Уравнения математической физики

Лабораторная работа по формуле Даламбера и Кирхгофа

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Задача №1

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 6$$

$\partial_t u = 4\,x$ при $t = 0$

$u = x^2$ при $t = 0$

$a = 1$

Решение ищем по формуле Даламбера.

$$u[x\_, t\_] = \frac{1}{2}\left((x+t)^2 + (x-t)^2\right) + \frac{1}{2}\int_{x-t}^{x+t} 4\,\xi\, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} 6\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

$\partial_{t,t} u[x, t] - \partial_{x,x} u[x, t]$

u[x, 0]

$\partial_t u[x, t]$ /. {t → 0}

Задача №2

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - 4\,\frac{\partial^2 u}{\partial x^2} = x\, t$$

$\partial_t u = x$ при $t = 0$

$u = x^2$ при $t = 0$

$a = 2$

Решение ищем по формуле Даламбера.

$$u[x\_, t\_] = \frac{1}{2}\left((x+2\,t)^2 + (x-2\,t)^2\right) + \frac{1}{4}\int_{x-2\,t}^{x+2\,t} \xi\, d\xi + \frac{1}{4}\int_0^t \int_{x-2\,(t-\tau)}^{x+2\,(t-\tau)} \xi\,\tau\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

$\partial_{t,t} u[x, t] - 4\,\partial_{x,x} u[x, t]$

u[x, 0]

$\partial_t u[x, t]$ /. {t → 0}

Задача №3

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \text{Sin}[x]$$

$\partial_t u = 0$ при $t = 0$

$u = \text{Sin}[x]$ при $t = 0$

$a = 1$

Решение ищем по формуле Даламбера.

$$u[x\_, t\_] = \frac{1}{2}\left(\text{Sin}[x+t] + \text{Sin}[x-t]\right) + \frac{1}{2}\int_{x-t}^{x+t} 0\, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \text{Sin}[\xi]\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

```
∂t,t u[x, t] - ∂x,x u[x, t]
u[x, 0]
∂t u[x, t] /. {t → 0}
```

Задача №4

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^x$$

$\partial_t u = \mathrm{Cos}[x] + x$ при $t = 0$

$u = \mathrm{Sin}[x]$ при $t = 0$

$a = 1$

Решение ищем по формуле Даламбера.

$$u[x\_, t\_] = \frac{1}{2}(\mathrm{Sin}[x+t] + \mathrm{Sin}[x-t]) + \frac{1}{2}\int_{x-t}^{x+t} (\mathrm{Cos}[\xi] + \xi)\, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} e^{\xi}\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

```
∂t,t u[x, t] - ∂x,x u[x, t] // FullSimplify
u[x, 0]
∂t u[x, t] /. {t → 0}
```

Задача №5

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = \mathrm{Sin}[x]$$

$\partial_t u = 1$ при $t = 0$

$u = 1$ при $t = 0$

$a = 3$

Решение ищем по формуле Даламбера.

$$u[x\_, t\_] = 1 + \frac{1}{6}\int_{x-3t}^{x+3t} 1\, d\xi + \frac{1}{6}\int_0^t \int_{x-3(t-\tau)}^{x+3(t-\tau)} \mathrm{Sin}[\xi]\, d\xi\, d\tau \text{ // FullSimplify}$$

$$1 + t + \frac{2}{9}\mathrm{Sin}\left[\frac{3t}{2}\right]^2 \mathrm{Sin}[x]$$

Проверка

```
∂t,t u[x, t] - 9 ∂x,x u[x, t] // FullSimplify
u[x, 0]
∂t u[x, t] /. {t → 0}
```

Задача №6

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = \mathrm{Sin}[\omega x]$$

$\partial_t u = 0$ при $t = 0$

$u = 0$ при $t = 0$

$a = a$

Решение ищем по формуле Даламбера.

$$u[x\_,\ t\_] = \frac{1}{2\,a}\int_{x-a\,t}^{x+a\,t} 0\, d\xi + \frac{1}{2\,a}\int_0^t \int_{x-a\,(t-\tau)}^{x+a\,(t-\tau)} \mathrm{Sin}[\omega\,\xi]\, d\xi\, d\tau \ //\ \mathrm{FullSimplify}$$

Проверка

```
∂t,t u[x, t] - a^2 ∂x,x u[x, t] // FullSimplify
u[x, 0]
∂t u[x, t] /. {t → 0}
```

Задача №7

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \mathrm{Sin}[\omega\, t]$$

$\partial_t u = 0$ при $t = 0$

$u = 0$ при $t = 0$

$a = a$

Решение ищем по формуле Даламбера.

$$u[x\_,\ t\_] = \frac{1}{2\,a}\int_{x-a\,t}^{x+a\,t} 0\, d\xi + \frac{1}{2\,a}\int_0^t \int_{x-a\,(t-\tau)}^{x+a\,(t-\tau)} \mathrm{Sin}[\omega\,\tau]\, d\xi\, d\tau \ //\ \mathrm{FullSimplify}$$

Проверка

```
∂t,t u[x, t] - a^2 ∂x,x u[x, t] // FullSimplify
u[x, 0]
∂t u[x, t] /. {t → 0}
```

Задача №8

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2$$

$\partial_t u = y$ при $t = 0$

$u = x$ при $t = 0$

```
<< Calculus`VectorAnalysis`

φ[x_, y_, z_] = x;
ψ[x_, y_, z_] = y;
a = 1;
f[x_, y_, z_, t_] = 2;

u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
     D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
           {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №9

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 6\,x\,y\,t$$

$\partial_t\, u = x\, y$ при $t = 0$

$u = x^2 - y^2$ при $t = 0$

```
<< Calculus`VectorAnalysis`

φ[x_, y_, z_] = x^2 - y^2;
ψ[x_, y_, z_] = x y;
a = 1;
f[x_, y_, z_, t_] = 6 x y t;

u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
         z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
        y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №10

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = x^3 - 3\,x\,y^2$$

$\partial_t\, u = e^y\, \mathrm{Sin}[x]$ при $t = 0$

$u = e^x\, \mathrm{Cos}[y]$ при $t = 0$

```
<< Calculus`VectorAnalysis`

φ[x_, y_, z_] = e^x Cos[y];
ψ[x_, y_, z_] = e^y Sin[x];
a = 1;
f[x_, y_, z_, t_] = x^3 - 3 x y^2;

ψ[x + a * t * Cos[phi] * Sin[theta],
  y + a * t * Sin[phi] * Sin[theta], z + a * t * Cos[theta]] // FullSimplify
```

```
φ[x + a * t * Cos[phi] * Sin[theta],
  y + a * t * Sin[phi] * Sin[theta], z + a * t * Cos[theta]] // FullSimplify
```

```
u[t_, x_, y_, z_] =
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
   D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
         {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
  Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
       y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
      (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]
```

Проверка

```
∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Mathematica не справляется с решением этого примера

Задача №11

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = t\,\mathrm{Sin}[y]$$

$\partial_t u = \mathrm{Sin}[y]$ при $t = 0$

$u = x^2$ при $t = 0$

```
φ[x_, y_, z_] = x^2;
ψ[x_, y_, z_] = Sin[y];
a = 1;
f[x_, y_, z_, t_] = t Sin[y];
```

```
u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
         z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
        y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

```
FullSimplify[t^2 + x^2 +
  ∫_0^t 1/2 (t - τ) τ If[t ≠ τ, (2 Sin[Abs[t - τ]])/(Abs[t - τ]), Integrate[BesselJ[0, Abs[t - τ] Sin[theta]]
      Sin[theta], {theta, 0, π}, Assumptions → t == τ]] Sin[y] dτ +
  1/2 t If[t ≠ 0, (2 Sin[Abs[t]])/(Abs[t]), Integrate[BesselJ[0, Abs[t] Sin[theta]] Sin[theta],
     {theta, 0, π}, Assumptions → t == 0]] Sin[y], {t > τ, t > 0}]

u[t_, x_, y_, z_] = t^2 + x^2 + t Sin[y];
```

Проверка

```
Expand[∂_{t,t} u[t, x, y, z] - (∂_{x,x} u[t, x, y, z] + ∂_{y,y} u[t, x, y, z])]

u[0, x, y, z]

∂_t u[t, x, y, z] /. {t → 0}
```

Задача №12

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$\partial_t u = 2x^2 + y^2$ при $t = 0$

$u = 2x^2 - y^2$ при $t = 0$

```
φ[x_, y_, z_] = 2 x^2 - y^2;
ψ[x_, y_, z_] = 2 x^2 + y^2;
a = √2 ;
f[x_, y_, z_, t_] = 0;

u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
     D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
           {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
Expand[∂_{t,t} u[t, x, y, z] - 2 (∂_{x,x} u[t, x, y, z] + ∂_{y,y} u[t, x, y, z])]

u[0, x, y, z]

∂_t u[t, x, y, z] /. {t → 0}
```

Задача №13

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - 3\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = x^3 + y^3$$

$\partial_t u = y^2$ при $t = 0$
$u = x^2$ при $t = 0$

```
φ[x_, y_, z_] = x^2;
ψ[x_, y_, z_] = y^2;
a = Sqrt[3];
f[x_, y_, z_, t_] = x^3 + y^3;
u[t_, x_, y_, z_] = FullSimplify[
  t/(4*π) * Integrate[ψ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
       z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4*π)
    D[t*Integrate[φ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
        z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
         {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4*π) * Integrate[f[x + a*(t - τ)*Cos[phi]*Sin[theta],
        y + a*(t - τ)*Sin[phi]*Sin[theta], z + a*(t - τ)*Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
Expand[∂t,t u[t, x, y, z] - 3 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №14

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = e^{3x+4y}$$

$\partial_t u = e^{3x+4y}$ при $t = 0$
$u = e^{3x+4y}$ при $t = 0$

```
φ[x_, y_, z_] = E^(3 x+4 y);
ψ[x_, y_, z_] = E^(3 x+4 y);
a = 1;
f[x_, y_, z_, t_] = E^(3 x+4 y);
```

Ищем решение в виде

```
u[t_, x_, y_, z_] = C1 E^(3 x+4 y);
```

Для решения задачи требуется замена переменных по знаком интеграла

Задача №15(замена переменных)

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$\partial_t u = \mathrm{Sin}[b x + c1\, y]$ при $t = 0$
$u = \mathrm{Cos}[b x + c1\, y]$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = Cos[b x + c1 y];
ψ[x_, y_, z_] = Sin[b x + c1 y];
a = a;
f[x_, y_, z_, t_] = 0;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
         z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
        y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

```
FullSimplify[t^2 + x^2 +
  ∫_0^t 1/2 (t - τ) τ If[t ≠ τ, 2 Sin[Abs[t - τ]]/Abs[t - τ], Integrate[BesselJ[0, Abs[t - τ] Sin[theta]]
       Sin[theta], {theta, 0, π}, Assumptions → t == τ]] Sin[y] dτ +
  1/2 t If[t ≠ 0, 2 Sin[Abs[t]]/Abs[t], Integrate[BesselJ[0, Abs[t] Sin[theta]] Sin[theta],
      {theta, 0, π}, Assumptions → t == 0]] Sin[y], {t > τ, t > 0}]
```

```
u[t_, x_, y_, z_] = t^2 + x^2 + t Sin[y];
```

Проверка

```
Expand[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №16

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$\partial_t u = \left(x^2 + y^2\right)^2$ при $t = 0$

$u = \left(x^2 + y^2\right)^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = (x^2 + y^2)^2 ;
ψ[x_, y_, z_] = (x^2 + y^2)^2 ;
f[x_, y_, z_, t_] = 0;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4*π) * Integrate[ψ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
        z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4*π)
    D[t*Integrate[φ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
         z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4*π) * Integrate[f[x + a*(t - τ)*Cos[phi]*Sin[theta],
        y + a*(t - τ)*Sin[phi]*Sin[theta], z + a*(t - τ)*Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - a^2 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №17

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \left(x^2 + y^2\right) e^t$$

$\partial_t u = 0$ при $t = 0$

$u = 0$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = 0;
ψ[x_, y_, z_] = 0;
f[x_, y_, z_, t_] = (x^2 + y^2) e^t;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4*π) * Integrate[ψ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
        z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4*π)
    D[t*Integrate[φ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
         z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4*π) * Integrate[f[x + a*(t - τ)*Cos[phi]*Sin[theta],
        y + a*(t - τ)*Sin[phi]*Sin[theta], z + a*(t - τ)*Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - a^2 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №18

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 2\,x\,y\,z$$

$\partial_t u = 1$ при $t = 0$

$u = x^2 + y^2 - 2z^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = x^2 + y^2 - 2 z^2;
ψ[x_, y_, z_] = 1;
f[x_, y_, z_, t_] = 2 x y z;
a = 1;

u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
         z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
        y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]

u[0, x, y, z]

∂t u[t, x, y, z] /. {t → 0}
```

Задача №19

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - 8\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = t^2\,x^2$$

$\partial_t u = z^2$ при $t = 0$

$u = y^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = y^2;
ψ[x_, y_, z_] = z^2;
f[x_, y_, z_, t_] = t^2 x^2;
a = √8 ;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
      z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
   D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
        {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
  Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
      y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
     (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - 8 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №20

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - 3\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 6\left(x^2 + y^2 + z^2\right)$$

$\partial_t u = x\,y\,z$ при $t = 0$

$u = x^2 y^2 z^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = x^2 y^2 z^2;
ψ[x_, y_, z_] = x y z;
f[x_, y_, z_, t_] = 6 (x^2 + y^2 + z^2);
a = √3 ;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
      z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
   D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
        {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
  Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
      y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
     (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - 3 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №21

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

$\partial_t u = \left(x^2 + y^2 + z^2\right)^2$ при $t = 0$

$u = \left(x^2 + y^2 + z^2\right)^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = (x^2 + y^2 + z^2)^2;
ψ[x_, y_, z_] = (x^2 + y^2 + z^2)^2;
f[x_, y_, z_, t_] = 0;

u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
     D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
           {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - a^2 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №22

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \left(x^2 + y^2 + z^2\right) e^t$$

$\partial_t u = 0$ при $t = 0$

$u = 0$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = 0;
ψ[x_, y_, z_] = 0;
f[x_, y_, z_, t_] = (x^2 + y^2 + z^2) e^t;
```

```
u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
     D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
            {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - a^2 (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
u[0, x, y, z]
∂t u[t, x, y, z] /. {t → 0}
```

Задача №23(заниматься не будем)

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = 0$ при $t = 0$

$u = \frac{1}{x}$ при $t = 0$

$x \neq 0,\ x^2 + y^2 + z^2 \neq t^2$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = 1/x;
ψ[x_, y_, z_] = 0;
f[x_, y_, z_, t_] = 0;
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
     D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
            {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}], {x ≠ 0, x^2 + y^2 + z^2 ≠ t^2}]
```

$$\frac{1}{4\pi}\left(t\int_0^{\pi}\left(\int_{-\pi}^{\pi}-(\text{Cos[phi] Sin[theta] Sin[theta]})\Big/(\text{x + t Cos[phi] Sin[theta]})^2\,d\text{phi}\right)d\text{theta}+\int_0^{\pi}\int_{-\pi}^{\pi}\frac{\text{Sin[theta]}}{\text{x + t Cos[phi] Sin[theta]}}\,d\text{phi}\,d\text{theta}\right)$$

```
$Aborted
```

$$\int_{-\pi}^{\pi}-(\text{Cos[phi] Sin[theta] Sin[theta]})\Big/(\text{x + t Cos[phi] Sin[theta]})^2\,d\text{phi}$$

```
JacobianDeterminant[Spherical[1, theta, phi]]
```

Проверка

$$\text{FullSimplify}\left[\partial_{t,t}\,u[t,x,y,z]-\left(\partial_{x,x}\,u[t,x,y,z]+\partial_{y,y}\,u[t,x,y,z]+\partial_{z,z}\,u[t,x,y,z]\right)\right]$$

```
Needs["VectorAnalysis`"]
```

$$\int_0^{\pi}\int_{-\pi}^{\pi}\text{Sin[theta]}\,d\text{phi}\,d\text{theta}$$

```
u[0, x, y, z] // FullSimplify
```

$$\partial_t\,u[t,x,y,z]\ /.\ \{t\to 0\}$$

Задача №24

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+b\,x^2$$

$\partial_t\,u = a$ при $t = 0$

$u = e^{-x}$ при $t = 0$

Решение ищем по формуле Даламбера.

$$u[x\_,t\_]=\frac{1}{2}\left(e^{-x+t}+e^{-x-t}\right)+\frac{1}{2}\int_{x-t}^{x+t}a\,d\xi+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}b\,\xi^2\,d\xi\,d\tau\ //\ \text{FullSimplify}$$

Проверка

$$\partial_{t,t}\,u[x,t]-\partial_{x,x}\,u[x,t]\ //\ \text{FullSimplify}$$

```
u[x, 0]
```

$$\partial_t\,u[x,t]\ /.\ \{t\to 0\}$$

Задача №25

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+a\,x\,t$$

$\partial_t\,u = \text{Sin}[x]$ при $t = 0$

$u = x$ при $t = 0$

Решение ищем по формуле Даламбера.

$$u[x\_,t\_]=-x+\frac{1}{2}\int_{x-t}^{x+t}\text{Sin}[\xi]\,d\xi+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}a\,\xi\,\tau\,d\xi\,d\tau\ //\ \text{FullSimplify}$$

Проверка

$$\partial_{t,t}\,u[x,t]-\partial_{x,x}\,u[x,t]\ //\ \text{FullSimplify}$$

```
u[x, 0]
```

$$\partial_t\,u[x,t]\ /.\ \{t\to 0\}$$

Задача №26

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + a\, e^{-t}$$

$\partial_t u = c\,\mathrm{Cos}[x]$ при $t = 0$

$u = b\,\mathrm{Sin}[x]$ при $t = 0$

Решение ищем по формуле Даламбера.

$$u[x\_,\, t\_] = \frac{1}{2}\,(b\,\mathrm{Sin}[x-t] + b\,\mathrm{Sin}[x+t]) + \frac{1}{2}\int_{x-t}^{x+t} c\,\mathrm{Cos}[\xi]\, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} a\, e^{-\tau}\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

$\partial_{t,t} u[x, t] - \partial_{x,x} u[x, t]$ // FullSimplify

u[x, 0]

$\partial_t u[x, t]$ /. {t → 0}

Задача №27

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + a\,\mathrm{Sin}[b\, t]$$

$\partial_t u = \mathrm{Sin}[x]$ при $t = 0$

$u = \mathrm{Cos}[x]$ при $t = 0$

Решение ищем по формуле Даламбера.

$$u[x\_,\, t\_] = \frac{1}{2}\,(\mathrm{Cos}[x-t] + \mathrm{Cos}[x+t]) + \frac{1}{2}\int_{x-t}^{x+t} \mathrm{Sin}[\xi]\, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} a\,\mathrm{Sin}[b\,\tau]\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

$\partial_{t,t} u[x, t] - \partial_{x,x} u[x, t]$ // FullSimplify

u[x, 0]

$\partial_t u[x, t]$ /. {t → 0}

Задача №28

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x\,\mathrm{Sin}[t]$$

$\partial_t u = \mathrm{Cos}[x]$ при $t = 0$

$u = \mathrm{Sin}[x]$ при $t = 0$

Решение ищем по формуле Даламбера.

$$u[x\_,\, t\_] = \frac{1}{2}\,(\mathrm{Sin}[x-t] + \mathrm{Sin}[x+t]) + \frac{1}{2}\int_{x-t}^{x+t} \mathrm{Cos}[\xi]\, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \xi\,\mathrm{Sin}[\tau]\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

$\partial_{t,t} u[x, t] - \partial_{x,x} u[x, t]$ // FullSimplify

u[x, 0]

$\partial_t u[x, t]$ /. {t → 0}

Задача №29

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g[x]\, f[t]$$

$\partial_t u = 0$ при $t = 0$

$u = \phi[x]$ при $t = 0$

Решение ищем по формуле Даламбера.

$$u[x\_, t\_] = \frac{1}{2}(\phi[x-t] + \phi[x+t]) + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} g[\xi]\, f[\tau]\, d\xi\, d\tau \text{ // FullSimplify}$$

Проверка

$\partial_{t,t} u[x, t] - \partial_{x,x} u[x, t]$ // FullSimplify

u[x, 0] // FullSimplify

$$\frac{1}{2}\int_0^0 \int_{x+\tau}^{x-\tau} f[\tau]\, g[\xi]\, d\xi\, d\tau + \phi[x] = 0 + \phi[x] = \phi[x]$$

$\partial_t u[x, t]$ /. {t → 0}

Задача №30

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = x^2 y^2 z^2$ при $t = 0$

$u = x\, y\, z$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = x y z;
ψ[x_, y_, z_] = x^2 y^2 z^2;
f[x_, y_, z_, t_] = 0;
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
      z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
         {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
       y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
      (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

FullSimplify$[\partial_{t,t} u[t, x, y, z] - (\partial_{x,x} u[t, x, y, z] + \partial_{y,y} u[t, x, y, z] + \partial_{z,z} u[t, x, y, z])]$

u[0, x, y, z]

$\partial_t u[t, x, y, z]$ /. {t → 0}

Задача №31

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = x y$ при $t = 0$
$u = r^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = r^2;
ψ[x_, y_, z_] = x y;
f[x_, y_, z_, t_] = 0;
a = 1;

u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
      z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
   D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
        {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
  Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
      y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
     (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]

u[0, x, y, z]

∂t u[t, x, y, z] /. {t → 0}
```

Задача №32*

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = x^2 - y^2$ при $t = 0$
$u = e^x \mathrm{Cos}[y]$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = e^x Cos[y];
ψ[x_, y_, z_] = x^2 - y^2;
f[x_, y_, z_, t_] = 0;
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
           {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

$Aborted

Задача №33

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = 1$ при $t = 0$

$u = x^2 + y^2$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = x^2 + y^2;
ψ[x_, y_, z_] = 1;
f[x_, y_, z_, t_] = 0;
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
   t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
          z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
           {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

$t + 2 t^2 + x^2 + y^2$

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
```

0

```
u[0, x, y, z]
```

$x^2 + y^2$

```
∂t u[t, x, y, z] /. {t → 0}
```

1

Задача №34

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = e^{-x}$ при $t = 0$

$u = e^{x}$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = e^x;
ψ[x_, y_, z_] = e^-x;
f[x_, y_, z_, t_] = 0;
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
   t/(4*π) * Integrate[ψ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
        z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4*π)
     D[t*Integrate[φ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
          z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
           {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4*π) * Integrate[f[x + a*(t - τ)*Cos[phi]*Sin[theta],
         y + a*(t - τ)*Sin[phi]*Sin[theta], z + a*(t - τ)*Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}], Re[t] > 0]
```

Cosh[t - x] + Sinh[t + x]

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
```

0

```
u[0, x, y, z] // TrigToExp
```

e^{x}

```
∂t u[t, x, y, z] /. {t → 0} // TrigToExp
```

e^{-x}

Задача №35*

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$\partial_t u = 0$ при $t = 0$

$u = \frac{1}{x}$ при $t = 0$

$x \neq 0$

$x^2 + y^2 + z^2 \neq t^2$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = 1/x;
ψ[x_, y_, z_] = 0;
f[x_, y_, z_, t_] = 0;
a = 1;
u[t_, x_, y_, z_] = FullSimplify[
  t/(4*π) * Integrate[ψ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
       z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4*π)
    D[t*Integrate[φ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
         z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4*π) * Integrate[f[x + a*(t - τ)*Cos[phi]*Sin[theta],
        y + a*(t - τ)*Sin[phi]*Sin[theta], z + a*(t - τ)*Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
, {x ≠ 0 ,
 x^2 + y^2 + z^2 ≠ t^2}
```

Проверка

```
FullSimplify[∂_{t,t} u[t, x, y, z] - (∂_{x,x} u[t, x, y, z] + ∂_{y,y} u[t, x, y, z] + ∂_{z,z} u[t, x, y, z])]
```

0

```
u[0, x, y, z]
```

$x^2 + y^2$

```
∂_t u[t, x, y, z] /. {t → 0}
```

1

Задача №36

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = a\,x + b\,t$$

$\partial_t u = x\,y + z$ при $t = 0$

$u = x\,y\,z$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = x y z;
ψ[x_, y_, z_] = x y + z;
f[x_, y_, z_, t_] = aa x + b t;
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
     {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
    D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
        z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
   Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
        y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
       (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

$$\frac{1}{6}\, t \left(b\, t^2 + 3\, aa\, t\, x + 6\, x\, y\right) + (t + x\, y)\, z$$

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
```

$b\, t + aa\, x$

```
u[0, x, y, z]
```

$x\, y\, z$

```
∂t u[t, x, y, z] /. {t → 0}
```

$x\, y + z$

Задача №37*

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{x}{1 + t^2}\, e^y \operatorname{Cos}[z]$$

$\partial_t u = 0$ при $t = 0$

$u = z \operatorname{Sin}\left[\sqrt{2}\,(x + y)\right]$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = z Sin[√2 (x + y)];
ψ[x_, y_, z_] = 0;
f[x_, y_, z_, t_] = x/(1 + t^2) e^y Cos[z];
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4*π) * Integrate[ψ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
      z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4*π)
   D[t*Integrate[φ[x + a*t*Cos[phi]*Sin[theta], y + a*t*Sin[phi]*Sin[theta],
       z + a*t*Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
        {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
  Integrate[(t - τ)/(4*π) * Integrate[f[x + a*(t - τ)*Cos[phi]*Sin[theta],
      y + a*(t - τ)*Sin[phi]*Sin[theta], z + a*(t - τ)*Cos[theta], τ] *
     (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
    {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}]]
```

```
$Aborted
```

Проверка

```
FullSimplify[∂t,t u[t, x, y, z] - (∂x,x u[t, x, y, z] + ∂y,y u[t, x, y, z] + ∂z,z u[t, x, y, z])]
```

$b\,t + aa\,x$

```
u[0, x, y, z]
```

$x\,y\,z$

```
∂t u[t, x, y, z] /. {t → 0}
```

$x\,y + z$

Задача №38

Решить задачу Коши

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{x\,t}{1 + t^2}$$

$\partial_t u = y\,\mathrm{Cos}[z]$ при $t = 0$

$u = x\,\mathrm{Sin}[y]$ при $t = 0$

```
Clear[a, φ, ψ, t, f, x, y, z, u];
φ[x_, y_, z_] = x Sin[y];
ψ[x_, y_, z_] = y Cos[z];
f[x_, y_, z_, t_] = x t/(1 + t^2);
a = 1;
```

```
u[t_, x_, y_, z_] = FullSimplify[
  t/(4 * π) * Integrate[ψ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
       z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
      {theta, 0, Pi}, {phi, -Pi, Pi}] + 1/(4 * π)
     D[t * Integrate[φ[x + a * t * Cos[phi] * Sin[theta], y + a * t * Sin[phi] * Sin[theta],
         z + a * t * Cos[theta]] * (JacobianDeterminant[Spherical[r, theta, phi]] /.
          {r → 1}), {theta, 0, Pi}, {phi, -Pi, Pi}], t] +
    Integrate[(t - τ)/(4 * π) * Integrate[f[x + a * (t - τ) * Cos[phi] * Sin[theta],
         y + a * (t - τ) * Sin[phi] * Sin[theta], z + a * (t - τ) * Cos[theta], τ] *
        (JacobianDeterminant[Spherical[r, theta, phi]] /. {r → 1}),
       {theta, 0, Pi}, {phi, -Pi, Pi}], {τ, 0, t}], t > 0]

Plot3D[u[t, 1, 1, z], {t, .1, 10}, {z, -10, 10}]

TrigFactor[(-2 + Log[1 - ⅈ t] + Log[1 + ⅈ t])]

-2 + Log[1 - ⅈ t] + Log[1 + ⅈ t]

FullSimplify[ComplexExpand[(-2 + Log[1 - ⅈ t] + Log[1 + ⅈ t]), t], t > 0]

-2 + ⅈ Arg[1 - ⅈ t] + ⅈ Arg[1 + ⅈ t] + Log[1 + t^2]

Plot[(-2 + Log[1 - ⅈ t] + Log[1 + ⅈ t]), {t, .1, 10}]
```

Проверка

```
FullSimplify[∂_{t,t} u[t, x, y, z] - (∂_{x,x} u[t, x, y, z] + ∂_{y,y} u[t, x, y, z] + ∂_{z,z} u[t, x, y, z])]

t x/(1 + t^2)

u[0, x, y, z]

x Sin[y]

∂_t u[t, x, y, z] /. {t → 0}

y Cos[z]
```