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Unexpected shortfalls of Expected Shortfall: Extreme default profiles and regulatory arbitrage

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ABSTRACT

The purpose of this paper is to dispel some common misunderstandings about capital adequacy rules based on Expected Shortfall. We establish that, from a theoretical perspective, Expected Shortfall based regulation can provide a misleading assessment of tail behavior, does not necessarily protect liability holders' interests much better than Value-at-Risk based regulation, and may also allow for regulatory arbitrage when used as a global solvency measure. We also show that, for a value-maximizing financial institution, the benefits derived from protecting its franchise may not be sufficient to disincentivize excessive risk taking. We further interpret our results in the context of portfolio risk measurement. Our results do not invalidate the possible merits of Expected Shortfall as a risk measure but instead highlight the need for its cautious use in the context of capital adequacy regimes and of portfolio risk control.

1. Introduction

The main objective of financial regulation has traditionally been to ensure the safety of individual financial institutions with the declared aim of protecting the institution's liability holders. Regulation with this focus is known as microprudential regulation. However, at the very latest since the financial crisis of 2007–2009, the scope of regulation has been enlarged to include macroprudential objectives, which aim at securing the stability of the financial system as a whole. Although regulators have a wide variety of instruments to help them meet their objectives, setting capital requirements continues to be a central pillar of prudential regulation. The two most prominent capital adequacy tests used by financial regulators are those based on Value-at-Risk (VaR) and Expected Shortfall (ES), respectively. In the banking sector, the Basel framework still relies mainly on VaR, even though the forthcoming third Basel Accord is set to adopt ES for the assessment of market risk. In the insurance sector, Solvency II prescribes a VaR test, while the Swiss Solvency Test is based on ES. Sometimes, depending on the situation, the emphasis may not be on controlling default behavior but rather on constraining portfolio risk. Hence, although this paper takes primarily a capital adequacy perspective for concreteness, we have made an effort to provide in various remarks an interpretation of our results in terms of portfolio risk.

There is widespread agreement that, from a theoretical perspective, VaR performs poorly in supporting regulatory objectives because of two fundamental deficiencies: the structural blindness to the "far tail" of capital positions and the lack of subadditivity. Here, by "far tail" of a capital position we mean the tail beyond the quantile at the relevant level of confidence. While the tail blindness allows a firm to assume uncontrolled exposures in the "far tail", the lack of subadditivity implies that a VaR test fails to give credit for diversification. The theory of *coherent* risk measures articulated in Artzner et al. (1999) was developed to provide a sound theoretical basis for designing capital adequacy tests beyond the VaR paradigm. Within this framework, ES has emerged as an attractive alternative to VaR because it is not blind to the "far tail" and because it captures diversification. In this paper we will not elaborate on the difference between VaR and ES in terms of their ability to give credit for diversification. Instead, we will focus exclusively on the way ES captures tail risk, which is essentially by taking the expectation of a capital position, or a portfolio's profit and loss, over the "far tail". By doing so, it is argued, ES accounts

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not only for the probability of default, but also for the average magnitude of potential losses. This constitutes one of the key reasons given by the Basel Committee on Banking Supervision (2012) for moving from VaR to ES, as stated in the Consultative Document issued in May 2012 (see p. 20):

> The current framework's reliance on VaR as a quantitative risk metric stems largely from historical precedent and common industry practice. This has been reinforced over time by the requirement to use VaR for regulatory capital purposes. However, a number of weaknesses have been identified with VaR, including its inability to capture "tail risk". The Committee therefore believes it is necessary to consider alternative risk metrics that may overcome these weaknesses. Expected Shortfall [...] is an example of a risk metric that considers a broader range of potential outcomes than VaR. Unlike VaR, ES measures the riskiness of an instrument by considering both the size and likelihood of losses above a certain threshold [...]. In this way, ES accounts for tail risk in a more comprehensive manner.

In the same document (p. 41) the following question was raised:

> What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?

The above question revived the discussion about the relative merits of VaR and ES. However, this time the focus was more on the statistical properties of VaR and ES and less on theoretical aspects. That discussion has not been conclusive and has led to the articulation of a wide range of partly contradictory assessments. We refer to the survey article by Embrechts et al. (2014) and the references therein for a comprehensive presentation of most of the recent contributions to this debate.

In this paper we would like to enlarge the scope of this debate by addressing the topic from a more financial theoretical perspective. In particular, we aim to assess the suitability of ES as a regulatory risk measure *per se* by asking the following question:

How good is Expected Shortfall in helping meet both the micro- and the macroprudential objectives of financial regulation?

We answer this question by highlighting a variety of features of ES that have not yet been addressed in the literature. While these features may not be surprising from a mathematical perspective, they do lead to surprising anomalies when interpreted from a financial perspective. The root cause of all of these anomalies is the particular way in which ES takes tail behavior into account: averages are poor indicators of risk, as exemplified by the adage of the man who drowned in a river with an average depth of one centimeter. These anomalies do not diminish the merits of ES as a risk measure but do provide an argument against its indiscriminate use. In the context of capital adequacy, our findings can be summarized in the following list of "unexpected shortfalls":

1. **An** ES **based test does not always distinguish between the interests of liability holders and those of the owners of the financial institution (Section 3).** Tail behavior should not be equated with default behavior since the far tail of the capital position of an institution may contain states in which the institution defaults and states where it has a surplus. Hence, by averaging across the entire far tail, ES mixes the interests of liability holders and owners of the institutions by allowing the compensation of losses suffered by the former by gains enjoyed by the latter.
2. **An** ES **based test may accept a position in which liability holders are worse off than in a position which is not accepted (Section 4).** It is possible that a capital position which is acceptable under ES exhibits a higher default probability and higher losses than a position which is deemed unacceptable. Hence, liability holders may be better off in the institution that does not pass the capital adequacy test and worse off in the one that passes it.
3. ES **based tests are compatible with the same default behaviors as VaR based tests at a higher level of confidence (Section 5).** Consider an ES test with a given level of confidence and a VaR test with a strictly higher level of confidence. Given any position that is acceptable under the VaR test, we can find a position with identical default behavior that is acceptable under the ES test.[1] Moreover, there are positions that are not accepted by the VaR test but are accepted by the ES test with arbitrarily negative capital requirements. In particular, the Swiss Solvency Test accepts more extreme default behaviors than Solvency II does. Similarly, the switch from VaR to ES in the assessment of market risk in Basel III allows for more extreme loss profiles.
4. ES **may allow for regulatory arbitrage when used as a global regulatory measure (Section 6).** The outcome of a capital adequacy test based on ES depends on the underlying unit of account. Hence, an unacceptable company may become acceptable by a mere change of the currency in which risks are aggregated. Unless the aggregation currency is fixed, otherwise identical institutions may be assessed differently only because they have adopted different aggregation currencies. Hence, a global solvency regime based on ES performed in the various local currencies would create regulatory arbitrage opportunities and incentivize unacceptable firms to move to a jurisdiction where they may become acceptable without the need of restructuring their balance sheet.
5. ES **may not prevent institutions from increasing firm value by engaging in excessive risk taking (Section 7).** By allowing extreme default profiles, ES makes it possible to increase the value of the institution by increasing the value of the default option which is an asset to the owners of the institution. It is sometimes argued that the franchise value – the value put on the ability of the institution to write profitable business in the future – acts as a deterrent for excessive risk taking. This is because, typically, franchise value can only be realized if the institution stays in operation and not if it defaults. In spite of this, the increase in value of the default option due to an increase in tail risk may more than outweigh the potential damage to the institution's franchise. Hence, the owners of the institution may have an incentive to build up an extreme default profile.

The "unexpected shortfalls" of ES described above highlight important aspects that had so far been absent from the "VaR vs ES" debate. While critical of ES, these "unexpected shortfalls" by no means invalidate the critique of VaR, as we try to emphasize throughout the paper. However, what the discussion above does is to qualify the benefits of switching from VaR to ES. Indeed, the implications of taking account of tail behavior through an averaging process seem to indicate that moving to ES does not offer a fundamentally better control on what liability holders may end up losing in case of default. At the same time, when assessing capital adequacy with ES some new, counterintuitive phenomena arise that are not immediately recognizable from the ES figures themselves and which may eventually undermine an effective regulation. It would seem that a VaR test is less "deceiving" than an ES test because it does not purport to contain any information about tail risk, while, by inadequately capturing tail risk, an ES test may lull us into a false sense of security. But, ultimately, whether to use

[1] Note that this does not mean that every position accepted by the VaR test is also accepted by the ES test, which is clearly not true.

VaR or ES remains a matter of judgement. Our work aims at enabling a more informed way of exercising that judgement by, on the one side, increasing the transparency with respect to the behaviors that are compatible with ES based regimes and, on the other, by qualifying from a theoretical perspective some of the key benefits typically associated with ES. In this paper we have worked in a stylized setting which blends out many of the concrete features of solvency regimes. Nevertheless, we believe that our work constitutes a necessary first step to systematically investigate the important issue of materiality, a task that requires taking into account the practical limitations to exploiting the above anomalies.

2. Testing for capital adequacy under VaR and ES

We consider a one-period economy with initial date $t = 0$ and terminal date $t = T$ where terminal uncertainty is modeled by a fixed nonatomic[2] probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The (terminal) *capital position* of a financial institution, i.e. the terminal value of the company's assets net of liabilities, is represented by a random variable $X : \Omega \to \mathbb{R}$. The set of capital positions is denoted by $\mathcal{X}$ and is assumed to be $L^1(\Omega, \mathcal{F}, \mathbb{P})$, the vector space of integrable $\mathcal{F}$-measurable random variables with the usual identification of functions that coincide $\mathbb{P}$-almost surely. It is important to bear in mind that financial positions are expressed in a *pre-specified unit of account*, e.g. some fixed currency. The *surplus* and the *option to default* of a capital position X are defined, respectively, by

$$S_X := \max\{X, 0\} \quad \text{and} \quad D_X := -\min\{X, 0\}.$$

We will only consider financial institutions with limited liability. Consequently, the surplus S_X is the excess of available funds over the amount needed to meet liabilities and thus represents the payoff to shareholders at time $t = T$. On the other side, the option to default D_X is the amount by which available funds fall short of the amount needed to meet liabilities. Hence, it represents the difference between the contractual payment and the actual payment to liability holders or the amount on which the institution will default on its obligations to liability holders. Since S_X and D_X are just the positive, respectively negative, part of the random variable X, we can always write

$$X = X1_{\{X>0\}} + X1_{\{X<0\}} = S_X - D_X,$$

where 1_A denotes the indicator function of $A \in \mathcal{F}$, i.e. 1_A takes the value 1 on the event A and the value 0 on its complement A^c.

Remark 2.1. The choice of the ambient space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is optimal in the sense that it is the largest space on which ES is finite. If we were interested only in VaR one could also work on $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the space of $\mathcal{F}$-measurable random variables, that is, without asking for any integrability properties. Note also that since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, the space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to support all distributions that are relevant in practice.

2.1. Capital adequacy under Value-at-Risk

For a fixed probability level $\alpha \in (0, 1)$, the *Value-at-Risk* (VaR) of a capital position X at the level α is defined by

$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R};\ \mathbb{P}[X + m < 0] \leqslant \alpha\}.$$

The *capital adequacy test* based on VaR at level α can be formalized in terms of the *acceptance set*

$$\mathcal{A}_{\mathrm{VaR}}(\alpha) := \{X \in \mathcal{X};\ \mathrm{VaR}_\alpha(X) \leq 0\}.$$

In other words, a financial institution with position X is said to be *adequately capitalized*, or *acceptable*, with respect to VaR at the level α provided that $X \in \mathcal{A}_{\mathrm{VaR}}(\alpha)$. This is equivalent to requiring that the institution is solvent with probability higher than the *confidence level* $1 - \alpha$ or, equivalently, that its default probability is lower than α, i.e.

$$X \in \mathcal{A}_{\mathrm{VaR}}(\alpha) \iff \mathbb{P}[X \geqslant 0] \geqslant 1 - \alpha \iff \mathbb{P}[X < 0] \leqslant \alpha.$$

The capital adequacy test defined by $\mathcal{A}_{\mathrm{VaR}}(\alpha)$ can be linked to capital requirements. Before doing so we recall three important properties of VaR, called monotonicity, cash-additivity, and positive homogeneity, respectively:

(M) $X \leqslant Y \Rightarrow \mathrm{VaR}_\alpha(X) \geqslant \mathrm{VaR}_\alpha(Y)$;
(CA) $\mathrm{VaR}_\alpha(X + m) = \mathrm{VaR}_\alpha(X) - m$ for every $m \in \mathbb{R}$;
(PH) $\mathrm{VaR}_\alpha(\lambda X) = \lambda \mathrm{VaR}_\alpha(X)$ for every $\lambda \geqslant 0$.

Using the definition, or the property of cash-additivity, the number $\mathrm{VaR}_\alpha(X)$ can be interpreted as a measure of *required capital* in the following sense: when positive, $\mathrm{VaR}_\alpha(X)$ is the minimum amount of cash that needs to be added to the capital position X to make it acceptable. When negative, it is the maximum amount of cash that can be extracted maintaining the acceptability of X. If we view $\mathrm{VaR}_\alpha(X)$ as a capital requirement, the above mentioned properties of VaR have a clear financial interpretation. Monotonicity reflects the basic intuition that a company with a worse net profile should be subject to a higher capital requirement. Cash-additivity implies that if the amount m of cash has already been added to the capital position X, it is only necessary to add the difference $\mathrm{VaR}_\alpha(X) - m$ to make the position X acceptable. Finally, positive homogeneity means that if the capital positions of two institutions differ by a multiple, then capital requirements also differ by the same multiple.

2.2. Capital adequacy under Expected Shortfall

For a fixed probability level $\alpha \in (0, 1)$, the *Expected Shortfall* (ES) of a position X at the level α is defined as[3]

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta.$$

For obvious reasons, Expected Shortfall is sometimes referred to as *Average Value-at-Risk*. As in the case of VaR, we refer to $1 - \alpha$ as the *level of confidence*.

There are several equivalent ways to express ES. A useful representation is[4]

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\mathbb{E}[-X1_{\{X<-\mathrm{VaR}_\alpha(X)\}}] + \frac{\alpha - \mathbb{P}[X < -\mathrm{VaR}_\alpha(X)]}{\alpha}\mathrm{VaR}_\alpha(X). \tag{2.1}$$

This representation highlights the link between $\mathrm{ES}_\alpha(X)$ and the conditional expectation of X "beyond" the level $-\mathrm{VaR}_\alpha(X)$, i.e. the so-called *Conditional Tail Expectation*[5] (CTE)

$$\mathrm{CTE}_\alpha(X) := \mathbb{E}[-X \mid X < -\mathrm{VaR}_\alpha(X)].$$

[2] The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called nonatomic if no atoms exist. Recall that $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ is an *atom* if for any measurable $B \subset A$ we have either $\mathbb{P}[B] = 0$ or $\mathbb{P}[B] = \mathbb{P}[A]$. Note that these spaces support random variables with arbitrary distribution functions. For a useful list of properties of nonatomic spaces see Delbaen (2012).

[3] Recall that $\mathrm{ES}_\alpha(X)$ is a well-defined real number if and only if the negative part X^- is integrable. For this reason we have opted to work with integrable positions.

[4] We refer to Acerbi and Tasche (2002), Föllmer and Schied (2011) or Pflug and Römisch (2007) for a detailed treatment of ES and other equivalent representations.

[5] For a review of properties of CTE see for instance Dhaene et al. (2006).

Indeed, we have

$$\mathrm{ES}_\alpha(X) = \frac{\mathbb{P}[X < -\mathrm{VaR}_\alpha(X)]}{\alpha}\mathrm{CTE}_\alpha(X) + \frac{\alpha - \mathbb{P}[X < -\mathrm{VaR}_\alpha(X)]}{\alpha}\mathrm{VaR}_\alpha(X).$$

Note that, for a position X with *continuous* distribution, the correction term on the right-hand side vanishes and the previous identity reduces to

$$\mathrm{ES}_\alpha(X) = \mathrm{TCE}_\alpha(X) = \frac{1}{\alpha}\mathbb{E}[-X1_{\{X<-\mathrm{VaR}_\alpha(X)\}}].$$

Remark 2.2. In our setting it is natural to define, as in Föllmer and Schied (2011), VaR and ES for (net) capital positions of financial institutions. However, it is also quite common, as done for instance in McNeil et al. (2005), to define VaR for the (net) loss L of an institution or of a portfolio. In that context, one defines $\mathrm{VaR}_\alpha(L) := \inf\{l \in \mathbb{R};\ \mathbb{P}[L \leqslant l] \geqslant \alpha\}$ and bases the definition of ES_α on this convention. In fact, this is the convention usually adopted in practice. As a result, what we call VaR or ES at the level α is often referred to as VaR or ES at the level $1-\alpha$ (for a position with the opposite sign). This should cause no confusion since throughout this paper we adhere to our terminology. However, when comparing with the literature, the reader needs to be mindful of this terminological difference.

Remark 2.3. We point out another situation where ES and CTE coincide, which will prove useful later on. Fix $\alpha \in (0,1)$ and a measurable set $A \subset \Omega$ with $\mathbb{P}[A] = \alpha$. Let X and Y be random variables in $\mathcal{X}$ satisfying $\mathrm{essinf}_{A^c}(Y) > \mathrm{esssup}_A(X)$, where

$$\mathrm{esssup}_A(X) := \inf\{c \in \mathbb{R};\ X1_A \leqslant c1_A \text{ almost surely}$$

and

$$\mathrm{essinf}_{A^c}(Y) := \sup\{c \in \mathbb{R};\ Y1_{A^c} \geqslant c1_{A^c} \text{ almost surely}.$$

Then, the position

$$Z := X1_A + Y1_{A^c}$$

satisfies

$$\mathrm{ES}_\alpha(Z) = \mathrm{CTE}_\alpha(Z) = \frac{1}{\alpha}\mathbb{E}[-X1_A]. \tag{2.2}$$

Indeed, if $y := \mathrm{essinf}_{A^c}(Y)$, then it is easy to see that $\mathbb{P}[Z < y] = \mathbb{P}(A) = \alpha$ and $\mathbb{P}[Z<y+\varepsilon]=\mathbb{P}(A)+\mathbb{P}[Y1_{A^c}<(y+\varepsilon)1_{A^c}]>\alpha$ for any $\varepsilon>0$. This implies that $\mathrm{VaR}_\alpha(Z)=-y$ and, hence, that $\{Z<-\mathrm{VaR}_\alpha(Z)\}=A$. The identities in (2.2) now follow from the ES representation (2.1).

As in the case of VaR, we can formalize the *capital adequacy test* based on ES at the level α in terms of the *acceptance set*

$$\mathcal{A}_{\mathrm{ES}}(\alpha) := \{X \in \mathcal{X};\ \mathrm{ES}_\alpha(X) \leq 0\}.$$

A financial institution with position X is deemed *adequately capitalized*, or *acceptable*, with respect to ES at the level α provided that $X \in \mathcal{A}_{\mathrm{ES}}(\alpha)$. By the above representation, for a position X with a continuous distribution we have

$$X \in \mathcal{A}_{\mathrm{ES}}(\alpha) \iff \mathbb{E}[X1_{\{X<-\mathrm{VaR}_\alpha(X)\}}] \geqslant 0,$$

showing that acceptability based on ES is equivalent to requiring that, on the left tail beyond the threshold $-\mathrm{VaR}_\alpha(X)$, the company is "solvent on average". Since $\mathrm{ES}_\alpha(X) \geqslant \mathrm{VaR}_\alpha(X)$ for every position X, acceptability based on ES is more stringent than VaR acceptability, i.e. $\mathcal{A}_{\mathrm{ES}}(\alpha) \subset \mathcal{A}_{\mathrm{VaR}}(\alpha)$.

Remark 2.4. Fix $\alpha \in (0,1)$. Since $\mathrm{VaR}_\alpha(X) \leqslant \mathrm{ES}_\alpha(X)$ for every $X \in \mathcal{X}$, any position that is accepted by the ES_α test is also accepted by the VaR_α test. In fact, any position X accepted by the ES_α test must also be accepted by a VaR_β test for some $\beta \in (0,\alpha)$. Indeed, if X is accepted by the ES_α test and $\mathbb{P}[X<0] \geqslant \alpha$, we immediately obtain that $\mathrm{VaR}_\beta(Y) > 0$ for every $\beta \in (0,\alpha)$. This implies that

$$\mathrm{ES}_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta > 0,$$

contradicting the fact that $X \in \mathcal{A}_{\mathrm{ES}}(\alpha)$. It follows that $\mathbb{P}[X<0] < \alpha$ proving that X is accepted by VaR_β with $\beta := \mathbb{P}[X<0] < \alpha$.

It is easy to see that ES shares with VaR the properties of monotonicity, cash-additivity and positive homogeneity introduced above. Hence, analogously to the case of VaR acceptability, the number $\mathrm{ES}_\alpha(X)$ can be interpreted as *required capital*: when positive, $\mathrm{ES}_\alpha(X)$ is the minimum amount of cash that needs to be added to the capital position X to make it acceptable. When negative, it is the maximum amount of cash that can be extracted maintaining the acceptability of X. In contrast to VaR, however, ES is also subadditive, i.e. we have:

(SA) $\mathrm{ES}_\alpha(X+Y) \leqslant \mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(X)$.

In the context of capital requirements, subadditivity means that the capital required after merging two companies is less than or equal to the sum of the capital requirements for the individual companies. In this sense one can say that ES gives credit for diversification. The failure of subadditivity for VaR constitutes one of the major criticisms of VaR and is one of the reasons why, for capital adequacy purposes, ES is viewed as being superior to VaR.

2.3. Capital adequacy tests in practice

In practice capital adequacy tests are often formulated in terms of available and required capital. To see that this is equivalent to the way we have defined them, assume $\rho : \mathcal{X} \to \mathbb{R}$ is either of the functionals VaR_α or ES_α for some $\alpha \in (0,1)$. The capital position $X \in \mathcal{X}$ of an institution can be written as

$$X = X_0 + \Delta X$$

where $X_0 \in \mathbb{R}$ represents the capital position at time 0 and $\Delta X : \Omega \to \mathbb{R}$ the profit and loss for the period from 0 to T. Using the cash additivity of ρ, the test

$$\rho(X) \leqslant 0$$

is easily seen to be equivalent to

$$\rho(\Delta X) \leqslant X_0.$$

In this context, X_0 is referred to as *available capital* and $\rho(\Delta X)$ as *required capital*. Hence, the capital adequacy test reduces to showing that available capital is greater or equal than required capital.

Remark 2.5 (*Portfolio risk control*). The above reformulation of capital adequacy tests establishes the link to what is typically done in the context of portfolio risk control. There, VaR, or ES, is applied not to the value of the portfolio at time T but rather to its profit and loss. Let $\rho : \mathcal{X} \to \mathbb{R}$ be either of the functionals VaR_α or ES_α for some $\alpha \in (0,1)$ and assume that $\Delta X : \Omega \to \mathbb{R}$ represents the profit and loss of a portfolio. Note that $D_{\Delta X}$ corresponds to the losses suffered by the portfolio and $S_{\Delta X}$ to the gains. In this context, $\rho(\Delta X)$ is interpreted as the *capital required* to support the profit and loss distribution. In various remarks throughout the paper we will recast our results as statements about the profit and loss of a portfolio.

2.4. Who uses what?

The capital rules adopted by most regulatory frameworks in the banking and insurance world are based on VaR or ES:

(a) **Banking:** the second Basel Accord essentially relies on a VaR test at the level $\alpha = 1\%$. However, the third Basel Accord is set to adopt ES at the level $\alpha = 2.5\%$ for market risk.
(b) **Insurance:** in the European Union, Solvency II relies on a VaR test at the level $\alpha = 0.5\%$. In Switzerland, the Swiss Solvency Test uses an ES test at the level $\alpha = 1\%$.

Remark 2.6. An important caveat relates to Basel III. While it has been decided to use ES for the assessment of market risk, ES is not being used for the global balance sheet as is the case of the Swiss Solvency Test and as we do in our investigation. This needs to be borne in mind when assessing the implications of our theoretical findings in a "practical" Basel III context.

2.4.1. Is ES *better than* VaR *at controlling extreme default behavior?*

Let $\alpha \in (0,1)$ be fixed. One of the most criticized shortcomings of using VaR to test for capital adequacy is that VaR only controls the probability, and not the magnitude, of default of a company: $\mathrm{VaR}_\alpha(X)$ is structurally blind to what X does on the *α-tail*, i.e. on the set $\{X < -\mathrm{VaR}_\alpha(X)\}$. Indeed, we can make the tail as negative as we wish without altering the VaR figure. For instance, if Y is an arbitrary nonzero, positive random variable and we set

$$Z := X - Y1_{\{X<-\mathrm{VaR}_\alpha(X)\}},$$

it is easy to check that

$$\mathrm{VaR}_\alpha(Z) = \mathrm{VaR}_\alpha(X). \tag{2.3}$$

This is certainly not the case if we replace VaR by ES since $\mathrm{ES}_\alpha(Z)$ is essentially an average of $-Z$ over the α-tail. Indeed, using (2.3) and noting that $\mathbb{P}[Z < -\mathrm{VaR}_\alpha(X)] = \mathbb{P}[X < -\mathrm{VaR}_\alpha(X)]$ it is easy to infer from representation (2.1) that

$$\mathrm{ES}_\alpha(Z) > \mathrm{ES}_\alpha(X),$$

since Y was assumed to be nonzero. Moreover, with increasing Y, $\mathrm{ES}_\alpha(Z)$ will become arbitrarily large.

Hence, in contrast to VaR, ES does take tail behavior into account and would seem to exert a more tight control on the default behavior of financial institutions. However, the fact that tail behavior is taken into account through an averaging procedure makes it a potentially deceiving measure of risk. The remainder of the paper is dedicated to illustrating this by displaying a variety of undesirable consequences in the context of capital adequacy tests and portfolio risk control.

3. How good is ES in capturing the tail behavior that matters to liability holders?

Recall that we can write $X = S_X - D_X$, where S_X denotes the surplus and D_X the default option. In our stylized model, if the institution does not default, liability holders obtain the exact amount owed to them. Hence, they should be indifferent to the size of the surplus, S_X, which benefits only the owners of the institution. They will, however, be sensitive to the likelihood and size of defaults. Hence, they should care about the default option, D_X, which encapsulates the losses they suffer. This fact should be reflected in any capital requirement framework that claims to take the interests of liability holders into account as follows: if X and Y are two capital positions and Y has a worse default behavior than X, we would expect the required capital associated with Y to be larger than that associated with X. The following example shows that ES does not exhibit this behavior, even to the extent of allowing the possibility that Y has a negative capital requirement and X a positive one. It follows that, from a liability holders' perspective, ES is not a reliable guide when ranking default behavior.

Example 3.1. Fix $\alpha \in (0,1)$ and consider a partition of Ω given by the measurable sets $A, B, C \subset \Omega$. Assume that $\mathbb{P}[A] = \frac{\alpha}{2}$ and $\mathbb{P}[B] = \frac{\alpha}{4}$ and consider the positions

$$X = \begin{cases} -2 & \text{on } A \\ 1 & \text{on } B \\ 1 & \text{on } C \end{cases} \quad \text{and} \quad Y = \begin{cases} -20 & \text{on } A \\ -4 & \text{on } B \\ 4a & \text{on } C, \end{cases}$$

where $a \geqslant 0$. Clearly, we have

$$\mathbb{P}[X < 0] = \frac{1}{2}\alpha < \frac{3}{4}\alpha = \mathbb{P}[Y < 0]$$

and

$$D_X = \begin{cases} 2 & \text{on } A \\ 0 & \text{on } B \\ 0 & \text{on } C \end{cases} \quad \text{and} \quad D_Y = \begin{cases} 20 & \text{on } A \\ 4 & \text{on } B \\ 0 & \text{on } C, \end{cases}$$

so that $D_Y \geqslant D_X$ holds. It is immediate to verify that

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}(2\,\mathbb{P}[A] - (\alpha - \mathbb{P}[A])) = 0.5.$$

Hence the position X is unacceptable and has a capital requirement of 0.5. On the other hand,

$$\mathrm{ES}_\alpha(Y) = \frac{1}{\alpha}(20\,\mathbb{P}[A] + 4\,\mathbb{P}[B] - 4a(\alpha - \mathbb{P}[A] - \mathbb{P}[B])) = 11 - a.$$

Hence, for $a \geqslant 10.5$ we have $\mathrm{ES}_\alpha(X) \geqslant \mathrm{ES}_\alpha(Y)$ so that the capital requirement for X is higher than that for Y even though both the default probability and the default size are strictly larger for Y. If $a \geqslant 11$, then Y even passes the capital adequacy test.

The critical point in the above example is that, within the ES averaging process, the higher losses and higher default probability of Y are compensated by its higher surplus. In order to provide a rigourous statement highlighting this feature of ES we first need the following simple result on VaR which is of independent interest. It says that if a position X is unacceptable under a VaR test, the amount of required capital depends solely on the default option D_X. By contrast, if X is acceptable, then the required capital (which in this case is negative and represents a capital extraction) depends only on the surplus S_X.

Proposition 3.2. *Let $\alpha \in (0,1)$ and $X \in \mathcal{X}$ be given. The following statements hold:*

(i) *If* $\mathrm{VaR}_\alpha(X) \geqslant 0$ *holds, then* $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(-D_X)$.
(ii) *If* $\mathrm{VaR}_\alpha(X) \leqslant 0$ *holds, then* $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(S_X)$.

Proof. Since $-D_X \leqslant X \leqslant S_X$ holds, the monotonicity of VaR_α implies that

$$\mathrm{VaR}_\alpha(S_X) \leqslant \mathrm{VaR}_\alpha(X) \leqslant \mathrm{VaR}_\alpha(-D_X). \tag{3.1}$$

To prove (i) note that for any $m \geqslant 0$ we have that $\{X + m < 0\} = \{-D_X + m < 0\}$. Hence, if $\mathrm{VaR}_\alpha(X) \geqslant 0$, we obtain

$$\mathbb{P}[-D_X + \mathrm{VaR}_\alpha(X) < 0] = \mathbb{P}[X + \mathrm{VaR}_\alpha(X) < 0] \leqslant \alpha.$$

It follows that $\mathrm{VaR}_\alpha(-D_X) \leqslant \mathrm{VaR}_\alpha(X)$ and, consequently, $\mathrm{VaR}_\alpha(-D_X) = \mathrm{VaR}_\alpha(X)$.

To prove (ii) note that $\mathrm{VaR}_\alpha(X) \leqslant 0$ together with inequality (3.1) implies that $\mathrm{VaR}_\alpha(S_X) \leqslant 0$. Moreover, for any $m \leqslant 0$ we have that $\{X + m < 0\} = \{S_X + m < 0\}$. Hence, it follows that $\mathbb{P}[X + \mathrm{VaR}_\alpha(S_X) < 0] = \mathbb{P}[S_X + \mathrm{VaR}_\alpha(S_X) < 0] \leqslant \alpha$.

As a result, $\mathrm{VaR}_\alpha(X) \leqslant \mathrm{VaR}_\alpha(S_X)$ proving that $\mathrm{VaR}_\alpha(S_X) = \mathrm{VaR}_\alpha(X)$. □

The above proposition allows us to shed light on the nature of the averaging process in ES and helps create transparency around the dependence of ES on both the default option and the surplus of a capital position.

Proposition 3.3. *For any $X \in \mathcal{X}$ we have*

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^{\mathbb{P}[X<0]} \mathrm{VaR}_\beta(-D_X)d\beta + \frac{1}{\alpha}\int_{\mathbb{P}[X<0]}^{\alpha} \mathrm{VaR}_\beta(S_X)d\beta.$$

In particular, if $\mathbb{P}[X < 0] \geqslant \alpha$, then

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}_\beta(-D_X)d\beta.$$

Proof. To see this we just need to note that, if $\beta < \mathbb{P}[X < 0]$, then $\mathrm{VaR}_\beta(X) > 0$. Similarly, if $\beta > \mathbb{P}[X < 0]$, we have $\mathrm{VaR}_\beta(X) \leqslant 0$. Hence, the preceding proposition immediately implies that

$$\begin{aligned}\mathrm{ES}_\alpha(X) &= \frac{1}{\alpha}\int_0^{\mathbb{P}[X<0]} \mathrm{VaR}_\beta(X)d\beta + \frac{1}{\alpha}\int_{\mathbb{P}[X<0]}^{\alpha} \mathrm{VaR}_\beta(X)d\beta \\ &= \frac{1}{\alpha}\int_0^{\mathbb{P}[X<0]} \mathrm{VaR}_\beta(-D_X)d\beta + \frac{1}{\alpha}\int_{\mathbb{P}[X<0]}^{\alpha} \mathrm{VaR}_\beta(S_X)d\beta,\end{aligned}$$

concluding the proof. □

However simple, the above result is interesting when interpreted from a financial perspective. Indeed, it highlights the fact that within ES the interests of liability holders may be mixed up with those of the owners of the financial institution: the averaging process in ES allows to compensate losses suffered by liability holders by gains enjoyed by the owners of the institution. Only when the tail does not contain "surplus states" does ES depend exclusively on the default probability. As we will see in the sections to follow, this circumstance has a variety of serious financial consequences which imply that statements on capital adequacy based on ES should be interpreted with due care.

Remark 3.4. One might feel tempted to rectify the entanglement of the interests of liability holders and owners by disregarding the part of the tail that corresponds to the surplus. This would amount to testing whether

$$\mathrm{ES}_{\mathbb{P}[X<0]}[X] \leqslant 0.$$

Note that, since $\mathrm{VaR}_\beta(-D_X) = 0$ for any $\beta \in (\mathbb{P}[X < 0], 1)$, we have

$$\begin{aligned}\mathrm{ES}_{\mathbb{P}[X<0]}[X] &= \frac{1}{\mathbb{P}[X < 0]}\int_0^{\mathbb{P}[X<0]} \mathrm{VaR}_\beta(-D_X)d\beta \\ &= \frac{1}{\mathbb{P}[X < 0]}\int_0^{1} \mathrm{VaR}_\beta(-D_X)d\beta = \frac{1}{\mathbb{P}[X < 0]}\mathbb{E}[D_X].\end{aligned}$$

Hence, the above test boils down to checking that $\mathbb{E}[D_X] \leqslant 0$ or, equivalently, $D_X = 0$. This, however, seems too restrictive, because it implies that for an institution to pass the test, it would need to be almost surely solvent.

Remark 3.5 (*Portfolio risk control*). In Remark 2.5 we described the framework for portfolio risk control. In that context, Proposition 3.3 reads

$$\mathrm{ES}_\alpha(\Delta X) = \frac{1}{\alpha}\int_0^{\mathbb{P}[\Delta X<0]} \mathrm{VaR}_\beta(-D_{\Delta X})d\beta + \frac{1}{\alpha}\int_{\mathbb{P}[\Delta X<0]}^{\alpha} \mathrm{VaR}_\beta(S_{\Delta X})d\beta.$$

This means that in the calculation of required capital, ES allows to compensate portfolio losses by gains, which in the context of risk control may be considered undesirable. On the other hand, at first sight this may seem desirable from a risk/return perspective. However, these risk control frameworks are not by any means designed to automatically make a tradeoff between risk and return. Indeed, whether gains compensating losses are adequate in a risk adjusted sense can be inferred neither from the VaR nor from the ES of the profit and loss distribution.

4. Are ES based tests aligned with the interests of liability holders?

Throughout this section we fix $\alpha \in (0, 1)$. In contrast to VaR, it would seem that by taking into account tail behavior, a capital adequacy test based on ES is well aligned with the primary objective of microprudential financial regulation, which is to ensure the safety of individual financial institutions with the declared aim of protecting the institution's liability holders. The problem is that liability holders are not necessarily interested in *tail* behavior but rather in *default* behavior. This is an important distinction since the tail may contain states where there is a positive surplus, which, as we have explained, should leave liability holders indifferent. Thus, for a capital adequacy test based on ES to be aligned with the interests of liability holders, the test should satisfy *surplus invariance*: if an institution X does not pass the test, then neither should any institution Y with a less favorable default profile, i.e. if X and Y are such that $D_Y \geqslant D_X$, then $X \notin \mathcal{A}_{\mathrm{ES}}(\alpha)$ should imply that $Y \notin \mathcal{A}_{\mathrm{ES}}(\alpha)$. This property is called surplus invariance because it implies that acceptability of a capital position X does not depend on the surplus S_X.

Remark 4.1. One may be bothered by the fact that in our definition of surplus invariance X and Y might have considerably different liabilities, so that the comparison of the respective options to default may not be too informative. However, any two capital positions X and Y satisfying $D_Y \geqslant D_X$ can be viewed as belonging to two institutions having precisely the same liabilities. Indeed, assume that $X = A_X - L_X$ and $Y = A_Y - L_Y$ for positive random variables A_X, A_Y and L_X, L_Y representing the corresponding assets and liabilities, respectively, and set $A'_X = X + L_Y$. Then, it is easy to see that $D_Y \geqslant D_X$ implies that

$$A'_X = X + L_Y = S_X - D_X + L_Y \geqslant S_X - D_Y + L_Y = S_X + \min\{A_Y, L_Y\} \geqslant 0.$$

Since by definition of A'_X we have $X = A'_X - L_Y$, this implies that, at least formally, both X and Y can always be interpreted as capital positions of institutions having the same liabilities.

In Example 3.1 we exhibited two capital positions X and Y such that $D_Y \geqslant D_X$ and $X \notin \mathcal{A}_{\mathrm{ES}}(\alpha)$ but $Y \in \mathcal{A}_{\mathrm{ES}}(\alpha)$. It follows that ES does not satisfy the requirement of surplus invariance. This has been recently pointed out, but not elaborated upon, in Koch-Medina et al. (2015). The following lemma provides the basis for a detailed characterization of when surplus invariance fails.

Lemma 4.2. *Fix $\alpha \in (0, 1)$ and consider a position $X \in \mathcal{X}$. Then, the following statements are equivalent:*

(i) *For every $m \geqslant 0$ there exists a position $Y \in \mathcal{X}$ satisfying $D_Y = D_X$ and $\mathrm{ES}_\alpha(Y) = \mathrm{ES}_\alpha(X) - m$.*
(ii) *The default probability of X is strictly smaller than α, i.e. $\mathbb{P}[X < 0] < \alpha$.*

Proof. To prove that "(i) $\Rightarrow$ (ii)" assume that (i) holds but $\mathbb{P}[X<0] \geqslant \alpha$. Since $D_Y = D_X$, it follows from Proposition 3.3 that

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(-D_X)d\beta = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(-D_Y)d\beta = \mathrm{ES}_\alpha(Y),$$

which is impossible if $m > 0$. Hence, (ii) must hold.

To prove the implication "(ii) $\Rightarrow$ (i)" assume that $\mathbb{P}[X<0] < \alpha$. If $m = 0$ we can always take $Y := X$ and the claim is trivially satisfied, hence assume that $m > 0$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, we find a measurable set $C \subset \{X \geqslant 0\}$ such that $\mathbb{P}[C] = \alpha - \mathbb{P}[X<0]$. Now, define for every $\lambda \geqslant 0$ the position

$$Y_\lambda := (\lambda 1_C - D_X)1_A + (2\lambda + S_X)1_{A^c},$$

where $A = \{X<0\} \cup C$. It is immediate to see that $D_{Y_\lambda} = D_X$. Moreover, Remark 2.3 implies

$$\mathrm{ES}_\alpha(Y_\lambda) = \mathrm{CTE}_\alpha(Y_\lambda) = \frac{1}{\alpha}\mathbb{E}[(D_X - \lambda 1_C)1_A] = \frac{1}{\alpha}\mathbb{E}[D_X] - \lambda\frac{\alpha - \mathbb{P}[X<0]}{\alpha}. \tag{4.1}$$

whenever $\lambda > 0$. Note that, by the continuity of ES, we have $\lim_{\lambda\downarrow 0}\mathrm{ES}_\alpha(Y_\lambda) = \mathrm{ES}_\alpha(Y_0)$ and that $\mathrm{ES}_\alpha(Y_0) \geqslant \mathrm{ES}_\alpha(X)$ since $Y_0 \leqslant X$. The assertion now follows from (4.1) once we recall that $\alpha - \mathbb{P}[X<0] > 0$. $\square$

The following proposition characterizes when surplus invariance fails.

Proposition 4.3. *Fix $\alpha \in (0,1)$ and consider a position $X \in \mathcal{X}$ that is not accepted by the ES_α test, i.e. $X \notin \mathcal{A}_{\mathrm{ES}}(\alpha)$. Then, the following statements are equivalent:*

(i) *There exists a position $Y \in \mathcal{X}$ satisfying $D_Y \geqslant D_X$ that is accepted by the ES_α test.*
(ii) *There exists a position $Y \in \mathcal{X}$ satisfying $D_Y = D_X$ that is accepted by the ES_α test.*
(iii) *There exists a position $Y \in \mathcal{X}$ satisfying $\mathbb{P}[Y<0] = \mathbb{P}[X<0]$ that is accepted by the ES_α test.*
(iv) *The default probability of X is strictly smaller than α, i.e. $\mathbb{P}[X<0] < \alpha$.*

Proof. To prove the implication "(i) $\Rightarrow$ (ii)" assume that Y satisfies $D_Y \geqslant D_X$ and $Y \in \mathcal{A}_{\mathrm{ES}}(\alpha)$. Note that $Z := Y1_{\{Y\geqslant 0\}} - D_X 1_{\{Y<0\}}$ satisfies $D_Z = D_X$ and $Z \geqslant Y$. It follows that $\mathrm{ES}_\alpha(Z) \leqslant \mathrm{ES}_\alpha(Y) \leqslant 0$, so that $Z \in \mathcal{A}_{\mathrm{ES}}(\alpha)$.

The implication "(ii) $\Rightarrow$ (iii)" is obvious. Moreover, Remark 2.4 immediately yields the implication "(iii) $\Rightarrow$ (iv)". Finally, the implication "(iv) $\Rightarrow$ (i)" follows directly from the preceding lemma. $\square$

Remark 4.4 (*Conditional Tail Expectation*). It follows from (4.1) that, whenever the default probability of a position $X \in \mathcal{X}$ is strictly smaller than $\alpha \in (0,1)$, we can always find a position $Y \in \mathcal{X}$ having the same default profile of X, i.e. such that $D_Y = D_X$, and showing an arbitrarily small $\mathrm{CTE}_\alpha(Y)$. As a consequence, a capital adequacy test based on CTE would also fail to be surplus invariant.

Remark 4.5 (*Portfolio risk control*). Lemma 4.2 implies that whenever the profit and loss ΔX of a portfolio satisfies $\mathbb{P}[\Delta X<0] < \alpha$, it is possible to find another portfolio with the same loss profile $D_{\Delta X}$ but arbitrarily lower capital requirement than the first one; in fact, $\mathbb{P}[\Delta X<0] < \alpha$ characterizes when this is possible. Clearly, this behavior is not desirable from a portfolio risk control perspective.

We conclude this section with a few remarks on surplus invariance and VaR acceptability.

Remark 4.6 (*Value-at-Risk*).

(i) Note that VaR acceptability is surplus invariant. To see this let $\alpha \in (0,1)$. Then, if a position X does not pass the VaR_α test, i.e. $X \notin \mathcal{A}_{\mathrm{VaR}}(\alpha)$, then neither can any position Y satisfying $D_Y \geqslant D_X$. Indeed, in this case we have

$$\mathbb{P}[Y<0] = \mathbb{P}[D_Y>0] \geqslant \mathbb{P}[D_X>0] = \mathbb{P}[X<0] > \alpha,$$

showing that $Y \notin \mathcal{A}_{\mathrm{VaR}}(\alpha)$.
(ii) Note that the above remark does not in the least imply that VaR is the better risk measure than ES. We have made the case that ES takes tail risk into account only inadequately, which leads to "anomalies" from a liability holders' perspective. However, if the capital adequacy test based on VaR does not exhibit these anomalies, it is only because it completely ignores tail risk.

5. Are default profiles under VaR more extreme than under ES?

Proposition 4.3 has the following immediate, but important, financial implication: for any position that is acceptable under VaR below the level α, we find a position with the same default behavior that is acceptable under ES at the level α. In fact the position can be chosen to have an arbitrarily negative capital requirement. This highlights from a different angle that there is virtually no difference in terms of the range of default behaviors allowed by VaR and ES.

Proposition 5.1. *Take $0 < \beta < \alpha < 1$. The following statements hold:*

(i) *Assume X is acceptable under VaR_β and let $m \geqslant 0$. Then, there exists a capital position Y that satisfies $D_Y = D_X$ as well as $\mathrm{ES}_\alpha(Y) = \mathrm{ES}_\alpha(X) - m$. In particular, if m is sufficiently large, then Y is acceptable under ES_α.*
(ii) *If X is not acceptable under VaR_β for any $\beta \in (0,\alpha)$, then no position Y that is acceptable under ES_α can satisfy $D_Y = D_X$.*

Proof. The first claim is a reformulation of the implication "(ii) $\Rightarrow$ (i)" in Lemma 4.2. The second claim is a straightforward implication of Remark 2.4. $\square$

Remark 5.2. Consider a position X which is rejected by ES at some level α such that $\mathbb{P}[X<0] < \alpha$. By Proposition 4.3, we can always find a position $Y \in \mathcal{A}_{\mathrm{ES}}(\alpha)$ such that $D_Y \geqslant D_X$. But how much larger than D_X can D_Y be? The preceding proposition immediately implies that for every $D \geqslant D_X$ with $\mathbb{P}[X<0] \leqslant \mathbb{P}[D<0] < \alpha$ and $m \geqslant 0$ there exists a capital position Y such that $D_Y = D$ and $\mathrm{ES}_\alpha(Y) \leqslant -m$. Hence, the answer to the above question is: Y can have a default probability larger than that of X (in fact, as close to α as we like) and, when it defaults, it can default by strictly larger amounts than X (in fact, as large as we choose). In addition, it can have an arbitrarily negative capital requirement.

The above result is interesting since it shows that the failure of capital adequacy tests based on ES to be surplus invariant is of practical relevance as illustrated in the following remark.

Remark 5.3. Assume $0 < \beta < \alpha < 1$. Then, the preceding proposition implies that any capital adequacy test based on ES_α is compatible with more extreme default behaviors than a test based on VaR_β. In fact, there exist capital positions that are not acceptable under VaR_β and yet have an arbitrarily negative capital requirement under ES_α. In particular, this means that the Swiss Solvency Test — an ES_α with $\alpha = 1\%$ — allows positions that have a more extreme default behavior than any of the positions accepted by Solvency II — a VaR_β test with $\beta = 0.5\%$. Moreover, switching from $\mathrm{VaR}_{1\%}$ to $\mathrm{ES}_{2.5\%}$ to measure market risk in Basel III also means allowing portfolios with a more extreme loss behavior.

Note, however, that for a position with an extreme default behavior to be accepted in the ES_α regime, it needs to have a sufficiently large contribution from the surplus in the tail. This poses practical limitations to the ability to achieve such capital positions and questions about materiality arise. Nevertheless, from a theoretical perspective, this calls for a qualification of the claim that ES based capital adequacy tests are more restrictive than VaR based ones.

Remark 5.4.

(i) From the previous proposition it immediately follows that the default profiles of positions that are accepted by the ES_α test are not uniformly bounded, i.e. there does not exist a $\lambda > 0$ such that $D_X \leqslant \lambda$ holds whenever $X \in \mathcal{A}_{\mathrm{ES}}(\alpha)$. In other words, ES_α-acceptable positions can default by arbitrary absolute amounts.
(ii) A capital position that is acceptable in terms of ES_α cannot only default by arbitrary absolute amounts, but the proportion of contractual payments liability holders may receive in case of default may also be arbitrarily small. We show this by way of example. Indeed, fix $\alpha \in (0,1)$ and take a measurable set $A \subset \Omega$ with $\mathbb{P}[A] = \frac{\alpha}{2}$. Consider for every $\lambda > 0$ an institution with assets and liabilities defined by

$$A_\lambda = 1_A + (2+\lambda)1_{A^c} \quad \text{and} \quad L_\lambda = (1+\lambda)1_A + 1_{A^c}.$$

The corresponding capital position is then given by

$$X_\lambda = -\lambda 1_A + (1+\lambda)1_{A^c}.$$

Note that

$$\mathrm{ES}_\alpha(X_\lambda) = \frac{1}{\alpha}(\lambda\mathbb{P}[A] - (1+\lambda)(\alpha - \mathbb{P}[A])) = -\frac{1}{2}.$$

It follows that $X_\lambda \in \mathcal{A}_{\mathrm{ES}}(\alpha)$ for every $\lambda > 0$. At the same time, we have $D_{X_\lambda} = \lambda 1_A$ so that the liability holders will receive $L_\lambda - D_{X_\lambda} = 1$ on A, i.e. should the institution default. Hence,

$$\frac{L_\lambda - D_{X_\lambda}}{L_\lambda} = \frac{1}{1+\lambda} \quad \text{on } A.$$

This means that, when the institution defaults, it will pay the proportion $\frac{1}{1+\lambda}$ of the contractual obligation which can get arbitrarily close to 0 as λ grows.

Remark 5.5 (*Portfolio risk control*). Proposition 5.1 tells us that controlling portfolio risk by ES_α allows to have loss profiles as extreme as those implied by any VaR_β with $\beta \in (0,\alpha)$ and with arbitrarily negative capital requirement.

6. Is ES suitable as a global capital adequacy standard?

So far we have always expressed capital positions in a fixed unit of account. In this section we allow to change the accounting unit and model the transition to a new currency by means of a random variable

$R : \Omega \to (0,\infty)$.

This random variable represents the *(stochastic) exchange rate* from the old currency to the new one. More precisely, if X is the capital position of an institution expressed in the old currency, then RX represents the same position but now denominated in the new currency. We assume that R is essentially bounded so that $RX \in \mathcal{X}$ for any $X \in \mathcal{X}$.

To provide a financially sound basis for testing for capital adequacy, a solvency test should be independent of the chosen currency used to aggregate risks. Indeed, the management of a badly capitalized company could otherwise be able to turn its unacceptable position into an acceptable position by simply choosing a suitable accounting currency. However, as put by Artzner et al. (2009), "changing the unit of account is essentially a different way to quote prices and should by itself have no impact on whether a financial position is accepted or not". In other words, acceptability should reflect a fundamental aspect of the balance sheet, i.e. the ability to cover liabilities, which is independent of the choice of any accounting unit.

The dependence of acceptability on the chosen accounting unit has another important financial implication. Indeed, consider two different jurisdictions that apply a capital adequacy test based on ES at the same level $\alpha \in (0,1)$ but each in its own currency (the domestic and the foreign currency). The possibility for an institution to be unacceptable when its capital position is expressed in domestic currency but acceptable when it is expressed in foreign currency creates a clear regulatory arbitrage opportunity. Indeed, ignoring all practical barriers to do so, unacceptable institutions would have the incentive to move to a jurisdiction where they would become acceptable without the need of restructuring their balance sheets.

In the remainder of this section we investigate the impact of a change of currency on the outcome of a capital adequacy test based on ES. We start by showing that ES acceptability is not invariant under a change of numéraire.

Example 6.1. Fix $\alpha \in (0,1)$ and consider a set $A \in \mathcal{F}$ with $\mathbb{P}[A] = \frac{\alpha}{2}$. The capital position of an institution expressed in domestic currency and the exchange rate are given, respectively, by

$$X = \begin{cases} -105 & \text{on } A \\ 100 & \text{on } A^c \end{cases} \quad \text{and} \quad R = \begin{cases} 1 & \text{on } A \\ 1.1 & \text{on } A^c. \end{cases}$$

Then, it is easy to see that

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}(105\,\mathbb{P}[A] - 100(\alpha - \mathbb{P}[A])) = 2.5.$$

Hence, the financial institution is unacceptable under the ES_α test in domestic currency. On the other hand,

$$\mathrm{ES}_\alpha(RX) = \frac{1}{\alpha}(105\,\mathbb{P}[A] - 110(\alpha - \mathbb{P}[A])) = -2.5,$$

so that the institution is acceptable under the ES_α test in foreign currency. This is because the favorable exchange rate on A^c will "increase" the surplus of X making the ES figure actually flip sign. Hence, this institution would not be able to operate in the domestic jurisdiction but could comfortably do so in the foreign jurisdiction.

Remark 6.2. The possibility that the outcome of a capital adequacy test may depend on the underlying currency was already pointed out, though not further pursued, by Artzner et al. (1999): "acceptance sets allow us to address a question of importance to an international regulator and to the risk manager of a multinational firm, namely the invariance of acceptability of a position with respect to a change of currencies". The same issue was raised again in Artzner et al. (2009) where the lack of "currency invariance" of ES acceptability was discussed by way of example in a finite state setting. The preceding (simpler) example works in general spaces.

In light of the above discussion, it is important to assess whether the dependence of ES acceptability on the underlying currency was the result of a "pathological" profile of the exchange rate or whether it is of a more fundamental nature. The next result provides a complete answer to this question: for every nonconstant exchange rate R we find a position which becomes acceptable under ES after switching to the new currency specified by R.

Proposition 6.3. *Fix $\alpha \in (0,1)$. For every nonconstant exchange rate R there exists a position that is not accepted by ES_α in the original currency, i.e. $X \notin \mathcal{A}_{\mathrm{ES}}(\alpha)$, but is accepted by ES_α after a change in aggregation currency, i.e. $RX \in \mathcal{A}_{\mathrm{ES}}(\alpha)$.*

Proof. Since R is not constant and almost surely strictly positive, we find $\lambda > 0$ satisfying $\mathbb{P}[R < \lambda] > 0$ and $\mathbb{P}[R \geqslant \lambda] > 0$. Assume that we can find $0 < \varepsilon < \frac{\lambda}{2}$ and measurable sets $A, B \subset \Omega$ satisfying

$$\{R \leqslant \varepsilon\} \subset A \subset \{R \leqslant \lambda - \varepsilon\}, \quad B \subset \{R \geqslant \lambda\}, \quad \text{and} \quad \mathbb{P}[A] < \alpha = \mathbb{P}[A] + \mathbb{P}[B].$$

Let C be the complement of $A \cup B$ and consider the position

$$X := -a1_A + b1_B + c1_C$$

for strictly positive numbers $a, b, c > 0$ satisfying

$$\frac{\mathbb{P}[B]}{\mathbb{P}[A]} < \frac{a}{b} < \frac{\lambda}{\lambda - \varepsilon} \cdot \frac{\mathbb{P}[B]}{\mathbb{P}[A]} \quad \text{and} \quad c > \frac{\lambda b}{\varepsilon}.$$

Then, it easily follows from (2.2) in Remark 2.3 that

$$\mathrm{ES}_\alpha(X) = \mathrm{CTE}_\alpha(X) = \frac{1}{\alpha}(a\,\mathbb{P}[A] - b\,\mathbb{P}[B]) > 0. \tag{6.1}$$

Moreover, since $R > 0$ almost surely, we have

$$RX \geqslant -(\lambda - \varepsilon)a1_A + \lambda b1_B + \varepsilon c1_C =: Y.$$

Hence, using the monotonicity of ES and relying again on (2.2) in Remark 2.3, it follows that

$$\mathrm{ES}_\alpha(RX) \leqslant \mathrm{ES}_\alpha(Y) = \mathrm{CTE}_\alpha(Y) = \frac{1}{\alpha}((\lambda - \varepsilon)a\,\mathbb{P}[A] - \lambda b\,\mathbb{P}[B]) < 0. \tag{6.2}$$

In conclusion, $X \notin \mathcal{A}_{\mathrm{ES}}(\alpha)$ but $RX \in \mathcal{A}_{\mathrm{ES}}(\alpha)$.

It remains to establish the existence of the measurable sets $A, B \subset \Omega$ with the desired properties. We will use the fact that

$$\lim_{\varepsilon \to 0} \mathbb{P}[R \leqslant \lambda - \varepsilon] = \mathbb{P}[R < \lambda] > 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \mathbb{P}[R \leqslant \varepsilon] = 0.$$

This implies that, by choosing $\varepsilon > 0$ small enough, we can always ensure that

$$\max\{\mathbb{P}[R \leqslant \varepsilon], \alpha - \mathbb{P}[R \geqslant \lambda]\} < \min\{\mathbb{P}[R \leqslant \lambda - \varepsilon], \alpha\}.$$

Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, we find a measurable set $A \subset \Omega$ satisfying

$$\{R \leqslant \varepsilon\} \subset A \subset \{R \leqslant \lambda - \varepsilon\}$$

and

$$\max\{\mathbb{P}[R \leqslant \varepsilon], \alpha - \mathbb{P}[R \geqslant \lambda]\} < \mathbb{P}[A] < \min\{\mathbb{P}[R \leqslant \lambda - \varepsilon], \alpha\}.$$

This follows, for instance, from Lemma 1 in Chapter 2 in Delbaen (2012). In particular, we have $\mathbb{P}[A] < \alpha$. Moreover, given that

$$\mathbb{P}[R \geqslant \lambda] > \alpha - \mathbb{P}[A] > 0,$$

we can also find a measurable subset $B \subset \{R \geqslant \lambda\}$ with probability equal to $\alpha - \mathbb{P}[A]$. Thus, the claim follows. □

Remark 6.4 (*Conditional Tail Expectation*). The above result remains valid if we replace Expected Shortfall by Conditional Tail Expectation. Indeed, using the same notation introduced in the above proof, we see that $\mathrm{CTE}_\alpha(X) > 0$ by (6.1) while (6.2) implies that $\mathrm{CTE}_\alpha(RX) \leqslant \mathrm{CTE}_\alpha(Y) < 0$. In other words, the outcome of a capital adequacy test based on CTE also depends on the currency used to aggregate risks.

As in the case of surplus invariance, a few remarks on numéraire invariance and VaR acceptability are in order.

Remark 6.5 (*Value-at-Risk*).

(i) Clearly, VaR acceptability is numéraire invariant. To see this let $\alpha \in (0,1)$. If a position X does not pass the VaR_α test, i.e. $X \notin \mathcal{A}_{\mathrm{VaR}}(\alpha)$, then neither does RX for any choice R of the exchange rate. Indeed, in this case we have

$$\mathbb{P}[RX < 0] = \mathbb{P}[X < 0] > \alpha$$

so that $RX \notin \mathcal{A}_{\mathrm{VaR}}(\alpha)$ holds as well.
(ii) The above remark does not in the least imply that VaR is the better risk measure than ES. Because it captures tail risk through averaging, when used as a global regulatory standard ES may lead to regulatory arbitrage opportunities. Global regulation based on a VaR regime would not allow this type of regulatory arbitrage, but, as was the case with surplus invariance, this is only because VaR completely ignores tail risk.

Remark 6.6 (*Portfolio risk control*). Proposition 6.3 tells us that, when the capital required to support a portfolio is determined using ES_α for some $\alpha \in (0,1)$, the sign of the capital requirement may depend on the currency used to aggregate positions. Indeed, given any two currencies, there exists a portfolio that has a positive capital requirement under one currency and negative under the other.

7. How likely are institutions to engage in excessive risk taking?

We have seen that ES acceptability is consistent with almost arbitrarily risky default behavior. Here we show that value-maximizing financial institutions will, in fact, have the incentive to actively seek these extreme default profiles. To do so we assume the existence of a linear *present value operator* V_0 with which we can determine the value at time $t = 0$ of any payoff in $\mathcal{X}$. We assume that V_0 is *strictly positive*, i.e. for any positive payoff $Z \in \mathcal{X}$ that is not identically zero we have $V_0(Z) > 0$.[6]

In our simple one-period model, the terminal value of a limited-liability financial institution with capital position X is equal to the surplus S_X. Hence, the value of the institution at time $t = 0$ is just the present value of S_X, i.e. the present value of the (random) payoff the owners would receive at time $t = T$. In reality, however, there is another component of value which is related to the ability of the firm to continue to operate and undertake profitable new business with positive net present value at time $t = T$. For concreteness, we will assume that the ability to write new business depends on the amount of capital the firm has available at time $t = T$ although nothing in the discussion changes if we let it depend on the full capital position or only on the institution's assets. Consequently, we will assume there exists a nonnegative function $F : \mathbb{R}_+ \to \mathbb{R}_+$, called the *franchise* of the institution, such that $F(S_X)$ represents the net present value at time $t = T$ of future opportunities.[7]

[6] For our purposes we do not need to be more specific about the particular form of V_0. Note that in the context of incomplete, arbitrage-free markets, a pricing functional π is naturally defined on the subspace $\mathcal{M}$ of $\mathcal{X}$ given by the payoffs that can be replicated by portfolios of traded instruments. In this case, candidates for a present value operator V_0 are the strictly positive extensions of π to $\mathcal{X}$. Under suitable technical conditions, these can be identified with risk-neutral probability measures.

[7] We only need to assume that F is a nonnegative function. Typical additional assumptions on the franchise function – necessary for other type of results – are for instance those in Froot et al. (1993) and Froot and Stein (1998): F is strictly increasing, concave, bounded from above and satisfies $F(0) = 0$.

At time $t = T$ the value available to the owners of the company will then be

$$S_X + F(S_X),$$

i.e. the amount of available funds at the end of the period increased by the net present value of the opportunities that can be exploited with this amount of funds. Note that, through S_X, the franchise is contingent on the state of the economy at time T.

The value of the financial institution to its owners at time $t = 0$ is then

$$V_0(S_X + F(S_X))$$

or, using that $S_X = X + D_X$,

$$V_0(X) + V_0(D_X) + V_0(F(S_X)).$$

The above expression shows that the value of the institution consists of three components: the *(net) tangible value* $V_0(X)$, the value of the option to default $V_0(D_X)$, and the *franchise value* $V_0(F(S_X))$.

The existence of a franchise value is often seen as a deterrent for excessive risk taking since the franchise value can only be realized if the institution stays in operation and vanishes whenever the institution fails. For instance, we refer to the standard references by Froot et al. (1993), by Froot and Stein (1998), and Helmann et al. (2000). Whether or not a value-maximizing financial institution chooses to increase risk depends on the impact of this additional risk on the value of the institution. We will explore this in the context of a simple example.

Fix $\alpha \in (0,1)$ and consider a financial institution with capital position $X \in \mathcal{A}_{ES}(\alpha)$. Assume the market does not admit arbitrage opportunities and there exists a fully leveraged portfolio of traded assets with payoff $Z \in \mathcal{A}_{ES}(\alpha)$, i.e. $\mathrm{ES}_\alpha(Z) \leq 0$. Here, fully leveraged means that $V_0(Z) = 0$ so that, by no arbitrage, we have $\mathbb{P}[Z < 0] \in (0,1)$. As explained in Jarrow (2013), current banking practice also allows for the possibility of shorting risky assets and, therefore, our model will also allow for this. Hence we will allow the management to acquire a multiple $\lambda > 0$ of the fully leveraged portfolio, which can be done at zero cost. After this transaction, the capital position will change to

$$X_\lambda = X + \lambda Z.$$

Note first that the new capital position X_λ continues to be ES_α acceptable. Indeed,

$$\mathrm{ES}_\alpha(X_\lambda) \leqslant \mathrm{ES}_\alpha(X) + \lambda \mathrm{ES}_\alpha(Z) \leqslant 0, \tag{7.1}$$

where we have used the subadditivity and the positive homogeneity of ES. We claim that

$$\lim_{\lambda\to\infty} V_0(D_{X_\lambda}) = \infty,$$

i.e. we can arbitrarily increase the value of the default option while remaining acceptable under ES. Indeed, since $\mathbb{P}[Z < 0] \in (0,1)$, there exists $a > 0$ satisfying $\mathbb{P}[Z \leqslant -a] > 0$. Clearly, we find $b > 0$ such that $\mathbb{P}[A \cap B] > 0$ where $A = \{Z \leqslant -a\}$ and $B = \{X \leqslant b\}$. If λ is large enough, we see that $D_{X_\lambda} \geqslant -b + \lambda a > 0$ on $A \cap B$. Hence, noting that $V_0(1_{A\cap B}) > 0$ by the strict positivity of V_0, the claim easily follows from

$$V_0(D_{X_\lambda}) \geqslant V_0((-b + \lambda a)1_{A\cap B}) = (-b + \lambda a)V_0(1_{A\cap B}).$$

Note that $V_0(X_\lambda) = V_0(X)$ since $V_0(Z) = 0$. Hence, the value of the institution after the zero-cost transaction is given by

$$V_0(X) + V_0(D_{X_\lambda}) + V_0(F(S_{X_\lambda})).$$

It follows that the value of the institution will tend to ∞ as λ tends to ∞. We conclude that the opportunities given by an ES based regime to take additional risks are also attractive because they allow increasing the value of the company by boosting the value of the option to default. The disadvantaged by such a transaction are liability holders since the value of the option to default is precisely the value of what will not be paid to them.

Remark 7.1. The previous discussion casts doubts on the claim, made in Jarrow (2013), that in contrast to a VaR regime, capital adequacy rules based on ES allow to exert control over the probability of catastrophic failure of financial institutions. Indeed, we have showed that what Jarrow calls catastrophic failure is also possible under an ES regime.

Example 7.2. Fix $\alpha \in (0,1)$. We provide a stylized example of an economy where a fully leveraged portfolio as described above exists. Consider a market where a risk-free bond and a defaultable bond are traded. The corresponding initial price and terminal payoff are given, respectively, by

$$B_0 = 1 \quad \text{and} \quad B_1 = 1_\Omega$$

and

$$B_0^d \in (0,1) \quad \text{and} \quad B_1^d = r1_A + 1_{A^c}$$

where $\mathbb{P}[A] \in (0,\alpha)$ and $r \in [0, B_0^d)$. Note that $\mathbb{P}[A]$ and r are, respectively, the default probability and the recovery rate of the defaultable bond. Now, consider the zero-cost portfolio which is one unit of the defaultable bond long and B_0^d units of the risk-free bond short. Its payoff is given by

$$Z = B_1^d - B_0^d B_1.$$

It is easy to see that

$$\mathrm{ES}_\alpha(Z) \leq 0 \iff \frac{1 - B_0^d}{1 - r} \geqslant \frac{\mathbb{P}[A]}{\alpha}.$$

Remark 7.3 (*Value-at-Risk*). Clearly, the above discussion also works in the context of a VaR based solvency regime. In fact, it is easy to show that the fully-leveraged portfolio with payoff Z in the above example is always acceptable under VaR_α without any additional conditions on the price and on the recovery rate of the defaultable bond. Hence, while ES and VaR based regimes both incentivize companies to take arbitrary amounts of risk in the tail by leveraging, it is much easier to do so in a VaR based regime.

8. Conclusion

Capital adequacy tests based on VaR have been widely criticized mainly for two reasons: they do not give credit for diversification and they do not capture tail risk. In particular because of the last point, VaR has been often described as being a shareholders' risk measure: it focuses on the probability and not the size of default. On the other hand, ES has been put forward as a risk measure that does not suffer from these shortcomings and which could thus be viewed as a risk measure that takes a liability holders' perspective. However, while ES does give credit for diversification, we have shown in this paper that the fact that ES captures tail risk essentially by averaging over the tail leads to unintended phenomena that are difficult to reconcile with one of the key regulatory objectives, namely the protection of liability holders. Fixing $\alpha \in (0,1)$, these unintended consequences can be described as follows:

(a) Whenever the α-tail $\{X < -\mathrm{VaR}_\alpha(X)\}$ contains states in which the surplus is strictly positive, ES_α compensates losses for liability holders by gains to the owners of the institution. Hence, ES_α mixes up the interests of liability holders and owners.

(b) It is possible for an ES_α-acceptable financial institution to display a more unfavorable default behavior than a company that has been declared unacceptable by the ES_α test. Hence, ES_α cannot be said to take a liability holders' perspective.
(c) An ES_α-acceptable position can have any default profile allowed by a VaR_β test for any $\beta \in (0, \alpha)$. Hence, when it comes to default behavior, the ES_α based capital adequacy test is less restrictive than the VaR_β test.
(d) An ES_α test allows making an unacceptable position acceptable by a mere change in the aggregation currency. Hence, using ES_α as a global solvency standard (applied in each jurisdiction in the respective local currency) opens up regulatory arbitrage opportunities.
(e) An institution's desire to protect its franchise is not sufficient to counterbalance the attractiveness of exploiting the opportunities that exist under an ES_α test to build up extreme default profiles. Under certain circumstances, this build up can be achieved through financial market transactions at zero cost. Hence, theoretically, institutions have an incentive to engage in extreme risk taking in the α-tail allowed by ES_α.

It follows that, although ES does capture tail behavior, the benefits of switching from a VaR to an ES regime need to be qualified. Because it averages across some α-tail which may contain default as well as surplus scenarios, ES tests may conceal the true risks luring in the tail, potentially giving wrong signals about how dangerous a position is for liability holders. This does not imply that VaR is the better risk measure. In fact, properties (c) and (e) are also present in the context of a VaR regime in an even more pronounced form. However, a VaR test may be less likely to inspire a false sense of security if only because it completely ignores the tail.

Given the above results, a natural question to ask is whether other coherent risk measures perform better in terms of surplus and numéraire invariance. This topic was addressed recently in an abstract context in Koch-Medina et al. (2015) where it was shown that a coherent acceptance set is surplus invariant if and only if it is numéraire invariant and that the only coherent capital adequacy tests that are surplus invariant are those based on simple scenarios, i.e. tests of the form

$$\mathcal{A}(S) = \{X \in \mathcal{X};\ X \geqslant 0 \text{ on } S\}$$

for some measurable scenario set $S \subset \Omega$. In particular, this implies that capital adequacy tests based on other well-known coherent risk measures — e.g. spectral risk measures or expectiles — are neither surplus not numéraire invariant and suffer from the same deficiencies as ES based tests, even tough a clear statement on the materiality of these deficiencies requires further investigation.

Note that although coherent, when $\mathbb{P}[S] < 1$ the capital adequacy test $\mathcal{A}(S)$ suffers from the fact that it is blind on the complement of S. Hence, it offers no control on default behavior on that set. On the other hand, when $\mathbb{P}[S] = 1$, passing the capital adequacy test requires an institution to be solvent with probability 1, which seems to be a tall order.

We also note that the "explosion" of the value of the default option discussed in Section 7 is the immediate consequence of inequality (7.1). Since this inequality just relies on ES being a coherent risk measure, it continues to hold if we substitute for ES any coherent risk measure ρ provided the payoff Z is acceptable under ρ. It follows that the "explosive" behavior of the default option is also true for general coherent risk measures.

In conclusion, choosing the risk measure on which to base a capital adequacy test remains a matter of judgement and sacrificing some desirable feature in favor of another feature which is deemed more important seems to be unavoidable.

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