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Probability and Time Trade-Off

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Probability and time are integral dimensions of virtually any decision. To treat them together, we consider the prospect of receiving outcome x with a probability p at time t. We define risk and time distance, and show that if these two distances are traded off linearly, then preferences are characterized by three functions: a value function, a probability discount rate function, and a psychological distance function. The concavity of the psychological distance function explains the common ratio and common difference effects. A decreasing probability discount rate accounts for the magnitude effect. The discount rate and the risk premium depend on the shape of these three functions.

Key words: risk preferences; time preferences; probability discount rate; subendurance; magnitude effect; psychological distance
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1. Introduction

Time and probability are fundamental attributes of virtually any decision. Decisions involving future risky consequences can be found in domains such as investment, saving, consumption, environmental preservation, and health. In all these situations, decision makers face a trade-off between an immediate and/or certain reward and a delayed and/or uncertain reward. A deeper understanding of their preferences may be useful to construct more realistic models of economic behavior and to supply prescriptions for decision making.

In this paper we provide a simple axiomatic model of preferences that formalizes the connection between probability and time. The idea that people behave similarly in the face of probability and delay has a long standing in psychology (Rachlin et al. 1991). This paper builds on the work of Prelec and Loewenstein (1991), who discussed general preference conditions that can be applied to risk and time. Similarities between risk and time preferences have also been discussed by Quiggin and Horowitz (1995) and Ebert and Prelec (2007). Our work is also related to that of Halevy (2008), who, following Chew and Epstein (1990), provided an axiom system for discounted utility with nonlinear probability weighting. Our axiom system integrates probability and time in a different way, accounting for magnitude effects.

We propose a preference model over triples of the form (x, p, t); that is, a gain x will be received with probability p at time t, and otherwise the payoff is zero. We adopt the framework of Fishburn and Rubinstein (1982), to which we add the probability dimension. The representation we derive involves a value function, v, together with a probability discount rate function, r, and a psychological distance function, d. Appropriate restrictions of these functions will imply the common ratio effect for risk (Kahneman and Tversky 1979), decreasing impatience (Loewenstein and Prelec 1992), and the magnitude effect for time preference (Frederick et al. 2002).

This paper has two central themes. The first is a preference condition called *subendurance*: Most subjects prefer a 50/50 chance of \$100 today to having \$100 for sure a year from now; but this preference is reversed if we increase the amount to \$1,000. Subendurance describes the growing willingness to wait for the increase in probability as the outcome gets larger. Subendurance is captured by the probability discount rate function which decreases with x. The name *probability discount rate* is in contrast with the traditional *monetary discount rate*. Whereas the monetary discount rate measures the trade-off between outcome and delay, the probability discount rate measures the trade-off between probability and delay. The probability discount rate is a key component of time preference and is necessary to accommodate the magnitude effect.

The second theme is risk and time as *distance*, and the associated psychological distance function. Although the psychological distance function is simply a recasting of the traditional probability weighting function, we are convinced it provides an insightful way to represent preferences. A nonlinear psychological distance function will bond nonlinear probability weighting with nonexponential discounting. The psychologi-

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cal distance function is useful to propose paired probability weighting and time discounting functions.

We do not directly address the source of time preference, such as impatience, savoring, planning constraints, timing of resolution, or changes in future preferences (Epstein and Hynes 1983, Loewenstein and Angner 2003). Instead, we characterize the shape of risk and time preferences when the trade-off between time and risk is done in a particular way.

2. Motivation from Experimental Evidence

Table 1 presents some experimental evidence on how subjects resolve the trade-offs in the (x, p, t) domain. We witness *five* preference patterns.

Pattern 1–2, *the common ratio effect*. This effect implies a violation of proportionality in probabilities. The common ratio effect is the origin of considerable literature on nonlinear probability weighting (Allais 1953).

Pattern 4–5, *the common difference effect*. This effect implies a violation of stationarity or constant impatience. Models incorporating such decreasing impatience have entered mainstream economic analysis (Laibson 1997).

Pattern 1–3, *the common ratio using delay*. Rather than multiplying probabilities by a common factor, the experimenters add a common delay. Preferences shift toward the larger, less likely reward. Comparing with 1–2 suggests that *time acts as probability*.

Pattern 4–6, *the common difference using probability*. Rather than adding a common delay, the experimenters multiply the probabilities by a common factor. Preferences shift toward the larger, later reward. Comparing with 4–5 suggests that *probability acts as time*.

Pattern 7–8, *subendurance*. Choice 7 presents a trade-off between a later more likely reward and an earlier less likely reward. This trade-off is central in this paper. The modal preference favors the later, more likely reward. But if the reward is made smaller, then preferences shift toward the earlier, less likely reward.

3. The Probability and Time Trade-Off Model

3.1. Setup

We consider preferences over triplets (x, p, t), which describe the prospect of receiving some reward x at time t with a probability p, where otherwise the payoff is zero. The reward, x, could be money or the quantity of a divisible good. The decision maker's preference increases in x, and 0 is interpreted as the neutral outcome (no gain, no loss). We assume that the lottery is resolved and payed at t.

Formally, our choice set is $\mathcal{M} = X \times P \times T$, where $X = [0, \infty)$, $P = [0, 1]$, and $T = [0, \infty]$. Note that the time interval includes "never" $(t = \infty)$. Let $\mathcal{M}^+$ be the set of nontrival prospects, those with $x > 0$, $p > 0$, and $t < \infty$; and let $\mathcal{M}^0$ be the set of trivial prospects, those with $xpe^{-t} = 0$. Let $\succsim$ be a preference relation over pairs in $\mathcal{M}$ possessing the following properties:

A1. (Ordering and Continuity). *$\succsim$ is a continuous weak order on $\mathcal{M}$.*

A2. (Monotonicity). *Let $(x, p, t), (y, q, s) \in \mathcal{M}$.*

A2.1. *If $(x, p, t), (y, q, s) \in \mathcal{M}^0$, then $(x, p, t) \sim (y, q, s)$.*

A2.2. *If $(x, p, t) \in \mathcal{M}^+$ and $x > y$, then $(x, p, t) \succ (y, p, t)$.*

A2.3. *If $(x, p, t) \in \mathcal{M}^+$ and $p > q$, then $(x, p, t) \succ (x, q, t)$.*

A2.4. *If $(x, p, t) \in \mathcal{M}^+$ and $t < s$, then $(x, p, t) \succ (x, p, s)$.*

A1 and A2 are standard and guarantee the representation of preferences by a continuous function $V(x, p, t)$ on $\mathcal{M}$. On $\mathcal{M}^0$, V is constant, and without a

Table 1 **Choices Between Prospects A and B**

	Prospect A	vs.	Prospect B	Response	N
1.	**(€9, for sure, now)**	vs.	(€12, with 80%, now)	**58%** vs. 42%	142
2.	(€9, with 10%, now)	vs.	**(€12, with 8%, now)**	22% vs. **78%**	65
3.	(€9, for sure, 3 months)	vs.	**(€12, with 80%, 3 months)**	43% vs. **57%**	221
4.	**(*f*100, for sure, now)**	vs.	(*f*110, for sure, 4 weeks)	**82%** vs. 18%	60
5.	(*f*100, for sure, 26 weeks)	vs.	**(*f*110, for sure, 30 weeks)**	37% vs. **63%**	60
6.	(*f*100, with 50%, now)	vs.	**(*f*110, with 50%, 4 weeks)**	39% vs. **61%**	100
7.	**(€100, for sure, 1 month)**	vs.	(€100, with 90%, now)	**81%** vs. 19%	79
8.	(€5, for sure, 1 month)	vs.	**(€5, with 90%, now)**	43% vs. 57%	79

Sources. Rows 1–3, Baucells and Heukamp (2010, Table 1); rows 4–6, Keren and Roelofsma (1995, Table 1) (*f*1 in 1995 equaled \$0.6); rows 7 and 8, Baucells et al. (2009).

Notes. Modal preferences are shown in bold. For rows 4–6, we do not know whether the authors employed real incentives. In all the rest, some subjects were selected at random and one of their choices was played out for real money. For rows 4 and 5, there is abundant evidence of the common difference effect using real incentives (Horowitz 1991, Loewenstein and Prelec 1992). Except for row 7, the modal preference was significantly higher than 50% using a binomial test. Hence, the five preference patterns are statistically significant. In all cases, the timing of resolution is set at t, except for row 6, in which it was not stipulated.

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loss of generality we set $V(x, p, t) = 0$. By continuity, V tends to zero whenever xpe^{-t} tends to zero. On $\mathcal{M}^+$, V is strictly increasing in the first and second components, and strictly decreasing in the third.

What shape may V have to accommodate the five preference patterns? To be compatible with Patterns 1–2 and 4–5, V cannot be linear in probabilities or discount time exponentially (Tversky and Wakker 1995, Loewenstein and Prelec 1992). One may be tempted to propose $V(x, p, t) = w(p)f(t)v(x)$. Unfortunately, this form is not appropriate because, by Patterns 1–3 and 4–6, probability and time cannot be separated. One may then propose the more general form $V(x, p, t) = g(p, t)v(x)$, but this fails to accommodate subendurance.

Proposition 1. *Assume V satisfies* A1 and A2. *Pattern* 1–2, *as well as* 4–5, *is incompatible with $V(x, p, t) = pe^{-rt}v(x)$. Pattern* 1–3, *as well as Pattern* 4–6, *is incompatible with $V(x, p, t) = w(p)f(t)v(x)$. Pattern* 7–8 *is incompatible with $V(x, p, t) = g(p, t)v(x)$.*

3.2. Probability and Time Trade-Off

Our goal is to provide axiomatic foundations of a utility form compatible with *all* five preference patterns. The key to our approach is to focus on the trade-off between probability and time. To be specific, consider the prospect of receiving 100 € with 80% in three months. Suppose that shortening the delay by $\Delta = 1$ month is offset by multiplying the probability by $\theta = 0.5$; that is, the decision maker is indifferent between $(100, 80\%, 3)$ and $(100, 40\%, 2)$. Consider a decision maker who, for all prospects involving $x = 100$ €, would agree to this same Δ for θ trade-off regardless of the moment in time and the level of probability. She would then agree to $(100, 20\%, 3) \sim (100, 10\%, 2)$, $(100, 80\%, 5) \sim (100, 40\%, 4)$, or $(100, 60\%, 10) \sim (100, 30\%, 9)$.

A3. (Probability–Time Trade-Off). *For all (x, p, t), $(x, q, s) \in \mathcal{M}^+$, $\Delta \in (0, \infty)$, and $\theta \in (0, 1)$, if $(x, p, t+\Delta) \sim (x, p\theta, t)$, then $(x, q, s+\Delta) \sim (x, q\theta, s)$.*

Although the form of A3 is routine, the axiom is novel in decision theory. A3 has been tested by Baucells et al. (2009). The evidence suggests that the axiom is plausible in some parts of the domain. A3 is compatible with Pattern 7–8, because the value of θ that offsets Δ may depend on x. It is compatible with the rest of patterns and creates relationships between them (see §4). If multiplying probabilities by a common factor produces insensitivity to probability ratios, then, by A3, it also produces insensitivity to time differences. Similarly, if adding a common delay produces insensitivity to time differences, then it also produces insensitivity to probability ratios. Lemma 1 in the appendix establishes that, under A1–A3, Patterns 1–2 and 4–6 are equivalent, and Patterns 1–3 and 4–5 are also equivalent.

In folk wisdom, sayings such as "a bird in the hand is worth two in the bush" refer to both risk and time. Likewise, when presented with the choice between a larger-later and a smaller-immediate reward, many subjects justify their preference for the smaller-immediate reward because "the future is always uncertain." Hence, time and risk seem to be interchangeable. A3 is a useful way to capture this connection.

3.3. Risk Distance, Time Distance, and the Probability Discount Rate

Both time and risk put distance between the decision maker and the outcome (Trope et al. 2007). The value of t, measured in some units, is an obvious measure of *time distance*. Let the *risk distance*, $h(p) \geq 0$, $p \in [0, 1]$, be any strictly decreasing and continuous function, with $h(0) > 0$ possibly unbounded, and $h(1) = 0$. Consider three prospects, with probabilities p, q, and pq, respectively. The interpretation is that the third prospect is determined by the independent success of the first two. It is reasonable to expect the risk distance of the third prospect to be the sum of the risk distances of the first two.

Proposition 2. *$h(p)$ is a risk distance that is additive with respect to the concatenation of independent events, $h(pq) = h(p) + h(q)$, $p, q \in (0, 1]$, if and only if (iff), up to choice of scaling, $h(p) = -\ln p$, $p \in [0, 1]$.*

Henceforth, we adopt $-\ln p$ as a measure of risk distance. A3 is instrumental to combining risk and time distance.

Proposition 3. A1–A3 *imply that for some continuous function $r_x > 0$, possibly unbounded at zero, $(x, p, t) \sim (x, pe^{-r_x t}, 0)$, $(x, p, t) \in \mathcal{M}$.*

We will call any such function a *probability discount rate*. Note that $pe^{-r_x t} = e^{-(-\ln p + r_x t)}$. Hence, the role of r_x is to adjust the units of time, so that risk and time distance can be added into $\tau = -\ln p + r_x t$, the *distance* of (x, p, t). To elicit r_x, choose some $(x, p, t) \in \mathcal{M}^+$ and $\Delta > 0$. Then, find $0 < \theta < 1$ such that $(x, p\theta, t) \sim (x, p, t+\Delta)$, and let

$$r_x = \frac{-\ln \theta}{\Delta}.$$

3.4. Subendurance

Recall that in the preference Pattern 7–8, the larger the reward, the more subjects are willing to wait in exchange for improved probabilities. We call this pattern subendurance and show that it is characterized by decreasing probability discount rates. Throughout, decreasing means nonincreasing, and positive means nonnegative.

A4. (Subendurance). *For all $(x, p, t) \in \mathcal{M}^+$, $\theta \in (0, 1)$, $\Delta \in (0, \infty)$, and $y \in (0, x)$, if $(x, p, t+\Delta) \sim (x, p\theta, t)$, then $(y, p, t+\Delta) \preccurlyeq (y, p\theta, t)$.*

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Subendurance guarantees outcome monotonicity, because larger outcomes will be less penalized by delay. Hence, it produces the following intuitive representation theorem.

Theorem 1. A1–A4 *hold iff, for some continuous function* $V(x, p, t)$ *satisfying* A1 and A2, *and some decreasing probability discount rate*, $r_x > 0$,

$$V(x, p, t) = V(x, pe^{-r_x t}, 0) = V(x, e^{-(-\ln p + r_x t)}, 0).$$

Define *isoendurance* by replacing "$\preccurlyeq$" with "$\sim$" in A4. It is immediate to show that isoendurance is equivalent to r_x constant.

The data from Baucells et al. (2009) provide evidence of subendurance. For t measured in months and x in euros, they find averages of $r_{50} = 8.3\%$, $r_{100} = 6.6\%$, and $r_{1,000} = 4.7\%$. Subjectively, time runs faster for smaller rewards than for larger rewards. For €1,000, the passage of one year is equivalent to multiplying probabilities by a factor of 57%. This factor decreases to 37% for prospects of €50. In behavioral applications, the probability discount rate can be made a function of factors that may change the perception of time (Read et al. 2005, Benhabib et al. 2010).

Simon (1978, p. 13) said that in "a world where attention is a major scarce resource, information may be an expensive luxury, for it may turn our attention from what is important to what is unimportant." Subendurance could be considered a mental habit of not thinking hard about small consequences in the future, reflecting the large degree of uncertainty associated with claiming and receiving small outcomes in the distant future.

3.5. Multiplicative Separability

It is convenient to obtain a representation where V is multiplicatively separable. Following Fishburn and Rubinstein (1982), we employ a standard separability condition, to be imposed on immediate prospects.

A5. (Outcome–Probability Separability). *For all* $x, y, z \in X$ *and* $p, q, u \in P$, *if* $(x, p, 0) \sim (y, q, 0)$ *and* $(y, u, 0) \sim (z, p, 0)$, *then* $(x, u, 0) \sim (z, q, 0)$.

Our representation will involve three functions: a *value function*, $v(x)$, $x \in X$, a continuous and strictly increasing function with $v(0) = 0$; a *probability discount rate*, r_x; and a *psychological distance function*, $d(\tau)$, $\tau \in T$, a continuous and strictly increasing function with $d(0) = 0$ and $\lim_{\tau\to\infty} d(\tau) = \infty$.

Given $d(\tau)$, we define its associated *weighting function* as $w(p) = e^{-d(-\ln p)}$, and its associated *discounting function* as $f(\tau) = e^{-d(\tau)}$.[1] For future reference, let $\epsilon_v(x) = xv'(x)/v(x)$ and $\epsilon_r(x) = xr'_x/r_x$ denote the elasticities of $v(x)$ and r_x, respectively.

Theorem 2. A1–A5 *hold if and only if, for some value function, some decreasing probability discount rate, and some psychological distance function*,

$$\begin{aligned} V(x, p, t) &= e^{-d(-\ln p + r_x t)} v(x) = w(pe^{-r_x t}) v(x) \\ &= f(-\ln p + r_x t) v(x), \quad (x, p, t) \in \mathcal{M}. \quad (1) \end{aligned}$$

Moreover, any such $\hat{v}$, $\hat{r}$, *and* $\hat{d}$ *represent this preference if and only if* $\hat{v} = \eta v^\kappa$, $\hat{r} = r$, *and* $\hat{d} = \kappa d$ *for some* $\eta, \kappa > 0$.

By definition, $v(0) = d(0) = 0$. The uniqueness condition allows us to set $v(1) = d(1) = 1$.

It is clarifying to consider the pure risk and time models contained in the general model. If $t = 0$, then we obtain a standard risk preference model:

$$V(x,p,0) = e^{-d(-\ln p)} v(x) = w(p) v(x) = f(-\ln p) v(x). \quad (2)$$

If $p = 1$, then we obtain a novel magnitude-dependent time preference model:

$$V(x,1,t) = e^{-d(r_x t)} v(x) = w(e^{-r_x t}) v(x) = f(r_x t) v(x). \quad (3)$$

Note that f is not a function of time, but of speed-adjusted time, $r_x t$.

A1–A5 are restrictive as to how magnitude effects are to be represented in time preference models. We may use a fixed cost component in the probability discount rate, $r_x = a + b/x$, $a, b \geq 0, a + b > 0$. However, A1–A5 are not compatible with a fixed cost component in the discount factor (Benhabib et al. 2010, p. 209), because $f(r_x t)$ cannot be written as $a\delta(t) - b/x$.

Halevy (2008) derives a model for preferences over consumption paths under risk. Restricted to our domain, his model evaluates (x, p, t) as $w(pe^{-rt}) \cdot \beta^t v(x)$. Setting $\beta = 1$, we obtain a version of our model restricted to a constant probability discount rate. A constant probability discount rate may be interpreted as a mortality rate that makes the passage of time objectively uncertain. This interpretation is not possible if r_x depends on x.

The probability discount rate, or relative increase in *probability* that offsets a delay, is different from the monetary discount rate, or relative increase in *reward* that offsets a delay. To see this, note that

$$r_x = \left.\frac{\partial p}{p\,\partial t}\right|_V \neq \left.\frac{\partial x}{x\,\partial t}\right|_V = \frac{r_x}{\epsilon_v(x)/d'(\tau) - xr'_x t}.$$

Whereas the first marginal rate of substitution is independent of p and t, and equal to r_x, the second is a complex function of (x, p, t).

Subendurance could be tightened to isoendurance, producing a version of the main result with a constant probability discount rate. The result is trivial. Alternatively, subendurance could be dispensed from the set of conditions (Theorem 3 in the appendix). The representation is identical, except that r_x is not necessarily

[1] Any such w is continuous, strictly increasing, with $w(0) = 0$ and $w(1) = 1$, and any such f is continuous, strictly decreasing, with $f(0) = 1$ and $\lim_{\tau\to\infty} f(\tau) = 0$. Given w, or f, we obtain $d(\tau) = -\ln w(e^{-\tau}) = -\ln f(\tau)$.

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decreasing, and v, r, and d must satisfy a joint condition to ensure that $V(x, p, t)$ is monotone in x. This joint condition is such that if d is sufficiently sensitive, e.g., $\lim_{\tau\to\infty} d'(\tau)\tau = \infty$, then subendurance must necessarily hold. Otherwise, subendurance may fail.

Example 1. A1–A3 and A5 hold, but A4 fails whenever preferences are represented by

$$v(x) = x^\alpha, \quad r_x = b\frac{x+1}{x+2}, \quad \text{and} \quad d(\tau) = \frac{\ln(1+\delta\tau)}{\ln(1+\delta)},$$
$$\alpha > 0,\ \delta \geq e^{1/\alpha} - 1,\ b > 0.$$

4. Diminishing Sensitivity to Distance

Consider a small and a large reward, $0 < y < x$, to be received at distances $\tau_x > \tau_y \geq 0$. By (1), these prospects are indifferent iff the difference between the psychological distances is equal to the log of their relative desirability, or

$$d(\tau_x) - d(\tau_y) = \ln\frac{v(x)}{v(y)}. \tag{4}$$

The psychological distance function allows for a nonlinear evaluation of risk and time distance. In this section, we explore the implications of *diminishing sensitivity to distance*, that is, of $d(\tau)$ being concave. The function $d(\tau)$ being concave is equivalent to $w(p)$ being subproportional and to $f(\tau)$ being substationary.[2] This, we will show, accounts for the following patterns.

A6. (Common Ratio Effect). *For all* $(x, p, t) \in \mathscr{M}^+$, $y \in (0, x)$, $\theta \in (0, 1)$, *and* $q < p$, *if* $(y, p, t) \sim (x, p\theta, t)$, *then* $(y, q, t) \preccurlyeq (x, q\theta, t)$.

A7. (Common Difference Effect). *For all* $(x, p, t) \in \mathscr{M}^+$, $y \in (0, x)$, $\Delta \in (0, \infty)$, *and* $s > t$, *if* $(y, p, t) \sim (x, p, t + \Delta)$, *then* $(y, p, s) \preccurlyeq (x, p, s + \Delta)$.

A8. (Common Ratio Using Delay). *For all* $(x, p, t) \in \mathscr{M}^+$, $y \in (0, x)$, $\theta \in (0, 1)$, *and* $s > t$, *if* $(y, p, t) \sim (x, p\theta, t)$, *then* $(y, p, s) \preccurlyeq (x, p\theta, s)$.

A9. (Common Difference Using Probability). *For all* $(x, p, t) \in \mathscr{M}^+$, $y \in (0, x)$, $\Delta \in (0, \infty)$, *and* $q < p$, *if* $(y, p, t) \sim (x, p, t + \Delta)$, *then* $(y, q, t) \preccurlyeq (x, q, t + \Delta)$.

A6 is also known as subproportionality, and A7 as decreasing impatience or substationarity (Kahneman and Tversky 1979, Prelec and Loewenstein 1991). Define *proportionality* and *stationarity* by replacing "$\preccurlyeq$" with "$\sim$" in A6 and A7, respectively. Recall that the probability and time trade-off axiom, A3, makes A6 and A9 equivalent and A7 and A8 equivalent (Lemma 1).

[2] Formally, the following are equivalent: d is concave [$\forall\tau \geq 0$, $\tau', \tau'' > 0$, $d(\tau+\tau') - d(\tau) \geq d(\tau+\tau'+\tau'') - d(\tau+\tau'')$], w is subproportional [$\forall p \in (0, 1]$, $p', p'' \in (0, 1)$, $w(pp')/w(p) \leq w(pp'p'')/w(pp'')$], and f is substationary [$\forall\tau \geq 0$, $\tau', \tau'' > 0$, $f(\tau+\tau')/f(\tau) \geq f(\tau+\tau'+\tau'')/f(\tau+\tau'')$].

The essence of A6 and A9 is that, for small probabilities, subjects become insensitive to probability ratios and to time differences. Hence, outcome comparisons become more preeminent. Viewing these effects in light of (4), it shows that adding common distance lowers the left-hand side of the equation, that is $d(\tau_x + \tau) - d(\tau_y + \tau)$ decreases with τ:

Proposition 4. *Under* A1–A5, *the following are equivalent*: (i) *the common ratio effect*, (ii) *the common difference using probability, and* (iii) *diminishing sensitivity to distance*.

The essence of A7 and A8 is that, for large delays, subjects become insensitive to time differences and probability ratios. The common difference effect, A7, follows directly from A1–A4 and A6 (Lemma 2 in the appendix). Hence, under A1–A5, diminishing sensitivity to distance implies A7. Under some conditions, the converse is true, producing a full equivalence among the four preference patterns.

Proposition 5. *Under* A1–A5, V *differentiable, and* $\inf_{x\in X}(\epsilon_r(x)/\epsilon_v(x)) = 0$, *the following are equivalent*: (i) *the common ratio effect*, (ii) *the common difference effect*, (iii) *the common ratio using delay*, (iv) *the common difference using probability, and* (v) *diminishing sensitivity to distance*.

There are two simple ways to verify that $\inf_{x\in X}(\epsilon_r(x)/\epsilon_v(x)) = 0$. The first is a weak version of isoendurance: $r'_x = 0$ for some $x > 0$. The second is to have $\lim_{x\to\infty}(\epsilon_r(x)/\epsilon_v(x)) = 0$, as, for example, when v is power, and $r_x = a + b/x$, $a > 0$, $b \geq 0$. The condition is shown to be tight.

Example 2. A1–A5 and A7 hold, but A6 fails whenever preferences are represented by

$$v(x) = x^\alpha, \quad r_x = \frac{b}{x}, \quad \text{and} \quad d(\tau) = \frac{e^{\delta\tau} - 1}{e^\delta - 1},$$
$$\alpha > 0,\ \delta \in (0, \ln(1 + 1/\alpha)],\ b > 0.$$

Here, the common difference effect is induced by subendurance, which offsets the increasing sensitivity to time distance. In this example, setting r_x constant, and keeping the same convex d results in a preference that satisfies A1–A5 but violates A6–A9.

Diminishing sensitivity to outcomes has been the cornerstone of economic analysis. We have shown that diminishing sensitivity to (risk and time) distance is equivalent to the common ratio effect and, under some conditions, to the common difference effect.

5. Implications for Time and Risk Preference

Let $(x, p, t) \in \mathscr{M}^+$. If $t > 0$, then one can consider the smaller reward $0 < y < x$ to be received now that would make $(x, p, t) \sim (y, p, 0)$. We then define the

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discount rate as $\rho = (1/t)\ln(x/y)$, so that $y = xe^{-\rho t}$ is the discounted value of x. Similarly, if $p < 1$, then one can consider the smaller reward $0 < y < x$ to be received for sure that would make $(x, p, t) \sim (y, 1, t)$. We then define the *risk premium* as $\pi = px - y$, the difference between the expected value of a prospect and its certainty equivalent. In this section, we explore how A1–A5, together with A6, are consistent with observed behavioral patterns of discount rates and risk premia.

5.1. The Discount Rate

Consider the benchmark model, $V(x, p, t) = pe^{-rt}x$, in which the discount rate ρ is equal to r and independent of p, t, and x. In contrast with this benchmark, it has been observed that the discount rate *decreases* as p decreases, as t increases, and as x increases (Thaler 1981, Keren and Roelofsma 1995, Frederick et al. 2002). We show how each of our conditions account for these three observations.

Proposition 6. *Assume* A1–A3, A5, *and that* V *is differentiable*.

1. A6 *is equivalent to* $\partial\rho/\partial p \geq 0$.
2. A6 *and decreasing elasticity of* v *imply* $\partial\rho/\partial t \leq 0$.
3. A4 *and increasing elasticity of* v *imply* $\partial\rho/\partial x \leq 0$.

That the discount rates decrease as p decreases is equivalent to the common difference using probability, A9. It implies that subjects become less patient once uncertainty partially resolves. Take a manager who considers two prospects, affected by specific risks and a common risk. One prospect pays off more, but later, whereas the second prospect pays less, but sooner. The manager may be patient and prefer the prospect that pays more later. If the common risk resolves favorably, then the probabilities of success for both prospects will increase. The manager will then apply a higher discount rate and find the project that pays off sooner relatively more attractive.

The second property, called hyperbolic discounting by Frederick et al. (2002), is the product of the concavity of d and the decreasing elasticity of v. The third property, known as *the magnitude effect*, is the product of subendurance and the increasing elasticity of v.

In the past, i.e., without subendurance, the magnitude effect has been attributed to the increasing elasticity of v (Loewenstein and Prelec 1992, p. 584). Noor (2011) rejects such an explanation using a calibration argument. Moreover, the parametric forms of v, elicited using risk, exhibit constant or decreasing elasticity (Holt and Laury 2002, p. 1653). Some may argue the value function for time and for risk are different.

We attribute the magnitude effect to subendurance, which, if sufficiently strong, can offset the possible effect of a decreasing elasticity of v. In fact, if the elasticity of v is constant, as often assumed, then $\partial\rho/\partial x < 0$ iff $r'_x < 0$. Hence, the desired signs for the three derivatives are compatible with A1–A6, with no need to assume a different value function for risk and time. Abdellaoui et al. (2011), who estimate the value function using delayed simple prospects, conclude that v for risk and time are the same, whereas the weighting function is time dependent. In our framework, the weighting function they elicited is $w_t(p) = w(pe^{-r_x t})$, which depends on time.

5.2. The Risk Premium

If time enters via a multiplicatively separable discount factor, then the risk premium does not depend on t. However, the common ratio using delay provides evidence that the risk premium decreases as t increases. Abdellaoui et al. (2011) found that when binary lotteries were played immediately, 77% of the subjects exhibited risk aversion. This percentage declines to 75% and 67% when the delay of lottery receipts is set to 6 and 12 months, respectively. Noussair and Wu (2006, p. 401) also found that "a substantial fraction of subjects exhibit a greater level of risk aversion for lotteries resolved and paid in the present rather than in the future." Subendurance and diminishing sensitivity to distance account for this observation. They also predict a new pattern of risk-seeking behavior, not for low, but for *high* probabilities and sufficiently large delays.

Proposition 7. *Assume* A1–A6 *and that* V *is differentiable*.

1. $\partial\pi/\partial t \leq 0$, *with strict inequality if either* d *is strictly concave or* r_x *is strictly decreasing*.
2. *Assume* r_x *is strictly decreasing and that, after scaling* $d(1) = 1$, v *is strictly concave. For any* $x > 0$, *there is a* $t^*_x > 0$ *such that* $\partial\pi/\partial p\,|_{p=1} < 0$ *iff* $t < t^*_x$.

The first part of the proposition has clear implications on how risk attitudes change with the passage of time. For example, a manager may commit to a risky project to be executed at some future calendar date, but cancel the project as time passes and t decreases, even though the probability of success remains unchanged. He has become more risk averse. This assumes consequentialism, that is, time distance is measured relative to "now," and not relative to some fix calendar date.

The second part of the proposition allow us to predict the sign of the risk premium for p close to 1. The risk premium at $p = 1$ is always zero. If t is zero, or less than t^*_x, the concavity of v induces risk aversion for probabilities close to 1. This is consistent with the fourfold pattern of risk attitudes for gains of moderate to high probability (Kahneman and Tversky 1979). However, the risk premium decreases with t, to the point of becoming negative if t is sufficiently large. Thus, large temporal psychological distance, $t > t^*_x$, induces risk-seeking behavior for probabilities close to 1 even if v is concave. The preference expressed in row 3 of

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Table 1, (€9, for sure, 3 months), $\preccurlyeq$ (€12, with 80%, 3 months), suggests evidence in this direction, as the expected value of both prospects is almost the same.

Evidence of negative risk premium for probabilities close to 1 is provided by the *peanuts effect* (Hershey and Schoemaker 1980). For very small rewards, subjects exhibit risk-seeking preferences for gains of high probability. Recall that the time distance depends on the magnitude. For small amounts, small delays may correspond to large time distances. Hence, perhaps the minutes that pass between making a choice and receiving the reward may be sufficient to induce risk-seeking preferences.

As p decreases, the risk premium will evolve depending on t and the shape of v, r, and d. In experiments with no delay, subjects' risk aversion decreases for prospects of small probability (Kahneman and Tversky 1979, Bruhin et al. 2010). A sufficient condition for π to eventually decrease as p becomes small is $\lim_{\tau\to\infty} d'(\tau) = 0$ and $\lim_{x\to\infty} \epsilon_v > 0$. However, this may not necessarily be the case.

Example 3. A1–A6 are compatible with a rich pattern of risk attitudes. Fix $t = 2$ months and $xp = 10$, and lower p while increasing x. Preferences given by

$$v(x) = \frac{1 - e^{-0.05x^{0.8}}}{0.05}, \quad r = 0.02, \quad \text{and} \quad d(\tau) = \tau^{0.7}$$

are risk averse for $p > 0.13$ and $p < 0.0023$, and risk seeking for $0.0023 < p < 0.13$.

6. Power and Constant Sensitivity Psychological Distance

The function d is useful to judge and suggest paired functional forms for probability weighting and time discounting. We propose two simple one-parameter specifications for $d(\tau)$. The single parameter $0 < \delta \le 1$ controls sensitivity, the extent to which people are able to discriminate between probability ratios and between differences in time delays. It corresponds to the curvature parameter of the weighting function, or the time-sensitivity parameter of the discount function. The lower the value of δ, the more w departs from linearity in probability, and the more f departs from exponential discounting. We scale $d(1) = 1$ and assume the function v is provided independently.

1. *Power*. $d(\tau) = \tau^\delta$, $\tau \ge 0$, and

$$V(x, p, t) = e^{-(-\ln p + r_x t)^\delta} v(x), \quad (x, p, t) \in \mathcal{M}.$$

The corresponding probability weighting is Prelec's (1998): $w(p) = e^{-(-\ln p)^\delta}$, $0 < p \le 1$. The paired time discount function is a magnitude-dependent version of Ebert and Prelec's (2007): $f(r_x t) = e^{-(r_x t)^\delta}$, $t \ge 0$. Both w and f fit the experimental data quite well (Ebert and Prelec 2007, Booij et al. 2010), and we expect the combined form to be a good candidate for parametric modeling.

2. *Constant sensitivity*. Let $d(\tau) = (1 - \delta) + \delta\tau$, $\tau > 0$, $d(0) = 0$. Let $\beta = e^{-(1-\delta)}$, $\delta_x = e^{-\delta r_x}$. Then,

$$V(x, p, t) = \beta p^\delta \delta_x^t v(x), \text{ either } p < 1 \text{ or } t > 0; \text{ and}$$
$$V(x, 1, 0) = v(x), \quad p = 1, \ t = 0.$$

This psychological distance function exhibits a discontinuity at $\tau = 0$, followed by constant sensitivity thereafter. Characterizing such preferences requires a weakening of continuity. They satisfy diminishing sensitivity to distance, exhibit present bias and certainty bias, are multiplicatively separable, and allow for magnitude effects. The associated discounting function is $f(r_x t) = \beta \delta_x^t$, $t > 0$, $f(0) = 1$. If r_x is constant, this form particularizes into the *beta-delta* time discounting model used extensively in economic modeling (Laibson 1997). The associated weighting function, $w(p) = \beta p^\delta$, $0 \le p < 1$, extends to the entire domain the low-probability segment of the CRS weighting function (Abdellaoui et al. 2010). We call it the *quasi-power* probability weighting function.[3]

3. *Linear*. If $\delta = 1$, then both models particularize into $d(\tau) = \tau$, and

$$V(x, p, t) = p e^{-r_x t} v(x).$$

This can be called the *magnitude-dependent expected discounted utility* model. It is characterized by A1–A5 and proportionality (Theorem 4 in the appendix). Adding isoendurance or replacing proportionality by stationarity characterizes the expected discounted utility model, $pe^{-rt}v(x)$ (Theorem 5 in the appendix). Figure 1 illustrates the weighting functions and discount functions associated with the power, constant sensitivity, and linear psychological distance functions.

7. Parameter-Free Elicitation of $V(x, p, t)$

We propose a parameter-free method to elicit values of v on any desired range $(0, x_0]$, the psychological distance function, and the probability discount rates for any desired values of x. The proposed method is chained, resembling the trade-off method used in rank-dependent models (Abdellaoui 2000). By setting p_1 closer to 1, we have a smaller step size and obtain a finer grid. This comes at the expense of having to use more steps, K, to approximately cover the same range of outcomes and risk distances.

Proposition 8. *Choose an outcome $x_0 > 0$, a probability $0 < p_1 < 1$, and a number of steps $K \ge 1$. Let $\tau_1 = -\ln p_1 > 0$. Fix the scales for $v(x_0) > 0$ and*

[3] The quasi power approximates the popular linear-in-log-odds form, $\beta p^\delta / [\beta p^\delta + (1-p)^\delta]$, for p near 0, and the neoadditive form, $w(p) = b + ap$, $a + b = \beta$, and $a = \beta\delta$, for p near 1. The linear-in-log odds, proposed by Goldstein and Einhorn (1987), are not subproportional.

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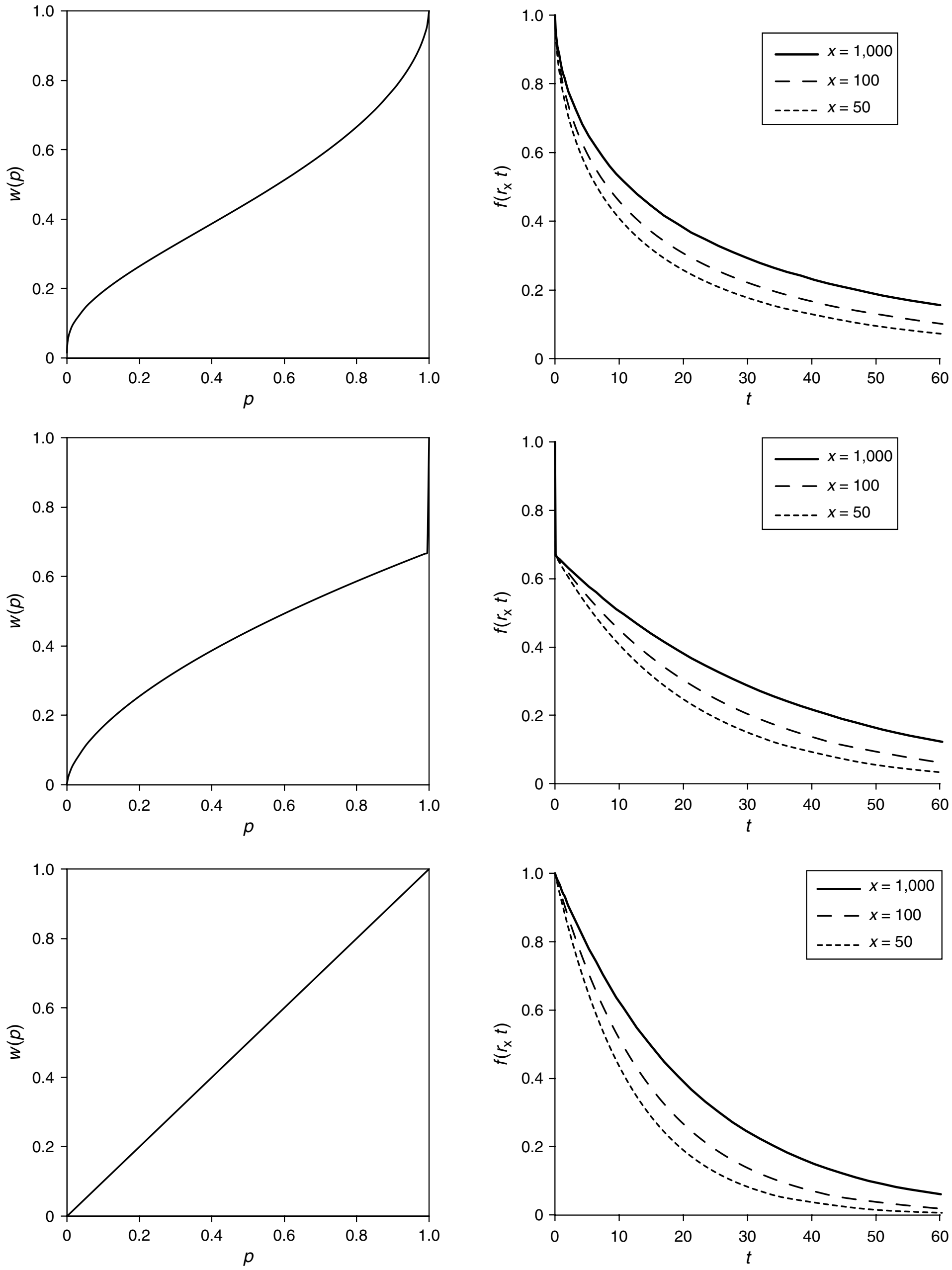
Figure 1 Probability Weighting Functions and Corresponding Time Discount Functions



Notes. Prelec (1998) weighting function (top left) and magnitude-dependent Ebert and Prelec (2007) discount function (top right), corresponding to $d(\tau) = \tau^{0.6}$, are shown. Quasi-power probability weighting function (middle left) and magnitude-dependent beta-delta discount function (middle right), corresponding to $d(\tau) = 0.4 + 0.6\tau$, $\tau > 0$, and $\beta = e^{-(1-\delta)} = 0.69$, are shown. Magnitude-dependent expected discounted utility (bottom), corresponding to $d(\tau) = \tau$, is shown. In all cases, $r_{50} = 8.3\%$, $r_{100} = 6.6\%$, and $r_{1,000} = 4.7\%$

$d(\tau_1) > 0$, *and let* $w(p_1) = f(\tau_1) = e^{-d(\tau_1)}$. *Under* A1–A5, *the chained elicitation proposed in* Table 2 *produces sequences* $x_0 > x_1 > \cdots > x_K$, $p_0 > p_1 > \cdots > p_K$, *and* $t_0 \geq t_1 \geq \cdots \geq t_K$ *such that*

$$d(-\ln p_k) = kd(\tau_1), \quad w(p_k) = w(p_1)^k,$$

$$f(-\ln p_k) = f(\tau_1)^k, \quad r_{x_k} = \frac{\tau_1}{t_k}, \quad \textit{and}$$

$$v(x_k) = e^{-kd(\tau_1)} v(x_0) = w(p_1)^k v(x_0) = f(\tau_1)^k v(x_0), \quad k = 0, \ldots, K.$$

The elicitation of r_{x_k} can done independently of p_1 and using prospect comparisons away from immediacy or certainty. To do so, choose $0 < q < p \leq 1$ and $s \geq 0$, find t_k such that $(x_k, p, t) \sim (x_k, q, s)$, and let $r_{x_k} = \ln(p/q)/(t-s)$.

Our method is different from existing alternatives of joint elicitation of risk and time preferences (Andersen et al. 2008). The usual way to elicit time preference is to assume a model of the form $f(t)v(x)$ and use a trade-off between a smaller-sooner and a larger-later

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Table 2 **Elicitation Procedure**

k	x	p	t
0	$x_0 > 0$ given	$p_0 = 1$	$(x_0, 1, \mathbf{t}_0) \sim (x_0, p_1, 0)$
1	$(\mathbf{x}_1, 1, 0) \sim (x_0, p_1, 0)$	p_1 given	$(x_1, 1, \mathbf{t}_1) \sim (x_1, p_1, 0)$
2	$(\mathbf{x}_2, 1, 0) \sim (x_1, p_1, 0)$	$(x_2, 1, 0) \sim (x_0, \mathbf{p}_2, 0)$	$(x_2, 1, \mathbf{t}_2) \sim (x_2, p_1, 0)$
3	$(\mathbf{x}_3, 1, 0) \sim (x_2, p_1, 0)$	$(x_3, 1, 0) \sim (x_0, \mathbf{p}_3, 0)$	$(x_3, 1, \mathbf{t}_3) \sim (x_3, p_1, 0)$
	…	…	…
K	$(\mathbf{x}_K, 1, 0) \sim (x_{K-1}, p_1, 0)$	$(x_K, 1, 0) \sim (x_0, \mathbf{p}_K, 0)$	$(x_K, 1, \mathbf{t}_K) \sim (x_K, p_1, 0)$

Notes. For each indifference, elicit the values in bold. The elicitation can proceed either by rows or by columns.

reward. The use of two outcome levels is problematic, because one needs to account for outcome preference. This includes the curvature of v, as well as anchoring effects (if subjects anchor on the larger reward, they might perceive the smaller-sooner reward as a loss). In contrast, we first elicit r_x using probability and time, and fixing one outcome level. This makes r_x independent of v and anchoring effects. However, we do elicit $f(-\ln p) = w(p)$ using risk and two outcome levels. Using calibrated outcomes makes the elicitation of f independent of v, but vulnerable to anchoring effects. It is an open question to devise an elicitation method that avoids certainty or immediacy effects for v and f.

Two methods have been examined recently that also bypass the effect of v. Takeuchi (2011) uses risk and time prospects, as we do, but assumes $V(x, p, t) = pf(t)v(x)$. This form is incompatible with all preference patterns, except for 4–5, and his $f(t)$ does not correspond to our $f(\tau)$. A more general method, based on temporal trade-offs, is proposed by Attema et al. (2010). They employ two outcome levels, $x > y$, and elicits a sequence t_k, $k = 0, \ldots, K$ such that, in our setup, $d(r_x t_k) - d(r_y t_{k-1}) = d(r_x t_1) - d(r_y t_0)$, $k = 2, \ldots, K$. If $r_x = r_y$, then their methods successfully elicits $d(\tau)$ and $f(\tau) = e^{-d(\tau)}$. But if $r_x \neq r_y$, then the intervals $[r_x t_0, r_y t_1)$, $[r_x t_1, r_y t_2)$, etc., are not contiguous, and the method requires some repair.

8. Conclusions

We have proposed a preference model that is parsimonious, and, in the domain of simple prospects, has rank-dependent utility, prospect theory, and nonexponential discounting as special cases. The model is compatible with the magnitude effect and other experimental evidence in the domain of risk and time.

Attema et al. (2010) show that to measure time preferences accurately one needs to account for the utility/value function. We go one step further and propose that risk and time preferences follow from the combined effect of the functions $v(x)$, r_x, and $d(\tau)$. The probability discount rate, r_x, is a key determinant of time preference. Baucells et al. (2009) elicit r_x and find it decreases with x, as well as with factors such as age and wealth.

The magnitude effect, although strongly documented, has not been readily accepted as part of standard models. The magnitude effect has been attributed to the curvature of v, an explanation that has empirical and theoretical difficulties. We attribute the magnitude effect to a preference condition called subendurance, which is characterized by r_x decreasing.

Our model insists on the ideas of risk and time distance and the associated psychological distance. Imposing diminishing sensitivity to distance produces the common ratio and common difference effects. In addition, we derive clean interactions between probability and time preference, and between delay and risk preference. For example, delay makes people insensitive to probabilities and lowers their risk aversion. Or, for events of small probability, subjects become more time patient.

In summary, A1–A5 offer a unified view of risk and time preference, setting a promising framework in which to build theory, connecting a large body of empirical evidence, as well as providing experimentalists with new sets of testable propositions.

Acknowledgments

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Appendix. Proofs

Proof of Proposition 1. Let $V = pe^{-rt}v(x)$. Row 1 in Table 1, $v(9) \geq 0.8v(12)$, implies $0.1v(9) \geq 0.08v(12)$, contradicting row 2. Row 4, $v(100) \geq e^{-r4}v(110)$, implies $e^{-r26}v(100) \geq e^{-r30}v(110)$, contradicting row 5.

Let $V = w(p)f(t)v(x)$. Row 1, $w(1)f(0)v(9) \geq w(0.8) \cdot f(0)v(12)$, implies $w(1)f(3m)v(9) \geq w(0.8)f(3m)v(12)$, contradicting row 3. Row 4, $w(1)f(0)v(100) \geq w(1)f(4w)v(110)$, implies $w(0.5)f(0)v(100) \geq w(0.5)f(4w)v(110)$, contradicting row 6.

Let $V = g(p, t)v(x)$. Row 7, $g(1, 1m)v(100) \geq g(0.9, 0) \cdot v(100)$, implies $g(1, 1m)v(5) \geq g(0.9, 0)v(5)$, contradicting row 8. □

Proof of Proposition 2. The condition $h(pq) = h(p) + h(q)$, $p, q \in (0, 1]$, is Cauchy's logarithmic equation on a restricted domain, which only admits the solution

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$h(p) = c \ln p$ (Aczél 1996, §2.1.2). Because h is strictly decreasing in p, $c < 0$, and we set $c = -1$. By continuity, $h(0) = \infty$. □

Proof of Proposition 3. If $t = 0$, then the result holds trivially. Let $(x_0, p_0, t_0) \in \mathscr{M}^+$, $t_0 > 0$ rational. Choose some rational Δ, $0 < \Delta \le t_0$. By A1 and A2, there is a unique θ, $0 < \theta < 1$, such that

$$(x_0, p_0, t_0) \sim (x_0, p_0\theta, t_0 - \Delta). \tag{5}$$

For any $(x_0, p, t) \in \mathscr{M}^+$, we wish to show that $(x_0, p, t) \sim (x_0, p\theta^{t/\Delta}, 0)$. Consider three cases.

Case 1. t/Δ is some integer $m \ge 1$. Equation (5) and A3 imply $(x_0, p, t) \sim (x_0, p\theta, t - \Delta)$. By A3, reducing the time dimension by Δ and multiplying the probability dimension by θ maintains indifference. Doing so $m - 1$ times produces $(x_0, p, t) \sim (x_0, p\theta^m, t - m\Delta) = (x_0, p\theta^{t/\Delta}, 0)$, as desired.

Case 2. $0 < t/\Delta < 1$, i.e., $t/\Delta = b/n$ for some integers $1 \le b < n$. By A1 and A2, there is a unique $\hat\theta$ such that $(x_0, p_0, t_0) \sim (x_0, p_0\hat\theta^{1/n}, t_0 - \Delta/n)$. By A3, reducing the time dimension by Δ/n and multiplying the probability dimension by $\hat\theta^{1/n}$ maintains indifference. Doing so $n - 1$ times produces

$$\begin{aligned}(x_0, p_0, t_0) &\sim (x_0, p_0\hat\theta^{1/n}, t_0 - \Delta/n)\\ &\sim (x_0, p_0\hat\theta^{2/n}, t_0 - 2\Delta/n) \sim \cdots \sim (x_0, p_0\hat\theta, t_0 - \Delta).\end{aligned}$$

Use (5) to conclude $(x_0, p_0\theta, t_0 - \Delta) \sim (x_0, p_0\hat\theta, t_0 - \Delta)$. Hence, $\hat\theta = \theta$, and $(x_0, p_0, t_0) \sim (x_0, p_0\theta^{b/n}, t_0 - \Delta b/n)$. By A3, replace p_0 for p and t_0 for $\Delta b/n = t$, producing $(x_0, p, t) \sim (x_0, p\theta^{b/n}, 0)$, as desired.

Case 3. $t/\Delta = m + b/n$, $m \ge 1$, $1 \le b < n$. The logic used in Case 1 yields $(x_0, p, t) \sim (x_0, p\theta^m, \Delta b/n)$. Case 2 now applies, producing $(x_0, p, t) \sim (x_0, p\theta^m, \Delta b/n) \sim (x_0, p\theta^{t/\Delta}, 0)$, as desired.

Given $(x, p_0, t_0) \in \mathscr{M}$, t_0 rational, and $\Delta > 0$ rational, let $0 < \theta_x < 1$ be such that $(x, p_0, t_0) \sim (x, p_0\theta_x, t_0 - \Delta)$. The previous argument leads to $(x, p, t) \sim (x, p\theta_x^{t/\Delta}, 0)$ for all $(x, p, t) \in \mathscr{M}^+$, t rational. Because such set is dense in $\mathscr{M}$ and the preference is continuous, the indifference holds for all $(x, p, t) \in \mathscr{M}$. Finally, let

$$r_x = \frac{-\ln \theta_x}{\Delta} \tag{6}$$

so that $\theta_x^{t/\Delta} = e^{-r_x t}$. By time monotonicity, $\theta_x < 1$, and $r_x > 0$. By A1, r_x is continuous. Use continuity to define r_0, which may be unbounded. □

Proof of Theorem 1. A1 holds because V is continuous. A2.1 holds because V takes value zero on $\mathscr{M}^0$. A2.2 holds because V is strictly increasing in both components, and r_x is decreasing: if $x > y$, then $e^{-r_x t} \ge e^{-r_y t}$ and $V(x, pe^{-r_x t}, 0) > V(y, pe^{-r_y t}, 0)$. A2.3 and A2.4 also follow. As for A3, let $(x, p, t) \in \mathscr{M}^+$, $\Delta > 0$, and $\theta \in (0, 1)$ be such that $(x, p, t + \Delta) \sim (x, p\theta, t)$, or $V(x, pe^{-r_x(t+\Delta)}, 0) = V(x, p\theta e^{-r_x t}, 0)$. Because V is strictly increasing, $pe^{-r_x t}e^{-r_x\Delta} = p\theta e^{-r_x t}$. Hence, $qe^{-r_x s}e^{-r_x\Delta} = q\theta e^{-r_x s}$, $(x, q, s) \in \mathscr{M}^+$, which implies $V(x, qe^{-r_x(s+\Delta)}, 0) = V(x, q\theta e^{-r_x s}, 0)$, or $(x, q, s + \Delta) \sim (x, q\theta, s)$, $(x, q, s) \in \mathscr{M}^+$, as required. A4 follows because $r_y \ge r_x$ and $(x, p, t + \Delta) \sim (x, p\theta, t)$ implies $(y, p, t + \Delta) \sim (y, pe^{-r_y\Delta}, t) \preccurlyeq (y, pe^{-r_x\Delta}, t) = (y, p\theta, t)$.

By A1–A3 and Proposition 3, $V(x, p, t) = V(x, pe^{-r_x t}, 0) = V(x, e^{-(-\ln p + r_x t)}, 0)$, which is a function of x and $-\ln p + r_x t$. Let $(x, p, t + \Delta) \sim (x, p\theta, t)$, or, by A1–A3, $\theta = e^{-r_x\Delta}$. By A4, $(y, p, t + \Delta) \preccurlyeq (y, p\theta, t)$, which implies $\theta \ge e^{-r_y\Delta}$, or $r_y \ge r_x$. □

Proof of Theorem 2. By Theorem 1, A1–A4 hold. The multiplicative separability beween $w(p)$ and $v(x)$ implies A5.

Conversely, let $\tau = -\ln p$, and define $g(x, \tau) = V(x, p, 0)$, $(x, p, 0) \in \mathscr{M}$. A1, A2.2, A2.3, and A5 allow us to apply Fishburn and Rubinstein's (1982) Theorem 3 to conclude that $g(x, \tau) = f(\tau)v(x)$, $(x, e^{-\tau}, 0) \in \mathscr{M}^+$, for some strictly positive and continuous functions $f(\tau)$ and $v(x)$. Moreover, $\hat f(\tau)\hat v(x)$ represent the preference iff $\hat v = \eta v^\kappa$ and $\hat f = \eta' f^\kappa$, $\eta, \eta', \kappa > 0$. Let $w(e^{-\tau}) = f(\tau)$, and let $d(\tau) = -\ln f(\tau)$. We use η' to set $f(0) = w(1) = 1$ and $d(0) = 0$. The uniqueness properties hold because $\hat d = -\ln f^\kappa = \kappa d$, and r_x is uniquely defined by (6). By A2.1 and continuity, $g(x, \tau) = 0$ if either $x = 0$ or $\tau = \infty$, implying $v(0) = 0$, $f(\infty) = w(0) = 0$, and $d(\infty) = \infty$. By A2.2, v is strictly increasing, and hence a value function. By A2.3, f is strictly decreasing, and w and d are strictly increasing. Hence, f is a discount function, w a weighting function, and d a psychological distance function. Thus, $g(x, \tau) = f(\tau)v(x) = w(e^{-\tau})v(x) = e^{-d(\tau)}v(x)$, $(x, e^{-\tau}, 0) \in \mathscr{M}$, which, combined with Theorem 1, produces the desired representation. □

Theorem 3. *Let V be a differentiable function representing preferences on $\mathscr{M}$. A1–A3 and A5 hold iff $V(x, p, t) = e^{-d(-\ln p + r_x t)}v(x)$ for some value function, probability discount rate, and psychological distance function satisfying $\epsilon_v(x) \ge \epsilon_r(x)d^*$, $x > 0$, where $d^* = \sup_{\tau\ge 0} d'(\tau)\tau$. Moreover, if $d^* = \infty$, then A4 holds.*

Proof of Theorem 3. In the proof of Theorems 1 and 2, A4 is only used to set r_x decreasing and to establish A2.2, outcome monotonicity. Hence, suffices to show that $\epsilon_v(x) \ge \epsilon_r(x)d^*$, $x > 0$, is equivalent to outcome monotonicity. Using elementary calculus, one can show that $\partial V(x, p, t)/\partial x \ge 0$, $x > 0$, iff $\epsilon_v(x) \ge \epsilon_r(x)d'(-\ln p + r_x t)r_x t$, $(x, p, t) \in \mathscr{M}^+$. By definition, $d^* \ge d'(r_x t - \ln p)(r_x t - \ln p) \ge d'(r_x t - \ln p)r_x t$, $(x, p, t) \in \mathscr{M}^+$. Hence, if $\epsilon_v(x) \ge \epsilon_r(x)d^*$, then $\epsilon_v(x) \ge \epsilon_r(x)d'(-\ln p + r_x t)r_x t$, (x, p, t), and A2.2 follows. Conversely, if $\epsilon_v(x) < \epsilon_r(x)d^*$ for some $x > 0$, then there is some τ such that $d'(\tau)\tau$ is sufficiently close to d^*, ensuring $\epsilon_v(x) < \epsilon_r(x)d'(\tau)\tau$. This implies $\partial V(x, 1, \tau/r_x)/\partial x < 0$, a contradiction.

If $d^* = \infty$, then A2.2 implies $\epsilon_r \le 0$, and hence $r_x' \le 0$ and A4. If $d^* < \infty$, then monotonicity can be satisfied by some nonconstant and increasing r_x. In Example 1, $v(x) = x^\alpha$, $d(\tau) = \ln(1 + \delta\tau)/\ln(1 + \delta)$, and $r_x = b(x + 1)/(x + 2)$, $b > 0$. We verify that $\epsilon_v = \alpha$, $\epsilon_r(x) = x/((x + 1)(x + 2)) \le 1$, and $d^* = 1/\ln(1 + \delta)$. Hence, if $\delta \ge e^{1/\alpha} - 1$, then $\epsilon_v(x) = \alpha \ge d^* \ge \epsilon_r(x)d^*$, and outcome monotonicity holds. □

Theorem 4. A1–A5 *and proportionality hold if and only if for some value function and decreasing probability discount rate, $V(x, p, t) = pe^{-r_x t}v(x)$, $(x, p, t) \in \mathscr{M}$.*

Proof of Theorem 4. In view of (1), it suffices to prove that proportionality is equivalent to $d(\tau) = \tau$. Proportionality can be written as $d(\tau + \tau') - d(\tau) = d(\tau + \tau' + \tau'') -$

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$d(\tau+\tau'')$, $\tau\geq 0$, $\tau',\tau''>0$. Setting $\tau=0$ yields $d(\tau')+d(\tau'')=d(\tau'+\tau'')$, $\tau,\tau''>0$. The unique solution to Cauchy's equation on the positives is $d(\tau)=c\tau$ (Aczél 1996, §2.1.1). Because d is strictly increasing, $c>0$, and we set $c=1$. Conversely, and using the prospects as in the proof of Proposition 4, if d is linear, then $d(\tau+\tau'+\tau'')-d(\tau+\tau'')=d(\tau+\tau')-d(\tau)=\ln(v(x)/v(y))$, which implies $(y,q,t)\sim(x,q\theta,t)$ and proportionality. □

Theorem 5. *Under* A1–A3 *and* A5, *the following are equivalent*: (i) *proportionality and isoendurance*, (ii) *stationarity, and* (iii) *for some value function*, $v(x)$, *and* $r>0$, $V(x,p,t)=pe^{-rt}v(x)$, $(x,p,t)\in\mathcal{M}$.

Proof of Theorem 5. In view of Theorem 4, that proportionality and isoendurance characterize V is trivial. Fishburn and Rubinstein (1982, Theorem 2) show that A1, A2, A5, and stationarity hold iff $V(x,1,t)=e^{-rt}v(x)$, $(x,1,t)\in\mathcal{M}$, and r_x is constant. A3 then holds iff $V(x,p,t)=pe^{-rt}v(x)$, $(x,p,t)\in\mathcal{M}$. □

Lemma 1. *Under* A1–A3, A6 *is equivalent to* A9, *and* A7 *is equivalent to* A8.

Proof of Lemma 1. A6 implies A9. Assume the premise of A9, $(y,p,t)\sim(x,p,t+\Delta)$. By A1–A3, $(x,p,t+\Delta)\sim(x,pe^{-r_x\Delta},t)$. Hence, $(y,p,t)\sim(x,pe^{-r_x\Delta},t)$, which we will use as the premise of A6. Using A6 and A1–A3, $(y,q,t)\preccurlyeq(x,qe^{-r_x\Delta},t)\sim(x,q,t+\Delta)$, and A9 follows. Adapting this same argument shows the remaining three implications. □

Proof of Proposition 4. In view of Lemma 1 it suffices to show that under A1–A5, A6 holds iff d is concave.

For any choice of $\tau\geq 0$, $\tau',\tau''>0$, let $t=0$, $p=e^{-\tau}$, $\theta=e^{-\tau'}$, and $q=pe^{-\tau''}$. Given $x>0$, find $0<y<x$ such that $(y,p,t)\sim(x,p\theta,t)$, or $d(\tau+\tau')-d(\tau)=\ln(v(x)/v(y))$. If A6 holds, then $(y,q,t)\preccurlyeq(x,q\theta,t)$, $d(\tau+\tau'+\tau'')-d(\tau+\tau'')\leq\ln(v(x)/v(y))$, which implies $d(\tau+\tau'+\tau'')-d(\tau+\tau'')\leq d(\tau+\tau')-d(\tau)$.

Conversely, let $(y,p,t)\sim(x,p\theta,t)$ as in the premise of A6. Let $\tau=\tau_y$, and let $\tau'=\tau_x-\tau_y$. By A1–A5, $d(\tau+\tau')-d(\tau)=\ln(v(x)/v(y))$. Let $\tau''=\ln p/q>0$ so that $\tau+\tau''=r_yt-\ln q$ and $\tau+\tau'+\tau''=r_xt-\ln q\theta$. If d is concave, then $d(\tau+\tau'+\tau'')-d(\tau+\tau'')\leq d(\tau+\tau')-d(\tau)=\ln(v(x)/v(y))$, which implies $(y,q,t)\preccurlyeq(x,q\theta,t)$ and A6. □

Lemma 2. *Under* A1–A4, A6 *implies* A7.

Proof of Lemma 2. Assume the premise of A7, $(y,p,t)\sim(x,p,t+\Delta)$. By A1–A3, $(y,p,t)\sim(x,pe^{-r_x\Delta},t)$, which we will use as the premise of A6. Let $q=pe^{-r_y(s-t)}$. Using A1–A3, A6, A4, and A1–A3, respectively, $(y,p,s)\sim(y,pe^{-r_y(s-t)},t)\preccurlyeq(x,pe^{-r_y(s-t)}e^{-r_x\Delta},t)\preccurlyeq(x,pe^{-r_x(s-t+\Delta)},t)\sim(x,p,s+\Delta)$, and A7 follows. □

Proof of Proposition 5. Given Lemmas 1 and 2, all that remains to be proven is A1–A5, A7, that V is differentiable, and that $\inf_{x\in X}(\epsilon_r/\epsilon_v)=0$ implies d is concave. Assume d is not concave: $d'(\tau+\tau'')>d'(\tau)$, for some $\tau,\tau''>0$. Let $p=e^{-\tau}$ and $s=\tau''/r_x$. For any $x>0$ and $\Delta>0$, find $0<y<x$ such that $(y,p,0)\sim(x,p,\Delta)$, or $d(\tau+r_x\Delta)-d(\tau)=\ln(v(x)/v(y))$. A7 implies $(y,p,s)\preccurlyeq(x,p,s+\Delta)$, or

$$d(\tau+r_x\Delta+\tau'')-d\left(\tau+\tau''\frac{r_y}{r_x}\right)\leq\ln\frac{v(x)}{v(y)}=d(\tau+r_x\Delta)-d(\tau).$$

Under isoendurance, the concavity of d follows. Otherwise, use differentiability and the intermediate value theorem: For some $\hat{\tau}$ between $\tau+\tau''r_y/r_x$ and $\tau+r_x\Delta+\tau''$,

$$d'(\hat{\tau})\left(1-\tau''\frac{r_y-r_x}{r_x^2\Delta}\right)=\frac{d(\tau+r_x\Delta+\tau'')-d(\tau+\tau''r_y/r_x)}{r_x\Delta}\leq\frac{d(\tau+r_x\Delta)-d(\tau)}{r_x\Delta}.$$

We will let $\Delta\to 0$, $y\to x$, and $\hat{\tau}\to\tau+\tau''$. By the implicit function theorem, $\partial y/\partial\Delta=-d'(\tau+r_x\Delta)r_xv(y)/v'(y)$. Combined with l'Hôpital's rule yields

$$\lim_{\Delta\to 0}\frac{r_y-r_x}{r_x^2\Delta}=\lim_{\Delta\to 0}\frac{r_y'\,\partial y/\partial\Delta}{r_x^2}=\lim_{\Delta\to 0}\frac{-r_y'd'(\tau+r_x\Delta)v(y)}{r_xv'(y)}=-d'(\tau)\frac{\epsilon_r(x)}{\epsilon_v(x)}\quad\text{and}$$

$$d'(\tau+\tau'')\left(1+\tau''d'(\tau)\frac{\epsilon_r(x)}{\epsilon_v(x)}\right)\leq d'(\tau),\quad\tau\geq 0,\ \tau''>0.\tag{7}$$

If $\inf_{x\in X}(\epsilon_r(x)/\epsilon_v(x))=0$, we can choose x^* so that $\epsilon_r(x^*)/\epsilon_v(x^*)$, and hence $\tau''d'(\tau)(\epsilon_r(x^*)/\epsilon_v(x^*))$, is arbitrarily small, contradicting $d'(\tau+\tau'')>d'(\tau)$. Hence, d is concave.

The joint condition is tight. Let $\inf(\epsilon_r/\epsilon_v)=-\sigma<0$, and consider the convex function $d(\tau)=(e^{\delta\tau}-1)/(e^{\delta}-1)$, $0<\delta\leq\ln(1+\sigma)$. To see that (7) holds, use $\sigma\geq e^{\delta}-1$, $e^{\delta\tau}\geq 1$, and $1-\delta\tau''\leq e^{-\delta\tau''}$ to yield

$$1+\tau''d'(\tau)\frac{\epsilon_r(x)}{\epsilon_v(x)}\leq 1-\delta\tau''\frac{e^{\delta\tau}\sigma}{e^{\delta}-1}\leq 1-\delta\tau''\leq e^{-\delta\tau''}=\frac{d'(\tau)}{d'(\tau+\tau'')}.$$

In Example 2, $\epsilon_r=-1$, $\epsilon_v=\alpha$, and $\sigma=1/\alpha$. □

Proof of Proposition 6. If $(x,p,t)\sim(xe^{-\rho t},p,0)$, then ρ is the implicit solution to

$$d(-\ln p+r_xt)-d(-\ln p)=\ln\frac{v(x)}{v(xe^{-\rho t})}.$$

Let $y=xe^{-\rho t}$, $\tau_x=-\ln p+r_xt$, and $\tau_y=-\ln p$. By the implicit function theorem,

$$\frac{\partial\rho}{\partial p}=\frac{d'(\tau_y)-d'(\tau_x)}{\epsilon_{v(y)}pt},\tag{8}$$

$$\frac{\partial\rho}{\partial t}=\frac{d'(\tau_x)(\tau_x-\tau_y)-\epsilon_v(y)\ln(x/y)}{\epsilon_v(y)t^2}=\frac{d'(\tau_x)(\tau_x-\tau_y)-[d(\tau_x)-d(\tau_y)]+\int_y^x((\epsilon_v(z)-\epsilon_v(y))/z)\,dz}{\epsilon_v(y)t^2},\tag{9}$$

$$\frac{\partial\rho}{\partial x}=\frac{d'(\tau_x)r_x'tx+\epsilon_v(y)-\epsilon_v(x)}{\epsilon_v(y)tx}.\tag{10}$$

To see (9), note that

$$\int_y^x\frac{\epsilon_v(z)-\epsilon_v(y)}{z}\,dz=\int_y^x\frac{v'(z)}{v(z)}\,dz-\epsilon_v(y)\int_y^x\frac{dz}{z}=\ln\frac{v(x)}{v(y)}-\epsilon_v(y)\ln\frac{x}{y}=d(\tau_x)-d(\tau_y)-\epsilon_v(y)\ln\frac{x}{y}.$$

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In (8), if d is concave, then $d'(\tau_y) \geq d'(\tau_x)$, and $\partial\rho/\partial p \geq 0$. In (9), if d is concave, then $d'(\tau_x)(\tau_x - \tau_y) - [d(\tau_x) - d(\tau_y)] \leq 0$, and if the elasticity of v is decreasing, then $\int_y^x ((\epsilon_v(z) - \epsilon_v(y))/z)\,dz \leq 0$ and $\partial\rho/\partial t \leq 0$. Finally, in (10), if $r_x^y \leq 0$ and $\epsilon_v(y) \leq \epsilon_v(x)$, $y < x$, then $\partial\rho/\partial x \leq 0$. □

Proof of Proposition 7. If $(x, p, t) \sim (px - \pi, 1, t)$, then π is the implicit solution to

$$d(-\ln p + r_x t) - d(r_{px-\pi} t) = \ln \frac{v(x)}{v(px - \pi)}.$$

Let $y = px - \pi$, $\tau_x = -\ln p + r_x t$, and $\tau_y = r_y t$. By the implicit function theorem,

$$\frac{\partial \pi}{\partial t} = \frac{d'(\tau_x) r_x - d'(\tau_y) r_y}{v'(y)/v(y) - d'(\tau_y) r_y' t} \quad \text{and}$$

$$\left.\frac{\partial \pi}{\partial p}\right|_{p=1} = \frac{\epsilon_v(x) - d'(r_x t)(1 + r_x' x t)}{v'(x)/v(x) - d'(\tau_x) r_x' t}.$$

Because $\tau_x \geq \tau_y$, d is concave and r is decreasing, $d'(\tau_y) r_y \geq d'(\tau_x) r_x$. Hence, $\partial\pi/\partial t \leq 0$, with a strict inequality if either d is strictly concave or r is strictly decreasing.

Note that $\partial\pi/\partial p|_{p=1} < 0$ iff $\epsilon_v(x) < d'(r_x t)(1 + r_x' x t)$. Because d is concave, $d(1) = 1$, and v is strictly concave, we have that $xv'(x)/v(x) < 1 \leq d'(0)$. Hence, the inequality holds at $t = 0$. If $r_x' < 0$ and d is concave, then $d'(r_x t)(1 + r_x' x t)$ strictly decreases from $d'(0)$ to past 0. Hence, there is a unique $t_x^* > 0$ for which $\partial\pi/\partial p|_{p=1} = 0$. It follows that $\partial\pi/\partial p|_{p=1} < 0$ iff $t < t_x^*$. □

Proof of Proposition 8. Evaluating column x produces $v(x_k) = w(p_1) v(x_{k-1})$, $k = 1, \ldots, K$. Hence, $v(x_k)/v(x_0) = w(p_1)^k$, $k = 0, \ldots, K$. Evaluating column p produces $v(x_k) = w(p_k) v(x_0)$, $k = 2, \ldots, K$. Hence, $w(p_k) = v(x_k)/v(x_0) = w(p_1)^k$, $k = 0, \ldots, K$. Finally, evaluating column t produces $w(e^{-r_{x_k} t_k}) = w(p_1)$, $k = 0, \ldots, K$. Hence, $r_{x_k} = -\ln p_1 / t_k$, $k = 0, \ldots, K$. □

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