

University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I
Assignment Solution

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Assignment 3 due on Monday September 22th at 5PM

Question 1

Show explicitly that the 4-vector current density for a collection of point charges satisfies $\partial_\mu J^\mu = 0$

Answer

In the class, we defined the 4-vector current density for a collection of point charges as

$$J^0(t, \mathbf{x}) = \rho(t, \mathbf{x}) = \sum_a q_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \quad (1)$$

$$\mathbf{J}(t, \mathbf{x}) = \sum_a q_a \mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)), \quad \mathbf{v}_a(t) = \frac{d\mathbf{x}_a(t)}{dt}. \quad (2)$$

Then we have

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \quad (3)$$

$$= \sum_a q_a \left[\frac{\partial}{\partial t} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))) \right] \quad (4)$$

$$= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))) \right] \quad (5)$$

$$= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \right] \quad (6)$$

$$= 0. \quad (7)$$

Question 2

Prove that the electromagnetic energy density squared minus the square of the Poynting vector is a Lorentz invariant for an electromagnetic field by expressing this quantity in terms of tensors. You might consider using the dual field strength tensor defined by $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Answer

In the EM class, we defined the electromagnetic energy density and the Poynting vector as

$$u = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \quad (8)$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}. \quad (9)$$

Also, we have the following relations

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad (10)$$

Besides, the EM field energy momentum strength tensor is given by (from Weinberg's GR book)

$$T_{EM}^{\mu\nu} = F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \quad (11)$$

$$u = T_{EM}^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad S^i = T_{EM}^{i0} = (\mathbf{E} \times \mathbf{B})^i, \quad (12)$$

where u is the EM energy density and \mathbf{S} is the Poynting vector. Hence, we can define a 4-vector U^μ as

$$U^\mu = (u, \mathbf{S}) = (T_{EM}^{00}, T_{EM}^{i0}) = T_{EM}^{\mu 0}. \quad (13)$$

Then we have

$$T_{EM}^{00} = F^{0\alpha}F^0{}_\alpha - \frac{1}{4}\eta^{00}F_{\alpha\beta}F^{\alpha\beta} = F^{0i}F^0{}_i + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad (14)$$

$$= \mathbf{E}^2 - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \quad (15)$$

$$= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = u \quad (16)$$

$$(17)$$

and

$$T_{EM}^{i0} = F^{i\alpha} F^0_{\alpha} - \frac{1}{4} \eta^{i0} F_{\alpha\beta} F^{\alpha\beta} = F^{ij} F^0_j \quad (18)$$

$$= \epsilon_{ijk} B_k E_j \quad (19)$$

$$= (\mathbf{E} \times \mathbf{B})^i = S^i. \quad (20)$$

Therefore, we have

$$U^\mu U_\mu = -\eta_{\mu\nu} U^\mu U^\nu \quad (21)$$

$$= -\eta_{\mu\nu} T_{EM}^{\mu 0} T_{EM}^{\nu 0} \quad (22)$$

$$= T_{EM}^{00} T_{EM}^{00} - T_{EM}^{i0} T_{EM}^{i0} = u^2 - \mathbf{S}^2. \quad (23)$$

We can claim that $U^\mu U_\mu$ is a Lorentz invariant since it is the contraction of two tensors. Hence, we conclude that $u^2 - \mathbf{S}^2$ is a Lorentz invariant.

Remark: We can also prove this by using the dual field strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. First, we have

$$\tilde{F}^{0i} = \frac{1}{2} \epsilon^{0ijk} F_{jk} = \frac{1}{2} \epsilon^{ijk} F_{jk} = \frac{1}{2} \epsilon^{ijk} \epsilon_{jkl} B_l = B^i, \quad (24)$$

$$\tilde{F}^{ij} = \frac{1}{2} \epsilon^{ij0k} F_{0k} = \frac{1}{2} (\epsilon^{ij0k} - \epsilon^{ji0k}) F_{0k} = \epsilon^{ij0k} E_k = \epsilon^{ijk} E_k. \quad (25)$$

Then we can calculate the following two Lorentz invariants:

$$F_{\mu\nu} F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2), \quad (26)$$

$$\tilde{F}_{\mu\nu} F^{\mu\nu} = -4\mathbf{E} \cdot \mathbf{B}. \quad (27)$$

Now we can calculate

$$(F_{\mu\nu} F^{\mu\nu})^2 + (\tilde{F}_{\mu\nu} F^{\mu\nu})^2 = 4(\mathbf{B}^2 - \mathbf{E}^2)^2 + 16(\mathbf{E} \cdot \mathbf{B})^2 \quad (28)$$

$$= 4[(\mathbf{B}^2 + \mathbf{E}^2)^2 - 4\mathbf{E}^2 \mathbf{B}^2 + 4(\mathbf{E} \cdot \mathbf{B})^2] \quad (29)$$

$$= 4[(\mathbf{B}^2 + \mathbf{E}^2)^2 - 4(\mathbf{E} \times \mathbf{B})^2] \quad (30)$$

$$= 4(u^2 - \mathbf{S}^2). \quad (31)$$

This quantity is Lorentz invariant since all indices are contracted. Hence, we conclude that $u^2 - \mathbf{S}^2$ is a Lorentz invariant.

Question 3

Calculate the scalar $T^\alpha{}_\alpha$ associated with the electromagnetic stress tensor.

Answer

Consider the energy momentum tensor with the EM field:

$$T_{total}^{\alpha\beta} = T^{\alpha\beta} + T_{EM}^{\alpha\beta} = \sum_n p_n^\alpha(t) \frac{dx_n^\beta}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)) + F^{\alpha\mu} F^\beta{}_\mu - \frac{1}{4} \eta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \quad (32)$$

$$= \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\mathbf{x} - \mathbf{x}_n(t)) + F^{\alpha\mu} F^\beta{}_\mu - \frac{1}{4} \eta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}, \quad (33)$$

We have

$$T^\alpha{}_\alpha = \eta_{\alpha\beta} T_{total}^{\alpha\beta} \quad (34)$$

$$= \sum_n \frac{p_n^\alpha p_{n\alpha}}{E_n} \delta^3(\mathbf{x} - \mathbf{x}_n(t)) + F^{\alpha\mu} F_{\alpha\mu} - \frac{1}{4} \eta_{\alpha\beta} \eta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \quad (35)$$

$$= \sum_n \frac{m_n^2}{E_n} \delta^3(\mathbf{x} - \mathbf{x}_n(t)) + 0, \quad \text{by } \eta^{\alpha\beta} \eta_{\alpha\beta} = 4 \quad (36)$$

$$= \sum_n \frac{m_n^2}{E_n} \delta^3(\mathbf{x} - \mathbf{x}_n(t)) \quad (37)$$

$$= \sum_n \frac{m_n^2}{E_n} \delta^3(\mathbf{x} - \mathbf{x}_n(t)). \quad (38)$$

Remark: In class, we have shown that

$$\frac{\delta^3(\mathbf{x} - \mathbf{x}_n(t))}{E_n} \quad (39)$$

is a Lorentz invariant. Hence, we can conclude that $T^\alpha{}_\alpha$ is a Lorentz invariant since m_n is also a Lorentz invariant. In other words, $T^\alpha{}_\alpha$ is a Lorentz scalar.

Remark 2 (after deadline): Actually, I don't know whether should I only consider the EM part or the total energy momentum tensor. If I only consider the EM part, then we have 0. *Seth* and I discussed this question and we think that the question might be ambiguous. So I just write both of them here.