Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Dynamic Pricing Strategies in the Presence of Demand Shifts

Omar Besbes, Denis Sauré

MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 16, No. 4, Fall 2014, pp. 513–528
ISSN 1523-4614 (print) | ISSN 1526-5498 (online)
http://dx.doi.org/10.1287/msom.2014.0489

Dynamic Pricing Strategies in the Presence of Demand Shifts

Omar Besbes
Columbia Business School, Columbia University, New York, New York 10027, ob2105@columbia.edu
Denis Sauré
Department of Industrial Engineering, University of Chile, 8370439 Santiago, Chile, dsaure@dii.uchile.cl

Many factors introduce the prospect of changes in the demand environment that a firm faces, with the specifics of such changes not necessarily known in advance. If and when realized, such changes affect the delicate balance between demand and supply and thus current prices should account for these future possibilities. We study the dynamic pricing problem of a retailer facing the prospect of a change in the *demand function* during a finite selling season with no inventory replenishment opportunity. In particular, the time of the change and the postchange demand function are unknown upfront, and we focus on the fundamental trade-off between collecting revenues from current demand and doing so for postchange demand, with the capacity constraint introducing the main tension. We develop a formulation that allows for isolating the role of dynamic pricing in balancing inventory consumption throughout the horizon. We establish that, in many settings, optimal pricing policies follow a monotone path up to the change in demand. We show how one may compare upfront the attractiveness of pre- and postchange demand conditions and how such a comparison depends on the problem primitives. We further analyze the impact of the model inputs on the optimal policy and its structure, ranging from the impact of model parameter changes to the impact of different representations of uncertainty about future demand.

Keywords: revenue management; dynamic pricing; nonstationary demand; model uncertainty
History: Received: June 12, 2012; accepted: March 23, 2014. Published online in *Articles in Advance* June 20, 2014.

1. Introduction

Pricing plays a fundamental role in managing demand in many industries and often in settings in which inventory replenishment decisions are infrequent. In this context, price-based revenue management arises naturally as a means for adjusting to the demand's intrinsic stochastic variability, which arises even when the fundamental relationship between price and demand, typically referred to as *demand function*, is stable and known. In many settings, however, unexpected disruptions to the landscape may result in shifts in this relationship. Consider, for example, the sales of NFL jersey replicas during playoff season, where sales of the jerseys of winning teams increase over time, until the moment the team loses and sales come to an almost abrupt halt (Parsons 2004). Similarly, the introduction of Apple's iPad mini lead to persistent speculation about the date of its release beginning in April 2012; yet it was officially released in October. In addition, despite predictions of a negative effect on sales of Amazon's Kindle Fire, those sales allegedly rose to record highs in the weeks following the announcement (AllThingsD.com 2012). More generally, shifts in demand might also arise when new products are introduced and old ones are phased out. In such situations, the release timing of a new product as well as the customer response to it might be uncertain, and a key question is how to price the remaining inventory of the old product during this transition phase (Li and Graves 2012).

As illustrated by these examples, although infrequent, shifts in demand might impact revenues significantly, and while in some settings there might be a general sense of their possible timing and of the resulting new demand environment, in other settings these aspects might be hard to predict. This paper is motivated by the challenges that arise in situations that share some of the salient features above. We aim to understand how the prospect of a change in the demand environment affects a firm's pricing decisions, the main factors influencing such decisions, and how these prescriptions vary according to different representations of future demand uncertainty.

To this end, we study a family of stylized prototypical problems with a finite selling horizon and fixed initial inventory in which the retailer seeks to maximize the cumulative revenue by adjusting prices in a dynamic fashion. Initially operating in a known demand environment, the decision maker anticipates that the demand function might change in the future; however, neither the time nor the postchange demand function are initially known. Our analysis is centered around the setting in which the retailer knows the joint distribution of the time and postchange demand function and maximizes the expected cumulative revenues collected over the sales horizon.

Main Results and Contributions. The first contribution lies in the formulation of the problem and the ability to crisply isolate and identify the underlying pricing structure that may be induced by future changes in demand. Our analysis is based on a semideterministic formulation of the retailer's problem that captures the key trade-offs associated with the problem of anticipating potential future changes in demand. It enables us to isolate the structure induced on dynamic pricing strategies from various other potential sources of dynamic price adjustments (such as experimentation and learning or reactions to stochastic fluctuations of demand). Our formulation can be seen as a relaxation of an associated stochastic problem in which real-time detection and learning of new demand conditions must also be performed. In particular, one may build on the policies obtained to construct near-optimal policies in a general stochastic environment when the time horizon and the inventory are large.

The second contribution lies in the characterization of the structure of optimal dynamic pricing strategies. When the time and the postchange demand function are independent, we establish that an optimal pricing policy will always follow a monotone path up to the change in demand. Pricing policies might be nonincreasing or nondecreasing. Intuitively, loosely speaking, when the demand conditions are more "attractive" postchange, the decision maker will attempt to save inventory for postchange periods compared with a setting where no change would occur and, hence, price higher than in such a setting. As time passes and while no change occurs, the decision maker saves fewer units for postchange periods and (weakly) decreases prices. We formalize the notion of a comparison between pre- and postchange demand conditions. We show that it is possible to compute a simple index on which one may, in some sense, declare whether pre- or postchange demand conditions are more attractive, and the sign of this index determines the monotonicity direction of the optimal pricing policy. The index depends on both pre- and postchange demand conditions but also on the initial inventory level, and it can be computed in terms of model primitives and without any optimization. Inventory level is a key input in quantifying the attractiveness of postchange demand conditions. From a managerial perspective, the results establish that monotone policies (up to a change) arise naturally in these settings. Furthermore, the comparison alluded to above highlights that it is not possible to look at forecasting of future demand conditions in isolation to decide on the pricing strategy structure (nonincreasing or nondecreasing), but it is necessary to integrate those with the operational constraints and objective.

The final contribution consists of analyzing the impact of the inputs of the model on the optimal pricing policy and its structure. We first characterize the impact of the marginal distributions of the time of change and the postchange demand function as well of the initial inventory level. Furthermore, at a high level, from both practical and theoretical perspectives, there is always a question of how to model uncertainty and to what extent the insights derived are a result of the modeling choice associated with the uncertainty. To address this, we analyze settings in which the retailer does not have access to the marginal distribution of either the time of change or the postchange demand function (or both). To do so, we approach the problem through a minimax regret formulation, assuming that the retailer anticipates an adversarial realization of the aspect of change for which no distributional information is available. The retailer attempts to minimize the worst-case difference between the revenues of an oracle with access to the realization of such aspect of change (but no a priori access to the other) and his or her performance. When the marginal distribution of the time of change is unknown, then an optimal policy is still monotone up to a change. In particular, we establish that the firm's problem may be reduced to one in which the firm has a "worst" prior on the time of change. Hence, an index can again be computed upfront, without any optimization, to determine the type of pricing strategy (nonincreasing or nondecreasing) that one ought to follow. When the marginal distribution of the postchange demand function is unknown, then an optimal policy is still monotone up to a change. However, the monotonicity direction of pricing policies may not be predicted via the computation of a simple index upfront. The comparison of pre- and postchange demand conditions now may require to solve an optimization problem.

Broadly speaking, our work contributes to the analysis of pricing strategies in nonstationary environments. Such environments are prevalent in practice and the literature pertaining to them is relatively scarce (see further discussion on §2).

The Remainder of the Paper. Section 2 reviews related work. Section 3 formulates the problem. Section 4 presents the main results. Section 5 considers extensions to other forms of uncertainty. Section 6 presents concluding remarks. Selected proofs and supporting materials are relegated to Appendices A and B. Additional proofs can be found in the online companion to this paper (available as supplemental material at http://dx.doi.org/10.1287/msom.2014.0489).

2. Literature Review

Dynamic pricing has spread to a variety of industries including airlines, hotels, and retailing; see Talluri and van Ryzin (2005) for a comprehensive overview of the practice of revenue management. A prototypical model is that of Gallego and van Ryzin (1994), where, facing a finite horizon with no replenishment opportunity, the seller maximizes cumulative revenues through dynamic pricing. Structural properties of optimal pricing policies have been established in the literature (see e.g., Bitran and Mondschein 1997), and these mainly characterize the role of pricing in adjusting to stochastic variability of demand.

Several revenue management studies have departed from the assumptions of known and stationary demand models but few lift both of them simultaneously. For the case of time-homogeneous demand settings, there is a growing number of studies that analyze heuristics for jointly learning and pricing and that provide fundamental lower bounds on performance. Araman and Caldentey (2011) provide a review of some of the approaches taken in the literature, including parametric and nonparametric ones (see also Wang et al. 2014 and the references therein for a recent study). In the absence of capacity constraints, Broder and Rusmevichientong (2012), den Boer and Zwart (2014), and Harrison et al. (2014) study the balance between exploration and exploitation.

The studies on nonstationary demand environments are much less common. Gallego and van Ryzin (1997) study dynamic pricing in a changing demand environment when the temporal evolution of the demand model is *known* in advance. In this setting, Zhao and Zheng (2000) establish a sufficient condition for the monotonicity of a pricing policy. Focusing on the timing of periodic pricing policies, Netessine (2006) derives conditions for uniqueness and monotonicity of the optimal pricing policy in a deterministic but nonstationary environment. Cao et al. (2012) explore structural properties of the optimal policy when rewards are discounted. In a related study, Levina et al. (2009) consider real-time demand learning in a dynamic pricing formulation in which strategic consumer behavior drives nonstationarity of demand. In general, uncertainty in the timing of a change brings forward the need for continuous tracking and monitoring. In the absence of capacity constraints, Keller and Rady (1999) and Besbes and Zeevi (2011) study such a problem, highlighting when the need above is most present.

Few studies consider the combination of temporal uncertainty in demand in conjunction with capacity constraints. Besbes and Maglaras (2009) formulate the dynamic pricing problem of a revenue-maximizing make-to-order manufacturer operating in an environment in which an unobservable market size varies stochastically over time. They devise near-optimal policies using a stochastic fluid model approximation. In a related study and closer to our setting, Chen and Farias (2013) analyze a stochastic fluid approximation of a variant of the model in Gallego and van Ryzin (1994), where the authors consider a Gaussian market size process. The authors propose heuristic reoptimized fixed price policies for the case in which the market size process is observed. The latter two studies focus on heuristic prescriptions to track (explicitly or implicitly) demand, when demand changes over time in a "continuous" manner. These studies restrict attention to the evolution of the market size parameter and do not consider, e.g., changes in price sensitivity.

The problem we study is also related to the so-called capacity booking problem. In its simplest instance, a decision maker with finite capacity sees two classes of customers arriving sequentially over time, each with a given (and known) willingness to pay distribution. Fares are fixed and the decision maker needs to decide how many seats to reserve for the high fare class. The solution of this problem is given by the classical Littlewood rule (Littlewood 1972). The current paper can be seen as a capacity-booking problem in which the decision maker has dynamic pricing capabilities, and neither the arrival time nor the willingness to pay distribution of the second type are known in advance. When viewed through that lens, a related study is that of Ball and Queyranne (2009) who analyze the capacity-booking problem when the order of arrivals may be generated in an adversarial fashion. The present formulation is less conservative with respect to the order of arrivals and gives the decision maker pricing flexibility.

3. Problem Formulation

Model Primitives. We consider the pricing problem faced by a retailer offering a single product to a stream of price-sensitive customers over a sales horizon with T periods. The retailer, initially endowed with x units of the product, selects the prices to charge consumers each time period. In particular, we let p_t denote the price charged during period $t \in \mathcal{T} := \{1, \ldots, T\}$. We assume that, under the initial demand environment, such a price results in a deterministic demand $d(p_t)$ during period t. We assume that the retailer can sell fractional quantities of the product. The retailer is familiar with the initial demand environment, i.e., he or she knows $d(\cdot)$, and anticipates that such a function will remain unchanged prior to some unknown period $\tau \in \mathcal{T}^+ := \mathcal{T} \cup \{T+1\}$. For convenience, we let $t = T+1$ denote a fictional final period reflecting that no change occurred. After that, the retailer expects to face a new postchange demand function that will remain constant for the rest of the horizon (i.e., from period $t = \tau$ to $t = T$).

The new demand function is *not known* a priori, but the retailer knows that it belongs to a class with elements that are indexed by $k \in \mathcal{K} := \{1, \ldots, K\}$ with $K < \infty$, and we use $d^k(p_t)$ to denote the demand during period t under the kth postchange demand function $d^k(\cdot)$. We let θ denote the realization of the index of the new demand function. We assume that both τ and θ are revealed to the retailer at time $t=\tau$. As is customary, we assume that $d(0)\le\bar{\lambda}$ and $d(\bar{p})=0$, and that $d^k(0)\le\bar{\lambda}$ and $d^k(\bar{p})=0$ for $k\in\mathcal{K}$, for some finite constant $\bar{\lambda}>0$ and price $\bar{p}>0$.

Remark 1 (Connection to Stochastic Formulation). The formulation above can be seen as a semideterministic relaxation of a model in which the arrival process of customers and their willingness to pay are stochastic, and in which, in addition, both τ and θ are not revealed at time $t=\tau$. In such an environment, a pricing policy needs to balance multiple roles ranging from detecting a change in demand with learning a new demand environment and properly spreading inventory consumption across the horizon to account for the possibility of future changes. These come in addition to the classical role of adjusting to the inherent variability of stochastic demand. The formulation we adopt has the key feature that it *preserves the order of information revelation* in the problem, in the sense that the decision maker still does not know τ nor θ up until the beginning of period $t=\tau$. Thus, the key uncertainty and trade-offs of the problem associated with pricing with the prospect of a change in demand are maintained. In this regard, one can build on existing results in the literature (see, e.g., Besbes and Zeevi 2011) to show that it is possible to use the prescriptions obtained from the semideterministic relaxation to construct near-optimal policies in a general stochastic environment as described above, where detection and learning have to be performed. In Appendix B, we describe the main components of such a prescription for the setting analyzed in §4 as well as guarantees of asymptotic optimality one may obtain under a suitably chosen regime.

Decision-Maker's Objective. We assume that the retailer seeks to maximize the cumulative revenues collected over the finite selling horizon by adjusting prices in a dynamic fashion. For $\lambda\in[0,\bar{\lambda}]$, let $p(\lambda)$ and $p^k(\lambda)$ denote the maximum prices that induce a demand of λ units during a time period under prechange and (kth) postchange demand conditions respectively, i.e.,

$$p(\lambda):=\sup\{p\ge 0\colon\ d(p)=\lambda\},$$
$$p^k(\lambda):=\sup\{p\ge 0\colon\ d^k(p)=\lambda\},\quad \forall\, k\in\mathcal{K}.$$

Similarly, for $\lambda\in[0,\bar{\lambda}]$, define $r(\lambda)$ and $r^k(\lambda)$ as the revenue collected on a period under prechange and (k-th) postchange demand environments, respectively, i.e., $r(\lambda):=\lambda p(\lambda)$, and $r^k(\lambda)=\lambda p^k(\lambda)$ for $k\in\mathcal{K}$. We assume that $r(\cdot)$ and $r^k(\cdot)$ for $k\in\mathcal{K}$ are twice differentiable and strictly concave on $[0,\bar{\lambda}]$. With this, we can express a pricing policy in terms of the demand it induces in each period.

Note that, once τ and θ are realized and contingent on the current inventory position, the retailer faces a stationary demand. For $t\in\mathcal{T}^+$, $k\in\mathcal{K}$, and $y\le x$, let $V^{t,k}(y)$ denote the optimal cumulative revenue generated by a retailer with an initial inventory of y units over a sales horizon with $(T-t+1)$ periods when facing demand function $d^k(\cdot)$. That is,

$$V^{t,k}(y):=\max\left\{\sum_{s=t}^{T} r^k(\lambda_s)\colon\ \sum_{s=t}^{T}\lambda_s\le y,\ \lambda_s\in[0,\bar{\lambda}]\right.$$
$$\left. s\in\{t,\dots,T\}\right\},\quad \forall\, k\in\mathcal{K},\ t\in\mathcal{T}^+,$$

where we have stated the revenue maximization problem directly in terms of the sales induced under a pricing policy. Let λ^* and λ^{*k} for $k\in\mathcal{K}$ denote the unique maximizers of $r(\cdot)$ and $r^k(\cdot)$, respectively, i.e.,

$$\lambda^*:=\arg\max\{r(\lambda)\colon\ \lambda\in[0,\bar{\lambda}]\},$$
$$\lambda^{*k}:=\arg\max\{r^k(\lambda)\colon\ \lambda\in[0,\bar{\lambda}]\},\quad \forall\, k\in\mathcal{K}.$$

Note that, under our assumptions, $\lambda^{*k}<\bar{\lambda}$ and $\lambda^*<\bar{\lambda}$, and thus $(r^k)'(\lambda^{*k})=r'(\lambda^*)=0$, for all $k\in\mathcal{K}$.

When the change occurs at time t and the postchange environment is given by $d^k(\cdot)$, it is possible to establish (see Gallego and van Ryzin 1994, Proposition 2) that the optimal revenue generated after the demand change is given by

$$V^{t,k}(y)=(T-t+1)r^k(\min\{\lambda^{*k},y/(T-t+1)\}),\tag{1}$$

for $t\le T$, and $V^{T+1,k}(y)=0$, for $k\in\mathcal{K}$. An optimal postchange sales rate is constant and given by

$$\lambda_s=\min\{\lambda^{*k},y/(T-t+1)\},\quad \forall\, s\ge t.\tag{2}$$

The paper is centered around the case in which both τ and θ are represented as random variables with a known distribution for the pair, and the retailer maximizes the expected cumulative revenue over the selling horizon. (We return to consider deviations from such a representation of uncertainty in §5.) Note that any optimal policy should maximize the postchange revenue, so we can restrict attention to policies that induce the sales in (2) after τ and θ are revealed.

Let $J^t(y)$ denote the maximum expected revenue collected by a retailer between periods t and T when it is known that demand has not changed up to and including period t, and there are y units in inventory. The following recursion characterizes $J^t(y)$:

$$\begin{aligned}J^t(y)=\max\{&r(\lambda_t)+J^{t+1}(y-\lambda_t)\,\mathbb{P}\{\tau>t+1\mid\tau>t\}\\&+\mathbb{E}\{V^{t+1,\theta}(y-\lambda_t)\mid\tau=t+1\}\\&\cdot\mathbb{P}\{\tau=t+1\mid\tau>t\}\colon 0\le\lambda_t\le y\},\qquad(3)\\&\qquad\forall\, t\in\mathcal{T},\ y\le x,\\J^{T+1}(y)=0,&\quad\forall\, y\le x.\end{aligned}$$

In the above, note that the decision maker is updating his belief about the time of change as time progresses. The optimal value of the problem is given by

$$J(x) := J^1(x)\,\mathbb{P}\{\tau > 1\} + \mathbb{E}\{V^{1,\theta}(x) \mid \tau = 1\}\,\mathbb{P}\{\tau = 1\}. \quad (4)$$

We remark that in the absence of the prospect of a change (i.e., if τ is known to be equal to $T+1$), the formulation above can be seen as a deterministic discrete-time version of the classical one in Gallego and van Ryzin (1994).

4. Main Results

Reformulating the Problem. Let $V^t(y)$ denote the expected optimal revenue collected by a retailer with initial inventory of y units over a horizon of length $(T-t+1)$ when the demand function remains constant during the entire horizon, but it is initially chosen randomly from the postchange class conditional on the event $\tau = t$. That is, for $t \in \mathcal{T}^+$ and $y \geq 0$,

$$V^t(y) := \mathbb{E}\{V^{t,\theta}(y) \mid \tau = t\}. \quad (5)$$

One can show that $V^t(\cdot)$ is concave and differentiable (see Lemma 1 in the appendix).

From (3), one sees that the optimal policy can be characterized by the sales rates that should be applied conditional on no change. This is so because, postchange, the value to go and the associated optimal sales rate are known (see (1) and (2)). This observation implies that any admissible policy can be mapped into a unique vector $\lambda := \{\lambda_t,\ t \in \mathcal{T}\}$ of induced *prechange* sales, where λ_t denotes the sales during period t induced by p_t, conditional on $\tau > t$ (i.e., before a change in demand is observed). The vector λ associated with an optimal pricing policy *depends on the demand change only through its distribution and not its realization*, because it only prescribes sales induced *in the absence of a change in demand*, understanding that were a change to occur it will be optimal to subsequently induce the sales prescribed by (2).

From (4) and the iterative use of (3), one may use backward induction to rewrite the firm's problem as

$$J(x) := \max\left\{\sum_{t\in\mathcal{T}}\left(r(\lambda_t)\,\mathbb{P}\{\tau > t\} + V^t\left(x - \sum_{s=1}^{t-1}\lambda_s\right)\mathbb{P}\{\tau = t\}\right):\ \lambda \in \mathcal{L}\right\}, \quad (6)$$

where $\mathcal{L}$ denotes the set of feasible sales vectors, i.e.,

$$\mathcal{L} := \left\{\lambda\colon\ \lambda_t \geq 0\ t \in \mathcal{T},\ \sum_{t\in\mathcal{T}}\lambda_t \leq x\right\}.$$

Note that it is possible to show that $\lambda_t \leq \lambda^*$ for all $t \in \mathcal{T}$ in an optimal solution, and hence, it is not necessary to impose $\lambda_t \leq \bar{\lambda}$ for $t \in \mathcal{T}$. This formulation maximizes a continuously differentiable concave function subject to affine constraints (inventory constraint and nonnegativity of sales), so there exists an optimal vector of prechange sales rates and the Karush–Kuhn–Tucker conditions (see, e.g., Boyd and Vandenberghe 2004) are necessary and sufficient to characterize this vector.

Remark 2 (Nature of Feasible Policies). Formulation (6) may appear at first as restricting attention to open-loop policies in the original problem. This is not the case. The problem above solves for the sequence of sales rates λ_t one should apply conditional on not detecting a change by time t. It is sufficient to do so because, once a change has been detected, the policy and value-to-go are fully characterized.

4.1. Optimal Policy for Independent Time of Change and Postchange Demand

An important case we analyze is that in which the time and the postchange demand realizations are independent, as formalized in the following assumption.

Assumption 1. *The time of change τ and the postchange demand index θ are such that*

$$\mathbb{P}\{\theta = k \mid \tau = t\} = \mathbb{P}\{\theta = k \mid \tau = 1\}, \quad \forall k \in \mathcal{K},\ t \in \mathcal{T}^+.$$

Assumption 1 implies independence through "stationarity" of the conditional distribution of θ. In §4.2, we return to relax this assumption.

Our first result characterizes the structure of optimal policies.

Theorem 1 (Monotonicity of the Optimal Strategy). *Suppose that Assumption 1 holds, and let λ denote an optimal solution to (6). Then $\{\lambda_t\colon 1 \leq t \leq T\}$ is monotonic. In addition, if $\lambda_1 \geq x/T$, then λ_t is nonincreasing in t. Conversely, if $\lambda_1 \leq x/T$, then λ_t is nondecreasing in t.*

The result establishes that, under Assumption 1, the possibility of a future change in demand will lead a firm to always use a monotone price path up to a change, independently of the marginal distributions of the time of change and the postchange demand.

If it were known that no demand change would take place (if almost certainly $\tau = T+1$), the optimal sales rate would be $\min\{\lambda^*, x/T\}$. We refer to x/T as the "depletion rate" because it would lead the firm to exactly deplete its inventory by the end of period T. In proving Theorem 1, we show that the optimal prechange sales sequence λ always stays either above or below the updated depletion rates $((x - \sum_{s=1}^{t-1}\lambda_s)/(T-t)$, $t \in \mathcal{T})$ and that these depletion rates are also always monotone (see Lemma 3 in the appendix).

Theorem 1 provides a characterization of the monotonicity direction that depends on the optimal sales rate in period one, which requires computing the optimal solution.

Next, we show that it is possible to determine the monotonicity direction of the optimal sales rates based on the model primitives and without computing the optimal solution. We do so by introducing a proper method to compare pre- and postchange demand environments upfront at time 0.

Recall that when no potential change in demand is possible, optimal sales are constant and given by $\lambda_t = \min\{\lambda^*, x/T\}$, $t \in \mathcal{T}$. Define ρ as the difference, at the beginning of the horizon, between the marginal value of a unit of inventory absent demand change and that expected under new demand conditions. That is,

$$\rho := r'(\min\{\lambda^*, x/T\}) - (V^1)'(x). \tag{7}$$

In the proof of Lemma 1 (available in the online companion), we establish that

$$(V^1)'(x) = \mathbb{E}\{(r^\theta)'(\min\{\lambda^{*\theta}, x/T\})\}.$$

Hence, ρ may be computed upfront without any optimization. Intuitively speaking, a nonnegative ρ implies that the marginal value of a unit in inventory under current demand conditions is higher than that under future conditions (in expectation). This suggests one should impose an initial sales rate above the depletion rate if one is to balance current and future instantaneous marginal revenues. Similarly, a nonpositive ρ suggests that one should deplete inventory at a slower pace than at the optimal rate when no change can take place. (Note that the latter case always takes place when $\lambda^* < x/T$.) The next proposition establishes that ρ is indeed the key driver for the monotonicity direction of an optimal policy.

Proposition 1 (Monotonicity Direction). *Suppose that Assumption* 1 *holds. Then, the sequence of optimal sales rates is nonincreasing if $\rho \geq 0$ and nondecreasing if $\rho \leq 0$.*

We establish the result by showing that the optimal sales rate in period 1, λ_1, is greater than or equal to the depletion rate x/T if and only if $\rho \geq 0$ and use Theorem 1 to conclude.

Based on the result, one may interpret the sign of ρ as indicating whether future demand conditions are more or less favorable than current ones. A positive value of ρ indicates less favorable future demand conditions and inventory consumption should take place at a faster rate than if there would be no possibility of a change. Remarkably, if one was to recompute ρ after each period (provided no change occurs), its sign would remain the same.

Proposition 1 implies that it is possible to evaluate the type of prechange pricing policy (nonincreasing versus nondecreasing) upfront through the computation of ρ, which does not require solving for the optimal prechange sale rates. The resulting policy type is independent of the retailer's belief about the timing of the change and only depends the belief about the postchange demand function. In this regard, our results indicate that to check whether current pricing policies are consistent with the prospect of a change, one should pay attention primarily to anticipating the postchange demand function rather than to the timing. Finally, the structure of the index ρ highlights the key inputs to compare pre- and postchange demand. In particular, the sign of ρ depends on the initial inventory, so demand environments (and forecasts) should not be analyzed in isolation but in conjunction with operational constraints.

For illustrative purposes, one can revisit the example in the introduction about sales of NFL jerseys during playoff season, where demand for jerseys of a team comes to a halt after elimination. Putting this in our context, one may envision a setting with periods of increasing length (so as to model increasing pre-elimination demand), where elimination triggers a drop in demand. In such a case, one would expect to have $\rho \geq 0$, and our results indicate that optimal pricing takes the form of (weak) price increases after each game up to elimination, independent of a team's chances of playing the Super Bowl. As for the introduction of the iPad mini, had Amazon anticipated a favorable future demand environment (to the extent that this corresponds to a nonpositive value of ρ), our results suggest that using decreasing prices prior to the introduction of the iPad mini would have been optimal. Such expectations might arise, for example, when an increase of the market for tablets resulting from an advertising campaign surpasses the effect of tighter competition.

4.1.1. Illustrative Examples.

Example 1 (The Case of Unknown Price Sensitivity). Consider the case in which the postchange demand function is given by

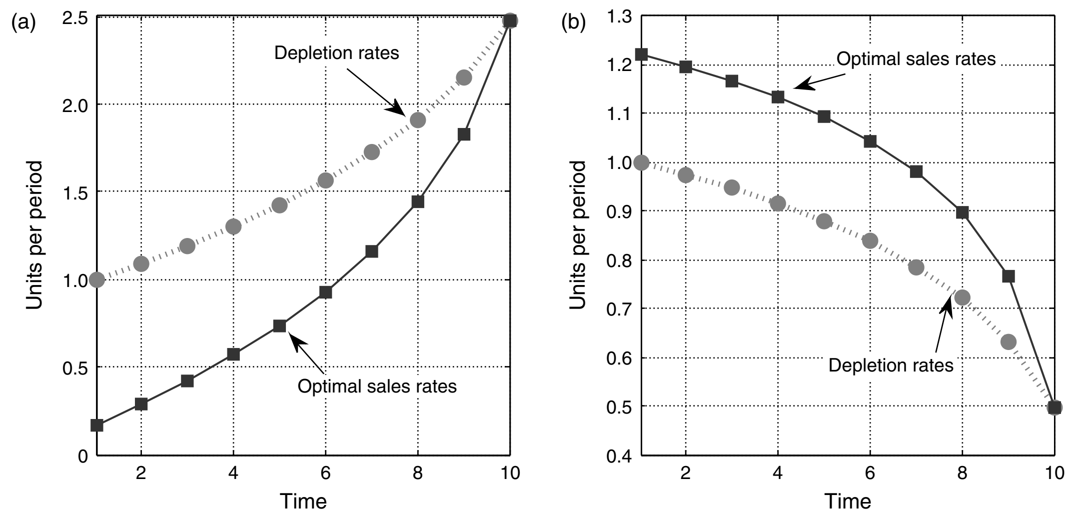
$$d^k(p) = f((\alpha - \beta^k p)^+), \quad \forall p \in \mathbb{R}_+,\ k \in \mathcal{K},$$

where $f(\cdot) \in [0, 1]$ is an arbitrary increasing continuously differentiable function, $\beta^k > 0$ for all $k \in \mathcal{K}$, and z^+ denotes $\max\{z, 0\}$, and where $d(p) = f((\alpha - p)^+)$. We assume that $f(0) = 0$, that $f(a) = \bar{\lambda}$, and that $r(\lambda)$ admits a unique maximizer $\lambda^* > x/T$. One can check that $r^k(\lambda) = r(\lambda)/\beta^k$ for all $k \in \mathcal{K}$ and $\lambda \in [0, \bar{\lambda}]$. Using Proposition 1, one can check that $\rho \leq 0$ if and only if

$$\mathbb{E}\{(\beta^\theta)^{-1}\} \leq 1. \tag{8}$$

The above implies that one can have both increasing or decreasing optimal sales sequences even when one constrains price sensitivity, for example, to increase on average (i.e., $\mathbb{E}\{\beta^\theta\} \geq 1$) in the event of a change in

Figure 1 **Prechange Monotonicity**



Notes. Graphs (a) and (b) illustrate optimal prechange sales rates for instances where $\mathbb{E}\{(\beta^\theta)^{-1}\}$ is greater and lower than 1, respectively, in the context of Example 1. The solid and dotted lines denote the prechange sales rates and inventory depletion rates, respectively.

demand conditions. Indeed, in such a case, the decision maker might still want to sell in a slow fashion initially and save units for postchange demand. Figure 1 illustrates the case $f(x) = x$ and $\alpha = \bar{\lambda} = 10$. The instances depicted in Figure 1 consider $x = 10$ and $T = 10$, with τ uniformly distributed in $\mathcal{T}^+$ and independent of θ. There, $K = 2$, $\beta^1 = 2.5$, and $\beta^2 = 0.5$. Graph (a) corresponds to the setting $\mathbb{P}\{\theta = 1\} = 1 - \mathbb{P}\{\theta = 2\} = 0.3$, and graph (b) corresponds to $\mathbb{P}\{\theta = 2\} = 1 - \mathbb{P}\{\theta = 1\} = 0.3$. The solid lines indicate the prechange optimal sales sequence, for both settings. The dotted lines depict the prechange depletion sales volume at each decision point. While both instances are such that $\mathbb{E}\{\beta^\theta\} \geq 1$, the one in panel (a) is such that $\mathbb{E}\{(\beta^\theta)^{-1}\} > 1$, and the nondecreasing pattern of the optimal sales sequence is consistent with $\rho \leq 0$. The instance in panel (b), on the other hand, is such that $\mathbb{E}\{(\beta^\theta)^{-1}\} < 1$ and results in a decreasing optimal sales sequence. In both cases, depletion sales volumes are either always above or below optimal volumes.

Example 2 (Common Demand Functions). Table 1 provides the value of ρ for some common demand functions as presented in, e.g., Talluri and van Ryzin (2005, Chap. 7.3.3), under the assumption that pre- and postchange demand functions belong to the same parametric family of functions. Observe that the setting in Example 1 with $f(x) = x$ corresponds to a special case of linear demand. Note that when $\alpha = \alpha^k$ for all $k \in \mathcal{K}$, and $\beta = 1$, the sign of ρ for linear, log-linear and logit demand also follows from checking the condition in (8).

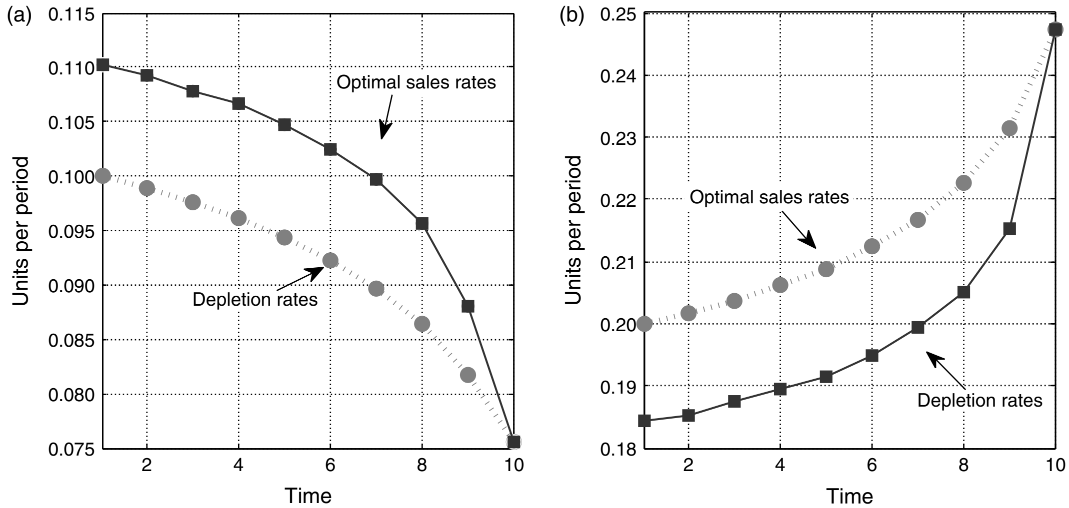
More generally, we observe that monotonicity direction (driven by the sign of ρ) not only depends on the distribution of θ, but also on the initial level of inventory. To illustrate this point, consider a setting with log-linear demand where $\beta = \beta^k = 1$ for all $k \in \mathcal{K}$ (this might correspond to a setting with unknown multiplicative factor/market size), $T = 10$, τ uniformly distributed in $\mathcal{T}^+$, $K = 2$ with $\alpha = 1$, $\alpha^1 = 2.5$, $\alpha^2 = 0.1$, and $\mathbb{P}\{\theta = 2\} = \mathbb{P}\{\theta = 1\} = 0.5$. Figure 2 illustrates the optimal prechange sales sequence for different values of x: graph (a) corresponds to $x = 1$ (for which $\rho > 0$), and graph (b) corresponds to $x = 2$ (for which $\rho < 0$). One can show that, in this setting, ρ changes sign at $x \approx 1.47$. The solid lines indicate the prechange sales sequence, for both settings. The dotted lines depict the optimal prechange depletion sales rate at each decision point.

Table 1 **Characterization of ρ for Common Demand Functions Presented in Talluri and van Ryzin (2005)**

	$d(p)$	$r(\lambda)$	ρ
Linear	$\alpha - \beta p$	$\lambda(\alpha - \lambda)/\beta$	$1/\beta(\alpha - 2(x/T))^+ - \mathbb{E}\{1/\beta^\theta(\alpha^\theta - 2(x/T))^+\}$
Log-linear	$\alpha e^{-\beta p}$	$\lambda/\beta \ln(\alpha/\lambda)$	$1/\beta(\ln(\alpha T/x) - 1)^+ - \mathbb{E}\{1/\beta^\theta(\ln(\alpha^\theta T/x) - 1)^+\}$
Constant elasticity	$\alpha p^{-\beta}$	$\alpha^{1/\beta}\lambda^{1-1/\beta}$	$(1 - 1/\beta)(\alpha T/x)^{1/\beta} - \mathbb{E}\{(1 - 1/\beta^\theta)(\alpha^\theta T/x)^{1/\beta^\theta}\}$
Logit	$e^{-\beta p}/(1 + e^{-\beta p})$	$\lambda/\beta \ln(\lambda^{-1} - 1)$	$(1/\beta - \mathbb{E}\{1/\beta^\theta\})(\ln(T/x - 1) - T/(x - T))^+$

Note. In the case of constant elasticity, it is assumed that $\beta_\theta < 1$ almost surely.

Figure 2 **Impact of Inventory on Pricing Structure**



Notes. Graphs (a) and (b) illustrate prechange sales volumes for instances where $x = 1$ and $x = 2$, respectively, in the context of Example 2. The solid and dotted lines denote the prechange sale and inventory depletion sale volumes, respectively.

4.1.2. Impact of Model Primitives on the Optimal Policy. *Impact of the Distribution of the Time of Change.* From Proposition 1, it follows that the monotonicity direction of the optimal sales sequence does not depend on the retailers' belief regarding the time of change. However, changes to the marginal distribution of τ will in general result in changes in prechange sales rates and inventory levels. Consider, for example, a situation in which current conditions compare favorably with the future (i.e., $\rho \geq 0$). If the retailer perceives a change might occur "sooner" than originally anticipated, then one should expect inventory to be depleted at a faster rate. This suggests that if one is to consider a stochastically larger ($\geq_{st}$) alternative marginal distribution for τ (leaving that of θ fixed) and $\rho \geq 0$, then one should observe higher prechange inventory levels. Similarly, when $\rho \leq 0$ one might expect to observe lower prechange inventories. We show this is indeed the case provided that τ remains stochastically larger *throughout the whole sales horizon*.

For a distribution $\mathbb{Q}$ of (τ, θ), let $\mathbb{Q}_\tau$ and $\mathbb{Q}_\theta$ denote the marginal distributions of τ and θ, respectively. For a pair of distributions $\mathbb{Q}$ and $\mathbb{P}$, one has that $\mathbb{Q}_\tau$ is larger than $\mathbb{P}_\tau$ in the hazard rate sense ($\mathbb{Q}_\tau \geq_{hr} \mathbb{P}_\tau$) if (see, e.g., Shaked and Shanthikumar 2007)

$$\frac{\mathbb{Q}_\tau\{\tau = t\}}{\mathbb{Q}_\tau\{\tau > t\}} \leq \frac{\mathbb{P}_\tau\{\tau = t\}}{\mathbb{P}_\tau\{\tau > t\}}, \quad \forall\, t \in \mathcal{T}.$$

Note that $\mathbb{Q}_\tau \geq_{hr} \mathbb{P}_\tau$ can be interpreted as $\mathbb{Q}_\tau(\cdot \mid \tau > t) \geq_{st} \mathbb{P}_\tau(\cdot \mid \tau > t)$ for all $t \in \mathcal{T}$, i.e., the hazard rate ordering preserves stochastic dominance (conditional on a change not occurring) throughout the horizon.

Proposition 2. *Suppose that Assumption* 1 *holds for both* $\mathbb{P}$ *and* $\mathbb{Q}$, *that* $\mathbb{Q}_\theta = \mathbb{P}_\theta$, *and that* $\mathbb{Q}_\tau \geq_{hr} \mathbb{P}_\tau$. *Let* y *and* y' *denote prechange optimal inventory paths under* $\mathbb{P}$ *and* $\mathbb{Q}$, *respectively. If* $\rho \geq (\leq)\ 0$, *then* $y'_t \geq (\leq)\ y_t$ *for all* $t \in \mathcal{T}$.

The condition $\mathbb{Q}_\tau \geq_{hr} \mathbb{P}_\tau$ not only formalizes the notion that a change in demand is expected later but also that such a belief is consistent across the whole horizon, so the intuition alluded to above applies consistently through time. Verifying the conditions of the result does not require any optimization and is only based on model primitives and the computation of the index ρ.

Impact of the Distribution of the Postchange Demand. Consider now a distribution $\mathbb{Q}$ that preserves the marginal distribution of τ. If future demand conditions are "better" under $\mathbb{Q}$, then prechange sales should be lower to save more units for the postchange environment. Postchange demand functions are not ordered, so there is no notion of stochastic dominance that makes future conditions more favorable. To compare distributions, for a distribution $\mathbb{P}$ define

$$V^t_{\mathbb{P}}(\cdot) := \mathbb{E}_{\mathbb{P}}\{V^{t,\theta}(\cdot) \mid \tau = t\}, \quad \forall\, t \in \mathcal{T},$$

and consider $\mathbb{Q}$ and $\mathbb{P}$ such that $V^t_{\mathbb{Q}}(\cdot) \geq V^t_{\mathbb{P}}(\cdot)$. From above, one would expect to observe higher prechange inventories under $\mathbb{Q}$ relative to $\mathbb{P}$. The next result formalizes this.

Proposition 3. *Suppose that Assumption* 1 *holds for* $\mathbb{Q}$ *and* $\mathbb{P}$ *and that these are such that* $\mathbb{Q}_\tau = \mathbb{P}_\tau$, *and let* y *and* y' *denote prechange optimal inventory paths under* $\mathbb{P}$ *and* $\mathbb{Q}$, *respectively. If* $(V^1_{\mathbb{Q}})' \geq (V^1_{\mathbb{P}})'$ *point wise, then* $y'_t \geq y_t$ *for all* $t \in \mathcal{T}$.

For the common demand functions in Table 1, when $\alpha^k = \alpha$ for all $k \in \mathcal{K}$, the condition in Proposition 3 is equivalent to $\mathbb{E}_{\mathbb{Q}}\{(\beta^\theta)^{-1}\} \geq \mathbb{E}_{\mathbb{P}}\{(\beta^\theta)^{-1}\}$ for settings with linear, log-linear and logit demand as well as for the setting in Example 1. In such settings, assume w.l.o.g. that $\beta^k \leq \beta^{k+1}$ (one can check that the condition is satisfied when $\mathbb{Q}_\theta \leq_{st} \mathbb{P}_\theta$). Similarly, when $\beta = \beta^k$ for all $k \in \mathcal{K}$ and we assume (w.l.o.g.) that $\alpha^k \leq \alpha^{k+1}$, the condition is satisfied when $\mathbb{P}_\theta \leq_{st} \mathbb{Q}_\theta$ for all common demand functions. More generally, the condition depends on the joint distribution of α^θ and β^θ.

Impact of the Initial Inventory Level. A close look at (7) and Figure 2 reveals that changes in the initial inventory x might in general result in changes in the monotonicity direction. However, it is still possible to characterize the impact of a change in initial inventory on prechange sales rates and inventory levels.

Proposition 4. *Suppose that Assumption 1 holds. Let (λ, y) and (λ', y') denote optimal sales and inventory sequences when the initial inventories are x and x', respectively. If $x \geq x'$, then $(\lambda, y) \geq (\lambda', y')$, component wise.*

4.2. Correlated Time of Change and Postchange Demand

When Assumption 1 fails to hold, optimal sales rates are not necessarily monotonic as the comparison between pre- and postchange demand (the sign of ρ) is not necessarily consistent throughout the selling horizon. This is illustrated in the following example.

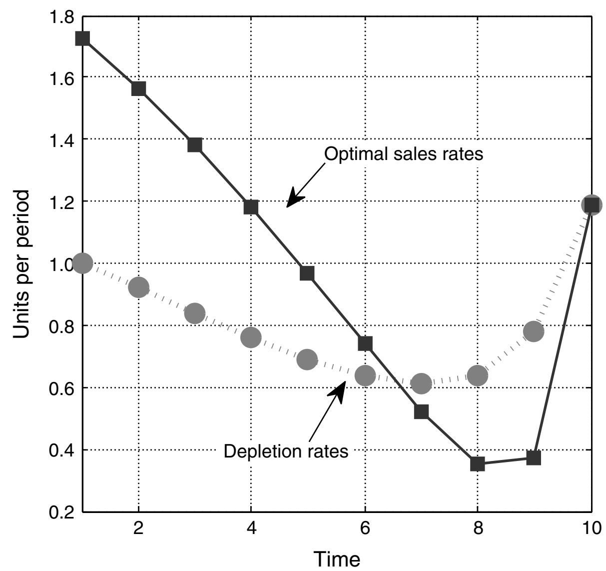
Example 3 (Nonmonotonicity). Consider the setting of Example 1, assuming that $d^k(p) = (10 - \beta^k p)^+$, $p \in \mathbb{R}_+$, $k \in \mathcal{K}$, and setting $x = T = 10$. In addition, suppose that τ is uniformly distributed in $\mathcal{T}^+$ and that $\mathbb{P}$ is such that

$$\mathbb{P}\{\beta^\theta = 1/2 + (T - t + 1)/5 \mid \tau = t\} = 1, \quad \forall\, t \in \mathcal{T}^+.$$

In this case, initially, prechange demand compares favorably with the expected future demand, but the situation (conditional on no change occurring) is gradually reversed throughout the horizon. Figure 3 depicts the prechange sales rates and the corresponding depletion rates in this setup. In this example, optimal sales rates are initially decreasing, when the possibility of a change in demand implies switching to a more elastic demand. When a change in demand implies switching to a less elastic demand, we observe the prechange sales rates increase.

While optimal sales rates need not to be monotonic in general, the above suggests that local monotonicity properties might be derived from those of the continuation functions $\{V^t,\ t \in \mathcal{T}^+\}$. In particular, Example 3 suggests that monotonicity of optimal sales rates will occur when the expectation about future demand gets progressively better or worse with the time of change, relative to an initially worse or better initial demand environment, respectively. With this in mind, consider the following definition.

Figure 3 **Nonmonotonicity with Correlation**



Notes. The graph illustrates the prechange sales rates and corresponding depletion rates in Example 3. The solid and dotted lines denote the prechange sales and inventory depletion rates, respectively.

Definition 1 (Monotone Expected Future Marginal Profit). Expected future marginal profit is said to be nondecreasing (nonincreasing) if, for any fixed $z \in \mathbb{R}_+$, $(V^t)'(z(T-t+1))$ is nondecreasing (nonincreasing) in t.

When τ and θ are independent, one can check that $(V^t)'(z(T-t+1))$ is independent of $t \in \mathcal{T}$ for all $z \in \mathbb{R}_+$ (see the proof of Lemma 2 in the online companion).

One can show that, for the common demand functions in Table 1 (with the exception of the constant elasticity case), expected future marginal profit is nonincreasing (nondecreasing) when $\mathbb{E}\{(\beta^\theta)^{-1} \mid \tau = t\}$ is nonincreasing (nondecreasing) in t and $\alpha = \alpha^k$ for all $k \in \mathcal{K}$, or when $\mathbb{E}\{\alpha^\theta \mid \tau = t\}$ is nonincreasing (nondecreasing) in t and $\beta = \beta^k$ for all $k \in \mathcal{K}$.

Our next result uses this notion of evolution on postchange demand priors to establish sufficient conditions for monotonicity of optimal sales rates.

Theorem 2 (Monotonicity of the Optimal Strategy Continued). *Suppose expected future marginal profit is nondecreasing and that $r'(\lambda) \geq (V^T)'(\lambda)$ for all $\lambda \in [0, \lambda^*]$, then the optimal prechange sales rate sequence is nonincreasing. Conversely, if expected future marginal profit is nonincreasing and $r'(\lambda) \leq (V^T)'(\lambda)$ for all $\lambda \in [0, \lambda^*]$, then the optimal prechange sales rate sequence is nondecreasing.*

When the conditions in Theorem 2 are not satisfied, monotonicity of the optimal sales rates is not guaranteed. It is possible in general to derive sufficient conditions for nonmonotonicity of the optimal prechange rates. For example, if expected future marginal profit is increasing for periods $t \geq 2$ and $r'(\lambda) > (V^T)'(\lambda)$ for all $\lambda \in [0, \lambda^*]$, one may conclude from the theorem that the sequence of prechange sales rates *starting from* $t=2$ is monotone nonincreasing. In addition, if $x<\lambda^*(T-1)$ and $(V^2)'((x-\lambda^*)/(T-1))\geq r'(0)$, then one can show that $\lambda_1<\lambda_2$ and $\lambda_2>\max\{\lambda_t\colon\, t\geq 3\}$. In other words, the sequence of sales rates starting from $t=2$ will necessarily be nonmonotone. In general, if conditions in Theorem 2 hold in a strict fashion for all $t>\hat{t}$ for some $\hat{t}\in\{2,\dots,T-2\}$, but the expected future marginal profit monotonicity condition is suitably violated at time $\hat{t}$, one can show that the resulting optimal sequence of sales rates will necessarily be nonmonotone.

5. Comparison with Other Forms of Uncertainty

This section analyzes the impact of having limited information about the change, and in particular of not having access to a prior distribution on τ or θ (or both), and only knowing the support of those, $\mathcal{T}$ and $\mathcal{K}$. We show how these settings are closely related to the one analyzed in §4 and that structural properties of optimal pricing policies identified there continue to hold.

5.1. Adversarial Selection of the Time of Change

Here, we assume that the retailer knows the marginal distribution of θ conditional on $\tau=t$, for all $t\in\mathcal{T}^+$, but does not know the marginal distribution of τ. Thus, from the retailer's perspective, τ is not a random variable but rather an unknown quantity with a value that lies in $\mathcal{T}^+$.

For a given feasible sales sequence $\lambda\in\mathcal{L}$, let $\tilde{\pi}(\lambda,t)$ denote the expected revenues generated by λ conditional on the event that demand changes at time $\tau=t$. That is,

$$\tilde{\pi}(\lambda,t):=\sum_{s=1}^{t-1}r(\lambda_s)+V^t\left(x-\sum_{s=1}^{t-1}\lambda_s\right).$$

(Note that computing the V^t's only requires knowing the conditional distribution of θ.) For $t\in\mathcal{T}^+$, define $\tilde{D}(t)$ as the maximal revenues attainable by a retailer that knows upfront that $\tau=t$. That is,

$$\tilde{D}(t):=\sup\{\tilde{\pi}(\lambda,t)\colon\,\lambda\in\mathcal{L}\},\quad \forall\, t\in\mathcal{T}^+.$$

We assume that the retailer, who does not possess such a piece of information, aims to minimize the revenue loss relative to this semi-oracle revenue while considering all possible realizations of τ. For that, define $\tilde{R}(x)$ as the minimum revenue loss (across all feasible sales rate vectors) relative to $\tilde{D}(\tau)$, when τ is selected in an adversarial fashion. That is,

$$\tilde{R}(x)=\min\left\{\max\left\{\tilde{D}(t)-\sum_{s=1}^{t-1}r(\lambda_s)-V^t\left(x-\sum_{s=1}^{t-1}\lambda_s\right)\colon\, t\in\mathcal{T}^+\right\}\colon\,\lambda\in\mathcal{L}\right\}.\qquad(9)$$

We refer to $\tilde{R}(x)$ as the retailer's minimax *regret* when the initial inventory is x. This formulation can be seen as a game between the decision maker and "nature," where the former first selects a policy and then the latter counters such a policy by selecting the worst possible time of change. In this regard, the regret should be interpreted carefully because knowledge about the time of change should provide a significant (and in some sense unfair) advantage. In particular, it is in many cases not realistic to expect to match the performance of the oracle. However, it is natural to attempt to get as close as possible to it.

The formulation above minimizes a convex function subject to affine constraints. We next show that one may derive further insights on the structure of an optimal policy by establishing a strong connection between Formulations (6) and (9). For that, consider the following equivalent formulation of (9).

$$\min\ z \qquad(10a)$$

$$\text{s.t.}\ \tilde{D}(t)-\sum_{s=1}^{t-1}r(\lambda_s)-V^t\left(x-\sum_{s=1}^{t-1}\lambda_s\right)\leq z,\quad t\in\mathcal{T}^+,\qquad(10b)$$

$$z\in\mathbb{R},\ \lambda\in\mathcal{L}.\qquad(10c)$$

Proposition 5. *The objective function in Problem* (9) *is convex, and any optimal solution coincides with an optimal solution of Problem* (6) *when*

$$\mathbb{P}\{\tau=t\}=\mu_t,\quad \forall\, t\in\mathcal{T}^+,\qquad(11)$$

where the probability distribution $\mu:=(\mu_t,\ t\in\mathcal{T}^+)$ *is the vector of shadow prices associated with the constraints* (10b).

In the current setting, the retailer cannot solve formulation (6) because the marginal distribution of τ is unknown. Proposition 5 essentially says that an adversarial view about τ is equivalent to assuming a worst-case marginal distribution for τ and solving (6) using such a proxy. The result implies that both Theorems 1 and 2 hold in this setting. (Note that those results are independent of the marginal distribution of τ.) In this context, the next corollary, which we state without proof, is a direct extension of Proposition 1 to the case of adversarial selection of the time of change.

Corollary 1 (Monotonicity of Optimal Strategy in an Adversarial Environment). *Suppose that Assumption 1 holds. Then all optimal prechange sales rate sequences associated with Problem* (9) *are monotonic. Moreover, if* $\rho\geq 0$, *then such sequences are nonincreasing* (*with* ρ *defined as in* (7)); *conversely, if* $\rho\leq 0$, *such sequences are nondecreasing*.

One would have $\mu_t=0$ when the corresponding constraint (10b) is not tight, and demand will not change at t under the worst-case marginal distribution. Thus, by the concavity of $r(\cdot)$, one will have that $\lambda_t = \lambda_{t-1}$ for such a t. Hence, optimal sales will be held constant in between periods an adversary would not *select* for changing demand, and μ only assigns positive probability to periods in which the regret is maximal under the optimal solution. This is illustrated in the following example.

Example 4 (Adversarial Time of Change). Consider the setting in Example 1 in which the retailer does not know the marginal distribution of τ. The optimal prechange sales rate sequence is now given by $\lambda = (0.45, 0.45, 0.45, 0.59, 0.93, 1.12, 1.29, 1.44, 1.58, 1.70)$. Optimal prechange sales are equal from periods $t = 1$ to $t = 3$. Inspection of the shadow prices of constraints (10b) reveals that $\mu_0 = \mu_1 = \mu_2 = 0$, as expected, i.e., nature does not put any mass on early time changes.

Consider the impact of not knowing the distribution of the time of change. In this example, when the distribution of the time of change is uniform, the prescription obtained through the minimax regret framework yields performance within 1% of the optimal performance (analyzed in §4) one would have obtained with knowledge of the distribution. In other words, while the performance of the decision maker might be significantly off that of an oracle that knows the time of change, the minimax regret approach is not necessarily overly conservative.

5.2. Adversarial Selection of the Postchange Demand

Suppose now that the retailer knows the marginal distribution of τ but does not know that of θ. For a given sales sequence $\lambda \in \mathcal{L}$, let $\hat{\pi}(\lambda, k)$ denote the expected revenues generated by λ conditional on the event that $\theta = k$, i.e., $\hat{\pi}(\lambda, k) := \mathbb{E}\{\sum_{s=1}^{\tau-1} r(\lambda_s) + V^{\tau,\theta}(x - \sum_{s=1}^{\tau-1} \lambda_s) \mid \theta = k\}$. For $k \in \mathcal{K}$, define $\hat{D}(k)$ as the maximal revenues attainable by a retailer that knows upfront that $\theta = k$, conditional on such an event occurring. That is,

$$\hat{D}(k) := \max\{\hat{\pi}(\lambda, k) \colon \lambda \in \mathcal{L}\}, \quad \forall\, k \in \mathcal{K}. \tag{12}$$

We assume that the retailer, with no prior information on θ, aims to minimize the revenue loss relative to this semi-oracle while considering all possible realization of θ. For that, define $\hat{R}(x)$ as the minimum revenue loss relative to $\hat{D}(\theta)$, when θ is selected in an adversarial fashion. That is,

$$\hat{R}(x) := \min\left\{\max\left\{\hat{D}(k) - \sum_{t=1}^{T}\left(r(\lambda_t)\,\mathbb{P}\{\tau > t\} + V^{k,t}\left(x - \sum_{s<t}\lambda_s\right)\mathbb{P}\{\tau = t\}\right) \colon k \in \mathcal{K}\right\} \colon \lambda \in \mathcal{L}\right\}. \tag{13}$$

As in the previous section, the formulation above can be interpreted as a game in which nature responds with the worst possible realization of θ for a given sequence of sales rates. Moreover, this formulation also admits an equivalent formulation that allows one to connect this setting to formulation (6).

$$\min\ z \tag{14a}$$

$$\text{s.t. } \hat{D}(k) - \sum_{t=1}^{T}\left(r(\lambda_t)\,\mathbb{P}\{\tau > t\} + V^{k,t}\left(x - \sum_{s<t}\lambda_s\right) \cdot \mathbb{P}\{\tau = t\}\right) \le z, \quad k \in \mathcal{K}, \tag{14b}$$

$$z \in \mathbb{R},\ \lambda \in \mathcal{L}. \tag{14c}$$

The next result, which we state without proof, is the equivalent of Proposition 5 for this setting.

Proposition 6. *The objective function in Problem* (13) *is convex, and any optimal solution coincides with an optimal solution of Problem* (6) *when*

$$\mathbb{P}\{\theta = k\} = \nu_k, \quad \forall\, k \in \mathcal{K},$$

where the probability distribution $\nu := \{\nu_k, k \in \mathcal{K}\}$ is the vector of shadow prices associated with the constraints (14b).

Proposition 6 states that the adversarial selection of the type of change is equivalent to letting nature select a worst-case marginal distribution for θ. This worst-case distribution is independent of τ, so according to Proposition 1, the optimal sales sequence is monotonic. We formalize this in the next corollary, which we state without proof.

Corollary 2 (Monotonicity of Optimal Strategy in an Adversarial Environment Continued). *Suppose that $\mathbb{P}\{\tau = t \mid \theta = k\}$ is independent of k for all $t \in \mathcal{T}^+$. Then all optimal sales sequences associated with Problem* (13) *are monotonic.*

Although monotonicity of the optimal sales sequence is guaranteed by the result above, the monotonicity direction is driven by the sign of

$$\rho = \sum_{k \in \mathcal{K}} (r'(\min\{x/T, \lambda^*\}) - (r^k)'(\min\{x/T, \lambda^{*k}\}))\nu_k.$$

Hence, the values of the ν_k's are required to compute ρ in this case, and thus, unlike the case of an adversarial realization of the time of change, one must solve the optimization problem to compute ρ. Nevertheless, in many cases, monotonicity direction can be predicted with little or no computation—for example, when the sign of $r'(\min\{x/T, \lambda^*\}) - (r^k)'(\min\{x/T, \lambda^{*k}\})$ is independent of k (e.g., when $\beta^k \ge (\le)\ 1$ for all k, as in Example 1).

5.3. Adversarial Selection of the Time of Change and Postchange Demand

A natural extension to the settings above is that in which the retailer does not have information about τ and θ beyond their support. As before, one can state a minimax regret formulation for this setting and show that its solution coincides with that of (6) when the joint distribution of time of change and the postchange demand function coincides with that emanating from the shadow prices associated with constraints in an alternative formulation of the minimax problem (see the online companion). This worst-case distribution, however, might include correlation between τ and θ as no structure can be imposed a priori on the shadow prices, and thus it might not satisfy the requisites of Theorem 2. This suggests that the optimal policy might fail to be monotonic. Indeed, in the online companion we exhibit an example where adversarial selection of the time of change and the postchange demand results in a nonmonotonic sequence of sales rates.

6. Conclusion

The present paper has analyzed the role price-based revenue management plays in distributing inventory consumption over the horizon in settings when there is uncertainty regarding the future demand function. To do so, we considered a semideterministic formulation of a prototypical revenue management setting that enabled to preserve the uncertainty surrounding the future demand function while isolating the role that dynamic pricing plays for trading off pre- and postchange revenues.

Our results shed light on dynamic pricing settings with nonstationary and uncertain demand. First and foremost, even though demand environments differ along many dimensions, a decision maker is able to quantify upfront the relative attractiveness (compared with current conditions) of future demand conditions based on her or his current knowledge (through, e.g., a prior). This comparison leads to the systematic monotonicity of prices up to change in a broad set of scenarios. Moreover, the notion of attractiveness depends critically on the available inventory. As a result, one may only evaluate future conditions in conjunction with the inventory levels, which highlights the need to connect forecasting with the ultimate operational objective one cares about. We further established that these insights hold under different models of uncertainty, and it is possible to show that the policies obtained are a key building block of any policy that would account for learning and detection in a more general stochastic environment.

We have focused on settings in which changes in demand are infrequent, motivated by the fact that some of these are often driven by particular events. This is one end of a spectrum of possibilities for temporal uncertainty, with the other one being that of gradual and continuous changes, such as, e.g., those considered by Chen and Farias (2013) for the market size parameter. Nonstationary environments are more the norm than the exception in practical revenue management settings and yet have been relatively unexplored in the literature. There are many avenues for future research, ranging from the appropriate modeling of temporal uncertainty to the delineation of the boundaries of what is possible in terms of performance when facing such environments.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/msom.2014.0489.

Acknowledgments

The authors thank Stephen Graves, the associate editor, and the three referees for thoughtful and constructive comments, which helped improve the quality of this work in various fronts. This research is supported in part by the National Science Foundation [Grant CMMI-1233441], the Chilean Millennium Institute of Complex Engineering Systems [ICM: P05-004-F FIN. ICM-FIC], and the Business Intelligence Research Center (CEINE) at the University of Chile.

Appendix A. Selected Proofs

A.1. Preliminaries

In proving our results, we use a few side lemmas, the proofs of which are in the online companion to this paper. The first of these establishes the concavity and differentiability of the continuation function V.

Lemma 1. *The function $V^t(\cdot)$ is concave and continuously differentiable, for all $t \in \mathcal{T}^+$.*

The second auxiliary result establishes a property of the continuation function key to establishing monotonicity of the optimal sales rates.

Lemma 2. *If the expected future marginal profit is nondecreasing (nonincreasing), then $(V^t)'(y(T-t+1)) \le (\ge) (V^s)'(z(T-s+1))$ when $y \ge (\le)\ z$ and $t < s$.*

The reader may verify that, when Assumption 1 holds, expected future marginal profit is both nonincreasing and nondecreasing. Our last preliminary result ties monotonicity of a sales sequence to that of the correspondent sequence of depleting sales volumes.

Lemma 3. *Consider $\lambda \in \mathcal{L}$ and let $y := \{y_t, t \in \mathcal{T}\}$ denote the inventory path associated with λ. If $\lambda_s \ge (\le)\ \lambda_{s+1}$ for all $s \ge t$ for some $t < T$ and $\lambda_T = (\le)\ y_T$, then*

$$\frac{y_s}{T-s+1} \ge (\le)\ \frac{y_{s+1}}{T-s}, \quad \forall\, s \ge t.$$

Note that it is always the case that $\lambda_T \le y_T$. Also, note that under the additional conditions in Lemma 3 one has that $\lambda_s \ge (\le)\ y_s/(T-s+1)$ for all $s \ge t$.

From §4.1, we have that one can solve for the optimal sales sequence by solving the following optimization problem.

$$J(x) := \max\left\{\sum_{t\in\mathcal{T}}\Big(r(\lambda_t)\,\mathbb{P}\{\tau > t\} + V^t(y_t)\,\mathbb{P}\{\tau = t\}\Big) \;:\; \sum_{t\in\mathcal{T}}\lambda_t \le x,\ \lambda \ge 0\right\},$$

where for a feasible λ we define

$$y_t := x - \sum_{s=1}^{t-1}\lambda_s, \quad \forall\, t\in\mathcal{T}.$$

As mentioned in §4, the formulation above maximizes a continuously differentiable concave function subject to affine constraints (inventory constraint and nonnegativity of sales). Thus, Karush–Kuhn–Tucker conditions are necessary and sufficient to characterize the optimal sales sequence. Moreover, by Weierstrass Theorem (see, e.g., Sundaram 1996), there exits at least one optimal sales sequence. The Karush—Kuhn–Tucker conditions are

$$r'(\lambda_t)\,\mathbb{P}\{\tau > t\} - \sum_{s>t}(V^s)'(y_s)\,\mathbb{P}\{\tau = s\} + \gamma_t = \gamma, \quad t\in\mathcal{T}, \tag{A1a}$$

$$\sum_{t\in\mathcal{T}}\lambda_t \le x, \tag{A1b}$$

$$\lambda_t \ge 0, \quad t\in\mathcal{T}, \tag{A1c}$$

$$\gamma_t\lambda_t = 0, \quad t\in\mathcal{T}, \tag{A1d}$$

$$\gamma\Big(\sum_{t\in\mathcal{T}}\lambda_t - x\Big) = 0, \tag{A1e}$$

where $\gamma_t \ge 0$ is the dual multiplier associated with nonnegativity of λ_t, $t\in\mathcal{T}$, and $\gamma \ge 0$ is the dual multiplier associated with the capacity constraint.

A.2. Proof of Theorem 1

Let $\lambda := (\lambda_t\colon t\in\mathcal{T})$ denote a set of optimal sales rates and y_t denote the inventory position at time t when the retailer uses such a policy, for $t\in\mathcal{T}$. We prove the result in four steps. In the first three steps, we show that optimal sales rates are always monotonic. Then, we show that the direction of the monotonicity depends on the relation between λ_1 and x/T.

Step 1: *Monotonicity of the optimal sales rate when* $\gamma = 0$. We will prove by induction on t that $\lambda_t \le \lambda_{t+1}$ for all $t < T$. Consider first the case of $t = T-1$. Condition (A1a) for T becomes

$$r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} + \gamma_T = 0.$$

Since $r'(0) > 0$ and $\gamma_T \ge 0$, one must have that $\gamma_T = 0$ and $\lambda_T = \lambda^*$.

If $\gamma_{T-1} > 0$, then $\lambda_{T-1} = 0$ and the base case holds. Therefore, assume that $\gamma_{T-1} = 0$. In such a case, condition (A1a) for $t = T-1$ becomes

$$r'(\lambda_{T-1})\,\mathbb{P}\{\tau > T-1\} - (V^T)'(y_T)\,\mathbb{P}\{\tau = T\} = 0.$$

Note that $(V^T)'(y_T) \ge 0$, hence

$$r'(\lambda_T)\,\mathbb{P}\{\tau > T-1\} - (V^T)'(y_T)\,\mathbb{P}\{\tau = T\} \le 0,$$

since $r'(\lambda_T) = 0$. The above and the concavity of $r(\cdot)$ imply that $\lambda_{T-1} \le \lambda_T$.

Suppose now that the induction hypothesis holds for $t+1 < T$. Again, we only consider the case when $\gamma_t = 0$; otherwise the induction hypothesis holds trivially for t. By condition (A1a) for $t+1$, one has that

$$r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1}(V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \le 0. \tag{A2}$$

In addition, the induction hypothesis and Lemma 3 imply that $y_s/(T-s+1) \le y_{s+1}/(T-s)$, for all $s > t$. Thus, using Lemma 2, one has that

$$(V^s)'(y_s) \ge (V^{s+1})'(y_{s+1}), \quad \forall\, s > t.$$

This observation and (A2) imply that

$$r'(\lambda_{t+1}) \le \sum_{s>t+1}(V^s)'(y_s)\,\mathbb{P}\{\tau = s \mid \tau > t+1\} \le (V^{t+1})'(y_{t+1});$$

therefore,

$$\big(r'(\lambda_{t+1}) - (V^{t+1})'(y_{t+1})\big)\,\mathbb{P}\{\tau = t+1\} + r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1}(V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \le 0.$$

The left-hand-side above is that of condition (A2) for t, but evaluated at λ_{t+1}. Using the concavity of $r(\cdot)$, we conclude that $\lambda_t \le \lambda_{t+1}$. This proves the induction hypothesis. We conclude that, in this case, optimal sales rates are nondecreasing.

Step 2: *Monotonicity of the optimal sales rate when* $\gamma > 0$ *and* $r'(\lambda_T) \le (V^T)'(y_T)$. We will prove by induction on t that $\lambda_t \le \lambda_{t+1}$ for all $t < T$. Consider first the case of $t = T-1$. Condition (A1a) for T becomes

$$r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} + \gamma_T = \gamma. \tag{A3}$$

If $\gamma_{T-1} > 0$, then $\lambda_{T-1} = 0$ and the base case holds. Therefore, assume that $\gamma_{T-1} = 0$. Condition (A1a) for $t = T-1$ becomes

$$r'(\lambda_{T-1})\,\mathbb{P}\{\tau > T-1\} - (V^T)'(y_T)\,\mathbb{P}\{\tau = T\} = \gamma. \tag{A4}$$

However, (A3) implies that

$$r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} + \big(r'(\lambda_T) - (V^T)'(y_T)\big)\,\mathbb{P}\{\tau = T\} \le \gamma,$$

because the first term above is lower than γ and the second term is nonpositive since we assumed that $r'(\lambda_T) \le (V^T)'(y_T)$. Once again, by the concavity of $r(\cdot)$, this and (A4) imply that $\lambda_{T-1} \le \lambda_T$.

Suppose now that the induction hypothesis holds for $t+1 < T-1$. Again, we only consider the case when $\gamma_t = 0$; otherwise the induction hypothesis holds trivially for t. By condition (A1a) for $t+1$ one has that

$$r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1}(V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \le \gamma. \tag{A5}$$

In addition, the induction hypothesis and Lemma 3 imply that $y_s/(T-s+1) \le y_{s+1}/(T-s)$, for all $s > t$. Thus using Lemma 2 one has that

$$(V^s)'(y_s) \ge (V^{s+1})'(y_{s+1}), \quad \forall\, s > t. \tag{A6}$$

We consider two cases.

- *Case* (a): $\gamma_T = 0$. In this case, (A3) implies that

$$\gamma = r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} \le (V^T)'(y_T)\,\mathbb{P}\{\tau = T+1\}.$$

This observation, together with (A6) and (A5), imply that $r'(\lambda_{t+1}) \le (V^{t+1})'(y_{t+1})$; therefore,

$$(r'(\lambda_{t+1}) - (V^{t+1})'(y_{t+1}))\,\mathbb{P}\{\tau = t+1\} + r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1} (V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \le \gamma,$$

because the first term above is nonpositive and the second term is lower than γ. Considering condition (A1a) for t, the concavity of $r(\cdot)$ implies that $\lambda_t \le \lambda_{t+1}$.

• *Case* (b): $\gamma_T > 0$. Using the induction hypothesis, one has that $\lambda_s = 0$ and $y_s = y_t$ for all $s > t$. Thus,

$$r'(\lambda_{t+1}) = r'(\lambda_T) \le (V^T)'(y_T) \stackrel{\text{(a)}}{=} (V^{t+1})'(y_{t+1}),$$

where (a) follows from Lemma 2. Following the reasoning in case (a), we have that

$$(r'(\lambda_{t+1}) - (V^{t+1})'(y_{t+1}))\,\mathbb{P}\{\tau = t+1\} + r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1} (V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \le \gamma,$$

because the first term above is nonpositive and the second term is lower than γ by Equation (A5). Considering condition (A1a) for t, the concavity of $r(\cdot)$ implies that $\lambda_t \le \lambda_{t+1}$.

This proves the induction hypothesis. We conclude that, in this case, optimal sales rates are nondecreasing.

Step 3: *Monotonicity of the optimal sales rate when* $\gamma > 0$ *and* $r'(\lambda_T) \ge (V^T)'(y_T)$. We will prove by induction on t that $\lambda_t \ge \lambda_{t+1}$ for all $t < T$. Consider first the case of $t = T-1$ and assume $\lambda_T > 0$ and, hence, that $\gamma_T = 0$; otherwise the base case holds trivially. Condition (A1a) for T becomes

$$r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} = \gamma.$$

If $\gamma_{T-1} > 0$, then $\lambda_{T-1} = 0$ and condition (A1a) for $t = T-1$ implies that

$$r'(0)\,\mathbb{P}\{\tau = T+1\} + (r'(0) - (V^T)'(y_T))\,\mathbb{P}\{\tau = T\} < r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\}.$$

However, $r'(0) \ge r'(\lambda_T) \ge (V^T)'(y_T)$, contradicting the above and therefore the fact that $\lambda_{T-1} = 0$. We conclude that $\gamma_{T-1} = 0$. Note that

$$r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} + (r'(\lambda_T) - (V^T)'(y_T))\,\mathbb{P}\{\tau = T\} \ge r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\}.$$

Considering condition (A1a) for $t = T-1$ and the concavity of $r(\cdot)$, the above implies that $\lambda_{T-1} \ge \lambda_T$.

Suppose now that the induction hypothesis holds for $t+1 < T$. Again, we only consider the case when $\gamma_{t+1} = 0$; otherwise, the induction hypothesis holds trivially for t. By condition (A1a) for $t+1$ one has that

$$r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1} (V^s)'(y_s)\,\mathbb{P}\{\tau = s\} = \gamma.$$

In addition, the induction hypothesis, the fact that $y_T = \lambda_T$ (since $\gamma > 0$), and Lemma 3 imply that $y_s/(T-s+1) \ge y_{s+1}/(T-s)$ for all $s > t$. Thus, using Lemma 2, one has that

$$(V^s)'(y_s) \le (V^{s+1})'(y_{s+1}), \quad \forall\, s > t.$$

In addition, condition (A1a) for $t = T$ implies that $\gamma \ge r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\}$. Combining the observations above and the fact that $r'(\lambda_T) \ge (V^T)'(y_T)$, one has that

$$\begin{aligned} r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} &= \gamma + \sum_{s>t+1} (V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \\ &\ge r'(\lambda_T)\,\mathbb{P}\{\tau = T+1\} + \sum_{s>t+1} (V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \\ &\ge (V^{t+1})'(y_{t+1})\,\mathbb{P}\{\tau > t+1\}. \end{aligned}$$

We conclude that $r'(\lambda_{t+1}) \ge (V^{t+1})'(y_{t+1})$. With this, one has that

$$(r'(\lambda_{t+1}) - (V^{t+1})'(y_{t+1}))\,\mathbb{P}\{\tau = t+1\} + r'(\lambda_{t+1})\,\mathbb{P}\{\tau > t+1\} - \sum_{s>t+1} (V^s)'(y_s)\,\mathbb{P}\{\tau = s\} \ge \gamma,$$

because the first term above is nonnegative and the second term is equal to γ. Considering condition (A1a) for n and the concavity of $r(\cdot)$, the above implies that $\lambda_t \ge \lambda_{t+1}$.

This proves the induction hypothesis. We conclude that, in this case, optimal sales rates are nonincreasing.

Step 4: *Putting things together*. In Steps 1 and 2, we have that optimal sales rates are nondecreasing, so $\lambda_1 \le x/T$ because otherwise one would violate condition (A1b). In Step 3, the optimal sales rates are nonincreasing, so $\lambda_1 \ge x/T$ because otherwise one would have that $\sum_{t\in\mathcal{T}} \lambda_t < x$, contradicting the fact that $\gamma > 0$. This proves the result. □

A.3. Proof of Proposition 1

The proof is organized in two steps.

Step 1. Suppose $\lambda_1 \ge x/T$. We next show that $\rho \ge 0$. In this case, Theorem 1 asserts that the optimal sales rates are nonincreasing. Remember that γ denotes the dual multiplier associated with the inventory constraint.

• *Case* (a): $\gamma = 0$. Based on Step 1 in the proof of Theorem 1, optimal sales rates are nondecreasing, which is only possible in this case (due to the inventory constraint) if λ_t is constant for all $t \in \mathcal{T}$. In this case, condition (A1a) for $t = T$ implies that $\lambda_t = \lambda^*$, and hence, $\lambda^* = x/T$. The same condition for $t = T-1$ implies that $(V^T)'(x/T) \stackrel{\text{(a)}}{=} (V^1)'(x) = 0$, where (a) follows from Lemma 2. Therefore, we conclude that $\rho = r'(x/T) - (V^1)'(x) = 0$.

• *Case* (b): $\gamma > 0$ and $r'(\lambda_T) \le (V^T)'(y_T)$. Based on Step 2 in the proof of Theorem 1 and Case (a) above, again $\lambda_t = x/T$ for all $t \in \mathcal{T}$. In addition, condition (A1a) for $t = T$ and $t = T-1$ imply that $r'(x/T) = (V^T)'(x/T) = (V^1)'(x) \ge 0$. By the concavity of $r(\cdot)$ we conclude that $\lambda^* \ge x/T$ and that $\rho = 0$.

• *Case* (c): $\gamma > 0$ and $r'(\lambda_T) \ge (V^T)'(y_T)$. Based on Step 3 in the proof of Theorem 1, $r'(x/T) \ge r'(\lambda_1) \ge (V^1)'(x) \ge 0$. Because $x/T \le \lambda_1 \le \lambda^*$, we conclude that $\rho \ge 0$.

Putting the above together, we conclude that $\rho \ge 0$ when $\lambda_1 \ge x/T$.

Step 2. Suppose now that $\lambda_1 \le x/T$. We next show that $\rho \le 0$. In this case, Theorem 1 asserts that the optimal sales rates are nondecreasing.

• *Case* (a): $\gamma = 0$. Based on Step 1 in the proof of Theorem 1, $(V^t)'(y_t)$ is nonincreasing in t, so condition (A1a) for $t = 1$ implies that $r'(\lambda_1) \le (V^1)'(x)$. Also, by condition (A1a) for $t = T$ and the monotonicity of the sales sequence, $\lambda_1 \le \lambda_T = \lambda^*$. Thus, we conclude that $\lambda_1 \le \min\{x/T, \lambda^*\}$ and thus that $\rho \le 0$.

- *Case* (b): $\gamma > 0$ and $r'(\lambda_T) \le (V^T)'(y_T)$. Note that the exact same arguments in Case (a) above apply. Based on Step 2 in the proof of Theorem 1, $r'(\lambda_1) \le (V^1)'(x)$; condition (A1a) for $t = T$ and the monotonicity of λ imply that $\lambda_1 \le \lambda_T \le \lambda^*$, and we conclude that $\rho \le 0$.
- *Case* (c): $\gamma > 0$ and $r'(\lambda_T) \ge (V^T)'(y_T)$. Theorem 1 asserts that optimal sales rate are nonincreasing, which is not possible due to the inventory constraint, unless $\lambda_t = x/T$ for all $t \in \mathcal{T}$. In addition, condition (A1a) for $t = T$ implies that $\lambda^* \ge x/T$ and, when combined with that for $t = T-1$, imply that $r'(x/T) = (V^T)'(x/T) = (V^1)'(x)$. We conclude that $\rho \le 0$.

Putting the above together, we conclude that $\lambda_1 \le x/T$ implies that $\rho \le 0$. □

Appendix B. Asymptotic Justification of the Semideterministic Relaxation

In this section, we show that $J(x, T)$, defined in (6) (in a slight abuse of notation we now make its dependence with respect to T explicit), represents the asymptotic benchmark one ought to aim for in the general stochastic version of the firm's problem outlined in Remark 1 when the capacity and time horizon grow proportionally large. Recall that in such a stochastic version of the problem, the demand change is not revealed to the firm when it occurs, but it rather should be detected based on the data collected and the new demand environment should be estimated. Here, we argue that the analysis in §4 is a key building block to design good policies in this more general setting.

General stochastic version of the problem. Consider the following stochastic version of the problem. The retailer knows the initial *mean* demand function $d(\cdot)$ and anticipates that such a function stays constant until some unknown time $\tau \in \mathcal{T}^+$, after which the mean demand function switches to $d^k(\cdot)$ for some $k \in \mathcal{K}$. The time of change τ and the new mean demand function are *not known* upfront, and only a prior is initially available. The assumptions on mean demand functions are the same as those outlined in §3. We denote the *random* demand in period t by D_t. For concreteness, we assume that $d(\cdot) \in [0, 1]$ and $d^k(\cdot) \in [0, 1]$ for $k \in \mathcal{K}$ and that D_t is a Bernoulli random variable with means $d(p_t)$ for $t < \tau$ and $d^\theta(p_t)$ for $t \ge \tau$.

We let p denote a *nonanticipating* pricing policy generating the set of prices $\{p_t,\ t \in \mathcal{T}\}$ and let I_t denote the inventory in the system at the beginning of period t (one has $I_1 = x$). In this stochastic setting, the retailer selects a pricing policy to maximize the expected cumulative revenues collected over the finite selling horizon. That is, the retailer solves

$$W(x, T) := \sup\left\{\mathbb{E}\left\{\sum_{t=1}^{T} p_t \min\{D_t, I_t\}\right\}: p \in \Pi\right\},$$

where Π denotes the set of feasible nonanticipating pricing policies. Note that, in this setting, a policy needs to detect a change and learn the new demand environment based on the collected sales observations.

One can show that $J(x, T)$ serves as an upper bound to the revenues achieved by any admissible policy in the stochastic setting, i.e., that for any positive x and T, $W(x, T) \le J(x, T)$.

A near-optimal policy. Consider a policy that charges $p_t := (d)^{-1}(\lambda_t)$ to a customer arriving at period t, where $\{\lambda_t: t \in \mathcal{T}\}$ denotes the solution to (6), but that periodically (every n_r periods) deviates from such a price trajectory, and charges suitably chosen prices p'_t to customers for n_e consecutive periods ($n_e < n_r$). Price p'_t is chosen to minimize the deviation from p_t while generating a minimum separation in terms of expected demand. (The existence of such "separating" prices is a common assumption in the literature; see, e.g., Besbes and Zeevi 2011.) At the end of each of these n_e periods, cumulative sales during such a time window is compared with that expected under the prechange demand function; if the absolute difference is above a suitably chosen static threshold, a change in demand is declared, and the next n_e periods are devoted to estimate new demand (a different set of suitably chosen prices that impose a minimum separation in mean demands across all postchange demand functions is used), after which the constant price prescribed by (2) (where y represents current capacity and k minimizes the difference between cumulative sales and expected sales during the window devoted to estimate new demand) is charged until the end of the sales horizon.

Asymptotic optimality. Consider a sequence of settings, indexed by n, along which both the inventory level and the number of periods grow proportionally large, and as a result, the key trade-offs remain at play. Specifically, the initial inventory for setting n is given by $x_n = nx$, and the number of periods is $T_n = nT$. We do not impose any structure on the sequence of probability measures associated with this sequence, only that it is such that $J(x_n, T_n) \ge \underline{c}n$ for some $\underline{c} > 0$ (e.g., when $\mathbb{P}_n\{\theta = k,\ \tau = (t-1)n + 1\} = \mathbb{P}_1\{\theta = k,\ \tau = t\}$ $t \le T$, $k \in \mathcal{K}$, $n \ge 1$, where $\mathbb{P}_n$ denotes the probability measure associated with instance n).

Adapting the proof techniques in Besbes and Zeevi (2011), it is possible to prove that selecting $n_r = O(\sqrt{n})$ and $n_e = O(\ln n)$ allows bounding policy performance so that

$$\mathbb{E}\left\{\sum_{t=1}^{T} q_t \min\{D_t, I_t\}\right\} \ge J(x_n, T_n) - C \log n\sqrt{n}, \quad \forall\, n \in \mathbb{N}, \quad \text{(B1)}$$

for some finite positive constant C, where $\{q_t: t \in \mathcal{T}\}$ denotes the sequence price generated by policy outlined above and I_t represents the capacity at time t. This and the fact that $W(x_n, T_n) \le J(x_n, T_n)$ establish the asymptotic optimality of policy outlined above. In particular, it establishes that it is possible to conduct detection and learning in conjunction with pre- and postchange inventory allocation efficiently in a general stochastic setting. Furthermore, it establishes that the dynamic allocation of capacity between pre- and postchange demand environments as prescribed by the analysis in §4 solves the key first order problem faced by the retailer in a general stochastic setting.

References

AllThingsD.com (2012) Amazon says kindle withstood ipad mini assault. (October 26), http://allthingsd.com/20121026/amazon-says-kindle-withstood-ipad-mini-assault/.

Araman V, Caldentey R (2011) *Revenue Management with Incomplete Demand Information* (John Wiley & Sons, Hoboken, NJ).

Ball M, Queyranne M (2009) Toward robust revenue management: Competitive analysis of online booking. *Oper. Res.* 57(4):950–963.

Besbes O, Maglaras C (2009) Revenue optimization of a make-to-order queue in an uncertain market environment. *Oper. Res.* 57(6):1438–1450.

Besbes O, Zeevi A (2011) On the minimax complexity of pricing in a changing environment. *Oper. Res.* 59(1):66–79.

Bitran G, Mondschein S (1997) Periodic pricing of seasonal products in retailing. *Management Sci.* 43(1):64–79.

Boyd S, Vandenberghe L (2004) *Convex Optimization* (Cambridge University Press, New York).

Broder J, Rusmevichientong P (2012) Dynamic pricing under a general parametric choice model. *Oper. Res.* 60(4):965–980.

Cao P, Li J, Yan H (2012) Optimal dynamic pricing of inventories with stochastic demand and discounted criterion. *Eur. J. Oper. Res.* 217(3):580–588.

Chen Y, Farias V (2013) Simple policies for dynamic pricing with imperfect forecasts. *Oper. Res.* 61(3):612–624.

den Boer A, Zwart B (2014) Simultaneously learning and optimizing using controlled variance pricing. *Management Sci.* 60(3): 770–783.

Gallego G, van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* 40(8):999–1020.

Gallego G, van Ryzin G (1997) A multiproduct dynamic pricing problem and its applications to network yield management. *Oper. Res.* 45(1):24–41.

Harrison J, Keskin B, Zeevi A (2014) Dynamic pricing with an unknown linear demand model: Asymptotically optimal semi-myopic policies. Working paper, Stanford University, Stanford, CA.

Keller G, Rady S (1999) Optimal experimentation in a changing environment. *Rev. Econom. Stud.* 66(3):475–507.

Levina T, Levin Y, McGill J, Nediak M (2009) Dynamic pricing with online learning and strategic consumers: An application of the aggregating algorithm. *Oper. Res.* 57(2):327–341.

Li H, Graves S (2012) Pricing decisions during inter-generational product transition. *Production Oper. Management* 12(1):14–28.

Littlewood K (1972) Forecasting and control of passenger booking. *Proc. 12th Annual AGIFORS Sympos., Nathanya, Israel*, 95–117.

Netessine S (2006) Dynamic pricing of inventory/capacity with infrequent price changes. *Eur. J. Oper. Res.* 174(1):553–580.

Parsons J (2004) Using a newsvendor model for demand planning of nfl replica jerseys. M. Eng Thesis, MIT, Cambridge, MA.

Shaked M, Shanthikumar J (2007) *Stochastic Orders* (Springer Science + Business Media, New York).

Sundaram R (1996) *A First Course in Optimization Theory* (Cambrige University Press, New York).

Talluri K, van Ryzin G (2005) *Theory and Practice of Revenue Management* (Kluwer Academic Publishers, Norwell, Massachusetts, USA).

Wang Z, Deng S, Ye Y (2014) Close the gaps: A learning-while-doing algorithm for a class of single-product revenue management problems. *Oper. Res.*62(2):318–331.

Zhao W, Zheng Y (2000) Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management Sci.* 46(3): 375–388.