Quantum Comeback - 2

Generalised Statistical interpretation

Enables you to figure out the possible results of any measurement and their probabilities

If you measure an observable $Q(x,p)$ on a particle in the state $\Psi(x,t)$, you are certain to get one of the eigen values of the Hermitian operator $\hat{Q}(x, -i\hbar\frac{d}{dx})$.

- If the spectrum of $\hat{Q}$ is discrete, the probability of getting the particle eigen value q_n associated with the orthonormalised eigen function $f_n(x)$ is

$$|c_n|^2 \rightarrow c_n = \langle f_n | \Psi \rangle$$

- If the spectrum is continuous, with real eigen values $q(z)$ and associated (Dirac orthonormalised) eigen functions $f_z(x)$, the probability of getting a result in the range dz is:

$$|c(z)|^2 dz \quad \text{where} \quad c(z) = \langle f_z | \Psi \rangle$$

Now, the statistical interpretation

Eigenfunctions of an observable operator are complete, Hence,

$$\Psi(x,t) = \sum_n c_n(t) f_n(x).$$

[Let's for now assume that spectrum is discrete]

Since $f_n(x)$ are orthonormal,

$$c_n(t) = \langle f_n | \Psi \rangle = \int f_n(x)^* \Psi(x,t)\, dx$$

c_n tells you "how much f_n is contained in Ψ".

Ofcourse: $\sum_n |c_n|^2 = 1.$

How? :

$$1 = \langle \Psi | \Psi \rangle = \left\langle \left(\sum_{n'} c_{n'} f_{n'} \right) \middle| \left(\sum_{n} c_n f_n \right) \right\rangle$$

$$= \sum_{n'} \sum_{n} c_{n'}^* c_n \langle f_{n'} | f_n \rangle$$

$$= \sum_{n'} \sum_{n} c_{n'}^* c_n \delta_{n'n} = \sum_{n} c_n^* c_n = \sum_{n} |c_n|^2.$$

Similarly, $\langle Q \rangle = \sum_n q_n |c_n|^2$ ... How?:

$$\langle Q \rangle = \langle \Psi | \hat{Q} \Psi \rangle = \left\langle \left(\sum_{m} c_m f_m \right) \middle| \hat{Q} \left(\sum_{n} c_n f_n \right) \right\rangle$$

But $\hat{Q} f_n = q_n f_n$.

$$\therefore \Rightarrow \sum_m \sum_n c_m^* c_n q_n \langle f_m | f_n \rangle = \sum_m \sum_n c_m^* c_n q_n \delta_{mn}$$

$$= \sum_n c_n^* c_n q_n = \sum_n q_n |c_n|^2.$$

Now, can we get the original statistical interpretation for position measurements? Let us see.

$$\hat{x} |\Psi\rangle = y |\Psi\rangle. \quad \therefore \quad g_y(x) = \delta(x-y).$$

$$\therefore c(y) = \langle g_y | \Psi \rangle = \int_{-\infty}^{\infty} \delta(x-y) \Psi(x,t)\, dx = \Psi(y,t).$$

$\therefore$ Probab of getting a result in the range dy : $|\Psi(y,t)|^2 dy$.

precisely the original statistical interpretation.

Then, momentum?

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$

$$\therefore c(p) = \langle f_p | \Psi \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx.$$

$-\infty$

This is of such great importance $\rightarrow$ It is called the momentum space wave function : $\Phi(p,t)$.

This essentially turns out to be the Fourier Transform of the (position space) wave function $\psi(x,t)$ $\rightarrow$ which is its inverse fourier transform by Plancherel's theorem.

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \psi(x,t)\, dx$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, \Phi(p,t)\, dp.$$

and the probab. that a measurement of momentum would yield a result in the range dp is : $|\Phi(p,t)|^2 dp$.

How to tackle questions :

A $V(x)$ would be given, from which we need to get $\psi(x)$ by solving TISE, then $\psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$.

For example, let $V(x) = -\alpha\, \delta(x)$ [delta function well]

What is the probability that a measurement of its momentum would be greater than $p_0 = m\alpha/\hbar$?

We know, $\psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar}\, e^{-m\alpha|x|/\hbar^2}\, e^{-iEt/\hbar}$

[where $E = -m\alpha^2/2\hbar^2$).

$$\therefore \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{m\alpha}}{\hbar}\, e^{-iEt/\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, e^{-m\alpha|x|/\hbar^2} dx$$

$$= \sqrt{\frac{2}{\pi}}\; \frac{p_0^{3/2}\, e^{-iEt/\hbar}}{\quad}$$ [looked up the integral]

$$\sqrt{\ } \quad p^2 + p_0^2$$

$\therefore$ Now, probability $= |\Phi(p,t)|^2 dp$,

$$\Rightarrow \frac{2}{\pi} p_0^3 \int_{p_0}^{\infty} \frac{1}{(p^2+p_0^2)^2} dp = \frac{1}{\pi}\left[\frac{p\,p_0}{p^2+p_0^2} + \tan^{-1}\left(\frac{p}{p_0}\right)\right]\Bigg|_{p_0}^{\infty}$$

$$= \frac{1}{4} - \frac{1}{2\pi} = 0.0908 \quad \text{[looked up the integral]}$$

The Uncertainty Principle

A beautiful yet abstract proof of a more general version of the uncertainty Principle

We know $\sigma_A^2 = \langle (\hat{A} - \langle A \rangle)\psi \,|\, (\hat{A} - \langle A \rangle)\psi \rangle \equiv \langle f | f \rangle$

$$\therefore \ f = (\hat{A} - \langle A \rangle)\psi$$

Similarly $\sigma_B^2 = \langle g | g \rangle, \quad g \equiv (\hat{B} - \langle B \rangle)\psi.$

$\therefore$ Invoking the Schwarz inequality,

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

Now, for any complex number z,

$$|z|^2 = [Re(z)]^2 + [Im(z)]^2 \geq [Im(z)]^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

Now, $\langle f | g \rangle \equiv z$. $\therefore \langle g | f \rangle = z^*$

$\therefore |z|^2 = |\langle f | g \rangle|^2$ and we know from above: $|z|^2 \geq \left[\frac{1}{2i}(z - z^*)\right]^2$.

$$\therefore \ |\langle f | g \rangle|^2 \geq \left(\frac{1}{2i}\left[\langle f | g \rangle - \langle g | f \rangle\right]\right)^2$$

$$\therefore \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\left[\langle f|g\rangle - \langle g|f\rangle\right]\right)^2$$

The generalised Uncertainty Principle

$$\therefore \sigma_A \sigma_B \geq \frac{1}{2i}\left[\langle f|g\rangle - \langle g|f\rangle\right]$$

Now, $\langle f|g\rangle = \langle (\hat{A} - \langle A\rangle)\Psi \,|\, (\hat{B} - \langle B\rangle)\Psi \rangle$

Using the hermiticity of $(\hat{A} - \langle A\rangle)$ operator,

$$\langle f|g\rangle = \langle \Psi \,|\, (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | (\hat{A}\hat{B} - \hat{A}\langle B\rangle - \hat{B}\langle A\rangle + \langle A\rangle\langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi\rangle - \langle \Psi | \hat{A}\Psi\rangle\langle B\rangle - \langle \Psi | \hat{B}\Psi\rangle\langle A\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi\rangle - \langle A\rangle\langle B\rangle - \langle B\rangle\langle A\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle$$

$|||^{ly}$, $\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle$

$$\therefore \langle f|g\rangle - \langle g|f\rangle = \langle \hat{A}\hat{B}\rangle - \cancel{\langle A\rangle\langle B\rangle} - \langle \hat{B}\hat{A}\rangle + \cancel{\langle A\rangle\langle B\rangle}$$
$$= \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle$$
$$= \langle [\hat{A},\hat{B}]\rangle. \leftarrow \text{Basically the Commutator!}$$

$$\therefore \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle [\hat{A},\hat{B}]\rangle\right)^2$$

Now, commutator of 2 Hermitian operators is anti-Hermitian $[\hat{Q}^{\dagger} = -\hat{Q}]$ and the expectation value of the commutator of that anti Hermitian operator, which is the commutator of 2 Hermitian operators is imaginary !.

$\therefore \langle [\hat{A},\hat{B}]\rangle =$ imaginary. $\therefore \frac{1}{2i}\langle [\hat{A},\hat{B}]\rangle =$ real !

and the square of it is positive ... ♡

Now, let $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$.

We know, $[\hat{x}, \hat{p}] = i\hbar$.

$$\therefore \quad \sigma_x^2 \sigma_p^2 \geq \left(\frac{\hbar}{2}\right)^2 \quad \text{or} \quad \boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}} \quad !!$$

Infact, there is an "uncertainty principle" for every pair of observables whose operators do not commute [incompatible observables]

Incompatible observables do not have shared eigen functions → cannot have a complete set of common eigen functions of simultaneous eigen functions (states that are determinate for both observables)

Examples of compatible operators:

In the Hydrogen atom for instance, Hamiltonian, magnitude of angular momentum and L_z are mutually compatible.

Uncertainty principle – not an extra assumption in QM, but rather a consequence of the statistical interpretation.

Corresponds to the fact that non-commuting matrices cannot be simultaneously diagonalised, whereas commuting matrices can be simultaneous diagonalised.

Minimum Uncertainty Wave-packet ?

Schwarz inequality becoming an equality → place where we would get minimum uncertainty right ?

that happens when $g(x) = c\, f(x)$. $c \in \mathbb{C}$.

Now, c is purely imaginary $= ia$, say. How?

We know, $|z|^2 = \mathrm{Re}^2(z) + \mathrm{Im}^2(z) \geq \mathrm{Im}^2(z)$.

$$|z|^2 = \mathrm{Im}^2(z) \text{ if } \mathrm{Re}(z) = 0 \quad \text{[Min. Uncertainty]}.$$

$$\therefore \mathrm{Re}\langle f|g\rangle = \mathrm{Re}\left(c\langle f|f\rangle\right) = 0.$$

Now, $\langle f|f\rangle$ is real. $\therefore$ c must be complex. Purely. So that $\mathrm{Re}\left(c\langle f|f\rangle\right) = 0$.

$$\therefore g(x) = ia\, f(x), \quad a \in \mathbb{R}$$

Let $f = (\hat{p} - \langle p\rangle)\psi$ and $g = ia(\hat{x} - \langle x\rangle)\psi$.

We know, $\langle f|f\rangle\langle g|g\rangle = \sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2$

Minimum Uncertainty $= \sigma_x^2\sigma_p^2 = \langle f|g\rangle^2$

$$\therefore \sigma_x^2\sigma_p^2 = \left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\psi = ia\left(x - \langle x\rangle\right)\psi$$

Differential equation for ψ as a function of x.

General solution (Guess what?) is:

$$\psi(x) = A e^{-a(x-\langle x\rangle)^2/2\hbar}\, e^{i\langle p\rangle x/\hbar} \rightarrow \text{GAUSSIAN!!}$$

Energy-time Uncertainty Principle

$$\Delta x\, \Delta p \geq \frac{\hbar}{2} \longrightarrow \Delta E\, \Delta t \geq \frac{\hbar}{2}.$$

In special Relativity, this is quite obvious. The energy-time form might be thought of as a consequence of position-momentum version.

x and t $\longrightarrow$ Position – time four vector
p and E $\longrightarrow$ energy – momentum four vector

But here we are dealing with non-relativistic QM!!

Now, let's derive the energy-time uncertainty principle!

Here, we treat time as an independent variable, of which dynamical quantities are functions.

To measure how fast the system is changing, let us compute the time-derivative of the expectation value of an observable $Q(x,p,t)$:

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle \Psi | Q\Psi\rangle = \left\langle \frac{\partial \Psi}{\partial t} \middle| Q\Psi\right\rangle + \left\langle \Psi \middle| \frac{\partial Q}{\partial t}\Psi\right\rangle + \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t}\right\rangle$$

$$\text{TDSE: } i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \quad , \quad H = \frac{p^2}{2m} + V$$

But $\hat{H}$ is hermitian

$$\therefore \frac{d}{dt}\langle Q\rangle = \left\langle \frac{1}{i\hbar}\hat{H}\Psi \middle| \hat{Q}\Psi\right\rangle + \left\langle \Psi \middle| \frac{\hat{Q}\hat{H}\Psi}{i\hbar}\right\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle.$$

$$= \frac{1}{-i\hbar}\langle \hat{H}\Psi|\hat{Q}\Psi\rangle + \frac{1}{i\hbar}\langle \Psi|\hat{Q}\hat{H}\Psi\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

$$= \cancel{\frac{1}{-i\hbar}\langle \hat{H}\Psi|\hat{Q}\Psi\rangle + \frac{1}{i\hbar}\langle \hat{H}\Psi|\hat{Q}\Psi\rangle} + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

$$= \left\langle \frac{\partial Q}{\partial t}\right\rangle.$$

For any operator, $\displaystyle \frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\left\langle [\hat{H},\hat{Q}]\right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$

The Generalised Ehrenfest Theorem !!

Now let's say, $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$

$A \to H$ and $B \to Q$,

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2$$

$$= \frac{1}{2i} \left[\frac{\hbar}{i} \frac{d}{dt} \langle Q \rangle \right]^2 = \left(\frac{\hbar}{2} \right)^2 \left(\frac{d\langle Q \rangle}{dt} \right)^2$$

or more simply, $\sigma_H \sigma_Q \geq \frac{\hbar}{2} \frac{d\langle Q \rangle}{dt}$.

Defining σ_H as ΔE and $\Delta t \equiv \frac{\sigma_Q}{|d\langle Q \rangle / dt|}$.

Δt represents the amount of time it takes the expectation value of Q to change by one standard deviation.

Then, $\Delta E \, \Delta t \, \cancel{|d\langle Q \rangle / dt|} \geq \frac{\hbar}{2} \cancel{\frac{d\langle Q \rangle}{dt}}$

$$\boxed{\Delta E \, \Delta t \geq \frac{\hbar}{2}}$$

Δt depends entirely on what observable (Q) you care to look at — the change might be rapid for one observable and slow for another.

But if ΔE is small, then the rate of change of all observables must be very gradual,

or, if any observable changes rapidly, the uncertainty in its energy must be large.

Caution: The energy-time uncertainty principle tells us only that $\Delta E \Delta t$ must be greater than $\hbar/2$, but it does not specify how large ΔE and Δt are. That will be determined by other properties of the system.

Put another way, if I have some system with a very large timescale Δt, then my system could also have a large energy uncertainty. That depends on the nature of my system. All the energy-time uncertainty principle tells us is that ΔE has a lower limit.

Vectors and Operators

1) Bases in Hilbert space:

A state of a system in Quantum Mechanics is a vector. $|S(t)\rangle$ that lives out there in Hilbert space.

Wave function is the 'x' component of the state vector in position basis: $\Psi(x,t) = \langle x|S(t)\rangle$

$|x\rangle \rightarrow$ eigen function of $\hat{x}$
$x \rightarrow$ eigen value of $\hat{x}$.

We can express $|S(t)\rangle$ in any basis.

Let us say we expand it in a basis of Energy eigen functions

$$C_n(t) = \langle n|S(t)\rangle$$

$|n\rangle \rightarrow$ n^{th} eigen function of $\hat{H}$ and $n \rightarrow$ corresponding eigen value.

Let us say we expand it in a basis of momentum eigen functions

$$\Phi(p,t) = \langle p|S(t)\rangle.$$

$|p\rangle \rightarrow$ eigen function of $\hat{p}$
$p \rightarrow$ eigen value of p.

All of $\Psi(x,t)$, $\Phi(p,t)$ and $C_n(t)$ are the same state!

It is the same (vector) information in different forms (bases)

$|S(t)\rangle$:

$$\int \Psi(y,t)\,\delta(x-y)\,dy = \int \Phi(p,t)\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}\,dp$$

$$= \sum c_n e^{-iE_n t/\hbar}\psi_n(x).$$

→ Operators are linear transformations on Hilbert space :
They transform one vector to another :

$$|\beta\rangle = \hat{Q}|\alpha\rangle.$$

Like how vectors have components w.r.t. orthonormal bases,
$|\alpha\rangle = \sum_n a_n|e_n\rangle$, $a_n = \langle e_n|\alpha\rangle$ and $|\beta\rangle = \sum_n b_n|e_n\rangle$, $b_n = \langle e_n|\beta\rangle$

Operators have matrix elements to be described : (w.r.t. a particular basis)

$$\langle e_m|\hat{Q}|e_n\rangle = Q_{mn}.$$

$\because$ $|\beta\rangle = \hat{Q}|\alpha\rangle$, (right ?)

$$|\beta\rangle = \sum_n b_n|e_n\rangle = \sum_n a_n\hat{Q}|e_n\rangle$$

Taking inner product of the above eqⁿ with $|e_m\rangle$,

$$\sum_n b_n\langle e_m|e_n\rangle = \sum_n a_n\langle e_m|\hat{Q}|e_n\rangle$$

We know, $\langle e_m|e_n\rangle = \delta_{mn}$,

$$b_n\delta_{mn} = a_n Q_{mn}$$ [Adopted Einstein Summation Convention]

$$\underline{b_m = a_n Q_{mn}}$$ → Would be seen in Linear Algebra!

But this is for a finite N dim. Vector space.

For infinite dimensional Vector space, $b_m = \sum_n a_n Q_{mn}.$

There are systems where there are finite (N) number of linearly independent states.

Simplest of Quantum systems → The most simplest being two-state system. Will discuss about this in examples notes.

Just like vectors change in different bases, so do operators!

$$\hat{x} \longrightarrow x \text{ in } |x\rangle \text{ and } i\hbar\frac{\partial}{\partial p} \text{ in } |p\rangle$$

$$\hat{p} \longrightarrow -i\hbar\frac{\partial}{\partial x} \text{ in } |x\rangle \text{ and } p \text{ in } |p\rangle.$$

Dirac Notation

$\langle\alpha|$ bra c? ket $|\beta\rangle$

Operator acts on a vector (ket) to give out a vector. Whereas, a bra acts on a vector to give a number!

Hence, bra is a linear function of vectors.

It is thought of as an instruction to integrate...

$$\langle f| = \int f^* [\dots] dx$$ → This is in IVS

In FVS, $\langle f|$ are row vectors (matrix) and kets are column vectors (matrix, again) while operators being square matrices.

Collection of all bras is called a Dual Space.

We will find out why to treat bras as separate entities is powerful:

let $\hat{P} \equiv |\alpha\rangle\langle\alpha|$

this picks out the portion of any other vector that lies along $|\alpha\rangle$.

Let us see $\hat{P}$ acting on a ket $|\beta\rangle$.

$$\hat{P}|\beta\rangle = |\alpha\rangle\langle\alpha|\beta\rangle = \underline{\langle\alpha|\beta\rangle}\,|\alpha\rangle.$$

We call it the projection of $|\beta\rangle$ onto the one-dimension subspace formed by $|\alpha\rangle$.

If $\{|e_n\rangle\}$ is a discrete orthonormal basis,

$$\langle e_m|e_n\rangle = \delta_{mn}.$$

$$|e_m\rangle\langle e_m|e_n\rangle = \delta_{mn}|e_m\rangle = |e_n\rangle$$

$\therefore$ The operator $|e_m\rangle\langle e_m|$ acting on $|e_n\rangle$ gives out $|e_n\rangle$ itself.

What if for all vectors?

$$|e_m\rangle\langle e_m|\alpha\rangle = \langle e_m|\alpha\rangle\,|e_m\rangle.$$

This is basically the vector $|\alpha\rangle$'s component along $|e_m\rangle$, or the representation of $|\alpha\rangle$ in $|e_m\rangle$ basis? Yes! both.

Now we are ready to look at something that we already know from Mathphy (But would make better sense >:))

Now, what if we start summing up all of the projection vectors [of all the bases that's there] of $|\alpha\rangle$?

Let us say, for simplicity, $|e_m\rangle$ & $|e_n\rangle$ are the basis vectors.

$|e_m\rangle\langle e_m|$ and $|e_n\rangle\langle e_n|$ are the projection vectors.

$|e_m\rangle\langle e_m|\alpha\rangle + |e_n\rangle\langle e_n|\alpha\rangle =$: [It is just arranged differently]

$$= \langle e_m|\alpha\rangle\, |e_m\rangle + \langle e_n|\alpha\rangle\, |e_n\rangle = |\alpha\rangle \;!!$$
$$(A_x \quad \hat{x} \quad + \quad A_y \quad \hat{y} \quad = \quad \vec{A} \quad !!)$$

$$\therefore \sum_n |e_n\rangle\langle e_n|\alpha\rangle = |\alpha\rangle.$$

So, there is a new operator formed from summing the projection operators → And this is cool!

The Identity Operator: $\sum_n |e_n\rangle\langle e_n| = \mathbb{1}$ [discrete basis]

If this operator acts on any vector $|\alpha\rangle$, we recover the expansion of $|\alpha\rangle$ in the $\{|e_n\rangle\}$ basis (really? Lets check)

$$\sum_n |e_n\rangle\langle e_n|\alpha\rangle = |\alpha\rangle \checkmark$$

In continuous basis, (Which is dirac orthonormalised)

$$\langle e_z|e_{z'}\rangle = \delta(z-z')$$

then, $\int |e_z\rangle\langle e_z|\, dz = 1.$ → How?

orthonormalised basis (it is something like dx)

$$\int |e_z\rangle\langle e_z|e_{z'}\rangle = \underbrace{\int |e_z\rangle\, \delta(z-z')} = 1.$$

So, integrating the delta over all space, gives us 1.

$\sum_n |e_n\rangle\langle e_n| = \mathbb{I}$ is the tidiest way to express completeness.

$\int |e_n\rangle\langle e_n|\, dn = I$ as well.

Some properties

$$\hat{Q}|\alpha\rangle = |\beta\rangle$$

$$(\hat{Q}+\hat{R})|\alpha\rangle = \hat{Q}|\alpha\rangle + \hat{R}|\alpha\rangle$$

$$\hat{Q}\hat{R}|\alpha\rangle = \hat{Q}(\hat{R}|\alpha\rangle)$$

Functions of operators:

(defined by power series expansion)

$$e^{\hat{Q}} \equiv 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \cdots$$

$$\frac{1}{1-\hat{Q}} \equiv 1 + \hat{Q} + \hat{Q}^2 + \hat{Q}^3 + \hat{Q}^4 + \cdots$$

$$\ln(1+\hat{Q}) \equiv \hat{Q} - \frac{1}{2}\hat{Q}^2 + \frac{1}{3}\hat{Q}^3 - \frac{1}{4}\hat{Q}^4 + \cdots$$

Changing Bases in Dirac Notation

Advantage of Dirac Notation: Frees us from working in any particular basis and makes transforming between bases seamless

$$1 = \int dx\, |x\rangle\langle x|$$

$$1 = \int dp\, |p\rangle\langle p|$$

$$1 = \sum |n\rangle\langle n|$$

$$\therefore\ |S(t)\rangle = \int dx\, |x\rangle\langle x|S(t)\rangle = \int \Psi(x,t)|x\rangle\, dx$$

$$|S(t)\rangle = \int dp\, |p\rangle\langle p|S(t)\rangle = \int \Phi(p,t)\, |p\rangle\, dp$$

$$|S(t)\rangle = \sum_n |n\rangle\langle n|S(t)\rangle = \sum c_n(t)\, |n\rangle.$$

Hence, we recognise that $\Psi(x,t)$, $\Phi(p,t)$ and $c_n(t)$ are just components of the state of the system $|S(t)\rangle$ on the respective bases.

$\hat{x} \longrightarrow x$ in position basis

$\hat{x} \longrightarrow i\hbar \dfrac{\partial}{\partial p}$ in momentum basis

Instead of using arrows, dirac notation allows us to use equalities in this way.

$$\langle x | \hat{x} | S(t) \rangle = x \Psi(x,t)$$

$$\langle p | \hat{x} | S(t) \rangle = i\hbar \frac{\partial \Phi}{\partial p}(p,t)$$
