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On the Interactions Between Routing and Inventory-Management Policies in a One-Warehouse N-Retailer Distribution System

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This paper examines the interactions between routing and inventory-management decisions in a two-level supply chain consisting of a cross-docking warehouse and N retailers. Retailer demand is normally distributed and independent across retailers and over time. Travel times are fixed between pairs of system sites. Every m time periods, system inventory is replenished at the warehouse, whereupon an uncapacitated vehicle departs on a route that visits each retailer once and only once, allocating all of its inventory based on the status of inventory at the retailers who have not yet received allocations. The retailers experience newsvendor-type inventory-holding and backorder-penalty costs each period; the vehicle experiences in-transit inventory-holding costs each period. Our goal is to determine a *combined* system inventory-replenishment, routing, and inventory-allocation policy that minimizes the total expected cost/period of the system over an infinite time horizon. Our analysis begins by examining the determination of the optimal *static* route, i.e., the best route if the vehicle must travel the same route every replenishment-allocation cycle. Here we demonstrate that the optimal static route is not the shortest-total-distance (TSP) route, but depends on the variance of customer demands, and, if in-transit inventory-holding costs are charged, also on mean customer demands. We then examine *dynamic*-routing policies, i.e., policies that can change the route from one system-replenishment-allocation cycle to another, based on the status of the retailers' inventories. Here we argue that in the absence of transportation-related cost, the optimal dynamic-routing policy should be viewed as balancing management's ability to respond to system uncertainties (by changing routes) against system uncertainties that are *induced by* changing routes. We then examine the performance of a *change-revert* heuristic policy. Although its routing decisions are not fully dynamic, but determined and fixed for a given cycle at the time of each system replenishment, simulation tests with $N = 2$ and $N = 6$ retailers indicate that its use can substantially reduce system inventory-related costs even if most of the time the chosen route is the optimal static route.

Key words: supply chain; routing; inventory allocation; inventory replenishment
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1. Introduction

Since 1990, about 25 published papers have examined the *joint determination* of transportation (e.g., routing, vehicle/customer assignment) and inventory-management policies in supply chains, some in complex business scenarios. See Schwarz et al. (2004). We believe that this paper is the first to explicitly examine the *interactions* between routing and inventory-management policies. Although the model we examine is very stylized, we are able to use it to get analytical expressions that represent the trade-off between routing and inventory-management considerations. Then, using simulation, we are able to substantiate the insights provided by the analytical model under more realistic assumptions.

Why should the manager of a warehouse-retailer distribution system, one with authority to determine policy for both vehicle routing and inventory management, care about the interaction between them? Most important, because vehicles are integral to the supply chain, transportation policy influences supply chain performance. Hence, in the same way that the basic interactions between buyer and supplier inventory-management policies should be understood in managing system inventories, we believe that the basic interactions between transportation and inventory-management policies should also be understood. Competitiveness, particularly in retailing, depends on providing high availability (e.g., high fill-rates) with low average inventory levels (i.e., high turnover). It is well known that "dynamic allocation" (i.e., postponing the allocation of vehicle inventory to the moment that delivery occurs) improves competitiveness given an arbitrary "static" vehicle route (i.e., one under which the route used is the same route in every replenishment-allocation cycle). See Kumar et al. (1995).

Technology *is* available to facilitate both dynamic allocation and dynamic routing. In particular, retailers already use bar codes and radio frequency ID (RFID) to monitor inventory (e.g., Wal-Mart), and transport companies already employ technology to route or reroute vehicles (e.g., Schneider trucking). Given that the technology is already in place, it behooves supply chain researchers to learn *when* and *how* to take advantage of such technology to implement dynamic allocation and dynamic routing.

Our work extends that of Kumar et al. (1995) by combining dynamic routing and dynamic allocation, and by considering in-transit inventory-holding cost, which that model ignores. Our work can also be viewed as an extension of Park et al. (2002) by permitting arbitrary fixed between-site travel times. These and other related papers are reviewed in §2. Section 3 describes our model and its assumptions in detail.

Our analysis begins in §4, where we examine the determination of the optimal static route. Section 5 examines dynamic routing, in particular, the interactions between dynamic routing and inventory-management decisions in detail, summarizes the Park et al. (2002) results for "symmetric" retailers, and lays the foundation for the change-revert dynamic-routing decision rule. Section 6 introduces the change-revert routing decision rule, and §7 describes our analytical model. Section 8 uses simulation to assess the validity of the analytical model and to assess the value of the change-revert heuristic under a variety of parameterizations. Although its routing decisions are not fully dynamic, but fixed for a given cycle at the time of system replenishment, simulation tests with $N = 2$ and $N = 6$ retailers indicate that the change-revert heuristic can substantially reduce system inventory-related costs even if most of the time the chosen route is the optimal static route. A modified, *threshold* change-revert heuristic is demonstrated to provide most of the savings of the original, but with significantly fewer change-route decisions. Section 9 concludes the paper and makes suggestions for further study.

2. Literature Review

To the best of our knowledge, Federgruen and Zipkin (1984) made the first attempt to integrate inventory-management and routing decisions in a single model. They use a heuristic nonlinear-programming formulation to make vehicle-assignment, routing, and inventory-replenishment decisions for a single replenishment-allocation cycle. Vehicle-travel cost between locations is considered, but transportation times are assumed to be zero. Our model can be viewed as extending Federgruen and Zipkin (1984) to account for fixed transportation times between sites, multiple replenishment cycles (i.e., an infinite time horizon), and to employ both *dynamic routing* (i.e., possibly changing the route associated with each replenishment cycle) and *dynamic allocation* (i.e., making allocation decisions sequentially as each retailer is visited).

The distribution of liquid propane is considered by Golden et al. (1984). Retailers use tanks with limited capacity to store inventory. Inventory-management costs are not explicitly modeled. Instead, the focus is on maintaining customer inventories above some specified level (e.g., 30% of capacity) and avoiding "premature" deliveries (i.e., deliveries when customer inventories are above 50%). Kleywegt et al. (2002a, b) examine a business scenario similar to that of Golden et al. (1984) with M vehicles, each with capacity C_V, and each customer with storage capacity C_n. They formulate this problem as a Markov decision process and propose approximate methods (i.e., customer decomposition) to find "good solutions with reasonable computational effort." Adelman (2001) examines a multi-item routing/inventory-management model with joint replenishment costs. His model is similar to the inventory-routing problem (IRP) of Kleywegt et al. except that in the IRP vehicle capacity is limited and only a single product is considered. Adelman also formulates the problem as a Markov decision process, but uses a price-directed heuristic control policy that approximates the future, using dual prices from linear-programming relaxations.

2.1.1. Allocation. Several methods for allocating vehicle inventory have been modeled in the literature. We find it useful to differentiate between those that use *static allocation*, under which the allocated quantities are determined for all retailers simultaneously, and *dynamic allocation*, under which these quantities are determined sequentially as the vehicle travels the route, based on system status. Kumar et al. (1995) examine the same supply chain model we examine here, operating under a static-routing policy, using static allocation. They assess the reduction in system variance and cost/time provided by dynamically allocating system inventory to each retailer based on the inventory status of all retailers' yet to be allocated inventory. Reiman et al. (1999) incorporate dynamic allocation in their heavy-traffic analysis of a similar supply chain model, but with a single capacitated vehicle. Berman and Larson (2001) incorporate dynamic allocation in their model of industrial-gas distribution. Their analysis does not incorporate inventory costs per se, but instead, incorporates linear costs associated with making deliveries either earlier or later than desired. Bassok and Ernst (1995) incorporate dynamic allocation in a multiproduct distribution setting. Their model also allocates space on the vehicle to different products.

2.1.2. Routing. Several methods for choosing the route of the vehicle have also been modeled. We find it useful to differentiate among four types. Under a *fixed-routing policy*, the route associated with each system replenishment is known prior to the start of the first cycle. A special case of fixed routing is *static routing*. Under a *static-routing policy*, the route used is the same whenever the same set of retailers is to be visited. Under a *dynamic-routing policy*, the route is permitted to change whenever the same set of retailers is to be visited, but once the vehicle has begun traveling, the route is completed as planned. We examine this form of routing and compare its performance to that of static routing. A *fully dynamic-routing policy* is a dynamic policy under which the route is permitted to change while the vehicle is traveling a previously planned route. Only a few published models explicitly address dynamic routing in conjunction with a corresponding inventory-management policy. Trudeau and Dror (1992) examine what they call the "stochastic vehicle routing problem," which is the traditional inventory-routing problem, but with customer demands not revealed until the vehicle arrives at a customer to make a delivery. Inventory costs are not modeled per se, but "route failures" (i.e., what happens when a vehicle is unable to satisfy all the demand on a given route) are examined. They compare several different solution procedures for this problem. Reiman et al. (1999) incorporate a limited form of dynamic routing in their model (described above). Savelsbergh and Goetschalckx (1995) compare the efficiency of static versus dynamic routes in a stochastic inventory-routing problem. Their objective is to minimize the total cost of travel for servicing all customers. As in Trudeau and Dror (1992), inventory-management costs are not modeled. Instead, it is assumed that every customer has its tank filled, even if this requires having the vehicle return to the depot for resupply.

Finally, Park et al. (2002) examine both dynamic allocation and fully dynamic routing in the same model as we examine here, but under the limiting assumption that all between-retailer travel times are the same and all warehouse-retailer travel times are the same. Their results are described more in §5.

3. Description of Model and Major Assumptions

The model we examine consists of a single cross-docking warehouse, N retailers, one uncapacitated vehicle, and a single item (SKU). This supply chain is centrally managed, using a periodic-review system. Each retailer i experiences normally distributed customer demand (μ_i, σ_i) each time period. Demand realizations are independent across time and across retailers. Excess demand is backordered at cost \$$p$/unit-period, and a holding cost of \$$h$/unit-period is charged on system (i.e., on-vehicle or at-retailer) inventory. Transportation times between system sites are fixed and known. Specifically, the travel time between the warehouse and retailer i is $r_{0i} > 0$, $i = 1, \ldots, N$; the travel time between retailer i and j is $r_{ij} > 0$, $i, j = 1, \ldots, N$. Transportation costs between system sites are considered in our analysis of static routing, but not in our analysis of dynamic routing. Because our objective is to quantify the decrease in inventory costs and to examine the interactions between routing decisions and inventory-management decisions, the differences in management and transportation costs associated with these policies are ignored.

The system operates as follows: Every m time periods the system places an order on an outside supplier, and the warehouse receives it instantaneously. In our model m is given. In practice, m could reflect a fixed cost of operating a route or be limited by vehicle capacity. Upon receipt, an uncapacitated vehicle departs immediately on a route that visits each retailer once and only once, dynamically allocating its inventory based on the status of inventory at the retailers who have not yet been allocated inventory. Retailer backorder-penalty costs and retailer and in-transit inventory-holding costs are charged at the end of each time period. Our goal is to determine a *combined* system inventory-replenishment, routing, and inventory-allocation policy that minimizes the total expected cost/period of the system over an infinite time horizon. In what follows we provide a few definitions that are helpful in understanding our results.

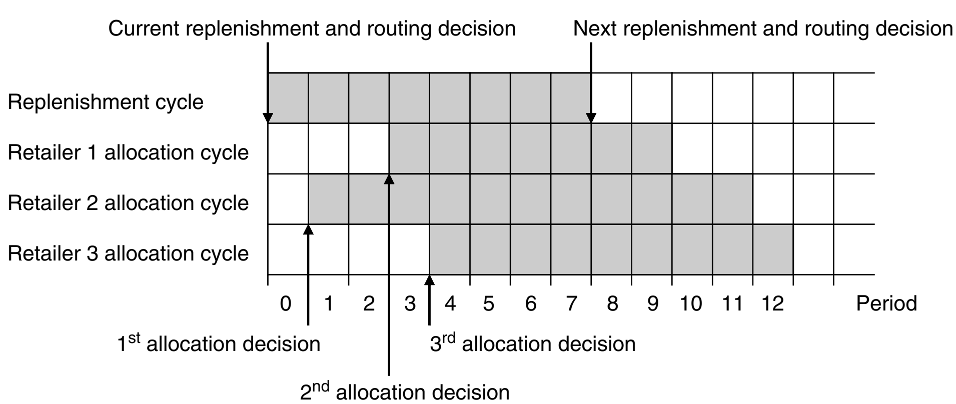
3.1.1. Time Periods, Replenishment Cycles, and Allocation Cycles. The m periods between system replenishments are defined to be the system *replenishment cycle*. The set of periods between successive allocations of vehicle stock to retailer i is defined to be retailer i's *allocation cycle*. We define the start of any given retailer's kth allocation cycle to be the time period when it receives its allocation from the kth system replenishment. Correspondingly, the end of the retailer's kth allocation cycle is the period immediately prior to receiving its allocation from the $(k+1)$st system replenishment. The set of N retailer allocation cycles, together with their associated system-replenishment cycle, is called the system *replenishment-allocation cycle*. These terms were first introduced by Kumar (1995). Figure 1 illustrates a typical replenishment-allocation cycle. More details are provided below.

3.1.2. An Overview of How Routing and Inventory Policy Interact. Let $[j]$ be the index of the jth retailer on the route, and let $B_{[j]}$ be its delivery lead time. Then $B_{[1]} = r_{0[1]}$, $B_{[2]} = r_{0[1]} + r_{[1][2]}$ and, in general, $B_{[j]} = \sum_{k=0}^{j-1} r_{[k][k+1]}$. Under any fixed-routing policy, the number of periods in every retailer's allocation cycle is known in advance. Under dynamic routing, delivery lead times and the *number* of periods in each retailer's allocation cycle become uncertain, i.e., subject to the quantities allocated, to future demand realizations, and to the specific routing decision rule, as described in detail below.

3.2. Major Assumptions

Three major assumptions are made to facilitate our analysis. All uses of the word "optimal" in our results below are, of course, subject to these assumptions. As we report in §8, we tested the validity of these assumptions via simulation and conclude that for the high-service-level systems we examine: (1) These assumptions are violated no more than 5% of the time, and (2) even when they are violated, the insights provided by our analysis appear to be insensitive to these assumptions.

Figure 1 **A Replenishment-Allocation Cycle**



3.2.1. The Allocation Assumption. The allocation assumption was introduced by Eppen and Schrage (1981) and has been widely adopted by others (e.g., Kumar et al. 1995). In the language of our model, it means that as the vehicle travels along its route allocating inventory, it is always possible to make a cost-minimizing allocation (i.e., one that equalizes the marginal expected costs) among the retailers not yet visited on that route. In effect, this assumption allows negative allocations. The role of the allocation assumption is that, for any given system-replenishment and routing policy, the allocation assumption decouples the allocation decisions in any given replenishment-allocation cycle from the allocation decisions made in any other replenishment-allocation cycle.

3.2.2. The Returns-Without-Penalty Assumption. In analyzing multiperiod newsvendor-type models, it is often assumed that, if desired, system stock can be reduced instantaneously and without penalty. In effect, this assumption allows negative replenishments. See, for example, Kahn (1987) and Lee et al. (1997). In our model, this assumption also decouples replenishment decisions in a given time period from those in all subsequent cycles. Although this assumption is seldom valid in practice, it is typically innocuous when costs and demand parameters are stationary and demands are nonnegative.

In combination, in the analysis that follows, the dynamic-allocation and returns-without-penalty assumptions decouple replenishment and allocation decisions made in any given replenishment-allocation cycle from the corresponding decisions in any other replenishment-allocation cycle. More specifically, under any fixed-routing policy (including a static-routing policy), the optimal replenishment and allocation policy for a single replenishment-allocation cycle yields the optimal replenishment and allocation policy for any arbitrary number of consecutive replenishment-allocation cycles. We summarize this single-cycle problem below.

3.2.3. The Last-Period Backorders Assumption. This assumption was introduced by Jonsson and Silver (1987b). In the language of our model, and in Kumar's, it means that retailer backorders, if they occur, only occur in the last period of that retailer's allocation cycle. From a modeling perspective, this assumption transforms each retailer from a "newsvendor" whose net inventory incurs either a holding cost h or penalty cost p at the end of every period into a "newsvendor" facing only holding costs over the first $(m-1)$ periods, and then either a holding cost h or penalty cost p, at the end of its allocation cycle. Jonsson and Silver (1987a), Kumar et al. (1995), and Park et al. (2002) argue and demonstrate that this assumption is innocuous for systems with high desired service levels because backorders, if they occur, are most likely to occur only in these last periods. Dynamic-routing policies, such as the change-revert policy introduced below, which are designed to make allocations "early" to retailers whose stock is low, are, all other things being equal, less likely to violate this assumption than fixed or static-routing policies.

3.3. The Single Cycle

In this subsection we will detail the determination of the systemwide holding and penalty costs associated with a single replenishment-allocation cycle. We do so with two goals in mind: First, to lay the foundation for understanding two important analytical results, based on the assumptions described above: (1) regardless of the replenishment policy, the optimal allocation policy minimizes the expected backorders associated with each replenishment-allocation cycle, and (2) the optimal system replenishment policy is a base-stock policy. Our second, more fundamental, goal is to illustrate the fact that to assess the costs incurred in any single replenishment-allocation cycle, it is necessary to know the route chosen for the given cycle and for the next cycle. This fact is what contributes to the complexity of determining the optimal dynamic-routing policy.

The shaded portion of Figure 1 illustrates a single replenishment-allocation cycle—henceforth called a "cycle"—in which $m = 8$ periods and $N = 3$ retailers. We have arbitrarily identified the first period as period $t = 0$. Note that the vehicle first goes to Retailer 2, then Retailer 1, then Retailer 3. Note, further, that in the next cycle the vehicle goes to Retailer 1 first. Hence, Retailer 1's allocation cycle starts in Period 3, because in this cycle $B_1 = 3$, and ends in Period 9, because in the following cycle $B_1 = 2$.

As background for what follows, consider a multiperiod "newsvendor" model. It is well known that the newsvendor's holding and penalty cost in period t, C_t, can be written as

$$C_t = hE_t + (h+p)S_t, \tag{1}$$

where E_t is the retailer's net inventory at the end of period t, where end-of-period backorders, S_t, count as negative net inventory.

We uniquely assign each element of (1) to a specific replenishment-allocation cycle, using the same cost assignment as in Park et al. (2002). The holding cost, hE_t, is assigned to the replenishment cycle containing period t. The backorder-related cost, $(h+p)S_t$, is assigned to the allocation cycle in which the backorders occur. Hence, the total cost assigned to a single cycle that starts in period t_0, and whose retailers' allocation cycles have known arbitrary starting and ending periods, is

$$h\sum_{t=t_0}^{t_0+m-1} E_t + (h+p)\left(\sum_{i=1}^{N} S_{i,\,e[i]}\right), \tag{2}$$

where E_t is the system total net inventory, $s_{i,\,t}$ are the backorders at retailer i in period t, and $e[i]$ is the end period in retailer i's allocation cycle. The $e[i]$s are determined by the route used in the next cycle. To illustrate, the total costs assigned to the shaded cycle in Figure 1 would be $h\sum_{t=0}^{7} E_t + (h+p)(s_{1,\,9} + s_{2,\,11} + s_{3,\,12})$.

Note, first, that the expectation of the $h\sum_{t=t_0}^{t_0+m-1} E_t$ is determined by the replenishment decision at time t_0. Hence, given the replenishment decision, the expectation of (2) is minimized by minimizing the expectation of $\sum_{i=1}^{n} S_{i,\,e[i]}$. Appendices A and B prove that this is minimized by dynamic allocations that equalize the runout probabilities of the retailer about to receive the allocation, and a hypothetical "composite" retailer that represents the set of retailers whose allocations will be determined later. Kumar proved the same result, but for static routes under a somewhat different cost structure. Our proof applies to any single cycle, provided that the retailers' allocation cycles have known starting and ending periods, as they would under any fixed-routing policy (but would not under a dynamic-routing policy).

4. Choosing the Optimal Static Route

We begin examining the choice of the optimal static route using the model of Kumar et al. (1995), henceforth labeled "Kumar," because its results are well known and because its analysis sheds light on the impact of on-vehicle inventory-holding costs on routing decisions. Kumar's static-routing model is identical to ours except that in Kumar, inventory-holding cost is charged *only* on retailer inventory. Initially, our analysis ignores transportation-related costs.

Given any static route, Kumar describes how to make optimal dynamic allocations (Theorem 5.1) and establishes the optimality of a system base-stock inventory-replenishment policy (Theorem 5.2). Both results are based on showing that these decisions are equivalent to the corresponding decisions for a hypothetical "composite" retailer. For example, Kumar establishes that each allocation equalizes the runout probabilities of two retailers: the one currently receiving an allocation and the other a composite retailer that represents the set of all retailers that have not yet received an allocation. Appendix A describes its construction. (Note that the construction of the composite retailer in Appendix A differs slightly from Kumar. However, the resulting allocations and replenishments are the same.) Similarly, the system-replenishment decision is shown to be equivalent to the replenishment of the single composite retailer representing all N retailers. In particular, the optimal base-stock is given by

$$Y^* = \mu_1^C + \sigma_1^C K^*, \tag{3}$$

where K^* is such that

$$\Phi(K^*) = (p - h(m-1))/(p+h), \tag{4}$$

where Φ is the standard normal c.d.f. and

$$\mu_1^C = \sum_{i=1}^{N} \mu_i(m + B_i). \tag{5}$$

σ_1^C is determined recursively as follows:

$$(\sigma_N^C)^2 = (m + b_{[N]})(\sigma_{[N]})^2$$

$$(\sigma_j^C)^2 = b_{[j]}\sum_{r=j}^{N}(\sigma_{[r]})^2 + \left(\sigma_{[j]}\sqrt{m} + \sigma_{j+1}^C\right)^2, \quad \text{for } 1 \le j \le N-1 \tag{6}$$

In the above, μ_1^C and σ_1^C are the mean and standard deviation of the composite retailer representing all N retailers, and $b_{[j]}$ is the incremental lead time to retailer $[j]$ (i.e., $b_{[j]} = B_{[j]} - B_{[j-1]}$). Note that (3)–(4) can be interpreted to be the optimal base stock for the standard newsvendor model with normally distributed demand (with mean μ_1^C and standard deviation σ_1^C), leftover cost hm, and backorder cost $p - h(m-1)$.

Although Kumar based (3) through (6) on the assumption that inventory-holding cost ($\$h$/unit-period) is charged only at the retailers (i.e., *retailer-only holding* cost), we show in Appendix B that they remain valid under our cost structure, in which inventory-holding cost is also charged on vehicle inventory (i.e., *system-based holding* cost). While Y^*, K^*, μ_1^C, and σ_1^C are identical under the two cost structures, Z^S, the optimal expected per cycle cost, differs. Specifically,

(retailer-only holding costs)

$$Z^S = H^0 + (p+h)\phi(K^*)\sigma_1^C \tag{7a}$$

(system-based holding costs)

$$Z^S = H^0 + H^F + (p+h)\phi(K^*)\sigma_1^C \tag{7b}$$

where $H^0 = h[m(m-1)(\sum_{i=1}^N \mu_i)/2]$ is the same for all routes, and $H^F = hm\sum_{i=1}^N \mu_i B_i$ depends upon the route. Equation (7a) is from Theorem 5.2B of Kumar, and (7b) is (B6) from Appendix B, specialized for a static route. H^0 is the expected holding cost of the inventory required to satisfy average demand over an m-period replenishment cycle. Under system-based holding costs (7b), Z^S has an additional term not present in (7a), H^F, equal to the average on-vehicle holding cost per replenishment cycle.

Under either cost structure, the optimal static route minimizes (7a) or (7b) over all possible $N!$ routes. With retailer-only holding costs, observe that the optimal static route minimizes σ_1^C in (6), and that σ_1^C depends only on the retailer delivery lead times and their per-period demand variances.

For the special case of $N = 2$ retailers, (7a) reduces to

$$H^0 = (\mu_1 + \mu_2)hm(m-1)/2 \quad \text{and} \tag{8}$$

$$\sigma_1^C = \sqrt{(r_{0[1]}+m)(\sigma_1^2+\sigma_2^2)+r_{[1][2]}\sigma_{[2]}^2+\sigma_1\sigma_2\sqrt{m(m+r_{[1][2]})}}. \tag{9}$$

Using (9), it is straightforward to show that if travel times are such that $r_{01} < r_{02}$ and $r_{12} = r_{21}$, then Retailer 1 is first on the optimal static route if and only if

$$(\sigma_2^2 - \sigma_1^2)/(\sigma_2^2 + \sigma_1^2) < (r_{02} - r_{01})/r_{12}. \tag{10}$$

More generally, a sufficient condition to visit the closest retailer first is for the closest retailer to also have the largest variance. Otherwise, the choice of the optimal static route depends on a trade-off between travel times (i.e., a TSP-oriented parameter) and per-period demand variance (i.e., an inventory-management parameter).

Now, consider the choice of the optimal static route under system-based holding costs; i.e., the minimization of (7b) for the same special case of $N = 2$ retailers. Equation (7b) becomes

$$Z^* = H^0 + (p+h)\phi(K^*)\sigma_1^C + hm(B_1\mu_1 + B_2\mu_2). \tag{11}$$

Under any static route the length of each retailer's allocation cycle is m period, hence, the *at-retailer* inventory required to satisfy average demand is the same for any route. Similarly, the average amount of inventory transported each replenishment-allocation cycle to satisfy each retailer's mean demand is unaffected by the route. However, the number of periods that this inventory must be transported is affected by the choice of route. Hence, when holding costs are also charged on the vehicle inventory, the retailers' average demands affect the choice of the route. From (11), one can see the impact that retailer *mean* demand has on the choice of the optimal static route: Everything else being equal, the pipeline inventory-holding cost component of (11) favors visiting the retailer with the largest mean demand first.

The role of retailer mean demand—which has no impact on the choice of the optimal static route given retailer-only holding costs—makes the determination of the optimal static route more complicated, even in the case of $N = 2$ retailers, because this choice no longer depends only on σ_1^C, but on a trade-off between σ_1^C and the mean per-period demand at each retailer. (For $N = 2$ and $r_{12} = r_{21}$, it is trivial to show that if the retailers are labeled so that Retailer 1 is closest to the warehouse (i.e., $r_1 \le r_2$), then a sufficient condition for Retailer 1 to be first on the optimal static route is for it to have the largest variance

($\sigma_1 \geq \sigma_2$) and the largest mean demand ($\mu_1 \geq \mu_2$).) More generally, if σ_1^C is viewed as a route-dependent measure of system variance, and H^F is viewed as a route-dependent measure of system pipeline mean inventory, then the choice of the static optimal route involves a trade-off between these system mean and variance values.

Because in general under either cost structure the optimal static route must be determined by enumeration of the $N!$ alternatives, it is straightforward to incorporate direct transportation costs (e.g., total mileage, total travel time) into the analysis. In particular, if T_k is the total transportation cost of route k, and Z_k^S is (7a) or (7b) under route k, then the optimal static route k^* solves

$$k^* = \text{Minimize}_{\text{all possible routes}, k}\{T_k + Z_k^S\}. \qquad (12)$$

We now turn to our analysis of dynamic routing. Note that the model being analyzed in all the subsequent sections ignores direct transportation cost and assumes a system-based holding-cost structure.

5. Dynamic-Routing Policies

In the business scenario we examine, dynamic routing provides the opportunity to expedite the allocation of vehicle inventory to retailer/s whose inventory is "running low," thereby avoiding backorders that otherwise might occur. However, such expediting necessarily postpones deliveries to other retailers, thereby possibly causing backorders that otherwise would not have occurred. Nonetheless, under certain circumstances it may be desirable to change routes to minimize total expected costs in some given *single* replenishment-allocation cycle. We will illustrate how such a change is evaluated in §6.

Of course, regardless of the inventory-management benefit of a change-route decision in some given replenishment-allocation cycle, a dynamic-routing policy can increase transportation-related costs. Further, because of its associated instabilities in routing and delivery schedules, dynamic routing introduces two types of uncertainties, neither of which is present under a fixed routing policy: (1) "U_{AL} uncertainty," i.e., uncertainty in each retailer's delivery lead time, and (2) "U_{PD} uncertainty," i.e., uncertainty in the number of periods of customer demand each retailer's allocation is to supply.

Therefore, a management decision to adopt a dynamic-routing policy should be viewed as a trade-off between reducing expected inventory-related costs (compared to the corresponding expected costs if a static route were used), in exchange for increased logistics-related (e.g., transportation) cost. Therefore, the obvious questions are: (1) how to determine the optimal dynamic-routing policy and (2) are the cost savings of dynamic routing versus static routing large enough to justify the increased costs associated with them (i.e., administrative costs and possibly increased direct transportation costs).

Unfortunately, the optimal dynamic-routing policy is known only for the very special case of "symmetric" retailers; see Park et al. (2002). We summarize some of their results below. For more general cases, the optimal dynamic-routing policy is unknown, due to complexities that we will also illustrate. Therefore, after reviewing the results of Park et al., we will turn to an examination of change-revert policies.

The next section summarizes some of the Park et al. results to examine the interactions between routing decisions and inventory-allocation decisions, and to provide additional motivation for the change-revert heuristic described above.

5.1. Dynamic Routing: The Symmetric Retailer Case

Park et al. (2002), henceforth labeled "Park," examined the same model as ours, but for the special case of N *symmetric* retailers. Symmetric retailers have identically distributed customer demand, identical travel time to/from the warehouse (i.e., $r_{01} = r_{02} = \cdots = r_{0N} = a > 0$), and identical between-retailer travel times for all retailer pairs (i.e., $r_{ij} = b > 0$ for all $i \neq j$). For a symmetric system (but otherwise subject to the same objective and assumptions as ours) Park proves that for *any* given inventory-allocation policy, *least-inventory first* (LIF) is an optimal dynamic-routing policy.

Under LIF, at the time of departure from the warehouse, the vehicle travels first to the retailer with the smallest inventory position, makes an allocation, then travels to the unvisited retailer that, at that instant, has the smallest inventory position, etc. Because LIF is the optimal dynamic-routing decision rule regardless of the allocation policy being used, the amount of

U_{AL} uncertainty (in each retailer's delivery lead time) and U_{PD} uncertainty (in the number of periods of customer demand each retailer's allocation is to supply) depends entirely on the inventory-allocation policy. For simplicity of exposition, consider $N = 2$ symmetric retailers.

Note first that, regardless of the route used in any given cycle, if the first retailer on this route is allocated none (all) of the vehicle inventory, then under LIF this retailer is very likely to be the first (last) retailer on the route chosen for the next cycle. Now consider two possible heuristic allocation policies: equal allocation and fixed-route allocation. Under an *equal-allocation policy*, both retailers are brought to the same inventory position. Equal allocation makes the next route a 50–50 coin toss, and, hence, induces high levels of U_{AL} and U_{PD} uncertainly. The *fixed-route allocation policy* allocates inventory to minimize total expected inventory-holding and backorder cost/period under the assumption that the routes in all subsequent replenishment-allocation cycles will be the same as the current route (although, in fact, the LIF-routing decision rule will be used to choose the route in all subsequent replenishment-allocation cycles).

Although neither fixed-route nor equal-allocation (combined with LIF routing) is designed to be the optimal routing-allocation policy, the proximity of either of these allocation policies to the optimal allocation policy (which also uses LIF) is an *indicator* of how frequently route changes occur in the optimal policy.

Park compared the simulated performance of the equal-allocation and fixed-route allocation heuristics with the optimal allocation policy (all using LIF routing) for 128 parameter sets. Park reported that the fixed-route heuristic increased the average cost/cycle compared with the optimal allocation policy, only 0.48%; the maximum increase was only 4.12%. On the other hand, the equal-allocation heuristic performed poorly, increasing cost/cycle by an average of 17% and a maximum of 95%. Park's results suggest that the optimal dynamic-routing and allocation policy allocates inventory to favor the long term over the short term in the trade-off described above (i.e., to support relatively few change-route decisions) and then, when a route change occurs, allocates inventory to re-establish a stable (i.e., near-static) route.

6. The Change-Revert Heuristic

As already noted (§3), the routing decision in any given replenishment cycle determines for each retailer both: (1) the first period in the corresponding allocation cycle for each of the retailers *and* (2) the last period in the allocation cycle corresponding to the immediately preceding (i.e., "last") replenishment cycle. In other words, the routing decision in the current replenishment cycle "ends" the allocation cycles associated with the last replenishment cycle, and "starts" the allocation cycles associated with the current replenishment cycle. Therefore, to determine the optimal dynamic-routing policy, one must evaluate the consequences of each alternative route on the backorder costs incurred in the previous cycle, the holding and backorder costs of the current cycle. The latter depends, in part, on the end periods of the allocation cycles, which are determined by the route chosen in the next cycle. Hence, the optimal route in this cycle depends on, among other things, the optimal route in the next cycle. Although this is conceptually straightforward, it is analytically impossible. This is the complexity that limits what is known about optimal dynamic-routing decisions to the symmetric case (because, in the symmetric case, routing decisions in all subsequent cycles are independent of the routing decision in the current cycle). Hence, to examine dynamic-routing decisions, one must examine heuristics.

In this section, we develop the change-revert heuristic introduced above. As already described, this heuristic is based on the premise that there is a preferred, "default" route (e.g., the optimal static route), and that there are long-term advantages in using this route most of the time. The heuristic's decision rule is applied before the vehicle leaves the warehouse, and yields one of two decisions: (a) use the default route in the current replenishment cycle or (b) choose another route. In either case, the decision rule *assumes* that the route in the next replenishment cycle will be the default route. This assumption fixes the ending periods of all the retailers' allocation cycles for the current replenishment cycle (i.e., the values of $e[i]$ in (2)). Appendices A and B prove that the optimal allocation policy—that is, the policy that minimizes (2) for any alternative change route—equalizes the runout probabilities of the retailer about to receive the allocation, and a hypothetical "composite" retailer that represents the set of retailers whose allocations will be determined in some future periods. It is important to note, however, that although it is assumed that the route chosen in the next replenishment cycle will be the default route, this choice will, in fact, be made at the beginning of the next replenishment cycle, using the same decision rule.

In choosing a route for the current replenishment cycle, the decision rule recognizes that, at the time it is being made, this decision affects total expected cost in two ways: (1) by determining the end of the retailers' allocation cycles associated with the last replenishment cycle, and, hence, their backorders-related cost, and (2) by determining the start of the allocation cycles associated with the current replenishment cycle. Because by assumption the route to be chosen in the next replenishment cycle is the default route, (2), in effect, determines the expected inventory costs in the current replenishment-allocation cycle. More specifically, (2) determines the number of periods, denoted m_i, in the allocation cycle for retailer i, $i = 1, \ldots, N$, in the current cycle. Appendix B provides the optimal replenishment and allocation decisions associated with any choice of allocation cycles (i.e., starting and ending periods) for all N retailers, and their expected cost.

Consider the routing decision in any given replenishment cycle. Each possible route has an associated set of travel times to each retailer i, B_i, and an associated set of starting and ending periods for each retailer's respective allocation cycle. From Appendix B, Equations (B6) and (B7), the single-cycle cost is

$$Z^* = H^0 + H^F + (p+h)\sigma_1^C \phi(K^*), \tag{13}$$

where both $H^0 + H^F$ are "unmanageable" (i.e., not affected by the current routing decision) pipeline inventory-holding costs associated with any m-period replenishment cycle. H^0 equals the at-retailer holding costs of the units needed to satisfy average demand over m periods, and is the same regardless of either the current or next route. H^F is the on-vehicle holding cost of the units needed to satisfy average demand over m periods in the next cycle, and depends on the route used in the next cycle. In the static-routing case considered in §4, H^F depends on the choice of the static route. Under the change-revert heuristic, the next route is assumed to be the default route, so H^F is independent of the route used in the current cycle. In either case, σ_1^C depends on the route used in the current cycle.

In addition, the expected shortages of retailer i in its last allocation cycle, as viewed from the start of the current replenishment cycle, are given by

$$s_i^P = \sigma_i \sqrt{B_i} R\big[(x_i - B_i\mu_i)/\sigma_i\sqrt{B_i}\big],$$

where x_i is retailer i's net inventory at the start of the replenishment cycle, and R is the standard-normal loss function. Define $S^P = \sum_{i=1}^N s_i^P$.

Given the expected costs of (2) and (3) above, the decision rule then chooses the route with the smallest value of C^*, where

$$C^* = Z^* + (p+h)S^P. \tag{14}$$

For purposes of illustration, consider the special case of two retailers having the same per-period mean, μ, and standard deviation, σ, of demand. Label the two retailers $R1$ and $R2$, assume that $r_{12} = r_{21}$ and, without loss of generality, assume that $r_{01} \le r_{02}$. As already noted, the optimal static route visits $R1$ first. Assume this is the default route. Hence, the route that visits $R2$ first is the *change* route. For these two alternative routes, Table 1 gives the values that are key in determining C^*, and in (13), $H^0 + H^F = hm\mu(m - 1 + 2r_{01} + r_{12})$ is the same for either route.

Finally, note that for $r_{01} < r_{02}$, Z^* is smaller for the change route. This is because the change route postpones the allocation of inventory between the two retailers for $(r_{02} - r_{01})$ additional time periods, thereby providing more risk pooling (represented by the lower value of σ_1^C). This fact alone suggests that a fixed-routing policy that alternated the default and change routes might have lower total expected cost/period than the optimal static route. However, this is not necessarily the case. Under an alternating route policy, the expected costs in each cycle that uses the default route will be greater than the Z^* given above, because in these cycles the change route (not the default route) determines the end periods of the allocation cycles.

7. The Analytical Model

The formulas (B6) and (B7) in Appendix B give the expected cost/cycle for any fixed-routing policy; i.e.,

Table 1 **Comparison of Key Values for the Two Routes when $N = 2$**

	Default	Change
B_1	r_{01}	$r_{02} + r_{12}$
B_2	$r_{01} + r_{12}$	r_{02}
m_1	m	$m - r_{02} - r_{12} + r_{01}$
m_2	m	$m - r_{02} + r_{01} + r_{12}$
μ_1^C	$\mu(2m + 2r_{01} + r_{12})$	$\mu(2m + 2r_{01} + r_{12})$
σ_1^C	$\sigma\sqrt{2m + 2r_{01} + r_{12} + 2\sqrt{m(m + r_{12})}}$	$\sigma\sqrt{2m + 2r_{01} + r_{12} + 2\sqrt{(m + r_{01} - r_{02})(m + r_{01} - r_{02} + r_{12})}}$

for any choice of B_is and m_is. These are estimates, of course, because of the assumptions (§3) used in their development. We examine the sensitivity of the estimates to these assumptions in the next section.

Furthermore, given any state-dependent routing decision rule, these formulas can be used to compute the expected cost/cycle of that routing policy, provided that the steady-state distribution of system status can be determined. For the case of $N = 2$ retailers, it can be shown that, given optimal replenishment and allocation, the distribution of system status (i.e., the joint distribution of (x_1, x_2), the inventory levels at the two retailers at the time of routing, used by the decision rule to choose routes using the change-revert heuristic) follows a joint normal distribution with known parameters. Appendix C shows how this joint distribution is derived when the two retailers have identical means and standard deviations. A similar derivation is possible for two nonidentical retailers. Given this distribution of system status at the time of the routing decision, the analytical model computes the steady-state probabilities of each alternative route, and the expected cost/period of the change-revert routing decision rule (by numerical integration using the Maple software package). (See Appendix C for details.) For any given set of parameters, the difference in the analytical model's corresponding estimates of the cost/cycle of the optimal static-routing policy and its estimates of the change-revert routing policy provides the analytical model's estimate of savings. The quality of these estimates is also examined below.

8. Simulation Tests

We conducted simulations for three reasons: first, to assess the ability of the analytical model (§7) to estimate the cost/period of static and change-revert routing policies and, in particular, to estimate the savings associated with a change-revert routing decision; second, to assess the value of the change-revert heuristic under a variety of parameterizations. This assessment was conducted using $N = 2$ and $N = 6$ retailers with normally distributed and negative binomially distributed demand. Third, we used the simulation to test the validity and impact of some of our model's major assumptions. Our observations follow a description of the simulation model.

8.1. The Simulation Model

The simulation was written in Visual Basic for Application (VBA) and run on a Pentium(R) 4, 2.40 GHz computer. Each replenishment-allocation cycle was simulated as described in §3. In each replenishment-allocation cycle, the system replenishment quantity is the optimal base stock (B5) less the current system net inventory. If a negative system replenishment is prescribed, then a replenishment of zero is made. Retailer allocations are prescribed using (B4). If a negative allocation is prescribed, then an allocation of zero is made. Backorders are assessed a penalty cost regardless of the period in which they occur. For any given parameterization, the same retailer-demand realizations occur regardless of the routing decision rule being simulated. In simulation tests involving the normal distribution, negative demand realizations are permitted.

The first 5,000 cycles of the simulation were used to allow the system to attain a steady state. Then, every next 10,000 cycles were used to compute a single observation of the simulated results. At least 10 observations were collected to compute the average results. Cost savings are reported with respect to manageable cost; i.e., the average dollar saving as a percentage of the cost/period of the default (i.e., optimal static route) policy minus its unmanageable pipeline inventory-holding cost.

8.2. Simulation Results for $N = 2$ Retailers

For an $N = 2$ retailer business scenario, the base case used the following parameters: $r_{01} = 1$, $r_{02} = 2$, and $r_{12} = r_{21} = 3$. Each replenishment cycle contains (m=) eight time periods. Holding cost, h, is \$1/unit-period; the backorder cost, p, is \$160/unit-period, which provides a composite newsvendor target fractile of $(p - h(m-1))/(p+h) = 95.03\%$. Period demand for each retailer is normally distributed with $\mu = 100$ units/period, $\sigma = 120$ units/period, so that the coefficient of variation in the last period of each static allocation cycle (eight periods) is 0.42.

8.2.1. Assessing the Analytical Model. First, as expected, the analytical model underestimates cost/period observed in the simulation, because its assumptions ignore constraints (e.g., no negative allocations) that can increase cost in reality and, hence, in the simulation. Nonetheless, its estimates are close to those observed in the simulation. For example, in the base case, its estimates of cost/period for either the optimal-static or change-revert routing policy are within 5% of the simulated mean. Second, and for the same reason as above, the change-revert decision rule underestimates the savings from changing routes, but these estimates are generally "good." For example, in the base case, its estimate of cost savings is 3.54%, while the simulation prescribes an average cost savings of 5.66%. Third, the change-revert decision rule provides "very good" estimates of the "change frequency," i.e., the percentage of replenishment cycles in which the change route is chosen. For example, in the base case, the analytical model estimates an 18.86% change frequency and the simulation reported a change frequency of 18.67%. The second and third observations support the assertion that the model's estimates are managerially meaningful; e.g., that at least (this) saving will be provided with approximately (this) change frequency.

As a direct test of the sensitivity of the analytical model's three major assumptions, the simulation also kept track of the percentage of replenishment cycles: (1) in which a negative allocation was prescribed, (2) in which a negative system replenishment was prescribed, and (3) in which backorders occurred before the last period of any retailer's allocation cycle. Negative allocations were prescribed in between 1.81% and 1.89%; negative replenishments were prescribed in between 0.049% and 0.055%; and backorders occurred before the end of any retailer's allocation cycle in between 0.00% and 4.75% of the replenishment cycles for each of the parameterizations. Hence, based on the simulations we conducted, we assert that the three major assumptions made in the analytical model are innocuous.

8.2.2. Assessing the Value of the Change-Revert Heuristic. First, and most important, the expected savings provided by the change-revert heuristic w.r.t. the optimal static route, measured as a percentage of manageable cost, are statistically significant and managerially meaningful. Savings are $5.66\% \pm 0.41\%$ in the base case. Further, these savings are provided with route changes in only 18.7% of the replenishment cycles. Virtually all of these savings are from reduced backorders. For example, in the base case, 93.1% of the savings provided by change-revert over the optimal static route were due to reduced backorders. This supports the assertion that change-revert manages (virtually) the same system inventory better.

Table 2 reports the 95% confidence intervals for the savings in manageable cost and the percentage of the time the change route was chosen (under the two columns labeled as "Original decision rule"). Results for the base-case parameterization are in bold. To assess sensitivity, the same is reported for alternative r_{01}, r_{02}, and r_{12} values. Starting with the base case, but increasing r_{02} from two to three periods reduces both the savings (to $2.34\% \pm 0.37\%$) and change-route frequency (to 11.2%). This is to be expected, because increasing r_{02} increases the "cost" of a change, i.e., increases the variance of both lead time and allocation-cycle length. Similarly, decreasing r_{02} from two periods to one period increases savings (to $8.74\% \pm 0.48\%$) and change-route frequency (to 18.8%). Note, further, that holding r_{01} and r_{02} fixed, but increasing r_{12}, first increases and then decreases the estimated cost savings. For example, using the base case as the starting point, decreasing r_{12} decreases estimated savings to $5.29\% \pm 0.30\%$ (for $r_{12} = 2$) and to $1.17\% \pm 0.21\%$ (for $r_{12} = 1$). However, increasing r_{12} to four and then five periods also causes a reduction in savings (to $4.61\% \pm 0.49\%$ and

Table 2 Sensitivity of the Simulated Results to the Lead Times

			Original decision rule		Threshold decision rule	
r_{01}	r_{02}	r_{12}	Savings (%)	Frequency of change (%)	Savings (%)	Frequency of change (%)
1	1	1	7.92 ± 0.35	37.47	6.97 ± 0.26	7.22
		2	9.39 ± 0.46	27.51	8.59 ± 0.37	6.99
		3	8.74 ± 0.48	18.81	7.95 ± 0.48	5.81
		4	6.75 ± 0.48	12.05	6.38 ± 0.45	4.32
		5	4.95 ± 0.48	7.24	4.59 ± 0.37	2.84
	2	1	1.17 ± 0.21	45.46	0.00	0.00
		2	5.29 ± 0.30	31.77	4.11 ± 0.23	4.75
		3	**5.66 ± 0.41**	**18.67**	**5.12 ± 0.32**	**4.22**
		4	4.61 ± 0.49	10.19	4.31 ± 0.30	2.94
		5	3.24 ± 0.49	5.20	3.23 ± 0.42	1.82
	3	2	0.55 ± 0.19	17.21	0.00	0.00
		3	2.34 ± 0.37	11.21	1.75 ± 0.14	1.86
		4	2.39 ± 0.42	5.97	2.37 ± 0.31	1.63
		5	2.05 ± 0.37	2.94	1.84 ± 0.35	1.03
2	2	1	7.00 ± 0.37	36.94	5.89 ± 0.26	6.73
		2	8.64 ± 0.48	25.86	7.67 ± 0.39	6.57
		3	7.39 ± 0.46	16.90	6.67 ± 0.34	5.04
		4	5.28 ± 0.48	10.17	4.91 ± 0.37	3.49
		5	3.76 ± 0.46	5.71	3.50 ± 0.45	2.14
	3	2	4.18 ± 0.38	26.57	3.12 ± 0.19	3.51
		3	4.39 ± 0.34	15.19	3.85 ± 0.28	3.31
		4	3.38 ± 0.45	7.86	3.27 ± 0.37	2.18
		5	2.50 ± 0.46	3.74	2.21 ± 0.38	1.24
3	3	2	7.08 ± 0.49	23.99	6.16 ± 0.41	5.55
		3	5.86 ± 0.49	14.48	5.02 ± 0.45	4.03
		4	3.82 ± 0.48	8.01	3.44 ± 0.36	2.53
		5	2.55 ± 0.39	4.08	2.28 ± 0.49	1.36

3.24% ± 0.49%, respectively). Change-route frequency decreases monotonically as r_{12} increases. Our interpretation of this follows: As r_{12} changes, we observe the combined impact from two effects. For small values of r_{12} there may be little or no reduction in the delivery lead time to Retailer 2 by using the change route, so the potential to reduce shortages at Retailer 2 is small. For example, in the base case, but with $r_{12} = 1$, the delivery lead time to Retailer 2 is 2 for either route. Hence, Retailer 2 shortages are never reduced by the change route. On the other hand, as r_{12} increases, the difference in the default route's delivery lead times between the two retailers increases, so the difference in the allocation quantities increases. As the difference in the allocation quantities increases, the frequency of route change decreases, along with the average savings.

8.2.3. Examining the Frequency of Route Changes. Note (Table 2, Column 5) that although the change frequency reported is less than 20% in 18 (66.7%) of the 27 parameterizations, it is over 30% in 4 (15%) of the parameterizations. We attribute this to the fact that the decision rule prescribes the change route whenever the estimated expected cost of the change route is smaller than that of the default route, regardless of the magnitude of the savings. Table 2 (Column 7) also provides the corresponding simulation results using a "threshold" decision rule under which a change is prescribed only if the estimated savings in that particular cycle is greater than or equal to 10% of the estimated costs under the default route. Note, for example, that in the base case, this threshold decision rule provides most of the cost savings (5.12% ± 0.32% versus 5.66% ± 0.41%) as the original rule, but reduces

Table 3 Sensitivity of the Simulated Results to the Standard Deviation of Retailer Demand/Period

	Original decision rule		Threshold decision rule	
σ	Savings (%)	Frequency of change (%)	Savings (%)	Frequency of change (%)
20	0.00	0.00	0.00	0.00
60	1.53 ± 0.37	2.56	1.35 ± 0.30	0.82
120	$\mathbf{5.66 \pm 0.41}$	**18.67**	$\mathbf{5.12 \pm 0.32}$	**4.22**
180	8.43 ± 0.47	30.79	7.68 ± 0.33	6.60
240	10.44 ± 0.47	38.53	9.34 ± 0.37	8.49

the frequency of change (from 18.7% to 4.2%). Similar results apply to other parameterizations.

Table 3 reports the sensitivity of the simulated results to the standard deviation of retailer demand/period. Results for the base case (with $\sigma = 120$ units/period), already reported, are in bold. As expected, decreasing σ reduces the estimated savings and change frequency (to zero for $\sigma = 20$), while increasing σ increases the savings and the change frequency. To examine the sensitivity of the simulated results to the overall level of system inventory, we also ran simulations for composite retailer fill-rates of 80.5% and 99% and compare them with the 95.03% fill-rate for the base case. Table 4 reports the results. As expected, systems with higher target fill-rates provide less opportunity for cost savings to change-revert, regardless of the decision rule. For the unmodified decision rule, more and more frequent change-route decisions are required to provide lower levels of savings. Again, the threshold decision rule provides most of the savings, and change frequency decreases as target fill-rate increases.

To verify that the effectiveness of the change-revert heuristic is not an artifact of the assumption that customer demand was normally distributed, we performed simulations for the base-case travel parameters using negative binomial customer demand/period with a mean of 100 and two coefficients of variation: 0.6 (low), and 1 (high).

Table 4 Sensitivity of the Simulated Results to the Composite Retailer Fill-Rates

	Original decision rule		Threshold decision rule	
Composite retailer fill-rate (%)	Savings (%)	Frequency of change (%)	Savings (%)	Frequency of change (%)
80.50	6.83 ± 0.35	15.04	5.10 ± 0.23	4.93
95.03	$\mathbf{5.66 \pm 0.41}$	**18.67**	$\mathbf{5.12 \pm 0.32}$	**4.22**
99	4.64 ± 0.48	23.61	4.36 ± 0.43	2.63

As expected, the analytical model (using the normal approximation to the negative binomial) underestimates the cost/period observed in the simulation, but except in the high-variance case, its estimate was within 5% of the simulated mean. Also, as in the scenarios with normally distributed demand, the major assumptions of the analytical model were seldom violated.

Finally, and most significantly, the change-revert heuristic generated observed savings/period of 2.01% (low) and 7.29% of manageable cost with change frequencies of 2.0% and 12.6%. As expected, the threshold heuristic provided most of these savings, 2.04% and 6.92%, respectively, with significantly reduced change frequencies of 0.9% and 4.2%.

8.3. Simulation Results for $N > 2$ Retailers

To assess the value of the change-revert heuristic for systems with more than two retailers, we conducted simulations for two configurations of a six-retailer system. The simulations were conducted as described above. In both configurations, retailer demand each period is independently, identically, normally distributed with $\mu = 100$ and $\sigma = 120$ units, $h = \$1$/unit-period. The number of periods in the replenishment cycle, m, was set equal to one period larger than the longest possible delivery route in that configuration (so that one vehicle could make all the deliveries), and p was chosen so that the critical ratio for the corresponding composite retailer, (4), which depends on m, was 0.95.

Configuration 1 was a star, with each retailer two time periods (travel time) from the warehouse (i.e., $r_{0i} = 2$), and either two, three, or four periods of travel time from the others. For this example, we used $m = 21$ periods and $p = \$420$/unit-period. The results are summarized in Column 2 of Table 5. Note that the change-revert heuristic saved 18.9% of the manageable cost compared with the default route, and had a change frequency of 83.9%. The threshold policy provided more than half of these savings (11.8% versus 18.9%), but dramatically reduced the change frequency to 10.5%. We also kept track of which non-default routes were chosen. For the original decision

Table 5 Simulation Results for $N = 6$ Retailers

	Configuration 1: star	Configuration 2: random
Savings (%): Original/threshold	18.9/11.8	14.3/8.0
Frequency of change (%): Original/threshold	83.9/10.5	78.7/6.1
No. of routes used: Original/threshold	561/307	413/160
No. of routes used with freq. ≥1%: Original/threshold	21/1	18/0
No. of routes used for 80% of the time: Original/threshold	70/54	35/20

rule, of the 719 ($N! - 1$) nondefault routes, 561 were chosen at least once (78%). Of these 561 nondefault routes, 21 (3%) were chosen in at least 1% of the cycles. Finally, 70 routes accounted for 80% of the nondefault choices made. The corresponding number of routes for the threshold decision rule were 307, 1, and 54, respectively. Configuration 2 was generated randomly, with travel times integer and uniformly distributed between one and five periods. The results are summarized in Column 3 of Table 5.

We believe that these example results support the following assertions, all of them consistent with the results for $N = 2$ retailers: First, dynamic routing provides statistically significant and managerially meaningful savings for systems with more than two retailers. Second, although the "original" routing decision rule might prescribe very frequent route changes, the threshold policy provides more than half of the manageable cost savings of the original, but with dramatically fewer route changes. Third, the statistics collected on the choices of nondefault routes suggest that change-revert can still be effective even if the number of "candidate" nondefault routes is relatively small. For example, note that in the random Configuration 2 (in Table 5), only 20 of the 719 possible nondefault routes accounted for 80% of the routes chosen by the threshold heuristic. As expected, these particular nondefault routes were among those with the smallest traveling-salesman (total) travel times (e.g., for Configuration 2, in which the minimum (maximum) TSP time was 13 (26) periods, these 10 routes had total travel times of 13–17 periods). Finally, these example results indicate that the potential for dynamic routing to provide savings increases with the number of retailers. This is to be expected because the opportunity for inventory imbalances increases with the number of retailers.

9. Conclusion and Suggestions for Further Work

In our research we have developed an analytical model to determine a combined system inventory-replenishment, routing, and inventory-allocation policy that minimizes the total expected cost/period of one-warehouse N-retailer distribution system over an infinite time horizon. We have employed this model: (1) to determine the optimal static route, (2) to examine the interactions between routing and inventory-management policies, particularly the interaction between dynamic routing and inventory allocation, and (3) to develop a heuristic routing policy: the change-revert heuristic.

We used simulation to examine the impact of the major assumptions used to develop the analytical model. The results indicate, first, that these assumptions are seldom violated; and, second, even when they are, the insights provided by the analytical model—in particular, the model's estimates of cost/period and the desirability of changing routes—appear to be insensitive to them. In particular, the simulation results provide evidence that the change-revert heuristic provides statistically significant and managerially meaningful savings for multiretailer systems.

Of course, to focus on the *potential* of dynamic routing to reduce costs, our analysis deliberately excluded explicit consideration of the *cost* of alternative routes (e.g., mileage-, or driver-time-related costs). In §4 we explained how such costs are easily accommodated in choosing the optimal static route. In a similar fashion, transportation-related cost can be added to the decision rule(s) used in the change-revert heuristic.

Our analysis also excluded consideration of other costs associated with dynamic routing. However, in some systems this involves no more than adding a feature to a system already used to monitor system status and communicate with its operatives (e.g., vehicle drivers). Our model also ignores other costs that might be associated with dynamic routing. The quality-management literature, for example, suggests that variety itself can increase cost and reduce quality. Hence, it is possible that by increasing *route* variety, dynamic routing may increase system costs. However, it is also possible, as indicated by our sample results for six retailers, that considerably fewer than $N!$ routes will provide most of the savings of $N!$ variety.

Our analysis only considered a single product. However, we believe the extension to multiple products is straightforward, provided that our assumption of a single (uncapacitated) vehicle remains. If multiple vehicles, each with possibly a different route, are used, an additional element is added to the problem: the assignment of products to vehicles.

Finally, we deliberately leave two questions open for future research: First is an examination of fully dynamic-routing policy. This form of dynamic routing would decide which retailer to visit *next* as part of the allocation decision made at every retailer (except the last) in each replenishment cycle. (The policy we examine makes these $N-1$ decisions once and for all at the time the vehicle first leaves the warehouse.) Second is a closer examination of change-revert routing policies. Perhaps the most interesting question here is "What is the best default route?" We arbitrarily chose the optimal static route to be the default route, but, given a change-revert policy, it is certainly possible that a different default route would provide even better results.

Appendix A. The Composite Retailer

This appendix defines "composite retailer" and gives its fundamental properties. The composite retailer concept was defined in Kumar (1995). The construction of the composite retailer given here differs slightly from that in Kumar, but is conceptually the same. In Kumar, the composite retailer was defined so that at the time of allocation it includes the retailer about to receive an allocation (along with the other retailers yet to receive an allocation). Our definition is such that, at the time of an allocation, the composite retailer does not include the retailer about to receive the allocation. We believe this new definition makes it clear that each allocation is a division of stock between two retailers (one of which is a composite). The composite retailer plays a key role in determining optimal allocations and replenishment under fixed routes given in Appendix B and referenced in §§4, 6, and 7.

Consider the following general case involving a system of two retailers. Initially, the system has U units in stock. A system demand, G_0, occurs that reduces the stock to $X = U - G_0$. X is then allocated to two retailers, denoted A and B, so as to equalize the resulting runout probabilities after each experiences additional demand. Retailer A experiences additional demand of G_A, and Retailer B experiences G_B. Let μ_i and σ_i be the mean and standard deviations of G_i, $i = 0$, A, and B. We wish to express the sum of the expected shortages at the two retailers as a function of U and these demand parameters.

Given X, let y_A and y_B be the allocations to the two retailers, where $y_A + y_B = X$. The allocation of X that equalizes runout probabilities is given by

$$y_A = \mu_A + z^*\sigma_A \quad \text{and} \quad y_B = \mu_B + z^*\sigma_B \quad \text{where} \tag{A1}$$

$$z^* = (X - (\mu_A + \mu_B))/(\sigma_A + \sigma_B) \tag{A2}$$

with expected shortages equal to

$$\sigma_A R(z^*) + \sigma_B R(z^*) = (\sigma_A + \sigma_B)R(z^*), \tag{A3}$$

where $R(z)$ is the standard normal loss function given by $R(z) = \int_z^\infty (x - z)\phi(x)\,dx$. That is, the expected shortages resulting from the allocation of X equal the expected shortages from a single "combined" retailer that has stock X prior to experiencing demand with

$$\text{mean} = (\mu_A + \mu_B) \tag{A4}$$

$$\text{standard deviation} = (\sigma_A + \sigma_B). \tag{A5}$$

Now consider the expected shortages, as a function of U, prior to the occurrence of G_0. Under the assumption that an allocation as prescribed above will be made after G_0 occurs, expected shortages are equivalent to the shortages from a single retailer that has inventory of U and experiences two "sets" of demands, G_0 and G^+, where G^+ has the mean and standard deviations given by (A4) and (A5). Convoluting these two normal variables gives a "total demand" that is normal with

$$\text{mean:} \quad \mu^C = (\mu_0 + \mu_A + \mu_B), \tag{A6}$$

$$\text{variance:} \quad (\sigma^C)^2 = \sigma_0^2 + (\sigma_A + \sigma_B)^2, \tag{A7}$$

and expected shortages given by

$$\sigma^C R((U - \mu^C)/\sigma^C).$$

That is, expected shortages, as a function of U, equal the expected shortages from a hypothetical single retailer that has stock U, and experiences normal demand characterized by μ^C and σ^C. We call this single retailer a "composite retailer."

Property 1. *In the definition of a composite retailer, the allocation of x that equalizes runout probabilities also minimizes the sum of the expected shortages from the two retailers.*

Comment. Property 1 is based on well-known results. It is derived from the first-order condition obtained by setting the derivative of expected shortages to zero. In particular, the derivative of $R(x)$ is $-\Phi(-x)$.

Appendix B. Optimal Replenishment and Dynamic Allocation with Fixed Routes

We consider the joint replenishment-allocation-routing problem described in §3 in the case when the routes are fixed (i.e., chosen in advance) for all cycles. We show that under the assumptions of §3, the infinite-horizon problem decomposes into independent single-cycle problems. For the single-cycle problem, we use the composite retailer defined in Appendix A to give the cost-minimizing replenishment and allocation policy. These optimal policies are the basis for the heuristic approach to the problem that includes route selection given in §6.

Relative to some particular replenishment-allocation cycle, denoted the "current" cycle, let $t = 0$ denote the period when the replenishment occurs, and we use the following:

B_i is the delivery lead time (start of the allocation cycle) for retailer i in the cycle.

Let B_i^F be the delivery lead time for retailer i in the next allocation cycle. Then $(m + B_i^F - 1)$ is the last period in retailer i's current allocation cycle, and $m_i = m + B_i^F - B_i$ is the number of periods in retailer i's current allocation cycle.

Let $[j]$ denote the index of the jth retailer on the current route, and $b_{[j]} = B_{[j]} - B_{[j-1]}$ is the incremental lead time to retailer $[j]$. $B_{[0]} = B_0 = 0$.

Let $d_{t,i}$ denote the demand at retailer i in period t. It is drawn from a normal distribution with mean and standard deviation of μ_i and σ_i.

B.1. The Single-Cycle Problem

Consider the operation of our system over an infinite horizon. We wish to determine the replenishment and allocation policies that will minimize the average cost/period. To this end, we divide the problem into a series of single-cycle problems. As described in §3, all costs over the infinite horizon are uniquely assigned to a cycle. The current cycle is assigned costs (§3.3)

$$h\sum_{t=0}^{m-1} E_t + (h+p)\sum_{i=1}^{N}\{s_{i,t}\colon t = B_i + m_i\}, \tag{B1}$$

where the $s_{i,t}$ terms are the backorders incurred by each retailer at the end of their allocation cycles. The single-cycle problem is to choose the replenishment quantity and the N (dynamic) allocation quantities within the current cycle that minimize the expected value of (B1).

The expectation of the first term in (B1) depends only on the system inventory after the replenishment quantity has been added. Expected backorders at retailer i depend only on the sum of retailer i's inventory at $t = 0$ and the amount allocated to retailer i in the current cycle. The only constraint on the replenishment and allocation quantities is that the sum of the allocations equal the replenishment quantity. Therefore, each single-cycle problem is equivalent to the problem in which (1) all initial inventories are zero, (2) the replenishment decision is to choose Y, the system inventory at $t = 0$, and (3) each allocation decision, y_i for $i = 1$ to N, is an allocation from Y, constrained only by $\sum y_i = Y$. In particular, given the model's assumptions that allow negative replenishment and delivery quantities, the replenishment and allocation decisions in previous cycles have no impact on the optimal solution, or its expected costs, in the current cycle problem. Hereafter, we denote this "equivalent" single-cycle problem as SCP.

B.2. Optimal Replenishment and Allocation Using the Composite Retailer in SCP

In each cycle, there is a sequence of N dynamic allocation decisions. Given any Y, optimal dynamic allocations minimize the sum of the retailers expected shortages, given the information available at the time of each allocation. We apply the composite retailer, defined in Appendix A, to derive expressions for optimal allocations and replenishment in SCP. The composite retailer was first defined in Kumar (1995) for dynamic allocation in a single-cycle problem with retailer-only holding costs and static routing. Although we use a slightly different definition of the composite retailer, the allocation quantities are identical with Kumar. However, one important difference is the allocation results given in Kumar apply only to static routes. Our results, for system-based costs, apply to any fixed routes. (We do not believe the composite retailer results are generalizable to fixed routes under retailer-only holding cost.) For $j = 1$ to $N - 1$, the jth allocation decision is made in period $B_{[j]}$, and is viewed as an allocation between two retailers; retailer $[j]$ and a composite retailer that represents retailers $[j+1]$ to $[N]$. The Nth allocation decision allocates $y_{[N]} = Y - \sum_{j=1}^{N-1} y_{[j]}$ to retailer $[N]$.

Consider the $N-1$st allocation decision. The values of $y_{[k]}$, $k = 1$ to $N-2$, have already been determined, so there are $Y - \sum_{k=1}^{N-2} y_{[k]}$ units to divide between retailers $[N-1]$ and $[N]$. At the time of allocation these retailers have experienced $B_{[N-1]}$ periods of demand, which in effect reduce the amount of units to allocate between the two retailers to

$$X_{[N-1]} = Y - \sum_{k=1}^{N-2} y_{[k]} - \sum_{t=0}^{B_{[N-1]}-1} d_{t,[N-1]} - \sum_{t=0}^{B_{[N-1]}-1} d_{t,[N]},$$

where the demand terms are now known realizations. Because the objective of the allocation is to minimize the resulting expected shortages, we use the composite retailer defined in Appendix A. Specifically, let retailer $[N-1]$ ($[N]$) play the role of Retailer A (B) with $X_{[N-1]}$ the value of X. This gives

$$\mu_A = m_{[N-1]}\mu_{[N-1]} \qquad (\sigma_A)^2 = m_{[N-1]}\sigma_{[N-1]}^2$$
$$\mu_B = (b_{[N]} + m_{[N]})\mu_{[N]} \qquad (\sigma_B)^2 = (b_{[N]} + m_{[N]})\sigma_{[N]}^2.$$

The optimal allocation to retailer $[N-1]$ is given by (A1) and (A2). The sum of the expected shortages from the two retailers is given by (A3), and is equivalent to the shortages of a single "combined" retailer with inventory $X_{[N-1]}$ facing normal demands characterized by (A4) and (A5). Further, given this optimal allocation in period $B_{[N-1]}$, the expected shortages of these two retailers as viewed in period $B_{[N-2]}$ are captured by the composite retailer with mean and variance given by (A6) and (A7) where

$$\mu_0 = b_{[N-1]}(\mu_{[N-1]} + \mu_{[N]}) \qquad (\sigma_0)^2 = b_{[N-1]}(\sigma^2_{[N-1]} + \sigma^2_{[N]}).$$

That is, the mean and variance of the $N-1$st composite retailer, as viewed in period $B_{[N-2]}$, is given by

$$\mu^C_{N-1} = (b_{[N-1]} + m_{[N-1]})\mu_{[N-1]} + (b_{[N-1]} + b_{[N]} + m_{[N]})\mu_{[N]}$$

$$(\sigma^C_{N-1})^2 = b_{[N-1]}(\sigma^2_{[N-1]} + \sigma^2_{[N]}) + \left(\sqrt{m_{[N-1]}}\sigma_{[N-1]} + \sqrt{b_{[N]} + m_{[N]}}\sigma_{[N]}\right)^2.$$

Given that an optimal allocation will be made in period $B_{[N-1]}$, the $N-2$nd allocation is equivalent to a two-retailer problem composed of retailer $[N-2]$ and the $N-1$st composite retailer. Given an optimal allocation in period $B_{[N-2]}$, the $N-2$nd composite retailer can be constructed to use in the $N-3$rd allocation decision, and so on. This leads to

$$\mu^C_j = \sum_{k=j}^{N}(B_{[k]} - B_{[j-1]} + m_{[k]})\mu_{[k]} \tag{B2}$$

and $(\sigma^C_j)^2$ defined recursively by

$$(\sigma^C_N)^2 = (b_{[N]} + m_{[N]})\sigma^2_{[N]} \tag{B3}$$

$$(\sigma^C_j)^2 = b_{[j]}\sum_{k=j}^{N}\sigma^2_{[k]} + \left(\sqrt{m_{[j]}}\sigma_{[j]} + \sigma^C_{j+1}\right)^2, \quad \text{for } i = 1 \text{ to } N-1.$$

To calculate the optimal dynamic allocation quantities, $y_{[j]}$, for $j = 1$ to $N-1$:

(1) Let $X_{[j]} = Y - \sum_{k=1}^{j-1} y_{[k]} - \sum_{p=j}^{N}\sum_{t=0}^{B_{[j]}-1} d_{t,[p]}$.

(2) Calculate $z^*_j = (X_{[j]} - m_{[j]}\mu_{[j]} - \mu^C_{j+1})/(\sqrt{m_{[j]}}\sigma_{[j]} + \sigma^C_{j+1})$.

(3) $y_{[j]} = m_{[j]}\mu_{[j]} + \sum_{t=0}^{B_{[j]}-1} d_{t,[j]} + z^*_j\sqrt{m_{[j]}}\sigma_{[j]}$. (B4)

$X_{[j]}$ is the total inventory at the vehicle and all the retailers that have not received an allocation yet at the time of the allocation to retailer j. It equals Y minus all allocations to those retailers already visited and the demands at the nonvisited retailers (which reduce the inventories of those retailers) that have occurred since replenishment. $y_{[j]}$ is the allocation to the $[j]$th retailer, which includes an amount to cover its demands since replenishment.

The first composite retailer represents the entire system of N retailers with μ^C_1 equal to the sum of the mean demands from the replenishment decision to the end of each retailer's allocation cycle. The optimal replenishment decision is equivalent to a single-retailer newsvendor problem with demand characterized by μ^C_1 and σ^C_1, an overstocking cost of hm (the holding cost per unit incurred by leftovers), and an understocking cost of $p-(h-1)m$ (an additional unit that prevents a backorder saves the $(p+h)$ of the second term in (B1), but incurs a cost of hm in the first term). Thus, the optimal replenishment is given by

$$Y^* = \mu^C_1 + K^*\sigma^C_1, \tag{B5}$$

where K^* is such that $\Phi(K^*) = (p - h(m-1))/(p+h)$.

The resulting expected backorders in the cycle are $R(K^*)\sigma^C_1$, where R is the standard normal loss function. Let $\mu^S = \sum_{j=1}^{N}\mu_j$. The expected values of the E_t terms in (B1) are $Y^* - (t+1)\mu^S$, for $t = 0$ to $m-1$. Hence, the optimal expected costs in the cycle are

$$Z^* = h[m(Y^* - \mu^S(m+1)/2)] + (p+h)\sigma^C_1 R(K^*);$$

$$Z^* = h[m(\mu^C_1 + K^*\sigma^C_1 - \mu^S(m+1)/2)] + (p+h)\sigma^C_1 R(K^*).$$

Using

$$R(K^*) = K^*\Phi(K^*) - K^* + \phi(K^*)$$

and

$$\Phi(K^*) = (p - h(m-1))/(p+h),$$

we get

$$\begin{aligned}(p+h)\sigma^C_1 R(K^*) &= \sigma^C_1(p+h)[K^*(p-h(m-1))/(p+h) - K^* + \phi(K^*)]\\ &= \sigma^C_1(K^*[p - h(m-1) - p - h] + (p+h)\phi(K^*))\\ &= \sigma^C_1[-hmK^* + (p+h)\phi(K^*)].\end{aligned}$$

This leads to

$$Z^* = hm[\mu^C_1 - \mu^S(m+1)/2] + (p+h)\phi(K^*)\sigma^C_1.$$

Using $\mu^C_1 = \sum_{j=1}^{N}(m_j + B^F_j)\mu_j = m\mu^S + \sum_{j=1}^{N} B^F_j\mu_j$,

$$Z^* = H^0 + H^F + (p+h)\phi(K^*)\sigma^C_1, \tag{B6}$$

where

$$H^0 = hm(m-1)\mu^S/2 \quad \text{and} \quad H^F = hm\sum_{j=1}^{N} B^F_j\mu_j. \tag{B7}$$

Note that H^0 is the holding cost of the units needed to satisfy average demand over m periods, and is the same regardless of either the current or next route. H^F depends only on the route used in the next cycle and is the additional holding cost incurred on on-vehicle inventory.

Appendix C. The Analytical Model ($N = 2$)

In the change-revert heuristic developed in §6, a routing decision, made simultaneously with a replenishment decision, selects the current (i.e., about to start) route with least expected cost given by (14), $C^* = Z^* + (p+h)S^P$, where Z^* is the expected cost of the current cycle (assuming the default route is used in the next cycle), and S^P are the expected backorders incurred at the end of the ongoing (i.e., last) allocation cycles. Because Z^* takes on a specific value for each route, but is independent of both the previous route or the inventory levels of two retailers at the time of routing, denote its route-specific value by Z_R^*, where R denotes the current route. For the $N=2$ case, there are only two routes, which we denote by $R=F$ (default) and $R=G$ (change). Similarly, because S^P is a function of both the current route and the retailer inventory levels at the time routing, denote its value by $S_R^P(x_1, x_2)$, where x_i is the inventory level of retailer i $(=1, 2)$ at the time of routing.

$\Delta Z^* = Z_G^* - Z_F^*$ is the difference in the current cycle's expected costs between the two routes. (Note that if the default route is the least-cost static route, then this difference is positive.) Also define $\Delta S^P(x_1, x_2) = S_F^P(x_1, x_2) - S_G^P(x_1, x_2)$ as the reduction in expected backorders achieved by choosing the change route versus the default route. The change route will be chosen if and only if $(x_1, x_2) \in Q$, where

$$Q = \left\{(x_1, x_2)\colon \left(\Delta S^P(x_1, x_2) \geq \frac{\Delta Z^*}{p+h}\right)\right\}.$$

In particular, the set Q remains the same for each cycle, so that given the joint distribution of (x_1, x_2), we can calculate the fraction of cycles in which each of the two routes are used, and the long-run average costs per cycle of using the change-revert heuristic.

Define $J_R(x_1, x_2)$ as the joint density function given previous route R. (At the end of this appendix we illustrate how $J_R(x_1, x_2)$ is derived.) Define P_{AB} as the probability that route A is selected given that route B was used in the previous cycle, A and $B = F$ or G.

$$P_{GR} = \int_Q J_R(x_1, x_2) \quad \text{and} \quad P_{FR} = 1 - P_{GR}, \quad \text{for } R = F \text{ or } G.$$

Denote by P_R the long-run fraction of cycles using route $R = F$ or G. Noting that $P_G = 1 - P_F$, we can calculate P_F using a simple Markov process given by: $P_F = (1-P_F)P_{FG} + P_F P_{FF}$. This leads to $P_F = P_{FG}/(1 + P_{FG} - P_{FF})$.

The long-run cost per cycle if route F is used each cycle is just Z_F^*. Under the change-revert heuristic the average cost per cycle equals

$$P_F(Z_F^* - (p+h)L_F) + P_G(Z_G^* - (p+h)L_G),$$

where L_R is the expected reduction in backorders, given current route R, from choosing the next route using the heuristic, as compared with using F in the next cycle (the assumption used to calculate Z_R^*). L_R is given by

$$L_R = \int_Q (J_R(x_1, x_2)\Delta S^P(x_1, x_2)), \quad R = F, G.$$

Hence, the long-run cost savings per cycle from using the change-revert heuristic versus always using default is given by $(p+h)(P_F L_F + P_G L_G) - P_G \Delta Z^*$.

C.1. Calculating the Joint Probability Distribution of (x_1, x_2)

To illustrate the derivation of $J_R(x_1, x_2)$, we derive it for a particular route, and for two "identical" retailers having the same mean (μ) and standard deviation (σ) per period. Assume the default route (F) visits Retailer 1 first. We show that $J_F(x_1, x_2)$ is a joint normal distribution with known means, standard deviations, and correlation coefficient. The derivation of $J_G(x_1, x_2)$ is similar, as is the derivation when the two retailers are not identical.

The systemwide total inventory at the time of the previous replenishment is $Y^* = \mu^C + \sigma^C K^*$, where

$$\mu^C = \mu(2m + 2r_{01} + r_{12})$$

and

$$\sigma^C = \sigma\sqrt{2m + 2r_{01} + r_{12} + 2\sqrt{m(m + r_{12})}}$$

are the mean and standard deviation of the composite retailer. *Note that μ^C and σ^C, and all the derivations below are route dependent*. The systemwide inventory at the time of the previous allocation is given by $Y^A = Y^* - D^A$, where D^A is the systemwide demand between replenishment and allocation. Using $N(*, *)$ to denote a normal random variable with mean and standard deviation parameters, D^A is $N(2r_{01}\mu, \sigma\sqrt{2r_{01}})$ and therefore Y^A is $N(m^A, s^A)$, where $m^A = \mu^C + \sigma^C K^* - 2r_{01}\mu$ and $s^A = \sigma\sqrt{2r_{01}}$.

Let x_1^A and x_2^A be the allocation amounts to Retailers 1 and 2, respectively, in the previous cycle. They satisfy

$$\frac{Y^A - \mu^A}{\sigma^A} = \frac{x_1^A - m\mu}{\sigma\sqrt{m}} = \frac{x_2^A - (m + r_{12})\mu}{\sigma\sqrt{m + r_{12}}},$$

where $\mu^A = \mu(2m + r_{12})$ and

$$\sigma^2 = \sigma\sqrt{m} + \sigma\sqrt{m + r_{12}}.$$

Hence, $x_1^A = (Y^A - \mu^A)(\sigma\sqrt{m}/\sigma^A) + m\mu$. x_1^A is normal because Y^A is normal, and has a mean of $m_1^A = (m^A - \mu^A)(\sigma\sqrt{m}/\sigma^A) + m\mu$. Noting that $\mu^A = \mu^C - 2r_1\mu$, we get x_1^A is $N(m_1^A, s_1^A)$, where $m_1^A = \sigma^C K^*(\sigma\sqrt{m}/\sigma^A) + m\mu$ and $s_1^A = s^A(\sigma\sqrt{m}/\sigma^A)$.

Similarly, $x_2^A = (Y^A - \mu^A)(\sigma\sqrt{m + r_{12}}/\sigma^A) + (m + r_{12})\mu$ and is $N(m_2^A, s_2^A)$, where $m_2^A = \sigma^C K^*(\sigma\sqrt{m + r_{12}}/\sigma^A) + (m + r_{12})\mu$ and $s_2^A = s^A(\sigma\sqrt{m + r_{12}}/\sigma^A)$. *Note that x_1^A and x_2^A are perfectly correlated.*

For $i = 1, 2$, $x_i = x_i^A - D_i$, where D_i is

$$N((m - r_{01})\mu, \sigma\sqrt{m - r_{01}})$$

and independent of x_i^A. Thus, x_1 is $N(\mu_{\text{one}}, \sigma_{\text{one}})$ and x_2 is $N(\mu_{\text{two}}, \sigma_{\text{two}})$, where

$$\mu_{\text{one}} = m_1^A - (m - r_{01})\mu = r_{01}\mu + \sqrt{m}\sigma K^*(\sigma^C/\sigma^A),$$

$$\sigma_{\text{one}} = \sigma\sqrt{2r_{01}m(\sigma/\sigma^A)^2 + m - r_{01}},$$

$$\mu_{\text{two}} = (r_{01} + r_{12})\mu + \sqrt{m + r_{12}}\sigma K^*(\sigma^C/\sigma^A),$$

$$\sigma_{\text{two}} = \sigma\sqrt{2r_{01}(m + r_{12})(\sigma/\sigma^A)^2 + m - r_{01}}.$$

To get the joint distribution of x_1 and x_2, we also need ρ, the correlation coefficient. x_1 and x_2 are two correlated normal variables that are based on three independent normal values (y_{FA}, D_1, and D_2) as given by: $x_i = x_i^A - D_i = a_i y^A + b_i - D_i$, $i = 1, 2$, where (from above)

$$a_1 = \sqrt{m}\sigma/\sigma^A \qquad b_1 = m\mu - a_1\mu^A$$

$$a_2 = \sqrt{m + r_{12}}\sigma/\sigma^A \qquad b_2 = (m + r_{12})\mu - a_2\mu^A$$

$$\begin{aligned}
\rho &= E[((x_1 - \mu_{\text{one}})/\sigma_{\text{one}})((x_2 - \mu_{\text{two}})/\sigma_{\text{two}})] \\
&= E[(x_1 - \mu_{\text{one}})(x_2 - \mu_{\text{two}})]/(\sigma_{\text{one}}\sigma_{\text{two}}) \\
&= E[(a_1(y^A - m^A) - (D_1 - \mu_D)) \\
&\qquad \cdot (a_2(y^A - m^A) - (D_2 - \mu_D))]/(\sigma_{\text{one}}\sigma_{\text{two}})
\end{aligned}$$

where μ_D is the mean of both D_1 and D_2. Note that b_i ($i = 1, 2$) does not appear in the expression for ρ because it is a constant that occurs in both x_i and its mean value. Because y^A, D_1, and D_2 are all independent, only one nonzero term emerges in the expectation, and we get

$$\rho = (a_1 a_2)E[(y^A - m^A)^2]/(\sigma_{\text{one}}\sigma_{\text{two}}) = (a_1 a_2)(s^A)^2/(\sigma_{\text{one}}\sigma_{\text{two}}).$$

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