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Trading book and credit risk: How fundamental is the Basel review?

Jean-Paul Laurent[a,*], Michael Sestier[a,b], Stéphane Thomas[b]

[a] PRISM, Université Paris 1 Panthéon-Sorbonne, 17 rue de la Sorbonne, & Labex ReFi, Paris 75005, France
[b] PHAST Solutions Group, 54–56 Avenue Hoche, & Labex ReFi, Paris 75008, France

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ABSTRACT

Within the new Basel regulatory framework for market risks, non-securitization credit positions in the trading book are subject to a separate default risk charge (formally incremental default risk charge). Banks using the internal model approach are required to use a two-factor model and a 99.9% VaR capital charge. This model prescription is intended to reduce risk-weighted asset variability, a known feature of internal models, and improve their comparability among financial institutions. In this paper, we analyze the theoretical foundations and relevance of these proposals. We investigate the practical implications of the two-factor and correlation calibration constraints through numerical applications. We introduce the Hoeffding decomposition of the aggregate unconditional loss to provide a systematic-idiosyncratic representation. In particular, we examine the impacts of a J-factor correlation structure on risk measures and risk factor contributions for long-only and long-short credit-sensitive portfolios.

1. Basel recommendations on credit risk

1.1. Credit risk in the Basel II, 2.5, and III agreements

Created in 1974 by 10 leading industrial countries and now including supervisors from 27 countries, the Basel Committee on Banking Supervision (BCBS) is responsible for strengthening the resilience of the global financial system, ensuring the effectiveness of prudential supervision and improving the cooperation among banking regulators. The Basel II agreements (BCBS, 2004) define regulatory capital through the concept of Risk-Weighted Assets (RWAs) and through the McDonough ratio. Under the Internal-Rating Based (IRB) approach, the RWAs in the banking book measure the exposure of a bank granting loans by applying a weight according to the intrinsic riskiness of each asset. An issuer's default probability and loss at default time are based on the bank's own internal estimates, though correlation parameters are regulatory prescribed. The BCBS also addresses portfolio risk by prescribing a model based on the Asymptotic Single Risk Factor model (ASRF).

The 2008 financial crisis revealed a major gap in the inability to adequately identify the credit risk of the trading book positions contained in credit-quality linked assets. Considering this deficiency, the BCBS (2009) revised the market risk capital requirements in the 2009 reforms, also known as Basel 2.5 agreements, which added two new capital requirements: the Incremental Risk Charge (IRC) and the Comprehensive Risk Measures (CRM). The former was designed to deal with long-term changes in credit quality, and the latter specifically controls for correlation products. More precisely, the IRC is a capital charge that captures default and migration risks through a VaR-type calculation at 99.9% on a one-year horizon. In contrast with the credit risk treatment in the banking book, the trading book model specification results from a complete internal model validation process, by which financial institutions are required to implement their own framework.

Recently, the reliability of internal risk measures has become a key issue in capital regulation, as underscored by Colliard (2014). Indeed, together with the new rules, the BCBS has investigated the RWAs comparability among institutions, for both the banking book (BCBS, 2013b) and the trading book (BCBS, 2013c; 2013d), through the Regulatory Consistency Assessment Program (RCAP). Using a set of hypothetical benchmark portfolios, studies have shown large discrepancies in risk measure levels among participating financial institutions, resulting from discrepancies in internal methodologies of risk calculation, especially for the trading book's RWAs. In particular, studies have argued that modeling choices that institutions make within their IRC model (e.g. whether it uses spread-based or transition matrix-based models, calibration of the transition matrix or that of the initial credit rating, correlations' assumptions

* Corresponding author.
E-mail addresses: laurent.jeanpaul@free.fr, jean-paul.laurent@univ-paris1.fr (J.-P. Laurent), michael.sestier@univ-paris1.fr (M. Sestier), stephane.thomas@phastsolutions.com (S. Thomas).

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across obligors) are one of the main contributors of this variability. These findings are in line with previous studies led by the International Monetary Fund (IMF) (see Lesle and Avramova, 2012), based on banks' density ratios.

1.2. Credit risk in the fundamental review of the trading book

In response to these shortcomings, the BCBS has been working since 2012 on a new post-crisis update of the market risk global regulatory framework, known as Fundamental Review of the Trading Book (FRTB) (BCBS, 2012; 2013a; 2014b; 2015a; 2015b; 2016). Main discussions revolve around the proposal on the transformation of the IRC in favor of a default-only risk capital charge (i.e. without migration feature), named DRC. With a one-year 99.9% VaR calculation, DRC for the trading book is based on a two-factor model:

"One of the key observations from the Committee's review of the variability of market risk weighted assets is that the more complex migration and default models were a relatively large source of variation. The Committee has decided to develop a more prescriptive IDR charge [amended to DRC for Default Risk Charge] *in the models-based framework. Banks using the internal model approach to calculate a default risk charge must use a two-factor default simulation model* [*"with two types of systematic risk factors*" according to BCBS (2015b; 2016)]*, which the Committee believes will reduce variation in market risk-weighted assets but be sufficiently risk sensitive as compared to multi-factor models*". (BCBS, 2013a).

In its report to the G20, BCBS (2014c) also mentioned the objective of constraining the default risk modeling choices by *"limiting discretion on the choice of risk factors*". The BCBS would also monitor model risk through correlation calibration constraints. The first consultative documents on the FRTB (BCBS, 2012; 2013a), prescribed the use of listed equity prices to calibrate the default correlations. In its trading book hypothetical portfolio exercise, BCBS (2014a) observes that the use of equity data was dominant among financial institutions, though some of them chose Credit Default Swap (CDS) spreads for the quantitative impact study. Indeed, equity-based prescribed correlations raise practical problems when data are not available[1], such as for sovereign issuers, resulting in the consideration of other data sources. Consequently, the third consultative report of the BCBS (2014b), the subsequent ISDA (2015) response, the instructions for the Basel III monitoring (BCBS, 2015a), and the final version of the FRTB by BCBS (2016), all recommend the joint use of credit spreads and equity data.

"Default correlations must be based on credit spreads or on listed equity prices. Banks must have clear policies and procedures that describe the correlation calibration process, documenting in particular in which cases credit spreads or equity prices are used. Correlations must be based on a period of stress, estimated over a 10-year time horizon and be based on a [one]-year liquidity horizon. [...] These correlations should be based on objective data and not chosen in an opportunistic way where a higher correlation is used for portfolios with a mix of long and short positions and a low correlation used for portfolio with long only exposures. [...] A bank must validate that its modeling approach for these correlations is appropriate for its portfolio, including the choice and weights of its systematic risk factors. A bank must document its modeling approach and the period of time used to calibrate the model." (BCBS, 2015a; 2016).

The current study investigates the practical implications of these recommendations and, in particular, examines the impact of factor models and their induced correlation structures on the trading book credit risk measurement. Our aim is to provide a comparative analysis of risk factor modeling to assess the relevance of the BCBS's proposals of prescribing model and calibration procedures to reduce the RWAs variability and enhance comparability among financial institutions. For this purpose, we focus our analysis on the correlation part of the modeling and therefore do not include probability of default or Loss Given Default (LGD) estimations for which the BCBS (2016) also provides instructions. Wilkens and Predescu (2015) describe a general modeling approach, compliant with the Basel recommendations and covering the correlation structure, the default probabilities and the recovery rates.

The paper proceeds as follows. In Section 2, we describe a two-factor default risk model within the usual Gaussian latent variables framework, also used in the current banking book setting (one-factor model) and in the IRC implementations (general J-factor ($J \geq 1$) models). Following the BCBS's recommendations, we discuss main correlation estimation methods. In Section 3, we use the Hoeffding decomposition of the aggregate loss to explicitly derive contributions of systematic and idiosyncratic risks of particular interest in the trading book. Section 4 is devoted to numerical applications on representative long-only and long-short credit-sensitive portfolios, in which we consider impacts of J-factor correlation structures on risk measures and risk factor contributions. In Section 5, we provide answers to the question raised in the title on the *fundamentalness* (in the sense of the relevance here) of prescribing a two-factor model to reduce the RWAs' variability and enhance comparability among financial institutions.

2. Two-factor default risk charge model

2.1. Model specification

The portfolio loss at a one-period horizon is modeled by a random variable L, defined as the sum of the individual losses on issuers' default over that period. We consider a portfolio with K positions: $L = \sum_{k=1}^{K} L_k$, where $L_k = w_k \times \mathbb{I}_k$ denotes the loss on the position k, with w_k as the positive or negative LGD-adjusted exposure[2] (assumed constant for conciseness) at the time of default and $\mathbb{I}_k$ as a random default indicator taking the value of 1 when default occurs and 0 otherwise.

To define the probability distribution of the L_ks as well as their dependence structure, we rely on a usual structural factor approach, introduced by Vasicek (1987, 2002) and based on seminal work of Merton (1974), which financial institutions and regulators widely use to model default risk. Thus, we define the default indicator as $\mathbb{I}_k := \mathbf{1}_{[-\infty, x_k]}(X_k)$, where X_k is a latent variable representing the obligor k's creditworthiness index, x_k is a predetermined threshold, and $\mathbf{1}_A(.)$ is the indicator function of the set A. Modeling the $\mathbb{I}_k$ s thus amounts to modeling the obligors creditworthiness indices ($X \in \mathbb{R}^{K\times 1}$), which evolve according to a J-factor Gaussian model:

$$X = \beta Z + \sigma(\beta)\varepsilon, \tag{1}$$

where $Z \sim \mathcal{N}(\mathbf{0}, \Sigma_Z)$ is a J-dimensional random vector of centered systematic factors, $\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$ is a K-dimensional random vector of centered and independent specific risks, $\beta \in \mathbb{R}^{K\times J}$ is the factor loading matrix, and $\sigma(\beta) \in \mathbb{R}^{K\times K}$ is a diagonal matrix with elements $\sigma_k = \sqrt{1 - \beta_k \Sigma_Z \beta_k^t}$, ($\beta_k \in \mathbb{R}^{1\times J}$). This setting ensures that the random vector X is standard normal, with a correlation matrix depending on the factor loadings:

$$\beta \mapsto C(\beta) = \beta \Sigma_Z \beta^t + \sigma^2(\beta). \tag{2}$$

[1] Likewise, no guidance has yet been formulated for the treatment of exposures depending on non-modellable risk factors.

[2] We define LGD-adjusted exposure of the position k as the product of the Exposure At Default (EAD_k) and the Loss Given Default (LGD_k). Formally, $w_k = EAD_k \times LGD_k$. Although we could use stochastic LGDs, the correct choices have not reached consensus, either regarding marginal LGDs or the joint distribution of LGDs and default indicators. The BCBS is not prescriptive at this stage, and it is more than likely that most banks will retain constant LGDs.

In the remainder of the paper, we note $\mathcal{Z} = \{Z_j | j = 1, \ldots, J\}$ as the set of all systematic factors and $\mathcal{E} = \{\varepsilon_k | k = 1, \ldots, K\}$ as the set of all idiosyncratic risks, such that $\mathcal{F} = \mathcal{Z} \cup \mathcal{E}$.

We choose the threshold x_k such that $\mathbb{P}(\mathbb{I}_k = 1) = p_k$, where p_k is the obligor k's marginal default probability. From normality of X_k, we have $x_k = \Phi^{-1}(p_k)$, with $\Phi(.)$ denoting the standard normal cumulative function. The portfolio loss[3] is then written as follows:

$$L = \sum_{k=1}^{K} w_k \mathbf{1}_{[-\infty, \Phi^{-1}(p_k)]}(\beta_k Z + \sigma_k \varepsilon_k). \tag{3}$$

The single factor variant of this model, for which correlation coefficients are regulatorily prescribed, is at the heart of the Basel II credit risk capital charge. In this model, known as the ASRF model (see Gordy, 2003), the latent systematic factor is usually interpreted as the state of the economy, that is a generic macroeconomic variable affecting all firms. In multi-factor models[4] ($J \geq 2$), factors can be either latent or observable. For the latter, a fine segmentation (into industrial sectors, geographical regions, ratings, and so on) of factors allows modelers to define a detailed operational representation of the portfolio correlation structure.

2.2. Calibration of correlations

Considering our modeling assumptions on the general framework, we now discuss the calibration of the correlation matrix of X. As previously mentioned, the BCBS recommends the joint use of equity and credit spread data (notwithstanding, such a combination may raise consistency issues because pairwise correlations computed from credit spreads and equity data are sometimes distant). Nevertheless, at that stage, the BCBS set aside concerns about which estimator of the correlation matrix was to be used. However, we stress that this recommendation neglects issues regarding the sensitivity of the estimation to the underlying calibration period and the processing of noisy information, both of which are essential to financial risk measurement.

In the following subsection, for ease of presentation we consider the latent factor models with $\Sigma_Z = Id$, where Id is the identity matrix. We introduce $\tilde{X}$, the $(K \times T)$-matrix of historical centered stock or CDS-spread returns (T is the time-series length), and the standard estimators of the sample covariance and correlation matrices:

$$\Sigma = T^{-1}\tilde{X}\tilde{X}^t \tag{4}$$

$$C = (diag(\Sigma))^{-\frac{1}{2}} \Sigma (diag(\Sigma))^{-\frac{1}{2}}. \tag{5}$$

It is well known that these matrices suffer some drawbacks. Indeed, when the number of variables (equities or CDS-spreads), K, is close to the number of historical returns, T, the total number of parameters is of the same order as the total size of the data set, which is problematic for estimator stability (see Michaud, 1989, for a proof of the instability of the empirical estimator)[5]. Moreover, when K is larger than T, the matrices are always singular[6]. In the vast literature dedicated to covariance/correlation matrix estimation from equities[7], Alexander and Leigh (1997) provide a review of covariance matrix estimators in VaR models and Disatnik and Benninga (2007) offer a brief review of covariance matrix estimators in the context of the shrinkage method.

Shrinkage methods are statistical procedures that involve imposing low-dimensional factor structure to a covariance matrix estimator to deal with the trade-off between bias and estimation error. Indeed, the sample covariance matrix can be interpreted as a K-factor model, in which each variable is a factor (no residuals) so that the estimation bias is low (the estimator is asymptotically unbiased) and the estimation error is large. Conversely, we may postulate a one-factor model that has a large bias from likely misspecified structural assumptions but little estimation error. According to Stein et al. (1956), an optimal trade-off can be reached by taking a properly weighted average of the biased and unbiased estimators: this is called shrinking the unbiased estimator. Within the context of default correlation calibration, we focus on the approach of Ledoit and Wolf (2003), who define a weighted average of the sample covariance matrix with the Sharpe's (1963) single-index model estimator: $\Sigma_{shrink} = \alpha_{shrink}\Sigma_J + (1 - \alpha_{shrink})\Sigma$, where Σ_J is the covariance matrix generated by a $(J = 1)$-factor model and the weight α_{shrink} controls how much inertia to impose. The authors show how to determine the optimal shrinking intensity (α_{shrink}) and, using historical data, illustrate their approach through numerical experiments in which the method out-performs all other standard estimators.

Subsequently, following the BCBS's proposal, we consider an initial correlation matrix C_0, estimated from historical stock or CDS spread returns. To determine the impact of the correlation structure on the levels of risk and factor contributions (see Section 4), we also view other candidates as the initial matrix, such as the *"shrinked"* correlation matrix (computed from Σ_{shrink}), the matrix associated with the IRB ASRF model, and the one associated with a standard J-factor model (e.g. the Moody's KMV model).

2.3. Nearest correlation matrix with J-factor structure

Factor models are popular in finance because they offer parsimonious explanations of asset returns and correlations. Following the BCBS's recommendation, financial institutions would need to build factor models (with a specified number of factors) generating a correlation matrix as close as possible to the pre-determined correlation matrix C_0. At this stage, the BCBS does not provide any guidance on the calibration of factors loadings β needed to pass from a $(J > 2)$-factor structure to a $(J = 2)$-factor structure. In this context, we present generic methods to calibrate a model with a J-factor structure from an initial $(K \times K)$-correlation matrix.

Among popular exploratory methods used to calibrate such models, PCA helps specify a linear factor structure between variables. Indeed, by using the spectral decomposition theorem on the initial correlation matrix, $C_0 = \Gamma\Lambda\Gamma^t$ (where Λ is the diagonal

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[3] Because $\mathbb{I}_k$ is discontinuous, L can take only a finite number of values in the set $\mathbb{L} = \{\sum_{a \in A} w_a | \forall A \subseteq \{1, \ldots, K\}\} \cup \{0\}$. In the homogeneous portfolio, where all weights are equal, $Card(\mathbb{L}) = K + 1$. In contrast, if the weights are different, the number of possible loss values can go up to 2^K, and the numerical computation of quantile-based risk measures may be more difficult.

[4] In its analysis of the trading book hypothetical portfolio exercise, BCBS (2014a) reports that most banks currently use an IRC model with three or fewer factors, and only 3% have more than three factors. Prior studies have examined multi-factor models for credit-risk portfolio and compared them with the one-factor model. For example, Düllmann et al. (2008) provide a comparison of the correlation and the VaR estimates among a one-factor model, a multi-factor model (based on the Moody's KMV model), and the Basel II IRB model.

[5] See also Laloux et al. (1999) for evidence of ill-conditioning and the curse of dimension within a random matrix theory approach, and Papp et al. (2005) for an application of random matrix theory to portfolio allocation.

[6] Note that this feature is problematic when considering the Principal Component Analysis (PCA) to estimate factor models because the method requires the invertibility of Σ or C. To overcome this problem, Connor and Korajczyk (1986; 1988) introduce the asymptotic PCA, which consists of applying PCA to the $(T \times T)$-matrix, $K^{-1}\tilde{X}^t\tilde{X}$, rather than to Σ. The authors show that asymptotic PCA is asymptotically equivalent to the PCA on Σ.

[7] Several studies also address the estimation of dynamic correlations. See, for example, the work of Engle (2002), which introduces the Dynamic Conditional Correlation (DCC), or that of Engle and Kelly (2012), which provides a brief overview of dynamic correlation estimation and the presentation of the Dynamic Equi-Correlation (DECO) approach. For practical purposes, we focus on techniques that can easily account for a credit universe that encompasses several hundreds or thousands of names.

matrix of ordered eigenvalues of C_0 and Γ is an orthogonal matrix whose columns are the associated eigenvectors), the principal components transform of X is $Y = \Gamma^t X$, where the random vector Y contains the ordered principal components. As this transformation is invertible, we can finally write $X = \Gamma Y$. In this context, an easy way to postulate a J-factor model is to partition Y according to $(Y_J^t, Y_{\bar{J}}^t)^t$, where $Y_J \in \mathbb{R}^{J\times 1}$ and $Y_{\bar{J}} \in \mathbb{R}^{(K-J)\times 1}$, and to partition Γ according to $(\Gamma_J, \Gamma_{\bar{J}})$, where $\Gamma_J \in \mathbb{R}^{K\times J}$ and $\Gamma_{\bar{J}} \in \mathbb{R}^{K\times (K-J)}$. This truncation leads to $X = \Gamma_J Y_J + \Gamma_{\bar{J}} Y_{\bar{J}}$. Thus, by considering Γ_J the factors loadings (comprised of the J first eigenvectors of C_0), Y_J the factors (comprised of the J first principal components of X), and $\Gamma_{\bar{J}} Y_{\bar{J}}$ the residuals, we obtain a J-factor model.

Nevertheless, according to Andersen et al. (2003), the specified factor structure in Eq. (1) cannot be merely calibrated by a truncated eigen-expansion, due to the specifications of residuals that depend on β. Here, we look for a $(J = 2)$-factor modeled X for which the correlation matrix $C(\beta) = \beta\beta^t + \sigma^2(\beta)$, with $\sigma^2(\beta) = Id - diag(\beta\beta^t)$, is as close to C_0 as possible. Thus, we define the following optimization problem[8]:

$$\begin{cases} \arg\min_\beta f(\beta) = \|C_0 - C(\beta)\|_F \\ \text{subject to: } \beta \in \Omega = \{\beta \in \mathbb{R}^{K\times J} | \beta_k \beta_k^t \le 1; k = 1, \dots, K\}, \end{cases} \quad (6)$$

where $\|.\|_F$ is the Froebenius norm $\forall A \in \mathbb{R}^{K\times K} : \|A\|_F = \mathrm{tr}(A^t A)$ (with $tr(.)$ denoting the trace of a square matrix). The constraint, $\beta \in \Omega$, ensures that $\beta\beta^t$ has diagonal elements bounded by 1, implying that $C(\beta)$ is positive semi-definite.

Previous researches have tackled the general problem of computing a correlation matrix of J-factor structure nearest to a given matrix. For example, in the context of credit basket securities, Andersen et al. (2003) determine a sequence $\{\beta_i\}_{i>0}$ in the following way. Given the spectral decomposition of $\left(C_0 - \sigma^2(\beta_i)\right)$, $\Gamma_\sigma \Lambda_\sigma \Gamma_\sigma^t$: $\beta_{i+1} = \Gamma_{\sigma,J} \Lambda_{\sigma,J}^{1/2}$; that is, they consider the eigenvectors associated with the J largest eigenvalues. The iterations stop when $\sigma^2(\beta_{i+1})$ is sufficiently close to $\sigma^2(\beta_i)$.

Borsdorf et al. (2010) show that this PCA-based method, which is not supported by any convergence theory, often performs surprisingly well on the one hand, partly because the constraints are often not active at the solution, but may fail to solve the constrained problem on the other hand. They acknowledge the Spectral Projected Gradient (SPG) method as being the most efficient to solve the constrained problem. This method allows minimizing $f(\beta)$ over the convex set Ω by iterating over β in the following way: $\beta_{i+1} = \beta_i + \alpha_i d_i$, where $d_i = \mathrm{Proj}_\Omega(\beta_i - \lambda_i \nabla f(\beta_i)) - \beta_i$ is the descent direction[9], with $\lambda_i > 0$ a pre-computed scalar, and α_i is a positive scalar chosen through a non-monotone line search strategy. Because Proj_Ω is not costly to compute, the algorithm is fast enough to enable the calibration of portfolios having a large number of positions. Birgin et al. (2001) provide a detailed presentation and algorithms.

An important point for the validity of a factor model is the correct specification of the number of factors. Until now, in accordance with the BCBS's specification, we have assumed arbitrary J-factor models, in which J is specified by the modeler ($J = 2$ for the BCBS). From the data, we may also consider the problem of determining the optimal number of factors. Some previous works have dealt with this issue, such as that by Bai and Ng (2002), who propose panel criteria to consistently estimate the optimal number of factor from historical data[10].

Finally, note that the presented methods, approximating the initial correlation matrix (methods based on PCA or SPG to find the nearest correlation matrix with a J-factor structure, or the shrinkage method to make the correlation matrix more robust), may sometimes smooth the pairwise correlations. For example, the shrinkage method does not specifically treat the case when the pairwise correlations are near or equal to one. Rather, it tends to reverse these correlations to the mean level even if, statistically, the variance of the correlation estimator with a value close to the unit is often low or even null.

3. Hoeffding decomposition of losses and risk contributions

In the banking book, the Basel II capital requirement formula (IRB modeling) is based on the assumption that the portfolio is infinitely fine grained; that is, it consists of a large number of credits with small exposures, so that only one systematic risk factor influences portfolio default risk (ASRF model). Thus, the aggregate loss can be approximated by the systematic part of the portfolio loss: $L \approx L_{\mathcal{Z}} = \mathbb{E}[L|\mathcal{Z}]$. Under this assumption, Wilde (2001) expresses a portfolio invariance property in which the required capital for any given loan does not depend on the portfolio to which it is added. This makes the IRB modeling appealing because it allows the straightforward calculation of risk measures and contributions. In contrast, the particularities of the trading book positions (actively traded positions, the presence of long-short credit risk exposures, heterogeneous and potentially small number of positions) make this assumption too restrictive and warrant analyzing the risk contribution of both the systematic factors and the non-systematic (idiosyncratic or specific) risks.

In the following subsections, we first represent the portfolio loss through the Hoeffding decomposition to show the impact of both the systematic factors and the idiosyncratic risks. After this, we presents analytics of factor contributions to the risk measure.

3.1. Hoeffding decomposition of the aggregate loss

As we defined previously, the portfolio loss is the sum of the individual losses (see Eq. (3)). We consider a representation of the loss as a sum of terms involving sets of factors through the Hoeffding decomposition, previously used in a credit context by Rosen and Saunders (2010). Formally, if $F_1, \dots, F_M$ and $L \equiv L[F_1, \dots, F_M]$ are square-integrable random variables, the Hoeffding decomposition[11] expresses the aggregate portfolio loss, L, as a sum of terms involving conditional expectations given factor sets:

$$\begin{aligned} L &= \sum_{S\subseteq\{1,\dots,M\}} \phi_S(L; F_m, m \in S) \\ &= \sum_{S\subseteq\{1,\dots,M\}} \sum_{\tilde{S}\subseteq S} (-1)^{|S|-|\tilde{S}|} \mathbb{E}\left[L | F_m; m \in \tilde{S}\right]. \end{aligned} \quad (7)$$

Although the Hoeffding decomposition may be impractical when the number of factors is large (because it requires the calculation of 2^M terms), computation for a two-factor model does not present any challenge, especially in the Gaussian framework (see Eqs. (10) and (11) for explicit analytical form of each term).

An appealing feature of the Hoeffding decomposition is that it can be applied to subsets of risk factors. Therefore, we use the

[8] Ω is a closed and convex set in $\mathbb{R}^{K\times J}$. Moreover, the gradient of the objective function is given by $\nabla f(\beta) = 4\left(\beta(\beta\beta^t) - C_0\beta + \beta + diag(\beta\beta^t\beta)\right)$.

[9] Note that $\beta_{i+1} = (1-\alpha_i)\beta_i + \alpha_i \mathrm{Proj}_\Omega(\beta_i - \lambda_i \nabla f(\beta_i))$. Thus, if $\beta_1 \in \Omega$, then $\beta_{i+1} \in \Omega, \forall i \ge 1$.

[10] The authors consider the sum of squared residuals, noted $V(j, Z_j)$, in which the j factors are estimated by PCA ($\forall j \in 1, \dots J$), and introduce the panel criteria and the information criteria for determining the optimal number of factors : $PC_m(j)$ and IC_m (for $m = 1, 2, 3$). According to the authors, in the strict factor model in which the idiosyncratic errors are uncorrelated as in our framework, the preferred criteria are the following: PC_1, PC_2, IC_1, and IC_2.

[11] The Hoeffding decomposition is usually applied to independent factors. If this assumption is fulfilled, all terms of the decomposition will be uncorrelated, easing the interpretation of each term (see Van der Vaart, 2000, for a detailed presentation of this decomposition).

decomposition on the set of systematic factors, $\mathcal{Z}$, and the set of specific risks, $\mathcal{E}$, to break the portfolio loss down in terms of aggregated systematic and idiosyncratic parts:

$$L = \phi_\emptyset(L) + \phi_1(L;\mathcal{Z}) + \phi_2(L;\mathcal{E}) + \phi_{1,2}(L;\mathcal{Z},\mathcal{E}), \tag{8}$$

where $\phi_\emptyset(L) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L;\mathcal{Z}) = L_\mathcal{Z} - \mathbb{E}[L]$ is the loss induced by the systematic factors Z_1 and Z_2, $\phi_2(L;\mathcal{E}) = \mathbb{E}[L|\mathcal{E}] - \mathbb{E}[L]$ is the loss induced by the K idiosyncratic terms ε_k, and $\phi_{1,2}(L;\mathcal{Z},\mathcal{E}) = (L - \mathbb{E}[L|\mathcal{Z}] - \mathbb{E}[L|\mathcal{E}] + \mathbb{E}[L])$ is the remaining risk induced by the interaction (cross-effect) between idiosyncratic and systematic risk factors. The relationship between unconditional and conditional portfolio losses is:

$$L - L_\mathcal{Z} = \phi_2(L;\mathcal{E}) + \phi_{1,2}(L;\mathcal{Z},\mathcal{E}). \tag{9}$$

Note that the Hoeffding decomposition is not an approximation. It is an equivalent representation of the same random variable in uncorrelated terms (due to the independence between the sets of idiosyncratic and systematic risk factors).

From a practical perspective, as we consider a Gaussian factor model, we can easily compute each term of the decomposition:

$$\mathbb{E}[L|\mathcal{Z}] = \sum_{k=1}^{K} w_k \Phi\left(\frac{\Phi^{-1}(p_k) - \beta_k Z}{\sigma_k}\right), \tag{10}$$

$$\mathbb{E}[L|\mathcal{E}] = \sum_{k=1}^{K} w_k \Phi\left(\frac{\Phi^{-1}(p_k) - \sigma_k \varepsilon_k}{\sqrt{\beta_k \Sigma_Z \beta_k^t}}\right). \tag{11}$$

Rosen and Saunders (2010) focus on the Hoeffding decomposition of the systematic part of the portfolio loss: $L_\mathcal{Z}$. In this context, they define the decomposition (with two factors: Z_1 and Z_2) of the conditional loss $L_\mathcal{Z}$ by:

$$L_\mathcal{Z} = \phi_\emptyset(L_\mathcal{Z}) + \phi_1(L_\mathcal{Z};Z_1) + \phi_2(L_\mathcal{Z};Z_2) + \phi_{1,2}(L_\mathcal{Z};Z_1,Z_2), \tag{12}$$

where $\phi_\emptyset(L_\mathcal{Z}) = \mathbb{E}[L]$ is the expected loss, $\phi_1(L_\mathcal{Z};Z_1) = \mathbb{E}[L_\mathcal{Z}|Z_1] - \mathbb{E}[L_\mathcal{Z}]$ and $\phi_2(L_\mathcal{Z};Z_2) = \mathbb{E}[L_\mathcal{Z}|Z_2] - \mathbb{E}[L_\mathcal{Z}]$ are the losses induced by the systematic factors Z_1 and Z_2, respectively, and the last term $\phi_{1,2}(L_\mathcal{Z};Z_1,Z_2) = L_\mathcal{Z} - \mathbb{E}[L_\mathcal{Z}|Z_1] - \mathbb{E}[L_\mathcal{Z}|Z_2] + \mathbb{E}[L]$ is the remaining loss induced by the interaction between systematic factors (cross-effect of Z_1 and Z_2).

Eq. (12) provides a factor-by-factor decomposition of the aggregate loss but raises questions about the significance of each term. Indeed, in the case of models with endogenous factors, Σ_Z is the identity matrix and the terms are uncorrelated. For exogenous factors, for which we account for their correlation structure, the elements in Eq. (12) are correlated as well, and the meaning of each term is unclear. More generally, the significance of the terms is not obvious because factor rotations[12] of the systematic factors leave the distribution of $X = \beta Z + \sigma(\beta)\varepsilon$ unchanged but affect the computation of the Hoeffding terms that depend on subsets included in $\mathcal{Z}$. For example, let us consider in a first case $\beta_{k,1} = \beta_{k,2} = b \in [-1,1], \forall k \in \{1,\ldots,K\}$. This implies that $\phi_1(L_\mathcal{Z};Z_1)$ and $\phi_2(L_\mathcal{Z};Z_2)$ are equal in distribution. Now consider the special case of a one-factor model providing the same loss distribution as in the first case and specified as follows: $\beta_{k,1} = \sqrt{2\times b^2} \neq \beta_{k,2} = 0$. It is clear that $\phi_1(L_\mathcal{Z};Z_1)$ and $\phi_2(L_\mathcal{Z};Z_2)$ are no longer equal in distribution, even though the loss distribution is similar in both cases. Because we further focus on systematic factors on the one hand and specific risks on the other hand, these two conundrums are not relevant in our case.

3.2. Systematic contributions to the risk measure

The portfolio risk is determined by means of a risk measure ϱ, which is a mapping of the loss to a real number: $\varrho : L \mapsto \varrho[L] \in \mathbb{R}$. For a given confidence level $\alpha \in [0,1]$, the VaR is the α-quantile of the loss distribution: $VaR_\alpha[L] = \inf\{l \in \mathbb{R} | \mathbb{P}(L \le l) \ge \alpha\}$. Because both IRC in Basel 2.5 and DRC in the FRTB require the use of a one-year 99.9% VaR, we further restrict ourselves to this risk measure, even though risk decompositions can readily be extended to the set of spectral risk measures.

By definition, the portfolio loss equals the sum of individual losses: $L = \sum_{k=1}^{K} L_k$. As we showed previously, it can also be defined as the sum of the Hoeffding decomposition terms: $L = \sum_{S\subseteq\{1,\ldots,M\}} \phi_S(L; F_m, m \in S)$. To understand risk origin in the portfolio, it is common to refer to a contribution measure $C_{L_k}^{VaR}[L,\alpha]$ ($C_{\phi_S}^{VaR}[L,\alpha]$, respectively) of the position k (of the Hoeffding decomposition term ϕ_S) to the total portfolio risk $VaR_\alpha[L]$. The position risk contribution is important for hedging, capital allocation, performance measurement, and portfolio optimization (for a detailed presentation, see Tasche, 2008).

As fundamental as the position risk contribution, the factor risk contribution helps unravel alternative sources of portfolio risk. For example, Cherny and Madan (2006) consider the conditional expectation of the loss with respect to the systematic factor and refer to it as *factor risk brought by that factor*. Martin and Tasche (2007) consider the same conditional expectation, but they apply the Euler's principle, taking the derivative of the portfolio risk in the direction of this conditional expectation and call it *risk impact*. Rosen and Saunders (2010) apply the Hoeffding decomposition of the loss with respect to sets of systematic factors, with the first several terms of this decomposition coinciding with the conditional expectations mentioned previously.

Several studies have analyzed the theoretical and practical aspects of various allocation schemes (see Dhaene et al., 2012, for a review). Among these, the marginal contribution method, based on Euler's allocation rule, is quite standard (see Tasche, 2007). To apply this method herein, we need to adapt it because we face discrete distributions (see Laurent, 2003, for an analysis of the technical issues at hand). Yet, under differentiability conditions, we can show (see also Gouriéroux et al., 2000) that the marginal contribution of the individual loss L_k to the risk associated with the aggregate loss $L = \sum_{k=1}^{K} L_k$ is given by $C_{L_k}^{VaR}[L,\alpha] = \mathbb{E}[L_k | L = VaR_\alpha[L]]$. Furthermore, computing this expectation does not involve any assumption other than integrability, and defining risk contributions along these lines fulfills the usual full allocation rule $L = \sum_{k=1}^{K} C_{L_k}^{VaR}[L,\alpha]$ (see Tasche, 2008, for details on this rule).

Similarly, we can compute the contributions of the different terms involved in the Hoeffding decomposition of the aggregate loss. For example, we can readily derive the contribution of the systematic term as $C_{\phi_1}^{VaR}[L,\alpha] = \mathbb{E}[\phi_1(L;\mathcal{Z}) | L = VaR_\alpha[L]]$. Likewise, we can easily write contributions of specific risk and interaction terms and add them afterward to the systematic term so as to retrieve the risk measure of the aggregate loss. The additivity property of risk contributions prevails when subdividing the vector of risk factors into multiple blocks. This parallels the convenient invariance property well known in the Basel ASRF framework (see Wilde, 2001).

Also note that although our approach could be deemed as belonging to the granularity adjustment corpus, the techniques as well as the mathematical properties involved here (such as differentiability, associated with smooth distributions) are completely different. Fermanian (2014) and Gagliardini and Gourieroux (2014) are recent studies involving this concept.

4. Empirical implications for diversification and hedge portfolios

This section is devoted to the empirical examination of the effects of the correlation structure on risk measure and risk factor

[12] For further details on this statistical procedure, see common statistical literature such as Kline (2014).

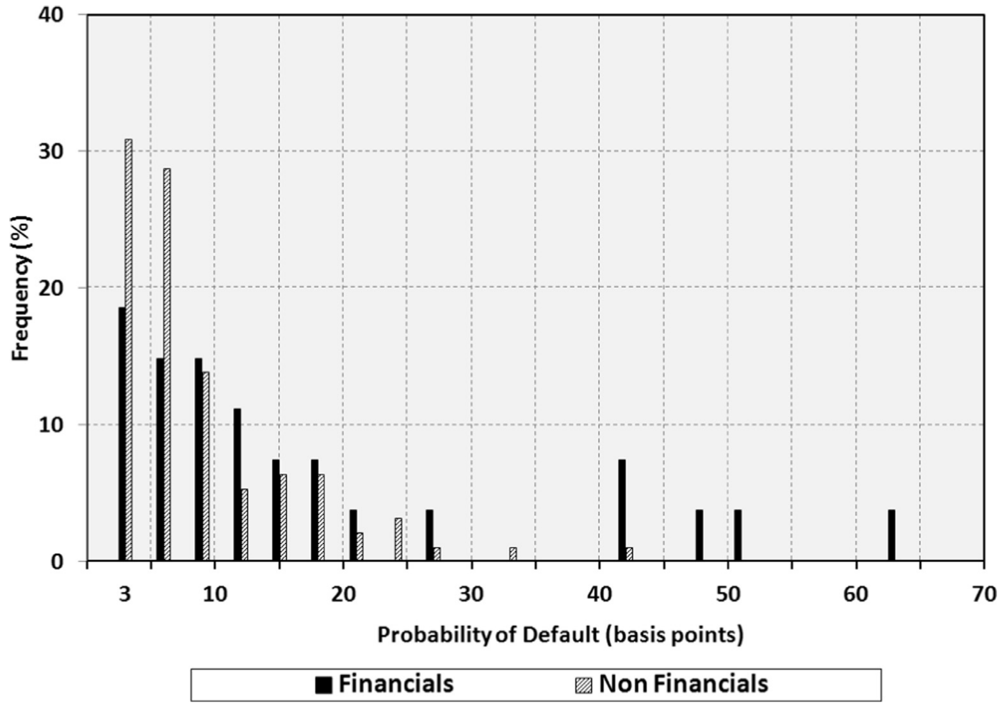

Fig. 1. Histogram of the default probabilities distribution.

contributions. In particular, we aim to analyze the impacts of modeling constraints for both the future DRC prescription and the current Basel 2.5 IRC built on constrained (factor-based) and unconstrained models.

We base our numerical analysis on representative long-only and long-short credit-sensitive portfolios. Because we want to focus on widely traded issuers, who represent a large portion of banks' exposures, we opt for portfolios with large investment grade companies. Specifically, we consider the names in the iTraxx Europe index, retrieved on 31 October 2014. The portfolio contains 121 European investment grade companies[13], 27 of which are Financials; we tag the remaining companies as Non-Financials.

We successively consider two types of portfolio: (i) a *diversification portfolio*, containing positive-only exposures (long-only credit risk), and (ii) a *hedge portfolio*, consisting of positive and negative exposures (long-short credit risk). The distinction parallels the one between the banking book, containing notably long-credit loans, and the trading book, usually including long and short positions (e.g. bonds, CDS); for the latter, an issuer's default on a negative exposure yields a gain. For conciseness, we set the LGD rate to 100% for each position of the two portfolio types.

Regarding the *diversification portfolio* ($K = 121$), we consider a constant and equally weighted effective exposure for each name so that $\forall k, w_k = 1/K$ and $\sum_{k=1}^{K} w_k = 1$. For the *hedge portfolio* ($K = 54$), we assume long exposure to 27 Financials and short exposure to 27 Non-Financials, selected such that the average default probability between the two groups is the same (approximately 0.16%). By considering $w_{k \in Financials} = 1/27$ and $w_{k \notin Financials} = -1/27$, the *hedge portfolio* is thus credit-neutral: $\sum_{k=1}^{K} w_k = 0$.

For the sake of numerical application, we use default probabilities provided by the Bloomberg Issuer Default Risk Methodology[14].

Fig. 1 illustrates default probabilities' frequencies of the portfolio companies grouped by Financials and Non-Financials.

Financials clearly show higher and more dispersed default probabilities, with a mean and standard deviation equal to 0.16% and 0.16%, respectively, compared with 0.08% and 0.07%, respectively, for Non-Financials. We also note that the floor value (at 0.03%) prescribed by the BCBS is restrictive for many (34 of the 121) Financials and Non-Financials.

Next, we discuss results on the calibration of both the initial correlation matrix (C_0) and the loading matrix ($\beta \in \mathbb{R}^{K \times J}$) of the J-factor models. We then consider the impact of these different models on the risk for both portfolios. By using a Hoeffding-based representation of the aggregate loss, we compute the contributions of the systematic and non-systematic (idiosyncratic risks and the interactions) parts of the loss to the risk.

4.1. Unconstrained and constrained correlation matrices

Following the BCBS's proposal, we use listed equity prices[15] of the 121 issuers, spanning a one-year period, to calibrate the initial default correlation matrix through the empirical estimator (see Eq. (5))[16]. To illustrate the sensitivity to the calibration window, we use two sets of equity time-series. Period 1 covers a time of high market volatility from 07/01/2008 to 07/01/2009, whereas Period 2 spans a comparatively lower market volatility window, from 09/01/2013 to 09/01/2014. For both periods, the computed unconstrained (i.e. with no factor structure) (121×121)-matrix consists

[13] The index generally contains 125 names. Nevertheless, because of a lack of data for initial correlation computation, we were unable to estimate a (125×125)-matrix.

[14] This methodology, referred to as Bloomberg DRSK methodology, is based on the model of Merton (1974). The model does not use credit market variables; rather it is an equity markets-based view of default risk. In addition to market data and companies' balance sheet fundamentals, the model also includes companies' income statements.

[15] Data provided by Bloomberg Professional.

[16] The BCBS suggests calibrating correlations over a 10-year period, which includes a one-year stress period, using annual co-movements. Wilkens and Predescu (2015) point out that the prescribed methodology may induce substantial noise in pairwise correlation estimates (i.e. with wide confidence intervals) due to a lack of data. In consequence, our approach differs from this proposal because it is based on daily returns on a one-year stress period selected over the last 10-year period, increasing the size of the data panel from 10 dates for the BCBS approach to approximately 250 in ours.

Table 1
Initial correlation matrix estimation and J-factor model calibration.

Configuration		Data for estimating C_0	Period	Estimation method for C_0	Calibration method for the J-factor models
(1)	**Equity - P1**	Equity returns	1	Sample correlation	PCA and SPG algorithms
(2)	**Equity - P1 Shrinked**	Equity returns	1	Shrinkage ($\alpha_{shrink} = 0.32$)	PCA and SPG algorithms
(3)	**Equity - P1 Exogenous Factors**	Equity returns	1	Sample correlation	Linear Regression
(4)	**Equity - P2**	Equity returns	2	Sample correlation	PCA and SPG algorithms
(5)	**Equity - P2 Shrinked**	Equity returns	2	Shrinkage ($\alpha_{shrink} = 0.43$)	PCA and SPG algorithms
(6)	**Equity - P2 Exogenous Factors**	Equity returns	2	Sample correlation	Linear Regression
(7)	**IRB**	–	-	IRB formula	PCA and SPG algorithms
(8)	**KMV - P2**	–	2	GCorr methodology	PCA and SPG algorithms
(9)	**CDS - P2**	CDS spreads	2	Sample correlation	PCA and SPG algorithms

Period 1: from 07/01/2008 to 07/01/2009. Period 2: from 09/01/2013 to 09/01/2014.

of a matrix of pairwise correlations that we adjust[17] to ensure semi-definite positivity. To limit the estimation error, we also apply the shrinkage methodology to the two periods.

Furthermore, we use Period 2 to define other initial correlation matrices to analyze the effects of changes in the correlation structure on the J-factor model as computed from different types of financial data. We consider three alternative sources: (i) the imposed IRB correlation formula[18], based on the issuer's default probability; (ii) the GCorr methodology of Moody's KMV; and (iii) the relative changes of issuers' CDS spreads (also advocated by the BCBS). For each initial correlation matrix, C_0, the optimization problem in Eq. (6) is solved with the PCA-based and the SPG-based algorithms. In addition to these uncorrelated latent factor models, we consider models based on two correlated exogenous factors: one region factor (identical across all issuers) and one sector factor. For both periods, we calibrate the factor loadings by projecting (through linear regression) each issuer's equity returns onto the returns of the MSCI Europe index and the returns of a MSCI sector index corresponding to the issuer's sector (10 indices overall). Table 1 reports all characteristics.

We also consider the optimization problem for the calibration of J^*-factor models (where J^* is the data-based "optimal number" of factors) for both the "(1) Equity - P1" and the "(4) Equity - P2" configurations, which we define as the integer part of the arithmetic average of the panel and information criteria (see Bai and Ng, 2002). Applying this methodology to the historical time-series, we obtain $J^* = 6$ for the "(1) Equity - P1" configuration and $J^* = 3$ for the "(4) Equity - P2" configuration[19]. To make the results comparable, we assume that these optimal numbers[20] are the same for the *diversification portfolio* and the *hedge portfolio*.

Table 2 shows the calibration results for models with endogenous factors for both the PCA-based and the SPG-based algorithms, and Fig. 2 displays histograms of the pairwise correlations frequencies for each configuration within each J-factor model ($J = 1, 2, J^*$ with PCA-based calibration, SPG-based algorithm providing similar results).

Fitting results among the endogenous factor models, in Table 2, suggests that the two considered nearest correlation matrix approaches (SPG-based and PCA-based) perform similarly and correctly. As expected, increasing the number of factors or shrinking the correlation matrix tends to produce a better fit to the unconstrained model (i.e. a smaller Froebenius norm).

Fig. 2 presents important disparities on average level and dispersion in pairwise correlations among the configurations, reflecting discrepancies in the data used to calibrate the correlation matrices. For example, the "(1) Equity - P1" shows large dispersion and high average levels (approximately 50%), whereas the "(4) Equity - P2" configuration shows more concentrated frequencies with a peak around 30%, much lower than that in stress Period 1. The shrinkage seems to have a small effect on the level of the pairwise correlations, but it slightly decreases disparities among the models. The "(7) IRB" configuration yields concentrated correlation levels quite close to the upper bound of 24%. The "(9) CDS - P2" configuration presents the most dispersed distribution of pairwise correlations.

The factor models accurately reproduce the underlying pairwise correlations distribution. In particular, combining Fig. 2 and Table 2, the less dispersed the distribution of pairwise correlations, the fewer is the number of factors needed to correctly reproduce the correlation structure. Nevertheless, we note that factor models tend to overestimate central correlations and underestimate tail correlations (see, for example, the "(9) CDS - P2" configuration).

4.2. Correlation impacts on regulatory VaR

We analyze the impacts of correlation matrices on portfolio risk. We compute the VaR using the Monte-Carlo method with 2×10^7 scenarios. Note that the discreteness[21] of L implies that the mapping $\alpha \mapsto VaR_\alpha[L]$ is piecewise constant, so that jumps in the risk measure are possible for small changes in the default probability.

For both the *diversification portfolio* (see Fig. 3) and the *hedge portfolio* (see Fig. 4), we simulate the $VaR_\alpha[L]$ for $\alpha \in \{0.99, 0.995, 0.999\}$ for each of the nine configurations and the unconstrained and factor-based models.

Figs. 3 and 4 underscore the relevance of the BCBS's prescribed model and provide guidance on the calibration procedure for reducing the RWAs variability and improving comparability between financial institutions. These numerical simulations clearly show

[17] We use a spectral projection method. Note that this treatment is not needed if we only consider the simulation of the J-factor model calibrated with the optimization problem in Eq. (6). For the unconstrained matrix, we could have used other approaches such as expectation-maximization algorithm to deal with missing data.

[18] The IRB approach is based on a one-factor model: $X_k = \sqrt{\rho_k} Z_1 + \sqrt{1-\rho_k}\varepsilon_k$. Thus, $Correl(X_k, X_j) = \sqrt{\rho_k \rho_j}$, where ρ_k is provided by a regulatory formula: $\rho_k = \lambda_k \times 0.12 + (1-\lambda_k) \times 0.24$, with $\lambda_k = \frac{1-\exp^{-50 p_k}}{1-\exp^{-50}}$. Note that the European implementation of the Basel formula applies a multiplier of 1.25 for all exposures to large financial sector entities and for all exposures to unregulated financial sector entities (see EBA, 2013). We chose to apply the general Basel recommendation to our case study.

[19] In particular, we obtain $IC_1 = 6$, $IC_2 = 5$, $PC_1 = 8$, and $PC_2 = 7$ for the "(1) Equity - P1" configuration and $IC_1 = 2$, $IC_2 = 2$, $PC_1 = 4$, and $PC_2 = 4$ for the "(4) Equity - P2" configuration.

[20] Note that these experimental results are consistent with the empirical conclusions of Connor and Korajczyk (1993), who find between 1 and 2 factors for "non-stressed" periods and between 3 to 6 factors for "stressed" periods for the monthly stock returns of the NYSE and the AMEX, during the period 1967–1991. The results are also in line with those of Bai and Ng (2002), who show the presence of two factors when analyzing the daily returns on the NYSE, the AMEX, and the NASDAQ, during the period 1994–1998.

[21] Because we deal with discrete distributions, we cannot rely on standard asymptotic properties of sample quantiles. At discontinuity points of VaR, sample quantiles do not converge. This can be solved with the asymptotic framework introduced by Ma et al. (2011) and the use of the mid-distribution function.

Table 2
Factor-model calibration over the 121 iTraxx issuers.

Configuration		Number of factors	Froebenius norm		Average correlation		Average correlation Financial		Average correlation Non-Financial	
			SPG	PCA	SPG	PCA	SPG	PCA	SPG	PCA
(1)	**Equity - P1**	C_0	0,00	0,00	0,46	0,46	0,62	0,62	0,46	0,46
		1 factor	8,75	8,73	0,47	0,46	0,54	0,54	0,45	0,45
		2 factors	6,10	6,01	0,47	0,46	0,60	0,59	0,46	0,46
		$(J^* = 6)$ factors	4,26	3,84	0,46	0,46	0,63	0,61	0,46	0,46
(2)	**Equity - P1 Shrinked**	C_0	0,00	0,00	0,46	0,46	0,60	0,60	0,45	0,45
		1 factor	592	5,88	0,47	0,46	0,55	0,55	0,45	0,45
		2 factors	4,18	4,05	0,47	0,46	0,59	0,58	0,46	0,45
(4)	**Equity - P2**	C_0	0,00	0,00	0,28	0,28	0,44	0,44	0,26	0,26
		1 factor	8,69	8,66	0,28	0,28	0,41	0,41	0,26	0,25
		2 factors	6,99	6,94	0,28	0,28	0,43	0,43	0,26	0,26
		$(J^* = 3)$ factors	6,36	6,24	0,28	0,28	0,44	0,43	0,26	0,26
(5)	**Equity - P2 Shrinked**	C_0	0,00	0,00	0,28	0,28	0,43	0,43	0,26	0,26
		1 factor	498	4,95	0,28	0,28	0,41	0,41	0,26	0,26
		2 factors	4,07	3,97	0,28	0,28	0,42	0,42	0,26	0,26
(7)	**IRB**	C_0	0,00	0,00	0,25	0,25	0,29	0,29	0,25	0,25
		1 factor	0,22	0,00	0,25	0,25	0,29	0,29	0,25	0,25
		2 factors	0,34	0,00	0,25	0,25	0,29	0,29	0,25	0,25
(8)	**KMV - P2**	C_0	0,00	0,00	0,29	0,29	0,47	0,47	0,27	0,27
		1 factor	4,14	4,09	0,29	0,29	0,43	0,43	0,26	0,26
		2 factors	2,29	2,10	0,29	0,29	0,47	0,47	0,27	0,26
(9)	**CDS - P2**	C_0	0,00	0,00	0,58	0,58	0,81	0,81	0,57	0,57
		1 factor	7,69	7,66	0,59	0,58	0,70	0,69	0,57	0,56
		2 factors	5,51	5,44	0,59	0,58	0,80	0,80	0,58	0,57

The column “Froebenius norm” corresponds to the optimal value of the objective function, and the three right-hand-side columns state the average pairwise correlations from, respectively, the overall portfolio matrix, the Financial sub-matrix, and the Non-Financial sub-matrix.

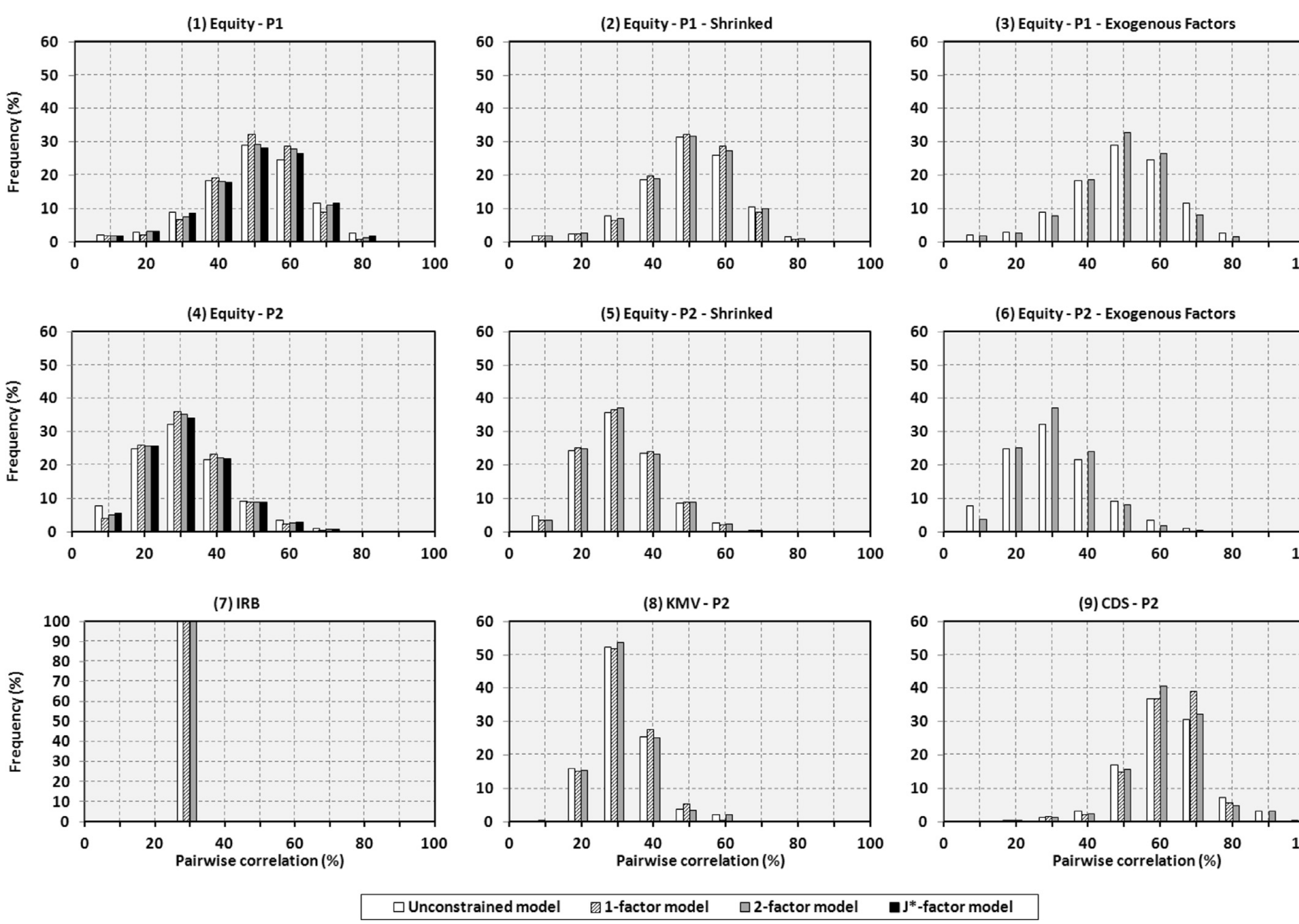

Fig. 2. Histogram of the pairwise correlations among the 121 iTraxx issuers (PCA-based calibration).

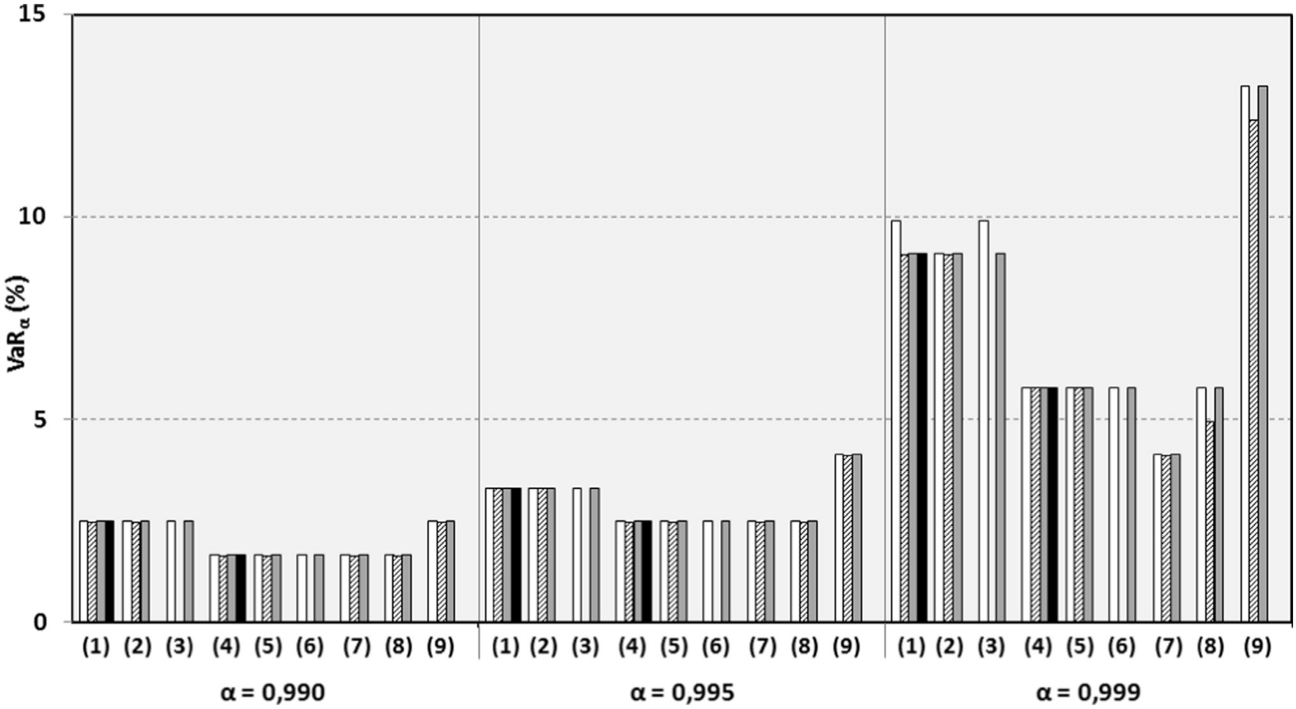

Fig. 3. Risk measure as a function of α for the *diversification portfolio* (PCA-based calibration).

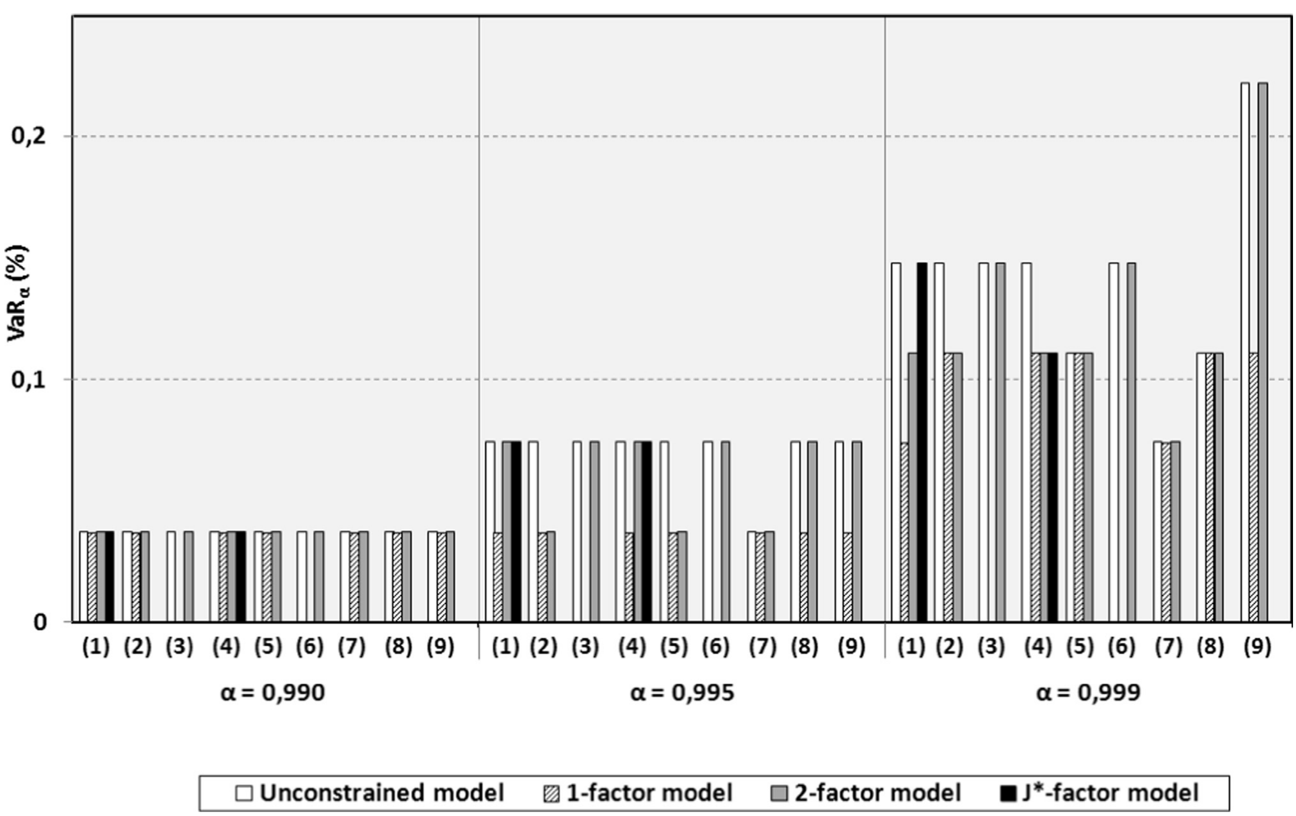

Fig. 4. Risk measure as a function of α for the *hedge portfolio* (PCA-based calibration).

that the VaR variability among configurations crucially depends to the level of α: increasing the confidence level results in higher variability.

In addition, for each level of α, the main sources of VaR dispersion are the differences in average correlation levels among configurations. For example, considering the regulatory confidence level ($\alpha = 0.999$), the prescribed "(1) Equity - P1" and "(9) CDS - P2" configurations lead to considerable VaR-level differences.

With these equally weighted portfolios, the constrained J-factor models (including the regulatory two-factor model) tend to produce lower tail risk measures (high level of α) than the unconstrained model. This phenomenon is even more pronounced when

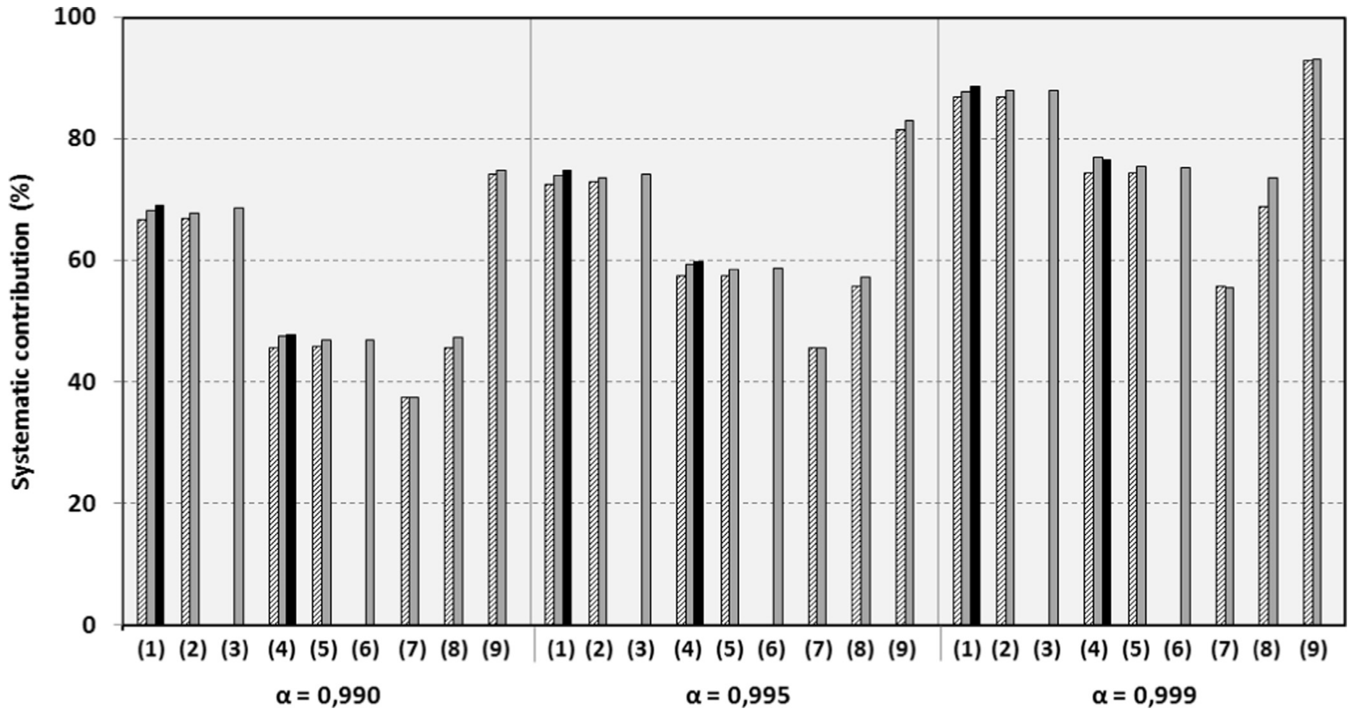

Fig. 5. Systematic contribution as a function of α for the *diversification portfolio* (PCA-based calibration).

considering dispersed correlation matrices (e.g. the configurations "(1) Equity - P1" and "(9) CDS - P2") and particularly the hedge portfolio in which constrained models (generating less dispersed correlation matrices) may mitigate substantial risk.

Overall, from our numerical simulations, we find that the principal sources of DRC variability are (i) the high confidence level of the regulatory risk measure and (ii) the differences in average correlations among configurations. Moreover, the two-factor constraint seems to be unsuitable for reducing the DRC variability and tends to underestimate risk measures in comparison with the ($J > 2$)-factor models or unconstrained ones. These results seem to refute the BCBS's (2013a) beliefs that prescribing a two-factor model reduces variability in market RWAs.

4.3. Systematic and idiosyncratic contributions to regulatory VaR

We use the Hoeffding-based representation of the loss (Eqs. (8) and (9)) to provide insights into the contributions of the systematic factors to the overall portfolio risk and to assess the quality of the systematic-risk conditional approach (also known as the large pool approximation). From iid replications of systematic and specific factors, let $\left(L^{(1)}, \dots, L^{(MC)}\right)$ and $\left(\phi_S^{(1)}, \dots, \phi_S^{(MC)}\right)$ be the corresponding replications of the portfolio loss and the projected loss onto the subset of factors S, respectively. By denoting v_α the estimator of the risk measure $VaR_\alpha[L]$, obtained from $\left(L^{(1)}, \dots, L^{(MC)}\right)$, we define the estimator of the contribution[1] $C^{VaR}_{\phi_S}[L;\alpha]$ as follows[22]:

$$C_{\alpha,S} := \frac{\sum_{n=1}^{MC} \phi_S^{(n)} \mathbf{1}_{\{v_\alpha\}}(L^{(n)})}{\sum_{n=1}^{MC} \mathbf{1}_{\{v_\alpha\}}(L^{(n)})} \tag{13}$$

Because the conditional expectation defining the risk contribution is conditioned on rare events, this estimator requires intensive simulations to reach an acceptable confidence interval[23]. Both Glasserman (2006) and Tasche (2009), among other authors, address the issue of computing credit risk contributions of individual exposures or sub-portfolios from numerical simulations. Recently, Liu (2015) developed a restricted importance sampling method to calculate VaR and VaR contributions that has the fastest convergence rate (MC^{-1}) achieved by Monte Carlo estimators. Our framework is similar to theirs, except that we focus on the contributions of the different terms involved in the Hoeffding decomposition of the aggregate risk. We are thus able to derive contributions of factors, idiosyncratic risks, and interactions. For both the *diversification portfolio* (see Fig. 5) and the *hedge portfolio* (see Fig. 6), we illustrate the influence of α on the systematic contribution to the risk by considering $c_{\alpha,S}/v_\alpha$ with $\alpha \in \{0.99, 0.995, 0.999\}$ for each of the nine configurations and the unconstrained and factor-based models.

In the *diversification portfolio*, regarding the systematic risk as a function of α (see Fig. 5), we observe a high level of systematic contribution to the overall risk for configurations with a high level of average pairwise correlations (see Fig. 2). Moreover, for all configurations, because extreme values of the systematic factors lead to the simultaneous default of numerous dependent issuers, $C^{VaR}_{\phi_S}[L,\alpha]$ is an increasing function of α (also true for the *hedge portfolio*).

In the *hedge portfolio*, regarding the systematic risk as a function of α (see Fig. 6), we observe much lower levels of systematic contributions than those in the *diversification portfolio*. Strikingly, the one and two-factor approximations may be inoperable and misleading, with the majority of the risk being explained by the other terms of the Hoeffding decomposition.

Overall, the risk contribution analysis offers insights into the modeling assumptions: according to our numerical simulations,

[22] Given the discrete nature of considered distributions, and similarly of the risk measure, the mapping $\alpha \mapsto C^{VaR}_{\phi_S}[L,\alpha]$ is piecewise constant. Note also that negative risk contributions may arise within the *hedge portfolio*.

[23] Numerical experiments indicate that complex correlation structures (e.g. in the "(1) Equity - P1" configuration) may induce noisy contribution estimation. This phenomenon is even more pronounced in the presence of a large number of loss combinations, which implies frequent changes in value for the mapping $\alpha \mapsto c_{\alpha,S}/v_\alpha$.

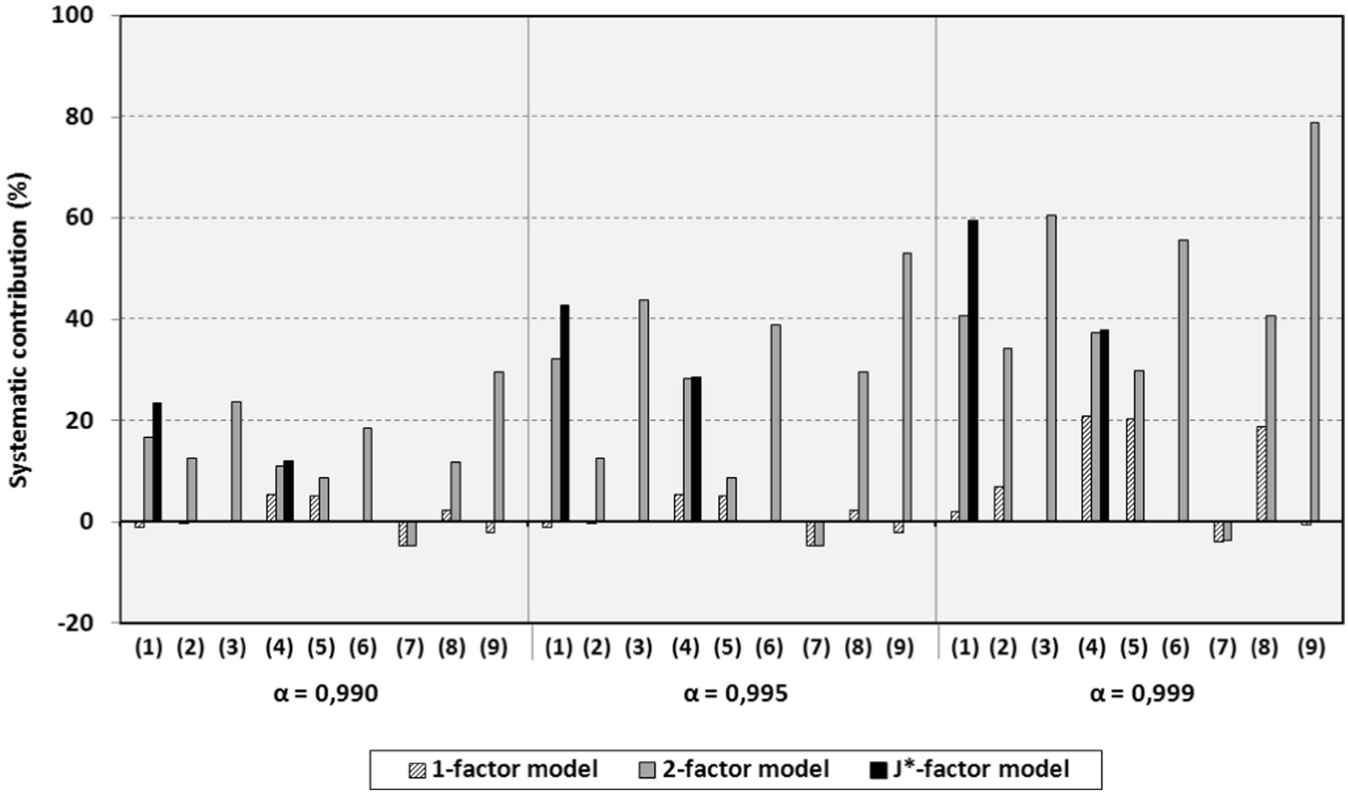

Fig. 6. Systematic contribution as a function of α for the *hedge portfolio* (PCA-based calibration).

the conditional approach used for the banking book should not be transposed with the trading book in which typical long-short portfolios may contain significant non-systematic risk components.

5. Conclusions

Assessment of default risk in the trading book is a key point in the FRTB. In the BCBS's approach, the dependence structure of defaults must be modeled through a systematic factor model with constraints on (i) the calibration data of the initial correlation matrix and (ii) the number of factors in the underlying correlation structure.

Considering representative long-only and long-short portfolios, this paper considers the practical implications of such modeling constraints for both the future DRC prescription and the current Basel 2.5 IRC built on constrained and unconstrained factor models. We considered various correlation structures. Using a structural-type credit model, we assessed the impacts on the VaR (at various confidence levels) of the calibration data as well as those of the estimation method of the correlation structures and the chosen number of factors. We introduced Hoeffding decomposition of portfolio exposures to factor and specific risks to quantify systematic contributions to the credit VaR.

The comparative analysis of risk factors modeling allows us to gauge the relevance of the BCBS's proposals of prescribing model and calibration procedure, to reduce the RWAs variability and increase comparability among financial institutions. The key insights of our empirical analysis are as follows:

- The main source of DRC variability is the high confidence level of the regulatory risk measure. As expected, $\alpha = 0.999$ gives rise to significant discrepancies among configurations (i.e. calibration data) and among the constrained models we tested. For $\alpha = 0.99$, risk measures are less dispersed and therefore less prone to model risk. Using this confidence level might be counterbalanced by a multiplier adjustment factor[24]. We suggest that this should be accounted for in future benchmarking exercises.
- Another important source of DRC variability is the disparities among correlation matrices. This may be due to the type of data (e.g. equity returns, CDS spread returns) and/or the calibration period. Therefore, within the current BCBS's approach, financial institutions could be led to take arbitrary choices regarding the calibration of the default correlation structure. This may then cause an ill-favored variability in the DRC, making the comparison among institutions more difficult. Apart from disclosure of the data retained for correlation estimates, one response could be to use IRB-like regulatory prescribed correlations, as in the banking book approach. Another response could be to disclose RWAs computed with such prescribed correlations, which would mitigate the issue of moving assets across the trading/banking books boundary.
- The strength of the two-factor constraint depends on the smoothness of the pairwise correlations frequencies in the initial correlation matrix: the more dispersed the underlying correlation structure, the greater is the number of factors needed to approximate it. In contrast, the estimation methods for both the initial correlation (standard or shrinked estimators) and the factor-based correlation matrices (SPG-based or PCA-based algorithms) have smaller effects, at least on the *diversification portfolio* (long-only exposures).
- The impact of the correlation structure on the risk measure mainly depends on the composition of the portfolio (long-only or long-short). For the particular case of a *diversification portfolio* (long-only exposures) with a smooth

[24] Earlier points in the Basel II agreements envisaged a confidence level of $\alpha = 0.995$, but then retained a confidence level of $\alpha = 0.999$. The motivation was that Tier 1 was most restricted to hard capital, which is no longer the case in the Basel III agreements.

correlation structure (e.g. estimated on non-stressed equity returns), constrained factor models (mostly when considering at least two factors) and unconstrained models produce almost the same risk measure. For the specific case of a *hedge portfolio* (long-short exposures), for which widely dispersed pairwise equity or CDS-spread correlations and far tail risks (99.9% VaR) are jointly considered, cliff effects arise from discreteness of loss: small changes in exposures or other parameters (default probabilities) may lead to significant changes in the credit VaR, jeopardizing the comparability of RWAs.

– When it comes to long/short portfolios, the use of the conditional (to systematic factors) approach is questionable, because it contributes poorly to the total credit VaR.

Overall, the relevance of the BCBS's prescribed two-factor constraint to reduce RWAs variability can be challenged. In our case study on diversified portfolios, the two-factor constraint decreases the VaR but does not reduce the RWAs variability because data sources and calibration methods are specific choices to each institution. However, it is not clear whether the two-factor constraint would be more relevant when considering bank-specific portfolios. Our analysis on the correlation structure is a first step toward assessing the *fundamentalness* of the whole default risk review and calls for further works on the other components of the DRC modeling, i.e. the default probabilities and the LGD, which should be other important sources of RWAs variability.

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