University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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HW4 Due to November 4 11:59 PM

Question 1

Problem 14.1

Derive a generalization of Feynman's formula,

$$\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)}\frac{1}{(n-1)!}\int dF_n \frac{\prod_i x_i^{\alpha_i-1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}. \tag{1}$$

$$\int dF_n = (n-1)!\int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \delta\left(\sum_{i=1}^n x_i - 1\right). \tag{2}$$

Hint: start with

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty dt\, t^{\alpha-1}e^{-tA}, \tag{3}$$

which defines the gamma function. Put an index on A, α and t, and take the product. Then multiply on the right-hand side by

$$1 = \int_0^\infty ds\delta(s - \sum_i t_i). \tag{4}$$

Make the change of variables $t_i = sx_i$ and carry out the integral over s.

Answer

By definesion of the gamma function, we have

$$\frac{1}{A_i^{\alpha_i}} = \frac{1}{\Gamma(\alpha_i)}\int_0^\infty dt_i\, t_i^{\alpha_i-1}e^{-t_iA_i}. \tag{5}$$

Then we have

$$\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_n^{\alpha_n}} = \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)}\int_0^\infty dt_i\, t_i^{\alpha_i-1}e^{-t_iA_i} \tag{6}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)}\int_0^\infty dt_1\int_0^\infty dt_2\cdots\int_0^\infty dt_n \prod_{i=1}^n t_i^{\alpha_i-1}e^{-t_iA_i} \tag{7}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)}\int_0^\infty dt_1\int_0^\infty dt_2\cdots\int_0^\infty dt_n \prod_{i=1}^n \left(t_i^{\alpha_i-1}e^{-t_iA_i}\right)\int_0^\infty ds\delta(s - \sum_{i=1}^n t_i). \tag{8}$$

We make the change of variables $t_i = sx_i$, then we have

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n \big((sx_i)^{\alpha_i - 1} e^{-sx_i A_i}\big) \delta(s - s\sum_{i=1}^n x_i) s \tag{9}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds\, s^{\sum_{i=1}^n \alpha_i - 1} e^{-s\sum_{i=1}^n x_i A_i} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \tag{10}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \int_0^\infty ds\, s^{\sum_{i=1}^n \alpha_i - 1} e^{-s\sum_{i=1}^n x_i A_i} \tag{11}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{(\sum_{i=1}^n x_i A_i)^{\sum_{i=1}^n \alpha_i}} \tag{12}$$

$$= \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_{i=1}^n x_i^{\alpha_i - 1}}{(\sum_{i=1}^n x_i A_i)^{\sum_{i=1}^n \alpha_i}}. \tag{13}$$

Hence proved the formula. $\square$

Question 2

Problem 14.2
Verify eq. (14.23).

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{14}$$

Answer

We start with the Gaussian integral in d dimensions,

$$I_d = \int d^d x \, e^{-\mathbf{x}^2}. \tag{15}$$

In catrtesian coordinates, we have

$$I_d = \left(\int_{-\infty}^{\infty} dx \, e^{-x^2} \right)^d = (\sqrt{\pi})^d = \pi^{d/2}. \tag{16}$$

In spherical coordinates, we have

$$I_d = \int_0^{\infty} dr \, r^{d-1} e^{-r^2} \int d\Omega_d = \Omega_d \int_0^{\infty} dr \, r^{d-1} e^{-r^2}. \tag{17}$$

Make the change of variable $t = r^2$, then we have

$$I_d = \frac{\Omega_d}{2} \int_0^{\infty} dt \, t^{d/2-1} e^{-t} = \frac{\Omega_d}{2} \Gamma(d/2), \tag{18}$$

where we have used the definition of the gamma function

$$\Gamma(\alpha) = \int_0^{\infty} dt \, t^{\alpha-1} e^{-t}. \tag{19}$$

Equating the two expressions for I_d, we have

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{20}$$

□

Question 3

Problem 14.5
Compute the $O(\lambda)$ correction ot the propagator in φ^4 theory (see problem 9.2) in $d = 4 - \epsilon$ spacetime dimensions, and compute the $O(\lambda)$ terms in A and B.

Answer

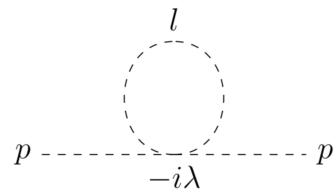

Figure 1: The Feynman diagram with the ϕ^4 propagator for 1-loop correction at $O(\lambda)$.

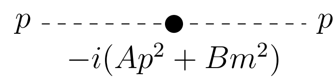

Figure 2: The Feynman diagram with the ϕ^4 propagator for 1-loop counter term at $O(\lambda)$.

First, we write down the Lagrangian for the φ^4 theory,

$$\mathcal{L} = \mathcal{L}_{,} + \mathcal{L}_I, \tag{21}$$

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2, \tag{22}$$

$$\mathcal{L}_I = -\frac{Z_\lambda}{4!}\lambda\varphi^4 + \mathcal{L}_{ct}, \tag{23}$$

$$\mathcal{L}_{ct} = -\frac{1}{2}(Z_\varphi - 1)\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2. \tag{24}$$

For the $O(\lambda)$ correction to the propagator, the Feynman diagram is shown in Figure 1. The corresponding amplitude is given by

$$i\Sigma(p) = \frac{1}{2}(-i\lambda)\frac{1}{i}\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2 + m^2 - i\epsilon} - i(Ap^2 + Bm^2) + O(\lambda^2), \tag{25}$$

where the factor $\frac{1}{2}$ is the symmetry factor for this diagram. Consider the wick rotation to Euclidean space ($d^4l \to id^4l_E$ and $l^2 \to l_E^2$), we have

$$\Sigma(p) = \frac{-\lambda}{2}\int \frac{d^dl_E}{(2\pi)^d}\frac{1}{l_E^2 + m^2} - (Ap^2 + Bm^2) + O(\lambda^2), \tag{26}$$

where the $m = m - i\epsilon$ prescription is implied. Using the formula derived in Problem 14.1, we have

$$\begin{aligned}
\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 + m^2} &= \int \frac{d^d l_E}{(2\pi)^d} \int_0^\infty dx\, e^{-x(l_E^2+m^2)} && (27)\\
&= \int_0^\infty dx\, e^{-xm^2} \int \frac{d^d l_E}{(2\pi)^d} e^{-xl_E^2} && (28)\\
&= \int_0^\infty dx\, e^{-xm^2} \frac{1}{(2\pi)^d} \left(\frac{\pi}{x}\right)^{d/2} && (29)\\
&= \frac{1}{(4\pi)^{d/2}} \int_0^\infty dx\, x^{-d/2} e^{-xm^2} && (30)\\
&= \frac{1}{(4\pi)^{d/2}} (m^2)^{d/2-1} \Gamma(1 - d/2). && (31)
\end{aligned}$$

Substituting $d = 4 - \epsilon$, we have

$$\begin{aligned}
\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 + m^2} &= \frac{1}{(4\pi)^{2-\epsilon/2}} (m^2)^{1-\epsilon/2} \Gamma(-1 + \epsilon/2) && (32)\\
&= \frac{m^2}{16\pi^2} \left(\frac{4\pi}{m^2}\right)^{\epsilon/2} \Gamma(-1 + \epsilon/2). && (33)
\end{aligned}$$

Also, when we consider dimension regularization, we need to introduce a mass scale μ to keep the coupling constant λ dimensionless. Thus, we have

$$\lambda \to \lambda \widetilde{\mu}^\epsilon. \tag{34}$$

Therefore, we have

$$\begin{aligned}
\Sigma(p) &= \frac{-\lambda \widetilde{\mu}^\epsilon}{2} \frac{m^2}{16\pi^2} \left(\frac{4\pi}{m^2}\right)^{\epsilon/2} \Gamma(-1 + \epsilon/2) - (Ap^2 + Bm^2) + O(\lambda^2) && (35)\\
&= \frac{-\lambda m^2}{32\pi^2} \left(\frac{4\pi \widetilde{\mu}^2}{m^2}\right)^{\epsilon/2} \Gamma(-1 + \epsilon/2) - (Ap^2 + Bm^2) + O(\lambda^2). && (36)
\end{aligned}$$

Using the expansion of the gamma function around the pole at -1 and the $4\pi\widetilde{\mu}^2/m^2$ around $\epsilon = 0$,

$$\begin{aligned}
\Gamma(-1 + \epsilon/2) &= -\frac{2}{\epsilon} - 1 + \gamma + O(\epsilon), && (37)\\
\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\epsilon/2} &= 1 + \frac{\epsilon}{2} \ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right) + O(\epsilon^2), && (38)
\end{aligned}$$

we have

$$\Sigma(p) = \frac{-\lambda m^2}{32\pi^2}\left[-\frac{2}{\epsilon}-1+\gamma\right]\left[1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)\right]-(Ap^2+Bm^2)+O(\lambda^2) \tag{39}$$

$$= \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma+1+\ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)\right]-(Ap^2+Bm^2)+O(\lambda^2) \tag{40}$$

$$= \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon}+1+\ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right]-(Ap^2+Bm^2)+O(\lambda^2) \tag{41}$$

To satisfy the renormalization conditions,

$$\left.\frac{d}{dp^2}\Sigma(p)\right|_{p^2=-m^2} = 0, \tag{42}$$

$$\Sigma(p^2=-m^2) = 0, \tag{43}$$

we have

$$A = 0+O(\lambda^2), \tag{44}$$

$$B = \frac{\lambda}{32\pi^2}\left[\frac{2}{\epsilon}+1+\ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right]+O(\lambda^2). \tag{45}$$

In summary, we have

$$\Sigma(p) = \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon}+1+\ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right]-(Ap^2+Bm^2)+O(\lambda^2) = 0+O(\lambda^2), \tag{46}$$

$$A = 0+O(\lambda^2), \tag{47}$$

$$B = \frac{\lambda}{32\pi^2}\left[\frac{2}{\epsilon}+1+\ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right]+O(\lambda^2). \tag{48}$$

□

Question 4

Problem 16.1

Compute the $O(\lambda^2)$ correction in $\mathbf{V}_4$ in φ^4 theory in $d = 4 - \epsilon$ spacetime dimensions. Take $\mathbf{V}_4 = \lambda$ when all four external momenta are on shell, and $s = 4m^2$. What is the $O(\lambda)$ contribution to C?

Answer

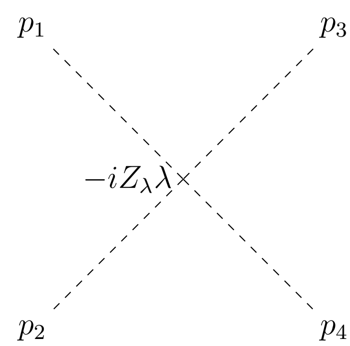

Figure 3: The Feynman diagram with the ϕ^4 vertex for tree level at $O(\lambda)$.

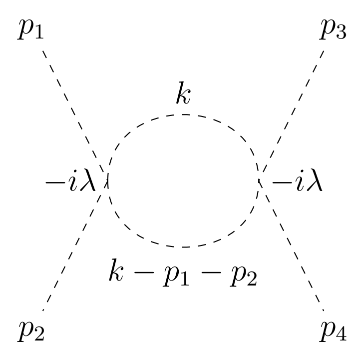

Figure 4: The Feynman diagram with the ϕ^4 vertex for 1-loop correction at $O(\lambda^2)$ in the s-channel.

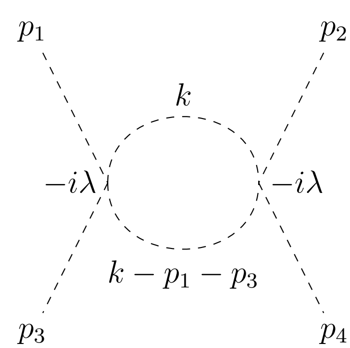

Figure 5: The Feynman diagram with the ϕ^4 vertex for 1-loop correction at $O(\lambda^2)$ in the t-channel.

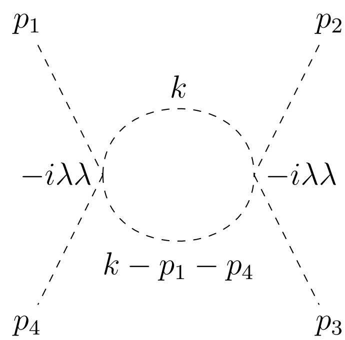

Figure 6: The Feynman diagram with the ϕ^4 vertex for 1-loop correction at $O(\lambda^2)$ in the u-channel.

At $O(\lambda^2)$, there are three Feynman diagrams contributing to the 1-loop correction to the ϕ^4 vertex, as shown in Figure 4, 5 and 6. The tree-level diagram is shown in Figure 3. The corresponding amplitude is given by

$$
\begin{aligned}
i\mathbf{V}_4 =& iV_{tree} + iV_4^{(s)} + iV_4^{(t)} + iV_4^{(u)} && (49)\\
=& -iZ_\lambda\lambda + \left(\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_2)^2+m^2} && (50)\\
&+\left(\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_3)^2+m^2} && (51)\\
&+\left(\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_4)^2+m^2}. && (52)
\end{aligned}
$$

Using Feynman parameterization, we have

$$
\begin{aligned}
&\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_i-p_j)^2+m^2} && (53)\\
=&\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-2xk\cdot(p_i+p_j)+x(p_i+p_j)^2+m^2]^2} && (54)\\
=&\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(k-x(p_i+p_j))^2+x(1-x)(p_i+p_j)^2+m^2]^2} && (55)\\
=&\int_0^1 dx\int\frac{d^4q}{(2\pi)^4}\frac{1}{[q^2+D_{ij}]^2} && (56)\\
=&i\int_0^1 dx\int\frac{d^4q_E}{(2\pi)^4}\frac{1}{[q_E^2+D_{ij}]^2}, && (57)
\end{aligned}
$$

where $D_{ij}=x(1-x)(p_i+p_j)^2+m^2$ and we have performed the wick rotation to Euclidean space ($d^4k\to id^4k_E$ and $k^2\to k_E^2$). Using the eq. (14.27) in textbook, we have

$$
\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{(\bar{q}^2)^a}{[\bar{q}^2+D_{ij}]^b}=\frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)}D_{ij}^{-(b-a-d/2)}. \qquad (58)
$$

Thus, with $a = 0$, $b = 2$ and $d = 4 - \epsilon$, we have

$$\int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{[q_E^2 + D_{ij}]^2} = \frac{\Gamma(2 - 0 - (4-\epsilon)/2)\Gamma(0 + (4-\epsilon)/2)}{(4\pi)^{(4-\epsilon)/2}\Gamma(2)\Gamma((4-\epsilon)/2)} D_{ij}^{-(2-0-(4-\epsilon)/2)} \tag{59}$$

$$= \frac{\Gamma(\epsilon/2)\Gamma(2-\epsilon/2)}{(4\pi)^{2-\epsilon/2} \cdot 1 \cdot \Gamma(2-\epsilon/2)} D_{ij}^{-\epsilon/2} \tag{60}$$

$$= \frac{1}{(4\pi)^{2-\epsilon/2}} \Gamma(\epsilon/2) D_{ij}^{-\epsilon/2} \tag{61}$$

$$= \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{ij}}\right)^{\epsilon/2} \Gamma(\epsilon/2). \tag{62}$$

Besides, when we consider dimension regularization, we need to introduce a mass scale μ to keep the coupling constant λ dimensionless. Thus, we have

$$\lambda \to \lambda \widetilde{\mu}^{\epsilon}. \tag{63}$$

Therefore, we have

$$i\mathbf{V}_4 = -iZ_\lambda \lambda + \left(\frac{1}{2}\right)(-i\widetilde{\mu}^{\epsilon}\lambda)^2 \left(\frac{1}{i}\right)^2 i \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{12}}\right)^{\epsilon/2} \Gamma(\epsilon/2) \tag{64}$$

$$+ \left(\frac{1}{2}\right)(-i\widetilde{\mu}^{\epsilon}\lambda)^2 \left(\frac{1}{i}\right)^2 i \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{13}}\right)^{\epsilon/2} \Gamma(\epsilon/2) \tag{65}$$

$$+ \left(\frac{1}{2}\right)(-i\widetilde{\mu}^{\epsilon}\lambda)^2 \left(\frac{1}{i}\right)^2 i \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{14}}\right)^{\epsilon/2} \Gamma(\epsilon/2) \tag{66}$$

$$= -iZ_\lambda \lambda - i(\widetilde{\mu}^{\epsilon}\lambda)^2 \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi}{D_{12}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{13}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{14}}\right)^{\epsilon/2}\right]. \tag{67}$$

In other words, we have

$$\mathbf{V}_4 = -Z_\lambda \lambda - (\widetilde{\mu}^{\epsilon}\lambda)^2 \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi}{D_{12}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{13}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{14}}\right)^{\epsilon/2}\right]. \tag{68}$$

To satisfy the renormalization condition,

$$\mathbf{V}_4 = -\lambda \quad \text{at} \quad s = 4m^2, \tag{69}$$

we have

$$Z_\lambda = 1 + \lambda \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi\widetilde{\mu}^2}{D_{12}}\right)^{\epsilon/2} + \left(\frac{4\pi\widetilde{\mu}^2}{D_{13}}\right)^{\epsilon/2} + \left(\frac{4\pi\widetilde{\mu}^2}{D_{14}}\right)^{\epsilon/2}\right] + O(\lambda^2). \tag{70}$$

For $s = 4m^2$ and $t = u = 0$ (by assuming all $p_i = (m, \mathbf{0})$), we have

$$\begin{aligned}
D_{12} =& x(1-x)(-s) + m^2 = (1-4x(1-x))m^2 = (1-2x)^2 m^2, &(71)\\
D_{13} =& x(1-x)(-t) + m^2 = m^2, &(72)\\
D_{14} =& x(1-x)(-u) + m^2 = m^2. &(73)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
Z_\lambda =& 1 + \lambda \frac{1}{2(4\pi)^2}\Gamma(\epsilon/2)\int_0^1 dx \left[\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2}\right)^{\epsilon/2} + 2\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\epsilon/2}\right] + O(\lambda^2) &(74)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\Gamma(\epsilon/2)\left[\int_0^1 dx \left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2}\right)^{\epsilon/2} + 2\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\epsilon/2}\right] + O(\lambda^2). &(75)
\end{aligned}$$

For small ϵ, we have

$$\begin{aligned}
\Gamma(\epsilon/2) =& \frac{2}{\epsilon} - \gamma + O(\epsilon), &(76)\\
\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2}\right)^{\epsilon/2} =& 1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2}\right) + O(\epsilon^2), &(77)\\
\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\epsilon/2} =& 1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right) + O(\epsilon^2). &(78)
\end{aligned}$$

Substituting these expansions into the expression of Z_λ, we have

$$\begin{aligned}
Z_\lambda =& 1 + \lambda \frac{1}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma\right)\left[\int_0^1 dx \left(1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2}\right)\right) + 2\left(1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)\right)\right] + O(\lambda^2) &(79)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\left[\int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2}\right)\right) + 2\left(\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)\right)\right] + O(\lambda^2) &(80)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\left[\int_0^1 dx \left(\frac{2}{\epsilon} + \ln\left(\frac{4\pi\widetilde{\mu}^2/e^\gamma}{(1-2x)^2 m^2}\right)\right) + 2\left(\frac{2}{\epsilon} + \ln\left(\frac{4\pi\widetilde{\mu}^2/e^\gamma}{m^2}\right)\right)\right] + O(\lambda^2) &(81)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\left[\int_0^1 dx \left(\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{(1-2x)^2 m^2}\right)\right) + 2\left(\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right)\right] + O(\lambda^2) &(82)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\left[\int_0^1 dx \left(\frac{2}{\epsilon} + \ln\left(\frac{\mu^2/m^2}{(1-2x)^2}\right)\right) + 2\left(\frac{2}{\epsilon} + 2\ln\left(\frac{\mu}{m}\right)\right)\right] + O(\lambda^2) &(83)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\left[\left(\frac{2}{\epsilon} + 2 + 2\ln(\mu/m)\right) + 2\left(\frac{2}{\epsilon} + 2\ln\left(\frac{\mu}{m}\right)\right)\right] + O(\lambda^2) &(84)\\
=& 1 + \lambda \frac{1}{2(4\pi)^2}\left(\frac{6}{\epsilon} + 2 + 6\ln(\mu/m)\right) + O(\lambda^2) &(85)\\
=& 1 + \frac{3\lambda}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{3} + \ln(\mu/m)\right) + O(\lambda^2). &(86)
\end{aligned}$$

In summary, we have

$$
\begin{aligned}
Z_\lambda - 1 =& \frac{3\lambda}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{3} + \ln(\mu/m)\right) + O(\lambda^2), && (87)\\
i\mathbf{V}_4 =& - i\lambda + i(Z_\lambda - 1)\lambda + O(\lambda^3) && (88)\\
=& - i\lambda + \frac{3i\lambda^2}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{3} + \ln(\mu/m)\right) + O(\lambda^3). && (89)
\end{aligned}
$$

$\square$