 5$. Within each data set, each of the 500 values for $F_i$ was generated from a normal distribution with mean $\mu(F) = 200$, and each of the 500 values for $U_i$ was generated from a normal distribution with mean $\mu(U) = k\mu(F)$, where $k = 0.2, 0.5, 1, 2, 5$, so $\mu(U) = 40, 100, 200, 400, 1000$. To create 15 data sets for each $k$, we used three sets of standard deviations: $\sigma(F) = 0.1\mu(F)$ and $\sigma(U) = 0.1\mu(U)$, $\sigma(F) = 0.25\mu(F)$ and $\sigma(U) = 0.25\mu(U)$, $\sigma(F) = 0.5\mu(F)$ and $\sigma(U) = 0.5\mu(U)$. Within each data set, tests were run for values of $e/h$ that ranged from 0.5 to 14 in increments of 0.5. Note that the range of values for $k$ and $e/h$ were chosen to reflect various practical scenarios.

For each of the 15 data sets and each of the 28 values of $e/h$, we tested 100 instances by comparing the cost of an optimal policy (obtained via complete enumeration) to that of the policy chosen by Algorithm Find Policy with $U = \sum_{i=1}^{5} U_i/5$ and $F = \sum_{i=1}^{5} F_i/5$. For each combination of data set and $e/h$, the average extra cost incurred by using Algorithm Find Policy was never more than 0.40% of the optimal cost. These results, averaged over $e/h$, are presented in Table 3. We conclude that it is reasonable for banks to base their policy decisions on weekly averages by using Algorithm Find Policy. Additionally, using the weekly averages should provide stability by smoothing daily variations in demand.

**Table 3** The Average Percentage Error Incurred by Using Algorithm Find Policy

| | Standard deviation percentages (%) | | |
|---|---|---|---|
| $k$ | 0.10 | 0.25 | 0.50 |
| 0.2 | 0.00 | 0.07 | 0.24 |
| 0.5 | 0.00 | 0.00 | 0.06 |
| 1 | 0.00 | 0.01 | 0.05 |
| 2 | 0.00 | 0.01 | 0.06 |
| 5 | 0.00 | 0.03 | 0.19 |
| Averages: | 0.00 | 0.02 | 0.12 |

## 4. Basic Model with Custodial Inventory

As described in §2, the Fed's custodial inventory program would allow banks to earn interest on used cash received from customers without facing a cross-shipping fee on withdrawals made in the same week. Custodial inventory, which may contain only fit cash, is resident at the bank's facility or at that of a Fed-approved third party, e.g., a secure logistics provider. Daily deposits to and withdrawals from custodial inventory are allowed. By offering the custodial inventory program, the Fed hopes to encourage banks to fit sort used cash at their own expense, thereby reducing the Fed's fit sorting costs.

The custodial inventory program, consequently, introduces two business opportunities for a third-party logistics provider: managing the secure custodial inventory facilities within its premises, and fit sorting used cash. The main purpose of our work with Brink's, Inc. was to assess the potential of these two opportunities. As will become clear in the next subsection, a complete characterization of the optimal policies in the presence of custodial inventories seems to be quite challenging. We first develop a model for obtaining an optimal policy within the subclass of one-week and two-week policies, with custodial inventory and fit sorting performed at the bank's request by a secure logistics provider. In this model, the logistics provider proposes to charge a per bundle fee to the bank for each deposit or withdrawal of currency from custodial inventory. It is anticipated that market forces will cause this fee to be small in comparison to the cross-shipping fee: either the logistics provider will reduce its fee to attract the bank's business, or the Fed will raise its fee to deter cross-shipping. In §4.2, we prove some structural results for optimal policies.

### 4.1. Finding an Optimal Cyclic Policy

We first describe the notation and the flow of currency for this model. We then derive the constraints and the objective. As in the previous model, the structure of the transportation costs allows us to consider only one denomination at a time. The following parameters will be part of this formulation for a two-week policy (or a one-week policy that is repeated) in which $i = 1, \ldots, 10$:

$F_i$: The amount of fit cash demanded by the bank's customers on day $i$.

$U_i$: The amount of used cash received from the bank's customers on day $i$.

$g$: The percentage of fit sorted used cash that is fit. Typically, this value is 75%.

$h$: The holding cost (incurred by the bank) per bundle per day for either fit or used cash in the bank's inventory.

$e$: The fee charged (to the bank by the Fed) per bundle of cross-shipped currency.

$r$: The fee charged (to the bank by the logistics provider) per bundle to fit sort used cash.

$q$: The fee charged (to the bank by the logistics provider) per bundle to deposit or withdraw currency from custodial inventory.

$a_1$: The fee charged (to the bank by the logistics provider) per bundle to ship currency from the bank to the logistics provider and vice versa.

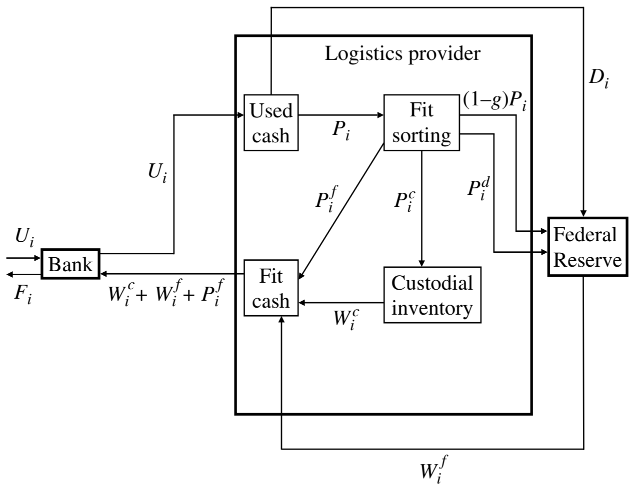
$a_2$: The fee charged (to the bank by the logistics provider) per bundle to ship currency from the logistics provider to the Fed and vice versa.

Note that this model can also capture the case in which the bank performs the fit sorting and handles the custodial inventory itself. It is anticipated that such a system would be feasible only for a few large banks that have economies of scale that are sufficient to justify the required initial investment. In this case, there is no shipment between the bank and a logistics provider, so $a_1 = 0$ and $a_2$ is the cost to ship a bundle of currency from the bank directly to the Fed. The parameter $r$ represents the cost of the labor required for fit sorting one bundle plus the per bundle allocation of the cost of the fit-sorting equipment. Similarly, the parameter $q$ represents the cost of the labor required for depositing or withdrawing one bundle from custodial inventory plus the per bundle allocation of the cost of acquiring and maintaining the custodial inventory's storage facility.

The variables used in the formulation are defined in the following description of how currency flows through the system. A diagram of the transportation of cash is presented in Figure 7.

At close of business on day $i$, all used cash received ($U_i$) from customers is separated by denomination and then transported to the logistics provider, where it is added to the used cash inventory. Some of this used

**Figure 7** **Transportation of Currency in the Model with Custodial Inventory and Fit Sorting**




cash ($P_i$) is fit sorted overnight; the bank may choose to fit sort less than all used cash because there is a charge of $r$ per bundle fit sorted. Some of that which is not fit sorted may become part of the next day's deposit of used cash ($D_{i+1}$), or remain in inventory to be either fit sorted later or deposited during the following week (to avoid cross-shipping charges). Therefore the balance constraint for used cash inventory is

$$I_i^u = I_{i-1}^u - P_{i-1} - D_i + U_i,$$

so $P_{i-1} + D_i \le I_{i-1}^u$, which implies

$$U_i \le I_i^u. \tag{8}$$

This equation also implies

$$\sum_{i=1}^{10} U_i = \sum_{i=1}^{10} D_i + \sum_{i=1}^{10} P_i, \tag{9}$$

since $I_0^u = I_{10}^u$. As in the previous model, $I_i^u$ is computed at close of business.

The fit cash generated by fit sorting, $gP_i$, is divided into that which is returned to the bank's fit cash inventory ($P_i^f$), that which is deposited into the custodial account ($P_i^c$), and that which is deposited to the Fed ($P_i^d$). Hence $gP_i = P_i^f + P_i^c + P_i^d$. These three transactions are completed on day $i+1$. A diagram of the schedule of the daily transactions effecting a bank's operations and inventory levels can be found in Figure 8.

The unfit cash recovered through sorting, $(1-g)P_i$, will also be deposited to the Fed on day $i+1$. Because the logistics provider certifies that this deposit is unfit and should be removed from circulation, it will not lead to a cross-shipping charge.

On day $i$, the custodial inventory increases by $P_{i-1}^c$ and decreases by $W_i^c$, where $W_i^c$ is the amount withdrawn from custodial inventory during business hours on day $i$. $I_i^c$ is computed at close of business:

$$I_i^c = I_{i-1}^c + P_{i-1}^c - W_i^c.$$

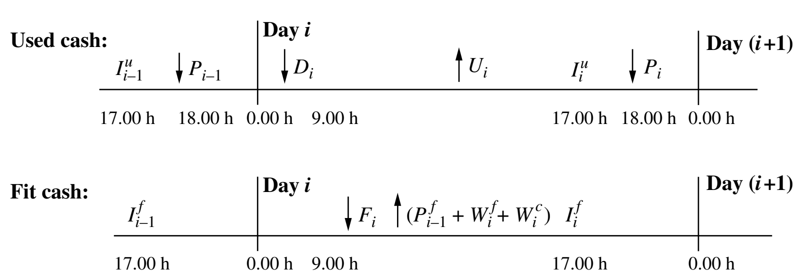
Because there is a transaction cost $q$ for depositing or withdrawing from custodial inventory, $P_{i-1}^c$ and $W_i^c$ will not both be positive for the same $i$, so $W_i^c \le I_{i-1}^c$. Note that if $2q = \eta h$ for some $\eta > 0$, then it is economical to use custodial inventory, rather than fit cash inventory, only for fit currency that the bank will hold for at least $\lceil \eta \rceil$ days.

The bank's inventory of fit cash is also computed at close of business, after receipt of fit cash recovered from the previous day's fit sorting ($P_{i-1}^f$), withdrawals from the Fed ($W_i^f$), and withdrawals from the custodial account ($W_i^c$). Fit cash is decreased by the day's demand ($F_i$):

$$I_i^f = I_{i-1}^f - F_i + P_{i-1}^f + W_i^f + W_i^c \quad \text{and} \quad F_i \le I_{i-1}^f,$$

so $I_i^f \ge P_{i-1}^f + W_i^f + W_i^c$. Since $I_0^f = I_{10}^f$, if we sum

**Figure 8** **Bank's Inventory Schedule with Custodial Inventory and Fit Sorting for Day *i***



the expression for $I_i^f$ over an entire two-week policy, we get

$$\sum_{i=1}^{10} F_i = \sum_{i=1}^{10} \left(P_i^f + W_i^f + W_i^c\right). \tag{10}$$

We now derive the transportation cost $C_t$ for this model. All used cash received by the bank is transferred to the logistics provider, which generates cost $a_1 U_i$. After fit sorting, some of the resulting fit cash is returned to the bank: $a_1 P_i^f$. Some of the fit cash may be deposited to the Fed, along with the unfit cash and some used cash: $a_2(P_i^d + (1-g)P_i + D_i)$. Because custodial inventory is held at the logistics provider's facility, there is no transportation charge for currency deposited into it. However, for currency withdrawn from custodial inventory there is a transportation charge of $a_1 W_i^c$. Currency withdrawn from the Fed travels via the logistics provider's location, so the transportation cost is $(a_1+a_2)W_i^f$. Hence the total transportation cost for this model is

$$C_t = a_1 \sum_{i=1}^{10} \left(U_i + P_i^f + W_i^c + W_i^f\right) + a_2 \sum_{i=1}^{10} \left(P_i^d + (1-g)P_i + D_i + W_i^f\right).$$

The bank's total cost is transportation cost, plus cross-shipping costs for week one ($R_1$) and for week two ($R_2$), plus inventory holding costs, plus fit-sorting costs, plus transaction costs for custodial inventory. Therefore we have the following mixed-integer program (MIP) to find an optimal two-week cyclic policy:

$$\min \; Z = C_t + R_1 + R_2 + \sum_{i=1}^{10} h\left(I_i^u + I_i^f\right) + \sum_{i=1}^{10} rP_i + \sum_{i=1}^{10} q(P_i^c + W_i^c) \tag{11}$$

$$\text{s.t.} \quad gP_i = P_i^f + P_i^c + P_i^d, \quad i=1,\ldots,10 \tag{12}$$

$$I_i^u = I_{i-1}^u - P_{i-1} - D_i + U_i, \quad i=1,\ldots,10 \tag{13}$$

$$I_i^c = I_{i-1}^c + P_{i-1}^c - W_i^c, \quad i=1,\ldots,10 \tag{14}$$

$$I_i^f = I_{i-1}^f - F_i + P_{i-1}^f + W_i^f + W_i^c, \quad i=1,\ldots,10 \tag{15}$$

$$I_i^c \le 0.25 \sum_{j=1}^{5} U_j, \quad i=1,\ldots,10 \tag{16}$$

$$P_i + D_{i+1} \le I_i^u, \quad i=1,\ldots,10 \tag{17}$$

$$W_i^c \le I_{i-1}^c, \quad i=1,\ldots,10 \tag{18}$$

$$F_i \le I_{i-1}^f, \quad i=1,\ldots,10 \tag{19}$$

$$ML_1 \ge D_1 + D_2 + D_3 + D_4 + D_5 + P_1^d + P_2^d + P_3^d + P_4^d + P_5^d \tag{20}$$

$$ML_2 \ge D_6 + D_7 + D_8 + D_9 + D_{10} + P_6^d + P_7^d + P_8^d + P_9^d + P_{10}^d \tag{21}$$

$$R_1 \ge e \sum_{i=1}^{5} W_i^f - M(1-L_1) \tag{22}$$

$$R_2 \ge e \sum_{i=6}^{10} W_i^f - M(1-L_2) \tag{23}$$

$$I_0^u = I_{10}^u, \qquad I_0^c = I_{10}^c, \qquad I_0^f = I_{10}^f \tag{24}$$

$$I_i^u, I_i^f, I_i^c, P_i, P_i^f, P_i^d, P_i^c, D_i, W_i^f, W_i^c, R_1, R_2 \ge 0, \quad i=1,\ldots,10 \tag{25}$$

$$L_1, L_2 \in \{0,1\}. \tag{26}$$

Here, $M$ is a large number. It is sufficient to set $M = \max\{e\sum_{i=1}^5 F_i, \sum_{i=1}^5 U_i\}$. Constraint (12) ensures that the amount of fit cash obtained from fit sorting on day $i$ is equal to the amount of fit cash that is returned to the bank, the amount of fit cash that is deposited to custodial inventory, and the amount of fit cash that is deposited to the Fed on day $i+1$. Constraints (13)–(15) are inventory balance equations for used, custodial, and fit cash, respectively, for each day. Constraint (16) states that the amount of currency held in custodial inventory should not be more than 25% of the total amount of used cash received from customers during one week. The Fed has proposed this 25% limit. Constraint (17) ensures that the amount of used cash either fit sorted by the logistics provider after close of business on day $i$ or deposited to the Fed on day $i+1$ is not more than the amount of used cash in inventory at close of business on day $i$. Constraint (18) ensures that the amount of fit cash withdrawn from custodial inventory during business hours on a given day is no more than the amount of fit cash in custodial inventory at the end of the previous day. Constraint (19) states that the fit cash demand for the current day will be met by the amount of fit cash inventory at the end of previous day.

In constraint (20) (constraint (21)), the boolean variable $L_1$ ($L_2$) assumes the value one only if a deposit of unsorted used cash or of fit cash is made during week one (two); otherwise, it takes the value zero. Constraint (22) (constraint (23)) computes the cost of obtaining fit cash bundles from the Fed if cross-shipping occurs during week one (two). Constraints (24) ensure that the currency flow cycle's ending inventory levels equal its starting inventory levels for used, custodial, and fit cash. Constraint (25) (constraint (26)) indicates nonnegative (boolean) variables. Note that the formulation is efficiently solvable (i.e., in time that is polynomial in the binary encoding of the problem) because it has only two binary variables. The formulation can be easily extended for computing $n$-week cycles, $n \geq 3$, however, the number of binary variables required will be $n$.

Before deriving additional results for this model, we provide some example policies and their costs. In policy $P^\alpha$, the bank has the logistics provider fit sort $P_i = F_{i+2}/g$ each day, so that $P_i^f = F_{i+2}$ is received before close of business on day $i+1$. Obviously, this policy requires that $gU_i \geq F_{i+2}$, $\forall\, i$. A deposit of $D_i = U_{i-1} - F_{i+1}/g$ in used cash is made to the Fed each day. This policy uses neither custodial inventory nor withdrawals from the Fed. Its cost over two weeks is

$$C(P^\alpha) = (a_1 + a_2 + h)\sum_{i=1}^{10} U_i + \left(a_1 - a_2 + h + \frac{r}{g}\right)\sum_{i=1}^{10} F_i.$$

Policy $P^\beta$ is the same as policy $P_15$ described in §3. The bank deposits $U_{i-1}$ in used cash and withdraws $F_{i+1}$ in fit cash at the Fed each day. It spends no money on fit sorting, but does pay a cross-shipping fee for each withdrawal. This policy's cost is

$$C(P^\beta) = (a_1 + a_2 + h)\sum_{i=1}^{10} U_i + (a_1 + a_2 + h + e)\sum_{i=1}^{10} F_i,$$

which differs from $C(P_15)$ only because we consider transportation costs in this model. In general, because only fit cash obtained by fit sorting can be deposited into custodial inventory, if the bank requests no fit sorting, then this model reduces to the basic model of §3. Policy $P^\beta$ has a lower cost than policy $P^\alpha$ if and only if $e < r/g - 2a_2$. However, policy $P^\beta$ is always feasible, though policy $P^\alpha$ is not if $gU_i < F_{i+2}$ for any $i$.

Another alternative is policy $P^\gamma$: policy $P^\gamma$ makes no withdrawals from the Fed ($W_i^f = 0$, $\forall\, i$) and always maintains just enough fit cash in inventory to meet demand ($I_i^f = F_{i+1}$, $\forall\, i$). To meet demand for days on which $U_i < F_{i+2}/g$, fit cash is withdrawn from custodial inventory: $W_{i+1}^c = F_{i+2} - gU_i$, so that $I_{i+1}^f = F_{i+2}$. Because $\sum_{l=1}^{10} W_l^c = \sum_{l=1}^{10} P_l^c$, there must be at least one day $j$ for which $gU_j > F_{j+2}$ and $P_j^c > 0$. This policy's cost is

$$C(P^\gamma) = (a_1 + a_2 + h)\sum_{i=1}^{10} U_i + \left(a_1 - a_2 + h + \frac{r}{g}\right)\sum_{i=1}^{10} F_i + 2q\sum_{i=1}^{10} \max\{0, F_{i+2} - gU_i\}.$$

It follows that policy $P^\gamma$ never has a lower cost than policy $P^\alpha$, though it is feasible over a larger set of input values: $g\sum_{i=1}^{5} U_i \geq \sum_{i=1}^{5} F_i$. Policy $P^\gamma$'s relation to policy $P^\beta$ depends on the relation between $(2a_2 + e)$ and $r/g$, and the amount that is deposited and withdrawn from custodial inventory.

### 4.2. Characteristics of Optimal Policies

We now derive some additional results to characterize optimal policies in this model. The first theorem shows that it is wasteful to fit sort currency that will be deposited to the Fed. Its corollaries further reduce the set of potentially optimal policies.

Theorem 8. *In any optimal policy, $P_i^d = 0$, $\forall\, i$.*

Proof. Suppose there is an optimal policy $Py$ in which $P_i^d(y) = g\lambda > 0$. Let $Px$ be a policy that is identical to $Py$ except that $P_i^d(x) = 0$, $D_{i+1}(x) = D_{i+1}(y) + \lambda$, and $P_i(x) = P_i(y) - \lambda$. It follows that the total cost of $Px$ is less than that of $Py$: $Z(x) = Z(y) - r\lambda$. This contradicts the optimality of policy $Py$ and proves the result. □

Note that combining Theorem 8 with Equations (10) and (12) yields

$$\sum_{i=1}^{10} F_i = \sum_{i=1}^{10} W_i^f + g\sum_{i=1}^{10} P_i \qquad (27)$$

in an optimal policy.

Corollary 1. *In an optimal policy, any used cash received from customers on day $i$ is either fit sorted that night or deposited as unsorted used cash either on day $i+1$ or on the following Monday, i.e., used cash will not be held to be fit sorted later. Thus $P_i \leq U_i$, $\forall\, i$.*

**Table 4 Policies for the Proof of Corollary 1**

| Policy | $I^u_{i-\eta+1}$ | $I^f_{i-\eta+1}$ | $\cdots$ | $I^u_i$ | $I^f_i$ |
|---|---|---|---|---|---|
| $Px$ | $I^u_{i-\eta+1}(x)$ | $I^f_{i-\eta+1}(x)$ | $\cdots$ | $I^u_i(x)$ | $I^f_i(x)$ |
| $Py$ | $I^u_{i-\eta+1}(x)-\lambda$ | $I^f_{i-\eta+1}(x)+g\lambda$ | $\cdots$ | $I^u_i(x)-\lambda$ | $I^u_i(x)+g\lambda$ |

Proof. Consider an optimal policy $Px$ in which $P_i = U_i - D_{i+1} + \lambda$, $\lambda > 0$, for some $i$. The extra $\lambda$ has been residing in used inventory for some number $\eta > 0$ of days. Consider another policy $Py$ in which this extra $\lambda$ was fit sorted on day $i-\eta$ and resided in fit cash inventory until day $i+1$ (see Table 4). The policies fit sort the same amount of cash over two weeks and have the same inventory levels on days $i-\eta$ and $i+1$. It follows that $C(Px) - C(Py) = h(\lambda - g\lambda)(i-\eta) > 0$, contradicting the optimality of $Px$. Either the unsorted used cash is deposited on day $i+1$ to minimize holding cost, or it will be deposited the following Monday to avoid cross-shipping charges. □

The following corollary is analogous to Lemma 3 in the basic model.

Corollary 2. *If there is a set of optimal policies in which $D_q > 0$ for some $q \in \{5j+1, \ldots, 5j+5\}$, $j \in \mathbb{Z}^+ \cup \{0\}$, then there is at least one element of that set in which $D_{5j+1} = I^u_{5j} - P_{5j}, \ldots, D_q = I^u_{q-1} - P_{q-1}, \ldots, D_{5j+5} = I^u_{5j+4} - P_{5j+4}$. Moreover, $D_i = U_{i-1} - P_{i-1}$, $i = 5j+2, \ldots, 5j+5$ for this policy.*

Proof. $D_q > 0$ for some $q \in \{5j+1, \ldots, 5j+5\}$ implies that all withdrawals during this week will be subject to cross-shipping charges. Therefore, any additional deposits will effect $\sum_{i=5j+1}^{5j+5} I^u_i h$ but have no bearing on any other term in the expression (11) for the policy's cost. According to constraint (13), $\sum_{i=5j+1}^{5j+5} I^u_i h$ is minimized by $D_i = I^u_{i-1} - P_{i-1}$, for $i = 5j+1, \ldots, 5j+5$. This implies $I^u_i = U_i$, so $D_i = U_{i-1} - P_{i-1}$, $i = 5j+2, \ldots, 5j+5$. □

An equivalent expression for the MIP's objective is helpful for finding optimal policies in special cases. By using Equations (9), (10), and (27), the expression for $C_t$ becomes

$$C_t = a_1 \sum_{i=1}^{10}(U_i + F_i) + a_2 \sum_{i=1}^{10}[(1-g)P_i + U_i - P_i + F_i - gP_i]$$
$$= (a_1 + a_2)\sum_{i=1}^{10}(U_i + F_i) - 2a_2 g P_i.$$

Substituting this into (11) yields

$$Z = (a_1 + a_2)\sum_{i=1}^{10}(U_i + F_i) + R_1 + R_2 + (r - 2a_2 g)\sum_{i=1}^{10} P_i + h\sum_{i=1}^{10}(I^u_i + I^f_i) + q\sum_{i=1}^{10}(P^c_i + W^c_i). \quad (28)$$

This statement of the objective is much simpler to analyze. It is composed of a constant part $(a_1 + a_2)\sum_{i=1}^{10}(U_i + F_i)$ that depends only on given data, plus a variable part. The variable part can be analyzed as a piece that measures the cost of producing the fit cash used to meet demand $(R_1 + R_2 + (r - 2a_2 g)\sum_{i=1}^{10} P_i)$ and a piece that measures the cost of storing currency: $h\sum_{i=1}^{10}(I^u_i + I^f_i) + q\sum_{i=1}^{10}(P^c_i + W^c_i)$. These pieces of the variable part are not independent, however, because withdrawals may be timed to avoid cross-shipping costs, in which case, the amount of currency stored increases.

Corollary 3 uses parameter values to limit the bank's actions in optimal policies.

Corollary 3. *The following are characteristics of optimal policies:*

(a) *If $r/g - 2a_2 > e$, then no fit sorting will be done ($\sum_{i=1}^{10} P_i = 0$), and hence custodial inventory will not be used.*

(b) *If $gU_i \geq F_{i+2}$, $\forall\, i$ and $r/g - 2a_2 < e$, then no cross-shipping will be done.*

(c) *If $r/g - 2a_2 + 2q < e$ and $g\sum_{i=1}^{5} U_i \geq \sum_{i=1}^{5} F_i$, then no cross-shipping will be done.*

(d) *If $r/g - 2a_2 + 2q < 0$ and $g\sum_{i=1}^{5} U_i \geq \sum_{i=1}^{5} F_i$, then no withdrawals from the Fed will be made.*

Proof. Equations (27) and (28) imply the following:

(a) If $r/g - 2a_2 > e$, then policy $P^\beta$, in which $W^f_i = F_{i+1}$, $\forall\, i$ will have lower total cost than any policy for which $\sum_{i=1}^{10} P_i > 0$.

(b) If $gU_i \geq F_{i+2}$, $\forall\, i$ and $r/g - 2a_2 < e$, then policy $P^\alpha$, in which $gP_i = P^f_i = F_i$, $\forall\, i$ will have lower total cost than any policy for which cross-shipping occurs.

(c) If $r/g - 2a_2 + 2q < e$ and $g\sum_{i=1}^{5} U_i \geq \sum_{i=1}^{5} F_i$, then policy $P^\gamma$, in which $g\sum_{i=1}^{10} P_i = \sum_{i=1}^{10} F_i$ will have lower total cost than any policy for which cross-shipping occurs.

(d) If $r/g - 2a_2 + 2q < 0$ and $g\sum_{i=1}^{5} U_i \geq \sum_{i=1}^{5} F_i$, then policy $P^\gamma$ will have lower total cost than any policy for which withdrawals from the Fed are made. □

Note that both cross-shipping and fit sorting may occur in an optimal policy if $e-2q<r/g-2a_2<e$ and $gU_i<F_{i+2}$ for some $i$. Additionally, part (a) of this corollary provides the bank's reservation price for fit sorting. This can be used by the logistics provider to determine the price it will charge per bundle fit sorted: $r<(e+2a_2)g$. The form (28) of the objective allows us to easily prove the following result that specifies an optimal policy in a special case.

Theorem 9. *If $gU_i\geq F_{i+2}$, $\forall i$ and $r-2a_2g\leq 0$, then policy $P^\alpha$ is an optimal policy.*

Proof. The inventory levels in this policy are $I_i^f=F_{i+1}$ and $I_i^u=U_i$, $\forall i$, each of which is its minimum value by inequalities (8) and (19). There is no cross-shipping, so $R_1=R_2=0$, and there is no use of custodial inventory. Therefore the cost of this policy is

$$C(P^\alpha)=(a_1+a_2+h)\sum_{i=1}^{10}(U_i+F_i)+(r-2a_2g)\sum_{i=1}^{10}P_i,$$

which is the constant part, plus the minimum inventory cost, plus a nonpositive number times $\sum_{i=1}^{10}P_i$, which is at its maximum value ($\sum_{i=1}^{10}F_i/g$), according to Equation (27). □

There are circumstances in which neither cross-shipping nor fit sorting is used. Consider the following example with $a_1=15$, $a_2=10$, $r=67$, $h=1$, $q=1$, and $e=70$; fit sorting and cross-shipping are each expensive, and none of the hypotheses of Corollary 3 is satisfied. In this scenario, policy $P_20$ described in §3 is optimal: $W_i^f=0$, $D_i>0$ in the first week, and $W_i^f>0$, $D_i=0$ in the second week. The values for customer deposits and demands, along with the variable values for the optimal policy, are in Table 5. The constant part of the cost for this system is \$134,000. The variable part of the cost for this optimal policy is \$16,950. The variable cost for policy $P^\alpha$ in this system is \$72,266. The variable cost for policy $P^\beta$ is \$72,800.

This optimal two-week policy clearly violates the spirit of the Fed's new guidelines. Although the bank does not cross-ship, it deposits all of its used cash ($\sum_{i=1}^{10}U_i=\sum_{i=1}^{10}D_i$) and withdraws fit cash to meet its entire demand ($\sum_{i=1}^{10}W_i^f=\sum_{i=1}^{10}F_i$). In this instance, the new guidelines do not reduce the Fed's currency management costs and do increase the bank's operating costs by forcing it to hold currency for longer periods. Hence the Fed's stated goal of "minimizing the societal cost of providing currency to the public" (Federal Reserve 2003) is not met, which suggests a weakness in the Fed's new policy.

It is easy to show that for the general case, $C(P_20)<C(P^\alpha)$ if and only if

$$2\left(\frac{r}{g}-2a_2\right)\sum_{i=1}^{5}F_i>(4U_1+3U_2+2U_3+U_4+5U_5+5F_1+F_2+2F_3+3F_4+4F_5)h. \tag{29}$$

Similarly, $C(P_20)<C(P^\beta)$ if and only if

$$2e\sum_{i=1}^{5}F_i>(4U_1+3U_2+2U_3+U_4+5U_5+5F_1+F_2+2F_3+3F_4+4F_5)h.$$

These inequalities state that policy $P_20$ is attractive when the holding cost $h$ is inexpensive relative to the cost of fit sorting and to the cost of cross-shipping. Therefore, when the Fed decreases the Federal Funds Rate (which affects the cost of capital, and therefore,

**Table 5** **Example Optimal Policy with $W_i^f=0$, $D_i>0$, $i=1,\ldots,5$, and $W_i^f>0$, $D_i=0$, $i=6,\ldots,10$**

| | Day 1 | Day 2 | Day 3 | Day 4 | Day 5 | Day 6 | Day 7 | Day 8 | Day 9 | Day 10 | Totals |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $U_i$ | 1,000 | 800 | 500 | 400 | 250 | 1,000 | 800 | 500 | 400 | 250 | 5,900 |
| $F_i$ | 30 | 50 | 80 | 120 | 120 | 30 | 50 | 80 | 120 | 120 | 800 |
| $D_i$ | 3,200 | 1,000 | 800 | 500 | 400 | 0 | 0 | 0 | 0 | 0 | 5,900 |
| $P_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_i^f$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_i^c$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $W_i^f$ | 0 | 0 | 0 | 0 | 0 | 50 | 80 | 120 | 120 | 430 | 800 |
| $W_i^c$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $I_i^u$ | 1,000 | 800 | 500 | 400 | 250 | 1,250 | 2,050 | 2,550 | 2,950 | 3,200 | 14,950 |
| $I_i^f$ | 400 | 350 | 270 | 150 | 30 | 50 | 80 | 120 | 120 | 430 | 2,000 |
| $I_i^c$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$h$), it may influence currency management policies of banks, and, if inequality (29) is satisfied, lead to less fit sorting by banks and more transactions at the Fed. This could cause the Federal Reserve to face the appearance of a conflict of interest: the monetary policy that is best for the economy may increase the Fed's operating expenses. However, the cost to the economy from a bad management policy is expected to greatly dominate the added labor costs incurred by the Fed.

## 5. Conclusions and Recommendations for Future Study

The new currency recirculation guidelines proposed by the Federal Reserve System of the United States stem from the Fed's desire to encourage banks to be more active in recycling currency and thereby reduce the work load on the Fed's currency processing services. The guidelines are expected to significantly impact the scheduling of deposits (to the Fed) and withdrawals (from the Fed) of currency by its member banks. This paper introduced two models to capture the flow of currency between a bank and the Fed. The first (basic) model helps us analyze the impact on the current operations of most banks; we expect this analysis to be useful to managers in the near term. The second model incorporates some of the more advanced offerings (e.g., the custodial inventory program) of the Fed's guidelines and is expected to be adopted by many banks. Our analysis of the second model should offer insights to managers as they decide whether or not to participate in the advanced programs. If a bank decides to participate, our analysis should be helpful in deciding whether or not to invest in fit-sorting equipment and a custodial inventory facility.

For each model, the emphasis of our analysis is primarily on operational issues. The problem we address is that of obtaining a bank's optimum schedule of deposits (to the Fed) and withdrawals (from the Fed) of currency under the new guidelines. For the basic model, we first show the dominance of cyclic schedules and then offer a detailed picture of the optimum policies under all possible cost structures and demand patterns. Because of its complex nature, our analysis of the second model is relatively limited: we focus on obtaining an optimum two-week policy and then provide some structural results for optimum policies in general. These results are applicable to a system in which fit sorting and administration of custodial inventory are performed by a third party or by the bank itself. Our analysis can also be useful to the Fed as it finalizes its new guidelines.

The analysis of the impact of the Fed's new guidelines is important—both strategically and operationally—for most banks. Our work in this paper is a first step in this direction. There are several topics that future research into the currency management policies of banks may consider. Our analysis was deterministic: although the values of the used cash received and the fit cash distributed by a bank were allowed to vary each day of a week, they were assumed to be known. A stochastic version, where these values come from a known distribution, is worth exploring. A logical extension would be to consider coordinating the branches of a depository institution within a Federal Reserve zone. In such a model, it is more likely to be economical for a bank to purchase fit-sorting equipment and to hold its own custodial inventory. If so, determining the number of centers for fit sorting and custodial inventory within a zone, and their locations, is an interesting problem. Another problem is that of formalizing a coordination scheme to transfer currency from branches with excess currency to those with a deficit. Such schemes would reduce deposits and withdrawals at the Fed and, therefore, reduce cross-shipping fees.

## Appendix

Proof of Lemma 2. ($\Leftarrow$) $I^f_{i-1} = F_i$ implies that current inventory will be exhausted at close of business on day $i$. To satisfy demand on day $i+1$, $W_i \geq F_{i+1} > 0$.

($\Rightarrow$) Suppose policy $Py$ is an optimal policy for which $W_i > 0$ and $I^f_{i-1} = F_i + \delta$ $(0 < \delta < F_{i+1})$. We compare $Py$ to policy $Px$, in which $W_i > 0$ and $I^f_{i-1} = F_i$, and to policy $Pz$, for which $W_i = 0$ and $I^f_{i-1} = F_i + F_{i+1}$, for some $i$. Note that policy $Pz$ satisfies the contrapositive of what is to be proven. We show that policy $Py$ is always worse than either policy $Px$ or policy $Pz$.

Each policy has the same schedule of deposits to the Fed, each receives fit cash from the Fed on day 0 (with no cross-shipping charge), each receives its next shipment of fit cash from the Fed on day $i$ ($Px$ and $Py$) or day $i+1$ ($Pz$), and each receives its third shipment on day $q$ (see Table 6). Let the costs of these three policies be $C(Px)$, $C(Py)$, $C(Pz)$, respectively. It follows that

$$C(Py) - C(Px) = i\delta h - \delta x_i e = \delta(ih - x_i e)$$

**Table 6** **Policies for the Proof of Lemma 2**

| Policy | $I_0^f$ | $I_{i-1}^f$ | $W_i$ | $I_i^f$ | $W_{i+1}$ | $I_{i+1}^f$ |
|---|---|---|---|---|---|---|
| $Px$ | $\sum_{j=1}^{i} F_j$ | $F_i$ | $\sum_{j=i+1}^{q} F_j$ | $\sum_{j=i+1}^{q} F_j$ | 0 | $\sum_{j=i+2}^{q} F_j$ |
| $Py$ | $\sum_{j=1}^{i} F_j+\delta$ | $F_i+\delta$ | $\sum_{j=i+1}^{q} F_j-\delta$ | $\sum_{j=i+1}^{q} F_j$ | 0 | $\sum_{j=i+2}^{q} F_j$ |
| $Pz$ | $\sum_{j=1}^{i+1} F_j$ | $F_i+F_{i+1}$ | 0 | $F_{i+1}$ | $\sum_{j=i+2}^{q} F_j$ | $\sum_{j=i+2}^{q} F_j$ |

$$C(Py)-C(Pz) = \sum_{j=i+2}^{q} F_j h - i(F_{i+1}-\delta)h + \left(\sum_{j=i+1}^{q} F_j - \delta\right)x_i e - \sum_{j=i+2}^{q} F_j x_{i+1} e.$$

If $x_i=0$, then $C(Py)-C(Px)>0$. If $x_i=1$, then $C(Py)-C(Px)\le 0 \Leftrightarrow e\ge ih$. Also, $x_i=1$ implies

$$\begin{aligned} C(Py)-C(Pz) &\ge \sum_{j=i+2}^{q} F_j h - i(F_{i+1}-\delta)h + (F_{i+1}-\delta)e \\ &\ge \left[\sum_{j=i+2}^{q} F_j - i(F_{i+1}-\delta) + (F_{i+1}-\delta)i\right]h \\ &= \sum_{j=i+2}^{q} F_j h > 0. \end{aligned}$$

Thus, either $C(Py)>C(Px)$ or $C(Py)>C(Pz)$, which contradicts the optimality of $Py$. □

Proof of Theorem 4. We determine the circumstances under which policy $P_n^m l$ has less per-week cost than policies $P_2 0$ and $P_{n-m+1} l$.

$$2C(P_n^m l) < nC(P_2 0) \Leftrightarrow \left(10n+2\sum_{q=1}^{5m} q\right)kh + \left(10n+2\sum_{q=1}^{5(n-m)-l} q\right)h + 2le < 25nkh+25nh$$

$$2l\frac{e}{h} < \left(15n-2\sum_{q=1}^{5m} q\right)k + 15n - 2\sum_{q=1}^{5(n-m)-l} q \tag{30}$$

$$(n-m+1)C(P_n^m l) < nC(P_{n-m+1} l)$$

$$\begin{aligned} &\Leftrightarrow (n-m+1)\left[\left(5n+\sum_{q=1}^{5m} q\right)kh + \left(5n+\sum_{q=1}^{5(n-m)-l} q\right)h + le\right] \\ &< n\left[(5(n-m+1)+15)kh + \left(5(n-m+1)+\sum_{q=1}^{5(n-m)-l} q\right)h + le\right] \end{aligned}$$

$$\left[(n-m+1)\sum_{q=1}^{5m} q - 15n\right]k - (m-1)\sum_{q=1}^{5(n-m)-l} q < (m-1)l\frac{e}{h} \tag{31}$$

By combining (30) and (31), we get

$$\begin{aligned} &2\left[(n-m+1)\sum_{q=1}^{5m} q - 15n\right]k - 2(m-1)\sum_{q=1}^{5(n-m)-l} q \\ &< (m-1)\left(15n-2\sum_{q=1}^{5m} q\right)k + 15n(m-1) - 2(m-1)\sum_{q=1}^{5(n-m)-l} q. \end{aligned}$$

This implies $2n\sum_{q=1}^{5m} qk - 15n(m+1)k < 15n(m-1)$, which, in turn, implies $k<3/(5m+3)$. □

Proof of Theorem 6. Suppose $n\ge m+2$.

$$\begin{aligned} &C(P_n^m l) < C(P_n^{n-1} 0) \\ &\Leftrightarrow \sum_{q=1}^{5m} qk + \sum_{q=1}^{5(n-m)-l} q + l\frac{e}{h} < \sum_{q=1}^{5(n-1)} qk + 15 \\ &[5m(5m+1) - 5(n-1)(5n-4)]k \\ &\qquad < 30 - (5n-5m-l)(5n-5m-l+1) - 2l\frac{e}{h} \end{aligned}$$

The coefficient of $k$ is negative by $n\ge m+2$, so

$$k > \frac{(n-m-l/5)(5n-5m-l+1) + (2/5)l(e/h) - 6}{(n-1)(5n-4) - m(5m+1)}. \tag{32}$$

Combining Theorem 4 and Lemma 6 with (32) yields

$$\frac{5n^2-10mn+5m^2+n-m-(l^2+l)/5-6}{5n^2-5m^2-m-9n+4} < \frac{3}{5m+3}$$

$$25m(n-m)^2+25m^2+30n-25mn-30m-30 < \left(m+\frac{3}{5}\right)(l^2+l)$$

$$[5m(n-m)+6](n-m-1) < \left(\frac{m}{5}+\frac{3}{25}\right)(l^2+l).$$

Because $n\ge m+2$ implies $l\le 4$ (by Theorem 3),

$$\left(\frac{m}{5}+\frac{3}{25}\right)(l^2+l) \le 20\left(\frac{m}{5}+\frac{3}{25}\right) \le 4m+\frac{12}{5}.$$

Because $n\ge m+2$, $[5m(n-m)+6](n-m-1) > 5m+6$. This yields the contradiction that proves the theorem. □

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