libby@iitm.ac.in

EP2210 – Intro to QM.

Introduction and UV Catastrophe

Preparatory topics → From 2102 :
a) Lagrangian Formalism
b) Conservation laws and symmetries
c) Hamiltonian formalism (Poisson brackets)

Objectives:

a) Why quantum mechanics was necessary?
b) Time - independent Schröedinger equation.– How to calculate in one dimension?
c) Essential mathematical formalism : What are the foundations?
d) How to apply to real systems?: H atom : angular momentum, spin, perturbation theory.

Books:

1) Quantum Mechanics by Griffiths
2) Rae Quantum Mechanics (5^{th} edition) A small book, has experimental stuff.
3) Notes by David Tong.

UV Catastrophe

Perfect Blackbody
(Actually doesn't exist)

What is the spectrum?

Total energy stored in the blackbody/Volume = U

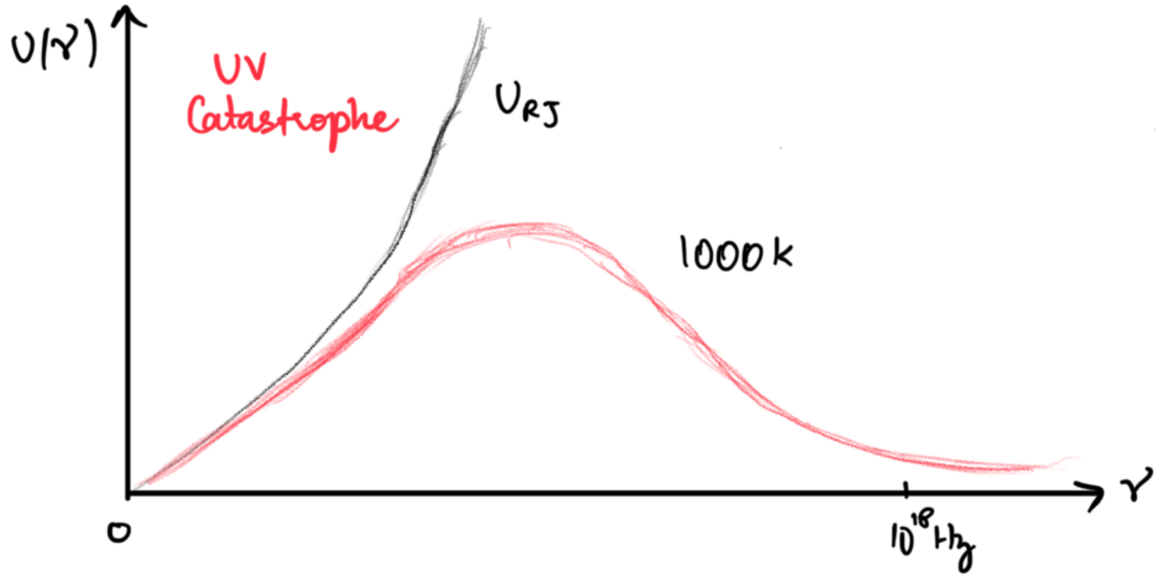
frequency of radiation

$U(\nu)\,d\nu$ = Energy stored from ν to $d\nu$.

Now, $U = \int_0^\infty U(\nu)\,d\nu$ → spectral function

Rayleigh Jeans: $U_{RJ} = \frac{8\pi\nu^2}{c^2} k_B T$

$k_B \rightarrow$ Boltzmann constant.



RJ model works only at low temperatures

Plank said energy of radiation is quantised in units of $h\nu$.

$\uparrow$ Plank's constant.

Plank gave:

$$U(\nu) = \frac{8\pi h\nu^3}{c^3} \times \frac{1}{e^{h\nu/k_BT} - 1} \sim RJ \text{ when } h\nu \ll k_BT$$

$h\nu$ is a "quantum" of energy.

Lecture - 2

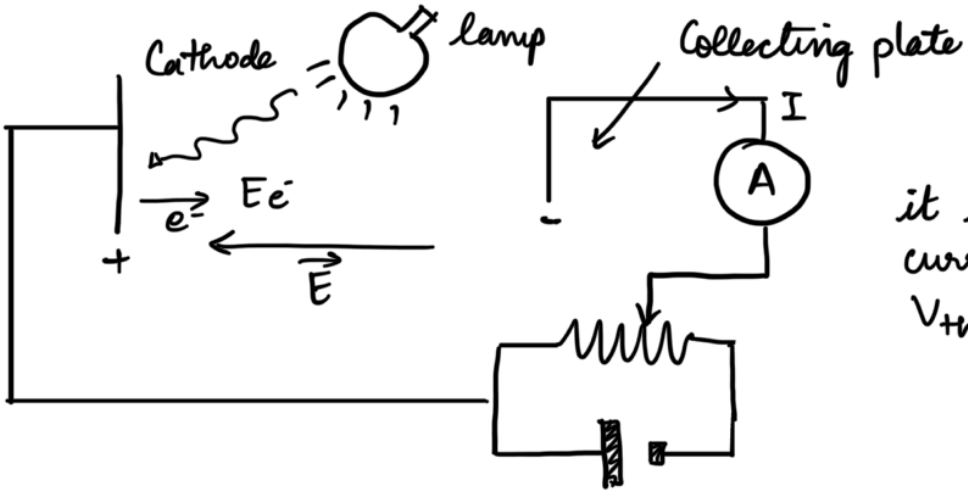
Photoelectric and Compton effect

Light at a frequency ν has a "quanta" of energy $h\nu$

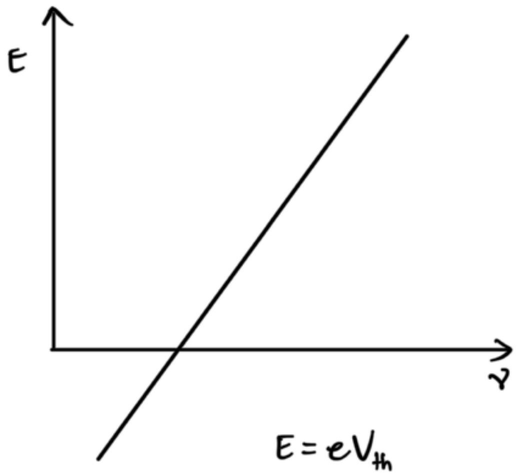
$h = 6.63 \times 10^{-34}$ Js

Nobody really bought the idea of light being an electromagnetic Radiation until they observed this phenomenon called Photoelectric effect.

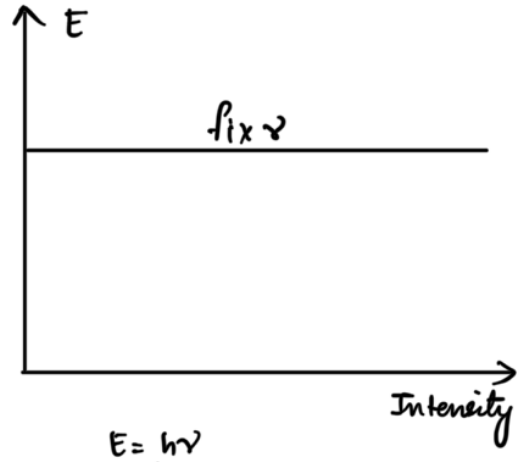
The experimental set up of P-E effect:



$Ee^- > eV$
it is collected
current flows:
V_{th} current stops.

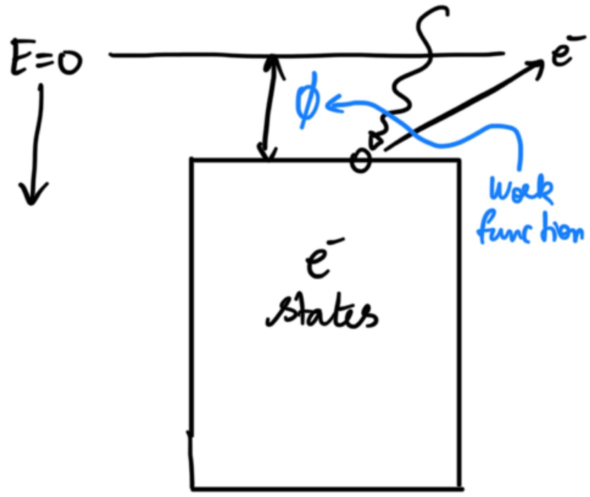


$E = eV_{th}$
↑ Threshold Voltage



$E = h\nu$
photons

The maximum kinetic energy $E = h\nu - \phi$ ↖ Work function



example :

In W(tungsten), the threshold is 230 nm wavelength λ_{th} light. E_{max} for $\lambda = 190\,nm$.

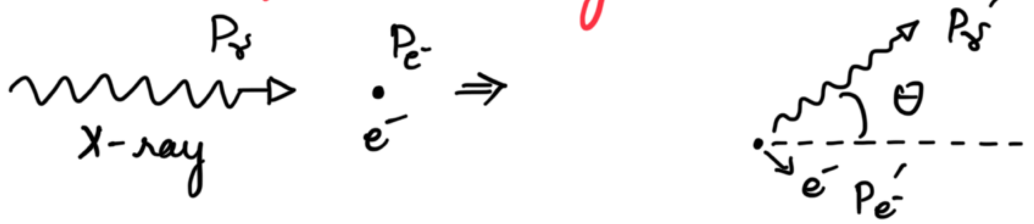
$$E = h\nu - \phi.$$

$$\phi = \frac{hc}{\lambda_{th}} \quad ; \quad h\nu = \frac{hc}{\lambda}. \quad \therefore E = \frac{hc}{\lambda} - \frac{hc}{\lambda_{th}}$$

$$hc = 1.99 \times 10^{-25}\ Jm. = 1243\ nm\,eV.$$

$\therefore$ $E = 1.1\,eV$ [1 to Kilo eV = right]
[milli eV or Giga eV → Nope].

Another phenomenon for sure made us to see light as a particle : Compton scattering



$$P = (\vec{E}, \vec{p}c) = (\vec{E}, p_x c, p_y c, p_z c)$$

$$P^\mu P_\mu = P^2 = E^2 - |\vec{p}|^2 c^2 \leftarrow \text{"length" of four vector}$$
$$= m^2 c^4$$
(But in 4 dimensions)
Invariant quantity.

[four vectors are those who obey lorentz transformation]

Now, $P_\gamma + P_{e^-} = P_\gamma' + P_{e^-}'$. [Energy / momentum Conservation]

Now, We care about P_γ'.

I do not need this.

$$\therefore (P_\gamma + P_{e^-} - P_\gamma') = P_{e^-}'.$$

$$(P_\gamma + P_{e^-} - P_\gamma')^2 = P_{e^-}'^2$$

$$P_\gamma^2 + P_\gamma'^2 + P_{e^-}^2 + 2P_e \cdot (P_\gamma - P_\gamma') = P_{e^-}'^2$$

$$\underset{0}{\overset{\shortparallel}{}} \qquad \underset{0}{\overset{\shortparallel}{}} \qquad \overset{\shortparallel}{m_e^2 c^4} + 2P_e\cdot(P_\gamma - P_\gamma') - 2P_\gamma P_\gamma' = m_e^2 c^4$$

(mass of photon = 0)

$\therefore m_p^2 c^4 = 0$

$$\Rightarrow m_e^2 \cancel{c^4} + 2P_e\cdot(P_\gamma - P_\gamma') - 2P_\gamma P_\gamma' = m_e^2 \cancel{c^4}$$

$$\Rightarrow \cancel{2}P_e\cdot(P_\gamma - P_\gamma') = \cancel{2}P_\gamma P_\gamma'$$

$$P_e\cdot(P_\gamma - P_\gamma') = P_\gamma\cdot P_\gamma'$$

$$P_e = (m_e c^2, 0) \quad \text{why?}$$

$E_\gamma = |\vec{P_\gamma}|\, c.$

$\frac{h\cancel{c}}{\lambda} = |\vec{P_\gamma}|\,\cancel{c}$

$\therefore |\vec{P_\gamma}| = \frac{h}{\lambda}.$

$$\therefore P_e(P_\gamma - P_\gamma') = m_e c^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)$$

$$\vec{P_\gamma}\cdot\vec{P_\gamma}' = E_\gamma E_\gamma' - |\vec{P_\gamma}||\vec{P_\gamma}'|\cos\theta\, c^2$$

I didn't understand how we got $E_\gamma E_\gamma'$, when it should be $|\vec{P_\gamma}||\vec{P_\gamma}'|$.

$$= \frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta).$$

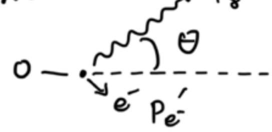
$$\Rightarrow (\lambda' - \lambda) = \frac{hc}{m_e c^2}(1-\cos\theta) = \lambda_c(1-\cos\theta)$$

λ_c: Compton wavelength.

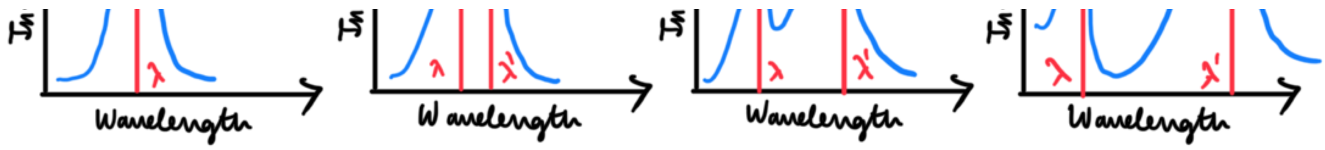
$$E_{\gamma c} = \frac{hc}{\lambda_c} = m_e c^2 \; ; \quad \lambda_c = 3.9\times 10^{-13}\,\text{m}.$$

Still didn't understand how compton effect solidifies the particle picture of light.

Compton scattering: X rays on graphite, recording the emitted photons' (inspired from photoelectric effect) wavelength and frequency, and plotting them.



Results:



$\theta = 0°$ $\theta = 45°$ $\theta = 90°$ $\theta = 135°$

Classical Electromagnetic Theory predicts that scattered wavelength = incident wavelength. [I.e, here, λ = incident wavelength so we expect only one peak in the above graphs, i.e, at λ]. This is when we consider light as a wave

But Nahhh... We see 2 peaks: λ and λ'. And the difference between λ and λ' ain't constant also → it increases with increasing scattering angle.

There is an explanation: When we consider light as a particle

Basically photons collide with the "free" electrons in Graphite and lose some of the momentum. Losing the momentum results in the release of a photon with lesser energy — meaning lesser frequency → greater wavelength → That's why λ' is larger than the incident λ, irrespective of θ.

But if this is the case, then why do we have 2 peaks? i.e., at λ and λ'?

↳ This is because, we think the electrons are all free. But in practice, they aren't. So, if the incident photon hits a tightly bound electron (bound to atom obviously) it is like, the photon hits the atom itself.

So here, $\lambda_c \neq \frac{h}{m_e c}$, but $\frac{h}{m_{atom} c}$. In case of Graphite,

$m_G = 22{,}000\, m_e$. $\therefore \lambda_c' = \frac{h}{22000 m_e c} \approx 0$ in the given

graph's wavelength's scale, where λ and λ' are comparable.

This is why the shift in λ is so small, it appears to us like there's a peak at λ.

Or we can say, this is an elastic collision with $m_1 \gg m_2$.

The photon with the incident wavelength essentially "bounces" off m_1 with almost no change in velocity. That's Why peak at λ.

The atom, which was stationary before, remains as it is because the momentum transfer is minimal, relative to its large mass.

$$P_e = m_e c^2$$
$$P_\gamma = hc/\lambda.$$
$$P_\gamma' = hc/\lambda'.$$

$$P_e = |\mathbb{P}_e| \rightarrow \mathbb{P}_e = (m_e c^2, 0)$$
$$\mathbb{P}_\gamma = \left(\frac{hc}{\lambda}, \vec{P}\right) \quad ; \quad \mathbb{P}_\gamma' = \left(\frac{hc}{\lambda'}, \vec{P}'\right)$$

$$P_e \cdot (P_\gamma - P_\gamma') = P_\gamma \cdot P_\gamma'$$

$\vec{P}^2 = E^2 - p^2$

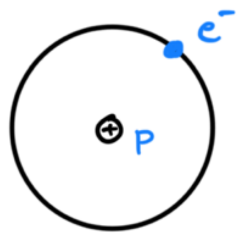
$$P_\gamma \cdot P_\gamma' = E_\gamma E_\gamma' - P_\gamma P_\gamma' \cos\theta \, c^2$$

[The four vector: "Energy-momentum" - Has units of "Energy" in Libby's Version, and in Sriramkumar's version, it has units of momentum].

I also learnt about the dot product of 2 four vectors:

Lecture - 3
Bohr model and wave nature.

Discrete spectra emitted by H and other elements?



Bohr model

Bohr's Postulates

plank's constant

integer

1) $\oint_0^{2\pi} P_\theta \, d\theta = nh \Rightarrow P_\theta = n\hbar.$

line integral of orbit

angular momentum

2) No classical radiation, but discrete radiation between n and m orbits : $h\nu = E_n - E_m$

Now, we borrow some ideas from Classical Mechanics:

$$E = T + V = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} = \frac{p_\theta^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}$$

$(p_\theta = mvr)$

The centrifugal barrier, remember?

Centripetal condition:

$\hbar = h/2\pi$ - reduced plank's constant

$$\frac{mv^2}{r} = \frac{p_\theta^2}{mr^3} = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

$$\Rightarrow \frac{(n\hbar)^2}{mr_n^3} = \frac{e^2}{4\pi\varepsilon_0 r_n^2}$$

r_n = n^{th} orbit radius

$$\Rightarrow r_n = \frac{n^2\hbar^2(4\pi\varepsilon_0)}{me^2}$$

$$E_n = \frac{-p_\theta^2}{2mr_n^2} = \frac{-R}{n^2}$$, where R = energy Rydberg constant

and $R = \dfrac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$

$$R = \frac{1}{2}\underbrace{(m_e c^2)}_{\text{rest mass of an electron}}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\frac{1}{(\hbar c)^2}$$

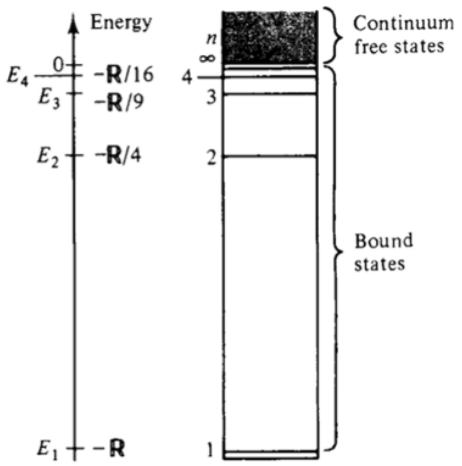
$$= \frac{1}{2}(5.11\times10^3\,eV)(1.44\ eV\,nm)^2 \times \frac{1}{(198\ eV\,nm)^2}$$

$\boxed{R = -13.6\ eV}$ → energy required to rip off an Hydrogen atom.

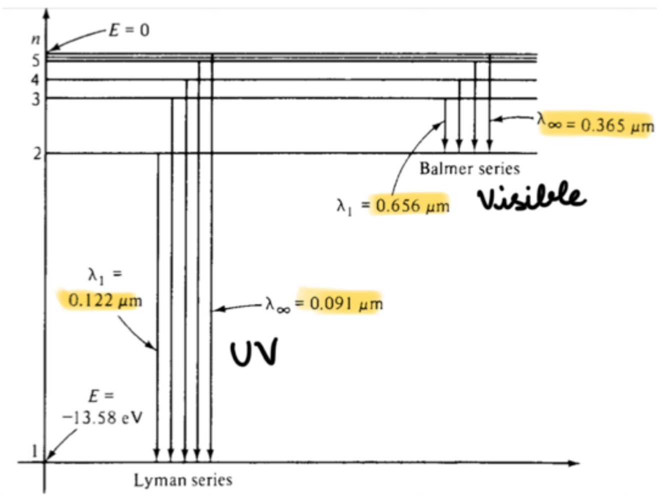
$n=1 \Rightarrow r_1 = a_0 = 5.3\times10^{-2}\,nm = 5.3\times10^{-11}\,m$

↓ Bohr Radius

Radius of the orbit corresponding to the lowest energy state (n=0)



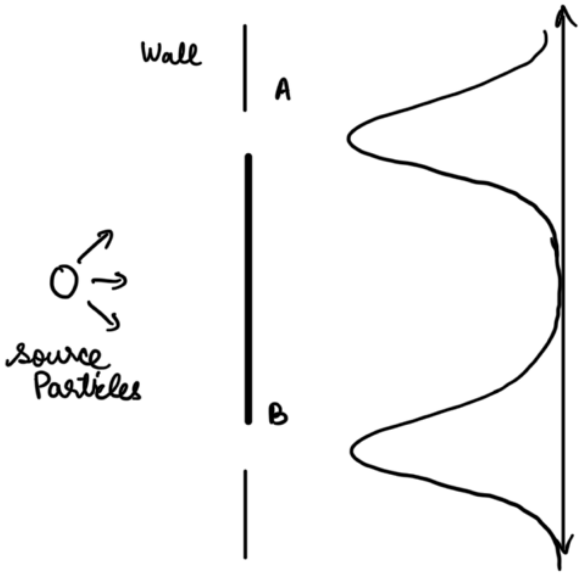
Bohr Spectrum



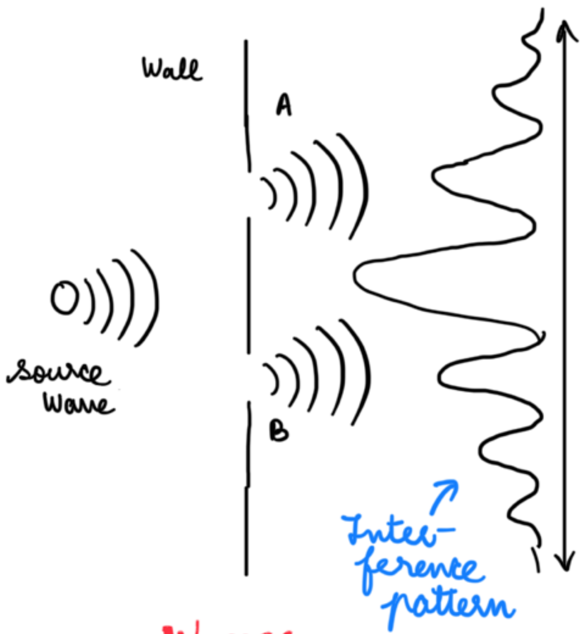
Hydrogen Emission Spectrum

Wave Nature

"Interference" is the property of waves that makes it so special, unlike particles.



Particles



Waves

Amp $= I$, $I_A + I_B = I_{TOT}$.

$I = |A|^2$, $I_{TOT} = |A_1 + A_2|^2$
(A = Amplitude of the wave)

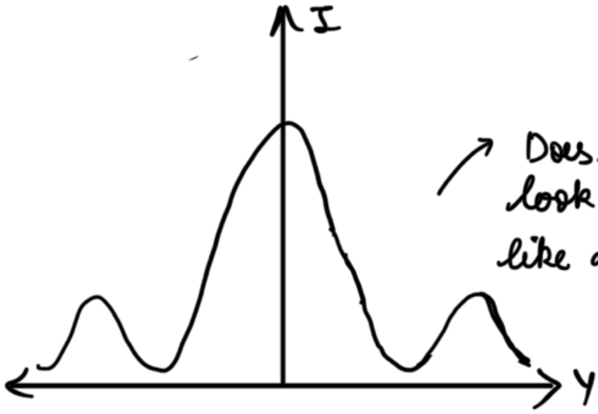
Example:

A open : $A_A = \sqrt{\frac{1}{2} e^{-y^2/2} e^{i(\omega t - ay)}}$

B open : $A_B = \sqrt{\frac{1}{2} e^{-y^2/2} e^{i(\omega t - ay - by)}}$

Both open? $I = |A_A + A_B| = |A_A| + |A_B| + 2\,\mathrm{Re}(A_A^* A_B)$

$$I = e^{-y^2} + e^{-y^2/2} \cos(by/2)$$



Does this look somewhat like a shifted sinc?

Photon : wave and particle

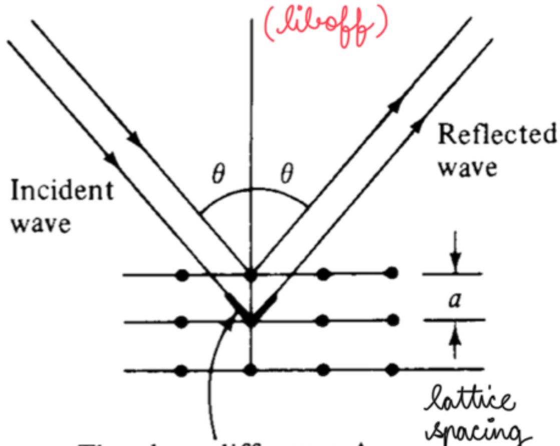
Standing waves

$$\lambda = \frac{h}{|\vec{p}|} = \frac{\hbar}{|\vec{k}|}, \quad |\vec{k}| = \frac{2\pi}{\lambda}$$

Lecture 4: Wave particle duality, wave function and probability

$\lambda = \frac{h}{p}$ → deBroglie relation

Bragg's diffraction with X Rays



The phase difference Δ

$\Delta = 2a\cos\theta = n\lambda$

Davisson - Germer Experiment

e^- on Ni found Bragg like pattern with $p \sim h/\lambda$.

neutron scattering off materials behaved in the same way.

↓

but neutron, since it is charge-neutral, is preferred - so that it is not being confused with electron or nuclei scattering.

Numerical :

$\lambda_n \cong d = 2\,nm$: what speed is required?

$$= \frac{h}{|P_n|} \Rightarrow |\vec{P_n}| = \frac{h}{\lambda_n} \rightarrow |\vec{P_n}|c = \frac{\hbar c}{\lambda_n} \times 2\pi = \frac{198}{2} \times 2\pi$$

$$\simeq 622\,eV.$$

$$(M_n = 940\ MeV/c^2)$$

$$\frac{|\vec{v}|}{c} = \frac{|\vec{P}|c}{M_n c^2} = 7\times10^{-7} \Rightarrow v = 200\,m/s.$$

What do you mean by "cold" neutrons → Explore more in here

Now, coming back to the double slit experiment,

To really know whether the electron is behaving as a wave or a particle when it passes through the slits.

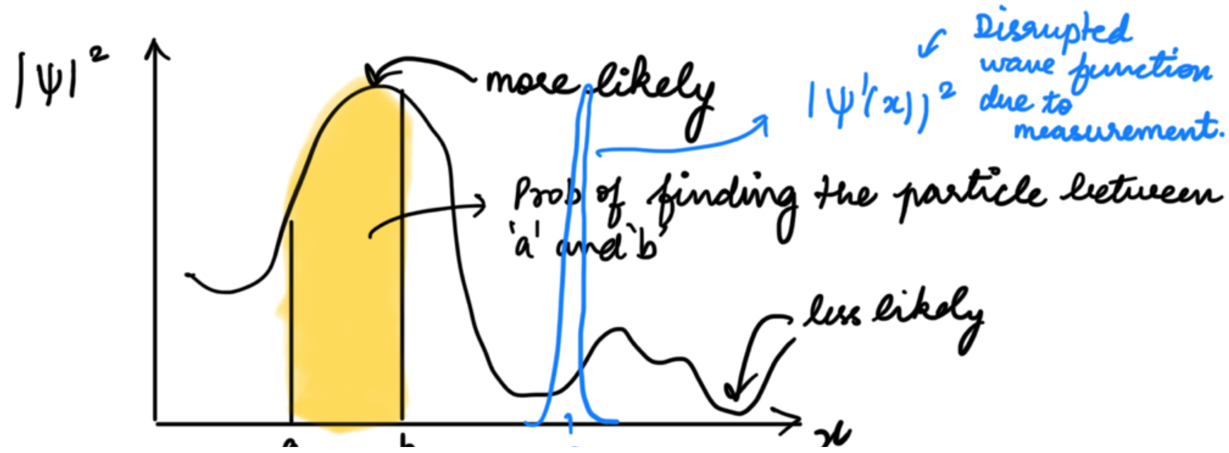
. . . .

Since we don't know the "position", we need to learn the math of unknown things:
Probability.

The wave function describes a quantum system : $\psi(x,t)$

Definition according to Born :

$$\int_a^b |\psi(x,t)|^2 dx = \text{Probability of finding a particle between } a \text{ \& } b$$



Normalisation $\rightarrow \displaystyle\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

Goes to zero at $\pm\infty$

$\downarrow$

Square integrable function $\in L_2(-\infty, \infty)$

Lecture 5 - Needs to be done.

Lecture 6 : Momentum & Uncertainty

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi(x,t)$$

Recall: $\langle x \rangle = \displaystyle\int_{-\infty}^{\infty} \Psi^* x \Psi \, dx.$

$$\frac{d\langle x \rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} x |\Psi|^2 dx.$$

$$= \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} |\Psi|^2 dx$$

$$= \int x \left(\frac{\partial \Psi^*}{\partial t} \Psi + \Psi^* \frac{\partial \Psi}{\partial t} \right) dx.$$

$\frac{\partial \Psi^*}{\partial t}$ and $\frac{\partial \Psi}{\partial t}$ substituted,

$$\Rightarrow \frac{d\langle x \rangle}{dt} = \frac{i\hbar}{2m} \int_{-\infty}^{\infty} x \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx.$$

$$= -\frac{i\hbar}{2m} \int_{-\infty}^{\infty} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \underbrace{\frac{\partial \Psi^*}{\partial x} \Psi}_{\text{integration by parts.}} \right) dx$$

$\Psi(\pm\infty) = 0$

$$= \frac{-i\hbar}{m} \int_{-\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, dx.$$

Now, $$m \frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx.$$

$\langle p \rangle$ = Expectation Value of momentum

$$\Rightarrow \langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{p}\, \Psi \, dx. \quad , \hat{p} = \text{momentum operator.}$$

General formula for kinematic Variables:

$$\hat{Q}(\hat{x}, \hat{p}) = \hat{Q}\left(x, -i\hbar \frac{\partial}{\partial x}\right)$$

eg. Ehrenfest theorem $\rightarrow$ What is $\frac{d\langle p \rangle}{dt}$?

$$\frac{d}{dt}\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \left(\frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial^2}{\partial x \partial t} \Psi \right) dx.$$

$\vdots$ [More integration by parts]

$$\frac{d\langle p \rangle}{dt} = -\int \Psi^* \left(\frac{\partial}{\partial x} V\right) \Psi \, dx = \left\langle -\frac{\partial V}{\partial x} \right\rangle = -\left\langle \frac{dV}{dx} \right\rangle.$$

Ehrenfest Theorem: "Expectation values follow classical laws".

$$\langle x \rangle, \langle x^2 \rangle \Rightarrow \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\langle p \rangle, \langle p^2 \rangle \Rightarrow \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

Heisenberg Uncertainty principle

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$ → Simultaneous measurements

Explained by a rope fixed on one end,.

→ Standing wave.
Know momentum,
but don't know position.

→ Pulse travelling
Know position,
but don't know momentum

Example: $\Psi(x) = \left(\frac{\pi}{\alpha}\right)^{-1/4} e^{-\alpha x^2/2}$.

What are $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, σ_x and σ_y ?
Also, what is $\sigma_x \sigma_p$?

$$\langle x \rangle \equiv \int_{-\infty}^{\infty} \Psi^* x \, \Psi \, dx \sim \int_{-\infty}^{\infty} \underset{\text{even}}{e^{-\alpha x^2}} \, \underset{\text{odd}}{x} \, dx = 0.$$

$$\langle x^2 \rangle = \left(\frac{\pi}{\alpha}\right)^{-1/2} \int_{-\infty}^{\infty} x^2 e^{\alpha x^2} dx$$

$$\int_0^{\infty} x^{2n} e^{-x^2/b^2} dx = \frac{\sqrt{\pi}\,(2n)!}{n!} \cdot \left(\frac{b}{2}\right)^{2n-1}$$

$$\Rightarrow \langle x^2 \rangle = \frac{1}{2\alpha} \Rightarrow \sigma_x = \frac{1}{\sqrt{2\alpha}} \quad \because \langle x \rangle = 0$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx. = 0.$$

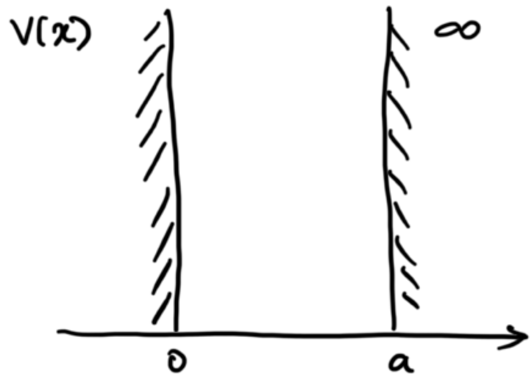
$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)\left(-i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx. = \frac{\alpha \hbar^2}{2}.$$

$$\Rightarrow \sigma_p = \frac{\sqrt{\alpha}\,\hbar^2}{\sqrt{2}}.$$

Lecture - 7 (Needs to be done)

Lecture - 8

Infinite Well



$$u(x) = A \sin kx$$

$$k = \sqrt{2mE}/\hbar$$

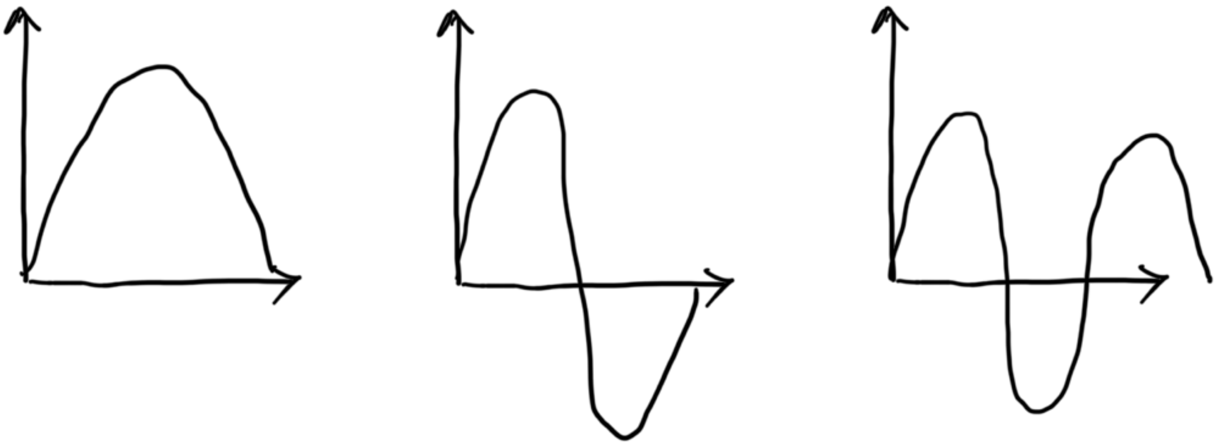
$$E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2$$ ← integer

Now Normalise:

$$\int_0^a |A|^2 \sin^2 kx \, dx = |A|^2 \frac{a}{2} \Rightarrow A = \sqrt{\frac{2}{a}}$$

Check for dimensions of the Normalisation Constant as well.

$$u_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$



A distinct difference between Classical and Quantum World:

Squaring U_2 up, we see that Prob. to find (let's say a quantum ball) at the centre of the well = 0, but in Classical, it is not so.

Nodes: $u_n(x) = 0$ → Increase with n.

The sol^{n}s are orthogonal:

$$\int_0^a U_m^* U_n \, dx = 0 \quad \text{if } m \neq n$$

$$\int_0^a U_m^* U_n = \delta_{mn} \qquad \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

Recall:

$$\psi(x,0) = \sum_{n=1}^{\infty} C_n U_n$$

$t=0$ for now

$$|\psi|^2 dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_m^* C_n \overbrace{\int U_m^* U_n \, dx}^{\delta_{mn}.} = 1$$

$$= \boxed{\sum_{n=1}^{\infty} |C_n|^2 = 1}$$

$\therefore$ The coefficient squared of each state is now, the probability of finding that particle in that state.

$$\int_{-\infty}^{\infty} U_n^* \psi(x,0) \, dx = \sum_{m=1}^{\infty} C_m \underbrace{\int_{-\infty}^{\infty} U_n^* U_m \, dx}_{\delta_{mn}} = C_n$$

Example: $(t=0)$

$$\psi(x,0) = \begin{cases} A(x/a) & 0 \leq x \leq a/2 \\ A(1 - x/a) & a/2 \leq x \leq a \end{cases}$$

What is the probability for E_n ?

Normalise to find A:

$$I = 2\int_0^{a/2} |A|^2 \left(\frac{x}{a}\right)^2 dx \quad \text{[Symmetric Potential]}$$

$$= \frac{2|A|^2}{3a} x^3\Big|_0^{a/2} = \frac{\cancel{2}}{3}\frac{|A|^2}{\cancel{a}} \cdot \frac{a^{\cancel{3}}}{\cancel{8}4}$$

$$\frac{|A|^2 a^2}{12} = 1, \quad |A|^2 = \frac{12}{a^2}; \quad \boxed{|A| = \frac{2\sqrt{3}}{a}}$$

$$C_m = \int_0^a U_m^* \psi(x,0)\, dx$$

via Symm,
$m = 2n+1$, $n = 0, 1, \ldots$

$$= \sqrt{\frac{24}{a}} \times 2 \int_0^{a/2} \frac{x}{a} \frac{\sin m\pi x}{a}\, dx \quad [\text{Int. By Parts}].$$

$$C_{2n+1} = \frac{2\sqrt{24}}{(2n+1)^2\pi^2}(-1)^n \Rightarrow |C_n|^2 = \frac{96}{\pi^4(2n+1)^4}$$

$$\langle E \rangle = \sum_{n=0}^{\infty} |C_n|^2 E_{n+1}$$

Is the well physical?

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ma} = \frac{\pi^2(\hbar c)^2 n^2}{2(m_e c^2)\, a}$$

$(\hbar c) = 198$ eVnm
$m_e c^2 = 511$ keV
$a = 1$ Å $= 0.1$ nm.

$$= 37.9\, n^2 \text{ eV}$$

$E_2 - E_1 = 28.3$ eV.

CPT Theorem ↙ Next Class on Symmetries?!
feat. Zibley? ♡
(Let's See)

Lecture - 9
Symmetry and free particle

Nah bro, not CPT !!

If $V(x)$ is even, i.e; $V(x) = V(-x)$, the $U(x)$ are odd or even function.

Proof:

$$\frac{\hbar^2}{2m} \frac{d^2 u(x)}{dx^2} + \left[V(x) - E\right] u(x) = 0$$

↑ Time Independent Schroedinger Equation

Change of Variables : $\omega = -x$.

$$\Rightarrow \frac{\hbar^2}{2m} \frac{d^2}{dx^2} u(-\omega) + \left[V(-\omega) - E\right] u(-\omega) = 0.$$

Recall: $\frac{d^2 f}{dx^2} = \left(\frac{d\omega}{dx} \frac{d}{d\omega} f\right) =$

$$\frac{\hbar^2}{2m} \frac{d}{dx^2} u(-x) + \left[V(-x) - E\right] u(-x) = 0$$

ω is a dummy variable

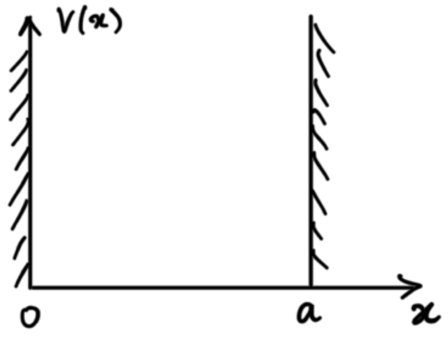
$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} u(-x) + \left[V(x) - E\right] u(-x) = 0$$

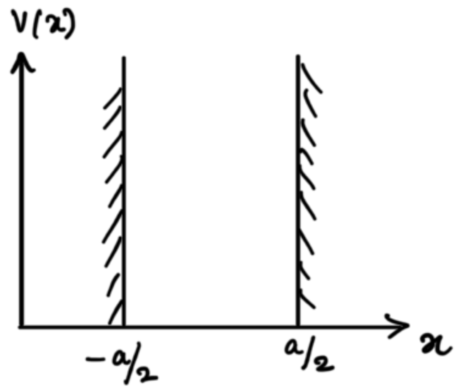
This implies $V(x) = V(-x)$

Compare (i) and (ii) and Conclude $u(x)$ and $u(-x)$ satisfy the Time Independent Schroedinger Equation.

$$u_{even}(x) = u(x) + u(-x) \; ; \; u_{odd} = u(x) - u(-x)$$

Infinite square well:





$$x' = x - a/2$$

$$V(x') = V(x)$$

$$\psi_n(x') = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2}\right)$$

$$= \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x'}{a}\right)\cos\left(\frac{n\pi}{2}\right)}_{\text{This part when } n = \text{odd}} + \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x'}{a}\right)}_{\text{This part when } n = \text{even}}$$

$\psi_n =$ This part when $n =$ odd — This part when $n =$ even

Free Particle $V(x) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = Eu \Rightarrow u \cdot Ae^{ikx} - Be^{-ikx} \quad \left(k = \frac{\sqrt{2mE}}{\hbar}\right)$$

$$\Rightarrow \psi(x,t) = \underbrace{Ae^{ik\left(x - \frac{\hbar k}{2m}t\right)}}_{\text{Wave in } +x \text{ direction}} + \underbrace{Be^{-ik\left(x - \frac{\hbar k}{2m}t\right)}}_{\text{Wave in } -x \text{ direction}}$$

$$\psi_t(x,t) = Ae^{i\left(hx - \frac{\hbar k^2}{2m}t\right)} \qquad -\infty < k < \infty.$$

"Stationary states are waves with $\lambda = 2\pi/k$."

But $\int |\psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} dx$?!?!

A free particle cannot be in a stationary state.

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n U_n e^{-iE_x t/\hbar} \qquad \text{eg. for infinite well.}$$

But for continuous Variable : $\sum_{n=0}^{\infty} \rightarrow \int_{-\infty}^{\infty} dk$, $C_n \rightarrow \frac{1}{\sqrt{2\pi}}\phi(k)$

$$\Rightarrow \boxed{\psi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk}$$

Is this Normalisable? — Depends on $\phi(k)$

Wave Packet.

If $\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{ikx}\, dx$

$$\Rightarrow \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx}\, dx$$ (Fourier Transform)

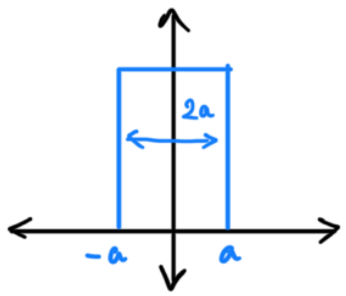
Eg.

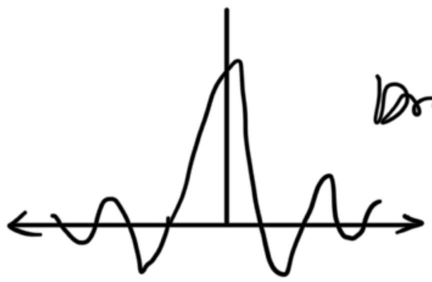
$$\Psi(x,0) = \begin{cases} A & (-a < x < a) \\ 0 & \text{everywhere else} \end{cases}$$

Normalising, we get:

$$|\Psi(x,0)|^2 = \int_{-a}^{a} |A|^2 dx = 1 \Rightarrow A = \frac{1}{\sqrt{2a}}.$$

$$\Rightarrow \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1}{\sqrt{2a}} e^{-ikx}\, dx = \sqrt{\frac{a}{\pi}}\, \frac{\sin ka}{ka}$$ ← Sinc (ka) !!





Draw properly