The Asymptotic Bode Diagram: Derivation of Approximations

| Overview | Freq Domain | Asymptotic plots | Making Plot | Examples | BodePlotGui | Rules Table | Printable |

Skip ahead to interactive demos.

Contents

Skip ahead to interactive demos.

Introduction

Given an arbitrary transfer function, such as hach frequency and displaying them. This is what a computer would naturally do. For example if you use MATLAB® and enter the commands

```
>> mySys=tf(100*[1 1],[1 110 1000])
mySys =
     100 s + 100
   ----------------
   s^2 + 110 s + 1000
>> bode(mySys)
```

you get a plot like the one shown below. The asymptotic solution is given elsewhere.
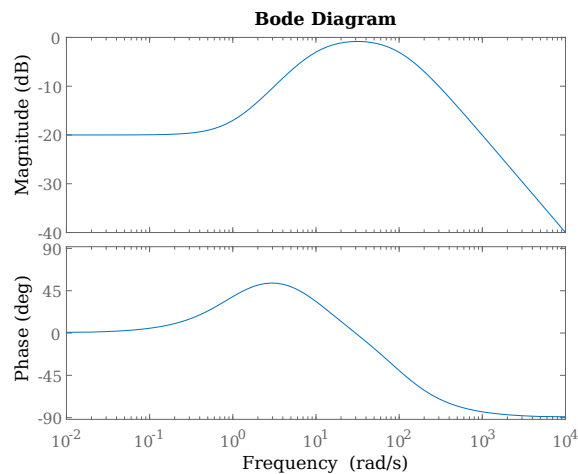
However, there are reasons to develop a method for sketching Bode diagrams manually. By drawing the plots by hand you develop an understanding about how the locations of poles and zeros effect the shape of the plots. With this knowledge you can predict how a system behaves in the frequency domain by simply examining its transfer function. On the other hand, if you know the shape of transfer function that you want, you can use your knowledge of Bode diagrams to generate the transfer function.

The first task when drawing a Bode diagram by hand is to rewrite the transfer function so that all the poles and zeros are written in the form $(1+s/\omega_0)$. The reasons for this will become apparent when deriving the rules for a real pole. A derivation will be done using the transfer function from above, but it is also possible to do a more generic derivation. Let's rewrite the transfer function from above.

$$H(s) = 100\frac{s+1}{(s+10)(s+100)} = 100\frac{1+s/1}{10\cdot(1+s/10)\cdot100\cdot(1+s/100)}$$

$$= 0.1\frac{1+s/1}{(1+s/10)(1+s/100)}$$

Now let's examine how we can easily draw the magnitude and phase of this function when $s=j\omega$.

First note that this expression is made up of four terms, a constant (0.1), a zero (at s=-1), and two poles (at s=-10 and s=-100). We can rewrite the function (with $s=j\omega$) as four individual phasors (i.e., magnitude and phase), each phasor is within a set of square brackets to make them more easily distinguished from each other..

$$H(j\omega) = 0.1\frac{1+j\omega/1}{(1+j\omega/10)(1+j\omega/100)}$$

$$= [|0.1|\angle(0.1)]\frac{[|1+j\omega/1|\angle(1+j\omega/1)]}{[|1+j\omega/10|\angle(1+j\omega/10)][|1+j\omega/100|\angle(1+j\omega/100)]}$$

We will show (below) that drawing the magnitude and phase of each individual phasor is fairly straightforward. The difficulty lies in trying to draw the magnitude and phase of the more complicated function, H($j\omega$). To start, we will write H($j\omega$) as a single phasor:

$$H(j\omega) = \left(|0.1|\frac{|1+j\omega/1|}{|1+j\omega/10||1+j\omega/100|}\right)(\angle(0.1)+\angle(1+j\omega/1)-\angle(1+j\omega/10)-\angle(1+j\omega/100))$$

$$= |H(j\omega)|\angle H(j\omega)$$

$$|H(j\omega)| = |0.1|\frac{|1+j\omega/1|}{|1+j\omega/10||1+j\omega/100|}$$

$$\angle H(j\omega) = \angle(0.1)+\angle(1+j\omega/1)-\angle(1+j\omega/10)-\angle(1+j\omega/100)$$

Drawing the phase is fairly simple. We can draw each phase term separately, and then simply add (or subtract) them. The magnitude term is not so straightforward because the magnitude terms are *multiplied*, it would be much easier if they were added - then we could draw each term on a graph and just *add* them. We can accomplish this by usinga logarthmic scale (so multiplication and division become addition and subtraction). Instead of a simple logarithm, we will use a deciBel (or dB) scale.

A Magnitude Plot

One way to transform multiplication into addition is by using the logarithm. Instead of using a simple logarithm, we will use a deciBel (named for Alexander Graham Bell).

(Note: Why the deciBel?) The relationship between a quantity, Q, and its deciBel representation, X, is given by:

$$X = 20 \cdot log_{10}\left(Q\right)$$

So if Q=100 then X=40; Q=0.01 gives X=-40; X=3 gives Q=1.41; and so on.

If we represent the magnitude of H(s) in deciBels several things happen.

$$|H(s)| = |0.1|\, \frac{|1 + j\omega/1|}{|1 + j\omega/10|\,|1 + j\omega/100|}$$

$$20 \cdot \log_{10}\left(|H(s)|\right) = 20 \cdot \log_{10}\left(|0.1|\, \frac{|1 + j\omega/1|}{|1 + j\omega/10|\,|1 + j\omega/100|}\right)$$

$$= 20 \cdot \log_{10}\left(|0.1|\right) + 20 \cdot \log_{10}\left(|1 + j\omega/1|\right) + 20 \cdot \log_{10}\left(\frac{1}{|1 + j\omega/10|}\right) + 20 \cdot \log_{10}\left(\frac{1}{|1 + j\omega/100|}\right)$$

$$= 20 \cdot \log_{10}\left(|0.1|\right) + 20 \cdot \log_{10}\left(|1 + j\omega/1|\right) - 20 \cdot \log_{10}\left(|1 + j\omega/10|\right) - 20 \cdot \log_{10}\left(|1 + j\omega/100|\right)$$

The advantages of using deciBels (and of writing poles and zeros in the form (1+s/ω₀)) are now revealed. The fact that the deciBel is a logarithmic term transforms the multiplications and divisions of the individual terms to additions and subtsractions. Another benefit is apparent in the last line that reveals just two types of terms, a constant term and terms of the form 20·log10(|1+jω/ω₀|). Plotting the constant term is trivial, however the other terms are not so straightforward. These plots will be discussed below. However, once these plots are drawn for the individual terms, they can simply be added together to get a plot for H(s).

A Phase Plot

If we look at the phase of the transfer function, we see much the same thing: The phase plot is easy to draw if we take our lead from the magnitude plot. First note that the transfer function is made up of four terms. If we want

$$\angle H(s) = \angle\left(0.1\right) + \angle\left(1 + j\omega/1\right) - \angle\left(1 + j\omega/10\right) - \angle\left(1 + j\omega/100\right)$$

Again there are just two types of terms, a constant term and terms of the form (1+jω/ω ₀). Plotting the constant term is trivial; the other terms are discussed below.

A more generic derivation

The discussion above dealt with only a single transfer function. Another derivation that is more general, but a little more complicated mathematically is here.

Making a Bode Diagram

Following the discussion above, the way to make a Bode Diagram is to split the function up into its constituent parts, plot the magnitude and phase of each part, and then add them up. The following gives a derivation of the plots for each type of constituent part. Examples, including rules for making the plots follow in the next document, which is more of a "How to" description of Bode diagrams.

A Constant Term

Consider a constant term: $H(s) = H(j\omega) = K$

Magnitude

Clearly the magnitude is constant as ω varies. $|H(j\omega)| = |K|$

Phase

The phase is also constant. If K is positive, the phase is 0° (or any even multiple of 180°, i.e., ±360°). If K is negative the phase is -180°, or any odd multiple of 180°. We will use -180° because that is what MATLAB® uses. Expressed in radians we can say that if K is positive the phase is 0 radians, if K is negative the phase is -π radians.

Example: Bode Plot of Gain Term

Bode Diagram

$$H(s) = H(j\omega) = 15$$
$$|H(j\omega)| = |15| = 15 = 23.5\,dB$$
$$\angle H(j\omega) = \angle 15 = 0°$$

The magnitude (in dB is calculated as

$$20 \cdot \log_{10}(15) = 23.5$$

.
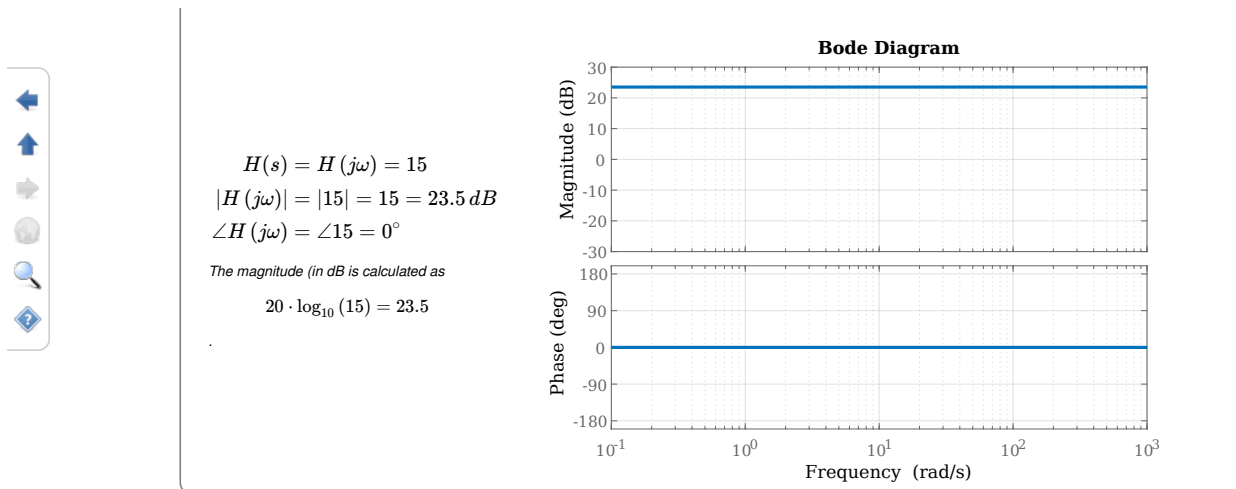
Key Concept: Bode Plot of Gain Term
- For a constant term, the magnitude plot is a straight line.
- The phase plot is also a straight line, either at 0° (for a positive constant) or ±180° (for a negative constant).

Interactive Demo

A Real Pole

Consider a simple real pole : $H(s) = \frac{1}{1+\frac{s}{\omega_0}}, \qquad H(j\omega) = \frac{1}{1+j\frac{\omega}{\omega_0}}$

The frequency ω_0 is called the break frequency, the corner frequency or the 3 dB frequency (more on this last name later).

Magnitude

The magnitude is given by

$$|H(j\omega)| = \left| \frac{1}{1+j\frac{\omega}{\omega_0}} \right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

$$|H(j\omega)|_{dB} = 20 \cdot \log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \right)$$

Let's consider three cases for the value of the frequency, and determine the magnitude in each case.:

Case 1) ω<<ω₀. This is the low frequency case with ω/ω₀→0. We can write an approximation for the magnitude of the transfer function:

$$\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx 1\$, and\$ |H(j\omega)|_{dB} \approx 20 \cdot \log_{10}\left(\frac{1}{1}\right) = 0$$

This low frequency approximation is shown in blue on the diagram below.

Case 2) ω>>ω₀. This is the high frequency case with ω/ω₀→∞. We can write an approximation for the magnitude of the transfer function:

$$\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx \sqrt{\left(\frac{\omega}{\omega_0}\right)^2} \approx \frac{\omega}{\omega_0}, \text{so}$$
$$|H(j\omega)|_{dB} \approx 20 \cdot \log_{10}\left(\frac{\omega_0}{\omega}\right)$$

The high frequency approximation is at shown in green on the diagram below. It is a straight line with a slope of -20 dB/decade going through the break frequency at 0 dB (if ω=ω₀ the approximation simplifies to 0 dB; ω=10·ω₀ gives an approximate gain of 0.1, or -20 dB and so on). That is, the approximation goes through 0 dB at ω=ω₀, and for every factor of 10 increase in frequency, the magnitude drops by 20 dB..

Case 3) ω=ω₀. At the break frequency

$$|H\left(j\omega_0\right)|_{dB} = 20 \cdot \log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}}\right) = 20 \cdot \log_{10}\left(\frac{1}{\sqrt{2}}\right) \approx -3\ dB$$

This point is shown as a red circle on the diagram.

    To draw a piecewise linear approximation, use the low frequency asymptote up to the break frequency, and the high frequency asymptote thereafter.
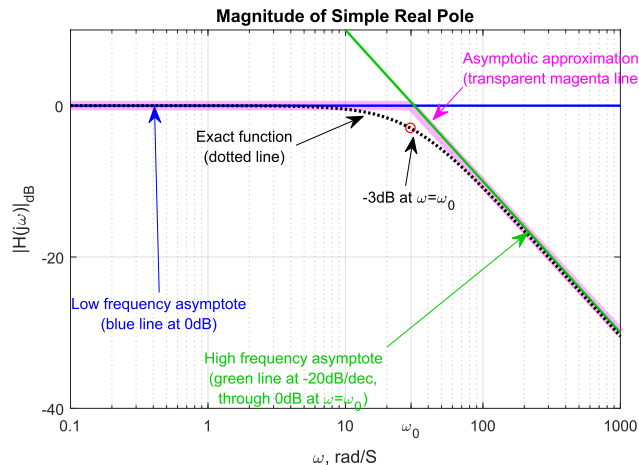
The resulting asymptotic approximation is shown highlighted in transparent magenta. The maximum error between the asymptotic approximation and the exact magnitude function occurs at the break frequency and is approximately -3 dB.

Magnitude of a real pole: The piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency and then drops at 20 dB per decade as frequency increases (i.e., the slope is -20 dB/decade).

Phase

    The phase of a single real pole is given by is given by

$$\angle H\left(j\omega\right) = \angle\left(\frac{1}{1 + j\frac{\omega}{\omega_0}}\right) = -\angle\left(1 + j\frac{\omega}{\omega_0}\right) = -\arctan\left(\frac{\omega}{\omega_0}\right)$$

Let us again consider three cases for the value of the frequency:

    Case 1) $\omega \ll \omega_0$. *This is the low frequency case with $\omega/\omega_0 \to 0$. At these frequencies We can write an approximation for the phase of the transfer function*

$$\angle H\left(j\omega\right) \approx -\arctan(0) = 0° = 0\ rad$$

The low frequency approximation is shown in blue on the diagram below.

    Case 2) $\omega \gg \omega_0$. *This is the high frequency case with $\omega/\omega_0 \to \infty$. We can write an approximation for the phase of the transfer function*

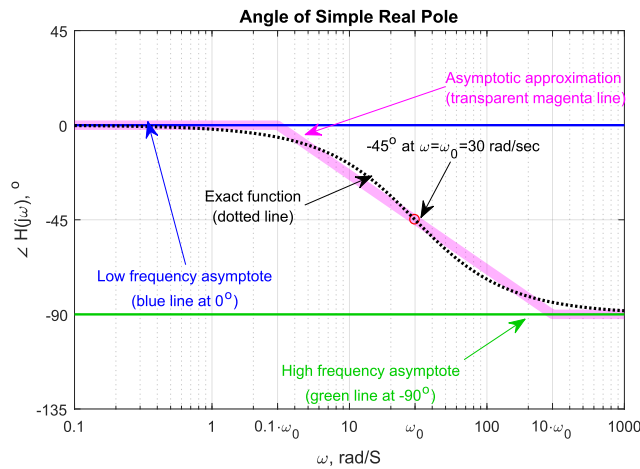$$\angle H\left(j\omega\right) \approx -\arctan(\infty) = -90° = -\frac{\pi}{2}\ rad$$

The high frequency approximation is at shown in green on the diagram below. It is a horizontal line at -90°.

    Case 3) $\omega = \omega_0$. *The break frequency. At this frequency*

$$\angle H\left(j\omega\right) = -\arctan(1) = -45° = -\frac{\pi}{4}\ rad$$

This point is shown as a red circle on the diagram.

Angle of Simple Real Pole

A piecewise linear approximation is not as easy in this case because the high and low frequency asymptotes don't intersect. Instead we use a rule that follows the exact function fairly closely, but is also somewhat arbitrary. Its main advantage is that it is easy to remember.

Phase of a real pole: *The piecewise linear asymptotic Bode plot for phase follows the low frequency asymptote at 0° until one tenth the break frequency $(0.1 \cdot \omega_0)$ then decrease linearly to meet the high frequency asymptote at ten times the break frequency $(10 \cdot \omega_0)$. This line is shown above. Note that there is no error at the break frequency and about 5.7° of error at $0.1 \cdot \omega_0$ and $10 \cdot \omega_0$ the break frequency.*

Example: Real Pole

The first example is a simple pole at 5 radians per second. The asymptotic approximation is magenta, the exact function is a dotted black line.
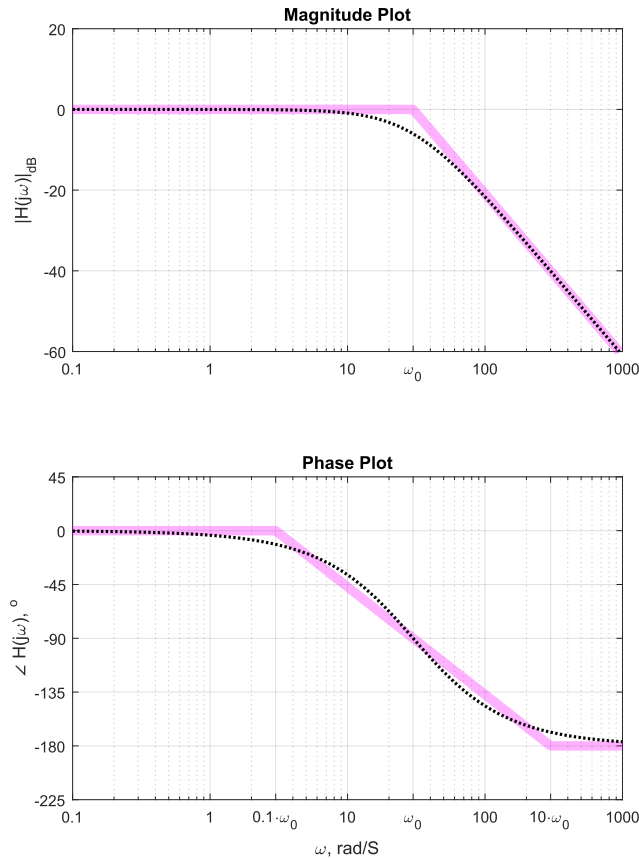
$$H(s) = \frac{1}{1 + \frac{s}{5}}$$

Magnitude Plot

Phase Plot
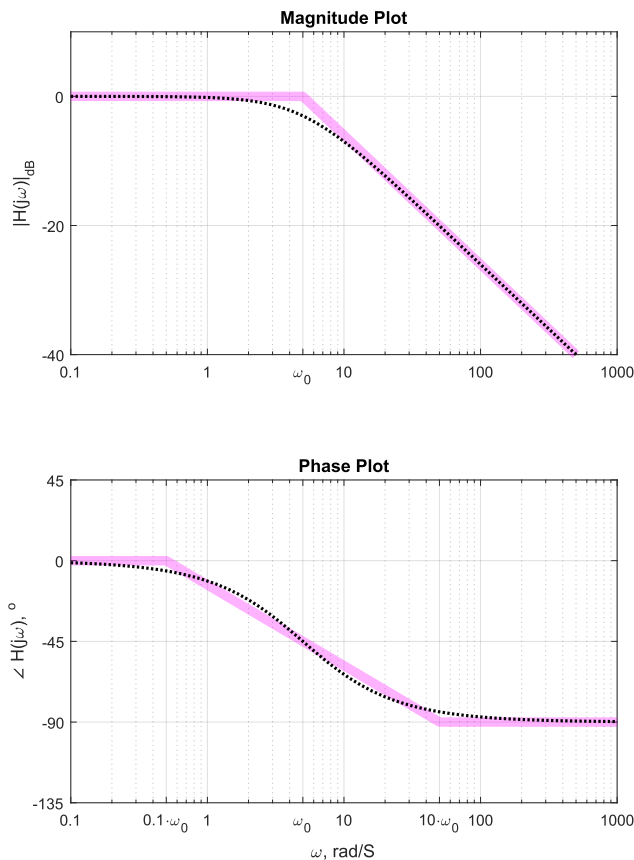
Example: Repeated Real Pole

The second example shows a double pole at 30 radians per second. Note that the slope of the asymptote is -40 dB/decade and the phase goes from 0 to -180°. The effect of repeating a pole is to double the slope of the magnitude to -40 dB/decade and the slope of the phase to -90°/decade.

$$H(s) = \frac{1}{\left(1 + \frac{s}{30}\right)^2}$$

**Magnitude Plot**

Phase Plot

Key Concept: Bode Plot for Real Pole

- For a simple real pole the piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency and then drops at 20 dB per decade (i.e., the slope is -20 dB/decade). An nth order pole has a slope of -20·n dB/decade.

- The phase plot is at 0° until one tenth the break frequency and then drops linearly to -90° at ten times the break frequency. An nth order pole drops to -90°·n.

Interactive Demo

Aside: a different formulation of the phase approximation

There is another approximation for phase that is occasionally used. The approximation is developed by matching the slope of the actual phase term to that of the approximation at $\omega=\omega_0$. Using math similar to that given here (for the underdamped case) it can be shown that by drawing a line starting at 0° at $\omega=\omega_0/e^{\pi/2}=\omega_0/4.81$ (or $\omega_0 \cdot e^{-\pi/2}$) to -90° at $\omega=\omega_0 \cdot 4.81$ we get a line with the same slope as the actual function at $\omega=\omega_0$. The approximation described previously is much more commonly used as is easier to remember as a line drawn from 0° at $\omega_0/5$ to -90° at $\omega_0 \cdot 5$, and easier to draw on semi-log paper. The latter is shown on the diagram below.
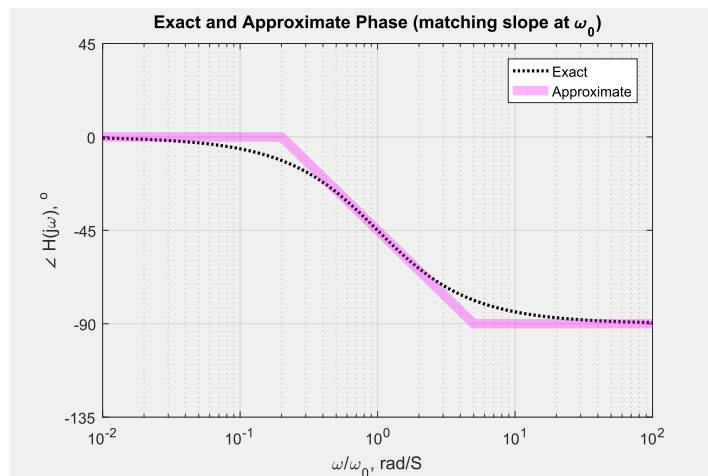
Although this method is more accurate in the region around ω=ω₀ there is a larger maximum error (more than 10°) near ω₀/5 and
ω₀·5 when compared to the method described previously.

A Real Zero

The piecewise linear approximation for a zero is much like that for a pole Consider a simple zero:

$$H(s) = 1 + \frac{s}{\omega_0}, \quad H(j\omega) = 1 + j\frac{\omega}{\omega_0}$$

Magnitude

The development of the magnitude plot for a zero follows that for a pole. Refer to the previous section for details. The magnitude of the zero
is given by

$$|H(j\omega)| = \left|1 + j\frac{\omega}{\omega_0}\right|$$

Again, as with the case of the real pole, there are three cases:

1. At low frequencies, $\omega \ll \omega_0$, the gain is approximately 1 (or 0 dB).
2. At high frequencies, $\omega \gg \omega_0$, the gain increases at 20 dB/decade and goes through the break frequency at 0 dB.
3. At the break frequency, $\omega = \omega_0$, the gain is about 3 dB.

Magnitude of a Real Zero: *For a simple real zero the piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency*
and then increases at 20 dB per decade (i.e., the slope is +20 dB/decade).

Phase
The phase of a simple zero is given by:

$$\angle H(j\omega) = \angle\left(1 + j\frac{\omega}{\omega_0}\right) = \arctan\left(\frac{\omega}{\omega_0}\right)$$

The phase of a single real zero also has three cases (which can be derived similarly to those for the real pole, given above):

1. At low frequencies, $\omega \ll \omega_0$, the phase is approximately zero.
2. At high frequencies, $\omega \gg \omega_0$, the phase is +90°.
3. At the break frequency, $\omega = \omega_0$, the phase is +45°.

Phase of a Real Zero: *Follow the low frequency asymptote at 0° until one tenth the break frequency (0.1 ω₀) then increase linearly to meet the*
high frequency asymptote at ten times the break frequency (10 ω₀).

Example: Real Zero

This example shows a simple zero at 30 radians per second. The asymptotic approximation is magenta, the exact function is the
dotted black line.

$$H(s) = 1 + \frac{s}{30}$$

Magnitude Plot

Phase Plot
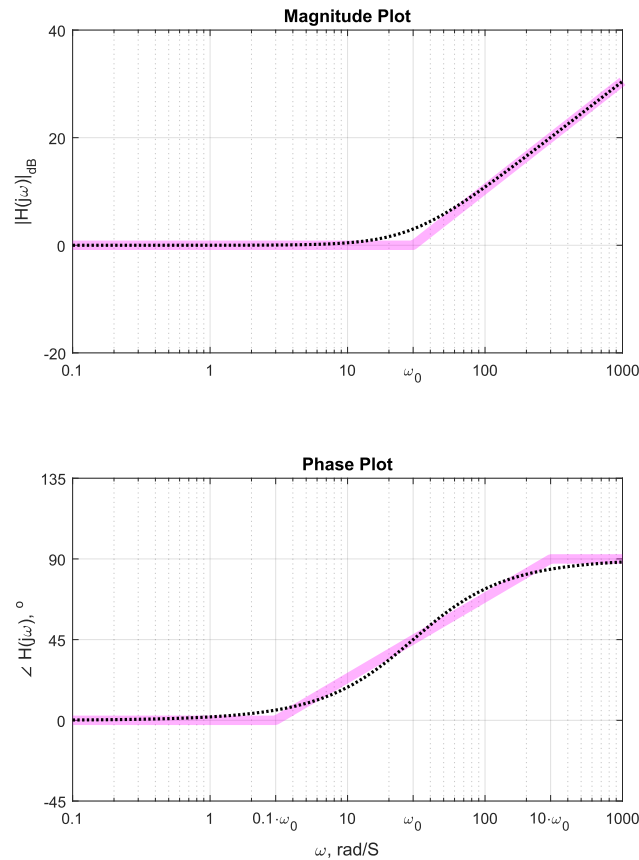
Key Concept: Bode Plot of Real Zero:

- *The plots for a real zero are like those for the real pole but mirrored about 0dB or 0°.*
- *For a simple real zero the piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency and then rises at +20 dB per decade (i.e., the slope is +20 dB/decade). An nth order zero has a slope of $+20 \cdot n$ dB/decade.*
- *The phase plot is at 0° until one tenth the break frequency and then rises linearly to +90° at ten times the break frequency. An nth order zero rises to $+90° \cdot n$.*

Interactive Demo

A Pole at the Origin

A pole at the origin is easily drawn exactly. Consider

$$H\left(s\right) = \frac{1}{s}, \quad H\left(j\omega\right) = \frac{1}{j\omega} = -\frac{j}{\omega}$$

Magnitude

The magnitude is given by

$$\left|H\left(j\omega\right)\right| = \left|-\frac{j}{\omega}\right| = \frac{1}{\omega}$$

$$\left|H\left(j\omega\right)\right|_{dB} = 20 \cdot \log_{10}\left(\frac{1}{\omega}\right) = -20 \cdot \log_{10}\left(\omega\right)$$

In this case there is no need for approximate functions and asymptotes, we can plot the exact funtion. The function is represented by a straight line on a Bode plot with a slope of -20 dB per decade and going through 0 dB at 1 rad/ sec. It also goes through 20 dB at 0.1 rad/sec, -20 dB at 10 rad/sec... Since there are no parameters (i.e., ω_0) associated with this function, it is always drawn in exactly the same manner.

Magnitude of Pole at the Origin: *Draw a line with a slope of -20 dB/decade that goes through 0 dB at 1 rad/sec.*
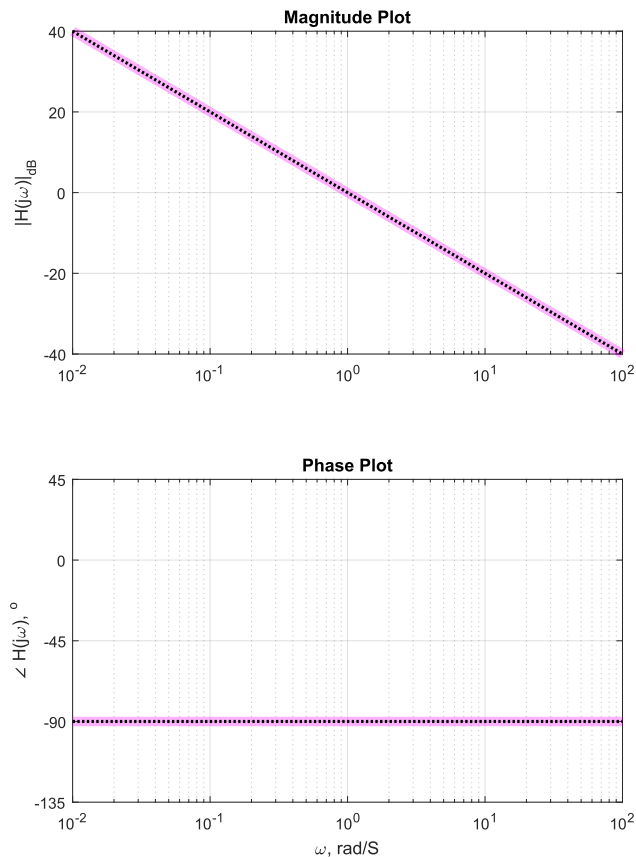
Phase

The phase of a simple zero is given by (H(jω) is a negative imaginary number for all values of ω so the phase is always -90°):

$$\angle H\left(j\omega\right) = \angle\left(-\frac{j}{\omega}\right) = -90°$$

Phase of pole at the origin: The phase for a pole at the origin is -90°.

Example: Pole at Origin

This example shows a simple pole at the origin. The exact (dotted black line) is the same as the approximation (magenta).

Magnitude Plot

Phase Plot

Key Concept: Bode Plot for Pole at Origin

No interactive demo is provided because the plots are always drawn in the same way.

- For a simple pole at the origin draw a straight line with a slope of -20 dB per decade and going through 0 dB at 1 rad/ sec.
- The phase plot is at -90°.
- The magnitude of an nth order pole has a slope of -20·n dB/decade and a constant phase of -90°·n.

A Zero at the Origin

A zero at the origin is just like a pole at the origin but the magnitude increases with increasing ω, and the phase is +90° (i.e. simply mirror the graphs for the pole around the origin around 0dB or 0°).

Example: Zero at Origin

This example shows a simple zero at the origin. The exact (dotted black line) is the same as the approximation (magenta).

Magnitude Plot

Phase Plot

Key Concept: Bode Plot for Zero at Origin

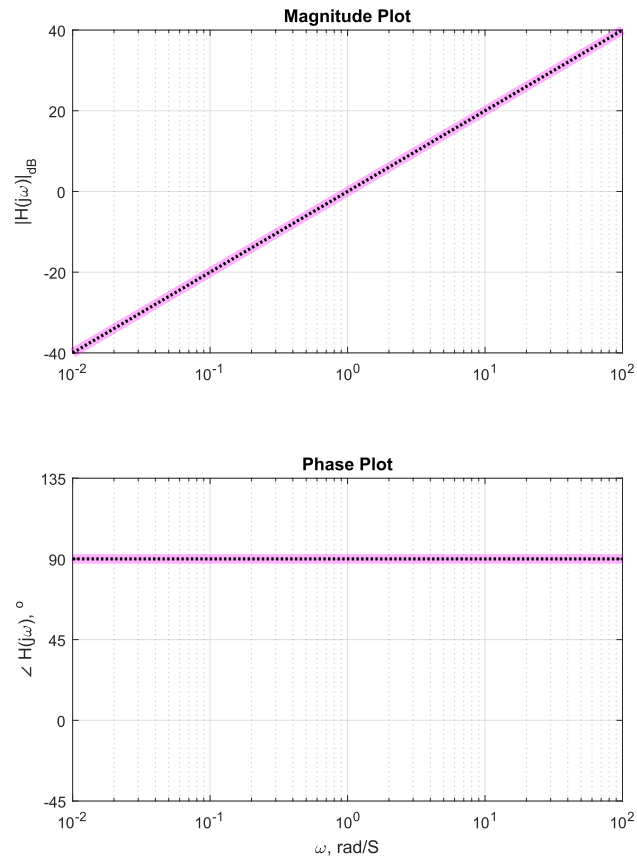- *The plots for a zero at the origin are like those for the pole but mirrored about 0dB or 0°.*
- *For a simple zero at the origin draw a straight line with a slope of +20 dB per decade and going through 0 dB at 1 rad/ sec.*
- *The phase plot is at +90°.*
- *The magnitude of an nth order zero has a slope of +20·n dB/decade and a constant phase of +90°·n.*

A Complex Conjugate Pair of Poles

The magnitude and phase plots of a complex conjugate (underdamped) pair of poles is more complicated than those for a simple pole. Consider the transfer function (with 0<ζ<1):

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$$

Magnitude

The magnitude is given by

$$|H(j\omega)| = \left| \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1} \right| = \left| \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_0}\right) + 1} \right| = \left| \frac{1}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) + j\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)} \right|$$

$$= \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}$$

$$|H(j\omega)|_{dB} = -20 \cdot \log_{10}\left(\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}\right)$$

As before, let's consider three cases for the value of the frequency:

　　Case 1) $\omega \ll \omega_0$. *This is the low frequency case. We can write an approximation for the magnitude of the transfer function*

$$|H(j\omega)|_{dB} = -20 \cdot \log_{10}(1) = 0$$

The low frequency approximation is shown in red on the diagram below.

　　Case 2) $\omega \gg \omega_0$. *This is the high frequency case. We can write an approximation for the magnitude of the transfer function*

$$|H(j\omega)|_{dB} = -20 \cdot \log_{10}\left(\left(\frac{\omega}{\omega_0}\right)^2\right) = -40 \cdot \log_{10}\left(\frac{\omega}{\omega_0}\right)$$

The high frequency approximation is at shown in green on the diagram below. It is a straight line with a slope of -40 dB/decade going through the break frequency at 0 dB. That is, for every factor of 10 increase in frequency, the magnitude drops by 40 dB.

　　Case 3) $\omega \approx \omega_0$. *It can be shown that a peak occurs in the magnitude plot near the break frequency. The derivation of the approximate amplitude and location of the peak are given* here. *We make the approximation that a peak exists only when*

$$0 < \zeta < 0.5$$

and that the peak occurs at ω_0 with height $1/(2\cdot\zeta)$.

　　To draw a piecewise linear approximation, use the low frequency asymptote up to the break frequency, and the high frequency asymptote thereafter. If $\zeta < 0.5$, then draw a peak of amplitude $1/(2\cdot\zeta)$ Draw a smooth curve between the low and high frequency asymptote that goes through the peak value.

　　As an example for the curve shown below $\omega_0 = 10$, $\zeta = 0.1$,

$$H(s) = \frac{1}{\frac{s^2}{100} + 0.02\zeta s + 1} = \frac{1}{\left(\frac{s}{10}\right)^2 + 0.2\left(\frac{s}{10}\right) + 1} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$$

The peak will have an amplitude of $1/(2\cdot\zeta) = 5.00$ or 14 dB.

Magnitude of Complex Conjugate Poles
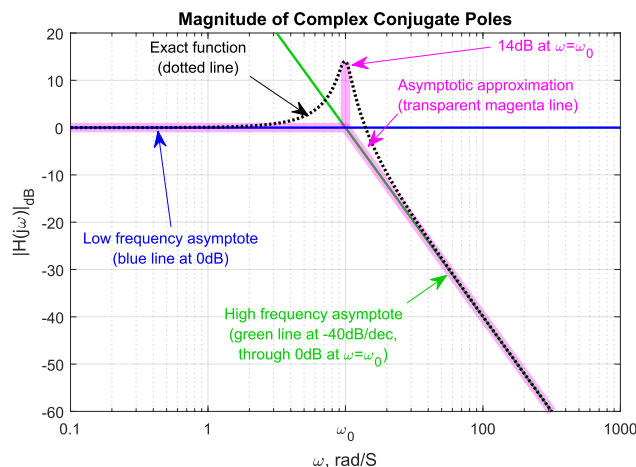
The resulting asymptotic approximation is shown as a black dotted line, the exact response is a black solid line.

Magnitude of Underdamped (Complex) poles: *Draw a 0 dB at low frequencies until the break frequency, ω_0, and then drops with a slope of -40 dB/decade. If $\zeta<0.5$ we draw a peak of height at ω_0, otherwise no peak is drawn.*

$$|H(j\omega_0)| \approx \frac{1}{2\zeta}, \quad |H(j\omega_0)|_{dB} \approx -20 \cdot \log_{10}(2\zeta)$$

Note: The actual height of the peak and its frequency are both slightly less than the approximations given above. An in depth discussion of the magnitude and phase approximations (along with some alternate approximations) are given here.

Phase

The phase of a complex conjugate pole is given by is given by

$$\angle H(j\omega) = \angle\left(\frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1}\right) = -\angle\left(\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1\right) = -\angle\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right)\right)$$

$$= -\arctan\left(\frac{2\zeta\frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2}\right)$$

Let us again consider three cases for the value of the frequency:

Case 1) $\omega<<\omega_0$. This is the low frequency case. At these frequencies We can write an approximation for the phase of the transfer function

$$\angle H(j\omega) \approx -\arctan\left(\frac{2\zeta\omega}{\omega_0}\right) \approx -\arctan(0) = 0° = 0 \ rad$$

The low frequency approximation is shown in red on the diagram below.

Case 2) $\omega>>\omega_0$. This is the high frequency case. We can write an approximation for the phase of the transfer function

$$\angle H(j\omega) \approx -180° = -\pi \ rad$$

Note: this result makes use of the fact that the arctan function returns a result in quadrant 2 since the imaginary part of H(jω) is negative and the real part is positive. The high frequency approximation is at shown in green on the diagram below. It is a straight line at -180°.

Case 3) $\omega=\omega_0$. The break frequency. At this frequency

$$\angle H(j\omega_0) = -90°$$

The asymptotic approximation is shown below for $\omega_0=10$, $\zeta=0.1$, followed by an explanation

$$H(s) = \frac{1}{\frac{s^2}{100} + 0.02\zeta s + 1} = \frac{1}{\left(\frac{s}{10}\right)^2 + 0.2\left(\frac{s}{10}\right) + 1} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$$
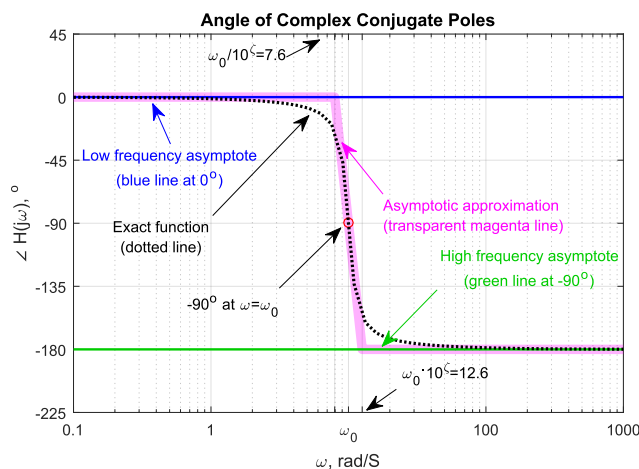
A piecewise linear approximation is a bit more complicated in this case, and there are no hard and fast rules for drawing it. The most common way is to look up a graph in a textbook with a chart that shows phase plots for many values of ζ. Three asymptotic approximations are given here. We will use the approximation that connects the the low frequency asymptote to the high frequency asymptote starting at

$$\omega = \frac{\omega_0}{10^\zeta} = \omega_0 \cdot 10^{-\zeta}$$

and ending at

$$\omega = \omega_0 \cdot 10^\zeta$$

Since ζ=0.2 in this case this means that the phase starts at 0° and then breaks downward at ω=ω$_0$/10$^\zeta$=7.9 rad/sec. The phase reaches -180° at ω=ω$_0$·10$^\zeta$=12.6 rad/sec.

As a practical matter If ζ<0.02, the approximation can be simply a vertical line at the break frequency. One advantage of this approximation is that it is very easy to plot on semilog paper. Since the number 10·ω$_0$ moves up by a full decade from ω$_0$, the number 10$^\zeta$·ω$_0$ will be a fraction ζ of a decade above ω$_0$. For the example above the corner frequencies for ζ=0.1 fall near ω$_0$ one tenth of the way between ω$_0$ and ω$_0$/10 (at the lower break frequency) to one tenth of the way between ω$_0$ and ω$_0$·10 (at the higher frequency).

Phase of Underdamped (Complex) Poles: Follow the low frequency asymptote at 0° until

$$\omega = \frac{\omega_0}{10^\zeta}$$

then decrease linearly to meet the high frequency asymptote at -180° at

$$\omega = \omega_0 \cdot 10^\zeta$$

Other magnitude and phase approximations (along with exact expressions) are given here.

Key Concept: Bode Plot for Complex Conjugate Poles

- For the magnitude plot of complex conjugate poles draw a 0 dB at low frequencies, go through a peak of height,

$$|H(j\omega_0)| \approx \frac{1}{2\zeta}, \quad |H(j\omega_0)|_{dB} \approx -20 \cdot \log_{10}(2\zeta)$$

  at the break frequency and then drop at 40 dB per decade (i.e., the slope is -40 dB/decade). The high frequency asymptote goes through the break frequency. Note that in this approximation the peak only exists for

  $$0 < \zeta < 0.5$$

- To draw the phase plot simply follow low frequency asymptote at 0° until

$$\omega = \frac{\omega_0}{10^\zeta} = \omega_0 \cdot 10^{-\zeta}$$

  then decrease linearly to meet the high frequency asymptote at -180° at

$$\omega = \omega_0 \cdot 10^\zeta$$

  If ζ<0.02, the approximation can be simply a vertical line at the break frequency.
- Note that the shape of the graphs (magnitude peak height, steepness of phase transition) are determined solely by ζ, and the frequency at which the magnitude peak and phase transition occur are determined solely by ω$_0$.

Note: Other magnitude and phase approximations (along with exact expressions) are given here.

Interactive Demo

A Complex Conjugate Pair of Zeros

Not surprisingly a complex pair of zeros yields results similar to that for a complex pair of poles. The magnitude and phase plots for the complex zero are the mirror image (around 0dB for magnitude and around 0° for phase) of those for the complex pole. Therefore, the magnitude has a dip instead of a peak, the magnitude increases above the break frequency and the phase increases rather than decreasing. The results will not be derived here, but closely follow those foe complex poles.

Example: Complex Conjugate Zero

The graph below corresponds to a complex conjugate zero with ω$_0$=3, ζ=0.25

$$H\left(s\right) = \left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1$$

The dip in the magnitude plot will have a magnitude of 0.5 or -6 dB. The break frequencies for the phase are at $\omega = \omega_0/10^\zeta = 1.7$ rad/sec and $\omega = \omega_0 \cdot 10^\zeta = 5.3$ rad/sec.

Magnitude Plot

Phase Plot
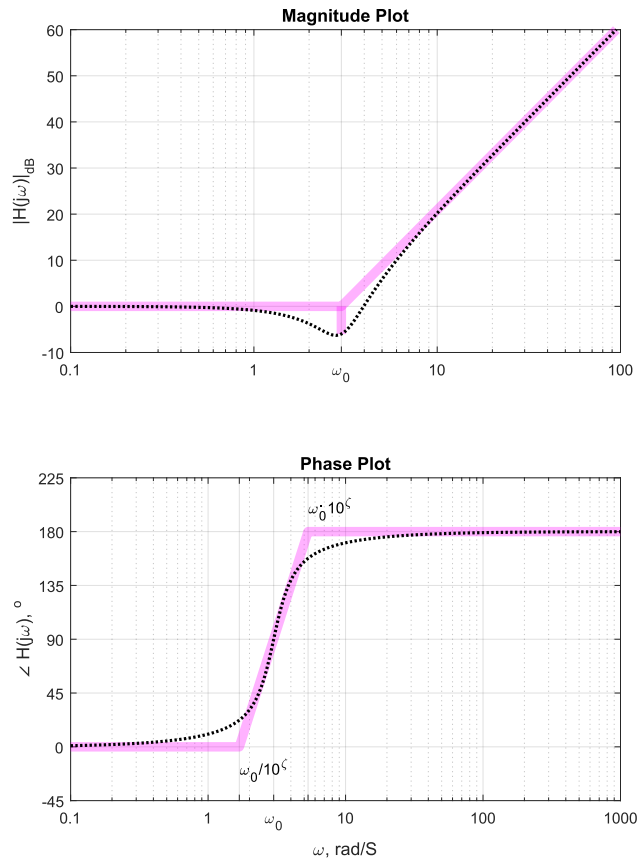
Key Concept: Bode Plot of Complex Conjugate Zeros

- *The plots for a complex conjugate pair of zeros are very much like those for the poles but mirrored about 0dB or 0°.*
- *For the magnitude plot of complex conjugate zeros draw a 0 dB at low frequencies, go through a dip of magnitude:*

$$|H(j\omega_0)| \approx 2\zeta, \quad |H(j\omega_0)|_{dB} \approx 20 \cdot \log_{10}(2\zeta)$$

  at the break frequency and then rise at +40 dB per decade (i.e., the slope is +40 dB/decade). The high frequency asymptote goes through the break frequency. Note that the peak only exists for

$$0 < \zeta < 0.5$$

- *To draw the phase plot simply follow low frequency asymptote at 0° until*

$$\omega = \frac{\omega_0}{10^\zeta} = \omega_0 \cdot 10^{-\zeta}$$

  then increase linearly to meet the high frequency asymptote at 180° at

$$\omega = \omega_0 \cdot 10^\zeta$$

- *Note that the shape of the graphs (magnitude peak height, steepness of phase transition) are determined solely by ζ, and the frequency at which the magnitude peak and phase transition occur are determined solely by ω_0.*

Note: Other magnitude and phase approximations (along with exact expressions) are given here.

Interactive Demo

Interactive Demos:

Below you will find interactive demos that show how to draw the asymptotic approximation for a constant, a first order pole and zero, and a second order (underdamped) pole and zero. Note there is no demo for a pole or zero at the origin because these are always drawn in exactly the same way; there are no variable parameters (i.e., ω_0 or ζ).

Interactive Demo: Bode Plot of Constant Term

This demonstration shows how the gain term affects a Bode plot. To run the demonstration either enter the value of K, or |K| expressed in dB, in one of the text boxes below. If you enter |K| in dB, then the sign of K is unchanged from its current value. You can also set |K| and $\angle K$ by either clicking and dragging the horizontal lines on the graphs themselve. The magnitude of K must be between 0.01 and 100 (-40dB and +40dB). The phase of K ($\angle K$) can only be 0° (for a positive value of K) or ±180° (for negative K).

Enter a value for gain, K: 1.00 *,*
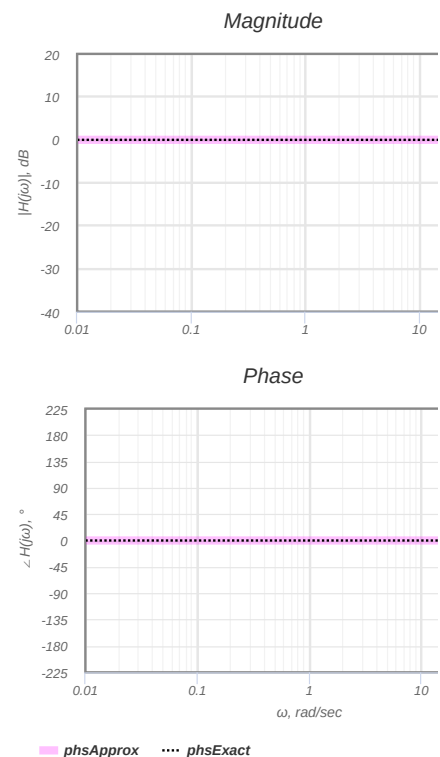
or enter |K| expressed in dB: 0.00 *dB.*

$K = 1.00$ so the value of $K_{dB} = 20 \cdot log_{10}(|K|) = 20 \cdot log_{10}(|1.00|) = 0.00$.

Or, given that $K_{dB} = 0.00$, $|K| = 10^{Kdb/20} = 10^{0.00/20} = 1$.

The sign of K depends on phase, in this case K is positive.and phase = 0°.

Note that for the case of a constant term, the approximate (magenta line) and exact (dotted black line) representations of magnitude and phase are equal.

Interactive Demo: Bode Plot of a Real Pole

This demonstration shows how a first order pole expressed as:

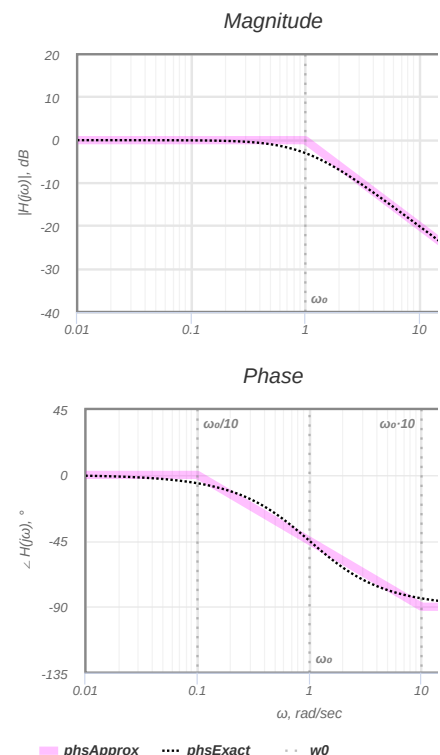$$H(s) = \frac{1}{1 + \frac{s}{\omega_0}} = \frac{1}{1 + j\frac{\omega}{\omega_0}},$$

is displayed on a Bode plot. To change the value of ω_0, you can either change the value in the text box, below, or drag the vertical line showing ω_0 on the graphs to the right. The exact values of magnitude and phase are shown as black dotted lines and the asymptotic approximations are shown with a thick magenta line. The value of ω_0 is constrained such that $0.1 \leq \omega_0 \leq 10$ rad/second.

Enter a value for ω_o: 1.000

Asymptotic Magnitude: *The asymptotic magnitude plot starts (at low frequencies) at 0 dB and stays at that level until it gets to ω_0. At that point the gain starts dropping with a slope of -20 dB/decade.*

Asymptotic Phase: *The asymptotic phase plot starts (at low frequencies) at 0° and stays at that level until it gets to $0.1 \cdot \omega_0$ (0.1 rad/sec). At that point the phase starts dropping at -45°/decade until it gets to -90° at $10 \cdot \omega_0$ (10 rad/sec), at which point it becomes constant at -90° for high frequencies. Phase goes through -45° at $\omega = \omega_0$.*

$\omega_0/10$	ω_0	$10 \cdot \omega_0$
0.10	1.00	10.00

Interactive Demo: Bode Plot of a Real zero

This demonstration shows how a first order zero expressed as:

$$H(s) = 1 + \frac{s}{\omega_0} = 1 + j\frac{\omega}{\omega_0},$$

is displayed on a Bode plot. To change the value of ω_0, you can either change the value in the text box, below, or drag the vertical line showing ω_0 on the graphs to the right. The exact values of magnitude and phase are shown as black dotted lines and the asymptotic approximations are shown with a thick magenta line. The value of ω_0 is constrained such that $0.1 \leq \omega_0 \leq 10$ rad/second.

Enter a value for ω_o: 1.000

Asymptotic Magnitude: The asymptotic magnitude plot starts (at low frequencies) at 0 dB and stays at that level until it gets to ω_0. At that point the gain starts rising with a slope of +20 dB/decade.

Asymptotic Magnitude: The asymptotic phase plot starts (at low frequencies) at 0° and stays at that level until it gets to $0.1 \cdot \omega_0$. At that point the phase starts rising at +45°/decade until it gets to +90° at $10 \cdot \omega_0$, at which point it becomes constant at +90° for high frequencies. Phase goes through +45° at $\omega = \omega_0$.

$\omega_0/10$	ω_0	$10 \cdot \omega_0$
0.10	1.00	10.00

Magnitude

Phase

▬ phsApprox	···· phsExact
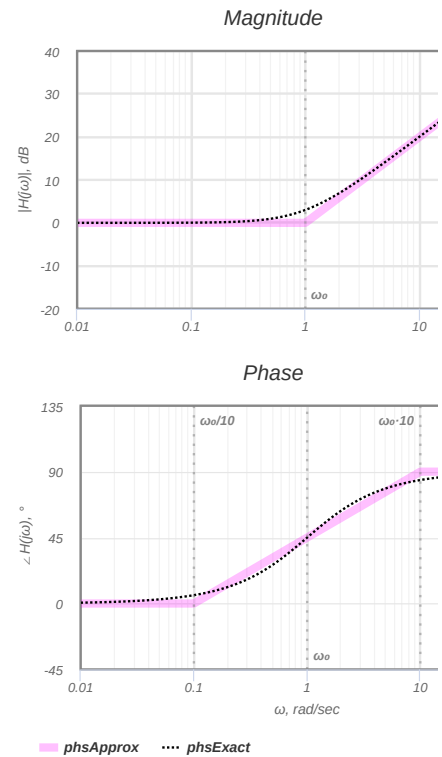
Interactive Demo: Bode Plot of a Pair of Complex Conjugate Poles

This demonstration shows how a second order pole (complex conjugate roots) expressed as:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j2\zeta\frac{\omega}{\omega_0}},$$

is displayed on a Bode plot. You can change ω_0, and ζ. The value of ω_0 is constrained such that $0.1 \leq \omega_0 \leq 10$ rad/second, and $0.05 \leq \zeta \leq 0.99$.

Enter value for ω_o: 1.000 or click and drag on graph to set ω_0., and use text-box or slider, below, for ζ.

0.200 ζ 0.05 ⬤   0.99

Asymptotic Magnitude: The asymptotic magnitude plot starts (at low frequencies) at 0 dB and stays at that level until it gets to ω_0. At that point the gain starts dropping with a slope of -40 dB/decade. *Note: it is -40 dB per decade because there are two poles in the denominator.* If $\zeta < 0.5$ we estimate the peak height as $|H(j\omega_{peak})| \approx \frac{1}{2\zeta}$ (exact height is $|H(j\omega_{peak})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$). We approximate the peak location at $\omega_{peak} \approx \omega_0$ (exact peak location is at $\omega_{peak} = \omega_0\sqrt{1 - 2\zeta^2}$).
However, if $\zeta \geq 0.5$, the peak is sufficiently small that we don't include it in our plot.

Since $\zeta < 0.5$, we draw a peak. Note how close together the approximate and exact values are for ω_{peak} and $|H(j\omega_{peak})|$.

	ω_{peak}	$|H(j\omega_{peak})|$	$|H(j\omega_{peak})|_{dB}$
Approximate	1.00	2.50	7.96
Exact	0.96	2.55	8.14

Asymptotic Phase: The asymptotic phase plot starts (at low frequencies) at 0° and stays at that level until it gets to $\omega_0/10^\zeta$. At that point the phase starts dropping at -90°/decade until it gets to -180°, at which point it becomes constant at -180° for high frequencies. Phase goes

Magnitude

Phase

▬ phsApprox	···· phsExact
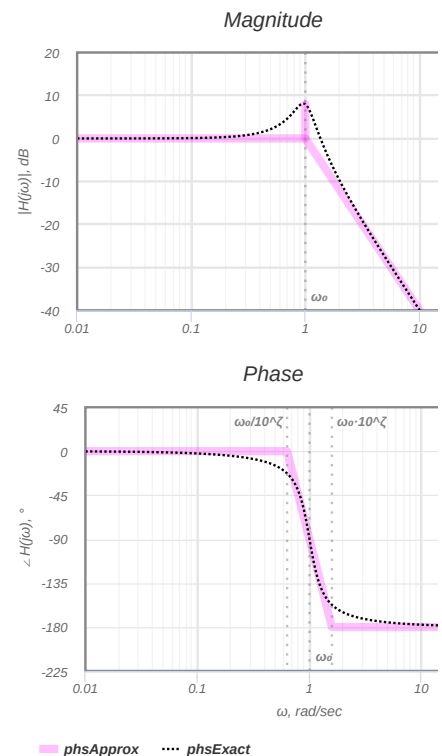
through -90° at ω=ω₀. If ζ<0.02 the phase transition between 0 and -180° can be approximated by a vertical line.

$\omega_0/10^\zeta$	ω_0	$\omega_0 \cdot 10^\zeta$
0.63	1.00	1.58

Interactive Demo: Bode Plot of a Pair of Complex Conjugate Zeros

This demonstration shows how a second order zero (complex conjugate roots) expressed as:

$$H(s) = \left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1 = 1 - \left(\frac{\omega}{\omega_0}\right)^2 + j2\zeta\frac{\omega}{\omega_0},$$

is displayed on a Bode plot. You can change ω_0, and ζ. The value of ω_0 is constrained such that $0.1 \leq \omega_0 \leq 10$ rad/second, and $0.05 \leq \zeta \leq 0.99$.

Enter value for ω_o: `1.000` *or click and drag on graph to set ω_0., and use text-box or slider, below, for ζ.*

`0.200` ζ *0.05* ⎯⎯⎯⦿⎯⎯⎯⎯⎯ *0.99*

Magnitude

Asymptotic Magnitude: *The asymptotic magnitude plot starts (at low frequencies) at 0 dB and stays at that level until it gets to ω_0. At that point the gain starts rising with a slope of +40 dB/decade.*

If $\zeta<0.5$ we estimate the peak height as $|H(j\omega_{peak})| \approx \frac{1}{2\zeta}$ (exact height is $|H(j\omega_{peak})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$). We approximate the peak location at $\omega_{peak} \approx \omega_0$ (exact peak location is at $\omega_{peak} = \omega_0\sqrt{1-2\zeta^2}$).
However, if $\zeta \geq 0.5$, the peak is sufficiently small that we don't include it in our plot.

Phase

Since $\zeta<0.5$, we draw a peak. Note how close together the approximate and exact values are for ω_{peak} and $|H(j\omega_{peak})|$.

	ω_{peak}	$|H(j\omega_{peak})|$	$|H(j\omega_{peak})|_{dB}$
Approximate	1.00	0.40	-7.96
Exact	0.96	0.39	-8.14

Asymptotic Phase: *The asymptotic phase plot starts (at low frequencies) at 0° and stays at that level until it gets to $\omega_0/10^\zeta$. At that point the phase starts rising at +90°/decade until it gets to +180°, at which point it becomes constant at +180° for high frequencies. Phase goes through +90° at ω=ω₀. If $\zeta<0.02$ the phase transition between 0 and +180° can be approximated by a vertical line.*

$\omega_0/10^\zeta$	ω_0	$\omega_0 \cdot 10^\zeta$
0.63	1.00	1.58
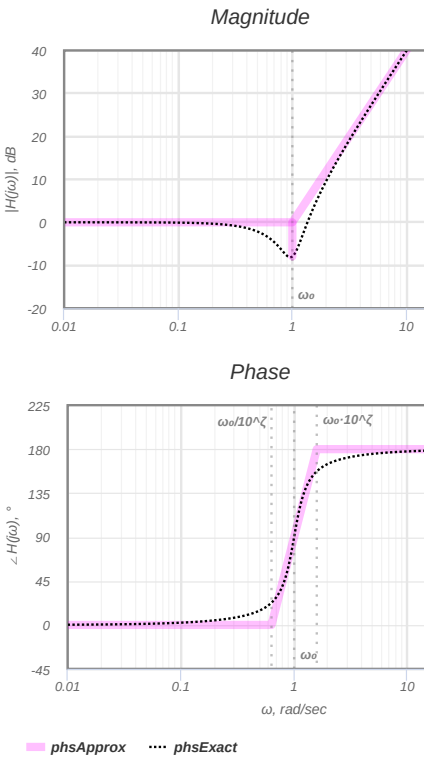
Brief review of page: *This document derived piecewise linear approximations that can be used to draw different elements of a Bode diagram. A synopsis of these rules can be found in a separate document.*

References