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Pricing, Production, and Inventory Policies for Manufacturing with Stochastic Demand and Discretionary Sales

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We study determining prices and production jointly in a multiple period horizon under a general, nonstationary stochastic demand function with a discrete menu of prices. We assume that the available production capacity is limited and that unmet demand is lost. We incorporate discretionary sales, when inventory may be set aside to satisfy future demand even if some present demand is lost. We analyze and compare partial planning or delayed strategies. In delayed strategies, one decision may be planned in advance, whereas a second decision is delayed until the beginning of each time period, after observing the results of previous decisions. For example, in delayed production (delayed pricing), pricing (production) is determined at the beginning of the horizon, and the production (pricing) decision is made at the beginning of each period before new customer orders are received. A special case is where a single price is chosen over the horizon. We describe policies and heuristics for the strategies based on deterministic approximations and analyze their performances. Computational analysis yields additional insights about the strategies, such as that delayed production is usually better than delayed pricing except sometimes when capacity is tight. On average, the delayed production (pricing) heuristic achieved 99.3% (99.8%) of the corresponding optimal strategy.

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1. Introduction

1.1. Motivation and Outline

In recent years, a number of industries have used innovative pricing strategies to manage their inventory effectively. For example, techniques such as revenue management have been applied in service industries as varied as the airlines, hotels, and rental car agencies—integrating price, inventory control, and quality of service.

However, the integration of pricing with production in a manufacturing setting is still in its early stages. There are a number of characteristics that distinguish general manufacturing industries from the industries mentioned previously, particularly the nonperishability of products. In addition, manufacturing has the control to vary production levels, although production capacity may be limited. Price may be used not only as a market clearing mechanism, but also as a tool to match demand with a limited and controllable supply.

Our objective in this research is thus to analyze situations in which pricing is considered jointly with production decisions, incorporating the relationship between price and demand, and taking into account capacity limitations. In particular, we are interested in partial planning strategies, where products are nonperishable or have a long selling season. We consider discretionary sales, where inventory may be saved to satisfy future demand. We also focus on situations in which prices are selected from a discrete set or from a menu of prices. An example of this is the simplified pricing policy used by Dollar General, which sells every item at a price chosen from a fixed menu of 17 prices.

In partial planning strategies, one decision (price or production) is made at the beginning of the horizon, and the second decision (production or price) is made at the beginning of each period—after the realization of demand uncertainty in previous periods but before demand is realized in the current period. The advance decision may be due to limited firm flexibility (e.g., fixed contracts) or in many cases may be based on forecasted (often deterministic) parameters. We focus on easy-to-implement policies for each strategy, the general performance of the strategy, and the effectiveness of the strategies as parameters vary. The strategies we analyze are driven by industrial examples such as the following:

1. A supplier of manufactured parts has variability in raw material supply and uses price to match demand and supply. The supplier contracts with a manufacturer and can offer time-varying prices but must commit to these in advance. The supplier adjusts production in each period based on previous inventory and expected orders.

2. A manufacturer needs to plan procurement decisions in advance to sign contracts with its suppliers for part delivery. Thus, the manufacturer determines a production schedule at the beginning of a time horizon but makes decisions regarding available inventory and price to customers on a period-by-period basis.

3. A manufacturer whose primary distribution channel is through catalogs determines a single constant price over the lifetime of the product and wants the profit maximizing price. Production decisions are determined period by period, based on expected demand in present and future periods as well as inventory from previous periods.

In the first example, the decision-making firm determines prices for a planning horizon a priori and makes the production decision based on the state of the system and forecasted demand. We refer to this strategy as *delayed production*. The second example illustrates the delayed pricing partial planning model. The manufacturer plans production at the beginning of the horizon but makes the price decision on a period-by-period basis. In other cases, a firm may believe that selecting a constant price for a product is the best strategy; however, procurement flexibility in each period may still be desired. This is illustrated in the last example, and the strategy is referred to as *fixed pricing*. This pricing strategy is often applied when goods are nonperishable or have a long selling season, for example, furniture products.

In all the examples described above, it may be profitable to apply *discretionary sales*, that is, to commit inventory to satisfy future demand, even if the decision means losing sales in the current period. Although choosing to lose sales may seem counter to making profit, the inventory is set aside in situations when it is likely to generate a larger income in the future. This would typically occur if the price in the future is higher or if the future production costs were high.

The following assumptions are common to all the models considered in this paper. We assume that the relationship between price and expected demand is known, although we do not assume a specific shape to the function. Demand is allowed to be nonstationary over time, and demand for a given price is assumed to be differentiable. Demand uncertainty in each period may vary with price or be price independent. Production is limited by the available capacity in each period, which is known and may vary over time. If product is not available to meet customer demand, we assume the sale is lost; inventory may be saved for the future even if this results in lost sales. We focus on a price-setter firm whose customers do not act strategically over time. We also assume the firm does not have sufficient power to manipulate a market through deceptive practices. The goal is to maximize the expected profit, which is composed of revenue, inventory holding cost, and production cost, over a finite horizon.

This paper is organized as follows. We describe the notation and problem assumptions in §2.1. In §2.2 we analyze delayed production. We show that given a price vector, the delayed production problem has a special structure that leads to a simple, optimal production and inventory policy. Specifically, the optimal policy is characterized by two parameters, the classic order-up-to level and a new save-up-to level, both of which are modified base-stock policies. For the save-up-to level, the policy is to set aside that amount of inventory for the future if it is available; otherwise, set aside as much inventory as possible for the future. Both policies are time dependent but are independent of the beginning inventory level and of the realized customer demand; thus, the decision to fulfill orders may be made as customers arrive. Our results also imply that a simple search on all possible prices finds the optimal fixed price policy. Finally, we describe a heuristic to generate a pricing vector for the general delayed production strategy and analyze its worst-case performance.

In §2.3, we focus on delayed pricing, in particular studying the relationship between optimal price and the beginning inventory level in each period. We show that the optimal period price is not necessarily a decreasing function of inventory, and simple threshold policies are not always optimal. Additionally, we show that determining the inventory available for sale in a given period before demand is realized is inferior to determining it after demand is realized in that period. Therefore, given a production schedule, we provide a dynamic program that generates the optimal pricing vector for delayed pricing and is useful for determining the policies for the strategy with full price and full production flexibility in each period. When both pricing and production are flexible, we refer to this as the *full-flexibility* strategy.

In §3, we report results from an extensive computational analysis using data from a manufacturing partner to analyze the delayed production, delayed pricing, and full-flexibility strategies. Our analysis indicates that all three strategies perform significantly better than fixed pricing when the available capacity is low—as much as a 7% profit increase. The performance of the three strategies depends on the type of product seasonality, the capacity level, and the level of uncertainty. The experiments also indicate that the heuristics can be quite close to the corresponding optimal partial planning strategy (average of >99% over all experiments) as well as to the fully flexible pricing and production strategy (average of 98.8% and 97.7% for delayed production and pricing, respectively). This suggests that full flexibility may not be necessary in all situations and that focusing on either better cost or revenue management may be sufficient. For example, when capacity is in high supply, a focus on production cost may be better (e.g., delayed production), but when capacity is tight, sometimes focusing on revenue management is more important (e.g., delayed pricing).

In the final section, §4, we describe some extensions to the stochastic pricing problem and make conclusions. We also suggest future directions for research in this area.

1.2. Literature Review

We review literature in a variety of areas, including inventory control, marketing, and revenue management that pertains to pricing models. Joint pricing and inventory control strategies in a single-period (newsvendor) manufacturing environment were first considered in Whitin (1955); see Petruzzi and Dada (1999) for a recent review of single-period models. Price determination and restocking in a multiperiod setting has also been considered by a number of researchers; for example, see Thomas (1970), Thomas (1974), Thowsen (1975), and Zabel (1972) for early examples, with more recent examples in Federgruen and Heching (1999) and Chen and Simchi-Levi (2004). Many researchers have assumed linear demand, and the stochastic demand models have mostly focused on backlogging of excess demand. A thorough review of both single and multiperiod models combining pricing and inventory strategies can be found in Chan et al. (2004).

Revenue management research explores the pricing and inventory control of products, particularly in transportation but also in the retail industry. For instance, Gallego and van Ryzin (1994) analyze the dynamic adjustment of price as a function of inventory and length of remaining sales period; the demand is stochastic, but no restocking is allowed. Gallego and van Ryzin (1997) analyze pricing of multiple products assembled from existing inventories of a set of components. They show that heuristics generated from a deterministic version of the problem are asymptotically optimal as the demand increases.

Most research on integrating pricing strategies with inventory control policies has ignored production capacity limitations. One exception is Lai (1990), who considers the issue of whether capacity constraints contribute to asymmetry in price behavior. The most notable exception, however, is the work by Federgruen and Heching (1999); they address the problem of determining optimal pricing and inventory control strategies. Their research shows that an order-up-to policy is optimal for a periodic review system with zero fixed ordering cost. One additional property they show is that price decreases with increasing initial inventory. Their model is somewhat similar to our model; however, they assume backlogging, whereas our model assumes lost sales. Another important distinction is that they assume "demand in each period is concave in the period's price," while we assume a general function. Finally, in our model we show that price does not necessarily decrease with increasing initial inventory, and we incorporate discretionary sales.

Some research has also been directed toward finding a single optimal price over a multiperiod horizon, particularly under the assumption of deterministic demand. Kunreuther and Schrage (1973) provide bounds on the optimal price, whereas Gilbert (1999) finds the optimal price under a less-general model of demand. Gilbert (2000) considers a problem with multiple products sharing common production capacity and demonstrates a procedure for finding the optimal fixed price for each product. In the first two papers, production has a set-up cost, whereas in the multiple product work, there is no set-up cost. Little work has been done on the fixed pricing problem with stochastic demand. This research is pertinent to the fixed pricing strategy we discuss. In our case, we provide a way to find the best fixed price and inventory policy for a single product with a general stochastic demand distribution when there is no fixed ordering cost.

Van Mieghem and Dada (1999) have another important paper relevant to our work; the authors explicitly consider price-postponement versus production-postponement strategies. They focus on a single-period, two-stage process with an initial decision, for example, production, followed by a realization of demand, followed by another decision, for example, pricing. Thus, price (production) postponement as outlined by Van Mieghem and Dada is different from delayed pricing (production) in our model, as in their case the postponed decisions are made after demand is realized. They find that conditions dictate whether price postponement or production postponement is more valuable to a firm. Specifically, they show that the former is likely to be more valuable if demand variability, marginal production, and holding costs are low. Their paper also addresses the decision of capacity investment and considers competition.

In addition to papers combining pricing and inventory, it is important to consider traditional inventory models (relevant to our delayed production strategy). A multiperiod inventory model with nonstationary demand, varying production cost, and lost sales is most closely related to the stochastic pricing problems considered in this paper (see Zipkin 2000, Chapter 9). Of course, in the traditional inventory models, demand is always satisfied when inventory exists, while in our model it is possible to decide not to satisfy demand even when inventory is available. This difference between our work and the traditional inventory literature leads to the concept of a save-up-to policy, which is an important strategy in our models, as well as leading to structurally optimal results when pricing is fixed in advance.

Recent work by Scarf (2000) addresses an inventory model in which sales are discretionary, as we also assume. Scarf considers a model with production capacity limits and fixed set-up costs where price is given and inventory policy is the decision factor. Scarf finds that the optimal policy for his problem is of the (s, S) form, and the optimal discretionary sales are dependent on realized demand. Our delayed production strategy after the pricing decision has been made is a special case of this Scarf model but with zero fixed ordering cost. We also find that a base stock policy is optimal (a modified $(S-1, S)$ policy for our problem), but we additionally show in our special case of the Scarf problem that the optimal discretionary sales decision is a threshold policy that is independent of initial inventory and realized demand.

2. The Stochastic Pricing Problems

We analyze a multiperiod single manufacturer model for a single product, where pricing and production decisions must be made for each period. We focus initially on partial planning strategies, where one decision (e.g., production) is made at the beginning of the horizon for all periods, and where a second decision (e.g., pricing) is made in each period based on the state of the system. We examine situations where there may be demand seasonality over time represented by time-dependent expected demand curves as well as an additional stochastic element that is a random component with a known distribution.

2.1. Notation and Assumptions

In all models, we assume the following: Periods are indexed consecutively from $1, 2, \ldots, T$. The production quantity in period t is limited by q_t for $t = 1, 2, \ldots, T$. The cost of producing a unit is k_t, and inventory holding cost, h_t, is charged to carry inventory from period $t-1$ to period t. The salvage value at the end of the horizon is represented as v, which we assume is less than the price charged in the last period.

In both delayed production and delayed pricing strategies, we assume that demand in each period is a nonstationary, general stochastic function, $d_t(P_t, \epsilon_t(P_t))$, that depends on the price in period t, P_t, and contains a random term, $\epsilon_t(P_t)$, with a known distribution that may also depend on price. Furthermore, we assume demands in successive periods are independent and that expected demand in period t for a price P_t, $\bar{d}_t(P_t)$, is decreasing in P_t. We do not assume a particular distribution on demand, although we do assume that demand is continuous and differentiable given price. The prices in period t are bounded below and above by $p_t^{\min}$ and $p_t^{\max}$ respectively, implying corresponding bounds on the expected demand in period t, $\bar{d}_t^{\max}$ and $\bar{d}_t^{\min}$. The prices must also be chosen from a menu of prices, $\underline{\bar{P}}_t$, that may differ by period. For example, these could be generated by discretizing a demand curve or by preordained price points. We assume the cost to change prices is negligible, such as when prices are set in a contract or offered over the Internet.

In each period, we need to decide on price, P_t, and production quantity, X_t. Let I_t represent the inventory available at the beginning of period t, and D_t is the satisfied demand in each period. Let the unindexed variables P, X, I, and D represent the vectors of price, production, inventory, and satisfied demand, respectively, over the entire horizon.

As in classic inventory literature, we define Y_t as the inventory level after production and before demand is realized. Finally, and unlike in the traditional inventory literature, we allow for discretionary sales, where the facility does not have to satisfy a unit of demand even if inventory is available. Specifically, we assume that at the beginning of every period, the facility can decide on the amount of inventory S_t to save for future sales. (In fact, we will also analyze the impact of allowing the decision maker to decide how many customers to reject after observing all or a portion of the period's demand.)

Thus, in each period, the inventory available for customers is determined by three parameters:

1. the initial inventory at the beginning of the period,
2. production level in this period, and
3. the amount of inventory to be saved for future periods.

The inventory available for customers in the current period is determined before the actual demand, d_t, is observed and hence for every period t, $D_t \le d_t$.

We assume that customers act myopically and do not buy strategically by anticipating inventory or price changes. The former is reasonable for the discretionary sales decision, as customers would not know whether a stockout is induced or unavoidable.

The assumption that customers do not anticipate price changes requires further discussion. It is certainly a common assumption in most recent pricing papers (e.g., see Federgruen and Heching 1999, Chen and Simchi-Levi 2004). There are some cases when customers are unlikely to strategically time their purchases, for example, if customers have limited inventory space or have demand that is time dependent. It is also reasonable to assume that customers do not necessarily buy strategically according to price. For example, recent studies of airline tickets, books, and CDs sold online showed that the average price difference was 20%–33% for the same item across numerous retailers (Smith et al. 1999), due in part to the effort required by consumers to find the low prices. If the effort required to find low prices online is too high for many customers, then the effort to ascertain uncertain price changes is even higher, resulting in few strategic customers. Finally, if price changes follow a historic pattern, then strategic buying may be estimated as part of the seasonal demand curve.

2.2. Delayed Production Strategy

In the delayed production strategy, the firm determines period-dependent prices at the beginning of the horizon. The firm determines production on a period-by-period basis, incorporating demand realizations in previous periods but not in the current period. For a given price vector P, the problem of determining production and sales is defined as a Markov decision problem, with the initial inventory (I) in a period as the state of the system.

Let $J_t(I)$ represent the expected profit from the beginning of period t until the end of the time horizon. We use the phrase "profit-to-go" to refer to profit in the current and future periods. Let $G_t(Y)$ represent the expected profit-to-go with Y units of product available. That is, $G_t(Y)$ includes revenue and inventory holding costs for the current period but does not include production cost; $G_t(Y)$ also includes the expected profit-to-go for future periods, J_{t+1}, and thus does include production costs in future periods. We denote the cumulative demand distribution for a given price P_t in period t as $\Psi_t^{P_t}$ and let the corresponding demand distribution be $d\Psi_t^{P_t}$ or $\psi_t^{P_t}$.

Given a price vector, the optimal expected profit in period t is as follows:

$$J_t(I_t) = \max_{Y_t: I_t \le Y_t \le I_t + q_t} \{-k_t(Y_t - I_t) + G_t(Y_t)\}, \quad \text{where} \quad (1)$$

$$\begin{aligned} G_t(Y_t) = \max_{S_t: 0 \le S_t \le Y_t} \bigg\{ \int \{ & P_t(\min(d_t, Y_t - S_t)) \\ & - h_{t+1}(S_t + \max(0, Y_t - S_t - d_t)) \\ & + J_{t+1}(S_t + \max(0, Y_t - S_t - d_t))\} \, d\Psi_t^{P_t}(d_t) \bigg\}. \end{aligned} \quad (2)$$

In the case where t is the last period in the horizon, the last term in Equation (2) is replaced with the expected salvage value of leftover inventory given by $\int v(\max(0, Y_T - d_T)) \, d\Psi_T^{P_T}(d_T)$. The first term in (1) is production cost, and the first term in (2) is the revenue, where sales is the minimum between realized demand and the available inventory after production in period t. The next two terms in (2) represent the inventory holding cost. The first of these is the holding cost for the inventory that has been set aside for the future, and the latter of the two accounts for holding cost when realized demand is less than the available inventory, $Y_t - S_t$. Finally, the last term in the second equation represents profit-to-go, given an inventory level at the beginning of period $t+1$ equal to the inventory set aside as well as inventory that was unsold because of low demand. We do not include a penalty cost other than lost profit when sales are not satisfied.

2.2.1. Analysis of Delayed Production for a Given Price Vector. Suppose that a vector of prices P is determined at the beginning of the horizon. The delayed production strategy then becomes a decision of scheduling production (X_t) and determining the inventory to set aside for the future, S_t, in each period in a way that maximizes profit. We shall first address the delayed production strategy assuming a given price vector, then discuss finding a price vector.

A simple example shows that in delayed production without discretionary sales, the expected profit function is not necessarily concave with respect to beginning inventory.

Example 1. Consider a two-period problem with no holding or production cost; demand is filled based on a first-come, first-served basis. In the second period, production capacity is zero. Without loss of generality, let demand be known exactly: D_1 units in the first period (at price P_1) and $D_2 < D_1$ units in the last period (at price $P_2 > P_1$). The profit is $P_1 \cdot I$ for $I \le D_1$, $P_1 \cdot D_1 + P_2 \cdot (I - D_1)$ for $D_1 \le I \le D_1 + D_2$, and $P_1 \cdot D_1 + P_2 \cdot D_2$ for $I > D_1 + D_2$. Clearly in this case the derivative of the cumulative profit increases at $I = D_1$, and the function is not concave.[1]

In classical inventory theory, where profit is maximized, additional assumptions are needed on the relationships between costs for the problem to have the structure such that a base stock policy is optimal. (See Zipkin 2000 for the case when production capacity is unlimited.) For instance, in Example 1, one key factor is that the price in the second period is higher than

[1] This can also be shown in general using derivatives. The function J_t will be concave with these assumptions only if $P_t \ge J'_{t+1}(0)$ for all periods, or in other words, if the current price is higher than the expected (marginal) profit for the first unit sent forward to the next period.

in the first period. In the results we will show for delayed production, we do not require any assumptions on costs except for the last period, where we assume that price is higher than salvage value.

The example suggests other cases where the total expected profit function may not be concave. In particular, if the revenue in the future is higher than the present (through an increase in price or a decrease in production cost) and if future capacity is limited (or very expensive), then this may cause the marginal profit to be nonmonotonic. However, the addition of discretionary sales allows more control over profit, and we show below that the profit function is concave in this case.

Let $J_t'(I)$ and $G_t'(I)$ be the first derivatives with respect to inventory of the functions $J_t(I)$ and $G_t(I)$; $J_t''(I)$ and $G_t''(I)$ are the second-order derivatives. We prove the following results for the delayed production strategy (see §A.1 in the appendix for the details):

Theorem 2.
- *For all* $t = 1, \ldots, T$, $G_t'(Y)$ *is nonincreasing in* Y, *and thus* $G_t(Y)$ *is concave.*
- *For all* $t = 1, \ldots, T$, $J_t(I)$ *is concave in* I.

The theorem thus implies the optimal policy for the delayed production strategy.

Corollary 3. *Given a vector of prices,* P, *there exists an optimal policy for the delayed production strategy with an optimal order-up-to level* (Y_t^*) *and an optimal save-up-to* (S_t^*).

The policy is implied in part because the objective function is concave. Indeed, the specific values for Y_t^* and S_t^* are those obtained by setting the derivatives of G and J, respectively, to 0 and solving. The optimal decisions also have an economic interpretation that relates to the derivatives of (or marginal) profit. In the case of Y, one should produce as long as the additional or marginal profit ($G_t'(Y)$) is better than the production cost (k_t). For S, one should save as long as the future marginal profit from the units ($J_t'(S) - h$) is better than the current price (P_{t-1}).

When production capacity is limited, the result is a modified order-up-to policy. If there is sufficient capacity, then produce enough to bring the inventory level up to the order-up-to level; otherwise, produce to maximum capacity. Similarly, the optimal policy for inventory to save for the future is to set aside the save-up-to amount if possible; otherwise, set aside the maximum available. The modified policies based on limited capacity also follow from the concavity of the profit functions. Clearly if one cannot achieve the maximizing point, then one should produce (save) as much as possible up to that value, as the profit is increasing prior to the maximum and decreasing after that point.

Furthermore, the structure of the delayed production problem is such that the assumption of when to reject customers, either before demand is observed or after it is observed, has no impact on the optimal inventory policy. Specifically,

Corollary 4. *Given a vector of prices,* P, *assume that the decision to reject or accept customers is made* after *the period demand has been observed. There exists an optimal policy for the delayed production strategy characterized by an order-up-to level* (Y_t^*) *and an optimal save-up-to* (S_t^*) *level.*

This is due to the fact that the optimal decision S_t^* found from the derivative of the profit function is independent of the demand that arrives. Knowing the amount of observed demand does not have any impact on how much inventory the firm should transfer for future periods. Thus, the decision to save (correspondingly to accept or reject customers) can be made anytime, including as the customers order.

Using discretionary sales can also improve profit. Consider applying this strategy to the inventory problem in Example 1.

Example 5. Since $J_t'(I) > P_1$ for $I \le D_2$, set $S_1^* = D_2$. The profit is $P_2 \cdot I$ for $I \le D_2$, $P_2 \cdot D_2 + P_1 \cdot (I - D_2)$ for $D_2 \le I \le D_1 + D_2$, and $P_2 \cdot D_2 + P_1 \cdot D_1$ for $I > D_1 + D_2$. The marginal profit in the first period is now P_2 for $I \le D_2$ and P_1 for $D_2 < I \le D_1$, and the total profit function is concave. The overall profit is also higher than in Example 1. The profit improvement from discretionary sales is $P_2 - P_1$ for each unit of inventory $I \le D_2$. Let the capacity in Period 1 be equal to D_2. The profit improvement with discretionary sales grows as a function of increasing P_2, decreasing P_1, or increasing D_2.

The results on delayed production can also be extended to the case where lost sales are penalized by some amount, say β_t. In this case, the discretionary sales decision should be determined by $S^*_{t-1} = 0$ if $P_{t-1} > J'_t(I) - h_t - \beta_t$ for all I, otherwise as $\max\{I\colon I > 0$ and $P_{t-1} \le J'_t(I) - h_t - \beta_t\}$.

2.2.2. Heuristic for Prices in Delayed Production. The policy we have developed thus far does not include prices as decisions. One common method of setting prices in advance is to do so based on forecasted mean demand. This heuristic is described below, together with an analysis of its worst-case performance.

The deterministic pricing problem we present has similar assumptions (e.g., limited production capacity and lost sales) as the stochastic problems we consider except for the relationship between demand and price. For the deterministic problem, we assume that demand is a nonincreasing function of the product's price and that the relationship between demand and price is known. As there is a known relationship between price and demand, it is sufficient to decide on sales in a particular period to determine the price in that period. This implies that the revenue function, R_t, can be expressed as a function of satisfied demand, D_t. Thus, $R_t(D_t) = P_t D_t$, and the pricing problem with deterministic demand, referred to as Problem PP, can be formulated as

$$\text{(PP)} \qquad \max \sum_{t=1}^{T} (R_t(D_t) - h_t I_t - k_t X_t)$$
$$\text{subject to } I_0 = 0$$
$$I_{t+1} = I_t + X_t - D_t, \quad t = 1, 2, \ldots, T$$
$$X_t \le q_t, \quad t = 1, 2, \ldots, T$$
$$I_t, X_t, D_t \ge 0, \quad t = 1, 2, \ldots, T.$$

The objective is to determine prices in each period and the production schedule for the horizon to maximize profit. The first constraint requires that the initial inventory for the first period is zero. The second constraint ensures that inventory is balanced and that enough units are produced to meet customer sales. Production is limited by the capacity of the system in the third constraint.

We consider now the problem of determining prices for delayed production, using the deterministic problem with mean demand. For any price P, $\bar{d}_t(P)$ is the corresponding expected demand at Period t, and $\bar{d}(P)$ is the vector of expected demands for price P. Since in the deterministic case, sales may be less than the demand as generated by the demand curve, we find prices in Problem PP by letting $R_t(D_t) = \max_{P_t}\{D_t P_t\colon \bar{d}_t(P_t) \ge D_t\}$. We refer to Problem PP with this revenue function as PP $(\bar{d})$. Also note that for deterministic demand, the delayed production dynamic program shown in Equations (1) and (2) is equivalent to Problem PP.

There are many ways to generate candidates for the discrete set of prices. One such candidate, for example, can be generated by solving Problem PP. Another method for generating a candidate might be by solving another deterministic problem. Alternatively, some predetermined policy might generate additional candidates. In any case, Problem PP produces one such element in the discrete set, and we analyze the worst-case performance of using this candidate from the discrete set as the set of prices.

To obtain prices based on expected demand, solve Problem PP$(\bar{d})$ and denote the resulting optimal profit as $F^*(\bar{d})$. Let D^H and X^H be the demand and production solution vectors to Problem PP$(\bar{d})$, and let P^H be the optimal prices resulting from a solution of D^H and X^H. The prices can be applied to the delayed production strategy to obtain the corresponding inventory policy.

We refer to the delayed production algorithm with prices chosen as above as the delayed production heuristic. To examine the effectiveness of the delayed production heuristic, we perform worst-case analysis and show there exist both upper and lower bounds on the performance of the heuristic. For this purpose, let Z^*_1 be the expected profit of an optimal policy for the delayed production strategy and Z^H_1 be the expected profit of the delayed production heuristic.

We begin by showing that the optimal profit from Problem PP$(\bar{d})$ is an upper bound on Z^*_1. We have

Lemma 6. $Z^H_1 \le Z^*_1 \le F^*(\bar{d})$.

For the proof of this result the reader is referred to §A.2 in the appendix. We also show in the appendix that a similar bound from a deterministic problem exists for the problem where demand in a period depends on the prices of all periods (such as strategic customers). Unfortunately, the following example shows that the bound in general may be quite weak.

Example 7. Consider a one-period problem with no holding or production cost. There is one possible price, 1, with distribution function $\psi^1(0) = 1 - 1/n$ and $\psi^1(n) = 1/n$, for $n > 0$. The available production capacity is one unit. Then $P^H = 1$ with $F^*(\bar{d}) = (1)(1) = 1$. Of course, the optimal price is also 1, and $Z_1^* = Z_1^H = 0 + (1)(1)(1/n) = 1/n$. Thus, in this case, $F^*(\bar{d})$ is an arbitrarily (infinitely) bad upper bound.

Observe that in this example, the deterministic strategy had the same price as the stochastic strategy and further that the expected profit of the heuristic was equal to the expected profit of the optimal solution for the stochastic problem. The following example shows that in general this is not true.

Example 8. Consider a one-period model with no holding or production cost. Price $P \in [1, 3/2]$ with $\bar{d}(P) = 4 - 2P$ and distribution function $\psi^P(0) = \psi^P(2\bar{d}(P)) = 1/2$. The production capacity is limited to two units. In this case, $P^H = 1$ with $F^*(\bar{d}) = (1)(2) = 2$ and $Z_1^H = (1)(2)(1/2) = 1$. However, the optimal price is $3/2$ with $Z_1^* = (3/2)(2)(1/2) = 3/2$.

So in general, choosing prices based on forecasted means may not be effective. However observe that in both these examples, demand has a unique structure: either zero or very high, and much higher than the capacity. The challenge thus is to characterize the effectiveness of the heuristic for more reasonable demand functions. Specifically, we next provide data-dependent worst-case performance analysis for the delayed production heuristic.

For any price P, let $d_t^{\min}(P)$ be the minimum possible demand at Period t, and let $\xi^{\min}(P) = \min_t[d_t^{\min}(P)/\bar{d}_t(P)]$. Consider $\xi^{\min} = \xi^{\min}(P^H)$, defined over the prices generated by the delayed production heuristic.

Lemma 9. *If the deterministic revenue curves, $R_t(D_t)$ are concave with respect to satisfied demand, then we have $Z_1^H \geq \xi^{\min} Z_1^*$.*

For details of the proof the reader is referred to §A.3 in the appendix. In the next example, we show that the bound is tight.

Example 10. Consider a one-period model with no holding cost but with production cost = \$2 per unit. Price $P \in [3, 4]$ with $\bar{d}(P) = 4/(P-2) - e$. The demand distribution function is $\psi^4(\bar{d}(P)) = 1$, and $\psi^P((P/4) \cdot \bar{d}(P)) = \psi^P(2\bar{d}(P) - (P/4)\bar{d}(P)) = 1/2$ for $3 \leq P < 4$. The production capacity is limited to 4 units. In this case, $P^H = 3$ with $F^*(\bar{d}) = (3-2)(4/(3-2) - e) = 4 - e$. Because the optimal production is $(3/4)(4 - e)$ for this price, we have $Z_1^H = (3-2)(3/4)(4-e) = 3 - (3/4)e$. However, the optimal price is 4 with an optimal production quantity of $2 - e$, and $Z_1^* = (4-2) \cdot (2-e) = 4 - 2e$. Hence, $\xi^{\min} = 3/4$ and as $e \to 0$, $Z_1^H = 3 = (3/4)(4) = \xi^{\min} Z_1^*$.

2.2.3. Special Case: Fixed Pricing Strategy. Although many firms may be interested in finding the optimal fixed price over an entire horizon, the problem of determining the best fixed price while allowing production flexibility under stochastic demand has received little attention in the literature. Indeed, the fixed price strategy is a special case of the delayed production strategy, except that *only a single price is allowed in the horizon*. Let P^{Fix} represent the single price over the horizon, and let the price in each Period $P_t = P^{\text{Fix}}$. Of course, the dynamic program described earlier holds for any price vector and therefore for a fixed price policy.

Determining the optimal policy is rather straightforward given a finite set of discrete prices: Let the set of prices be $\underline{\bar{P}}$; then the optimal fixed price is $P^{\text{Fix}*} = \arg\max_{P_t \in \underline{\bar{P}}} J_0(0)$. Along with the optimal price, the strategy defines order-up-to and save-up-to targets that determine production and sales decisions.

2.3. Delayed Pricing Strategy

Many firms need to plan production in advance but would like to take advantage of price flexibility. Price flexibility provides a mechanism for clearing inventory from previous periods as well as helping to match supply and demand. As in delayed production, we define the problem using the initial inventory as the state of the system, and again the demand distribution, $d\Psi_t^{P_t}$, depends on the price in Period t. Prices are discrete values chosen from a set of prices ($\underline{\bar{P}}_t$) that may depend on the period.

As before, let Y_t be the inventory level after production and before demand is realized, and S_t is the inventory that is set aside to satisfy demand in future periods. Again, we begin by assuming the decision on the amount of inventory to save for future periods has been determined at the beginning of a period before demand is realized. Let $J_t(I)$ represent the expected profit from the beginning of Period t until the end of the horizon with an initial inventory level of I; let $G_t(Y)$ represent the expected profit-to-go with Y units of product available. Given a vector of production values, the optimal expected profit in Period t for delayed pricing is

$$J_t(I_t) = -k_t(Y_t - I_t) + G_t(Y_t), \quad \text{where} \tag{3}$$

$$G_t(Y_t) = \max_{S_t: 0 \le S_t \le Y_t, P_t \in \bar{P}_t} \bigg\{ \int \{P_t(\min(d_t, Y_t - S_t)) - h_{t+1}(S_t + \max(0, Y_t - S_t - d_t)) + J_{t+1}(S_t + \max(0, Y_t - S_t - d_t))\} \, d\Psi_t^{P_t}(d_t) \bigg\}. \tag{4}$$

When t is the last period in the horizon, the last term in the equation is replaced with the expected salvage value of leftover inventory given by $\int v(\max(0, Y_T - d_T)) \, d\Psi_T^{P_T}(d_T)$.

As before, the first term in (3) is production cost, and the first term in (4) is the revenue, where sales is the minimum between realized demand and the available inventory after production in Period t. The next two terms in (4) represent the inventory holding cost and, finally, the profit for the future is accounted for in the last term of (4), as a function of the inventory that is carried forward.

2.3.1. Analysis of Delayed Pricing for a Given Production Schedule. Given an initial production schedule X, the delayed pricing strategy becomes a question of the pricing policy to set in each Period (P_t) and the optimal discretionary sales policy (S_t). Unfortunately, this problem does not possess the same properties as the delayed production strategy, and the optimal policy is more complex. Specifically, a result similar to Theorem 2 does not hold; that is, the expected profit-to-go function in Period t, $J_t(I)$ shown in Equation (3), is not necessarily concave with I. Thus, threshold policies (e.g., save-up-to) are no longer optimal.

The lack of concavity also implies that deciding when to reject customers (either before the period demand is realized or after) can have an impact on expected profit (unlike the delayed production strategy). These results are summarized in the next property, and an example follows.

Proposition 11. *The expected profit-to-go function $J_t(I)$ is not necessarily concave with I; the optimal amount of inventory to save for the future, S_t, depends on the realized demand in Period t.*

Example 12. Consider only the last two periods of a multiperiod problem. The price in the second-to-last period is fixed to be \$0.6, and the inventory level after production for that period is 8 units. There are two possible prices in the last period: \$1 with demand ~Uniform $[4, 8]$, and \$1.4 with demand ~Uniform $[2, 4]$. All other costs are equal to zero. Demand in Period $T-1$ can take values between 0 and 8. Table 1 shows profit and price results evaluated for various scenarios. Observe that $J_T(I)$ is not concave in I; the marginal value of this function decreases (from 1.4 to 0.35) and then increases (to 0.675). Also observe that in Period $T-1$, the amount of inventory to save for Period T depends on demand. If demand is low (e.g., 4 units), clearly saving six with a profit of 6.7 is better than saving 4 for the future with a profit of 6.6; the exact best saving amount in this range is 5.6 with a profit of 6.72. However, clearly higher profit can be achieved by saving a lower amount if demand is sufficiently high (e.g., saving three for the future and selling five now for a profit of 6.85). The overall best saving policy with eight units of beginning inventory is to save 3.14 with a profit of 6.86 as long as the remaining inventory is less than 4, otherwise save 5.6.

The example shows that for delayed pricing, deciding on the amount to save for the future *before* demand is observed, yields an inferior performance relative to determining the amount to save for the future after observing customer demand, because the marginal profit contribution is not necessarily decreasing with decreased demand. Further, the

Table 1 Expected Profit by Realized Demand for Example 12

Period T				Period $T-1$		
Initial inventory in T	$J_T(I)$ given $P_2 = 1.4$	$J_T(I)$ given $P_2 = 1$	Best $J_T(I)$	Amount sold in $T-1$	Revenue in $T-1$	$G_{T-1}(8)$ for amount sold
0	0	0	0	8	4.8	4.8
1	1.4	1	1.4	7	4.2	5.6
2	2.8	2	2.8	6	3.6	6.4
3	3.85	3	3.85	5	3	6.85
4	4.2	4	4.2	4	2.4	6.6
5	4.2	4.875	4.875	3	1.8	6.675
6	4.2	5.5	5.5	2	1.2	6.7
7	4.2	5.875	5.875	1	0.6	6.475
8	4.2	6	6	0	0	6

example shows that the expected profit of delayed pricing may not be concave in inventory even if the amount to save is determined after demand arrives (as $J_T(I)$ is not concave in I).

Now consider the optimal pricing policy for delayed pricing. Intuition suggests that as the initial inventory level, I_t, increases, the corresponding optimal price should decrease, as price is often used as a market-clearing tool. Unfortunately, this property may not hold in general (even when the optimal saving value is used), as is illustrated in the following.

Proposition 13. *In a Period t, the optimal price in that Period, P_t, does not always decrease with increasing initial inventory.*

Example 14. The last period is the same as in Example 12. However, an additional price is now available in Period $t-1$. The first price, $P^1 = 0.60$, has corresponding (known) demand of 6 units. The second price, $P^2 = 0.63$, has corresponding (known) demand of 4 units. Table 2 shows the best amount to save and maximum expected profit evaluated at several possible beginning inventory levels for Period $T-1$. Note that when the beginning inventory level is 8 units, the best amount to save using P^2 is 3.1 if demand is high, or 5.48 when demand is relatively low (i.e., remaining inventory is >4). It is clear in this case that price does not always decrease with increasing inventory.

This observation about nonmonotonic price is due to the lack of concavity of the objective function.

Table 2 Expected Profit and Optimal Price by Inventory Level for Example 14

Inventory in $T-1$		$P = 0.60$ (Dem $= 6$)	$P = 0.63$ (Dem $= 4$)	Best price
14	Sell	6	4	P^1
	Save	8	10	
	$G_{T-1}(14)$	9.6	8.52	
10	Sell	4.4	4	P^2
	Save	5.6	6	
	$G_{T-1}(10)$	7.92	8.02	
8	Sell	4.857	2.52	P^1
	Save	3.143	5.480	
	$G_{T-1}(8)$	6.857	6.794	
4	Sell	0.86	0.9	P^2
	Save	3.143	3.100	
	$G_{T-1}(4)$	4.457	4.484	

Specifically, if the future profit is not concave, then it is not true that more inventory (until the maximum) is always better. The ability to change price is a way to control how much inventory is sent forward, and price is used to control for the nonsmooth future profit function.

The result is that the optimal policy in our case is not a list price policy. A list price policy is a policy in which price is a nonincreasing function of the inventory level at the beginning of the period. List price policies have been proven to be optimal for models with backlogged demand and price and production flexibility under some assumptions on the revenue curve and holding costs (see Federgruen and Heching 1999). Our model, on the other hand, is a lost sales model that may have discrete prices, which implies that there may be jumps in the marginal profit.

Of course, the implication of this observation and the lack of concavity of the expected profit function is that each beginning inventory level may have a different price, and it is necessary to consider all prices in each period to determine the price that maximizes expected profit.

2.3.2. Heuristic for Production in Delayed Pricing. In this section, we describe a heuristic to set production for the delayed pricing model. Given a production schedule, the dynamic program represented by Equations (3) and (4) generates an optimal pricing and discretionary sales policy. Because the state space is large, and because determining discretionary sales prior to demand realization may not be the best strategy, we force $S_t = 0$ for all t.

In many situations, a deterministic forecast is used to plan a production schedule. Thus, Problem PP, as outlined in §2.2.2, can be solved to generate a production vector. Call the optimal production quantities X^H, and apply the production schedule X^H to Equations (3) and (4) to generate a pricing and sales policy for delayed pricing. We call this approach the delayed pricing heuristic. The performance of this heuristic is also bounded by data-dependent functions (see Swann 2001 for details).

It is also possible to generate the policy for the full-flexibility strategy, including the optimal production decision. In this case, the production decisions must be evaluated from zero to capacity for each price to determine the profit of the complete strategy, with the best combination chosen. Unfortunately, because the policy may be complex, this may be computationally intensive.

3. Computational Analysis

3.1. Performance Measures

We first solve the delayed production and pricing heuristics described in the paper, in addition to the best fixed pricing strategy. We also find the optimal policies for delayed production and delayed pricing by enumerating all the possible pricing or production vectors within a range. Finally, we solve the full-flexibility problem with period-specific price and production decisions; this is an obvious upper bound on the other strategies, as in those a decision is fixed in advance. We use expected profit in the full-flexibility strategy as our common comparison benchmark for the partial planning strategies.[2] The ratio we use is the following: (expected profit of partial planning heuristic)/(expected profit of full flexibility).

Using the same ratio, we also measure the additional gain of the optimal partial planning strategy to full flexibility. We also use this type of benchmark ratio to look at the impact of discretionary sales, by comparing the profit of the delayed production heuristic or optimal strategy with $S_t = 0$ in each period to the profit where S_t is unrestricted.

3.2. Computational Details

Our automotive industrial partner provided us with demand curves and variable production costs. In the experiments below, we use (transformed) data for a typical mid-size car. An example of our base data appears in Table 3.

The basic demand curves used by our partner are linear. We study both deterministic variability, described by seasonality factors (γ_t), as well as stochastic variability, or uncertainty, (ϵ_t), which we assume is additive to the demand curve. The resulting curve in a Period t is thus $d_t = \gamma_t(ap_t + b) + \epsilon_t$; we use this type of seasonality unless otherwise indicated.

[2] Because of the difficulties in determining discretionary sales for full flexibility (as in delayed pricing), we set $S_t = 0$ for all t in this strategy. To ensure fair comparisons, delayed production and fixed pricing are also solved without discretionary sales for comparisons to full flexibility and delayed pricing.

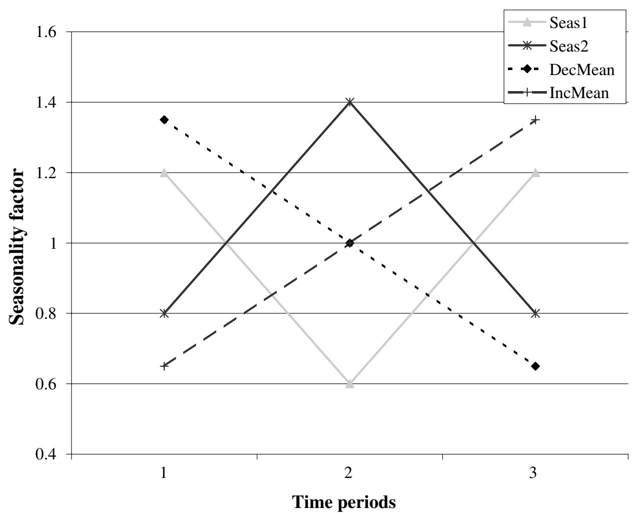
Table 3 **Computational Analysis Data Without Seasonality**

Prod cost	Holding cost
\$100	\$5
Capacity levels	
Level (C/D^*)	Value
Low (50%)	102
Med (75%)	153
High (100%)	205
Demand curve	
a	b
−2.67	676
(with Prod Cost)	
D^*	P^*
205	\$176.59
Uncertainty levels for Capacity = Med	
CV_U	σ for avg ϵ
0.2	0.2(153) = 31
0.4	0.4(153) = 61
0.6	0.6(153) = 92
0.8	0.8(153) = 123
Uncertainty levels for $CV_U = 0.2$	
Capacity	σ for avg ϵ
Low	0.2(102) = 20
Med	0.2(153) = 31
High	0.2(205) = 41

We also ran all experiments with alternate applications of seasonality, where $d_t = ap_t + \gamma_t(b) + \epsilon_t$. This allows a shifting of the demand curve without changing the sensitivity to price.

To measure and determine the demand uncertainty, we define the coefficient of variation of demand in a given period as follows: $CV_U = \mathrm{stdev}(d_t)/\mathrm{mean}(d_t)$. For a given test case, CV_U is constant across the periods. We consider $CV_U = 0.2$ to be our base case, and higher values of CV_U were also tested but are not shown. We use a normal distribution with a mean of 0 for ϵ_t. The distribution is truncated to be ≤ 2 standard deviations from the mean, and we do not allow demands to be negative. The standard deviation of the random component, ϵ_t, can be calculated as a function of CV_U, and the value indirectly depends on the price in each period. Alternatively, we also tested

Figure 1 **Product Seasonality Factors**



all cases with ϵ the same in each period and independent of price; the first form of uncertainty is depicted unless otherwise indicated.

The seasonality factors we use (Figure 1) represent different product types or industries over three time periods. Seasonality 1 (Seas1) has seasonality low at first, high in the middle, and low at the end of the horizon; it is similar to demand for cars. Seasonality 2 (Seas2) is the opposite of Seas1, that is, high-low-high. Decreasing mean (DecMean) has decreasing demand over the horizon, as in fashion items or products that experience obsolescence. Increasing mean (IncMean) has increasing demand, similar to products with word-of-mouth effects, such as a musical CD. The mean of each set of seasonality factors is 1, and the coefficient of variation is approximately 0.35. The factors are multiplied against the demand curve above.

We set capacity levels to be low (88 units), medium (153), and high (205); these values correspond to ratios of $C/D^* = 50\%, 75\%$, and 100%, where C is the capacity and D^* is the average optimal demand satisfied in the uncapacitated case. Specifically, D^* is obtained by taking the derivative of profit $(=D(P-K))$ with respect to D giving $D^* = (-\gamma b)/(2(1-k\gamma a))$ for the main form of seasonality. Holding cost is 5% of the production cost in each period. The salvage value is zero in all scenarios, and all parameters other than demand are stationary over the time horizon.

To find the optimal strategies we enumerate the pricing and production vectors. We allow 10 different price and production values in each time period; this implies that 10^3 stochastic problems are solved to find each optimal partial planning strategy. In comparison, our heuristic solution can be found by solving one deterministic optimization problem and one stochastic problem, where the latter grows as a function of the number of periods times the number of values allowed in each period. Full flexibility may be easier to solve than the optimal partial planning strategies (but harder than the heuristics), as one stochastic problem is solved in time that grows as a function of the number of periods times the number of production values times the number of pricing values.

3.3. Insights from Computational Analysis

Observation 15. As capacity becomes more constrained, the benefit of dynamic pricing tends to increase.

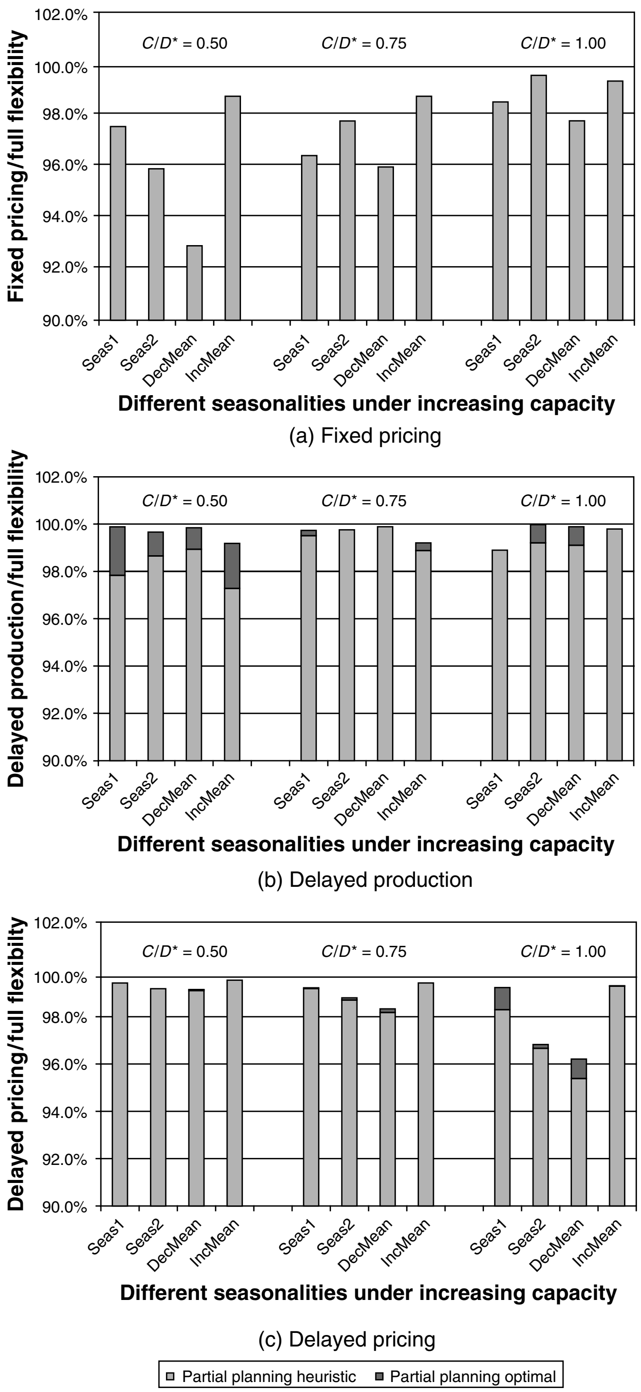
Both partial planning strategies (and full flexibility) can use nonstationary price to match supply and demand, though in the case of delayed production the price is planned in advance. In contrast, fixed pricing only has production and inventory decisions to match demand. We examine the impact of price flexibility on profit and the conditions under which it is most beneficial.

Consider Figure 2(a), which displays fixed pricing relative to full flexibility. Overall, the relative profit of fixed pricing declines when capacity is tight. In contrast, the relative profit of delayed production compared to full flexibility remains about the same as capacity becomes tighter. The benefit of dynamic pricing relative to fixed pricing also was large under large uncertainty (graphs not shown).

This capacity result can be explained as follows. When capacity is tight, no excess inventory is available to serve as a buffer, and pricing is the primary lever that can be used. With demand seasonality, variable pricing can be used to match supply and demand. Indeed, both of the partial planning strategies have pricing as a flexibility tool to match changes in demand, while the fixed pricing strategy does not.

Note that fixed pricing occasionally outperforms the partial planning strategies. For Seas2 at the biggest capacity, fixed price (Figure 2(a)) is slightly better than the delayed production heuristic (Figure 2(b)), but not the optimal delayed production strategy. It is better than the delayed pricing heuristic and optimal solutions (see Figure 2(c)) sometimes, particularly when capacity is high. This is because fixed pricing has more flexibility in production than delayed pricing does, and this production flexibility is important when capacity is high.

Figure 2 **Effect of Increasing Capacity** ($CV_U = 0.2$)



Of course, we assume that the cost to change prices is negligible. In some industries or situations this is not true, for example, when sales are primarily through catalog channels, when changing price requires significant labor to mark every item, or when advertising costs are high. Further, there may be an intangible cost associated with the loss of customer goodwill because of a more complex pricing policy. In these situations, the costs of implementing dynamic pricing may outweigh the potential benefit from increased revenue.

Observation 16. The benefit of all of the pricing strategies depends on the type of seasonality.

The type of seasonality (i.e., when in the horizon the high and low demand points occur) clearly impacts the benefit of dynamic pricing (see Figures 2(b) and 2(c)). For example, delayed pricing and fixed pricing typically perform best under IncMean, while delayed production is more effective under other types of seasonality. Similar results hold under different levels of uncertainty. This suggests that a company should understand the nature of their seasonality when deciding where flexibility is most useful.

Observation 17. Delayed production is often more effective than delayed pricing.

This is particularly true for high uncertainty (graphs not shown). An exception is the case of tight capacity and IncMean seasonality, in which delayed pricing is more effective than delayed production (see Figures 2(b) and 2(c)). Clearly, when capacity is tight, production quantities are mostly determined by the available capacity. In this case, pricing flexibility is more important than production flexibility, so delayed pricing may have a higher profit than delayed production.

The result concerning the performance of delayed pricing versus delayed production is somewhat different than earlier results obtained for a related two-stage problem by van Mieghem and Dada (1999). The authors consider a single-period model where the postponed decisions are made after demand is realized. They found that in many instances pricing postponement outperformed production postponement, usually when demand variability and production costs were low. Although we also found variability and production costs to be factors for delayed pricing as a strategy, in most cases delayed production dominated delayed pricing. Of course, in our models demand is realized after the second decision (i.e., pricing in delayed pricing), so price may not be used to completely clear inventory. Two other differences between our model and the one in their work are that we consider a longer time horizon and that we include the impact of different types of seasonality factors.

Observation 18. The delayed pricing heuristic described in this paper performs very well compared to the optimal delayed pricing solutions; the delayed production heuristic often performed well. The level of capacity can influence the performance of both, and the type of seasonality function can impact delayed production particularly.

In the graphs shown, the biggest gap between the delayed production heuristic and the delayed production optimal solutions is 2.1 percentage points (see Figure 2(b)). Clearly, when capacity is low, delayed production can be improved by using the optimal partial planning decisions over the heuristic ones. This is intuitive, as when capacity is low, price is the only lever remaining to shift demand, so it is more important to use the optimal price vector. Uncertainty is also a factor: The gap between the optimal and heuristic delayed production strategies grows with uncertainty.

Overall, the delayed production heuristic did reasonably well using prices planned from deterministic demand and sometimes achieved the same profit as the optimal delayed production strategy. For the cases with randomness (ϵ) dependent on price, the optimal improved the profit on average by only 0.4 points under increasing capacity and 0.2 points under increasing uncertainty, averaging over both forms of seasonalities. However, the performance of the heuristic is worse for the cases with randomness independent of price; the average (highest) increase in the profit ratio due to the optimal was 0.4 (2.1) points under increasing capacity and 2.0 (7.1) points under increasing uncertainty with the alternate form of epsilon, where both averages are taken over the two forms of seasonality factors.

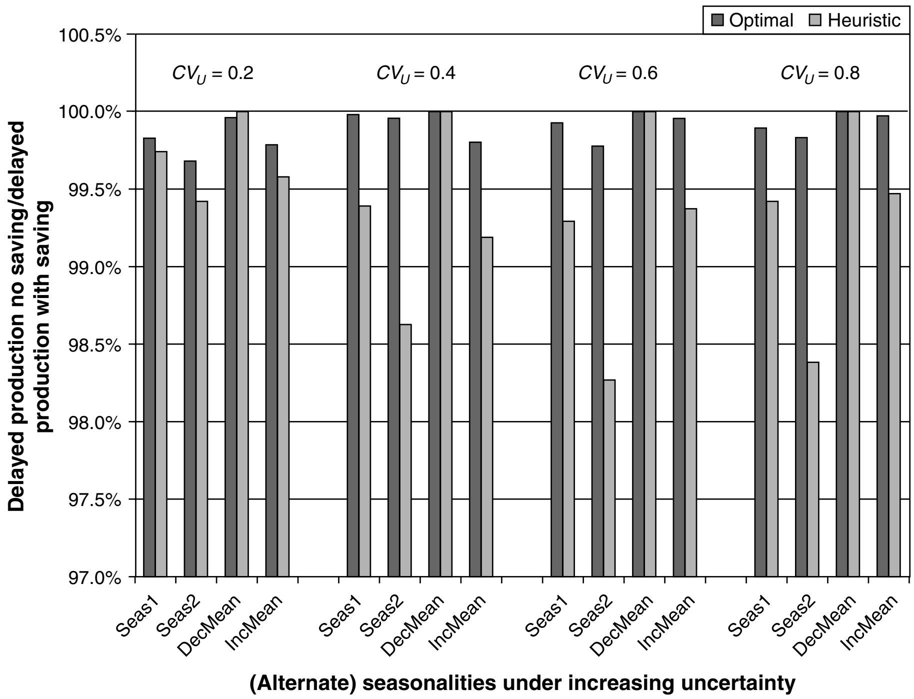
The delayed pricing heuristic performed even better compared to its optimal solution. In the worst case of the graphs shown, the optimal is 0.9 points higher on the ratio (see Figure 2(c)). Interestingly, this occurs when capacity is high, in contrast to the heuristic performance of delayed production. This makes sense when considering the specific strategy though. When capacity is low, the deterministic forecast is sufficient, as the firm produces as much as possible. However, when capacity is high, using the best production amount in each period offers an advantage, even when planned in advance. The average increase in performance for the optimal delayed pricing (considering both forms of seasonalities) is 0.2 points for increasing capacity, 0.1 for increasing uncertainty, and 0.1 points for increasing capacity or increasing uncertainty when uncertainty is independent of price.

Observation 19. Discretionary sales can have an impact of several percentage points over strategies without discretionary sales.

In about half the experiments, discretionary sales had little to no impact. This is in part due to the nature of the seasonality factor, γ_t (and the lack of other time-dependent factors). Observe that the deterministic profit function for a single period is revenue − production cost, or $d_t(P_t - K) = P_t(\gamma_t)(aP_t + b) - K(\gamma_t)(aP_t + b)$. The single-period price that optimizes this, $P_t^* = (Ka - b)/2a$, is the same in every period if the production cost is stationary. However, in the second type of seasonality we consider, where $d_t = aP_t + \gamma_t b$, the price that optimizes the single-period profit, $(P_t^* = (Ka - b\gamma_t)/2a)$, depends in part on the seasonality factor. It is for seasonalities of the second type that we found discretionary sales to have an impact on the profit of delayed production. See Figure 3, where the uncertainty is independent of price and the alternate form of seasonality was used. In the figure, discretionary sales has an impact of up to two percentage points over the strategies that do not allow for this decision. This is particularly true when capacity is low, which is intuitive, as resources should be used more carefully when they are scarce. Stronger behavior was observed for additional tests run with lower capacity and higher uncertainty.

For the experiments with the alternate seasonality, which were half of all experiments run, the level of discretionary sales was positive for both the heuristic

Figure 3 **Impact of Using Discretionary Sales in Delayed Production Under Increasing Uncertainty** ($C/D^* = 0.75$ and $\epsilon_t = \epsilon$)



and optimal delayed production solution in 75% of these experiments (or 37.5% of all experiments). The seasonality types Seas1 and Seas2 each had positive S_t (discretionary sales level) values for one time period, DecMean values were zero in all cases, and IncMean had positive values in two time periods most of the time. On average, in the experiments with positive S_t values, 25.1% of the total capacity was allocated to reserve inventory (inventory sold in the future) in the case of the heuristic, and 22.1% was allocated in the optimal delayed production strategy. As the averages suggest, in almost all cases the heuristic reserved higher inventory for the future than the corresponding optimal strategy. This makes sense, as the optimal strategy is optimizing over price as well, while the heuristic can only use inventory to match supply and demand. Thus, discretionary sales can be seen as a tool for shifting resources to manage revenue more effectively.

4. Extensions and Conclusions

In this paper, we present a number of partial planning strategies, motivated by the variety of problems that firms must address. For all of the strategies, delayed production, delayed pricing, full flexibility, and fixed pricing, we provide methods to determine pricing and production decisions. We introduce the general concept of discretionary sales, where inventory is set aside to satisfy future demand even if sales are lost in the current period.

In the case of delayed production (and thus fixed pricing as well), we show that the structure of the problem leads to an optimal policy characterized by two parameters: an order-up-to level, Y_t, and a save-up-to level, S_t. If the inventory level at the beginning of the Period, I_t, is below Y_t, an order of size $\min\{Q_t, Y_t - I_t\}$ is placed; otherwise, no order is made. Interestingly, the amount to save for the future, S_t, is independent of realized demand and thus can be determined at the beginning of the planning horizon. In other words, the decision of how many customers to reject to save inventory for future periods can be done as demand arrives in a particular period.

In computational studies, dynamic pricing strategies, that is, delayed production, delayed pricing, and full flexibility, outperformed fixed pricing strategies significantly (up to 7%), and this difference often increased with tightening capacity. The relative benefit of each strategy depends particularly on the type of seasonality but could also depend on production cost or uncertainty level.

In general, the optimal delayed production strategy outperforms the optimal delayed pricing strategy, and our partial planning heuristics each performed well compared to their respective optimal solutions (>99% of their respective optimal, averaged over the experiments) and compared to the full-flexibility strategy (˜98% of that strategy). In addition, we showed that discretionary sales can result in modest improvements, particularly when capacity is tight. Indeed, discretionary sales can be seen as another way to manage demand by shifting resources to where they are most needed.

There are a number of future directions suggested by the results in this paper and by industry-motivated situations. Further exploration of production and inventory models where inventory may be set aside for the future would be useful, particularly when capacity is limited. For instance, it would be interesting to see how the results change when customers are known to act strategically, taking into account potential price changes and stockouts. Also of significant interest is a problem with multiple products and multiple parts, where there is a limited supply of parts and limited common production capacity.

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Appendix

A.1. Proof of Theorem 2

Proof. 1. In the last period, we see from Equation (2) that

$$G_T(Y) = \int \{P_t(\min(d_t, Y_t - S_t)) + v(\max(0, Y_t - d_t))\}\, d\Psi_t^{P_t}(d_t).$$

Using Leibnitz's rule, we obtain $G'_T(Y) = P_T[1 - \Psi_T^{P_T}(Y)] + v\Psi_T^{P_T}(Y)$, which gives us $G''_T(Y) = -P_T\psi_T^{P_T}(Y) + v\psi_T^{P_T}(Y)$. Since by assumption $v \le P_T$, $G''_T(Y) \le 0$ and $G_T(Y)$ is concave.

2. Given t, $t = 1, \ldots, T$, assume that $G'_t(Y)$ is nonincreasing. We show that J_t is concave.

- Define Y_t^* as 0 if $k_t > G'_t(Y)$ for all Y, otherwise as $\max\{Y\colon Y > 0 \text{ and } k_t \le G'_t(Y)\}$.
- Hence,

$$J_t(I) = \begin{cases} -k_t q_t + G_t(I + q_t) & \text{if } I \le Y_t^* - q_t \\ -k_t(Y_t^* - I) + G_t(Y_t^*) & \text{if } Y_t^* - q_t < I \le Y_t^* \\ G_t(I) & \text{if } Y_t^* < I. \end{cases}$$

- Because $G'_t(Y)$ is nonincreasing and thus G_t is concave, and by the choice of Y_t^*, Y_t^* maximizes $J_t(I_t)$ with production of $\max(0, \min(q_t, Y_t^* - I_t))$.
- We consider these cases and show that $J'_t(I)$ is nonincreasing within each of them:

  (a) *Case* 1. $I \le Y_t^* - q_t$: $J'_t(I) = G'_t(I + q_t) \ge k_t$ because of the choice of Y_t^*. As $G'_t(Y)$ is nonincreasing, so is $J'_t(I)$ for this case.

  (b) *Case* 2. $Y_t^* - q_t < I \le Y_t^*$: $J'_t(I) = k_t$. $J'_t(I)$ is constant and thus is nonincreasing in this case as well.

  (c) *Case* 3. $I > Y_t^*$: $J'_t(I) = G'_t(I) \le k_t$ because of the choice of Y_t^*. Again, because $G'_t(Y)$ is nonincreasing, $J'_t(I)$ is nonincreasing.
- To see that $J'_t(I)$ is nonincreasing for all I, we compare the three cases as I increases. We have that $J'_t(I)$ is biggest when I is small ($\ge k_t$); it decreases to k_t in the second case, and when I is largest, $J'_t(I)$ is smallest ($\le k_t$). Thus, $J'_t(I)$ is nonincreasing, and $J_t(I)$ is concave.

3. Given that $G'_t(Y)$ is nonincreasing and $J_t(I)$ is concave for $t = t, \ldots, T$, we now show that $G'_{t-1}(Y)$ is nonincreasing.

- Define S_{t-1}^* as 0 if $P_{t-1} > J'_t(I) - h_t$ for all I, otherwise as $\max\{I\colon I > 0 \text{ and } P_{t-1} \le J'_t(I) - h_t\}$.
- Hence,

$$G'_{t-1}(Y) = \begin{cases} J'_t(Y) - h_t \quad \text{if } Y \le S_{t-1}^* \\ P_{t-1}[1 - \Psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*)] \\ \quad + \int_{u < Y - S_{t-1}^*} [J'_t(Y - u) - h_t]\psi_{t-1}^{P_{t-1}}(u)\, du \\ \qquad \text{if } Y > S_{t-1}^*. \end{cases}$$

- As $J_t(I)$ is concave and $J'_t(I)$ is nonincreasing, $\min(S_{t-1}^*, Y_{t-1})$ is the S that maximizes $G_{t-1}(Y_{t-1})$.
- We consider $G'_{t-1}(Y)$ within several cases:

  (a) *Case* 1. $Y < S_{t-1}^*$: $G''_{t-1}(Y) = J''_t(Y) \le 0$ due to the concavity of J_t.

  (b) *Case* 2. $Y > S_{t-1}^*$:

$$G''_{t-1}(Y) = -P_{t-1}\psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) + [J'_t(S_{t-1}^*) - h_t]\psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) + \int_{u < Y - S_{t-1}^*} [J''_t(Y - u)]\psi_{t-1}^{P_{t-1}}(u)\, du \le 0.$$

This is true because

(i) $-P_{t-1}\psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) + [J'_t(S_{t-1}^*) - h_t]\psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) \le 0$ because of the choice of S_{t-1}^*, and

(ii) $\int_{u<Y-S^*_{t-1}}[J''_t(Y-u)]\psi^{P_{t-1}}_{t-1}(u)\le 0$ for $Y>S^*_{t-1}\le 0$ due to the concavity of J_t.

(c) *Case* 3. $Y=S^*_{t-1}$:

$$\begin{aligned}G''_{t-1}(S^*_{t-1})&\\ &=P_{t-1}[1-\psi^{P_{t-1}}_{t-1}(0)]+[J'_t(S^*_{t-1})-h_t]\psi^{P_{t-1}}_{t-1}(0)-[J'_t(S^*_{t-1})-h_t]\\ &=[P_{t-1}-[J'_t(S^*_{t-1})-h_t]][1-\psi^{P_{t-1}}_{t-1}(0)]\\ &\quad+[J'_t(S^*_{t-1})-J'_t(S^*_{t-1})]\psi^{P_{t-1}}_{t-1}(0)\ge 0\end{aligned}$$

due to the choice of S^*_{t-1}.

- As $G''_{t-1}\le 0$ for all Y, we have that $G_{t-1}(Y)$ is concave. □

A.2. Proof of Lemma 6

In this section of the appendix we prove that the performance of the delayed production strategy (Z^*_1) has an upper bound of the profit from a deterministic pricing problem using expected demand ($F^*(\bar{d})$). We begin by defining an additional deterministic pricing problem, and we show additional lemmas based on this deterministic problem that lead to the final upper bound.

For any given price vector P and realized demand vector d, the problem of finding the production quantities, X, and sales quantities, D, that maximize profit under deterministic demand can be written as

$$\begin{aligned}\mathrm{LSP}(P,d):\qquad &\max\ F^{P,d}(D)=\sum_{t=1}^{T}[R^{P,d}_t(D_t)-h_tI_t-k_tX_t]\\ &\text{subject to } I_0=0\\ &I_{t+1}=I_t+X_t-D_t,\quad t=1,2,\ldots,T\\ &X_t\le q_t,\quad t=1,2,\ldots,T\\ &I_t,X_t,D_t\ge 0,\quad t=1,2,\ldots,T,\end{aligned}$$

with

$$\begin{cases}R^{P,d}_t(D_t)=P_tD_t & \text{for } D_t\le d_t\\ P_td_t-(D_t-d_t) & \text{for } D_t>d_t\end{cases}$$

and other parameters are as described previously. Observe that if $D_t>d_t$, then the revenue function includes a penalty proportional to (D_t-d_t). Thus the optimal solution for LSP(P,d) is such that $D_t\le d_t$. Other choices of penalty functions are also possible; in all cases, R_t should be a decreasing function of D_t for $D_t\ge d_t$.

Let $\tilde{F}^{P,d}$ be the optimal objective function value of problem LSP(P,d) and $\tilde{X}^{P,d}$, $\tilde{D}^{P,d}$ be the corresponding optimal production schedule and demand to be satisfied. As before, for any price P, let $\bar{d}_t(P)$ be the corresponding expected demand at Period t, and let $\bar{d}(P)$ be the corresponding vector of demand curves. Let $E_d(\tilde{F}^{P,d})$ be the expected profit for Problem LSP for a price P over all corresponding realized demands d.

We now show an upper bound from Problem $SP(P,d)$ with expected demand:

Lemma 20. *For any price P, $\tilde{F}^{P,\bar{d}(P)}\ge E_d(\tilde{F}^{P,d})$; that is, the optimal profit for Problem LSP(P,d) with expected demand is an upper bound on the expected profit for any price P over the realized demand d.*

Proof. For any price P, let the vector of expected values of the production, sales, and inventory variables be defined as

$$\bar{X}=E_d(\tilde{X}^{P,d}),\quad \bar{D}=E_d(\tilde{D}^{P,d}),\quad\text{and}\quad \bar{I}=E_d(\tilde{I}^{P,d}).$$

Clearly the optimal production and sales values, $(\tilde{X}^{P,d},\tilde{D}^{P,d})$, are feasible for LSP$(P,d)$:

$$\begin{aligned}\tilde{I}^{P,d}_0&=0\\ \tilde{I}^{P,d}_{t+1}&=\tilde{I}^{P,d}_t+\tilde{X}^{P,d}_t-\tilde{D}^{P,d}_t\ge 0,\quad t=1,2,\ldots,T\\ \tilde{X}^{P,d}_t&\le q_t,\quad t=1,2,\ldots,T\quad\text{and}\\ \tilde{X}^{P,d}_t,\tilde{D}^{P,d}_t&\ge 0,\quad t=1,2,\ldots,T.\end{aligned}$$

Taking expected values, we obtain the following equations:

$$\begin{aligned}\bar{I}_0&=0\\ \bar{I}_{t+1}&=\bar{I}_t+\bar{X}_t-\bar{D}_t=E_d(\tilde{I}^{P,d}_t)+E_d(\tilde{X}^{P,d}_t)-E_d(\tilde{D}^{P,d}_t)\\ E_d(\tilde{I}^{P,d}_{t+1})&=E_d(\tilde{I}^{P,d}_t+\tilde{X}^{P,d}_t-\tilde{D}^{P,d}_t)\ge 0,\quad t=1,2,\ldots,T\\ \bar{X}_t&=E_d(\tilde{X}^{P,d}_t)\le q_t,\quad t=1,2,\ldots,T\quad\text{and}\\ \bar{X}_t&=E_d(\tilde{X}^{P,d}_t)\ge 0,\quad \bar{D}_t=E_d(\tilde{D}^{P,d}_t)\ge 0,\quad t=1,2,\ldots,T.\end{aligned}$$

Hence, the set of vectors of expected values $(\bar{X},\bar{D})$ is a feasible solution of the Problem LSP$(P,\bar{d}(P))$.

As $R^{P,d}_t(D_t)$ is decreasing for $D_t\ge d_t$, by the optimality of $\tilde{D}^{P,d}$ we know that $\tilde{D}^{P,d}_t\le d_t$; hence, $\bar{D}_t=E_d(\tilde{D}^{P,d}_t)\le E_d(d_t)=\bar{d}_t(P)$ for all t. Because $\bar{D}_t\le\bar{d}(P)_t$ and $\tilde{D}^{P,d}_t\le d_t$, and because $\bar{D}_t=E_d(\tilde{D}^{P,d}_t)$, then $R^{P,\bar{d}(P)}_t(\bar{D}_t)=E_d[R^{P,d}_t(\tilde{D}^{P,d}_t)]$ for all t. Thus,

$$\begin{aligned}&\sum_{t=1}^{T}[R^{P,\bar{d}(P)}_t(\bar{D}_t)-h_t\bar{I}_t-k_t\bar{X}_t]\\ &\qquad=\sum_{t=1}^{T}E_d[R^{P,d}_t(\tilde{D}^{P,d}_t)+h_t\tilde{I}^{P,d}_t-k_t\tilde{X}^{P,d}_t]=E_d(\tilde{F}^{P,d}),\end{aligned}$$

and the optimal objective function value of

$$\mathrm{LSP}(P,\bar{d}(P))\ge\sum_{t=1}^{T}[R^{P,\bar{d}(P)}_t(\bar{D}_t)-h_t\bar{I}_t-k_t\bar{X}_t]=E_d(\tilde{F}^{P,d}).\quad\square$$

Now consider a specific choice of price vector. Set price $\underline{P}$ such that $\tilde{F}^{\underline{P},\bar{d}(\underline{P})}=\max_P\tilde{F}^{P,\bar{d}(P)}$. That is, for each possible price vector P, solve Problem LSP$(P,\bar{d}(P))$, and we set $\underline{P}$ equal to the prices that maximize profit for Problem LSP$(P,\bar{d}(P))$ over all possible price vectors.

Under this choice of P, the deterministic profit from Problem LSP(P, d) is an upper bound on the performance of the delayed production strategy under any choice of price vector, as shown in the following lemma.

Lemma 21. $Z_1^* \le \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$.

Proof. For any price P, apply the delayed production algorithm and identify the corresponding optimal order-up-to and save-up-to policies. Given a vector of realized demand d, let $Z^1(P, d)$ be the profit resulting from following these identified policies. Then,

$$\begin{aligned} Z_1^* &= \max_P E_d[Z^1(P, d)] \\ &\le \max_P E_d[\tilde{F}^{P, d}], \quad \text{since } \tilde{F}^{P, d} \ge Z^1(P, d) \text{ for each } P \\ &\le \max_P \tilde{F}^{P, \bar{d}(P)}, \quad \text{by Lemma 20} \\ &\le \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}, \quad \text{by the choice of } \underline{P}. \quad \square \end{aligned}$$

Of course, this also implies that the expected profit from the delayed production heuristic is bounded according to this lemma. Indeed, the upper bound based on the specific choice of prices from Problem PP$(\bar{d})$, $F^*(\bar{d})$, is equivalent to the upper bound developed in Lemma 21, $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$. Recall that the revenue function in PP$(\bar{d})$ is $R_t(D_t) = \max_{P_t}\{D_t P_t \colon \bar{d}_t(P_t) \ge D_t\}$. The optimal solution is D^H, X^H, P^H, with $F^*(\bar{d}) = F^H(D^H)$ and $R_t^H(D_t^H) = D_t^H P_t^H$. Thus, we have

Corollary 22. $F^*(\bar{d}) = \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$.

Proof. Because (D^H, X^H) is a feasible solution of LSP$(P^H, \bar{d}(P^H))$, $D^H \le \bar{d}(P^H)$. This implies $R_t^{P^H, \bar{d}(P^H)}(D_t^H) = D_t^H P_t^H = R_t^H(D_t^H)$, and so $F^*(\bar{d}) \le \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$.

Because $R_t^{\underline{P}, \bar{d}(\underline{P})}(D_t)$ is decreasing for $D_t \ge \bar{d}_t(\underline{P})$, $\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})} \le \bar{d}_t(\underline{P})$ by the optimality of $\tilde{D}^{\underline{P}, \bar{d}(\underline{P})}$. Hence, $(\tilde{D}^{\underline{P}, \bar{d}(\underline{P})}, \tilde{X}^{\underline{P}, \bar{d}(\underline{P})})$ is a feasible solution of PPD. We know that $\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})} \underline{P}_t \le D_t^H \underline{P}_t$ (by the optimality of D_t^H) $\le \max_{P_t}\{D_t^H P_t \colon D_t^H \le \bar{d}_t(P_t)\}$ (as this is maximized over all P_t) $= R_t^H(D_t^H)$. Thus, $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} \le F^*(\bar{d})$, and the proof is complete. $\square$

Given Lemma 21 and Corollary 22, we have that $Z_1^* \le F^*(\bar{d})$, and thus Lemma 6 holds.

Extension: Alternative Demand Assumption

The results we have shown can also be extended to another demand scenario, in which demand in a Period (d_t) may depend not only on the price in Period t (P_t), but also on the entire price vector, $\underline{P}$. In this case, the prices for the delayed production heuristic are set as described above, specifically: Set price $\underline{P}$ such that $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} = \max_P \tilde{F}^{P, \bar{d}(P)}$. The performance of the delayed production heuristic under this demand scenario has an upper bound of $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$, as described by Lemma 21.

A.3. Proof of Lemma 9

The structure of Problem PP (and thus LSP(P, d)) places it into a class of problems that can sometimes be solved with a greedy algorithm, or marginal allocation algorithm (MAA). The greedy algorithm applied to Problem PP would be as follows:

Greedy Algorithm or MAA

Step 0. Set $D_t = 0$ in each period.

Step 1. Choose a Period t to increase demand by one unit such that the contribution to profit is maximized and the feasibility of the production schedule is maintained.

Step 2. If no such period exists, stop.

The algorithm may also be modified to take into account lower and upper bounds on demand and price.

Biller et al. (2005) analyzed conditions under which a greedy algorithm solves Problem PP. Specifically, their main result is the following:

Theorem 23 (Biller et al. 2005 Main Result). *If the revenue functions R are concave in pricing Problem PP, then a greedy algorithm provides the optimal solution.*

When the revenue functions are not concave, other solution methods, namely nonlinear optimization methods, may be used to solve the deterministic pricing problem.

In the proof below, we show that $Z_1^H \ge \xi^{\min} Z_1^*$, where $\xi^{\min} = \xi^{\min}(P^H) = \min_t[d_t^{\min}(P^H)/\bar{d}_t(P^H)]$, and $d_t^{\min}(P)$ is the minimum possible demand at Period t, as described in Lemma 9.

Proof. Apply MAA as described above to solve LSP$(\underline{P}, \bar{d}(\underline{P}))$. Let $F_t(k)$ be the marginal profit of satisfying the kth unit of demand at Period t. Then F_t is nonincreasing and

$$\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} = \sum_{t=1}^{T} \sum_{k \le \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k).$$

Let $D_t^{\min} = \min\{d_t^{\min}(\underline{P}), \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}\}$. As the vector $D^{\min} \le \tilde{D}^{\underline{P}, \bar{d}(\underline{P})}$, it is a feasible solution for LSP$(\underline{P}, \bar{d}(\underline{P}))$. Consider satisfying $D^{\min}$ in the same order $\tilde{D}^{\underline{P}, \bar{d}(\underline{P})}$ is satisfied using MAA; we have

$$\begin{aligned} F^{\underline{P}, \bar{d}(\underline{P})}(D^{\min}) &= \sum_{t=1}^{T} D_t^{\min} \sum_{k \le D_t^{\min}} \frac{F_t(k)}{D_t^{\min}} \\ &\ge \sum_{t=1}^{T} \left(\frac{D_t^{\min}}{\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} \sum_{k \le \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k) \right), \end{aligned}$$

since $\displaystyle\sum_{k \le D_t^{\min}} \frac{F_t(k)}{D_t^{\min}} \ge \sum_{k \le \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} \frac{F_t(k)}{\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}}$,

$$= \sum_{t=1}^{T} \left(\min\left\{1, \frac{d_t^{\min}(\underline{P})}{\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}}\right\} \sum_{k \le \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k) \right),$$

by the definition of $D^{\min}$

$$
\begin{aligned}
&\geq \sum_{t=1}^{T} \min\left\{1, \frac{d_t^{\min} \underline{P}}{\bar{d}_t(\underline{P})}\right\} \sum_{k \leq \tilde{D}_t^{\underline{P},\, \bar{d}(\underline{P})}} F_t(k), \\
&\qquad\qquad \text{since } \tilde{D}_t^{\underline{P},\, \bar{d}(\underline{P})} \leq \bar{d}_t(\underline{P}) \text{ by the} \\
&\qquad\qquad \text{optimality of } \tilde{D}^{\underline{P},\, \bar{d}(\underline{P})} \\
&\geq \sum^{T} \frac{d_t^{\min}(\underline{P})}{\bar{d}_t(\underline{P})} \sum_{k \leq \tilde{D}_t^{\underline{P},\, \bar{d}(\underline{P})}} F_t(k) \\
&\geq \xi^{\min} \sum_{t=1}^{T} \sum_{k \leq \tilde{D}_t^{\underline{P},\, \tilde{D}(\underline{P})}} F_t(k), \\
&\qquad\qquad \text{by the definition of } \xi^{\min} \text{ and} \\
&\qquad\qquad \text{since } \tilde{D}_t^{\underline{P},\, \bar{d}(\underline{P})} \leq \bar{d}_t(\underline{P}) \\
&= \xi^{\min} \tilde{F}^{\underline{P},\, \bar{d}(\underline{P})}.
\end{aligned}
$$

As $R_t^{\underline{P},\, \bar{d}(\underline{P})}(D_t) = R_t^{\underline{P},\, d^{\min}(\underline{P})}(D_t)$ for $D_t \leq d_t^{\min}(\underline{P})$, then

$$
\begin{aligned}
\tilde{F}^{\underline{P},\, d^{\min}(\underline{P})} &\geq F^{\underline{P},\, d^{\min}(\underline{P})}(D^{\min}), \quad \text{by optimality} \\
&= F^{\underline{P},\, \bar{d}(\underline{P})}(D^{\min}), \quad \text{since the revenue is the same} \\
&\geq \xi^{\min} \tilde{F}^{\underline{P},\, \bar{d}(\underline{P})}, \quad \text{proof above.}
\end{aligned}
$$

As $R_t^{\underline{P},\, \bar{d}(\underline{P})}(D_t)$ is decreasing for $D_t \geq d_t^{\min}(\underline{P})$, then $\tilde{D}^{\underline{P},\, d^{\min}(\underline{P})} \leq d_t^{\min}(\underline{P})$. Hence, $(\tilde{D}^{\underline{P},\, d^{\min}(\underline{P})}, \tilde{X}^{\underline{P},\, d^{\min}(\underline{P})})$ is a feasible policy for the delayed production strategy with expected profit the same as the profit from the deterministic problem, $\tilde{F}^{\underline{P},\, d^{\min}(\underline{P})}$. As Z_1^H is the maximum expected profit for price $= \underline{P}$ obtained by following the optimal policy for D and X, $Z_1^H \geq \tilde{F}^{\underline{P},\, d^{\min}(\underline{P})} \geq \xi^{\min} \tilde{F}^{\underline{P},\, \bar{d}(\underline{P})} \geq \xi^{\min} Z_1^*$ (from Lemma 21). □

References

Biller, S., L. M. A. Chan, D. Simchi-Levi, J. Swann. 2005. Dynamic pricing and the direct-to-customer model in the automotive industry. *Electronic Commerce Res.* **5**(2) 309–334.

Chan, L. M. A., Z. J. Shen, D. Simchi-Levi, J. Swann. 2004. Review of dynamic pricing and on-line pricing research to improve supply chain performance. Z. J. Shen, D. Simchi-Levi, D. Wu, eds. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era*. Kluwer Academic Publishers, Boston/Dordrecht/London.

Chen, X., D. Simchi-Levi. 2004. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Oper. Res.* **52**(6) 887–896.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Oper. Res.* **47**(3) 454–475.

Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* **40**(8) 999–1020.

Gallego, G., G. van Ryzin. 1997. A multiproduct dynamic pricing problem and its applications to network yield management. *Oper. Res.* **45**(1) 24–41.

Gilbert, S. M. 1999. Coordination of pricing and multi-period production for constant priced goods. *Eur. J. Oper. Res.* **114**(2) 330–337.

Gilbert, S. M. 2000. Coordination of pricing and multiple-period production across multiple constant priced goods. *Management Sci.* **46**(12) 1602–1616.

Kunreuther, H., L. Schrage. 1973. Joint pricing and inventory decisions for constant priced items. *Management Sci.* **19**(7) 732–738.

Lai, K. S. 1990. Price smoothing under capacity constraints. *Southern Econom. J.* **57**(1) 150–159.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Oper. Res.* **47**(2) 183–194.

Scarf, H. E. 2000. Optimal inventory policies when sales are discretionary. Working paper, Cowles Foundation Discussion Paper 1270, Yale University.

Smith, M., J. Bailey, E. Brynjolfsson. 1999. Understanding digital markets: Review and assessment. E. Brynjolfsson, B. Kahin, eds. *Understanding the Digital Economy*. MIT Press, Boston, MA.

Swann, J. 2001. *Dynamic Pricing Models to Improve Supply Chain Performance*. Unpublished doctoral dissertation, Northwestern University, Chicago, IL.

Thomas, L. J. 1970. Price-production decisions with deterministic demand. *Management Sci.* **16**(11) 747–750.

Thomas, L. J. 1974. Price and production decisions with random demand. *Oper. Res.* **22** 513–518.

Thowsen, G. T. 1975. A dynamic, nonstationary inventory problem for a price/quantity setting firm. *Naval Res. Logist.* **22** 461–476.

Van Mieghem, J. A., M. Dada. 1999. Price versus production postponement: Capacity and competition. *Management Sci.* **45**(12) 1631–1649.

Whitin, T. M. 1955. Inventory control and price theory. *Management Sci.* **2**(1) 61–68.

Zabel, E. 1972. Multi-period monopoly under uncertainty. *Econom. Theory* **5** 524–536.

Zipkin, P. 2000. *Foundations of Inventory Management*. McGraw-Hill, Burr Ridge, IL.