University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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HW1 Due to September 23 11:59 PM

Question 1

Problem 1.2

With the Hamiltonian of eq. (1.32), show that the state defined in eq. (1.33) obeys the abstract Schrodinger equation, eq. (1.1), if and only if the wave function obeys eq. (1.30). Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations of eq. (1.31), and to fermions, where the particle creation and annihilation operators obey the anti-commutation relations of eq. (1.38).

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle \tag{1.1}$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U\left(\mathbf{x}_j\right)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V\left(\mathbf{x}_j - \mathbf{x}_k\right)\right]\psi \tag{1.30}$$

$$\begin{aligned}
\left[a(\mathbf{x}), a\left(\mathbf{x}'\right)\right] &= 0 \\
\left[a^\dagger(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] &= 0 \\
\left[a(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] &= \delta^3\left(\mathbf{x} - \mathbf{x}'\right)
\end{aligned} \tag{1.31}$$

$$\begin{aligned}
H = &\int d^3x a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right)a(\mathbf{x}) \\
&+ \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})
\end{aligned} \tag{1.32}$$

$$|\psi,t\rangle = \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{1.33}$$

Answer

We first consider boson case, and then we have

$$LHS = i\hbar\frac{\partial}{\partial t}|\psi,t\rangle \tag{1}$$

$$= i\hbar\frac{\partial}{\partial t}\int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{2}$$

$$= \int d^3x_1 \ldots d^3x_n i\hbar\frac{\partial}{\partial t}\psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{3}$$

$$= \int d^3x_1 \ldots d^3x_n\left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U\left(\mathbf{x}_j\right)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V\left(\mathbf{x}_j - \mathbf{x}_k\right)\right]\psi a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{4}$$

$$
\begin{aligned}
&RHS = H|\psi, t\rangle && (5)\\
&= \left[\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) + \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x})\right] |\psi, t\rangle && (6)\\
&= \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle && (7)\\
&+ \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle && (8)
\end{aligned}
$$

For the term in Equation 7, by considering $[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}')$, we have

$$
\begin{aligned}
&a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) && (9)\\
=&[a^\dagger(\mathbf{x}_1) a(\mathbf{x}) + \delta^3(\mathbf{x}-\mathbf{x}_1)] a^\dagger\left(\mathbf{x}_2\right) \dots a^\dagger\left(\mathbf{x}_n\right) && (10)\\
=&a^\dagger(\mathbf{x}_1) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_2\right) \dots a^\dagger\left(\mathbf{x}_n\right) + \delta^3(\mathbf{x}-\mathbf{x}_1) a^\dagger\left(\mathbf{x}_2\right) \dots a^\dagger\left(\mathbf{x}_n\right) && (11)\\
=&a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) a(\mathbf{x}) + \sum_{j=1}^{n} \delta^3(\mathbf{x}-\mathbf{x}_j) a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) && (12)\\
=&0 + \sum_{j=1}^{n} \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) && (13)
\end{aligned}
$$

we can drop the first term in Equation 12 since this term will act on the $|0\rangle$, giving 0. Hence, we have

$$
\begin{aligned}
&\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle && (14)\\
=&\sum_{j=1}^{n} \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j |0\rangle && (15)\\
=&\sum_{j=1}^{n} \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j |0\rangle && (16)\\
=&\sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \mathcal{O}_j |0\rangle && (17)\\
=&\sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \mathcal{O}_j |0\rangle && (18)\\
=&\sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle && (19)\\
=&\int d^3x_1 \dots d^3x_n \sum_{j=1}^{n} \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle, && (20)
\end{aligned}
$$

where $a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n) = a^\dagger(\mathbf{x}_j)\mathcal{O}_j$ since they (boson fields) commute. Now, we do the same thing for the term in Equation 8, we have

$$
\begin{aligned}
& a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n) && (21)\\
=& a(\mathbf{y})\sum_{j=1}^{n}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n) && (22)\\
=& \sum_{i\neq j}^{n}\sum_{j=1}^{n}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}, \quad \mathcal{T}_{ij} = \prod_{k\neq i,j}^{n} a^\dagger(\mathbf{x}_k). && (23)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\int d^3x_1\dots d^3x_n\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle && (24)\\
=& \frac{1}{2}\int d^3x d^3y\int d^3x_1\dots d^3x_n V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle && (25)\\
=& \frac{1}{2}\int d^3x d^3y\int d^3x_1\dots d^3x_n V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\sum_{i\neq j}^{n}\sum_{j=1}^{n}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle && (26)\\
=& \sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}\int d^3x d^3y\int d^3x_1\dots d^3x_n V(\mathbf{x}-\mathbf{y})\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})\mathcal{T}_{ij}|0\rangle && (27)\\
=& \sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}\int d^3x_1\dots d^3x_n V(\mathbf{x}_j-\mathbf{x}_i)\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij}|0\rangle && (28)\\
=& \int d^3x_1\dots d^3x_n\sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}V(\mathbf{x}_j-\mathbf{x}_i)\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle && (29)\\
=& \int d^3x_1\dots d^3x_n\sum_{j=1}^{n}\sum_{k=1}^{j-1}V(\mathbf{x}_j-\mathbf{x}_k)\,\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle, && (30)
\end{aligned}
$$

where $a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij} = a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)$ for the same reason. Hence, we have proved the $LHS = RHS$ and Equation 1.1 for the boson field case.

For fermion fields, the only difference is the anti-commutation relation. We start from Equation 7 again, by considering$\{a(\mathbf{x}), a^\dagger(\mathbf{x})\} = \delta^3(\mathbf{x}-\mathbf{x}')$, and we have

$$
\begin{aligned}
& a(\mathbf{x})a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n) && (31)\\
=& [-a^\dagger(\mathbf{x}_1)a(\mathbf{x}) + \delta^3(\mathbf{x}-\mathbf{x}_1)]a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_n) && (32)\\
=& (-1)^n a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)\,a(\mathbf{x}) + \sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n) && (33)\\
=& 0 + \sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n), && (34)
\end{aligned}
$$

where the 0 term comes from the same reason. Then the term in Equation 7 is given by

$$\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{35}$$

$$= \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{36}$$

$$= \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \sum_{j=1}^{n} (-1)^j \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{37}$$

$$= \sum_{j=1}^{n} \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) (-1)^j \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{38}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) (-1)^j \mathcal{O}_j |0\rangle \tag{39}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) (-1)^j a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{40}$$

$$= \int d^3x_1 \dots d^3x_n \sum_{j=1}^{n} \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle, \tag{41}$$

where $a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) = (-1)^{j-1} a^\dagger(\mathbf{x}_j) \mathcal{O}_j$ by anti-commutation relation of fermion fields. Next, given the term in Equation 8, we have

$$a(\mathbf{y}) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) \tag{42}$$

$$= a(\mathbf{y}) \sum_{j=1}^{n} (-1)^{j-1} \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \tag{43}$$

$$= \sum_{i<j}^{n} \sum_{j=1}^{n} (-1)^{i-1} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij} + \sum_{i>j}^{n} \sum_{j=1}^{n} (-1)^{i-2} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij}, \tag{44}$$

where $\mathcal{T}_{ij} = \prod_{k\neq i,j}^{n} a^\dagger(\mathbf{x}_k)$. Now, we can simplify Equation 8 and it is given by

$$
\begin{aligned}
&\frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right) a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (45)\\
=&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \ldots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi a(\mathbf{y}) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (46)\\
=&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \ldots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i<j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle \\
+&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \ldots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i>j}^{n}\sum_{j=1}^{n}(-1)^{i-2}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle && (47)\\
=&\frac{1}{2}\int d^3x_1 \ldots d^3x_n \sum_{i<j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1}V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle \\
+&\frac{1}{2}\int d^3x_1 \ldots d^3x_n \sum_{i>j}^{n}\sum_{j=1}^{n}(-1)^{i-2}(-1)^{j-1}V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle && (48)\\
=&\frac{1}{2}\int d^3x_1 \ldots d^3x_n \sum_{i\neq j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1}V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle && (49)\\
=&\frac{1}{2}\int d^3x_1 \ldots d^3x_n \sum_{i\neq j}^{n}\sum_{j=1}^{n} V(\mathbf{x}_j-\mathbf{x}_i)\psi a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (50)\\
=&\int d^3x_1 \ldots d^3x_n \sum_{j=1}^{n}\sum_{k=1}^{j-1} V\left(\mathbf{x}_j-\mathbf{x}_k\right)\psi a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle, && (51)
\end{aligned}
$$

where $(-1)^{i-1}(-1)^{j-1}a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij} = a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)$ by anti-commutation relation. In summary, we have proved both cases for boson fields and fermion fields. $\square$

Question 2

Problem 2.3

Verify that eq. (2.16) follows from eq. (2.14).

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \tag{2.14}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma) \tag{2.16}$$

$$= i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{2.16}$$

Answer

Considering an infinitesimal transformation in $U(\Lambda) = 1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}$ and $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$, now we get

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) \tag{52}$$

$$\Rightarrow \delta\omega_{\alpha\beta}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\alpha\beta}] = \delta\omega_{\rho\sigma}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\rho\sigma}] \tag{53}$$

$$RHS = (\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma})M^{\rho\sigma} \tag{54}$$

$$\rightarrow \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} = \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} \tag{55}$$

$$= g^{\nu\rho}\delta\omega_{\rho\sigma}M^{\mu\sigma} + g^{\sigma\mu}\delta\omega_{\sigma\rho}M^{\rho\nu} = \delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\nu}\right), \tag{56}$$

we only consider the linear term $\delta\omega$. We can further simplify it to

$$[M^{\mu\nu}, M^{\rho\sigma}] = \frac{2\hbar}{i}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\nu}\right) = 2i\hbar\left(g^{\sigma\mu}M^{\rho\nu} - g^{\nu\rho}M^{\mu\sigma}\right) = 2i\hbar\left(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right) \tag{57}$$

$$= -[M^{\nu\mu}, M^{\rho\sigma}] = 2i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma}\right) \tag{58}$$

Finally, we have

$$[M^{\mu\nu}, M^{\rho\sigma}] \tag{59}$$

$$= \frac{1}{2}\left([M^{\mu\nu}, M^{\rho\sigma}] - [M^{\nu\mu}, M^{\rho\sigma}]\right) \tag{60}$$

$$= i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right). \tag{61}$$

□

Question 3

Problem 2.8

(a) Let $\Lambda = 1 + \delta\omega$ in eq.(2.26), and show that

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x),$$

where

$$\mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

(b) Show that $[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x)$.

(c) Prove the *Jacobi identity*, $[[A,B],C]+[[B,C],A]+[[C,A],B] = 0$. Hint: write out all the commutations.

(d) Use your results from parts (b) and (c) to show that

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \tag{2.31}$$

(e) Simplify the right-hand side of eq. (2.31) as much as possible.

(f) Use your results from part (e) to verify eq. (2.16), up to the possibility of a term on the right-hand side that commutes with $\varphi(x)$ and its derivatives. (Such a term, called a *central charge*, in fact does not arise for the Lorentz algebra.)

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \tag{2.26}$$

Answer

(a)

$$LHS = U^{-1}(\Lambda)\varphi(x)U(\Lambda) \tag{62}$$

$$= \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \tag{63}$$

$$\to \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\varphi(x), M^{\mu\nu}] \tag{64}$$

$$RHS = \varphi(\Lambda^{-1}x) = \varphi((\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)x^\nu) = \phi(x) - \delta\omega^\mu{}_\nu x^\nu\partial_\mu\phi(x) \tag{65}$$

$$\to -\delta\omega^\mu{}_\nu x^\nu\partial_\mu\varphi(x) = -\delta\omega_{\mu\nu}x^\nu\partial^\mu\varphi(x) = \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\varphi(x), \tag{66}$$

we only focus on the linear term $\delta\omega$. Now we have

$$[\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu)\varphi(x) = \mathcal{L}^{\mu\nu}\varphi(x). \tag{67}$$

(b)

$$[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\varphi(x)M^{\rho\sigma} - M^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{68}$$

$$= \mathcal{L}^{\mu\nu}[\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x). \tag{69}$$

(c)

$$[[A, B], C] + [[B, C], A] + [[C, A], B] \tag{70}$$

$$= (CAB - CBA - ABC + BAC) + (ABC - ACB - BCA + CBA) + (BCA - BAC - CAB + ACB) \tag{71}$$

$$= 0. \tag{72}$$

(d)

By the Jacobi identity, we have

$$0 = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\mu\nu}, M^{\rho\sigma}], \varphi(x)] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \tag{73}$$

$$\rightarrow [\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \tag{74}$$

$$= [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] - [[\varphi(x), M^{\rho\sigma}], M^{\mu\nu}] \tag{75}$$

$$= \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{76}$$

$$= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \tag{77}$$

(e)

For the result in Equation 77, considering the relation $\partial^\mu x^\nu \varphi(x) = (g^{\mu\nu} + x^\nu \partial^\mu)\varphi(x)$, we can have

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \frac{\hbar}{i}\frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu)(x^\rho \partial^\sigma - x^\sigma \partial^\rho)\varphi(x) \tag{78}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[x^\mu (g^{\nu\rho} + x^\rho \partial^\nu)\partial^\sigma - x^\nu (g^{\mu\rho} + x^\rho \partial^\mu)\partial^\sigma - x^\mu (g^{\nu\sigma} + x^\sigma \partial^\nu)\partial^\rho + x^\nu (g^{\mu\sigma} + x^\sigma \partial^\mu)\partial^\rho\right]\varphi(x) \tag{79}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[g^{\nu\rho} x^\mu \partial^\sigma - g^{\mu\rho} x^\nu \partial^\sigma - g^{\nu\sigma} x^\mu \partial^\rho + g^{\mu\sigma} x^\nu \partial^\rho + x^\mu x^\rho \partial^\nu \partial^\sigma - x^\nu x^\rho \partial^\mu \partial^\sigma - x^\mu x^\sigma \partial^\nu \partial^\rho + x^\nu x^\sigma \partial^\mu \partial^\sigma\right]\varphi(x). \tag{80}$$

$$\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{81}$$

$$= \left(\frac{\hbar}{i}\right)^2 [g^{\sigma\mu}\, x^\rho\partial^\nu - g^{\rho\mu}\, x^\sigma\partial^\nu - g^{\sigma\nu}\, x^\rho\partial^\mu + g^{\rho\nu}\, x^\sigma\partial^\mu + \, x^\rho x^\mu\, \partial^\sigma\partial^\nu - x^\rho x^\nu\, \partial^\sigma\partial^\mu - x^\sigma x^\mu\, \partial^\rho\partial^\nu + x^\sigma x^\nu\, \partial^\rho\partial^\mu]\, \varphi(x). \tag{82}$$

Using simpler forms to express:

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[g^{\nu\rho}\, x^\mu\partial^\sigma - g^{\mu\rho}\, x^\nu\partial^\sigma - g^{\nu\sigma}\, x^\mu\partial^\rho + g^{\mu\sigma}\, x^\nu\partial^\rho$$
$$+ \, x^\mu x^\rho\, \partial^\nu\partial^\sigma - x^\mu x^\sigma\, \partial^\nu\partial^\rho - x^\nu x^\rho\, \partial^\mu\partial^\sigma + x^\nu x^\sigma\, \partial^\mu\partial^\rho\Big]\varphi(x), \tag{83}$$

$$\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[g^{\sigma\mu}\, x^\rho\partial^\nu - g^{\rho\mu}\, x^\sigma\partial^\nu - g^{\sigma\nu}\, x^\rho\partial^\mu + g^{\rho\nu}\, x^\sigma\partial^\mu$$
$$+ \, x^\rho x^\mu\, \partial^\sigma\partial^\nu - x^\sigma x^\mu\, \partial^\rho\partial^\nu - x^\rho x^\nu\, \partial^\sigma\partial^\mu + x^\sigma x^\nu\, \partial^\rho\partial^\mu\Big]\varphi(x). \tag{84}$$

Combining those two results together, it gives

$$(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[g^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) - g^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu)$$
$$- g^{\nu\sigma}(x^\mu\partial^\rho - x^\rho\partial^\mu) + g^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu)\Big]\varphi(x) \tag{85}$$

$$= \frac{\hbar}{i}\Big(g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\sigma}\mathcal{L}^{\mu\rho} + g^{\mu\sigma}\mathcal{L}^{\nu\rho}\Big)\varphi(x) \tag{86}$$
$$= i\hbar\Big(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\Big)\varphi(x). \tag{87}$$

Actually, it looks similar to

$$[M^{\mu\nu}, M^{\rho\sigma}]$$
$$= i\hbar\,(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \tag{2.16}$$

(f)

Now we assume there is a non trivial term $\mathcal{C}$ on $[M^{\mu\nu}, M^{\rho\sigma}]$, giving that

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\,(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C})\,, \tag{88}$$

where C can commutes with φ and its derivatives. Now, we have

$$
\begin{aligned}
&[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] && (2.31)\\
=&i\hbar[\varphi(x), (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C})] && (89)\\
=&i\hbar\Big(g^{\nu\sigma}[\varphi(x), M^{\mu\rho}] + g^{\mu\rho}[\varphi(x), M^{\nu\sigma}] - g^{\mu\sigma}[\varphi, M^{\nu\rho}] - g^{\nu\rho}[\varphi(x), M^{\mu\sigma}] + [\varphi(x), \mathcal{C}]\Big) && (90)\\
=&i\hbar\big(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\big)\varphi(x) = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) && (91)\\
=&[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\varphi(x). && (92)
\end{aligned}
$$

Hence, with the central charge $\mathcal{C}$, the relation still holds. $\square$

Question 4

Problem 2.9

Let us write

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau + \frac{i}{2\hbar}\delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\tau, \tag{2.32}$$

where

$$(S_V^{\mu\nu})^\rho{}_\tau \equiv \frac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau) \tag{2.33}$$

are matrices which constitute the *vector representation* of the Lorentz generators.

(a) Let $\Lambda = 1 + \delta\omega$ in eq. (2.27), and show that

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x) \tag{2.34}$$

(b) Show that the matrices $(S_V^{\mu\nu})$ must have the same commutation relations as the operators $M^{\mu\nu}$. Hint: see the previous problem.

(c) For a rotation by an angle θ about the z axis, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Show that

$$\Lambda = \exp\left(-i\theta S_V^{12}/\hbar\right). \tag{2.36}$$

(d) For a boost by *rapidity* η in the z direction, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}. \tag{2.37}$$

Show that

$$\Lambda = \exp\left(+i\eta S_V^{30}/\hbar\right). \tag{2.38}$$

$$U^{-1}(\Lambda)\partial^\rho\varphi(x)U(\Lambda) = \Lambda^\rho{}_\mu\overline{\partial}^\mu\varphi(\Lambda^{-1}x), \quad \overline{x}^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu, \quad \overline{\partial}^\mu = (\Lambda^{-1})^\mu{}_\nu\partial^\nu \tag{2.27}$$

Answer

(a)

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \tag{93}$$

$$= \partial^\rho\varphi(x) + \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\partial^\rho\varphi(x), M^{\mu\nu}] \tag{94}$$

$$RHS = \Lambda^\rho{}_\mu\overline{\partial}^\mu\varphi(\Lambda^{-1}x) \tag{95}$$

$$= (\delta^\rho{}_\mu + \delta\omega^\rho{}_\mu)\overline{\partial}^\mu\varphi((\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)x^\nu) \tag{96}$$

$$= (\delta^\rho{}_\mu + \delta\omega^\rho{}_\mu)(\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)\partial^\nu(\varphi(x) - \delta\omega^\alpha{}_\beta x^\beta\partial_\alpha\varphi(x)) \tag{97}$$

$$= \delta^\rho{}_\nu(\partial^\nu\varphi(x) - \delta\omega^\alpha{}_\beta g^{\nu\beta}\partial_\alpha\varphi(x) - \delta\omega^\alpha{}_\beta x^\beta\partial^\nu\partial_\alpha\varphi(x)) \tag{98}$$

$$= \partial^\rho\varphi(x) - \delta\omega^\alpha{}_\beta g^{\rho\beta}\partial_\alpha\varphi(x) - \delta\omega^\alpha{}_\beta x^\beta\partial^\rho\partial_\alpha\varphi(x) \tag{99}$$

$$= \partial^\rho\varphi(x) - \delta\omega_{\alpha\beta}g^{\rho\beta}\partial^\alpha\varphi(x) - \delta\omega_{\alpha\beta}x^\beta\partial^\rho\partial^\alpha\varphi(x) \tag{100}$$

$$= \partial^\rho\varphi(x) - \delta\omega_{\mu\nu}g^{\rho\nu}\partial^\mu\varphi(x) - \delta\omega_{\mu\nu}x^\nu\partial^\rho\partial^\mu\varphi(x) \tag{101}$$

$$= \partial^\rho\varphi(x) + \delta\omega_{\mu\nu}\left(-g^{\rho\nu}\partial^\mu - x^\nu\partial^\rho\partial^\mu\right)\varphi(x) \tag{102}$$

$$= \partial^\rho\varphi(x) + \delta\omega_{\mu\nu}\frac{1}{2}\left(-g^{\rho\nu}\partial^\mu - x^\nu\partial^\rho\partial^\mu + g^{\rho\mu}\partial^\nu + x^\mu\partial^\rho\partial^\nu\right)\varphi(x) \tag{103}$$

Hence, we have

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i}\left(-g^{\rho\nu}\partial^\mu - x^\nu\partial^\rho\partial^\mu + g^{\rho\mu}\partial^\nu + x^\mu\partial^\rho\partial^\nu\right)\varphi(x) \tag{104}$$

$$= \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\partial^\rho\varphi(x) + \frac{\hbar}{i}(g^{\rho\mu}\delta^\nu{}_\tau\partial^\tau - g^{\rho\nu}\delta^\mu{}_\tau\partial^\tau)\varphi(x) \tag{105}$$

$$= \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\partial^\rho\varphi(x) + \frac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)\partial^\tau\varphi(x) \tag{106}$$

$$= \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x), \tag{107}$$

which is exactly the result we want to prove.

(b)

$$[(S_V^{\mu\nu}),(S_V^{\rho\sigma})]^\alpha{}_\beta \tag{108}$$
$$=(S_V^{\mu\nu})^\alpha{}_\tau(S_V^{\rho\sigma})^\tau{}_\beta-(S_V^{\rho\sigma})^\alpha{}_\tau(S_V^{\mu\nu})^\tau{}_\beta \tag{109}$$
$$=\frac{\hbar}{i}(g^{\mu\alpha}\delta^\nu{}_\tau-g^{\nu\alpha}\delta^\mu{}_\tau)\frac{\hbar}{i}(g^{\rho\tau}\delta^\sigma{}_\beta-g^{\sigma\tau}\delta^\rho{}_\beta)-\frac{\hbar}{i}(g^{\rho\alpha}\delta^\sigma{}_\tau-g^{\sigma\alpha}\delta^\rho{}_\tau)\frac{\hbar}{i}(g^{\mu\tau}\delta^\nu{}_\beta-g^{\nu\tau}\delta^\mu{}_\beta) \tag{110}$$
$$=\left(\frac{\hbar}{i}\right)^2\Big[g^{\mu\alpha}g^{\rho\nu}\delta^\sigma{}_\beta-g^{\mu\alpha}g^{\sigma\nu}\delta^\rho{}_\beta-g^{\nu\alpha}g^{\rho\mu}\delta^\sigma{}_\beta+g^{\nu\alpha}g^{\sigma\mu}\delta^\rho{}_\beta \tag{111}$$
$$-g^{\rho\alpha}g^{\mu\sigma}\delta^\nu{}_\beta+g^{\rho\alpha}g^{\nu\sigma}\delta^\mu{}_\beta+g^{\sigma\alpha}g^{\mu\rho}\delta^\nu{}_\beta-g^{\sigma\alpha}g^{\nu\rho}\delta^\mu{}_\beta\Big] \tag{112}$$
$$=i\hbar\left(\frac{\hbar}{i}\right)\Big[g^{\mu\rho}(g^{\nu\alpha}\delta^\sigma{}_\beta-g^{\sigma\alpha}\delta^\nu{}_\beta)-g^{\nu\rho}(g^{\mu\alpha}\delta^\sigma{}_\beta-g^{\sigma\alpha}\delta^\mu{}_\beta)-g^{\mu\sigma}(g^{\nu\alpha}\delta^\rho{}_\beta-g^{\rho\alpha}\delta^\nu{}_\beta)+g^{\nu\sigma}(g^{\mu\alpha}\delta^\rho{}_\beta-g^{\rho\alpha}\delta^\mu{}_\beta)\Big] \tag{113}$$
$$=i\hbar\left(g^{\mu\rho}(S_V^{\nu\sigma})^\alpha{}_\beta-g^{\nu\rho}(S_V^{\mu\sigma})^\alpha{}_\beta-g^{\mu\sigma}(S_V^{\nu\rho})^\alpha{}_\beta+g^{\nu\sigma}(S_V^{\mu\rho})^\alpha{}_\beta\right) \tag{114}$$
$$=i\hbar\Big(g^{\mu\rho}(S_V^{\nu\sigma})-g^{\nu\rho}(S_V^{\mu\sigma})-g^{\mu\sigma}(S_V^{\nu\rho})+g^{\nu\sigma}(S_V^{\mu\rho})\Big)^\alpha{}_\beta. \tag{115}$$

This looks exactly the same as

$$\begin{aligned}&[M^{\mu\nu},M^{\rho\sigma}]\\ =&i\hbar\left(g^{\nu\sigma}M^{\mu\rho}+g^{\mu\rho}M^{\nu\sigma}-g^{\mu\sigma}M^{\nu\rho}-g^{\nu\rho}M^{\mu\sigma}\right)\end{aligned} \tag{2.16}$$

(c)

$$\frac{i}{\hbar}(S_V^{12})^\mu{}_\nu=(g^{1\mu}\delta^2{}_\nu-g^{2\mu}\delta^1{}_\nu)=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\equiv\mathcal{R}, \tag{116}$$
$$\mathcal{R}^2=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}, \tag{117}$$
$$\mathcal{R}^3=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}=-\mathcal{R}, \tag{118}$$
$$\mathcal{R}^4=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}=-\mathcal{R}^2. \tag{119}$$

Hence, we have

$$\begin{aligned}
\Lambda =& \exp\left(-i\theta S_V^{12}/\hbar\right) = \exp\left(-\theta\mathcal{R}\right) && (120)\\
=& \sum_{n=0}^{\infty}\frac{(-\theta\mathcal{R})^n}{n!} = \sum_{m=0}^{\infty}\frac{(-\theta\mathcal{R})^{2m}}{(2m)!} + \sum_{m=0}^{\infty}\frac{(-\theta\mathcal{R})^{2m+1}}{(2m+1)!} && (121)\\
=& I + \sum_{m=1}^{\infty}\frac{(-1)^m\theta^{2m}\mathcal{R}^{2m}}{(2m)!} + \sum_{m=0}^{\infty}\frac{(-1)\theta^{2m+1}\mathcal{R}^{2m+1}}{(2m+1)!} && (122)\\
=& I + \sum_{m=1}^{\infty}\frac{(-1)^m\theta^{2m}}{(2m)!}\mathcal{R}^2 - \sum_{m=0}^{\infty}\frac{(-1)^m\theta^{2m+1}}{(2m+1)!}\mathcal{R} && (123)\\
=& I + (\cos\theta - 1)\mathcal{R}^2 - \sin\theta\mathcal{R} && (124)\\
=& \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \cos\theta - 1 & 0 & 0\\ 0 & 0 & \cos\theta - 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -\sin\theta & 0\\ 0 & \sin\theta & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} && (125)\\
=& \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. && (126)
\end{aligned}$$

This is exactly the same as the result in Equation (2.35).

(d)

$$\frac{i}{\hbar}(S_V^{30})^\mu{}_\nu = (g^{3\mu}\delta^0{}_\nu - g^{0\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \equiv \mathcal{B}, \tag{127}$$

$$\mathcal{B}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{128}$$

$$\mathcal{B}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathcal{B}, \tag{129}$$

$$\mathcal{B}^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathcal{B}^2. \tag{130}$$

Hence, we have

$$\Lambda = \exp(+i\eta S_V^{30}/\hbar) = \exp(\eta\mathcal{B}) \tag{131}$$

$$= \sum_{n=0}^{\infty} \frac{(\eta\mathcal{B})^n}{n!} = \sum_{m=0}^{\infty} \frac{(\eta\mathcal{B})^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(\eta\mathcal{B})^{2m+1}}{(2m+1)!} \tag{132}$$

$$= I + \sum_{m=1}^{\infty} \frac{\eta^{2m}\mathcal{B}^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{\eta^{2m+1}\mathcal{B}^{2m+1}}{(2m+1)!} \tag{133}$$

$$= I + \sum_{m=1}^{\infty} \frac{\eta^{2m}}{(2m)!}\mathcal{B}^2 + \sum_{m=0}^{\infty} \frac{\eta^{2m+1}}{(2m+1)!}\mathcal{B} \tag{134}$$

$$= I + (\cosh\eta - 1)\mathcal{B}^2 + \sinh\eta\,\mathcal{B} \tag{135}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cosh\eta - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh\eta - 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \sinh\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\eta & 0 & 0 & 0 \end{pmatrix} \tag{136}$$

$$= \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}. \tag{137}$$

This is exactly the same as the result in Equation (2.37). □

Question 5

Problem 3.1

Derive eq. (3.29) from eqs. (3.21), (3.24), and (3.28).

$$a(\mathbf{k}) = \int d^3x e^{-ikx} \left[i\partial_0\varphi(x) + \omega\varphi(x) \right] \tag{3.21}$$

$$\Pi(x) = \dot{\varphi}(x) = \partial_0\varphi(x) \tag{3.24}$$

$$[\varphi(\mathbf{x},t), \varphi(\mathbf{x}',t)] = 0, \quad [\Pi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = 0, \quad [\varphi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{3.28}$$

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \quad [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0, \quad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{3.29}$$

Answer

Since $a(\mathbf{x})$ is independent of time, we can set $x^0 = y^0$, meaning all time variables are the same. Now, we compute

$$\begin{aligned}
&[a(\mathbf{k}), a(\mathbf{k}')] && (138)\\
=&\int d^3xd^3ye^{-ikx}e^{-ik'y}[i\partial_0\varphi(x) + \omega\varphi(x), i\partial_0\varphi(y) + \omega\varphi(y)] && (139)\\
=&\int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(- [\partial_0\varphi(x), \partial_0\varphi(y)] + i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)] + \omega^2[\varphi(x), \varphi(y)]\Big) && (140)\\
=&\int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)]\Big) && (141)\\
=&\int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(i\omega(-i\delta^{(3)}(\mathbf{x}-\mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\Big) && (142)\\
=&0, && (143)
\end{aligned}$$

where we have used the commutation relations in Equation (3.28). Similarly, we can show that

$$\begin{aligned}
&[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] && (144)\\
=&\int d^3xd^3ye^{ikx}e^{ik'y}[i\partial_0\varphi(x) - \omega\varphi(x), i\partial_0\varphi(y) - \omega\varphi(y)] && (145)\\
=&\int d^3xd^3ye^{ikx}e^{ik'y}\Big(- [\partial_0\varphi(x), \partial_0\varphi(y)] - i\omega[\partial_0\varphi(x), \varphi(y)] - i\omega[\varphi(x), \partial_0\varphi(y)] + \omega^2[\varphi(x), \varphi(y)]\Big) && (146)\\
=&\int d^3xd^3ye^{ikx}e^{ik'y}\Big(- i\omega[\partial_0\varphi(x), \varphi(y)] - i\omega[\varphi(x), \partial_0\varphi(y)]\Big) && (147)\\
=&\int d^3xd^3ye^{ikx}e^{ik'y}\Big(- i\omega(-i\delta^{(3)}(\mathbf{x}-\mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\Big) && (148)\\
=&0. && (149)
\end{aligned}$$

Now, we compute

$$
\begin{aligned}
&[a(\mathbf{k}), a^\dagger(\mathbf{k}')] && (150)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y} [i\partial_0\varphi(x) + \omega\varphi(x), i\partial_0\varphi(y) - \omega\varphi(y)] && (151)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y} \Big(- [\partial_0\varphi(x), \partial_0\varphi(y)] - i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)] - \omega^2[\varphi(x), \varphi(y)]\Big) && (152)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y} \Big(- i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)]\Big) && (153)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y} \Big(- i\omega(-i\delta^{(3)}(\mathbf{x}-\mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\Big) && (154)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y} (2\omega\delta^{(3)}(\mathbf{x}-\mathbf{y})) && (155)\\
=&\int d^3x e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} (2\omega) && (156)\\
=&(2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}'). && (157)
\end{aligned}
$$

□

Question 6

Problem 3.5

Consider a complex (that is, non-hermitian) scalar field φ with Lagrangian density

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0.$$

(a) Show that φ obeys the Klein-Gordon equation.

(b) Treat φ and $\varphi^\dagger$ as independent fields, and find the conjugate momentum for each. Compute the Hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).

(c) Write the mode expansion of φ as

$$\varphi(x) = \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}].$$

Express $a(\mathbf{k})$ and $b(\mathbf{k})$ in terms of φ and $\varphi^\dagger$ and their time derivatives.

(d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates.

(e) Express the Hamiltonian in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?

$$\widetilde{dk} = \frac{d^3k}{(2\pi)^3 2\omega}, \quad \omega = \sqrt{\mathbf{k}^2 + m^2} \tag{3.11}$$

Answer

(a)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \varphi} &= -\, m^2 \varphi^\dagger, && (158)\\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} &= -\, \partial^\mu \varphi^\dagger, && (159)\\
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) &= -\, \partial_\mu \partial^\mu \varphi^\dagger. && (160)
\end{aligned}$$

Hence, the Euler-Lagrange equation gives that

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0 \tag{161}$$

$$\Rightarrow - m^2 \varphi^\dagger + \partial_\mu \partial^\mu \varphi^\dagger = 0 \tag{162}$$

$$\Rightarrow (\partial_\mu \partial^\mu - m^2)\varphi^\dagger = 0. \tag{163}$$

Similarly, we can show that

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} = -m^2 \varphi, \tag{164}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)} = -\partial^\mu \varphi, \tag{165}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)} \right) = -\partial_\mu \partial^\mu \varphi. \tag{166}$$

Hence, the Euler-Lagrange equation gives that

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)} \right) = 0 \tag{167}$$

$$\Rightarrow - m^2 \varphi + \partial_\mu \partial^\mu \varphi = 0 \tag{168}$$

$$\Rightarrow (\partial_\mu \partial^\mu - m^2)\varphi = 0. \tag{169}$$

Therefore, both φ and $\varphi^\dagger$ obey the Klein-Gordon equation.

(b)

$$\Pi_\varphi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = -\partial^0 \varphi^\dagger = -\frac{-\partial}{\partial t}\varphi = +\dot{\varphi}^\dagger, \tag{170}$$

$$\Pi_{\varphi^\dagger} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^\dagger)} = -\partial^0 \varphi = -\frac{-\partial}{\partial t}\varphi = +\dot{\varphi}. \tag{171}$$

The Hamiltonian density is given by

$$\mathcal{H} = \Pi_\varphi \dot{\varphi} + \Pi_{\varphi^\dagger} \dot{\varphi}^\dagger - \mathcal{L} \tag{172}$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger + \partial^\mu \varphi^\dagger \partial_\mu \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{173}$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger - \dot{\varphi}^\dagger \dot{\varphi} + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{174}$$

$$= \dot{\varphi} \dot{\varphi}^\dagger + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{175}$$

$$= \Pi_{\varphi^\dagger} \Pi_\varphi + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0. \tag{176}$$

(c)

$$\varphi(x) = \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] \tag{177}$$

$$\Rightarrow \dot{\varphi}(x) = \int \widetilde{dk}[i\omega a(\mathbf{k})e^{ikx} - i\omega b^\dagger(\mathbf{k})e^{-ikx}] \tag{178}$$

$$\varphi^\dagger(x) = \int \widetilde{dk}[a^\dagger(\mathbf{k})e^{-ikx} + b(\mathbf{k})e^{ikx}] \tag{179}$$

$$\Rightarrow \dot{\varphi}^\dagger(x) = \int \widetilde{dk}[-i\omega a^\dagger(\mathbf{k})e^{-ikx} + i\omega b(\mathbf{k})e^{ikx}] \tag{180}$$

Hence, we have

$$\int d^3x e^{-ikx}\varphi(x) \tag{181}$$

$$= \int d^3x e^{-ikx} \int \widetilde{dk'}[a(\mathbf{k'})e^{ik'x} + b^\dagger(\mathbf{k'})e^{-ik'x}] \tag{182}$$

$$= \int \widetilde{dk'} \int d^3x[a(\mathbf{k'})e^{i(k'-k)x} + b^\dagger(\mathbf{k'})e^{-i(k'+k)x}] \tag{183}$$

$$= \int \widetilde{dk'} \left[a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} - \mathbf{k})e^{-i(\omega'-\omega)t} + b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} + \mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{184}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} - \mathbf{k})e^{-i(\omega'-\omega)t} + b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} + \mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{185}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} - \mathbf{k})e^{-i(\omega'-\omega)t} + b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} + \mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{186}$$

$$= \frac{1}{2\omega}a(\mathbf{k})e^{-i(\omega-\omega)t} + \frac{1}{2\omega}b^\dagger(-\mathbf{k})e^{i(\omega+\omega)t} \tag{187}$$

$$= \frac{1}{2\omega}a(\mathbf{k}) + \frac{1}{2\omega}b^\dagger(-\mathbf{k})e^{i2\omega t}. \tag{188}$$

Next, we compute

$$\int d^3x e^{-ikx}\dot{\varphi}(x) \tag{189}$$

$$= \int d^3x e^{-ikx} \int \widetilde{dk'}[-i\omega' a(\mathbf{k'})e^{ik'x} + i\omega' b^\dagger(\mathbf{k'})e^{-ik'x}] \tag{190}$$

$$= \int \widetilde{dk'} \int d^3x[-i\omega' a(\mathbf{k'})e^{i(k'-k)x} + i\omega' b^\dagger(\mathbf{k'})e^{-i(k'+k)x}] \tag{191}$$

$$= \int \widetilde{dk'} \left[-i\omega' a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} - \mathbf{k})e^{-i(\omega'-\omega)t} + i\omega' b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} + \mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{192}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[-i\omega' a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} - \mathbf{k})e^{-i(\omega'-\omega)t} + i\omega' b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} + \mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{193}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[-i\omega' a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} - \mathbf{k})e^{-i(\omega'-\omega)t} + i\omega' b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'} + \mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{194}$$

$$= \frac{-i\omega}{2\omega}a(\mathbf{k})e^{-i(\omega-\omega)t} + \frac{i\omega}{2\omega}b^\dagger(-\mathbf{k})e^{i(\omega+\omega)t} \tag{195}$$

$$= \frac{-i}{2}a(\mathbf{k}) + \frac{i}{2}b^\dagger(-\mathbf{k})e^{i2\omega t}. \tag{196}$$

Therefore, we have

$$\int d^3xe^{-ikx}[i\partial_0\varphi(x)+\omega\varphi(x)] = \int d^3xe^{-ikx}[i\dot{\varphi}(x)+\omega\varphi(x)] \tag{197}$$

$$= \left[i\frac{-i}{2}a(\mathbf{k})+i\frac{i}{2}b^\dagger(-\mathbf{k})e^{i2\omega t}\right]+\left[\frac{\omega}{2\omega}a(\mathbf{k})+\frac{\omega}{2\omega}b^\dagger(-\mathbf{k})e^{i2\omega t}\right] \tag{198}$$

$$= a(\mathbf{k}). \tag{199}$$

Similarly, we can show that

$$\varphi^\dagger(x) = \int \widetilde{dk}[a^\dagger(\mathbf{k})e^{-ikx}+b(\mathbf{k})e^{ikx}] \tag{200}$$

$$\Rightarrow b(\mathbf{x}) = \int d^3xe^{-ikx}[\omega\varphi^\dagger(x)+i\partial_0\varphi^\dagger(x)], \tag{201}$$

by exchanging $a(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ with $a^\dagger(\mathbf{k})$ and $b(\mathbf{k})$, respectively.

(d)

Again, since $a(\mathbf{x})$ and $b(\mathbf{x})$ are independent of time, we can set $x^0 = y^0$, meaning all time variables are the same. Also, we rewrite the $a(\mathbf{k})$ and $b(\mathbf{k})$ for convention.

$$a(\mathbf{k}) = \int d^3xe^{-ikx}\left[i\partial_0\varphi(x)+\omega\varphi(x)\right] = \int d^3xe^{-ikx}\left[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x)\right], \tag{202}$$

$$b(\mathbf{k}) = \int d^3xe^{-ikx}[\omega\varphi^\dagger(x)+i\partial_0\varphi^\dagger(x)] = \int d^3xe^{-ikx}\left[i\Pi_{\varphi}(x)+\omega\varphi^\dagger(x)\right]. \tag{203}$$

Now, we compute

$$[a(\mathbf{k}), a(\mathbf{k}')] \tag{204}$$

$$= \int d^3xd^3ye^{-ikx}e^{-ik'y}[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x), i\Pi_{\varphi^\dagger}(y)+\omega\varphi(y)] \tag{205}$$

$$= \int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(-[\Pi_{\varphi^\dagger}(x),\Pi_{\varphi^\dagger}(y)]+i\omega[\Pi_{\varphi^\dagger}(x),\varphi(y)]+i\omega[\varphi(x),\Pi_{\varphi^\dagger}(y)]+\omega^2[\varphi(x),\varphi(y)]\Big) \tag{206}$$

$$= \int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(0\Big) \tag{207}$$

$$=0, \tag{208}$$

where we have used the canonical commutation relations and the fact that φ and $\varphi^\dagger$ are independent fields. Similarly, we can show that

$$[b(\mathbf{k}), b(\mathbf{k}')] = 0, \tag{209}$$

by exchanging $a(\mathbf{k})$, φ and $\Pi_{\varphi^\dagger}$ with $b(\mathbf{k})$, $\varphi^\dagger$ and Π_φ, respectively. Now, we compute

$$\begin{aligned}
&[a(\mathbf{k}), b(\mathbf{k}')] && (210)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y}[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x), i\Pi_\varphi(y)+\omega\varphi^\dagger(y)] && (211)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y}\Big(-[\Pi_{\varphi^\dagger}(x),\Pi_\varphi(y)]+i\omega[\Pi_{\varphi^\dagger}(x),\varphi^\dagger(y)]+i\omega[\varphi(x),\Pi_\varphi(y)]-\omega^2[\varphi(x),\varphi^\dagger(y)]\Big) && (212)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y}\Big(i\omega[\varphi(x),\Pi_\varphi(y)]-i\omega[\varphi^\dagger(y),\Pi_{\varphi^\dagger}(x),]\Big) && (213)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y}\Big(i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))-i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\Big) && (214)\\
=&0. && (215)
\end{aligned}$$

Next, we compute

$$\begin{aligned}
&[a(\mathbf{k}), b^\dagger(\mathbf{k}')] && (216)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y}[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x), -i\Pi_{\varphi^\dagger}(y)+\omega\varphi(y)] && (217)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y}\Big([\Pi_{\varphi^\dagger}(x),\Pi_{\varphi^\dagger}(y)]+i\omega[\Pi_{\varphi^\dagger}(x),\varphi(y)]-i\omega[\varphi(x),\Pi_{\varphi^\dagger}(y)]+\omega^2[\varphi(x),\varphi(y)]\Big) && (218)\\
=&\ 0, && (219)
\end{aligned}$$

by the same reason as $[a(\mathbf{k}), a(\mathbf{k}')]$. Also, the result leads to

$$[a^\dagger(\mathbf{k}), b(\mathbf{k}')] = 0. \tag{220}$$

Finally, we compute

$$\begin{aligned}
&[a(\mathbf{k}), a^\dagger(\mathbf{k}')] && (221)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y}[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x), -i\Pi_\varphi(y)+\omega\varphi^\dagger(y)] && (222)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y}\Big([\Pi_{\varphi^\dagger}(x),\Pi_\varphi(y)]+i\omega[\Pi_{\varphi^\dagger}(x),\varphi^\dagger(y)]-i\omega[\varphi(x),\Pi_\varphi(y)]+\omega^2[\varphi(x),\varphi^\dagger(y)]\Big) && (223)\\
=&-i\int d^3x d^3y e^{-ikx} e^{ik'y}\Big(\omega[\varphi(x),\Pi_\varphi(y)]+\omega[\varphi^\dagger(y),\Pi_{\varphi^\dagger}(x)]\Big) && (224)\\
=&-i\int d^3x d^3y e^{-ikx} e^{ik'y}\Big(\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))+\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\Big) && (225)\\
=&\int d^3x d^3y e^{-ikx} e^{ik'y}(2\omega\delta^{(3)}(\mathbf{x}-\mathbf{y})) && (226)\\
=&\int d^3x e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}(2\omega) && (227)\\
=&(2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}'). && (228)
\end{aligned}$$

Similarly, we can show that

$$[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{229}$$

by exchanging $a(\mathbf{k})$,$a^\dagger(\mathbf{k})$, φ and $\Pi_{\varphi^\dagger}$ with $b(\mathbf{k})$,$b^\dagger(\mathbf{k})$, $\varphi^\dagger$ and Π_φ, respectively.

(e)

First, we know that

$$H = \int d^3x \mathcal{H} = \int d^3x \left(\Pi_{\varphi^\dagger}\Pi_\varphi + \nabla\varphi^\dagger \cdot \nabla\varphi + m^2\varphi^\dagger\varphi - \Omega_0 \right). \tag{230}$$

Before we proceed, we remind each term in the Hamiltonian density.

$$\begin{aligned}
\varphi =& \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] = \int \widetilde{dk}[a_\mathbf{k} e^{ikx} + b^\dagger_\mathbf{k} e^{-ikx}], && (231)\\
\varphi^\dagger =& \int \widetilde{dk}[a^\dagger(\mathbf{k})e^{-ikx} + b(\mathbf{k})e^{ikx}] = \int \widetilde{dk}[a^\dagger_\mathbf{k} e^{-ikx} + b_\mathbf{k} e^{ikx}], && (232)\\
\Pi_\varphi =& \dot{\varphi}^\dagger = \int \widetilde{dk}[i\omega a^\dagger_\mathbf{k} e^{-ikx} - i\omega b_\mathbf{k} e^{ikx}], && (233)\\
\Pi_{\varphi^\dagger} =& \dot{\varphi} = \int \widetilde{dk}[-i\omega a_\mathbf{k} e^{ikx} + i\omega b^\dagger_\mathbf{k} e^{-ikx}], && (234)\\
\nabla\varphi =& \int \widetilde{dk}[i\mathbf{k} a_\mathbf{k} e^{ikx} - i\mathbf{k} b^\dagger_\mathbf{k} e^{-ikx}], && (235)\\
\nabla\varphi^\dagger =& \int \widetilde{dk}[-i\mathbf{k} a^\dagger_\mathbf{k} e^{-ikx} + i\mathbf{k} b_\mathbf{k} e^{ikx}]. && (236)\\
& && (237)
\end{aligned}$$

Now, we compute each term in the Hamiltonian.

$$\int d^3x \Pi_{\varphi^\dagger}\Pi_\varphi \tag{238}$$

$$= \int d^3x \int \widetilde{dk}[-i\omega a_{\mathbf{k}} e^{ikx} + i\omega b_{\mathbf{k}}^\dagger e^{-ikx}] \int \widetilde{dk'}[i\omega' a_{\mathbf{k'}}^\dagger e^{-ik'x} - i\omega' b_{\mathbf{k'}} e^{ik'x}] \tag{239}$$

$$= \int \widetilde{dk}\widetilde{dk'} \int d^3x \left(\omega\omega' a_{\mathbf{k}} a_{\mathbf{k'}}^\dagger e^{i(k-k')x} + \omega\omega' b_{\mathbf{k}}^\dagger b_{\mathbf{k'}} e^{-i(k-k')x} - \omega\omega' a_{\mathbf{k}} b_{\mathbf{k'}} e^{i(k+k')x} - \omega\omega' b_{\mathbf{k}}^\dagger a_{\mathbf{k'}}^\dagger e^{-i(k+k')x} \right) \tag{240}$$

$$= \int \widetilde{dk}\widetilde{dk'} \Big(\omega\omega' a_{\mathbf{k}} a_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{-i(\omega-\omega')t} + \omega\omega' b_{\mathbf{k}}^\dagger b_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{i(\omega-\omega')t} \tag{241}$$

$$- \omega\omega' a_{\mathbf{k}} b_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{-i(\omega+\omega')t} - \omega\omega' b_{\mathbf{k}}^\dagger a_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{i(\omega+\omega')t} \Big) \tag{242}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(\omega\omega' a_{\mathbf{k}} a_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{-i(\omega-\omega')t} + \omega\omega' b_{\mathbf{k}}^\dagger b_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{i(\omega-\omega')t} \tag{243}$$

$$- \omega\omega' a_{\mathbf{k}} b_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{-i(\omega+\omega')t} - \omega\omega' b_{\mathbf{k}}^\dagger a_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{i(\omega+\omega')t} \Big) \tag{244}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{1}{2\omega} \left(\omega^2 a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + \omega^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \omega^2 a_{\mathbf{k}} b_{-\mathbf{k}} e^{-i2\omega t} - \omega^2 b_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{i2\omega t} \right) \tag{245}$$

$$= \int \widetilde{dk} \frac{1}{2\omega} \left(\omega^2 a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + \omega^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \omega^2 a_{\mathbf{k}} b_{-\mathbf{k}} e^{-i2\omega t} - \omega^2 b_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{i2\omega t} \right) \tag{246}$$

$$\int d^3x \nabla\varphi^\dagger \cdot \nabla\varphi \tag{247}$$

$$= \int d^3x \int \widetilde{dk}[-i\mathbf{k} a_{\mathbf{k}}^\dagger e^{-ikx} + i\mathbf{k} b_{\mathbf{k}} e^{ikx}] \cdot \int \widetilde{dk'}[i\mathbf{k'} a_{\mathbf{k'}} e^{ik'x} - i\mathbf{k'} b_{\mathbf{k'}}^\dagger e^{-ik'x}] \tag{248}$$

$$= \int \widetilde{dk}\widetilde{dk'} \int d^3x \left(\mathbf{k}\cdot\mathbf{k'} a_{\mathbf{k}}^\dagger a_{\mathbf{k'}} e^{i(k'-k)x} + \mathbf{k}\cdot\mathbf{k'} b_{\mathbf{k}} b_{\mathbf{k'}}^\dagger e^{-i(k'-k)x} - \mathbf{k}\cdot\mathbf{k'} a_{\mathbf{k}}^\dagger b_{\mathbf{k'}}^\dagger e^{-i(k+k')x} - \mathbf{k}\cdot\mathbf{k'} b_{\mathbf{k}} a_{\mathbf{k'}} e^{i(k+k')x} \right) \tag{249}$$

$$= \int \widetilde{dk}\widetilde{dk'} \Big(\mathbf{k}\cdot\mathbf{k'} a_{\mathbf{k}}^\dagger a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{-i(\omega'-\omega)t} + \mathbf{k}\cdot\mathbf{k'} b_{\mathbf{k}} b_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{i(\omega'-\omega)t} \tag{250}$$

$$- \mathbf{k}\cdot\mathbf{k'} a_{\mathbf{k}}^\dagger b_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{i(\omega+\omega')t} - \mathbf{k}\cdot\mathbf{k'} b_{\mathbf{k}} a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{-i(\omega+\omega')t} \Big) \tag{251}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(\mathbf{k}\cdot\mathbf{k'} a_{\mathbf{k}}^\dagger a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{-i(\omega'-\omega)t} + \mathbf{k}\cdot\mathbf{k'} b_{\mathbf{k}} b_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{i(\omega'-\omega)t} \tag{252}$$

$$- \mathbf{k}\cdot\mathbf{k'} a_{\mathbf{k}}^\dagger b_{\mathbf{k'}}^\dagger (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{i(\omega+\omega')t} - \mathbf{k}\cdot\mathbf{k'} b_{\mathbf{k}} a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{-i(\omega+\omega')t} \Big) \tag{253}$$

$$= \int \widetilde{dk} \frac{1}{2\omega} \left(\mathbf{k}^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \mathbf{k}^2 a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t} \right), \quad \text{by } \mathbf{k'} = -\mathbf{k}. \tag{254}$$

$$\int d^3xm^2\varphi^\dagger\varphi \tag{255}$$

$$= \int d^3xm^2\int \widetilde{dk}[a^\dagger_{\mathbf{k}}e^{-ikx} + b_{\mathbf{k}}e^{ikx}]\int \widetilde{dk'}[a_{\mathbf{k}'}e^{ik'x} + b^\dagger_{\mathbf{k}'}e^{-ik'x}] \tag{256}$$

$$= \int \widetilde{dk}\widetilde{dk'}\int d^3xm^2\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}e^{i(k'-k)x} + b_{\mathbf{k}}b^\dagger_{\mathbf{k}'}e^{-i(k'-k)x} + a^\dagger_{\mathbf{k}}b^\dagger_{\mathbf{k}'}e^{-i(k+k')x} + b_{\mathbf{k}}a_{\mathbf{k}'}e^{i(k+k')x}\right) \tag{257}$$

$$= \int \widetilde{dk}\widetilde{dk'}m^2\Big(a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{-i(\omega'-\omega)t} + b_{\mathbf{k}}b^\dagger_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i(\omega'-\omega)t} \tag{258}$$

$$+ a^\dagger_{\mathbf{k}}b^\dagger_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{i(\omega+\omega')t} + b_{\mathbf{k}}a_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{-i(\omega+\omega')t}\Big) \tag{259}$$

$$= \int \frac{d^3k}{(2\pi)^32\omega}\frac{d^3k'}{(2\pi)^32\omega'}m^2\Big(a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{-i(\omega'-\omega)t} + b_{\mathbf{k}}b^\dagger_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i(\omega'-\omega)t} \tag{260}$$

$$+ a^\dagger_{\mathbf{k}}b^\dagger_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{i(\omega+\omega')t} + b_{\mathbf{k}}a_{\mathbf{k}'}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{-i(\omega+\omega')t}\Big) \tag{261}$$

$$= \int \widetilde{dk}m^2\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}}b^\dagger_{-\mathbf{k}}e^{i2\omega t} + b_{\mathbf{k}}a_{-\mathbf{k}}e^{-i2\omega t}\right) \tag{262}$$

Therefore, the Hamiltonian is given by

$$H = \int \widetilde{dk}\frac{1}{2\omega}\left(\omega^2a_{\mathbf{k}}a^\dagger_{\mathbf{k}} + \omega^2b^\dagger_{\mathbf{k}}b_{\mathbf{k}} - \omega^2b^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}e^{i2\omega t} - \omega^2a_{\mathbf{k}}b_{-\mathbf{k}}e^{-i2\omega t}\right) \tag{263}$$

$$+ \int \widetilde{dk}\frac{1}{2\omega}\left(\mathbf{k}^2a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \mathbf{k}^2b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + \mathbf{k}^2a^\dagger_{\mathbf{k}}b^\dagger_{-\mathbf{k}}e^{i2\omega t} + \mathbf{k}^2b_{\mathbf{k}}a_{-\mathbf{k}}e^{-i2\omega t}\right) \tag{264}$$

$$+ \int \widetilde{dk}\frac{1}{2\omega}\left(m^2a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + m^2b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + m^2a^\dagger_{\mathbf{k}}b^\dagger_{-\mathbf{k}}e^{i2\omega t} + m^2b_{\mathbf{k}}a_{-\mathbf{k}}e^{-i2\omega t}\right) - \int d^3x\Omega_0 \tag{265}$$

Also, we can change varible $\mathbf{k} \to \mathbf{k}' = -\mathbf{k}, d\mathbf{k} = d\mathbf{k}'$, but $\omega \to \omega' = \omega$. Then, the third and fourth terms in the first integral become $b^\dagger_{\mathbf{k}} \to b^\dagger_{-\mathbf{k}}$ and so on. Hence, we have

$$\begin{aligned}
H &= \int \widetilde{dk} \frac{1}{2\omega}\left(\omega^2 a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + \omega^2 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \omega^2 b^\dagger_{-\mathbf{k}} a^\dagger_{\mathbf{k}} e^{i2\omega t} - \omega^2 a_{-\mathbf{k}} b_{\mathbf{k}} e^{-i2\omega t}\right) && (266)\\
&+ \int \widetilde{dk} \frac{1}{2\omega}\left(\mathbf{k}^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \mathbf{k}^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) && (267)\\
&+ \int \widetilde{dk} \frac{1}{2\omega}\left(m^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + m^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + m^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + m^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) - \int d^3x \Omega_0 && (268)\\
&= \int \widetilde{dk} \frac{1}{2\omega}\left(\omega^2 a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + \omega^2 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \omega^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} - \omega^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) && (269)\\
&+ \int \widetilde{dk} \frac{1}{2\omega}\left(\mathbf{k}^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \mathbf{k}^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) && (270)\\
&+ \int \widetilde{dk} \frac{1}{2\omega}\left(m^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + m^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + m^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + m^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) - \int d^3x \Omega_0 && (271)\\
&= \int \widetilde{dk} \frac{1}{2\omega}\omega^2\left(a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b_{\mathbf{k}} b^\dagger_{\mathbf{k}}\right) - \int d^3x \Omega_0, \quad \text{by } \omega^2 = \mathbf{k}^2 + m^2 && (272)\\
&= \int \widetilde{dk} \frac{\omega}{2}\left(2a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + 2b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + 2(2\pi)^3 2\omega \delta^{(3)}(0)\right) - \int d^3x \Omega_0, \quad \text{by commutation relation} && (273)\\
&= \int \widetilde{dk} \omega\left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right) + \int \frac{d^3k}{(2\pi)^3 2\omega}\omega(2\pi)^3 2\omega \delta^{(3)}(0) - \int d^3x \Omega_0 && (274)\\
&= \int \widetilde{dk} \omega\left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right) + 2\frac{1}{2(2\pi)^3}\int d^3k \omega (2\pi)^3 \delta^{(3)}(0) - \Omega_0 V, \quad \text{by } \int d^3x = V && (275)\\
&= \int \widetilde{dk} \omega\left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right) + 2\mathcal{E}_0 V - \Omega_0 V, \quad \text{by } \mathcal{E}_0 = \frac{1}{2(2\pi)^3}\int d^3k \omega, V = (2\pi)^3 \delta^{(3)}(0). && (276)
\end{aligned}$$

In order to make the vacuum energy zero, we set $\Omega_0 = 2\mathcal{E}_0$. Hence, the Hamiltonian is given by

$$H = \int \widetilde{dk} \omega\left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right). \tag{277}$$

□