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Aging Population, Retirement, and Risk Taking

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The increase in life expectancy spells disaster at retirement. One can solve this problem by investing in the maximum geometric mean (MGM) portfolio, which is empirically composed from equity. For a $T = 30$ year horizon or more, the MGM portfolio dominates other investment strategies by almost first-degree stochastic dominance. The MGM portfolio also maximizes the expected value of the commonly employed preferences and prospect theory value function, for various loss aversion parameters and various reference points, for $T \geq 10$. Life-cycle funds would increase virtually all investors' welfare by shifting to the MGM portfolio so long as the investment horizon is at least 10 years.

Keywords: first-degree stochastic dominance; asymptotic stochastic dominance; almost stochastic dominance; maximum geometric mean; FSD violation area; life-cycle funds; prospect theory
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1. Introduction

The prevailing trend of the increase in life expectancy accompanied by the current low interest rates has profound consequences corresponding to the future available consumption at retirement period, which requires a dramatic change in life-cycle economic planning. In his comprehensive study, Poterba (2014) provides a compelling argument that unless some economic changes are made, the longer lifespan implies a substantially lower level of future consumption for a relatively large segment of the population. It is to the analysis of the optimal investment for retirement and to the probability that stocks outperform bonds in the long run that we address this study.

The main related results of previous studies are as follows. Previous studies related to the present study are divided into two groups: theoretical studies and empirical studies. We mention here first the two main theoretical opponents' views. Kelly (1956), Latané (1959), Breiman (1960), Bernstein (1976), and Markowitz (1976, 2006) advocate that the portfolio with the maximum geometric mean (MGM) almost surely will end up with a higher terminal wealth than any other investment strategy in the indefinitely long run, and hence it is optimal. Merton and Samuelson (1974) disagree with this assertion and claim that for myopic preference the MGM strategy does not necessarily maximize expected utility, regardless of the length of the investment horizon. This theoretical debate remains unsolved. The empirical studies also do not reveal a clear-cut result regarding this issue. Siegel (2007) argues that for a sufficiently long period, stocks are less risky than bonds, where risk is defined by the standard deviation of the *annual* return. Bali et al. (2009) show that even for a relatively long investment horizon there is always a nagging little left tail distribution area, called "violation area," where bonds outperform stocks. Thus, there is no first-degree stochastic dominance (FSD) of stocks over bonds. However, they employ the *almost* stochastic dominance (ASD)[1] rules developed by Leshno and Levy (2002) to show that stocks almost stochastically dominate bonds for long horizon, implying that the dominance holds for all utility functions after eliminating some pathological preferences.

The contributions of this study are as follows:

(a) We first define asymptotic first-degree stochastic dominance (AFSD) and resolve the theoretical debate corresponding to the optimality of the MGM portfolio for the very long run.

(b) We analyze the optimal investment for the long run with the commonly employed normal distribution of terminal wealth and compare the results to the optimal investment corresponding to the more relevant log-normal distribution.

(c) We investigate the goodness of fit of the empirical distribution to the log-normal distribution and the optimal investment for long but finite horizon. We

[1] We employ in this paper the FSD and almost FSD rules. FSD asserts that with two cumulative distribution functions, F and G, the expected utility of prospect F is larger or equal to the expected utility of prospect G if and only if $F(x) \leq G(x)$ for all values x and there is at least one strict inequality (for a survey of stochastic dominance rules, see Levy 2006). Almost FSD is intact when the two distributions intersect, but F is located above G only over a small range of the returns. The area enclosed between F and G over this range is called the violation area.

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focus on the MGM portfolio and examine the existence of almost FSD (FSD^{A}) of this portfolio over other portfolios for a relatively long horizon, up to 30 years, which is relevant for investment for retirement.

(d) We solve for the optimal stock–bond portfolio mix for various finite horizons for some important commonly employed preferences and for the prospect theory value function and analyze whether for these utility functions the preference of stocks over bonds enhances with an increase in the investment horizon, and for what horizons the MGM portfolio is empirically optimal. As the probability that bonds will outperform stocks in the long run approaches zero, but is still strictly positive for any finite horizon, we suggest buying put options to protect the investment for retirement even against the negligible probability (related to the violation area) that bonds will outperform stocks. Issuing such put options for the very long run by institutional investors creates a win-win situation.

(e) We analyze the actual investment of "life-cycle" mutual funds around the world and show the gain induced by shifting from the commonly employed diversification by these funds to the MGM portfolio advocated in this study for various investment horizons.

2. MGM Portfolio and Expected Utility

2.1. MGM Portfolio and Terminal Wealth

Kelly (1956), Latané (1959), Breiman (1960), Bernstein (1976), and Markowitz (1976, 2006) show that under certain conditions the investor for the indefinitely long run should invest each period aiming to maximize $E(\log(1+R))$, where R stands for the one-period portfolio rate of return. The terminal wealth W_T is given by $W_T = W_0 \prod_{i=1}^{T}(1+R_t)$, where W_0 stands for the initial invested wealth, and R_t is the portfolio rate of return corresponding to period t. Assuming independent and identically distributed and finite mean and variance, by the weak law of large numbers, we have

$$Probability\left[\left|\left(\frac{1}{T}\right)\log(W_T/W_0) - E\log(1+R)\right| > \epsilon\right] \to 0. \qquad (1)$$

And by the strong law, we have that

$$\lim_{T\to\infty}\frac{1}{T}\log\left(\frac{W_T}{W_0}\right) = E\log(1+R)$$

with probability 1 (see Markowitz 1976). Thus, the geometric mean is the growth rate of the portfolio. Therefore,

$$Pr_{T\to\infty}[(W_T(MGM) > W_T(OTHER)] \to 1, \qquad (2)$$

implying that the MGM portfolio will end up, *almost surely*, with more wealth that any other portfolio.[2] This does not imply that the MGM portfolio necessarily maximizes expected utility for all preferences, as Merton and Samuelson (1974) show that for the myopic preferences the investment strategy is independent of the assumed investment horizon. Thus, despite the correct statement given in Equation (1), the MGM portfolio does not necessarily provide higher expected utility than all other possible investment strategies, and hence it does not dominate by FSD (see Footnote 1) all other possible investment strategies. This is, in a nutshell, the long-standing debate about the optimality of the MGM portfolio in the very long run for all investors.

We define and employ in this section of the paper the AFSD rule rather than the FSD rule and derive the precise conditions for the optimality of the MGM portfolio in the very long run. Because in the very long run, by the central limit theorem, the terminal wealth distribution is log-normal, we first employ in this study FSD rule corresponding to the most relevant log-normal distributions.

2.2. AFSD and the MGM Portfolio: Log-Normal Distributions

We first provide the definition of AFSD and some properties of the log-normal distribution, which will be employed in Theorem 1 given below.

Definition 1 (AFSD). Denote by $F_T(w)$ and by $G_T(w)$ the two cumulative distributions of prospects F and G, respectively, where w denotes the terminal wealth accumulated after T investment periods. Then, prospect $F_T(w)$ dominates prospect $G_T(w)$ by AFSD if and only if,

$$\lim_{T\to\infty}[EU(w_{F_T}) - EU(w_{G_T})] \geq 0 \quad \text{for all } U \text{ with } U' \geq 0,$$

and for some nondecreasing U there is a strict inequality.

Denote by R the portfolio *rate* of return (a random variable) and by $x = (1+R)$ the portfolio *return* (end of period value). Furthermore, assume that R cannot be lower than −100%. If $\log(x)$ is normally distributed, then by definition x is log-normally distributed. The parameters μ_x and σ_x^2 stand for the mean and the variance of the return x, and μ and σ^2 stand for the mean and variance of $\log(x)$. Of course, the higher μ is, the higher the geometric mean. Two

[2] Of course, one needs to impose some regulatory conditions. For example, $1+R \geq 0$, as for negative values the log function, is not defined. This means that some constraints on short selling must be imposed; otherwise, by shorting the bonds and levering the portfolio, $1+R$ may be negative. For an extension of the conditions under which the MGM portfolio is optimal, see Martin (2012).

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log-normal cumulative distributions, like two cumulative normal distributions, intersect at most once. We denote the intersection point of the normal distributions by z_0 and the corresponding intersection point of the log-normal distributions by x_0. We have the following relation:

$$z_0 \to \infty \quad \Leftrightarrow \quad x_0 \to \infty \quad \text{and} \quad z_0 \to -\infty \quad \Leftrightarrow \quad x_0 \to 0$$

(see Aitchison and Brown 1963 and Footnote 3).

The terminal wealth after T investment periods is given by

$$\text{Log} W_t = \sum_{t=1}^{T} \log(1+R_t).$$

Hence,

$$E_F \log W_T = T\mu_F \quad \text{and} \quad \text{Var}_F(\log W_T) = T\sigma_F^2;$$
$$E_G \log W_T = T\mu_G \quad \text{and} \quad \text{Var}_G(\log W_T) = T\sigma_G^2.$$

For very long T, we have log-normal distribution of terminal wealth. It can be shown that the two multiperiod log-normal cumulative distributions with unequal variances have one intersection point given by (see Levy 2006)

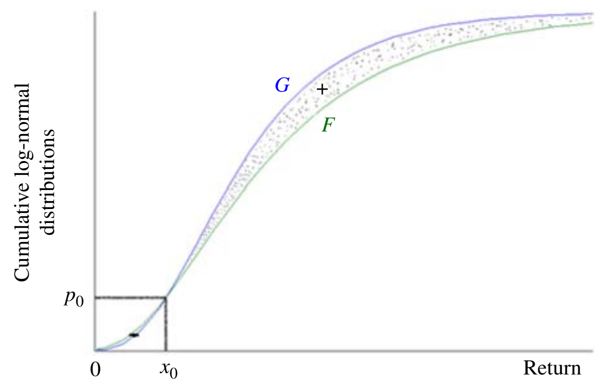
$$\begin{aligned} z_0 &\equiv T(\mu_F - \mu_G)/\sqrt{T}(\sigma_G - \sigma_F) \\ &= \sqrt{T}(\mu_F - \mu_G)/(\sigma_G - \sigma_F), \end{aligned} \tag{3}$$

where z_0 is the intersection point of the corresponding normal distribution, from which one can derive the intersection point x_0 corresponding to the log-normal variable.[3]

Thus, because F and G intersect, the MGM portfolio does not dominate by FSD all other portfolios in the multiperiod case as long as T is finite. The important feature emerging from Equation (3) is that the intersection point of the log-normal distributions, x_0, is a function of T. Hence, as the number of periods increases, the intersection point shifts either to the extreme left tails or to the extreme right tails of the two distributions under consideration, and hence the violation area where bonds have an edge over stocks may become negligible when T increases. In other words, when F has higher mean and a higher variance than G, we have (see Equation (3) and Footnote 3) $T \to \infty \Rightarrow z_o \to -\infty$ and $x_0 \to 0$, and when F has a higher mean and a smaller variance than G, we have $T \to \infty \Rightarrow z_o \to \infty$ and $x_0 \to \infty$, where x_0 is the intersection point of the two multiperiod log-normal distributions. Finally, note that to have these shifts of the intersection point to the extreme tails, F must have a *strictly* higher geometric mean than G, as for the case $\mu_F = \mu_G$ the intersection point is unaffected by T (see Equation (3)).

[3] The quantiles p of the normal distribution $z(p)$ and log-normal distribution $x(p)$ are related as follows: $x(p) = e^{\mu + z(p)\sigma}$ (see Levy 2006).

Figure 1 **(Color online) Two Log-Normal Cumulative Distributions with Parameters $\mu_F > \mu_G$, $\sigma_F > \sigma_G$ and with an Intersection Point at (x_0, p_0)**



Theorem 1 (AFSD). *Assume an investment horizon of T periods and that $W_T(F)$ and $W_T(G)$ are log-normally distributed. Assume that $\mu_F > \mu_G$; namely, F has a higher geometric mean than G. Then, for $T \to \infty$, $W_T(F)$ and $W_T(G)$ are log-normally distributed, and in this case F (with the MGM) dominates G by AFSD iff $\mu_F > \mu_G$ and $\sigma_F \geq \sigma_G$, provided that the marginal utility is bounded.*

Proof. When $\mu_F > \mu_G$ and $\sigma_G = \sigma_F$, we trivially have FSD for all horizons T, let alone for an infinite horizon, because F is located in the whole range below G (see Footnote 1). Thus, in this case we have FSD as well as AFSD. It can be easily shown that in the case $\mu_F > \mu_G$ and $\sigma_F < \sigma_G$, the AFSD of F over G is not guaranteed even if the marginal utility is bounded.[4] Therefore, to complete the proof what remains is to show the existence of AFSD of F over G in the case where $\mu_F > \mu_G$ and $\sigma_F > \sigma_G$. This is the most important case, because empirically stocks have a higher mean and a higher variance of log-returns than bonds. Therefore, we establish the conditions for the stock portfolio to dominate the bond portfolio in the very long run.

In this case, G intersects F from below, and as T approaches infinity, the normal intersection point z_0 approaches $-\infty$, and the corresponding log-normal intersection point denoted by point x_0 (see Figure 1) approaches zero. Figure 1 presents the intersection point of two log-normal cumulative distributions. Thus, we have a left tail intersection as described by Figure 1, and the stochastic dominance violation area (denoted by — in Figure 1) approaches zero as T

[4] The intuitive explanation of this result is that whereas with a left tail intersection the returns are bounded by zero, with a right tail intersection the returns are unbounded. Specifically, when the return is unbounded, it can be shown that the term $\lim_{x_0 \to \infty} \int_{x_0}^{\infty} [G(w) - F(w)]\, dw$ is not necessarily equal to the term $\int_{\infty}^{\infty} [G(w) - F(w)]\, dw = 0$. For the sake of brevity, this technical proof is omitted from the paper and it is available on request.

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approaches infinity. By the standard stochastic dominance formula with nonnegative returns (see Levy 2006), we have

$$E_F U(w) - E_G U(w) = \int_0^{+\infty} [G(w) - F(w)] U'(w)\, dw.$$

The difference in the expected utility of the two log-normal prospects can be rewritten as

$$E_F U(w) - E_G U(w) = \int_0^{x_0} [G(w) - F(w)] U'(w)\, dw + \int_{x_0}^{\infty} [G(w) - F(w)] U'(w)\, dw, \quad (4)$$

where the first integral is negative (the violation area) and the second one is positive (see the minus and plus signs in Figure 1). For the bounded first derivative, with the upper bound M, we have

$$\left| \int_0^{x_0} [G(w) - F(w)] U'(w)\, dw \right| < \left| M \int_0^{x_0} [G(w) - F(w)]\, dw \right|,$$

and therefore when $T \to \infty \Rightarrow x_0 \to 0$, we have

$$\lim_{x_0 \to 0} \int_0^{x_0} [G(w) - F(w)]\, dw = \int_0^0 [G(w) - F(w)]\, dw = 0.$$

Hence, for the finite upper bound on the marginal utility, M, the first term on the right-hand side of Equation (4) is equal to zero for $T \to \infty$. This means that the violation area component of the expected utility difference given by Equation (4) approaches zero. Note that the above claim is valid only if the integral is finite, which is intact for the case of left tail intersection, but not the case of right tail intersection (see Figure 1 and Footnote 4).

Because the first integral on the right-hand side of Equation (4) is equal to zero, what left is only the second integral. As for $x > x_0$, the cumulative distribution F is below G (see Figure 1), the second integral is nonnegative, and, therefore $E_F U(w) \geq E_G U(w)$ for all preferences[5] with a bounded first derivative,[6] which completes the proof. □

The above results corresponding to an infinite horizon are theoretically interesting and provide the first indication of the desired changes in the portfolio composition as T increases. However, these results are irrelevant for investment for retirement in practice because the investment horizon, albeit relatively large, is generally finite. We shall see below that for a finite and long horizon, which is relevant for investment for retirement, the MGM portfolio dominates all other investments for virtually all investors because we have dominance by almost FSD. Moreover, the MGM portfolio (which is empirically composed from stocks) provides the highest expected utility for some important preferences and the prospect theory value function, albeit not for all possible preferences, for $T \geq 10$ years.

2.3. The Rationale of Bounded Marginal Utility

We proved above that the MGM portfolio is asymptotically optimal by AFSD as long as the marginal utility is bounded. This of course does not characterize the myopic preference, because $U'(w) = 1/w^\alpha$, and because the range of log-normal distribution is $(0, \infty)$, at $w = 0$ the derivative of the myopic preference approaches infinity.[7] Therefore, for the myopic preference, the MGM portfolio is not necessarily optimal (see also Samuelson 1989, 1994; Merton and Samuelson 1974).

One may argue that the myopic function is important and should not be excluded because there is a rationale for the fact that $U'(x) \to \infty$ when $x \to 0$. The reason is that $x = 0$ implies zero consumption with severe implications, say, death. We claim that bounding the first derivative is generally reasonable because when investment is considered, we cannot take seriously the claim that when the terminal value of the investment approaches zero the damage that is equivalent to death (as with the myopic preference) exists, because in practice virtually all people have mental accounting (see, e.g., Thaler 1994, 1999), where they have wealth W_1 allocated to consumption and W_2 allocated to investment, and we are dealing in this study solely with W_2. Thus, we may have a modified preference with a similar mathematical myopic form,

$$U(W_1, W_2) = (W_1 + W_2(1 + R))^{1-\alpha} / (1 - \alpha). \quad (5)$$

Therefore, even in the extreme case where $R = -1$, we still have $U(W_1)^{1-\alpha}/(1-\alpha)$ with $W_1 > 0$, and therefore the first derivative is finite. Finally, although the above function with mental accounting is not myopic, because of the similar structure to the myopic function, in this paper we call it the *modified* myopic preference.

[5] Note that for some preferences we may have $E_F U(w) = E_G U(w)$. For example, for any function with zero first derivative for $x > x_0$, we have $E_F U(w) = E_G U(w)$.

[6] Note that if $\mu_F = \mu_G$, the intersection point is not affected by T (see Equation (3)); hence the mathematical argument is not intact. Therefore, we require that F will have a strictly larger geometric mean than G.

[7] Note that as T approaches infinity, the violation area approaches zero. However, with the myopic function the derivative approaches infinity as wealth approaches zero (a point that is included in the violation area), and hence the product of the violation area and the marginal utility is not necessarily zero. Therefore, the MGM portfolio is not necessarily optimal for the classic myopic preference. However, when we impose an upper bound the marginal utility, the fact that the violation area approaches zero dominates the result and the MGM portfolio is optimal.

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2.4. Normal Distributions of Terminal Wealth: No AFSD

Although the log-normal distribution best fits the long-run empirical distribution and is theoretically justified by the central limit theorem, investors may mistakenly employ the inappropriate mean-variance rule also for the long-run investment. Thus, one may wonder whether a similar AFSD is obtained with the most commonly employed mean-variance rule. Specifically, we assume that investors behave "as if" the terminal distribution is normal (although there is no theoretical justification for this assumption) and examine whether with the assumed normality there is AFSD of the MGM portfolio. Unlike the log-normal case, with normal distributions it can be proved that as T increases, the shift of the intersection point to the left tails does not necessarily occur. The direction of the shift rather depends on the assumed parameters. Indeed, using the empirical one-period parameters and assuming normality, we find that the intersection point does not shift to the left as needed for the proof of the AFSD. However, recall that log-normality is theoretically justified and empirically best fits the data. Furthermore, we find that with log-normal distribution, the optimal weight of equity in the portfolio is very high. Therefore, if investors mistakenly think that the distributions are normal, and because there is no AFSD of stocks over bonds with normality, they may mistakenly invest a relatively high proportion of the portfolio in bonds, which incurs expected utility loss.

3. Empirical Findings: Saving for Retirement with a Finite Investment Horizon

In practice, the investment horizon for retirement is generally long but finite. Therefore, in this section we examine empirically the following issues:

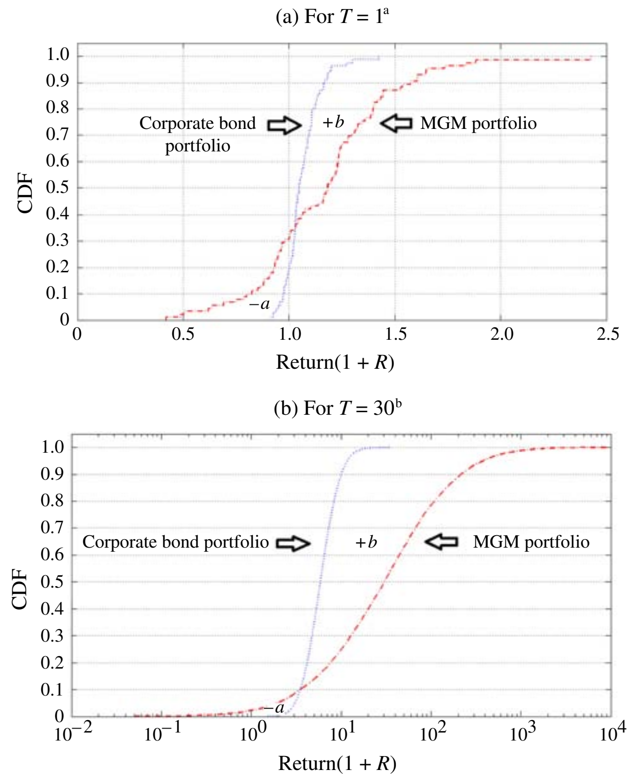
(a) the goodness of fit of the empirical distributions to the log-normal distribution for various finite investment horizons;

(b) whether there is almost FSD of the MGM portfolio over other relevant portfolios for horizons that are relevant for saving for retirement, despite the fact that there is no FSD;

(c) the composition of the optimal diversification between six categories of assets and the optimality of the MGM portfolio as a function of the assumed investment horizon for some commonly employed preferences and for the prospect theory value function.

Leshno and Levy (2002) establish the almost FSD. They show that the two distributions F (stocks) and G (bonds) may intersect (hence there is no FSD; see Footnote 1), yet F dominates G by FSD^A so long F has

Figure 2 (Color online) Cumulative Distributions of the Return on the MGM Portfolio and the Corporate Bond Portfolio for $T = 1$ and $T = 30$



Note. CDF, cumulative distribution function.
[a] $\varepsilon = |a|/(|a| + b) = 24.02\%$.
[b] $\varepsilon = |a|/(|a| + b) = 0.12\%$.

a higher mean and the stochastic dominance violation area (defined as the enclosed area between F and G in the range over which F is located above G; see Figure 1) is relatively small. Specifically, there is a direct relation between the allowed violation area and the imposed constraint on the marginal utility, such that FSD^A prevails. They prove that

$$\text{Sup}U' \leq \text{Inf}U'\left[\frac{1}{\varepsilon} - 1\right] \quad \text{for all } x \epsilon [0.1], \tag{6}$$

where ε is given by the absolute value enclosed between F and G over the FSD violation range of outcomes divided by the total absolute area enclosed between F and G. (In Figure 2 it is given by the negative area divided by the total area enclosed between the two distributions, when all areas are measured in absolute terms.) Obviously, if $\varepsilon = 0$, the above condition (see Equation (6)) is always fulfilled and we have FSD because the FSD^A and the FSD coincide in this case (for further analysis of FSD^A, see Levy 2009).

Levy et al. (2010) estimated the allowed violation area. They showed experimentally with a sample of 200 subjects who had to choose between F and G, where F had a higher mean than G, that the smaller

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the violation area, the larger the proportion of the subjects that preferred F over G, with 100% of them preferring F over G provided that and the violation area, ε, was *no larger* than 5.9%. (Of course, for a smaller violation area all subjects preferred F over G.) Thus, for a 5.9% violation area or smaller, we have FSD^{A} of F over G because all 200 subjects select F despite the fact that there is no FSD because the two distributions under consideration intersect. We find empirically (see below) that although for all horizons there is no FSD, for a 30-year horizon or more, the violation area is very small (much smaller than the experimentally allowed 5.9%); hence there is FSD^{A} of the MGM portfolio over other portfolios under consideration

3.1. Data and Methodology

To analyze the above issues, we employ annual rates of return corresponding to a relatively long period (1926–2012) of small stocks, large stocks, Treasury (T)-bills, T-bonds, corporate bonds, and the riskless asset with return denoted by R_F. The source of the raw data for T-bond and T-bill returns is the economic database of the Federal Reserve in St. Louis (FRED). The rates of return correspond to large stocks (the S&P 500 index), small stocks,[8] and corporate bonds portfolios. The Treasury bill rate is a three-month rate, and the Treasury bond is the constant maturity 10-year bond (Damodaran 2014). Note that the T-bills are for three months; hence, for any investment horizon longer than three months, this asset is a risky asset because the future T-bill interest rate is not known in advance. Therefore, we also add an artificial riskless asset with a constant annual rate of return of 3.65% reflecting the average annual return on T-bills for the period 1926–2012.

By the bootstrapping method we created rates of return corresponding to various assumed portfolio revision horizons, up to a horizon of 100 years. For each $T = 1, 2, 3, \ldots, 100$ (T-year horizon), we draw T observations from the annual data series (with replacement), and by calculating the value $\prod_1^T (1+R_t)$ we obtain the terminal wealth resulting from investment for T years. We repeat this procedure 10,000 times for each horizon to obtain the estimate of the T-year horizon distribution of terminal wealth. Of course, we draw at random one year that provides the rates of return for all assets; hence we maintain the correlation between the assets under consideration. We conduct this procedure for very unrealistic and economically irrelevant long horizons up to 100 years. However, because the results for a horizon longer than 30 years are quite stable, we report below the results up to the 30-year investment horizon, a period relevant for saving for retirement.

[8] For the definition of the small stock portfolio and corporate bond portfolio, see Ibbotson Associates (2014).

3.2. The Results

Considering the six assets discussed above, we find that the small stock portfolio has the highest arithmetic mean and the highest geometric mean (hence, a higher growth rate), as well as highest variance measured in both returns and log-returns. For example, for $T = 1$, the geometric annual mean of the small stock portfolio is 12.5%. Moreover, because the small stock index has the highest μ and σ, it also dominates by AFSD all other assets under consideration for the indefinitely long run (see Theorem 1). Note that when diversification is allowed, the MGM portfolio theoretically may be a diversified portfolio.

Not surprisingly, we also find that the mean and standard deviation of stocks become much larger than those of bonds as T increases, implying that, regardless of the assumed horizon, stocks do not dominate bonds by the mean-variance rule. Actually, for any finite horizon there is also no FSD of the stock portfolio over the other less risky asset portfolios because regardless of the employed investment horizon, the empirical cumulative distributions of stocks and bonds intersect as demonstrated in Figures 1 and 2.

Because in Theorem 1 we employ the log-normal distribution, we first conduct a goodness-of-fit test of the empirical distribution of the stock returns to 20 possible theoretical distributions. In this horse race, for $T \geq 20$, the log-normal distribution provides the best fit. Moreover, for the 20-year horizon or longer, the deviations between the theoretical log-normal distribution and the empirical distribution are negligible. Thus, pension funds and other long-term investors can safely assume a log-normal distribution of terminal wealth in seeking the optimal investment.

Figure 2 illustrates the cumulative distribution of the MGM portfolio (which is composed from 100% stocks) and corporate bond portfolio for $T = 1$ and $T = 30$. Here we do not assume log-normality and simply use the distribution-free stochastic dominance analysis. As can be seen, there is no FSD because the two distributions in both cases intersect. However, for a horizon of $T = 1$ year, there is also no FSD^{A} because the FSD violation area is relatively large, 24.02%. However, for the $T = 30$ years horizon, we have FSD^{A} of stocks over bonds because the violation area negligible—only 0.12%. (Recall that experimentally we have FSD^{A} even with a 5.9% violation area, because 100% of the subjects selected F, let alone with 0.12%; see Levy et al. 2010.) Thus, for a long horizon, although there is no FSD of the MGM portfolio over the bond portfolio, there is FSD^{A}, implying that all long horizon investors with the exception of those who have pathological and economically irrelevant preferences would be better off by holding the MGM portfolio rather than the bond portfolio.

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We turn now to deriving the optimal diversification for various specific preferences for a wide range of risk parameters for various assumed horizons. Using the six types of assets discussed above and the 86 year annual rates of return, we employ the bootstrapping methodology described above to derive the various distributions of returns from which we derive the optimal portfolio for various preferences and for various parameters of these preferences.

We solve the optimal diversification as a function of the assumed investment horizon for the following preferences:

(a) The modified myopic form utility function and its modified form given by[9]

$$U(w) = (w+A)^{1-\alpha}/(1-\alpha), \tag{7}$$

where $\alpha \geq 0$. We calculated the expected utility for various values α ranging from 0.5 to 10. This is the modified myopic preference, which takes into account metal accounting, where the amount $A > 0$ is not considered in the wealth allocated for investment. The marginal utility of this function is bounded as required by the AFSD rule. When $A = 0$, we have the classic myopic preference with no bound on the marginal utility.

(b) The log function given by

$$U(w) = \log w, \tag{8}$$

which is a special case of the classic myopic function where $\alpha \to 1$. This function is important because it implies choosing the portfolio with the highest geometric mean (MGM portfolio), and one wishes to know whether the MGM portfolio is specialized or diversified.

(c) The prospect theory value function given by

$$U(R) = R^{\gamma} \quad \text{if } R > 0 \quad \text{and} \quad -\lambda(-R)^{\gamma} \quad \text{if } R < 0, \tag{9}$$

where R is the *change* of wealth (namely, $w-1$), and $\lambda = 2.25$ (see Kahneman and Tversky 1979). We first assume constant loss aversion parameter $\lambda = 2.25$ and analyze the sensitivity of the optimum diversification to changes in the risk tolerance parameter given by γ. Specifically, we calculate the optimum diversification for values γ ranging from 0.5 to 1, where for the value $\gamma = 1$ this function collapses to the bilinear function given by

$$U(R) = R \quad \text{if } R > 0 \quad \text{and} \quad -\lambda(-R) = \lambda R \quad \text{if } R < 0$$

(see Benartzi and Thaler 1995).

In practice the loss aversion is presumably more profound than what is estimated in laboratory experiments; hence, we also calculate expected utility value (for a given $\gamma = 0.88$) with λ ranging from 2.25 up to 5. Another modification is related to the reference point. Whereas Kahneman and Tversky (1979) use zero wealth as the reference point more, recent studies (see Kőszegi and Rabin 2006, 2007) advocate that the reference point should be larger. We use here two alternate reference points: the average annual inflation rate (3.06%) and the average interest rate (3.57%) prevailing during the studied period. Obviously, for a horizon of T years the expected return is higher than the one-year expected return; hence the reference points also increase to $(1.0306)^T - 1$ and $(1.0357)^T - 1$, respectively.[10] We also examine the joint effect of a positive reference point and a higher loss aversion parameter.

The optimization for all preferences but the classic myopic function (with $A = 0$) is conducted for various investment horizons. For the classic myopic function only the results corresponding to $T = 1$ are presented, because in principle, apart from sampling errors of the bootstrapping procedure discussed above, the optimal diversification is invariant to the changes in the investment horizon (see Merton and Samuelson 1974).

Table 1 presents the optimal diversification for the myopic preference and for its modified form. Panel (a) of Table 1 reveals that for the myopic preference up to risk aversion $\alpha \leq 1$, investing 100% in small stocks is optimal. Because for $\alpha = 1$ the myopic preference is equal to the log function and because for this function maximizing expected utility is tantamount to maximizing the geometric mean, we find that maximizing the geometric mean implies specialization, because it is optimal to invest 100% in the small stock portfolio. As expected, for a high degree of risk aversion, the myopic investor tilts toward the less risky assets, and this optimal diversification is invariant to the assumed horizon. Although we have calculated the optimal diversifications for various α's, we discuss below the relevant range of this risk aversion parameter.

We calculated the optimal diversification also for the modified myopic form preference with various levels of the constant value A. For $A = 100$ or larger, the optimal investment is 100% in the small stock portfolio for all α, even for the unrealistically large value of $\alpha = 10$; with this function, holding the MGM portfolio is optimal. Panel (b) of Table 1 reports the result for the relatively small value $A = 1$, a case where for each \$1 invested the individual holds \$1

[9] We call it the modified myopic function because of the similar power form. However, recall that by adding the constant A it is not myopic anymore.

[10] Recall that with prospect theory the value function is written in terms of the change in wealth rather than the wealth; hence we subtract 1.

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Table 1 Optimal Investment Weights as a Function of the Risk Aversion Parameter and the Investment Horizon

(a) Myopic function $U(W) = W^{1-\alpha}/(1-\alpha)$ with the horizon $T = 1$

	Risk aversion parameter								
α	0	0.5	1	1.5	2	3	4	5	10
R_F	0	0	0	0	0	0	0	0	0.27
T-bills	0	0	0	0	0	0	0	0	0.05
T-bonds	0	0	0	0.20	0.16	0.10	0.07	0.28	0.01
Corp. bonds	0	0	0	0.01	0.22	0.45	0.59	0.40	0.51
S&P 500 index	0	0	0	0.01	0.04	0.06	0.07	0.13	0.05
Small stock portfolio	1	1	1	0.78	0.58	0.38	0.28	0.19	0.11

(b) Modified myopic function $U(W) = (A+W)^{1-\alpha}/(1-\alpha)$ with $A = 1$

	Years							
Risk aversion parameter	1	2	3	4	5	10	20	30
$\alpha = 0$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0	0	0	0	0	0	0	0
Corp. bonds	0	0	0	0	0	0	0	0
S&P 500 index	0	0	0	0	0	0	0	0
Small stock portfolio	1	1	1	1	1	1	1	1
$\alpha = 0.5$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0	0	0	0	0	0	0	0
Corp. bonds	0	0	0	0	0	0	0	0
S&P 500 index	0	0	0	0	0	0	0	0
Small stock portfolio	1	1	1	1	1	1	1	1
$\alpha = 1.5$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0	0	0	0	0	0	0	0.05
Corp. bonds	0	0	0	0	0	0	0	0
S&P 500 index	0	0	0	0	0	0	0.07	0.13
Small stock portfolio	1	1	1	1	1	1	0.93	0.82
$\alpha = 2$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0	0	0	0	0.05	0.08	0.17	0.16
Corp. bonds	0	0	0	0	0	0	0	0.08
S&P 500 index	0	0	0	0.04	0.02	0.10	0.20	0.22
Small stock portfolio	1	1	1	0.96	0.94	0.82	0.63	0.55
$\alpha = 3$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0.20	0.22	0.25	0.25	0.30	0.24	0.14	0.09
Corp. bonds	0	0	0	0	0	0.09	0.29	0.40
S&P 500 index	0.04	0.15	0.10	0.15	0.11	0.16	0.20	0.20
Small stock portfolio	0.76	0.63	0.65	0.60	0.59	0.50	0.37	0.31
$\alpha = 4$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0.19	0.37	0.31	0.22	0.24	0.18	0.09	0.04
Corp. bonds	0.18	0	0.09	0.20	0.22	0.32	0.48	0.57
S&P 500 index	0.06	0.17	0.13	0.15	0.12	0.15	0.17	0.18
Small stock portfolio	0.56	0.46	0.47	0.43	0.42	0.36	0.26	0.21
$\alpha = 5$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0	0
T-bonds	0.16	0.29	0.24	0.16	0.17	0.13	0.06	0.01
Corp. bonds	0.32	0.19	0.27	0.37	0.38	0.45	0.59	0.67
S&P 500 index	0.07	0.15	0.12	0.14	0.12	0.14	0.15	0.16
Small stock portfolio	0.45	0.36	0.37	0.34	0.33	0.28	0.20	0.16
$\alpha = 10$								
R_F	0	0	0	0	0	0	0	0
T-bills	0	0	0	0	0	0	0.05	0.06
T-bonds	0.05	0.10	0.10	0.03	0.04	0.04	0.04	0.00
Corp. bonds	0.66	0.61	0.62	0.71	0.72	0.73	0.73	0.75
S&P 500 index	0.08	0.11	0.10	0.11	0.10	0.11	0.10	0.11
Small stock portfolio	0.21	0.17	0.17	0.15	0.15	0.13	0.09	0.08

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not invested in the capital market. First note that even for such a small value of A, the marginal utility is bounded, yet it is relatively very high over the violation area. Nevertheless, up to $\alpha = 2$, the dominating weight in the optimal portfolio is composed from stocks. For a very large and presumably irrelevant risk aversion parameter, the optimal portfolio tilts to a cooperate bond portfolio even for very large investment horizons.

Because the myopic preference is employed to theoretically refute the optimality of the MGM portfolio, it is interesting to discuss the relevant range of the risk aversion parameter characterizing this function. Most of the evidence regarding the risk aversion parameter profoundly supports the MGM optimal portfolio also for the myopic preferences. Arrow (1971) argues on theoretical grounds that α should be approximately 1 (hence, we obtain log function for which the MGM portfolio is optimal). Kydland and Prescott (1982) report a parameter between 1 and 2. Altug (1983) estimates this parameter to be near 0, justifying empirically the optimality of 100% investment in the small stock portfolio. Kehoe (1983) estimates it to be about 1, Hildreth and Knowles (1982) estimate this parameter to be about 1, and Tobin and Dolde (1971) estimate it to be about 1.5. Friend and Blume (1975) present empirical evidence that this parameter is about 2. Thus, most evidence supports the optimality of the MGM portfolio even with the myopic preference. With an estimate of $\alpha = 2$ we have a deviation from the MGM optimality. However, even in this case equity dominates the portfolio, as about 62% is still invested in stocks (for more detail on the risk aversion parameter, see Mehra and Prescott 1985). Moreover, for the modified form of the myopic preference the results are even stronger in favor of equity: for the most relevant $\alpha \leq 1.5$ even with the small value of $A = 1$ (let alone for a larger A), 95%–100% of the optimal portfolio is composed of stocks (see panel (b) of Table 1) for all horizons, supporting the importance of stocks for the long run.

Table 2 provides the optimal diversification corresponding to the prospect theory value function with loss aversion parameter $\lambda = 2.25$, for the range $0.5 \leq \gamma \leq 1$ (recall that this parameter is estimated by Kahneman and Tversky 1979 as 0.88) and for three alternate reference points. Note that when $\gamma = 1$ we obtain the bilinear value function employed by Benartzi and Thaler (1995).

As we can see from this table, increasing the reference point from zero to the inflation rate and alternatively to the riskless interest rate counterintuitively does not necessarily induce a reduction in the equity weights in the optimal portfolio. The reason is that some of the returns of the fixed income assets, particularly the corporate bond index, also fall below the reference points and hence reduce the attractiveness of these assets. As expected, the smaller the parameter γ, the smaller the risky asset weight in the optimal portfolio. Yet, even for $\gamma = 1/2$, for $T \geq 10$ years the optimal investment is 100% in the small stock portfolio index, implying that the MGM portfolio is optimal. For larger values of γ, the optimality of the MGM portfolio is achieved even for a smaller horizon. For the parameter estimated by Kahneman and Tversky (1979), $\gamma = 0.88$, and for the bilinear value function, the MGM portfolio is optimal even for $T \geq 2$ (see Table 2).

Table 3 reports the sensitivity of the optimal diversification to the assumed loss aversion parameter. As can be seen from this table, the effect of the loss aversion parameter on the optimal diversification is quite dramatic for short investment horizons: for example, for $T = 1$ and $\lambda = 2.25$, the equity weight is 60%–80% (depending on the reference point), and for $T = 1$ and $\lambda = 5$ it is reduced to 13%–29%. Thus, if indeed the loss aversion parameter in practice is larger than what is estimated in laboratory experiments, it will certainly have a strong effect on the optimal diversification for short investment horizons. Yet the most astonishing result corresponding to the main issue analyzed in this study is that even for $\lambda = 5$, the MGM portfolio is optimal for horizons that are considered to be very small for investment for retirement, as for $T \geq 5$ the optimal portfolio is composed of 100% small stocks. Because this portfolio has the highest geometric mean, maximizing the geometric mean is optimal for $T = 5$, let alone for a longer horizon that is more relevant for investment for retirement. Thus, although to prove the dominance of the MGM portfolio by AFSD we need an infinite horizon, for the prospect theory investors, five years are sufficient to justify this choice.

To sum up, for the indefinitely long run, the MGM portfolio dominates all other diversification strategies, and for a horizon $T \geq 10$ we have almost FSD of equity over fixed income assets. For the commonly employed preference and for the prospect theory value function (with various risk aversion and loss aversion parameters and various reference points), for a horizon of $T \geq 10$ and in some cases also for a horizon of $T \geq 5$ years, specialization in the MGM portfolio is optimal.

4. Pension Funds Diversification in Practice and the Gain from Shifting to the MGM Portfolio

Table 4 reports the pension fund asset allocation around the world. The equity proportion ranges from 5% in South Korea to 65% in Hong Kong. In all countries, the equity investment proportion falls short relative to what is recommended in this study, at least

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Table 2 **Kahneman and Tversky (1979) Utility Function** $U(R)=R^{\gamma}$ **if** $R>$ *Reference Point* **and** $-\lambda(-R)^{\gamma}$ **if** $R<$ *Reference Point* **with** $\lambda=2.25$ **and** $\gamma=0.5, 0.75, 0.88, 1$

Risk aversion parameter	Years:	1			2			3			4			5			10			20			30		
	Reference point:	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F
$\gamma=0.5$																									
R_F		0.29	0.08	0.09	0	0.08	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills		0.27	0.23	0.20	0	0.09	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds		0.08	0.26	0.21	0.15	0.23	0.12	0.21	0.02	0.15	0.17	0.03	0.13	0.25	0	0.02	0	0	0	0	0	0	0	0	0
Corp. bonds		0.25	0.10	0.19	0.57	0.13	0.43	0.44	0.03	0.16	0.45	0.03	0.04	0.24	0	0.03	0	0	0	0	0	0	0	0	0
S&P 500 index		0.06	0.17	0.13	0.12	0.18	0.26	0.12	0.66	0.24	0.03	0.24	0.22	0.04	0.15	0.20	0	0	0	0	0	0	0	0	0
Small stock portfolio		0.05	0.17	0.18	0.15	0.29	0.19	0.22	0.28	0.45	0.35	0.71	0.61	0.47	0.85	0.75	1	1	1	1	1	1	1	1	1
$\gamma=0.75$																									
R_F		0	0.04	0.09	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills		0	0.10	0.13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds		0.44	0.25	0.17	0	0	0.03	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds		0.12	0.26	0.22	0	0	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index		0	0.12	0.16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio		0.43	0.23	0.24	1	1	0.93	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\gamma=0.88$																									
R_F		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills		0	0	0.12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds		0.24	0.19	0.09	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds		0	0	0.19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index		0	0	0.14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio		0.76	0.81	0.46	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\gamma=1$																									
R_F		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds		0	0.06	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds		0	0.06	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index		0	0.56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio		1	0.32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

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Table 3 **Kahneman and Tversky (1979) Utility Function** $U(R) = R^{\gamma}$ **if** $R > \textit{Reference Point}$ **and** $-\lambda(-R)^{\gamma}$ **if** $R < \textit{Reference Point}$ **with** $\lambda = 2.25, 3, 4, 5$ **and** $\gamma = 0.88$

Risk aversion parameter — Years:	1			2			3			4			5			10			20			30		
Reference point:	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F	Zero	Inflation	R_F
$\lambda = 2.25$																								
R_F	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills	0	0	0.12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds	0.24	0.19	0.09	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds	0	0	0.19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index	0	0	0.14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio	0.76	0.81	0.46	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\lambda = 3$																								
R_F	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills	0	0.01	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds	0.30	0.39	0.06	0.25	0.12	0.01	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds	0.34	0.20	0.54	0	0.05	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index	0	0.13	0.02	0	0	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio	0.36	0.27	0.38	0.75	0.83	0.97	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\lambda = 4$																								
R_F	0	0	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills	0.12	0.30	0.23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds	0.02	0.25	0.28	0.18	0.32	0.29	0.25	0.11	0.08	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds	0.68	0.14	0.14	0.42	0.18	0.25	0	0.07	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index	0.09	0.10	0.14	0.04	0.05	0.02	0	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio	0.09	0.21	0.19	0.37	0.45	0.44	0.75	0.82	0.88	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\lambda = 5$																								
R_F	0.23	0.16	0.04	0	0.01	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bills	0.29	0.19	0.29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T-bonds	0.08	0.19	0.22	0.06	0.13	0.34	0.16	0.17	0.16	0.11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Corp. bonds	0.27	0.21	0.15	0.67	0.41	0.26	0.37	0.17	0.16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S&P 500 index	0.08	0.10	0.12	0.05	0.24	0.04	0	0.03	0.01	0	0	0.03	0	0	0	0	0	0	0	0	0	0	0	0
Small stock portfolio	0.05	0.16	0.17	0.23	0.21	0.36	0.48	0.62	0.67	0.89	1	0.97	1	1	1	1	1	1	1	1	1	1	1	1

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Table 4 A Summary of Pension Fund Asset Allocation Around the World

Country	Equities (%)	Fixed interest (%)	Property and other (%)
Australia	50	25	25
Canada	35	40	25
Chile	40	45	15
China	20	80	—
Denmark	20	65	15
Hong Kong	65	35	—
Japan	30	50	20
Korea (South)	5	95	—
Netherlands	20	70	10
Switzerland	35	45	20
United Kingdom	40	45	15
United States	45	40	15

Source. Financial Services Council (2014).

for a 30-year investment horizon or longer. However, because the managed assets in each country represent the savings of various age groups, we cannot claim that the held portfolios are not optimal without detailed information on the retirement savers' age distributions. For example, it is possible theoretically that the investment in equity in the United States (45%) is optimal in the hypothetical case where the proportion of old savers for retirement in the saver population is relatively small. Yet, it is obvious from the international comparison that if in the United States the equity investment is optimal, it is far from being optimal in some other countries reported in Table 4, despite some possible differences in the age distribution across these countries. For example, we can safely conclude without precise information on age distribution that the percentages of equity investment in South Korea (5%), China (20%), and probably also in Denmark (20%) are far below the optimal figure. However, to refine the analysis and neutralize the impact of the age distribution factor, we focus below on the life-cycle mutual funds (known also as target date mutual funds), which invest in equity an investment proportion that is a function of the ages of the savers, generally with five-year bracket increments. The investment in such mutual funds is very popular in the United States with the 401(k) saving plan.

First, let us examine the common wisdom asserting that the equity percentage in the life-cycle portfolio should be "100 less your age." We find that this investment strategy is far from being optimal. For example, by this rule, a 35-year-old person who has about 30 years until retirement should invest 65% in equity, and we find that for such a person 100% equity is optimal.

We turn now to the actual diversification of life-cycle funds. Analyzing the actual investment of life-cycle funds, Idzorek (2008) reports on the equity investment of various life-cycle U.S. mutual funds, where for each group of investors the percentage of equity investment is different. For relatively long investment horizon (about 25–30 years) it is reported that the maximum difference in the equity investment of the funds included in the survey is about 35%, (the equity weight ranges from 65% to 100%). Because a relatively long horizon is considered, we calculate for these life-cycle funds the gain that they could achieve by shifting from the stock–bond portfolio mix they actually hold to the MGM portfolio. We first assume that the life-cycle mutual fund under consideration holds the corporate bond index and the S&P 500 index. For example, a mutual fund that invests 65% in equity is assumed to hold 65% in the S&P 500 index and 35% in the corporate bond index. We calculate the profit that this mutual fund would earn by shifting to the MGM portfolio. Because there are many funds in the 65%–100% equity range, we conduct the calculations with a 5% increment in the equity held in the portfolio, and hence analyze the benefit of shifting to the MGM portfolio of various life-cycle funds.

Figure 3 reports the accumulated wealth for the $T = 10$ and $T = 30$ years investment horizons corresponding to various portfolios and the MGM portfolio. The horizontal axis shows the starting investment year. For example, for the case $T = 10$, 1928 implies that one started accumulating money for retirement in 1928: \$1 is invested in 1928 and then the saver for retirement invests one additional dollar every year, and we examine the accumulated wealth after 10 years. The year 1929 on the horizontal axis indicates the same savings policy with the exception that the starting investment year is 1929. Because our study covers the period 1926–2012, the last year on the horizontal axis for $T = 10$ is 2002, because with this starting year we end savings after 10 years at 2012. Similarly, for the $T = 30$ years horizon, the last starting point is 1982, ending after 30 years in 2012.

For $T = 10$, the vertical axis denotes the accumulated wealth after 10 years of investment. Thus, the corresponding vertical value above 1928 is the accumulated wealth of saving one dollar every year for 10 years to retirement, which is in this case is 1938.

As we can see from Figure 3, for $T = 10$ (see Figure 3(a)) the MGM portfolio is located mostly above the other curves (which refer to life-cycle funds with various equity-fixed asset portfolios), but there are some starting years where the MGM portfolio yields a lower terminal value (see, e.g., 1968—1970). Yet, note that when the MGM portfolio underperforms, the underperformance is only by a small margin, but when it performs better than the funds it is generally by a large margin. However, for $T = 30$ (see Figure 3(b)), the MGM curve is located almost entirely above the fund curves, indicating that much more

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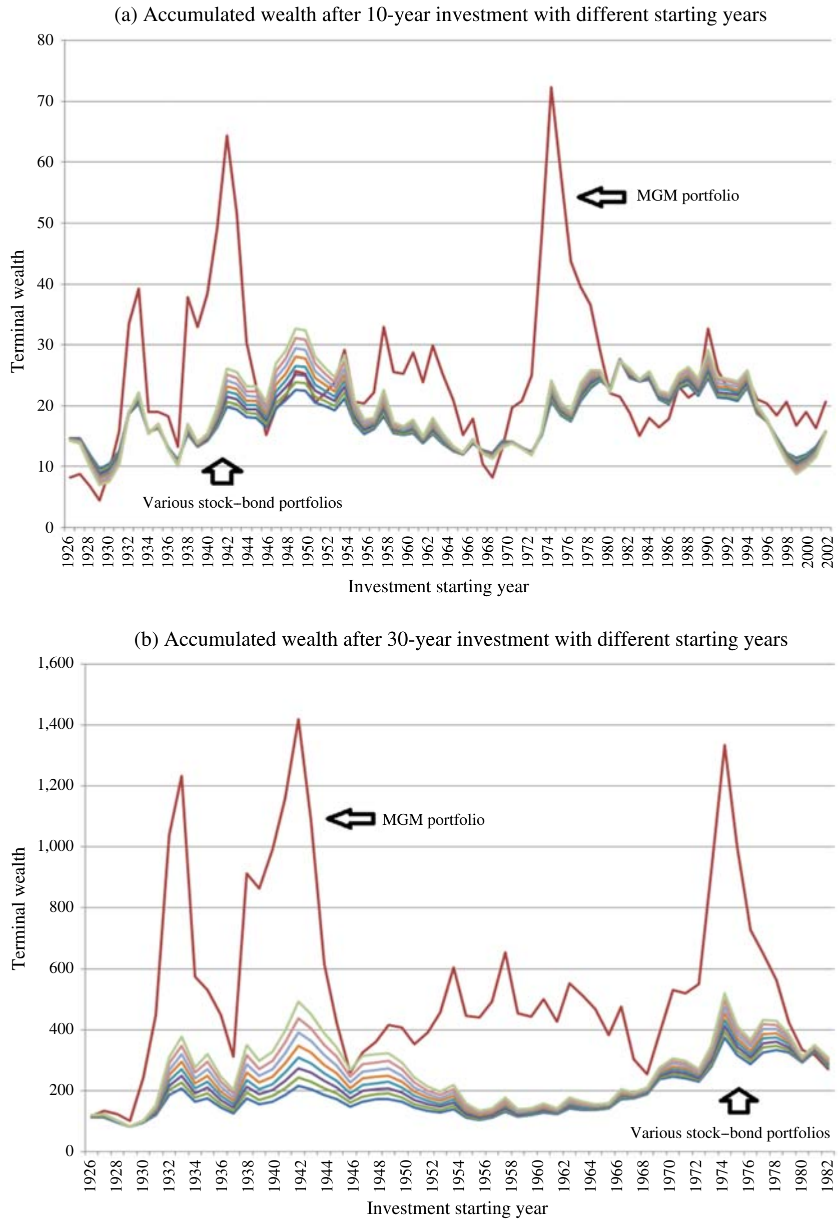
Figure 3 (Color online) Accumulated Wealth of \$1 Invested Every Year After T Years ($T = 10$ and $T = 30$) Corresponding to the MGM Portfolio and Various Life-Cycle Portfolios



Note. The life-cycle portfolios' investment in the S&P 500 index ranges from 65% to 100%.

wealth can be accumulated by investing in the MGM portfolio.

As shown above, the performance results, particularly for $T = 10$, depend on the starting investment year. In practice the investor does not know in advance which cycle she will face. Therefore, we next assume that each starting year has equal probability and draw the cumulative distribution of the terminal wealth of the MGM portfolio and of the 65% equity, 35% bond portfolio of the life-cycle fund. For example, with $T = 10$, we have 78 starting points, where the first one is 1926 and the last one 2002 (corresponding to the period 2002–2012). Thus, each cycle gets a probability of $1/78$. Similarly for $T = 30$, we have 58 cycles where the last one is for the period 1982–2012; hence we assign to each cycle a probability of $1/58$. Figure 4, (a) and (b), provides the cumulative distributions of the two alternate investment strategies. As we can see, consistent with previous results with $T = 30$, the MGM portfolio dominates the life-cycle fund portfolio by FSD. For $T = 10$, we do not have FSD, but we have almost FSD with a violation area of 3.59%, which is smaller than the 5.9% allowed as obtained in the experimental analysis (see Levy et al. 2010).

Finally, one may correctly advocate the following claims: (a) Generally, the life-cycle funds decrease

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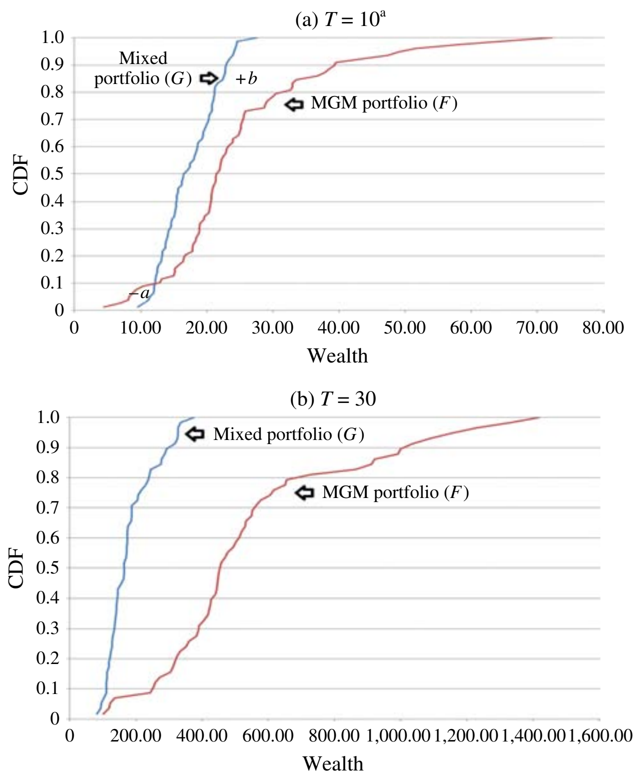
Figure 4 (Color online) Cumulative Distribution Function (CDF) of the MGM Portfolio and the Life-Cycle Fund's Mixed Portfolio (65% Equity, 35% Bonds) for $T = 10$ and $T = 30$ Investment Horizons



[a]$\varepsilon = |a|/(|a| + b) = 3.59\%$, F dominates G by almost FSD.

every time interval the equity exposure; hence, for the right performance comparison, the 65%–35% investment policy should be changed over time, where the equity proportion should decrease as the investment horizon becomes shorter. (b) The small stock index, despite being the MGM portfolio, may have small volume relative to the assets held for investment for retirement. Therefore, if 100% investment in equity is recommended, investment in the S&P 500 index is practically more appropriate. (c) One should reduce the equity exposure also in the MGM portfolio as time elapses because the MGM portfolio does not dominate other relevant investment strategies for relatively short horizons.

Figure 5 reports the results which take into account all these issues for an investment horizon of $T = 30$ years. Figure 5(a) presents the accumulated wealth (left-hand side) and the cumulative distributions (right-hand side) of the MGM portfolio and the life-cycle portfolio drawn as before (see explanation to the derivation of Figures 3 and 4) with one exception: here it is assumed that the investment weight of equity of the life-cycle fund is reduced every five years by 5%. Thus, after five years, 60% is invested in equity and 40% in bonds. In the last five-year investment period (out of a 30 year investment horizon), the bond weight in the portfolio increases to 60% and the equity weight is reduced to 40%. Note that we have various investment starting years, 1926 for the first 30-year investment cycle and 1982 for the last 30-year investment cycle. The cumulative distributions are calculated as in Figure 4. As we can see from Figure 5(a), also with this change, we have FSD of the MGM portfolio over the life-cycle portfolio. In Figures 5(b)–5(d), we keep this characteristic of the reduction in the equity exposure of the life-cycle fund as the horizon becomes shorter and analyze some other characteristics of the investment strategies.

We turn now to the second issue raised above: Suppose that due to the relatively small volume of the MGM portfolio (the small stock index), we decide to invest in the S&P 500 index. Thus, we give up the optimal MGM portfolio and recommend investing in the S&P 500 index instead. Figure 5(b) reveals that even in this compromising case, the S&P 500 index dominates the life-cycle fund by FSD.

Responding to the third issue given above, we also reduce the equity exposure with our recommended investment policy (the MGM portfolio) in the last five-year cycle. Note that with a $T = 10$ years investment horizon, we still have almost FSD of the MGM portfolio over the bond portfolio. With a five-year horizon, there is no FSD and no almost FSD because the violation area is quite large, 9.7%. (For brevity's sake, the figure corresponding to $T = 5$ is not presented in this paper.) Therefore, we examine the following policy: hold the MGM portfolio, but in the last five years before retirement, change the diversification to 50% small stock index and 50% corporate bond portfolio. Figure 5(c) reports the results. Again, in this case the MGM strategy with a reduction in the last five years' equity exposure dominates the life-cycle fund by FSD. Finally, Figure 5(d) is the same as Figure 5(c), but this time the S&P 500 index replaces the MGM portfolio. Again, we have that the suggested investment strategy dominates the life-cycle fund by FSD.

5. Concluding Remarks

The aging population accompanied by the prevailing relatively low interest rate spells economic disaster at retirement for a relatively large segment of the population (see Poterba 2014). We advocate that one can substantially increase the future accumulated wealth and overcome (partially or completely) the retirement economic crisis by investing in Stocks, which on average yields much higher returns than bonds with no additional risk.

We show empirically that the small stock portfolio is the MGM portfolio, and we show theoretically that the MGM portfolio dominates all other investment strategies by asymptotic first degree Stochastic dominance. AFSD implies that as the horizon T approaches

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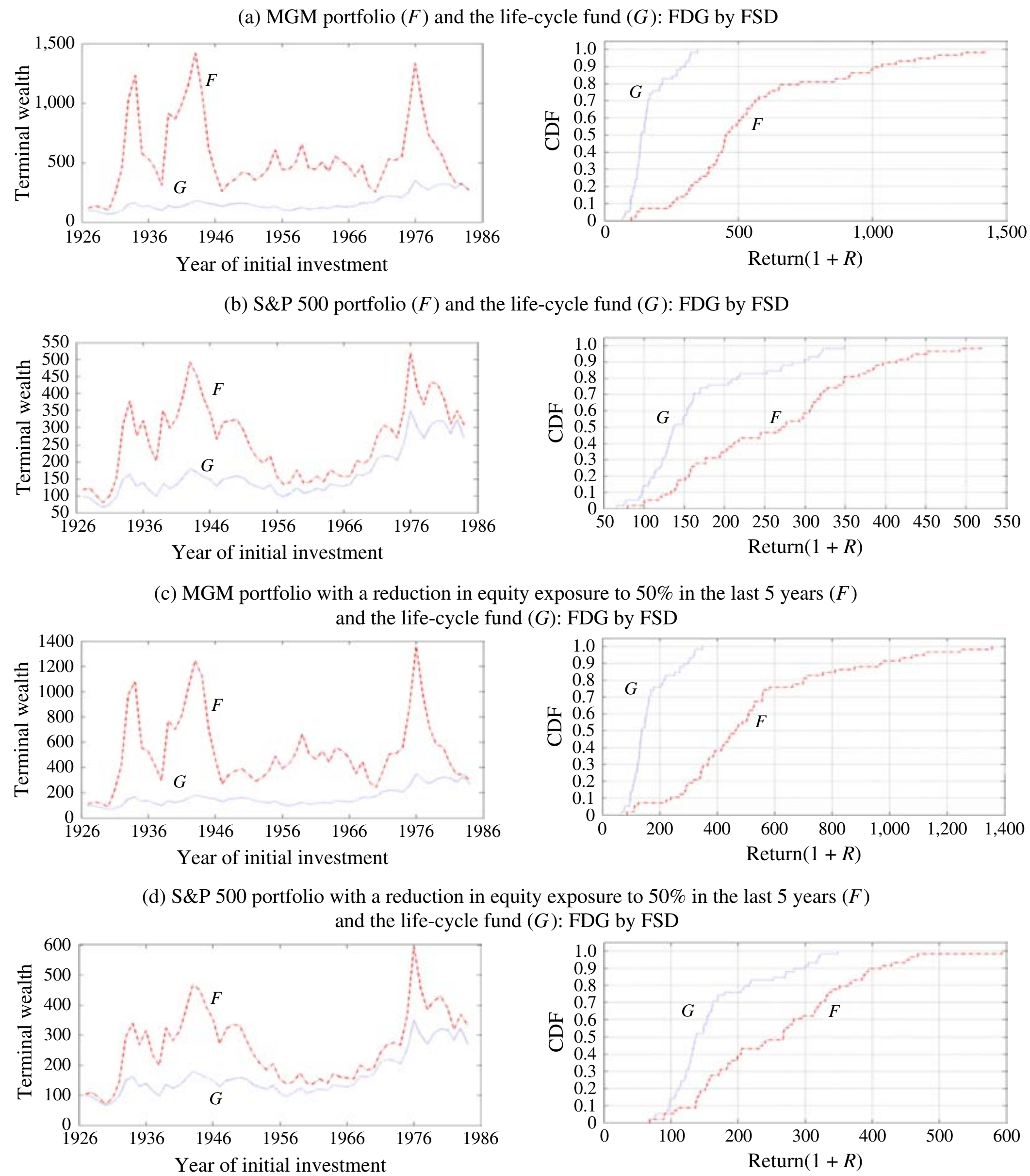
Figure 5 (Color online) A Comparison of the Accumulated Terminal Wealth (Left Panels) and Cumulative Distributions (Right Panels) of Various Portfolios



Notes. In all cases, the investment weight in the life-cycle mutual fund is reduced by 5% every five-year interval. The investment horizon is $T = 30$; hence, the last investment starting year is 1982, ending in 2012. CDF, cumulative distribution function; FDG, F dominates G.

infinity, the MGM portfolio (small stock portfolio) provides the highest expected utility for all preferences as well as for the prospect theory value function, provided that the marginal utility is bounded. As a byproduct of this analysis, we shed more light on the long-standing debate corresponding to the optimality of the MGM portfolio, and hence to the optimality of a portfolio with heavy weight in stocks for the long run investment. The MGM portfolio, which is optimal for the very long run for virtually all preferences, is not optimal for the classic myopic preference (which has unbounded marginal utility). However, it is optimal for the modified version of this function, which conforms to the mental accounting argument. This issue, although theoretical, also has important implications for institutional investors like pension funds.

The case $T \to \infty$ is theoretically interesting, but the investors who save for retirement are more interested in a long but finite horizon, e.g., 30–40 years. We show the following results for this long but finite horizon:

(a) The MGM portfolio dominates the bond portfolio by almost FSD.

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(b) We solve for the optimal diversification for various commonly employed preferences and for the prospect theory value function for a wide range of risk aversion and loss aversion parameters, for various reference points, and for various assumed investment horizons. We find that diversification is optimal for a relatively short investment horizon, but for a horizon of 10 years or longer (and in some cases even for five years and longer), the MGM portfolio is optimal; thus, investing 100% in the small stock index is once again optimal.

(c) Analyzing the actual investment of life-cycle mutual funds, we demonstrate the welfare gain that these funds can achieve by shifting from the commonly employed diversification to the MGM portfolio.

Finally, a word of caution is called for: In the empirical analysis, we assume that the years corresponding to the investment horizon are selected at random. Of course, if one buys the small stock index at its peak and holds it for a long time period, she may be worse off compared to investing in bonds. However, recall that normally people save every year some amount for retirement, which is consistent with our analysis in which we select the investment years at random. Thus, our analysis is intact for annual saving for retirement where the risk is averaged out across many years.

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