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Sponsored Search Marketing: Dynamic Pricing and Advertising for an Online Retailer

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Consider a retailer using sponsored search marketing together with dynamic pricing. The retailer's bid on a search keyword affects the retailer's rank among the search results. The higher the rank, the higher the customer traffic and the customers' willingness to pay will be. Thus, the question arises: When a retailer bids higher to attract more customers, should the accompanying price also decrease (to strengthen the bid's effect on demand) or increase (to take advantage of higher willingness to pay)? We find that the answer depends on how fast the retailer increases its bid. In particular, as the end of the season approaches, the optimal bid exhibits smooth increases followed by big jumps. The optimal price increases only when the optimal bid increases sharply, including the instances where the bid jumps up. Such big jumps arise, for example, when the customer traffic is an S-shaped function of the retailer's bid.

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1. Introduction

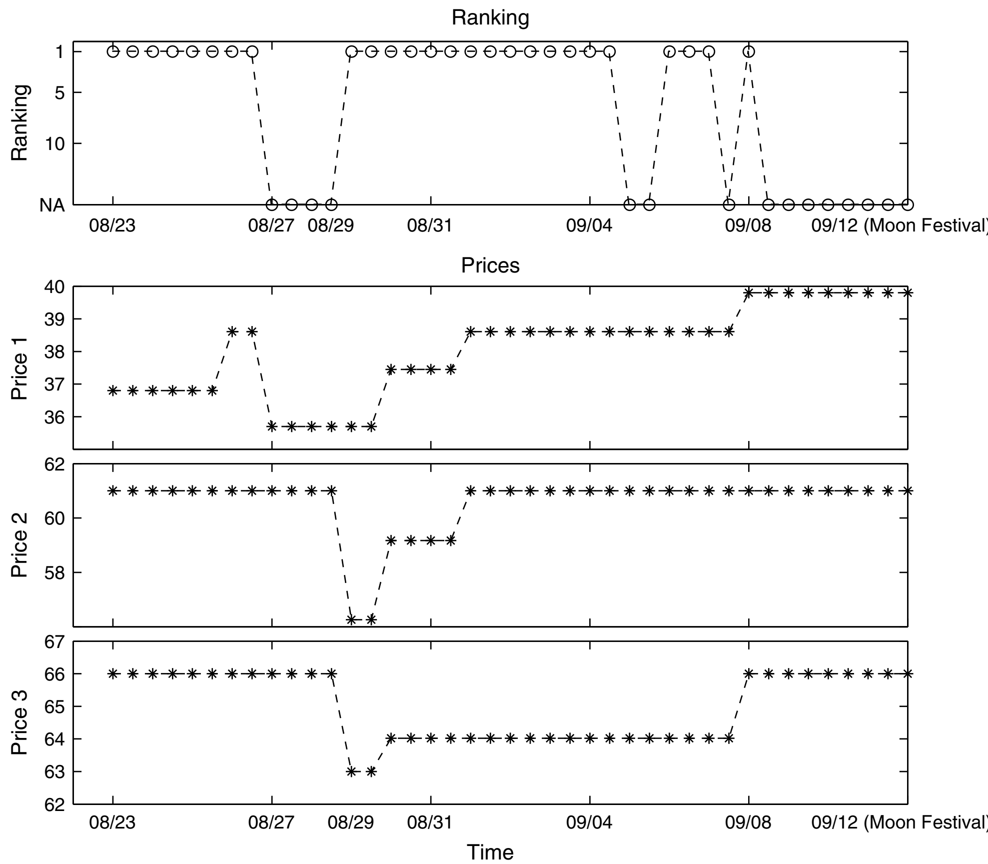
When using sponsored search marketing, the advertiser pays a search engine to display the advertiser's link among search results. Sponsored search is an increasingly influential advertising tool for many different sellers, especially for online retailers. According to an industry survey conducted by PricewaterhouseCoopers, sponsored search spending in the United States hit $7.3 billion in the first half of 2011, up 27% compared to the same half of 2010. The largest contributors have been retailers, who account for 23% of total sponsored search spending (PwC and IAB 2011). Many of these retailers also use dynamic pricing, especially online. When an online retailer uses dynamic pricing and sponsored search simultaneously, the retailer has two levers to shape the demand. The first lever is the price. The second lever is the *bid*, which indicates how much the retailer is willing to pay the search engine every time the retailer's link is clicked. The bid affects the *rank* of the retailer's link in the search results page. The rank, in turn, influences the customer traffic to the website and the customers' willingness to pay. The leverage enabled by the bid, namely the ability to shape the customer traffic and willingness to pay, is not well-studied in the dynamic pricing literature. In this paper, we study an online retailer's dual use of sponsored search and dynamic pricing. In so doing, we study a novel dynamic pricing problem in which the retailer influences the customer traffic and willingness to pay by deciding how much to bid on a keyword.

There is anecdotal evidence that online retailers are adjusting the two levers—the price and the bid—simultaneously and dynamically. Consider the online branch of a Chinese bakery based in Los Angeles.[1] A signature product of this bakery is the mooncake, a dessert served during the Chinese Moon Festival. Mooncake sales have strong seasonality because consumers start buying mooncakes approximately one month before the festival and rarely buy any after the festival is over. In 2011 the Moon Festival was on September 12. In the days approaching the Moon Festival, we tracked the retailer's rank among Google's sponsored links (for the keyword "mooncake"), as well as the prices of three different mooncake packages on its website. Figure 1 shows the tracking results.

The figure suggests that the retailer used sponsored search and also adjusted the prices of the three products dynamically. First, observe from the figure that the retailer's ranking displayed an "on-and-off" pattern; that is, its link either appeared at the top of the sponsored links or disappeared from the sponsored list. Although we do not observe the retailer's bids on the keyword mooncake, it is likely that the retailer

[1] The bakery is called Kee Wah, and its online branch is http://keewah.us.

Figure 1 The Anecdotal Evidence: Online Bakery



Notes. We recorded the rank and price at 10:00 A.M. and 7:00 P.M. for each day from August 28, 2011, to September 12, 2011. The ranking "NA" means that the retailer's link did not appear among the sponsored links.

was managing its bid so that its link, if it appeared, would be among the top search results.[2] Second, the retailer did not simply mark down its prices as the Moon Festival neared. In fact, for the three products we tracked, most discounts were offered in the middle of the selling season, indicating that the retailer utilized dynamic pricing.

Two features of the online bakery's business make it sensible to engage in both sponsored search and dynamic pricing. This bakery is not a household name like Amazon. This makes it unlikely that customers, especially first-time buyers, will land directly onto the bakery's website. Sponsored search is thus the bakery's primary method to attract Internet customer traffic. In addition, the mooncake is a highly perishable product, and the demand ceases after the Moon Festival. In the course of the limited selling season, the bakery might schedule additional production, but it likely faces some supply rigidity caused by, for example, the limited inventory of ingredients procured overseas (in this case, the filling paste for mooncakes). Hence, dynamic pricing is a natural choice to profitably match demand with supply. Much like this online bakery, there exist many online retailers that sell perishable products and rely on sponsored search to attract the bulk of their customers. These retailers would benefit from the dual use of sponsored search and dynamic pricing. In fact, the U.S. patent by Shah and Conine (2008) attests to the practical importance of making these decisions jointly. They outline a detailed procedure for updating prices and bids.

When using sponsored search and dynamic pricing simultaneously, a retailer must be mindful of how the two strategies interact. Consider the potential profit earned by the retailer when a customer clicks the retailer's link and lands on the retailer's website. The retailer's cost is simply the bid. The retailer's expected revenue from this customer is the price multiplied by the conversion rate, i.e., the fraction of clicks that eventually convert to a sale. Hence, the bid and the price must be determined together to generate a healthy profit margin. This decision is more complicated than it appears at first blush, because the conversion rate itself

[2] Of course, it is possible that some of the movement in the retailer's rank was due to changes in others' decisions. Notably, Google might have increased its reserve price for a link to appear, thus causing the bakery's link to drop from the search results. Alternatively, competitors might have collectively increased their bids, pushing down the online bakery's rank.

depends on both the price and the bid. Everything else being equal, the conversion rate decreases in the price and increases in the bid. The latter effect, namely that a higher bid leads to a higher conversion rate, is suggested by a recent empirical study (Ghose and Yang 2009) that observes that as the bid increases and the retailer's link moves up, the conversion rate improves. A possible explanation for this observation is that the customer mix attracted by links at different positions might be different. For instance, a higher-ranked link might attract impatient customers, who tend to be less price-sensitive. Given all these interactions between sponsored search and dynamic pricing, an online retailer would do well to coordinate its bidding and pricing decisions.

The research questions of this paper center on a retailer's dual use of sponsored search and dynamic pricing in the presence of inventory considerations. In particular, consider an online retailer that is selling a limited inventory of a perishable product for which the selling season has a predetermined deadline. Suppose the inventory of the item is high relative to the time to go; i.e., the retailer finds itself "overstocked." In such a case, the retailer needs to take action to increase the demand. If the retailer is using dynamic pricing without sponsored search, the retailer's course of action is clear: decrease the price to sell more. Likewise, if the retailer is using sponsored search without dynamic pricing, the retailer would choose to increase its bid, which increases both the customer traffic and the conversion rate. If, however, the retailer is using sponsored search and dynamic pricing simultaneously, the retailer has a larger set of choices. Namely, we expect that the retailer would take one of three actions: (1) The retailer could decrease the price and increase the bid, both of which would increase the demand. (2) The retailer could decrease the price, and if this were to cause a sufficient increase in demand, then the retailer could decrease its bid to cut down its advertising cost. (3) The retailer could increase the bid, thereby increasing the traffic and the conversion rate, which the retailer could exploit by increasing the price. Note that the first of these three actions uses the bid and the price to reinforce the respective effect of each on demand. In that sense, the first action uses the price and bid as complements. The latter two actions use only one lever to increase the demand, and they use the other lever in the opposite direction, with an eye toward reducing cost or increasing revenue. In that sense, the latter two actions use the price and bid as substitutes. These observations lead to our first research question:

(Q1) *Given that both the bid and the price can shape the demand, should the retailer use them as complements or substitutes*?

When an overstocked retailer uses the price and the bid as complements, the lower price and higher bid combine to increase the conversion rate. However, when the price and the bid are substitutes, it is not clear how the conversion rate changes, because the bid and the price push the conversion rate in opposite directions. This motivates our second research question:

(Q2) *How does the conversion rate change as the retailer reacts to changes in inventory/time*?

In addition, although the extant literature is rich in insights regarding dynamic pricing, not much is known about the optimal bidding policy for an inventory-constrained retailer. As observed in Figure 1, the retailer's ranking exhibited an on-and-off pattern that might have resulted from the following on-and-off bidding policy: Either bid high enough to appear among the top links or do not bid at all. Indeed, earlier literature suggested that similar bidding strategies may arise due to competition (e.g., Zhang and Feng 2011). We investigate whether there may be other explanations for the use of on-and-off bidding strategies. In particular, our third research question is as follows:

(Q3) *Under what circumstances is it optimal to use on-and-off bidding*?

Next, we position our work with respect to the extant literature. We include a summary of our results in the conclusion.

2. Literature Review

Given our focus on an online retailer's dual use of sponsored search and dynamic pricing, our work is ultimately a study of dynamic pricing and advertising decisions for a product with limited inventory. Thus, our work is related to the streams of literature on (i) dynamic pricing, (ii) dynamic advertising, and (iii) interactions among pricing, advertising, and capacity.

Dynamic Pricing. The dynamic pricing component of our work follows the literature on the dynamic pricing of a seasonal product with fixed inventory. This stream of literature, pioneered by Gallego and van Ryzin (1994) and Bitran and Mondschein (1997), studies how to adjust the price based on the remaining inventory and time until the end of the selling season. The models of Gallego and van Ryzin (1994) and Bitran and Mondschein (1997) have been extended in several directions, however, much of the earlier literature on dynamic pricing maintains the following assumption: Neither the customer arrival rate nor the customers' reservation price distribution depends on the retailer's decisions. Recently, two groups of papers have modified this assumption. The first group includes papers that study dynamic pricing in the presence of strategic customer behavior (e.g., Su 2007, Aviv and Pazgal 2008, Elmaghraby et al. 2008, Levin et al. 2009, Liu and van Ryzin 2011). In such models, strategic customers anticipate the retailer's actions and they decide if and when to buy. In that sense, the effective willingness to pay and arrival rate are influenced by the retailer's decisions. The second group includes papers that study dynamic pricing in the presence of reference price effects (e.g., Popescu and Wu 2007, Ahn et al. 2007, and Nasiry and Popescu 2011). In such models, the retailer's earlier pricing decisions lead the customers to form a reference price, which then influences the customers' future willingness to pay.

Similarly, we model a situation where the retailer can influence the arrival rate and reservation price of customers. In our model, such influence arises from the retailer's ability to advertise the product. In particular, the retailer in our model can improve both the arrival rate and the reservation price by bidding higher to move up its sponsored link. MacDonald and Rasmussen (2009) also consider a dynamic pricing problem where the retailer can influence the arrival rate through advertising expenditure. A fundamental difference is that in our model the retailer is able to influence the reservation price distribution, whereas in their model the reservation price distribution is exogenous. In addition, the retailer in their paper has full control over its advertising expenditure. This is not the case for sponsored search: The retailer in our model pays a fee whenever a customer clicks its link; hence its expenditure depends also on stochastic customer arrivals. As we show, these differences fundamentally change the trade-offs and deliver insights that are in sharp contrast to those in MacDonald and Rasmussen (2009).

Dynamic Advertising. There is a rich literature on dynamic advertising; see Feichtinger et al. (1994) for an extensive review of earlier work. Our work focuses on a specific form of dynamic advertising, namely sponsored search. There is a growing body of research on how a retailer can use sponsored search effectively. The economics literature on sponsored search treats the advertisers' problem as bidding in an auction for "position" and focuses on the bidding behavior that will arise in equilibrium (e.g., Aggarwal et al. 2007, Edelman et al. 2007 and Varian 2007). More recent papers by Edelman and Schwarz (2010) and Zhang and Feng (2011) consider the equilibrium behavior of advertisers in dynamic auctions. Our focus is different because we do not aim to study the equilibrium behavior of many advertisers. Instead, we adopt the perspective of an individual retailer that must make joint pricing and bidding decisions in the presence of inventory considerations. To enable this focus, we distill the effect of competition into a monotonic relationship between the bid and the arrival rate: To attract more customers, the retailer needs to bid higher.

The marketing literature on sponsored search studies advertisers' bidding decisions using models that incorporate specific customer behavior issues. For example, Athey and Ellison (2011) and Jerath et al. (2011) investigate advertisers' optimal bidding strategies by taking into account consumers' navigating and clicking behaviors. Li et al. (2010) consider the advertiser's bidding strategy when the click rate is unknown in advance. Katona and Sarvary (2010) incorporate customers' choices between the list of organic search results and the list of sponsored links. Our work differs from this literature in two ways. First, in this literature, the capacity constraint is often neglected: A customer's demand is guaranteed to be fulfilled once the customer is attracted to the retailer's website. In our model, this is not the case since the retailer has limited inventory with no replenishment opportunity. Second, most papers in this stream assume an exogenous revenue collected per click. In contrast, in our model, the price of the product, hence the revenue per click, is endogenously determined by the retailer.

Because the conversion rate is an important performance metric in sponsored search marketing, it appears as a factor in many of the above papers. For example, in the models of Athey and Ellison (2011) and Katona and Sarvary (2010), the conversion rate is an input parameter. Our paper adopts a different perspective on this important metric: In our model the conversion rate depends on the retailer's bid and price, which in turn are decisions that depend on the retailer's inventory and time until the end of the horizon.

Interactions Among Pricing, Advertising, and Capacity. Broadly speaking, our paper explores the interactions among pricing, advertising, and capacity (given our focus on a retailer's pricing and bidding decisions in the presence of inventory considerations). Others have studied the interaction between advertising and capacity, but in the absence of pricing decisions. For example, Tan and Mookerjee (2005) consider a budget-constrained electronic retailer that must trade off between sponsored search expenditure and website processing capacity. Chen et al. (2007) investigate the problem wherein an online retailer sells a perishable product and dynamically chooses whether to promote the product by listing it on a third-party channel. Swami and Khairnar (2006) focus on a diffusion-type demand model for a product with limited availability, and they study the retailer's optimal advertising and, separately, optimal pricing.

Another group of papers study the interaction between pricing and advertising, but in the absence of capacity considerations. For example, Albright and Winston (1979) investigate a firm's optimal advertising and pricing policy based on the firm's market position, which is an abstract notion that could capture, for example, the market share or the current sales level. In addition, Xu et al. (2011) consider a setting wherein two advertisers compete over both link position and price, focusing on the equilibrium behavior.

Other Related Research. A related stream of research takes the perspective of a search engine. The literature in this category generally considers two approaches to optimize the search engine's revenue. One approach is optimal auction design; see for example, Ostrovsky and Schwarz (2011), Liu et al. (2010), and Farboodi (2013). The main takeaway of these papers is that the search engine should rank advertisers not only by their bids, but also by other factors such as click-through rate. Another approach is more operations-oriented, considering the allocation of limited advertising space to random advertising demand. The literature belonging to this category includes Goel et al. (2010) and Najafi-Asadolahi and Fridgeirsdottir (2014), the latter of which provides a review.

Besides theoretical papers, empirical work is also emerging. For example, Yao and Mela (2011) empirically confirm that online retailers are practicing dynamic bidding, providing a validation for our research motivation. Others investigate how click-through rate and conversion rate change with respect to the rank of the link (Ghose and Yang 2009, Agarwal et al. 2011), the interdependence between organic and sponsored search results (Yang and Ghose 2010), the effect of an ad's obtrusiveness on purchase intent (Goldfarb and Tucker 2011), and so on. Some of these papers inform our modeling choices, and they will be discussed later in more detail.

3. Model Description

Suppose the retailer has a fixed inventory of the product to sell within a predetermined selling season. The retailer does not have the opportunity to replenish the inventory during the selling season, and the leftover inventory has zero value. Following the approach first adopted by Bitran and Mondschein (1997), we divide the selling season into discrete periods, each of which is short enough that at most one customer will click the retailer's link during the period. Hereafter, the *click rate* refers to the probability that a customer clicks the retailer's link within one period. The click rate is analogous to the arrival rate in the earlier literature on dynamic pricing (e.g., Bitran and Mondschein 1997), with the very important exception that in our model the retailer can influence the click rate by changing its bid on the search keyword.

At the beginning of each period t, the retailer chooses a keyword bid b_t for its sponsored link and submits it to the search engine. At the same time, the retailer determines a price p_t for the product, and posts it on the website. If a customer clicks the link during the period, the retailer pays the search engine its bid b_t.[3] Subsequently, the customer makes a purchase if his *reservation price* is above the posted price p_t. From the retailer's perspective, the reservation price of customers who follow the link is a random variable with a known distribution. A key feature of our model is that this reservation price distribution depends on the retailer's bid on the search keyword. This dependence arises because the bid influences the retailer's rank on the search results page, thereby affecting the customer mix attracted by the retailer.

Next, in §§3.1 and 3.2, we discuss our specific assumptions on how the retailer's bid influences, respectively, the click rate and the reservation price distribution. In §3.3, we formulate the retailer's profit maximization problem as a dynamic program and we state preliminary results.

3.1. The Click Rate

The click rate of the retailer's link depends on the link's rank on the search results page. Most search engines rank the links by taking into account both the retailer's bid and the retailer's relevance to the search keyword. For instance, Google AdWords defines "relevance" to mean "how well your keyword matches the message in your ads."[4] In particular, Google AdWords uses the so-called "Quality Score" to measure a retailer's relevance to a given search keyword. The quality score is known to a potential advertiser, who can find that information in the AdWords account statistics. For an online retailer that has a sufficiently long history with Google AdWords, little can be done to change the quality score in the short term. Thus, the retailer's main instrument for moving up its rank is the bid. Through a higher bid, the retailer can achieve a higher rank, which then results in a higher click rate. Instead of explicitly modeling the rank as an intermediary, we use the abstraction formalized in the next assumption.

Assumption 1. *In each period, the click rate $\lambda(b)$ is a strictly positive, increasing, and twice differentiable function in the retailer's bid b.*[5]

Of course, the retailer's rank in the search results page depends not only on its own bid, but also on competing bids from other advertisers. However, we do not model competition explicitly. Our goal is not to capture the equilibrium bids of many advertisers who

[3] This assumption corresponds to the case where the search engine uses generalized first-price auction (i.e., the bidder's per-click fee is equal to its bid). Other search engines use auctions similar to the generalized second-price auction (i.e., the bidder's per-click fee is no more than its bid, and it depends on the bidder whose link is placed immediately below). In the latter case, the per-click fee paid by the retailer is a stochastically increasing function of the retailer's bid. Our model could accommodate this case because there is a one-to-one correspondence between the bid and the expected per-click fee. Thus, we could extend our model by defining b_t as the expected per-click fee.

[4] See Google AdWords' online help page, https://support.google.com/adwords/answer/1659752 (accessed June 23, 2014).

[5] In this paper, we use the terms positive/negative, increasing/decreasing in the weak sense unless otherwise stated.

might be bidding on the same search keyword. Instead, given our focus on a single retailer's optimal bidding and pricing policy, it is more meaningful to assume that a retailer, when making its bidding decision, relies on an estimate of how the click rate depends on b. Hence, we adopt Assumption 1, which amounts to assuming that the retailer knows the empirical relationship between its bid and the click rate. In fact, search engines are assisting advertisers to estimate just such a relationship. For example, Google AdWords provides a tool called "Bid Simulator," which enables an advertiser to see retrospectively the results it would have obtained if it had used a different bid. Similarly, Yahoo provides a forecast of how the number of clicks and impressions will change if the advertiser adjusts its bid.

3.2. The Reservation Price Distribution

We assume that the customer decides whether to purchase only after clicking the retailer's link. In particular, upon following the link, the customer purchases the product if its posted price is less than the customer's reservation price—the customers in our model do not strategically choose their purchase time in anticipation of the retailer's prices. A key feature of our model is that the retailer's bid influences the reservation price distribution of customers who click the retailer's link. We adopt this assumption because the bid influences the rank, which then influences the mix of customers the retailer attracts. Later in this section, we will provide evidence from empirical papers regarding the connection between the bid and the reservation price, but first we specify our formulation for customers' reservation prices. Letting $Y(b)$ denote a customer's reservation price for a given bid b, we assume that $Y(b)$ is given by

$$Y(b)=\mu(b)+\sigma(b)X, \quad \text{with } \mu(0)=0 \text{ and } \sigma(0)=1. \quad (1)$$

Here, X is the customer's reservation price when the retailer bids zero and we assume that X takes values between $\underline{x}$ and $\bar{x}$ (where $\underline{x}\geq 0$, and $\bar{x}$ is allowed to approach infinity). Let $G(\cdot)$ and $g(\cdot)$ denote the cumulative distribution function (cdf) and probability density function (pdf) of X.[6] Equivalently, if we denote the cdf of $Y(b)$ as $F(\cdot \mid b)$, we may write

$$F(p\mid b)=G\left(\frac{p-\mu(b)}{\sigma(b)}\right).$$

Intuitively, this formulation incorporates both the additive and multiplicative effects of the retailer's bid on the customer's reservation price distribution.

[6] For ease of exposition, we assumed that the functions $\lambda(b)$, $\mu(b)$, and $\sigma(b)$ are stationary. Following the approach in Zhao and Zheng (2000), we could extend our analysis to cover the case where these functions are time-dependent. The key insights about the optimal bidding and pricing policy will remain the same.

In sponsored search marketing, the term "conversion rate" refers to the fraction of clicks that lead to purchases. One can influence the conversion rate through the bid and the price. Therefore, in the context of our model, which allows periodic bid and price adjustment, we define the *conversion rate* in a given period as the probability that a customer who clicked the retailer's link in that period purchases. Note that, with this definition, the conversion rate is essentially the probability that the reservation price $Y(b)$ exceeds the posted price; i.e., $\bar{F}(p\mid b):=1-F(p\mid b)$.

We impose the following assumption on the reservation price $Y(b)$.

Assumption 2. *$Y(b)$ satisfies the following properties*:

(a) *The standard deviation of $Y(b)$ is increasing in b. Equivalently, $\sigma(b)$ is increasing.*

(b) *The coefficient of variation of $Y(b)$ is decreasing in b. Equivalently, $\sigma(b)/\mu(b)$ is decreasing.*

Given Assumption 2, as the bid increases, both the mean and the standard deviation of the reservation price increase. However, Assumption 2(b) implies that the mean increases faster than the standard deviation. In fact, Assumption 2(b) leads to an intuitive outcome, namely that the reservation price $Y(b)$ is stochastically increasing in the bid b. The following lemma states this implication more formally.

Lemma 1. *Assumption 2 implies that the reservation price, $Y(b)$, is stochastically increasing in the bid, b (in the sense of first-order stochastic dominance).*

We next discuss the empirical and intuitive underpinnings of Assumption 2. Note that the reservation price depends on the retailer's bid only because the bid influences the rank of the retailer's link. To justify Assumption 2, let us first summarize two effects of the rank on the customer mix attracted by the link—similar effects are also discussed in Ghose and Yang (2009, §5.1.2):

- The first effect is that a higher-ranked link attracts *high-value customers*, i.e., customers with high reservation prices. There are many intuitive reasons for this effect. For example, some customers may see a higher-ranked link as a sign of a more reliable retailer, so they may be willing to pay more to purchase from that retailer. In addition, higher-ranked links attract impatient customers, who tend to be less price sensitive. Furthermore, customers attracted by higher-ranked links tend to visit fewer competing websites and, therefore, have fewer "outside options."
- The second effect runs counter to the first one. In addition to attracting high-value customers, high-ranked links also attract *window shoppers*, who might click the top links without a serious intent to purchase. Essentially, these are customers who have low reservation prices.

Assumption 2(a) follows from these two effects. High bids lead to high-ranked links, which attract a diverse group of customers, including both high-value customers and window shoppers. Thus, a higher bid leads to a customer mix whose reservation price exhibits a larger standard deviation.

Assumption 2(b) implies that the reservation price tends to increase in the bid (as Lemma 1 indicates); i.e., the first effect outweighs the second effect. In other words, the downward shift that window shoppers inflict on the reservation price of the customer mix is more than canceled out by the upward shift caused by high-value customers. Given our modeling framework, this assumption is in line with the empirical results of Ghose and Yang (2009, p. 1605), who find that "conversion rates are highest at the top and decrease with rank as one goes down the search engine results page." In our model, the conversion rate is given by $\bar{F}(p \mid b)$, the probability that a customer's reservation price $Y(b)$ exceeds the posted price. According to Lemma 1, as the retailer increases its bid and its link moves up, this probability also increases, thus improving the conversion rate. Therefore, Assumption 2(b) is consistent with Ghose and Yang's empirical observation.

We should note that, in another recent empirical paper, Agarwal et al. (2011) observe lower conversion rates for higher ranks. That is, in their data set, the second effect (the influence of window shoppers) dominates the first effect (the influence of high-value customers) and, hence, the customers who click on higher-ranked links might actually have lower reservation prices on average. Agarwal et al. (2011) attribute the difference between their results and those of Ghose and Yang (2009) primarily to the difference in the ranges of ranks studied by the two papers. Specifically, Ghose and Yang (2009) evaluate conversion rates for ranks ranging from the top all the way down to 131. In contrast, Agarwal et al. (2011) focus on the top seven positions. Given that we wish to model a retailer that is not restricted to remain among the top links, we believe that the empirical observations of Ghose and Yang (2009) are more applicable in our setting. Nonetheless, we will extend our study to accommodate the observation in Agarwal et al. (2011) in §7.2.

Assumption 3 below is needed to ensure the quasiconcavity of the profit function.

Assumption 3. *$G(\cdot)$ has an increasing failure rate (IFR); i.e., $g(\cdot)/\bar{G}(\cdot)$ is increasing.*

We make the following technical assumption to ensure the smoothness of the profit function.

Assumption 4. *The following regularity conditions hold*:

(a) *The density of X is zero at its lower bound; i.e., $g(\underline{x}) = 0$.*

(b) *$G(\cdot)$ has a continuous first-order derivative within its support.*

(c) *$\mu(b)$ and $\sigma(b)$ have continuous first-order derivatives.*

Assumptions 3 and 4 are satisfied by many probability distributions, including Gamma and Weibull distributions with shape parameter strictly greater than one, the power function distribution $g(x) = kx^{k-1}$ for $x \in [0, 1]$ with $k > 1$, and the Beta distribution $g(x) = x^{\alpha-1}(1-x)^{\beta-1}/\mathrm{B}(\alpha, \beta)$ for $x \in [0, 1]$ with $\alpha > 1$ and $\beta \geq 1$ (where $\mathrm{B}(\alpha, \beta)$ is the Beta function).

3.3. The Maximization Problem

Next we formulate the dynamic program for the retailer's profit maximization problem. Our convention is to count down the periods, with period T corresponding to the beginning of the horizon and period 1 being the last period. Let $\Pi(y, t)$ denote the retailer's optimal expected profit when starting period t with y units of inventory. The optimality equations are given by (we omit the subscript t of b_t and p_t)

$$\begin{aligned}\Pi(y, t) = \max_{b\geq 0,\, p\geq 0} \big\{&(1-\lambda(b))\Pi(y, t-1) \\ &+ \lambda(b)F(p \mid b)(\Pi(y, t-1) - b) \\ &+ \lambda(b)\bar{F}(p \mid b)(p + \Pi(y-1, t-1) - b)\big\}, \\ &\qquad \text{for } y > 0,\ t = 1, \ldots, T, \\ \Pi(0, t) = 0, &\quad \text{for } t = 1, \ldots, T, \\ \Pi(y, 0) = 0, &\quad \text{for } y > 0.\end{aligned}$$

The first term inside the maximization is the retailer's expected profit to go when no customer clicks the link. The second term corresponds to the case wherein a customer clicks but does not purchase the product. The third term is for the case wherein a customer clicks and purchases the product. The retailer chooses its optimal posted price p and bid b at the beginning of period t to maximize its profit to go. The terminal conditions indicate that no replenishment is possible and that the salvage value of the item is zero.

After some algebraic manipulation, $\Pi(y, t)$ can be rearranged as follows:

$$\Pi(y, t) = \Pi(y, t-1) + \max_{b\geq 0,\, p\geq 0} \big\{\lambda(b)[\bar{F}(p \mid b)(p - \Delta(y, t)) - b]\big\}, \quad (2)$$

where we use the notation $\Delta(y, t) := \Pi(y, t-1) - \Pi(y-1, t-1)$. Here, $\Delta(y, t)$ can be interpreted as the expected value of carrying one more unit of the product into period $t-1$ when there are already $y-1$ units in inventory. In other words, $\Delta(y, t)$ denotes the marginal value of the y-th unit of inventory in period t or, alternatively, the retailer's opportunity cost of selling the y-th unit in period t. Define

$$\pi(b, p, \Delta) = \lambda(b)[\bar{F}(p \mid b)(p - \Delta) - b]. \quad (3)$$

Here, $\pi(b, p, \Delta)$ can be interpreted as the expected marginal contribution from selling one unit of the product in period t, given the retailer's bid b, the price p, and the marginal value of inventory Δ. Using (2) and (3), the retailer's optimization problem becomes

$$\Pi(y, t) = \Pi(y, t-1) + \max_{b\geq 0,\, p\geq 0} \pi(b, p, \Delta(y, t)). \tag{4}$$

The following results about $\Delta(y, t)$ are standard in dynamic pricing models and we therefore present them as preliminary properties of the model.

Lemma 2. *The marginal value of inventory, $\Delta(y, t)$, is decreasing in the inventory level y and increasing in the time to go t.*

We define the following notation to refer to the retailer's optimal actions:

$$\langle b^*(\Delta), p^*(\Delta)\rangle := \arg\max_{b\geq 0,\, p\geq 0} \pi(b, p, \Delta),$$

$$\langle b^*(y, t), p^*(y, t)\rangle := \langle b^*(\Delta(y, t)), p^*(\Delta(y, t))\rangle.$$

Following the interpretation of $\pi(b, p, \Delta)$, $\langle b^*(\Delta), p^*(\Delta)\rangle$ is the optimal bid and posted price when the retailer's marginal value of inventory is some arbitrary number Δ. In particular, with y units in inventory and t periods to go, the retailer's marginal value of inventory is $\Delta(y, t)$, and the retailer's optimal bid and posted price are $\langle b^*(y, t), p^*(y, t)\rangle$.

Our analysis will focus on the optimal policy as a function of Δ, i.e., $\langle b^*(\Delta), p^*(\Delta)\rangle$, since it is more convenient to establish the properties of the optimal policy in the context of the continuous variable Δ. (We restrict Δ so that $\Delta \in [0, \Delta(1, T)]$, where $\Delta(1, T)$ yields the largest possible value for the marginal value of inventory, by Lemma 2.) We use the structure of this optimal policy to explain the behavior of $\langle b^*(y, t), p^*(y, t)\rangle$, which is simply a discretization of $\langle b^*(\Delta), p^*(\Delta)\rangle$.

4. Structure of the Optimal Policy

In this section, we first establish several properties of the optimal bidding and pricing policy. These properties allow us to identify two distinct structures of the optimal policy, both of which may arise under our current assumptions. Later, in §5, we will identify how the assumptions on the click rate and the reservation price are responsible for the differences between the two structures.

4.1. The Optimal Bidding Policy

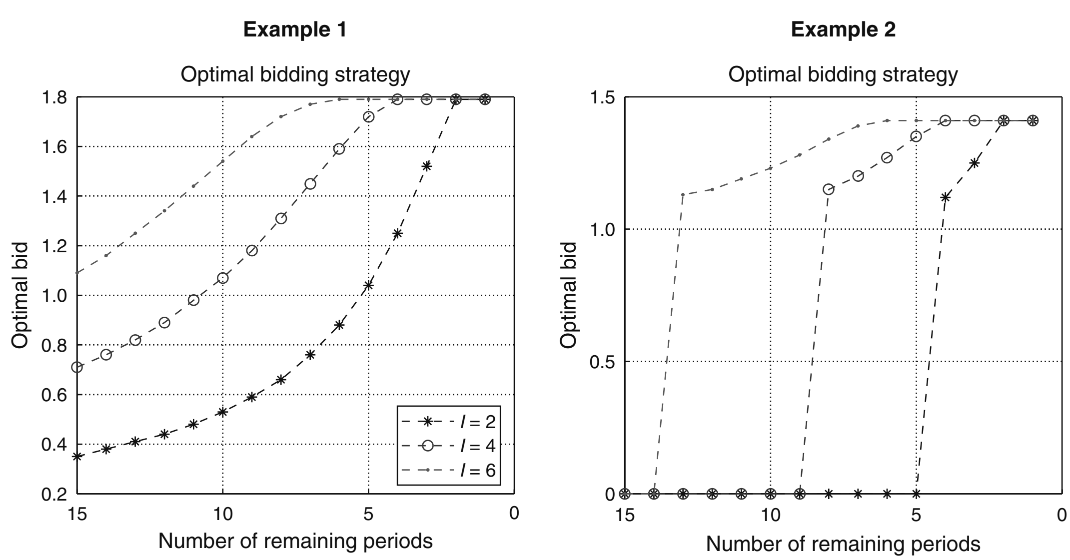
The next proposition describes the effects of time and inventory on the retailer's optimal bid.

Proposition 1. *The optimal bid as a function of the marginal value of inventory, i.e., $b^*(\Delta)$, decreases in Δ. Consequently, when the inventory increases or time-to-go decreases, the optimal bid as a function of inventory and time to go, $b^*(y, t)$, increases.*

This monotonic result is intuitive. With shorter time to go or higher inventory, the marginal value of inventory decreases; i.e., the opportunity cost of selling one unit of product in the current period decreases. Thus, the retailer has more incentive to sell the unit within the current period. As a result, the retailer increases the bid, thereby increasing the sponsored link's click rate and the reservation prices of the arriving customers.

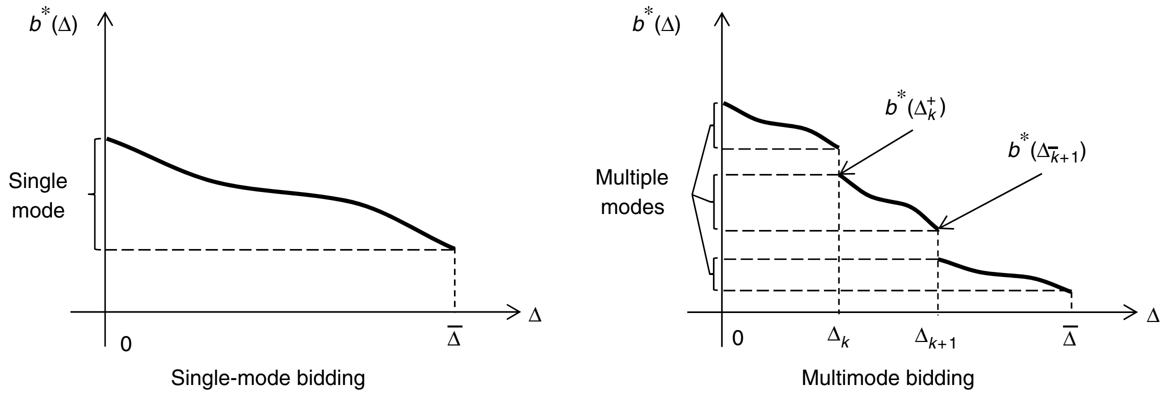
Figure 2 depicts two examples of the optimal bidding policy. Although both examples in Figure 2 exhibit the

Figure 2 **Two Distinct Examples of the Optimal Bidding Policy**



Notes. Parameter setting: In both examples, $\sigma(b) = 1 + 0.1b$, and X is distributed with Gamma(6, 0.5). In Example 1, we set $\mu(b) = 0.6b$ and $\lambda(b) = 0.8 - 0.7e^{-b}$. In Example 2, we set $\mu(b) = 0.4b$ and $\lambda(b) = (1 + 0.4e^{4-4b})/(1 + e^{4-4b})$. The three curves correspond to different values of inventory, $I = 2, 4, 6$.

Figure 3 Illustration of Two Structures of Bidding Policy



monotonic properties established in Proposition 1, there is a marked difference between them. In Example 1, for a fixed level of inventory, the optimal bid changes "smoothly" in the time to go t. In Example 2, however, the optimal bid switches between two modes that can be clearly distinguished: An "on" mode wherein the retailer sets a series of high bids and an "off" mode wherein the retailer simply bids 0.

The two examples in Figure 2 correspond to two distinct possibilities. The difference between them is not cosmetic, but structural. In particular, the difference between these two examples is rooted in the behavior of $b^*(\Delta)$: In Example 1 the optimal bid $b^*(\Delta)$ turns out to be continuous in Δ, whereas in Example 2 it is discontinuous. In Example 2, as the time to go decreases, the marginal value Δ also decreases and passes through a discontinuity point of $b^*(\Delta)$, thus leading to the jumps observed. Note that the discontinuity arises although all the input functions (e.g., the click rate, the reservation price distribution) are continuous and differentiable.

Hereafter, to distinguish between these two cases, we will refer to the smooth bidding policy illustrated in Example 1 as *single-mode bidding*, and we will refer to the on-and-off bidding policy illustrated in Example 2 as *multimode bidding*. Definition 1 introduces the formal distinction between these two types of bidding policies.

Definition 1. The optimal bidding policy follows one of two structures:

Single-mode bidding: This refers to the case where $b^*(\Delta)$ is continuous.

Multimode bidding: This refers to the case where $b^*(\Delta)$ is discontinuous at a set of points $\{\Delta_k\}_{k=1}^N$ (see Figure 3 for an illustration). In this case, we refer to each interval $(b^*(\Delta_{k+1}^-), b^*(\Delta_k^+)]$ as a "mode" of the bidding policy.

4.2. The Bid and the Price: Complements or Substitutes?

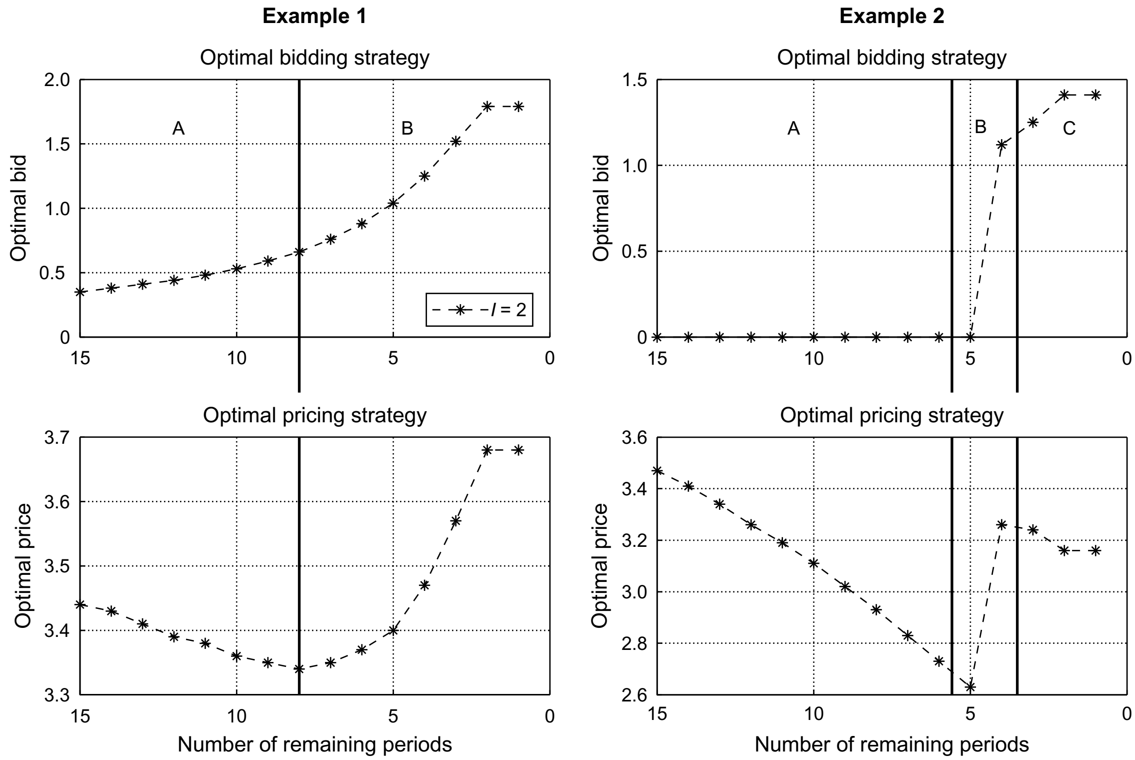
In this section, we address our first research question: (*Q*1) *Given that both the bid and the price can shape the demand, should the retailer use them as complements or substitutes*? Figure 4 revisits the numerical examples of Figure 2, and shows the optimal bid and the corresponding optimal price as a function of time to go when the inventory level is fixed at $I=2$.

In region A of Example 1 and in regions A and C of Example 2, as the end of the season approaches, the retailer's strategy is to increase its bid and to decrease its price (for a fixed level of inventory). The higher bid and lower price both serve to increase the demand, and they act as complements. In contrast, in region B of Examples 1 and 2, the optimal price increases as the end of the season approaches. This is the region where the bid and price are used as substitutes: The bid and the price both increase, thereby pushing demand in opposite directions.

Note that, in both Examples 1 and 2, the optimal price is nonmonotonic in the number of remaining periods. The nonmonotonicity arises from two opposing effects. As the end of the season approaches, on one hand, the risk of excess inventory looms large and the marginal value of inventory decreases. This gives the retailer an incentive to reduce the price. On the other hand, the retailer's optimal bid continues to increase, which leads to an increase in the reservation price of customers. This gives the retailer an incentive to increase its price. In region A of Example 1 and in regions A and C of Example 2, the retailer's concern for excess inventory outweighs its wish to exploit higher reservation prices. Hence, as the end of season approaches, the retailer's optimal price decreases—this is where the bid and price are used as complements. However, in region B of Examples 1 and 2, the retailer's wish to exploit higher reservation prices overrides its concern for excess inventory. Thus, the optimal price starts to increase in the number of remaining periods—this is where the bid and price are used as substitutes.

Intuitively, the bid and the price are more likely to be used as substitutes when the retailer increases its bid sharply, because a sharp increase in the bid leads

Figure 4 **Two Distinct Examples of the Optimal Policy**



Note. The parameter setting here is the same as for Figure 2.

to a significant increase in the reservation price. One instance in which the bid shows a sharp increase is when the retailer switches to a higher mode of bidding. This is what happens, for instance, in region B of Example 2. The following proposition proves that, in such instances (i.e., whenever the retailer's bid switches to a higher mode), the optimal price will also increase.

Proposition 2. *Suppose either Assumption* 2(a) *holds in the strict sense, or Assumptions* 2(b) *and* 3 *hold in the strict sense. Then, when a decrease in* Δ *(which may happen when inventory increases or time to go decreases) triggers the optimal bid to switch to a higher mode (i.e., the optimal bid jumps up at a point of discontinuity), the optimal price as a function of the marginal value of inventory, i.e.,* $p^*(\Delta)$*, jumps up.*[7]

As in region B of Example 2, the price increases as the end of the season approaches in region B of Example 1. The intuition behind this behavior is the same. Comparing regions A and B in Example 1, observe that the bid increases much faster in region B. This fast increase in the bid induces a fast increase in the reservation price, thereby resulting in an increase in the optimal price.

As discussed in the introduction, there exists yet another way in which the bid and the price could be used as substitutes: In response to a decrease in the marginal value, which indicates the retailer is now more overstocked, the retailer could reduce the price (to increase the demand) while also reducing the bid (to cut down on the advertising costs). However, we now know that such a pair of decisions would never arise in the optimal policy; Proposition 1 guarantees that when the marginal value of inventory decreases, the optimal bid increases. Intuitively, reducing the bid would reduce the reservation prices of the clicking customers and, thus, the retailer would have to take unattractively large price cuts to stimulate more demand.

It is interesting that the online bakery's bidding and pricing decisions on August 27 and September 8 are in compliance with our optimal policy (see Figure 1). According to our optimal policy, when the bid jumps (as it apparently did on August 27 and September 8), the price should jump in the same direction. This is indeed what happened with the retailer's price on those two days. Of course, we do not know how the retailer's inventory changed on those days, so we cannot say if the retailer's actions were indeed optimal according to our model. We should also note that there are instances where the retailer's decisions violate our optimal policy. For example, on August 29, it appears that the retailer "turned on" its bid and reduced the prices of two products. Such a coupling of bidding and pricing decisions would never be optimal under our model. This might be due to reasons outside of

[7] If both Assumptions 2(a) and 3 hold in the weak sense, then there might exist instances when mode switches do not trigger a jump in price.

our model, e.g., the risk attitude of the decision maker, the actions of a competitor, the additional production of mooncake, or lower-than-expected sales through other channels.

4.3. The Conversion Rate

Next we answer our second research question: (Q2) *How does the conversion rate change as the retailer reacts to changes in inventory/time*? Consider an increase in inventory (at a fixed time) or a decrease in time to go (at a fixed inventory). Both changes put more pressure on the retailer to move the inventory more quickly. When the bid and the price are complements, it is clear what happens to the conversion rate: The bid increases and the price decreases, both of which improve the conversion rate. However, when the bid and the price are substitutes, they will both increase, and the net effect on the conversion rate is not clear. The higher bid lifts up the customers' reservation prices and, hence, increases the conversion rate, whereas the higher price decreases the conversion rate. Nevertheless, we find that the effect of the bid always wins out over the effect of the price. Hence, whenever the inventory increases or time to go decreases, the conversion rate increases.

Proposition 3. *If the marginal value Δ decreases, the optimal bid $b^*(\Delta)$ and the optimal price $p^*(\Delta)$ change so that the conversion rate, $\bar{F}(p^*(\Delta) \mid b^*(\Delta))$, increases. Thus, in the optimal strategy, when the inventory increases or time to go decreases, the conversion rate increases.*

This proposition implies that, even when the retailer is using the bid and price as substitutes, the change in price should not be so large as to override the effect of the bid on the conversion rate.

5. When Is the Multimode Bidding Policy Optimal?

In this section, we turn to our third research question: (Q3) *Under what circumstances is it optimal to use on-and-off bidding*? In Figure 2, we already observed a numerical example (Example 2), in which an on-and-off bidding policy was optimal. The on-and-off bidding policy is a special case of the multimode bidding policy (see Definition 1), which arises when $b^*(\Delta)$ is discontinuous. Next we show that this discontinuity and, thus, the optimality of the multimode bidding policy may owe to the shape of the click-rate function and properties of the reservation price distribution.

In what follows, we first focus on the case wherein only the click rate depends on the bid (in §5.1) and then on the case wherein only the reservation price depends on the bid (in §5.2). By focusing on the click rate and the reservation price, respectively, we are able to derive clear-cut conditions for the optimality of single-mode and multimode bidding policies. Through these conditions, the drivers of the multimode bidding policy are easier to appreciate. In §5.3, we summarize by providing a generalized condition that incorporates the bid's effects on both the click rate and the reservation price.

5.1. The Bid's Effect on the Click Rate

Consider the case wherein the customer's reservation price, $Y(b)$, does not depend on the bid, b, while the click rate, $\lambda(b)$, does. As the next proposition states, the multimode bidding policy would never arise if the click rate were a concave function of the bid.

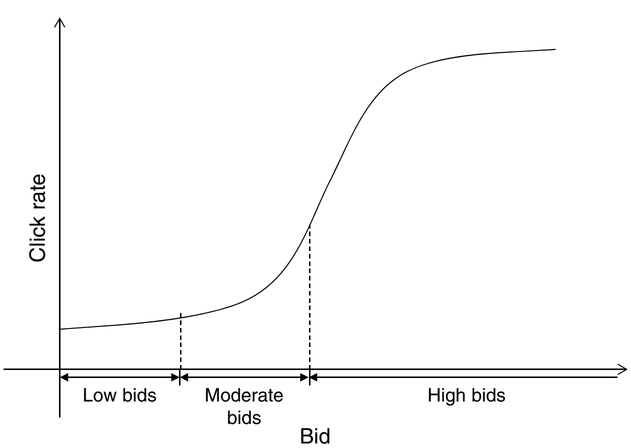
Proposition 4. *If the click rate, $\lambda(b)$, is strictly concave in the bid, b, then the optimal bid, $b^*(\Delta)$, is continuous and a single-mode bidding policy is optimal.*

Thus, when the click rate is concave increasing and the reservation price does not depend on the bid, the optimal bid's behavior will be similar to what we observed in Example 1 of Figure 2. Interestingly, in that figure also, the click rate was concave increasing in the bid.

When $\lambda(b)$ is concave, the improvement in the click rate diminishes as the bid increases. This assumption is realistic for high-end bids. After all, once the retailer's link is the top-ranked link, any further increase in the bid will not improve the click rate. However, one would expect that this assumption would not hold at low to moderate bids. When the retailer sticks to low bids, the link will appear toward the bottom of the page or, worse yet, the link will fall off the first page. Few customers will scroll far down the page or go beyond the first page of search results. Hence, we expect the click rate to be low and rather flat in the region of low bids. As the bid keeps increasing, it will reach a moderate level that will catapult the link to the easily visible range of search results, thus turning the click rate from flat to sharply increasing. Hence, as depicted in Figure 5, we expect the click rate to have a convex region, corresponding to low to moderate bids.

Suppose now that the click-rate function has the shape shown in Figure 5. If the retailer's inventory is low relative to the time remaining until the end

Figure 5 The Likely Shape of the Click-Rate Function



of the selling season (i.e., if Δ is large), the retailer will not advertise heavily and the optimal bid will be found in the region of low bids. As the end of the season approaches, if the sales proceed slowly and the inventory becomes larger relative to the remaining time (i.e., if Δ starts to decrease), the retailer will now find it attractive to increase its bid. However, as the retailer starts to increase its optimal bid, it would skip the region of bids where the click rate is turning from flat to sharply increasing (the region labeled as "moderate bids" in Figure 5). This is why multimode bidding may arise in the optimal solution: Every time there is a range of bids the retailer would never utilize, the optimal bid will jump from one side of that range to the other as time and inventory change.

Why would the retailer skip the moderate bids? To see the intuition behind this, first note that, in that region, the number of additional clicks brought by a marginally higher bid is increasing very fast. Therefore, to fully exploit the fast increase in the click rate, the retailer would always find it attractive to push its bid all the way to the end of that region. Hence, the retailer's optimal bid will skip the moderate bids and will move from the region of low bids into the region of high bids. The following proposition formalizes this result, as it shows that a bid b is never optimal if the click rate is sufficiently convex at that bid.

Proposition 5. *Consider a bid b such that $1/\lambda(b)$ is strictly concave at b. Equivalently, $\lambda(b)$ is convex enough that $\lambda''(b) > 2(\lambda'(b))^2/\lambda(b)$. Such a bid b is never optimal at any given Δ.*

The click-rate function depicted in Figure 5 is essentially an S-shaped curve. Such curves have been widely used in the literature to model the relationship between customers' attention and advertising spending (e.g., Little 1979 and Villas-Boas 1993). One particular S-shaped curve used in that domain is the logistic function (see, e.g., Johansson 1979, Tan and Mookerjee 2005):

$$\lambda(b) = \frac{\lambda_\infty + \lambda_0 e^{\beta-\alpha b}}{1+e^{\beta-\alpha b}}. \tag{5}$$

When interpreting this function as the click rate, λ_∞ stands for the market size, i.e., the click rate the retailer would achieve if it set the bid arbitrarily high. The parameter λ_0 calibrates the click rate we would see if the retailer did not bid at all. The parameter α measures the effectiveness of the bid. Finally, β captures the effect of all variables other than the bid that influence the click rate (e.g., the website's relevance to the search keyword). If we specialize our model to the click-rate function in (5), Proposition 5 boils down to the following corollary, which states that the retailer would never pick a certain range of bids.

Corollary 1. *Suppose the click-rate function $\lambda(b)$ is given by* (5). *Then, it is never optimal to pick a bid $b \in (0, (\beta+\ln(\lambda_0/\lambda_\infty))/\alpha)$.*

Therefore, when the click rate is given by (5), the optimal bid will be either zero or a range of high bids above a certain threshold (given by $(\beta+\ln(\lambda_0/\lambda_\infty))/\alpha$). This result sheds light on the behavior we observed in Example 2 of Figure 2: The click rate in that example is an S-shaped curve, which follows (5) with $\lambda_0 = 0.4$, $\lambda_\infty = 1$, $\beta = 4$, and $\alpha = 4$. Hence, as the inventory or time to go changes, the retailer's optimal bid is either zero or is found among a range of high bids.

5.2. The Bid's Effect on Customers' Reservation Price

Next consider the case where the click rate, $\lambda(b)$, is independent of the bid, b. We focus on the the case where the randomness of the reservation price $Y(b)$ is additive; i.e., we assume the bid influences the reservation price only through its effect on $\mu(b)$, which shifts the mean, so that $Y(b) = \mu(b) + X$. A result analogous to Proposition 5 holds.

Proposition 6. *A bid b is never optimal for any marginal value Δ if $\mu''(b) > 0$ at b.*

Proposition 6 suggests that the optimal bid never falls in the region where a marginally higher bid shifts the reservation price with increasing speed. When the mean reservation price follows the logistic form (analogous to the S-shaped curve in (5)), Proposition 6 leads to the following corollary, which specifies the range of bids that will never be optimal.

Corollary 2. *Suppose $\mu(b)$ is given by the logistic function, i.e., $\mu(b) = (\mu_\infty + \mu_0 e^{\beta-\alpha b})/(1+e^{\beta-\alpha b})$. Then, it is never optimal to pick a bid $b \in (0, \beta/\alpha)$.*

Consequently, when the mean reservation price follows the logistic function, the retailer's optimal bid will be either zero or a range of high bids above a certain threshold.

5.3. The General Case

To summarize, we consider the general case when the bid influences both the click rate and the reservation price. In other words, all the functions—$\lambda(b)$, $\mu(b)$, and $\sigma(b)$—depend on b. Under this general case, our earlier Propositions 5 and 6 extend to the following result.

Proposition 7. *A bid $b > 0$ is never optimal for any marginal value of inventory Δ if* (i) *$\lambda''(b) > 2(\lambda'(b))^2/\lambda(b)$, and* (ii) *$\sigma''(b) > 0$ and $(\mu(b)/\sigma(b))'' > 0$.*

The insight obtained through Propositions 5 and 6 are reinforced by Proposition 7: The optimal bid never falls in the region where the click rate and the mean of the reservation price are sufficiently convex. That is, the optimal bid will not be found in a region where a marginally higher bid shifts both the click rate and the reservation price with increasing speed.

6. A Heuristic: Static On-and-Off Bidding

We observed in §5 that on-and-off bidding might arise in the optimal solution as a special case of the multimode bidding policy. When following the on-and-off bidding policy, the retailer either bids zero (the "off" mode) or updates its bid within a high range (the "on" mode) as the inventory and time change. Observe from Example 2 of Figure 2 that the optimal bid lands within a narrow band when the bid is on. Motivated by this observation, it is tempting to consider a static version of the on-and-off bidding policy as a heuristic. In our heuristic, the retailer picks a fixed bid for the "on" mode at the beginning of the season. In the course of the season, at the beginning of every period, the retailer decides whether to use this fixed bid or to bid zero. In addition, at the beginning of every period, the retailer adjusts its price. The fixed bid for the "on" mode is picked so that the expected profit through this policy is maximized. In this section, we analyze the performance of this heuristic, which we refer to as the *static on-and-off bidding policy*.

To evaluate the performance of the heuristic policy, we conduct a numerical study. We consider a retailer with 10 units in inventory and 15 periods to go. We let X follow a Gamma distribution with shape parameter k and scale parameter θ. To generate problem instances, we let k take on one of five different values ($\{1,3,5,7,9\}$) with θ chosen accordingly so that X has a mean of three. We adopt the S-shaped click-rate function in (5) with λ_∞ normalized to be 1 and with each of λ_0, α and β taking one of six different values. In addition, we let

$$\mu(b) = \gamma_\mu b^\kappa \quad \text{and} \quad \sigma(b) = 1 + \gamma_\sigma b^\kappa,$$

where each of γ_μ and γ_σ can take one of four different values. Note the significance of κ: When $\kappa < 1$, $\mu(b)$ and $\sigma(b)$ functions are concave. When $\kappa > 1$, the two functions become convex. For κ, we test six different values ranging from 0.4 to 1.4, which cover both the convex and concave cases for $\mu(b)$ and $\sigma(b)$ functions. The parameter values used in our numerical study are summarized in Table 1. All in all, we are testing 103,680 instances. Table 2 provides the summary statistics.

For each instance, we evaluate the optimality gap, i.e., the difference between the optimal expected profit and the profit from the heuristic, expressed as a percentage of the optimal expected profit. Table 2 provides summary statistics observed across 103,680 instances. (To avoid giving unfair advantage to the heuristic, we removed instances where the optimality gap is zero; those are likely to be the trivial instances where the optimal policy is not to make a bid throughout the season.) As the summary statistics show, the static on-and-off bidding policy comes very close to the optimal solution. In particular, in 95% of the instances, the optimality gap is less than 1%. On average, the optimality gap is only 0.27%.

In addition, we observe that the parameter κ influences the optimality gap in an interesting way. Table 3 shows the average optimality gap for different values of κ. Note from the table that when $\kappa < 1$, the optimality gap is relatively unaffected by the magnitude of κ. This is the region where $\mu(b)$ and $\sigma(b)$ are concave. Once we reach $\kappa > 1$ and beyond, the functions $\mu(b)$ and $\sigma(b)$ become increasingly convex. It is in this region that the optimality gap is smaller, and it becomes even smaller as κ increases further. This behavior can be understood based on the insights obtained in §5. When $\mu(b)$ and $\sigma(b)$ are sufficiently convex, multimode bidding is likely to be optimal, and multimode bidding can be approximated by the static on-and-off heuristic. This explains why the heuristic performs better once $\mu(b)$ and $\sigma(b)$ are convex.

Table 1 Parameter Settings for the Numerical Test

Parameter	Set of values
k	$\{1,3,5,7,9\}$
α	$\{1,2,3,4,5,6\}$
β	$\{1,2,3,4,5,6\}$
λ_0	$\{0.1,0.2,0.3,0.4,0.5,0.6\}$
γ_μ	$\{0.1,0.2,0.3,0.4\}$
γ_σ	$\{0.1,0.2,0.3,0.4\}$
κ	$\{0.4,0.6,0.8,1.0,1.2,1.4\}$

Table 2 Statistics of the Optimality Gap

Statistics[a]	Value (%)
Mean	0.27
Standard deviation	0.25
Maximum and minimum	$(0, 2.89]$
95% confidence interval	$(0, 0.82]$

[a]When calculating the statistics, we removed all instances with zero optimality gap.

Table 3 The Average Optimality Gap for Different Values of κ

The value of κ	0.4	0.6	0.8	1.0	1.2	1.4
Average optimality gap* (%)	0.28	0.27	0.27	0.28	0.23	0.17

*When calculating the statistics, we removed all instances with zero optimality gap.

7. Extensions

So far we have assumed that the reservation price, $Y(b)$, is given by $Y(b) = \mu(b) + \sigma(b)X$. In the first extension, we drop the assumption that $Y(b)$ follows this specific functional form, but we continue to assume that higher bids tend to increase the reservation price. Our earlier model is a special case. In the second extension, we reverse the effect of the bid and we analyze the case where higher bids tend to decrease the reservation price. This extension is aligned with the empirical observations of Agarwal et al. (2011), who find that conversion rates are lower for higher-positioned links.

7.1. Generalizing the Relationship Between the Bid and Reservation Price

In this section, we make the following assumptions about the customers' reservation price $Y(b)$—the base model we specified in §3.2 is a special case of this assumption.

Assumption 5. *The reservation price $Y(b)$ and its distribution $F(p \mid b)$ satisfy the following*:

(a) *At any given bid b, $Y(b)$ has an increasing failure rate; i.e., $f(p \mid b)/\bar{F}(p \mid b)$ increases in p.*

(b) *The reservation price $Y(b)$ stochastically increases (in failure rate ordering) as b increases; i.e., $f(p \mid b)/\bar{F}(p \mid b)$ decreases in b.*

(c) *Define $\eta(p, b) = |(p\partial_p \bar{F}(p \mid b))/(b\partial_b \bar{F}(p \mid b))|$. Then, $\eta(p, b)$ increases in p.*

Assumption 5(a) is technical in that it is satisfied by many standard probability distributions. Assumption 5(b) captures the main relationship between the bid and reservation price: As the bid increases, the reservation price tends to increase. Assumption 5(c) has a meaningful interpretation in the context of "the price elasticity of the conversion rate" and "the bid elasticity of the conversion rate." To interpret Assumption 5(c), we first rewrite $\eta(p, b)$ as follows:

$$\eta(p, b) = \left| \frac{(\partial_p \bar{F}(p \mid b)/\bar{F}(p \mid b))/(1/p)}{(\partial_b \bar{F}(p \mid b)/\bar{F}(p \mid b))/(1/b)} \right|.$$

Note that the numerator and denominator correspond to elasticity terms. Specifically, the numerator is the absolute price elasticity of the conversion rate, which increases in price p (by Assumption 5(a)). The denominator is the absolute bid elasticity of the conversion rate, which also increases in price p.[8] Given these observations, Assumption 5(c) requires that the absolute price elasticity increase faster than the absolute bid elasticity.

Under Assumption 5, all the results in §4 (i.e., Propositions 1–3, which describe the structure of the optimal policy) continue to hold. Based on this extension, it appears that our results are quite robust as long as higher bids tend to increase the reservation price.

7.2. When Higher Bids Lead to Lower Reservation Prices

To model this case, we use the assumption that $Y(b)$ stochastically decreases in b (in failure rate ordering). More precisely, we make the following assumption.

Assumption 6. *The reservation price $Y(b)$ and its distribution $F(p \mid b)$ satisfy the following*:

(a) *At any given bid b, $Y(b)$ has an increasing failure rate; i.e., $f(p \mid b)/\bar{F}(p \mid b)$ increases in p.*

(b) *$Y(b)$ stochastically decreases (in failure rate ordering) as b increases; i.e., $f(p \mid b)/\bar{F}(p \mid b)$ increases in b.*

One property of the optimal policy has a natural extension to this new model. In the base model, higher bids lead to higher reservation prices and, therefore, the optimal price jumps up whenever the optimal bid jumps up (this result is stated in Proposition 2). In the new model, higher bids lead to lower reservation prices and, hence, this result is reversed, as stated in the following proposition.

Proposition 8. *Suppose Assumption 6(b) holds in the strict sense. Then, whenever a decrease in Δ triggers the optimal bid to switch to a higher mode, the optimal price as a function of the marginal value of inventory, i.e., $p^*(\Delta)$, jumps down.*

In the remainder of this section, we show that, under an additional plausible assumption, the behavior of the optimal bid remains the same as before whereas the behavior of the optimal price becomes much more predictable than it was in the base model.

To facilitate the remainder of the discussion, let us first define "the probability of making a sale" in a period when the price is p and the bid is b. This probability is given by the click rate (i.e., $\lambda(b)$) multiplied by the conversion rate (i.e., $\bar{F}(p \mid b)$). Under the model of the present section, the effect of the bid on this probability is ambiguous. On one hand, higher bids lead to higher click rates. On the other hand, higher bids lead to lower conversion rates (because the reservation price stochastically decreases in the bid). The net effect of the bid on the probability of making a sale now depends on which of these two effects dominates. In the remainder of this section, we assume that the positive effect on the click rate dominates the negative effect on the conversion rate. Thus, everything else being equal, the probability of making a sale increases in the bid. This assumption is reasonable. After all, if higher bids do not lead to higher probabilities of making a sale, then the appeal of sponsored search marketing is questionable.

Under this additional assumption, we obtain the following result about the optimal policy.

Proposition 9. *Assume the probability of making a sale; i.e., $\lambda(b)\bar{F}(p \mid b)$ is increasing in b for all values of p. Then, the following properties hold for the optimal policy*:

(a) *The optimal bid as a function of the marginal value of inventory, i.e., $b^*(\Delta)$, decreases in Δ. Consequently, when the inventory increases or time to go decreases, the optimal bid as a function of inventory and time to go, $b^*(y, t)$, increases.*

(b) *The optimal price as a function of the marginal value of inventory, i.e., $p^*(\Delta)$, increases in Δ. Consequently, when the inventory increases or time to go decreases, the optimal price as a function of inventory and time to go, $p^*(y, t)$, decreases.*

[8] To see why, note that $\partial_p((\partial_b \bar{F}(p \mid b))/\bar{F}(p \mid b)) = \partial^2_{p,b} \ln(\bar{F}(p \mid b)) = -\partial_b(f(p \mid b)/\bar{F}(p \mid b))$, which is positive by Assumption 5(b).

In the base model, when the inventory increases or time to go decreases (i.e., whenever the retailer faces more pressure to move inventory quickly), the retailer increases its bid (see Proposition 1). Under the new model, this earlier result remains intact because of our assumption that higher bids lead to higher probabilities of making a sale—a retailer that is under pressure to sell quickly can do so by increasing its bid.

In the base model, when the retailer faces more pressure to move inventory, the optimal price can go either way, because of two opposing effects: (i) On one hand, the marginal value of inventory decreases, thus giving the retailer an incentive to reduce the price. (ii) On the other hand, the retailer's optimal bid increases, which leads to higher reservation prices, thus giving the retailer an incentive to increase the price. Under the new model, however, the behavior of price is now simpler. In particular, effect (i) is still present in the new model and effect (ii) is now reversed as higher bids lead to lower reservation prices, thus giving the retailer yet another reason to reduce the price. Thus, whenever the inventory increases or time to go decreases, the optimal price decreases.

8. Conclusion

Sponsored search marketing has become a reliable marketing tool for online retailers to attract customer traffic. In this paper, we investigate an online retailer's dual use of sponsored search marketing and dynamic pricing. Such decisions involve complicated trade-offs. On one hand, the retailer faces the usual dynamic pricing trade-off: Too high a price may reduce the sales and lead to leftover inventory, yet too low a price leaves money on the table. On the other hand, the retailer needs to choose the right bid: It is unwise to bid so high as to unnecessarily sacrifice profit margin, yet the bid should be high enough to generate sufficiently high traffic. However, the real challenge is that there is an interaction between the pricing and bidding trade-offs: The retailer's bid on the keyword moves the rank of its sponsored link and, therefore, influences the customer's reservation price, which factors into the retailer's pricing decision.

We develop a model that incorporates the above trade-offs and interactions. In particular, in the base model, we assume that higher bids lead to higher reservation prices. In an extension, we analyze the case wherein higher bids lead to lower reservation prices.

In the base model, where higher bids lead to higher reservation prices, we identify the following structure for the optimal bidding policy. (Here we summarize how the optimal bid changes with time; we obtained similar results for the effect of inventory.) For a fixed level of inventory, as the retailer gets closer to the end of the season, the optimal bid increases. In particular, the optimal bid exhibits smooth increases followed by big jumps. The monotonicity of the optimal bid is intuitive: As the end of season approaches, the retailer is under increasing pressure to move inventory, thus giving the retailer an incentive to advertise more aggressively. On the other hand, the big jumps in the optimal bid occur when the click rate or the reservation price "takes off" once the bid reaches a threshold. In such a case, the retailer will choose a bid beyond the take-off region. This happens, for example, when the click rate is an S-shaped function of the bid.

As for the optimal pricing strategy, the accompanying optimal price may increase or decrease as the end of the season approaches and the bid increases. The case wherein the optimal price decreases is intuitive—that is the case wherein the retailer is using both the bid and the price as complements to move its inventory quickly as the end of the season is approaching. However, the case wherein the optimal price increases is in contrast to the conventional wisdom of dynamic pricing. In those instances, the retailer is now using the bid and price as substitutes. More specifically, those are instances where a sharp increase in the bid (sometimes in the form of an upward jump) causes a significant increase in the customers' reservation prices, and the retailer increases the price to exploit the higher willingness to pay.

Our results also provide implementation notes for retailers. When a retailer takes a break from bidding on a keyword, the retailer might want to mark down its price simultaneously. Symmetrically, when a retailer starts bidding again, it might want to increase its price. However, the increase in price should never be so high as to cancel the improvement in conversion rate caused by the higher bid. In addition, our numerical results indicate that, instead of adjusting the bid around the clock, the retailer can achieve similarly good results using a static on-and-off bidding heuristic.

In the extension, where higher bids lead to lower reservation prices, the results regarding the optimal bid are largely the same as in the base model. As for the optimal price, its behavior no longer violates the conventional wisdom of dynamic pricing: If the inventory increases or the remaining time decreases, then the optimal price decreases as well.

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Appendix. Proofs

Additional Notation

We first define additional notation to be used throughout our proofs. In particular, in some instances, it will be convenient to use the following change of variable to replace the price p:

$$\xi(b, p) := \frac{p - \mu(b)}{\sigma(b)}.$$

Given this definition, $F(p \mid b)$ can be written as $G(\xi(b,p))$. Hence, through the change of variable ($x = \xi(b,p)$), the profit function in (3) can be rewritten as

$$\tilde{\pi}(b,x,\Delta) := \lambda(b)[\bar{G}(x)(\mu(b)+\sigma(b)x-\Delta)-b], \tag{6}$$

where x is treated as a decision variable. Define $x^*(b,\Delta)$ as the optimal value of x given b and Δ; i.e.,

$$x^*(b,\Delta) := \arg\max_x \tilde{\pi}(b,x,\Delta). \tag{7}$$

In addition, define $x^*(\Delta)$ as the optimal value of x when the retailer uses the optimal bid $b^*(\Delta)$:

$$x^*(\Delta) = x^*(b^*(\Delta),\Delta).$$

Note that the relationship between $x^*(\Delta)$ and the optimal price $p^*(\Delta)$ is given by

$$p^*(\Delta) = \mu(b^*(\Delta)) + \sigma(b^*(\Delta))x^*(\Delta). \tag{8}$$

Further, define the induced profit function, $\pi(b,\Delta)$, as the profit obtained by maximizing $\tilde{\pi}(b,x,\Delta)$ over x:

$$\pi(b,\Delta) = \max_x \tilde{\pi}(b,x,\Delta) = \tilde{\pi}(b,x^*(b,\Delta),\Delta).$$

Alternatively, the induced profit function $\pi(b,\Delta)$ can also be obtained by maximizing $\pi(b,p,\Delta)$ over p:

$$\pi(b,\Delta) = \max_p \pi(b,p,\Delta).$$

Supporting Lemmas
Next we develop two supporting lemmas about $\xi(b,p)$ and $x^*(b,\Delta)$ that will be used by later proofs.

Lemma A.1. *$\xi(b,p)$ is increasing in p and decreasing in b.*

Proof of Lemma A.1. The monotonicity of $\xi(b,p)$ in p is clear from its definition. It is decreasing in b since $p/\sigma(b)$ is decreasing in b by Assumption 2(a), and $\mu(b)/\sigma(b)$ is increasing in b by Assumption 2(b). $\square$

Lemma A.2. *In the optimal solution of the dynamic program* (2), *for any given y and t, we have the following*:
(a) $x^*(\Delta(y,t)) < \bar{x}$.
(b) $\Delta(y,t) < \mu(b^*(y,t)) + \sigma(b^*(y,t))\bar{x}$.

Proof of Lemma A.2. To prove part (a), first note that setting $x^*(\Delta(y,t)) = \bar{x}$ means not selling in period t. Any policy $\mathcal{P}$ with such an action is dominated by a modified policy $\mathcal{P}'$. Here we provide a sketch of the proof. Clearly, such an action is not optimal in the last period. Suppose now $t \geq 2$. Suppose policy $\mathcal{P}'$ replicates in periods t through 2 the actions of policy $\mathcal{P}$ in periods $t-1$ through 1. Then, the expected profit achieved by $\mathcal{P}'$ over periods t through 2 is the same as the expected profit achieved by $\mathcal{P}$ over periods $t-1$ through 1. In addition, the policy $\mathcal{P}'$ has another period to go, in which there is a non-zero probability of making profit. Hence, $\mathcal{P}'$ dominates $\mathcal{P}$.

We prove part (b) by contradiction. Suppose that, in period t with y units of inventory, $\Delta(y,t)$ and $b^*(y,t)$ are such that $\Delta(y,t) \geq \mu(b^*(y,t)) + \sigma(b^*(y,t))\bar{x}$. From the preamble to the proofs, we know that the retailer must choose x to maximize (see Equation (6))

$$\begin{aligned}\tilde{\pi}(b^*(y,t),x,\Delta(y,t)) = \lambda(b^*(y,t))\{&\bar{G}(x)[\mu(b^*(y,t))\\ &+\sigma(b^*(y,t))x-\Delta(y,t)]-b^*(y,t)\}.\end{aligned}$$

Observe from above that for any $x < \bar{x}$, the term in square brackets is strictly negative. Therefore, the best the retailer can do is to set $x = \bar{x}$ so that $\bar{G}(x)[\mu(b^*(y,t)) + \sigma(b^*(y,t))x - \Delta(y,t)] = 0$. From part (a), we know that $x = \bar{x}$ cannot arise in the optimal solution of the dynamic program. Therefore, we conclude that $\Delta(y,t)$ and $b^*(y,t)$ cannot satisfy $\Delta(y,t) \geq \mu(b^*(y,t)) + \sigma(b^*(y,t))\bar{x}$. $\square$

Lemma A.3. *Define $\mathcal{L} = \{(b,\Delta): \Delta < \mu(b)+\sigma(b)\bar{x}\}$. Given $(b,\Delta) \in \mathcal{L}$, $x^*(b,\Delta)$ satisfies the following*:
(a) *$x^*(b,\Delta)$ is the unique value of x that satisfies $\partial_x \tilde{\pi}(b,x,\Delta) = 0$. Equivalently, $x^*(b,\Delta)$ is the unique solution to $\kappa(b,x,\Delta) = 1$, where*

$$\kappa(b,x,\Delta) := \frac{g(x)}{\bar{G}(x)}\left(x + \frac{\mu(b)-\Delta}{\sigma(b)}\right). \tag{9}$$

(b) *$x^*(b,\Delta)$ is decreasing in b.*
(c) *$x^*(b,\Delta)$ is increasing in Δ.*
(d) *$x^*(b,\Delta)$ is continuously differentiable with respect to both b and Δ.*

Proof of Lemma A.3. To prove (a), we first show that $x^*(b,\Delta)$ is an interior solution; i.e., $x^*(b,\Delta) \in (\underline{x},\bar{x})$.

To see why $x^*(b,\Delta) < \bar{x}$, recall that the retailer must choose x to maximize

$$\tilde{\pi}(b,x,\Delta) = \lambda(b)\{\bar{G}(x)[\mu(b)+\sigma(b)x-\Delta]-b\}.$$

Observe from above that if the retailer sets $x = \bar{x}$, then $\tilde{\pi}(b,x,\Delta) = -\lambda(b)b$. Observe also that the term in square brackets is strictly positive when $x = \bar{x}$ (because $(b,\Delta) \in \mathcal{L}$). Therefore, by decreasing x slightly below $\bar{x}$, we could ensure that the term in brackets remains positive and $\bar{G}(x)$ becomes strictly positive. Consequently, we attain $\tilde{\pi}(b,x,\Delta) > -\lambda(b)b$. Thus, it is never optimal to set $x = \bar{x}$.

To see why $x^*(b,\Delta) > \underline{x}$, note that the derivative of $\tilde{\pi}(b,x,\Delta)$ with respect to x is

$$\partial_x \tilde{\pi}(b,x,\Delta) = \lambda(b)\sigma(b)\bar{G}(x)[1-\kappa(b,x,\Delta)].$$

At $x = \underline{x}$, $\kappa(b,x,\Delta)$ is zero, so this derivative is strictly positive. Hence, it is never optimal to set $x = \underline{x}$.

Because $x^*(b)$ is an interior solution, it must satisfy the first-order condition, which boils down to $\kappa(b,x,\Delta) = 1$. It remains to show that the solution to $\kappa(b,x,\Delta) = 1$ is unique. Observe that once $\kappa(b,x,\Delta)$ becomes positive, it is monotonically increasing (because $G(\cdot)$ is IFR by Assumption 3). Hence, there exists a unique x at which $\kappa(b,x,\Delta) = 1$.

To see parts (b) and (c), first note that $\kappa(b,x^*(b,\Delta),\Delta) \equiv 1$ by part (a). Note that $\kappa(b,x,\Delta)$ is increasing in b (because $-1/\sigma(b)$ and $\mu(b)/\sigma(b)$ are both increasing in b by Assumption 2(a) and (b), respectively). Furthermore, whenever $\kappa(b,x,\Delta) > 0$, we have $\kappa(b,x,\Delta)$ is increasing in x (as already argued in the last paragraph). Therefore, when b increases, $x^*(b,\Delta)$ must decrease to recover $\kappa(b,x^*(b,\Delta),\Delta) = 1$. Similarly, since $\kappa(b,x,\Delta)$ is decreasing in Δ, if Δ increases, then $x^*(b,\Delta)$ must increase.

To see part (d), apply the Implicit Function Theorem to $\kappa(b,x^*(b,\Delta),\Delta) \equiv 1$. $\square$

Proofs of the Results for the Base Model

Proof of Lemma 1. As is stated in Lemma A.1, $F(p \mid b) = G(\xi(b, p))$. By Lemma A.1, $\xi(b, p)$ decreases in b at any $p \geq 0$. Hence, $F(p \mid b)$ decreases in b at any $p \geq 0$, which, by definition, means $Y(b)$ stochastically increases in b (in the sense of first-order stochastic dominance). □

Proof of Lemma 2. We omit the proof, which is similar to that in Bitran and Mondschein (1997). □

Proof of Proposition 1. Consider the set defined in Lemma A.3, $\mathscr{L} = \{(b, \Delta): \Delta < \mu(b) + \sigma(b)\bar{x}\}$, which is a lattice. In this proof, we restrict our attention to $(b, \Delta) \in \mathscr{L}$ because we know from Lemma A.2(b) that the optimal value of b is in this region. We will show that $\partial^2_{b,\Delta}\pi(b, \Delta)$ is continuous and negative for $(b, \Delta) \in \mathscr{L}$, thus concluding that $\pi(b, \Delta)$ is submodular, which is sufficient to conclude that $b^*(\Delta)$ is decreasing in Δ. Taking the cross-partial derivative, we obtain the following:

$$\begin{aligned}\partial^2_{b,\Delta}\pi(b, \Delta) &= \partial^2_{b,\Delta}\tilde{\pi}(b, x^*(b, \Delta), \Delta)\\ &= \partial_b[\partial_\Delta \tilde{\pi}(b, x, \Delta)|_{x^*(b,\Delta)}]\\ &\quad \text{by Lemma A.3(a) and the Envelope Theorem]}\\ &= -\partial_b[\lambda(b)\bar{G}(x^*(b, \Delta))],\end{aligned}$$

which is continuous because: (i) $\lambda(b)$ and $G(\cdot)$ have continuous derivatives by Assumptions 1 and 4(c), and (ii) $x^*(b, \Delta)$ is continuously differentiable by Lemma A.3(d). $\partial^2_{b,\Delta}\pi(b, \Delta)$ is also negative since $\lambda(b)$ is increasing in b, and $\bar{G}(x^*(b, \Delta))$ is increasing in b by Lemma A.3(b).

Furthermore, $\Delta(y, t)$ decreases in y and increases in t by Lemma 2. Therefore, $b^*(y, t) = b^*(\Delta(y, t))$ increases in y and decreases in t. □

Proof of Proposition 2. Recall from Lemma A.3(a) that the optimal b and x, denoted by $b^*(\Delta)$ and $x^*(\Delta)$, must satisfy $\kappa(b^*(\Delta), x^*(\Delta), \Delta) = 1$ at any given Δ. Therefore, using the change of variable $x = \xi(b, p)$, the optimal $b^*(\Delta)$ and $p^*(\Delta)$ must satisfy the following equation at any given Δ:

$$\frac{1}{\sigma(b)}\frac{g(\xi(b, p))}{\bar{G}(\xi(b, p))}(p - \Delta) = 1, \tag{10}$$

To prove the desired result, it is sufficient to show that the left side of (10) is increasing in p and is strictly decreasing in b. These properties guarantee that whenever $b^*(\Delta)$ jumps up at a point of discontinuity, $p^*(\Delta)$ must jump up to recover the equality.

The left side of (10) is increasing in p because $g(\xi(b, p))/\bar{G}(\xi(b, p))$ is increasing in p, by Lemma A.1 and Assumption 3.

The left side of (10) is decreasing in b because $1/\sigma(b)$ is decreasing in b (by Assumption 2(a)) and $g(\xi(b, p))/\bar{G}(\xi(b, p))$ is decreasing in b (since $G(\cdot)$ is IFR and $\xi(b, p)$ decreases in b by Lemma A.1). Furthermore, the left side is strictly decreasing in b whenever $1/\sigma(b)$ is strictly decreasing in b (i.e., when Assumption 2(a) holds in the strict sense) or $g(\xi(b, p))/\bar{G}(\xi(b, p))$ is strictly decreasing in b (which would be the case when Assumption 2(b) and 3 hold in the strict sense). □

Proof of Proposition 3. Note that $\bar{F}(p^*(\Delta) \mid b^*(\Delta)) = \bar{G}(x^*(\Delta))$. It is sufficient to show that $x^*(\Delta)$ is increasing in Δ. As Δ increases, $b^*(\Delta)$ decreases by Proposition 1. Therefore, $x^*(\Delta) = x^*(b^*(\Delta), \Delta)$ increases in Δ since $x^*(b, \Delta)$ decreases in b and increases in Δ by Lemma A.3(b) and (c). □

Proof of Proposition 4. It is assumed in §5.1 that the customers' reservation price is independent of the bid; i.e., $\mu(b) \equiv 0$ and $\sigma(b) \equiv 1$. Therefore, we can write $\pi(b, \Delta) = \max_p \pi(b, p, \Delta)$ in the following way:

$$\pi(b, \Delta) = \lambda(b)(v^*(\Delta) - b),\quad \text{where } v^*(\Delta) = \max_p \bar{G}(p)(p - \Delta). \tag{11}$$

Taking the first and second derivatives of this objective function with respect to b, we have

$$\partial_b \pi(b, \Delta) = \lambda'(b)(v^*(\Delta) - b) - \lambda(b), \tag{12}$$

$$\partial^2_{b,b}\pi(b, \Delta) = \lambda''(b)(v^*(\Delta) - b) - 2\lambda'(b). \tag{13}$$

Since we assumed $\lambda(b)$ to be strictly concave, the second derivative is strictly negative when $b < v^*(\Delta)$. Hence, the first derivative is strictly decreasing for $b < v^*(\Delta)$. In addition, the first derivative is continuous in b, and $v^*(\Delta)$ is continuous in Δ. Therefore, the optimal solution, $b^*(\Delta)$, is continuous. □

Proof of Propositions 5 and 6 and Corollaries 1 and 2. Propositions 5 and 6 are special cases of Proposition 7, whose proof will come later. To prove Corollary 1, we show that when $\lambda(b)$ is the S-shaped function defined in (5), $(1/\lambda(b))'' < 0$ for $0 < b < (\beta + \ln(\lambda_0/\lambda_\infty))/\alpha$. Note that

$$\left(\frac{1}{\lambda(b)}\right)' = -\frac{\alpha(\lambda_\infty - \lambda_0)}{2\lambda_\infty\lambda_0 + \lambda_\infty^2 e^{\alpha b - \beta} + \lambda_0^2 e^{\beta - \alpha b}}.$$

Therefore, $(1/\lambda(b))'' < 0$ if and only if the denominator's derivative with respect to b is strictly negative, which is true if and only if $0 < b < (\beta + \ln(\lambda_0/\lambda_\infty))/\alpha$.

To prove Corollary 2, we show that when $\mu(b)$ is given by the logistic function, $\mu''(b) > 0$ for any $0 < b < \beta/\alpha$. This holds since the logistic function is strictly convex for $0 < b < \beta/\alpha$. □

Proof of Proposition 7. To prove the proposition, we show that, at such a b value and any given Δ, whenever $\partial_b\pi(b, \Delta) = 0$, we have $\partial^2_{b,b}\pi(b, \Delta) > 0$ and, therefore, such b is never optimal.

Define $v(b, x, \Delta) := \bar{G}(x)(\mu(b) + \sigma(b)x - \Delta)$. Then, we have $\pi(b, \Delta) = \lambda(b)(v(b, x^*(b, \Delta), \Delta) - b)$. In the following, we will omit the arguments of $\lambda(b)$, $v(b, x, \Delta)$ and $x^*(b, \Delta)$ for expositional convenience. Hence, $\partial_b\pi(b, \Delta)$ and $\partial^2_{b,b}\pi(b, \Delta)$ can be expressed as

$$\partial_b\pi(b, \Delta) = [\lambda'(v - b) + \lambda(\partial_b v - 1)]|_{x=x^*},$$

$$\begin{aligned}&\partial^2_{b,b}\pi(b, \Delta)\\ &\quad = [\lambda''(v - b) + 2\lambda'(\partial_b v - 1) + \lambda\partial^2_{b,b}v + \lambda\partial^2_{b,x}v\partial_b x^*]|_{x=x^*}.\end{aligned}$$

We first show that the last term of $\partial^2_{b,b}\pi(b, \Delta)$ is positive. Note that $x^*(b, \Delta)$ satisfies the first-order condition of $v(b, x, \Delta)$ with respect to x. Therefore, for any value of b,

$\partial_x v(b, x, \Delta)|_{x=x^*} = 0$. Taking the derivative of this identity with respect to b provides $\partial^2_{b,x} v + \partial^2_{x,x} v \partial_b x^* = 0$, which implies

$$\partial_b x^* = -\frac{\partial^2_{b,x} v}{\partial^2_{x,x} v}\bigg|_{x=x^*} \implies \lambda \partial^2_{b,x} v \partial_b x^* = -\lambda \frac{(\partial^2_{b,x} v)^2}{\partial^2_{x,x} v}\bigg|_{x=x^*},$$

which is positive since $\partial^2_{x,x} v < 0$ at x^*. Hence,

$$\partial^2_{b,b} \pi(b, \Delta) \geq [\lambda''(v-b) + 2\lambda'(\partial_b v - 1) + \lambda \partial^2_{b,b} v]|_{x=x^*}.$$

Note that when $\partial_b \pi(b, \Delta) = 0$, we have $\partial_b v - 1 = -(\lambda'(v-b))/\lambda$. Substituting this into the above inequality, we obtain the following:

$$\partial^2_{b,b} \pi(b, \Delta)|_{\partial_b \pi(b, \Delta)=0} \geq \left[(v-b)\left(\lambda'' - \frac{2\lambda'^2}{\lambda}\right) + \lambda \partial^2_{b,b} v\right]\bigg|_{x=x^*}.$$

Substituting for $\partial^2_{b,b} v$ yields

$$\partial^2_{b,b} \pi(b, \Delta)|_{\partial_b \pi(b, \Delta)=0} \geq \left[(v-b)\left(\lambda'' - \frac{2\lambda'^2}{\lambda}\right) + \lambda \bar{G}(x^*)(\mu'' + \sigma'' x^*)\right]\bigg|_{x=x^*}.$$

If $(v(b, x^*(b, \Delta), \Delta) - b) \leq 0$, then such b is never optimal. On the other hand, when $(v(b, x^*(b, \Delta), \Delta) - b) > 0$, condition (a) ensures that the first term is positive. Condition (b) ensures that the second term is positive. To see why, first note that, by the first-order condition (9), $x^*(b, \Delta) > -\mu/\sigma$. Therefore, since $\sigma'' > 0$,

$$\mu'' + \sigma'' x^*(b, \Delta) > \mu'' - \frac{\sigma'' \mu}{\sigma} \overset{\text{sign}}{=} \frac{\mu'' \sigma - \sigma'' \mu}{\sigma^2} = \left(\frac{\mu}{\sigma}\right)'' + \frac{2\sigma'(\sigma\mu' - \mu\sigma')}{\sigma^3} > 0.$$

The last inequality follows from condition (b) and Assumption 2. □

Proofs of Results in §7.1
Here, we generalize Propositions 1–3 under the general setting of §7.1. First, define $p^*(b, \Delta)$ as the optimal value of p given b and Δ; i.e.,

$$p^*(b, \Delta) := \arg\max_p \pi(b, p, \Delta).$$

We then prove the following lemma, which will be used for later proofs.

Lemma A.4. *Under Assumption* 5, (a) $\bar{F}(p^*(b, \Delta) \mid b)$ *decreases in* Δ, *and* (b) $\bar{F}(p^*(b, \Delta) \mid b)$ *increases in* b.

Proof of Lemma A.4. To prove part (a), we will show that, for any given b, $p^*(b, \Delta)$ increases in Δ. Note that $p^*(b, \Delta)$ needs to satisfy the first-order condition of $\pi(b, p, \Delta)$ with respect to p:

$$\frac{f(p \mid b)}{\bar{F}(p \mid b)}(p - \Delta) = 1. \tag{14}$$

The left side of (14) is increasing in p by Assumption 5(a) and is decreasing in Δ. Therefore, when Δ increases, $p^*(b, \Delta)$ must increase to recover the equality.

To prove (b), we will examine the derivative $\partial_b \bar{F}(p^*(b, \Delta) \mid b)$. To do this, we first need to obtain an expression for $\partial_b p^*(b, \Delta)$. Replacing p with $p^*(b, \Delta)$, (14) becomes an identity. Taking the derivative of this identity with respect to b, and after some algebra, we obtain the following expression for $\partial_b p^*(b, \Delta)$. (For convenience, we use $\bar{F}$ as a shorthand notation for $\bar{F}(p \mid b)$ for the rest of this proof.)

$$\partial_b p^*(b, \Delta) = \frac{\bar{F}\partial^2_{b,p}\bar{F} - \partial_b \bar{F} \partial_p \bar{F}}{2(\partial_p \bar{F})^2 - \bar{F}\partial^2_{p,p}\bar{F}}.$$

We now use the expression above to obtain the expression for $\partial_b \bar{F}(p^*(b, \Delta) \mid b)$, and all values in the expression are evaluated at the optimum $p^*(b, \Delta)$:

$$\begin{aligned}
&\partial_b \bar{F}(p^*(b, \Delta) \mid b) \\
&\quad = \partial_p \bar{F} \partial_b p^*(b, \Delta) + \partial_b \bar{F} \quad \text{[by Chain Rule]} \\
&\quad = \frac{\bar{F}\partial_p \bar{F} \partial^2_{b,p}\bar{F} + (\partial_p \bar{F})^2 \partial_b \bar{F} - \bar{F}\partial_b \bar{F} \partial^2_{p,p}\bar{F}}{2(\partial_p \bar{F})^2 - \bar{F}\partial^2_{p,p}\bar{F}} \\
&\qquad\qquad \text{[substitute the expression for } \partial_b p^*(b, \Delta)] \\
&\quad = \partial_p\left(\frac{\bar{F}\partial_b \bar{F}}{\partial_p \bar{F}}\right) \frac{(\partial_p \bar{F})^2}{2(\partial_p \bar{F})^2 - \bar{F}\partial^2_{p,p}\bar{F}} \quad \text{[by algebra]} \\
&\quad \overset{\text{sign}}{=} \partial_p\left(\frac{\bar{F}\partial_b \bar{F}}{\partial_p \bar{F}}\right).
\end{aligned}$$

The "equality in sign" is true because the denominator of the second term, $2(\partial_p \bar{F})^2 - \bar{F}\partial^2_{p,p}\bar{F}$, is equal to $f^2(1 - \partial_p(\bar{F}/f)) > 0$. Therefore, it is sufficient to show that $(\bar{F}\partial_b \bar{F})/\partial_p \bar{F}$ increases in p at the optimum $p^*(b, \Delta)$. Note that

$$\frac{\bar{F}\partial_b \bar{F}}{\partial_p \bar{F}} = \frac{\partial_b \bar{F}}{p \partial_p \bar{F}}(p\bar{F}). \tag{15}$$

The first term of (15) is negative by Assumption 5(b), and its absolute value is decreasing in p by Assumption 5(c). The second term is positive and decreasing in p at the optimum $p^*(b, \Delta)$ since

$$\partial_b(p\bar{F})|_{p=p^*(b,\Delta)} = (\bar{F} - fp)|_{p=p^*(b,\Delta)} \leq (\bar{F} - f(p-\Delta))|_{p=p^*(b,\Delta)} = 0$$
$$\text{[by the first-order condition (14)]}.$$

Therefore, $(\bar{F}\partial_b \bar{F})/\partial_p \bar{F}$ is increasing in p at $p^*(b, \Delta)$. This implies that $\bar{F}(p^*(b, \Delta) \mid b)$ is increasing in b. □

Generalization of Proposition 1: The proof follows the same scheme as the proof for Proposition 1. That is, we will check the sign of $\partial^2_{b,\Delta} \pi(b, \Delta)$. Taking the cross-partial derivative, we obtain the following:

$$\begin{aligned}
\partial^2_{b,\Delta} \pi(b, \Delta) &= \partial^2_{b,\Delta} \pi(b, p^*(b, \Delta), \Delta) \\
&= \partial_b[\partial_p \pi(b, p, \Delta)|_{p^*(b,\Delta)} \partial_\Delta p^*(b, \Delta) \\
&\qquad + \partial_\Delta \pi(b, p, \Delta)|_{p^*(b,\Delta)}],
\end{aligned}$$

where the first term vanishes since $\partial_p \pi(b, p, \Delta)|_{p^*(b,\Delta)} = 0$. Therefore,

$$\begin{aligned}
\partial^2_{b,\Delta} \pi(b, \Delta) &= \partial_b[\partial_\Delta \pi(b, p, \Delta)|_{p^*(b,\Delta)}] \\
&= -\partial_b[\lambda(b)\bar{F}(p^*(b, \Delta) \mid b)].
\end{aligned} \tag{16}$$

By Assumption 1, $\lambda(b)$ is increasing. By Lemma A.4(b), $\bar{F}(p^*(b, \Delta) \mid b)$ is increasing in b. Therefore, the above expression is negative and, hence, the optimal bid $b^*(\Delta)$ decreases in Δ. □

Generalization of Proposition 2: Note that the optimal bid $b^*(\Delta)$ and the optimal price $p^*(\Delta)$ need to satisfy (14), the first-order condition of $\pi(b, p, \Delta)$ in p:

$$\frac{f(p \mid b)}{\bar{F}(p \mid b)}(p-\Delta)=1. \tag{17}$$

By Assumption 5(a), the left side of (17) is increasing in p. At the same time, the left side of (17) is strictly decreasing in b by Assumption 5(b) and, therefore, whenever $b^*(\Delta)$ jumps up at a point of discontinuity, $p^*(\Delta)$ must jump up to recover the equality. □

Generalization of Proposition 3: The optimal conversion rate given Δ can be expressed as $\bar{F}(p^*(b^*(\Delta), \Delta) \mid b^*(\Delta))$. Taking the derivative of this expression with respect to Δ yields

$$\begin{aligned}&\partial_\Delta \bar{F}(p^*(b^*(\Delta), \Delta) \mid b^*(\Delta))\\&\quad=[\partial_b \bar{F}(p^*(b, \Delta) \mid b)\partial_\Delta b^*(\Delta)+\partial_\Delta \bar{F}(p^*(b, \Delta) \mid b)]|_{b=b^*(\Delta)} \leq 0,\end{aligned}$$

where the inequality follows because $\partial_\Delta \bar{F}(p^*(b, \Delta) \mid b) \leq 0$ by Lemma A.4(a), $\partial_b \bar{F}(p^*(b, \Delta) \mid b) \geq 0$ by Lemma A.4(b), and $\partial_\Delta b^*(\Delta) \leq 0$ by our generalization of Proposition 1. □

Proofs of Results in §7.2

Proof of Proposition 8. The proof follows the same approach as in the generalization of Proposition 2. The only difference is that the left side of (17) now strictly increases in b by Assumption 6(b). □

Proof of Proposition 9. To prove part (a), we will check the sign of $\partial^2_{b,\Delta}\pi(b, \Delta)$. In the generalization of Proposition 1, we obtained an expression for $\partial^2_{b,\Delta}\pi(b, \Delta)$ as shown in Equation (16). For convenience, we rewrite it as follows:

$$\begin{aligned}\partial^2_{b,\Delta}\pi(b, \Delta) &= -\partial_b[\lambda(b)\bar{F}(p^*(b, \Delta) \mid b)]\\&= \lambda(b) f(p^*(b, \Delta) \mid b)\partial_b p^*(b, \Delta)\\&\quad -\partial_b[\lambda(b)\bar{F}(p \mid b)]|_{p=p^*(b,\Delta)} \quad \text{[by Chain Rule]}.\end{aligned}$$

The first term is negative since $\partial_b p^*(b, \Delta) \leq 0$ under Assumption 6(b) (i.e., when b increases, the optimal price $p^*(b, \Delta)$ decreases in the model of §7.2). Furthermore, the second term is positive due to the assumption made in the proposition statement. Therefore, $\partial^2_{b,\Delta}\pi(b, \Delta)$ is negative. Hence, $\pi(b, \Delta)$ is submodular in b and Δ, and the optimal bid $b^*(\Delta)$ decreases in Δ.

To see part (b), note that the optimal price $p^*(\Delta)$ solves the first-order condition (14) at $b=b^*(\Delta)$. That is, $p^*(\Delta)$ solves

$$\frac{f(p \mid b^*(\Delta))}{\bar{F}(p \mid b^*(\Delta))}(p-\Delta)=1.$$

Provided with part (a) and Assumption 6(b), the left side of the above equation decreases in Δ. Therefore, when Δ increases, $p^*(\Delta)$ must increase to recover the equality. □

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