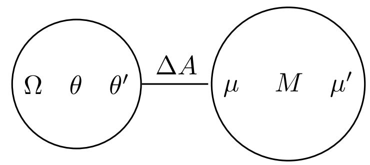
Sum-Product notes

November 5, 2012



$\mu = \Delta A \Theta$ Given any point μ there exists a θ. Given any point μ' there is a θ'.
When is the map 1-1? When the exponential family is not minimal. Variational methods are about moving from one representation to another.
When it is 1-1, $(\Delta A)^{-1}$ exists.
$A\theta = \sup_{\mu \in M} \{\langle \theta, \mu \rangle - A^*(\mu)\}$
This is discrete over complete sufficient statistics.
$P_\theta(x) = \exp\{\sum_s \theta_s x_s + \sum_(s, t \in E)(x_s x_t)$
$\theta_s = \sum_{j,i} \theta_{s_{ij}} \perp_{sj} (x, s)$

Pseudo-marginals

$$\mu = \begin{cases} ll\mu_s(x_s) & \text{Nodewise} \\ \mu_{st} & (x_s, x_t\text{–Pairwise marginals of some distribution } P \end{cases}$$

μ_{st} is only defined over edges of G.

$$\mathbb{L}_G = \begin{cases} llT_s(x_s) & s \in V \\ T_{st} & (s,t) \in E \end{cases}$$

Norm $\sum_{x_s} T_s x_s = 1$
Marginals $\sum_{x_s} T_{st}(x_s x_t) = T_t(x_t)$

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Prop$(I)\mathbb{L}G \supseteq M(G)$
If $\mu \in M(G)$ it satisfies norm and marginal constraints $\rightarrow \mu \in \mathbb{L}G$
There are exponential constraints for μ and linear ones for $\mathbb{L}(G)$

$$\begin{aligned} P(x_1, \ldots, x_n)\mu_i(x_i) &= P(x_i) \\ \mu_{i,j}(x_i x_j) &= P(x_i, x_j) \end{aligned}$$

If graph G is a tree T, then $\mathbb{L}(T) = \mu(T)$ because nodewise and pairwise marginals are all we need.

Proof
$L(G) \supseteq \mu_G$ (Shown). We want to show that $\mathbb{L}(T) \subseteq \mu_T$. We will take a $\mu \in \mathbb{L}(T)$ such that it satisfies marginal constaints and show that it is consistent with regard to a joint distribution.

$\mu_s(x_s), s \in V$
$\mu_{st}(x_s)(x_t), (s,t) \in T$
$P_\mu(x) = \Pi_{s\in V}\mu_s(x_s)\Pi_{s,t\in E}\frac{\mu_{st}(x_s x_t)}{\mu_s(x_s)\mu_t(x_t)}$ such that μ is consistent with regard to P_μ.

We can compute the Bethe-entropy of the tree-structure distribution. Suppose we have a tree-structured distribution with mean parameters μ. Then we can write down the distribution in terms of μ.

$$\begin{aligned} A^*_(\mu) &= \text{entropy}(P_\mu) \\ &= \mathbb{E}_{pm}|\log P_\mu(x)| \\ &= \mathbb{E}_{pm}|\textstyle\sum_{s\in V}\log\mu_s(x_s) + \sum_{s,t\in E}\log\frac{\mu_{st}(x_s,x_t)}{\mu_s\mu_t}| \\ &= -\textstyle\sum_s \mathbb{H}_s < \mu_s right > - \sum_{s,t\in E} I_{s,t}(\mu_{s,t}) \\ \mathbb{H}_{Bethe}(\mu) &= \textstyle\sum_{s\in V}\mathbb{H}_s(\mu_s) - \sum_{s,t\in E} I_{s,t}(\mu+s,t) \\ &= -A^*(\mu)\text{Exactly when } G \text{ is a tree} \end{aligned}$$

Computing entropy in the general case is NP-hard, but we can approximate it when G is not a tree using the variaitonal principle.

We can aproximate to $A(\theta)$ as so: $A_{Bethe}(\theta) = \sup_{\mu\in\mathbb{L}(G)}\{<\theta,\mu> + \mathbb{H}_{Bethe}(\mu) - I\}$

We get sum-product from the fixed point update driven solution of optimization problem I.

$\mu_{st}(x_t) = \Psi_{st}(x_s,x_t)\Pi_{v\in N(S)_t}\mu_{v_s)}(x_s)$
This is a nonconvex problem. If we do coordinate descent on the dual, we can get to a *local* minimum. Regular sum-product often oscillates between very large $(O(v^2))$ parameters.

Reparamaterization

Suppose we solve sum-product to a local minimum τ^* (a local minimum of J). Then $P_{\tau^*}(x) = I\Pi_{s\in V}T^*_s(x_s)\Pi_{s,t\in E}\frac{T^*_{st}(x_s,x_t)}{T^*_s(x_s)T^*_t(x_t)}$

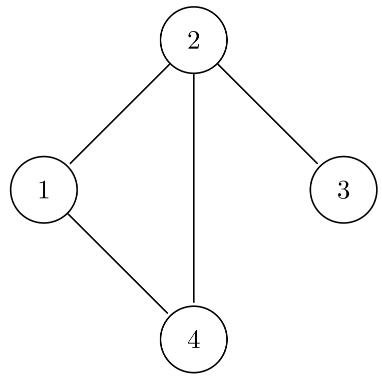
We know that if G is a tree and $T^* = \mu^*$, then $P_{T^*} = P_\theta$ and $P_{T^*}(x) = P_\theta(x)$

So it always holds that $P_\theta(x) = P_{T^*}(x)$

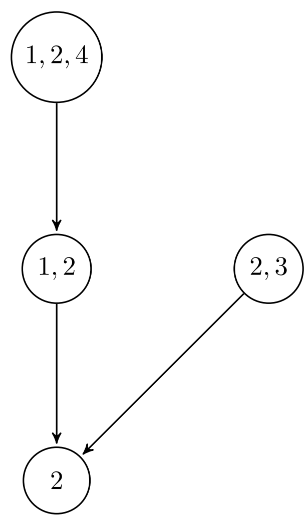
Hypergraphs

Define a hypergraph $G \in (V, E)$ with nodes $v \in V$ and hyperedges $e \in E$ such that $e \subseteq E$. A normal graph is a hypergraph with 2-node edges.

We can write the following graph as $E = (1,2)(2,3)(1,4)(2,4)$ or we can write $E = (1,2,4),(2,3)$



Suppose we have the hypergraph with edges $E = (1,2,4),(2,3),(1,2),(2)$. Then we have a ńodeŕfor each superedge and we connect $h \to g$ if $g \in h$. These form a partially ordered set with regard to set inclusion.

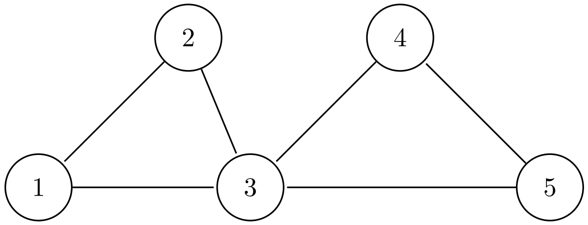


A general graphical model has subsets over cliques. The hypergraph edges correspond to cliques. We can do inference on these cliques.

Given a set of sufficient statistics $\mathbb{L}(G) = T_n(x_n)$, such that $\sum_n T_n(x_n) = 1$ we have g, h such that $f \subseteq h$:

$\sum_{x_h} |x_g T_n(x_n) = T_g(x_g)|$

For the following graph



(1, 2, 3) and (3, 4, 5) must be consistent.

$H_{hypertree}(\mu) = \sum_{n \in E} c(h) H_h(\mu_k)$ where $c(h)$ are overcounting numbers.
$\sup_{\mu \in \mathbb{L}_G} \{< \theta, \mu > + H_{HT}(\mu)\}$

Message passing here is generalized beleif propagation.