University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

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Question 1

In lecture we studied the time dilation of a slowly moving object in a weak stationary gravitational field. We found that the frequency difference of identical clocks located at points $\mathbf{x}_1$ and $\mathbf{x}_2$ and at rest in the gravitational field is

$$\frac{\nu_2 - \nu_1}{\nu_0} = \frac{\Delta\nu}{\nu_0} = \phi_2 - \phi_1, \tag{1}$$

where ϕ is the gravitational potential. Generalize this formula to the case when they are moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which are small compared to the speed of light. This results in contributions from both gravity and special relativity.

Answer

In a weak stationary gravitational field, the metric can be approximated as

$$\begin{aligned} d\tau^2 =& -\left(g_{\mu\nu} dx^\mu dx^\nu\right) & (2) \\ =& (1+2\phi)dt^2 - (1-2\phi)(dx^2 + dy^2 + dz^2), & (3) \end{aligned}$$

where ϕ is the gravitational potential and $|\phi| \ll 1$. For a clock moving with velocity $\mathbf{v} = (v_x, v_y, v_z)$, we have $dx = v_x dt$, $dy = v_y dt$, and $dz = v_z dt$. Therefore, the line element becomes

$$d\tau^2 = (1+2\phi)dt^2 - (1-2\phi)(v_x^2 + v_y^2 + v_z^2)dt^2. \tag{4}$$

This simplifies to

$$d\tau^2 = \left[(1+2\phi) - (1-2\phi)v^2\right] dt^2 \tag{5}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. Hence, if we consider the $dt/d\tau$, we will have

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{1}{\sqrt{(1+2\phi) - (1-2\phi)v^2}} = \frac{1}{\sqrt{1+2\phi - v^2 + 2\phi v^2}} \approx \frac{1}{\sqrt{1+2\phi - v^2}} & (6) \\ &= (1+2\phi - v^2)^{-\frac{1}{2}}, & (7) \end{aligned}$$

where we have used the binomial expansion. Therefore, the frequency of the clock moving with velocity $\mathbf{v}$ in a weak stationary gravitational field is

$$\nu_i = \frac{1}{dt_i} = \frac{1}{d\tau}\frac{d\tau}{dt_i} = \nu_0\sqrt{1+2\phi_i - v_i^2} \approx \nu_0\left(1+\phi_i - \frac{v_i^2}{2}\right), \quad i=1,2. \tag{8}$$

where $\nu_0 = 1/d\tau$ is the frequency of the clock at rest in the absence of gravitational field. Thus, we have

$$\frac{\Delta\nu}{\nu_0} = \frac{\nu_2-\nu_1}{\nu_0} = (\phi_2-\phi_1) - \frac{1}{2}(v_2^2 - v_1^2). \tag{9}$$

□

Question 2

Prove that $V_{\mu;\nu} \equiv \partial V_\mu / \partial x^\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$ transforms as a second rank tensor, similar to how we proved it for $V^\mu{}_{;\nu}$ in class, assuming that V transforms as a 4 -vector.

Answer

First, by the definition, we have

$$
\begin{aligned}
V'_\mu &= \frac{\partial x^\nu}{\partial x'^\mu} V_\nu, && (10)\\
\frac{\partial V'_\mu}{\partial x'^\nu} &= \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x'^\mu} V_\beta\right) = \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial^2 x^\beta}{\partial x^\alpha \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial V_\beta}{\partial x^\alpha} && (11)\\
&= \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial V_\beta}{\partial x^\alpha} && (12)\\
\Gamma'^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} && (13)\\
&= \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} - \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}, \quad \text{by } \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu}\right) = 0, && (14)\\
\Gamma'^\lambda_{\mu\nu} V'_\lambda &= \left(\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right) \frac{\partial x^\beta}{\partial x'^\lambda} V_\beta && (15)\\
&= \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\beta}{\partial x'^\lambda} V_\beta + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\beta}{\partial x'^\lambda} V_\beta && (16)\\
&= \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} \delta^\beta{}_\rho V_\beta + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \delta^\beta{}_\rho V_\beta && (17)\\
&= \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} V_\rho + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho. && (18)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
V'_{\mu;\nu} &= \frac{\partial V'_\mu}{\partial x'^\nu} - \Gamma'^\lambda_{\mu\nu} V'_\lambda && (19)\\
&= \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial V_\beta}{\partial x^\alpha} - \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} V_\rho - \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho && (20)\\
&= \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial V_\beta}{\partial x^\alpha} - \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} V_\rho && (21)\\
&= \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \left(\frac{\partial V_\beta}{\partial x^\alpha} - \Gamma^\rho_{\alpha\beta} V_\rho\right) = \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\alpha}{\partial x'^\mu} \left(\frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\rho_{\alpha\beta} V_\rho\right) && (22)\\
&= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} V_{\alpha;\beta}. && (23)
\end{aligned}
$$

This shows that $V_{\mu;\nu}$ transforms as a second rank tensor. $\square$

Question 3

Show that

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = A_{\mu\nu,\lambda} + A_{\lambda\mu,\nu} + A_{\nu\lambda,\mu}, \tag{24}$$

when $A_{\mu\nu}$ is an anti-symmetric tensor.

Answer

By the definition of covariant derivative, we have

$$A_{\mu\nu;\lambda} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho}, \tag{25}$$

$$A_{\lambda\mu;\nu} = \frac{\partial A_{\lambda\mu}}{\partial x^\nu} - \Gamma^\rho_{\lambda\nu} A_{\rho\mu} - \Gamma^\rho_{\mu\nu} A_{\lambda\rho}, \tag{26}$$

$$A_{\nu\lambda;\mu} = \frac{\partial A_{\nu\lambda}}{\partial x^\mu} - \Gamma^\rho_{\nu\mu} A_{\rho\lambda} - \Gamma^\rho_{\lambda\mu} A_{\nu\rho}. \tag{27}$$

Adding them together, we have

$$\begin{aligned} A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = &\frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} \\ &- \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho} - \Gamma^\rho_{\lambda\nu} A_{\rho\mu} - \Gamma^\rho_{\mu\nu} A_{\lambda\rho} - \Gamma^\rho_{\nu\mu} A_{\rho\lambda} - \Gamma^\rho_{\lambda\mu} A_{\nu\rho}. \end{aligned} \tag{28}$$

Since $A_{\mu\nu}$ is an anti-symmetric tensor, we have $A_{\mu\nu} = -A_{\nu\mu}$. Therefore, we have

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} \tag{29}$$

$$= A_{\mu\nu,\lambda} + A_{\lambda\mu,\nu} + A_{\nu\lambda,\mu}. \tag{30}$$

□