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We consider a stochastic, capacitated production-inventory model in which the customer provides information about the expected timing of future orders to the supplier. We allow for randomness in customer order arrivals as well as the quantity demanded, but work under the assumption that the customer is making every effort to follow the schedule provided. We term this as a target reverting policy. This gives rise to an interesting nonstationary inventory control model at the supplier. After characterizing the optimal policy, we develop solution procedures to compute the optimal parameters. An extensive computational study provides insights into the behavior of this model at optimality. Further, comparing the cost of the optimal policy to the cost of simple policies that either ignore the customer's information or the capacity constraint, we are able to provide insights as to when these simplifications could be costly.
(*Supply Chain; Capacitated Production-Inventory Model; Optimal Policy; Simulation-Based Optimization*)

1. Introduction

The modern manufacturing environment has become more predictable due to implementation of strategies such as demand smoothing, production smoothing, preventive maintenance, quality control, and setup reduction which are all important parts of the Just-In-Time (JIT) manufacturing philosophy. This has had a significant impact on the industrial supplier-customer relationships. Because of the improved predictable nature of their manufacturing, customers are able to provide their suppliers the future expected ordering pattern well ahead of time. Car manufacturers such as Ford and General Motors have such relations with their component manufactures (such as Lear for seats). However, since some variability still exists, the customer may not be able to meet the order pattern exactly. Even if a particular order is either delayed or comes early the customer will make every effort to place the future orders according to the given pattern: we term this as a target reverting policy. A simple target reverting policy is described below.

Consider a situation when the customer tells the supplier that an order will be placed every Monday (i.e., once a week). Suppose an order which was supposed to have come on Monday was in fact delayed and came on Tuesday. A traditional demand model, with an assumption of independence, works on the belief that the next order will on the average arrive on Tuesday since the mean interarrival time is one week. However, under the target reverting policy the next order will arrive on the average at its preplanned time, i.e., Monday. Thus under the target reverting policy the consecutive interarrival times of the demands are no longer independent, but negatively correlated. This is typical when the customer is a manufacturer, as we describe below.

We consider a model in which a single customer is ordering a single product from a supplier who has a

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finite production capacity. The supplier thus has to decide inventory levels that minimize the total holding and penalty costs in the presence of stochastic demands (when they occur), with the timing of the demands generated from the target reverting policy and where the quantity demanded is random. The three supply chain settings that motivated this work are as follows: We are modeling the final stage of a supply process, where capacity has been allocated to each customer (since the products are differentiated across customers) and significant semifinished inventory exists before this stage. At a division of TRW, in the automotive industry, the customer is an American plant of a Japanese manufacturer, following a planned schedule as best as it can, and reveals to TRW the target order dates and a band in which the order quantity is most likely to be. At a division of GE Plastics, the product line consists of Printed Circuit Boards whose industrial customers (such as Zycon) supply electrical subassemblies to automotive OEMs (Original Equipment Manufacturers) such as Ford. After a week long supplier-customer management meeting at GE Corporate School, it was decided that the customers would reveal to GE a target timing of orders, along with past data on order quantity (which was variable because of some yield losses at the customer). In return, GE would provide a better service and a lower price. The third motivation is from Blazer Diamond Products, a supplier of diamond-tipped cutting blades and bits for use in the construction industry. An operational analysis of their supply chain revealed that 70% of their demand arose from 6 major customers in 13 product families (out of 250 active customers and hundreds of products), and that these large customers tended to order (more or less) in a regular pattern.

In the presence of this information about the customer ordering policy, the inventory control problem faced by the supplier is nonstandard. We formulate this problem (in a discrete-time production-inventory framework), establish the structure of the optimal policies in the finite and the infinite horizon settings, and develop solution procedures to compute these policies efficiently. It is well known that many qualitative features of a policy dealing with a nonstationary demand process can be significantly different from those of a stationary policy: See Graves (1998) for example, who notes that under a nonstationary process, the familiar square root rule of setting inventories (based on stationary demand analysis) does not hold. We perform a computational study that reveals insights as to how the average cost per period and the inventory levels are affected by system parameters in this model. We notice qualitative differences between our model and the one studied in Gavirneni et al. (1999) where the nonstationarity is different and motivated by behavior of distributors who follow an (s, S) policy. We are also interested in seeing when these insights are managerially important. Another computational study compares simple policies (that either ignore the customer ordering pattern or the capacity constraint) with the optimal one and reveals when these insights, if ignored, can be costly.

The research in this paper contributes to the growing area that studies nonstationary demand processes and evaluates the benefit of using available information: Scraf (1959), Iglehart (1964), Silver (1978), Azoury (1985), Lovejoy (1990), Zheng and Zipkin (1990), Zipkin (1995), Song and Zipkin (1993, 1996), Hariharan and Zipkin (1995), Chen (1995), Cachon (1996), Aviv and Federgruen (1997), Lee et al. (1998) and Gavirneni et al. (1999) are some examples. We use ideas from Federgruen and Zipkin (1986a, b) (where the optimal policy for a capacitated system facing stationary demand with backlogging is developed), Glasserman and Tayur (1994, 1995) (where Infinitesimal Perturbation Analysis derivatives for determining sensitivities with respect to base stock levels in a stationary environment with backlogging are developed), and Kapuscinski and Tayur (1998) (where the optimal policy for a capacitated system facing periodic demand with backlogging is derived) for our analysis.

Some of our qualitative results are described briefly below: The uncapacitated solution (while not ignoring the target reverting policy) is a good heuristic when the penalty cost is small or the demand quantity variance is high. After accounting for the target reverting policy, the variance of interarrival times has very little effect on the total cost unless the quantity variance is very low. Similarly, the demand quantity variance should be considerably small to realize significant reduction in cost due to increase in capacity. Ignoring the customer ordering pattern is not that costly when the demand variance is high or the capacity is low. However, with respect to penalty cost, the relative cost of ignoring the pattern initially increases and then drops off. Thus, in a high service level situation with low capacity and high variance, we may ignore the insights of the refined model while at moderate to high capacities, modest demand variance and service levels, the benefits of using a sophisticated policy are high. We will provide intuition for these findings in the computational section.

The rest of the paper is organized as follows. In §2, we describe the target reverting process formally and study some of its properties. In §3, we formulate the demand process faced by the supplier and develop optimal inventory control policies for the supplier. Section 4 contains the solution procedure for the capacitated and uncapacitated situations. Section 5 details the insights gained from our computational studies. We conclude in §6.

2. Target Reverting

Our modeling is motivated by the "due-date process"; see Sengupta (1989). We have renamed the process primarily to avoid the obvious misunderstanding with the concept of due dates. Since we will be analyzing a periodic review situation, we use a discrete version. The sequence $\{R_n, n = 0, 1, \ldots\}$ is called a target schedule if R_0 has some known distribution and $(R_{n+1} - R_n)$ for all n are i.i.d. nonnegative random variables with cumulative distribution function G. The sequence of interest to us has $R_0 = 0$ and $(R_{n+1} - R_n)$ a mean of $c \geq 1$ and variance 0. That is, the reference sequence is $\{0, c, 2c, 3c, \ldots\}$ and it represents the points in time at which we expect the customer to order something. This reference sequence is achieved through past experience, supply contracts, or good coordination. Note that this implicitly assumes that the customer is facing a stable and stationary demand.

Despite the extensive advanced planning, we can expect some unforeseen events (at the customer site) to occur that would cause a deviation in the customers ordering process. The actual ordering times will be represented by the stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$. This process is called a target reverting process associated with the reference sequence $\{R_n\}$ if there exists a family of conditional distributions H_{ij} satisfying the property

$$\mathrm{Prob}\{X_{n+1} - X_n = j \mid X_n - R_n = i\} = H_{ij}.$$

The value $(X_n - R_n)$ represents the current deviation from the ordering process. H_{ij} is the probability that the time interval between orders is j given that the last order resulted in a deviation of i. We assume that $H_{ij} = 0$ for $j \leq 0$, so that $\{X_n\}$ is an increasing sequence. We further assume that the time of the next order X_{n+1} depends only on this deviation $(X_n - R_n)$ and is independent of the future due dates. That is $(X_{n+1} - X_n)$ is independent of $(R_{k+1} - R_k)$ for all $k \geq n$. Clearly, $\{X_n\}$ is a Markov chain.

If we define $D_n = X_n - R_n$, then the sequence $\{D_n, n = 0, 1, 2, \ldots\}$, called the Deviation Process, is also a Markov chain. The deviation D_n can take both negative (when the orders are early) and positive (when the orders are late) values. The transition probabilities are

$$\begin{aligned}
\mathrm{Prob}\{D_{n+1} = l \mid D_n = k\} &= \mathrm{Prob}\{X_{n+1} - R_{n+1} \\
&= l \mid X_n - R_n = k\} = \mathrm{Prob}\{X_{n+1} - X_n + R_n - R_{n+1} \\
&= l - k \mid X_n - R_n = k\} = \mathrm{Prob}\{X_{n+1} - X_n \\
&= l - k + (R_{n+1} - R_n) \mid X_n - R_n = k\} \\
&= H_{k(l-k+(R_{n+1}-R_n))}.
\end{aligned}$$

Since $R_{n+1} - R_n = c$, the transition probabilities are

$$\mathrm{Prob}\{D_{n+1} = l \mid D_n = k\} = H_{k(l-k+c)}.$$

Let P_{dk} be the probability that the customer will place an order in the kth period since the last order, given that the previous order resulted in a deviation of d. These probabilities determine the inventory levels at the supplier level. These probabilities can be calculated from the transition probabilities of the deviation process as follows:

$$\begin{aligned}
P_{dk} &= \mathrm{Prob}\{(n + 1)\text{th order comes in } k\text{th period} \mid \\
&\qquad X_n - R_n = d \text{ and no order in the last} \\
&\qquad k - 1 \text{ periods}\} \\
&= \frac{H_{dk}}{\sum_{j=k}^{\infty} H_{dj}}.
\end{aligned}$$

It is easily noted that P_{dk} has the functional form of a hazard rate as described in reliability theory (Barlow and Proschan, 1975).

We have the following assumptions.

Assumption A1. $\lim_{k\to\infty} P_{dk} = 1\ \forall d$. *That means that the customer will eventually place an order sometime in the future. In fact, it is reasonable to expect that the deviation will not be more than one cycle length and so we will assume that the deviation will always lie between* $-c$ *and* c. *Thus, for any deviation* d, $P_{d(2c-d)} = 1$.

Recall that d can be negative, and so when we write "low deviation" below, we mean (abusing the English language) low values of d. For example, $d = -2$ is lower deviation than $d = 1$.

Assumption A2. $P_{d(k+1)} \geq P_{dk}\ \forall d$. *We know that if the probabilities* H_{ij} *have an increasing failure rate, then the probabilities* P_{dk} *are nondecreasing in k (see Kijima 1989). This property implies that, for any deviation, there is a higher chance of an order as more periods go by without a demand.*

Assumption A3. $P_{d_1k} \leq P_{d_2k}\ \forall k$, *if* $d_1 < d_2$. *We should expect that when the deviation is larger, then the probability of an order in the kth period will be higher. This relation between* $\{P_{d_1j}\}$ *and* $\{P_{d_2j}\}$ *holds when the probabilities* $\{H_{d_1j}\}$ *are uniformly smaller than* $\{H_{d_2j}\}$ *(see Kijima 1989).*

Strictly speaking, the next assumption is not required for many of our results in the paper. However, an intuitively appealing property (Corollary 1, which follows from Property 8, in §3.4 below) requires it. Furthermore, the proof of the validity of our computational procedure based on simulation (§4.2 below) is greatly simplified with this assumption.

Assumption A4. $P_{d(k+1)} \leq P_{(d+1)k}$ *for all* d, k. *This condition indicates that at lower deviations, the next demand will occur after a larger number of periods (and so later) and at larger deviations the demands will occur sooner. Note that when this condition is satisfied, the system has a higher tendency for stability.*

3. The Production-Inventory Model

We work in a discrete-time, production-inventory framework. We use (d, k) to represent the state of the system where d is the deviation in the ordering process and it is the kth period since the last demand. In state (d, k) a demand is realized with probability P_{dk}. If a demand is realized, then the next period is in state $(d + k - c, 1)$. If the demand is not realized the state of the next period will be $(d, k + 1)$. The probabilities $\{P_{dk}\}$ satisfy the assumptions of the previous section.

Some of our other assumptions are as follows. The demands that cannot be satisfied from inventory at the supplier are lost and result in a linear penalty cost of b dollars per unit per period. (This assumption is quite common and keeps the analysis clean. A back-logging assumption can be handled directly as long as the backlogs do not affect $\{P_{dk}\}$. Alternately, if this is not reasonable, a larger state-space can be used.) The holding costs are also linear at h dollars per unit per period. Furthermore, there is finite capacity, C units per period, for production at the supplier. There are no lead-times, purchase costs,[1] salvage costs, or salvage values.[2] The random demand from the customer is independent[3] of the state of the system and has a cumulative distribution function $\Psi(.)$ and density function $\psi(.)$. To differentiate between the occurrence and nonoccurrence of demand at the supplier, we will assume that when the demand occurs it is at least $\Xi_l > 0$. That is, $\Psi(\Xi_l) = 0$.

The sequence of events in every period is as follows: 1) The supplier makes the production decision for the period after reviewing the state of the system. 2) The demand from the customer may or may not be realized. 3) Depending on the occurrence or nonoccurrence of the demand, the supplier updates the inventories, pays holding cost on the excess inventory or pays penalty cost on the lost demand. 4) The supplier updates the state of the system and we go to the next period.

[1]A nonzero production or purchase cost per unit can be added. We do not, however, consider fixed cost of purchasing or production. Current inventory theory literature is yet to characterize the optimal policy of a model, for stationary demand, with fixed cost and capacity constraints.

[2]The assumption on salvage basically means that there is no opportunity to get rid of inventory in any period, and in a finite horizon setting, there is no cost for disposing the excess, if any, at the end of the horizon.

[3]State dependent demands can also be accommodated.

As is common in inventory control literature, we first analyze the one period problem, then the finite horizon, and later the infinite horizon problem. We use x to denote the inventory before production and y to denote the inventory after production. The finite capacity restriction implies that $y \in [x, x + C]$. Note that although the results we develop here follow along the lines of Federgruen and Zipkin (1989 a,b) and Kapuscinski and Tayur (1998), they are not implied by them.

3.1. One-Period Problem

This forms the building block for the finite and the infinite horizon cases. Let $J_1(d, k, y)$ be the cost of a 1-period problem in state (d, k) and inventory y. Let $V_1(d, k, x)$ be the optimal cost of a 1-period problem in state (d, k) when the initial inventory is x units. The notation x^+ stands for max(0,x). The objective is to find $y \in [x, x + C]$ such that $V_1(d, k, x) = \min_{y\in[x,x+C]} J_1(d, k, y)$ where $J_1(d, k, y)$ is defined as:

$$J_1(d, k, y) = P_{dk}L(y) + (1 - P_{dk})yh$$

where

$$L(y) = h \int_{-\infty}^{y} (y - \xi)d\Psi(\xi) + b \int_{y}^{\infty} (\xi - y)d\Psi(\xi).$$

As defined in Federgruen and Zipkin (1989a,b), the modified order up-to policy with level z is a policy where if the inventory level is less than z, we raise it to z; if this level cannot be reached, we exhaust the available capacity; if the inventory level is above z, we produce nothing.

Property 1. *For the one period problem in state (d, k), a modified order up-to policy is optimal.*

Proof. Because the holding and penalty costs are linear, $L(y)$ is convex in y. Being a linear combination of convex functions $J_1(d, k, y)$ is convex in y. Taking the derivative of $J_1(d, k, y)$ with respect to y and setting it equal to zero, we get

$$P_{dk}[(h + b)\Psi(y) - b] + (1 - P_{dk})h = 0.$$

That means that there exists an inventory level y^1_{dk}, that satisfies the relation

$$\Psi(y^1_{dk}) = \left(l - \frac{h}{P_{dk}(h + b)}\right)^+,$$

that minimizes the one period cost. Notice that if $P_{dk} = 1$, this condition simplifies to the newsvendor relation. Because the capacity is finite, it is not always possible to reach the inventory level y^1_{dk}. Due to convexity of $J_1(d, k, y)$, the optimal policy is modified order up-to with level y^1_{dk}:

$$y = \begin{cases} x & \text{if } x \geq y^1_{dk}, \\ \min(x + C, y^1_{dk}) & \text{otherwise.} \end{cases} \quad \square$$

3.2. *n*-period Problem

We count down from period n; period zero is the end of the horizon where excess inventory if any can be disposed off at no cost. Let $J_n(d, k, y)$ be the cost of an n-period problem with starting state (d, k) and inventory y. Let $V_n(d, k, x)$ be the minimal n-period cost if the starting state is (d, k) and the initial inventory is x. For the n-period problem, the objective is to find $y \in [x, x + C]$ such that $V_n(d, k, x) = \min_{y\in[x,x+C]} J_n(d, k, y)$ where $J_n(d, k, y)$ is defined as:

$$J_n(d, k, y) = P_{dk}[L(y) + E_\xi[V_{n-1}(d + k - c, 1, (y - \xi)^+)]] + (1 - P_{dk})[yh + V_{n-1}(d, k + 1, y)].$$

Because of our assumption of no salvage cost, we have that $V_0(d, k, y) = 0\ \forall d, k, y$. We use ′,″ to denote first and second derivatives respectively with respect to y.

Property 2. *For all values of n and all states (d, k):*

(i) *$J_n(d, k, y)$ is convex in y;*
(ii) *$V_n(d, k, x)$ is convex in x; and*
(iii) *a modified order up-to policy is optimal.*

Proof. We use induction to prove these properties. Because $V_0(d, k, x) \equiv 0\ \forall x$, we know that $V_0(d, k, x)$ is convex in x and $V_0'(d, k, x) \geq -b$. $J_0(d, k, y)$ is convex in y. Assume that for all states (d, k), $J_{n-1}(d, k, y)$ is convex in y, $V_{n-1}(d, k, x)$ is convex in x, and $V_{n-1}'(d, k, x) \geq -b\ \forall x$. Notice that $J_n(d, k, y)$ is

$$\begin{aligned} J_n(d, k, y) = {} & P_{dk}L(y) + P_{dk} \int_0^y V_{n-1} \\ & (d + k - c, 1, y - \xi)\psi(\xi)d\xi \\ & + P_{dk} \int_y^\infty V_{n-1}(d + k - c, 1, 0)\psi(\xi)d\xi \\ & + (1 - P_{dk})[yh + V_{n-1}(d, k + 1, y)]. \end{aligned}$$

Using Leibniz's rule, the derivatives of these cost functions, $V'_n(d, k, x)$ and $J'_n(d, k, y)$, are:

$$J'_n(d, k, y) = P_{dk}L'(y) + P_{dk}\int_0^y V'_{n-1}(d + k - c, 1, y - \xi)\psi(\xi)d\xi + (1 - P_{dk})[h + V'_{n-1}(d, k + 1, y)]$$

$$\geq P_{dk}(h + b)\Psi(y) - P_{dk}b - P_{dk}b\Psi(y) + (1 - P_{dk})[h - b] \geq -b;$$

$$J''_n(d, k, y) = P_{dk}L''(y) + P_{dk}\int_0^y V''_{n-1}(d + k - c, 1, y - \xi)\psi(\xi)d\xi + P_{dk}V'_{n-1}(d + k - c, 1, 0)\psi(y) + (1 - P_{dk})V''_{n-1}(d, k + 1, y) \geq 0;$$

$$V'_n(d, k, x) = \max\{J'_n(d, k, x), \min\{0, J'_n(d, k, x + C)\}\}.$$

The nonnegativity of $J''_n(d, k, y)$ follows from the convexity of $V_{n-1}(d, k, x)$ and the fact that

$$P_{dk}[L''(y) + V'_{n-1}(d + k - c, 1, 0)\psi(y)] \geq P_{dk}[(h + b)\psi(y) - b\psi(y)] \geq 0.$$

Therefore $J_n(d, k, y)$ is convex in y. If $f(t)$ is convex then $g(s) = \min_{t\in]s,s+C]} f(t)$ is convex, and $V_n(d, k, x)$ is also convex. Because $V'_n(d, k, x) \geq J'_n(d, k, x)$, we have $V'_n(d, k, x) \geq -b$. Since $J_n(d, k, y)$ is convex in y, it is optimal to order up-to y^n_{dk} that minimizes $J_n(d, k, y)$. If we do not have enough capacity to reach y^n_{dk} it is optimal to go as close to y^n_{dk} as possible. □

It is natural to expect that if the capacity is lower, the optimal order up-to levels will be higher. This is proved next.

Property 3. *Let* ${}^{C_1}J_n(.)$, ${}^{C_1}V_n(.)$, ${}^{C_1}y^n_{dk}$, *and* ${}^{C_2}J_n(.)$, ${}^{C_2}V_n(.)$, ${}^{C_2}y^n_{dk}$ *be the quantities defined above when the capacities are* C_1 *and* C_2 *respectively. If* $C_2 \leq C_1$ *then*

(i) ${}^{C_1}J'_n(d, k, x) \geq {}^{C_2}J'_n(d, k, x)$;
(ii) ${}^{C_1}V'_n(d, k, x) \geq {}^{C_2}V'_n(d, k, x)$; *and*
(iii) ${}^{C_1}y^n_{dk} \leq {}^{C_2}y^n_{dk}$.

Proof. We first prove (i) and (ii) by induction; these proofs are used to prove (iii). They are obviously true for $n = 0$. Assume they are true for $n - 1$. After comparing ${}^{C_1}J'_n(d, k, x)$ and ${}^{C_2}J'_n(d, k, x)$, and using (ii) for $n - 1$, it is easily established that ${}^{C_1}J'_n(d, k, x) \geq {}^{C_2}J'_n(d, k, x)$. This, along with convexity of J_n, gives ${}^{C_1}J'_n(x + C_1) \geq {}^{C_1}J'_n(x + C_2) \geq {}^{C_2}J'_n((x + C_2)$. Using the expression for $V'_n(d, k, x)$ from the proof of Property 2, and the observation that if $A \geq B$ then $\min(0, A) \geq \min(0, B)$ and $\max(0, A) \geq \max(0, B)$, it is easily established that ${}^{C_1}V'_n(d, k, x) \geq {}^{C_2}V'_n(d, k, x)$. So, by induction, parts (i) and (ii) of the property are true for all n. Because both ${}^{C_1}V_n(d, k, x)$ and ${}^{C_2}V_n(d, k, x)$ are convex and ${}^{C_1}V'_n(d, k, x) \geq {}^{C_2}V'_n(d, k, x)$, we have ${}^{C_1}y^n_{dk} \leq {}^{C_2}y^n_{dk}$. This proves part (iii), which implies that at lower capacities the order up-to levels will be higher. □

The next result shows that the optimal levels are always finite, independent of the number of periods. This result is crucial for showing the optimality of the policy in the infinite horizon average cost setting that is discussed later.

Property 4. $\limsup\{y^n_{dk}, n \in N\} < \infty$ *as* $n \to \infty$.

Proof. Recall that ${}^0y^n_{dk}$ is the optimal order up-to level when the capacity is zero. We establish this result by first showing that $\limsup\{{}^0y^n_{dk}, n \in N\} < \infty$ as $n \to \infty$. We prove this by contradiction. Assume that there exists a sequence $\{n_t\}$ such that ${}^0y^{n_t}_{dk} \to \infty$ as $n_t \to \infty$ and ${}^0y^n_{dk} < {}^0y^{n_t}_{dk}$ for all $n < n_t$. Let us choose a particular n_t and compare the cost of the optimal policy to the cost of a policy that is order up-to zero and starts with zero inventory. The cost of these policies is different only for a few periods early in the horizon when the inventory at the beginning of a period is positive. As soon as the inventory at the beginning of a period reaches zero, both the policies operate the same way, resulting in equal costs. Let $\xi_1, \xi_2, \ldots, \xi_{l_d}$ be the demands observed when the inventory is positive. We assume that these demands occurred in consecutive periods. Notice that this assumption results in lower holding costs than when the demands are not in consecutive periods. Thus, the resulting cost will be a lower bound on the actual cost of this policy.

The demands $\xi_1, \xi_2, \ldots, \xi_{l_d}$ are i.i.d. and let μ be their mean. These demands satisfy the condition $\Sigma_{j=1}^{l_d-1}\xi_j < {}^0y^{n_t}_{dk}$ and $\Sigma_{j=1}^{l_d}\xi_j \geq {}^0y^{n_t}_{dk}$. This implies that

$${}^0y^{n_t}_{dk} + \xi_{l_d} > \sum_{j=1}^{l_d} \xi_j \geq {}^0y^{n_t}_{dk}.$$

Taking expectations, we get

$${}^0y^{n_t}_{dk} + \mu > E\left[\sum_{j=1}^{l_d}\xi_j\right] \geq {}^0y^{n_t}_{dk}.$$

Because $\xi_1, \xi_2, \ldots, \xi_{l_d}$ are i.i.d. and l_d is a stopping time, from Wald's equality, we have $E[\Sigma_{j=1}^{l_d}\xi_j] = \mu E[l_d]$. Therefore

$$\frac{{}^0y_{dk}^{n_t}}{\mu} + 1 > E[l_d] \geq \frac{{}^0y_{dk}^{n_t}}{\mu}.$$

The total inventory over these l_d periods is

$$\begin{aligned}&({}^0y_{dk}^{n_t} - \xi_1) + ({}^0y_{dk}^{n_t} - \xi_1 - \xi_2) + \cdots + \\ &({}^0y_{dk}^{n_t} - \xi_1 - \xi_2 - \cdots - \xi_{l_d-1}) \\ &= {}^0y_{dk}^{n_t}(l_d - 1) - \xi_1(l_d - 1) - \cdots - \\ &\xi_{j0}(l_d - j_0) - \cdots - \xi_{l_d-1}.\end{aligned}$$

The expected total inventory is

$${}^0y_{dk}^{n_t}E(l_d - 1) - \mu E[(l_d - 1) + (l_d - 2) + \cdots + 1].$$

Rewriting the terms in the expectation, the total expected inventory is

$${}^0y_{dk}^{n_t}[E(l_d) - 1] - \frac{1}{2}\mu E(l_d)[E(l_d) - 1].$$

The total expected inventory cost over these l_d periods is greater than or equal to

$$\begin{aligned}&h\ {}^0y_{dk}^{n_t}E(l_d) - h\frac{1}{2}\mu E(l_d)E(l_d - 1) \\ &\geq h\ {}^0y_{dk}^{n_t}\frac{{}^0y_{dk}^{n_t}}{\mu} - h\frac{1}{2}\mu\left(\frac{{}^0y_{dk}^{n_t}}{\mu} + 1\right)\frac{{}^0y_{dk}^{n_t}}{\mu} \\ &= \frac{h}{2\mu}({}^0y_{dk}^{n_t})^2 - \frac{1}{2}h\ {}^0y_{dk}^{n_t}.\end{aligned}$$

The expected cost of order up-to zero policy with zero initial inventory is at most $b({}^0y_{dk}^{n_t} + \mu)$. The difference in the cost of the optimal policy and the cost of the order up-to zero policy is greater than or equal to

$$\frac{h}{2\mu}({}^0y_{n_t}^{n_t})^2 - \frac{1}{2}h\ {}^0y_{dk}^{n_t} - b\ {}^0y_{dk}^{n_t} - b\mu$$

which is quadratic and is positive for large values of ${}^0y_{dk}^{n_t}$, thus contradicting our assumption of optimality. This proves that $\lim\sup\{{}^0y_{dk}^n\}$ cannot go to infinity as n goes to infinity. Therefore $\lim\sup\{{}^0y_{dk}^n\} < \infty$. Since the sequence $\{y_{dk}^n\}$ is dominated from above by the sequence $\{{}^0y_{dk}^n\}$ (Property 3), we obtain $\lim\sup\{y_{dk}^n\} < \infty$. □

3.3. Infinite Horizon

We consider both the discounted cost criterion as well as the average cost criterion.

3.3.1. Discounted Cost. Under this criterion, the costs in the future are discounted by a factor $0 < \beta < 1$. The finite horizon recursive relation for $V_n(d, k, x)$ is modified as below to include β:

$$\begin{aligned}V_n(d, k, x) &= \min_{y\in[x,x+C]} J_n(d, k, y), \\ J_n(d, k, y) &= P_{dk}[L(y) + \beta E_\xi[V_{n-1}(d + k - c, 1, \\ &\quad (y - \xi)^+)]] + (1 - P_{dk}) \\ &\quad [yh + \beta V_{n-1}(d, k + 1, y)].\end{aligned}$$

Let us define $V_*(d, k, x) = \lim_{n\to\infty} V_n(d, k, x)$; the objective is to minimize $V_*(d, k, x)$.

PROPERTY 5. *For any finite x, $V_*(d, k, x)$ is finite.*

PROOF. Because the single period cost $L(y)$ is nonnegative for all y, $V_n(d, k, x)$ is nondecreasing in n. Consider the policy that is order up-to zero. For this policy, the holding cost in any period is less than hx and the penalty cost in any period is less than $b\xi$. The total expected cost of this policy is bounded above by

$$\sum_{i=0}^{\infty}\beta^i(hx + b\mu) = \frac{hx + b\mu}{1 - \beta}$$

which is finite. Because $\{V_n(d, k, x)\}$ is a nondecreasing sequence with a finite upper bound, $\lim_{n\to\infty} V_n(d, k, x)$ exists and is finite. Therefore, the expected cost of the optimal policy, if it exists, must be finite. □

PROPERTY 6. *For each state (d, k), the modified order up-to policy with level y_{dk}^* is optimal.*

PROOF. Because $V_*(d, k, x)$ is finite, from pp. 210–212 in Bertsekas (1988), the optimal policy exists and satisfies the following recursion:

$$\begin{aligned}V_*(d, k, x) = \min_{y\in[x,x+C]} & P_{dk}[L(y) + \beta E_\xi \\ &[V_*(d + k - c, 1(y - \xi_i)^+)]] \\ &+ (1 - P_{dk})[yh + \beta V_*(d, k + 1, y)]. \quad (1)\end{aligned}$$

Because $L(y)$ and yh are continuous, $V_*(d, k, x)$ is convex in x as it is a pointwise limit of convex functions. Therefore the right-hand side of equation (1) is convex and there exists $y_{dk}^* \in [0, \infty]$ that minimizes it. As $y_{dk}^* =$

$\lim_{n\to\infty} y^n_{dk}$ and because $\limsup_{n\to\infty} \{y^n_{dk}$ is finite (Property 4), y^*_{dk} must be finite. □

3.3.2. Average Cost. Here the objective is to minimize $\hat{V}_*(d, k, x = \lim_{n\to\infty} 1/n\, V_n(d, k, x)$. To analyze this situation we use the discrete version of the problem (i.e., the demand is discrete).[4] This implies that the possible inventory levels are also discrete. Furthermore, since we know that $\limsup_{n\to\infty}\{y^n_{dk}\}$ is finite, the possible inventory levels are $\{0, 1, \ldots, A\}$ where $A = \max_{(d,k)}\{\limsup_{n\to\infty}\{y^n_{dk}\}\}$. The complete state space is $\Gamma = \{(-c, 1), (-c, 2), \ldots, (c, 3c - 1), (c, 3c)\} \times \{0, 1, \ldots, A\}$, which is finite.

PROPERTY 7. *For each state (d, k), }the modified order up-to policy with level $\hat{y}^*_{dk}$ is optimal.*

PROOF. Because the state space is finite, the result follows from Bertsekas (1988, pp. 310–313). □

3.4. Structural Properties

In this section we establish structural properties of the optimal order up-to levels. These properties will be useful in developing effective solution procedures to compute the optimal order up-to levels. Recall that, by Assumptions (A2)–(A4), $P_{dk} \le P_{d(k+1)}$, $P_{dk} \le P_{(d+1)k}$, and $P_{d(k+1)} \le P_{(d+1)k}$. Because the distribution of the demand (if it occurs) does not change from one state to another, we can expect the order up-to levels to satisfy $y^*_{dk} \le y^*_{d(k+1)}$ and $y^*_{dk} \le y^*_{(d+1)k}$. This result follows from Property 8 described below.

PROPERTY 8. *For all inventory levels y, and every state (d, k) the following properties hold:*

(i) $V'_n(d, k, y) - V'_{n-1}(d, k, y) \le h$;
(ii) $J'_n(d, k, y) \ge J'_n(d, k + 1, y)$; *and*
(iii) $J'_n(d, k, y) \ge J'_n(d + 1, k, y)$.

PROOF. We prove this by induction on n. Notice that $J'_0(d, k, y) = 0$ and $J'_1(d, k, y) = P_{dk}L'(y) + (1 - P_{dk})h$ for all values of d, k, y. Therefore, properties (i)–(iii) hold for the case $n = 1$. Assume they hold for all values less than n. We establish that they will hold for n also and thus by induction they hold for all n.

[4]Recent results of Beyer and Sethi (1997) (who consider an uncapacitated model with fixed costs) can be used to show that the optimality result continues to hold when the demands are continuous. However, the technical machinery is different, and quite involved. We provide the proof for the discrete demand case here.

First we compare $J'_n(d, k, y)$ and $J'_{n-1}(d, k, y)$ as defined below:

$$J'_n(d, k, y) = P_{dk}[L'(y) + E_\xi[V'_{n-1}(d + k - c, 1, (y - \xi)^+)]] + (1 - P_{dk})[h + V'_{n-1}(d, k + 1, y)]; \text{ and}$$

$$J'_{n-1}(d, k, y) = P_{dk}[L'(y) + E_\xi[V'_{n-2}(d + k - c, 1, (y - \xi)^+)]] + (1 - P_{dk})[h + V'_{n-2}(d, k + 1, y)].$$

Because property (i) holds for $n - 1$, it must follow that $J_n(d, k, y) - J'_{n-1}(d, k, y) \le h$. Consider the following possibilities for $V'_n(d, k, y)$:

1. $V'_n(d, k, y) = J'_n(d, k, y)$. Since $V'_{n-1}(d, k, y) \ge J'_{n-1}(d, k, y)$, we obtain $V'_n(d, k, y) - V'_{n-1}(d, k, y) \le J'_n(d, k, y) - J'_{n-1}(d, k, y) \le h$.

2. $V'_n(d, k, y) \le 0$. If $V'_{n-1}(d, k, y)$ is either equal to $J'_{n-1}(d, k, y)$ or 0, then $V'_{n-1}(d, k, y) \ge 0$ implying that $V'_n(d, k, y) - V'_{n-1}(d, k, y) \le 0 \le h$. If $V'_{n-1}(d, k, y) = J'_{n-1}(d, k, y + C)$, then $V'_n(d, k, y) - V'_{n-1}(d, k, y) \le J'_n(d, k, y + C) - J'_{n-1}(d, k, y + C) \le h$.

This allows us to say that $V'_n(d, k, y) - V'_{n-1}(d, k, y) \le h$, thus establishing property (i) for n. Since $J'_n(d, k, y) \le V'_n(d, k, y)$, we get that $J'_n(d, k, y) \le h + V'_{n-1}(d, k, y)$, which we need for establishing properties (ii) and (iii).

Compare $J'_n(d, k, y)$ and $J'_n(d, k + 1, y)$ as defined below:

$$J'_n(d, k, y) = P_{dk}[L'(y) + E_\xi[V'_{n-1}(d + k - c, 1, (y - \xi)^+)]] + (1 - P_{dk})[h + V'_{n-1}(d, k + 1, y)]; \text{ and}$$

$$J'_n(d, k + 1, y) = P_{d(k+1)}[L'(y) + E_\xi[V'_{n-1}(d + k + 1 - c, 1, (y - \xi)^+)]] + (1 - P_{d(k+1)})[h + V'_{n-1}(d, k + 2, y)].$$

Using property (ii) for $n - 1$, we have $J'_{n-1}(d, k + 1, y) \ge J'_{n-1}(d, k + 2, y)$, and by considering the following cases we obtain the relation $V'_{n-1}(d, k + 1, y) \ge V'_{n-1}(d, k + 2, y)$.

1. $V'_{n-1}(d, k + 1, y) = J'_{n-1}(d, k + 1, y)$. If $V'_{n-1}(d, k + 2, y) = J'_{n-1}(d, k + 2, y)$ then $V'_{n-1}(d, k + 1, y) \ge V'_{n-1}(d, k + 2, y)$. On the other hand if $V'_{n-1}(d, k + 2, y) \le 0$, then because $V'_{n-1}(d, k + 1, y) \ge 0$, we have $V'_{n-1}(d, k + 1, y) \ge V'_{n-1}(d, k + 2, y)$.

2. $V'_{n-1}(d, k + 1, y) = 0$. In this case $J'_{n-1}(d, k + 1, y) \le 0$ which implies that $J'_{n-1}(d, k + 2, y) \le 0$. Thus

$V'_{n-1}(d, k + 2, y) \leq 0$ implying that $V'_{n-1}(d, k + 1, y) V'_{n-1}(d, k + 2, y)$.

3. $V'_{n-1}(d, k + 1, y) = J'_{n-1}(d, k + 1, y + C)$. In this case $J'_{n-1}(d, k + 1, y + C) \leq 0$ which implies that $J'_{n-1}(d, k + 2, y + C) \leq 0$. Thus $V'_{n-1}(d, k + 2, y) = J'_{n-1}(d, k + 2, y + C)$ implying that $V'_{n-1}(d, k + 1, y) \geq V'_{n-1}(d, k + 2, y)$.

Using property (iii) for $n - 1$, it follows that $J'_{n-1}(d + k - c, 1, (y - \xi)^+) \geq J'_{n-1}(d + k - c + 1, 1, (y - \xi)^+)$, and using similar arguments as above we establish that $V'_{n-1}(d + k - c, 1, (y - \xi)^+) \geq V'_{n-1}(d + k - c + 1, 1, (y - \xi)^+)$.

Because $P_{dk} \leq P_{d(k+1)}$, if $L'(y) + E_\xi[V'_{n-1}(d + k - c, 1, (y - \xi)^+)] \leq h + V'_{n-1}(d, k + 1, y)$, it follows that $J'_n(d, k, y) \geq J'_n(d, k + 1, y)$. However, if $L'(y) + E_\xi[V'_{n-1}(d + k - c, 1, (y - \xi)^+)] > h + V'_{n-1}(d, k + 1, y)$, since $h + V'_{n-1}(d, k + 1, y) \geq J'_n(d, k + 1, y)$, it follows that $J'_n(d, k, y) \geq J'_n(d, k + 1, y)$, thus establishing property (ii) for n.

Comparing $J'_n(d, k, y)$ with $J'_n(d + 1, k, y)$ where

$$J'_n(d, k, y) = P_{dk}[L'(y) + E_\xi[V'_{n-1}(d + k - c, 1, (y - \xi)^+)]] + (1 - P_{dk})[h + V'_{n-1}(d, k + 1, y)]; \text{ and}$$

$$J'_n(d + 1, k, y) = P_{(d+1)k}[L'(y) + E_\xi[V'_{n-1}(d + 1 + k - c, 1, (y - \xi)^+)] + (1 - P_{(d+1)k})[h + V'_{n-1}(d + 1, k + 1, y)],$$

and starting with $J'_n(d, 2c - d, y) = L'(y) + E_\xi[V'_{n-1}(c, 1, (y - \xi)^+)]$ and $J'_n(d + 1, 2c - d - 1, y) = L'(y) + E_\xi[V'_{n-1}(c, 1, (y - \xi)^+)]$, using reverse induction on k, the assumption that $P_{d(k+1)} \leq P_{(d+1)k}$, and arguments similar to the ones presented above, we can establish property (iii) for all values of d and k. □

COROLLARY 1. *For all values of d and k, $y^*_{dk} \leq y^*_{d(k+1)}$ and $y^*_{dk} \leq y^*_{(d+1)k}$.*

This monotonicity result is reminiscent of those in Soug and Zipkin (1993) where uncapacitated models are studied.

4. Solution Methods

In this section, we develop solution procedures for both the capacitated and the uncapacitated situations. To find the optimal solution in closed form in a capacitated setting is difficult even for the stationary newsboy problem in a general setting, and simple nonstationary cases are analytically intractable. However, for the uncapacitated case, easy solution methods have been presented by Karlin (1960) and Zipkin (1989) for the nonstationary problem with periodic data. In the first part of this section, we present a simple procedure to estimate the optimal order up-to levels for the uncapacitated case and later provide a method for the capacitated case.

4.1. Uncapacitated System, $C = \infty$

We know from Equation (1) (proof of Property 6) that the derivative of the infinite horizon cost will follow the recursive relation

$$V'_*(d, k, y) = P_{dk}[L'(y) + E_\xi[V'_*(d + k - c, 1(y - \xi)^+)]] + (1 - P_{dk})[h + V'_*(d, k + 1, y)].$$

The procedure will recursively estimate the derivative of the infinite horizon cost function for all values of x in all states (d, k).

Let $\pi_j = \text{Prob}\{\xi = j\}$ be the discretized demand distribution. Recall that when a demand occurs, it is at least Ξ_l. The algorithm to estimate the derivatives is described below.

Step 1. Set $V'_*(d, k, 0) = 0$ for all states (d, k).
Step 2. For x starting at 1 and increasing by 1:
For d starting at c and decreasing by 1 to $-c$:
compute

$$V'_*(d, 2c - d, x) = L'(x) + \sum_{j=1}^{x} V'_*(c, 1, x - j)\pi_j; \quad V'_*(d, k, x) = P_{dk}L'(x) + P_{dk} \sum_{j=\Xi_l}^{x} V'_*(d + k - c, 1, x - j)\pi_j + (1 - P_{dk})[h + V'_*(d, k + 1, x)],$$
$$k \in [2c - d - 1, 1].$$

At any point during the recursion, if $V'_*(d, k, x)$ is negative, we set it equal to zero, indicating that x is below the optimal solution and the future cost does not depend on x. The recursion can be stopped when

$V'_*(c, c, x)$ becomes positive. However, if the derivatives at higher inventory levels are needed for additional analysis, the recursions can be continued until x reaches a predefined limit.

Step 3. For each state (d, k), find the optimal order up-to level $y^*_{dk} = \inf\{x \mid V'_*(d, k, x) > 0\}$.

The recursion procedure has complexity $O(R^2 c^2)$, where R is the newsvendor solution $\Psi^{-1}(b/b + h)$, and c is cycle time. In the experiments the average computation time for this procedure was 0.73 seconds (0.43 for uniform distribution, 0.70 for normal distribution and 1.07 for exponential distribution). All the times are on a SGI Onyx 2 workstation (comparable to SPARC2 machine).

4.2. Capacitated System

For the capacitated situation, we develop a solution procedure using Infinitesimal Perturbation Analysis (IPA). To use standard nonlinear gradient methods for optimization, we need to find the derivative of expected cost with respect to the order up-to levels. Instead, using simulation we are able to find the expected value of the derivative of cost by using sample path recursions, differentiating them and averaging across many periods. We can prove that the latter equals the former. Now, a simple gradient search provides us with optimal parameters because we are dealing with a convex optimization problem.

Let C_n, Y_n, I_n, ξ_n, d_n, k_n be the cost at the end of period n (sum of holding and penalty costs), inventory level after (any) production, inventory level after demand, demand, deviation, and the number of periods since last demand in period n, respectively. If y_{dk} is the order up-to level in state (d, k), the simulation recursions are:

$$
\begin{aligned}
d_{n+1} &= d_n \quad \text{if there is no demand,} \\
&= d_n + k_n - c \quad \text{if there is a demand;} \\
k_{n+1} &= k_n + 1 \quad \text{if there is no demand,} \\
&= 1 \quad \text{if there is a demand;} \\
Y_{n+1} &= I_n^+ + \min\{C, [y_{d_{n+1}k_{n+1}} - I_n^+]^+\}; \\
I_{n+1} &= Y_{n+1} - \xi_{n+1}; \\
C_{n+1} &= hI_{n+1}^+ + bI_{n+1}^-.
\end{aligned}
$$

The derivative recursions are

$$
\begin{aligned}
\frac{dI_{n+1}}{dy_{dk}} &= \frac{dI_n}{dy_{dk}} \quad \text{if } Y_{n+1} < y_{d_{n+1}k_{n+1}} \text{ or} \\
&\qquad Y_{n+1} > y_{d_{n+1}k_{n+1}}, \\
&= 0 \quad \text{if } Y_{n+1} = y_{d_{n+1}k_{n+1}}, \text{ and } (d, k) \\
&\qquad \neq (d_{n+1}, k_{n+1}), \\
&\qquad = 1 \quad \text{if } Y_{n+1} = y_{d_{n+1}k_{n+1}} \text{ and } (d, k) \\
&\qquad = (d_{n+1}, k_{n+1}); \\
\frac{dC_{n+1}}{dy_{dk}} &= hI'_{n+1} \quad \text{if } I_{n+1} \geq 0, \\
&= -bI'_{n+1} \quad \text{if } I_{n+1} < 0.
\end{aligned}
$$

To validate the derivatives for the finite horizon and the infinite horizon discounted, the approach is similar to Glasserman and Tayur (1994, 1995). To show that the IPA derivatives are valid for the infinite horizon average cost case, using the approach similar to Kapuscinski and Tayur (1998), we show that each of the order up-to levels $\{y_{dk}\}$ are reached an infinite number of times. To show this we assume that the demand, when it occurs, is continuous above Ξ_l and there exists a large value $D^u > y_{cc}$ such that $\Psi(y_{cc}) < 1$. When we say $\{y_{dk}\}$ is an order up-to policy, we assume that $y_{d0} = 0$, $y_{di} \leq y_{d(i+1)}\ \forall i < 2c - d$.

Definition. y_{dk} communicates with $y_{d'k'}$ if we can reach inventory level $y_{d'k'}$ in state (d', k') while starting with inventory y_{dk} in state (d, k).

Definition. An order up-to policy $\{y_{dk}\}$ is regenerative if for all pairs $\{(d, k), (d', k')\}$, and y_{dk} communicates with $y_{d'k'}$.

Property 9. *An order up-to policy $\{y_{dk}\}$ is regenerative if and only if for every state with $k < 2c - d$, $y_{d(k+1)} \leq \max\{y_{dk} + C, [y_{cd} - \Xi_l]^+ + C\}$ where $y_{d0} = 0$ as defined above.*

Proof. Consider a state $(d, k + 1)$. In that state, to reach its order up-to level there exist two possibilities. We are at the corresponding order up-to level in state (d, k), there is no demand, and the capacity is sufficient to reach the order up-to level in state $(d, k + 1)$. That is, $y_{d(k+1)} \leq y_{dk} + C$. However, if the capacity is not sufficient, then it must be that the inventory in state $(d,$

k) is above the corresponding order up-to level and from that higher inventory level the capacity is sufficient to reach $y_{d(k+1)}$. This can happen when a demand occurs in state (c, d) resulting in state (d, 1) and since Ξ_l is the smallest possible demand, then it must be that for regeneration we must have $y_{d(k+1)} \leq [y_{cd} - \Xi_l]^+ + C$. Putting these two conditions together, we obtain $y_{d(k+1)} \leq \max\{y_{dk} + C, [y_{cd} - \Xi_l]^+ + C\}$. □

These properties, along with the reasoning in Glasserman and Tayur (1994) and Kapuscinski and Tayur (1998) establish that the IPA procedure is valid. The average computational times across our experiments described below, were 41.5 seconds for uniform distribution, 78.1 seconds for normal distribution, and 114.7 seconds for exponential distribution. All the times are on a SGI Onyx 2 workstation.

5. Computational Results

In this section we present the computational results found by implementing both solution procedures described in the previous section. We want to understand how this inventory control problem behaves under different capacity restrictions, penalty costs, and demand distributions. The following parameters were used in our study. The cycle time c was set equal to 5. The deviations were allowed to range from -2 to $+2$. The probability matrix $[P_{dk}, d \in \{-2, -1, 0, 1, 2\}, k \in \{1, 2, \ldots, 9\}]$ was

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 1 \\ 0 & 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 1 & 1 \\ 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 1 & 1 & 1 \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Notice that these probabilities satisfy the monotonicity assumptions of § 3.1. Further, under this probability transition matrix the average deviation will be zero because for every deviation d, the next order will on the average arrive at time $c - d$ resulting in an average deviation of zero. The holding cost in all our experiments was set equal to 1. Since an order arrives once in five periods and we assumed that the average demand per period was 20 units, the demand distribution that we used (Exp(100), Uniform [0,200], and Nor(100,30))[5] were set to have mean 100. The production capacity in a period was set to range from 25 to 45 in increments of 5 units.

To understand the intuition behind our findings below, it may help to think of total cost T of any cycle between two consecutive demands, as consisting of two components: (1) a cost, P, of holding material for those periods where a demand does not occur, and (2) a cost L consisting of holding or penalty in the period when demand does occur. Therefore, if we had infinite capacity and no variance in demand quantity nor any deviation from target, we would have $T = P + L = 0 + 0 = 0$. Similarly, if we had infinite capacity, no deviation from the target but demand variance in quantity, we would have $T = P + L = 0 + L = L$. As a final illustration, if we had finite capacity, no quantity or timing variance, we would have $T = P + L = P + 0 = P$ as we may have to produce ahead of the time. Clearly, the higher the variance of demand quantity, the higher is L; the lower the capacity per period, the higher is P. And, for a given capacity, the higher the variance of demand, the higher are both P (as we need to produce more, and perhaps start production earlier) and L components.

5.1. Effectiveness of the Uncapacitated Solution

First we study the effectiveness of using the uncapacitated solution (as it is easier to compute) as a heuristic for the capacitated problem under two different penalty costs, 5 and 10. (We do not ignore the target reverting process.) Recall that the uncapacitated solution plans for lower up-to levels (Property 3). Furthermore, to be able to reach these inventory levels under finite capacity we should plan for it well ahead of time, which the uncapacitated solution does not plan for. The question is: When is this costly?

The resulting costs for these cases are given in Tables 1 and 2. The columns labeled OCost contain the optimal costs and the columns labeled UCost contain the costs when the uncapacitated solution is used under finite capacity. Columns labeled % Diff. in Tables 1 and 2 contain the percentage difference between the optimal cost and the cost of using the uncapacitated solution in the presence of finite capacity.

[5]The standard deviations are 100, 57.3 and 30, respectively.

Table 1 Performance of the Uncapacitated Solution as a Heuristic for the Capacitated Problem When $b = 5$

	Uniform(0,200)			Exp(100)			Nor(100,30)		
Cap.	OCost	UCost	% Diff.	OCost	UCost	% Diff.	OCost	UCost	% Diff.
25	69.43	69.62	0.27	80.87	80.94	0.00	55.94	55.94	0.01
30	69.20	69.23	0.05	80.80	80.82	0.02	54.95	54.95	0.01
35	69.17	69.18	0.01	80.79	80.50	0.00	54.56	54.57	0.02
40	69.17	69.18	0.01	80.79	80.80	0.00	54.38	54.40	0.03
45	69.17	69.18	0.01	80.79	80.80	0.00	54.28	54.30	0.03

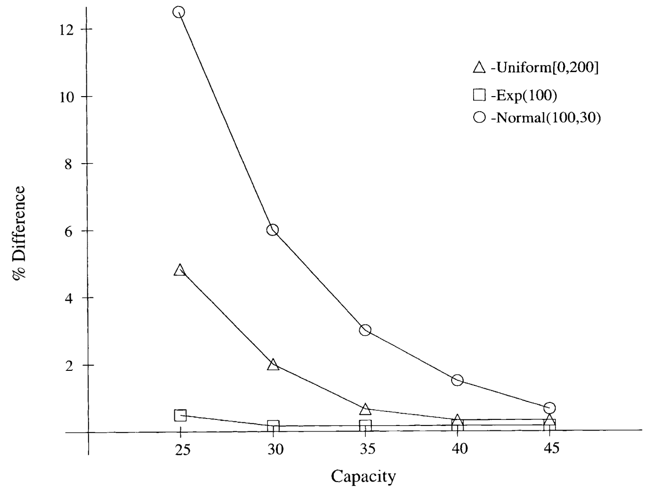
Table 2 Performance of the Uncapacitated Solution as a Heuristic for the Capacitated Problem When $b = 10$

	Uniform(0,200)			Exp(100)			Nor(100,30)		
Cap.	OCost	UCost	% Diff.	OCost	UCost	% Diff.	OCost	UCost	% Diff.
25	99.44	104.43	5.01	128.91	129.32	0.32	75.10	84.68	12.75
30	97.08	98.79	1.77	127.65	127.67	0.02	72.87	77.26	6.03
35	96.07	95.28	0.22	127.34	127.36	0.02	71.71	73.85	2.98
40	95.36	95.37	0.01	127.22	127.24	0.01	70.82	71.81	1.41
45	94.87	94.88	0.02	127.15	127.17	0.02	70.02	70.35	0.47

Notice that in Table 1 when the penalty cost was equal to 5, the cost of the uncapacitated solution was very close to the optimal cost with the maximum deviation being only 0.27% and the average deviation less than 0.04%. However, when the penalty cost is increased to 10 (Table 2), the uncapacitated solution is not as effective. The percentage difference was as high as 12%. This is because at low penalty cost, the contribution of the L term is low, which is what is affected by ignoring the capacity. At higher values of penalty cost, while the uncapacitated solution saves a little on the P term, it pays heavily on the L term.

Figure 1 contains the plot of the performance of the uncapacitated solution when the penalty cost is 10. Observe that as the capacity is increased from 25 to 45, the performance improves for all three demand distributions (as it should). An interesting observation is that the uncapacitated solution performs poorly when the demand variance is low, and better when the demand has a higher variance. This is because at high

Figure 1 Performance of Uncapacitated Solution when $b = 10$



variance, the L term is high even at optimality. At lower variances, however, the L term at optimality is low, and therefore not being able to be at optimal levels is relatively very costly. From these observations we may conclude that the uncapacitated solution is not an effective heuristic for finite capacity when the demand variance is low.

5.2. Cost per Period

Next we study the behavior of (optimal) cost per period as capacity, penalty cost, and variance are changed. Observe that the cost per period increased with increase in penalty cost, increase in demand variance, and decrease in capacity. These results are expected and have been described in detail in earlier literature. We will not elaborate on them here.

However, let us study the relative reduction in cost per period when the capacity is increased or the variance is decreased. Figure 2 contains the plot of the optimal cost when the penalty cost is 5 for the three demand distributions for various capacities. Observe that at high variances (Uniform[0,200] has variance 3,333.33 and Exp(100) has variance 10,000) the cost per period decreases very little when the capacity is increased. For the case of Nor(100,30) with a variance of 900, a much higher reduction in cost is observed as the capacity is increased. The intuition is as follows. When the variance is high, the L term is large even at optimality, therefore having a higher capacity does reduce the P and L terms, but the relative benefit is low.

Figure 2 Plot of Optimal Cost Versus Capacity when $b = 5$

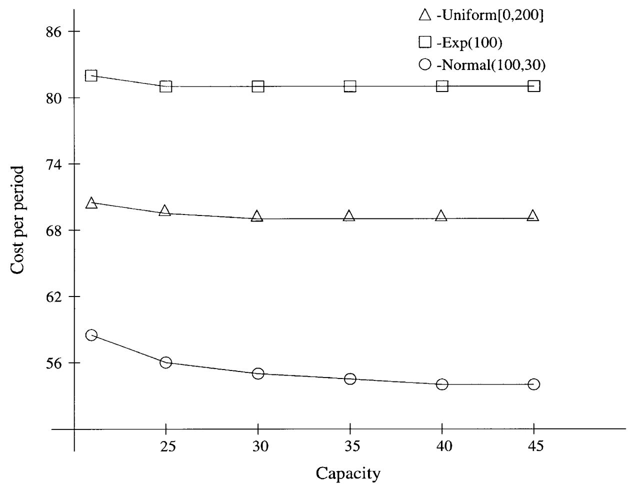
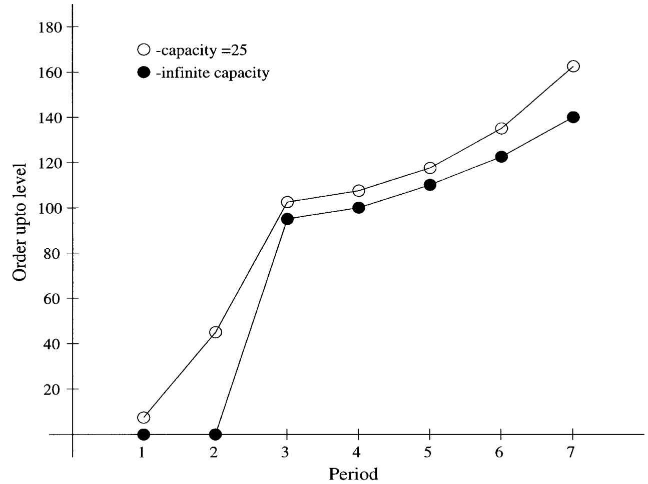


Figure 3 Optimal Order Up-To Levels for Nor(100, 30), and $b = 10$



At the higher penalty cost of 10, notice from Table 2 that cost per period decreased significantly with increase in capacity for Unif[0,200] and Nor(100,30) but not for Exp(100). The intuition is as follows. At higher penalty costs, we need higher inventory levels, and therefore at lower capacities we need to produce more, and start earlier. This increases the P component and we notice the benefits of increased capacity, at moderate demand variances as well (and not just at low demand variances), when the penalty cost is higher. From these observations we conclude that the demand variance should be low if we want to reduce the cost per period by increasing capacity. However, penalty cost plays an important role in deciding how low the variance should be.

5.3. Inventory Levels

Figure 3 contains the optimal order up-to levels (two cases, capacity = 25, and uncapacitated) for the Nor(100,30) demand distribution when the penalty cost was 10. Notice that as established by Property 3, the order up-to levels when the capacity is 25 are greater than the order up-to levels when the capacity was infinity. Also as established in Property 7 the order up-to levels increase as the number of periods without a demand increase. Notice the difference between this figure and that observed in Gavirneni et al. (1999) where the ordering process was also nonstationary, but from a distributor following an (s, S) policy. (Therefore the consecutive interarrival times are independent—and not negatively correlated as is our case—but the order quantity is stochastically increasing—and not i.i.d. as is our case here—with the number of periods since last demand.) There, the order up-to levels under finite capacity could be very well approximated from the uncapacitated order up-to levels, see Gavirneni et al. (1999) for details. That does not seem to be the case here. This is an example to show that qualitative insights differ as the type of nonstationarities differ and amplifies the observation of Graves (1998) that one cannot transfer the insights from a stationary demand model to a nonstationary demand model.

5.4. Effect of Maximum Deviation

In all the previous experiments, the cycle time c was set at 5 and the maximum deviation was 2. That is, d could take values $\{-2, -1, 0, 1, 2\}$. In this section we want to study the effect of this maximum deviation on the cost per period. To achieve this we set up a problem in which the cycle time was set to 9. The maximum deviation was allowed to range from 1 to 4. The costs per period for the case of Unif[0,200] as a function of the maximum deviation are given in Figure 4. Notice that a unit increase in the maximum deviation resulted in a change of approximately 3–4% in the cost per period. Observe the behavior of the cost per period (in Figure 5) for the case of Unif[80,120]. First notice that

Figure 4 Plot of Cost per Period Versus Maximum Deviation for Unif[0, 100]

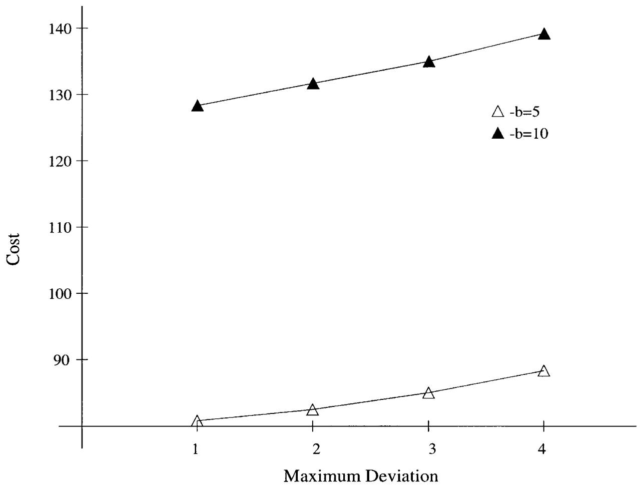
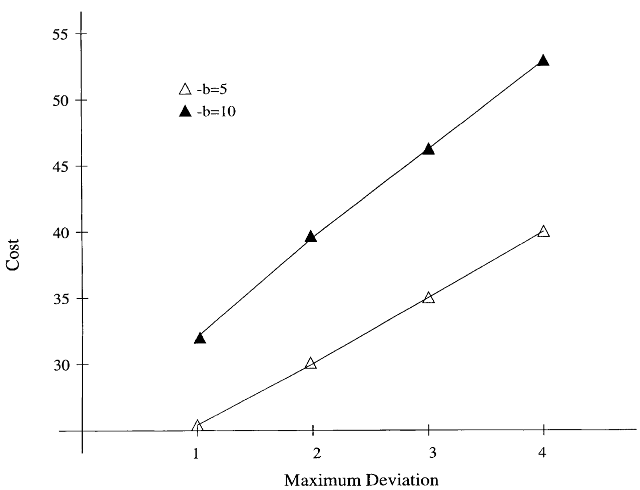


Figure 5 Plot of Cost per Period Versus Maximum Deviation for Unit[80, 120].



the costs under this demand are significantly lower than the costs in the Unif[0,200] situation (20–50 versus 80–140). Also under this situation a unit increase in the maximum deviation resulted in an increase of approximately 15–20% in the cost per period. That means that as the variance of the order quantity decreases, the variance of order timing plays an increasingly important role in determining the cost per period. The intuition is as follows: at lower demand quantity variance, the L term is small, and any change in the P term, which is affected by the deviation process, significantly impacts the total cost. In conclusion, the variance of order quantity must be considerably low before expecting a significant drop in cost per period due to reduction of variance in order timing.

5.5. Value of Using the Target Schedule Information

In this section we compute the value of information and study how the various system parameters such as capacity, demand variance, and penalty cost affect it. The information being provided is that the customer is using a target reverting policy to place orders. If we did not know this, we observe that there are a few periods of no demand followed by a period with demand. The number of periods without demand appears to be random. A simple strategy is to assume that the order intervals are i.i.d. Under this assumption a reasonable solution is:

1. Calculate an order up-to level, $\hat{z}$, using the newsboy formula on the demand distribution for the given values for h and b.
2. Set $z_j = \hat{z}\ \forall_j \geq l$ for some value of $l > 0$. Adjust the order up-to levels for finite capacity using the following rule: If an order up-to level in period $l \geq i > 1$ is $z_i > C$, then set the order up-to level $z_{i-1} = z_i - C$; otherwise $z_{i-1} = 0$.
3. Simulate the system for various values of l and choose the value that gives the lowest cost.

When the information that the customer is using a target reverting policy is provided, then the problem is solved optimally using the solution procedures developed earlier. The difference between the costs of these two solution procedures can be primarily attributed to information. We compute the % benefit of information flow as follows:

$$\% \text{ Benefit} = \frac{\text{Cost without information} - \text{Cost with information}}{\text{Cost with information}} \times 100.$$

Our main objective is to study how this % Benefit is affected by capacity, demand variance, and penalty cost. We should note that the cost with information was always lower than the cost without information.

This is not a surprising result as the availability of information only increases the options available to the supplier. Our other observations are detailed below.

1. *Effect of Capacity.* Figure 6 contains the plot of percentage benefit as a function of capacity for the three demand distributions. Observe that as the capacity increases the percentage benefit also increases. This increase is mainly due to the flexibility that comes with the presence of higher capacity. For example, when the capacity is low (close to the mean demand per period), the supplier must produce every period to meet the customer demand. However, when the capacity is high (compared to the mean demand per period), then the supplier can decide to delay production in a period or produce a larger quantity. This ability to react to customer demand is the main reason for observing higher benefits of information at higher capacities.

2. *Effect of Penalty Cost.* Figure 7 contains the plot of percentage benefit as a function of penalty cost for the Exp(100) demand distribution. Notice that as the penalty cost is increased from 5 to 20 in increments of 5, the percentage benefit initially increases from 6.97% to 8.23% and then decreases to 7.73% and further drops to 6.83%. The explanation for this is as follows. When the penalty cost is low the system has low inventories and the penalty of lost sales is not that large, and as a result the information is not very useful. When the penalty cost is high the system has high inventories which need to be planned for ahead of time (because of finite capacity); the additional information of target reversion does not substantially reduce them. As a result, information is most beneficial at moderate values of penalty cost.

3. *Effect of Demand Variance.* Figure 8 contains the plot of percentage benefit as a function of the standard deviation of demand. Observe that as the demand variance is increased the percentage benefit is steadily decreasing. Again, this is different from the observation

Figure 7 **Value of Information as a Function of Penalty Cost**

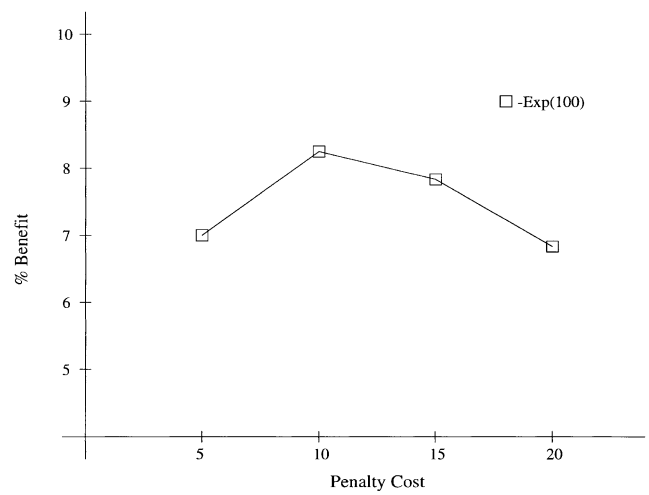


Figure 6 **Value of Information as a Function of Capacity**

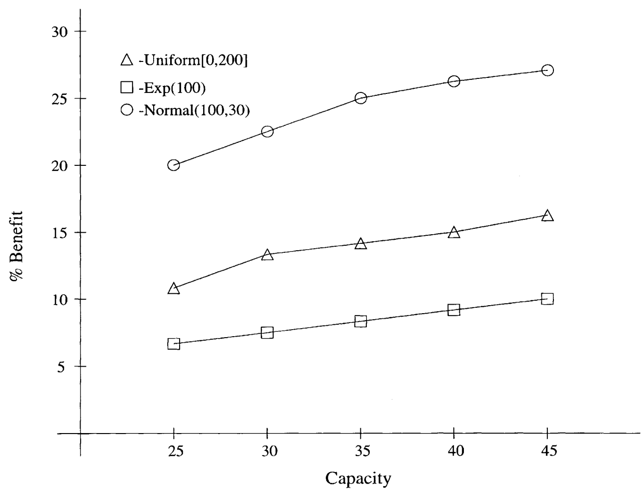
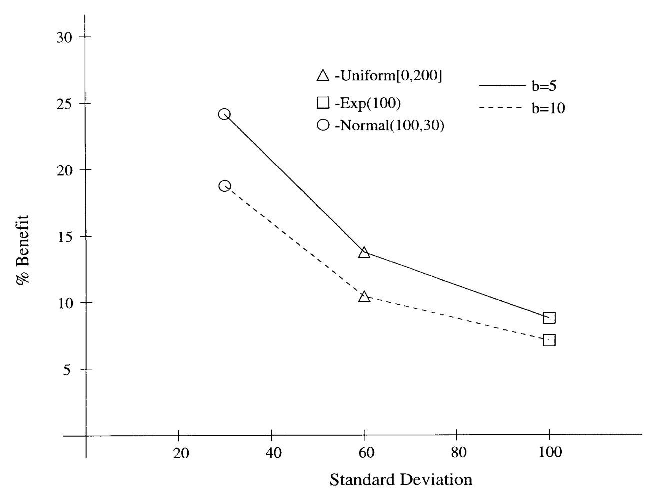


Figure 8 **Value of Information as a Function of Standard Deviation of Demand**



in Gavirneni et al. (1999). There, as the demand variance increased, the percentage benefit initially increased and then started to decrease. This difference is due to the fact that in Gavirneni et al. (1999) when the demand variance was low, it also resulted in a low variance in the order interarrival time as well. That is not so in the target reverting policy. Here, even if the demand variance is reduced, the variance in the order intervals is not affected. But at low demand variances the information provided relative to the uncertainty in the demand process (in order timing and order quantity) is higher and thus results in higher savings in cost. (As before, the value of L being low or high at optimality affects the relative value of information.) Thus information is more useful at lower quantity variances.

6. Conclusions

Motivated by common situations in industrial supply chains, we have modeled and studied a situation, from a supplier's perspective, where a customer is using a target reverting policy. This gave rise to interesting nonstationary demand processes. Optimal policy structures are found to be order up-to policies. We are able to prove certain monotonicity properties of these levels. We have provided effective solution procedures to compute the optimal parameters in both the capacitated (using simulation based optimization) and the uncapacitated settings (using a recursive procedure). An extensive computational study provided us with insights into how the optimal cost and levels behave, and on the savings and relative benefits of using the optimal policy as against simple policies that ignore either the capacity constraint or the target reverting policy. Most importantly, insights regarding inventory levels, value of information, the behavior of system under different parameters and sensitivity analysis differ from one nonstationary model to another. Briefly, we repeat our main insights here.

1. The uncapacitated solution (while not ignoring the target reverting policy) is a good heuristic when the penalty cost is small or the demand quantity variance is high.

2. After accounting for the target reverting policy, the variance of interarrival times has very little effect on the total cost unless the quantity variance is very low.

3. Similarly, the demand quantity variance should be considerably small to realize significant reduction in cost due to increase in capacity.

4. Ignoring the customer ordering pattern is not that costly when the demand variance is high or the capacity is low.

5. With respect to penalty cost, the relative cost of ignoring the pattern initially increases and then drops off.

Thus, in a high service level (high penalty cost) situation with low capacity and high quantity variance, we may ignore the insights of the refined model while at moderate to high capacities, modest demand variance and service levels, the benefits of using a sophisticated policy are high.

Future modeling incorporates having multiple customers, and studies the issue of capacity allocation to these different customers. Current technical effort is focused on proving the optimal policy structure for general Markov chains, and determining the structure of a special class of Markov chains (see Song and Zipkin 1993 for example) for which the monotonicity properties (Corollary 1) proved hold.

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