University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

Lecture Instructor: Professor Zhen Liu

Zong-En Chen
chen9613@umn.edu

December 3, 2025

HW6 Due to December 4 11:59 PM

Question 1

Problem 36.3

(a) Prove the *Fierz identities*

$$(\chi_1^\dagger \overline{\sigma}^\mu \chi_2)(\chi_3^\dagger \overline{\sigma}_\mu \chi_4) = -2(\chi_1^\dagger \chi_3^\dagger)(\chi_2 \chi_4) \tag{36.58}$$
$$(\chi_1^\dagger \overline{\sigma^\mu} \chi_2)(\chi_3^\dagger \overline{\sigma_\mu} \chi_4) = (\chi_1^\dagger \overline{\sigma}^\mu \chi_4)(\chi_3^\dagger \overline{\sigma}_\mu \chi_2) \tag{36.59}$$

(b) Define the Dirac fields

$$\Psi_1 \equiv \begin{pmatrix} \chi_i \\ \xi_i^\dagger \end{pmatrix}, \quad \Psi_i^C \equiv \begin{pmatrix} \xi_i \\ \chi_i^\dagger \end{pmatrix} \tag{36.60}$$

Use eqs. (36.58) and (36.59) to prove the Dirac form of the Fierz identities,

$$(\overline{\Psi}_1 \gamma^\mu P_L \Psi_2)(\overline{\Psi}_3 \gamma_\mu P_L \Psi_4) = -2(\overline{\Psi}_1 P_R \Psi_3^C)(\overline{\Psi}_4^C P_L \Psi_2) \tag{36.61}$$
$$(\overline{\Psi}_1 \gamma^\mu P_L \Psi_2)(\overline{\Psi}_3 \gamma_\mu P_L \Psi_4) = (\overline{\Psi}_1 \gamma^\mu P_L \Psi_4)(\overline{\Psi}_3 \gamma_\mu P_L \Psi_2) \tag{36.62}$$

(c) By writing both sides out in terms of Weyl fields, show that

$$\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = -\overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C \tag{36.63}$$
$$\overline{\Psi}_1 P_L \Psi_2 = \overline{\Psi}_2^C P_L \Psi_1^C \tag{36.64}$$
$$\overline{\Psi}_1 P_R \Psi_2 = \overline{\Psi}_2^C P_R \Psi_1^C. \tag{36.65}$$

Combining equations (36.63–36.65) with equations (36.61–36.62) yields more useful forms of the Fierz identities.

Answer

(a)

We start from the left-hand side of equation (36.58):

$$\begin{aligned}(\chi_1^\dagger \overline{\sigma}^\mu \chi_2)(\chi_3^\dagger \overline{\sigma}_\mu \chi_4) &= (\chi_1^\dagger)_{\dot a} (\overline{\sigma}^\mu)^{\dot a b} (\chi_2)_b (\chi_3^\dagger)_{\dot c} (\overline{\sigma}_\mu)^{\dot c d} (\chi_4)_d & (1)\\ &= (\chi_1^\dagger)_{\dot a} (\chi_2)_b (\chi_3^\dagger)_{\dot c} (\chi_4)_d (\overline{\sigma}^\mu)^{\dot a b} (\overline{\sigma}_\mu)^{\dot c d} & (2)\end{aligned}$$

Using the identity in equations (35.4),(35.19)

$$(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} = -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \tag{35.4}$$
$$(\overline{\sigma}^\mu)^{\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}(\sigma_\mu)_{b\dot{b}} \tag{35.19}$$

we have

$$\begin{aligned}
(\overline{\sigma}^\mu)^{\dot{a}b}(\overline{\sigma}_\mu)^{\dot{c}d} &= \epsilon^{be}\epsilon^{\dot{a}\dot{f}}(\sigma^\mu)_{e\dot{f}}\epsilon^{dg}\epsilon^{\dot{c}\dot{h}}(\sigma_\mu)_{g\dot{h}} && (3)\\
&= \epsilon^{be}\epsilon^{dg}\epsilon^{\dot{a}\dot{f}}\epsilon^{\dot{c}\dot{h}}(\sigma^\mu)_{e\dot{f}}(\sigma_\mu)_{g\dot{h}} && (4)\\
&= -2\epsilon^{be}\epsilon^{dg}\epsilon^{\dot{a}\dot{f}}\epsilon^{\dot{c}\dot{h}}\epsilon_{eg}\epsilon_{\dot{f}\dot{h}} && (5)\\
&= -2\epsilon^{be}\delta^d_e\epsilon^{\dot{a}\dot{f}}\delta^{\dot{c}}_{\dot{f}} && (6)\\
&= -2\epsilon^{bd}\epsilon^{\dot{a}\dot{c}} && (7)
\end{aligned}$$

Substituting this back, we get

$$\begin{aligned}
(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) &= -2(\chi_1^\dagger)_{\dot{a}}(\chi_2)_b(\chi_3^\dagger)_{\dot{c}}(\chi_4)_d\epsilon^{bd}\epsilon^{\dot{a}\dot{c}} && (8)\\
&= -2(\chi_1^\dagger)_{\dot{a}}(\chi_3^\dagger)_{\dot{c}}\epsilon^{\dot{a}\dot{c}}(\chi_2)_b(\chi_4)_d\epsilon^{bd} && (9)\\
&= -2(\chi_1^\dagger)^{\dot{c}}(\chi_3^\dagger)_{\dot{c}}(\chi_2)_d(\chi_4)^d && (10)\\
&= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4)\times(-1)(-1) && (11)\\
&= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) && (12)
\end{aligned}$$

This proves equation (36.58). Similarly, we can prove equation (36.59):

$$\begin{aligned}
(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) &= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) && (13)\\
&= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_4\chi_2) && (14)\\
&= (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) && (15)
\end{aligned}$$

This proves equation (36.59).

(b)

First $\Psi = \begin{pmatrix} \chi_a \\ (\xi^\dagger)^{\dot{a}} \end{pmatrix}$, so $\overline{\Psi} = (\xi^a, (\chi^\dagger)_{\dot{a}})$. Also, $P_L\Psi = \begin{pmatrix} \chi_a \\ 0 \end{pmatrix}$ and $P_R\Psi = \begin{pmatrix} 0 \\ (\xi^\dagger)^{\dot{a}} \end{pmatrix}$. Thus,

$$\begin{aligned}
\overline{\Psi}_1\gamma^\mu P_L\Psi_2 &= (\xi_1^a, (\chi_1^\dagger)_{\dot{a}})\begin{pmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \overline{\sigma}^{\mu\dot{a}b} & 0 \end{pmatrix}\begin{pmatrix} (\chi_2)_b \\ 0 \end{pmatrix} && (16)\\
&= \xi_1^a\sigma^\mu_{a\dot{b}}(\chi_2)_b && (17)\\
&= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2) && (18)
\end{aligned}$$

Similarly,

$$\overline{\Psi}_3\gamma_\mu P_L\Psi_4 = (\chi_3^\dagger\overline{\sigma}_\mu\chi_4) \tag{19}$$

Therefore,

$$\begin{aligned}(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) && (20)\\ &= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) && \text{(from (a))}\end{aligned}$$

Now, for the right-hand side of equation (36.61):

$$\begin{aligned}\overline{\Psi}_1 P_R\Psi_3^C &= (\xi_1^a, (\chi_1^\dagger)_{\dot{a}})\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}(\xi_3)_b\\ (\chi_3^\dagger)^{\dot{b}}\end{pmatrix} && (21)\\ &= (\chi_1^\dagger)_{\dot{a}}(\chi_3^\dagger)^{\dot{a}} && (22)\\ &= (\chi_1^\dagger\chi_3^\dagger) && (23)\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\Psi}_4^C P_L\Psi_2 &= ((\xi_4^a, (\chi_4^\dagger)_{\dot{a}})\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}(\chi_2)_b\\ (\xi_2^\dagger)^{\dot{b}}\end{pmatrix} && (24)\\ &= (\xi_4^a)(\chi_2)_a && (25)\\ &= (\chi_4\chi_2) = (\chi_2\chi_4) && (26)\end{aligned}$$

Thus,

$$-2(\overline{\Psi}_1 P_R\Psi_3^C)(\overline{\Psi}_4^C P_L\Psi_2) = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{27}$$

This proves equation (36.61). Similarly, we can prove equation (36.62):

$$\begin{aligned}(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) && (28)\\ &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) && \text{(from (a))}\\ &= (\overline{\Psi}_1\gamma^\mu P_L\Psi_4)(\overline{\Psi}_3\gamma_\mu P_L\Psi_2) && (29)\end{aligned}$$

This proves equation (36.62).

(c)

First, we compute the left-hand side of equation (36.63):

$$\begin{aligned}\overline{\Psi}_1\gamma^\mu P_R\Psi_2 &= (\xi_1^a, (\chi_1^\dagger)_{\dot{a}})\begin{pmatrix}0 & \sigma^\mu_{a\dot{b}}\\ \overline{\sigma}^{\mu\dot{a}b} & 0\end{pmatrix}\begin{pmatrix}0\\ (\xi_2^\dagger)^{\dot{b}}\end{pmatrix} && (30)\\ &= \xi_1^a\sigma^\mu_{a\dot{b}}(\xi_2^\dagger)^{\dot{b}} && (31)\end{aligned}$$

Next, we compute the right-hand side of equation (36.63):

$$\overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \overline{\sigma}^{\mu\dot{a}b} & 0 \end{pmatrix} \begin{pmatrix} (\xi_1)_b \\ 0 \end{pmatrix} \tag{32}$$

$$= (\xi_2^\dagger)_{\dot{a}} \overline{\sigma}^{\mu\dot{a}b} (\xi_1)_b \tag{33}$$

Using the identity

$$(\overline{\sigma}^\mu)^{\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}(\sigma_\mu)_{b\dot{b}} \tag{35.19}$$

we have

$$(\xi_2^\dagger)_{\dot{a}} \overline{\sigma}^{\mu\dot{a}b} (\xi_1)_b = (\xi_2^\dagger)_{\dot{a}} \epsilon^{bc}\epsilon^{\dot{a}\dot{b}} (\sigma_\mu)_{c\dot{b}} (\xi_1)_b \tag{34}$$

$$= -\xi_1^c \sigma^\mu_{c\dot{b}} (\xi_2^\dagger)^{\dot{b}} \tag{35}$$

$$= -\xi_1^a \sigma^\mu_{a\dot{b}} (\xi_2^\dagger)^{\dot{b}} \tag{36}$$

$$= \overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C = -\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 \tag{37}$$

This proves equation (36.63). Similarly, we can prove equations (36.64) and (36.65):

$$LHS = \overline{\Psi}_1 P_L \Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\chi_2)_a \\ (\xi_2^\dagger)^{\dot{a}} \end{pmatrix} \tag{38}$$

$$= \xi_1^a (\chi_2)_a = (\xi_1 \chi_2) \tag{39}$$

$$RHS = \overline{\Psi}_2^C P_L \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\xi_1)_a \\ (\chi_1^\dagger)^{\dot{a}} \end{pmatrix} \tag{40}$$

$$= (\chi_2)_a \xi_1^a = (\chi_2 \xi_1) = (\xi_1 \chi_2) \tag{41}$$

This proves equation (36.64).

$$LHS = \overline{\Psi}_1 P_R \Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\chi_2)_a \\ (\xi_2^\dagger)^{\dot{a}} \end{pmatrix} \tag{42}$$

$$= (\chi_1^\dagger)_{\dot{a}} (\xi_2^\dagger)^{\dot{a}} = (\chi_1^\dagger \xi_2^\dagger) \tag{43}$$

$$RHS = \overline{\Psi}_2^C P_R \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\xi_1)_a \\ (\chi_1^\dagger)^{\dot{a}} \end{pmatrix} \tag{44}$$

$$= (\xi_2^\dagger)_{\dot{a}} (\chi_1^\dagger)^{\dot{a}} = (\xi_2^\dagger \chi_1^\dagger) = (\chi_1^\dagger \xi_2^\dagger) \tag{45}$$

This proves equation (36.65). □

Question 2

38.1

Use equation (38.12) to compute $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ explicitly. Hint: Show that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1, and that, for any matrix A with eigenvalues ± 1, $\exp(cA) = \cosh(c) + A\sinh(c)$, where c is an arbitrary complex number.

Extra question: What is the expression in the large energy limit $E_{\mathbf{p}} \gg m$? Please write down the result.

$$u_s(\mathbf{p}) = \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)u_s(\mathbf{0}), \quad v_s(\mathbf{p}) = \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)v_s(\mathbf{0}) \tag{38.12}$$

Answer

We start by showing that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1. The boost generators $\mathbf{K}$ in the Dirac representation are given by

$$K^j = \frac{i}{2}\gamma^j\gamma^0 = \frac{i}{2}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \tag{46}$$

Thus,

$$2i\hat{\mathbf{p}}\cdot\mathbf{K} = 2i\sum_{j=1}^{3}\hat{p}_j K^j = 2i\sum_{j=1}^{3}\hat{p}_j\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} = -\begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & -\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix} \tag{47}$$

Now we want to prove $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ has eigenvalues ± 1, since $\hat{\mathbf{p}}$ is a unit vector. The characteristic polynomial of $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ is given by

$$\det(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} - \lambda I) = \det\begin{pmatrix} \hat{p}_3 - \lambda & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 - \lambda \end{pmatrix} = (\hat{p}_3 - \lambda)(-\hat{p}_3 - \lambda) - (\hat{p}_1 - i\hat{p}_2)(\hat{p}_1 + i\hat{p}_2) \tag{48}$$

Simplifying this, we get

$$\det(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} - \lambda I) = \lambda^2 - (\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) = \lambda^2 - 1 \tag{49}$$

Setting the determinant to zero, we find the eigenvalues:

$$\lambda^2 - 1 = 0 \implies \lambda^2 = 1 \implies \lambda = \pm 1 \tag{50}$$

Thus, $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ has eigenvalues ± 1. Consequently, the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1 as well. Now we can use the identity for any matrix A with eigenvalues ± 1:

$$\exp(cA) = \cosh(c) + A\sinh(c) \tag{51}$$

where c is an arbitrary complex number. Applying this to our case with $A = 2i\hat{\mathbf{p}} \cdot \mathbf{K}$ and $c = i\eta/2$, we have

$$\begin{aligned}
\exp(i\eta\hat{\mathbf{p}} \cdot \mathbf{K}) &= \cosh\left(\frac{i\eta}{2}\right) + (2i\hat{\mathbf{p}} \cdot \mathbf{K})\sinh\left(\frac{i\eta}{2}\right) && (52) \\
&= \cos\left(\frac{\eta}{2}\right) + i(2i\hat{\mathbf{p}} \cdot \mathbf{K})\sin\left(\frac{\eta}{2}\right) && (53) \\
&= \cos\left(\frac{\eta}{2}\right) - 2(\hat{\mathbf{p}} \cdot \mathbf{K})\sin\left(\frac{\eta}{2}\right) && (54)
\end{aligned}$$

Question 3

45.2

Use the Feynman rules to write down (at tree level) $i\mathcal{T}$ for the processes: $e^+e^+ \to e^+e^+$ and $\varphi\varphi \to e^+e^-\varphi$.

Remark: Do not write $\varphi\varphi \to e^+e^-$. Also, please draw Feynman diagrams when doing this problem. Remember the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + g\varphi\overline{\Psi}\Psi. \tag{55}$$

Answer

Question 4

48.2

Compute $\langle|\mathcal{T}|^2\rangle$ for $e^+e^- \to \varphi\varphi$. You should find that your result is the same as that for $e^-\varphi \to e^-\varphi$, but with $s \leftrightarrow t$, and an extra overall minus sign. This relationship is known as *crossing symmetry*. There is an overall minus sign for each fermion that is moved from the initial to the final state.

Remark: Please compute for $e^-\varphi \to e^-\varphi$, do not compute for $e^+e^- \to \varphi\varphi$. Please also draw Feynman diagrams. Remember the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + g\varphi\overline{\Psi}\Psi. \tag{56}$$

Answer