APPENDIX

In this appendix we will solve $\arg\min_{\boldsymbol{X}} \lambda \|\boldsymbol{X}\|_* + \frac{1}{2}\|\boldsymbol{Z}-\boldsymbol{X}\|_F^2$. To derive the solution, we need a definition and some properties of inner product on $\mathbb{R}[G]^{N\times M}$.

Definition 1. $\langle \boldsymbol{X}, \boldsymbol{Z}\rangle := \operatorname{tr}(\boldsymbol{X}\boldsymbol{Z}^*)_{K_1K_2\ldots K_D}$, *i.e. the "real part" of the trace of* $\boldsymbol{X}\boldsymbol{Z}^*$.

Proposition 1. $\|\boldsymbol{X}\|_F^2 = \langle \boldsymbol{X}, \boldsymbol{X}\rangle$.

Proof.

$$\begin{aligned}\langle \boldsymbol{X}, \boldsymbol{X}\rangle &= \sum_{n,m} \left(\boldsymbol{X}_{nm}\overline{\boldsymbol{X}}_{nm}\right)_{K_1K_2\ldots K_D}\\ &= \sum_{n,m} |\boldsymbol{X}_{nm}|^2 = \|X\|_F^2\end{aligned}$$

□

Proposition 2. $\langle \boldsymbol{X}, \boldsymbol{X}\rangle = \langle \boldsymbol{\Delta}, \boldsymbol{\Delta}\rangle$, *where* $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{Q}^*$ *is the SVD in AGA.*

Proof.

$$\begin{aligned}\langle \boldsymbol{X}, \boldsymbol{X}\rangle &= \langle \boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{Q}^*, \boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{Q}^*\rangle\\ &= \operatorname{tr}(\boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{Q}^*\boldsymbol{Q}\boldsymbol{\Delta}^*\boldsymbol{P}^*)_{K_1K_2\ldots K_D}\\ &= \operatorname{tr}(\boldsymbol{\Delta}\boldsymbol{\Delta}^*)_{K_1K_2\ldots K_D} = \langle \boldsymbol{\Delta}, \boldsymbol{\Delta}\rangle\end{aligned}$$

□

Lemma 1. $\langle \boldsymbol{X}, \boldsymbol{Z}\rangle \le \langle \boldsymbol{\Delta}, \boldsymbol{\Sigma}\rangle$, *where* $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{Q}^*$ *and* $\boldsymbol{Z} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$ *are the SVD in AGA.*

Proof. We enumerate the components of $\mathcal{F}(\boldsymbol{x})$ and the frontal slices of $\mathcal{F}(\boldsymbol{X})$, and denote the k-th one as $\mathcal{F}(\boldsymbol{x})_k$ and $\mathcal{F}(\boldsymbol{X})_{:,:,k}$. Then,

$$\begin{aligned}\mathcal{F}(\operatorname{tr}(\boldsymbol{X}\boldsymbol{Z}^*))_k &= \operatorname{tr}\left(\mathcal{F}\left(\boldsymbol{X}\boldsymbol{Z}^*\right)_{:,:,k}\right)\\ &= \operatorname{tr}\left(\mathcal{F}\left(\boldsymbol{X}\right)_{:,:,k}\mathcal{F}\left(\boldsymbol{Z}^*\right)_{:,:,k}\right) \ .\end{aligned} \tag{1}$$

We expand and rewrite the inner product in the Fourier domain, using properties of Fourier transform on the second line and (1) on the last line.

$$\begin{aligned}\langle \boldsymbol{X}, \boldsymbol{Z}\rangle &= \operatorname{tr}(\boldsymbol{X}\boldsymbol{Z}^*)_{K_1K_2\ldots K_D}\\ &= \frac{1}{G}\sum_k \Re\left(\mathcal{F}\left(\operatorname{tr}(\boldsymbol{X}\boldsymbol{Z}^*)\right)_k\right)\\ &= \frac{1}{G}\sum_k \Re\left(\operatorname{tr}\left(\mathcal{F}\left(\boldsymbol{X}\right)_{:,:,k}\mathcal{F}\left(\boldsymbol{Z}^*\right)_{:,:,k}\right)\right)\end{aligned} \tag{2}$$

Our last step is to observe that

$$\left|\operatorname{tr}\left(\mathcal{F}\left(\boldsymbol{X}\right)_{:,:,k}\mathcal{F}\left(\boldsymbol{Z}^*\right)_{:,:,k}\right)\right| \le \operatorname{tr}\left(\mathcal{F}\left(\boldsymbol{\Delta}\right)_{:,:,k}\mathcal{F}\left(\boldsymbol{\Sigma}^*\right)_{:,:,k}\right) , \tag{3}$$

which follows from von Neumann's trace inequality and the definition of SVD over AGA. The lemma is proved by combining (2) and (3), using the facts that the real part is always not greater than the absolute value of a complex number and that $\mathcal{F}(\boldsymbol{\Delta}), \mathcal{F}(\boldsymbol{\Sigma})$ have non-negative entries. □

Lemma 2. $\|\boldsymbol{X}-\boldsymbol{Z}\|_F^2 \ge \|\boldsymbol{\Delta}-\boldsymbol{\Sigma}\|_F^2$, *where* $\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{\Delta}, \boldsymbol{\Sigma}$ *are as in Lemma 1.*

Proof.

$$\begin{aligned}\|\boldsymbol{X}-\boldsymbol{Z}\|_F^2 &= \langle \boldsymbol{X}-\boldsymbol{Z}, \boldsymbol{X}-\boldsymbol{Z}\rangle\\ &= \langle \boldsymbol{X}, \boldsymbol{X}\rangle - 2\langle \boldsymbol{X}, \boldsymbol{Z}\rangle + \langle \boldsymbol{Z}, \boldsymbol{Z}\rangle\\ &\ge \langle \boldsymbol{\Delta}, \boldsymbol{\Delta}\rangle - 2\langle \boldsymbol{\Delta}, \boldsymbol{\Sigma}\rangle + \langle \boldsymbol{\Sigma}, \boldsymbol{\Sigma}\rangle\\ &= \|\boldsymbol{\Delta}-\boldsymbol{\Sigma}\|_F^2\end{aligned}$$

□

Lemma 3. $\left(1-\frac{\lambda}{|\boldsymbol{z}|}\right)_+\boldsymbol{z} = \arg\min_{\boldsymbol{x}} \lambda|\boldsymbol{x}| + \frac{1}{2}|\boldsymbol{x}-\boldsymbol{z}|^2$, *where* $(x)_+ = \max(x, 0)$.

Proof. Let $f(\boldsymbol{x}) = \lambda|\boldsymbol{x}| + \frac{1}{2}|\boldsymbol{x}-\boldsymbol{z}|^2$. We think of it as a multivariate function with $N = K_1K_2\ldots K_D$ variables, and we find out the minimum by looking for zeros of its partial derivatives $\frac{\partial f}{\partial x_i} = \lambda\frac{x_i}{|x|} + x_i - z_i$ for all $i \in [1, N]$. This leads to the equation

$$\left(1+\frac{\lambda}{|x|}\right)x = z \ . \tag{4}$$

If $|z| > \lambda$, we have $|x| = |z| - \lambda$ by taking the absolute value on both sides of (4). Then,

$$x = \left(1-\frac{\lambda}{|z|}\right)z.$$

If $|z| \le \lambda$, we prove that f attains its minimum at $x = 0$, i.e.

$$f(0) = \frac{1}{2}|z|^2 \le \lambda|x| + \frac{1}{2}|x-z|^2 = f(x), \text{ for all } x \ .$$

The inequality holds because

$$\begin{aligned}|z|^2 - 2\lambda|x| &\le |z|^2 - 2|z||x|\\ &\le (|x|-|z|)^2\\ &\le |x-z|^2 ,\end{aligned}$$

where the last line follows from the triangle inequality. Combining the result of both cases, we conclude that f is minimized at $\left(1-\frac{\lambda}{|\boldsymbol{z}|}\right)_+\boldsymbol{z}$. □

Theorem 1. *The solution to*

$$\arg\min_{\boldsymbol{X}} \lambda\|\boldsymbol{X}\|_* + \frac{1}{2}\|\boldsymbol{Z}-\boldsymbol{X}\|_F^2 \tag{5}$$

is

$$\boldsymbol{X} = \boldsymbol{U}\cdot\mathcal{S}(\boldsymbol{\Sigma})\cdot\boldsymbol{V}^* , \tag{6}$$

where $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$ *is the SVD of the matrix* $\boldsymbol{Z}$ *over AGA, and* $\mathcal{S}(\boldsymbol{\Sigma})_{nm} = \left(1-\frac{\lambda}{|\boldsymbol{\Sigma}_{nm}|}\right)_+\boldsymbol{\Sigma}_{nm}$ *is the soft thresholding operator,* $(x)_+ = \max(x, 0)$.

Proof. Let $\boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{Q}^*$ be the SVD of $\boldsymbol{X}$.

$$\begin{aligned}\min_{\boldsymbol{X}} \lambda\|\boldsymbol{X}\|_* + \frac{1}{2}\|\boldsymbol{X}-\boldsymbol{Z}\|_F^2 &\ge \min_{\boldsymbol{\Delta}} \lambda\|\boldsymbol{\Delta}\|_* + \frac{1}{2}\|\boldsymbol{\Delta}-\boldsymbol{\Sigma}\|_F^2\\ &= \min_{\boldsymbol{\Delta}} \sum_i \lambda|\boldsymbol{\delta}_i| + \frac{1}{2}|\boldsymbol{\delta}_i-\boldsymbol{\sigma}_i|^2\\ &= \sum_i \min_{\boldsymbol{\delta}_i} \lambda|\boldsymbol{\delta}_i| + \frac{1}{2}|\boldsymbol{\delta}_i-\boldsymbol{\sigma}_i|^2\end{aligned}$$

When $\boldsymbol{X} = \boldsymbol{U}\cdot\mathcal{S}(\boldsymbol{\Sigma})\cdot\boldsymbol{V}^*$, the inequality above becomes equality and the minimum is achieved for all i. This proves the theorem. □

THEOREM 2. *The solution to*

$$\arg\min_{\boldsymbol{X}} \|\boldsymbol{Z} - \boldsymbol{X}\|_F^2\,, s.t.\ \pi(\boldsymbol{X}) = 0 \tag{7}$$

is

$$\boldsymbol{X} = \boldsymbol{Z} - \pi\left(\boldsymbol{Z}\right)\,. \tag{8}$$

PROOF. This should be easy. ☺