Chapter 2: A review of some basic concepts

2.1 Light as an electromagnetic wave.

A review of complex numbers:

To add

2.1 Light as an electromagnetic wave.

Solution to the maxwell equations

To add

2.2 The monochromatic, time-harmonic plane wave

Construction of the plane-wave solution

Speed of the EM wave $= \dfrac{c}{\sqrt{\epsilon\mu}} = \dfrac{c}{n}$

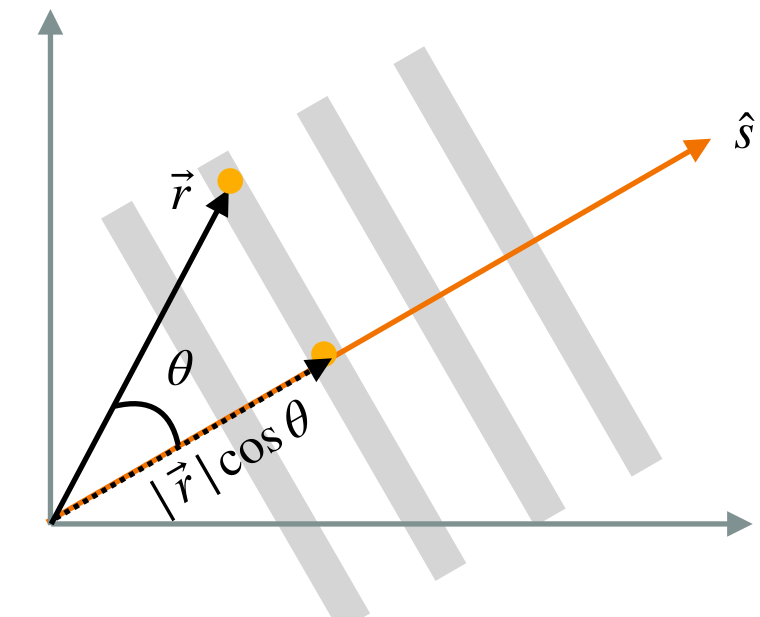
Direction of propagation

Wave vector: $\overrightarrow{k} = \dfrac{2\pi}{\lambda}\hat{s}$

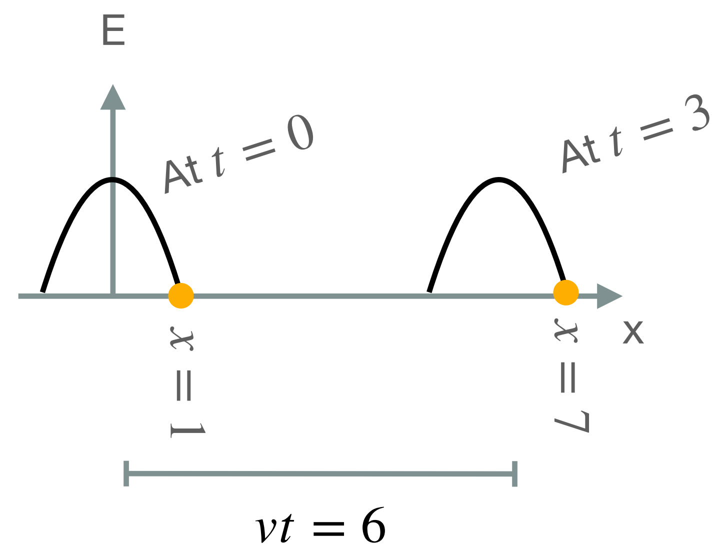
Wavelength

Angular frequency: $\omega = 2\pi\nu = 2\pi\dfrac{v}{\lambda}$

Every point in space and time with the same $\overrightarrow{u} = (\vec{r}\cdot\hat{s} - vt)\hat{s} = \dfrac{1}{k}(\overrightarrow{k}\cdot\vec{r} - \omega t)$ has the same electric field



$E(\vec{r}\,) = E(\vec{r}\cdot\hat{s})$



$E(x = 7, t = 3) = E(x = x - vt, t = 0) = E(x = 1, t = 0)$

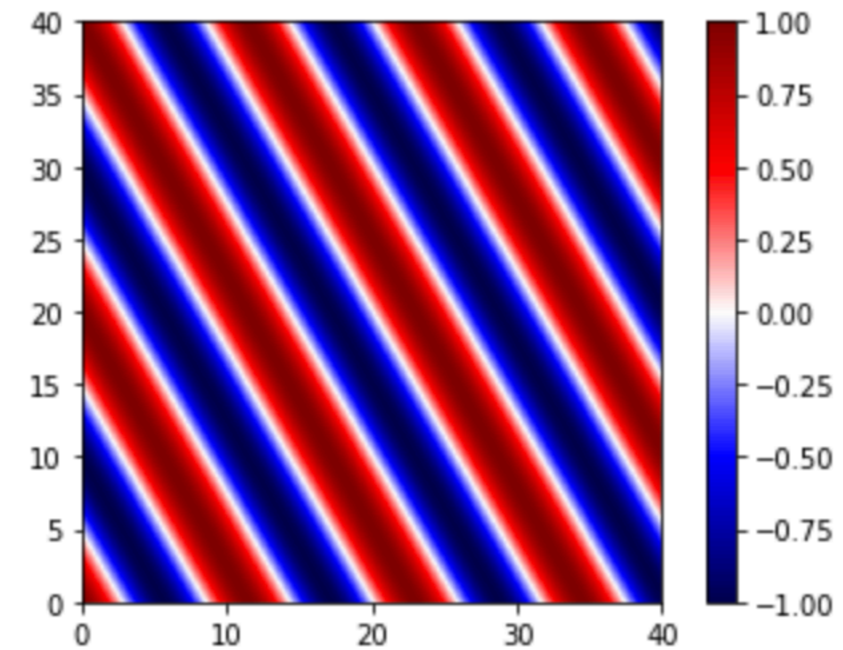
Simple harmonic solution:

$$(2.7)\qquad E_j(\vec{r}, t) = a_j e^{i(ku+\delta_j)}$$

Replacing u in favor of $\overrightarrow{k}$ and ω:

$$(2.9)\qquad E_j(\vec{r}, t) = a_j e^{i(\overrightarrow{k}\cdot\vec{r})} e^{-i(\omega t+\delta_j)}$$

2.2 The monochromatic, time-harmonic plane wave



$$(2.9) \quad \boxed{E_j(\vec{r}, t) = a_j e^{i(\overrightarrow{k} \cdot \vec{r})}} e^{-i(\omega t + \delta_j)}$$

Exercise in Python:

1. Illustration of the plane wave in space

As we have seen in class, a plane wave can be expressed as

$$(2.7) \quad E_j(\vec{r}, t) = a_j e^{i(\vec{k} \cdot \vec{r})} e^{-i(\omega t + \delta_j)}$$

Let's start with the spatial dependence of the plane wave. If we set $t = 0$, and we consider a plane wave for which the phase term $\delta_j = 0$, we simplify our expression below because the second exponential term is 1.0.

In the graph below, we will make a color map that illustrated the magnitude of the electric field as a function of position.

I have already defined two variables `XX` and `YY` which are 2D array that contains the x and y coordinates, respectively, of a cartesian grid of points.

1. Let's make the wavelength $\lambda = 10$ distance units, and let's make the light wave propagate in a direction $\hat{s}$ that is oriented at 30° with

2.3 The polarization tensor

Construction of the polarization tensor

Now let's make the wave propagate in the z direction

$$(2.10)\quad E_x = a_x \mathrm{e}^{i\vec{k}\cdot\vec{r}}\mathrm{e}^{-i(\omega t-\delta_x)}$$

$$(2.10)\quad E_y = a_y \mathrm{e}^{i\vec{k}\cdot\vec{r}}\mathrm{e}^{-i(\omega t-\delta_y)}$$

$$(2.10)\quad E_z = 0$$

We cannot measure the electric field directly. Our detectors are sensitive to electromagnetic energy (averaged over the period of the wave)

Square of the magnitude of the electric field vector

$(z = x + iy,\, zz^* = x^2 + y^2 = |z|^2)$

The volume energy density: $$(2.12)\quad w = \frac{\epsilon}{8\pi}\boxed{EE^*}$$

Our equations have separated the x and y components of the electric field. The magnitude of the electric vector (with time) will depend on how these two components are synchronized with each other.
e.g.: are they both zero at the same time? Is one zero while the other at its maximum?

$$(2.16)\quad \delta \equiv \delta_x - \delta_y$$

Magnitude of the x component

$$(2.14)\quad C \equiv \begin{pmatrix} \boxed{E_xE_x^*} & E_xE_y^* \\ E_yE_x^* & \boxed{E_yE_y^*} \end{pmatrix} \qquad (2.15)\quad C \equiv \begin{pmatrix} \boxed{a_x^2} & a_xa_y\boxed{e^{i\delta}} \\ a_xa_y\boxed{e^{-i\delta}} & \boxed{a_y^2} \end{pmatrix}$$

Magnitude of the y component

Complex: not measurable

2.3 The polarization tensor

Properties of the polarization tensor

$$(2.16)\quad \delta \equiv \delta_x - \delta_y$$

Magnitude of the x component

$$(2.14)\quad C \equiv \begin{pmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{pmatrix} \qquad (2.15)\quad C \equiv \begin{pmatrix} a_x^2 & a_x a_y e^{i\delta} \\ a_x a_y e^{-i\delta} & a_y^2 \end{pmatrix}$$

Magnitude of the y component

Complex: not measurable

As we can think of the electric vector as the addition of two vectors corresponding to the x and y components, it means that we can also think of the EM wave as being a sum of a wave in the x-direction and a wave in the y-direction. In this case, a_x and a_y corresponds to the amplitude of these two waves, and therefore a_x^2 and a_y^2 are proportional to the intensity of these two waves. As the intensity cannot be negative, it means that:

$$(2.17)\quad \mathrm{Tr}(C) = a_x^2 + a_y^2 \geq 0$$

We can also see that the determinant of the coherence matrix (2.15) is zero, such that:

$$(2.18)\quad \det(C) = 0$$

As we will see later, this is always true of mono-chromatic plane waves. But, this will not be always the case for polychromatic light. Light with $\det(C) = 0$ is called totally (or completely) polarized light.

2.3 The polarization tensor

Path of the electric vector in the E_x/E_y plane

We can describe the "path" of the electric vector of totally polarized light in the E_x vs E_y plane. This will be at a single point in space, as a function of time.

Set $\vec{r} = 0$

Because I am looking for the path over a whole period of oscillation, I can recast the phase shift this way, without loosing the information I am looking for (i.e. I am not interested in where the path starts at $t = 0$).

$$(2.10) \quad E_x = a_x e^{i\vec{k}\cdot\vec{r}} e^{-i(\omega t - \delta_x)}$$

$$(2.10) \quad E_y = a_y e^{i\vec{k}\cdot\vec{r}} e^{-i(\omega t - \delta_y)}$$

$$(2.10) \quad E_z = 0$$

Won't need this

$$E_x(t) = a_x e^{-i(\omega t)}$$

$$E_y(t) = a_y e^{-i(\omega t - \delta)}$$

$$(2.16) \quad \delta \equiv \delta_x - \delta_y$$

$$E_{x,\mathrm{R}}(t) = a_x \cos(\omega t)$$

$$E_{y,\mathrm{R}}(t) = a_y \cos(\omega t - \delta)$$

$$E_{x,\mathrm{R}}(\theta) = a_x \cos(\theta)$$

$$E_{y,\mathrm{R}}(\theta) = a_y \cos(\theta - \delta)$$

During one oscillation period, ωt will go from 0 to 2π, so I'll just substitute to an angle θ

A simple example: let's have $a_x = a_y = a$ and $\delta = \pi$.

$$E_{x,\mathrm{R}}(\theta) = a\cos(\theta)$$

$$E_{y,\mathrm{R}}(\theta) = a\cos(\theta - \pi)$$

$$E_{x,\mathrm{R}}(\theta) = a\cos(\theta)$$

$$E_{y,\mathrm{R}}(\theta) = a\sin(\theta)$$

That is the x and y components of a circle of radius a

This is how we get the cartesian circle equation from these:

$$\cos(\theta) = \frac{E_{x,\mathrm{R}}}{a}$$

$$\sin(\theta) = \frac{E_{y,\mathrm{R}}}{a}$$

We know that:

$$\cos^2(\theta) + \sin^2\theta = 1$$

$$\frac{E_{x,\mathrm{R}}^2}{a^2} + \frac{E_{y,\mathrm{R}}^2}{a^2} = 1$$

2.3 The polarization tensor

A simple case with $a_x = a_y = a$ and $\delta = \pi/2$

2. Illustration of the electric field vector with time in a single plane.

In section 2.3, we have seen that we can express the electric field vector at a single point in space for a wave propagating in the z-direction as:

$$E_{x,\mathrm{R}}(t) = a_x \cos(\omega t)$$

and

$$E_{y,\mathrm{R}}(t) = a_y \cos(\omega t - \delta),$$

where $\delta \equiv \delta_x - \delta_y$

Remember that as here we will be interested in the path taken by the electric vector in the $E_{x,\mathrm{R}} - E_{y,\mathrm{R}}$ plane when ωt goes from 0 to 2π, so we do not care about the position of the electric field at $t = 0$.

a. The simple case where $a_x = a_y = a$ and $\delta = \pi/2$

For this case, we have seen that the electric field simplifies to:

$$E_{x,\mathrm{R}}(\theta) = a\cos(\theta)$$

and

$$E_{y,\mathrm{R}}(\theta) = a\sin(\theta)$$

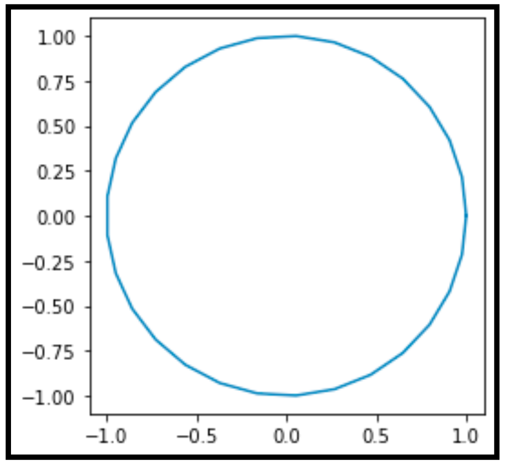
In the code cell below, I have already define an array of θ values.

Make a graph of the path of the electric field vector in the $E_{x,\mathrm{R}} - E_{y,\mathrm{R}}$ plane

[4]

```
fig, ax = plt.subplots(1,1)
ax.set_aspect('equal')

a = 1.0
theta = np.linspace(0,2*np.pi, 30)
```

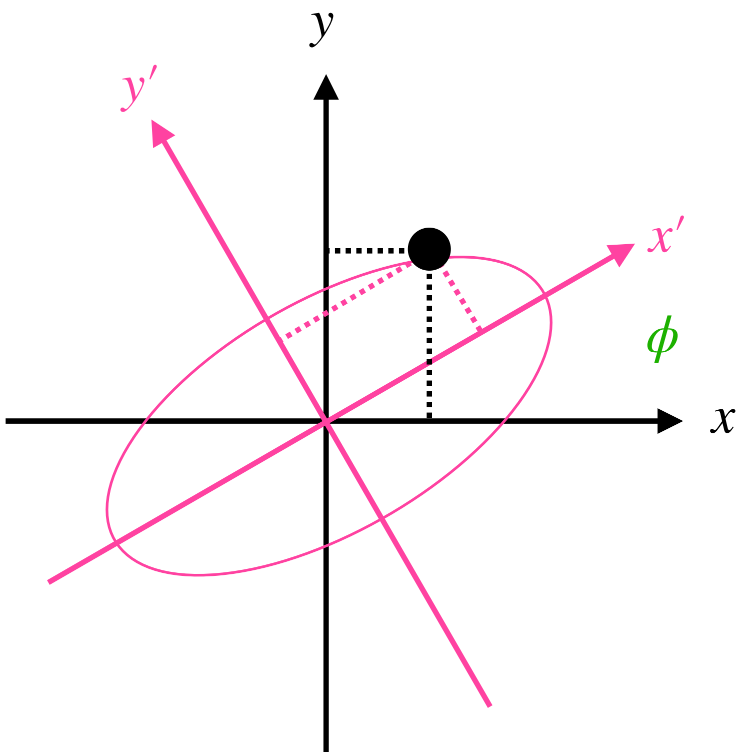


2.3 The polarization tensor

The cartesian formula for a tilted ellipse

Useful for the next slide: The cartesian formula for a tilted ellipse

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x\cos\phi + y\sin\phi$$

$$y' = y\cos\phi - x\sin\phi$$

$$\frac{(x\cos\phi + y\sin\phi)^2}{a^2} + \frac{(y\cos\phi - x\sin\phi)^2}{b^2} = 1$$

$$x^2\left[\frac{\cos^2\phi}{a^2} - \frac{\sin^2\phi}{b^2}\right] + y^2\left[\frac{\sin^2\phi}{a^2} - \frac{\cos^2\phi}{b^2}\right] + 2xy\cos\phi\sin\phi\left[\frac{1}{a} - \frac{1}{b}\right] = 1$$

If $\phi = 0$, the xy cross term vanishes,
and we recover the normal ellipse equation

2.3 The polarization tensor

The general case for the path of the electric vector

1 $E_{x,\mathrm{R}} = a_x \cos(\theta)$ $\Rightarrow$ $\cos(\theta) = \dfrac{E_{x,\mathrm{R}}}{a_x}$ and $\sin(\theta) = \left[1 - \dfrac{E_{x,\mathrm{R}}^2}{a_x^2}\right]^{1/2}$

Substitute

2 $E_{y,\mathrm{R}} = a_y \cos(\theta - \delta)$

$E_{y,\mathrm{R}} = a_y \left[\cos\theta\cos\delta + \sin\theta\sin\delta\right]$ $\Rightarrow$ $\dfrac{E_{y,\mathrm{R}}}{a_y} = \left[\cos\theta\cos\delta + \sin\theta\sin\delta\right]$

$$\frac{E_{y,\mathrm{R}}}{a_y} = \frac{E_{x,\mathrm{R}}}{a_x}\cos\delta + \left[1 - \frac{E_{x,\mathrm{R}}^2}{a_x^2}\right]^{1/2}\sin\delta$$

$$\frac{E_{y,\mathrm{R}}}{a_y} - \frac{E_{x,\mathrm{R}}}{a_x}\cos\delta = \left[1 - \frac{E_{x,\mathrm{R}}^2}{a_x^2}\right]^{1/2}\sin\delta$$

$$\left[\frac{E_{y,\mathrm{R}}}{a_y} - \frac{E_{x,\mathrm{R}}}{a_x}\cos\delta\right]^2 = \left[1 - \frac{E_{x,\mathrm{R}}^2}{a_x^2}\right]\sin^2\delta$$

$$\frac{E_{y,\mathrm{R}}^2}{a_y^2} + \frac{E_{x,\mathrm{R}}^2}{a_x^2}\cos^2\delta - 2\frac{E_{x,\mathrm{R}}E_{y,\mathrm{R}}}{a_x a_y}\cos\delta = \sin^2\delta - \frac{E_{x,\mathrm{R}}^2}{a_x^2}\sin^2\delta$$

$$\frac{E_{y,\mathrm{R}}^2}{a_y^2} + \frac{E_{x,\mathrm{R}}^2}{a_x^2}(\cos^2\delta + \sin^2\delta) - 2\frac{E_{x,\mathrm{R}}E_{y,\mathrm{R}}}{a_x a_y}\cos\delta = \sin^2\delta$$

$$\frac{E_{y,\mathrm{R}}^2}{a_y^2} + \frac{E_{x,\mathrm{R}}^2}{a_x^2} - 2\frac{E_{x,\mathrm{R}}E_{y,\mathrm{R}}}{a_x a_y}\cos\delta = \sin^2\delta \tag{2.19}$$

This equation has the form of a tilted ellipse

$$x^2\left[\frac{\cos^2\phi}{a^2} - \frac{\sin^2\phi}{b^2}\right] + y^2\left[\frac{\sin^2\phi}{a^2} - \frac{\cos^2\phi}{b^2}\right] + 2xy\cos\phi\sin\phi\left[\frac{1}{a} - \frac{1}{b}\right] = 1$$

2.3 The polarization tensor

b. The general case

Now make a graph for the general case, where a_x, a_y, and δ can vary.

Then, let's play a detective game, using your code. By playing with the paramters, figure out what is needed for the path of the electric field to be

1. A perfect circle

   > Answer:

2. A line that is tilted:

   > Answer:

3. A line that is purely vertical or horizontal

   > Answer:

4. An elipse that is not titlted (so horizontal or vertical)

   > Answer:

```
fig, ax = plt.subplots(1,1)
ax.set_aspect('equal')

a_x = 1.0
a_y = 1.5
delta = np.pi/3
theta = np.linspace(0,2*np.pi, 30)

######## Solution ###########



######## Solution for the part below about the Stokes parameters.
```

2.3 The polarization tensor

Explaining analytically the empirical results from the notebook exercise

$$\frac{E_{y,\mathrm{R}}^2}{a_y^2} + \frac{E_{x,\mathrm{R}}^2}{a_x^2} - 2\frac{E_{x,\mathrm{R}}E_{y,\mathrm{R}}}{a_x a_y}\cos\delta = \sin^2\delta \tag{2.19}$$

$$x^2\left[\frac{\cos^2\phi}{a^2} - \frac{\sin^2\phi}{b^2}\right] + y^2\left[\frac{\sin^2\phi}{a^2} - \frac{\cos^2\phi}{b^2}\right] + 2xy\cos\phi\sin\phi\left[\frac{1}{a} - \frac{1}{b}\right] = 1$$

If $\delta = \pi/2$, the xy cross term vanishes, and we recover the normal ellipse equation

If $\phi = 0$, the xy cross term vanishes, and we recover the normal ellipse equation

If $\delta = 0$ (or similarly with $\delta = \pi$):

$$\frac{E_{y,\mathrm{R}}^2}{a_y^2} + \frac{E_{x,\mathrm{R}}^2}{a_x^2} - 2\frac{E_{x,\mathrm{R}}E_{y,\mathrm{R}}}{a_x a_y} = 0$$

$$\left[\frac{E_{y,\mathrm{R}}}{a_y} - \frac{E_{x,\mathrm{R}}}{a_x}\right]^2 = 0$$

$$E_{y,\mathrm{R}} = \frac{a_y}{a_x}\,E_{x,\mathrm{R}}$$

Equation of a line with slope a_y/a_x

2.4 The Stokes parameters of a monochromatic, time-harmonic plane wave

$$(2.14)\quad C \equiv \begin{pmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{pmatrix} \qquad \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} \qquad (2.15)\quad C \equiv \begin{pmatrix} a_x^2 & a_x a_y e^{i\delta} \\ a_x a_y e^{-i\delta} & a_y^2 \end{pmatrix}$$

$$I \equiv \kappa(C_{11} + C_{22}) = \kappa(a_x^2 + a_y^2) \qquad \begin{pmatrix} \boxed{C_{11}} & C_{21} \\ C_{12} & \boxed{C_{22}} \end{pmatrix} + \qquad \begin{pmatrix} \boxed{a_x^2} & a_x a_y e^{i\delta} \\ a_x a_y e^{-i\delta} & \boxed{a_y^2} \end{pmatrix} +$$

$$Q \equiv \kappa(C_{11} - C_{22}) = \kappa(a_x^2 - a_y^2) \qquad \begin{pmatrix} \boxed{C_{11}} & C_{21} \\ C_{12} & \boxed{C_{22}} \end{pmatrix} - \qquad \begin{pmatrix} \boxed{a_x^2} & a_x a_y e^{i\delta} \\ a_x a_y e^{-i\delta} & \boxed{a_y^2} \end{pmatrix} -$$

$$\Rightarrow \quad (2.21)\quad C \equiv \begin{pmatrix} I + Q & \\ & I - Q \end{pmatrix}$$

$$\cos\delta = \frac{e^{i\delta} + e^{-i\delta}}{2}$$

$$U \equiv \kappa(C_{12} + C_{21}) = 2\kappa a_x a_y \cos\delta \qquad \begin{pmatrix} C_{11} & \boxed{C_{21}} \\ \boxed{C_{12}} & C_{22} \end{pmatrix} + \qquad \begin{pmatrix} a_x^2 & \boxed{a_x a_y e^{i\delta}} \\ \boxed{a_x a_y e^{-i\delta}} & a_y^2 \end{pmatrix} +$$

$$V \equiv i\kappa(C_{21} - C_{12}) = 2\kappa a_x a_y \sin\delta \qquad \begin{pmatrix} C_{11} & \boxed{C_{21}} \\ \boxed{C_{12}} & C_{22} \end{pmatrix} - \qquad \begin{pmatrix} a_x^2 & \boxed{a_x a_y e^{i\delta}} \\ \boxed{a_x a_y e^{-i\delta}} & a_y^2 \end{pmatrix} -$$

$$\Rightarrow \quad (2.21)\quad C \equiv \begin{pmatrix} & U + iV \\ U - iV & \end{pmatrix}$$

$$\sin\delta = \frac{e^{i\delta} - e^{-i\delta}}{2i}$$

2.4 The Stokes parameters of a monochromatic, time-harmonic plane wave

In the following sets of slides, we have seen that the Stokes parameters are defined as:

$$I = \kappa(a_x^2 + a_y^2)$$
$$Q = \kappa(a_x^2 - a_y^2)$$
$$U = 2\kappa a_x a_y \cos\delta$$
$$V = 2\kappa a_x a_y \sin\delta$$

In your code above, calculate the Stokes paramters, and display the values of Q/I, U/I, and V/I in the graph.

Then, let's play the detective again, using your code. For the cases we discussed above, what are the values of the Stokes parameters?

1. A perfect circle

    > Answer:

2. A line that is tilted:

    > Answer:

3. A line that is purely vertical or horizontal

    > Answer:

4. An elipse that is not titlted (so horizontal or vertical)

    > Answer:

2.4 The Stokes parameters of a monochromatic, time-harmonic plane wave

$$(2.21) \quad C \equiv \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

In the previous section, we have seen that for a monochromatic plane wave:

$$(2.18) \quad \det(C) = 0$$

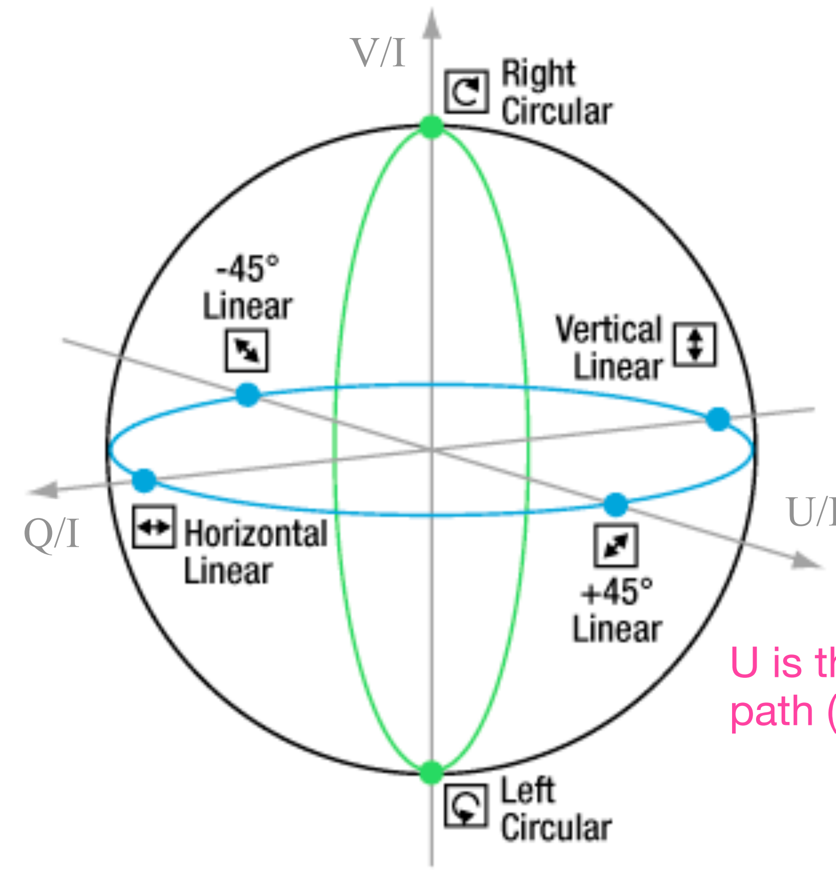
Therefore:

$$(2.22) \quad I^2 - Q^2 - U^2 - V^2 = 0$$

$$(2.24) \quad \frac{Q^2}{I^2} + \frac{U^2}{I^2} + \frac{V^2}{I^2} = 1$$

Stokes V is the ‘roundness’ of the path

Q is the ‘horizontalness/ verticalness’ of the path



U is the ‘tiltness’ of the path (with max tilt = 45°)