University of Minnesota
School of Physics and Astronomy

# 2025 Fall Physics 8501
# General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

September 21, 2025

# Assignment 3 due on Monday September 22th at 5PM

# Question 1

Show explicitly that the 4-vector current density for a collection of point charges satisfies $\partial_\mu J^\mu = 0$

# Answer

In the class, we defined the 4-vector current density for a collection of point charges as

$$J^0(t, \mathbf{x}) = \rho(t, \mathbf{x}) = \sum_a q_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{1}$$

$$\mathbf{J}(t, \mathbf{x}) = \sum_a q_a \mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)), \quad \mathbf{v}_a(t) = \frac{d\mathbf{x}_a(t)}{dt}. \tag{2}$$

Then we have

$$\begin{aligned}
\partial_\mu J^\mu &= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} && (3)\\
&= \sum_a q_a \left[ \frac{\partial}{\partial t} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot \left( \mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \right) \right] && (4)\\
&= \sum_a q_a \left[ -\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot \left( \mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \right) \right] && (5)\\
&= \sum_a q_a \left[ -\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \nabla \cdot \mathbf{v}_a(t) \right] && (6)\\
&= 0. && (7)
\end{aligned}$$

# Question 2

Prove that the electromagnetic energy density squared minus the square of the Poynting vector is a Lorentz invariant for an electromagnetic field by expressing this quantity in terms of tensors. You might consider using the dual field strength tensor defined by $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

# Answer

# Question 3

Calculate the scalar $T^{\alpha}{}_{\alpha}$ associated with the electromagnetic stress tensor.

# Answer