Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Multimodularity and Its Applications in Three Stochastic Dynamic Inventory Problems

Qing Li, Peiwen Yu

MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 16, No. 3, Summer 2014, pp. 455–463
ISSN 1523-4614 (print) | ISSN 1526-5498 (online)

http://dx.doi.org/10.1287/msom.2014.0488

Multimodularity and Its Applications in Three Stochastic Dynamic Inventory Problems

Qing Li
School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, imqli@ust.hk

Peiwen Yu
School of Management, Fudan University, 200433 Shanghai, China, freeypw@ust.hk

We apply the concept of multimodularity in three stochastic dynamic inventory problems in which state and decision variables are economic substitutes. The first is clearance sales of perishable goods. The second is sourcing from multiple suppliers with different lead times. The third is transshipment under capacity constraints. In all three problems, we establish monotone optimal polices with bounded sensitivity. Multimodularity proves to be an effective tool for these problems because it implies substitutability, it is preserved under minimization, and it leads directly to monotone optimal policies with bounded sensitivity.

Keywords: dynamic programming; multimodularity; substitutability and complementarity; stochastic inventory models
History: Received: June 9, 2013; accepted: February 11, 2014. Published online in *Articles in Advance* May 21, 2014.

1. Introduction

In stochastic dynamic programs, one is often interested in the question of whether and when the optimal policies are monotone in the states. Take, for example, a cost minimization problem. A standard approach is to show inductively that (a) if the minimal cost function is submodular, then the objective function is also submodular; and (b) the submodularity of the minimal cost function is preserved after optimization. One can then conclude that the objective function is indeed submodular and hence the optimal policies are monotone in the states.

The above standard approach works well for complementarity. We say two variables are complements (substitutes) if increasing one variable decreases (increases) the marginal cost of the other variable. In inventory management, there are many models in which the state and decision variables are economic substitutes (e.g., inventories at different ages, inventories that will arrive at different times, and capacities at different locations). To use the above standard approach in these models, one may start the induction with the hypothesis that the minimal cost function is supermodular. However, minimization does not in general preserve supermodularity and hence the standard approach above is not directly useful. In this study, we show that multimodularity, which is known to imply substitutability, is preserved after optimization. We use three examples from inventory management to illustrate its applications.

The concept of multimodularity, first introduced by Hajek (1985), has been a useful tool in the study of queuing systems (e.g., Hajek 1985, Glasserman and Yao 1994, Altman et al. 2000). We show that multimodularity propagates through dynamic programming recursion, which implies monotonicity of the optimal policies. We also establish bounds for the marginal effects of each state variable to the optimal policies.

For the models in which the state and decision variables are economic substitutes, there seem to be two other approaches to show monotone optimal policies in the existing literature. One is to directly take the derivatives of the optimal actions and optimal value functions with respect to the state variables (Fries 1975, Yang and Qin 2007, Hu et al. 2008). This approach requires twice differentiability. In addition, the analysis with this approach is typically very tedious, especially when the state and/or action space are large. The second approach is to rely on a tool called $L^\natural$-convexity. The concept of $L^\natural$-convexity was introduced into inventory management by Lu and Song (2005). $L^\natural$-convexity is a stronger notion of complementarity than submodularity. To use $L^\natural$-convexity to show structural properties, one must first transform the original variables into complementary variables, then show structural properties with respect to the new variables through showing $L^\natural$-convexity, and finally transform the properties back to those with respect to the original variables. The second approach has been used in the analysis of various inventory models with substitutable variables (Zipkin 2008, Huh and Janakiraman 2010, Pang et al. 2012, Chen et al. 2014).

Multimodularity and $L^\natural$-convexity can be related through a unimodular coordinate transformation

(Murota 2005). In spite of their mathematical equivalence, they represent two conceptually different paths to the same destination. Whereas one tackles the problems directly, the other takes a detour by transforming them into problems of complementarity.

The remainder of this paper is organized as follows. In §2, we define multimodularity and discuss its properties. We show that multimodularity is preserved after minimization and that if the objective function is multimodular, then the optimal actions are nonincreasing in the state variables with bounded sensitivity. In §§3–5, we illustrate the applicability of multimodularity with examples. We conclude the paper in §6.

The analysis and results below do not require functions to be differentiable. For ease of exposition, we use the notation $\Delta_{x_i} f(\mathbf{x})$ to represent $(f(\mathbf{x}+\delta\mathbf{e}_i)-f(\mathbf{x}))/\delta$, where $\mathbf{e}_i$ is a vector with one in its ith component and zero in all the other components and δ is a small positive number. When $f(\mathbf{x})$ is differentiable, then $\Delta_{x_i} f(\mathbf{x})$ means $\partial f(\mathbf{x})/\partial x_i$. The proofs that are not given in the paper can be found in the online appendix (available as supplemental material at http://dx.doi.org/10.1287/msom.2014.0488).

2. Multimodularity and Optimization Properties

Multimodularity has been traditionally defined in integer variables (i.e., Murota 2005, Hajek 1985). In this paper, we define it in real space. A set $V\subseteq\mathbb{R}^n$ is called a polyhedron if there exist $\mathbf{a}_i\in\mathbb{R}^n$ and $b_i\in\mathbb{R}$, such that $V=\{\mathbf{v}\in\mathbb{R}^n \mid \mathbf{a}_i\cdot\mathbf{v}\geq b_i,\ i=1,2,\ldots,m\}$. We shall focus on the following special polyhedral form:

(P1) Each n-dimensional vector $\mathbf{a}_i$ has the form $\pm(0,\ldots,0,1,\ldots,1,0,\ldots,0)$; that is, the nonzero components of $\mathbf{a}_i$ are either consecutive ones or consecutive negative ones.

Let $V\subseteq\mathbb{R}^n$ and $W\subseteq\mathbb{R}$ be polyhedra satisfying (P1). We say a function $g\colon V\to\mathbb{R}$ is *multimodular* (*antimultimodular*) if $\psi(\mathbf{x},y)=g(x_1-y,x_2-x_1,\ldots,x_n-x_{n-1})$ is submodular (supermodular) on $S=\{(\mathbf{x},y)\in\mathbb{R}^n\times\mathbb{R}\mid y\in W,(x_1-y,x_2-x_1,\ldots,x_n-x_{n-1})\in V\}$. Because V and W satisfy (P1), S is a polyhedron. Moreover, each inequality that defines S involves either only one variable or two variables with opposite signs. Hence, S is a lattice (Topkis 1998, Example 2.2.7(b)).

The following lemma shows that multimodularity implies increasing difference and convexity.

Lemma 1. *Suppose that $g(\mathbf{v})$ is multimodular.*

(i) *$g(\mathbf{v})$ has increasing difference and component-wise convexity.*

(ii) *If $g(\mathbf{v})$ is continuous, then it is jointly convex.*

(iii) *If $g(\mathbf{v})$ is twice differentiable, then*

$$\frac{\partial^2 g}{\partial v_i^2}\geq\frac{\partial^2 g}{\partial v_i\partial v_{i+1}}\geq\frac{\partial^2 g}{\partial v_i\partial v_{i+2}}\geq\cdots\geq\frac{\partial^2 g}{\partial v_i\partial v_n}\geq 0,$$

and

$$\frac{\partial^2 g}{\partial v_i^2}\geq\frac{\partial^2 g}{\partial v_i\partial v_{i-1}}\geq\frac{\partial^2 g}{\partial v_i\partial v_{i-2}}\geq\cdots\geq\frac{\partial^2 g}{\partial v_i\partial v_1}\geq 0\quad\textit{for all } i.$$

Lemma 1(iii) means that the marginal value of v_i is more sensitive to a change in a variable that is closer to v_i. For a two-dimensional function, this property is equivalent to diagonal dominance.

A multimodular function of discrete variables is defined similarly by replacing the space $\mathbb{R}^n$ with $\mathbb{Z}^n$ in the definitions above, where $\mathbb{Z}$ is the set of all integers. As a multimodular function has increasing difference, multimodularity implies substitutability. A multimodular function is not necessarily supermodular, however, because it is not defined on a lattice. The relationship between multimodularity and convexity in integer space is discussed in Altman et al. (2000). If convexity is defined as *convexity in the sense of linear interpolation extension*, then mutimodularity implies (in fact, is equivalent to) convexity in $\mathbb{Z}^n$ (Altman et al. 2000) and it has been recently discussed in the context of discrete batch ordering inventory systems by Ang et al. (2013).

We now present some basic operations that preserve multimodularity.

Lemma 2. (i) *If $g(\mathbf{v})$ is multimodular and $\alpha>0$, then $\alpha g(\mathbf{v})$ is multimodular.*

(ii) *If $g(\mathbf{v})$ is multimodular, then $g(-\mathbf{v})$ is multimodular.*

(iii) *If $f(\mathbf{v})$ and $g(\mathbf{v})$ are multimodular, then $f(\mathbf{v})+g(\mathbf{v})$ is multimodular.*

(iv) *If $g(\mathbf{v},d)$ is multimodular in $\mathbf{v}$ for any given d and D is a random variable, then $\mathsf{E}g(\mathbf{v},D)$ is multimodular in $\mathbf{v}$.*

(v) *If $f\colon R\to R$ is convex, then $g(\mathbf{v})=f(v_1+v_2+\cdots+v_n)$ is multimodular.*

(vi) *If $g(\mathbf{v})$ is multimodular, then $\tilde{g}(\mathbf{v})=g(v_n,v_{n-1},\ldots,v_1)$ is multimodular.*

(vii) *If $g(\mathbf{v})$ is multimodular, then $g(v_1,\ldots,v_{i-1},w_1+\cdots+w_m,v_{i+1},\ldots,v_n)$ is multimodular in $(v_1,\ldots,v_{i-1},w_1,\ldots,w_m,v_{i+1},\ldots,v_n)$.*

The above lemma also holds if we replace multimodular with antimultimodular and convex with concave. It is obvious from Lemma 1 and Lemma 2(v) that a one-dimensional function is multimodular if and only if it is convex. To see that Lemma 2(vi) is nontrivial, suppose that $\sigma(\mathbf{v})$ is a permutation of $\mathbf{v}$, $g(\mathbf{v})$ is multimodular, and $\tilde{g}(\mathbf{v})=g(\sigma(\mathbf{v}))$. Then $\tilde{g}(\mathbf{v})$ may not be multimodular in general. Therefore, unlike supermodularity and convexity, the order of variables is important for multimodular functions.

In what follows, we discuss the implication of multimodularity on parametric sensitivity and preservation of multimodularity under minimization.

Theorem 1. (i) *If $\mathscr{C}=\{(\mathbf{v},\mathbf{w})\mid\mathbf{v}\in S\subseteq\mathbb{R}^n,\mathbf{w}\in A(\mathbf{v})\subseteq\mathbb{R}^m\}$ is a polyhedron satisfying* (P1), *$g(\mathbf{v},\mathbf{w})$ is multimodular in $(\mathbf{v},\mathbf{w})$ on $\mathscr{C}$, then $f(\mathbf{v})=\min\{g(\mathbf{v},\mathbf{w})\mid\mathbf{w}\in A(\mathbf{v})\}$ is multimodular in $\mathbf{v}$ on S.*

(ii) *Suppose that* $g(\mathbf{v}, \zeta)$ *is multimodular on* $\mathscr{C}$, *where* $\mathscr{C} \subseteq \mathbb{R}^n \times \mathbb{R}$ *is a polyhedron satisfying* (P1). *Let* $\zeta^*(\mathbf{v})$ *denote the largest value of* ζ *that minimizes* $g(\mathbf{v}, \zeta)$. *Then,* $\zeta^*(\mathbf{v})$ *is nonincreasing in* $\mathbf{v}$, *and*

$$-1 \le \Delta_{v_n}\zeta^* \le \Delta_{v_{n-1}}\zeta^* \le \cdots \le \Delta_{v_1}\zeta^* \le 0.$$

Theorem 1 still holds for multimodular functions defined on a discrete state space. The theorem states that multimodularity is preserved under minimization when the objective function is defined on a polyhedron that satisfies (P1). With this structural property, we can show that the optimal action is nonincreasing in each state variable and the marginal effect of each state variable on the optimal action is bounded by -1. The change in response is smaller in magnitude than a change in a state. This reflects the stability of the system. We can also compare the marginal effects of different state variables on the optimal action. In parts (i) and (ii), if the objective functions are $g(\mathbf{w}, \mathbf{v})$ and $g(\zeta, \mathbf{v})$, respectively, the conclusions continue to hold. These results are formally presented in Corollary 1 in the appendix.

For discrete two-dimensional state space, it is known that multimodularity is preserved through dynamic programming iteration. Multimodularity implies increasing difference and diagonal dominance, both of which are preserved through dynamic programming iterations (e.g., Zhuang and Li 2012, van Wijk et al. 2009). This approach is essentially the same as differentiation in real space and becomes tedious when the state space increases.

Multimodularity is closely related to $L^\natural$-convexity. The definition of an $L^\natural$-convex function requires the following special form of polyhedron $V = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{a}_i \cdot \mathbf{v} \ge b_i,\ i = 1, 2, \ldots, m\}$:

(P2) Each n-dimensional vector $\mathbf{a}_i$ has the form $\pm(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$ or $\pm(0, \ldots, 0, 1, 0, \ldots, 0)$; that is, each entry of $\mathbf{a}_i$ is either zero, one, or negative one. There can be at most two nonzero entries. When there are two nonzero entries, they have opposite signs.

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}$ be polyhedra satisfying (P2), and let $\mathbf{e}$ denote an n-dimensional vector of ones. We say that a function $f\colon V \to \mathbb{R}$ is $L^\natural$-*convex* ($L^\natural$-*concave*) if $\psi(\mathbf{x}, y) = f(\mathbf{x} - y\mathbf{e})$ is submodular (supermodular) on $S = \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \in W, \mathbf{x} - y\mathbf{e} \in V\}$. $L^\natural$-convexity implies decreasing difference (i.e., complementarity) and minimization preserves $L^\natural$-convexity (e.g., Zipkin 2008). The following lemma shows that multimodular functions and $L^\natural$-convex functions are related through some unimodular coordinate transformations.

Lemma 3. *Let* U_i *be the* $i \times i$ *upper triangular matrix where all the entries on and above the main diagonal are one, and let* L_i *be the* $i \times i$ *lower triangular matrix where all the entries on and below the main diagonal are one. Define* $M_{n,i}$ *as the following block diagonal matrix*:

$$M_{n,i} = \begin{vmatrix} -U_i & 0 \\ 0 & L_{n-i} \end{vmatrix}.$$

Then, for any $0 \le i \le n$, *a function* $f(\mathbf{v})$ *is* $L^\natural$-*convex if and only if* $f(\pm M_{n,i}\mathbf{v})$ *is multimodular in* $\mathbf{v}$.

Lemma 3 allows us to transform substitutable variables into complementary ones, and vice versa, and is a generalization of Murota (2005). According to Murota (2005), a function $g\colon V \to \mathbb{R}$ is multimodular if and only if it can be represented as $g(\mathbf{v}) = f(v_1, v_1 + v_2, \ldots, v_1 + \cdots + v_n)$ for some $L^\natural$-convex function f. This corresponds to the case when the transformation matrix in Lemma 3 is $M_{n,0}$. This relationship between the two concepts has also been remarked upon in Pang et al. (2012). By using Lemma 3, we can show the preservation of multimodularity and the preservation of $L^\natural$-convexity imply each other (i.e., Theorem 1(i) can be shown by using a slightly modified version of Lemma 2 in Zipkin 2008, and vice versa).

Because both multimodularity and $L^\natural$-convexity can propagate through dynamic program iteration, we have potentially two paths to analyze problems in which state and decision variables are economic substitutes. One is to transform the original variables into complementary variables, then show structural properties with respect to the new variables through showing $L^\natural$-convexity, and finally transform the properties back to those with respect to the original variables. The other is to show multimodularity with respect to the original variables directly. For example, in Zipkin (2008), the original state variables $x = (x_1, x_2, \ldots, x_n)$ represent inventories that will arrive at different times in the future, and the decision variable is the order quantity q. He transformed the state variables and the decision variable by $v_j = x_j + x_{j+1} + \cdots + x_n$, for $1 \le j \le n$, and $\xi = -q$ and showed $L^\natural$-convexity with respect to the new variables. Alternatively, one could have directly established structural properties by showing multimodularity with respect to the original variables x.

In defining state and decision variables, we typically choose those that are most natural and intuitive. The transformed variables, which are essential in the $L^\natural$-convexity path, are not always natural and intuitive, especially in cases with complex original state and decision variables (e.g., the dual sourcing example in §4). Using $L^\natural$-convexity for problems with substitutable variables is a detour, so is using multimodularity for problems with complementary variables. (For example, in Gong and Chao 2013, the original variables are complementary. The authors have chosen the direct path by establishing $L^\natural$-convexity.) In the three examples in this paper, the state and decision variables we define, which are natural and intuitive, are all economic substitutes. Multimodularity, therefore, is a direct path for all of them.

In the next three sections we discuss the applications of multimodularity in three inventory problems in which state and decision variables are economic substitutes. The first and the third examples demonstrate that, to apply Theorem 1, the feasible region must form a polyhedron satisfying (P1). The second example highlights the importance of defining the right order of variables. Throughout the three sections, we let α be the discount rate and assume that at the end of the planning horizon, any unsold inventory has no value.

3. Clearance Sales of Perishable Inventory

Clearance sales can be an effective strategy to reduce mismatch between supply and demand for perishable goods. We consider a firm that sells perishable goods with an n-period lifetime. The firm purchases the goods at a cost c per unit. The goods can be sold at either a regular price, r, or a clearance sale price, s. Under a regular price, the demand in a period is random. Let D represent the demand and Φ its distribution function. Unmet demand is lost. Demand under the clearance sales price is abundant so that the firm can control at will how many it wants to sell under the price. Without loss of generality, items have zero value after they expire. The items that expire incur an outdating cost θ per unit to be removed from the shelf and disposed of. The items that are carried over to the next period cost a holding cost h per unit. We assume that $r > c$ and $s < \alpha c - h$. When these conditions are not met, the optimal policies are obvious.

The model is applicable to many business scenarios where the firm controls issuing. For example, blood banks supply perishable blood products to hospitals. The demand from hospitals is uncertain. When blood products approach the end of their life times, blood banks may sell them to research labs at a discounted price (Ballou 2004). The problem has been pursued concurrently by Xue et al. (2012), but their focus is not on monotonicity and bounded sensitivity.

At the beginning of each period, the firm decides an order quantity, q, of new items. Then, after the regular demand is realized, the firm decides the issuing policy to meet demand. At the end of each period, the items that expire in the period will be removed and disposed of. For the remaining inventory, if any, the firm decides how much of it should be carried over to the next period and how much should be sold at a clearance sale price. We use $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})$ to describe the initial inventory in the current period, where x_i is the number of units on hand with i periods of life remaining. Let d_i denote the amount of the regular demand that is met by using inventories with a remaining lifetime of i periods. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ and $\hat{O}(D) = \{\mathbf{d}\colon \sum_{i=1}^n d_i \le D, 0 \le d_n \le q, 0 \le d_i \le x_i \text{ for } 1 \le i \le n-1\}$.

The dynamic programming formulation is as follows:

$$V_t(\mathbf{x}) = \max_{q\ge 0} -cq + \mathsf{E} \max_{\mathbf{d}\in\hat{O}(D)} \bigg\{ r\sum_{i=1}^n d_i - \theta(x_1 - d_1) + \pi_{t+1}(x_2 - d_2, \ldots, x_{n-1} - d_{n-1}, q - d_n) \bigg\}, \quad (1)$$

where

$$\pi_t(\mathbf{y}) = \max_{\mathbf{0}\le\mathbf{z}\le\mathbf{y}} \bigg\{ s\sum_{i=1}^{n-1}(y_i - z_i) - h\sum_{i=1}^{n-1} z_i + \alpha V_t(\mathbf{z}) \bigg\}, \quad (2)$$

and $V_{T+1}(\mathbf{x}) = 0$. In (2), $\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})$, where z_i represents the amount of inventory with a remaining life time of i periods that is carried over to the next period.

Because there is no information updating between clearance sale decision at the end of a period and the ordering decision in the next period, we can redefine time periods. Combining (1) and (2) yields

$$\pi_t(\mathbf{y}) = s\sum_{i=1}^{n-1} y_i + \max_{\mathbf{0}\le\mathbf{z}\le\mathbf{y},\, q\ge 0} u_t(\mathbf{z}, q), \quad (3)$$

where

$$u_t(\mathbf{z}, q) = -(s+h)\sum_{i=1}^{n-1} z_i - \alpha c q + \alpha\mathsf{E} \max_{\mathbf{d}\in O(D)} \bigg\{ r\sum_{i=1}^n d_i - \theta(z_1 - d_1) + \pi_{t+1}(z_2 - d_2, \ldots, z_{n-1} - d_{n-1}, q - d_n) \bigg\},$$

and $\pi_{T+1}(\mathbf{y}) = s\sum_{i=1}^{n-1} y_i$. The constraint set $O(D) = \{\mathbf{d}\colon \sum_{i=1}^n d_i \le D, 0 \le d_n \le q, 0 \le d_i \le z_i \text{ for } 1 \le i \le n-1\}$. We will analyze the optimization problem (3) henceforth. The state variables now are represented by $\mathbf{y}$, the inventory levels after regular demand is fulfilled but before the clearance sales. In (3), both the clearance sale decision and the order decision are made at the beginning of a period.

Denote

$$(\mathbf{z}_t^*, q_t^*) = \arg\max_{\mathbf{0}\le\mathbf{z}\le\mathbf{y},\, q\ge 0} u_t(\mathbf{z}, q),$$

where $\mathbf{z}_t^* = (z_{t,1}^*, z_{t,2}^*, \ldots, z_{t,n-1}^*)$. When there are multiple maximizers, $(\mathbf{z}_t^*, q_t^*)$ is defined as the smallest in lexicographical order. Before we discuss the optimal policies, we introduce the following lemma:

Lemma 4. (i) *The optimal inventory issuing rule for regular sales is first-in, first-out* (*FIFO*);
(ii) $s \le \Delta_{y_1}\pi_t \le \Delta_{y_2}\pi_t \le \cdots \le \Delta_{y_{n-1}}\pi_t \le \alpha c - h$;
(iii) $\Delta_{z_1} u_t \le \Delta_{z_2} u_t \le \cdots \le \Delta_{z_{n-1}} u_t$.

The inequalities in Lemma 4(ii) and (iii) mean that a newer unit is more valuable than an older one. Fries (1975), Nahmias (1975), and Nandakumar and Morton (1993) have shown similar inequalities, but only for the ordering region, under a cost minimizing objective, and without considering clearance sales. They have all assumed the FIFO issuing rule, which may not be optimal under their model setups. Multimodularity can be established in Fries (1975) and Nandakumar and Morton (1993) if the parameters are such that FIFO is optimal. The objective function in Nahmias (1975) is obviously not multimodular because it is not even convex. Define

$$k_t = \max\{i\colon z^*_{t,i} < y_i\},$$

and if the set on the right-hand side is empty, we let $k_t = 0$. Here k_t represents the remaining lifetime of the newest inventory that is sold in clearance sales. The following theorem shows that older items are always chosen for clearance sales before newer ones.

Theorem 2. (i) *$z^*_{t,j} = 0$ for $j \le k_t - 1$ and $z^*_{t,j} = y_j$ for $j \ge k_t + 1$.*

(ii) *If $k_t \ge 2$, then $q^*_t = 0$.*

According to Theorem 2(i), all items with a remaining lifetime strictly less than k_t will be sold in clearance sales and will not be carried to the next period. All items with a remaining lifetime strictly greater than k_t will be carried to the next period. The marginal value of newer inventory is always greater than that of older inventory, hence older items should be cleared before newer ones. That is, FIFO is also optimal for the clearance sales. This policy is similar to the disposal saturation policy proposed for a multiechelon inventory system by Angelus (2011) under which the firm sells off all stock upstream of some threshold stage k_t and there are no disposals downstream of k_t. But unlike in Angelus (2011), we show the optimality of the policy here.

If any items that have a remaining lifetime of two periods or longer are sold in clearance sales, then no order of new items should be placed. Otherwise, one could order one unit less and keep one more unit of existing inventory. The immediate cost saving of doing this is $\alpha c - s - h$. If this additional existing unit is used to meet the regular demand in the current period, then the total saving is also $\alpha c - s - h$. If this additional unit is not used in the current period, then in the next period, we sell it at clearance sales and at the same time increase the order quantity by one. The total discounted saving in this case is $(1-\alpha)(\alpha c - s - h)$. Ordering and clearance sale of the inventory that will perish in one period, however, may take place at the same period because of the risk of outdating.

In what follows, we show that both the maximal profit function and the objective function in our model are antimultimodular, which means that inventories at different ages are economic substitutes. The monotonicity and bounded sensitivity of the optimal inventory levels follow naturally from the antimultimodularity properties. Formally, we have the following:

Theorem 3. (i) *The functions $u_t(\mathbf{z}, q)$ and $\pi_t(\mathbf{y})$ are antimultimodular.*

(ii) *The optimal policy for clearance sales is characterized by $\bar{z}_{t,i}$, where $\bar{z}_{t,i}$ is a decreasing function of $y_{i+1}, y_{i+2}, \ldots, y_{n-1}$, and independent of $y_1, y_2, \ldots, y_i$. The optimal policy is*

$$z^*_{t,i}(\mathbf{y}) = \begin{cases} y_i & \text{if } y_i \le \bar{z}_{t,i}, \\ \bar{z}_{t,i} & \text{if } y_i > \bar{z}_{t,i}. \end{cases}$$

Besides, the following inequalities hold:

$$-1 \le \Delta_{y_{i+1}} \bar{z}_{t,i} \le \Delta_{y_{i+2}} \bar{z}_{t,i} \le \cdots \le \Delta_{y_{n-1}} \bar{z}_{t,i} \le 0. \tag{4}$$

(iii) *The optimal replenishment quantity $q^*_t(\mathbf{y})$ is a decreasing function of $y_1, y_2, \ldots, y_{n-1}$, and the following inequalities hold:*

$$-1 \le \Delta_{y_{n-1}} q^*_t \le \Delta_{y_{n-2}} q^*_t \le \cdots \le \Delta_{y_1} q^*_t \le 0.$$

The quantities $\bar{z}_{t,i}$ are state-dependent thresholds, and they depend only on the inventories that are newer than i. Specifically, the more inventories whose remaining lifetime is longer than i, the less inventory with an i-period remaining lifetime should be carried to the next period. In addition, the thresholds $\bar{z}_{t,i}$ are more sensitive to the inventory whose remaining lifetime is closer to i. Similarly, the inequalities about the optimal order quantity confirm that the order quantity and existing inventory of any age are economic substitutes and the order quantity is more sensitive to the newer inventory than to the older one.

4. Dual Sourcing

Firms often order from multiple suppliers with different lead times and costs. This stream of research was started by Daniel (1963) several decades ago. When there are only two suppliers and they have consecutive lead times, Fukuda (1964) and Whittemore and Saunders (1977) showed that base-stock policies are optimal. Generalization in any direction appears to be challenging. When there are three suppliers with consecutive lead times, Feng et al. (2006) showed by using a counterexample that the base stock policies are not optimal. When there are two suppliers and one of them has zero lead time, Gong et al. (2014) and Zhou and Chao (2014) remarked that the structural properties can be derived by using the tool of $L^\natural$-convexity. When both suppliers have arbitrary lead times, heuristics have been developed by Chiang and Gutierrez (1995), Veeraraghavan and Scheller-Wolf (2008), and Sheopuri et al. (2010).

We use the model in Sheopuri et al. (2010) as an example to show the application of multimodularity. For $i = 1, 2$, let L_i and c_i denote the lead time and unit ordering cost of supplier i. We assume $L_1 > L_2$ and $c_1 < c_2$. Let u be the amount of inventory on hand after the orders that are due in the current period are received. The system starts with an initial inventory vector $\mathbf{u} = (x_{L_1-1}, \ldots, x_1, u, y_1, \ldots, y_{L_2-1})$, where x_i and y_i represent the number of units that will arrive i periods later from supplier 1 and 2, respectively. Then the firm needs to decide the ordering quantity q_i from supplier i before a random demand D occurs. After the demand has been realized, the system state in the next period is given by

$$\mathbf{U} = (q_1, x_{L_1-1}, \ldots, x_2, x_1 + u + y_1 - D, y_2, \ldots, y_{L_2-1}, q_2).$$

Let $h(\cdot)$ denote the convex holding and backordering cost. Define $V_t(\mathbf{u})$ to be the total minimum discounted cost function from period t to the end of the planning horizon, then the dynamic programming formulation is as follows:

$$V_t(\mathbf{u}) = \mathsf{E}h(u - D) + \min_{q_1 \geq 0,\, q_2 \geq 0} J_t(q_1, \mathbf{u}, q_2),$$

where

$$J_t(q_1, \mathbf{u}, q_2) = \sum_{i=1}^{2} c_i q_i + \alpha \mathsf{E} V_{t+1}(\mathbf{U}).$$

Let $s_{L_2} = u + x_{L_2} + \sum_{i=1}^{L_2-1}(x_i + y_i)$, which denotes the sum of the initial inventory and the aggregated orders that will arrive within L_2 periods.

Theorem 4. (i) *The functions $V_t(\mathbf{u})$ and $J_t(q_1, \mathbf{u}, q_2)$ are multimodular.*

(ii) *The optimal ordering quantities $q^*_{t,1}(\mathbf{u})$ and $q^*_{t,2}(\mathbf{u})$ satisfy the following inequalities*:

$$-1 \leq \Delta_{x_{L_1-1}} q^*_{t,1} \leq \cdots \leq \Delta_{x_{L_2+1}} q^*_{t,1} \leq \Delta_{s_{L_2}} q^*_{t,1} \leq 0,$$
$$-1 \leq \Delta_{s_{L_2}} q^*_{t,2} \leq \Delta_{x_{L_2+1}} q^*_{t,2} \leq \cdots \leq \Delta_{x_{L_1-1}} q^*_{t,2} \leq 0.$$

When there are three suppliers with consecutive lead times, similar structural properties can be shown. In the above analysis, we have assumed that unmet demand is backlogged. For the models with lost sales, similar structural properties can also be established for the case when there are only two suppliers with consecutive lead times. It is an open question whether the results can be generalized.

We have established multimodularity with respect to the state and decision variables $(q_{L_1}, x_{L_1-1}, x_{L_1-2}, \ldots, x_1, u, y_1, \ldots, y_{L_2-1}, q_{L_2})$. Each variable has a clear physical meaning and the order of variables is also natural. The structural results can also be established by using $L^\natural$-convexity. Any transformation suggested by Lemma 3 will create the right complementary variables, but none with the same intuitive appeal.

5. Transshipment Under Capacity Constraints

Transshipment is a common practice in both manufacturing and services and has been studied in the literature under different modeling assumptions. Capacity constraints add an additional challenge to the already hard problem. Before we introduce the specific model, we first present the following theorem, which is needed for the preservation of multimodularity and the parametric analysis in the model that we consider.

Theorem 5. (i) *Suppose $V(x_1, x_2)$ is multimodular. If $\Delta_{x_i} V(x_1, x_2) \geq -p_i$ for $i = 1, 2$, then $\sum_{i=1}^{2} p_i(-x_i)^+ + V(x_1^+, x_2^+)$ is multimodular.*

(ii) *Let $V(x_1, x_2) = \min\{J(y_1, y_2) \mid x_1 \leq y_1 \leq x_1 + C_1, x_2 \leq y_2 \leq x_2 + C_2\}$, where C_1 and C_2 are nonnegative constants. If $J(y_1, y_2)$ is multimodular, so is $V(x_1, x_2)$. Let y_1^* and y_2^* denote the largest minimizers, then the following inequalities hold*:

$$-1 \leq \Delta_{x_1} y_1^* - 1 \leq \Delta_{x_2} y_1^* \leq 0 \quad \textit{and}$$
$$-1 \leq \Delta_{x_2} y_2^* - 1 \leq \Delta_{x_1} y_2^* \leq 0.$$

Theorem 5(ii) is not an immediate result of Theorem 1. The reason is that although the objective function $J(y_1, y_2)$ is multimodular in (x_1, x_2, y_1, y_2), the set $\{(x_1, x_2, y_1, y_2) \mid x_1 \leq y_1 \leq x_1 + C_1, x_2 \leq y_2 \leq x_2 + C_2\}$ does not form a polyhedron that satisfies (P1). We suspect that besides transshipment, Theorem 5 may have applications in other inventory models under capacity constraints.

The literature on transshipment is large (see, e.g., Yang and Qin 2007, van Wijk et al. 2009, Hu et al. 2008, and the literature therein). The specific transshipment model that we shall examine here is adopted from Hu et al. (2008) by removing capacity uncertainty. There are two facilities owned by one firm, each serving demands in its own region. Let $i, j = 1, 2$ denote the facilities. At the beginning of each period, the firm must decide an order-up-to inventory level y_i. Unit order cost is c_i and the order quantity at facility i is constrained by a capacity level C_i. At the end of each period, after the demand at both facilities is realized but before the holding and penalty costs are charged, the firm can transship inventory from one facility to the other, if that can reduce cost. Unmet demand after transshipment is lost. Let u_{ij} be the transshipment quantity from facility i to facility j. Let D_i be the random demand at facility i and d_i its realization. Let $z_i = y_i - d_i + u_{ji} - u_{ij}$, then at the end of period after transshipment, if any, facility i's inventory level is z_i^+.

Let s_{ij} be the unit transshipment cost from facility i to facility j. Let h_i and p_i be the unit holding and penalty costs, respectively, at facility i. We assume that it is more beneficial to meet demand with inventory from the same facility than with inventory from the other facility. That is, $p_1 \geq p_2 - s_{12}$ and $p_2 \geq p_1 - s_{21}$.

Denote $V_t(x_1, x_2)$ as the minimum discounted cost function from period t to the end of the planning horizon when the initial inventory is (x_1, x_2), then we have the following dynamic programming formulation:

$$V_t(x_1, x_2) = \min_{x_1 \leq y_1 \leq x_1 + C_1,\, x_2 \leq y_2 \leq x_2 + C_2} \left\{ \sum_{i=1}^{2} c_i(y_i - x_i) + \mathsf{E} J_t(y_1, y_2, D_1, D_2) \right\},$$

where

$$J_t(y_1, y_2, d_1, d_2) = \min_{0 \leq u_{12} \leq y_1,\, 0 \leq u_{21} \leq y_2} \left\{ s_{12}u_{12} + s_{21}u_{21} + \sum_{i=1}^{2} h_i z_i^+ + \sum_{i=1}^{2} p_i(-z_i)^+ + \alpha V_{t+1}(z_1^+, z_2^+) \right\}.$$

Because transshipment is costly, it would be suboptimal to have transshipment in both directions in the same period. We let $w_{12} = u_{12} - u_{21}$ and use w_{12} as the decision variable. A positive w_{12} indicates that an amount of w_{12} is being transshipped from facility 1 to facility 2, and a negative w_{12} indicates that an amount of $-w_{12}$ is being transshipped from facility 2 to facility 1. We can rewrite $J_t(y_1, y_2, d_1, d_2)$ as

$$J_t(y_1, y_2, d_1, d_2) = \min_{-y_2 \leq w_{12} \leq y_1} \left\{ |w_{12}|(s_{12} + s_{21})/2 + w_{12}(s_{12} - s_{21})/2 + \sum_{i=1}^{2} h_i z_i^+ + \sum_{i=1}^{2} p_i(-z_i)^+ + \alpha V_{t+1}(z_1^+, z_2^+) \right\},$$

where $z_1 = y_1 - d_1 - w_{12}$ and $z_2 = y_2 - d_2 + w_{12}$. We can now state the main result.

Theorem 6. (i) *The function $V_t(x_1, x_2)$ is multimodular in (x_1, x_2), and the function $J_t(y_1, y_2, d_1, d_2)$ is multimodular in (y_1, y_2) and (d_1, d_2).*

(ii) *The optimal transshipment level $w_{12t}^*(y_1, y_2, d_1, d_2)$ satisfies the following inequalities:*

$$-1 \leq \Delta_{y_1} w_{12t}^* - 1 \leq \Delta_{y_2} w_{12t}^* \leq 0 \quad \textit{and}$$
$$-1 \leq \Delta_{d_2} w_{12t}^* - 1 \leq \Delta_{d_1} w_{12t}^* \leq 0.$$

(iii) *The optimal produce-up-to levels y_{1t}^* and y_{2t}^* satisfy the following inequalities:*

$$-1 \leq \Delta_{x_1} y_{1t}^* - 1 \leq \Delta_{x_2} y_{1t}^* \leq 0 \quad \textit{and}$$
$$-1 \leq \Delta_{x_2} y_{2t}^* - 1 \leq \Delta_{x_1} y_{2t}^* \leq 0.$$

Let the optimal order quantities in period t be q_{it}^*. Then Theorem 6(iii) implies that for $i = 1, 2$ and $j = 3 - i$,

$$-1 \leq \Delta_{x_i} q_{it}^* \leq \Delta_{x_j} q_{it}^* \leq 0.$$

Theorem 6 confirms that inventories are economic substitutes, regardless of their location and whether they are initial inventories or new items.

We have used Hu et al. (2008) with certain capacity constraints as an example to show the applicability of multimodularity in transshipment under capacity constraints. However, the main results continue to hold true under some other model setups (e.g., Yang and Qin 2007) and the analysis is similar.

6. Conclusion

Although multimodularity is an elegant and useful tool, it is a strong property and there are models in which the optimal value functions and/or the objective functions are not multimodular. For example, in inventory management, when ordering is constrained by a random capacity (e.g., Hu et al. 2008), the objective function is not even convex, let alone multimodular. In the perishable inventory model, if the outdating cost is accounted for by the approach in Nahmias (1975), then convexity, and consequently multimodularity, no longer hold. For these difficult problems, there seems to be no other tool besides direct differentiation.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/msom.2014.0488.

Acknowledgments

The authors thank Stephen Graves, Jingsheng Song, and an anonymous associate editor for their many suggestions that helped to improve the paper. Part of this research was supported by a grant from the Research Grants Council of Hong Kong [Grant HKUST 643211].

Appendix

Proof of Theorem 3. (i) The proof is by induction. The function $\pi_{T+1}(\mathbf{y})$ is obviously antimultimodular. Suppose it is true for $t+1$. To prove the antimultimodularity of $u_t(\mathbf{z}, q)$, it suffices to show that for any given d,

$$\tilde{u}_t(\mathbf{z}, q, d) = \max_{\mathbf{d} \in O(d)} \left\{ r \sum_{i=1}^{n} d_i - \theta(z_1 - d_1) + \pi_{t+1}(z_2 - d_2, \ldots, z_{n-1} - d_{n-1}, q - d_n) \right\} \tag{5}$$

is antimultimodular in $(\mathbf{z}, q)$ because antimultimodularity is preserved by expectation (Lemma 2(iv)). Let

$$g(\mathbf{z}, q, \mathbf{d}) = r \sum_{i=1}^{n} d_i - \theta(z_1 - d_1) + \pi_{t+1}(z_2 - d_2, \ldots, z_{n-1} - d_{n-1}, q - d_n).$$

Recall the constraint set

$$O(d) = \left\{ \mathbf{d}\colon \sum_{i=1}^{n} d_i \leq d, 0 \leq d_n \leq q, 0 \leq d_i \leq z_i \text{ for } 1 \leq i \leq n-1 \right\}.$$

We define a new constraint set,

$$T(d)=\left\{\mathbf{d}\colon \sum_{i=1}^{n} d_i\le d,\, d_n\le q,\, d_i\le z_i,\, \sum_{k=1}^{i} d_k\le d \quad \text{for all } 1\le i\le n-1\right\}.$$

In creating the new constraint set $T(d)$, we have dropped the nonnegativity constraints in $O(d)$ and changed the other set of constraints from $\sum_{i=1}^{n} d_i\le d$ to a more restrictive one: $\sum_{k=1}^{i} d_k\le d$ for all $1\le i\le n$. By Lemma 4(ii), we have

$$\Delta_{d_1}g\ge\Delta_{d_2}g\ge\cdots\ge\Delta_{d_n}g\ge 0.$$

This means that one should try to satisfy demand as much as possible and use older inventories as much as possible before using newer ones. Consequently, the optimal solutions to (5) are the same whether the maximization is subject to $O(d)$ or $T(d)$, and the maximizers are given by

$$d_i=\min\left(z_i,\left(d-\sum_{k=1}^{i-1}z_k\right)^+\right)\quad \text{for } 1\le i\le n-1,$$

and

$$d_n=\min\left(q,\left(d-\sum_{k=1}^{n-1}z_k\right)^+\right).$$

Henceforth, we will work with the new constraint set $T(d)$.

Let $\tilde{d}_i=d_i-z_i$ for $1\le i\le n-1$, $\tilde{d}_n=d_n-q$ and denote $\tilde{\mathbf{d}}=(\tilde{d}_1,\tilde{d}_2,\dots,\tilde{d}_n)$, $\mathbf{z}'_+=(q,z_{n-1},\dots,z_1)$ and $\tilde{u}'_t(\mathbf{z}'_+,d)=\tilde{u}_t(\mathbf{z},q,d)$, then the optimization problem (5) becomes

$$\tilde{u}'_t(\mathbf{z}'_+,d)=\max_{\tilde{\mathbf{d}}}\tilde{g}(\mathbf{z}'_+,\tilde{\mathbf{d}}),$$

where

$$\tilde{g}(\mathbf{z}'_+,\tilde{\mathbf{d}})=r\sum_{i=1}^{n-1}(\tilde{d}_i+z_i)+r(\tilde{d}_n+q)+\theta\tilde{d}_1+\pi_{t+1}(-\tilde{d}_2,-\tilde{d}_3,\dots,-\tilde{d}_n),$$

subject to the following constraints: $\tilde{\mathbf{d}}\le 0$, $\tilde{d}_n+q+\sum_{k=1}^{n-1}(\tilde{d}_k+z_k)\le d$ and $\sum_{k=1}^{i}(\tilde{d}_k+z_k)\le d$ for $1\le i\le n-1$.

Obviously, the set $\{(\mathbf{z}'_+,\tilde{\mathbf{d}})\mid \tilde{\mathbf{d}}\le 0,\, \tilde{d}_n+q+\sum_{k=1}^{n-1}(\tilde{d}_k+z_k)\le d,\, \sum_{k=1}^{i}(\tilde{d}_k+z_k)\le d \textit{ for } 1\le i\le n\}$ is a polyhedron satisfying (P1). As we can see, the original feasible set $O(d)$ is not a polyhedron satisfying (P1). The change of constraint set from $O(d)$ to $T(d)$ is a key step in the proof and it allows us to apply the preservation properties of multimodularity. In the expression for $\tilde{g}(\mathbf{z}'_+,\tilde{\mathbf{d}})$, the first three terms are linear, and therefore antimultimodular. The last term is antimultimodular by the induction hypothesis and Lemma 2(ii). Hence, $\tilde{g}(\mathbf{z}'_+,\tilde{\mathbf{d}})$ is a sum of antimultimodular functions and is therefore antimultimodular by Lemma 2(iii). The antimultimodularity of $\tilde{u}'_t(\mathbf{z}'_+,d)$ follows from Theorem 1(i). Hence, $\tilde{u}_t(\mathbf{z},q,d)$ is antimultimodular by Lemma 2(vi).

To prove that $\pi_t(\mathbf{y})=s\sum_{i=1}^{n-1}y_i+\max_{0\le\mathbf{z}\le\mathbf{y},\,q\ge0}u_t(\mathbf{z},q)$ is antimultimodular, it suffices to show that $\max_{0\le\mathbf{z}\le\mathbf{y},\,q\ge0}u_t(\mathbf{z},q)$ is antimultimodular in $\mathbf{y}$. Similar to the change of constraints above, we relax the feasible region to $\{q\ge0,\mathbf{z}\ge0,\sum_{k=i}^{n-1}z_k\le\sum_{k=i}^{n-1}y_k$ for all $1\le i\le n-1\}$. To see that the relaxation of the feasible region is without loss of optimality, suppose that there exists an $i\in[1,n-2]$, such that the optimal $z_i>y_i$. Then, we can always find some $j>i$ such that the optimal $z_j<y_j$. By Lemma 4(iii), the value of $u_t(\mathbf{z},q)$ can be increased if we reduce z_i and raise z_j by the same amount until $z_i\le y_i$. Therefore, there is no optimal z_i such that $z_i>y_i$.

Now let

$$f(\mathbf{y})=\max_{\mathbf{z},q}u_t(\mathbf{z},q),$$

subject to the new constraints $\{q\ge0,\mathbf{z}\ge0,\sum_{k=i}^{n-1}z_k\le\sum_{k=i}^{n-1}y_k$ for all $1\le i\le n-1\}$. The function $u_t(\mathbf{z},q)$ can be maximized sequentially. Let

$$\hat{u}_t(\mathbf{z})=\max_{q\ge0}u_t(\mathbf{z},q),$$

then $\hat{u}_t(\mathbf{z})$ is antimultimodular in $\mathbf{z}$ by Theorem 1(i). Denote $\mathbf{z}'=(z_{n-1},z_{n-2},\dots,z_1)$, $\tilde{\mathbf{y}}=(\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_{n-1})=(-y_1,-y_2,\dots,-y_{n-1})$ and let $\tilde{f}(\tilde{\mathbf{y}})=f(\mathbf{y})$ and $\hat{u}'_t(\mathbf{z}')=\hat{u}_t(\mathbf{z})$. Then we have the following optimization problem:

$$\tilde{f}(\tilde{\mathbf{y}})=\max_{\mathbf{z}'}\hat{u}'_t(\mathbf{z}'),$$

subject to the constraints: $z_i\ge0$, and $\sum_{k=i}^{n-1}(z_k+\tilde{y}_k)\le0$ for $1\le i\le n-1$. The function $\hat{u}'_t(\mathbf{z}')$ is antimultimodular by Lemma 2(vi). The antimultimodularity of $\tilde{f}(\tilde{\mathbf{y}})$ comes from Theorem 1(i). Finally, by Lemma 2(ii), $f(\mathbf{y})$ is antimultimodular in $\mathbf{y}$. This completes the induction.

(ii) From the proof of Theorem 3(i), $\pi_t(\mathbf{y})=s\sum_{i=1}^{n-1}y_i+\max_{0\le\mathbf{z}\le\mathbf{y}}\hat{u}_t(\mathbf{z})$. Denote

$$\bar{z}_{t,i}(y_{i+1},\dots,y_{n-1})=\arg\max_{z_i\ge0}\hat{u}_t(0,0,\dots,0,z_i,y_{i+1},y_{i+2},\dots,y_{n-1}).$$

Then $\bar{z}_{t,i}$ is decreasing and the inequalities (4) hold because of the antimultimodularity of $\hat{u}_t(\mathbf{z})$ and Theorem 1. The structure of the optimal policy for clearance sale follows because of the concavity of $\hat{u}_t(\mathbf{z})$, which is implied by its antimultimodularity.

(iii) From the proof of Theorem 3(i), we know that the optimal ordering quantity $q_t^*(\mathbf{y})$ is the solution to the following optimization problem:

$$\max_{\mathbf{z},q}u_t(\mathbf{z},q),$$

subject to the constraints $\{q\ge0,\mathbf{z}\ge0,\sum_{k=i}^{n-1}z_k\le\sum_{k=i}^{n-1}y_k$ for all $1\le i\le n-1\}$. Suppose we are maximizing over $\mathbf{z}$ first, and let

$$f(\mathbf{y},q)=\max_{\mathbf{z}}u_t(\mathbf{z},q),$$

subject to the constraints $\{\mathbf{z}\ge0,\sum_{k=i}^{n-1}z_k\le\sum_{k=i}^{n-1}y_k$ for all $1\le i\le n-1\}$. The result follows if $f(\mathbf{y},q)$ is antimultimodular. Define $z_{n-1}=\tilde{z}_{n-1}+\sum_{k=1}^{n-1}y_k$, then it is easy to verify that $u_t(\mathbf{z},q)$ is antimultimodular in $(z_1,\dots,z_{n-2},\tilde{z}_{n-1},\mathbf{y},q)$ and the constraint set

$$\left\{(z_1,\dots,z_{n-2},\tilde{z}_{n-1},\mathbf{y},q)\mid -y_{n-1}\le\tilde{z}_{n-1}+\sum_{k=1}^{n-2}y_k\le0,\, z_i\ge0,\ \sum_{k=i}^{n-2}z_k+\tilde{z}_{n-1}+\sum_{k=1}^{i-1}y_k\le0 \textit{ for } 1\le i\le n-2\right\}$$

is a polyhedron satisfying (P1). The antimultimodularity of $f(\mathbf{y},q)$ comes from Theorem 1(i).

Corollary 1. (i) *If* $\mathscr{C} = \{(\mathbf{w}, \mathbf{v}) \mid \mathbf{v} \in S \subseteq \mathbb{R}^n, \mathbf{w} \in A(\mathbf{v}) \subseteq \mathbb{R}^m\}$ *is a polyhedron satisfying* (P1), $g(\mathbf{w}, \mathbf{v})$ *is multimodular in* $(\mathbf{w}, \mathbf{v})$ *on* $\mathscr{C}$, *then* $f(\mathbf{v}) = \min\{g(\mathbf{w}, \mathbf{v}) \mid \mathbf{w} \in A(\mathbf{v})\}$ *is multimodular in* $\mathbf{v}$ *on* S.

(ii) *Suppose that* $g(\zeta, \mathbf{v})$ *is multimodular on* $\mathscr{C}$, *where* $\mathscr{C} \subseteq \mathbb{R} \times \mathbb{R}^n$ *is a polyhedron satisfying* (P1). *Let* $\zeta^*(\mathbf{v})$ *denote the largest value of* ζ *that minimizes* $g(\zeta, \mathbf{v})$. *Then,* $\zeta^*(\mathbf{v})$ *is nonincreasing in* $\mathbf{v}$, *and*

$$-1 \le \Delta_{v_1} \zeta^* \le \Delta_{v_2} \zeta^* \le \cdots \le \Delta_{v_n} \zeta^* \le 0.$$

Proof. The result follows from Theorem 1 and Lemma 2(vi). □

References

Altman E, Gaujal B, Hordijk A (2000) Multimodularity, convexity, and optimization properties. *Math. Oper. Res.* 25(2):324–347.

Ang M, Song J-S, Wang M, Zhang H (2013) On properties of discrete (r, q) and (s, T) inventory systems. *Eur. J. Oper. Res.* 229(1):95–105.

Angelus A (2011) A multiechelon inventory problem with secondary market sales. *Management Sci.* 57(12):2145–2162.

Ballou RH (2004) Case study: American Red Cross: Blood services. *Business Logistics/Supply Chain Management*, 5th ed. (Pearson Prentice Hall, NJ).

Chen X, Pang Z, Pan L (2014) Coordinating inventory control and pricing strategies for perishable products. *Oper. Res.* 62(2): 284–300.

Chiang C, Gutierrez G (1995) A periodic review inventory system with two supply modes. *Eur. J. Oper. Res.* 94:527–547.

Daniel KH (1963) A delivery-lag inventory model with emergency order. Scarf H, Gilford D, Shelly M, eds. *Multistage Inventory Models and Techniques*, Chap. 2 (Stanford University Press, Stanford, CA), 32–46.

Feng Q, Sethi S, Yan H, Zhang H (2006) Are base-stock policies optimal in inventory problems with multiple delivery modes? *Oper. Res.* 54(4):801–807.

Fries B (1975) Optimal ordering policy for a perishable commodity with fixed lifetime. *Oper. Res.* 23(1):46–61.

Fukuda Y (1964) Optimal policies for the inventory problem with negotiable leadtime. *Management Sci.* 10(4):690–707.

Glasserman P, Yao D (1994) Monotone optimal control of permutable GSMPs. *Math. Oper. Res.* 19(2):449–476.

Gong X, Chao X (2013) Optimal control policy for capacitated inventory systems with remanufacturing. *Oper. Res.* 61(3): 603–611.

Gong X, Chao X, Zheng S (2014) Dynamic pricing and inventory management with dual suppliers of different lead times and disruption risks. *Production Oper. Management*, ePub ahead of print June 2, http://onlinelibrary.wiley.com/doi/10.1111/poms.12221/abstract.

Hajek B (1985) Extremal splitting of point processes. *Math. Oper. Res.* 10(4):543–556.

Hu X, Kapuscinski R, Duenyas I (2008) Optimal joint inventory and transshipment control under uncertain capacity. *Oper. Res.* 56(4):881–897.

Huh WT, Janakiraman G (2010) On the optimal policy structure in serial inventory systems with lost sales. *Oper. Res.* 58(2):486–491.

Lu Y, Song J (2005) Order-based cost optimization in assemble-to-order systems. *Oper. Res.* 53(1):151–169.

Murota K (2005) Note on multimodularity and L-convexity. *Math. Oper. Res.* 30(3):658–661.

Nahmias S (1975) Optimal ordering policies for perishable inventory–II. *Oper. Res.* 23(4):735–749.

Nandakumar P, Morton TE (1993) Near myopic heuristics for the fixed-life perishability problem. *Management Sci.* 39(12): 1490–1498.

Pang Z, Chen F, Feng Y (2012) A note on the structure of joint inventory-pricing control with leadtimes. *Oper. Res.* 60(3):581–587.

Sheopuri A, Janakiraman G, Seshadri S (2010) New policies for the stochastic inventory control problem with two supply sources. *Oper. Res.* 58(3):734–745.

Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press, Princeton, NJ).

van Wijk ACC, Adan IJBF, van Houtum GJ (2009) Optimal lateral transshipment policy for a two location inventory problem. External report, Eurandom Report 2009-027, Eurandom, Eirdhoven, The Netherlands.

Veeraraghavan S, Scheller-Wolf A (2008) Now or later: Dual index policies for capacitated dual sourcing systems. *Oper. Res.* 56(4): 850–864.

Whittemore AS, Saunders SC (1977) Optimal inventory under stochastic demand with two supply options. *SIAM J. Appl. Math.* 32(2):293–305.

Xue Z, Ettl M, Yao DD (2012) Managing freshness inventory: Optimal policy, bounds and heuristics. Working paper, IEOR Department, Columbia University, NY.

Yang J, Qin Z (2007) Capacitated production control with virtual lateral transshipments. *Oper. Res.* 55(6):1104–1119.

Zhou SX, Chao X (2014) Dynamic pricing and inventory management with regular and expedited supplies. *Production Oper. Management* 23(1):65–80.

Zhuang W, Li MZ (2012) Monotone optimal control for a class of Markov decision processes. *Eur. J. Oper. Res.* 217(2):342–350.

Zipkin P (2008) On the structure of lost-sales inventory models. *Oper. Res.* 56(4):937–944.