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The Dynamic Pricing Problem from a Newsvendor's Perspective

George E. Monahan, Nicholas C. Petruzzi
Department of Business Administration, College of Business, University of Illinois at Urbana-Champaign, 1206 South Sixth Street, Champaign, Illinois 61820 {gmonahan@uiuc.edu, petruzzi@uiuc.edu}

Wen Zhao
Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, 1206 West Green Street, Urbana, Illinois 61801, wenzhao@uiuc.edu

The dynamic pricing problem concerns the determination of selling prices over time for a product whose demand is random and whose supply is fixed. We approach this problem in a novel way by formulating a dynamic optimization model in which the demand function is isoelastic but the random demand process is quite general. Ultimately, what we find is a strong parallel between the dynamic pricing problem and dynamic inventory models. This parallel leads to a reinterpretation of the dynamic pricing problem as a price-setting newsvendor problem with recourse, which is useful not only because it yields insights into the optimal solution, but also because it leads to additional insights into how pricing recourse affects the actions and profits of a price-setting newsvendor. We make contributions in three areas: First, we develop structural properties that define an optimal pricing strategy over a finite horizon and investigate how that policy impacts a newsvendor's optimal procurement policy and optimal expected profit. Second, we establish a practical and efficient algorithm for computing the optimal prices. Third, we examine how market parameters affect the optimal solution through a series of numerical experiments that utilize the algorithm.

Key words: revenue management; dynamic pricing; newsvendor model; pricing recourse
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1. Introduction

The dynamic pricing problem concerns the determination of selling prices over time for a product whose demand is random and whose supply is fixed. This problem tends to present significant computational challenges. To overcome these difficulties, heuristics are often employed to compute solutions that may not be optimal. In general, the problem is to dynamically adjust selling prices as the fixed inventory is depleted to maximize the expected revenue stream over a finite planning horizon. A prototypical example is the pricing of seats on an airline flight: although an aircraft is committed to a flight in advance, the price of the seats can change dynamically right up until take-off. Other examples now include the pricing of hotel rooms, fashion goods, discontinued or leftover products, golf course tee times, and even financial aid packages (Virshup 1997). The payoff functions in these problems are typically not concave, which creates a computational challenge.

Because of the strategic importance of a firm's pricing decisions and the technical challenges inherent in the modeling and computation of optimal decisions, the dynamic pricing problem has generated a fair amount of interest in recent years among researchers across such fields as operations research, marketing, and economics. In this paper, we approach the problem in a novel way by formulating a dynamic program in which the relationship between demand and price is somewhat specific, but the random demand process is quite general. Our focus is three-fold: First, we develop structural properties that define an optimal pricing strategy over a finite horizon and investigate how that policy impacts the firm's optimal procurement decision and optimal expected profit. Second, we establish a practical and efficient algorithm for computing the optimal prices for each period of the finite horizon. Third, by implementing the algorithms for a variety of problem instances, we examine how market parameters affect the optimal solution. Ultimately what we find is a strong parallel between the dynamic pricing problem and dynamic inventory models; thus, we can draw on the rich insights from inventory theory to better understand the structure of the dynamic pricing policy and its efficacy for higher-level managerial decision making.

We approach the problem within the following modeling framework: A retailer has a single opportunity to establish an inventory (or capacity) level prior to the start of a selling season that consists of multiple periods. Demand in each period is random and depends on price, but the elasticity of demand is known and is independent of price. At the beginning of each period, a price is announced that depends both on the amount of inventory that remains and on the number of periods that remain to sell it. Then demand is realized, the period ends, and the next one begins.

Our model stipulates a specific class of demand functions, those functions in which price elasticity of demand is constant, but places no requirements on the form of the probability distribution that is used to characterize the uncertainty in demand. This model offers three benefits that, collectively, constitute its primary contribution to the literature. One benefit of our model is its tractability. In particular, we find that with an appropriate transformation of variables, the dynamic pricing problem that we formulate can be reduced to a sequence of state-independent, static, single-variable optimization problems. Moreover, each of the resulting optimization problems is of the same form, differing from the others only by the magnitude of a single coefficient. As a result, determination of the optimal pricing policy does not require the solution of a dynamic program; and, the complexity involved in computing the optimal price for each period does not increase significantly as the number of periods comprising the finite horizon increases. The model's tractability also leads to new insights. Perhaps most notably, we find a strong parallel between the dynamic pricing problem and dynamic inventory models. This parallel leads to a reinterpretation of the dynamic pricing problem as a price-setting newsvendor problem with recourse, which is useful not only because it yields insights into the optimal solution, but also because it yields additional insights into how pricing recourse affects the actions and profits of a price-setting newsvendor.

A second benefit of our model is its robustness: it can be applied using any demand distribution. In contrast, most pricing models appearing in the yield management literature require demand to be characterized by the Poisson distribution to obtain tractability. One limitation of the Poisson approach is that it is a single-parameter distribution; hence there is no freedom to estimate separately the mean and variance (and higher moments) of demand. Another limitation is that the coefficient of variation of demand decreases as the mean of demand increases; hence, for applications in which mean demand is very large, demand uncertainty is "modeled away" and, as a result, the business environment is effectively treated as a deterministic one. A compound-Poisson distribution can be used to overcome these limitations, but then the resulting solution procedure typically becomes enough of an onus that heuristics need to be employed. Our model allows us to address these issues without having to revert to heuristics.

A third benefit of our model is that it has practical and intuitive appeal. Specifically, because it is formulated as a periodic pricing model, it can be applied directly to business scenarios in which it either is not practical or is not desirable to adjust the selling price after each sale of a unit of inventory (which might be the case, for example, for fashion goods). Yet, because the computational burden of the solution procedure does not increase significantly as the number of periods in the planning horizon is increased, our model also can be applied to business scenarios in which it is acceptable practice to adjust the price after each unit sale (e.g., for airline seats) simply by defining the "periods" as appropriately small intervals of time. The intuitive appeal derives from the fact that properties of the optimal pricing policy are sensible. For example, our model yields the result that, everything else being equal, the optimal price to set in a given period is a decreasing function of the amount of stock that is available for sale in that period.

The implicit "cost" of our modeling approach is the assumption that demand elasticity is independent of price. However, the isoelastic form of demand, which is also referred to as the multiplicative form or the double-log-linear form, is widely considered to be the most frequently used demand specification among both econometricians and marketing empiricists. (See, for example, Leeflang et al. 2000, p. 74; Intriligator 1978, p. 220; Montgomery and Bradlow 1999, p. 570.) In a meta-analysis of econometric studies conducted between 1960 and 1985 to estimate the price elasticity of demand for specific product brands, Tellis (1988) indicates that a vast majority of the models he surveyed were constructed from a multiplicative specification of demand. In a recent update of the Tellis survey, Kalyanam (1996) confirms a similar trend. Besides its analytical appeal, four reasons emerge to explain the popularity of the isoelastic form of demand. First, the functional form is consistent with consumer-utility-maximization theory; thus, it is a reasonable candidate for model building (Varian 1992, p. 211). Second, by explicitly accounting for the effects of price elasticity on demand, the functional form has an unambiguous economic interpretation. Third, its log-linearity is particularly amenable to empirical analysis because its parameters can be estimated using well-established linear regression techniques (Intriligator 1978, Hirshleifer and Glazer 1992). Fourth, and perhaps most importantly, it typically provides a good statistical fit with available sales data. For example, the well-known SCAN∗PRO model, which is loglinear in price and other marketing variables, has been used worldwide in over 2,000 different commercial applications, predominately in the retail and grocery industries (Leeflang et al. 2000, p. 168). For additional examples of the statistical applicability of isoelastic demand, see Montgomery and Rossi (1999), Krishnamurthi and Raj (1991), Ghosh et al. (1984), and Brodie and de Kluyver (1984).

In our case, by employing an isoelastic demand function in our analysis of the dynamic pricing problem, we ultimately end up with a state-space reduction technique that leads to a practical algorithm, regardless of whether the randomness in demand is stationary or nonstationary. Thus, our model offers a useful alternative for actually computing the optimal price for each of an arbitrary number of periods and for exploring new insights, while still maintaining qualitative results that are consistent with the existing literature.

The remainder of this paper is organized as follows. In §2, we position the paper in relation to the literature. In §3, we formulate the dynamic decision problem and demonstrate the state-space reduction that allows the model to be solved as a series of static problems rather than as a dynamic program. Then, in §4, we establish the structure of the optimal pricing policy, discuss corresponding implications on the initial inventory decision, and develop corresponding insights. In §5, we develop properties to improve the efficiency of the solution procedure for the special case in which the uncertainty in demand has finite support. In §6, we implement the algorithm to further investigate implications and interpretations of the optimal solution. We conclude the paper with §7.

2. Relationship to the Literature

The model developed in this paper spans several streams of literature. One stream is the literature on dynamic pricing. Single-product dynamic pricing models were first studied by Kincaid and Darling (1963), who formulated a continuous-time stochastic dynamic program and developed properties of the revenue function. Gallego and van Ryzin (1994) later derived useful structural properties of the optimal price and proposed a deterministic heuristic for solving the problem. Zhao and Zheng (2000) then derived structural properties under more general conditions and found a closed-form solution for the case of a discrete price set. Gallego and van Ryzin (1997) extended their single-product model and asymptotically optimal heuristic to the multiple-product case. Common to these models is the formulation of the problem as a continuous-time stochastic program in which demand uncertainty is characterized by a Poisson process with a price-dependent intensity. The solutions for these models are both time and state dependent. In contrast, we approach the problem using a discrete-time model in which demand uncertainty can be characterized by a generic distribution, and we establish a solution procedure that is state independent. Bitran and Mondschein (1997) considered a discrete-time pricing model for a retail setting. However, they assumed that demand is Poisson and developed an optimal solution that is state dependent.

A second stream related to our model is the literature on dynamic inventory models with pricing and stochastic demand. Petruzzi and Dada (1999) provide a recent review of this literature, but representative papers most related to our model include Zabel (1972), Federgruen and Heching (1997), and Petruzzi and Dada (2002). Petruzzi and Dada (2002) were particularly instrumental in the development of our model because they specifically considered a case in which demand is modeled as a constant-elastic function of price. Their focus, however, is on learning the demand distribution when lost sales are not observable. Like the other papers in this literature stream, their model differs from ours because it applies to scenarios in which inventory can be replenished each period.

More similar to our model is the model by Chan et al. (2001), who consider a manufacturing setting in which production occurs every period, but each period's production decision is determined ahead of time, at the beginning of the finite horizon. Then, the price is determined dynamically at the start of each period. However, Chan et al. (2001) maintain discretion over each period's inventory decision by allowing some or all of the predetermined production for a given period to be "set aside" for future periods. In effect, this discretion replaces production as the mechanism to achieve a desired inventory level in a given period.

A third related literature stream is that on yield management, which is a set of problems that can be more generally classified as perishable asset revenue management (PARM) models. The basic problem of yield management is how to sell a finite inventory over a finite horizon to maximize the total revenue. Models of this sort have developed a rich history in recent years. They typically are formulated under the assumption that demand is segmented into classes, and that each class has associated with it a fixed price that is determined exogenously. However, there is a number of variations on this theme that appear in the literature. One variation involves settings in which demands for the different classes arrive sequentially. In these cases, the fundamental question that must be answered each time a demand occurs is whether to accept the demand or to reserve the unit of inventory for possible sale later to a potentially higher-paying customer. Models of this variety include Littlewood (1972), Belobaba (1989), Brumelle and McGill (1993), and Robinson (1995).

A second variation involves settings in which demands for different classes occur concurrently, and inventory can be made available to multiple classes simultaneously. In these cases, the key question is how much inventory to allocate to each demand class at any given time. Models of this type include Gerchak et al. (1985), Lee and Hersh (1993), Subramanian et al. (1999), and Zhao and Zheng (2001).

Finally, a third variation on this theme involves settings in which demands for different classes, in effect, occur concurrently, but inventory cannot be made available to multiple classes (at multiple prices) simultaneously. The basic decision in these models is when to switch from one demand class (usually characterized as higher paying customers) to another class (usually characterized as lower paying customers). Models of this type include Feng and Gallego (1995), Feng and Xiao (2000), and Petruzzi and Monahan (2003).

It is worth noting that this third variation also can be interpreted as a class of the dynamic pricing problem in which a discrete price set is given. Thus, this class of models bridges the dynamic pricing and yield management literatures. In general, however, yield management models differ from the dynamic pricing model in their assumption that price is exogenous; thus, their primary focus, in effect, is on inventory control. McGill and van Ryzin (1999) offer an excellent survey of not only yield management and dynamic pricing models, but also of related models on airline overbooking and demand forecasting.

3. Model and Solution Procedure

A finite-horizon selling season consists of T periods, indexed so that period t represents the number of periods *remaining* in the selling season; i.e., chronological periods are labeled $T, T-1, \dots, 1$. Demand in period t is random and depends on price as follows: $D_t(p) = A_t p^{-b}$, where $b > 1$ and $A_1, \dots, A_T$ are independent, identically distributed (iid) random variables, each with known cumulative distribution function (cdf) F and corresponding probability density function (pdf) f. In other words, the randomness in demand is price independent and multiplicative in nature, and the parameter b denotes the price elasticity of demand. (See, for example, Petruzzi and Dada 1999.) Note that if the stationary distribution F were replaced by the nonstationary process F_t, then all but one of our results (Proposition 3) would continue to hold. We discuss the implications of such a replacement in §7. Note also that, like a preponderance of the literature adopting an isoelastic demand form for analysis, we assume that b is stationary over time. Nonetheless, it is conceivable that situations exist in which the price elasticity of a given product might, indeed, change over time, which suggests that a time-dependent b would make an interesting extension to our model. We discuss the challenges and limitations of such an extension in §7.

Prior to the beginning of the season, an initial stock of S units is acquired (e.g., procured, produced) at a per-unit cost of c and is made available for sale over the finite horizon. At the beginning of each period, as long as some of the initial stock remains, a selling price is chosen. If no stock remains at the beginning of period t, the dynamic pricing problem ends. In general, p_t, the selling price set for period t, depends both on how much of the initial stock remains at the beginning of period t and on how many selling periods still remain.

Let $R_t(I_t)$ denote the maximum expected revenue-to-go function at the beginning of period t, given that I_t is the stock remaining at the beginning of period t, and that an optimal dynamic pricing policy is followed for the remaining t periods. The observation that I_t is equivalent to the number of leftovers that remain at the end of period $t+1$ leads to a recursive formula for $R_t(I_t)$:

$$\begin{aligned} R_t(I_t) &= \max_{p_t}\{p_t \mathrm{E}[\text{period-}t \text{ sales}] \\ &\qquad + \mathrm{E}[R_{t-1}(\text{period-}t \text{ leftovers})]\} \\ &= \max_{p_t}\{p_t(I_t - \mathrm{E}[I_t - A_t p_t^{-b}]^+) \\ &\qquad + \mathrm{E}[R_{t-1}([I_t - A_t p_t^{-b}]^+)]\}, \qquad (1) \end{aligned}$$

where $I_T = S$ and $R_0(I) = 0$ for all I.

Let p_t^* denote the optimal price for period t and S^* denote the optimal starting stock level. Then (1) implies that

$$\begin{aligned} p_t^* &= \arg\max_{p_t}\{p_t(I_t - \mathrm{E}[I_t - A_t p_t^{-b}]^+) \\ &\qquad + \mathrm{E}[R_{t-1}([I_t - A_t p_t^{-b}]^+)]\}, \qquad (2) \\ S^* &= \arg\max_S\{R_T(S) - cS\}. \qquad (3) \end{aligned}$$

Note from (2) that the optimization problem required to compute p_t^* depends on I_t, the state of the system at the beginning of period t. This problem, however, can be simplified to a state-*in*dependent optimization problem through the following transformation of variables. Let $z_t = I_t/p_t^{-b}$ denote the period-t *stocking factor* (Petruzzi and Dada 1999). Then, the period-t expected sales function can be written as the product of I_t and a term that is independent of I_t:

$$\begin{aligned} \mathrm{E}[\text{period-}t \text{ sales}] &= I_t - \mathrm{E}[I_t - A_t p_t^{-b}]^+ \\ &= I_t \cdot \left(\frac{z_t - \mathrm{E}[z_t - A_t]^+}{z_t}\right). \qquad (4) \end{aligned}$$

Following the interpretation from Petruzzi and Dada (1999), the *stocking* factor is a unit-normalized representation of the *safety* factor (which itself serves as a proxy for safety stock). To illustrate, let σ_t denote the standard deviation of A_t, and let $\mathrm{SD}_t(D_t(p_t)) = p^{-b}\sigma_t$ denote the standard deviation of $D_t(p_t)$. Then, the stocking factor can be written as $z_t = \mathrm{E}[A_t] + \mathrm{SF}_t \cdot \sigma_t$, where SF_t denotes the safety factor, which is defined by Silver and Peterson (1985) as the number of standard deviations that inventory deviates from expected demand:

$$\mathrm{SF}_t = \frac{I_t - \mathrm{E}[D_t(p_t)]}{\mathrm{SD}_t(D_t(p_t))}.$$

Even though the notion of stocking factor has more intuitive application to a scenario in which inventory can be replenished dynamically, it also has the following interpretation in the context of the dynamic pricing problem: In each period of the dynamic pricing problem, the fundamental decision is to determine how much demand to “match,” in expectation, to the remaining inventory (by controlling expected demand through the pricing policy variable). However, this decision is equivalent to determining how much of the remaining inventory *not* to match to expected demand. In other words, one way of thinking of the dynamic pricing problem is as a problem in which each period the question is to determine how much of the remaining inventory to designate this period as expected *leftovers*. In effect, it is this perspective of the dynamic pricing problem that is represented by the stocking factor transformation.

As a result of the stocking-factor transformation, $R_t(I_t)$, the maximum expected-revenue-to-go-function, can be written as a multiplicatively separable function of I_t, which we demonstrate by the following proposition.

Proposition 1. *Let $m = 1 - 1/b$ serve as a proxy for the elasticity of demand. Moreover, let $r_t^* = \max_z r_t(z)$ be defined as a constant that can be interpreted as an optimal revenue factor, where $r_0^* = 0$ and*

$$r_t(z) = \frac{z - E[(z-A_t)^+] + r_{t-1}^* E[((z-A_t)^+)^m]}{z^m}. \quad (5)$$

Then, $R_t(I_t) = r_t^* I_t^m$.

Proof. The proof follows by induction on t, given the induction hypothesis $R_t(I_t) = r_t^* I_t^m$. If $t = 1$, then, from (1), (4), the definition of z, and (5),

$$R_1(I_1) = \max_z \left\{ \left(\frac{I_1}{z}\right)^m (z - \mathrm{E}[(z-A_1)^+]) \right\}$$
$$= I_1^m \max_z r_1(z) = r_1^* I_1^m,$$

which establishes that the result is true for $t = 1$. Assume that the induction hypothesis is true for $t = i$, so that $R_i(I_i) = r_i^* I_i^m$, and consider the case $t = i + 1$. From (1) and the induction hypothesis,

$$\begin{aligned} &R_{i+1}(I_{i+1}) \\ &= \max_z \left\{ \left(\frac{I_{i+1}}{z}\right)^m (z - \mathrm{E}[(z-A_{i+1})^+]) \right. \\ &\qquad \left. + \mathrm{E}\left[R_i\left(I_{i+1}\frac{(z-A_{i+1})^+}{z}\right)\right]\right\} \\ &= I_{i+1}^m \cdot \max_z \left\{ \frac{z - E(z-A_{i+1})^+ + r_i^* \mathrm{E}[((z-A_{i+1})^+)^m]}{z^m} \right\} \\ &= r_{i+1}^* I_{i+1}^m. \end{aligned}$$

Thus, if the induction hypothesis is true for $t = i$, then it is true for $t = i + 1$. Because it is true for $t = 1$, it is therefore true for all t, which completes the proof. □

Thus, the computation of $R_t(I_t)$ requires only a maximization of $r_t(z)$, a function that does not depend upon I_t. Let $z_t^* = \arg\max_z r_t(z)$ denote the optimal stocking factor for period t. Clearly, this value depends on t, the number of periods remaining in the season, but is independent of I_t, the number of items available for sale in period t. Consequently, z_t^* can be computed for all t at the beginning of the finite horizon, without first having to observe I_t, by iteratively solving T "single-period" problems using the following algorithm: First set $r_0 = 0$. Then, for $t = 1, \ldots, T$, find $z_t^* = \arg\max_z r_t(z)$ and set $r_t^* = r_t(z_t^*)$, where $r_t(z)$ is given by (5). In the worst case, maximizing the function $r_t(z)$ requires an exhaustive search over z's domain. However, in §§4 and 5 we develop properties of $r_t(z)$ that give rise to more efficient search algorithms for computing z_t^*.

Given the state-independent sequence of optimal stocking factors z_t^*, the optimal price for period t can be recovered from the definition of z, once I_t is observed:

$$p_t^* = \left(\frac{z_t^*}{I_t}\right)^{1-m}. \quad (6)$$

Thus, the optimal price in period t is decreasing and convex as a function of I_t, the amount of inventory that remains at the time that the period-t price is chosen (because $0 < m < 1$). This intuitive result helps validate the model and is consistent with the literature. (See, for example, Federgruen and Heching 1997.) Moreover, from Proposition 1, $R_T(S) = r_T^* S^m$, which is increasing and concave in S. Therefore, (3) implies

$$S^* = \left(\frac{m r_T^*}{c}\right)^{1/(1-m)} = \left(\frac{m r_T^*}{c}\right)^b \quad (7)$$

and

$$r_T^* \cdot (S^*)^m - cS^* = \frac{1-m}{m} cS^* = \frac{1-m}{m} c \left(\frac{m r_T^*}{c}\right)^b, \quad (8)$$

which provide convenient closed-form expressions for the retailer's optimal stocking level and optimal expected profit for the season. Note that S^* is decreasing and convex as a function of c, and optimal profit is decreasing as function of c. These properties again provide intuitive validation for the model.

4. Properties of the Optimal Solution

In §3, we demonstrated how the dynamic pricing problem can be reduced to an iterative procedure involving the solution of T single-variable optimization problems by reformulating the problem as a dynamic stocking factor problem. The result was an independence between periods wherein the tth iteration of the problem can be constructed from the $(t-1)$st iteration of the problem by replacing a single constant term from the $(t-1)$st iteration with the computed solution of the $(t-1)$st iteration.

In this section, we establish and discuss the following properties of the retailer's optimal dynamic pricing problem, which are useful for developing insights as well as more efficient solution procedures:

- If the distribution of A_t has an *increasing generalized failure rate* (IGFR), then the optimal stocking factor for the final period of the selling season is characterized uniquely by an implicit function.
- The optimal stocking factor is increasing in the number of periods remaining in the selling season.
- If the random variable A_t is rescaled to nA_t, the resulting optimal stocking factor for period t will become rescaled by the same factor n.
- The optimal stocking level is at least as large as the optimal stocking level for the price-setting newsvendor problem, which serves as a benchmark case that is otherwise equivalent to the dynamic pricing problem except that the pricing decision is not a dynamic one, but instead is made once, at the beginning of the selling season.
- For the special case of deterministic demand, the optimal pricing policy is a single-price policy.

We now formally establish and discuss each of these properties.

Proposition 2. *Let $g(a)=af(a)/[1-F(a)]$ denote the generalized failure rate. If $dg(a)/da>0$, then z_1^*, the optimal stocking factor for the last period of the selling season, is the unique solution to the following equation*:

$$\frac{z[1-F(z)]}{z-\Lambda(z)}=m,$$

where $\Lambda(z)=\int_0^z F(a)\,da$.

Proof. By definition, $z_1^* = \arg\max_z r_1(z)$, where, from (5), $r_1(z) = (z-\Lambda(z))/z^m$. Thus, the first-order condition is

$$\frac{dr_1(z)}{dz}=\frac{z[1-F(z)]-m[z-\Lambda(z)]}{z^{m+1}}=0,$$

which is satisfied if and only if $z[1-F(z)]/[z-\Lambda(z)]=m$. Thus, if we let

$$L(z)=\frac{z[1-F(z)]}{z-\Lambda(z)},$$

then the proof is complete if we show that $L(z)=m$ has exactly one solution. To that end, consider the behavior of $L(z)$:

(i) $\lim_{z\to 0} L(z)=1>m$,

(ii) $\lim_{z\to\infty} L(z)=0<m$,

(iii)
$$\begin{aligned}\frac{dL(z)}{dz}&=\frac{[z-\Lambda(z)][1-F(z)-zf(z)]-z[1-F(z)]^2}{[z-\Lambda(z)]^2}\\&=\frac{L(z)}{z}[1-g(z)-L(z)],\end{aligned}$$

(iv)
$$\left.\frac{d^2L(z)}{dz^2}\right|_{dL(z)/dz=0}=\frac{-L(z)}{z}g'(z)<0.$$

From (iv), $dL(z)/dz$ can change sign at most once, from positive to negative. However, from (iii), $\lim_{z\to 0}(dL(z)/dz)\le 0$. Thus, (iii) and (iv) together imply that $L(z)$ is a decreasing function of z. Moreover, from (i) and (ii), $L(z)>m$ for some range of z, and $L(z)<m$ for some range of z. Therefore, (i)–(iv) together imply that $L(z)=m$ has exactly one solution, namely z_1^*. □

Lariviere (1999) describes the robustness of the IGFR condition: IGFR applies to many common classes of probability distributions including, but not limited to, the gamma, Weibull, and normal distributions. Thus, Proposition 2 is important because it ensures an efficient start to the iterative procedure used to compute the sequence of optimal stocking factors under quite general conditions. Proposition 3, which is presented next, provides a useful complement to Proposition 2 by establishing a monotone relationship between the optimal stocking factors.

Proposition 3. *$z_t^*>z_{t-1}^*$ for all t.*

Proof. The proof is by induction on t, given the following two induction hypotheses: (i) $r_{t+1}^*>r_t^*$, and (ii) $z_{t+1}^*>z_t^*$. To begin, note from (5) that

$$r_{t+1}(z)=r_t(z)+(r_t^*-r_{t-1}^*)\int_0^z\left(1-\frac{a}{z}\right)^m f(a)\,da. \quad (9)$$

Thus,

$$\begin{aligned}\frac{dr_{t+1}(z)}{dz}&=\frac{dr_t(z)}{dz}+m(r_t^*-r_{t-1}^*)\\&\quad\cdot\int_0^z\left(\frac{a}{z^2}\right)\left(\frac{z}{z-a}\right)^{1-m}f(a)\,da. \quad (10)\end{aligned}$$

If $t=1$, then, by definition, $r_0(z)=0$ for all z. Thus, from (9),

$$r_2^*\ge r_2(z_1^*)=r_1(z_1^*)+r_1^*\int_0^{z_1^*}\left(1-\frac{a}{z_1^*}\right)^m f(a)\,da>r_1(z_1^*)=r_1^*,$$

where $r_2^* \geq r_2(z_1^*)$ because $r_2^* \geq r_2(z)$ for all z, by the definition of r_2^*. This establishes that induction hypothesis (i) is true for $t=1$. Moreover, from (10), $dr_2(z)/dz > dr_1(z)/dz$ for all z; thus, induction hypothesis (ii) is also true for $t=1$. Therefore, assume that both induction hypotheses are true for $t=i$ and consider the case $t=i+1$.

From (9), the definition of optimality, and induction hypothesis (i), $r_{i+1}^* \geq r_{i+1}(z_i^*) > r_i(z_i^*) = r_i^*$. From (10) and induction hypothesis (i), $dr_{i+1}(z)/dz > dr_i(z)/dz$, which implies that $z_{i+1}^* > z_i^*$. Thus, induction hypotheses (i) and (ii) are true for $t=i+1$ if they are true for $t=i$. Because they are true for $t=1$, they are true for all t, which completes the proof. □

By establishing a monotone relationship between the optimal stocking factors, Proposition 3 provides a useful lower bound for the iteration procedure that serves as the general search algorithm for the optimal pricing policy. Accordingly, Proposition 3 can be applied together with Proposition 2 as follows: Beginning with the value of z_1^*, which, from Proposition 2, can be computed easily under rather general conditions, the search procedure for determining z_2^* need only consider values of $z > z_1^*$. Then, for $t=2,\ldots,T$, the procedure is repeated iteratively, each time using z_{t-1}^* as the lower bound for the search for z_t^*. Moreover, with each iteration, the search requires less effort because it is conducted over a progressively smaller region of z values.

From a managerial perspective, Proposition 3 and (6) establish that, for a fixed inventory level, the optimal price is increasing as a function of the amount of time remaining in the selling season, a characteristic that is intuitive and is consistent with results of related dynamic pricing models (e.g., Gallego and van Ryzin 1997). More generally, however, Proposition 3 establishes that the optimal stocking factor decreases as the number of periods remaining in the season decreases. Recall, in our setting, the stocking factor is a proxy for expected leftovers. Thus, Proposition 3 can be interpreted to mean that it is optimal to plan for less leftovers as the end of the selling season approaches. This is completely consistent with stochastic inventory theory, which indicates that if price is fixed at the beginning of a finite selling season (usually exogenously), but the stocking quantity can be adjusted at the beginning of each new period, the optimal stocking quantity (which, because price is fixed, corresponds to the optimal "expected leftover level") decreases as the number of periods remaining decreases. Intuitively, if a leftover occurs when there are many periods remaining in the finite horizon, there is a greater chance that the leftover will eventually be sold. However, the chance of eventually selling a given leftover diminishes over time. Consequently, as the number of remaining periods decreases, the overage cost associated with having a leftover increases, thereby resulting in a lower optimal expected leftover level. This in turn translates into a lower optimal stocking *quantity* in a dynamic inventory problem and into a lower optimal stocking *factor* in a dynamic pricing problem. This parallel with inventory theory is another appealing feature that helps validate the model.

It is interesting to note that if the A_t were not identically distributed, then Proposition 3 is not necessarily true. The following example illustrates this point.

Example. Suppose $T=2$ and $m=0.5$. Let $A_1 \sim \text{uniform}(0,100)$ and $A_2 \sim \text{uniform}(0,10)$. Then,

$$\frac{z[1-F_1(z)]}{z-\Lambda_1(z)} = \frac{z(1-z/100)}{z-z^2/200} = \frac{200-2z}{200-z}.$$

Thus, from Proposition 2, $z_1^* = 200(1-m)/(2-m) = 66.667$. Correspondingly, from (5), $r_1^* = 5.443$ and

$$r_2(z) = \begin{cases} \sqrt{z}\left(1-\dfrac{z}{20}\right) + \dfrac{zr_1^*}{15} & \text{if } z < 10, \\[2ex] \dfrac{5}{\sqrt{z}} + \dfrac{zr_1^*}{15}\left[1-\left(\dfrac{z-10}{z}\right)^{1.5}\right] & \text{if } z \geq 10, \end{cases}$$

which is maximized at $z_2^* = 36.432$. Thus, for this example, $z_2^* < z_1^*$.

This example demonstrates that if the increase in demand (over time) is large enough, a future optimal stocking factor (e.g., z_1^*) can be greater than a nearer-term stocking factor (e.g., z_2^*). Thus, if A_{t-1} is large relative to A_t, it may be that the optimal price in period $t-1$ is higher than the optimal price in period t, even if the available inventory levels are the same for both periods. Although this is not necessarily intuitive, it is again consistent with inventory theory: given a fixed price, if the distribution of demand is not stationary, then the optimal stocking quantities need not necessarily decrease as the number of periods remaining in the finite horizon decreases. Basically, if the standard deviation of demand for a future period is greater than the standard deviation of demand for a nearer-term period (as is the case in the above illustration), then the resulting upward pressure on safety stock could outweigh the downward pressure created by the reduced overage cost (associated with the future period).

Proposition 4 provides additional insight.

Proposition 4. *Suppose $D_t(p)=\tilde{A}_t p^{-b}$, where $\tilde{A}_t=nA_t$ and define $\tilde{z}_t^*$ as the corresponding optimal stocking factor for period t. Then, $\tilde{z}_t^*=nz_t^*$ for all t.*

Proof. First, note that by definition, $\tilde{z}_t^*=\arg\max_z \tilde{r}_t(z)$, where, analogous to (5),

$$\begin{aligned}\tilde{r}_t(z) &= \frac{z-\mathrm{E}[(z-\tilde{A}_t)^+]+\tilde{r}_{t-1}^*\mathrm{E}[((z-\tilde{A}_t)^+)^m]}{z^m}\\ &= \frac{1}{z^m}\left[z-\int_0^{z/n}(z-na)f(a)\,da\right]\\ &\quad +\tilde{r}_{t-1}^*\int_0^{z/n}\left(1-\frac{na}{z}\right)^m f(a)\,da, \qquad (11)\end{aligned}$$

and $\tilde{r}_0^*=0$. Then, the proof follows by induction on t, given the induction hypothesis $\tilde{r}_t^*=n^{1-m}r_t^*$.

If $t=1$, then, from (11) and (5),

$$\begin{aligned}\tilde{r}_1(z) &= \frac{n}{n^m(z/n)^m}\left[\frac{z}{n}-\int_0^{z/n}\left(\frac{z}{n}-a\right)f(a)\,da\right]\\ &= n^{1-m}r_1\left(\frac{z}{n}\right),\end{aligned}$$

which implies that $\tilde{z}_1^*=nz_1^*$. Hence, $\tilde{r}_1^*=n^{1-m}r_1^*$ and the induction hypothesis is true for $t=1$. Therefore, assume that the induction hypothesis is true for $t=i$, so that $\tilde{r}_i^*=n^{1-m}r_i^*$. When $t=i+1$, (11) and (5) imply

$$\begin{aligned}\tilde{r}_{i+1}(z) &= \frac{n}{n^m(z/n)^m}\left[\frac{z}{n}-\int_0^{z/n}\left(\frac{z}{n}-a\right)f(a)\,da\right]\\ &\quad +\tilde{r}_i^*\int_0^{z/n}\left(1-\frac{a}{z/n}\right)^m f(a)\,da=n^{1-m}r_{i+1}\left(\frac{z}{n}\right).\end{aligned}$$

Thus, $\tilde{z}_{i+1}^*=nz_{i+1}^*$ and $\tilde{r}_{i+1}^*=n^{1-m}r_{i+1}^*$, which completes the proof. □

One implication of Proposition 4 is that the optimal pricing policy is independent of the scale of the problem. To demonstrate this, suppose that for a given problem scenario, the parameter A_t is replaced with $\tilde{A}_t=nA_t$. Then, the optimal stocking factors change from z_t^* to $\tilde{z}_t^*=nz_t^*$ for all t. Moreover, the optimal stocking level changes from S^* to $\tilde{S}^*=nS^*$ (because, from (7), $\tilde{S}^*=(m\tilde{r}_T^*/c)^{1/(1-m)}$, and, from the proof of Proposition 4, $\tilde{r}_T^*=n^{1-m}r_T^*$). Thus, a change in problem parameters from A_t to $\tilde{A}_t$ leaves the optimal prices unchanged in expectation: from (6),

$$\begin{aligned}\tilde{p}_T^* &= \left(\frac{nz_T^*}{nS^*}\right)^{(1-m)}=p_T^*,\\ \mathrm{E}[\tilde{p}_{T-1}^*] &= \mathrm{E}\left[\left(\frac{nz_{T-1}^*}{(nS^*-nA_T\tilde{p}_T^{*-b})^+}\right)^{1-m}\right]\\ &= \mathrm{E}[p_{T-1}^*],\ \text{etc.}\end{aligned}$$

This is in direct contrast to related results derived from dynamic pricing models in which demand is formulated as a Poisson process. In those models, if the demand rate and the initial inventory level both are scaled by a common factor, the optimal pricing policy changes as a result. This is because, in Poisson models, "scale" is not an independent parameter: an increase in the scale of demand results in a corresponding decrease in the demand coefficient of variation, which, in turn, affects the optimal price trajectory.

On a technical note, Proposition 4 is useful because it provides the flexibility to model any $\tilde{A}_t$ of finite support $[0,w]$, simply by defining $\tilde{A}_t=wA_t$, where A_t is a random variable defined over $[0,1]$. We comment further on this observation, and the practical convenience it ensures, in §5.

The next proposition, together with its corollary, offer insight on how recourse affects the optimal stocking level, which is a one-time decision made at the beginning of the selling season, and the optimal expected profit. The insights come from comparing S^* against the optimal S for a price-setting newsvendor, which serves as the benchmark stocking level.

Proposition 5. *The optimal stocking level, S^*, is at least as large as the benchmark quantity S_B, which is defined as the optimal stocking level for an otherwise equivalent decision scenario in which price is set only at the beginning of the selling season and then held fixed for the duration of the season.*

Proof. First, define $v_t(z)$ as follows:

$$v_t(z)=\frac{1}{z^m}\left(z-\mathrm{E}\left[\left(z-\sum_{j=1}^{t}A_j\right)^+\right]\right). \qquad (12)$$

Then, consider the following two lemmas, which we will establish in turn:

Lemma 1. $S_B=(mv_T^*/c)^{1/(1-m)}$, *where* $v_T^*=\max_z v_T(z)$.

Lemma 2. $r_T^*\geq v_T^*$.

Note that if Lemmas 1 and 2 are true, then, from (7),

$$S^*=\left(\frac{mr_T^*}{c}\right)^{1/(1-m)}\geq\left(\frac{mv_T^*}{c}\right)^{1/(1-m)}=S_B.$$

Thus, to complete the proof of Proposition 5, it suffices to show that Lemmas 1 and 2 are true. □

Proof of Lemma 1. If price is set only once and then held constant for the duration of the selling season, then, given S, the total revenue function for the season is

$$\begin{aligned}V_T(p\,|\,S)&=p(S-\mathrm{E}[\text{leftovers at the end of the season}])\\&=p\left(S-\mathrm{E}\left[\left(S-p^{-b}\sum_{t=1}^{T}A_t\right)^+\right]\right).\end{aligned}$$

Correspondingly, the total profit function for the season is $V_T(p\,|\,S)-cS$. Defining $k=S/p^{-b}$ as the stocking factor for the price-setting newsvendor, and substituting it into the expression for $V_T(p\,|\,S)$, yields

$$\begin{aligned}V_T(p\,|\,S)&=V_T\left(\left(\frac{k}{S}\right)^{1-m}\middle|\,S\right)\\&=\left(\frac{k}{S}\right)^{1-m}\left(S-\mathrm{E}\left[\left(S-\frac{S}{k}\sum_{t=1}^{T}A_t\right)^+\right]\right)\\&=S^m v_T(k),\end{aligned}$$

where $v_T(k)$ is given by (12). Thus, S_B is the value of S that maximizes the concave function $v_T^*S^m-cS$, where $v_T^*=\arg\max_k v_T(k)$. That is, $S_B=(mv_T^*/c)^{1/(1-m)}$, thereby establishing Lemma 1. □

Proof of Lemma 2. This proof is by induction on t, given the induction hypothesis $r_t(z)\geq v_t(z)$ for all z. If $t=1$, then, from (5) and (12), $r_1(z)=v_1(z)$. Therefore, the induction hypothesis is true for $t=1$. Assume, then, that the hypothesis is true for $t=i$, so that $r_i(z)\geq v_i(z)$, and consider the case $t=i+1$. From (5),

$$r_{i+1}(z)=\frac{z-\mathrm{E}[(z-A_{i+1})^+]+\mathrm{E}[r_i^*\cdot(z-A_{i+1})^{+m}]}{z^m}.\tag{13}$$

However, by the definition of optimality, $r_i^*\geq r_i(z)$ for all z. Thus, $r_i^*\geq r_i(z-A_{i+1})$ for all realizations of $A_{i+1}\leq z$. Consequently, for all $A_{i+1}\leq z$, $r_i^*\geq r_i(z-A_{i+1})\geq v_i(z-A_{i+1})$ by the induction hypothesis. Applying this inequality and (12) to (13) yields

$$\begin{aligned}r_{i+1}(z)&\geq\frac{z-\mathrm{E}[(z-A_{i+1})^+]+\mathrm{E}[v_i(z-A_{i+1})\cdot(z-A_{i+1})^{+m}]}{z^m}\\&=\frac{z-\mathrm{E}[(z-\sum_{j=1}^{i+1}A_j)^+]}{z^m}=v_{i+1}(z),\end{aligned}$$

which implies that the induction hypothesis is true for all t. Let $k_B=\arg\max_k v_T(k)$ so that $v_T^*=v_T(k_B)$. Then, $r_T^*\geq r_T(k_B)\geq v_T(k_B)=v_T^*$, where the first inequality follows by the definition of optimality and the second inequality follows because the induction hypothesis is true for all t. □

Intuitively, from an inventory-theory perspective, the flexibility to change selling price each period decreases the price-setting newsvendor's cost of having leftovers in the following sense. If, coming into a new period, the actual stocking level exceeds the anticipated stocking level, a price change can stimulate demand, thereby establishing, in effect, a salvage market for the excess units. Thus, recourse in the form of flexibility to adjust prices allows the newsvendor to salvage leftovers that otherwise would not be salvaged. As a result, this recourse flexibility implicitly reduces the cost of having leftovers, which results in an increased optimal stocking level. Interestingly, similar intuition cannot be extended directly to the newsvendor's initial pricing decision: the relationship between p_T^* and p_B (which is defined analogously to S_B) is an ambiguous one. We investigate this ambiguity as part of our numerical study in §6.

With the following corollary, we demonstrate that the relative value of the newsvendor's recourse flexibility is equivalent to the corresponding relative increase in stocking level, which itself can be expressed in succinct fashion.

Corollary 1. *The relative expected value of being able to adjust prices dynamically over the course of a single selling season, which is defined as the ratio of the expected optimal profit of the dynamic pricing problem to the expected optimal profit of the price-setting newsvendor problem, can be expressed as follows*:

$$\begin{aligned}\mathrm{E}[\textit{value of recourse flexibility}]&\equiv\frac{r_T^*\cdot(S^*)^m-cS^*}{v_T^*S_B^m-cS_B}\\&=\left(\frac{r_T^*}{v_T^*}\right)^b,\end{aligned}\tag{14}$$

where r_T^* is the maximum of (5), *v_T^* is the maximum of* (12), *and b is the price elasticity of demand.*

Proof. From (7), $S^* = (mr_T^*/c)^{1/(1-m)}$; and from Lemma 1, $S_B = (mv_T^*/c)^{1/(1-m)}$. Thus, the corollary follows from elementary algebra. □

Note that the expected value of recourse flexibility can be measured independent of c. This is a convenience that traces to (8), which indicates that the optimal expected profit for the selling season is log linear in c. We further explore the expected value of recourse in §6.

We conclude this section by establishing the optimality of a single-price policy for the special case where demand each period is deterministic. We use this result in §6 to examine the loss of performance resulting from the employment of a commonly used heuristic that replaces random variables by their means to facilitate the computation of a pricing policy.

Proposition 6. *For a given initial stocking level S, the pricing policy that is optimal for the deterministic case in which $d_t(p) \equiv \mathrm{E}[A_t]p^{-b}$ is a single-price policy: $p_t^* = p_d(S)$ for all t, where*

$$p_d(S) \equiv \left(\frac{\sum_{t=1}^T \mathrm{E}[A_t]}{S}\right)^{1-m}.$$

Proof. The deterministic problem can be written as the static optimization problem

$$\max \sum_{t=1}^T \mathrm{E}[A_t] p_t^{-b} p_t,$$

$$\text{subject to } \sum_{t=1}^T \mathrm{E}[A_t] p_t^{-b} \le S.$$

The Lagrangean function is $L = \sum_{t=1}^T \mathrm{E}[A_t]p_t^{-b}p_t + \lambda(S - \sum_{t=1}^T \mathrm{E}[A_t]p_t^{-b})$. The optimal price vector p_t must satisfy the Kuhn-Tucker conditions for some λ:

$$\mathrm{E}[A_t](p_t - \lambda)(-b)p_t^{-b-1} + \mathrm{E}[A_t]p_t^{-b} = 0. \quad (15)$$

That is, $p_t = \lambda/(1-1/b) = \lambda/m$. Note that p_t is independent of t so that the same price is optimal in each period.

The decision problem becomes

$$\max \sum_{t=1}^T \mathrm{E}[A_t] p^{-b+1},$$

$$\text{subject to } \sum_{t=1}^T \mathrm{E}[A_t] p^{-b} \le S.$$

Because the objective function is decreasing in price, the optimal price should be set such that all units are sold:

$$p_d(S) = \left(\frac{\sum_{t=1}^T \mathrm{E}[A_t]}{S}\right)^{1/b}. \quad \square$$

In other words, if the demand is isoelastic and deterministic, there is no need for price adjustments even if such opportunities exist.

5. Additional Properties for the Finite-Support Case

Note from (5) that for $t > 1$, $r_t(z)$ is a single-variable function having a "static" form (because for a given $t > 1$, r_{t-1}^* is a constant that is strictly greater than zero). As a result, the procedure for solving the T optimization problems required to yield the complete set of $\{z_t^*\}$ is no more difficult, computationally, than solving for z_2^*. Thus, the degree of difficulty associated with solving an instance of the dynamic pricing problem boils down to determining the shape of $r_t(z)$ for a given value of r_{t-1}^*. In implementing our algorithm for the numerical study presented in §6, we observed that $r_t(z)$ typically is quasi-concave for a variety of continuous distributions used to characterize A_t. Unfortunately, however, we have not been able to establish a proof to that effect; thus, as a worst-case scenario, determining z_t^* requires an exhaustive search over the feasible domain for z. Fortunately, however, Propositions 2 and 3 limit the magnitude of the searches by establishing a lower bound for z_t^*.

In this section, we further pare the space over which an exhaustive search is required for the special case in which A_t has finite support. (A_t is considered to have finite support if $A_t \in [0, w]$ for $w < \infty$.) In particular, we establish two key results for this case. The first result effectively establishes that an exhaustive search is required only over the domain $[z_{t-1}^*, 1]$; if this domain either is empty or does not yield z_t^*, then z_t^* can be determined uniquely by the first-order condition $dr_t(z)/dz = 0$. The second result establishes that no exhaustive search is required if A_t can be characterized by the power distribution, $F(a) = a^k$ for $a \in [0, 1]$ and $k > 0$. (Note that the power distribution is a subset of the Beta distribution; see, for example, Bagnoli and Bergstrom 1989 for an introduction to the power distribution.) The proofs for both results are provided in the Appendix.

Proposition 7. *If A_t has finite support so that $A_t \in [0, w]$, then $r_t(z)$ is quasi-concave for $z \geq w$.*

Proposition 7 is important because, used in conjunction with Propositions 3 and 4, it significantly reduces the domain over which an exhaustive search is necessary. To demonstrate, suppose that the random component of demand is given as $\tilde{A}_t \in [0, w]$. Then, because of the scaling property established as Proposition 4, the problem first can be rescaled so that $\tilde{A}_t = wA_t$, where $A_t \in [0, 1]$. Correspondingly, from Proposition 7, $r_t(z)$ is quasi-concave for $z \geq 1$. Moreover, from Proposition 3, $z^*_{t+1} \geq z^*_t$. Thus, for a given value of z^*_t, if $z^*_t \geq 1$, then z^*_{t+1} can be found simply as the unique solution to $dr_{t+1}(z)/dz = 0$. In other words, once it is determined that $z^*_t \geq 1$ for some t, then for all $j > 0$, z^*_{t+j} can be computed directly and efficiently from its first-order condition. (Our numerical investigation suggests that, typically, $z^*_t \geq 1$ even for relatively low values of t.) If $z^*_t < 1$, then an exhaustive search for z^*_{t+1} is required, but only over the region $[z^*_t, 1]$; the best candidate from this region then can be compared with the best candidate from the region $z \geq 1$, which can be determined efficiently because $r_{t+1}(z)$ is guaranteed to be well behaved for $z \geq 1$. Finally, once z^*_t is determined, $\tilde{z}^*_t$, the optimal period-t stocking factor for the original problem (in which $\tilde{A}_t \in [0, w]$ is given) can be recovered, from Proposition 4, as $\tilde{z}^*_t = wz^*_t$.

We conclude this section with Proposition 8, which establishes that, for the special class of Beta distributions in which one of the two parameters is identically equal to 1, $r_t(z)$ is quasi-concave over all z. Thus, z^*_t can be found directly from its first-order condition for all t.

Proposition 8. *If A_t has a power distribution so that $F(a) = a^k$ for $a \in [0, 1]$ and $k > 0$, then $r_t(z)$ is quasi-concave for all z.*

6. Numerical Results

In this section, we develop additional insights to complement those established in earlier sections by implementing our solution procedure for a wide variety of probability distributions and problem parameters. Specifically, we report the findings of three experiments designed to develop some intuition with regard to the following questions:

(1) What is the "cost" of using well-established heuristics for solving the dynamic pricing problem in lieu of computing the optimal solution?

(2) How does the recourse flexibility that is inherent to the dynamic pricing problem affect the optimal price chosen to start the selling season (when compared to the price-setting newsvendor benchmark case); and what is the magnitude of the value of that recourse?

(3) How does the optimal price sequence behave over time, in expectation?

We also explore how the answers to these questions are affected by changes in demand elasticity, coefficient of variation, and season length.

In the results presented here, the Gamma distribution $\Gamma(\alpha, \beta)$ is used for the distribution of A_t. The Gamma distribution is a two-parameter distribution, whose mean is $\alpha\beta$ and variance is $\alpha\beta^2$.

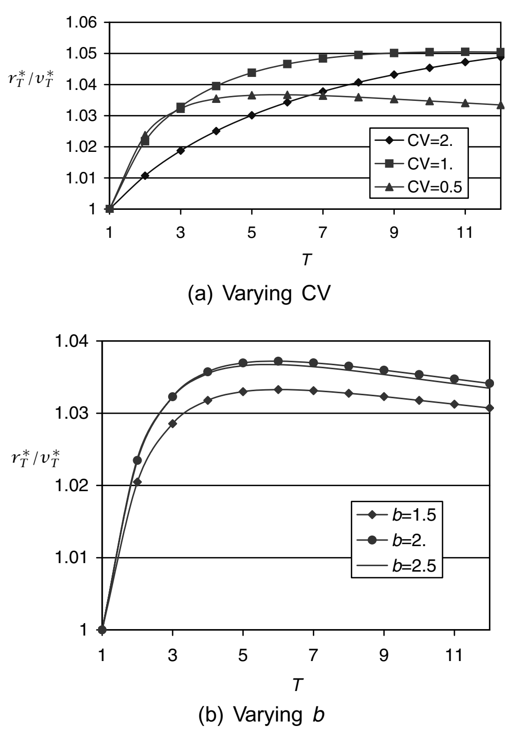
6.1. Loss of Performance Due to Static Pricing

As indicated in the introduction, optimal dynamic pricing policies typically are difficult to compute. As a consequence, heuristics often are developed and applied. In this section, we use our model to investigate how financial performance might suffer if one such heuristic is used in lieu of optimally solving the dynamic problem, for a given (fixed) amount of inventory.

The *deterministic heuristic*, in which random variables are replaced by their means, is commonly employed to compute approximately optimal prices in dynamic pricing problems. From Proposition 6, we know that the deterministic heuristic yields a single-price policy. Accordingly, we suggest that a better price to use is $p_B(S)$, the price set by an optimizing price-setting newsvendor whose initial stock is S. From Proposition 5, $p_B(S) = (k_B/S)^{1/b}$, where k_B is the price-setting newsvendor's optimal safety factor. (See the proof of Proposition 5 for details.) Because this price is optimal in the original stochastic setting of the problem with the added stipulation that price can be set only once, it dominates all other single-price policies, including the deterministic heuristic. Moreover, it is effectively as easy to implement as the deterministic heuristic. Thus, to better understand the performance loss resulting from applying a heuristic to the dynamic pricing problem, we compare $v^*_T S^m$, the expected total revenue associated with trying to sell a fixed supply S over a finite horizon T using the single-price policy $p_B(S)$, to $r_T^* S^m$, the expected total revenue associated with trying to sell the same S units over the same time T using the optimal pricing policy p_t^*. That is, we compute the ratio r_T^*/v_T^*, which is independent of S.

Figure 1(a) shows graphs of r_T^*/v_T^* as a function of T when $\alpha\beta = 10$ and the per-period coefficient of variation ($\mathrm{CV} = \alpha^{-1/2}$) takes on values $\mathrm{CV} = 2.0$, 1.5, and 0.5. We find that the percentage loss resulting from using the single-price policy $p_B(S)$ in the dynamic pricing problem is as high as 5%. Note that the performance loss is not monotone in CV: when the season is short ($T \le 3$), performance losses are greater for small values of CV ($\mathrm{CV} = 0.5$) than they are for larger values of CV. On the other hand, when the season is longer ($T \ge 7$), performance losses are smallest for $\mathrm{CV} = 0.5$ and largest for $\mathrm{CV} = 1.0$. Intuitively, if we compared the optimal algorithm to the deterministic heuristic (which does not account for demand variance), then we might expect to see the performance loss associated with the heuristic increase with CV. However, in Figure 1 we are using the price-setting newsvendor policy as our heuristic for comparison. Because the price-setting newsvendor actually accounts for demand variance, the relationship between the performance loss of the heuristic and the coefficient of variation of demand is ambiguous.

Figure 1 Revenue Performance Loss Comparisons



(a) Varying CV

(b) Varying b

Figure 1(b) shows graphs of r_T^*/v_T^* as a function of T for three values of b ($b = 1.5$, 2.0, and 2.5), the elasticity parameter. We see that performance losses first increase and then slowly begin to decline as the length of the selling season increases. Although this observation is not particularly intuitive, it can be explained by noting, first, that *both* r_T^* and v_T^* increase in T, and second, that Figure 1(b) plots the ratio of r_T^* to v_T^*. Therefore, even if the magnitude of the increase in r_T^* exceeds the increase in v_T^*, it is possible that the new ratio could decline.

6.2. Value and Effect of Recourse

Because it includes only one stocking opportunity, the dynamic pricing problem studied in this paper can be thought of as a price-setting newsvendor problem *with recourse*. The recourse comes in the form of flexibility to adjust prices dynamically as demand is observed. Accordingly, it is interesting to explore how the amount of recourse (i.e., the number of price changes associated with a given season's worth of demand) affects a price-setting newsvendor's optimal policy and corresponding optimal profit.

Recall that S_B, p_B, k_B, and v_T^* denote the price-setting newsvendor's optimal stocking quantity, optimal selling price, optimal stocking factor, and optimal revenue factor, respectively; and that these quantities serve as the benchmarks for comparison. Recall also that Proposition 5 and its corollary already establish that $S^* \ge S_B$, and that the relative value of recourse can be measured simply as that ratio of S^* to S_B. Here, we further investigate the ratio $S^*/S_B = (r_T^*/v_T^*)^b$, to develop keener insight into the relative value that pricing recourse provides to the price-setting newsvendor as well as how that value depends on the problem primitives. In addition, we explore the ratio

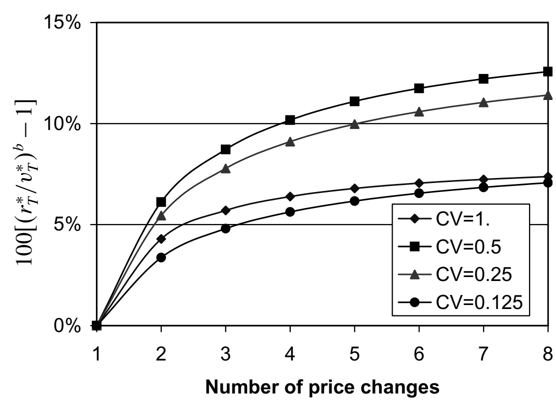
$$\frac{p_T^*}{p_B} = \left(\frac{z_T^* S_B}{k_B S^*}\right)^{1-m} = \frac{v_T^*}{r_T^*}\left(\frac{z_T^*}{k_B}\right)^{1-m}, \tag{16}$$

to develop an understanding of how a newsvendor's optimal pricing strategy is affected by the amount of recourse available.

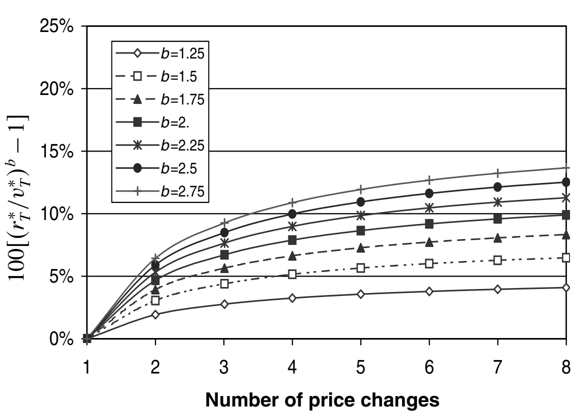
In this context, T no longer represents the length of the selling season. Instead, T denotes the number of price adjustments, equally spaced in time, that a newsvendor makes during the course of a given season. Accordingly, in Figures 2 and 3, we assume that the uncertainty associated with seasonal demand, $\sum_{t=1}^{T} A_t$, has a Gamma distribution $\Gamma(\alpha, \beta)$, so that $A_t \sim \Gamma(\alpha/T, \beta)$, and we let $T=1,2,\ldots,8$. Figure 2 shows that $100((r_T^*/v_T^*)^b - 1)$, the percentage gain in *profit* resulting from pricing recourse (see (14)), is increasing in T for all values of $\mathrm{CV}=\alpha^{-1/2}$ and b. Therefore, a newsvendor's optimal profit increases as the number of price adjustments over the course of a season increases. However, the marginal return of each additional price change diminishes. Thus, when T is large, the marginal benefit of making an additional price change may not make up for the cost of doing so.

Figure 3 shows that $100(p_T^*/p_B - 1)$, the percentage change in the (initial) optimal price resulting from pricing recourse, seems to depend primarily on the uncertainty in demand. Interestingly, if the demand coefficient of variation is relatively low, then the newsvendor with recourse tends to start the season with a lower price than a newsvendor without recourse (i.e., $100(p_T^*/p_B - 1) < 0$). Otherwise, the newsvendor with recourse tends to start the season with a higher price than the newsvendor without recourse. To help develop one possible interpretation of this observation, note that a higher level of risk may be associated with a high demand coefficient of variation which increases the challenge of coordi-

Figure 2 **Expected Value of Recourse Flexibility (as a Percentage Gain) with a Fixed Distribution of Seasonal Demand**

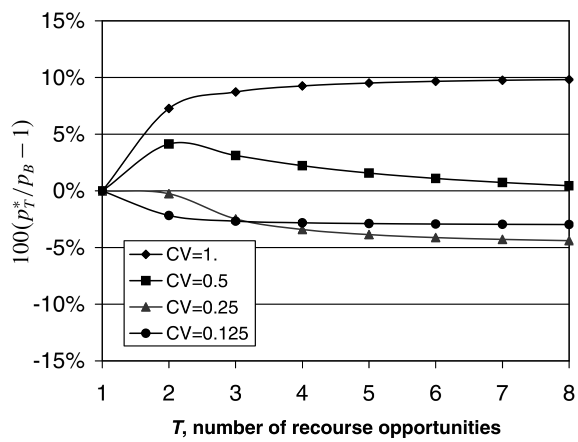


(a) Varying CV

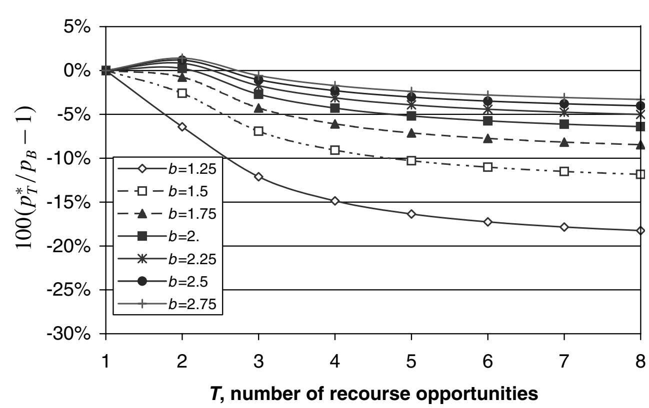


(b) Varying b

Figure 3 **The Impact of Recourse on the Newsvendor's Choice of Price (as a Percentage Change)**



(a) Varying CV



(b) Varying b

nating supply and demand. Because recourse represents one way to mitigate such risk, it is reasonable to expect that a newsvendor with recourse might be more aggressive than a newsvendor without recourse when it comes to setting policy. In other words, for higher relative demand CV's, a newsvendor with recourse is in a better position to risk the possibility of having higher than expected leftovers, and thus is in a better position to set price higher (i.e., set expected demand lower) than a newsvendor who does not have the luxury of recourse.

6.3. Optimal Price Sequence

From (6), the *actual* period-t optimal price depends on I_t, the observed level of inventory available at the beginning of the period (which depends on the period-$(t+1)$ realization of demand). However, the amount of inventory available at the beginning of period t is equivalent to the number of leftovers from period $t+1$, which is a function of the period-$(t+1)$ optimal price:

$$I_t = \frac{I_{t+1}}{z^*_{t+1}}(z^*_{t+1} - A_{t+1})^+ = \frac{(z^*_{t+1} - A_{t+1})^+}{(p^*_{t+1})^{1/(1-m)}}.$$

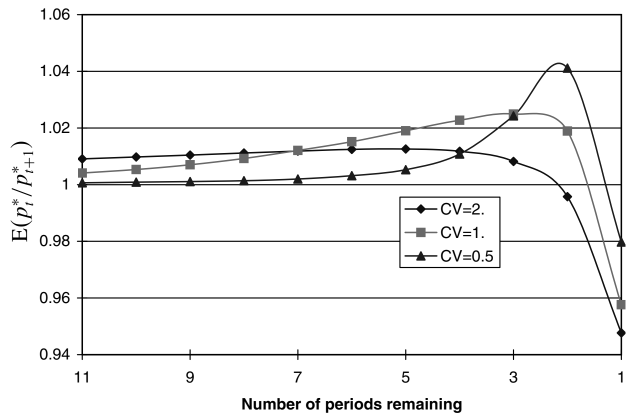
Therefore, in expectation, the ratio of prices from one period to the next throughout the selling season is a straightforward comparison that is independent of the stocking level:

$$\begin{aligned} \mathrm{E}\left[\frac{p^*_t}{p^*_{t+1}}\right] &= \mathrm{E}\left[\frac{(z^*_t/I_t)^{1-m}}{p^*_{t+1}}\right] = \mathrm{E}\left[\left(\frac{z^*_t}{(p^*_{t+1})^{1/(1-m)} I_t}\right)^{1-m}\right] \\ &= \mathrm{E}\left[\left(\frac{z^*_t}{z^*_{t+1} - A_{t+1}}\right)^{1-m}\right] \end{aligned} \tag{17}$$

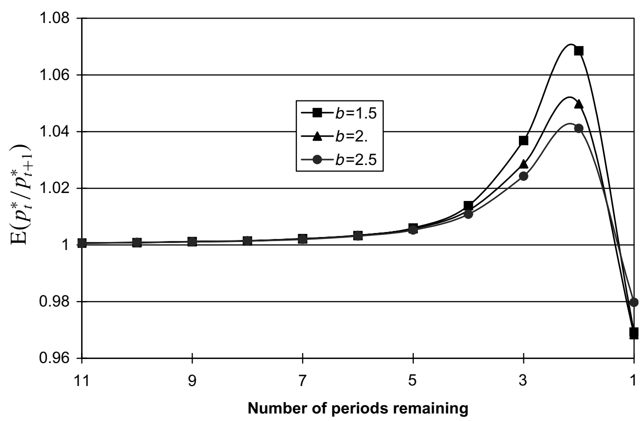
for all t. Hence, (17) provides a convenient expression for testing the behavior of optimal prices over time for a fixed season length. When (17) yields a ratio that is greater than 1, the indication is an expected price increase from period $t+1$ to t; and when (17) yields a ratio that is less than 1, the indication is an expected price decrease from period $t+1$ to t.

Figure 4 shows typical results. In this representative case, we again use the Gamma distribution $\Gamma(\alpha, \beta)$ as the distribution of A_t; and we consider a 12-period horizon. For various values of $\mathrm{CV} = \alpha^{-1/2}$ and b, we find that p^*_t typically increases during earlier periods of the season (i.e., when there are many periods remaining), and tends to decrease sharply as the finite horizon draws to an end. This likely is indicative of end-of-season salvage pricing.

Figure 4 $\mathrm{E}[p^*_t/p^*_{t+1}]$



(a) Varying CV



(b) Varying b

7. Conclusion

We have considered the problem of determining the optimal prices to set in each period of a finite selling season when the stocking level can be chosen only at the beginning of the season and demand is modeled as an isoelastic function of price with a multiplicative uncertainty term characterized by any given probability distribution. In general, the period-t optimal price depends on the inventory state of the system. However, we show that this dynamic pricing problem can be reformulated as a dynamic stocking-factor problem that is independent of the inventory state of the system. As a result, the problem can be solved with relative ease using an iterative routine in which a stand-alone revenue function is maximized for T separate specifications that differ only by the magnitude of a single coefficient term. In effect, the solution procedure requires solving T single-period problems, each involving a time-dependent "revenue parameter" that is exogenous to the particular iteration of the solution routine.

We further show that the first iteration of the solution routine (which corresponds to the trivial case in which the "revenue parameter" is identically equal to zero) is quasi-concave under the robust condition of an increasing generalized failure rate for the distribution of the A_t. Thus, the final-period optimal stocking factor can be found directly from its first-order condition under quite general circumstances. For cases in which the uncertainty in demand is bounded by finite support w (so that $A_t \in [0, w]$), we establish sufficient conditions indicating when the optimal stocking factor for any iteration can be determined directly from its first-order condition, namely for situations in which it is known that the optimal stocking factor is no less than w or in which A_t can be characterized by a power distribution. We also demonstrate that the optimal solution is independent of the scale of the problem; hence, large problems may be rescaled to a more convenient size before implementing the solution routine.

For the case in which the A_t are identically distributed, we show that the optimal stocking factors are monotone increasing as a function of the number of periods remaining in the finite horizon. This result thus captures the essence of the relationship between the dynamic pricing problem and dynamic inventory theory, thereby inspiring an interpretation of the dynamic pricing problem as a price-setting newsvendor problem with recourse. This resulting interpretation is useful not only because it yields insights into the optimal solution, but also because it leads to additional insights into how pricing recourse affects the actions and profits of a price-setting newsvendor. Consequently, we find that a price-setting newsvendor will choose a higher stocking level when recourse is available. However, the effect that the recourse flexibility has on the newsvendor's corresponding selling price depends on the level of uncertainty inherent in the problem: a relatively lower demand coefficient of variation tends to drive the newsvendor with recourse to choose a lower initial price than the newsvendor without recourse, but relatively higher demand coefficients of variation tend to drive the newsvendor with recourse to choose a higher initial price. Finally, the relative value of the recourse flexibility can be expressed simply as the ratio of the optimal stocking level for the price-setting newsvendor with recourse to the optimal stocking level for the price-setting newsvendor without recourse.

Although we have assumed that the A_t are identically distributed, this assumption is not essential in the analysis. If, instead, the A_t were nonstationary, then all of the propositions except for Proposition 3 would continue to hold. Thus, the general solution procedure and the insights developed in this paper would still apply. The only effect of having nonstationary demand, therefore, is a technical one: if demand is not stationary, then the monotone relationship between the optimal stocking factors is not guaranteed. Hence, in the worst case, an exhaustive search would be necessary at each iteration of the solution procedure. Note, however, if the demand process is not stationary, then one possible approach would be to redefine the "periods" of the finite horizon so that expected demand in each period is roughly the same. This approach leads to periods that are of unequal length (in terms of time), but it does not affect our solution technique.

In contrast, the assumption of a stationary price elasticity is central to the analysis of this paper. If, instead, b were time dependent, then Proposition 1 would hold only for $t=1$. This would complicate the analysis because it means that the optimal stocking factors for periods $t \geq 2$ would depend on the inventory state of the system; hence, it would significantly increase the complexity of the solution procedure. We expect that many of the qualitative results of this paper would continue to hold, but we leave this as a possible extension.

A more interesting extension, perhaps, is the development of a model in which the stocking level can be replenished at representative intervals, though not as frequently as price can be adjusted. Extrapolating the description of our model, this situation would be characterized as one in which there were multiple selling seasons (say, N), each composed of T periods (i.e., opportunities) for setting the price. Solving this extension seems to require more than a trivial adaptation of our solution procedure because the resulting optimal safety factor decisions appear, in general, not to be independent of the inventory state of the system as they are when there is only a single stocking opportunity.

To provide insight into this phenomenon, consider a two-season model. At the end of the first season, the decision maker is faced with a single-season, dynamic pricing problem like the one studied here, except that some number of units, say, x, will be available before the stocking decision is settled upon. As to be expected, the resulting implication is for the decision maker to replenish stock only if x is small; if x is relatively large, then it behooves the decision maker not to replenish and to operate in the second season with x units. As a result, the T-period optimal value function associated with the second of the two selling seasons turns out to be concave in x. This function, in turn, effectively represents a terminal-value salvage function for leftovers associated with the last period of the first selling season. Consequently, the decision problem at the beginning of the first selling season is similar to the single-season dynamic pricing problem studied here except that there is a concave end-of-season salvage function associated with leftovers. Unfortunately, this embellishment to the dynamic pricing problem then appears to lead to optimal stocking factors for the first selling season that depend on the inventory state of the system.

Given that the dynamic pricing problem studied in this paper can be reinterpreted as a price-setting newsvendor problem with (pricing) recourse, the multiple-season extension introduced in the previous paragraphs can thus be thought of as a dynamic pricing and inventory model with (pricing) recourse. From a theoretical point of view, this extension is potentially appealing because it stands to continue the interpretive analogy between dynamic pricing theory and dynamic inventory theory. Indeed, we conjecture that the reflective symmetries between the two theories will continue to exist. Consequently, just as the newsvendor model provides the fundamental building block for dynamic inventory theory, and just as the price-setting newsvendor model offers a stepping stone for studying joint (periodic) inventory and pricing decisions, a reinterpretation of the dynamic pricing problem as a price-setting newsvendor model with recourse has the potential to stimulate the merging of insights from dynamic pricing and dynamic inventory models. In turn, this could lead to a more basic decision-analytic framework for understanding the coordination of supply and demand in an uncertain world.

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Appendix

Proof of Proposition 7. Assume that $z \geq w$. This implies that $(z-A_t)^+ = (z-A_t)$; thus, from (5),

$$r_t(z) = \frac{1}{z^m}\left(\mathrm{E}[A_t] + r^*_{t-1}\int_0^w (z-a)^m f(a)\,da\right).$$

Consequently,

$$\begin{aligned}\frac{dr_t(z)}{dz} &= \frac{mr^*_{t-1}}{z^m}\int_0^w \frac{1}{(z-a)^{1-m}} f(a)\,da - \frac{mr_t(z)}{z}\\ &= \frac{m}{z}\left[r^*_{t-1}\int_0^w \left(1+\frac{a}{z-a}\right)^{1-m} f(a)\,da - r_t(z)\right],\end{aligned}$$

which implies

$$\left.\frac{d^2 r_t(z)}{dz^2}\right|_{dr_t(z)/dz=0} = \frac{-m(1-m)r^*_{t-1}}{z^{1+m}}\int_0^w \frac{af(a)}{(z-a)^{2-m}}\,da < 0.$$

Therefore, $dr_t(z)/dz$ can change sign at most once, from positive to negative, over the region $z \geq w$. In other words, $r_t(z)$ is quasi-concave for $z \geq w$. □

Proof of Proposition 8. Assume that $F(a) = a^k$ for $a \in [0,1]$ and $k > 0$. This implies that $f(a) = ka^{k-1}$ for $a \in [0,1]$; and $f(a) = 0$ otherwise. Thus,

$$\begin{aligned}\mathrm{E}\left[\left(1-\frac{A_t}{z}\right)^{+m}\right] &= \int_0^{\min\{z,1\}} k\left(1-\frac{a}{z}\right)^m a^{k-1}\,da\\ &= \int_0^{\min\{1,1/z\}} kz^k(1-v)^m v^{k-1}\,dv,\end{aligned}$$

which, from (5), implies

$$r_t(z) = \begin{cases} z^{1-m}\left[1-\int_0^1 kz^k(1-v)v^{k-1}\,dv\right] & \\ \quad + r^*_{t-1}\int_0^1 kz^k(1-v)^m v^{k-1}\,dv & \text{if } z<1,\\ \dfrac{\mathrm{E}[A_t]}{z^m} + r^*_{t-1}\int_0^{1/z} kz^k(1-v)^m v^{k-1}\,dv & \text{if } z \geq 1,\end{cases}$$

or

$$r_t(z)=\begin{cases} z^{1-m}\left[1-\dfrac{z^k}{k+1}\right]+r^*_{t-1}\dfrac{\Gamma(m+1)\Gamma(k+1)}{\Gamma(k+m+1)}z^k & \text{if } z<1,\\[2ex] \dfrac{k/(k+1)}{z^m}+r^*_{t-1}\displaystyle\int_0^{1/z}kz^k(1-v)^m v^{k-1}\,dv & \text{if } z\ge 1,\end{cases}$$

where Γ denotes the Gamma function. Note that $r_t(z)$ is continuous. Moreover,

$$\frac{dr_t(z)}{dz}=\begin{cases}\dfrac{1}{z^m}\left[(1-m)-\dfrac{(1-m+k)z^k}{k+1}\right]\\[1ex] \quad +kr^*_{t-1}\dfrac{\Gamma(m+1)\Gamma(k+1)}{\Gamma(k+m+1)}z^{k-1} & \text{if } z<1,\\[2ex] kr^*_{t-1}\left[kz^{k-1}\displaystyle\int_0^{1/z}(1-v)^m v^{k-1}\,dv-\dfrac{1}{z}\left(1-\dfrac{1}{z}\right)^m\right]\\[1ex] \quad +\dfrac{-mk/(k+1)}{z^{m+1}} & \text{if } z\ge 1.\end{cases}$$

Thus, $r_t(z)$ also is differentiable.

Because $r_t(z)$ is continuous and differentiable, and because $r_t(z)$ is quasi-concave for $z\ge 1$ (by Proposition 7), it suffices to show that $r_t(z)$ is quasi-concave for $z<1$ to complete the proof. For $z<1$,

$$\frac{dr_t(z)}{dz}=\frac{1}{(k+1)z^m}[(1-m)(k+1)-(1-m+k)z^k+\theta z^{k-1+m}], \tag{18}$$

where $\theta=kr^*_{t-1}\Gamma(m+1)\Gamma(k+2)/\Gamma(k+m+1)$. This implies

$$\frac{d^2r_t(z)}{dz^2}=\frac{1}{(k+1)z^m}[-k(k+1-m)z^{k-1}+(k-1+m)\theta z^{k-1+m-1}]-\frac{m}{z}\frac{dr_t(z)}{dz}. \tag{19}$$

There are two cases to consider: $k\le 1-m$ and $k>1-m$. If $k\le 1-m$, then

$$\left.\frac{d^2r_t(z)}{dz^2}\right|_{dr_t(z)/dz=0}<0,$$

which follows directly from (19). Likewise, if $k>1-m$, then

$$\left.\frac{d^2r_t(z)}{dz^2}\right|_{dr_t(z)/dz=0}<0,$$

because (18) applied to (19) yields

$$\begin{aligned}\left.\frac{d^2r_t(z)}{dz^2}\right|_{dr_t(z)/dz=0} &= \frac{1}{(k+1)z^m}\left[-k(k+1-m)z^{k-1}\right.\\ &\quad\left.+\frac{k-1+m}{z}((1-m+k)z^k-(1-m)(k+1))\right]\\ &< -(1-m)\frac{k+1-m}{k+1}z^{k-1-m}<0.\end{aligned}$$

Thus,

$$\left.\frac{d^2r_t(z)}{dz^2}\right|_{dr_t(z)/dz=0}<0$$

is true for both cases, which implies that $r_t(z)$ is quasi-concave for $z<1$. This completes the proof. □

References

Bagnoli, M., T. Bergstrom. 1989. Log-concave probability and its applications. Technical report, University of Michigan, Ann Arbor, MI.

Belobaba, P. P. 1989. Application of a probabilistic decision model to airline seat inventory control. *Oper. Res.* **37** 183–197.

Bitran, G. R., S. V. Mondschein. 1997. Periodic pricing of seasonal products in retailing. *Management Sci.* **43** 64–79.

Brodie, R., C. A. de Kluyver. 1984. Attraction versus linear and multiplicative market share models: An empirical evaluation. *J. Marketing Res.* **21** 194–201.

Brumelle, S. L., J. I. McGill. 1993. Airline seat allocation with multiple nested fare classes. *Oper. Res.* **41** 127–137.

Chan, L. M., D. Simchi-Levi, J. Swann. 2001. Dynamic pricing strategies for manufacturing with stochastic demand and discretionary sales. Technical report, University of Toronto, Toronto, Ontario, Canada.

Federgruen, A., A. Heching. 1997. Combined pricing and inventory control under uncertainty. *Oper. Res.* **47** 454–475.

Feng, Y., G. Gallego. 1995. Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Sci.* **41** 1371–1391.

Feng, Y., B. Xiao. 2000. A continuous-time yield management model with multiple prices and reversible price changes. *Management Sci.* **46** 644–657.

Gallego, G., G. J. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* **40** 999–1020.

Gallego, G., G. J. van Ryzin. 1997. A multi-product dynamic pricing problem and its application to network yield management. *Oper. Res.* **45** 24–41.

Gerchak, Y., M. Parlar, T. K. M. Yee. 1985. Optimal rationing policies and production quantities for products with several demand classes. *Canadian J. Admin. Sci.* **2** 161–176.

Ghosh, A., S. Neslin, R. Shoemaker. 1984. A comparison of market share models and estimation procedures. *J. Marketing Res.* **2** 202–210.

Hirshleifer, J., A. Glazer. 1992. *Price Theory and Applications*, 5th ed. Prentice Hall, Englewood Cliffs, NJ.

Intriligator, M. D. 1978. *Econometric Models, Techniques, and Applications*. Prentice Hall, Englewood Cliffs, NJ.

Kalyanam, K. 1996. Pricing decisions under demand uncertainty: A Bayesian mixture model approach. *Marketing Sci.* **15** 172–183.

Kincaid, W. M., D. Darling. 1963. An inventory pricing problem. *J. Math. Anal. Appl.* **7** 83–208.

Krishnamurthi, L., S. Raj. 1991. An empirical analysis of the relationship between brand loyalty and consumer price elasticity. *Marketing Sci.* **10** 172–183.

Lariviere, M. A. 1999. Supply chain contracting and coordination with stochastic demand. S. Tayur, R. Ganeshan, M. Magazine, eds. *Quantitative Models for Supply Chain Management*. Kluwer Academic Publishers, Boston, MA, 234–268.

Lee, T. C., M. Hersh. 1993. A model for dynamic airline seat inventory control with multiple seat bookings. *Transportation Sci.* **27** 252–265.

Leeflang, P. S., D. R. Wittink, M. Wedel, P. A. Naert. 2000. *Building Models for Marketing Decisions*. Kluwer Academic Publishers, Boston, MA.

Littlewood, K. 1972. Forecasting and control of passenger bookings. *AGIFORS Sympos. Proc.*, Vol. 12. Nathanya, Israel.

McGill, J. I., G. J. van Ryzin. 1999. Revenue management: Research overview and prospects. *Transportation Sci.* **33** 233–251.

Montgomery, A. L., E. T. Bradlow. 1999. Why analyst overconfidence about the functional form of demand models can lead to overpricing. *Marketing Sci.* **18** 569–583.

Montgomery, A. L., P. E. Rossi. 1999. Estimating price elasticities with theory-based priors. *J. Marketing Res.* **36** 413–423.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Oper. Res.* **47** 183–194.

Petruzzi, N. C., M. Dada. 2002. Dynamic pricing and inventory control with learning. *Naval Res. Logist.* **49** 304–325.

Petruzzi, N. C., G. E. Monahan. 2003. Managing fashion goods inventories: Dynamic recourse for retailers with outlet stores. *IIE Trans.* **35** 1033–1047.

Robinson, L. W. 1995. Optimal and approximate control policies for airline booking with sequential nonmonotonic fare classes. *Oper. Res.* **43** 252–263.

Silver, E., R. Peterson. 1985. *Decision Systems for Inventory Management and Production Planning*. John Wiley and Sons, New York.

Subramanian, J., J. S. Stidham, C. Lautenbacher. 1999. Airline yield management with overbooking, cancellations, and no-shows. *Transportation Sci.* **33** 147–167.

Tellis, G. J. 1988. The price elasticity of selective demand: A meta-analysis of econometric models of sales. *J. Marketing Res.* **25** 331–341.

Varian, H. R. 1992. *Microeconomic Analysis*, 3rd ed. Norton & Co., New York.

Virshup, A. 1997. Have they got a deal for you. *Smart Money Magazine* **6** 120–131.

Zabel, E. 1972. Multiperiod monopoly under uncertainty. *J. Econom. Theory* **5** 524–536.

Zhao, W., Y. Zheng. 2000. Optimal dynamic pricing for perishable assets. *Management Sci.* **46** 375–388.

Zhao, W., Y. Zheng. 2001. General dynamic models for airline seat allocation with diversion and no-shows. *Transportation Sci.* **35** 80–98.