Quantum Comeback

Finale - II

Spin

In Classical Mechanics, a rigid body could have 2 kinds of angular momentum :

a) Orbital : $\vec{L} = \vec{r} \times \vec{p}$ → Motion of centre of mass

b) Spin : $\vec{S} = I\vec{\omega}$ → Motion about the centre of mass

In Classical world, spin of earth is basically the total "orbital" angular momenta of all the rocks and dirt clouds that go to make up the earth, as they circle around the axis.

But in the quantum world, elementary particles carry intrinsic angular momentum (S) in addition to their extrinsic angular momentum (L).

Let us see what Spin has got for us!

$$[S_x, S_y] = i\hbar S_z \quad ; \quad [S_y, S_z] = i\hbar S_x \quad ; \quad [S_z, S_x] = i\hbar S_y$$

Now, eigen states of spin are not functions. So, we'll depict them with kets $|sm\rangle$ [No idea why $|sm\rangle$ but it is depicted this way in Griffiths, so.].

Like L_z and L^2,

$$S_z |sm\rangle = \hbar^2 s(s+1) |sm\rangle \quad ; \quad S^2 |sm\rangle = \hbar^2 m |sm\rangle.$$

and,

$$S_\pm |sm\rangle = \hbar^2 \sqrt{s(s+1) - m(m\pm 1)} \, |s(m\pm 1)\rangle$$

where $S_\pm = S_x \pm i S_y$.

Remember, we excluded the half integer values for λ and μ for L^2 and L_z because the eigen state was a function — Spherical harmonics — allowing only whole numbers, not half integers.

This time, we will include the half integers as well, for s and m.

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \; ; \qquad m = -s, -s+1, \dots, s-1, s.$$

Every elementary particle has a specific and immutable value of s and we call it "spin".

π mesons	:	0 spin
electrons	:	1/2 spin
photons	:	1 spin
Δ Baryons	:	3/2 spin
gravitons	:	2 spin

The orbital angular momentum number (l) can take on any integer value, but <u>s is fixed</u>.

Let us now study the most important spin 1/2 systems.

Spin 1/2 particles

$s = 1/2 \rightarrow$ protons, neutrons, electrons, all quarks, leptons

Only 2 eigen states are there:

- Spin up $\uparrow$ $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle$
- Spin down $\downarrow$ $\left|\frac{1}{2}\left(-\frac{1}{2}\right)\right\rangle$

$\rightarrow$ Only the m's change.

Spinor (general state of a spin 1/2 vector): $\chi = \begin{bmatrix} a \\ b \end{bmatrix}$

Basis vectors: $\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\boxed{\chi = a\chi_+ + b\chi_-}$. Hence, spin operators became 2×2 matrix.

$$S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \quad \text{and} \quad S^2\chi_- = \frac{3}{4}\hbar^2\chi_-$$

gotten from the $S^2|sm\rangle = \hbar^2 s(s+1)|sm\rangle$

let $S = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$

$$\begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{4}\hbar^2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{4}\hbar^2 \end{bmatrix}$$

$\therefore d = 0, \; f = \frac{3}{4}\hbar^2.$

|||ly, $\begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{4}\hbar^2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c \\ e \end{bmatrix} = \begin{bmatrix} 3/4\,\hbar^2 \\ 0 \end{bmatrix}$

$\therefore c = 3/4\,\hbar^2 \; ; \; e = 0.$

$$\therefore S^2 = \frac{3}{4}\hbar^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

|||ly, $S_z\chi_+ = \frac{\hbar}{2}\chi_+$, $S_z\chi_- = -\frac{\hbar}{2}\chi_-$.

$$\therefore S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$S_+\chi_- = \hbar\chi_+ \; ; \; S_-\chi_+ = \hbar\chi_-, \; S_+\chi_+ = S_-\chi_- = 0.$

$$\therefore S_+ = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \; S_- = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Now, $S_\pm = S_x \pm i S_y$

$\therefore S_x = (1/2)(S_+ + S_-) \; ; \; S_y = (1/2i)(S_+ - S_-)$

$\therefore\ S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$

Since S_x, S_y, S_z, all carry $\frac{\hbar}{2}$, $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

where $\vec{\sigma} \rightarrow$ Pauli Spin Matrices.

$$\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} ;\ \sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} ;\ \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$S^2, \vec{\sigma} \longrightarrow$ All are Hermitian (As they give observables)

S_+ & S_- are not Hermitian [Makes sense, right?]

$S^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \rightarrow$ Eigen values: $\frac{3}{4}\hbar^2$,
Eigenvectors: $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

For S_z?: $\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \frac{\hbar}{2}$, $\begin{bmatrix}0\\1\end{bmatrix} \rightarrow -\frac{\hbar}{2}$. χ_+, χ_-

$\boxed{S_y \chi_\pm = \pm\frac{\hbar}{2}\chi_\pm}$

$\boxed{\sigma_y \cdot \chi_\pm = \pm\frac{\hbar}{2}\chi_\pm}$

$$\begin{vmatrix}-\lambda & -i\hbar/2\\ i\hbar/2 & \lambda\end{vmatrix} = 0$$

$\Rightarrow i^2 = -1$
$\Rightarrow i = \pm 1$

$\begin{matrix}1 & -1\\ -i & i\end{matrix}$

$-\lambda^2 + \frac{\hbar^2}{4} = 0$; $\pm\frac{\hbar}{2} = \lambda$.

$$\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \pm\frac{\hbar}{2}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$\beta = i\alpha$ $\begin{pmatrix}1\\ i\end{pmatrix}$
$\beta = -i\alpha$

$$\begin{bmatrix}-i\beta\\ i\alpha\end{bmatrix} = \pm\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$\begin{bmatrix}\alpha\\ i\alpha\end{bmatrix}$ $\begin{bmatrix}\alpha\\ -i\alpha\end{bmatrix}$

$\alpha = -i\beta \Rightarrow i\alpha = \beta$: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$, $\frac{\hbar}{2}$.

$i\alpha = -\beta \Rightarrow -i\alpha = \beta$: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}$, $-\frac{\hbar}{2}$.

$$\frac{1}{2}\underset{}{\overset{\downarrow\ \text{Transpose}}{\begin{bmatrix}1 & -i\\ 1 & +i\end{bmatrix}}}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\overset{\downarrow\ \text{Correct}}{\begin{bmatrix}1&1\\i&-i\end{bmatrix}} = \begin{bmatrix}i & -i\\1&1\end{bmatrix}\begin{bmatrix}1&i\\1&-i\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}1 & -i\\ -1 & -i\end{bmatrix}\begin{bmatrix}1&1\\i&-i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2&0\\0&-2\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix} \rightarrow \sigma_y \text{ in } y \text{ basis!!}$$

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}.$$

$$\chi = \begin{pmatrix}a\\b\end{pmatrix} = \alpha\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} + \beta\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$\frac{\alpha+\beta}{\sqrt{2}} = a \quad ; \quad i\,\frac{(\alpha-\beta)}{\sqrt{2}} = b.$$

$$\alpha = a\sqrt{2}-\beta \quad ; \quad -i\sqrt{2}\,b+\beta = \alpha.$$

$$\therefore a\sqrt{2}-\beta = -i\sqrt{2}\,b+\beta$$

$$(a-ib)\sqrt{2} = 2\beta$$

$$\therefore \beta = \left(\frac{a-ib}{\sqrt{2}}\right)$$

$$|||^{ly},\ \alpha = a\sqrt{2}-\left(\frac{a-ib}{\sqrt{2}}\right)$$

$$= \frac{2a-a+ib}{\sqrt{2}}$$

$$\alpha = \left(\frac{a+ib}{\sqrt{2}}\right)$$

$$\therefore \chi = \left(\frac{a+ib}{\sqrt{2}}\right)\chi_+^{(y)} + \left(\frac{a-ib}{\sqrt{2}}\right)\chi_-^{(y)}.$$

$$\boxed{C_+ = \left(\chi_+^{(y)}\right)^\dagger \chi} = \frac{1}{\sqrt{2}}(1 \;\; -i)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a - ib)$$

$\therefore$ $S_y \to +\frac{\hbar}{2}$ with probability $\frac{|a-ib|^2}{2}$

|||ly, $C_- = \left(\chi_-^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}(1 \;\; i)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a+ib)$

$\therefore$ $-\frac{\hbar}{2}$ with probab $\frac{|a+ib|^2}{2}$

$$\therefore \quad \chi = \frac{(a-ib)}{\sqrt{2}}\chi_+^{(y)} + \frac{(a+ib)}{\sqrt{2}}\chi_-^{(y)}$$

Electron in a Magnetic Field

A spinning, charged particle constitutes a magnetic dipole.

Magnetic dipole moment — Spin angular momentum

$$\vec{\mu} = \gamma \vec{S}$$

Gyromagnetic ratio

Torque experienced by a magnetic dipole when placed in a magnetic field $\vec{B}$: $= \vec{\mu} \times \vec{B}$

Energy associated with the torque $= \vec{H} = -\vec{\mu} \cdot \vec{B}$.

$= \vec{H} = -\gamma \vec{B} \cdot S$ Spin matrix.

Larmor Precession

Particle (spin 1/2) in Uniform Magnetic Field $\vec{B} = B_0 \hat{k}$.

The hamiltonian $= H = -\gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Eigen states of $H \to$ same as those of S_z:

χ_+ with $E_+ = -(\gamma B_0 \hbar/2)$

χ_+ , with $E_+ = (-\gamma B_0 \hbar/2)$
χ_- , with $E_- = (\gamma B_0 \hbar/2)$

Since H is time independent,

$$i\hbar \frac{\partial \chi}{\partial t} = H\chi ,$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{bmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{bmatrix}$$

$\chi(0) = \begin{bmatrix} a \\ b \end{bmatrix}$ and $|a^2| + |b^2| = 1.$

if $a = \cos(\alpha/2)$, b then is $\sin(\alpha/2)$. [α's significance will be evident soon].

$$\therefore \chi(t) = \begin{bmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{bmatrix}$$

Not sure what is happening here? Let's calculate the expectation value of $\vec{S}$ as a function of time.

$$\langle S_x \rangle = \chi(t)^\dagger S_x \chi(t).$$

$$\begin{bmatrix} \cos(\alpha/2) e^{-i\gamma B_0 t/2} & \sin(\alpha/2) e^{i\gamma B_0 t/2} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{bmatrix}$$

$$= \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t).$$

$|||^{ly}$, $\langle S_y \rangle = \chi(t)^\dagger S_y \chi(t) = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B_0 t)$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\alpha.$$

$\therefore \langle S \rangle$ is tilted about z axis at a constant angle α, and precesses about the field by Larmor frequency.

$$\omega = \gamma B_0.$$

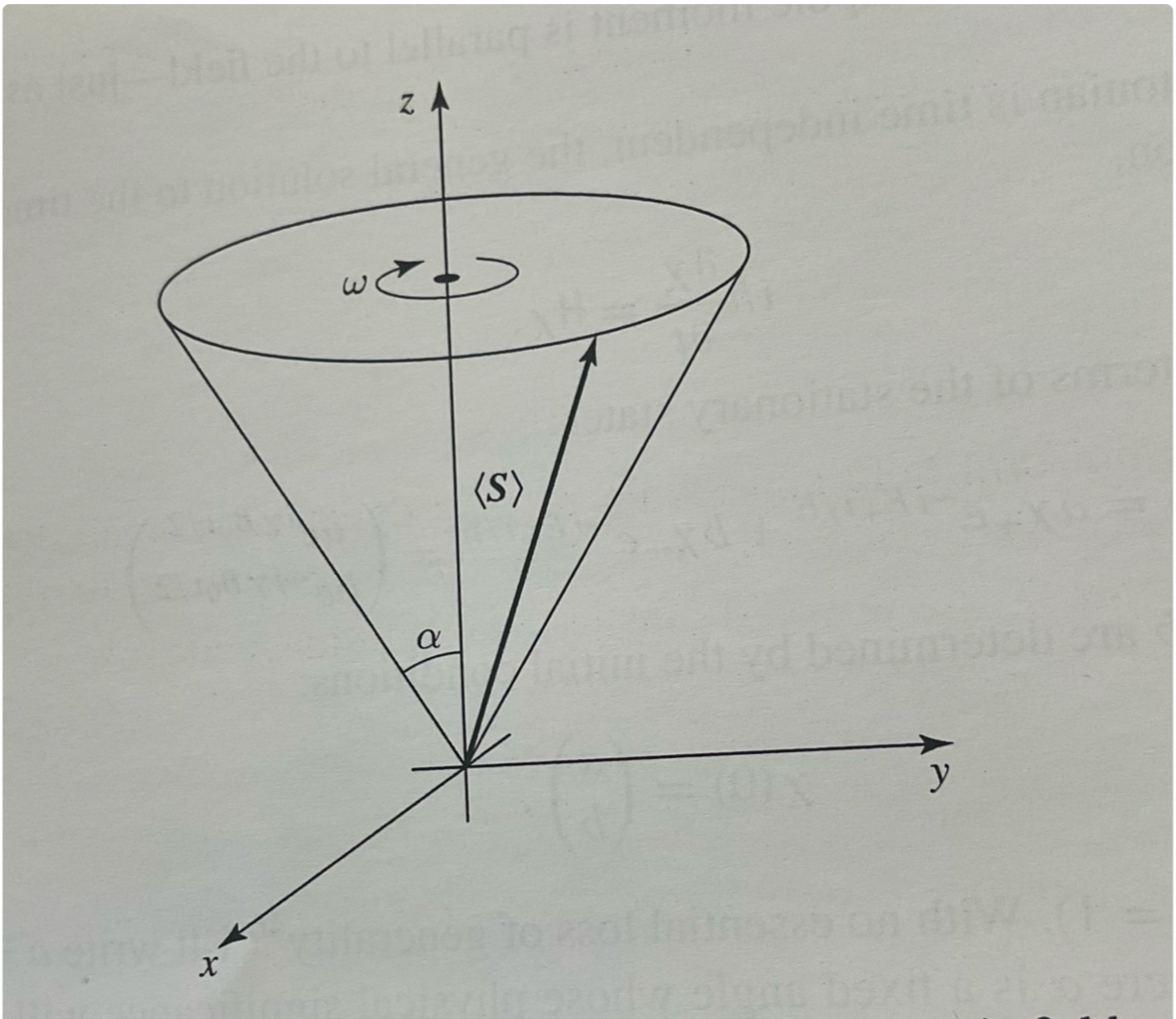

Figure 4.14: Precession of ⟨S⟩ in a uniform magnetic field.

The Stern-Gerlach experiment

Inhomogeneous magnetic field : Torque + Force.

$$\vec{F} = \nabla(\vec{\mu}\cdot\vec{B})$$

This force can be used to separate out particles with a particular spin orientation (Like Polariser?).

Let $B(x,y,z) = -\alpha x\hat{i} + (B_0 + \alpha z)\hat{k}$ [Inhomogeneous but static]

We'd love if we can have just the z component, but it would then violate $\vec{\nabla}\cdot\vec{B}$...

$\therefore$ Now, $\vec{F} = \gamma\alpha\left(-S_x\hat{i} + S_z\hat{k}\right)$

But since Larmor precession is about Bo direction, S_x oscillates rapidly and cancells out to zero.

$\therefore$ The net force is in the z-direction:

$$F_z = \gamma\alpha S_z.$$

The beam is now deflected up or down, depending on its spin.

Classically we would expect a smear. [As S_z wouldn't be quantised].

But turns out, the beam splits into $2s+1$ separate beams, beautifully demonstrating the quantisation of angular momentum.

Addition of angular Momenta

Now if we have 2 particles with spins s_1 and s_2, their states being $|s_1 m_1\rangle$, $|s_2 m_2\rangle$,

The composite state would be $|s_1 s_2 m_1 m_2\rangle$:

$$S^{(1)^2}|s_1 s_2 m_1 m_2\rangle = s_1(s_1+1)\hbar^2|s_1 s_2 m_1 m_2\rangle$$

$$S^{(2)^2}|s_1 s_2 m_1 m_2\rangle = s_2(s_2+1)\hbar^2|s_1 s_2 m_1 m_2\rangle$$

$$S_z^{(1)}|s_1 s_2 m_1 m_2\rangle = m_1\hbar|s_1 s_2 m_1 m_2\rangle$$

$$S_z^{(2)}|s_1 s_2 m_1 m_2\rangle = m_2\hbar|s_1 s_2 m_1 m_2\rangle.$$

$$\therefore \vec{S} = S^{(1)} + S^{(2)} = ??$$

$$S_z|s_1 s_2 m_1 m_2\rangle = m_1\hbar|s_1 s_2 m_1 m_2\rangle + m_2\hbar|s_1 s_2 m_1 m_2\rangle$$

$$= (m_1+m_2)\hbar|s_1 s_2 m_1 m_2\rangle$$

$$\therefore \quad m = m_1 + m_2.$$

$$S^2 |s_1 s_2 m_1 m_2\rangle = \left(s_1(s_1+1) + s_2(s_2+1)\right)\hbar^2 |s_1 s_2 m_1 m_2\rangle.$$

$$\therefore \; s = \left[s_1(s_1+1) + s_2(s_2+1)\right]$$

$\therefore$ Generally,

$m = m_1 + m_2 + \cdots + m_i$ (i = no. of particles)

$s =$ Is it there for us or not?

Electromagnetic Interactions

Minimal coupling:

Lorentz Law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

This force cannot be expressed as a gradient of a scalar potential, hence the Schrödinger equation can only be written in this form:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi.$$

The Classic Hamiltonian for a particle of charge 'q' and momentum $\vec{p}$, in the presence of $\vec{E}$:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi.$$

where $\vec{A}$ is the vector potential and φ is the scalar potential.

$$E = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}.$$

Making the standard substitution of $\vec{p} \rightarrow i\hbar\nabla$ we now

Making the standard substitution of $p \to -i\hbar\nabla$, we now obtain the Hamiltonian operator:

$$\hat{H} = \left[\frac{1}{2m}\left(-i\hbar\nabla - qA\right)^2 + q\varphi\right].$$

$\therefore$ Schrödinger equation now becomes:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - qA\right)^2 + q\varphi\right]\Psi.$$

This is the quantum implementation of the Lorentz Law, and is also called the minimum coupling rule.

Aharonov - Bohm Effect

$\varphi' \equiv \varphi - \frac{\partial A}{\partial t}$; $A' \equiv A + \nabla\Lambda.$ $\to$ Gauge Transformations

Gauge Invariant!

Classically, $\vec{A}$ has no meaning and no effect too. But in Quantum mechanics, turns out $\vec{A}$ has some effect. But $\vec{A}$ can't be measured and.

It is easy to show that $\Psi' \equiv e^{iq\Lambda/\hbar}\Psi \longrightarrow$ satisfies the minimum coupling rule schrödinger equation.

Since Ψ and Ψ' differ only by a phase factor, it represents the same physical state. These are indeed gauge invariant.

But turns out, Aharonov and Bohm showed that the vector potential can affect the quantum behaviour of a charged particle, even when the particle is confined to a region where the field itself is zero.

Suppose a particle is moving through a region where $\vec{B}$ is zero. $\therefore \nabla\times\vec{A} = 0$. But $\vec{A}$ is static, let's say (can be generalized to time-dependent too.

∴ The Schrödinger equation ⇒

$$\left[\frac{1}{2m}(-i\hbar\vec{\nabla}-q\vec{A})^2\right]\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

This can be simplified by writing $\psi = e^{ig}\psi'$.

where $g(\vec{r}) = \frac{q}{\hbar}\int_0^{\vec{r}}\vec{A}(\vec{r}\,')\cdot d\vec{r}$.

[As you can see, $\nabla\times\vec{A}$ must be zero, or else the integral depends on the path taken from O to $\vec{r}$.]

Now, $\nabla\psi$ in terms of ψ':

$$\nabla\psi = e^{ig}(i\nabla g)\psi' + e^{ig}(\nabla\psi'),$$

But $\nabla g = (q/\hbar)\vec{A}$.

$$\therefore\ (-i\hbar\nabla - q\vec{A})\psi = -i\hbar e^{ig}\nabla\psi'.$$

and it follows that

$$(-i\hbar\nabla - q\vec{A})^2\psi = -\hbar^2 e^{ig}\nabla^2\psi'.$$

Putting this result in the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi' = i\hbar\frac{\partial\psi'}{\partial t}.$$

As you can see, ψ' satisfies the Schrödinger equation without $\vec{A}$.

The Aharonov and Bohm experiment splits a beam of electrons into two and pass through either side of a long solenoid before recombining.

Even though $\vec{B}=0$ in this region, since $\vec{A}\neq 0$, the even though $B = 0$ in this region, since $A \neq 0$, the two beams arrive with different phases.

$$g = \frac{q}{\hbar}\int \vec{A}\cdot d\vec{r} = \frac{q\Phi}{2\pi\hbar}\int\left(\frac{1}{r}\hat{\phi}\right)\cdot\left(r\hat{\phi}\, d\phi\right) = \pm\frac{q\Phi}{2\hbar}.$$

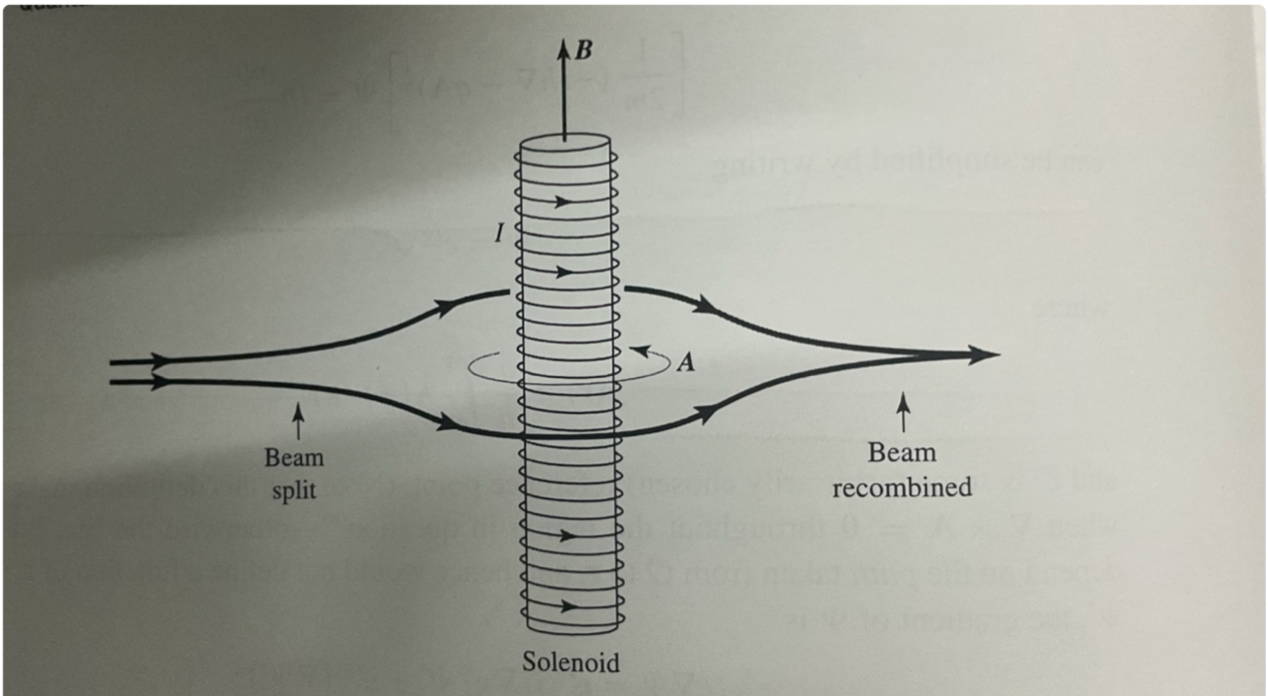

Figure 4.17: The Aharonov–Bohm effect: The electron beam splits, with half passing either side of a long solenoid.

$g = +\frac{\phi q}{2\hbar}$ → applies to electrons travelling in the same direction as A (as current).

$$\text{Phase difference} = \frac{q\Phi}{\hbar}.$$

What do we make out of the Aharonov - Bohm Effect ?

There can be electromagnetic effects in regions where the fields are zero.

However, this does not make $\vec{A}$