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Asset Pricing in a Monetary Economy with Heterogeneous Beliefs

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In this paper, we shed new light on the role of monetary policy in asset pricing by examining the case in which investors have heterogeneous expectations about future monetary policy. This case is realistic because central banks are typically less than perfectly open about their intentions. Accordingly, surveys of economists reveal that they frequently disagree in their expectations. Under heterogeneity in beliefs, investors place speculative bets against each other on the evolution of the money supply, and as a result the sharing of wealth in the economy evolves stochastically. Employing a continuous-time equilibrium model, we show that these fluctuations majorly affect the prices of all assets, as well as inflation. Our model could help explain some empirical puzzles. In particular, we find that the volatility of bond yields and stock market volatility could be significantly increased by the heterogeneity in beliefs, a conclusion supported by our empirical analyses.

Keywords: finance; asset pricing; heterogeneous beliefs; monetary economics
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1. Introduction

Monetary policy is arguably one of the main factors driving asset returns (Thorbecke 1997, Bernanke and Kuttner 2005), yet most asset pricing models solely focus on the real side of the economy. A growing body of literature explores the connections between asset pricing (the term structure especially) and monetary policy, but not much attention has been devoted to the effect of the trading opportunities that arise in the presence of heterogeneous investors.

Specifically, in the presence of heterogeneous beliefs, agents' speculative trading generates extra risk. Such trading risk has proven fruitful in furthering our understanding of some empirical puzzles (e.g., David 2008), but, to the best of our knowledge, the impact of heterogeneous beliefs on monetary policy has not been yet examined in an asset pricing framework (even though Basak 2005 sketches a model to do so). In this paper, we build a general equilibrium model that incorporates investors' disagreement on monetary policy. We show that this heterogeneity in beliefs has important implications. In particular, the stock market and bond yield volatility are much increased. Hence, our work shows that heterogeneous beliefs on monetary policy could help resolve long-standing excessive volatility puzzles (Shiller 1979, 1981).

Our basic premise—that market participants hold different beliefs on future monetary policy—is realistic because central banks' policies are typically hard to anticipate. Part of the monetary economics literature suggests that transparency in monetary policy could compromise its effectiveness, and this school of thought seems to still exert some influence. This is the case even though there has been a trend toward more transparency and the implementation of monetary policy rules, such as the Taylor rule, which could make monetary policy easier to forecast. (According to a survey by van der Cruijsen and Eijffinger 2007, because in most theoretical models "only unanticipated monetary policy has an effect on output" (p. 265), "the debate on the desirability of central bank transparency continues to be a lively one" (p. 283).) Although many central banks have recently switched to following more formal rules, "the U.S. system remains opaque and relies more on the judgment and credibility of its chair" (Barro 2005). The Bernanke era (from 2006 on) has been characterized by increased transparency, but whether it has led to easier predictability of the Fed's actions is subject to debate. What transparency has led to is often for the Fed to say that they do not know what they are going to do with interest rates, which does not make the market less uncertain. According to some market participants, "Chairman Bernanke's penchant for 'transparency' has caused more uncertainties than clarity over the years since adopted" (Harding 2012).

The fact that there is imperfect information on monetary policy makes it likely that market participants hold heterogeneous beliefs on its future evolution. Disagreement among traders, and its importance for understanding prices, is well established in other areas of finance. For example, Anderson et al. (2005) show that there exists significant disagreement among stock analysts about expected earnings, and that this matters for asset pricing. We are not aware of any academic study directly documenting heterogeneous beliefs on monetary policy, but Mankiw et al. (2003) establish substantial disagreement in expectations on inflation, a quantity closely related to monetary policy. Li and Li's (2010) findings on the relationship between dispersion of beliefs about future inflation and stock trading volume suggest that investors trade on these differences of opinion. In addition, examples of disagreement on monetary policy abound in the press.[1]

Our work builds on the monetary economy of Bakshi and Chen (1996). Their model is extended by Basak and Gallmeyer (1999) to an international context, and by Lioui and Poncet (2004) and Buraschi and Jiltsov (2005) to a production economy. (None of these studies include heterogeneity in beliefs.) Our modeling framework for heterogeneous beliefs is based on the approach of Detemple and Murthy (1994, 1997). Basak (2005) provides a model that synthesizes the main implications present in this literature, and in §5.2 of his paper, he sketches a monetary economy with heterogeneous beliefs. Our work builds directly on his model and explores its implications in detail.

In our model, money plays a role because investors' utilities are assumed to be a function of money holdings in addition to being a function of consumption—one of the modeling strategies used by monetary economists—and it enters the economy by being endowed to the investors by the central bank. The money supply is assumed to follow an Itô process, whose drift is chosen by the central bank. There are two investors, who observe the changes in the money supply, but have incomplete information on its dynamics. We assume that the two investors have heterogeneous beliefs (which they update over time via Bayesian updating) so that, even though they have symmetric information, they disagree on the expected money supply growth. We also consider, in an extension, the case in which agents also disagree on consumption growth.

The most important mechanism driving our results is that, under heterogeneous beliefs, agents place speculative bets (using financial markets) against each other on the money supply. Thus, shocks in the money supply affect the distribution of wealth. The extra "trading risk" that is generated leads to an extra factor in the pricing of assets. To derive sharp implications, we assume that investors exhibit separable constant relative risk aversion utility functions, as well as simple dynamics for economic fundamentals. As a result, providing explicit formulas for real quantities, including the stock price and its volatility, is possible. In particular, heterogeneity in beliefs on monetary policy generates a much higher volatility for the stock.

To provide an explicit computation of inflation and nominal interest rates, we further narrow our work to the case of logarithmic preferences. Nominal interest rates are driven by investors' expectations of future monetary policy and are decreased by heterogeneity in beliefs. Since the economy is more uncertain and heterogeneity in beliefs is greater during recessions (as demonstrated by Patton and Timmermann 2010 for the case of inflation forecasts), our model generates procyclical nominal interest rates, consistently with the data, as well as an increased volatility for interest rates. Heterogeneity in beliefs also increases the volatility of inflation: When a positive shock to the money supply occurs, not only does the extra amount of money cause inflation but those investors who expect higher money supply growth and higher future inflation "win their bet," and so their weight in the economy increases, generating extra inflation. This implication is consistent with recent empirical research (Dincer and Eichengreen 2007), which suggests that increased transparency of monetary policy (which can be expected to reduce heterogeneity in beliefs) reduces the volatility of inflation.

The level and time-varying behavior of market volatility is one of the key puzzles facing financial economists (see, e.g., Schwert 1989). As Engle and Rangel (2008, p. 1187) note, "[t]he number of models that have been developed to predict volatility based on time series information is astronomical, but models that incorporate economic variables are hard to find." Thus, highlighting the significant impact of heterogeneous beliefs on monetary policy on volatility may be the key contribution of our paper, and we empirically investigate this relationship. Using forecasts provided by the Survey of Professional Forecasters (SPF), this prediction of our model is supported by the data, with a significant positive relationship between heterogeneity in beliefs and both bond and stock market volatility. One interesting finding is that heterogeneity in beliefs on real output does not appear to have quite as robust an impact on volatility; this suggests that disagreement on the monetary side of the economy plays a specific role, as predicted by our model.

In addition, we perform a calibration of the level of heterogeneity in beliefs needed to explain, using our model, the observed asset returns and volatilities. We find that the model does a good job of matching

[1] For example, the Associated Press (2008) reported the following: "Could the US central bank keep rates unchanged for a considerable period? Yes, many analysts say, predicting that the Fed would leave rates alone until next spring. However, other economists are still worried that the Fed could start ratcheting up rates much sooner than that."

the level of stock and bond yield volatilities, and that the amount of disagreement needed for this is not extreme and seems plausible. The model can also generate a much increased equity premium relative to a benchmark economy; however, our results on the equity premium are not very robust because to generate a high value the model requires very specific assumptions on agents' beliefs (e.g., the investors who are optimistic on money growth are wealthier and the wealth-weighted average beliefs are lower than the true value). Results on the model's ability to match the level of interest rates—both real and nominal—are less favorable, with values significantly higher than in the data.

In addition to furthering our understanding of asset prices, our model provides a natural framework by which to assess certain aspects of monetary policy, particularly how much transparency on the part of central banks is optimal, a topical and controversial issue. Our model is appropriate for this because it is intuitive that one of the effects of increased transparency should be a drop in the amount of heterogeneity in beliefs: when the monetary policy is more transparent, subjective prior beliefs enter forecasts less and individual forecasts should be more similar. Hence, our model suggests that increased transparency could reduce the volatilities of inflation, nominal interest rates, and stock prices, as well as the level of real interest rates. Another implication of our model is that, in the presence of heterogeneous beliefs on money growth, a relationship similar to a Taylor rule holds between inflation and nominal rates: all other things being equal, high inflation leads to a rise in the nominal short rate. This suggests that, if heterogeneous beliefs are present, exogenously imposing a Taylor rule could be less effective than expected.

Our work complements the recent literature examining asset prices in the context of New Keynesian monetary models (described by Woodford 2003). Most of this research, including that by Gallmeyer et al. (2005), Rudebusch and Wu (2008), and Bekaert et al. (2010), focuses on examining the properties of the term structure. We complement this literature by examining important issues that are not (or are little) addressed, such as heterogeneity among investors and stock market volatility. Within this literature, a paper by Palomino (2010), whose work shows that monetary policy credibility reduces bond volatility, is closely related to ours. Using a completely different framework and economic mechanism, we establish a parallel result by linking volatility and heterogeneity in beliefs (which presumably would be reduced by increased credibility).

Our work is closely related to that of Xiong and Yan (2010), in which heterogeneous beliefs (on inflation) also generate increased bond yield volatility, as well as other empirically observed features of bond yields. We adopt a slightly different general equilibrium approach in which we endogenize inflation, which is exogenous and independent of economic conditions in Xiong and Yan's work. Our work complements theirs by additionally addressing the impact of heterogeneous beliefs on the stock market (which is nil in Xiong and Yan's work because of the assumption of logarithmic utility) and inflation and providing empirical analyses.

2. The Economic Setup

Our model builds directly on Basak and Gallmeyer (1999) and Basak (2005, §5.2). We consider a continuous-time, pure exchange, finite horizon ($[0, T]$) economy with two investors $i = 1, 2$. The uncertainty is generated by a Brownian motion, $w = (w_\varepsilon, w_M)^\top$. There is a single consumption good (the numeraire). In addition to their consumption, agents derive utility from holding money balances, capturing in reduced form the services rendered by money.

2.1. Aggregate Consumption, Money Supply, and the Information Structure

The aggregate endowment of consumption ε and the money supply M are assumed to follow geometric Brownian motions:

$$\frac{d\varepsilon(t)}{\varepsilon(t)} = \mu_\varepsilon dt + \sigma_\varepsilon dw_\varepsilon(t),$$

$$\frac{dM(t)}{M(t)} = \mu_M dt + \sigma_{M\varepsilon} dw_\varepsilon(t) + \sigma_{MM} dw_M(t).$$

Although w_ε (consumption risk) and w_M (monetary risk) are independent, we allow for correlation between aggregate consumption and money supply, given by $\rho = \sigma_{M\varepsilon}/(\sigma_{M\varepsilon}^2 + \sigma_{MM}^2)^{1/2}$. We denote the total volatility of the money supply by $\sigma_M = (\sigma_{M\varepsilon}^2 + \sigma_{MM}^2)^{1/2}$.

The investors commonly observe the processes ε, w_ε, and M. Assuming that agents can observe the consumption risk factor w_ε is tantamount to assuming that they know the true value of expected consumption growth μ_ε. On the other hand, investors have incomplete information on the dynamics of the money supply.[2] They observe the volatility coefficients $\sigma_{M\varepsilon}$, σ_{MM} (from the quadratic variation and covariation with ε), but must estimate μ_M. We denote agent i's estimate of μ_M by μ_M^i. Even though investors have symmetric information, because of their heterogeneous prior beliefs, they may disagree in their estimates of μ_M. We assume that investors have normally distributed prior beliefs on μ_M with (individual-specific) mean $\mu_M^i(0)$ and (common)

[2] We choose to focus on heterogeneity in beliefs on the money supply growth, a novelty of this work, and so, to not obscure its implications, we assume homogeneous beliefs on consumption growth. In §3.5, we extend the model to jointly consider heterogeneous beliefs on consumption.

variance $v(0)$ and that they update these estimates in a Bayesian fashion, given their observation of M and ε. Standard arguments (Liptser and Shiryaev 2000) imply that agent i's time t estimate for μ_M, $\mu_M^i(t)$ is given by

$$\mu_M^i(t) = \frac{\sigma_M^2}{2} + \frac{v(t)}{v(0)}\left(\mu_M^i(0) - \frac{\sigma_M^2}{2}\right) + \frac{v(t)}{\sigma_{MM}^2}\left(\ln\frac{M(t)}{M(0)} - \sigma_{M\epsilon}w_\varepsilon(t)\right),$$

where $v(t)$ denotes the variance of this estimate and is given by

$$v(t) = \frac{v(0)\sigma_{MM}^2}{\sigma_{MM}^2 + v(0)t}. \tag{1}$$

This implies that the (normalized) difference in their estimates, $\bar{\mu}_M$, equals

$$\bar{\mu}_M(t) = \frac{\sigma_{MM}^2\bar{\mu}_M(0)}{\sigma_{MM}^2 + v(0)t}.$$

The amount of heterogeneity in beliefs is thus revealed to be deterministic. Agent i's innovation process (or estimate for the monetary risk factor w_M) is given by

$$w_M^i(t) = \int_0^t \frac{1}{\sigma_{MM}}\left(\frac{dM(s)}{M(s)} - \mu_M^i(s)ds - \sigma_{M\varepsilon}dw_\varepsilon(s)\right),$$

which reconciles the investor's estimate of μ_M with his observation of the money supply. The investors' innovation processes are related by

$$dw_M^1(t) = dw_M^2(t) - \bar{\mu}_M(t)dt. \tag{2}$$

By Girsanov's theorem, w_M^i is a Brownian motion under agent i's subjective beliefs, and his perceived dynamics for the money supply are as follows:

$$dM(t) = M(t)[\mu_M^i(t)dt + \sigma_{M\varepsilon}dw_\varepsilon(t) + \sigma_{MM}dw_M^i(t)].$$

It will be verified that, in equilibrium, the price of money p, expressed in units of the consumption good (the numeraire), follows an Itô process with dynamics

$$\begin{aligned} dp(t) &= p(t)[\mu_p(t)dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M(t)] \\ &= p(t)[\mu_p^i(t)dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M^i(t)], \\ &\qquad i = 1, 2, \end{aligned}$$

under the "objective" probability and as perceived by the investors, respectively.

2.2. Investment Opportunities

There are three securities available for continuous trading. The first is a zero-net supply, riskless (in real terms) bond paying-off the real interest rate r. Its price follows $dB(t) = B(t)r(t)dt$. There is also a zero-net supply, nominally riskless bond paying-off the nominal interest rate R. An application of Itô's lemma shows that the real price of the nominal bond, B_m, has the following dynamics:

$$\begin{aligned} &dB_m(t) \\ &= B_m(t)[(\mu_p(t)+R(t))dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M(t)] \\ &= B_m(t)[(\mu_p^i(t)+R(t))dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M^i(t)], \\ &\qquad i = 1, 2. \end{aligned}$$

The nominal bond is risky in real terms. Finally, there is a risky stock paying off the aggregate consumption ε, with a supply of one share, whose price S (in real terms) has the following dynamics:

$$\begin{aligned} &dS(t) + \varepsilon(t)dt \\ &= S(t)[\mu_S(t)dt + \sigma_{S\varepsilon(t)}dw_\varepsilon(t) + \sigma_{SM}(t)dw_M(t)] \\ &= S(t)[\mu_S^i(t)dt + \sigma_{S\varepsilon(t)}dw_\varepsilon(t) + \sigma_{SM}(t)dw_M^i(t)], \quad i = 1, 2. \end{aligned}$$

Agents facing the same price processes S, B_m, and Equation (2) imply that the perceived expected returns are related by

$$\begin{aligned} \mu_p^1(t) - \mu_p^2(t) &= \sigma_{pM}(t)\bar{\mu}_M(t), \\ \mu_S^1(t) - \mu_S^2(t) &= \sigma_{SM}(t)\bar{\mu}_M(t). \end{aligned} \tag{3}$$

Assuming that the (endogenous) volatility coefficients $\sigma_{p\varepsilon}$, σ_{pM}, $\sigma_{S\varepsilon}$, and σ_{SM} are nonzero, the market is complete. Investor i faces unique market prices of risk, θ_ε^i and θ_M^i, associated with consumption and monetary uncertainty, respectively. The market prices of risk θ_ε^i and θ_M^i solve the following:

$$\begin{pmatrix} \sigma_{p\varepsilon}(t) & \sigma_{pM}(t) \\ \sigma_{S\varepsilon}(t) & \sigma_{SM}(t) \end{pmatrix}\begin{pmatrix} \theta_\varepsilon^i(t) \\ \theta_M^i(t) \end{pmatrix} = \begin{pmatrix} \mu_p^i(t) + R(t) - r(t) \\ \mu_S^i(t) - r(t) \end{pmatrix}.$$

Equation (3) implies that the investors face the same market price of consumption risk, $\theta_\varepsilon \equiv \theta_\varepsilon^1 = \theta_\varepsilon^2$. Agent i's perceived state-price density is given by

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \theta_\varepsilon(t)dw_\varepsilon(t) + \theta_M^i(t)dw_M^i(t)]. \tag{4}$$

No-arbitrage implies that, under standard regularity conditions, the price of any asset is given by a present value formula. In particular, the stock and money prices are given by

$$\begin{aligned} S(t) &= \frac{1}{\xi^i(t)}E_t^i\left[\int_t^T \xi^i(s)\varepsilon(s)\,ds\right], \\ p(t) &= \frac{1}{\xi^i(t)}E_t^i\left[\int_t^T \xi^i(s)R(s)p(s)\,ds\right], \end{aligned} \qquad i = 1, 2, \tag{5}$$

where E_t^i denotes the time t conditional expectation under agent i's beliefs. The first equation is standard. The second is intuitive, if one thinks of one unit of money as an asset worth p, and paying-off a continuous dividend equal to the nominal interest rate, R (worth Rp in real terms).[3]

[3] As in Basak and Gallmeyer (1999), the money price equation assumes that the terminal value of money, $p(T)$, equals zero, which is

2.3. Investors' Optimization

Investor i is endowed with $a^i > 0$ share of the stock, and $b^i > 0$ share of the money supply (with $a^1 + a^2 = b^1 + b^2 = 1$), so that his initial wealth is given by $W^i(0) = a^i S(0) + b^i p(0) M(0)$. He then chooses his consumption $c^i \geq 0$, money balance $m^i \geq 0$, and portfolio $\pi^i = (\pi_0^i, \pi_B^i, \pi_S^i)^\top$, where π_0^i, π_B^i, and π_S^i denote the amounts of the numeraire invested in the real riskless bond, the nominal bond, and the stock, respectively, so as to maximize his cumulated lifetime expected utility $E^i[\int_0^T u^i(c^i(t), p(t)m^i(t,t))\,dt]$, where

$$u(c^i, pm^i, t) = e^{-\delta t}\left[\phi \frac{(c^i)^{1-\alpha}}{1-\alpha} + (1-\phi)\frac{(pm^i)^{1-\beta}}{1-\beta}\right],$$
$$\text{where } \delta > 0,\ \alpha, \beta > 1 \text{ and } \phi \in (0,1), \quad \text{or} \tag{6}$$
$$u^i(c^i, pm^i, t) = e^{-\delta t}[\phi \log(c^i) + (1-\phi)\log(pm^i)],$$
$$\text{where } \delta > 0,\ \phi \in (0,1).$$

For the dynamic budget constraint and the solution, which can be characterized by standard martingale techniques, we follow Basak and Gallmeyer (1999), and refer interested readers to them.

The No-Updating Case. To obtain clearer expressions, we sometimes make the simplifying assumption that agents have zero prior belief variance ($v(0) = 0$, implying that $v(t) = 0$, $\forall\, t$). In such a case, investors do not update their beliefs over time, and μ_M^1, μ_M^2, and $\bar{\mu}_M$ are constant. This is not an appealing assumption: A classical objection would be that agents who do not update their beliefs will incur heavy losses and be driven out of the economy. Recent research (e.g., Yan 2008), however, shows that this may not be true, because investors with incorrect beliefs could survive (and remain wealthy enough to affect prices) for long periods. We refer to this case as the "no-updating" case. The expressions are made simpler and easier to interpret, but our qualitative insights are not affected (although the magnitude of the effects changes).

3. Equilibrium

Definition 3.1 (Competitive Equilibrium). An *equilibrium* is a price system (r, R, S, p) and admissible policies (c^i, m^i, θ^i), $i = 1, 2$ such that (1) agents choose their optimal policies given their beliefs and (2) markets for consumption, money, and securities clear: $c^1(t) + c^2(t) = \varepsilon(t)$, $m^1(t) + m^2(t) = M(t)$, $\pi_B^1(t) + \pi_B^2(t) = 0$, $\pi_S^1(t) + \pi_S^2(t) = S(t)$, $W^1(t) + W^2(t) = S(t) + p(t)M(t)$.

intuitive since money provides no utility beyond the time horizon T. Without this assumption, there would be a continuum of solutions for the money price. See Remark 3.1 in Basak and Gallmeyer (1999) for more detail on this issue.

3.1. General Properties of Equilibrium

Following Basak and Cuoco (1998), we introduce a representative investor with utility

$$U(c, pm, t; \lambda) \equiv \max_{c^1+c^2=c,\, m^1+m^2=m} u^1(c^1, pm^1, t) + \lambda u^2(c^2, pm^2, t). \tag{7}$$

Under heterogeneous beliefs, the equilibrium allocation only solves the representative agent's problem if the relative weight λ is allowed to be stochastic; its dynamics are given by

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M(t)\,dw_M^1(t) = -\bar{\mu}_M(t)\left(\frac{\mu_M - \mu_M^1(t)}{\sigma_{MM}}dt + dw_M(t)\right). \tag{8}$$

The relative weight of the two agents is driven by the heterogeneity in beliefs $\bar{\mu}_M$. The intuition for this is simple: The optimistic investor (with the higher estimate for μ_M) invests in a portfolio that is more positively correlated with the money supply than the pessimistic investor's. Effectively, he is betting (against the pessimistic investor) that the money supply will grow by a lot. If the realization of the Brownian motion representing pure monetary risk is high, he wins his bet, and his weight in the economy increases, at the expense of the other investor.

The general case (arbitrary utility functions) is sketched by Basak (2005), to whom we refer interested readers for more detail. A significant finding is that the market price of monetary risk (equal to $\theta_M^1(t) = (A^1(t)/(A^1(t) + A^2(t)))\bar{\mu}_M(t)$, $\theta_M^2(t) = -(A^2(t)/(A^1(t) + A^2(t)))\bar{\mu}_M(t)$, where A^i denotes agent i's absolute risk aversion) is only nonzero in the presence of heterogeneity in beliefs because only then do monetary shocks impact agents' consumptions (via the "betting" mechanism described above). The more optimistic agent facing a higher market price of monetary risk entices him to invest in a portfolio that is more positively correlated with monetary risk, which is the mechanism he uses to "bet" that the money supply will grow a lot. This shows that, under heterogeneous beliefs, there is a "spillover" of the monetary sphere into the real economy, whereas models assuming homogeneous beliefs might erroneously conclude that there is no effect.

In general (with relative risk aversion different than one), none of the monetary asset prices can be solved in closed form. This is because the price of money p obeys a backward stochastic differential equation (second equation in (5)) involving future values of p, which is very intractable. Real asset prices, however, can be solved explicitly. In particular, the real interest rate is given by

$$r(t) = \alpha\mu_\varepsilon - \frac{1}{2}\alpha(\alpha+1)\sigma_\varepsilon^2 + \delta + \frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)\frac{\lambda(t)^{1/\alpha}}{(1+\lambda(t)^{1/\alpha})^2}\bar{\mu}_M(t)^2.$$

The real interest rate is increased by the heterogeneity in beliefs (for relative risk aversion greater than one), as is typical under heterogeneous beliefs (Basak 2005).

3.2. Stock Price, Volatility and the Equity Premium

For this subsection, we assume, with little loss of generality, that the investors' relative risk aversion, α, is an integer.[4] The stock price is then as in Proposition 3.1.

Proposition 3.1. *If α is an integer, the stock price is given by*

$$S(t) = \frac{\varepsilon(t)}{(1+\lambda(t)^{1/\alpha})^\alpha}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T H(j,t,s)\,ds, \quad (9)$$

where $\binom{\alpha}{j}$ denotes the binomial coefficient $\alpha!/((\alpha-j)!j!)$ and

$$H(j,t,s) = \exp\left\{\left[(1-\alpha)\left(\mu_\varepsilon - \frac{1}{2}\alpha\sigma_\varepsilon^2\right) - \delta - \frac{\sigma_{MM}^4\bar{\mu}_M(0)^2 j(\alpha-j)}{2\alpha^2(\sigma_{MM}^2+v(0)s)(\sigma_{MM}^2+v(0)t)}\right](s-t)\right\}.$$

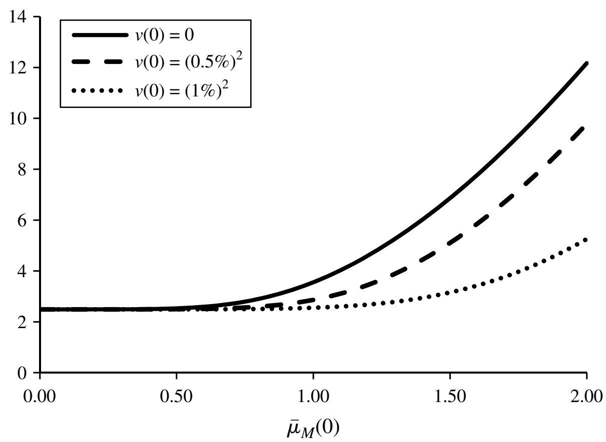
The stock volatility coefficients are given by

$$\sigma_{S\varepsilon}(t) = \sigma_\varepsilon,$$

$$\sigma_{SM}(t) = \left(\frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}} - \frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}(j/\alpha)\int_t^T H(j,t,s)\,ds}{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T H(j,t,s)\,ds}\right) \cdot \frac{\sigma_{MM}^2\bar{\mu}_M(0)}{\sigma_{MM}^2+v(0)t}. \quad (10)$$

The stock price is reduced (for relative risk aversion greater than one) by the presence of heterogeneity in beliefs. The intuition can be expressed as follows: both agents expect to profit from the heterogeneity in beliefs and "win their bets," leading to higher expected consumption growth and a lower value being placed on future dividends; hence, a decrease in the stock price occurs. The total stock volatility ($(\sigma_\varepsilon^2 + \sigma_{SM}(t)^2)^{1/2}$) is unambiguously raised by the heterogeneity in beliefs because $\sigma_{SM} = 0$ in its absence.[5] Under homogeneous beliefs, the stock volatility would equal that of aggregate consumption, but the heterogeneity in beliefs generates stochastic volatility, as "trading risk" affects consumptions, stochastic discount factors, and eventually the stock price. Agents' updating of their beliefs, however, reduces the magnitude of these effects: with updating, the amount of heterogeneity in beliefs decreases monotonically over time, so the expected profits from investors' "betting" against each other are reduced, leading to a price that is closer to a standard economy.

Figure 1 **Stock Volatility (%)**



The magnitude of this effect is significant, as demonstrated by Figure 1, which depicts the stock volatility as a function of the initial amount of disagreement $\bar{\mu}_M(0)$, for different values of the prior variance $v(0)$ (the case $v(0) = 0$ is the no-updating case).

The following parameter values (estimated, for the period 1986–2011, based on data from the Fed and Robert Shiller's website[6]) are used in all of our numerical analyses: $\mu_\varepsilon = 1.856\%$, $\sigma_\varepsilon = 2.49\%$, $\mu_M = 5.201\%$, $\sigma_M = 1.261\%$, $\rho = 0$,[7] $T = 50$ years, $\delta = 0.02$, $\alpha = 3$, $\lambda = 1/3$.[8] For example, for $\bar{\mu}_M = 2$, the total stock volatility is increased by a factor of approximately two to five (depending on the variance of prior beliefs) relative to the case of homogeneous beliefs, in which it equals 2.49%. This amount of heterogeneity in beliefs seems plausible and not an extreme case: with our parameter values, $\bar{\mu}_M = 2$ means that $\mu_M^1 - \mu_M^2 = 2.52\%$, a value

[4] In the case in which α is not an integer, one could use a Taylor series approximation around these results.

[5] We have not been able to formally show that the total stock volatility is always increasing in $\bar{\mu}_M$, but, based on our numerical investigations, this seems to be the case.

[6] http://www.econ.yale.edu/~shiller/data.htm.

[7] In our sample, the correlation is small (−0.06) and thus can be ignored without significantly affecting our results.

[8] In the case in which the two agents are equally wealthy, that is $\lambda = 1$, $\sigma_{SM} = 0$, and stock volatility is as in a standard economy. This is because the sign of σ_{SM} depends on which agent is wealthier; when both agents are equally wealthy, their impacts on stock volatility offset each other and $\sigma_{SM} = 0$. This issue is purely a byproduct of our model having only two agents and would disappear in an economy with more agents. As a result, we assume $\lambda = 1/3$ in our numerical analyses.

that should be compared with the true $\mu_M = 5.20\%$. Even though the mitigating effect of updating (when $v(0) > 0$) is significant, the impact of heterogeneous beliefs remains large even with a relatively large prior variance ($v(0) = (1\%)^2$; that is, the width of agents' 95% confidence band for μ_M is approximately 5%), when the effect of updating is strongest.

The equity premium equals, adopting the "full information" perspective of an observer who would know the true expected money supply growth,[9]

$$\mu_S(t) - r(t) = \alpha\sigma_\varepsilon^2 + \frac{\sigma_{SM}(t)}{\sigma_{MM}}\left\{\mu_M - \left[\frac{1}{1+\lambda(t)^{1/\alpha}}\mu_M^1(t) + \frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}}\mu_M^2(t)\right]\right\}. \quad (11)$$

The first term equals the value of the equity premium in a benchmark economy without heterogeneity in beliefs. The term in the square brackets is essentially a wealth-weighted average of the agents' beliefs. Thus, how the equity premium in our model departs from this standard value depends on two things: the sign of σ_{SM}, on one hand, and the difference between the average beliefs and the true expected money growth, on the other. The equity premium is higher than in a benchmark economy if either the investors who are optimistic on money supply growth are wealthier (so that $\sigma_{SM} > 0$) and the wealth-weighted average beliefs on μ_M are lower than the true value, or if the pessimistic investors are wealthier and the average beliefs are higher. This is intuitive. Take the case in which $\sigma_{SM} > 0$. If the agents' average beliefs are lower than the true value of μ_M, it is more likely that agents will see (given their overly pessimistic beliefs) positive shocks to the money supply, leading to increases in the stock price. And conversely when $\sigma_{SM} < 0$. Thus, our model could potentially play a role in explaining the equity premium puzzle, as the second term in the right-hand side of (11) is typically of a much higher magnitude than the first standard term. For example, for the above parameters, assuming that the more optimistic agents have correct beliefs (i.e., $\mu_M^1(0) = \mu_M = 5.20\%$), the heterogeneity in beliefs (for $\bar{\mu}_M = 2$ and $v(0) = (1\%)^2$) makes the equity premium about 24 times higher than in a benchmark economy (4.49% versus 0.19%). Equation (11) shows that a large equity premium, in our model, can only arise if the wealth-weighted average belief is very different from its true value; in other words, it is mostly caused by agents' biased expectations, not by heterogeneous beliefs. Thus, our results on the equity premium are not very robust because they critically depend on the specific assumptions made about each agent's beliefs.

[9] In other words, we compute the stock expected return under the objective, "historical" probability measure, as opposed to the agents' subjective beliefs. This is the value of the equity premium that should be compared to the data.

3.3. The Logarithmic Preferences Case

For this subsection, to be able to compute nominal quantities, we further narrow our work to the special case of logarithmic preferences (Equation (6)), in which the price of money and the nominal interest rate are as provided in Proposition 3.2.

Proposition 3.2. *In equilibrium, the real money price and nominal interest rate are as follows*:

$$p(t) = \frac{1-\phi}{\phi}\left(\int_t^T e^{-\delta(s-t)} h^1(M(t), t, s)\, ds + \lambda(t)\int_t^T e^{-\delta(s-t)} h^2(M(t), t, s)\, ds\right) \cdot (1+\lambda(t))^{-1}\frac{\varepsilon(t)}{M(t)}, \quad (12)$$

$$R(t) = (1+\lambda(t)) \cdot \left(\int_t^T e^{-\delta(s-t)} h^1(M(t), t, s)\, ds + \lambda(t)\int_t^T e^{-\delta(s-t)} h^2(M(t), t, s)\, ds\right)^{-1}, \quad (13)$$

where

$$h^i(M(t), t, s) = E_t^i\left[\frac{M(t)}{M(s)}\right] = \exp\left[(\sigma_M^2 - \mu_M^i(t))(s-t) + \frac{1}{2}v(t)(s-t)^2\right], \quad i = 1, 2. \quad (14)$$

Both money price and nominal interest rate are equal to wealth-weighted averages of their values in an (otherwise identical) homogeneous beliefs economy populated only with agents of each type, 1 and 2, as is typical with heterogeneous beliefs (Detemple and Murthy 1994). In the homogeneous beliefs economy, the price of money—or the present value of future services rendered by money, measured by the marginal utility of money holdings—is given by $p(t) = ((1-\phi)/\phi)(\varepsilon(t)/M(t))E^i[\int_t^T (e^{-\delta s}/M(s))\, ds]/(e^{-\delta t}/M(t))$. $E^i[\int_t^T (e^{-\delta s}/M(s))\, ds]/(e^{-\delta t}/M(t))$ is a measure of the expected future marginal utilities of money (relative to the current marginal utility), which are driven by money supply growth. The real price of money is decreasing in the current quantity of money, but it is also decreasing in μ_M^i: the more the money supply is expected to grow in the future, the less valuable money becomes.

The nominal short rate is the payoff an investor is willing to give up to hold a money balance, which equals, at the agent's optimum, the marginal utility of holding money relative to marginal utility of consumption. In the logarithmic case, we have $R(t) = ((1-\phi)/\phi)\varepsilon(t)/(p(t)M(t))$. Substituting $p(t)$, we see that, in the homogeneous belief economy with agents of type i, $R(t) = 1/E^i[\int_t^T e^{-\delta(s-t)}(M(t)/M(s))\, ds]$: the nominal interest rate is driven by the expected future growth of money. In our heterogeneous beliefs economy, the nominal short rate is additionally affected by the distribution of wealth, which fluctuates in response to monetary shocks. Interestingly, despite the myopic behavior of logarithmic investors, future expectations of money supply growth play a key role in the pricing of nominal assets.

3.3.1. Inflation. Whereas its effect on expected inflation (omitted for brevity) is ambiguous, the heterogeneity in beliefs is revealed to increase the volatility of the price of money, whose diffusion coefficients are

$$\begin{aligned}\sigma_{pM}(t) = -\sigma_{MM} - &\left(\int_t^T e^{-\delta(s-t)} h^2(M(t),t,s)\,ds - \int_t^T e^{-\delta(s-t)} h^1(M(t),t,s)\,ds\right)\\ &\cdot\left(\int_t^T e^{-\delta(s-t)} h^1(M(t),t,s)\,ds + \lambda(t)\int_t^T e^{-\delta(s-t)} h^2(M(t),t,s)\,ds\right)^{-1}\\ &\cdot\left(\frac{\lambda(t)}{1+\lambda(t)}\right)\bar{\mu}_M(t) - (s-t)\frac{v(t)}{\sigma_{MM}},\\ \sigma_{p\varepsilon}(t) = \sigma_\varepsilon - \sigma_{M\varepsilon}&.\end{aligned}$$

The higher the heterogeneity in beliefs, the more the price of money drops in response to unexpected increases in the money supply (as the numerator in the second term in the expression for σ_{pM} is positive, and increases with heterogeneity in beliefs). This is intuitive: When such a shock occurs, not only does the quantity of money increase, but the investor who expects higher money growth wins his bet, and so his weight in the economy increases; because he is also the one who expects higher inflation, the price of money decreases further. In addition, such a shock leads both agents to increase their estimates of expected money growth, further reducing the value of money. These effects reinforce each other, leading to more volatile inflation. This suggests that the volatility of inflation could be reduced by greater transparency of monetary policy (leading to reduced heterogeneity in beliefs and prior belief variance), as is consistent with empirical studies (Dincer and Eichengreen 2007).

3.3.2. Nominal Interest Rates. Proposition 3.3 characterizes the nominal term structure in our economy.

Proposition 3.3. *The time t zero-coupon yield with maturity τ is given by*

$$\begin{aligned}R(t,\tau) = &-\frac{1}{\tau}\ln\left[\int_{t+\tau}^T e^{-\delta s}(h^1(M(t),t,s) + \lambda(t)h^2(M(t),t,s))\,ds\right]\\ &+\frac{1}{\tau}\ln\left[\int_t^T e^{-\delta s}(h^1(M(t),t,s) + \lambda(t)h^2(M(t),t,s))\,ds\right]. \quad (15)\end{aligned}$$

Most comparative statics are ambiguous, because when heterogeneity in beliefs increases, it could be either that the more optimistic agent becomes more optimistic or the less optimistic agent becomes less optimistic, or both, and the effect on the term structure is different in each case. Nonetheless, our numerical analyses suggest that the yield curve is generally increasing, and that all three term structure factors—level (average level of bond yields for maturities between 0 and 30 years), slope (difference between the 30-year yield and the instantaneous short rate), and curvature (twice the 15-year yield minus the sum of the instantaneous short rate and the 30-year yield)—are generally decreased by heterogeneity in beliefs. Bomberger and Frazer (1981) empirically show that a higher dispersion of inflation forecasts has a negative impact on interest rates, consistently with our results. The effects are not very large: In our example, assuming that the wealth-weighted average beliefs are equal to the true value, raising the amount of disagreement $\bar{\mu}_M(0)$ from 0 to 2 causes respective drops of approximately 20, 14, and 3 basis points in the level, slope, and curvature of the yield curve. Raising the prior variance causes an additional drop of approximately 15 basis points in the level (and smaller drops in slope and curvature). Interestingly, unlike the case of the stock market, increasing the level of disagreement ($\bar{\mu}_M(0)$) and increasing the prior variance ($v(0)$) have similar effects on the shape of the term structure. This is intuitive: The higher the variance of prior beliefs, the more the agents update their beliefs. And, in this case, updating reinforces the impact of differences in beliefs: on one hand, a shock in the money supply increases the weight of the agents who just "won their bet," leading to a change of the average wealth-weighted expected money growth; on the other hand, with updating, the shock leads both agents to update their beliefs in the same direction.

Whereas the effects on the shape of the nominal term structure are not very large, this is not the case for the volatilities of nominal yields, which are as provided by Proposition 3.4.

Proposition 3.4. *The volatility of the zero-coupon yield with maturity τ, $\sigma_R(t,\tau)$ (which is such that $dR(t,\tau) = \mu_R^i(t,\tau)dt + \sigma_R(t,\tau)dw_M^i(t)$) is given by*

$$\begin{aligned}\sigma_R(t,\tau) &= \frac{1}{\tau}\left(\int_{t+\tau}^T e^{-\delta s}\left[\frac{v(t)}{\sigma_{MM}}(s-t)h^1(M(t),t,s)\right.\right.\\ &\left.\left.+\lambda(t)h^2(M(t),t,s)\left(\frac{v(t)}{\sigma_{MM}}(s-t)+\bar{\mu}_M(t)\right)\right]ds\right)\end{aligned}$$

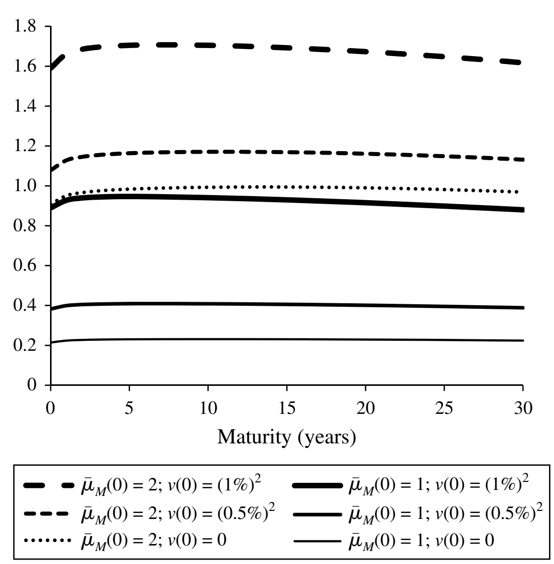
$$\cdot\left(\int_{t+\tau}^{T} e^{-\delta s}(h^1(M(t),t,s)+\lambda(t)h^2(M(t),t,s))\,ds\right)^{-1}$$
$$-\frac{1}{\tau}\left(\int_t^T e^{-\delta s}\left[\frac{v(t)}{\sigma_{MM}}(s-t)h^1(M(t),t,s)\right.\right.$$
$$\left.\left.+\lambda(t)h^2(M(t),t,s)\left(\frac{v(t)}{\sigma_{MM}}(s-t)+\bar{\mu}_M(t)\right)\right]ds\right)$$
$$\cdot\left(\int_t^T e^{-\delta s}(h^1(M(t),t,s)+\lambda(t)h^2(M(t),t,s))\,ds\right)^{-1}. \quad (16)$$

Bond yield volatility is increased by heterogeneity in beliefs as, under full information and homogeneous beliefs ($\bar{\mu}_M(0)=v(0)=0$), yield volatilities would equal zero. This could help explain the excessive volatility of long-term bond yields (relative to expectations of short rates), a puzzle pointed out by Shiller (1979). To help assess the magnitude of this phenomenon, Figure 2 provides the term structure of yield volatilities in our example. Our results demonstrate that heterogeneous beliefs on money growth could play a significant role in generating the high volatility. In addition, as in the case of the shape of the term structure, the effects of increased heterogeneity in beliefs ($\bar{\mu}_M(0)$) and increased prior variance ($v(0)$) are similar, both leading to increased yield volatility. The curve exhibits a hump, which is realistic (see, e.g., Figure 1 in Dai and Singleton 2003), and for sufficiently high heterogeneity and prior variance (e.g., $\bar{\mu}_M(0)=2$, $v(0)=(1\%)^2$, which are not implausible), the bond yield volatilities generated by the model are close to real-life levels (167 basis points on average across maturities).

3.4. Implications for Monetary Policy

A recent vein of literature has focused on interest rate rules, such as the Taylor rule. Even though we model

Figure 2 **Annualized Yield Volatility (%)**



monetary policy differently, with a constant average growth of money supply (as in Friedman's k-percent rule; see Evans and Honkapohja 2003 for a comparison), we can provide new insights relevant for the evaluation of the Taylor rule. In our model, both interest rate and inflation are endogenous, and we find that their relationship bears some similarity with a Taylor rule. This can be seen by substituting the price of money (Equation (12)) into the nominal short rate formula (13), which yields the following:

$$R(t)=\frac{1-\phi}{\phi}\frac{\varepsilon(t)}{M(t)}\frac{1}{p(t)}.$$

Ceteris paribus, the nominal interest rate is proportional to the inverse of the price of money, which itself can be thought of as the accumulated past inflation. Any inflation that is not directly caused by fluctuations in the money supply or real output (which also appear in the equation) will automatically lead to a higher nominal interest rate. (Note that this effect is different from nominal rates rising to offset higher expected inflation because it is caused by past inflation.) And any shock (in beliefs and/or distribution of wealth) that affects both inflation and nominal interest rate will do so in a way consistent with the Taylor rule (i.e., it will affect these two variables in opposite ways). The intuition is as follows: The same type of shock (unexpected increase in money supply) that increases inflation also gives more weight to agents who expect money supply to grow more in the future. These agents demand a higher nominal interest rate to offset future inflation. Thus, high inflation causes a rise in nominal rates.

This relationship holds only in the presence of heterogeneous beliefs because, in our model, under homogeneous beliefs the nominal interest rate is deterministic and realized inflation has no impact on it. This suggests that the effect of implementing a Taylor rule could be less than anticipated because a similar negative relationship between inflation and interest rate already holds in equilibrium. Because our equilibrium bears some similarity to one in which a Taylor rule is enforced, it is not surprising that the overall impact of heterogeneous beliefs on nominal interest rates, a drop in their level, is somewhat similar to that caused by such a rule, as demonstrated by Gallmeyer et al. (2009).

3.5. Extension: Heterogeneous Beliefs on Consumption Growth

We now assume that investors hold individual-specific beliefs not just on money supply, but also on consumption growth. The perceived dynamics of consumption and money supply are

$$d\varepsilon(t)=\varepsilon(t)[\mu_\varepsilon^i(t)dt+\sigma_\varepsilon dw_\varepsilon^i(t)],$$
$$dM(t)=M(t)[\mu_M^i(t)dt+\sigma_{M\varepsilon}dw_\varepsilon^i(t)+\sigma_{MM}dw_M^i(t)].$$

The normalized difference in beliefs is now the vector process

$$\bar{\mu}(t) = \begin{bmatrix} \bar{\mu}^i_\varepsilon(t) \\ \bar{\mu}^i_M(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\mu^1_\varepsilon(t) - \mu^2_\varepsilon(t)}{\sigma_\varepsilon} \\ \dfrac{\mu^1_M(t) - \mu^2_M(t)}{\sigma_{MM}} \end{bmatrix}.$$

Standard arguments (Liptser and Shiryaev 2000) imply that, assuming that investors' initial variances are equal ($v^1(0) = v^2(0) = v(0)$), the normalized difference of beliefs solves an ordinary (matrix) differential equation, and is therefore deterministic, as in the case of updating expected money growth only. It can be shown easily that the dynamics of the stochastic weighting obey

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_\varepsilon(t)dw^1_\varepsilon(t) - \left(\bar{\mu}_M(t) - \frac{\sigma_{M\varepsilon}}{\sigma_{MM}}\bar{\mu}_\varepsilon(t)\right)dw^1_M(t).$$

The investors now bet on both the fluctuations of the money supply and those of consumption. The distribution of wealth, accordingly, fluctuates based on both sources of risk. Despite the extra factor of risk affecting the stochastic weighting, its volatility can be either lowered or increased. This is because speculative bets made on consumption could either increase or offset the effect of those made on money supply. For example, if $\sigma_{M\varepsilon} > 0$, the volatility of the distribution of wealth is increased if the agent who is more optimistic on money growth is more pessimistic on consumption growth (i.e., $\bar{\mu}_M$ and $\bar{\mu}_\varepsilon$ have opposite signs) and decreased otherwise. As a result, the effects of disagreement on consumption growth are generally ambiguous.

For clarity, we assume no updating (i.e., $v^1(0) = v^2(0) = 0$) for the remainder of this section. (The formulas with updating that are used in the calibration of §4.2 are provided in Appendix B. The only substantial changes relative to the no-updating case described below concern the magnitude of the effects, in a way that is similar to the effects of updating described above.) Results are presented without proof as they are only slight variations on earlier ones.

3.5.1. Stock Pricing Under Power Preferences. Assuming that α is an integer, the stock price is given by

$$S(t) = \frac{\varepsilon(t)}{(1+\lambda(t)^{1/\alpha})^\alpha} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \lambda(t)^{j/\alpha} \frac{\exp[G(j)(T-t)]-1}{G(j)},$$

where

$$G(j) = ((j(j-\alpha))/(2\alpha^2))[\bar{\mu}^2_\varepsilon + (\bar{\mu}_M - (\sigma_{M\varepsilon}/\sigma_{MM})\bar{\mu}_\varepsilon)^2] + (1-\alpha)(\mu_\varepsilon - \tfrac{1}{2}\alpha\sigma^2_\varepsilon) - ((\alpha j - j^2)/(2\alpha^2))\bar{\mu}^2_M - \delta.$$

The stock volatility coefficients are given by

$$\sigma_{S\varepsilon}(t) = \sigma_\varepsilon + \left(\frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}} - \frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}(j/\alpha)((\exp[G(j)(T-t)]-1)/(G(j)))}{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}((\exp[G(j)(T-t)]-1)/(G(j)))}\right)\bar{\mu}_\varepsilon$$

$$\sigma_{SM}(t) = \left(\frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}} - \frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}(j/\alpha)((\exp[G(j)(T-t)]-1)/(G(j)))}{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}((\exp[G(j)(T-t)]-1)/(G(j)))}\right) \cdot \left(\bar{\mu}_M - \frac{\sigma_{M\varepsilon}}{\sigma_{MM}}\bar{\mu}_\varepsilon\right).$$

Both types of disagreement have a similar effect in that they raise volatility (and reduce the price) compared with a standard economy (where $\bar{\mu}_M = \bar{\mu}_\varepsilon = 0$). However, relative to an economy with heterogeneous beliefs on money only (i.e., $\bar{\mu}_\varepsilon = 0$), the effect of disagreement on consumption is ambiguous, as the positions taken to bet on consumption could partially offset those taken to bet on money growth.

3.5.2. Nominal Asset Pricing Under Logarithmic Preferences. To be able to analyze the monetary side of the economy, we now focus on logarithmic preferences (Equation (6)). Because of separable preferences, expressions for nominal interest rates ((12) and (15)) are unaffected by heterogeneous beliefs on consumption. However, because the dynamics of λ are affected, bond yields are affected by real shocks (fluctuations in w_ε). The volatility coefficients of bond yields (that are such that $dR(t,\tau) = \mu^i_R(t,\tau)dt + \sigma_{RM}(t,\tau)dw^i_M(t) + \sigma_{R\varepsilon}(t,\tau)dw^i_\varepsilon(t)$) are

$$\sigma_{RM}(t,\tau) = \frac{\sigma_R(t,\tau)}{\bar{\mu}_M}\left(\bar{\mu}_M - \frac{\sigma_{M\varepsilon}}{\sigma_{MM}}\bar{\mu}_\varepsilon\right),$$

$$\sigma_{R\varepsilon}(t,\tau) = \frac{\sigma_R(t,\tau)}{\bar{\mu}_M}\bar{\mu}_\varepsilon,$$

where $\sigma_R(t,\tau)$ is as in (16) (with 0 substituted for $v(t)$ given the no-updating assumption).

The total volatility of bond yields, $(\sigma^2_{RM} + \sigma^2_{R\varepsilon})^{1/2}$, is multiplied by $[(\bar{\mu}_M - (\sigma_{M\varepsilon}/\sigma_{MM})\bar{\mu}_\varepsilon)^2 + \bar{\mu}^2_\varepsilon]^{1/2}/|\bar{\mu}_M|$ compared with the case without heterogeneous beliefs on consumption (Equation (16)). As a result, the shape of the term structure of volatilities is not affected by heterogeneous beliefs on consumption; only its level is affected. For example, taking into account the correlation between money and consumption in our sample (-0.06), when $\bar{\mu}_M = 2$, bond yield volatilities are multiplied by 1.145 if $\bar{\mu}_\varepsilon = 1$ and 1.091 if $\bar{\mu}_\varepsilon = -1$.

4. Empirical Analyses

In this section, we empirically examine some of the implications of our model, by assessing the impact of heterogeneous beliefs on money growth on stock and bond market volatility, and by performing a calibration of the amount of heterogeneity in beliefs that could, based on our model, explain the observed behavior of asset prices.

4.1. Heterogeneous Beliefs on Money Growth and Market Volatility

4.1.1. Data on Heterogeneous Beliefs. Because data on heterogeneity in beliefs are not readily available, we use the SPF, published by the Philadelphia Fed, to construct a quarterly index of heterogeneity in beliefs. The SPF provides forecasts by 30 economists (on average) of various economic variables. Because these do not include money supply growth, we use changes in the GDP deflator (defined as Nominal GDP/Real GDP) as a proxy.[10]

This approximation can be justified as follows: In our paper, the GDP deflator is $1/p$ because p denotes the price of money in terms of the consumption good. A simple application of Itô's lemma and Equation (3) show that the difference in expected changes in $1/p$ across agents is equal to $(\sigma_{pM}/\sigma_{MM})(\mu_M^1 - \mu_M^2)$. So the difference in expected changes in the GDP deflator across investors is proportional to the amount of disagreement $\mu_M^1 - \mu_M^2$.[11]

Our index of heterogeneity in beliefs is constructed as follows. Given individual i's forecast for the GDP deflator, we calculate the growth rate, $GDEF_t^i$, and the median $GDEF_t^m$ of all forecasts. If $GDEF_t^i$ is greater than $GDEF_t^m$, we call individual i "optimistic"; if not, he is "pessimistic." We calculate the median forecast for all optimistic (pessimistic) investors' forecasts, denoted by $GDEF_t^O$ ($GDEF_t^P$). The differences in beliefs index is then given by $DB_{M,t} = GDEF_t^O - GDEF_t^P$ (our proxy for $\mu_M^1 - \mu_M^2$ in our theoretical model). We conduct our analysis (with quarterly data) for the period lasting from the first quarter of 1986 (given the availability of VIX data that we use to control for overall uncertainty) to the last quarter of 2011.

To further verify that we are testing the effect of disagreement on money supply (rather than the general level of uncertainty in the economy), we will also investigate the effect of disagreement on real GDP (evaluated, based on SPF data, using the same procedure as disagreement on money growth), as well as volatility of dividend growth (based on CRSP data and estimated because of wide swings in daily dividend data and the presence of volatility clustering, using an EGARCH model as in Li and Yang 2013), volatility of money growth (based on Fed data of M2 money supply and also estimated using an EGARCH model given the low frequency of available data), and VIX index of implied market volatility.[12] In our model, market volatility is an endogenous variable that is affected by the level of heterogeneous beliefs, so to isolate the information not related to these and avoid an endogeneity issue, we employ in the regressions the residual of a regression of the VIX index on heterogeneous beliefs on money and real output growth. Our goal will be to determine whether disagreement on money supply provides some extra, nonredundant information, in addition to these variables, for understanding volatility.

4.1.2. Bond Volatility. According to our model with logarithmic preferences, bond yield volatilities (Equation (16)) are approximately linear in heterogeneity in beliefs ($\bar{\mu}_M$). Accordingly, a linear regression model should capture the effect of heterogeneous beliefs on yield volatility. We run a regression of quarterly U.S. Treasury bond yield standard deviation (using daily yields provided by the Federal Reserve) on differences in beliefs (Model 1a).

We also run a multiple regression of yield volatilities on differences in beliefs on money growth and real output, volatilities of dividend and money supply growth, and the VIX index residual (Model 2a). This will allow us to check that our results are not simply due to the general level of uncertainty and disagreement in the economy. Finally, because of the persistence in volatility, we run regressions additionally involving one quarter lagged volatility (Models 1b and 2b). Our results are provided in Table 1 for several bond maturities (1, 5, 10, and 30 years). For brevity, intercept terms are omitted.

[10] Results obtained using the CPI are qualitatively similar, although the significance is slightly lower.

[11] This is an approximation because the coefficient σ_{pM} is endogenous and changes over time. Although our goal is not to perform a full empirical test of the model, verifying that this approximation does not qualitatively invalidate our conclusions is important. For the bond volatility case, we can verify this is in the following way. From Equation (15), we find that, for sufficiently large levels of heterogeneity ($\bar{\mu}_M > 0.4$ approximately), σ_{pM} is approximately proportional to $\bar{\mu}_M^{2/3}$, implying that $\bar{\mu}_M$ is approximately proportional to the proxy we are using, $\mu_p^1 - \mu_p^2$, raised to the power 1/3. This approximation can be shown algebraically and verified numerically. Such levels of disagreement are not implausible because in our model $\bar{\mu}_M = (\mu_p^1 - \mu_p^2)/\sigma_{pM}$, and in the SPF data we observe that the amount of disagreement is typically around 0.5 to 1 times the standard deviation of inflation (about 1% per year). Based on this, as a robustness check we additionally ran the regressions on the disagreement on inflation expectations raised to the power 1/3 and found that this did not affect our conclusions qualitatively (based on R^2 and levels of significance).

[12] Following Ang et al. (2006), we use the pre-2003 VIX index, denoted by the symbol VXO, because it is available (from the CBOE) from 1986 and onward, whereas the new index has been constructed by backfilling only to 1990, and the correlation between both indices is very close to one.

Table 1 Regression of Bond Yield Volatilities on Differences in Beliefs

	Model 1a	Model 1b	Model 2a	Model 2b
$\tau = 1$ year				
DB on money growth	1.247 (1.734)*	1.054 (1.379)	1.035 (1.497)	0.830 (1.175)
DB on real output			0.817 (1.276)	0.692 (1.043)
VIX residual			1.275 (1.036)	1.094 (0.879)
Dividend volatility			−0.047 (−0.859)	−0.069 (−1.315)
Money volatility			0.247 (0.591)	0.195 (0.446)
Lagged yield volatility		0.324 (3.531)***		0.125 (1.996)**
R^2	0.025	0.093	0.080	0.153
$\tau = 5$ years				
DB on money growth	1.639 (1.974)**	1.387 (1.792)*	1.573 (1.835)*	1.281 (1.467)
DB on real output			0.857 (1.493)	0.659 (1.087)
VIX residual			1.846 (1.659)	1.625 (1.498)
Dividend volatility			−0.067 (−1.081)	−0.088 (−1.558)
Money volatility			0.315 (0.849)	0.286 (0.762)
Lagged yield volatility		0.148 (4.866)***		0.189 (2.154)**
R^2	0.039	0.194	0.218	0.274
$\tau = 10$ years				
DB on money growth	2.437 (2.669)***	1.762 (2.087)**	1.976 (2.185)**	1.639 (1.812)*
DB on real output			1.038 (1.602)	0.945 (1.491)
VIX residual			2.475 (1.854)*	2.036 (1.502)
Dividend volatility			−0.095 (−1.269)	−0.117 (−1.652)
Money volatility			0.482 (1.197)	0.360 (0.903)
Lagged yield volatility		0.187 (3.873)***		0.205 (2.791)***
R^2	0.055	0.213	0.240	0.296
$\tau = 30$ years				
DB on money growth	3.317 (3.259)***	2.531 (2.315)**	2.256 (2.196)**	1.989 (1.941)*
DB on real output			1.516 (1.901)*	1.429 (1.705)*
VIX residual			3.045 (1.981)**	2.872 (1.798)*
Dividend volatility			−0.127 (−1.352)	−0.152 (−1.694)*
Money volatility			0.506 (1.253)	0.440 (1.095)
Lagged yield volatility		0.347 (3.154)***		0.314 (2.901)***
R^2	0.128	0.249	0.291	0.388

Notes. t-Statistics are reported in parentheses. DB, differences in beliefs.
*, **, and *** indicate that the coefficients are statistically significant at the 10%, 5%, and 1% levels, respectively.

In a simple regression, we find that differences in beliefs on money growth have an impact on all yield volatilities. The longer the maturity, the more significant this effect is at the 10% level for bonds of all maturities, at the 5% level for all maturities equal to or longer than 5 years, and at the 1% level for all maturities equal to or longer than 10 years.

Turning to the impact of other variables (disagreement on consumption growth, volatilities of dividend and money supply, and VIX residual), we find that among all variables, the differences in beliefs on money growth variable is clearly the one with the most significant impact.

Lagged volatility, once it has been included, becomes the most significant variable, because of persistence in volatility. This reduces the significance of other variables, but heterogeneous beliefs on money remain significant for maturities of 10 years or more, and they also remain the most significant among all other variables. In addition, the magnitude of the associated coefficient is not affected dramatically when extra variables are included, suggesting a robust effect.

In short, we find that differences in beliefs on money growth provide some valuable information for understanding fluctuations in bond yield volatility, and that this information is clearly more valuable than differences in beliefs on real economic growth. For longer maturities especially, the variable for differences in beliefs on money growth remains significant after controlling for overall economic uncertainty and persistence in volatility.

4.1.3. Stock Volatility. We now turn to the impact of heterogeneous beliefs on money growth on stock volatility. According to our theoretical model, the stock volatility is given by $(\sigma_{S\varepsilon}^2 + \sigma_{SM}^2)^{1/2}$, where $\sigma_{S\varepsilon}$ and σ_{SM} are as in Equation (10). Because, based on our numerical analyses, we expect the effect of consumption volatility ($\sigma_{S\varepsilon}$) to be relatively small compared with that of heterogeneous beliefs and σ_{SM} is linear in heterogeneity in beliefs (Equation (10)), a linear regression model should, again, capture the effect of heterogeneous beliefs on stock volatility. Accordingly, we perform a linear regression of quarterly realized S&P 500 daily return volatility (obtained from CRSP) on differences in beliefs (Model 3a). As in the case of bond yield volatility, we also run a multiple regression including differences in beliefs on money growth and real output,

Table 2 Regression of Stock Volatility on Differences in Beliefs

	Model 3a	Model 3b	Model 4a	Model 4b
DB on money growth	12.125 (2.367)**	8.527 (1.984)**	11.293 (2.035)**	9.381 (1.754)*
DB on real output			6.395 (1.943)*	5.619 (1.739)*
VIX residual			1.367 (1.758)*	1.274 (1.711)*
Dividend volatility			0.681 (0.937)	0.593 (0.823)
Money volatility			8.246 (1.317)	7.356 (1.169)
Lagged stock volatility		0.357 (3.581)***		0.254 (2.397)**
R^2	0.057	0.219	0.237	0.313

Notes. t-Statistics are reported in parentheses. DB, differences in beliefs.
*, **, and *** indicate that the coefficients are statistically significant at the 10%, 5%, and 1% levels, respectively.

volatilities of dividend and money supply growth, and the VIX index residual (Model 4a), and regressions additionally involving one quarter lagged volatility (Models 3b and 4b). Our results are provided in Table 2.

The results are qualitatively similar to the case of bond yields: The amount of heterogeneous beliefs on money growth has a significant impact on stock volatility. The degree of significance is comparable to long-term bond yields. That impact persists once other measures of disagreement and uncertainty are taken into account, heterogeneous beliefs on money growth remaining the most significant variable, and the value of its coefficient does not change a lot. As in the case of bond yields, because of persistence in volatility, lagged volatility becomes the most significant variable once added, but heterogeneous beliefs on money growth remain the most significant among other variables. The impact of heterogeneous beliefs on the growth of the real economic growth is more significant than in the case of bond yields, but remains less significant than that of heterogeneous beliefs on money.

Although these results are no doubt preliminary and imperfect (and R^2 are not very high, evidence that heterogeneity in beliefs only accounts for a limited share of volatility), they suggest that disagreement on money growth plays a significant role in generating volatility, especially for long maturity securities—stocks and bonds with maturities of 10 years and longer. More interestingly perhaps, this role seems specific to money growth, in that disagreement on real output growth, as well as several measures of economic uncertainty, does not provide the same information and does not appear to play as robust a role.

4.2. A Calibration

To assess whether our model generates plausible values for economic variables (stock and bond volatilities, shape of the term structure, and equity premium), we perform a calibration exercise in which, using our model, we assess the level of disagreement that could account for the observed values of these variables. For added realism, we use the extension of the model with heterogeneous beliefs on both money and consumption (formulas in Appendix B).

Our technique involves the following steps:

Step 1. Based on the fluctuations in money supply and consumption over the estimation period (1986–2011), we infer the quarterly fluctuations in the two underlying Brownian motions w_ε and w_M:

$$\Delta w_\varepsilon(t) = \frac{1}{\sigma_\varepsilon}\left[\ln\left(\frac{\varepsilon(t)}{\varepsilon(t-\Delta(t))}\right) - \left(\mu_\varepsilon - \frac{1}{2}\sigma_\varepsilon^2\right)\Delta t\right],$$

where ε refers to historical data and μ_ε, σ_ε are historical estimates and $\Delta t = 0.25$ year. The trajectory of w_M is estimated in a similar fashion.

Step 2. For any set of initial beliefs parameters $\Theta = (\bar{\mu}_\varepsilon(0), v_\varepsilon(0), \bar{\mu}_M(0), v_M(0))$, we can then compute, for $j = \varepsilon, M$, the evolution of beliefs over time:

$$\mu_j^i(\Theta, t) = \mu_j^i(\Theta, t-\Delta t) + \frac{v_j(\Theta, t)}{\sigma_j} \cdot \left[\frac{\mu_j - \mu_j^i(\Theta, t-\Delta t)}{\sigma_j}\Delta t + \Delta w_j(t)\right],$$

where v_j is as in Equation (1). Given initial beliefs that are chosen so that the wealth-weighted average of individual beliefs is equal to the true value (average growth rate over the whole estimation period), we deduce the evolution of the stochastic weight:

$$\begin{aligned}\lambda(\Theta, t) &= \lambda(\Theta, t-\Delta t)\exp\left\{-\frac{1}{2}[(\bar{\mu}_\varepsilon(\Theta, t))^2 + (\bar{\mu}_M(\Theta, t))^2]\Delta t\right.\\ &\quad - \bar{\mu}_\varepsilon(\Theta, t)\left(\frac{\mu_\varepsilon - \mu_\varepsilon^1(\Theta, t-\Delta t)}{\sigma_\varepsilon}\Delta t + \Delta w_\varepsilon(t)\right)\\ &\quad \left. - \bar{\mu}_M(\Theta, t)\left(\frac{\mu_M - \mu_M^1(\Theta, t-\Delta t)}{\sigma_M}\Delta t + \Delta w_M(t)\right)\right\}.\end{aligned}$$

Step 3. We then deduce, from the formulas in Appendix B, for any set of belief parameters Θ, interest rates and stock prices implied by the model.[13]

[13] Our calibration procedure requires an approximation because we can only obtain explicit expressions for nominal interest rates in the logarithmic utility case, but real asset prices are unaffected by heterogeneity in beliefs in that case. Thus, our calibration uses the

Table 3 Calibration Results

	Data	Model with heterogeneous beliefs	Benchmark economy
Stock volatility	0.1549	0.1462 (1.5994)	0.0249
Stock average return	0.0638	0.0797 (−0.6049)	0.0738
Real interest rate	0.0393	0.0619 (−26.4541)	0.0720
1-year nominal yield volatility	0.0143	0.0115 (2.1723)	0
5-year nominal yield volatility	0.0163	0.0150 (1.4614)	0
30-year nominal yield volatility	0.0183	0.0167 (1.4743)	0
Nominal yield curve level	0.0535	0.0851 (−20.9921)	0.0769
Nominal yield curve slope	0.0192	0.0105 (10.5743)	0.0061
Nominal yield curve curvature	0.0188	0.0049 (20.2229)	−0.0013
$\bar{\mu}_M(0)$		0.5482	0
$\bar{\mu}_\varepsilon(0)$		0.2628	0
$v_M(0)$		$(0.2388\%)^2$	0
$v_\varepsilon(0)$		$(0.7034\%)^2$	0

Note. t-Statistics are reported in parentheses.

Step 4. Finally, we choose the optimal set of belief parameters Θ so as to fit the following variables: stock return and volatility, real short interest rate, level, slope, and curvature of the nominal term structure, volatilities of 1-, 5-, and 30-year nominal interest rates. Specifically, the optimal beliefs solve the following problem:

$$\min_{\Theta} e(\Theta)'\Sigma^{-1}e(\Theta),$$

where $e(\Theta)$ is the column vector containing, for each of the nine variables, the sum across all periods of the absolute values of the relative errors committed by the model relative to the data. For example, the element of the vector $e(\Theta)$ corresponding to the stock return is

$$e(r_S(\Theta)) = \sum_{t=\Delta t}^{T_1} \left| \frac{\ln(S(\Theta,t)/S(\Theta,t-\Delta t))}{\ln(S(t)/S(t-\Delta t))} - 1 \right|,$$

where T_1 denotes the end of the calibration period (final quarter of 2011), $S(\Theta, t)$ the model-implied stock price, and $S(t)$ the actual stock price. The matrix Σ is the Newey-West covariance matrix of estimation errors.

In short, assuming that the model is true, we find the values of the initial beliefs parameters that lead to asset prices as close as possible to those observed in the calibration period, given the evolution of aggregate consumption and money supply in the data.

expressions derived under logarithmic utility for nominal interest rates and those obtained under CRRA utility (with relative risk aversion equal to three) for real asset prices. Mixing two levels of risk aversion in our calibration is an approximation, but it should be pointed out that much of the calibration procedure (Steps 1 and 2 above) is independent of the value of risk aversion. Thus, the calibrated values we obtain for asset prices are the actual ones generated by the model, not approximations. Our results are conservative in that we do not calibrate the level of risk aversion; results likely would be more favorable to the model if we could solve for all variables under general CRRA utility and calibrate the level of risk aversion, but doing so presents a formidable technical challenge.

Our results are summarized in Table 3. The values in parentheses are t-statistics for testing equality of the means of actual and calibrated values. For comparison, we also provide the values in a benchmark economy with full information and homogeneous beliefs (i.e., $\bar{\mu}_M(0) = \bar{\mu}_\varepsilon(0) = v_M(0) = v_\varepsilon(0) = 0$). The parameter values $(\mu_M, \sigma_M, \mu_\varepsilon, \sigma_\varepsilon, \rho, \delta, \alpha, T)$ are as in our previous numerical examples. The initial relative weight of the two agents $\lambda(0)$ is set to one.

We find that our model generates realistic volatility levels for both stock and bond markets. This does not require assuming implausibly large levels for the heterogeneity in beliefs: the initial value for the normalized disagreement, $\bar{\mu}_M(0)$, means that the difference in beliefs in money growth rate, $\mu_M^1(0) - \mu_M^2(0)$, equals 0.6913%, which should be compared with the value of average money growth (5.20% per year in the 1986–2011 period). (The calibrated level of heterogeneous beliefs on consumption also seems reasonable: it implies that $\mu_\varepsilon^1(0) - \mu_\varepsilon^2(0)$ equals 0.6544%.)

Although the improvement in matching empirical volatilities over a standard economy is considerable, and the t-statistics for volatilities are low (indicating that the model-generated averages are not significantly different from those in the data), the model has mixed success explaining the average level of asset returns. Results are encouraging for the stock market: The level of the equity premium is almost 10 times higher in our model relative to a homogeneous beliefs, full information economy (1.78% versus 0.19%) and relatively close to the level observed in the data (2.45% for our sample period). Furthermore, it should be noted that, as in our other numerical illustrations, we are assuming that the wealth-weighted average beliefs of investors are equal to real long-term average money growth, which (based on Equation (11)) effectively constrains the equity premium, preventing it from being large. The model would likely do even better in matching the equity premium if we removed this constraint and considered the case of agents holding biased average expectations on money growth. This would also, however, confuse the effects.

The model fails at matching observed levels for interest rates, both nominal and real, with large t-statistics. The real interest rate is considerably higher than realistic values. It has been noted before (Basak 2005, David 2008) that heterogeneous beliefs alone fail to resolve the interest rate puzzle, especially for relative risk aversion greater than one (when they tend to make it worse).[14] This issue likely could be solved by assuming nonseparable preferences, such as recursive utility (giving us one extra degree of freedom to adjust the interest rate), but doing so would eliminate the analytical tractability that is one of our model's most appealing features.

Nominal interest rates are also higher than in the data, even more so than in a homogeneous beliefs economy, as nominal rates seem to be affected by heterogeneous beliefs in a way similar to real rates. The model also cannot match the slope and curvature of the term structure because, consistent with our earlier observation, it generates small values for these factors; however, these two factors are significantly increased (and closer to empirically observed values) relative to a homogeneous beliefs economy.

In short, the model does an encouraging job of matching both bond and stock market volatilities, whereas the results for the level of asset returns are more mixed; in particular, the model generates excessively high interest rates. Other variables (equity premium and nominal term structure slope and curvature) are increased relative to a standard economy but are lower than in the data. This is not surprising given our many simplifying assumptions, required for tractability.

5. Conclusion

Heterogeneous beliefs on monetary policy have profound implications on asset pricing because they make agents' consumptions and stochastic discount factors more volatile and correlated with the money supply. In some cases, they can break the neutrality of money. In particular, inflation, bond yields, and stock returns are all made more volatile. This conclusion is supported by our empirical analyses. Most interestingly perhaps, these suggest a specific and valuable role for disagreement on monetary policy in understanding and forecasting volatility, adding valuable information on top of that provided by disagreement on the growth of the real economy, as well of several measures of overall economic uncertainty.

An interesting extension would be to examine the impact of heterogeneous beliefs in the presence of a rule-based monetary policy. We conjecture that our main mechanism would survive in that case because no matter the specific type of monetary policy in place agents who disagree will still use security markets to bet against each other, but the quantitative implications would be different. As far as empirical work is concerned, a particularly interesting investigation could consist in performing international comparisons that take into account differences in the transparency of monetary policy, which are likely to lead to differences in the level of heterogeneity in beliefs; our model provides tools to understand the impact of these differences.

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Appendix A. Proofs

Proof of Proposition 3.1. We first compute the state price density ξ^1. It can be easily checked that, for $\lambda = u_c^1/u_c^2$, the solution of the representative agent's problem coincides with the equilibrium allocation (as agents 1 and 2's optimality conditions and clearing in the good market hold). The stochastic weighting dynamics (8) then follow from applying Itô's lemma to the state-price density dynamics (4). The envelope theorem (applied to the optimization problem in the definition of the representative agent (7)) shows that $U_c = u_c^1 = y^1\xi^1$, and after substituting power utility, the expression for the state-price density follows:

$$\xi^1(t) = e^{-\delta t}\phi \frac{(1+\lambda(t)^{1/\alpha})^\alpha}{\varepsilon(t)^\alpha}. \tag{A1}$$

The stock price obeys a standard present value formula,

$$\begin{aligned} S(t) &= \frac{1}{\xi^1(t)} E_t^1\left[\int_t^T \xi^1(s)\varepsilon(s)\,ds\right] \\ &= \frac{\varepsilon(t)^\alpha}{(1+\lambda(t)^{1/\alpha})^\alpha} E_t^1\left[\int_t^T e^{-\delta(s-t)}(1+\lambda(s)^{1/\alpha})^\alpha \varepsilon(s)^{1-\alpha}\,ds\right] \\ &= \frac{\varepsilon(t)^\alpha}{(1+\lambda(t)^{1/\alpha})^\alpha} \int_t^T e^{-\delta(s-t)} E_t^1[(1+\lambda(s)^{1/\alpha})^\alpha] E_t^1[\varepsilon(s)^{1-\alpha}]\,ds, \end{aligned}$$

[14] In our case, the risk-free rate generated by the model is lower, and closer to the data, than in a benchmark economy, but this is caused by the agents' belief updating given the specific data that is used, not by the presence of heterogeneous beliefs. The same explanation accounts for why we obtain higher values for the term structure factors in the calibration than in the benchmark.

where the third equality follows from Fubini's theorem and the fact that λ and ε are independent. When α is an integer, we have

$$(1+\lambda(s)^{1/\alpha})^{\alpha}=\sum_{i=0}^{\alpha}\binom{\alpha}{i}\lambda(s)^{i/\alpha}.$$

Algebraic manipulation then yields (9). Applying Itô's lemma yields the volatility coefficients. Q.E.D.

Proof of Proposition 3.2. Agents' first-order conditions reveal that the nominal short rate is given by

$$R(t)=\frac{U_m(\varepsilon(t),p(t)M(t),t;\lambda(t))}{U_c(\varepsilon(t),p(t)M(t),t;\lambda(t))}.$$

Substituting logarithmic utility shows that

$$R(t)p(t)=\frac{1-\phi}{\phi}\frac{\varepsilon(t)}{M(t)}. \tag{A2}$$

Substituting logarithmic utility into the state-price density ξ^1 (Equation (A1)), and (A2) into the second equation in (5) yields

$$p(t)=\frac{(1-\phi)e^{\delta t}\varepsilon(t)}{\phi(1+\lambda(t))}E_t^1\left[\int_t^T e^{-\delta s}\frac{1+\lambda(s)}{M(s)}\right].$$

λ and M have stochastic Itô coefficients (because of agents' belief updating), but the expectation can be evaluated using the techniques in Ziegler (2000), leading to (12). The nominal interest rate then obtains readily from (A2). Q.E.D.

Proof of Proposition 3.3. The bond pays one unit of money, worth $p(t+\tau)$ in real terms, at time $t+\tau$. Applying the standard present value formula, its nominal price is given by

$$\begin{aligned}B_m(t,\tau)&=\frac{1}{p(t)}E_t^1\left[\frac{\xi^1(t+\tau)}{\xi^1(t)}p(t+\tau)\right]\\&=\frac{M(t)}{\int_t^T e^{-\delta s}h^1(M(t),t,s)\,ds+\lambda(t)\int_t^T e^{-\delta s}h^2(M(t),t,s)\,ds}\\&\quad\cdot\left\{\int_{t+\tau}^T e^{-\delta s}h^1(M(t),t,s)E_t^1\left[\frac{1}{M(t+\tau)}\right]ds\right.\\&\quad\left.+\int_{t+\tau}^T e^{-\delta s}h^2(M(t),t,s)E_t^1\left[\frac{\lambda(t+\tau)}{M(t+\tau)}\right]ds\right\},\end{aligned}$$

where the second equality follows by substituting the state-price density ξ^1 and the price of money expression in Proposition 3.2. The expectations can be computed explicitly using the techniques in Ziegler (2000). Q.E.D.

Proof of Proposition 3.4. The expressions follow from applying Itô's lemma to the bond price formula in (15). Q.E.D.

Appendix B. Formulas Used in the Calibration

This appendix reports the expressions for asset prices obtained in the presence of heterogeneous beliefs and Bayesian updating on both consumption and money, but under the assumption that money and consumption are not correlated (the correlation in our 1986–2011 data is small (−0.06) and would have a small quantitative impact on the results). These formulas are omitted from the main text for brevity because they provide little additional insight.

Stochastic Weighting Dynamics:

$$\frac{d\lambda(t)}{\lambda(t)}=-\bar{\mu}_\varepsilon(t)dw_\varepsilon^1(t)-\bar{\mu}_M(t)dw_M^1(t).$$

Stock Price:

$$S(t)=\frac{\varepsilon(t)}{(1+\lambda(t)^{1/\alpha})^{\alpha}}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T F(j,t,s)\,ds,$$

where $F(j,t,s)=\exp\{[(1-\alpha)(\mu_\varepsilon^1(t)-\frac{1}{2}\alpha\sigma_\varepsilon^2)-\delta-((i(\alpha-1))/(2\alpha^2))(\bar{\mu}_\varepsilon(t)\bar{\mu}_\varepsilon(s)+\bar{\mu}_M(t)\bar{\mu}_M(s))-((i(1-\alpha))/\alpha)\bar{\mu}_\varepsilon(t)\sigma_\varepsilon](s-t)+((1-\alpha)^2/2)v_\varepsilon(t)(s-t)^2\}$ and $v_\varepsilon(t)=(v_\varepsilon(0)\sigma_\varepsilon^2)/(\sigma_\varepsilon^2+v_\varepsilon(0)t)$.

Stock Volatility:

$$\sigma_S(t)=[\sigma_{S\varepsilon}(t)^2+\sigma_{SM}(t)^2]^{1/2},\quad\text{where}$$

$$\begin{aligned}\sigma_{S\varepsilon}(t)&=\sigma_\varepsilon+\left(\frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}}-\frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}(j/\alpha)\int_t^T F(j,t,s)\,ds}{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T F(j,t,s)\,ds}\right)\\&\quad\cdot\bar{\mu}_\varepsilon(t)+(1-\alpha)\frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T F(j,t,s)(s-t)\,ds}{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T F(j,t,s)\,ds}\frac{v_\varepsilon(t)}{\sigma_\varepsilon},\end{aligned}$$

$$\begin{aligned}\sigma_{SM}(t)&=\left(\frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}}-\frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}(j/\alpha)\int_t^T F(j,t,s)\,ds}{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\lambda(t)^{j/\alpha}\int_t^T F(j,t,s)\,ds}\right)\\&\quad\cdot\bar{\mu}_M(t).\end{aligned}$$

Real Interest Rate:

$$\begin{aligned}r(t)&=\alpha\left(\frac{1}{1+\lambda(t)^{1/\alpha}}\mu_\varepsilon^1(t)+\frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}}\mu_\varepsilon^2(t)\right)-\frac{1}{2}\alpha(\alpha+1)\sigma_\varepsilon^2\\&\quad+\delta+\frac{1}{2}\frac{\alpha-1}{\alpha}\frac{\lambda(t)^{1/\alpha}}{(1+\lambda(t)^{1/\alpha})^2}\left(\bar{\mu}_\varepsilon(t)^2+\bar{\mu}_M(t)^2\right).\end{aligned}$$

Nominal Bond Yields:
Nominal bond yields are as in Equation (15).

Nominal Bond Yield Volatility:

$$\sigma_R(t,\tau)=[\sigma_{R\varepsilon}(t,\tau)^2+\sigma_{RM}(t,\tau)^2]^{1/2},\quad\text{where}$$

$$\begin{aligned}\sigma_{R\varepsilon}(t,\tau)&=\left(\frac{\int_t^T e^{-\delta s}h^1(t,s)\,ds}{\int_t^T e^{-\delta s}[h^1(t,s)+\lambda(t)h^2(t,s)]\,ds}\right.\\&\quad\left.-\frac{\int_{t+\tau}^T e^{-\delta s}h^1(t,s)\,ds}{\int_{t+\tau}^T e^{-\delta s}[h^1(t,s)+\lambda(t)h^2(t,s)]\,ds}\right)\frac{\bar{\mu}_\varepsilon(t)}{\tau},\end{aligned}$$

$$\begin{aligned}\sigma_{RM}(t,\tau)&=\frac{1}{\tau}\left(\int_{t+\tau}^T e^{-\delta s}\left[\frac{v_M(t)}{\sigma_{MM}}(s-t)h^1(M(t),t,s)\right.\right.\\&\quad\left.\left.+\lambda(t)h^2(M(t),t,s)\left(\frac{v_M(t)}{\sigma_{MM}}(s-t)+\bar{\mu}_M(t)\right)\right]ds\right)\\&\quad\cdot\left(\int_{t+\tau}^T e^{-\delta s}(h^1(M(t),t,s)+\lambda(t)h^2(M(t),t,s))\,ds\right)^{-1}\\&\quad-\frac{1}{\tau}\left(\int_t^T e^{-\delta s}\left[\frac{v_M(t)}{\sigma_{MM}}(s-t)h^1(M(t),t,s)\right.\right.\\&\quad\left.\left.+\lambda(t)h^2(M(t),t,s)\left(\frac{v_M(t)}{\sigma_{MM}}(s-t)+\bar{\mu}_M(t)\right)\right]ds\right)\\&\quad\cdot\left(\int_t^T e^{-\delta s}(h^1(M(t),t,s)+\lambda(t)h^2(M(t),t,s))\,ds\right)^{-1},\end{aligned}$$

and h^i is as in Equation (14) and $v_M(t)=(v_M(0)\sigma_{MM}^2)/(\sigma_{MM}^2+v_M(0)t)$.

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