Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 6, No. 2, Spring 2004, pp. 195–198
ISSN 1523-4614 | EISSN 1526-5498 | 04 | 0602 | 0195
informs
DOI 10.1287/msom.1040.0042

Technical Note
Evolution of ARMA Demand in Supply Chains

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This paper shows that an ARMA demand generates an ARMA order history when ordering decisions are made based on an order-up-to policy. The order history preserves the autoregressive structure of the demand and transforms its moving average structure according to a simple algorithm. We apply this ARMA-in-ARMA-out property to examine the evolution of the demand signal in supply chains. Its practical implications are discussed in the context of quantifying the bullwhip effect, coordinating forecasting, and evaluating information sharing.

Key words: supply chain; time-series forecasting; demand propagation
History: Received: April 15, 2003; accepted: February 19, 2004. This paper was with the author 5 months for 3 revisions.

1. Introduction

Time series models have long been used in the inventory literature to describe correlated consumer demand. Johnson and Thompson (1975) establish the existence of an optimal policy for ARIMA (autoregressive and integrated moving average) demand processes. Aviv (2003) characterizes the order process for a state-space demand model, which encompasses ARIMA as a special case (De Jong and Penzer 2000, Hamilton 1994). Low order ARIMA processes can be found in works that study the bullwhip effect (Lee et al. 1997; Chen et al. 2000a, b; Graves 1999), estimate the value of information sharing (Lee et al. 2000, Raghunathan 2001, Cachon and Fisher 2000), and derive the benefits from coordinated forecasting (Seifert et al. 2003; Aviv 2001, 2002).

In this paper, we study an ARMA (autoregressive and moving average) demand process and show that it generates an ARMA ordering process. This ARMA-in-ARMA-out (AIAO) property is derived in the next section for a single inventory installation. Section 3 then applies it to examine demand propagation through supply networks. Section 4 discusses the managerial implications of the AIAO property. Section 5 concludes the paper.

2. The AIAO Property

Consider a periodic-review inventory system with a single installation. The installation is supplied with a constant leadtime L; there is no fixed ordering cost, and all shortages are backordered. The sequence of events during a replenishment period is as follows: At the beginning of each period, orders made L periods ago are received first; the demand is then observed, and finally the order decision is made. Let D_t represent the demand that the installation observes in period t, and $D_t^L = \sum_{\tau=1}^{L} D_{t+\tau}$ be the cumulative demand during leadtime. We assume that D_t is described by an ARMA(p, q) process (Box et al. 1994) such as:

$$D_t = \mu + \phi_1 D_{t-1} + \phi_2 D_{t-2} + \cdots + \phi_p D_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t-q}, \tag{1}$$

where μ is a constant that determines the mean of the process, ε_t is an i.i.d. normal error with a zero mean and a standard deviation of σ, p is the autoregressive order of the process, q is the moving average order of the process, ϕ_j is the autoregressive coefficient, and θ_j denotes the moving average coefficient. We assume that the above ARMA(p, q) process is invertible and covariance stationary, i.e., the roots of the following characteristic equations lie outside the unit circle:

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0$$
$$1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q = 0.$$

These assumptions necessarily imply that ϕ_j and θ_j coefficients do not add up to one. To have a unique representation of the demand process, we also assume that the two characteristic equations share no common roots.

The inventory replenishment decision is based on a simple order-up-to policy that keeps the inventory position y_t at a level determined by:

$$y_t = \widehat{D}_t^L + S, \tag{2}$$

where $\widehat{D}_t^L$ is the minimum mean squared error forecast for leadtime demand D_t^L, which corresponds to the anticipation stock, and S is the level of safety stock. The level of S is set to achieve a desired service level and to account for uncertainties in meeting customer demand, such as forecasting errors and supply delays (Aviv 2003). The sequence of events and conservation of flow imply that the order quantity is given by:

$$q_t = y_t - y_{t-1} + D_t. \tag{3}$$

Substituting Equation (2) into Equation (3) yields

$$q_t = \left(\widehat{D}_t^L - \widehat{D}_{t-1}^L\right) + D_t. \tag{4}$$

The simple ordering policy requires that we order enough to replace the demand and to compensate for an anticipated change in inventory position.[1]

Following standard time-series methods (Box et al. 1994, Hamilton 1994), we define $d_t = D_t - \mu_d$ as the mean-centered demand where $\mu_d = \mu/(1 - \phi_1 - \phi_2 \cdots - \phi_p)$ is the unconditional mean of the demand process. If we recursively substitute the lagged demand into the ARMA process and continue this calculation, we see that a stationary ARMA(p, q) demand process admits an infinite moving average representation (IMAR) such that:

$$d_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \tag{5}$$

[1] The order quantity can be negative, which is undesirable. Johnson and Thompson (1975) added additional bounds on the random shock to the demand system to address the issue. If the probability of negative order quantity is negligible, the inventory policy specified in this paper is a reasonable one (Zipkin 2000).

where $\psi_k = 0$ for $k < 0$, $\psi_0 = 1$, and ψ_k for $k \geq 1$ can be calculated recursively from the following:

$$\psi_k = \sum_{j=1}^{p} \phi_j \psi_{k-j} - \theta_k. \tag{6}$$

We can now state the AIAO property (see Gilbert 2002, Li et al. 2003 for independent parallel developments).

The AIAO Property. *Let $\beta = \sum_{j=0}^{L} \psi_j$, $\delta_1 = -\psi_{L+1}/\beta$, and $\delta_k = (\sum_{j=1}^{k-1} \phi_j \psi_{L+k-j} - \psi_{L+k})/\beta$ for $1 < k \leq p$. The order quantity follows an ARMA(p, m) process with $m = \mathrm{Max}(p, q - L)$ such as:*

$$\begin{aligned} q_t &= \mu + \phi_1 q_{t-1} + \phi_2 q_{t-2} + \cdots + \phi_p q_{t-p} + \tilde{\varepsilon}_t \\ &\quad - \tilde{\theta}_1 \tilde{\varepsilon}_{t-1} - \tilde{\theta}_2 \varepsilon_{t-2} - \cdots - \tilde{\theta}_m \tilde{\varepsilon}_{t-m}, \end{aligned} \tag{7}$$

where $\tilde{\varepsilon}_t = \beta \varepsilon_t$ and the moving average coefficients $\tilde{\theta}_k$ are determined from:

Case 1. $m = p$ when $p > q - L$

$$\tilde{\theta}_k = \phi_k + \delta_k. \tag{8}$$

Case 2. $m = q - L$ when $p \leq q - L$

$$\tilde{\theta}_k = \begin{cases} \phi_k + \delta_k & k \leq p \\ \theta_{L+k}/\beta & p < k \leq q - L. \end{cases} \tag{9}$$

Proof. See the Appendix. □

The AIAO property reveals that the order quantity process shares an AR structure identical to that in the demand process. Their different moving average structures are linked in two ways. First, the error terms associated with the order quantity process are proportional to the errors for the demand process, i.e., $\tilde{\varepsilon}_t = \beta \varepsilon_t$. Second, the moving average coefficients $\tilde{\theta}_k$ in the order quantity process are simple functions of autoregressive coefficients ϕ_k and the IMAR ψ_k coefficients.

3. Demand Evolution in Supply Chains

The AIAO property provides a convenient means for tracking ARMA demand propagation in supply chain networks. In a serial system linked by a demand and supply relationship, we can repeatedly apply the AIAO property for each installation to link the demand signals through the entire chain, assuming that the demand at the upstream consists entirely of orders made from the immediate downstream installations. As a result, the demand at each stage shares the same AR structure, and the corresponding moving average coefficients can be calculated recursively from Equations (8) and (9).

Combined with a well-known result from time-series analysis (Hamilton 1994, pp. 102–108), that the sum of uncorrelated ARMA processes remains ARMA, the AIAO property shows that the ARMA structure propagates through more complex supply chain settings. For example, if upstream installations in a serial system also have external ARMA demand streams that are stochastically independent of their order process, the arguments in the above paragraph remain valid. In a distribution network with one distribution center and an arbitrary number of retailers facing uncorrelated ARMA demands, their order histories are uncorrelated ARMA processes as well. Then the total demand for the distribution center is the sum of uncorrelated ARMA processes, which again is an ARMA process. Other network structures can be considered in a similar spirit in which the ARMA demand evolves through the chain as dictated by the AIAO property.

4. Managerial Implications

The AIAO property provides a convenient means for quantifying the bullwhip effect. Expanding the ARMA order process into its IMAR, we obtain $q_t = \sum_{k=0}^{\infty} \tilde{\psi}_k \tilde{\varepsilon}_{t-k}$. The ratio of the unconditional variance of the order process to that of demand process, namely, $\beta^2 \sum_{k=0}^{\infty} \tilde{\psi}_k^2 / \sum_{k=0}^{\infty} \psi_k^2$, measures the bullwhip effect. For an AR(1) demand, this ratio yields an expression identical to Lee et al. (1997). In general, explicit formulas for calculating this ratio are difficult to obtain, but the AIAO property yields a simple algorithm for obtaining numerical results.

In addition to quantifying the bullwhip effect, the AIAO property facilitates implementing the coordinated forecasting. First, it reduces the task of demand estimation at the upstream echelons. The structures for order streams can be recovered once the customer demand is estimated. In practice, only a few demand parameters are needed because the demand can usually be represented by an ARMA process with p, $q \leq 2$. This parsimony can represent a substantial cost saving when extensive data is required for demand estimation. Second, sampling errors associated with estimating the order history processes are eliminated. Lastly, the AIAO property makes it easier to coordinate inventory control policies in a supply network. When the parameters of the demand process shift, upstage installations can anticipate the impact of this shift on their leadtime demand forecasts and make adjustments to their inventory positions accordingly.

Resorting to the AIAO property, an upstream supplier can identify and recover the underlying demand of the immediate downstream customer from the order history. This ability implies that sharing demand information from the downstream installation is not likely to add significant value to the upstream supplier, beyond the savings from eliminating the supplier's need to estimate demand. Raghunathan (2001) confirms this observation for an AR(1) demand process.

5. Conclusions

This paper shows the existence of a simple AIAO property linking the demand process between two stages of a supply chain under a reasonable inventory control policy. As an ARMA demand evolves through a supply chain, this property reveals that the autoregressive portion of the demand process is preserved and the moving average portion is updated with a straightforward algorithm. The practical value of this theoretical finding lies in its ability to facilitate measuring the bullwhip effect, coordinating forecasts, and understanding the value of information sharing in supply chains.

Acknowledgments

The author thanks Roger Koppl, Rose Twomey, two anonymous referees, and the Senior Editor for their valuable comments on earlier versions of this paper. This research was supported in part by a summer research grant from the Silberman College of Business, Fairleigh Dickinson University.

Appendix

Proof of the AIAO Property. If we substitute the IMAR of the ARMA(p, q) demand in Equation (5) into leadtime demand $D_t^L = L\mu_d + \sum_{\tau=1}^{L} d_{t+\tau}$ and combine the same error terms, we obtain:

$$D_t^L - L\mu_d = (\omega_0 + \omega_1 B + \omega_2 B^2 + \cdots)\varepsilon_{t+L}$$

where B is the backward shift operator and coefficients ω_k are given by:

$$\omega_k = \begin{cases} \omega_{k-1} + \psi_k & 0 < k < L \\ \omega_{k-1} + \psi_k - \psi_{k-L} & k \geq L. \end{cases} \tag{10}$$

The leadtime demand forecast is its conditional expectation given the observed demand:

$$\widehat{D}_t^L - L\mu_d = E(D_t^L \mid d_t, d_{t-1}, \ldots) = \sum_{j=0}^{\infty} \omega_{L+j}\varepsilon_{t-j}. \tag{11}$$

Substituting the above into Equation (4) and simplifying with the recursive relation for ω_k in Equation (10) and the IMAR for the ARMA(p, q) demand, we have

$$q_t = \mu_d + (1+\omega_L)\varepsilon_t + \sum_{j=1}^{\infty} \psi_{L+j}\varepsilon_{t-j}. \tag{12}$$

Recognizing that $\beta = 1 + \omega_L$ and $\mu_d = \mu/(1 - \phi_1 - \phi_2 \cdots - \phi_p)$, we can obtain the following after substituting from Equation (12) and combining the same error terms:

$$q_t - \phi_1 q_{t-1} - \phi_2 q_{t-2} - \cdots - \phi_p q_{t-q} = \mu + \beta\varepsilon_t - \sum_{j=1}^{\infty} \lambda_j \varepsilon_{t-j} \tag{13}$$

where

$$\lambda_k = \begin{cases} \phi_1\beta - \psi_{L+1} & k = 1 \\ \phi_k\beta + \sum_{j=1}^{k-1} \phi_j \psi_{L+k-j} - \psi_{L+k} & 1 < k \leq p \\ \sum_{j=1}^{p} \phi_j \psi_{L+k-j} - \psi_{L+k} & k > p. \end{cases} \tag{14}$$

Consider the two cases when $p > q - L$ and when $p \leq q - L$. In the first case, $\lambda_k = 0$ for $k > p$ because the ψ_{L+k} coefficients satisfy the difference equation $\psi_{L+k} = \sum_{j=1}^{p} \phi_j \psi_{L+k-j}$ as shown in Equation (6). Equation (13) reduces to the ARMA(p, p) process stated in the AIAO property. In the second case, $\lambda_k = 0$ for $k > q - L$. For $p < k \leq q - L$, we have $\sum_{j=1}^{p} \phi_j \psi_{L+k-j} - \psi_{L+k} = \theta_{L+k}$ from Equation (6). This set of λ_k coefficients reduces Equation (13) to an ARMA(p, $q - L$) process stated in the AIAO property. □

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