University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

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Assignment 1 due on Wednesday September 10th at 5PM

Question 1

In lecture we found that $\Lambda^0{}_0 = \gamma$ and $\Lambda^i{}_0 = \gamma v_i$ for boosts. The other components should be of the form $\Lambda^i{}_j = a(v)\delta_{ij} + b(v)v_i v_j$ and $\Lambda^0{}_j = c(v)v_j$, where a, b, and c can be functions of the speed v. Determine these functions from the constraint $\eta_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta = \eta_{\gamma\delta}$.

Answer

We can use tensor notation to reduce the work. From the constraint

$$\eta_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta = \eta_{\gamma\delta}, \tag{1}$$

and the relation $\gamma = \frac{1}{\sqrt{1-v^2}}$ (or $\gamma^2 - \gamma^2 v^2 = \gamma^2(1-v^2) = 1$), we can consider different γ, δ, meaning that

- $(\gamma, \delta) = (0, 0)$:

$$-(\Lambda^0{}_0)^2 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 = -\gamma^2 + \gamma^2\sum_{i=1}^{3}(v_i)^2 = -\gamma^2 + \gamma^2 v^2 = -1 = \eta_{00} \tag{2}$$

- $(\gamma, \delta) = (0, i)$:

$$-\Lambda^0{}_0\Lambda^0{}_i + \sum_{k=1}^{3}\Lambda^k{}_0\Lambda^k{}_i \tag{3}$$

$$= -\gamma c v_i + \sum_{k=1}^{3}\gamma v_k(a\delta_{ki} + b v_k v_i) \tag{4}$$

$$= -\gamma c v_i + \gamma a v_i + \gamma b v^2 v_i = \gamma v_i(-c + a + bv^2) = \eta_{0i} = 0 \tag{5}$$

  Then we have $c = a + bv^2$.

- $(\gamma, \delta) = (i, 0)$

$$-\Lambda^0{}_i\Lambda^0{}_0 + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_0 \tag{6}$$

$$= \gamma v_i(-c + a + bv^2) \tag{7}$$

- $(\gamma, \delta) = (i, i)$

$$
\begin{aligned}
& -\Lambda^0{}_i\Lambda^0{}_i + \sum_{k=1}^{3} \Lambda^k{}_i\Lambda^k{}_i && (8)\\
=& -c^2v_i^2 + \sum_{k=1}^{3}(a\delta_{ki} + bv_kv_i)^2 && (9)\\
=& -c^2v_i^2 + \sum_{k=1}^{3}(a^2\delta_{ki} + 2ab\delta_{ki}v_kv_i + b^2v_i^2v_k^2) && (10)\\
=& -c^2v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 = \eta_{ii} = 1 && (11)
\end{aligned}
$$

- $(\gamma, \delta) = (i, j), i \neq j$

$$
\begin{aligned}
& -\Lambda^0{}_i\Lambda^0{}_j + \sum_{k=1}^{3} \Lambda^k{}_i\Lambda^k{}_j && (12)\\
=& -c^2v_iv_j + \sum_{k=1}^{3}(a\delta_{ki} + bv_kv_i)(a\delta_{kj} + bv_kv_j) && (13)\\
=& -c^2v_iv_j + \sum_{k=1}^{3}(a^2\delta_{ki}\delta_{kj} + ab(\delta_{ki}v_kv_j + \delta_{kj}v_kv_i) + b^2v_k^2v_iv_j) && (14)\\
=& -c^2v_iv_j + 2abv_iv_j + b^2v^2v_iv_j = \eta_{ij} = 0 && (15)
\end{aligned}
$$

Then we have $c^2 = b^2v^2 + 2ab$

Combining the above information, we can have

$$
\begin{aligned}
& c = a + bv^2 && (16)\\
& c^2 = a^2 + b^2v^4 + 2abv^2 = b^2v^2 + 2ab. && (17)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
& -c^2v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 = 1 && (18)\\
\rightarrow 1 &= -(b^2v^2 + 2ab)v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 && (19)\\
&= a^2 && (20)
\end{aligned}
$$

Hence, from Eq. 17 and Eq. 20, we have

$$
\begin{aligned}
& c^2 = 1 + b^2v^4 + 2abv^2 = b^2v^2 + 2ab && (21)\\
\rightarrow & a = \pm 1 = \frac{1 + b^2v^4 - b^2v^2}{2b(1 - v^2)} = \frac{1 + b^2v^2(v^2 - 1)}{2b(1 - v^2)} = \frac{\gamma^2 - b^2v^2}{2b} && (22)\\
\rightarrow & b^2v^2 \pm 2b - \gamma^2 = 0 && (23)
\end{aligned}
$$

For $a = +1$, we have

$$b = \frac{-1 \pm \sqrt{1 + \gamma^2 v^2}}{v^2} = \frac{-1 \pm \gamma}{v^2}, \tag{24}$$

and for $a = -1$, we have

$$b = \frac{1 \pm \gamma}{v^2}. \tag{25}$$

Hence, we choose $a = 1$ and $b = \frac{-1+\gamma}{v^2}$ by convention and have

$$c = a + bv^2 = 1 + (-1 + \gamma) = +\gamma \tag{26}$$

$$\tag{27}$$

Finally, we derive

$$\Lambda^i{}_j = a(v)\delta_{ij} + b(v)v_i v_j = \delta_{ij} + \frac{\gamma - 1}{v^2} v_i v_j \tag{28}$$

$$\Lambda^0{}_j = c(v)v_j = \gamma v_j, \tag{29}$$

which are mentioned in the Week 1 lecture. $\square$

Question 2

A rod of length L lies at rest along the x-axis in frame $\mathcal{O}$. An observer located along a perpendicular axis in frame $\mathcal{O}'$ sees frame $\mathcal{O}$ moving with speed v in the positive x direction. The observer in frame $\mathcal{O}'$, which may be an eye or a camera, looks in the direction of the rod. When the center of the rod is at the distance of closest approach D to the observer, lights at the ends of the rod send out flashes of light. When $v \ll c$, the opening angle θ between the light flashes would be seen as $\tan\theta/2 = L/2D$. Show that for any value of $v < c$, the opening angle is the same, meaning that the observer does not see length contraction. Drawing a picture will help.

Answer

In the $\mathcal{O}$ frame, we set the condition for the right-hand side of the rod:

$$\begin{aligned}
&x^\mu_{emit} = (0, L/2, D, 0), x^\mu_{receive} = (t, vt, 0, 0),\\
&\Delta x^\mu = x^\mu_{receive} - x^\mu_{emit} = (t, vt - L/2, -D, 0)\\
&\Delta x^\mu \Delta x_\mu = 0 = t^2 - (L/2 - vt)^2 - D^2.
\end{aligned}$$

Also, with $dt = t, dx = vt - L/2, dy = -D$, we have

$$\begin{aligned}
\Delta t' &= \gamma(t + v(vt - L/2))\\
\Delta x' &= \gamma(v + (vt - L/2))\\
\Delta y' &= -D
\end{aligned}$$

We can substitute the solution t to solve the dx' and dt', and define the opening angle

$$\tan\theta'/2 = \frac{\Delta x'}{\Delta y'}. \tag{30}$$

Note that this is the calculation for right-hand side of the rod. Hence, it should be denoted as θ'_R. Now we can apply the procedure to the left-hand side of the rod. After plugging all information into `Mathematica` and using the equation, we have

$$\tan(\theta'_R + \theta'_L) = \frac{\tan\theta'_R + \tan\theta'_L}{1 - \tan\theta'_R \tan\theta'_L} = \frac{4DL}{4D^2 - L^2} = \frac{2\frac{L}{2D}}{1 - \left(\frac{L}{2D}\right)^2} = \tan\theta. \tag{31}$$

Hence, we prove that the opening angle keeps the same value. $\square$

Assignment 2 due on Monday September 15th at 5PM

Question 1

Humans have finally been able to design and build a spaceship that can travel distances up to 100 light-years away. The propulsion system is capable of providing a constant acceleration $g = 9.8$ m/s in the rest frame of the spaceship; this simulates gravity so that people on board are as comfortable as on Earth. The spaceship leaves Earth from rest, accelerates towards its destination, and, halfway there, it reverses its engines so that it will come to rest when it reaches its destination. Find the position x and velocity v as functions of both Earth time t and proper time τ. How long does it take to reach a destination 10 light-years away according to a clock on Earth versus a clock in the spaceship? Repeat the calculation for a destination 100 light-years away. It is interesting to note that the characteristic time c/g is almost identically equal to one year.

Answer

Let's define the acceleration a^μ,

$$a^\mu = \frac{du^\mu}{d\tau}, a^\mu a_\mu = g^2, u^\mu = \frac{dx^\mu}{d\tau}, \tag{32}$$

where u^μ is the 4-velocity. We can re-parameterize to easily define the velocity: let y and z to be zero for convention,

$$u^\mu = (\cosh\eta, \sinh\eta, 0, 0), \quad \eta = \eta(\tau), \tag{33}$$

where η is the rapidity. Now we have

$$\frac{du^\mu}{d\tau} = \frac{d\eta}{d\tau}(\sinh\eta, \cosh\eta, 0, 0) \tag{34}$$

$$\frac{du^\mu}{d\tau}\frac{du^\mu}{d\tau} = -(\sinh^2\eta - \cosh^2\eta)\left(\frac{d\eta}{d\tau}\right)^2 = \left(\frac{d\eta}{d\tau}\right)^2 = g^2 \tag{35}$$

$$\rightarrow \left(\frac{d\eta}{d\tau}\right) = g \rightarrow \eta = g\tau + C \tag{36}$$

By initial condition, C should be 0, and then we have

$$u^\mu = (\cosh(g\tau), \sinh(g\tau), 0, 0), \tag{37}$$

$$x^\mu = \frac{1}{g}(\sinh(g\tau) + C_1, \cosh(g\tau) + C_2, 0, 0) \tag{38}$$

$$= \frac{1}{g}(\sinh(g\tau), \cosh(g\tau) - 1, 0, 0), \tag{39}$$

where C_1 and C_2 are determined by initial condition $t(\tau=0)=0, x(\tau=0)=0$. Finally, we have

$$t = x^0 = \frac{1}{g}\sinh{(g\tau)} = \frac{c}{g}\sinh{(\frac{g\tau}{c})} \tag{40}$$

$$x = x^1 = \frac{1}{g}\cosh{(g\tau)} - 1 = \frac{c^2}{g}\left(\cosh{(\frac{g\tau}{c})} - 1\right) \tag{41}$$

Considering the half of total traveling time $t_{1/2}$ when $x = L/2$, we have

$$t = \frac{1}{c}\sqrt{\left(x + \frac{c^2}{g}\right)^2 - \frac{c^4}{g^2}} \tag{42}$$

$$t_{1/2} = \frac{1}{c}\sqrt{\left(\frac{L}{2} + \frac{c^2}{g}\right)^2 - \frac{c^4}{g^2}} \tag{43}$$

$$\tag{44}$$

For $L/2 = 5$ light-years $= 5 \times \frac{c}{g} \times c = \frac{5c^2}{g}$, we have $t_{tatal} = 2t_{1/2} = 2\sqrt{35}c/g \approx 11.8$ years. For $L/2 = 50$ light-years $= 5 \times \frac{c}{g} \times c = \frac{50c^2}{g}$, we have $t_{tatal} = 2t_{1/2} = 20\sqrt{26}c/g \approx 102$ years. This is the time for the people on the earth (in $\mathcal{O}$ frame).

On the other hand, for people on the spaceship, the (proper) time would be τ, given by

$$\tau_{1/2} = \frac{c}{g}\sinh^{-1}(\frac{t_{1/2}}{c/g}) \tag{45}$$

Then for $L/2 = 5$ light-years $= 5 \times \frac{c}{g} \times c = \frac{5c^2}{g}$, we have $\tau_{total} = 2\tau_{1/2} = 4.96c/g \approx 4.96$ years. Also, for $L/2 = 50$ light-years, we have $\tau_{total} = 2\tau_{1/2} = 9.25c/g \approx 9.25$ years. $\square$

Question 2

Prove component by component that $\varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon^{\alpha\beta\gamma\delta}$, and evaluate the scalar $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta}$.

Answer

$$\varepsilon_{\alpha\beta\gamma\delta} = \eta_{\alpha\rho}\eta_{\lambda\beta}\eta_{\gamma\kappa}\eta_{\zeta\delta}\varepsilon^{\rho\lambda\kappa\zeta}$$

If we choose $(\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)$ as an example, we will have

$$\varepsilon_{0123} = \eta_{0\rho}\eta_{1\beta}\eta_{2\kappa}\eta_{3\delta}\varepsilon^{\rho\lambda\kappa\zeta} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\varepsilon^{0123} = -1.$$

Also, the value is zero if any two of the variables α, β, γ, or δ are equal. Last, if the permutation of $(\alpha, \beta, \gamma, \delta)$ is even, the value is -1 due to the property of $\varepsilon^{\rho\lambda\kappa\zeta}$. Once the permutation of $(\alpha, \beta, \gamma, \delta)$ is odd, the value is $+1$ due to the property of $\varepsilon^{\rho\lambda\kappa\zeta}$. Hence, we prove the statement:

$$\varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon^{\alpha\beta\gamma\delta}.$$

Last,

$$\begin{aligned}
&\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta} \\
=& -\left(\varepsilon^{\alpha\beta\gamma\delta}\right)^2 \\
=& -1 \times (4!) = -24,
\end{aligned}$$

where 4! means the number of possible permutation leaving non-vanishing terms. $\square$

Assignment 3 due on Monday September 22th at 5PM

Question 1

Show explicitly that the 4-vector current density for a collection of point charges satisfies $\partial_\mu J^\mu = 0$

Question 2

Prove that the electromagnetic energy density squared minus the square of the Poynting vector is a Lorentz invariant for an electromagnetic field by expressing this quantity in terms of tensors. You might consider using the dual field strength tensor defined by $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}$.

Question 3

Calculate the scalar ${T^\alpha}_\alpha$ associated with the electromagnetic stress tensor.