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Impact of Bayesian Learning and Externalities on Strategic Investment

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We investigate the interplay between learning effects and externalities in the problem of competitive investments with uncertain returns. We examine a game theoretic duopoly investment model in which (i) a firm can learn about the profitability of the investment by observing the performance of the first mover and (ii) externalities exist between the investments of two firms. We find a region of a war of attrition between the two firms in which the interplay between externalities and learning gives rise to counterintuitive effects on investment strategies and payoffs. In particular, we find that, contrary to the conventional war of attrition where an increase in benefits for the follower generally delays the first move, an increase in the rate of learning—which tends to benefit the follower—can hasten the first investment.

Keywords: games; group decisions; stochastic; decision analysis; sequential; dynamic programming; optimal control
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1. Introduction

Investment decisions in business, such as the introduction of products in unproven markets or the adoption of unproven technologies, often involve significant uncertainty associated with the market reception of new products or the performance of technologies. In competitive scenarios, returns on investments can also depend upon the timing of the investments. For example, it may be beneficial to delay an investment if time will bring more information from which a firm can learn about the future prospects of the investment, provided the opportunity to invest does not disappear (Carruth et al. 2000, Dixit 1992). Returns on investments can also depend upon the investment decisions made by other firms in the market. If there are positive externalities, a firm's returns can improve with an increase in the number of firms in the market, and if there are negative externalities, a firm's returns can diminish with an increase in the number of firms. The existing literature on investment under uncertainty has examined these two factors, i.e., learning effects and externalities, separately.

In this paper, we examine a duopoly game of investment with uncertain profitability where learning effects and externalities coexist. In our model, if one firm enters the market as a leader, then the other firm (the follower) has the opportunity to observe the leader's performance in the market and learn about the true profitability of the market. By observing the leader's performance, the follower can also make an investment to enter the market. If the two firms are in the market at the same time, their profit streams exhibit externalities. We study the following comparative statics in detail: (1) how the follower's time to invest changes with the rate of learning, (2) how the leader's payoff changes with the rate of learning, and (3) how the time to first investment changes with the rate of learning. We find the presence of a ubiquitous *single-crossing property* (property of changing its sign at most once) of the derivatives of the payoff and the time to investment with respect to the rate of learning. This result is driven by the interplay between externalities and learning, which are the salient features of the model that we investigate.

Under appropriate conditions, our model reduces to a war of attrition, which is a game in which the leader's payoff is less than the follower's payoff. In a mixed strategy equilibrium in the conventional war of attrition, each player may delay their investments in an attempt to be the follower (§4.2). It is well known that in such an equilibrium, an increase in the follower's reward delays both players' decisions to move first (Hendricks et al. 1988). Hence, one would expect that an increased rate of learning, which enables the follower to learn faster about the profitability of the investment, would induce both firms to delay their investments. However, we find that the effect of an increased rate of learning is more nuanced.

In this study, there are three main findings. The first finding is that an increased rate of learning has two opposing effects on the follower's time of investment: (a) it can hasten the follower's investment because the follower can acquire more meaningful information within a shorter time, or (b) it can delay the follower's investment because of increased value of waiting (to collect more information on the profitability of the investment). Either effect can dominate; we find that the derivative of the follower's time to investment with respect to the rate of learning exhibits a single-crossing property. To understand the underlying mechanism of this result, we note that the follower's optimal policy is to invest when p (the probability that the market has a high profit) exceeds a threshold θ_F (Proposition 1). As is well known in the literature (see, e.g., Proposition 2 of Kwon and Lippman 2011), the threshold θ_F increases in the learning rate of the follower because a higher learning rate increases the value of waiting and learning, which delays the follower's investment. Thus, when p is sufficiently close to θ_F, the follower's time of investment is strongly influenced by the comparative statics of θ_F, and, consequently, effect (b) dominates. In contrast, when p is sufficiently far away from θ_F (for sufficiently low values of p), the follower's time of investment is less influenced by the comparative statics of θ_F. In this case, due to the influence of effect (a), the follower's investment is hastened by an increased rate of learning. Thus, the comparative statics of the follower's time to investment changes as p increases: effect (a) dominates for low values of p, and effect (b) dominates for high values of p. Indeed, our analysis shows a single-crossing property of the comparative statics as shown in Theorem 1.

The second finding is that the interplay of externalities with learning drives the single-crossing property of the comparative statics of the leader's payoff. Under positive externalities, an earlier investment of the follower improves the leader's payoff because of the time value of money (i.e., discounting). Therefore, for p sufficiently close to θ_F when effect (b) dominates, the leader's payoff (mixed strategy equilibrium payoff) decreases with the rate of learning. On the other hand, for sufficiently low p when effect (a) dominates, the leader's payoff increases with the rate of learning. Thus, the comparative statics of the leader's payoff also exhibits a single-crossing property. Under negative externalities, the comparative statics are reversed since an early investment of the follower diminishes the leader's payoff. These results are reported as Theorem 2.

The third finding is that the comparative statics of the time to the first investment also has a single-crossing property. For example, in the case of positive externalities, if the leader's payoff increases (decreases) in the learning rate, then the time to the first investment tends to decrease (increase) in the learning rate. Thus, the time to the first investment tends to increase in the learning rate for sufficiently high values of p due to effect (b), and it tends to decrease in the learning rate for sufficiently low values of p due to effect (a). Thus, the derivative of the time to first investment with respect to the learning rate also has a single-crossing property. These results are reported as Theorem 3(i). We also report similar results for the case of a second-mover advantage in Theorem 4.

Investment problems with both positive and negative externalities are often seen in real life. Positive externalities can arise from a number of factors, including network externalities, product complementarity, and economies of scale. An example of positive externalities through economies of scale is illustrated by the introduction of organic cotton garments in the mid-1990s. At the time, introducing organic cotton garments was risky, as organic cotton garments are indistinguishable by sight and touch from garments made of conventionally grown cotton (Casadesus-Masanell et al. 2009), and so it was not clear whether customers would be willing to pay more for such products. It would also cost more to procure organic cotton, as firms would need to invest in growers to support their adoption of specific practices for cultivating organic cotton. Despite the risk, firms like Patagonia entered the market as first movers, and second movers such as Gap were able to observe the performance of the first movers and learn how the garments were received by consumers. However, Patagonia and Gap were not in competition with each other since they targeted different markets: Patagonia made garments for mountaineering-related activities, whereas Gap made garments for casual wear. As more firms introduced organic cotton garments, the growers of organic cotton benefited from the additional investments made by the new entrants, which helped lower the procurement costs of organic cotton due to economies of scale. The lower procurement costs translated to lower retail prices for organic cotton garments and thus increased the uptake of such products by consumers. Thus, the firms' investments had positive externalities.

Negative externalities can often be observed in the context of new product launches or adoption of new technologies. For instance, in the mid-1980s, the steel maker Nucor was pondering the difficult question of whether to adopt a new thin slab casting technology called compact strip production (CSP). There was a significant upside profitability potential if the technology was successful; however, there was also significant uncertainty about the viability of the technology (Ghemawat and Stander 1998). Moreover, even if Nucor's adoption of the new technology turned out to be successful, it was unclear how large the first-mover advantage would be. Other steelmakers were bound to notice the performance of CSP and would follow suit within a few years if Nucor successfully adopted the technology, which could drive down the profits due to competition. Here, the firms' investments had negative externalities that could potentially disrupt the leader's efforts to appropriate profits from its investments or deter the investments of followers. These examples illustrate that when learning effects and externalities combine, investment decisions under uncertainty become inherently complex.

The proofs of all mathematical statements in this paper are provided in Appendix B.

2. Related Literature

Our work builds on and contributes to several streams of literature: Bayesian decision models in investment under uncertainty, learning effects in investment games, externalities and complementarities in investment games, and the war of attrition.

Jensen (1982) was one of the first to apply sequential Bayesian decision models to investment decisions under uncertainty when he examined technology adoption under uncertain profitability. McCardle (1985) and Ulu and Smith (2009) studied the problem of technology adoption coupled with exit decisions when it is costly to acquire information about the technology's profitability. Using the continuous-time model of Shiryaev (1967) for Bayesian sequential decisions, Ryan and Lippman (2003) investigated the exit decision of a firm operating a project with uncertain underlying profitability. Using a similar framework, Kwon and Lippman (2011) examined the problem of a firm facing a choice between exit and expansion of a pilot project with uncertain profitability. These papers examine a single decision maker's problems. In contrast, this paper investigates investment decisions in a duopoly under uncertainty.

Another strand of literature has examined the role of externalities in investment games involving competing firms. Dybvig and Spatt (1983) and Katz and Shapiro (1986) viewed technology adoptions as providing complementarities to other firms. Nielsen (2002) studied a duopoly stochastic entry game in which the return on an investment depends on the number of firms in the market, through positive or negative externality. Femminis and Martini (2011) studied a similar stochastic entry game where profit improvement spills over from the leader to the follower at a Poisson time. Mamer and McCardle (1987) studied a Bayesian technology adoption game with positive or negative externalities, but their model separates the technology adoption stage from the product launch and competition stage. Weeds (2002) studied an extreme form of first-mover advantage in a winner-takes-all game of research and development competition. In her model, the economic value of the patent follows a stochastic process, and technological success is random. She found that investments are more delayed in a symmetric equilibrium than in a cooperative equilibrium because firms hold back on investing for fear of starting a patent race.

When multiple firms consider similar investment decisions under uncertainty, they can learn from the investment decisions of the competing firms. A follower can learn from the investment decisions the performance of a leader, which gives firms an incentive to delay their investments. Such behavior reflects a war of attrition, which was first introduced by Smith (1974) and has subsequently seen widespread application in economics and game theory, particularly in the context of investment games. Kapur (1995) studied how the adoption decisions of other firms facilitate learning in a game of technology adoption between multiple players whose private payoffs are independent of the technological progress of other firms. Hoppe (2000) also studied a duopoly game of new technology adoption and showed that when the probability of success is low, it results in a war of attrition because information externalities delay adoption.

In contrast to most papers on investment games, Décamps and Mariotti (2004) incorporated both externalities and Bayesian learning in a single model. They consider the investment decisions of two firms with respect to a project with unknown profitability, where the firms have private information about the cost of investment. The follower learns about the profitability by observing the leader's performance, and the resulting game is a war of attrition, analogous to the one that we identify, for which they find a unique symmetric equilibrium. Although the paper by Décamps and Mariotti (2004) is closely related to our paper in that they also study the impact of learning on a game of investment with externality, there are some notable differences from our paper. Their focus is on the interplay of the information externality and private information on costs, whereas we focus on the interplay of externality and learning. Another important difference is that their model assumes that the leader's payoff is independent of the follower's time of investment once the leader–follower relationship has been established, and consequently the leader's payoff is independent of the follower's learning rate. In contrast, in our model, the leader's payoff depends on the follower's time of investment, which drives our main results.

Another paper that incorporated both externalities and Bayesian learning is that by Thijssen et al. (2006), which studied a preemption or an attrition equilibrium in a game of competitive investment with Bayesian learning about the profitability of the project. Although their model is similar to ours, it assumes that the leader's investment immediately reveals the true profitability to the follower, and hence, the follower's learning process is not incorporated. Hence, their model is not designed to address the question that we investigate.

Choi (1997) also studied a technology adoption process where there is an interplay between informational externalities and payoff interdependency through network externalities. However, his study was focused on the description of a herd behavior through a model of sequential technology choice between two new technologies. Frisell (2003) developed a market entry model in which payoff externalities and informational externalities coexist, and he found that stronger payoff externalities weaken the second-mover advantage and reduce the delay to market. In his model, each firm receives a private signal regarding the market demand and enters the market only if the market demand is favorable. Due to the information asymmetry between the firms, one firm's entry is considered a favorable signal for the other firm as well, and hence it causes an information spillover. In contrast, the information externalities in our model arise from Bayesian learning based on observing the other firm's profit streams, and our focus is on examining the combined impact of learning and externalities on equilibrium strategies.

3. The Game of Externality and Bayesian Learning

We consider two firms indexed as $i \in \{1, 2\}$, and we use j as an index to denote the opponent of firm i. Each firm has a one-time irreversible option to make an irreversible investment to enter a new market with unknown demand. The investments made by the two firms have mutually positive or mutually negative externalities. The time-averaged market demand can be either high or low, but neither firm knows the true state of the demand. If one firm enters the market first, then the other firm can observe its performance and learn about the true state of the market demand.

To formulate the game, we specify the strategy space, the payoff function, and the objective of each firm. Let $T_i \in [0, \infty]$ denote firm i's time of investment. Then $(T_1, T_2) \in [0, \infty] \times [0, \infty]$ is the strategy profile of the game. In this section, we assume without loss of generality that firm 1 is the leader and firm 2 is the follower, so that $T_1 \le T_2$. We let $V_i(p; T_1, T_2)$ denote the *payoff* (defined as the expected cumulative discounted profit) for firm i given a strategy profile (T_1, T_2) conditional on the prior probability p (the initial belief of the firms) that the market demand is *high*. We define $\tau_2 = T_2 - T_1$, which represents the elapsed time between the leader's investment time T_1 and the follower's investment time T_2. Let $X = \{X_t \colon t \in [T_1, T_2]\}$ denote the process of the leader's cumulative profit before the follower invests, and let $r > 0$ denote the discount rate for both firms. We model the process X as a Brownian motion that satisfies

$$dX_t = \mu\, dt + \sigma\, dW_t \quad \text{for } t \in [T_1, T_2],$$

where $\sigma > 0$ is the noise level of the leader's income stream before the follower invests, and the drift μ represents the time-averaged profit per unit time. The process W_t is a Wiener process that represents the white noise in the profit stream. The true value of μ is unknown, but it is publicly known to be either h, if the demand is high, or l, if the demand is low. We assume that both firms share the same prior belief about μ.

We assume that each firm wants to maximize its expected cumulative discounted profit. Using the notation $E^p[\cdot]$ for the expectation conditional on the prior probability p, we express the objective function $V_i(p; T_1, T_2)$ for each firm i as follows:

$$\begin{aligned} V_1(p; T_1, T_2) &= e^{-rT_1} E^p\left[-k + \int_0^{\tau_2} e^{-rt}\, dX_t + e^{-r\tau_2} \hat{U}_L\right], \\ V_2(p; T_1, T_2) &= e^{-rT_1} E^p[e^{-r\tau_2}(\hat{U}_F - k)], \end{aligned} \tag{1}$$

where k is the up-front cost of investment for each firm. The random variables $\hat{U}_L$ and $\hat{U}_F$, defined in (2) and (3) below, respectively denote the leader's and follower's expected cumulative discounted incomes after the follower invests, conditional on the value of μ. Each firm's objective is to maximize its objective function by choosing the optimal time T_i given its opponent's strategy T_j.

We first consider the case where $\tau_2 > 0$, or equivalently, $T_2 > T_1$. For notational convenience, we use an index $I \in \{L, F\}$ to denote the role of each firm; $I = L$ represents the leader, and $I = F$ the follower. In this case (i.e., $T_2 > T_1$), the random variable $\hat{U}_I$ is given as follows:

$$\begin{aligned} \hat{U}_I &\equiv E\left[\int_0^\infty \mu(1+\alpha_I) e^{-rt}\, dt + \int_0^\infty e^{-rt} \sigma_I\, dW_t^I \,\middle|\, \mu\right] \\ &= \frac{\mu}{r}(1+\alpha_I) \quad \text{if } T_1 < T_2. \end{aligned} \tag{2}$$

Here $\hat{U}_I$ is essentially the present value of a perpetual income of $\mu(1+\alpha_I)$ per unit time with a discount rate r. After the follower invests at time T_2, the income for the role I during an infinitesimal time dt is given by $\mu(1+\alpha_I)dt+\sigma_I dW_t^I$. The processes W_t^I is a Wiener process that represents the white noise in the income streams. For instance, if $\alpha_I > 0$ ($\alpha_I < 0$) for $I \in \{L, F\}$, then positive (negative) externality exists between the investments of the two firms. (Because the externality from each firm's investment will have a similar directional impact for the other firm, we assume that the signs of α_L and α_F coincide.) Mixed signs of the externalities such as $\alpha_L > 0 > \alpha_F$ or $\alpha_F > 0 > \alpha_L$ are also possible.

In the second case, we consider simultaneous investment where $T_1 = T_2$. We assume that each player has an equal (50%) chance of being the leader or the follower, and hence the degree of externality is effectively $\alpha_S \equiv (\alpha_L+\alpha_F)/2$. Thus, $\hat{U}_L$ and $\hat{U}_F$ in the case of a simultaneous investment case can be expressed as follows:

$$\hat{U}_L = \hat{U}_F = \hat{U}_S \equiv E\left[\int_0^\infty \mu(1+\alpha_S)e^{-rt}\,dt + \int_0^\infty e^{-rt}\sigma_S\, d(W_t^L+W_t^F)/2 \,\middle|\, \mu\right] = \frac{\mu}{r}(1+\alpha_S) \quad \text{if } T_1 = T_2. \tag{3}$$

Except for §5.4, we make the following assumption in the rest of this paper:

Assumption 1. *$\alpha_I \in (-1, \infty)$ for $I \in \{L, F\}$, $\alpha_L \geq \alpha_F$, $0 < l/r < k < h/r$, and $0 < (1+\alpha_I)l/r < k < (1+\alpha_I)h/r$ for each I.*

This assumption implies that investment without learning would be profitable when $\mu = h$ and unprofitable when $\mu = l$. If $\alpha_F \leq -1$, then there is no incentive for the follower to invest, and the problem becomes trivial. Hence, we assume $\alpha_I \in (-1, \infty)$ for $I \in \{L, F\}$. The condition $\alpha_L \geq \alpha_F$ is based on the assumption of the first-mover advantage commonly observed in competitive contexts. In §5.4, we consider the case of a second-mover advantage.

Next we construct the Bayesian updating process for the posterior probability of $\mu = h$ for time $t \in (T_1, T_2)$, when the follower observes the leader's profit stream and learns about the market demand. Assume that X_t and μ belong to the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\{\mathcal{F}_t^X\colon t \geq 0\}$ denote the natural filtration with respect to the observable cumulative profit process X. We assume that the two firms share common prior and posterior probabilities concerning the profitability. Let $P_t = \mathcal{P}^p(\mu = h \mid \mathcal{F}_t^X) = E^p[\mathbf{1}_{\{\mu=h\}} \mid X_t]$ denote the posterior probability of $\mu = h$ at time t, conditional on the initial prior probability p (here $\mathbf{1}_{\{\cdot\}}$ is the indicator function). In particular, if $T_1 = 0$ and $X_0 = 0$, then P_t can be expressed in terms of X_t and t as follows:

$$\begin{aligned} P_t &= \frac{\mathcal{P}[\{\mu=h\}\cap\{X_t\}\mid P_0=p]}{\mathcal{P}[\{\mu=h\}\cap\{X_t\}\mid P_0=p]+\mathcal{P}[\{\mu=l\}\cap\{X_t\}\mid P_0=p]} \\ &= \left(p\,\exp\left\{-\frac{(X_t-ht)^2}{2\sigma^2 t}\right\}\right)\cdot\left(p\,\exp\left\{-\frac{(X_t-ht)^2}{2\sigma^2 t}\right\} + (1-p)\exp\left\{-\frac{(X_t-lt)^2}{2\sigma^2 t}\right\}\right)^{-1} \\ &= \left[1+\frac{1-p}{p}\exp\left\{-\frac{(h-l)}{\sigma^2}\left[X_t-\frac{h+l}{2}t\right]\right\}\right]^{-1}, \end{aligned}$$

which can be derived from Bayes' rule (Peskir and Shiryaev 2006, p. 288) and the fact that $X_t - \mu t = \sigma W_t$ is normally distributed with mean zero and variance $\sigma^2 t$. Furthermore, the process $P = \{P_t\colon t \geq 0\}$ can be shown (Peskir and Shiryaev 2006, pp. 288–289) to be the unique strong solution to the stochastic differential equation:

$$dP_t = P_t(1-P_t)\frac{h-l}{\sigma}\,d\hat{W}_t, \quad \text{where } \hat{W}_t = \frac{1}{\sigma}\left(X_t - \int_0^t E[\mu \mid \mathcal{F}_s^X]\,ds\right).$$

Here $\hat{W}_t$ is an observable Wiener process constructed purely from the observable process X. Last, note that P_t is defined only within the interval $[T_1, T_2]$, i.e., before the follower invests. Once the follower invests, it has no incentive to learn about the true value of μ. Since σ is the amplitude of the noise, the follower learns more quickly if $1/\sigma$ is higher (Bergemann and Valimaki 2000). Thus, for the remainder of this paper, we use $\beta \equiv 1/\sigma$ to represent the *rate of learning.*

We are now in a position to express the payoff functions in terms of P_t. Before the leader invests, neither firm receives any profit stream, and therefore neither firm receives any information with which to update the posterior probability. Thus the probability of $\{\mu = h\}$ coincides with p for all $t \in [0, T_1]$. Furthermore, if $T_1 = \infty$ and $T_2 = \infty$, then we have $V_i(p; \infty, \infty) = 0$ for both $i = 1, 2$ because there is no profit when neither firm invests. Next, for notational convenience, we define

$$m(p) \equiv E^p[\mu] = hp + l(1-p),$$

so that $E^p[\mu \mid \mathcal{F}_t^X]$ can be expressed as $m(P_t)$. Then we obtain the expressions for $V_i(p; T_1, T_2)$ in terms of the process P when $\tau_2 = T_2 - T_1$ is a stopping time. For ease of presentation, we consider the cases $\tau_2 > 0$ and $\tau_2 = 0$ separately; in Proposition 1, we show that the follower's optimal policy reduces to one of these two cases.

First, let us consider the case where $\tau_2 > 0$, or, equivalently, $T_2 > T_1$. The leader's payoff reduces to

$$\begin{aligned} V_1(p;T_1,T_2) &= e^{-rT_1}E^p\left[\left(\frac{\mu}{r}-k\right)+\alpha_L\frac{\mu}{r}e^{-r\tau_2}\right] \\ &= e^{-rT_1}\left\{\frac{1}{r}m(p)-k+\frac{\alpha_L}{r}E^p[e^{-r\tau_2}m(P_{\tau_2})]\right\}, \quad (4) \end{aligned}$$

where we have used the equality $E^p[\mu e^{-r\tau_2}] = E^p[E^p[\mu e^{-r\tau_2} \mid \mathcal{F}^X_{\tau_2}]]$. Note that the term $m(p)/r$ is the expected value of μ/r, which is the cumulative discounted stream of time-averaged profit μ per unit time. The term $\alpha_L E^p[e^{-r\tau_2}m(P_{\tau_2})]/r$ is the expected value of the additional profit from the follower's investment at time $T_1+\tau_2$. Similarly, the payoff to the follower is given by

$$\begin{aligned} V_2(p;T_1,T_2) &= e^{-rT_1}E^p\left[\left(\frac{\mu}{r}(1+\alpha_F)-k\right)e^{-r\tau_2}\right] \\ &= e^{-rT_1}E^p\left\{e^{-r\tau_2}\left[(1+\alpha_F)\frac{m(P_{\tau_2})}{r}-k\right]\right\}. \quad (5) \end{aligned}$$

Here the term $(\mu/r)(1+\alpha_F)$ is the cumulative discounted stream of the average profit $\mu(1+\alpha_F)$ per unit time.

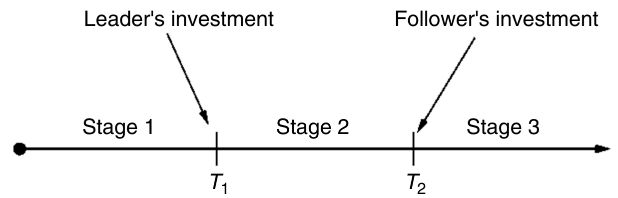
Second, let us consider the case of simultaneous investment, or $\tau_2 = 0$ (i.e., $T_2 = T_1$), in which firms 1 and 2 obtain identical payoffs:

$$\begin{aligned} V_1(p;T_1,T_2) = V_2(p;T_1,T_2) &= e^{-rT_1}E^p[\hat{U}_S-k] \\ &= e^{-rT_1}\left[(1+\alpha_S)\frac{m(p)}{r}-k\right], \end{aligned}$$

where $(1+\alpha_S)m(p)/r$ is the expected value of the cumulative discounted stream of profit $(1+\alpha_S)\mu$ per unit time, which originates from the assumption that each firm has an equal chance of being a leader or a follower.

In summary, our model can be viewed as a three-stage game (see Figure 1 and Table 1). The first stage is the time period $t < T_1$, i.e., before the first investment. The firms are simply waiting for the first investment to happen, and neither firm earns any profit stream in this stage, so the probability of the event $\{\mu = h\}$ remains constant. The second stage is the time period $t \in [T_1, T_2)$ in the case where $T_2 > T_1$.

Figure 1 The Three-Stage Game



(The second stage is absent if $T_1 = T_2$.) In this stage, the leader (firm 1) earns a cumulative profit stream X, and the follower (firm 2) updates the posterior process P based upon the observed process X. In our model, the processes X and P are terminated at the end of the second stage, i.e., as soon as the follower invests at time T_2. The third stage is the period after the follower's investment, i.e., $t \geq T_2$. In this stage, neither firm actively updates the probability of the event $\{\mu = h\}$ because neither firm has any investment decision to make, and both firms earn their final profit streams in perpetuity. The most frequently used notations and mathematical symbols are provided in Table 2.

4. Classification of Equilibria

In this section, we consider the cases of both positive externalities ($\alpha_I > 0$ for $I \in \{L, F\}$) and negative externalities ($-1 < \alpha_I < 0$) with the constraint that $\alpha_L \geq \alpha_F$, to obtain pure strategy (§4.1) and mixed strategy (§4.2) subgame perfect equilibria.

4.1. Pure Strategy Subgame Perfect Equilibria

In the spirit of backward induction, we first obtain the follower's optimal policy and its associated payoff. As in §3, we suppose that firm 1 is the leader in the sense that $T_2 \geq T_1$ (although we also allow for the possibility of simultaneous investment). Once the leader invests at time T_1, the objective of firm 2 (the follower) is to maximize its payoff (given by (5)) with respect to the stopping time $\tau_2 = T_2 - T_1$. We let $V_F(p) \equiv \sup_{\tau_2 \geq 0} V_2(p; 0, \tau_2)$ denote the optimal payoff for the follower for $T_1 = 0$ and $T_2 \geq 0$.

To obtain $V_F(p)$ and the optimal τ_2, we utilize the well-established verification theorem (see, for example, Theorem 3(A) of Alvarez 2001) that stipulates a number of sufficient conditions an optimal value function must satisfy.

Table 1 The Evolution of the Posterior Probability and the Profit Stream

	Stage 1	Stage 2	Stage 3
Probability of $\{\mu = h\}$	p (no evolution over time)	P_t evolves through Bayesian learning	No Bayesian learning takes place
Time-averaged profit stream			
Leader	No profit stream	μ per unit time	$\mu(1+\alpha_L)$ per unit time
Follower	No profit stream	No profit stream	$\mu(1+\alpha_F)$ per unit time

Table 2 Frequently Used Notations

Notation	Definition
k	Cost of investment
$m(p)$	$m(p) \equiv E^p[\mu] = ph + (1-p)l$
P_t	The posterior probability of $\mu = h$ at time t
p	The initial belief of the firms that the market demand is high
r	Discount rate
T_i	The strategy of firm i (firm i's time of investment)
$\hat{T}_i$	Firm i's stage 1 strategy in the mixed strategy game
$V_i(p; T_1, T_2)$	Payoff to firm i with a prior p given a strategy profile (T_1, T_2)
$V_F(p)$	The follower's optimal payoff
$V_L(p)$	The leader's equilibrium payoff
$V_M(p)$	Symmetric mixed strategy equilibrium payoff
α_L, α_F, α	The degree of externality
β	The rate of learning ($\beta \equiv \sigma^{-1}$)
γ	$\gamma \equiv \sqrt{1 + 8r\sigma^2/(h-l)^2}$
θ_c	The boundary between the preemption and the war of attrition regime
θ_F	The follower's optimal threshold of investment in stage 2 when $T_2 > T_1$
θ_0	$\theta_0 \equiv \lim_{\beta \to 0} \theta_F$
θ_L	The leader's equilibrium threshold
θ_S	The lower boundary of the region of simultaneous investment
$\mu \in \{h, l\}$	The time-averaged profit per unit time
$\Pi_L(p)$	The leader's payoff from an immediate investment
$\Pi_S(p)$	The payoff from simultaneous investment
σ	Magnitude of the noise
$\bar{\tau}_M(p)$	Inverse of the arrival rate of $\hat{T}_i$ in the mixed strategy equilibrium
τ_2	$\tau_2 \equiv T_2 - T_1$
τ_F	The follower's optimal stopping time in stage 2 when $T_2 > T_1$

Proposition 1. (i) *At time T_1, the follower's optimal payoff is given by $V_F(p) = \max\{\Pi_F(p), \Pi_S(p)\}$, where*

$$\Pi_F(p) = \begin{cases} \dfrac{\psi(p)}{\psi(\theta_F)}\left[(1+\alpha_F)\dfrac{m(\theta_F)}{r} - k\right] & \text{for } p < \theta_F, \\ \dfrac{1}{r}(1+\alpha_F)m(p) - k & \text{otherwise}, \end{cases} \tag{6}$$

$$\Pi_S(p) = \frac{1}{r}(1+\alpha_S)m(p) - k, \tag{7}$$

and θ_F and $\psi(x)$ are defined by (A1) and (A3) in Appendix A. Furthermore, the follower's optimal policy is to invest immediately at T_1 if $\Pi_S(p) \geq \Pi_F(p)$ and to wait and invest as soon as P_t hits the upper threshold θ_F if $\Pi_S(p) \leq \Pi_F(p)$.

(ii) *There exists $\theta_S \leq \theta_F$ such that $\Pi_S(p) > \Pi_F(p)$ if and only if $p > \theta_S$.*

Proposition 1 establishes that the follower's optimal strategy is to invest at time $T_2 = T_1 + \tau^*$, where

$$\tau^* = \begin{cases} \tau_F \equiv \inf\{t > 0: P_t \geq \theta_F\} & \text{if } \Pi_F(p) \geq \Pi_S(p), \\ 0 & \text{if } \Pi_F(p) < \Pi_S(p). \end{cases} \tag{8}$$

Here $\Pi_F(p)$ represents the optimal value function for the follower under the constraint $T_2 > T_1$, and $\Pi_S(p)$ is the value function for $T_1 = T_2$, i.e., from simultaneous investment. The functional form of $\Pi_F(p)$ for $p < \theta_F$ gives the payoff for waiting until P_t hits the threshold θ_F, whereas $\Pi_F(p)$ for $p \geq \theta_F$ gives the payoff for immediate investment. Proposition 1 asserts that when $\Pi_F(p) \geq \Pi_S(p)$, the follower's optimal policy is to invest as soon as P_t hits the optimal upper threshold θ_F given by (A1) in Appendix A. This optimal policy can be understood as the intuitive notion that a follower begins to learn about the market demand once the leader invests, and it invests only when its profit prospect (posterior probability P_t of a high profitability) hits a sufficiently high value θ_F.

Intuitively, the optimal threshold θ_F of the follower's investment can be obtained as follows. Let us define $\tau_\theta = \inf\{t > 0: P_t \geq \theta\}$ as the hitting time for some threshold θ. By the theory of stopping (Chap. 9 of Oksendal 2003), it is known that the random discount factor $e^{-r\tau_\theta}$ has the expected value

$$E^p[\exp(-r\tau_\theta)] = \frac{\psi(p)}{\psi(\theta)}. \tag{9}$$

This leads to $V_2(p; 0, \tau_\theta) = [(1+\alpha_F)m(\theta)/r - k] \cdot \psi(p)/\psi(\theta)$ because the follower's payoff from investment at time τ_θ is $(1+\alpha_F)m(\theta)/r - k$. The optimal threshold θ_F can be obtained from the necessary first-order condition $dV_2(p; 0, \tau_\theta)/d\theta = 0$.

In the next proposition, we obtain the leader's (firm 1's) best response T_1 conditional on the follower's optimal stopping time τ^* given by (8). Similar to the follower's payoff, let $V_L(p) \equiv \sup_{T_1} V_1(p; T_1, T_1 + \tau^*)$ denote the optimal payoff for the leader. We establish that there exists a critical value θ_L such that the leader invests immediately if $p \geq \theta_L$ and never invests if $p < \theta_L$.

Proposition 2. *Given the follower's time of investment τ^*, the leader's optimal payoff is given by*

$$V_L(p) = \max\{\Pi_L(p), 0\}, \tag{10}$$

where

$$\Pi_L(p) = \frac{m(p)}{r} - k + \frac{\alpha_L m(\theta_F)\psi(p)}{r\psi(\theta_F)} \quad \text{for } p < \theta_S, \tag{11}$$

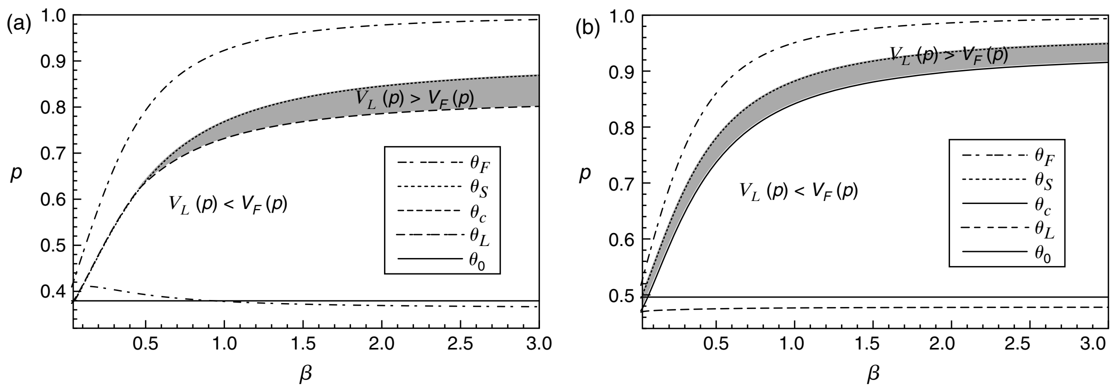
$$= \frac{1}{r}(1+\alpha_S)m(p) - k \quad \text{otherwise}. \tag{12}$$

The leader's best response is to invest at $T_1 = 0$ if $p \geq \theta_L$ and at $T_1 = \infty$ if $p < \theta_L$, where $\theta_L \in (0, \theta_S]$ is defined by

$$\theta_L = \inf\{p: \Pi_L(p) > 0\}. \tag{13}$$

The function $\Pi_L(p)$ represents the leader's payoff from an immediate investment when the follower is expected to invest at time τ^*. Note that $\Pi_L(p)$ can be negative, whereas $V_L(p)$ is nonnegative because the leader would not invest when $\Pi_L(p)$ is negative. The right-hand side of (11) is the leader's payoff

Figure 2 Values of θ_0, θ_L, θ_c, θ_S, and θ_F When $h = 2$, $l = 0.12$, $r = 0.1$, and $k = 10$



Note. Panel (a) shows the case $\alpha_L = 0.3$ and $\alpha_F = 0.2$, and panel (b) shows the case $\alpha_L = -0.02$ and $\alpha_F = -0.05$.

from an immediate investment when the follower is expected to invest after time τ_F. On the other hand, Equation (12) is the payoff from investment when the follower is expected to invest at the same time. The intuition behind Proposition 2 is that the leader immediately invests if and only if its net payoff from investment exceeds zero; otherwise, the leader never invests.

Next, we obtain the strategies in the pure strategy subgame perfect Nash equilibria.

Proposition 3. (i) *If $p \in [0, \theta_L)$, neither player invests in the equilibrium.*

(ii) *If $p \in [\theta_L, \theta_S)$, there are two pure strategy subgame perfect equilibria, each of which has a leader and a follower. The leader invests at $\tau_L = 0$, and the follower invests at a stopping time $\tau_F = \inf\{t > 0\colon P_t \geq \theta_F\} > 0$, where θ_F is given by (A1).*

(iii) *If $p \in [\theta_S, 1]$, then there exists a symmetric pure strategy Nash equilibrium in which players invest immediately at the same time at $t = 0$.*

Under the pure strategy equilibria obtained in Proposition 3(ii) for $p \in [\theta_L, \theta_S)$, a leader and a follower exist. The roles of the leader and the follower may be determined through the common expectation that both firms will choose to play a predetermined role to attain a specific equilibrium (Fudenberg and Tirole 1991, p. 18). For example, the firm that is publicly known to be a more proactive investor will be the leader in this game. However, it is not essential for a natural leader and a follower to exist in our model, and the indeterminacy may result in a mixed strategy equilibrium. Under a mixed strategy equilibrium, each firm's strategy for its first investment is to invest at a *random* time with a probability distribution specified by its strategy. In this case, the leader is randomly determined, but the follower's best response reduces to that of the pure strategy equilibrium in Proposition 1. The details are discussed in §4.2.

Next, we report that both a war of attrition region ($V_F(p) > V_L(p)$) and a preemption region ($V_L(p) > V_F(p)$) can exist within the interval $[\theta_L, \theta_S)$.

Lemma 1. *There exists $\theta_c \in (\theta_L, \theta_S]$ at which $V_L(p) > V_F(p)$ for $p \in (\theta_c, \theta_S)$ and $V_F(p) > V_L(p)$ for $p \in (\theta_L, \theta_c)$, with the understanding that (θ_c, θ_S) is empty whenever $\theta_c = \theta_S$.*

The relative values of $\theta_0 \equiv \lim_{\beta \to 0} \theta_F$, θ_L, θ_c, θ_S, and θ_F are illustrated in Figure 2.

For the remainder of this paper, we call the interval $[\theta_L, \theta_c)$ a *war of attrition* (WA) region, the interval (θ_c, θ_S) a *preemption* (PE) region, and the interval $[\theta_S, 1]$ a *simultaneous move* (SM) region.

4.2. Mixed Strategy Subgame Perfect Equilibrium in the War of Attrition Region

In this subsection, we obtain mixed strategy equilibria in the WA region $[\theta_L, \theta_c)$ by employing the results of Hendricks et al. (1988) pertaining to a war of attrition in continuous time. Unlike in the previous sections, we do not assume that any one firm is predetermined to take the leader's role.

In the WA region, a mixed strategy profile is completely characterized by (i) each firm's stopping time for investment as a follower in stage 2 in case the other firm invests first and (ii) each firm i's probability distribution of the random time $\hat{T}_i$ of investment for stage 1. Note that the random strategy $\hat{T}_i$ is applicable only in the first stage of the game. For example, if $\hat{T}_i < \hat{T}_j$, then firm i becomes the leader at time $\hat{T}_i$, at which point stage 1 is terminated. In this case, firm j's initial strategy $\hat{T}_j$ is never realized because it becomes the follower in stage 2. As shown in §4.1, a subgame perfect equilibrium requires that the follower's best response should be to invest at time τ_F given by (8). Hence, we focus on specifying $G_p^{(i)}(\cdot)\colon \mathbb{R}_+ \to [0, 1]$, which denotes firm i's cumulative probability distribution function for time $\hat{T}_i$ given the prior probability p. In what follows, to keep the notation brief, we let $G_p^{(i)}(\cdot)$ denote the strategy of firm i with the understanding that the follower's time of investment is τ_F. By our convention, the strategy profile is represented by $(G_p^{(1)}, G_p^{(2)})$.

Here we adopt the convention that $G_p^{(i)}(\cdot)$ is right-continuous with left limits. We let $q_p^{(i)}(t) = G_p^{(i)}(t) - \lim_{s\uparrow t} G_p^{(i)}(s)$ denote the discontinuity of $G_p^{(i)}$ at time t. Intuitively, $q_p^{(i)}(t)$ represents the probability that firm i will invest exactly at time t. Given a strategy profile $(G_p^{(1)}, G_p^{(2)})$, the payoff for firm i is given by

$$V_i(p; G_p^{(1)}, G_p^{(2)}) = E[\mathbf{1}_{\{\hat{T}_i<\hat{T}_j\}} e^{-r\hat{T}_i} V_L(p) + \mathbf{1}_{\{\hat{T}_i>\hat{T}_j\}} e^{-r\hat{T}_j} V_F(p) + \mathbf{1}_{\{\hat{T}_i=\hat{T}_j\}} e^{-r\hat{T}_i} \Pi_S(p) \mid (G_p^{(1)}, G_p^{(2)})] \tag{14}$$

$$= \int_0^\infty \left\{ e^{-rt}[1 - G_p^{(j)}(t)] V_L(p) + \left[\lim_{u\uparrow t} \int_0^u V_F(p) e^{-rs}\, dG_p^{(j)}(s)\right] + e^{-rt}\Pi_S(p) q_p^{(j)}(t) \right\} dG_p^{(i)}(t). \tag{15}$$

In (14), the term $\mathbf{1}_{\{\hat{T}_i<\hat{T}_j\}} e^{-r\hat{T}_i} V_L(p)$ represents the payoff for the event that firm i happens to invest before firm j, in which case firm i expects $V_L(p)$ at time $\hat{T}_i$. Note that firm j will not invest at time $\hat{T}_j$ if $\hat{T}_i < \hat{T}_j$, because $\hat{T}_j$ is j's investment time conditional on j being the first one to invest; if i happens to have invested first, then j will invest at time $\hat{T}_i + \tau_F$, which is j's best response. Analogously, $\mathbf{1}_{\{\hat{T}_i>\hat{T}_j\}} e^{-r\hat{T}_j} V_F(p)$ represents the payoff for the event that firm j invests before firm i, in which case firm i's expected payoff is $V_F(p)$ at time $\hat{T}_j$ because firm i will invest at time $\hat{T}_j + \tau_F$. Last, $\mathbf{1}_{\{\hat{T}_i=\hat{T}_j\}} e^{-r\hat{T}_i} \Pi_S(p)$ represents the payoff for the event of simultaneous investment in the case where $\hat{T}_i = \hat{T}_j$.

Note that there are no dynamics or updating of the probability p until the time $\min\{\hat{T}_i, \hat{T}_j\}$, because the time $t < \min\{\hat{T}_i, \hat{T}_j\}$ belongs to stage 1 of the game (see Figure 1 and Table 1). Therefore, at time $\min\{\hat{T}_i, \hat{T}_j\}$, the leader's payoff is $V_L(p)$ and the follower's payoff is $V_F(p)$ without any dependence on $\min\{\hat{T}_i, \hat{T}_j\}$. It follows that the mixed strategy game for $t \le \min\{\hat{T}_i, \hat{T}_j\}$ reduces to a *static* game of a war of attrition in the sense that the state variable p has no dynamics before the first move from either player.

Equation (15) is the integral representation of (14) with respect to the investment times of the two firms. Given firm i's investment time t, the probability that i will be the leader is $1 - G_p^{(j)}(t)$, and the probability that both firms invest at t is $q_p^{(j)}(t)$, which explains the terms $e^{-rt}[1 - G_p^{(j)}(t)]V_L(p)$ and $e^{-rt}\Pi_S(p) q_p^{(j)}(t)$ within the curly brackets in (15). The term $\lim_{u\uparrow t} \int_0^u V_F(p) e^{-rs} dG_p^{(j)}(s)$ is the integral over the payoff in the event that firm j invests before time t. Now we use the payoff function described above to characterize the mixed strategy equilibria through the following proposition:

Proposition 4. (i) *For $p \in [\theta_L, \theta_c)$, a strategy profile $(G_p^{(1)}, G_p^{(2)})$ with $q_p^{(1)}(0) < 1$ and $q_p^{(2)}(0) < 1$ is a subgame perfect mixed strategy equilibrium if and only if the following two conditions are satisfied*:

(a) $(q_p^{(1)}(0), q_p^{(2)}(0)) \in [0, 1)^2$ *and* $q_p^{(1)}(0) q_p^{(2)}(0) = 0$.

(b) *For both* $i = 1$ *and* 2,

$$G_p^{(i)}(t) = 1 - [1 - q_p^{(i)}(0)] \exp[-t/\bar{\tau}_M(p)], \tag{16}$$

$$\textit{where} \quad \bar{\tau}_M(p) = \frac{V_F(p) - V_L(p)}{rV_L(p)}. \tag{17}$$

(ii) *Under the subgame perfect mixed strategy equilibrium, the payoff for firm i is given by*

$$V_i(p; G_p^{(1)}, G_p^{(2)}) = q_p^{(j)}(0) V_F(p) + [1 - q_p^{(j)}(0)] V_L(p). \tag{18}$$

Furthermore, the expected time to the first investment is given by

$$E[\min\{\hat{T}_1, \hat{T}_2\}] = [1 - q_p^{(1)}(0) - q_p^{(2)}(0)] \frac{\bar{\tau}_M(p)}{2}. \tag{19}$$

Note that the equilibria are parameterized by the initial probabilities of the firms' entry, i.e., $q_p^{(1)}(0)$ and $q_p^{(2)}(0)$. Note also that $q_p^{(i)}(t) = 0$ for all $t > 0$. In other words, one of the firms may strategically allocate a positive probability of being the leader at time $t = 0$, but once the time has elapsed beyond $t = 0$, the two firms' strategies are characterized by a continuous probability distribution $G_p^{(i)}(\cdot)$.

The nonzero values of $q_p^{(i)}(t)$ are confined to $t = 0$ for the following reason: In a mixed strategy equilibrium for all $t > 0$, each player i must be indifferent regarding to the time $\hat{T}_i$ of investment; otherwise, the mixing of all strategies $\hat{T}_i > 0$ would not be feasible in a mixed strategy equilibrium. It implies that the equilibrium strategy of distribution $G_p^{(i)}(t)$ must be time invariant in the sense that at any time $t > 0$, the game must look exactly the same as at any other time $t' > 0$ for any $t' \neq t$. If, however, $q_p^{(i)}(\tau) > 0$ for some deterministic time $\tau > 0$, then the time invariance is broken because the game before τ and after τ looks different to player j; in this hypothetical case, player j would prefer to invest after time τ due to the advantage of being the follower. Therefore, $q_p^{(i)}(t) = 0$ must hold for all $t > 0$ for a mixed strategy equilibrium. The time invariance is not necessary at $t = 0$ because the players do not need to consider a strategy before $t = 0$, so either $q_p^{(1)}(0) > 0$ or $q_p^{(2)}(0) > 0$ is permissible even in a mixed strategy equilibrium.

Note also that at least one of $q_p^{(1)}(0)$ and $q_p^{(2)}(0)$ must be zero. If firm 1 chooses $q_p^{(1)}(0) > 0$, it is taking the role of the leader with a probability of $q_p^{(1)}(0)$ at time $t = 0$, so the equilibrium probabilistically takes a characteristic of a pure strategy equilibrium. In this case, because there is a nonzero probability that firm 1 will be a leader, firm 2 has no incentive to place any positive probability of investing at time $t = 0$ since being a follower is more profitable than being a leader; it would rather first wait to see if firm 1 does invest at time $t = 0$. Hence, when $q_p^{(1)} > 0$, firm 2's best response is to set $q_p^{(2)} = 0$. By symmetry of the game, it follows that there is no equilibrium in which $q_p^{(1)}(0) > 0$ and $q_p^{(2)}(0) > 0$ at the same time.

If we focus on a completely symmetric equilibrium between the two firms, we can set $q_p^{(1)}(0) = q_p^{(2)}(0) = 0$. Let $V_M(p)$ denote the mixed strategy equilibrium payoff for $q_p^{(1)}(0) = q_p^{(2)} = 0$. It is worth noting that by (18), $V_M(\cdot)$ coincides with $V_L(p)$ because the payoff for investment at any time for either firm is identically given when the opponent plays the equilibrium strategy given by (16). In this case, each firm's investment time $\hat{T}_i$ is exponentially distributed with a rate $1/\bar{\tau}_M(p)$. Thus, we can interpret the hazard rate $1/\bar{\tau}_M(p)$ as the rate of each firm's investment at any moment in time. From the property of exponential distributions, the expected time of the first investment from *any* firm is given by $E[\min\{\hat{T}_1, \hat{T}_2\}] = \bar{\tau}_M(p)/2$. Thus, $\bar{\tau}_M(p)/2$ characterizes how long it takes for the first investment to occur. In general, due to Equation (19), $\bar{\tau}_M(p)$ is the single most important quantity that characterizes the expected time to the first investment, even if $q_p^{(1)}(0)$ or $q_p^{(2)}(0)$ is nonzero. We defer the investigation of the comparative statics of $\bar{\tau}_M(p)$ until §5.

Last, we briefly comment on the possibility of a mixed strategy equilibrium in the PE region. Just as in the case of the WA game, if there is no natural leader, then a mixed strategy equilibrium makes more sense in practice even in a PE game. We refer the reader to Thijssen et al. (2012) for details on mixed strategy equilibria when $V_L(p) > V_F(p) > \Pi_S(p)$. Nevertheless, it is worth noting that even in a mixed strategy equilibrium, the first investment occurs at time $t = 0^+$ (as soon as the game begins) in a preemption game. Since the focus of our paper is on a war of attrition, we forgo further discussion of the preemption equilibria.

5. Impact of Learning

In this section, we explore the impact of learning on the equilibrium strategies and show that there exists an interplay between learning and externalities due to strategic interactions between the firms. In §5.1, we first study a benchmark model in which externalities do not exist while Bayesian learning does, and we obtain a benchmark result regarding the impact of learning. Then we return to the model presented in §4 to obtain the comparative statics for the firms' equilibrium strategies and payoff with respect to learning. We study the comparative statics of the follower's payoff and strategy in §5.2. In §5.3, we study the impact of learning on $V_M(\cdot)$ and $\bar{\tau}_M(p)$. (As noted in §4.2, the comparative statics of $\bar{\tau}_M(p)$ coincide with that of the expected time to the first investment.) Last, in §5.4, we study the case of a second-mover advantage.

5.1. Benchmark Model

In our benchmark model, we show that the expected time to the first investment monotonically increases[1] with the rate of learning. This agrees with the intuition that an increase in benefits to the follower delays the first investment in a war of attrition.

Suppose that each firm can invest once, but there is no externality between the two investments. Then the mixed strategy equilibrium reduces to the one considered in §4 with $\alpha_L = \alpha_F = 0$. In this case, the leader's payoff is independent of the follower's action, so we have $V_L(p) = m(p)/r - k$ for $p \geq \theta_L$, which does not depend on β or the follower's strategy. The follower's optimal payoff is given by

$$V_F(p) = E^p\left[e^{-r\tau_F}\left(\frac{\mu}{r} - k\right)\right] = \left(\frac{m(\theta_F)}{r} - k\right)\frac{\psi(p)}{\psi(\theta_F)}.$$

By Proposition 2 of Kwon and Lippman (2011), $V_F(p)$ increases with β, which is consistent with the intuition that a higher rate of learning improves the follower's profit. Furthermore, from (17), it follows that $\bar{\tau}_M(p)$ increases with β, which is consistent with the intuition that a player will delay investment if the followers payoff improves with a higher rate of learning.

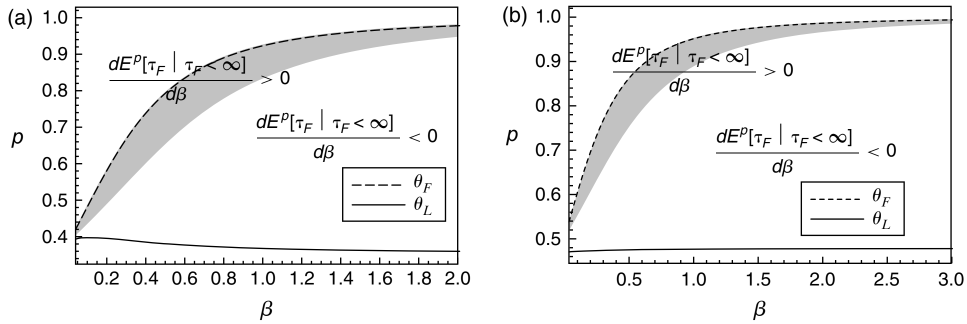
5.2. The Follower's Payoff and Strategy

In this section, we study the comparative statics of $V_F(\cdot)$, θ_F, and $E^p[\tau_F \mid \tau_F < \infty]$ with respect to β for nonzero α_L and α_F. (We do not study the comparative statics of $E^p[\tau_F]$ because $E^p[\tau_F] = \infty$ for any $p < \theta_F$.) We will use these results to provide intuitive explanations for the main results in 5.3.1 and 5.3.2.

We first establish the comparative statics of $V_F(\cdot)$ and θ_F. By Proposition 2 of Kwon and Lippman (2011), the value function and the investment threshold are both nonincreasing in σ, so $V_F(p)$ and θ_F are nondecreasing with β; $V_F(p)$ increases with β because a higher learning rate improves the follower's payoff. Because of the improved value of waiting and learning before investment, the follower delays its

[1] Throughout this paper, we make no distinction between increasing and nondecreasing functions; similarly, we do not distinguish between decreasing and nonincreasing functions.

Figure 3 **The Impact of β on $E^p[\tau_F \mid \tau_F < \infty]$**



Notes. The shaded (unshaded) area represents the region in which $E^p[\tau_F \mid \tau_F < \infty]$ increases (decreases) with β. Panel (a) shows the case where $\alpha_F = 0.2$ and $\alpha_L = 0.3$, and panel (b) shows the case where $\alpha_F = -0.05$ and $\alpha_L = -0.02$. Here we set $h = 2$, $l = 0.12$, $r = 0.1$, and $k = 10$.

investment as the learning rate increases. This intuition explains why θ_F increases with β. This result is consistent with the conventional result that the signal-to-noise ratio $(h-l)/\sigma$ increases the value of waiting as well as the upper threshold of investment (Bergemann and Valimaki 2000).

Next, we obtain the form of $E^p[\tau_F \mid \tau_F < \infty]$:

Lemma 2.

$$E^p[\tau_F \mid \tau_F < \infty] = \log\left(\frac{\theta_F}{p}\frac{1-p}{1-\theta_F}\right)\frac{4\sigma^2}{(h-l)^2}. \tag{20}$$

For notational convenience, we define

$$\theta_0 \equiv \lim_{\beta \to 0} \theta_F = \frac{kr - l(1+\alpha_F)}{(1+\alpha_F)(h-l)}. \tag{21}$$

By virtue of Proposition 2 of Kwon and Lippman (2011), $\theta_F > \theta_0$ for all values of β. Note also that θ_L may or may not be larger than θ_0, and there is no general ordering between θ_L and θ_0 as illustrated by Figure 2. Now we characterize the regions in which $E^p[\tau_F \mid \tau_F < \infty]$ increases or decreases with β.

Theorem 1. *For fixed $\beta \in (0, \infty)$, there exists $\hat{\theta}_F(\beta) \in (\theta_0, \theta_F)$ such that $E^p[\tau_F \mid \tau_F < \infty]$ decreases with β for $p \in (0, \hat{\theta}_F(\beta))$ and increases with β for $p \in (\hat{\theta}_F(\beta), \theta_F)$. For fixed $p \in (\theta_0, \theta_F)$, there exists $\hat{\beta}_F(p)$ such that $E^p[\tau_F \mid \tau_F < \infty]$ increases with β for $\beta \in (0, \hat{\beta}_F(p))$ and decreases with β for $\beta \in (\hat{\beta}_F(p), \infty)$. For fixed $p < \theta_0$, $E^p[\tau_F \mid \tau_F < \infty]$ decreases with β.*

A noteworthy feature of Theorem 1 is the single-crossing property: the sign change of the derivative $dE^p[\tau_F \mid \tau_F < \infty]/d\beta$ occurs at most once as β increases with fixed p or as p increases with fixed β (see Figure 3 for a numerical illustration). Theorem 1 reflects the fact that an increase in the rate of learning has two countervailing effects on the follower's time to investment (Kwon and Lippman 2011). On the one hand, such an increase may hasten the follower's investment because the follower acquires more meaningful information within a shorter time when the learning rate increases. On the other hand, an increase in the rate of learning may delay the follower's investment because of the increased value of waiting to collect more information. Either effect can be dominant, depending on the values of p and β.

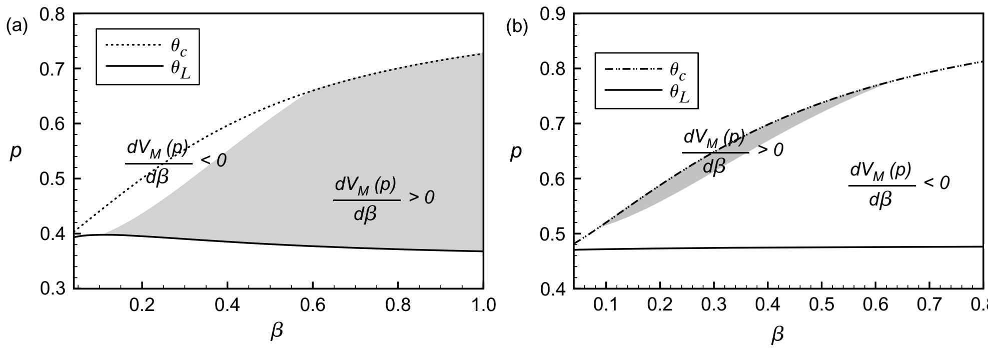
When p is sufficiently close to θ_F, the comparative statics of $E^p[\tau_F \mid \tau_F < \infty]$ is strongly influenced by the comparative statics of θ_F. For instance, as the learning rate β increases, the investment threshold θ_F increases due to the increase in the value of waiting, so the follower's time to investment also increases if p is very close to θ_F. On the other hand, if p is sufficiently far away from θ_F, the comparative statics of θ_F has little effect on $E^p[\tau_F \mid \tau_F < \infty]$ since it takes a long time for P_t to reach the vicinity of θ_F. Furthermore, in this regime, the higher learning rate hastens the follower's investment because it takes less time to collect sufficient information to make a decision. For instance, if θ_F hypothetically did not have any dependence on β, then $E^p[\tau_\theta \mid \tau_\theta < \infty]$ can be shown to decrease in β. Thus, the comparative statics of the follower's time to investment changes as p increases from small values to the vicinity of θ_F.

The single-crossing property in p also explains the single-crossing property in β because θ_F strictly increases with β: For a fixed p, the threshold θ_F is sufficiently close to p for sufficiently low values of β. Thus, $E^p[\tau_F \mid \tau_F < \infty]$ increases with β for sufficiently low β. For sufficiently large values of β, θ_F takes a high value, so it is far from p. It follows that $E^p[\tau_F \mid \tau_F < \infty]$ decreases with β for large β. Thus, the comparative statics of $E^p[\tau_F \mid \tau_F < \infty]$ changes as β increases.

5.3. The Impact of Learning on the Mixed Strategy Equilibrium

In this section, we investigate the impact of learning on the equilibrium payoff $V_M(\cdot)$ and $\bar{\tau}_M(\cdot)$. We also supplement our analyses with selected numerical examples.

Figure 4 **The Impact of β on $V_M(\cdot)$, θ_F, θ_L, and θ_c**



Notes. The shaded (unshaded) area represents the region in which $dV_M(p)/d\beta > 0$ ($dV_M(p)/d\beta < 0$). Panel (a) shows the case where $\alpha_L = 0.3$ and $\alpha_F = 0.2$, and panel (b) shows the case where $\alpha_L = -0.02$ and $\alpha_F = -0.05$. Here we set $h = 2$, $l = 0.12$, $r = 0.1$, and $k = 10$.

The dependencies of θ_F, θ_S, θ_L, and θ_c on β play an important role, so we will denote these dependencies as $\theta_F(\beta)$, $\theta_S(\beta)$, $\theta_L(\beta)$, and $\theta_c(\beta)$. We define the open set of pairs (p, β) in the WA region as

$$\mathscr{W} = \{(p, \beta) \in (0, 1) \times (0, \infty): \theta_L(\beta) < p < \theta_c(\beta)\},$$

and we also define $\underline{\beta}_p \equiv \inf\{\beta: \theta_c(\beta) > p > \theta_L(\beta)\}$, which is the smallest value of β at which a given value of p belongs to the WA region. For a fixed value of p, if $\beta < \underline{\beta}_p$, then p may belong to the SM region or the PE region.

5.3.1. Comparative Statics of the Mixed Strategy Equilibrium Payoff. Now we obtain the comparative statics of $V_M(\cdot)$ and θ_L with respect to β under positive and negative externalities.

Theorem 2. (i) *Suppose $\alpha_I > 0$ for $I \in \{L, F\}$. For $p > \theta_0$, there exists $\hat{\beta}_M(p) \in [\underline{\beta}_p, \infty)$ such that $V_M(p)$ decreases with β for $\{\beta: (p, \beta) \in \mathscr{W}, \beta < \hat{\beta}_M(p)\}$ and increases with β for $\{\beta: (p, \beta) \in \mathscr{W}, \beta > \hat{\beta}_M(p)\}$. Furthermore, for a fixed β, there exists $\hat{\theta}_M(\beta) \in [\theta_L(\beta), \theta_c(\beta)]$ such that $V_M(p)$ increases with β if $\theta_L(\beta) < p < \hat{\theta}_M(\beta)$ and decreases with β if $\hat{\theta}_M(\beta) < p < \theta_c(\beta)$. If $\theta_L(\beta) < p < \theta_0$, then $V_M(p)$ increases with β for sufficiently large or sufficiently small values of β.*

(ii) *Suppose $\alpha_I < 0$ for $I \in \{L, F\}$. For $p > \theta_0$, there exists $\hat{\beta}_M(p) \in [\underline{\beta}_p, \infty)$ such that $V_M(p)$ increases with β for $\{\beta: (p, \beta) \in \mathscr{W}, \beta < \hat{\beta}_M(p)\}$ and decreases with β for $\{\beta: (p, \beta) \in \mathscr{W}, \beta > \hat{\beta}_M(p)\}$. Furthermore, for a fixed β, there exists $\hat{\theta}_M(\beta) \in [\theta_L(\beta), \theta_c(\beta)]$ such that $V_M(p)$ decreases with β if $\theta_L(\beta) < p < \hat{\theta}_M(\beta)$ and increases with β if $\hat{\theta}_M(\beta) < p < \theta_c(\beta)$. If $\theta_L(\beta) < p < \theta_0$, then $V_M(p)$ decreases with β for sufficiently large or sufficiently small values of β.*

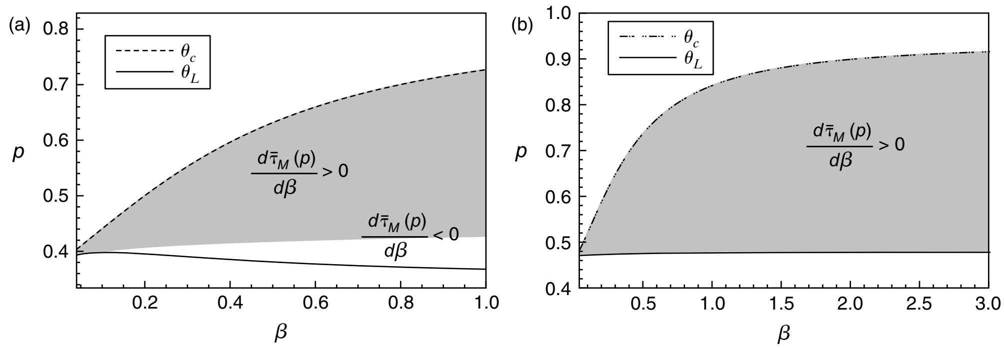
Figure 4 provides a numerical illustration of Theorem 2. The salient feature of this theorem is the single-crossing property of $dV_M(p)/d\beta$: the sign change of $dV_M(p)/d\beta$ occurs at most once, as p increases for fixed β or as β increases for fixed $p > \theta_0$. For fixed β, the sign change of $dV_M(p)/d\beta$ occurs when p crosses $\hat{\theta}_M(\beta)$, and for fixed $p > \theta_0$, the sign change of $dV_M(p)/d\beta$ occurs when β crosses $\hat{\beta}_M(p)$. We do not have analytical results for $p < \theta_0$, but numerical examples suggest that the sign of $dV_M(p)/d\beta$ does not change as β increases for fixed $p < \theta_0$.

Interestingly, the single-crossing property of the comparative statics of $E^p[\tau_F \mid \tau_F < \infty]$ (Theorem 1) provides an intuitive explanation for the single-crossing property of $dV_M(p)/d\beta$. First, we examine the case where $\alpha_I > 0$ for $I \in \{L, F\}$, as illustrated by Figure 4(a). For small values of β, Theorem 1 implies that an increase in β increases $E^p[\tau_F \mid \tau_F < \infty]$, which tends to decrease the leader's payoff $V_L(\cdot) = V_M(\cdot)$ because a delayed investment of the follower diminishes the leader's payoff due to the positive externality. This is reflected by Theorem 2(i) and Figure 4(a) for small values of β. For large values of β, Theorem 1 implies that $E^p[\tau_F \mid \tau_F < \infty]$ decreases (increases) with β for low (high) values of p. This tendency is exactly reflected in Theorem 2(i) and Figure 4(a) for large β: $V_L(\cdot) = V_M(\cdot)$ increases (decreases) with β for low (high) values of p. Then we examine the case where $\alpha_I < 0$ and find a similar result except that the sign of the comparative statics is opposite due to the opposite sign of α_L. This is illustrated in Figure 4(b).

5.3.2. Comparative Statics of $\bar{\tau}_M(p)$. Now we examine the impact of learning on $\bar{\tau}_M(p)$ and focus on the comparative statics of $\bar{\tau}_M(p)$ for large values of β. (The WA region shrinks to an almost null set in the limit $\beta \to 0$, so it is difficult to obtain meaningful analytical results in the small-β limit.)

Theorem 3. (i) *Suppose $\alpha_I > 0$ for $I \in \{L, F\}$. For sufficiently high values of p in the interval (θ_L, θ_c), $\bar{\tau}_M(p)$ increases with β. Furthermore, whenever $\beta > \beta_c$ for some $\beta_c > 0$, there exists $q(\beta) \in (\theta_L, \theta_c)$ such that $\bar{\tau}_M(p)$ decreases with β for $p \in (\theta_L, q(\beta))$ and increases with β for $p \in (q(\beta), \theta_c)$.*

Figure 5 **The Impact of β on $\bar{\tau}_M(\cdot)$**



Notes. The shaded (unshaded) area represents the region in which $d\bar{\tau}_M(p)/d\beta > 0$ ($d\bar{\tau}_M(p)/d\beta < 0$). Panel (a) shows the case where $\alpha_L = 0.3$ and $\alpha_F = 0.2$, and panel (b) is where $\alpha_L = -0.02$ and $\alpha_F = -0.05$. Here we set $h = 2$, $l = 0.12$, $r = 0.1$, and $k = 10$.

(ii) *If $\alpha_I < 0$ for $I \in \{L, F\}$, then for each fixed value of p, there exists $\beta_c(p) > 0$ such that $\bar{\tau}_M(p)$ increases with β whenever $\beta > \beta_c(p)$.*

Theorem 3 states that the comparative statics of $\bar{\tau}_M(p)$ also has a single-crossing property (changes the sign at most once). Most importantly, contrary to the naïve expectation that a higher rate of learning induces the firms to delay their first investment (based on the analysis of §5.1), Theorem 3 establishes that $\bar{\tau}_M(p)$ decreases with β under certain conditions.

Figure 5(a) numerically confirms Theorem 3. For positive externalities in Figure 5(a), $d\bar{\tau}_M(p)/d\beta$ changes its sign once as p increases. In contrast, even though there exists no general result for the intermediate values for negative externalities, Figure 5(b) suggests that $d\bar{\tau}_M/d\beta$ is always positive for p between θ_L and θ_c, which is consistent with our asymptotic results in Theorem 3.

In the presence of positive externalities, the parameter region (β, p) in which $E^p[\tau_F \mid \tau_F < \infty]$ decreases with β tends to coincide with the region in which $dV_M(p)/d\beta > 0$ and $d\bar{\tau}(p)/d\beta < 0$, i.e., for small p and large β. In the presence of negative externalities, we need to take into account another effect from the strategic behavior of the follower. On the one hand, an increase in β can increase (decrease) τ_F, which would increase (decrease) the value of the leader's investment as the follower delays (advances) the investment. On the other hand, an increase in β can also influence the follower's opportunistic behavior: If the follower learns faster, then it can selectively invest whenever the profit potential is high. Thus, the follower's selective action diminishes the leader's payoff when the profit prospect is good. As a result, a higher rate of learning can have a negative impact on the leader's payoff for high values of p. Hence, even though the leader's payoff can increase with β for higher values of p, the increase in $V_F(\cdot)$ tends to overshadow the increase in $V_L(\cdot)$, so we only observe $d\bar{\tau}_M(p)/d\beta > 0$. This phenomenon is illustrated in Figure 5(b).

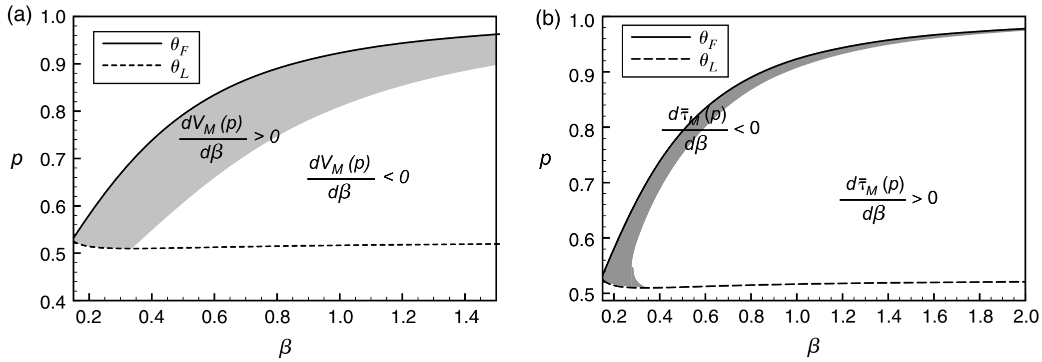
5.3.3. Discussion on the Interplay of Externality and Learning. Overall, the impact of learning on the equilibrium payoffs and the time to the first investment is nontrivial. In the WA region, the follower has the opportunity to learn about the unknown profitability of the investment by observing the leader's performance. Therefore, it is reasonable to infer that firms will tend to delay their investments with an increased rate of learning because learning tends to benefit the follower. However, our results show that an increased rate of learning may improve the leader's payoff and hence hasten the firms' investments. In fact, we obtain a single-crossing property in the comparative statics of the payoff and the expected time to the first investment.

This finding is driven by the following two conditions: (a) *The value of the leader's investment decreases (increases) with the follower's time to investment under positive (negative) externalities*. (b) *The follower's time to investment depends on the learning rate β*. For example, in the case of positive externalities of our model, the follower's earlier investment improves the leader's payoff, so it satisfies condition (a). In addition, the follower's time to investment increases in β for high p and decreases in β for low p, which implies condition (b). The combined effect of these two conditions causes the leader's payoff to decrease with β for high p and increase with β for low p. Incidentally, condition (b) is satisfied by the benchmark model ($\alpha_L = \alpha_F = 0$) since Theorem 1 holds even when $\alpha_L = \alpha_F = 0$. However, as shown by §5.1, $\bar{\tau}_M(p)$ always increases with β because condition (a) is absent. We conclude that the combined effect of (a) and (b) leads to our results on the single-crossing property of the comparative statics of the payoff and the time to investment.

5.4. Case of Second-Mover Advantage

Next we consider an interesting case when $\alpha_F > 0 > \alpha_L$, which represents situations with second-mover

Figure 6 The Impact of β on $V_M(p)$ and $\bar{\tau}_M(p)$



Note. Here we set $\alpha_L = -0.1$, $\alpha_F = 0.2$, $h = 2$, $l = 0.12$, $r = 0.1$, and $k = 10$.

advantage. For example, even though Apple was the first mover in the smartphone market, its position as the world's most profitable mobile phone maker was soon taken over by Samsung, which was the second mover (Garside 2013). Furthermore, this was a situation with both learning and externalities. Samsung's smartphones benefited from Apple's pioneering efforts on the development of the smartphone, and hence it enjoyed positive externalities from Apple's entry into the smartphone market. Moreover, by observing Apple's initial performance, Samsung was able to learn that there was very high demand in the smartphone market.

First, note that Theorem 1 always holds irrespective of the sign of α_F. Next, we establish the following:

Proposition 5. *If $\alpha_F > 0 > \alpha_L$, then $(\theta_L, 1)$ is the WA region.*

In other words, the regions of PE and SM do not exist because of the second-mover advantage. Furthermore, $V_M(\cdot)$ and $\bar{\tau}_M(\cdot)$ have no dependence on β for $p > \theta_F$ because $V_F(p) = (1/r)(1 + \alpha_F)m(p) - k$ and $V_L(p) = (1/r)(1+\alpha_L)m(p) - k$ for $p > \theta_F$. Thus, we focus on the comparative statics within the interval $(0, \theta_F)$.

It is straightforward to prove that the statements of Theorem 2(ii) exactly apply for the case $\alpha_F > 0 > \alpha_L$. In other words, the sign of $dV_M(p)/d\beta$ changes at most once as p or β increases, and hence the single-crossing property holds. This is illustrated in Figure 6(a). The proof is essentially identical to that for Theorem 2(ii), and hence omitted.

Last, we obtain the following comparative statics of $\bar{\tau}_M(p)$:

Theorem 4. *Suppose $\alpha_F > 0 > \alpha_L$. For sufficiently high values of p in the interval (θ_L, θ_F), $\bar{\tau}_M(p)$ decreases with β. Furthermore, there exist $\beta_c > 0$ and a function $p_c(\beta) \in (\theta_L, \theta_F)$ such that $\bar{\tau}_M(p)$ increases with β whenever $(\beta, p) \in (\beta_c, \infty) \times (\theta_L, p_c(\beta))$.*

This establishes that the comparative statics of $\bar{\tau}_M(p)$ changes in p at least once (an odd number of times) for large values of β. In fact, the numerical example in Figure 6(b) shows that $d\bar{\tau}_M(p)/d\beta$ changes its sign exactly once in p for sufficiently large β (for $\beta > 0.335$). For small values of β, the sign change of $d\bar{\tau}_M(p)/d\beta$ may not happen at all, as shown in Figure 6(b) for $\beta < 0.281$. For the intermediate values of β $(0.281 < \beta < 0.335)$, Figure 6(b) shows that the sign change of $d\bar{\tau}_M(p)/d\beta$ happens twice as p increases. Even if we account for a regime of small to intermediate values of β, our numerical examples indicate that $d\bar{\tau}_M(p)/d\beta$ exhibits a single-crossing property as β changes from small to large values with a fixed value of p. Overall, Figure 6(b) demonstrates that the qualitative behavior of $d\bar{\tau}_M(p)/d\beta$ is the same as the case of the first-mover advantage in the sense that there exists at most one single boundary between the regions of $d\bar{\tau}_M(p)/d\beta > 0$ and $d\bar{\tau}_M(p)/d\beta < 0$.

In summary, the main effect of the second-mover advantage with different signs of α_F and α_L is that there is no PE region. This is because both the externalities and the opportunity of learning favor the follower's payoff. Nevertheless, the single-crossing property largely remains true for this case as well.

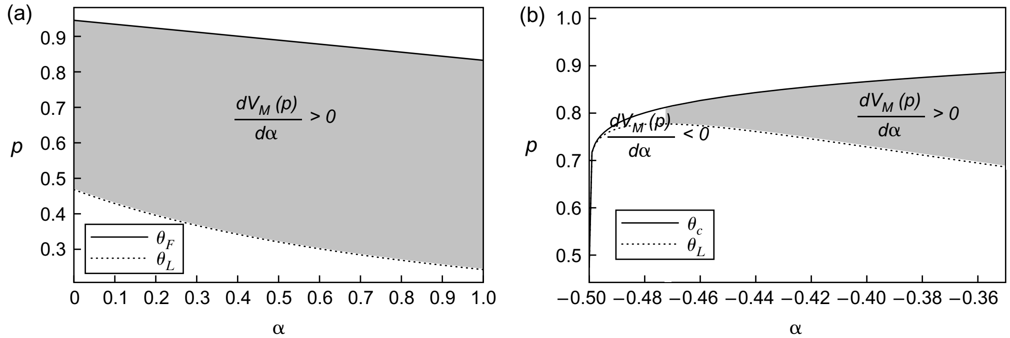
6. The Impact of Externality

In this section, we briefly discuss the impact of externality. For simplicity, we consider the case where $\alpha_L = \alpha_F = \alpha$ and illustrate examples of numerical comparative statics with respect to a single parameter α.[2] Here we remark that *an increase in the externality* means an increase in the value of α. This implies that when $\alpha < 0$, an increase in externality implies *a decrease in the magnitude of the externality* (a decrease in $|\alpha|$).

We first illustrate the comparative statics of $V_M(\cdot)$ in Figure 7. With the positive externalities in Figure 7(a), $V_M(p)$ always increases with α. If $\alpha > 0$, then the equilibrium payoff $V_M(\cdot)$ increases with α because

[2] The mathematical proofs of the main results illustrated in this section are available, although they are not presented in this paper.

Figure 7 **The Impact of α on $V_M(\cdot)$**



Notes. The shaded (unshaded) area represents the region in which $dV_M(p)/d\alpha > 0$ ($dV_M(p)/d\alpha < 0$). Here we set $h = 2$, $l = 0.12$, $r = 0.1$, $k = 10$, and $\beta = 1$.

the leader's profit stream after the follower's investment is proportional to $(1+\alpha)$, combined with the fact that the follower's expected time of investment $E^p[\tau_F \mid \tau_F < \infty]$ decreases with α. It follows that θ_L decreases with α. In contrast, if $\alpha < 0$, then an increase in α has two countervailing effects: On the one hand, an increase in α improves the leader's payoff after the follower invests. On the other hand, the strategic behavior of the follower diminishes the leader's payoff. More specifically, the follower tends to be discouraged from investing in the presence of negative externality; hence, if α increases, the follower is more encouraged to invest, and so the leader's payoff may decrease. Thus, if $\alpha < 0$, the comparative statics of θ_L or $V_M(\cdot)$ with respect to α are not clear a priori, based on intuitive reasoning alone. Figure 7(b) demonstrates the insight about the two countervailing effects of the negative externality. For the negative externalities in Figure 7(b), $V_M(p)$ is nonmonotonic and may increase or decrease with α.

Next, Figure 8 provides a numerical illustration of the comparative statics for $\bar{\tau}_M(p)$. It illustrates that whereas an increase in positive externality encourages firms to invest earlier, the same is not necessarily true in the presence of negative externality. In the presence of the positive externality as in Figure 8(a), firms tend to invest earlier when α increases because greater externality definitely improves firms' payoff. In contrast, in the presence of the negative externality as in Figure 8(b), the time to the first investment does not necessarily decrease with α, since $\bar{\tau}_M(p)$ increases with α near θ_c, whereas it decreases with α near θ_L. For high values of p close to θ_c, due to the possibility of high profit, the follower has a greater incentive to invest as α increases, but the leader's payoff may not improve much as α increases because of the higher likelihood that the follower will invest. By the functional form of $\bar{\tau}_M(p)$ in (17), if the follower's payoff increases with α to a larger extent than does the leader's payoff, then it makes intuitive sense that $\bar{\tau}_M(p)$ increases with α. The intuition for p close to θ_L coincides with that for the case of positive externality.

Overall, the comparative statics results indicate that a higher degree of positive externality encourages firms to invest earlier. In contrast, the impact of negative externality is more nuanced.

7. Some Related Models

In this section, we briefly discuss two related models and check whether our main results hold.

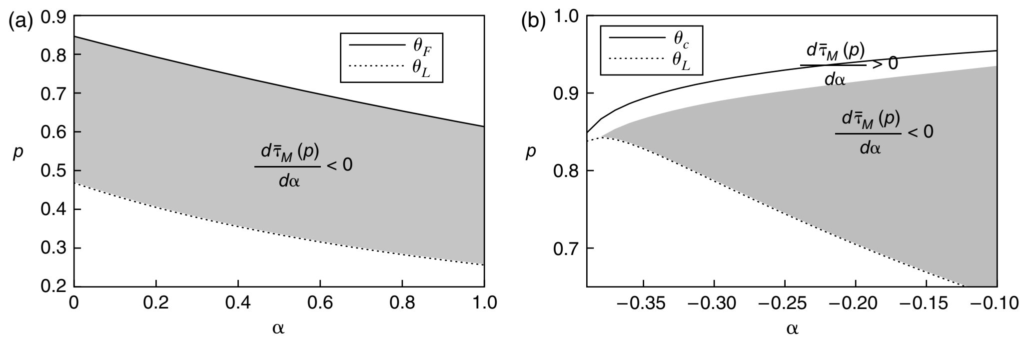
7.1. Nonzero Cost of Learning

We consider the case in which it is costly for the follower to collect information and learn the true market demand in stage 1. Let $V_i(p; T_1, T_2; k, c)$ denote firm i's payoff, where k is the cost of investment, and c is the follower's per-unit time cost of observing the leader's payoff in stage 1. For example, without loss of generality, suppose that firm 1 is the leader and firm 2 is the follower. Defining $k' \equiv k - c/r$, we obtain

$$\begin{aligned}
V_1(p;0,\tau_2;k,c) &= E^p\left[-k+\int_0^{\tau_2} e^{-rt}\,dX_t+e^{-r\tau_2}\hat{U}_L\right]\\
&= -\frac{c}{r}+E^p\left[-k'+\int_0^{\tau_2} e^{-rt}\,dX_t+e^{-r\tau_2}\hat{U}_L\right]\\
&= V_1(p;0,\tau_2;k',0)-\frac{c}{r},\\
V_2(p;0,\tau_2;k,c) &= E^p\left[\int_0^{\tau_2}(-c)e^{-rt}\,dt+e^{-r\tau_2}(\hat{U}_F-k)\right]\\
&= -\frac{c}{r}+E^p\left[e^{-r\tau_2}(\hat{U}_F-k')\right]\\
&= V_2(p;0,\tau_2;k',0)-\frac{c}{r}.
\end{aligned}$$

Thus, the game with a nonzero c can be conveniently transformed into another game with $k' = k - c/r$ and its associated payoff reduced by c/r for each player. Thus, all the results of the previous sections continue to hold for this model, except that $\theta_L = \inf\{p\colon V_1(p;0,\tau_2;k,c) > 0\}$ takes a higher value than $\inf\{p\colon V_1(p;0,\tau_2;k',0) > 0\}$. We conclude that the follower's cost of collecting information does not alter the essential impact of learning and externalities.

Figure 8 The Impact of α on $\bar{\tau}_M(\cdot)$



Notes. The shaded (unshaded) area represents the region in which $d\bar{\tau}_M(p)/d\alpha < 0$ ($d\bar{\tau}_M(p)/d\alpha > 0$). Here we set $h = 2$, $l = 0.1$, $r = 0.1$, $k = 12$, and $\beta = 1$.

7.2. Learning from a Public Signal

In some cases, the signal of the market demand is exogenous and public. Hence, it is useful to consider a model with learning from a public signal and to compare its results with our main findings. Let X denote the cumulative public signal that satisfies $dX_t = \mu\,dt + \sigma\,dW_t$ for $t \in [0, \infty)$, where $\mu \in \{h, l\}$ with $h > l > 0$, and μ is unknown. For example, X can be the cumulative profit stream from a closely related industry.

In stage 1, no one has invested, but the signal process X evolves, and the posterior probability P_t evolves as

$$P_t = \left[1 + \frac{1-p}{p}\exp\left\{-\frac{(h-l)}{\sigma^2}\left[X_t - \frac{h+l}{2}t\right]\right\}\right]^{-1}.$$

Suppose that firm 1 is the leader who invests at some time T_1. In stage 2, we assume that the leader's profit for the duration dt is simply ϕdX_t for some positive constant ϕ. This assumption ensures that the public signal is the only source of information regarding the quality of the market and that the leader's profit stream does not add any extra information.

Suppose that firm 2 invests at $T_2 \geq T_1$. Then in stage 3, we assume that both the leader and the follower earn $\phi(1+\alpha)\,dX_t$ for some α that represents the degree of externality.

Based on the model assumptions, the payoffs to the firms are given by the following:

$$V_1(p; T_1, T_2) = E^p\left[\int_{T_1}^{T_2} e^{-rt}\phi\,dX_t + \int_{T_2}^{\infty} e^{-rt}\phi(1+\alpha)\,dX_t\right],$$

$$V_2(p; T_1, T_2) = E^p\left[\int_{T_2}^{\infty} e^{-rt}\phi(1+\alpha)\,dX_t\right].$$

We do not present a detailed analysis, but it can be shown that in this model of a purely public signal, a war of attrition never happens. This is because in such a model the signal is already being generated, and neither player would need to wait for the other to invest first to learn about the market demand. In contrast, in the model of §3, the war of attrition occurs because each firm wants the other to invest first and start producing the signal, which generates a mixed strategy equilibrium in which both wait for the other to invest first. Since a war of attrition does not take place with purely public signals, we conclude that this model lies outside the scope of this paper.

8. Conclusions

Investments in new unproven projects in competitive situations are fraught with uncertainty. In general, returns from such investments are governed by positive or negative externalities from investments made by competing firms. Moreover, firms often have the opportunity to learn the potential value of investing in similar projects by observing the performance of their competitors' investments. In this paper, we investigate the impact of learning and externalities on equilibrium investment strategies. We find that due to the strategic interactions, externalities and learning opportunities have counterintuitive effects on investment strategies and on the time to the first investment. In particular, a single-crossing property in p and β is exhibited by the comparative statics of the payoff and the expected time to investment with respect to the rate of learning. Thus, depending on the values of p and β, a higher learning rate may hasten the first investment, which is in contrast to the conventional result from the benchmark model without externalities.

Overall, our results suggest that firms facing entry into an unproven market need to consider the strategic effects arising from the interplay between externalities and learning. In particular, the effect of externalities needs to be incorporated when modeling a competitive investment problem with learning opportunities as a war of attrition.

Acknowledgments

The authors are grateful to the two anonymous reviewers, the associate editor, and the department editor for their very helpful comments and suggestions.

Appendix A. Some Preliminaries for Proposition 1

Here we provide the expressions for the notations used in Proposition 1:

$$\theta_F \equiv \left[1+\frac{\gamma-1}{\gamma+1}\cdot\frac{h(1+\alpha_F)-kr}{kr-l(1+\alpha_F)}\right]^{-1}, \tag{A1}$$

$$\gamma \equiv \sqrt{1+8r\frac{\sigma^2}{(h-l)^2}} = \sqrt{1+\frac{8r}{\beta^2(h-l)^2}}, \tag{A2}$$

$$\psi(x) \equiv x^{(\gamma+1)/2}(1-x)^{(1-\gamma)/2}. \tag{A3}$$

In particular, the function $\psi(x)$ is an increasing fundamental solution to the differential equation $\mathcal{A}\psi(x)=0$ (see, e.g., Kwon and Lippman 2011), where

$$\mathcal{A} \equiv \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 p^2(1-p)^2\partial_p^2 - r \tag{A4}$$

is the r-excessive *characteristic operator* (Alvarez 2003) for the process P.

Appendix B. Proofs of Mathematical Statements

Proof of Proposition 1. (i) We first consider the case $\tau_2 > 0$ to obtain $\sup_{\tau_2>0} V_2(p; T_1, T_1+\tau_2)$, i.e., the optimal policy under the condition that firm 1 already invested. We employ Theorem 3(A) of Alvarez (2001) to prove this proposition. For convenience, we follow the convention of Alvarez (2001) and define $g(p)=(1+\alpha_F)m(p)/r-k$, which is the follower's value of investing immediately. We also define a function $\Pi_1(p)=g(p)/\psi(p)$, where $\psi(p)$ is defined in (A3). Our goal is to prove that $\tau_F=\inf\{t>0\colon P_t\ge\theta_F\}$ is the optimal stopping time and that the function $\Pi_1(\theta_F)\cdot\psi(p)=\Pi_F(p)$ in (6) is the optimal value function. To prove it, by virtue of Theorem 3(A) of Alvarez (2001), we only need to prove that $g(\cdot)$ is nondecreasing, that $\Pi_1(\cdot)$ attains a unique global interior maximum at θ_F, and that $\Pi_1(\cdot)$ is nonincreasing for $p>\theta_F$.

The derivative of $\Pi_1(\cdot)$ is given as follows:

$$\Pi_1'(p) = \frac{\psi'(p)\{(1-\gamma)[h(1+\alpha_F)-kr]+(1+\gamma)[l(1+\alpha_F)-kr]\}}{r\psi^2(p)(\gamma+1-2p)} \cdot(p-\theta_F).$$

Note that $\gamma>1$, $h(1+\alpha_F)>kr>l(1+\alpha_F)$, and $\gamma+1>2p$ for all $p\in(0,1)$. Hence, $\Pi_1'(p)>0$ for $p<\theta_F$ and $\Pi_1'(p)<0$ for $p>\theta_F$, and it follows that $\Pi_1(p)$ attains its global maximum at θ_F and that $\Pi_1(p)$ is decreasing for $p>\theta_F$. We conclude that $\Pi_F(p)=\sup_{\tau_2>0}V_2(p;0,\tau_2)=V_2(p;0,\tau_F)$.

If $\Pi_S(p)>\Pi_F(p)$, however, then the follower's optimal policy is to invest immediately at T_1 when the leader invests. For $p\ge\theta_F$, the inequality $\Pi_S(p)>\Pi_F(p)$ is always satisfied because $\alpha_S>\alpha_F$. Also, note that $\tau_F>0$ a.s. if $p<\theta_F$. Thus, the optimal value function is given by $V_F(p)=\sup_{\tau_2\ge0}V_2(p;0,\tau_2)=\max\{\Pi_F(p),\Pi_S(p)\}$, and the optimal stopping time is τ_F if $\Pi_F(p)\ge\Pi_S(p)$ and 0 if $\Pi_S(p)>\Pi_F(p)$.

(ii) Note that $\Pi_S(\theta_F)>\Pi_F(\theta_F)$ because $\alpha_S>\alpha_F$. Furthermore, $\Pi_S(0)<\Pi_F(0)$ because $\Pi_S(0)=l(1+\alpha_S)/r-k<0$ by Assumption 1 whereas $\Pi_F(p)>0$ for all $p\in(0,1)$. We also note that $\Pi_S(p)$ is linear whereas $\Pi_F(p)$ is convex. We conclude that $\Pi_S(p)$ and $\Pi_F(p)$ can cross only once in the interval $(0,\theta_F)$. The statement of the proposition follows. □

Proof of Proposition 2. First, we consider $p<\theta_S$, in which case the follower's strategy is to invest at τ_F with the threshold θ_F. Consider $\tau_\theta=\inf\{t>0\colon P_t\ge\theta\}$ for some $\theta\in(0,1)$ and the quantity $f(p)=E^p[e^{-r\tau_\theta}m(P_{\tau_\theta})]$. From the well-known theory of stopping values (see, e.g., Chap. 9 of Oksendal 2003) $f(\cdot)$ satisfies $\mathcal{A}f(p)=0$ for $p<\theta$ with the boundary condition $f(\theta)=m(\theta)$. It follows that $E^p[e^{-r\tau_\theta}m(P_{\tau_\theta})]=\psi(p)m(\theta)/\psi(\theta)$ for any $p\le\theta$. Hence, from (4), we obtain (11) for $p<\theta_S$. Second, if $p\ge\theta_S$, then the follower will also immediately invest at the same time, so both the leader and the follower expect a payoff given by $\Pi_S(p)$ in (12).

Given the follower's strategy of investment at τ^* given by (8), the leader's optimal policy is then to invest immediately if $V_1(p;0,\tau^*)\ge0$ and never invest if $V_1(p;0,\tau^*)<0$. Hence, the leader must compare $V_1(p;0,\tau^*)$ given by (11) and (12) with $V_1(p;\infty,\tau^*)=0$ and choose the maximum of the two. It follows that (10) holds.

Note that $\Pi_L(p)$ increases in p because both $m(\cdot)$ and $\psi(\cdot)$ are increasing functions. Also, because $\Pi_L(0)=l/r-k<0$ and $\Pi_L(\theta_S)=V_F(\theta_S)>0$, θ_L defined by (13) must satisfy $\theta_L\in(0,\theta_S]$. Thus, $V_L(p)>0$ if and only if $p>\theta_L$. □

Proof of Proposition 3. This proposition follows from Propositions 1 and 2, which detail the best responses of the leader and the follower.

(i) In $[0,\theta_L)$, the firm that invests first (the leader) expects a negative payoff, so none of the players invest.

(ii) In the region $[\theta_L,\theta_S)$, consider the strategy profile (T_1,T_2) in which firm 1 takes the leader's role with an investment threshold θ_L and firm 2 takes the follower's role with a threshold θ_F. We now prove that this strategy profile is a Nash equilibrium and that it is subgame perfect.

To show that it is a Nash equilibrium, we only need to show that each firm's strategy is the best response given the other firm's strategy. We first consider firm 1's best response. Since it is already known that firm 2 will wait until the probability P_t reaches θ_F before investment, firm 1's best response is to invest with the upper threshold of θ_L as was established in Proposition 2.

Now we suppose that firm 1's strategy is to invest immediately. Then firm 2's optimal policy (best response) is to wait until P_t reaches the threshold θ_F by virtue of Proposition 1. It follows that (T_1,T_2) is a Nash equilibrium.

Now we prove that (T_1,T_2) is a subgame perfect equilibrium. After firm 1 invests, firm 2's optimal policy is a stationary Markov policy, and hence (T_1,T_2) is still a Nash equilibrium. Before firm 1 invests, even if firm 1 (the leader) waits for any length of time, because the prior probability never changes, (T_1,T_2) still remains a Nash equilibrium. Therefore, the Nash equilibrium (T_1,T_2) is subgame perfect because it remains a Nash equilibrium at any point in time.

Finally, because firms 1 and 2 are symmetric, there exists another subgame perfect equilibrium (T_2,T_1) in which firm 2's threshold is θ_L while firm 1's threshold is θ_F.

(iii) In $[\theta_S,1]$, because $\Pi_S(p)\ge\Pi_F(p)>0$, both firms invest immediately. □

Proof of Lemma 1. Define the function

$$f(p)=\frac{m(p)}{r}-k+\frac{\psi(p)}{\psi(\theta_F)}\left[(\alpha_L-1-\alpha_F)\frac{m(\theta_F)}{r}+k\right]$$

so that $f(p) = \Pi_L(p) - \Pi_F(p)$ for $p < \theta_S$. (The function $f(\cdot)$ is defined above for any value of p, however.) Note that $f(0) = l/r - k < 0$, $f(\theta_L) < 0$ (because $V_L(\theta_L) = 0$ and $V_F(p) > 0$ for all p), but $\lim_{p\to\theta_S} f(p)$ may or may not be positive.

If $(\alpha_L - 1 - \alpha_F)(m(\theta_F)/r) + k \geq 0$, then $f(\cdot)$ is a strictly increasing convex function. Hence, even if $\lim_{p\to\theta_S} f(p) > 0$, there exist $\theta_c \in (\theta_L, \theta_S)$ such that $f(p) > 0$ if and only if $p \in (\theta_c, \theta_S)$.

If $(\alpha_L - 1 - \alpha_F)(m(\theta_F)/r) + k < 0$, then $f(\cdot)$ is a concave function. Then, using the fact that $V_F'(\theta_F) = (\alpha_F + 1)\cdot m'(\theta_F)/r = (\psi'(\theta_F)/\psi(\theta_F))[(1+\alpha_F)m(\theta_F)/r - k]$, we obtain

$$\begin{aligned} f'(\theta_F) &= \frac{m'(\theta_F)}{r} + \frac{\psi'(\theta_F)}{\psi(\theta_F)}\left[(\alpha_L - 1 - \alpha_F)\frac{m(\theta_F)}{r} + k\right] \\ &= \frac{m'(\theta_F)}{r}\,\frac{(\alpha_L - \alpha_F)(1+\alpha_F) + k\alpha_F}{(1+\alpha_F)m(\theta_F)/r - k}. \end{aligned}$$

Suppose that $f'(\theta_F) \geq 0$. From the concavity of $f(\cdot)$, we deduce that $f'(p) > 0$ for all $p \in (0, \theta_S)$. Thus, even if $\lim_{p\to\theta_S} f(p) > 0$, there exist $\theta_c \in (\theta_L, \theta_S)$ such that $f(p) > 0$ if and only if $p \in (\theta_c, \theta_S)$. Now suppose that $f'(\theta_F) < 0$. We also know that $f'(0) > 0$, $f(\theta_L) < 0$, and $f(\theta_F) = (\alpha_L - \alpha_F)\cdot m(\theta_F)/r \geq 0$, so $f(p)$ crosses zero (from negative to positive) only once in the interval (θ_L, θ_F) and at most once in the interval (θ_L, θ_S) because $\theta_S \leq \theta_F$. Thus, there exists $\theta_c \in (\theta_L, \theta_S]$ such that $f(p) > 0$ if and only if $p \in (\theta_c, \theta_S)$. □

Proof of Proposition 4. (i) The proof is based on the analytical results from Hendricks et al. (1988). To utilize the results of Hendricks et al. on the war of attrition for our problem, it is necessary to make a change of variable $z = t/(t+1)$ so that $z \in [0,1]$, where $z = 1$ is understood as the limit $t \to \infty$. We also define the following: $L(z) \equiv \Pi_L(p)\exp(-r(z/(1-z)))$, $F(z) \equiv V_F(p)\exp(-r(z/(1-z)))$, $S(z) \equiv \Pi_S(p)\exp(-r(z/(1-z)))$, and $I(z_1, z_2) \equiv \exp\int_{z_1}^{z_2} dL(z)/(F(z) - L(z))$. Here $L(\cdot)$ and $F(\cdot)$ are the discounted payoffs to the leader and the follower, respectively, and $S(\cdot)$ is the discounted payoff of simultaneous entry.

First, we note that $L(\cdot)$, $F(\cdot)$, and $S(\cdot)$ are continuous on $[0,1]$. Second, by definition of WA region, $V_F(p) > \Pi_S(p)$ and $V_F(p) > V_L(p)$ for p within WA. We also note that $L(z)$ is strictly decreasing because $\Pi_L(p) > 0$ for $p > \theta_L$, and $\exp[-rz/(1-z)]$ is strictly decreasing in z. Therefore, all the assumptions made by Hendricks et al. (1988) are satisfied here.

Now we check the conditions for a Nash equilibrium. Theorem 2 of Hendricks et al. (1988) stipulates the necessary and sufficient condition for the existence of equilibrium with $q_p^{(i)}(0) < 1$. From the definition of $I(z_1, z_2)$ given by Hendricks et al. (1988), we have $I(z_1, z_2) = \exp[-\int_{z_1}^{z_2}(1-z)^{-2}/\bar{\tau}_M(p)\,dz]$, where $\bar{\tau}_M(p)$ is given in (17). We note that $I(0,0) = 1$ and that $I(z,1) = 0$ so that $I(1,1) \equiv \lim_{z\uparrow 1} I(z,1) = 0$. Thus, our model satisfies the conditions of Theorem 2 of Hendricks et al. (1988), and it allows for an equilibrium.

Next, Theorem 3 of Hendricks et al. (1988) provides the necessary and sufficient conditions for a strategy profile to be an equilibrium. The theorem essentially states that if $L(z) > S(1) = 0$ for any $z < 0$, which is satisfied by our model, then the only possible form of equilibrium strategy profile is one in which $(q_p^{(1)}(0), q_p^{(2)}(0)) \in [0,1) \times [0,1)$ and $q_p^{(1)}(0)q_p^{(2)}(0) = 0$, and $G_p^{(i)}$ is exactly given by (16).

Then we show that the Nash equilibrium we obtained is subgame perfect. We only need to prove that at any time $s > 0$, the conditional probability distributions (conditional on the fact that neither firm has invested yet by time s) constitute a Nash equilibrium. Let $G_p^{(i)}(t \mid s)$ be the conditional distribution for time $t > s$. Then

$$G_p^{(i)}(t \mid s) = 1 - \frac{1 - G_p^{(i)}(t)}{1 - G_p^{(i)}(s)} = 1 - \exp\left[-\frac{t-s}{\bar{\tau}_M(p)}\right],$$

which reduces to $G_p^{(i)}(t-s)$ when $q_p^{(i)}(0) = 0$. Therefore, the strategy profile $(G_p^{(1)}(t \mid s), G_p^{(2)}(t \mid s))$ is a Nash equilibrium. We conclude that the equilibria we obtained in the proposition are subgame perfect.

(ii) First, we obtain (15) from when $q^{(i)} = 0$ and $q^{(j)}(0) > 0$, where $G_p^{(i)}$ and $G_p^{(j)}$ are given by (16). We also obtain (15) when $q^{(i)} > 0$ and $q^{(j)}(0) = 0$. (Note that $\Pi_L(p) = V_L(p)$ for $p \geq \theta_L$.)

Second, suppose that $q_p^{(i)}(0) = \mathbb{P}(\{\hat{T}_i = 0\}) \geq 0$ and $q_p^{(j)}(0) = 0$. Then

$$\begin{aligned} E[\min\{\hat{T}_i, \hat{T}_j\}] &= E[\min\{\hat{T}_i, \hat{T}_j\} \mid \hat{T}_i > 0]\,\mathbb{P}(\{\hat{T}_i > 0\}) \\ &= \frac{\bar{\tau}_M(p)}{2}[1 - q_p^{(i)}(0)], \end{aligned}$$

because $E[\min\{\hat{T}_i, \hat{T}_j\} \mid \hat{T}_i > 0] = E[\min\{\hat{T}_i, \hat{T}_j\}] = \bar{\tau}_M(p)/2$ from the fact that $\hat{T}_i$ and $\hat{T}_j$ are exponentially distributed. In general, because $q_p^{(i)}(0)q_p^{(j)}(0) = 0$, Equation (19) holds. □

Proof of Lemma 2. As a preliminary step, we study $E^p[\tau_\theta \mid \tau_\theta < \infty]$, where $\tau_\theta = \inf\{t > 0\colon P_t \geq \theta\}$ for some fixed value of θ. Note that for any $r > 0$, we can express $\mathbf{1}_{\{\tau_\theta < \infty\}} = \mathbf{1}_{\{\tau_\theta < \infty\}} + e^{-r\tau_\theta}\mathbf{1}_{\{\tau_\theta = \infty\}}$. Hence, we can express

$$\begin{aligned} \mathcal{P}(\tau_\theta < \infty) &= E^p(\mathbf{1}_{\{\tau_\theta < \infty\}}) = E^p[\mathbf{1}_{\{\tau_\theta < \infty\}} + e^{-r\tau_\theta}\mathbf{1}_{\{\tau_\theta = \infty\}}] \\ &= E^p\left(\lim_{r\to 0} e^{-r\tau_\theta}\right) = \lim_{r\to 0} E^p(e^{-r\tau_\theta}), \end{aligned}$$

where the last equality is due to the bounded convergence theorem. From (9), we have $\mathcal{P}(\tau_\theta < \infty) = \lim_{r\to 0}\psi(p)/\psi(\theta) = p/\theta$. Similarly, we obtain

$$\begin{aligned} E^p[\tau_\theta 1_{\{\tau_\theta < \infty\}}] &= E^p\left\{\lim_{r\to 0}[\tau_\theta e^{-r\tau_\theta}\mathbf{1}_{\{\tau_\theta < \infty\}} + \tau_\theta e^{-r\tau_\theta}\mathbf{1}_{\{\tau_\theta = \infty\}}]\right\} \\ &= E^p\left[-\lim_{r\to 0}\frac{d(e^{-r\tau_\theta})}{dr}\right] = -\lim_{r\to 0} E^p\left[\frac{d(e^{-r\tau_\theta})}{dr}\right], \end{aligned}$$

where the last inequality is due to the bounded convergence theorem. Interpreting $d(e^{-rt})/dr = \lim_{r'\to r}(e^{-r't} - e^{-rt})/(r' - r)$, we can express $E^p[d(e^{-r\tau_\theta})/dr] = dE^p[e^{-r\tau_\theta}]/dr$ from the bounded convergence theorem, which allows us to exchange the order of the limit $r' \to r$ and the expectation $E^p[\cdot]$. Thus,

$$\begin{aligned} E^p[\tau_\theta 1_{\{\tau_\theta < \infty\}}] &= -\lim_{r\to 0}\frac{dE^p[e^{-r\tau_\theta}]}{dr} = -\lim_{r\to 0}\frac{d[\psi(p)/\psi(\theta)]}{dr} \\ &= \frac{p}{\theta}\log\left(\frac{\theta}{p}\,\frac{1-p}{1-\theta}\right)\frac{4\sigma^2}{(h-l)^2}. \end{aligned}$$

From the Bayes' rule, we finally obtain

$$E^p[\tau_\theta \mid \tau_\theta < \infty] = \frac{E^p[\tau_\theta \mathbf{1}_{\{\tau_\theta < \infty\}}]}{\mathcal{P}(\tau_\theta < \infty)} = \log\left(\frac{\theta}{p}\,\frac{1-p}{1-\theta}\right)\frac{4\sigma^2}{(h-l)^2}. \quad \text{(B1)}$$

It follows that $E^p[\tau_F \mid \tau_F < \infty]$ is given by the right-hand side of (B1) with θ replaced by θ_F. □

Proof of Theorem 1. From the equality $\sigma^2/(h-l)^2 = (\gamma^2-1)/(8r)$, we can express $E^p[\tau_F \mid \tau_F < \infty]$ as a function of γ as follows:

$$f(\gamma,p) = \log\left(\frac{\theta_F}{1-\theta_F}\cdot\frac{1-p}{p}\right)\frac{\gamma^2-1}{2r},$$

where θ_F has γ dependence as in (A1). From the expression of θ_F in (A1), the partial derivative of $f(\gamma,p)$ with respect to γ can be expressed as follows:

$$\begin{aligned} f_1(\gamma,p) &\equiv \frac{\partial f(\gamma,p)}{\partial\gamma} = \frac{\gamma^2-1}{2r\theta_F(1-\theta_F)}\frac{d\theta_F}{d\gamma} + \frac{\gamma}{r}\log\left(\frac{\theta_F}{1-\theta_F}\cdot\frac{1-p}{p}\right) \\ &= -\frac{1}{r} + \frac{\gamma}{r}\log\left(\frac{\theta_F}{1-\theta_F}\cdot\frac{1-p}{p}\right), \end{aligned} \quad \text{(B2)}$$

where $d\theta_F/d\gamma$ is given by

$$\frac{d}{d\gamma}\theta_F = \frac{-2\theta_F^2[(1+\alpha_F)h - kr]}{(\gamma+1)^2[kr-(1+\alpha_F)l]} < 0.$$

To prove the theorem, we need to determine the sign of $f_1(\gamma,p)$. Because γ decreases in β, if $f_1(\gamma,p) > 0$, then $E^p[\tau_F \mid \tau_F < \infty]$ decreases in β, and vice versa.

We consider the following:

$$\frac{\partial(r\gamma^{-1}f_1(\gamma,p))}{\partial\gamma} = \frac{1}{\gamma^2} + \left(\frac{1}{\theta_F} + \frac{1}{1-\theta_F}\right)\frac{d\theta_F}{d\gamma} = -\frac{\gamma^2+1}{(\gamma^2-1)\gamma^2} < 0.$$

Furthermore, for a given $p \in (0,1)$, $\lim_{\gamma\to 1} f_1(\gamma,p) > 1$ because $\lim_{\gamma\to 1}\theta_F = 1$. Hence, $f_1(\gamma,p)$ crosses zero (from positive to negative) at most once as γ increases. If $p > \lim_{\gamma\to\infty}\theta_F = \theta_0$, then θ_F eventually coincides with p at a sufficiently large value of γ, at which point $f_1(\gamma,p) = -r^{-1} < 0$. Therefore, if $p > \lim_{\gamma\to\infty}\theta_F$, then $f_1(\gamma,p)$ crosses zero exactly once as γ increases from 1 to ∞.

If, on the other hand, $p < \theta_0$, then $\lim_{\gamma\to\infty} f_1(\gamma,p) > 0$ because the logarithmic term is positive, so $f_1(\gamma,p)$ never crosses zero as γ increases. It follows that $f_1(\gamma,p) > 0$ for all $p < \theta_0$.

Last, note that $f_1(\gamma,p)$ is strictly decreasing in p and that $\lim_{p\to 0} f_1(\gamma,p) > 0$ and $f_1(\gamma,\theta_F) < 0$. Thus, $f_1(\gamma,p)$ crosses zero exactly once as p increases from 0 to θ_F. □

Proof of Theorem 2. As a preliminary step, we obtain the comparative statics of $\Pi_L(p)$ with respect to β when $(p,\beta)\in\mathscr{W}$. After some algebra, $d\Pi_L(p)/d\beta$ can be expressed as follows:

$$\frac{d\Pi_L(p)}{d\beta} = \frac{d\gamma}{d\beta}\cdot\frac{d\Pi_L(p)}{d\gamma} = \frac{d\gamma}{d\beta}\cdot\alpha_L\frac{m(\theta_F(\beta))\psi(p)}{r\psi(\theta_F(\beta))}\cdot g(\theta_F(\beta)),$$

$$\text{where}\quad g(x) \equiv \frac{m(\theta_0)}{2\theta_0(1-\theta_0)}\cdot\frac{x-\theta_0}{m(x)} - \frac{1}{2}\log\frac{x(1-p)}{(1-x)p}. \quad \text{(B3)}$$

Here we used the fact that $\gamma = 1 + 2\theta_0(1-\theta_F)/(\theta_F-\theta_0)$ and expressed all the β-dependencies in terms of $\theta_F(\beta)$. Because it is already established that $d\gamma/d\beta < 0$ and $m(\theta_F) > 0$, the sign of $d\Pi_L(p)/d\beta$ depends on the signs of $g(\theta_F(\beta))$ and α_L. Note that $\theta_F(\beta)$ is strictly increasing in β by Proposition 2 of Kwon and Lippman (2011) and that $\lim_{\beta\to 0}\theta_F(\beta) = \theta_0$ and $\lim_{\beta\to\infty}\theta_F(\beta) = 1$.

As a preliminary step, we consider $p > \theta_0$, in which case the possible value of β is within $(\underline{\beta}_p, \infty)$ for some $\underline{\beta}_p > 0$, and the possible values of $\theta_F(\beta)$ are within the interval $(\theta_F(\underline{\beta}_p), 1)$, where $\theta_F(\underline{\beta}_p) \geq \theta_c(\underline{\beta}_p) = p$. For now, we extend the domain of the function $g(\cdot)$ to the interval $[p,1)$ and establish that this extended function $g(\cdot)$ crosses zero exactly once as x increases within the interval $[p,1)$. $g(\cdot)$ does cross zero because $\lim_{x\to 1} g(x) < 0$, and $g(p) > 0$ because $p > \theta_0$. We note that $dg(x)/dx = g_1(x)/[2m(x)^2\cdot\theta_0(1-\theta_0)x(1-x)]$, where $g_1(x) = m(\theta_0)^2x(1-x) - m(x)^2\cdot\theta_0(1-\theta_0)$. Note that $g_1(\cdot)$ is strictly concave quadratic function, and $g_1(x) = 0$ yields two roots: $x_1 = \theta_0$ and $x_2 = (1-\theta_0)l^2/[\theta_0h^2 + (1-\theta_0)l^2] < 1$. The first root x_1 is outside the interval $[p,1)$ because $p > \theta_0$, so we focus on the location of the second root x_2.

Suppose that $x_2 \leq p$. Then $g_1(x) < 0$ for $x\in[p,1)$, and hence $g(\cdot)$ strictly decreases in the interval $[p,1)$. It follows that $g(\cdot)$ changes sign only once.

Suppose that $p < x_2$. Then $g_1(x) > 0$ for $x < x_2$ and $g_1(x) < 0$ for $x > x_2$, so $g(\cdot)$ is strictly increasing in the interval $[p,x_2)$ and strictly decreasing in $(x_2,1)$. From the observation $g(p) > 0$ and $\lim_{x\to 1} g(x) < 0$, it follows that $g(x)$ changes sign only once in the interval $[p,1)$ as x increases.

Now we restrict the domain of $g(\cdot)$ to $(\theta_F(\underline{\beta}_p),1)$. By virtue of the analysis above, there is $\hat\theta\in[\theta_F(\underline{\beta}_p),1)$ such that $g(x)$ changes sign at $\hat\theta$ as x increases within the domain $(\theta_F(\underline{\beta}_p),1)$. (If $g(\cdot)$ does not change sign anywhere within $(\theta_F(\underline{\beta}_p),1)$, then $\hat\theta = \theta_F(\underline{\beta}_p)$.) Thus, $d\Pi_L(p)/d\beta$ changes sign at some $\hat\beta_M(p)\in[\underline{\beta}_p,\infty)$ as β increases within the domain $\{\beta\colon (p,\beta)\in\mathscr{W}\}$. Note also that $V_M(p) = \Pi_L(p)$ within the WA region.

Next, we note that the sign of $\lim_{\beta\to\infty} d\Pi_L(p)/d\beta$ coincides with the sign of α_L because $d\gamma/d\beta < 0$ and $\lim_{x\to 1} g(x) < 0$. Thus, the statements of the theorem follow regarding the comparative statics of $V_M(p)$ with respect to β for $p > \theta_0$.

We also note that $g(x)$ is strictly increasing in p, so the sign of $g(\cdot)$ can change at most once as p increases from $\theta_L(\beta)$ to $\theta_c(\beta)$ for a fixed β. If the sign change happens within $(\theta_L(\beta),\theta_c(\beta))$, then $d\Pi_L(p)/d\beta$ in (B3) changes from positive to negative as p increases if $\alpha_L > 0$, and from negative to positive as p increases if $\alpha_L < 0$.

Last, consider $p < \theta_0$. In the limit $x\downarrow\theta_0$ (or in the small-β limit) and in the limit $x\uparrow 1$ (or in the large-β limit), $g(x)$ is negative. Thus, $V_M(p)$ increases (decreases) in β for small or large values of β if α_L is positive (negative). □

Proof of Theorem 3. As the first step, we obtain an analytical expression for the determinant of the sign of $d\bar\tau_M(p)/d\beta$. From (17), we obtain

$$\frac{d\bar\tau_M(p)}{d\gamma} = \frac{1}{r\Pi_L^2(p)}\left[\Pi_L(p)\frac{d}{d\gamma}V_F(p) - V_F(p)\frac{d}{d\gamma}\Pi_L(p)\right].$$

Here, for $p < \theta_S$, $dV_F(p)/d\gamma$ is given by

$$\frac{dV_F(p)}{d\gamma} = \left[\frac{1}{r}(1+\alpha_F)m(\theta_F) - k\right]\frac{\psi(p)}{\psi(\theta_F)}\frac{1}{2}\ln\left(\frac{p}{1-p}\frac{1-\theta_F}{\theta_F}\right) < 0,$$

so we only need to compute $d\Pi_L(p)/d\gamma$. From the expression (11) and the equality

$$\left[\frac{1}{r}(1+\alpha_F)m(\theta_F) - k\right]\frac{\psi'(\theta_F)}{\psi(\theta_F)} = \frac{1}{r}(1+\alpha_F)m'(\theta_F), \quad \text{(B4)}$$

which is derived from the continuous differentiability of $V_F(p)$ at $p=\theta_F$, we obtain

$$\frac{d}{d\gamma}\Pi_L(p) = \frac{\alpha_L}{r}\frac{\psi(p)}{\psi(\theta_F)}\left[-\frac{d\theta_F}{d\gamma}\cdot\frac{\psi'(\theta_F)kr}{\psi(\theta_F)(1+\alpha_F)} + \frac{m(\theta_F)}{2}\log\left(\frac{p}{1-p}\cdot\frac{1-\theta_F}{\theta_F}\right)\right]$$

for $p<\theta_S$. Then it follows that, for $p<\theta_S$,

$$\begin{aligned}\Pi_L(p)\frac{d}{d\gamma}V_F(p) - V_F(p)\frac{d}{d\gamma}\Pi_L(p) &= \frac{V_F(p)}{r}\left[\frac{1}{2}(m(p)-kr)\log\left(\frac{p}{1-p}\cdot\frac{1-\theta_F}{\theta_F}\right)\right.\\ &\quad\left.-\frac{2\alpha_L kr\theta_F^2\theta_0}{(1+\alpha_F)(\gamma+1)^2}\cdot\frac{\psi'(\theta_F)\psi(p)}{\psi^2(\theta_F)}\right].\end{aligned}$$

Once we substitute $\gamma = 1+2\theta_0(1-\theta_F)/(\theta_F-\theta_0)$, which is an equality that can be verified from the definition of θ_F and θ_0, we conclude that $\bar{\tau}_M(p)$ increases (decreases) in β if and only if

$$\frac{1}{2}(m(p)-kr)\log\left(\frac{p}{1-p}\cdot\frac{1-\theta_F}{\theta_F}\right) - \frac{\alpha_L kr\theta_0(\theta_F-\theta_0)^2}{2(1+\alpha_F)(1-\theta_0)^2}\cdot\frac{\psi'(\theta_F)\psi(p)}{\psi^2(\theta_F)} \tag{B5}$$

is negative (positive).

Let us write (B5) as $A(p)+B(p)$, where

$$A(p) = \frac{1}{2}(m(p)-kr)\log\left(\frac{p}{1-p}\cdot\frac{1-\theta_F}{\theta_F}\right),$$

$$B(p) = -\frac{\alpha_L kr\theta_0(\theta_F-\theta_0)^2}{2(1+\alpha_F)(1-\theta_0)^2}\cdot\frac{\psi'(\theta_F)\psi(p)}{\psi^2(\theta_F)}.$$

(i) Suppose $\alpha_L>0$. Then $B(p)<0$ for all p, but the sign of $A(p)$ depends on the sign of $m(p)-kr$. (The logarithmic term is always negative because $p<\theta_F$.) Specifically, for sufficiently high p, either $m(p)>kr$ or $p=\theta_F$ is satisfied, so $A(p)\le 0$. Thus, for sufficiently high p, (B5) is negative.

Now consider the limit $\beta\to\infty$. From the expression of (A1) and (A3), $\psi'(\theta_F)\psi(p)/\psi^2(\theta_F)$ converges to a finite value in the limit $\gamma\to 1$ ($\beta\to\infty$), so $\lim_{\beta\to\infty}|B(p)|<\infty$. On the other hand, $\lim_{\beta\to\infty}\ln[(1-\theta_F)/\theta_F] = -\infty$ because $\lim_{\beta\to\infty}\theta_F=1$. Thus, $\lim_{\beta\to\infty}A(p)+B(p)>0$ if $m(p)-kr<0$, and $\lim_{\beta\to\infty}A(p)+B(p)<0$ if $m(p)-kr>0$. From (11), we find that $m(\theta_L)-kr = -((\alpha_L m(\theta_F)\psi(\theta_L))/(r\psi(\theta_F)))<0$. Because $m(\cdot)$ is strictly increasing, we conclude that $m(p)-kr<0$ for sufficiently low p that satisfies $p>\theta_L$.

(ii) Suppose $\alpha_L<0$. From (11), we find that $m(\theta_L)-kr = -((\alpha_L m(\theta_F)\psi(\theta_L))/(r\psi(\theta_F)))>0$, so $m(p)-kr>0$ for all $p>\theta_L$. By an analogous argument above, we conclude that $A(p)+B(p)<0$ for sufficiently large β. □

Proof of Proposition 5. We consider the intervals $I_1=(\theta_L,\theta_F)$ and $I_2=[\theta_F,1)$ separately below. (We define $I_1=\varnothing$ in case $\theta_L>\theta_F$.

(i) We first study the interval I_1. First, we prove that $\Pi_F(p)>\Pi_S(p)$ for all $p\le\theta_F$. By (6) and (7) and the definition $\alpha_S=(\alpha_L+\alpha_F)/2\le\alpha_F$, we can derive

$$\Pi_S(p)=\frac{1}{r}(1+\alpha_S)m(p)-k<\frac{1}{r}(1+\alpha_F)m(p)-k\le\Pi_F(p)$$

for all $p\le\theta_F$. Here the second inequality holds because $\Pi_F(\cdot)$ is convex, $m(\cdot)$ is linear, and $\Pi_F'(\theta_F)=(1+\alpha_F)\cdot m(\theta_F)/r$. It follows that the SM region does not exist within $(0,\theta_F)$.

Second, we prove that $V_F(p)\ge V_L(p)$ for all $p\le\theta_F$. Because $\alpha_L<0$, we have $\Pi_L(p)\le m(p)/r-k\le(1+\alpha_F)\cdot m(p)/r-k\le V_F(p)$ for all $p\le\theta_F$. Since $V_F(p)>0$ for all p, it follows that $V_F(p)\ge\max\{\Pi_L(p),0\}=V_L(p)$.

(ii) Next, we consider the interval $I_2=[\theta_F,1)$. In this interval, if the leader invests first, then the follower can choose to invest an infinitesimal time later to be the follower. Thus, $V_F(p)=(1/r)(1+\alpha_F)m(p)-k$ and $V_L(p)=(1/r)(1+\alpha_L)m(p)-k$. Because $\alpha_L<\alpha_F$, we have $V_F(p)>V_L(p)$. □

Proof of Theorem 4. We use the same notation employed in the proof of Theorem 3. Since $\bar{\tau}_M(p)$ has no dependence on β for $p\in(\theta_F,1)$, we focus on the interval (θ_L,θ_F) in case $\theta_L<\theta_F$.

For p sufficiently close to θ_F and for any fixed value of β, $A(p)$ is negligible compared to $B(p)$. Since $B(p)>0$ for $\alpha_L<0$, we have $A(p)+B(p)>0$ in the limit $p\to\theta_F$, which implies that $\bar{\tau}_M(p)$ decreases with β for p sufficiently close to θ_F.

Let us now consider a fixed $p<\theta_F$ and for sufficiently large values of β. By the same argument used in the proof of Theorem 3(ii), $m(p)-kr>0$ for all $p>\theta_L$, so we have $A(p)<0$. Furthermore, in the limit $\beta\to\infty$, $|A(p)|>|B(p)|$ by the argument used in the proof of Theorem 3(i). It follows that $A(p)+B(p)<0$ for a fixed $p<\theta_F$ and for sufficiently large values of β. Thus, we conclude that $\bar{\tau}_M(p)$ increases with β when p is not too close to θ_F and for sufficiently large values of β. □

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