# One-sided performance measures under Gram-Charlier distributions

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ABSTRACT

We derive closed-form expressions for risk measures based on partial moments by assuming the Gram-Charlier (GC) density for stock returns. As a result, the lower partial moment (LPM) measures can be expressed as linear functions on both skewness and excess kurtosis. Under this framework, we study the behavior of portfolio rankings with performance measures based on partial moments, that is, both Farinelli-Tibiletti (FT) and Kappa ratios. Contrary to previous results, significant differences are found in ranking portfolios between the Sharpe ratio and the FT family. We also obtain closed-form expressions for LPMs under the semi non-parametric (SNP) distribution which allows higher flexibility than the GC distribution.

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## 1. Introduction

An adequate risk-adjusted return performance measure (PM) is essential for selecting investment funds. The Sharpe ratio (Sharpe, 1966; 1994) has become the bechmark PM by adjusting the expected excess fund return by the symmetric risk measure or standard deviation. Although this ratio is still a reference indicator for assessing the accuracy of investment strategies, its use becomes rather doubtful when the fund return distribution is beyond the class of elliptical distributions (Owen and Rabinovitch, 1983) that include the normal distribution. As a result, several one-sided type measures of risk have been proposed and the associated PMs are known as one-sided PMs. In fact, some of these PMs are also characterized by one-sided reward measures.

Some examples of one-sided PMs are the adjusted for skewness Sharpe ratio (ASSR) proposed by Zakamouline and Koekebakker (2009a), the Generalized Rachev family based on the conditional Value at Risk (Biglova et al., 2004), the Farinelli-Tibiletti (FT) family based on both upper and lower partial moments (Farinelli and Tibiletti, 2008) and the Kappa or S-S family (Sortino and Satchell, 2001) based on lower partial moments. Other alternative

reward-to-variability ratios are well documented in Caporin et al. (2014) and the references therein. We will also implement PMs based on the certainty equivalent amount as a function of both prudence and temperance coefficients. These coefficients are related to the investor's appetite for asymmetry and aversion to leptokurtocity of fund returns. For details, see Ebert (2013); Eeckhoudt and Schlesinger (2006) and references therein.

Some very interesting papers as Eling (2008); Eling and Schuhmacher (2007), and Auer (2015) find that choosing different PMs is not critical to the portfolio evaluation. More specifically, the PM choice does not matter because any PM generates the same rank ordering as the Sharpe ratio (SR). Guo and Xiao (2016) agree with this result whenever the selected PMs satisfy the monotonic increasing property regarding the SR when the fund return distributions belong to the location-scale (LS) family.[3]

In contrast, our paper shows that some PMs like the FT family can generate different rank scores meaning that the choice of the PM is a relevant issue. This is in line with some empirical evidence such as the US mutual fund ranking indicated in Haas et al. (2012), where a member of the FT family (the Upside Potential ratio) exhibits lower Spearman's correlations (with respect to the SR)

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[3] See also Schuhmacher and Eling (2011); 2012. Some probability density functions that satisfy the LS condition are: Beta, Cauchy, Exponential, Extreme Value, Gamma, Logistic, Normal, t-Student, Uniform, Weibull, and Normal Inverse Gaussian (under some parametrizations). On the contrary, the lognormal density does not verify this condition.

than other PMs. Our results also agree with the evidence shown in Cogneau and Hübner (2009) and Zakamouline (2011).

We get closed-form expressions for the FT measures by assuming a return distribution that does not belong to the LS family. To be more precise, we consider the Gram-Charlier (GC) expansion as the probability density function (pdf). The GC distribution has been implemented, among others, by Corrado and Su (1996); Jondeau and Rockinger (2001) and Jurczenko and Maillet (2006). The advantage of this distribution is that both skewness ($s$) and excess kurtosis ($ek$) appear directly as the pdf's parameters. We previously get the closed-form expressions for the lower partial moment (LPM) measures as simple linear functions of both parameters. As a consequence, we can easily understand the behavior of these risk measures regarding changes in these higher moments. By expressing the upper partial moment (UPM) measures in terms of LPMs, we can focus just on this kind of downside risk measure[4] and analyze its properties under the GC distribution when studying the FT measures. Similarly, we obtain closed-form expressions for the Kappa measures under the previous distribution.[5]

Finally, the GC restriction to capture higher levels of $s$ and $ek$ suggests some other candidate distributions to seize better these higher moments but, unfortunately, leading to more complex expressions for the PMs. For instance, a more flexible distribution to the restricted higher moments under the GC distribution can be the semi-nonparametric (SNP) density proposed by Gallant and Nychka (1987). We also obtain the LPM analytical expressions under the SNP distribution.

The rest of the paper is organized as follows. In Section 2 we present different PMs based on either the Expected Utility Theory (EUT) or the Prospect Theory/Cumulative Prospect Theory (PT/CPT), see Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Section 3 shows the GC distribution and some properties. In Section 4 we obtain closed-form expressions for LPM and UPM measures under GC and hence, the expressions for both FT and Kappa ratios. We also analyze the behavior of the Kappa ratios regarding the levels of $s$ and $ek$ and obtain the isocurves for the Kappa measures. In Section 5 we conduct a simulation study on the performance evaluation. Section 6 shows the SNP distribution and the corresponding LPM expressions. Finally, Section 7 summarizes and provides the main conclusions. The proofs of propositions and corollaries are deferred to a final technical Appendix.

## 2. Performance measures (PMs)

Let $U(W)$ denote the investor's utility function where $W$ is the amount of wealth. The investor faces a capital allocation problema that is solved by maximizing his expected utility of wealth $E[U(W)]$, where $E[\,\cdot\,]$ is the expectation operator. The market includes a risky asset and a risk-free one. Assume that the initial wealth is $W_I$ and the capital allocation aims to invest an amount $a$ in the risky asset and, hence, $W_I - a$ in the risk-free asset. Thus, the investor's final wealth is

$$W(r, a) = a(r - r_f) + W_I(1 + r_f), \tag{1}$$

where $r$ is a random variable that denotes the return of the risky asset and $r_f$ is the risk-free rate of return that is assumed to be a constant. Assuming that $a \geq 0$ (short-selling is not allowed), the

investor's objective is selecting $a$ to maximize the expected utility:

$$E[U(W(r, a^*))] = \max_a E[U(W(r, a))], \tag{2}$$

where $a^*$ denotes the optimal amount invested in the risky asset from the maximization of the expected utility on the final wealth in (1). Besides EUT as the benchmark model of choice under uncertainty, we are interested in those models under PT/CPT where the utility function is defined over gains and losses relative to some reference point (kink), as opposed to wealth in EUT.

By using the *maximum principle* method,[6] we can rewrite (2) as $E[U(W(r, a^*))] = h(\pi(r))$ where $h(\cdot)$ is a strictly increasing function and $\pi(r)$ represents the PM.[7] More specifically, the investor prefers the risky portfolio $r_1$ to the risky portfolio $r_2$ if $\pi(r_1) > \pi(r_2)$. Hence, the aim at maximizing the investor's expected utility can alternatively be formulated as the maximization of a particular PM. In addition, a rational utility-based PM must be consistent with the stochastic-dominance principles that will be analyzed later.

Finally, a GC probability distribution for the returns of the risky asset will be assumed to obtain closed-form PM expressions under both EUT and PT/CPT. The reason for this specific distribution is because we can get a very easy interpretation in terms of the implied distribution parameters which are both skewness and kurtosis.

### 2.1. PMs based on EUT

These PMs will be obtained by implementing the maximum principle method and using the certainty equivalent (CE) amount corresponding to $E[U(W(r, a^*))]$ in (2) for ranking portfolios. Thus,

$$E[U(W(r, a^*))] = \max_a U(CE), \tag{3}$$

such that $CE = \mu_W - \xi$, where $\mu_W = E[W(r, a)]$ represents the expected final wealth and $\xi$ denotes the risk premium.

#### 2.1.1. Certainty equivalent as PM

Let $U$ be a utility function with desirable properties, that is, $U^{(1)} > 0, U^{(2)} < 0, U^{(3)} > 0$ and $U^{(4)} < 0$, where $U^{(i)}$ denotes the $i$-th derivative of the utility function. We start from the equation defining the CE amount given by

$$U(E[W(r, a)] - \xi) = E[U(W(r, a))]. \tag{4}$$

First, on the right-hand side in (4), approximate the utility function $U(x)$ by a fourth-order Taylor expansion around the point $x_0 = \mu_W$, where $x = W$, and take expectations. Second, on the left-hand side in (4), apply a first-order Taylor expansion around the same point $x_0$ but now $x = \mu_W - \xi$. Then, the maximum CE amount satisfying (3) is given by

$$CE^* \simeq \max_a \left\{ W_0 + a(\mu - r_f) - \frac{1}{2}\gamma\sigma^2 a^2 + \frac{1}{3!}s\psi_3\sigma^3 a^3 \right.$$
$$\left. - \frac{1}{4!}k\psi_4\sigma^4 a^4 \right\}, \tag{5}$$

where $\mu, \sigma, s,$ and $k$ denote, respectively, the mean, standard deviation, skewness and kurtosis of the risky asset return and $W_0 = W_I(1 + r_f)$. Let $\omega_k = U^{(k)}(\mu_W)$, then $\gamma = -\omega_2/\omega_1$ is the traditional absolute risk aversion coefficient, $\psi_3 = \omega_3/\omega_1$ is the coefficient of *appetite toward asymmetry* and $\psi_4 = -\omega_4/\omega_1$ is the coefficient of

aversion to leptokurticity.[8] Note that both $\psi_3$ and $\psi_4$ are connected with the popular coefficients of *prudence* and *temperance*. In particular, $\psi_3 = \gamma \delta_P$, where $\delta_P = -\omega_3/\omega_2$, is the coefficient of (absolute) prudence introduced by Kimball (1990), and $\psi_4 = \gamma \delta_P \delta_T$, where $\delta_T = -\omega_4/\omega_3$, is the coefficient of temperance presented in Kimball (1992).

### 2.1.2. Adjusted for skewness Sharpe ratio (ASSR)

By applying the maximum principle method, Zakamouline and Koekebakker (2009a) obtain a closed-form expression for $\pi(r)$ by using a third-order Taylor expansion of $U(\cdot)$ around the point $W_0$ and considering the HARA utility function. More specifically, the adjusted for skewness Sharpe ratio (*ASSR*) is defined as

$$ASSR = SR\sqrt{1 + \varphi \frac{s}{3} SR}, \tag{6}$$

such that *SR* denotes the Sharpe ratio, defined as $SR = (\mu - r_f)/\sigma$, where $\mu$ and $\sigma$ denote, respectively, the mean and standard deviation of the risky asset return, the coefficient $\varphi = \delta_P/\gamma$ is related to the investor's preferences through the coefficients of prudence and risk aversion, and $s$ is the skewness coefficient for the return of the risky asset.

Some properties of the *ASSR* are the following:

1. If all the investors have the same HARA utility (i.e., the parameter $\varphi$ does not change), the PM value is the same for all of them.
2. Note that (6) is just *SR* times a skewness adjusted factor and so, the *ASSR* nests the *SR* for symmetric return distributions.
3. It holds that $\varphi = 0$ for quadratic utility, $\varphi = 1$ for CARA utility (i.e., negative exponential utility), and $\varphi = 2$ for logarithmic utility. The case for CRRA utility (i.e., the power utility defined as $U(x) = \frac{x^{1-\lambda}}{1-\lambda}$ for $\lambda \neq 1$, where $\lambda$ is the coefficient of relative risk aversion) leads to $\varphi = (\lambda + 1)/\lambda$. Note that $\partial \varphi / \partial \lambda < 0$ for any $\lambda$. These types of utility functions are nested in the HARA utility.
4. Finally, the *ASSR* can be interpreted as a particular case of the generalized Sharpe ratio (*GSR*) introduced by Hodges (1998).[9]

### 2.2. PMs based on PT/CPT

As it was mentioned at the beginning of this Section, under PT/CPT, the utility function is defined over gains and losses relative to a reference point. It means that the utility function exhibits a kink at the reference point, with the slope of the loss function steeper than that of the gain function. This is called *loss aversion*. In this new framework, the investor can show three different types of aversion: (i) aversion to loss, $\lambda_L$, (ii) aversion to uncertainty in gains, $\gamma_+$, and (iii) aversion to uncertainty in losses, $\gamma_-$. Indeed, there is no unique definition of $\lambda_L$. For instance, Kahneman and Tversky (1979) define it in a global sense while Köbberling and Wakker (2005) do it in a local sense (around the reference point for the wealth, $\overline{W}$). Note that a utility function can show loss aversion according to one definition, but not to another.[10] Finally, He

and Zhou (2011) propose a new definition of $\lambda_L$ that is markedly different from the previous ones as it is relevant for large (instead of small) gains and losses.[11]

In particular, the utility investor has the generalized form of a piecewise linear function plus a power one:

$$U(W) = \begin{cases} \mathbf{1}_+ (W - \overline{W}) - (\gamma_+/q)(W - \overline{W})^q, & \text{if } W \geq \overline{W}, \\ -\lambda_L \left[ \mathbf{1}_- (\overline{W} - W) + (\gamma_-/m)(\overline{W} - W)^m \right], & \text{if } W < \overline{W}, \end{cases} \tag{7}$$

where $\mathbf{1}_+$ and $\mathbf{1}_-$ take values in $\{0, 1\}$, $\gamma_+$ and $\gamma_-$ are real numbers, and $\lambda_L, q, m > 0$. We can rewrite (7) as

$$U(W(r, a)) = \begin{cases} \mathbf{1}_+ a(r - \overline{\tau}) - (\gamma_+/q)a^q (r - \overline{\tau})^q, & \text{if } r \geq \overline{\tau}, \\ -\lambda_L \left[ \mathbf{1}_- a(\overline{\tau} - r) + (\gamma_-/m)a^m (\overline{\tau} - r)^m \right], & \text{if } r < \overline{\tau}, \end{cases} \tag{8}$$

where $\overline{\tau} = r_f - (W_0 - \overline{W})/a$. See Zakamouline (2014) for more details. Eq. (8) implies that $E[U(W(r, a))]$ can be expressed in terms of lower and upper partial moments.

A lower partial moment (LPM) measures risk by negative deviations of the stock returns, $r$, with respect to a minimal acceptable return (return threshold), $\tau$. Fishburn (1977) defines the LPM of order $m$ for a stock return as

$$LPM(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\, dr, \tag{9}$$

where $f(\cdot)$ denotes the probability density function (pdf) of $r$. Similarly, the upper partial moment (UPM) is defined as

$$UPM(\tau, q) = \int_{\tau}^{\infty} (r - \tau)^q f(r)\, dr. \tag{10}$$

If we set $\overline{W} = W_0$ in (8), then $\overline{\tau} = r_f$ and the investor's expected utility can be rewritten as

$$E[U(W(r))] = \mathbf{1}_+ a UPM(r_f, 1) - (\gamma_+/q)a^q UPM(r_f, q) \tag{11}$$
$$- \lambda_L \left[ \mathbf{1}_- a LPM(r_f, 1) + (\gamma_-/m)a^m LPM(r_f, m) \right].$$

If we maximize (11) with respect to $a$, the first order condition is given by

$$\mathbf{1}_+ UPM(r_f, 1) - \lambda_L \mathbf{1}_- LPM(r_f, 1)$$
$$- a^{q-1}\left[ \gamma_+ UPM(r_f, q) + a^{m-q}\lambda_L \gamma_- LPM(r_f, m) \right] = 0. \tag{12}$$

Next, some closed-form expressions for $a$ are obtained by considering two particular cases in (8). By applying the maximum principle, the PM will be obtained as a function of $E[U(W(r, a^*))]$ in (2).

### 2.2.1. The Kappa measures

Consider the following case in (8): $\mathbf{1}_+ = \mathbf{1}_- = 1$, $\lambda_L = 1$, $\gamma_+ = 0$, $\gamma_- > 0$ and $q = m > 1$. This setting leads to a piecewise linear (in both sides of the return threshold $r_f$) plus a power function (in one side, just for $r < r_f$). In this case, the investor exhibits loss aversion. Note that the excess expected return can be expressed as $\mu - r_f = UPM(r_f, 1) - LPM(r_f, 1)$. Then, (12) simplifies to $(\mu - r_f) - a^{m-1}\gamma_- LPM(r_f, m) = 0$. Solving for $a$ in this equation and applying the maximum principle, we obtain the group of PMs

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known as the S-S or Kappa ratios (see Sortino and Satchell, 2001), defined for any return threshold $\tau$ as

$$K(\tau, m) = \frac{\mu - \tau}{\sqrt[m]{LPM(\tau, m)}}, \tag{13}$$

where $\mu - \tau$ is the excess expected return with respect to $\tau$. Some popular PMs are:

1. The Omega–Sharpe ratio (see Kaplan and Knowles, 2004), which can be considered as the limit case of $K(r_f, m)$ in (13) for $m \rightarrow 1$ (see Zakamouline, 2014).
2. The Sortino ratio (see Sortino and Van der Meer, 1991) for $m = 2$.
3. The Kappa 3 ratio (see Kaplan and Knowles, 2004) for $m = 3$.

### 2.2.2. The Farinelli-Tibiletti (FT) measures

Consider now the following case in (8): $\mathbf{1}_+ = \mathbf{1}_- = 0$, $\gamma_+ = -q$, $\gamma_- = m$ such that $m > q > 0$. These restrictions lead to a piecewise power utility function. In this case, the equation (12) can be rewritten as $q\, UPM(r_f, q) - a^{m-q} m \lambda_L LPM(r_f, m) = 0$. Following the same steps as in the Kappa measures, we get the group of PMs known as the Farinelli–Tibiletti (FT) ratios, see Farinelli and Tibiletti (2008). For any return threshold $\tau$, the FT ratio is defined as

$$FT(\tau, q, m) = \frac{\sqrt[q]{UPM(\tau, q)}}{\sqrt[m]{LPM(\tau, m)}}. \tag{14}$$

Note that (14) nests two popular PMs:

1. The Omega ratio[12] which is obtained as the limit of $FT(\tau, 1, m)$ in (14) when $m \rightarrow 1$. This ratio can also be expressed (see Keating and Shadwick, 2002) as:

$$FT(\tau, 1, 1) = \frac{\int_\tau^\infty (1 - F(r))dr}{\int_{-\infty}^\tau F(r)dr}, \tag{15}$$

where $F(\cdot)$ is the cumulative distribution function of $r$.
2. The Upside Potential ratio (see Sortino et al., 1999) when $q = 1$, $m = 2$.

### 2.2.3. Relationship between Kappa and FT measures

If we rewrite $UPM(\tau, q)$ in terms of $LPM(\tau, m)$ (see Eqs. (9) and (10)), we get an alternative expression for the FT ratio (that will be useful later), as shown in the following Corollary.

**Corollary 1.** *Let $\psi(\tau, q) = E[(r - \tau)^q]$ and let $f(\cdot)$ be the pdf of the portfolio stock return $r$. Then, (14) can be expressed as*

$$FT(\tau, q, m) = \frac{\sqrt[q]{\psi(\tau, q) + (-1)^{q+1} LPM(\tau, q)}}{\sqrt[m]{LPM(\tau, m)}}. \tag{16}$$

Using the relationship between $LPM(\tau, m)$ and the Kappa measures (see (13)), Eq. (16) becomes

$$FT(\tau, q, m) = \frac{\sqrt[q]{\psi(\tau, q) + (-1)^{q+1}(\mu - \tau)^q K(\tau, q)^{-q}}}{(\mu - \tau)K(\tau, m)^{-1}}. \tag{17}$$

Note that, for $q = 1$, Eq. (17) simplifies to $FT(\tau, 1, m) = K(\tau, m)\left[1 + K(\tau, 1)^{-1}\right]$.

### 2.3. Properties of utility-based PMs

The above two groups of PMs (i.e., based on EUT and PT/CPT) will be consistent with the first-order stochastic dominance (FSD) when the underlying utility function is everywhere increasing. The

Kappa and FT measures will satisfy FSD when $q, m > 0$ in (8). Note that this restriction holds since $q = m > 1$ for Kappa measures (see (13)) whereas $m > q > 0$ for FT measures (see (14)).

Regarding the second-order stochastic dominance (SSD), a specific PM will be consistent with SSD if the utility function is everywhere increasing and concave. Under the EUT framework, this property is held for PMs like *ASSR* and *GSR*. The Kappa ratios are also consistent with SSD because the restriction $m > 1$ guarantees that the utility function (8) is concave. Nevertheless, under the utility function (8), the FT ratios (where $m > q$) are consistent with SSD when $q < 1 < m$ since (8) is concave everywhere. On the contrary, for other values of $q$ and $m$, the stochastic dominance (SD) principles cannot be applied.

Levy and Levy (2002) extend the classical SD principles by developing Prospect and Markowitz SD (PSD and MSD) theories with S-shaped and reverse S-shaped utility functions, respectively. Thus, a PM is consistent with second-order PSD (MSD) when the utility function is increasing, convex (concave) below the reference point and concave (convex) above the reference point. In particular, FT measures with $0 < q, m < 1$ are consistent with second-order PSD since (8) becomes a S-shaped utility function. FT measures with $q, m > 1$ verify the second-order MSD where (8) is a reverse S-shaped utility function. In brief, it is straightforward to prove the following Corollary that summarizes the main results for the above PMs based on PT/CPT.

**Corollary 2.** *Consider the utility function $U(\cdot)$ in (8) where $q, m > 0$, $\lambda_L > 0$, and $a > 0$ (i.e., short selling is not allowed). Then, the following properties are verified:*

1. *$U(\cdot)$ is increasing everywhere and, hence, Kappa and FT measures are consistent with FSD.*
2. *Kappa measures are consistent with SSD when $m > 1$.*
3. *By setting $m > q$, then FT measures are consistent with*
   (a) *SSD when $q < 1 < m$.*
   (b) *Second-order PSD when $q, m < 1$ (S-shaped utility function).*
   (c) *MSD when $q, m > 1$ (reverse S-shaped utility function).*

## 3. GC density and properties

We assume that the stock return $r$ is a random variable defined as

$$r = \mu + \sigma z, \qquad z \sim GC(0, 1, s, ek), \tag{18}$$

such that the pdf of the (standardized) random variable $z$ is the Gram-Charlier expansion with zero mean, unit variance, skewness $s$ and excess kurtosis $ek$. Hence, the return in (18) is just an affine transformation of a random variable with GC expansion as pdf. The GC pdf, denoted as $g(z)$, is defined as[13]

$$g(z) = p(z)\phi(z); \quad p(z) = 1 + \frac{s}{\sqrt{3!}}H_3(z) + \frac{ek}{\sqrt{4!}}H_4(z), \tag{19}$$

where $\phi(\cdot)$ denotes the pdf of the standard normal variable and $H_k(z)$ is the normalized Hermite polynomial of order $k$. Kendall and Stuart (1977) shows that

$$H_k(z) = \sum_{n=0}^{[k/2]} a_{k,n} z^{k-2n}, \tag{20}$$

where $[\cdot]$ rounds its argument to the nearest (smaller) integer and

$$a_{k,n} = \left(-\frac{1}{2}\right)^n \frac{\sqrt{k!}}{(k-2n)!n!}. \tag{21}$$

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[12] Note that $K(\tau, 1) = FT(\tau, 1, 1) - 1$. The ratio introduced by Bernardo and Ledoit (2000) is just the Omega measure for $\tau = 0$ (or gain-to-loss ratio).

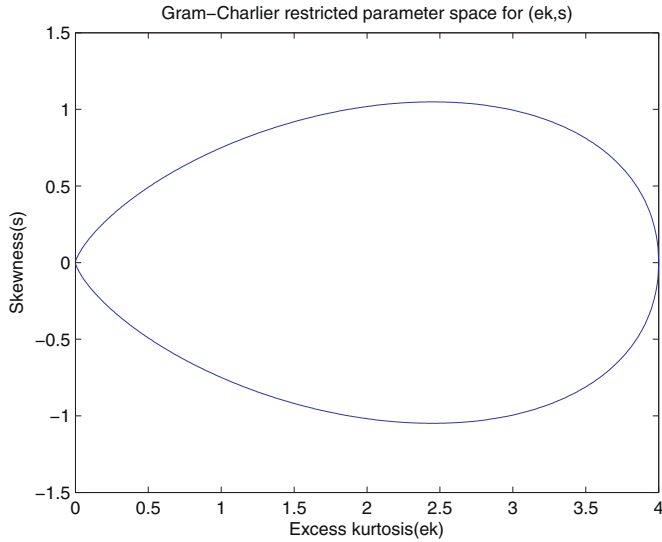[13] The pdf of $r$ in (18) is obtained as $f(r) = g(z)/\sigma$.

**Fig. 1.** Space containing for stock returns the points $(ek, s)$ with the excess kurtosis level, $ek$, in the x-axis and the skewness level, $s$, in the y-axis. This space is limited by a frontier (envelope) verifying that the Gram-Charlier (GC) density is well defined for the points on and inside the envelope. Thus, the GC density will be restricted to this space for $(ek, s)$. Note that $ek \in [0, 4]$ while $s \in [-1.0493, 1.0493]$. The range of $s$ depends on the level of $ek$. See Jondeau and Rockinger (2001) for more details about how to obtain this frontier.

It holds that $\{H_k(z)\}_{k \in \mathbb{N}}$, with $H_0(z) = 1$, constitutes an orthonormal basis with respect to the weighting function $\phi(z)$, that is,

$$E_\phi[H_k(z)H_l(z)] = \mathbf{1}_{(k=l)}, \tag{22}$$

where $\mathbf{1}_{(\cdot)}$ is the usual indicator function and the operator $E_\phi[\,\cdot\,]$ denotes the expectation of its argument regarding $\phi(\cdot)$. Note that $E_\phi[H_k(z)] = 0$ for $k \geq 1$ and then, (22) is just the covariance between $H_k(z)$ and $H_l(z)$.

The pdf $g(z)$ in (19) can take negative values for certain values of both centered moments. Jondeau and Rockinger (2001) obtain numerically a restricted space $\Gamma$ for possible values of $(ek, s)$ that guarantees the positivity of $g(z)$. The constrained GC expansion restricted to $\Gamma$ will be referred as the true GC density. Fig. 1 exhibits $\Gamma$ with a frontier (the envelope) delimiting the oval domain. $\Gamma$ is a compact and convex set. Note that $ek \in [0, 4]$ while $|s| \leq 1.0493$ verifying that the range of $s$ depends on the level of $ek$. For instance, if $|s| = 0.6$ then $ek$ ranges from 0.6908 to 3.7590, while for $s = 0$, $ek$ ranges from 0 to 4. The maximum size for skewness is reached for $ek = 2.4508$. Obviously, the case for the normal distribution corresponds to the origin.

### 3.1. Moments of r and z

The first four (non-central) moments of $z$ with pdf $g(\cdot)$ in (19) can be obtained by using the relationship between the powers of $z$ and the Hermite polynomials in (20) and the condition in (22):

$$E_g[z] = 0, \quad E_g[z^2] = 1, \quad E_g[z^3] = s, \quad E_g[z^4] = ek + 3.$$

The following Proposition provides a general expression for $E_g[z^k]$ where $k \geq 5$.

**Proposition 1.** Consider the pdf $g(\cdot)$ as given in (19) Then, for $k \geq 5$, we can obtain a general expression for $E_g[z^k]$. In more detail, we have the following:

• For the even powers of $z$, we get

$$E_g[z^{2i+4}] = \eta_{2i+4,0} + \eta_{2i+4,1}\, ek, \quad i = 1, 2, \ldots,$$

where, for $i = 1, 2, \ldots$, we have

$$\eta_{2i+4,0} = (-1)^i i\, \frac{2a_{2i+4,i+1} + (i+1)a_{2i+4,i+2}}{2a_{2i+4,0}}$$

$$\eta_{2i+4,1} = (-1)^i \frac{a_{2i+4,i}}{a_{2i+4,0}}.$$

• The expected value for any odd power of $z$ is given by

$$E_g[z^{2i+1}] = \eta_{2i+1,2}\, s, \quad i = 1, 2, \ldots,$$

with

$$\eta_{2i+1,2} = (-1)^{i-1} \frac{a_{2i+1,i-1}}{a_{2i+1,0}}, \quad i = 1, 2, \ldots$$

where $a_{k,\,n}$ in the above expressions are defined in (21).

Since $r$ in (18) is an affine transformation of $z$, the non-central moments of $r$ are obtained as

$$E_f[r^k] = \sum_{n=0}^{k} \binom{k}{n} \mu^{k-n} \sigma^n E_g[z^n], \tag{23}$$

where $\binom{k}{n} = \frac{k!}{n!(k-n)!}$.

**Proof.** See the Appendix. $\square$

Note that, if $k$ is even (odd), $E_g[z^k]$ depends only on the excess kurtosis (skewness).

### 3.2. Cumulative density function

For $m = 0$, the LPM expression in (9) becomes the distribution function for the standardized return $z$ in (18).

**Proposition 2.** The cumulative density function of $r$ in (18) is given by

$$F(r) = \Phi(r^*) - \frac{s}{3\sqrt{2!}} H_2(r^*)\phi(r^*) - \frac{ek}{4\sqrt{3!}} H_3(r^*)\phi(r^*), \tag{24}$$

where $r^* = (r - \mu)/\sigma$.

**Proof.** See the Appendix. $\square$

The following Corollary shows the behavior of $F(r)$ with respect to the parameters $s$, $ek$, $\mu$ and $\sigma$.

**Corollary 3.** Let $F(r)$ in (24) and $r^* = (r - \mu)/\sigma$. Then, it holds that

1. $\partial F(r)/\partial s > 0 \Leftrightarrow |r^*| < 1$.
2. $\partial F(r)/\partial ek > 0 \Leftrightarrow r^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right)$.
3. For $r^* \in \left(-\sqrt{3 - \sqrt{6}}, 0\right) \cup \left(\sqrt{3 + \sqrt{6}}, +\infty\right)$, then $\partial F(r)/\partial r^* > 0$ and so,
   (a) $\partial F(r)/\partial \mu < 0$.
   (b) $\partial F(r)/\partial \sigma > 0$ iff $\mu > r$.

**Proof.** See the Appendix. $\square$

Hence, for $r^* \in (0, 1)$, we have that both $\partial F(r)/\partial s$ and $\partial F(r)/\partial ek$ are positive.

## 4. LPMs And related PMs with GC distribution

We start providing the closed-form expressions of $LPM_f(\tau, m)$, where the stock return is driven by (18), and the related expressions for $K_f(\tau, m)$ and $FT_f(\tau, q, m)$. Later, we show that these LPMs are linear functions of both $s$ and $ek$ and analyze the behavior of the above PMs. Finally, we obtain some Kappa isocurves.

### 4.1. Closed-form expressions of $LPM_f(\tau, m)$

**Proposition 3.** *Let $z$ be the standardized return of $r$ in (18). The lower partial moment of order $m \in \mathbb{N}_+$ for the stock return $r$ can be expressed as*

$$LPM_f(\tau, m) = LPM_n(\tau, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m}, \qquad (25)$$

*where $LPM_n(\tau, m)$ is the LPM by assuming a normal distribution $n(\cdot)$ for $r$ in (18) with*

$$LPM_n(\tau, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k}(\tau - \mu)^{m-k}\sigma^k B_k, \qquad (26)$$

*and*

$$\theta_{j,m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k}(\tau - \mu)^{m-k}\sigma^k A_{kj}, \qquad (27)$$

*such that $B_k = B_k(\tau^*)$ and $A_{kj} = A_{kj}(\tau^*)$, with $\tau^* = (\tau - \mu)/\sigma$, can be seen in the Appendix.*

**Proof.** See the Appendix. $\square$

The Kappa measures are easily obtained by using (13) and (25). It can be seen that the behavior of $\theta_{j,m}$ depends on the expected return, the volatility and the return threshold, i.e., $\theta_{j,m} = \theta_{j,m}(\mu, \sigma, \tau)$. Note that $\theta_{2,m}$ and $\theta_{3,m}$ quantify the sensitivity of $LPM_f(\tau, m)$ to changes in $s$ and $ek$, respectively. The following Corollary shows the general expression for the FT measures in (16) under the GC density for the standardized return.

**Corollary 4.** *Let $z$ be the standardized return of $r$ in (18). The performance measure $FT_f(\tau, q, m)$ in (16) for $q, m \in \mathbb{N}_+$ can be expressed as*

$$FT_f(\tau, q, m) = \frac{\sqrt[q]{\psi_f(\tau, q) + (-1)^{q+1}LPM_f(\tau, q)}}{\sqrt[m]{LPM_f(\tau, m)}}, \qquad (28)$$

*where $LPM_f(\tau, \cdot)$ is given in (25) and $\psi_f(\tau, q) = E_f[(r - \tau)^q]$ is obtained as*

$$\psi_f(\tau, q) = \sum_{k=0}^{q} \binom{q}{k}(\mu - \tau)^{q-k}\sigma^k E_g[z^k], \qquad (29)$$

*where $E_g[z^k]$ is given in Proposition 1.*

Many studies about performance evaluation focus on some popular Kappa measures such as the Omega–Sharpe, Sortino, and Kappa 3 ratios and the Upside Potential ratio from the FT family. As a consequence, we are very interested in the expressions of $LPM_f(\tau, m)$ for $m = 1, 2, 3$. This is shown in the following Corollary.

**Corollary 5.** *The expressions of $LPM_n(\tau, m)$ and $\theta_{j,m}$ (see (26)-(27)) for $m = 1, 2, 3$ are given by*

$$LPM_n(\tau, 1) = (\tau - \mu)\Phi(\tau^*) + \sigma\phi(\tau^*),$$

$$LPM_n(\tau, 2) = (\tau - \mu)^2\Phi(\tau^*) + (\tau - \mu)\sigma\phi(\tau^*) + \sigma^2\Phi(\tau^*),$$

$$LPM_n(\tau, 3) = (\tau - \mu)^3\Phi(\tau^*) + (\tau - \mu)^2\sigma\phi(\tau^*)$$
$$+ 3(\tau - \mu)\sigma^2\Phi(\tau^*) + 2\sigma^3\phi(\tau^*). \qquad (30)$$

*and*

$$\theta_{j,1} = (\tau - \mu)A_{0j} - \sigma A_{1j},$$

$$\theta_{j,2} = (\tau - \mu)^2 A_{0j} - 2(\tau - \mu)\sigma A_{1j} + \sigma^2 A_{2j},$$

$$\theta_{j,3} = (\tau - \mu)^3 A_{0j} - 3(\tau - \mu)^2\sigma A_{1j}$$
$$+ 3(\tau - \mu)\sigma^2 A_{2j} - \sigma^3 A_{3j}, \qquad (31)$$

*where the values for $A_{kj} = A_{kj}(\tau^*)$, $\tau^* = (\tau - \mu)/\sigma$, can be seen in the Appendix.[14]*

**Proof.** See the Appendix. $\square$

### 4.2. Behavior of Kappa measures with respect to $s$ and $ek$

We analyze the effects of the higher moments on the performance ratios. We set the parameter vector $(\mu, \sigma, \tau)$ equal to $(\mu_0, \sigma_0, \tau_0)$. To shorten, $LPM_f(\tau_0, m)$ is denoted by $LPM_m$. Note that $LPM_m$ is a function, $g_m$, on both $s$ and $ek$. Let $\Delta LPM_m$ and $dLPM_m$ denote, respectively, the increment and the total differential of $LPM_m$ with respect to its arguments.[15] The next Corollary inmediately arises.

**Corollary 6.** *If we approximate $\Delta LPM_m$ by $dLPM_m$, we get*

$$\Delta LPM_m = \frac{\partial g_m}{\partial s}\Delta s + \frac{\partial g_m}{\partial ek}\Delta ek$$

*holding that*

$$\Delta LPM_m > 0 \Leftrightarrow \Delta s < \varphi_m \Delta ek$$

*where*

$$\varphi_m = -\frac{1}{2}\frac{\theta_{3,m}}{\theta_{2,m}}. \qquad (32)$$

Table 1 exhibits the behavior of the popular Kappa measures by changing either $s$ or $ek$, that is, we provide these measures for alternative portfolios with the same $\mu$ and $\sigma$ but different values for $s$ and $ek$ such that $(ek, s) \in \Gamma$. Consider, for instance, the monthly return values of $\mu = 0.86\%$, $\sigma = 2.61\%$, $\tau = r_f$ ($r_f = 0.39\%$), and three possible values of skewness ($s = -0.7, 0, 0.4$). The Sharpe ratio is equal to 0.1796. Plugging these parameters in (32), we get $\varphi_1 = -1.3471$, $\varphi_2 = 0.0499$ and $\varphi_3 = 0.2289$.

Columns 2 to 10 show the three Kappa measures for the different combinations $(ek, s)$ according to different levels of skewness. The main results can be summarized as follows:

1. Consider the portfolio $\pi_1$ with $(ek, s) = (0.8996, 0)$ and build new portfolios by only increasing $ek$. We see that $K_f(r_f, 1)$ increases but $K_f(r_f, 2)$ and $K_f(r_f, 3)$ decrease.
2. Take a new portfolio $\pi_2$ with $(ek, s) = (0.8996, 0.4)$. It holds that $K_f(r_f, m)$ increases if we only increase $s$ by changing $\pi_1$ for $\pi_2$. The same behavior holds for alternative values of $ek$.
3. Suppose now that $ek$ increases and $s$ decreases. Consider either portfolio $\pi_3$, with $(ek, s) = (1.2048, -0.7)$, or $\pi_4$, with $(ek, s) = (2.1205, -0.7)$. It is verified that both $K_f(r_f, 2)$ and $K_f(r_f, 3)$ decrease when going from $\pi_1$ to either $\pi_3$ or $\pi_4$. Meanwhile, there are opposite effects about the behavior of $LPM_f(r_f, 1)$. Note that $K_f(r_f, 1)$ decreases (increases) when moving to $\pi_3$ ($\pi_4$).
4. Finally, we can see that (32) holds under these examples and so, the behavior of their related Kappa measures is verified. Thus, changing $\pi_1$ for $\pi_2$ leads to $\Delta s = 0.4, \Delta ek = 0$. The case of changing $\pi_1$ for $\pi_3$ leads to $\Delta s = -0.7, \Delta ek = 0.3052$. Finally, changing $\pi_1$ for $\pi_4$ leads $\Delta s = -0.7, \Delta ek = 1.2209$.

In short, from the above results, we can suggest that $\partial LPM_f(r_f, m)/\partial s < 0$ for $m = 1, 2, 3$ and that $\partial LPM_f(r_f, 1)/\partial ek < 0$ and $\partial LPM_f(r_f, m)/\partial ek > 0$ for $m = 2, 3$.[16]

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[14] $LPM_n(\tau, m)$ is the same as $\theta_{1,m}$ when $j = 1$ in (31). We use $LPM_n(\tau, m)$ instead of $\theta_{1,m}$ for an easier interpretation of (25).

[15] That is, $\Delta LPM_m = g_m(s + \Delta s, ek + \Delta ek) - g_m(s, ek)$, where $\Delta x$ represents a small increment in $x$.

[16] These results are also supported by studying the behavior of $\theta_{2,m}$ and $\theta_{3,m}$ in (31) from many simulated parameters of $\mu$ and $\sigma$. The simulation results confirm the previous conclusions.

**Table 1**
Sensitivity analysis for the Kappa measures under a GC distribution. Effects of skewness ($s$) and excess kurtosis ($ek$).

| $ek$ | Panel A: Omega | | | Panel B: Sortino | | | Panel C: Kappa 3 | | |
|---|---|---|---|---|---|---|---|---|---|
| | $s = -0.7$ | $s = 0$ | $s = 0.4$ | $s = -0.7$ | $s = 0$ | $s = 0.4$ | $s = -0.7$ | $s = 0$ | $s = 0.4$ |
| 0.8996 | 0.5801 | 0.5959 | 0.6054 | 0.2622 | 0.2923 | 0.3150 | 0.1825 | 0.2070 | 0.2292 |
| 1.2048 | 0.5893 | 0.6057 | 0.6154 | 0.2617 | 0.2916 | 0.3141 | 0.1806 | 0.2040 | 0.2246 |
| 1.5100 | 0.5988 | 0.6157 | 0.6258 | 0.2612 | 0.2909 | 0.3133 | 0.1788 | 0.2011 | 0.2204 |
| 1.8153 | 0.6086 | 0.6261 | 0.6365 | 0.2607 | 0.2903 | 0.3124 | 0.1771 | 0.1983 | 0.2165 |
| 2.1205 | 0.6187 | 0.6368 | 0.6476 | 0.2602 | 0.2896 | 0.3116 | 0.1754 | 0.1957 | 0.2128 |
| 2.4257 | 0.6292 | 0.6479 | 0.6591 | 0.2598 | 0.2889 | 0.3107 | 0.1738 | 0.1932 | 0.2094 |
| 2.7309 | 0.6400 | 0.6594 | 0.6710 | 0.2593 | 0.2882 | 0.3099 | 0.1723 | 0.1909 | 0.2062 |
| 3.0361 | 0.6513 | 0.6713 | 0.6833 | 0.2588 | 0.2876 | 0.3091 | 0.1708 | 0.1887 | 0.2031 |
| 3.3414 | 0.6629 | 0.6836 | 0.6961 | 0.2583 | 0.2869 | 0.3083 | 0.1693 | 0.1865 | 0.2003 |
| 3.6466 | 0.6749 | 0.6964 | 0.7094 | 0.2578 | 0.2863 | 0.3074 | 0.1680 | 0.1845 | 0.1976 |

This Table exhibits the values of the closed-form formulas for the Kappa measures (Omega–Sharpe, Sortino and Kappa 3) by using the LPM expressions from Corollary 5 for monthly returns. All the portfolios in this Table have $\mu = 0.86\%$ and $\sigma = 2.61\%$ but different values for $s$ and $ek$ such that $(ek, s) \in \Gamma$. The Sharpe ratio is $SR = 0.1796$ and the return threshold is $r_f = 0.39\%$.

### 4.3. Iso-curves for performance measures

We obtain now the points $(ek, s)$ that provide the same value for the selected Kappa measure given fixed levels of $\tau$, $\mu$ and $\sigma$. To shorten, let $\Psi$ denote the vector $(\mu, \sigma, \tau)$ and let $\Psi_0$ be a fixed value for $\Psi$. Thus, the iso-curve associated for any Kappa measure, or isoKappa, corresponds to the set of points $\Pi$ defined as

$$\Pi(m, \Psi_0) = \left\{ (ek, s) \in \Gamma : \overline{K}_f(\tau_0, m) \right.$$

$$\left. = \frac{\mu_0 - \tau_0}{\sqrt[m]{LPM_n(\tau_0, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m}}} \right\}, \quad (33)$$

where $\overline{K}_f(\tau_0, m)$ denotes a fixed value for the Kappa ratio given by Eqs. (13) and (25). These spaces are easily obtained according to the following Corollary.

**Corollary 7.** *The isoKappa (33) implies a linear relation between $s$ and $ek$. Thus, $s = a_m + \varphi_m ek$ such that the slope $\varphi_m$ is defined in (32) and*

$$a_m = \frac{\sqrt{6}}{\theta_{2,m}}[\xi_{0,m} - LPM_n(\tau_0, m)], \quad \xi_{0,m} = \left[\frac{\mu_0 - \tau_0}{\overline{K}_f(\tau_0, m)}\right]^m, \quad (34)$$

*with $LPM_n(\tau_0, m)$ as in (26).*

The isoKappa in (34) will be labeled as 'iso-Omega–Sharpe', 'iso-Sortino' and 'iso-Kappa3' respectively for $m = 1, 2, 3$. Since $\partial\xi_{0,m}/\partial\overline{K}_f(\tau_0, m) < 0$ for $\mu_0 > \tau_0$, then $\partial a_m/\partial\overline{K}_f(\tau_0, m) > 0$ iff $\theta_{2, m} < 0$. Let $\Psi_0 = (0.86\%, 2.61\%, 0.39\%)$ be the parameter set used to obtain Table 1. Then, the slopes $\varphi_m$ for the different isoKappas (see the values of $\varphi_m$ in Subsection 4.2) verify that $\varphi_1 < 0$, $\varphi_2 > 0$, $\varphi_3 > 0$. So, an increase in $ek$ leads to a decrease (increase) in $s$ when moving along the isoOmega–Sharpe (iso-Sortino or isoKappa 3) curve.

By setting $s = -0.7$ and taking higher levels of $ek$, Table 1 shows that $K_f(0.39\%, 1)$ increases ($\xi_{0, 1}$ decreases) but $K_f(0.39\%, m)$ decreases ($\xi_{0, m}$ increases) for $m = 2, 3$. This means that the iso–Omega–Sharpe curves with negative slopes move in paralell to the right with higher levels of $\overline{K}_f(0.39\%, 1)$ since $a_1$ in (34) increases because $\theta_{2, 1} < 0$. Nevertheless, both the iso-Sortino and isoKappa3 curves with positive slopes move in paralell to the right with lower levels of $\overline{K}_f(0.39\%, m)$ since $a_1$ decreases because $\theta_{2, m} < 0$.[17]

Note that, on one hand, the isoKappas from Corollary 7 become very restrictive since we are fixing both the mean and volatility parameters for the portfolio returns but, on the other hand, we obtain linear equations which can help to understand the behavior of the iso-Kappas.[18] Finally, the isocurves under FT measures do not imply a linear relation between $s$ and $ek$ but we can obtain a linear approximation.

## 5. Simulation analysis

We implement a simulation analysis based on the closed-form expressions for the PMs by assuming a GC distribution for the standardized stock returns in (18). We start our analysis by obtaining the Spearman's rank correlation between the $SR$ and alternative PMs for different portfolios. The higher this rank correlation, the lower difference in ranking between the $SR$ and the selected PM. A deeper study will also be carried out by assessing how skewness and excess kurtosis can affect the portfolio composition. Finally, we also analyze the impact of both prudence and temperance coefficients on the PMs based on $CE$.

### 5.1. Simulation of parameters and performance ratios

To begin with, we simulate (for monthly returns) the parameter vectors $\vartheta_i = (\mu_i, \sigma_i, s_i, ek_i)$, $i = 1, \ldots, N_T$.[19] Let $x_{\min}$ and $x_{\max}$ denote the extreme values of $\sigma$, $s$ and $SR$. In particular, we set $\sigma_{\min} = 0.963\%$, $\sigma_{\max} = 2.163\%$, $s_{\min} = -0.798$, $s_{\max} = 0.987$, $SR_{\min} = 1\%$, and $SR_{\max} = 22.3\%$.[20] We need to generate four independent uniform random variables $U_j$, $j = 1, \ldots, 4$ on the interval $(0, 1)$, each with sample size $N = 10,000$. The realizations of these variables will be denoted as $u_{ji}$, $i = 1, \ldots, N$. We implement the following two steps for each portfolio $i$:

1. We get $\sigma_i = \sigma_{\min} + (\sigma_{\max} - \sigma_{\min})u_{1i}$ and $SR_i = SR_{\min} + (SR_{\max} - SR_{\min})u_{2i}$. Then, the mean is obtained as $\mu_i = r_f + \sigma_i SR_i$.
2. The skewness is obtained as $s_i = s_{\min} + (s_{\max} - s_{\min})u_{3i}$. Hence, $ek_i = ek_{i,\min} + (ek_{i,\max} - ek_{i,\min})u_{4i}$ such that $(ek_{i, \min}, s_i)$ and

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[17] The values for $\theta_{2, m}$ are, respectively, $\theta_{2,1} = -7.53 \times 10^{-4}$, $\theta_{2,2} = -2.18 \times 10^{-4}$ and $\theta_{2,3} = -1.87 \times 10^{-5}$. The curves for the isoKappas are not exhibited here for the sake of brevity.

[18] Suppose that we change the value for $\mu$ in Corollary 7. Then, the isoKappas depend on $\mu$, $s$ and $ek$. By using the implicit function Theorem, we can obtain the corresponding partial derivatives (evaluated at a certain point) to analyze the behavior of $\mu$ with respect to $s$ and $ek$. This extension is not shown here but it is available upon request.

[19] $N_T = N \times T$ where, as we will see later, $N$ denotes the sample size per regression and $T$ is the number of regressions. We set $N = 10,000$ and $T = 100$.

[20] The numbers for extreme values for standard deviation, skewness and $SR$ come from monthly returns of hedge funds from Mausser et al. (2006).

($ek_{i,\,\max}$, $s_i$) belong to the restricted space $\Gamma$, that is, ($ek_i$, $s_i$) $\in \Gamma$.[21]

Finally, we obtain the values for the PMs presented in Section 2 by inserting $\vartheta_i$ into the corresponding formulas and fixing $\tau = r_f$.

## 5.2. Rank correlations

We obtain the average of one hundred Spearman's rank correlations between $\pi_i$ and $SR_i$, such that each correlation is obtained through $N$ vectors ($\pi_i$, $SR_i$) computed for portfolios characterized by the vector $\vartheta_i$, as explained in Subsection 5.1. The mains results are as follows:

1. We compute the correlation between *SR* and $FT(r_f, q, m)$ in (28) for integer values $q$, $m \leq 6$ such that $m > q$ (reverse S-shaped utility function, see Corollary 2). Accordingly, the correlations never exceed 25%. For instance, the correlations for $FT(r_f, 2, 3)$ and $FT(r_f, 3, 4)$ are, respectively, 24.81% and 17.46%. Therefore, these PMs lead to quite different portfolio rankings with respect to the Sharpe ratio. These results are also supported by Eling et al. (2011) who analyze, among others, the behavior of the FTs. Nevertheless, for the Upside Potential ratio ($FT(r_f, 1, 2)$) which does not verify that $m > q$, the rank correlation is 62.95%.
2. The Sharpe–Omega ratio, $K(r_f, 1)$, exhibits a very high correlation of 97.90%. The same happens to both Sortino ratio, $K(r_f, 2)$, and Kappa 3, $K(r_f, 3)$, with high correlations of, respectively, 94.29% and 91.18%. This evidence may suggest no ranking difference with respect to the Sharpe ratio. The following Subsection provides a more robust analysis by splitting the total sample in two subsamples depending on the *SR* value. For instance, it will be shown that the ranking difference between $K(r_f, 1)$ and the *SR* increases with the level of *SR*.
3. The correlation between the *ASSR* (with $\varphi = 1$ in (6) assuming a CARA utility) and the *SR* is extremely high (99.75%). Similar results are also obtained by assuming a CRRA utility function.
4. Assuming a CRRA utility (with $\lambda$ as the relative risk aversion parameter), the correlation between the PMs based on the *CE* amount in (5) and the *SR* is 66.52% when $\lambda = 1$, 71.80% for $\lambda = 2$, 87.48% for $\lambda = 5$, and 94.35% for $\lambda = 7$. Similar results hold under the CARA utility function. In brief, for low risk averse investors, we find significant differences in ranking portfolios when comparing the *CE* and the *SR*.

A possible reason for the above high correlations, in particular those from the *ASSR* and Kappa measures, is that their numerator ($\mu - \tau$) is the same as in the *SR*. Hence, we might conclude that the risk adjustment included in the denominator of each measure does not change the final rankings in our first analysis. This evidence is also supported by Haas et al. (2012) for the same PMs but not for the Upside Potential ratio, whose numerator is not defined as $\mu - \tau$. This is why the above FT measures may exhibit much lower rank correlations.[22]

## 5.3. Effects of skewness and kurtosis on portfolio evaluation

We start studying the PMs from the S-S and FT families with higher rank correlations according to the results in Subsection 5.2. Later, we will consider the PMs based on *CE* and analyze the effects

of the higher moments on ranking portfolios for different types of risk-averse investors.

### 5.3.1. Results for PMs based on ASSR, both S-S and FT families
Following Zakamouline (2011), we propose two models:

1. The first model is defined as

$$\pi_i = \alpha_\pi SR_i^{\beta_\pi}, \quad \alpha_\pi, \beta_\pi > 0, \quad i = 1, \cdots, N, \tag{35}$$

where $\pi$ is a specific PM (for all the portfolios $i$) such that $\pi_i > 0$.[23] The portfolio $i$ is characterized by the parameter vector $\vartheta_i$. We estimate by ordinary least squares (OLS) the logarithm of Eq. (35) and obtain $R_{\pi,0}^2$, the (adjusted) $R^2$ statistic. If the estimates of $\alpha_\pi$ and $\beta_\pi$ are positive, $\pi_i$ is equivalent to $SR_i$ in the sense that both produce the same ranking. A high equivalence between both measures will be indicated by a high value of $R_{\pi,0}^2$.
2. The second model is given as

$$\pi_i = \alpha_\pi SR_i^{\beta_\pi} \exp\left(\beta_\pi^s s_i + \beta_\pi^{ek} ek_i + \varepsilon_{\pi,i}\right), \quad i = 1, \ldots, N, \tag{36}$$

where $\varepsilon_{\pi,\,i}$ is the error term according to $\pi$ and $\vartheta_i$. Note that $\beta_\pi^s$ and $\beta_\pi^{ek}$ are, respectively, the (relative) sensitivity of $\pi$ to the skewness and excess kurtosis of the portfolio return distribution. That is, $\beta_\pi^s = (\partial \pi / \partial s)/\pi$ and $\beta_\pi^{ek} = (\partial \pi / \partial ek)/\pi$. We estimate by OLS the following expression, obtained after applying logarithms to (36):

$$\log(\pi_i) = \log(\alpha_\pi) + \beta_\pi \log(SR_i) + \beta_\pi^s s_i + \beta_\pi^{ek} ek_i$$
$$+ \varepsilon_{\pi,i}, \quad i = 1, \ldots, N. \tag{37}$$

Let $R_{\pi,1}^2$ denote the (adjusted) $R^2$ statistics of the OLS regression in (37). If the estimates of $\beta_\pi^s$ and $\beta_\pi^{ek}$ were statistically significant, then both $s$ and $ek$ can affect the behavior of the PM given by $\pi$. Thus, this measure would produce a different portfolio ranking than that from the *SR*. It also means that $R_{\pi,1}^2 - R_{\pi,0}^2$ would become large and the Spearman's rank correlation coefficient between $\pi$ and *SR*, denoted as $R_S(\pi, SR)$, would be small. Otherwise, if the estimates of both parameters were not significant, then $R_{\pi,0}^2$ would be similar to $R_{\pi,1}^2$ and the Spearman's rank correlation would be larger.

Table 2 provides the OLS estimates of $\beta_\pi$, $\beta_\pi^s$ and $\beta_\pi^{ek}$, the rank correlation $R_S(\pi, SR)$ and the statistics $R_{\pi,0}^2$ and $R_{\pi,1}^2$ by considering five different PMs. Zakamouline (2011) shows[24] that a larger *SR* implies a lower $R_S(\pi, SR)$. A possible reason might be that the larger the Sharpe ratio, the larger the adjustment for non-normality of the portfolio return distribution by the selected PM. Hence, our simulation analysis aims to test this behavior by using two non-overlapping ranges for *SR* for each PM. Specifically, we take $SR_{\min}, SR_{\max}$ (see Subsection 5.1), we compute the mean for both values and set the intervals $J_1 \equiv [1\%, 11.65\%]$ and $J_2 \equiv (11.65\%, 22.3\%]$. We split each sample size $N$ in two parts, $N_1$ and $N_2$, and run two regressions with $\pi_i$ as dependent variable:

1. In the first regression, the independent variables are the vectors $(1, SR_i, s_i, ek_i)$ such that both $s_i$ and $ek_i$ come from $\vartheta_i$ and $SR_i = \left(\mu_i - r_f\right)/\sigma_i \in J_1$ where $\mu_i$ and $\sigma_i$ also belong to $\vartheta_i$.
2. The second regression is based on the remaining $N_2$ points such that each $SR_i \in J_2$.

The last column of this Table displays the Chow test (and its *p*-value). That is, testing the null hypothesis of no structural break (one regression) against the alternative one of structural

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**Table 2**
Some results and statistics from the regression analysis by using samples of simulated monthly returns from a GC distribution.

| log (Perf. Measure) | Sharpe ratio interval | beta log(Sharpe) | beta Skewness | beta Exc. Kurt. | Rank Corr. | R2 | R20 | Chow Test (p-value) |
|---|---|---|---|---|---|---|---|---|
| Omega–Sharpe | [1%, 11.65%] | 1.1888 | 0.0286 | 0.0536 | 0.9509 | 0.9996 | 0.8999 | 10484.41 |
|  | (11.65%, 22.3%] | 1.2855 | 0.0458 | 0.0570 | 0.8965 | 0.9993 | 0.7941 | (0.00) |
| Sortino | [1%, 11.65%] | 1.0993 | 0.1672 | -0.0046 | 0.8728 | 0.9984 | 0.7540 | 1012.26 |
|  | (11.65%, 22.3%] | 1.1480 | 0.1820 | -0.0077 | 0.7489 | 0.9966 | 0.5625 | (0.00) |
| Kappa 3 | [1%, 11.65%] | 1.0685 | 0.1974 | -0.0417 | 0.8059 | 0.9916 | 0.6439 | 95.29 |
|  | (11.65%, 22.3%] | 1.1017 | 0.2060 | -0.0457 | 0.6583 | 0.9853 | 0.4429 | (0.00) |
| FT(1,2) | [1%, 11.65%] | 0.2802 | 0.1365 | 0.0498 | 0.3842 | 0.9979 | 0.1642 | 9798.84 |
|  | (11.65%, 22.3%] | 0.4199 | 0.1437 | 0.0520 | 0.3736 | 0.9985 | 0.1558 | (0.00) |
| FT(2,3) | [1%, 11.65%] | 0.1634 | 0.3150 | 0.0222 | 0.1277 | 0.9984 | 0.0179 | 1823.73 |
|  | (11.65%, 22.3%] | 0.2426 | 0.3228 | 0.0164 | 0.1286 | 0.9986 | 0.0182 | (0.00) |

This Table exhibits the results of estimating by OLS the model in (37). All the regressions contain the same explanatory variables: a constant, the skewness, the excess kurtosis and the log of the Sharpe ratio. All the beta estimates (except the constant) are shown, respectively, in columns 3 to 5. Consider different performance measures (PM) (dependent variable) in column 1, then the regression is run twice. The total sample size (N=10,000) is divided in two parts having one regression per subsample. The criterion followed to split N depends on the size of the Sharpe ratio ($SR$) of the portfolio $i \in \{1, \cdots, N\}$, henceforth $SR_i$. The column 2 is the $SR$ interval, $SR_i \in [1\%, 11.65\%]$ or $SR_i \in (11.65\%, 22.3\%]$. There is a total of 10 different regressions. The column 6 is the Spearman's rank correlation between a certain PM and the $SR$ for each subsample. Columns 7 and 8 correspond, respectively, to the adjusted $R^2$ statistics of equaiton (37) and the logarithm of Eq. (35). The last column represents both the value and the $p$-value (in parenthesis) for the Chow-test with null hypothesis of no structural break (one regression) against the alternative one of structural break (two regressions). Each portfolio is characterized by the vector $(\mu_i, \sigma_i, s_i, ek_i)$ and simulated according to the procedure described in Subsection 5.1. Note that we repeat the above process a total of 100 times, so any value (or estimation) of the Table (from columns 3 to 9) is indeed the *mean* computed over 100 estimates.

break (two regressions). This experiment is repeated 100 times and Table 2 exhibits the *mean* values of the parameter estimates, rank correlations, etc.

The main results from this Table are as follows:

- $R^2$ **statistics, rank correlation and Chow test**
  Running two regressions is better than just one since the $p$-values for the Chow test are null for all the PMs. In other words, the larger the $SR$, the larger the sensitivity of any $\pi$ to the higher moments. For the PMs with very high rank correlations in Subsection 5.2 (*ASSR*, Omega–Sharpe ratio, Sortino, and Kappa 3), their rank correlations are lower under $J_2$. It holds that the *ASSR* (not reported in the Table) is the PM with the lowest difference between $J_1$ and $J_2$. Finally, $R^2_{\pi,1} - R^2_{\pi,0}$ becomes higher under $J_2$. On the contrary, as expected, the PMs with low rank correlation (such as $FT(r_f, 1, 2)$ and $FT(r_f, 2, 3)$) show correlations that are quite small and similar in both intervals. Furthermore, $R^2_{\pi,1} - R^2_{\pi,0}$ is now extremely large.
- **Behavior of OLS beta estimates**
  First, for all the PMs, the OLS estimates for the three betas are statistically significant at the 1% level and so, both skewness and excess kurtosis play significant roles in these measures. Note that we only study the effects of skewness (but not kurtosis) when the *ASSR* is considered. Second, $\widehat{\beta}_\pi$ and $\widehat{\beta}^s_\pi$ are always positive and larger in the regression under $J_2$. So, we can conclude that (a) a higher value for the $SR$ leads to a higher value for $\pi$ and (b) all the PMs are, as expected, in favour of positive skewness. Third, $\widehat{\beta}^{ek}_\pi$ is negative in the Sortino and Kappa 3 measures. Finally, $|\widehat{\beta}^s_\pi|$ is much higher than $|\widehat{\beta}^{ek}_\pi|$ in all the PMs except for the Omega–Sharpe ratio.

### 5.3.2. Results for PMs based on CE

We assume a CRRA utility function and analyze the impact of both higher moments on the behavior of the percentage change of the *CE* measure in (5) with respect to $CE_0$, a benchmark *CE* obtained under a normal distribution. We consider several types of risk-averse investors by setting $\lambda = 1, 2, \ldots, 8$ and denote the related *CE* as $CE(\lambda)$. In summary, we consider the following equation for any portfolio $i$, characterized by the vector $\vartheta_i$:

$$\frac{CE_i(\lambda) - CE_{0,i}(\lambda)}{CE_{0,i}(\lambda)} = \gamma_0 + \gamma_s s_i + \gamma_k k_i + \varepsilon_i, \quad i = 1, \cdots, N, \quad (38)$$

where $CE_i(\lambda)$ is obtained by considering both $s_i$ and $k_i$ ($=ek_i + 3$) from the simulated portfolio $i$ in Subsection 5.1, $CE_{0,i}$ is obtained

by assuming $s_i = ek_i = 0$ in $\vartheta_i$ and $\varepsilon_i$ denotes the error component. We estimate by OLS the equation (38), a different one to each type of investor (eight regressions). In the same way as in Subsection 5.3.1, we also test if there is a structural change, the experiment is repeated 100 times, we compute the mean values, etc. We obtain the following results:[25]

- The OLS estimates for $\gamma_s$ and $\gamma_k$ are always, respectively, positive and negative. Thus, a higher level of skewness (kurtosis) implies a higher (lower) difference between $CE_i(\lambda)$ and $CE_{0,i}(\lambda)$.
- The Chow test is rejected when $\lambda \geq 6$. In contrast, the $p$-value for $\lambda = 5$ is 9.84% and the p-values for lower values of $\lambda$ exhibit even more evidence of no structural change. In conclusion, the size of the Sharpe ratio really matters to higher risk-averse investors when ranking portfolios.
- The previous results can be reinforced as follows. We show that the mean of the ratio $\gamma_s^1/\gamma_s^2$ over the one hundred regressions goes from 0.83 ($\lambda = 6$) to 0.55 ($\lambda = 8$). Thus, the impact of skewness is higher in the interval $J_2$ where $SR$ is larger and it becomes higher when the investor is more risk-averse. A similar analysis shows that the ratio $\gamma_k^1/\gamma_k^2$ goes from 0.76 ($\lambda = 6$) to 0.43 ($\lambda = 8$) and we get the same conclusions as before.

### 5.4. Impact of investors' higher-order risk preferences on CE

In contrast to the previous analysis based on the impact of the higher-order moments on PMs, we study now the effects of higher-order risk preferences on the *CE*. Consider the following equation:

$$CE_i(\lambda) = \xi_0 + \xi_P \psi_{3i}(\lambda) + \xi_T \psi_{4i}(\lambda) + \epsilon_i, \ \lambda = 1, 2, \ldots, 8, \ i = 1, \ldots, N, \quad (39)$$

where $CE_i(\lambda)$ is the same dependent variable as in (38), $\lambda$ is the CRRA risk aversion parameter, the independent variables $\psi_{3i}(\lambda)$ and $\psi_{4i}(\lambda)$ denote, respectively, the prudence and temperance coefficients based on the power utility function, and $\epsilon_i$ is the error component. The coefficients for prudence and temperance are denoted as $\xi_P$ and $\xi_T$, respectively. We estimate by OLS the Eq. (39), a different one for each investor (eight different investors). More precisely, we run 100 regressions for each investor. It is verified that the higher the coefficient $\lambda$, the higher (lower) the sensitivity of the CE regarding prudence (temperance). For instance,

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[25] The details are not reported here but available upon request.

the average estimates are $\xi_P = -4.02$, $\xi_T = 0.85$ for $\lambda = 1$, $\xi_P = -0.74$, $\xi_T = 0.07$ for $\lambda = 5$, and $\xi_P = 0.66$, $\xi_T = -0.04$ for $\lambda = 8$. All the estimates are significant and the (adjusted) $R^2$ statistics are very high and decrease with $\lambda$. Thus, $R^2 = 0.99, 0.87$, and $0.86$ for, respectively, $\lambda = 1, 5$, and 8.

## 6. LPMs under the SNP density

We aim to provide closed-form expressions for LPMs under the semi-nonparametric (SNP) distributions introduced by Gallant and Nychka (1987).[26] Both SNP and GC densities are just the product of a standard normal density times a finite number of Hermite polynomials. We can be interested in this alternative distribution for two main reasons:

1. The SNP density shares the analytical tractability of the GC density, but it is always positive. By comparison, the GC density is restricted to be positive into the parameter set $\Gamma$ (see Section 3).
2. The SNP nests the GC density in the sense that it allows a higher flexibility in terms of skewness and excess kurtosis. Thus, $\Gamma$ is contained into a higher space under the SNP.[27]

However, in contrast to the GC density, the parameters implied in the SNP pdf do not correspond directly to the skewness and kurtosis of the distribution. In fact, this drawback is shared by other non-Gaussian distributions.

The SNP density of a random variable $x$ is defined as

$$h(x) = \frac{\phi(x)}{v'v} \left( \sum_{i=0}^{p} v_i H_i(x) \right)^2, \tag{40}$$

where $v = (v_0, v_1, \ldots, v_p)' \in \mathbb{R}^{p+1}$, $\phi(\cdot)$ denotes the pdf of a standard normal variable and $H_i(x)$ are the normalized Hermite polynomials from (20). Since $h(\cdot)$ is homogeneous of degree zero in $v$, we can either impose $v'v = 1$ or $v_0 = 1$ to solve the scale indeterminacy. By expanding the square term expression in (40), we arrive at Proposition 1 in León et al. (2009). For $p = 2$, we get

$$h(x) = \phi(x) \sum_{k=0}^{4} \gamma_k(v) H_k(x), \tag{41}$$

$$\gamma_0(v) = 1, \quad \gamma_1(v) = \frac{2v_1(v_0 + \sqrt{2}v_2)}{v'v},$$

$$\gamma_2(v) = \frac{\sqrt{2}(v_1^2 + 2v_2^2 + \sqrt{2}v_0 v_2)}{v'v},$$

$$\gamma_3(v) = \frac{2\sqrt{3}v_1 v_2}{v'v}, \quad \gamma_4(v) = \frac{\sqrt{6}v_2^2}{v'v}. \tag{42}$$

We are interested in an affine transformation $z^* = a(v) + b(v)x$ with $g_*(\cdot)$ as the density of $z^*$ verifying that $E_{g_*}[z^*] = 0$, $E_{g_*}[z^{*2}] = 1$. Hence, the location and scale parameters $a(v)$ and $b(v)$ are obtained as

$$a(v) = -\frac{E_h[x]}{\sqrt{V_h[x]}}, \quad b(v) = \frac{1}{\sqrt{V_h[x]}}, \tag{43}$$

where $E_h[x]$ and $V_h[x]$ denote, respectively, the mean and variance of $x$ with $h(\cdot)$ (defined in (41)) as pdf. Finally, we can express the stock return $r$ as

$$r = \mu + \sigma z^* = \mu + a\sigma + b\sigma x, \tag{44}$$

such that $l(r) = h(x)/(b\sigma)$ is the pdf of $r$. The mean and variance of $r$ are, respectively, $E_l[r] = \mu$ and $V_l[r] = \sigma^2$. The next Proposition shows the general LPM expression of $r$ with the SNP density.

**Proposition 4.** *Let $r$ be the stock return in (44) with pdf $l(r) = h(x)/(b\sigma)$ such that $h(x)$ is the SNP density in (41). The lower partial moment $LPM_l(\tau, m)$ of $r$ is given as*

$$LPM_l(\tau, m) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \kappa_0^{m-j} \kappa_1^j C_j, \tag{45}$$

*where $\kappa_0 = \tau - \mu - a\sigma$, $\kappa_1 = b\sigma$, the parameters $a$ and $b$ are defined in (43) and*

$$C_j = \sum_{i=0}^{4} \xi_i(v) B_{j+i}, \tag{46}$$

*such that $B_j = \int_{-\infty}^{\tau_+} x^j \phi(x) dx$, with general solution in (60) (see the Appendix), where $\tau_+ = \kappa_0/\kappa_1$ and $\xi_i(v)$ is given by the following expressions:*

$$\xi_0(v) = 1 - \gamma_2(v)/\sqrt{2} + 3\gamma_4(v)/\sqrt{4!},$$
$$\xi_1(v) = \gamma_1(v) - 3\gamma_3(v)/\sqrt{3!},$$
$$\xi_2(v) = \gamma_2(v)/\sqrt{2} - 6\gamma_4(v)/\sqrt{4!}, \tag{47}$$
$$\xi_3(v) = \gamma_3(v)/\sqrt{3!},$$
$$\xi_4(v) = \gamma_4(v)/\sqrt{4!},$$

*where the coefficients $\gamma_k(v)$ are defined in (42).*

**Proof.** See the Appendix. □

Given this result, it is easy to obtain the expressions of the S-S and FT families when the stock returns are driven by the SNP distribution.

## 7. Conclusions

We derive closed-form expressions for lower partial moment (LPM) and upper partial moment (UPM) measures under the Gram-Charlier (GC) density for stock returns. Since LPMs can be expressed as linear functions of both skewness and excess kurtosis ($s$ and $ek$), the behavior of the related performance measure (PM) indexes can be easily understood in most cases. Both the Farinelli–Tibiletti (FT) and Kappa measures are studied here.

A simulation analysis is also carried out for portfolio evaluation. We show that one-sided PMs can affect the portfolio ranking differently to the Sharpe ratio (SR) because of the PM sensitivity to the levels of $s$ and $ek$ implied in the portfolio returns. Note that the SR is fully compatible with normally distributed returns and, more general, with elliptical distributions of returns. Hence, the SR can lead to incorrect evaluations due to asymmetry and heavy tails under the GC distribution, see Jondeau and Rockinger (2001).

Finally, the semi-nonparametric (SNP) distribution by Gallant and Nychka (1987) is introduced here as a more flexible density than the GC distribution in terms of capturing more levels of $s$ and $ek$. Closed-form expressions for LPMs are also obtained under this distribution.

Several issues are left for further research. We could obtain efficient frontiers based on LPMs as alternative risk measures, see Cumova and Nawrocki (2011); 2014), and compare them with the Markowitz (1991) mean-variance approach. Alternative copula models, see Cherubini et al. (2004), could be assumed for dependence among the different stock returns under GC or SNP for their marginal distributions. Finally, in a similar way to Farinelli et al. (2008), we could also obtain optimal asset allocations with different PMs under a multivariate distribution framework.

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[26] See also Fenton and Gallant (1996) and Gallant and Tauchen (1999). León et al. (2009) analyze the statistical properties of the SNP distributions.

[27] See the skewness-kurtosis frontiers under the SNP distribution in Fig. 1 in León et al. (2009).

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## Appendix. Proofs

**Proof of Proposition 1.** Taking expectations in (20) with pdf $g(\cdot)$ (see equation (19)), we obtain

$$E_g[H_k(z)] = \sum_{n=0}^{[k/2]} a_{k,n} E_g[z^{k-2n}]. \tag{48}$$

We can easily show that, for $k \geq 5$, $E_g[H_k(z)] = 0$ and, then, we obtain recursively the values for $E_g[z^k]$ from this equation. After some algebra,[28] for the even powers of $z$, we obtain

$$E_g[z^{2i+4}] = \eta_{2i+4,0} + \eta_{2i+4,1}\,ek, \quad i = 1, 2, \ldots,$$

where, for $i = 1, 2, \ldots,$, we have

$$\eta_{2i+4,0} = (-1)^i i \, \frac{2a_{2i+4,i+1} + (i+1)a_{2i+4,i+2}}{2a_{2i+4,0}}$$

$$\eta_{2i+4,1} = (-1)^i \frac{a_{2i+4,i}}{a_{2i+4,0}}.$$

In a similar way, we get that the expected value for any odd power of $z$ is given by

$$E_g[z^{2i+1}] = \eta_{2i+1,2}\,s, \quad i = 1, 2, \ldots,$$

with

$$\eta_{2i+1,2} = (-1)^{i-1} \frac{a_{2i+1,i-1}}{a_{2i+1,0}}, \quad i = 1, 2, \ldots$$

where $a_{k,n}$ in the above expressions are defined in (21). Finally, we obtain the non-central moments of $r$ by using the binomial expansion. □

**Proof of Proposition 2.** Let $z$ be the standardized return of $r$ in (18) with pdf $g(z)$ in (19). Its distribution function is

$$
\begin{aligned}
F_{GC}(z; s, ek) &= \int_{-\infty}^{z} g(x)dx = \int_{-\infty}^{z} p(x)\phi(x)dx \\
&= \Phi(z) + \frac{s}{\sqrt{3!}} \int_{-\infty}^{z} H_3(x)\phi(x)dx \\
&\quad + \frac{ek}{\sqrt{4!}} \int_{-\infty}^{z} H_4(x)\phi(x)dx \\
&= \Phi(z) - \frac{s}{3\sqrt{2!}} H_2(z)\phi(z) - \frac{ek}{4\sqrt{3!}} H_3(z)\phi(z). \quad (49)
\end{aligned}
$$

where the last equality arises from the following relationship (see León et al. (2009) for details):

$$\int_{-\infty}^{a} H_k(x)\phi(x)dx = -\frac{1}{\sqrt{k}} H_{k-1}(a)\phi(a), \quad k \geq 1. \tag{50}$$

---

Let $f(r) = g(z)/\sigma$ be the pdf of $r$ and $F(r)$ be the associated cdf, then $F(r) = F_{GC}(r^*; s, ek)$ where $r^* = (r - \mu)/\sigma$ completes the proof. □

**Proof of Corollary 3.** We can rewrite (24) as

$$F(r) = B_0 + \frac{s}{\sqrt{3!}} A_{02} + \frac{ek}{\sqrt{4!}} A_{03} \tag{51}$$

where

$$B_0 = \Phi(r^*), \quad A_{02} = -\frac{1}{\sqrt{3}} H_2(r^*)\phi(r^*), \quad A_{03} = -\frac{1}{\sqrt{4}} H_3(r^*)\phi(r^*). \tag{52}$$

It holds that $B_0 > 0 \, \forall r^*$, $A_{02} > 0$ iff $|r^*| < 1$ and $A_{03} > 0$ iff $r^* \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$. Moreover, using the relations (a) $d\phi(x)/dx = -x\phi(x)$ and (b) $dH_k(x)/dx = \sqrt{k}H_{k-1}(x)$, we get

$$\frac{\partial A_{02}}{\partial r^*} > 0 \Leftrightarrow r^* \in (-\sqrt{3}, 0) \cup (\sqrt{3}, +\infty),$$

$$\frac{\partial A_{03}}{\partial r^*} > 0 \Leftrightarrow r^* \in (-\infty, -r_1^*) \cup (-r_2^*, r_2^*) \cup (r_1^*, +\infty),$$

where $r_1^* = \sqrt{3 + \sqrt{6}}$ and $r_2^* = \sqrt{3 - \sqrt{6}}$. Hence,

$$\frac{\partial F(r)}{\partial s} > 0 \Leftrightarrow |r^*| < 1,$$

$$\frac{\partial F(r)}{\partial ek} > 0 \Leftrightarrow r^* \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}),$$

and $\partial F(r)/\partial r^* > 0$ if $r^* \in (-r_2^*, 0) \cup (r_1^*, +\infty)$. Finally, the signs of the partial derivatives of $F(r)$ with respect to $\mu$ and $\sigma$ are obtained by applying the chain rule and using that $\partial r^*/\partial \mu < 0, \partial r^*/\partial \sigma > 0$ iff $\mu > r$. □

**Proof of Proposition 3.** Let $f$ and $g$ denote, respectively, the pdfs for $r$ and $z$ in (18). So, it is verified that $f(r) = g(z)/\sigma$ where $z = (r - \mu)/\sigma$. Then, we can rewrite (9) as

$$LPM_f(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\, dr = \int_{-\infty}^{\tau^*} (\tau - \mu - \sigma z)^m g(z)\, dz \tag{53}$$

where $\tau^* = (\tau - \mu)/\sigma$. If we apply the binomial expansion to $(\tau - \mu - \sigma z)^m$ in (53), then

$$LPM_f(\tau, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k I_k, \tag{54}$$

where $I_k = E_g[z^k | z < \tau^*]$ denotes the conditional expected value. Thus, for $k \geq 1$, we get

$$I_k = \int_{-\infty}^{\tau^*} z^k g(z)\, dz = B_k + \frac{s}{\sqrt{3!}} A_{k2} + \frac{ek}{\sqrt{4!}} A_{k3}, \tag{55}$$

where

$$B_k = \int_{-\infty}^{\tau^*} z^k \phi(z)dz, \quad A_{k2} = \int_{-\infty}^{\tau^*} z^k H_3(z)\phi(z)dz,$$

$$A_{k3} = \int_{-\infty}^{\tau^*} z^k H_4(z)\phi(z)dz. \tag{56}$$

By using (50), $A_{k2}$ and $A_{k3}$ can be expressed as

$$A_{k2} = \frac{1}{\sqrt{3!}} (B_{k+3} - 3B_{k+1}), \quad A_{k3} = \frac{1}{\sqrt{4!}} (B_{k+4} - 6B_{k+2} + 3B_k). \tag{57}$$

By plugging (55) into (54), we get

$$LPM_f(\tau, m) = LPM_n(\tau, m) + \frac{s}{\sqrt{3!}} \theta_{2,m} + \frac{ek}{\sqrt{4!}} \theta_{3,m},$$

where

$$\theta_{j,m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k A_{kj},$$

where $LPM_n(\tau, m) = E_n[(\tau - r)^m_+]$ such that $n(\cdot)$ is the pdf of the normal distribution with mean $\mu$ and standard deviation $\sigma$. Thus,

$$LPM_n(\tau, m) = \int_{-\infty}^{\tau^*} (\tau - \mu - \sigma z)^m \phi(z) dz$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k B_k. \quad (58)$$

Finally, to get the expression for $B_k$ in (56), we need the following result (available upon request):

$$z^k = \sum_{n=0}^{[k/2]} c_{k,n} H_{k-2n}(z), \quad (59)$$

where $c_{k,n} \in \mathbb{R}$ and $H_i(\cdot)$ is a Hermite polynomial. Note that (59) is just the inversion formula of (20). If we take (50) and (59), then

$$B_k = \int_{-\infty}^{\tau^*} z^k \phi(z) dz = \sum_{n=0}^{[k/2]} c_{k,n} \int_{-\infty}^{\tau^*} H_{k-2n}(z) \phi(z) dz$$

$$= \sum_{n=0}^{[k/2]} c_{k,n} \left[ \Phi(\tau^*) \mathbf{1}_{(k-2n=0)} \right.$$

$$\left. + \frac{1}{\sqrt{k-2n}} H_{k-2n-1}(\tau^*) \phi(\tau^*) \mathbf{1}_{(k-2n\geq 1)} \right], \quad (60)$$

where $\mathbf{1}_{(\cdot)}$ is the usual indicator function. $\square$

**Proof of Corollary 5.** The expressions of $LPM_n(\tau, m)$ in (30) and $\theta_{j,m}$ in (31) are easily obtained for $m \leq 3$ by using, respectively, the Eqs. (26) and (27) from Proposition 3. Since we need to obtain $A_{k2}$ and $A_{k3}$ in (57) for $k = 0, 1, 2, 3, 4$, then we must previously get $B_j$ in (56) for $j = 1, \ldots, 7$. This is straightforward by applying the condition (50). $\square$

**Proof of Proposition 4.** Let $h$ and $l$ denote, respectively, the pdfs for $x$ and $r$ in (41) and (44). So, it is verified that $l(r) = h(x)/(b\sigma)$. Then, we can rewrite (9) as

$$LPM_l(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m l(r) dr = \int_{-\infty}^{\tau_+} (\kappa_0 - \kappa_1 x)^m h(x) dx, \quad (61)$$

where $\kappa_0 = \tau - \mu - a\sigma, \kappa_1 = b\sigma > 0$ and $\tau_+ = \kappa_0/\kappa_1$. The expressions of $a$ and $b$ in (43) can be obtained from the first two unconditional moments of $x$, denoted as $\mu'_x(1)$ and $\mu'_x(2)$, from Lemma 1 in León et al. (2009). Thus, $E_h[x] = \mu'_x(1)$ is just $\gamma_1(v)$ in (42) and

$$\mu'_x(2) = \frac{2(v_1^2 + 2v_2^2 + \sqrt{2} v_2 v_0)}{v'v} + 1.$$

Then, $V_h[x] = \mu'_x(2) - \mu'_x(1)^2$. If we apply the binomial expansion to $(\kappa_0 - \kappa_1 x)^m$ in (61) and consider (41), we have

$$LPM_l(\tau, m) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \kappa_0^{m-j} \kappa_1^j \left[ B_j + \sum_{k=1}^{4} \gamma_k(v) G_{jk} \right] \quad (62)$$

where $B_j = \int_{-\infty}^{\tau_+} x^j \phi(x) dx$ and $G_{jk} = \int_{-\infty}^{\tau_+} x^j H_k(x) \phi(x) dx$. If we consider (20), then the expressions of $G_{jk}$, for $0 \leq j \leq m$ and $1 \leq k \leq 4$, are:

$$G_{j1} = B_{j+1}, \qquad G_{j2} = \frac{1}{\sqrt{2}} (B_{j+2} - B_j),$$
$$G_{j3} = \frac{1}{\sqrt{3!}} (B_{j+3} - 3B_{j+1}), \quad G_{j4} = \frac{1}{\sqrt{4!}} (B_{j+4} - 6B_{j+2} + 3B_j).$$
$$(63)$$

Note that, in the above expressions, $B_j$ are evaluated at the point $\tau_+$. By plugging (63) into (62), we arrive at the expressions (45)–(47). $\square$

# References

Auer, B.R., 2015. Does the choice of performance measure influence the evaluation of commodity investments? Int. Rev. Financial Anal. 38, 142–150.

Bawa, V.S., 1975. Optimal rules for ordering uncertain prospects. J. Financ. Econ. 2 (1), 95–121.

Bawa, V.S., Lindenberg, E.B., 1977. Capital market equilibrium in a mean-lower partial moment framework. J. Financ. Econ. 5 (2), 189–200.

Bernardo, A.E., Ledoit, O., 2000. Gain, loss and asset pricing. J. Political Econ. 108 (1), 144–172.

Biglova, A., Ortobelli, S., Rachev, S., Stoyanov, S., 2004. Different approaches to risk estimation in portfolio theory. J. Portfolio Manage. 31 (1), 103–112.

Caporin, M., Jannin, G.M., Lisi, F., Maillet, B.B., 2014. A survey on the four families of performance measures. J. Econ. Surv. 28 (5), 917–942.

Cherubini, U., Luciano, E., Vecchiato, W., 2004. Copula Methods in Finance. John Wiley & Sons.

Cogneau, P., Hübner, G., 2009. The (more than) 100 ways to measure portfolio performance. part 2: special measures and comparison. J. Perform. Meas. 14 (1), 56–69.

Corrado, C., Su, T., 1996. Skewness and kurtosis in s&p 500 index returns implied by option prices. J. Financial Res. 19 (2), 175–192.

Cumova, D., Nawrocki, D., 2011. A symmetric LPM model for heuristic mean-semi-variance analysis. J. Econ. Bus. 63 (3), 217–236.

Cumova, D., Nawrocki, D., 2014. Portfolio optimisation in an upside potential and downside risk framework. J. Econ. Bus. 71, 68–89.

Ebert, S., 2013. Moment characterization of higher-order risk preferences. Theory Decis. 74 (2), 267–284.

Eeckhoudt, L., Schlesinger, H., 2006. Putting risk in its proper place. Am. Econ. Rev. 96 (1), 280–289.

Eling, M., 2008. Does the measure matter in the mutual fund industry? Financial Analysts J. 64 (3), 54–66.

Eling, M., Farinelli, S., Rossello, D., Tibiletti, L., 2011. One-size or tailor-made performance ratios for ranking hedge funds. J. Derivatives Hedge Funds 16 (4), 267–277.

Eling, M., Schuhmacher, F., 2007. Does the choice of performance measure influence the evaluation of hedge funds? J. Banking Finance 31 (9), 2632–2647.

Farinelli, S., Ferreira, M., Rossello, D., Thoeny, M., Tibiletti, L., 2008. Beyond sharpe ratio: optimal asset allocation using different performance ratios. J. Banking Finance 32 (10), 2057–2063.

Farinelli, S., Tibiletti, L., 2008. Sharpe thinking in asset ranking with one-sided measures. Eur. J. Oper. Res. 185 (3), 1542–1547.

Fenton, V., Gallant, A.R., 1996. Qualitative and asymptotic performance of SNP density estimators. J. Econ. 74 (1), 77–118.

Fishburn, P.C., 1977. Mean-risk analysis with risk associated with below-target returns. Am. Econ. Rev. 67 (2), 116–126.

Gallant, A.R., Nychka, D.W., 1987. Semi-nonparametric maximum likelihood estimation. Econometrica 55 (2), 363–390.

Gallant, A.R., Tauchen, G., 1999. The relative efficiency of method of moment estimators. J. Econ. 92 (1), 149–172.

Guo, B., Xiao, Y., 2016. A note on why doesn't the choice of performance measure matter? Finance Res. Lett. 16, 248–254.

Haas, J.R., Silva, A.F., Barros, J.L., 2012. Yes, the choice of performance measure does matter for ranking of US mutual funds. Int. J. Finance Econ. 17 (1), 61–72.

Harlow, W., 1991. Asset allocation in a downside risk framework. Financial Analysts J. 47 (5), 28–40.

Harlow, W.V., Rao, R.K.S., 1989. Asset pricing in a generalised mean-lower partial moment framework: theory and evidence. J. Financial Quant. Anal. 24 (3), 285–311.

He, X.D., Zhou, X.Y., 2011. Portfolio choice under cumulative prospect theory: an analytical treatment. Manage. Sci. 57 (2), 315–331.

Hodges, S., 1998. A Generalization of the Sharpe Ratio and Its Applications to Valuation Bounds and Risk Measures. Working paper Financial Options Research Centre, 88. University of Warwick.

Holthausen, D.M., 1981. A risk-return model with risk and return measured as deviations from a target. Am. Econ. Rev. 71 (1), 182–188.

Jondeau, E., Rockinger, M., 2001. Gram-charlier densities. J. Econ. Dyn. Control 25 (10), 1457–1483.

Jurczenko, E., Maillet, B., 2006. Multi-moment Asset Allocation and Pricing Models. Wiley Finance.

Kahneman, D., Tversky, A., 1979. Prospect theory: an analysis of decision under risk. Econometrica 47 (2), 263–291.

Kaplan, P.D., Knowles, J.A., 2004. Kappa: a generalized downside risk adjusted performance measure. J. Perform. Meas. 8 (3), 42–54.

Keating, C., Shadwick, W.F., 2002. A universal performance measure. J. Perform. Meas. 6 (3), 59–84.

Kendall, M., Stuart, A., 1977. The Advanced Theory of Statistics, Volume 1: Distribution Theory, fourth edition Griffin.

Kimball, M.S., 1990. Precautionary saving in the small and in the large. Econometrica 58 (1), 53–73.

Kimball, M.S., 1992. Precautionary motives for holding assets. The New Palgrave Dictionary of Money and Finance.

Köbberling, V., Wakker, P.P., 2005. An index of loss aversion. J. Econ. Theory 122 (1), 119–131.

Le Courtois, O., 2012. On prudence and temperance. Illustration with Monoperiodic Portfolio Optimization. Cahier de Recherche Working Paper 2012/01, EMLYON Business School.

León, A., Mencía, J., Sentana, E., 2009. Parametric properties of semi-nonparametric distribution, with applications to option valuation. J. Bus. Econ. Stat. 27 (2), 176–192.

Levy, M., Levy, H., 2002. Prospect theory: much ado about nothing? Manage. Sci. 48 (10), 1334–1349.

Markowitz, H., 1991. Portfolio Selection. Efficient Diversification of Investments, second edition John Wiley & Sons.

Mausser, H., Saunders, D., Seco, L., 2006. Risk. 88–92.

Owen, J., Rabinovitch, R., 1983. On the class of elliptical distributions and their applications to the theory of portfolio choice. J. Finance 38 (3), 745–752.

Passow, A., 2005. Omega portfolio construction with johnson distributions. Risk 18 (4), 85–90.

Pedersen, C.S., Satchell, S.E., 2002. On the foundation of performance measures under asymmetric returns. Quant. Finance 2 (3), 217–223.

Schuhmacher, F., Eling, M., 2011. Sufficient conditions for expected utility to imply drawdown-based performance rankings. J. Banking Finance 35 (9), 2311–2318.

Schuhmacher, F., Eling, M., 2012. A decision-theoretic foundation for reward-to-risk performance measures. J. Banking Finance 36 (7), 2077–2082.

Sharpe, W.F., 1966. Mutual fund performance. J. Bus. 39 (1), 119–138.

Sharpe, W.F., 1994. The sharpe ratio. J. Portfolio Manage. 21 (1), 49–58.

Sortino, F.A., Van der Meer, R., 1991. Downside risk. J. Portfolio Manage. 17 (4), 27–31.

Sortino, F.A., Van der Meer, R., Platinga, A., 1999. The dutch triangle. J. Portfolio Manage. 26 (1), 50–57.

Sortino, F.A., Satchell, S., 2001. Managing Downside Risk in Financial Markets. Butterworth Heinemann, Oxford.

Tversky, A., Kahneman, D., 1992. Advances in prospect theory: cumulative representation of uncertainty. J. Risk Uncertain. 5 (4), 297–323.

Zakamouline, V., 2011. The performance measure you choose influences the evaluation of hedge funds. J. Perform. Meas. 15 (3), 48–64.

Zakamouline, V., 2014. Portfolio performance evaluation with loss aversion. Quant. Finance 14 (4), 699–710.

Zakamouline, V., Koekebakker, S., 2009a. Portfolio performance evaluation with generalized sharpe ratios: beyond the mean and variance. J. Banking Finance 33 (7), 1242–1254.

Zakamouline, V., Koekebakker, S., 2009b. A generalization of the mean-variance analysis. Eur. Financial Manage. 15 (5), 934–970.