University of Minnesota

School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Problem Set 1 due 9 AM, Monday, September 15th

Question 1

(a) In quantum mechanics, what is the dimension of a wave function $\phi(\mathbf{x})$ of a particle the norm of which is $\int d^3x \phi^*(\mathbf{x})\phi(\mathbf{x})$?

(b) The transition rate ω_{fi} for $i \to f$ is given by Fermi's golden rule

$$\omega_{fi} = |\langle \phi_f | H_{int} | \phi_i \rangle|^2 \rho,$$

where H_{int} is the interaction Hamiltonian, ρ is the number of final states per unit of energy, and $\phi_{i,f}$ are the wave functions of the initial and final states. Restore the appropriate factors of $\hbar$ and c to have ω_{fi} in number of events per second.

(c) Similar to quantum mechanics, the Hamiltonian in quantum field theory is the energy operator, which is equal to the spatial integral of the Hamiltonian density $\mathcal{H}$, so that in the natural unit $[\mathcal{H}] = [E]^4$. Given that the Hamiltonian density for a boson field ϕ contains terms such as $(\partial_\mu \phi)^2$ and $m^2\phi^2$, find the dimension of ϕ. Similarly, for a fermion field, $\mathcal{H}$ contains terms like $m\bar{\psi}\psi$, find the dimension of the fermion field ψ.

Answer

(a) Since the norm, $\int d^3x \phi^*(\mathbf{x})\phi(\mathbf{x})$, is dimensionless, then

$$1 = \left[\int d^3x \phi^*(\mathbf{x})\phi(\mathbf{x}) \right] = [d^3x]\,[\phi^*]\,[\phi] = [L^3]\,[\phi^2],$$

where $[\phi^*] = [\phi]$ and it is just complex conjugate. Hence, we have $[\phi] = \left[1/L^{\frac{3}{2}}\right] = [E]^{\frac{3}{2}}$.

(b) We can do it in the dimension analysis, meaning

$$[\omega_{fi}] = [1/T] = [|\langle \phi_f | H_{int} | \phi_i \rangle|^2 \rho] \tag{1}$$
$$= [E^2 \times 1/E] = [E] = [ML^2/T^2]. \tag{2}$$

We know $[\hbar] = [ML^2/T]$. That means we can just put the $\hbar$ in the denominator, giving

$$\omega_{fi} = \frac{1}{\hbar} |\langle \phi_f | H_{int} | \phi_i \rangle|^2 \rho. \tag{3}$$

In fact, in Sakurai's QM textbook, the result is $\omega_{fi} = \frac{2\pi}{\hbar} |\langle \phi_f | H_{int} | \phi_i \rangle|^2 \rho$.

(c) Since $[\mathcal{H}] = [E]^4 = [(\partial_\mu \phi)^2] = [1/L^2]\,[\phi^2] = [E]^2\,[\phi]^2$. Hence, we have $[\phi] = [E]$. Besides, we can

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check the mass term, meaning that $[E]^4 = [m^2\phi^2] = [E]^2[\phi]^2$, and we get the same result for a scalar boson field $[\phi] = [E]$.

For a fermion field ψ, we can solve it in the same way, meaning that $[E]^4 = [m\overline{\psi}\psi] = [E][\psi]^2$. For the same reason, $[\overline{\psi}] = [\psi]$. Hence, we get $[\psi] = [E]^{3/2}$. $\qquad\square$

Question 2

It is often useful to have rough estimates of physical quantities using dimensional analysis. In each process below, estimate the cross-section in GeV or in barn ($1b = 10^{-24} cm^2$), assuming a high energy limit where only the coupling and reaction energy are relevant.

(a) The total cross section for proton-proton elastic scattering.

(b) The total cross section for the electromagnetic annihilation process $e^+ e^- \to \mu^+ \mu^-$.

(c) The weak interaction scattering $\nu_e + \text{proton} \to \nu_e + \text{proton}$.

Answer

In Table 1, we can use these effective couplings and reaction energy to estimate the cross sections. Note that these couplings are dimensionless. Also, in natural units, the unit of cross-section is $[1/E]^2$.

	notation	Effective coupling
Electromagnetism	$g_e = e^2/4\pi$	$1/137$
Weak force	g_W	10^{-5}
Strong	g_S	1

Table 1: The interactions and their effective couplings.

(a) Since this is elastic scattering, the energy of reaction cannot be too large. Otherwise, the Strong interaction might dominate, leading to inelastic scattering. The only relevant interaction will be electromagnetism. For the Feynman diagram in Figure 1, each vertex gives a e in the matrix amplitude $|A|$. Hence, the cross section will be proportional to $|A|^2$. That is e^4. In other words, it is proportional to g_e^2. Next, we can consider the reaction energy E_R is roughly $\mathcal{O}(100)$ GeV, which is large enough to ignore the mass of a proton but not enough to let strong interaction dominate. Finally, we can estimate the cross-section for proton-proton elastic scattering now, and it is given by

$$\sigma \approx g_e^2 \times \frac{1}{E_R^2} = \frac{1}{137^2} \frac{1}{100^2 \text{ GeV}^2} \tag{4}$$

$$= 5.32 \times 10^{-9} \text{ GeV}^{-2} \tag{5}$$

$$= 2.13 \times 10^{-40} \text{ m}^2 = 2.13 \times 10^{-12} \text{ b} \tag{6}$$

(b) Again, the dominating interaction in this process will be electromagnetism, see Figure 2. Therefore, we can do it in the same way. I choose 10 GeV to be the reaction energy so that it is large enough
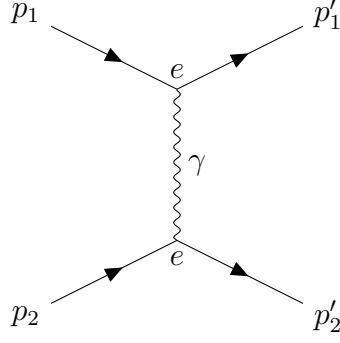
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Figure 1: The Feynman diagram for the proton-proton elastic scattering.

to produce a pair of muons and ignore the masses of muons and electrons. Hence, the cross-section is given by

$$\sigma \approx g_e^2 \times \frac{1}{E_R^2} = \frac{1}{137^2} \frac{1}{10^2 \text{ GeV}^2} \tag{7}$$

$$= 5.32 \times 10^{-7} \text{ GeV}^{-2} \tag{8}$$

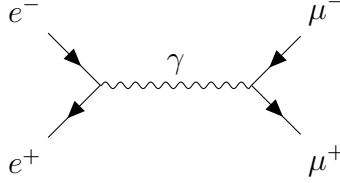$$= 2.13 \times 10^{-36} \text{ m}^2 = 2.13 \times 10^{-10} \text{ b} \tag{9}$$

Figure 2: The Feynman diagram for $e^+e^- \to \mu^+\mu^-$.

(c) I pick the Fermi's constant $G_F \approx 1.16 \times 10^{-5}$ GeV^{-2} and choose the reaction energy E_R to be 1 GeV. In Figure 3, the cross-section is proportional to G_F^2. In order to make the cross-section match the area dimension, it is now given by

$$\sigma \approx G_F^2 \times E_R^2 \tag{10}$$

$$= 1.34 \times 10^{-10} \text{ GeV}^{-2} \tag{11}$$
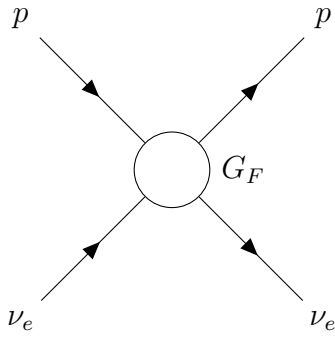
$$= 5.38 \times 10^{-14} \text{ b} \tag{12}$$

$\square$

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Figure 3: The Feynman diagram for $p + \nu_e \to p + \nu_e$ (effective 4-fermion contact, low-energy description).

Question 3

(a) Using natural units, determine the mass dimension of Newton's constant $G_N = 6.67 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$ and obtain a value for the mass scale in GeV.

(b) Suppose that the theory of Quantum Gravity does not conserve baryon number and thus gives rise to proton decay. Using dimensional arguments, estimate the lifetime of the proton if the only relevant parameters determining the gravitational decay amplitude and kinematical scale are Newton's constant, G_N, and the proton mass, respectively. Express your estimate for the lifetime in years. How does your estimate compare with the age of the universe $t_U = 13.8$ billion years?

Answer

(a) In the class, we already know that

$$1 \sec^{-1} = 6.6 \times 10^{-16} \text{ eV} = 6.6 \times 10^{-25} \text{ GeV} \tag{13}$$

$$1 \text{ m} = \frac{1}{2 \times 10^{-7} \text{ eV}} = \frac{1}{2 \times 10^{-16} \text{ GeV}} \tag{14}$$

$$1 \text{ kg} = 5.61 \times 10^{35} \text{ eV} = 5.61 \times 10^{26} \text{ GeV}, \tag{15}$$

in natural units. Hence, we just put everything together,

$$G_N = 6.67 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$$

$$= 6.67 \times 10^{-11} \left(\frac{1}{2 \times 10^{-7} \text{ GeV}} \right)^3 \left(\frac{1}{5.61 \times 10^{26} \text{ GeV}} \right) \left(6.6 \times 10^{-25} \text{ GeV} \right)^2$$

$$= 6.41 \times 10^{-39} \text{ GeV}^{-2}.$$

I have googled the answer, the answer is $G_N \approx 6.7076 \times 10^{-39} \text{ GeV}^{-2}$. It is pretty closed.

(b) Since we have already know the G_N and mass of proton, $m_p \approx 1$ GeV, we can combine these two things together. Besides, the dimension of time in natural units is $[T] = [L] = [E]^{-1}$. We should calculate the decay width Γ_p first. In the proton decay process, if we only consider gravity as our interaction, the matrix amplitude $|A|$ is proportional to G_N, see Figure 4. Hence, the decay width Γ is proportional to $|A|^2$ as well as G_N^2. Last, the dimension of decay width is $[E]$. In order to

match the dimension, now we have

$$\Gamma_p \approx G_N^2 \times m_p^5 \tag{16}$$

$$= 4.11 \times 10^{-77} \text{ GeV} \tag{17}$$

$$t_p = \frac{1}{\Gamma_p} = 2.43 \times 10^{76} \text{ GeV}^{-1} \tag{18}$$

$$= 5.09 \times 10^{44} \text{ years} \tag{19}$$
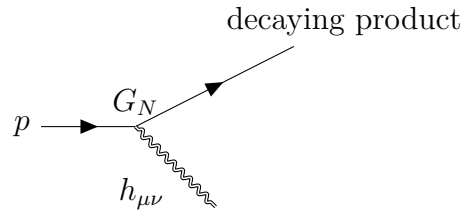
$$= 3.69 \times 10^{34} \times t_U \tag{20}$$

$\square$

Figure 4: The Feynman diagram for the decay of a proton due to a graviton.

Question 4

Consider a complex scalar field $\phi(\mathbf{x})$ described by the Lagrangian density:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi.$$

(a) Show that the Lagrangian is invariant under the global $U(1)$ transformation, $\phi \to e^{i\alpha}\phi$, where α is a real constant.

(b) Using Noether's theorem, derive the conserved current j^μ associated with this symmetry and compute the conserved Noether charge Q. What is the physical interpretation of Q?

Answer

(a) By the global $U(1)$, $\phi \to e^{i\alpha}\phi$, $\phi^\dagger \to e^{-i\alpha}\phi^\dagger$, we then have

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$
$$\to \mathcal{L}' = (\partial_\mu e^{-i\alpha}\phi^\dagger)\partial^\mu e^{i\alpha}\phi - m^2 e^{-i\alpha}\phi^\dagger e^{i\alpha}\phi$$
$$= e^{-i\alpha}e^{i\alpha}\left[\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi\right] = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi = \mathcal{L}.$$

That means that this Lagrangian is invariant under the global $U(1)$ transformation.

(b) We can now apply Noether's theorem, giving

$$0 = \delta S = \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right) + \left(\frac{\partial \mathcal{L}}{\partial \phi^*}\delta\phi^\dagger + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\delta(\partial_\mu \phi^\dagger)\right)\right] \tag{21}$$

$$= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \left(\frac{\partial \mathcal{L}}{\partial \phi^\dagger} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\right)\delta\phi^\dagger\right] + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\delta\phi^\dagger\right] \tag{22}$$

$$= \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\delta\phi^\dagger\right] = \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\frac{\delta\phi^\dagger}{\delta\alpha}\right]\delta\alpha, \tag{23}$$

where we have apply integration by part in Equation 21 and the Euler-Lagrange Equation in Equation 22. Now, with the variation of $\delta\phi = i\delta\alpha\phi$, $\delta\phi^\dagger = -i\delta\alpha\phi^\dagger$, we have

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^\dagger)}\frac{\delta\phi^\dagger}{\delta\alpha} \tag{24}$$

$$= i\left((\partial^\mu \phi^\dagger)\phi - (\partial^\mu \phi)\phi^\dagger\right). \tag{25}$$

We have $\partial_\mu j^\mu = i(\Box\phi^\dagger \phi - \Box\phi\phi^\dagger) = -m^2 i(\phi\phi^\dagger - \phi^\dagger\phi) = 0$. Now, we can define the conserved

Noether charge, Q,

$$Q = \int d^3x\, j^0 = i \int d^3x \left(\frac{\partial \phi^\dagger}{\partial t}\phi - \frac{\partial \phi}{\partial t}\phi^\dagger \right). \tag{26}$$

With the formula of the quantized complex scalar field, $\phi(x)$ is given by

$$\phi(x) = \sum_{\mathbf{p}} C_{\mathbf{p}} \left[a_{\mathbf{p}} e^{-ip\cdot x} + b_{\mathbf{p}}^\dagger e^{ip\cdot x} \right] \tag{27}$$

$$\phi^\dagger(x) = \sum_{\mathbf{p}} C_{\mathbf{p}} \left[a_{\mathbf{p}}^\dagger e^{ip\cdot x} + b_{\mathbf{p}} e^{-ip\cdot x} \right], \tag{28}$$

where $C_{\mathbf{p}}$ is a normalized constant, a and b are annihilation operators for anti-particle and particle, respectively, and $a^\dagger$ and $b^\dagger$ are creation operators for anti-particle and particle, respectively. From Equation 26, the conserved charge is now given by

$$Q = i \int d^3x \sum_{\mathbf{p},\mathbf{p}'} \left[\left(iE_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger e^{ip'\cdot x} - iE_{\mathbf{p}'} b_{\mathbf{p}'} e^{-ip'\cdot x} \right) \left(a_{\mathbf{p}} e^{-ip\cdot x} + b_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \right.$$

$$\left. + \left(iE_{\mathbf{p}} a_{\mathbf{p}} e^{-ip\cdot x} - iE_{\mathbf{p}} b_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \left(a_{\mathbf{p}'}^\dagger e^{ip'\cdot x} + b_{\mathbf{p}'} e^{-ip'\cdot x} \right) \right] \tag{29}$$

$$= \sum_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \right) = \sum_{p} (N_{\mathbf{p}} - \overline{N}_{\mathbf{p}}), \tag{30}$$

where $N_{\mathbf{p}}$ and $\overline{N}_{\mathbf{p}}$ are the number operators for antiparticle and particle, respectively. This result comes from Equation (2.112) in Ho-Kim. Hence, Q is the total conserved charge. $\square$

Problem Set 2 due 9:30 AM, Monday, September 29th

Question 1

The $\tau - \theta$ Puzzle

In the 1950's, two particles τ, θ were discovered with the same mass and lifetime that decayed differently. At the time, physicists believed that parity was conserved in all interactions.

(a) Consider the decay $\theta \to \pi^+\pi^0$. Assuming parity invariance and zero for the spin of θ, find the parity of θ.

(b) Now consider the decay process $\tau \to \pi^+\pi^+\pi^-$. (This is an old symbol for the K meson.) Let l be the orbital angular momentum of $\pi^+\pi^+$ and l' the orbital angular momentum of π^- relative to the center-of-mass of $\pi^+\pi^+$. Assuming parity invariance and the spin of τ equal to zero, find its parity.

(c) What resolved the $\tau - \theta$ puzzle?

Answer

(a)

First, we note that the intrinsic parity of a pion is -1. The parity of a two-particle system is given by

$$P_\theta = P_1 P_2 (-1)^l, \tag{31}$$

where P_1 and P_2 are the intrinsic parities of the two particles, and l is their relative orbital angular momentum. Since the θ and pions have spin 0, the system of two pions must have total angular momentum $j = s + l = 0$, in order to satisfy the conservation of total angular momentum. This means that l must be 0, too. Therefore, we have $P_\theta = 1$. It is even parity.

(b)

The parity of a three-particle system is given by

$$P_\tau = P_1 P_2 P_3 (-1)^{l+l'}, \tag{32}$$

where P_1, P_2, and P_3 are the intrinsic parities of the three particles, and l and l' are their relative orbital angular momenta. Since the τ and pions have spin 0, the system of three pions must have total angular momentum $j = s + l + l' = 0$, in order to satisfy the conservation of total angular momentum. This means that $l + l'$ must be 0, too. Therefore, we have $P_\tau = -1$. It is odd parity.

(c)

Since the τ and θ have the same mass and lifetime, they are actually the same particle, now known as the K meson. The resolution of the $\tau - \theta$ puzzle was the discovery that parity is not conserved in weak interactions, which is how the K meson decays. $\qquad\square$

Question 2

List all applicable conservation laws that are or would be violated in the following decays:

1. $\rho^0 \to \pi^0 \pi^0$

2. $\rho \to \gamma\gamma$

3. $K^+ \to \pi^+ \pi^0$

4. $\pi^0 \to 5\gamma$

(Look up the corresponding parities from the Particle Data Group at http://pdg.lbl.gov.)

Answer

Before we analyze each decay, we list the conservation laws that we will check for each decay:

- Conservation of electric charge

- Conservation of angular momentum (total angular momentum J).

- Conservation of isospin

- Conservation of parity

- Conservation of C-parity

- Conservation of G-parity

1. The ρ^0 has quantum numbers $I^G(J^{PC}) = 1^+(1^{--})$, the π^0 has quantum numbers $I^G(J^{PC}) = 1^-(0^{-+})$.

    - Electric charge: The ρ^0 has charge 0, and the two π^0's have charge $0+0 = 0$. Electric charge is conserved.

    - Angular momentum: The ρ^0 has spin 1, and the two π^0's have spin $0 \otimes 0 = 0$. To conserve total angular momentum, the two-pion system must have orbital angular momentum $l = 1$. Total angular momentum is conserved.

    - Isospin: The ρ^0 has isospin $I = 1$, and the two π^0's can form isospin $I = 1 \otimes 1 = 0, 1, 2$. I also check the C-G coefficients, and find that the state $|I = 1, I_3 = 0\rangle$ is given by

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|\pi^+\pi^-\rangle - |\pi^-\pi^+\rangle), \tag{33}$$

$$|\pi^0\pi^0\rangle = \sqrt{\frac{2}{3}}|2, 0\rangle - \sqrt{\frac{1}{3}}|0, 0\rangle. \tag{34}$$

    In other words, the two π^0's cannot form isospin $I = 1$ state. **Therefore, the decay violated isospin**.

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- Parity: The parity of the two-pion system is given by

$$P_\rho = P_\pi P_\pi (-1)^l = (-1)(-1)(-1)^l = (-1)^l. \tag{35}$$

Since the ρ^0 has spin 1 and the pions have spin 0, the two-pion system must have orbital angular momentum $l = 1$ to conserve total angular momentum. Therefore, the parity of the two-pion system is $P = -1$, which massches the parity of the ρ^0. Parity is conserved.

- C-parity: The C-parity of the two-pion system is given by

$$C_\rho = C_\pi C_\pi (-1)^l = (+1)(+1)(-1)^l = (-1)^l. \tag{36}$$

Since the ρ^0 has spin 1 and the pions have spin 0, the two-pion system must have orbital angular momentum $l = 1$ to conserve total angular momentum. Therefore, the C-parity of the two-pion system is $C = -1$, which matches the C-parity of the ρ^0. C-parity is conserved.

- G-parity: The G-parity of the two-pion system is given by

$$G_\rho = G_\pi G_\pi = (-1)(-1) = 1. \tag{37}$$

Hence, the G parity is $G = +1$, which matches the G-parity of the ρ^0. G-parity is conserved.

The isospin is not conserved.

2. The ρ has quantum numbers $I^G(J^{PC}) = 1^+(1^{--})$, the photon has quantum numbers $I^G(J^{PC}) = 0^-(1^{--})$.

Remark: Actually, PDG show that the photon has isospin $I = 0, 1$. Here, I choose $I = 0$ because the photon do not involve in strong interactions. In other words, the photon is a singlet under the strong interaction. I think $I = 1$ case is for the weak interaction, but I am not sure.

- Electric charge: The ρ has charge 0, and the two photons have charge $0 + 0 = 0$. Electric charge is conserved.

- Angular momentum: The ρ has spin 1, and the two photons have spin $s = 1 \otimes 1 = 0, 1, 2$. To conserve total angular momentum, the two-photon system must have orbital angular momentum $l = 0$(when $s = 1$), $l = 1$(when $s = 0, 1, 2$), $l = 2$(when $s = 1$). *Hence, the total angular momentum might be conserved.* However, by the **Landau-Yang theorem**, a massive spin-1 particle cannot decay into two photons. Therefore, the decay cannot occur. **Angular momentum is not conserved**.

- Isospin: The ρ has isospin $I = 1$, and the two photons can couple to isospin $I = 0$. Therefore, the decay cannot proceed through any isospin channel. **Isospin is not conserved**.

- Parity: The parity of the two-photon system is given by

$$P_\rho = P_\gamma P_\gamma (-1)^l = (-1)(-1)(-1)^l = (-1)^l. \qquad (38)$$

By checking the possible values of l above, we know the l should be $1, 3, 5$ and so on to conserve the parity.

- C-parity: The C-parity of the two-photon system is given by

$$C_\gamma C_\gamma = (-1)(-1) = 1 \neq -1 = C_\rho. \qquad (39)$$

Hence, the C parity is $C = +1$, which does not match the C-parity of the ρ. **C-parity is not conserved**.

The angular momentum, isospin, and C-parity are not conserved.

Remark: If we check the decay mode of $\rho \to \gamma\gamma$, we find that the branch ratio is 0%.

3. The K^+ has quantum numbers $I(J^P) = \frac{1}{2}(0^-)$, the π^+ has quantum numbers $I(J^P) = 1(0^-)$, and the π^0 has quantum numbers $I(J^P) = 1(0^-)$.

   - Electric charge: The K^+ has charge $+1$, and the two pions have charge $1 + 0 = +1$. Electric charge is conserved.

   - Angular momentum: The K^+ has spin 0, and the two pions have spin $0 \otimes 0 = 0$. To conserve total angular momentum, the two-pion system must have orbital angular momentum $l = 0$. Total angular momentum is conserved.

   - Isospin: The K^+ has isospin $I = \frac{1}{2}$, and the two pions can form isospin $I = 0, 1, 2$. **Therefore, the decay cannot proceed through any isospin channel**.

   - Parity: The parity of the two-pion system is given by

$$-1 = P_K = P_\pi P_\pi (-1)^l = (-1)(-1)(-1)^l = (-1)^l. \qquad (40)$$

Since the K^+ has spin 0 and the pions have spin 0, the two-pion system must have orbital angular momentum $l = 0$ to conserve total angular momentum. Therefore, the parity of the two-pion system is $P = +1$, which does not match the parity of the K^+. **Parity is not conserved**.

   - C-parity: Not applicable, since the particles are not neutral.

The isospin and parity are not conserved.

Remark: If we check the decay mode of $K^+ \to \pi^+\pi^0$, we find that the branch ratio is 21.13%. This is a **weak decay**, in which isospin and parity are not conserved.

4. The π^0 has quantum numbers $I^G(J^{PC}) = 1^-(0^{-+})$, the photon has quantum numbers $I^G(J^{PC}) = 0^-(1^{--})$.

- Electric charge: The π^0 has charge 0, and the five photons have charge $0 + 0 + 0 + 0 + 0 = 0$. Electric charge is conserved.

- Angular momentum: The π^0 has spin 0, and the five photons can have total spin $1, 2, 3, 4, 5$. To conserve total angular momentum, the five-photon system must have orbital angular momentum $l = 1, 2, 3, 4, 5$ to form the correct combinations. Hence, the total angular momentum might be conserved.

- Isospin: The π^0 has isospin $I = 1$, and the five photons can form isospin $I = 0$. **Therefore, the decay cannot proceed through any isospin channel**.

- Parity: The parity of the five-photon system is given by

$$-1 = P_\pi = P_\gamma P_\gamma P_\gamma P_\gamma P_\gamma (-1)^l = -1 \times (-1)^l. \tag{41}$$

By checking the possible values of l above, we know the l should be $0, 2, 4$ to conserve the parity.

- C-parity: The C-parity of the five-photon system is given by

$$+1 = C_\pi \neq C_\gamma C_\gamma C_\gamma C_\gamma C_\gamma = (-1)^5 = -1, \tag{42}$$

Hence, the C parity is $C = -1$, which does not match the C-parity of the π^0. **C-parity is not conserved**.

The isospin and C-parity are not conserved.

Remark: If we check the decay mode of $\pi^0 \to 5\gamma$, we find that the branch ratio is 0%. The dominant decay mode is $\pi^0 \to 2\gamma$, which has a branch ratio of 98.823%. This is consistent with our analysis that the decay cannot occur.

□

Question 3

List all states (J^{PC}) with total spin $J = 0, 1, 2$ and P, C parities that cannot be realized as a fermion-antifermion system (i.e., as e^+e^- or quark-antiquark). (Hypothetical particles with such combinations of quantum numbers are called exotic, and are being sought for in experiments, so far unsuccessfully.)

Answer

First we note that a fermion-antifermion system has the following properties:

- The intrinsic parity of a fermion is $+1$, and the intrinsic parity of an antifermion is -1. Therefore, the intrinsic parity of a fermion-antifermion system is -1. Hence the parity of a fermion-antifermion system is given by

$$P = P_f P_{\bar{f}}(-1)^l = -(-1)^l = (-1)^{l+1}, \tag{43}$$

- The C-parity of a fermion-antifermion system is given by

$$C = (-1)^{l+s}, \tag{44}$$

where s is the total spin of the fermion-antifermion system, which can be 0 or 1.

Based on the above properties, we can list all possible states (J^{PC}) with total spin $J = 0, 1, 2$ for a fermion-antifermion system:

- For $J = 0$:

  - When $l = 0$, $s = 0$: $P = (-1)^{0+1} = -1$, $C = (-1)^{0+0} = +1$, so $J^{PC} = 0^{-+}$.
  - When $l = 1$, $s = 1$: $P = (-1)^{1+1} = +1$, $C = (-1)^{1+1} = +1$, so $J^{PC} = 0^{++}$.

- For $J = 1$:

  - When $l = 0$, $s = 1$: $P = (-1)^{0+1} = -1$, $C = (-1)^{0+1} = -1$, so $J^{PC} = 1^{--}$.
  - When $l = 1$, $s = 0$: $P = (-1)^{1+1} = +1$, $C = (-1)^{1+0} = -1$, so $J^{PC} = 1^{+-}$.
  - When $l = 1$, $s = 1$: $P = (-1)^{1+1} = +1$, $C = (-1)^{1+1} = +1$, so $J^{PC} = 1^{++}$.
  - When $l = 2$, $s = 1$: $P = (-1)^{2+1} = -1$, $C = (-1)^{2+1} = -1$, so $J^{PC} = 1^{--}$.

- For $J = 2$:

  - When $l = 1$, $s = 1$: $P = (-1)^{1+1} = +1$, $C = (-1)^{1+1} = +1$, so $J^{PC} = 2^{++}$.
  - When $l = 2$, $s = 0$: $P = (-1)^{2+1} = -1$, $C = (-1)^{2+0} = +1$, so $J^{PC} = 2^{-+}$.
  - When $l = 2$, $s = 1$: $P = (-1)^{2+1} = -1$, $C = (-1)^{2+1} = -1$, so $J^{PC} = 2^{--}$.

– When $l = 3$, $s = 1$: $P = (-1)^{3+1} = +1$, $C = (-1)^{3+1} = +1$, so $J^{PC} = 2^{++}$.

Therefore, the states (J^{PC}) with total spin $J = 0, 1, 2$ and P, C parities that cannot be realized as a fermion-antifermion system are:

– 0^{+-}, 0^{--}

– 1^{-+}

– 2^{+-}

□

Question 4

State which of the following combinations can or cannot exist in a state of isospin $I = 1$, and give the reasons:

1. $\pi^0 \pi^0$

2. $\pi^+ \pi^-$

3. $\pi^+ \pi^+$

4. $\Sigma^0 \pi^0$

5. $\Lambda \pi^0$

Answer

First, we note the isospin quantum numbers of the particles involved:

- The π^0 has isospin $I = 1$, $I_3 = 0$.

- The π^+ has isospin $I = 1$, $I_3 = +1$.

- The π^- has isospin $I = 1$, $I_3 = -1$.

- The Σ^0 has isospin $I = 1$, $I_3 = 0$.

- The Λ has isospin $I = 0$, $I_3 = 0$.

Now we analyze each combination:

1. $\pi^0 \pi^0$: The two π^0's can form isospin $I = 0, 1, 2$. Therefore, the combination can exist in a state of isospin $I = 1$. But since the two pions are identical bosons, their total wavefunction must be symmetric under exchange. When the isospin state is $I = 1$ (which is antisymmetric), the spatial part must be antisymmetric (odd orbital angular momentum) to make the total wavefunction symmetric. Hence, the combination can exist in a state of isospin $I = 1$ **with odd orbital angular momentum** $L = 1, 3, 5$ and so on.
   Remark: I also check the C-G coefficients, and find that the state $|I = 1, I_3 = 0\rangle$ is given by

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|\pi^+ \pi^-\rangle - |\pi^- \pi^+\rangle), \tag{45}$$

$$|\pi^0 \pi^0\rangle = \sqrt{\frac{2}{3}}|2, 0\rangle - \sqrt{\frac{1}{3}}|0, 0\rangle. \tag{46}$$

   The $|1, 0\rangle$ state does not contain the $|\pi^0 \pi^0\rangle$ component. This means that the two π^0's cannot form isospin $I = 1$ state. This is consistent with our analysis above that the two π^0's cannot exist in a

state of isospin $I = 1$. In my opinion, I think we cannot consider orbital angular momentum when we analyze the isospin state. Therefore, I think the two π^0's cannot form isospin $I = 1$ state. But I am not sure about this point.

2. $\pi^+\pi^-$: The π^+ and π^- can form isospin $I = 0, 1, 2$. Therefore, the combination can exist in a state of isospin $I = 1$. Since the two pions are not identical particles, there is no symmetry requirement on their total wavefunction. Hence, the combination can exist in a state of isospin $I = 1$.
   Remark: I also check the C-G coefficients, and find that the state $|I = 1, I_3 = 0\rangle$ is given by

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|\pi^+\pi^-\rangle - |\pi^-\pi^+\rangle). \tag{47}$$

   The $|1, 0\rangle$ state contains the $|\pi^+\pi^-\rangle$ component. This means that the π^+ and π^- can form isospin $I = 1$ state. This is consistent with our analysis above.

3. $\pi^+\pi^+$: The two π^+'s can only form isospin $I = 2$ since $I_3 = +2$. Therefore, the combination cannot exist in a state of isospin $I = 1$.
   Remark: I also check the C-G coefficients, and find that the state $|I = 2, I_3 = 2\rangle$ is given by

$$|2, 2\rangle = |\pi^+\pi^+\rangle. \tag{48}$$

   The $|2, 2\rangle$ state contains the $|\pi^+\pi^+\rangle$ component. This means that the two π^+'s can only form isospin $I = 2$ state. This is consistent with our analysis above.

4. $\Sigma^0\pi^0$: The Σ^0 has isospin $I = 1$, and the π^0 has isospin $I = 1$. For the same realized reason and the C-G coefficients in part (1), this cannot exist in a state of isospin $I = 1$.
   Remark: I also check the C-G coefficients, and find that the state $|I = 1, I_3 = 0\rangle$ is given by

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|\Sigma^+\pi^-\rangle - |\Sigma^-\pi^+\rangle), \tag{49}$$

$$|\Sigma^0\pi^0\rangle = \sqrt{\frac{2}{3}}|2, 0\rangle - \sqrt{\frac{1}{3}}|0, 0\rangle. \tag{50}$$

   The $|1, 0\rangle$ state does not contain the $|\Sigma^0\pi^0\rangle$ component. This means that the Σ^0 and π^0 cannot form isospin $I = 1$ state. This is consistent with our analysis above.

5. $\Lambda\pi^0$: The Λ has isospin $I = 0$, and the π^0 has isospin $I = 1$. The combination can only form isospin $I = 1$. Therefore, the combination can exist in a state of isospin $I = 1$.
   Remark: I also check the C-G coefficients, and find that the state $|I = 1, I_3 = 0\rangle$ is given by

$$|1, 0\rangle = |\Lambda\pi^0\rangle. \tag{51}$$

   The $|1, 0\rangle$ state contains the $|\Lambda\pi^0\rangle$ component. This means that the Λ and π^0 can form isospin $I = 1$ state. This is consistent with our analysis above. $\qquad\square$

Problem Set 3 due 9:30 AM, Monday, October 13th

Question 1

p-d reactions

Consider the reactions

$$p + d \to \pi^+ + {}^3\text{H}, \quad p + d \to \pi^0 + {}^3\text{He}. \tag{52}$$

Since the deuteron is in a ${}^3S_1$ state, it must be an isospin singlet. Therefore, the initial state $p + d$ is a pure $I = \frac{1}{2}$ state. Given that ${}^3\text{H}$ and ${}^3\text{He}$ form an isodoublet, write down the isospin decomposition of the final states, and from this, the ratio of the two cross sections.

Answer

First, we can use I_3 to decide the isospin for ${}^3\text{H}$ and ${}^3\text{He}$. See the initial state $p + d$ has $I_3 = +\frac{1}{2}$, so the final state must also have $I_3 = +\frac{1}{2}$. Since π^+ has $I_3 = +1$ and π^0 has $I_3 = 0$, we can conclude that ${}^3\text{H}$ has $I_3 = -\frac{1}{2}$ and ${}^3\text{He}$ has $I_3 = +\frac{1}{2}$. Therefore, ${}^3\text{H}$ and ${}^3\text{He}$ form an isodoublet with $I = \frac{1}{2}$. Now we can write down the isospin decomposition of the final states. For the first reaction, we have

$$|\pi^+ + {}^3\text{H}\rangle = |\pi^+\rangle \otimes |{}^3\text{H}\rangle = |1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle. \tag{53}$$

Using the Clebsch-Gordan coefficients, we can decompose this into total isospin

$$|1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{54}$$

For the second reaction, we have

$$|\pi^0 + {}^3\text{He}\rangle = |\pi^0\rangle \otimes |{}^3\text{He}\rangle = |1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{55}$$

Using the Clebsch-Gordan coefficients, we can decompose this into total isospin

$$|1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}, \tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{56}$$

Now, since the initial state $p + d$ is a pure $I = \frac{1}{2}$ state, only the $I = \frac{1}{2}$ component of the final states will contribute to the cross sections. Therefore, we can write the amplitudes for the two reactions as

$$\mathcal{A}(p + d \to \pi^+ + {}^3\text{H}) \propto \sqrt{\tfrac{2}{3}}, \tag{57}$$

$$\mathcal{A}(p + d \to \pi^0 + {}^3\text{He}) \propto -\tfrac{1}{\sqrt{3}}. \tag{58}$$

The cross sections are proportional to the square of the amplitudes, so we have

$$\sigma(p + d \to \pi^+ + {}^3\text{H}) \propto \left| \sqrt{\tfrac{2}{3}} \right|^2 = \tfrac{2}{3}, \tag{59}$$

$$\sigma(p + d \to \pi^0 + {}^3\text{He}) \propto \left| -\tfrac{1}{\sqrt{3}} \right|^2 = \tfrac{1}{3}. \tag{60}$$

Finally, the ratio of the two cross sections is

$$\frac{\sigma(p + d \to \pi^+ + {}^3\text{H})}{\sigma(p + d \to \pi^0 + {}^3\text{He})} = \frac{\tfrac{2}{3}}{\tfrac{1}{3}} = 2. \tag{61}$$

□

Question 2

Particle production by strong interactions

Explain why the processes $\pi^- + p \to \pi^+ + \Sigma^-$, $\quad \pi^- + p \to K^0 + n$, $\quad \pi^- + p \to \Sigma^+ + K^-$ cannot be observed.

Answer

Before we analyze the processes, let's summarize the quantum numbers of the particles involved:

- π^-: $I = 1$, $I_3 = -1$, $S = 0$, $B = 0$

- π^+: $I = 1$, $I_3 = +1$, $S = 0$, $B = 0$

- p: $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$, $S = 0$, $B = 1$

- n: $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, $S = 0$, $B = 1$

- Σ^-: $I = 1$, $I_3 = -1$, $S = -1$, $B = 1$

- Σ^+: $I = 1$, $I_3 = +1$, $S = -1$, $B = 1$

- K^0: $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$, $S = +1$, $B = 0$

- K^-: $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, $S = -1$, $B = 0$

Now, let's analyze each process:

(a) $\pi^- + p \to \pi^+ + \Sigma^-$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$
- Final state: $I_3 = +1 - 1 = 0$, $S = 0 - 1 = -1$, $B = 0 + 1 = 1$

The strangeness S changes from 0 to -1, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to 0. Therefore, this process cannot be observed.

(b) $\pi^- + p \to K^0 + n$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$
- Final state: $I_3 = +\frac{1}{2} - \frac{1}{2} = 0$, $S = +1 + 0 = +1$, $B = 0 + 1 = 1$

The strangeness S changes from 0 to +1, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to 0. Therefore, this process cannot be observed.

(c) $\pi^- + p \to \Sigma^+ + K^-$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$

- Final state: $I_3 = +1 - \frac{1}{2} = +\frac{1}{2}$, $S = -1 - 1 = -2$, $B = 1 + 0 = 1$

The strangeness S changes from 0 to -2, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to $+\frac{1}{2}$. Therefore, this process cannot be observed.

□

Question 3

SU(2) invariants and pseudoreal representations

(a) Show that $\delta^a{}_b$ and ϵ_{ab} are invariant tensors under SU(2) transformations.

(b) The nucleon doublet $N^a = \binom{p}{n}, a = 1, 2$ transforms as the fundamental **2** of SU(2), while its conjugate $\bar{N}_a \equiv (N^a)^\dagger = (\bar{p}, \bar{n})$ transforms as $\bar{\mathbf{2}}$. Use $\delta^a{}_b$ to form an SU(2) invariant with $N, \bar{N}$ and write it explicitly in terms of the proton and neutron fields.

(c) Define $\tilde{N}^b = \epsilon^{bc} \bar{N}_c^T$ which maps the $\bar{\mathbf{2}}$ representation (lower index) into **2** (upper index). Construct an SU(2) invariant with $N, \tilde{N}$ using ϵ_{ab}, and write it in terms of the components. Verify that the result is identical to part (b), demonstrating that the **2** and $\bar{\mathbf{2}}$ representations are equivalent (or pseudoreal) in SU(2) and that any invariant constructed with $\delta^a{}_b$ can be rewritten using ϵ_{ab}.

(d) Consider SU(3), with the quark triplet $q^a (a = 1, 2, 3)$ transforming as **3** and its conjugate $\bar{q}_a \equiv (q^a)^\dagger$ transforming as $\bar{\mathbf{3}}$. Discuss why a similar mapping using the SU(3) invariant ϵ_{abc} does not make **3** and $\bar{\mathbf{3}}$ equivalent. Write down the possible SU(3) invariants involving $q, \bar{q}$.

Answer

(a)

$$\delta^a{}_b \to = \delta'^a{}_b = U^a{}_c \delta^c{}_d (U^\dagger)^d{}_b = U^a{}_c (U^\dagger)^c{}_b = \mathbf{1}^a{}_b = \delta^a{}_b, \tag{62}$$

$$\epsilon_{ab} \to \epsilon'_{ab} = (U^\dagger)^c{}_a (U^\dagger)^d{}_b \epsilon_{cd} = \det(U^\dagger)\epsilon_{ab} = \epsilon_{ab}. \tag{63}$$

(b)

Using $\delta^a{}_b$, we can form the invariant

$$\bar{N}_a N^a = \delta^a{}_b \bar{N}_a N^b = \bar{p}p + \bar{n}n. \tag{64}$$

We can verify that this is indeed invariant under SU(2) transformations:

$$\bar{N}_a N^a \to \bar{N}'_a N'^a = \bar{N}_b (U^\dagger)^b{}_a U^a{}_c N^c = \bar{N}_b \delta^b{}_c N^c = \bar{N}_a N^a. \tag{65}$$

(c)

Using ϵ_{ab}, we can form the invariant

$$\epsilon_{ab} N^a \tilde{N}^b = \epsilon_{ab} N^a \epsilon^{bc} \bar{N}_c^T = \delta_a{}^c N^a \bar{N}_c^T = N^a \bar{N}_a^T = \bar{N}_a N^a = \bar{p}p + \bar{n}n. \tag{66}$$

We can verify that this is indeed invariant under SU(2) transformations:

$$\epsilon_{ab}N^a\tilde{N}^b \rightarrow \epsilon_{ab}N'^a\tilde{N}'^b = \epsilon_{ab}U^a{}_cN^cU^b{}_d\tilde{N}^d = \det(U)\epsilon_{cd}N^c\tilde{N}^d = \epsilon_{cd}N^c\tilde{N}^d. \tag{67}$$

This demonstrates that the $\mathbf{2}$ and $\overline{\mathbf{2}}$ representations are equivalent (or pseudoreal) in SU(2) and that any invariant constructed with $\delta^a{}_b$ can be rewritten using ϵ_{ab}.

(d)

The possible SU(3) invariants involving q and $\bar{q}$ are:

$$\bar{q}_aq^a, \quad \epsilon_{abc}q^aq^bq^c, \quad \epsilon^{abc}\bar{q}_a\bar{q}_b\bar{q}_c. \tag{68}$$

We can start with the q^aq^b,

$$q^aq^b = \frac{1}{2}(q^aq^b + q^bq^a) + \frac{1}{2}(q^aq^b - q^bq^a) = S^{ab} + A^{ab}, \tag{69}$$

where S^{ab} is symmetric and A^{ab} is antisymmetric. Moreover, for the antisymmetric part, we can use ϵ_{abc} to lower an index and get

$$\theta_c = \epsilon_{abc}A^{ab} = \epsilon_{abc}\frac{1}{2}(q^aq^b - q^bq^a) = \epsilon_{abc}q^aq^b. \tag{70}$$

Now we can see that θ_c transforms as $\overline{\mathbf{3}}$. In order to see this, we can apply an SU(3) transformation:

$$\theta'_c = \epsilon_{abc}q'^aq'^b = \epsilon_{abc}U^a{}_{a'}U^b{}_{b'}q^{a'}q^{b'} = \epsilon_{abc'}\delta^{c'}{}_cU^a{}_{a'}U^b{}_{b'}q^{a'}q^{b'} \tag{71}$$

$$= \epsilon_{abc'}U^{c'}{}_k(U^\dagger)^k{}_cU^a{}_{a'}U^b{}_{b'}q^{a'}q^{b'} = \det(U)(U^\dagger)^k{}_c\epsilon_{a'b'k}q^{a'}q^{b'} \tag{72}$$

$$= (U^\dagger)^k{}_c\epsilon_{a'b'k}q^{a'}q^{b'} = (U^\dagger)^k{}_c\theta_k = (U^\dagger)^{c'}{}_c\theta_{c'}. \tag{73}$$

Therefore, q^aq^b can be decomposed into a symmetric part transforming as $\mathbf{6}$ and an antisymmetric part transforming as $\overline{\mathbf{3}}$. This shows that there is no way to map $\overline{\mathbf{3}}$ back to $\mathbf{3}$ using ϵ_{abc}, unlike the case in SU(2) where we could use ϵ_{ab} to map between $\mathbf{2}$ and $\overline{\mathbf{2}}$. Hence, the representations $\mathbf{3}$ and $\overline{\mathbf{3}}$ are not equivalent in SU(3). $\qquad\square$

Question 4

Applications of U-spin

(a) Show that $U_{\pm} = t_6 \pm it_7$ and $U_3 = \left(\sqrt{3}t_8 - t_3\right)/2$ satisfy the SU(2) algebra

$$[U_3, U_{\pm}] = \pm U_{\pm}, \quad [U_+, U_-] = 2U_3.$$

(b) Show that the charge operator $Q = t_3 + t_8/\sqrt{3}$ is a U-scalar i.e. it has U-spin $U = 0$ or $[Q, U_i] = 0$ for $i = \pm, 3$. Write the electromagnetic current operator in terms of quark fields.

(c) Show that for the meson octet, the ($U_3 = 0$) component of the U-triplet is $\pi_U^0 = \left(-\pi^0 + \sqrt{3}\eta\right)/2$, and the U-singlet is $\eta_U^0 = \left(\sqrt{3}\pi^0 + \eta\right)/2$. Since π_U^0 is a U -spin vector component it cannot couple to the electromagnetic current. Show that for the 2γ decay mode, $\langle \pi^0 \mid 2\gamma \rangle = \sqrt{3}\langle \eta \mid 2\gamma \rangle$. How does this U-spin prediction compare with the experimental decay widths?

Answer

(a)

Here we recap the Gell-Mann matrices t_3, t_6, t_7, t_8:

$$t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{74}$$

$$t_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{75}$$

Hence, we have

$$U_+ = t_6 + it_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_- = t_6 - it_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{76}$$

$$U_3 = \frac{\sqrt{3}t_8 - t_3}{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}. \tag{77}$$

Now we can verify the SU(2) algebra:

$$[U_3, U_+] = U_3 U_+ - U_+ U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{78}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = U_+, \tag{79}$$

$$[U_3, U_-] = U_3 U_- - U_- U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{80}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -U_-, \tag{81}$$

$$[U_+, U_-] = U_+ U_- - U_- U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{82}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 2U_3. \tag{83}$$

(b)

First, we write down the charge operator:

$$Q = t_3 + \frac{t_8}{\sqrt{3}} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{84}$$

Now we can verify that Q is a U-scalar:

$$[Q, U_+] = QU_+ - U_+Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{85}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} = 0, \tag{86}$$

$$[Q, U_-] = QU_- - U_-Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{87}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix} = 0, \tag{88}$$

$$[Q, U_3] = QU_3 - U_3Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{89}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} = 0. \tag{90}$$

Thus, we have shown that $[Q, U_i] = 0$ for $i = \pm, 3$, confirming that Q is a U-scalar. In QFT, the electormagnetic current operator in terms of fermion fields is given by

$$J_\mu^{\text{em}} = \sum_f Q_f \overline{\psi}_f \gamma_\mu \psi_f, \tag{91}$$

where the sum runs over all fermion flavors f, Q_f is the electric charge of the fermion in units of the elementary charge, ψ_f is the fermion field, and γ_μ are the gamma matrices. For the quark fields u, d, s, the electromagnetic current operator can be explicitly written as

$$J_\mu^{\text{em}} = \frac{2}{3}\overline{u}\gamma_\mu u - \frac{1}{3}\overline{d}\gamma_\mu d - \frac{1}{3}\overline{s}\gamma_\mu s. \tag{92}$$

(c)

First, we can express π^0 and η in terms of quark content:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d}), \tag{93}$$

$$\eta = \frac{1}{\sqrt{6}}(u\overline{u} + d\overline{d} - 2s\overline{s}). \tag{94}$$

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Now, we can construct the U-triplet and U-singlet components:

$$\pi_U^0 = \frac{-\pi^0 + \sqrt{3}\eta}{2} = \frac{-\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) + \sqrt{3}\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}{2} \tag{95}$$

$$= \frac{-\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) + \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d} - 2s\bar{s})}{2} = \frac{2d\bar{d} - 2s\bar{s}}{2\sqrt{2}} = \frac{d\bar{d} - s\bar{s}}{\sqrt{2}}, \tag{96}$$

$$\eta_U^0 = \frac{\sqrt{3}\pi^0 + \eta}{2} = \frac{\sqrt{3}\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) + \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}{2} \tag{97}$$

$$= \frac{1}{2\sqrt{6}}(3u\bar{u} - 3d\bar{d} + u\bar{u} + d\bar{d} - 2s\bar{s}) = \frac{4u\bar{u} - 2d\bar{d} - 2s\bar{s}}{2\sqrt{6}} = \frac{2u\bar{u} - d\bar{d} - s\bar{s}}{\sqrt{6}} \tag{98}$$

$$= \frac{2(u\bar{u} + d\bar{d} + s\bar{s})}{\sqrt{6}} - \frac{3d\bar{d} + 3s\bar{s}}{\sqrt{6}}. \tag{99}$$

In order to see they are indeed U-triplet and U-singlet, we can analyze their quark content with U-spin quantum numbers:

- u quark: $U = 0$ (U-spin singlet)

- d quark: $U = \frac{1}{2}, U_3 = +\frac{1}{2}$

- s quark: $U = \frac{1}{2}, U_3 = -\frac{1}{2}$

- $\bar{u}$ quark: $U = 0$ (U-spin singlet)

- $\bar{d}$ quark: $U = \frac{1}{2}, U_3 = -\frac{1}{2}$

- $-\bar{s}$ quark: $U = \frac{1}{2}, U_3 = +\frac{1}{2}$

For $\bar{s}$ quark, we need to add a minus sign due to the same reason in the isospin $SU(2)$ case. Hence, we can see that π_U^0 is a U-triplet component with $U = 1, U_3 = 0$, and η_U^0 is a U-singlet with $U = 0$.

Since π_U^0 is a U-spin vector component, it cannot couple to the electromagnetic current. Therefore, we have

$$\langle \pi_U^0 | 2\gamma \rangle = 0 \implies \left\langle \frac{-\pi^0 + \sqrt{3}\eta}{2} | 2\gamma \right\rangle = 0 \implies -\frac{1}{2}\langle \pi^0 | 2\gamma \rangle + \frac{\sqrt{3}}{2}\langle \eta | 2\gamma \rangle = 0. \tag{100}$$

$$\implies \langle \pi^0 | 2\gamma \rangle = \sqrt{3}\langle \eta | 2\gamma \rangle. \tag{101}$$

The decay width Γ is proportional to the square of the amplitude, so we have

$$\Gamma(\pi^0 \to 2\gamma) \propto |\langle \pi^0 | 2\gamma \rangle|^2, \quad \Gamma(\eta \to 2\gamma) \propto |\langle \eta | 2\gamma \rangle|^2. \tag{102}$$

Using the relation we derived, we find

$$\frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta \to 2\gamma)} \propto \frac{|\langle \pi^0 | 2\gamma \rangle|^2}{|\langle \eta | 2\gamma \rangle|^2} = 3. \tag{103}$$

Here I ignore the mass difference between π^0 and η for simplicity. Experimentally, the decay widths are approximately:

$$\Gamma(\pi^0 \to 2\gamma) \approx 7.8 \text{ eV}, \tag{104}$$

$$\Gamma(\eta \to 2\gamma) \approx 0.51 \text{ keV} = 510 \text{ eV}. \tag{105}$$

Thus, the experimental ratio is

$$\frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta \to 2\gamma)} \approx \frac{7.8 \text{ eV}}{510 \text{ eV}} \approx 0.0153, \tag{106}$$

which is significantly different from the U-spin prediction of 3. $\qquad\square$

Remark: I think the U-singlet should be $\eta_U^0 = \frac{d\bar{d}+s\bar{s}}{\sqrt{2}}$, which is different from the one given in the question. It is a more natural choice since it is orthogonal to the U-triplet component $\pi_U^0 = \frac{d\bar{d}-s\bar{s}}{\sqrt{2}}$. If we use this definition, we can see that η_U^0 does not contain any $u\bar{u}$ component. However, using this definition, we cannot express η_U^0 in terms of π^0 and η as given in the question.

Remark: I check the decay widths formula for π^0 (I quote eq. (30.14) from Schwarz's QFT book):

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \approx 7.73 \text{ eV}. \tag{107}$$

It show that Γ is proportional to m^3, so the mass difference between π^0 and η cannot be ignored. Including the mass difference, we have

$$\frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta \to 2\gamma)} = 3\left(\frac{m_\pi}{m_\eta}\right)^3 = 3\left(\frac{139.6 \text{ MeV}}{547.862 \text{ MeV}}\right)^3 \approx 0.049, \tag{108}$$

which is still significantly different from the experimental value of approximately 0.0153.

Problem Set 4 due 9:30 AM, Monday, October 27th

Question 1

Gell-Mann Okubo for the baryon octet

The generators $t_a(a = 1, \ldots, 8)$ of SU(3) are normalized as $t_a = \frac{\lambda_a}{2}$ with $\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab}$, where λ_a are the Gell-Mann matrices. They satisfy $[t_a, t_b] = i f_{abc} t_c$ and $\{t_a, t_b\} = \frac{1}{3}\delta_{ab}\mathbf{1} + d_{abc} t_c$, where f_{abc} are totally antisymmetric and d_{abc} are totally symmetric structure constants.

Let B and $\bar{B}$ be the baryon octet 3×3 traceless matrices, expanded in the generator basis as $B = B^i t_i$ and $\bar{B} = \bar{B}^i t_i$ where $B^i, \bar{B}^i$ are the adjoint components. Define the two bilinear combinations $O_A \equiv [\bar{B}, B] = \bar{B}B - B\bar{B}$ and $O_S \equiv \{\bar{B}, B\} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}B)$.

(a) Show that both O_A and O_S are traceless and therefore transform in the adjoint (octet) representation.

(b) Expand O_A and O_S in components using the generator basis and show that $O_A = i\left(\bar{B}^i B^j\right) f_{ijk} t_k$ and $O_S = \left(\bar{B}^i B^j\right) d_{ijk} t_k$, so that $(O_A)^k = i f_{ijk} \bar{B}^i B^j$ and $(O_S)^k = d_{ijk} \bar{B}^i B^j$.

(c) Introduce a flavor-breaking spurion $H_8 = H_8^i t_i$, with real components H_8^i. Construct the two independent SU(3)-invariant mass terms:

$$S_f = (O_A)^a_b \,(H_8)^b_a, \quad S_d = (O_S)^a_b \,(H_8)^b_a \tag{109}$$

Assuming H_8 points in the 8 -direction (i.e. $H_8^i \propto \delta_{i8}$), argue that S_f and S_d correspond to the f-type and d-type symmetry breaking terms in the baryon mass operator, respectively.

(d) Given that an adjoint operator O_8 acts on an octet state B as $O_8(B) = [O_8, B]$, show that the invariant scalars in (c) are equivalent to the matrix elements $S_f \propto \langle \bar{B}|t_8|B\rangle \equiv \mathrm{Tr}\left(\bar{B}\,[t_8, B]\right)$ and $S_d \propto \langle \bar{B}|d_{8ij}t_i t_j|B\rangle \equiv \mathrm{Tr}\left(\bar{B}\,[d_{8ij}\,[t_i, [t_j, B]]\right)$.

(e) Hence, argue that for each entry B_{ij} of the baryon octet matrix, $S_f \propto Y$ and $S_d \propto I(I+1) - Y^2/4$ where I, Y are the isospin and hypercharge of the baryon B, respectively, thereby reproducing the Gell-Mann-Okubo mass formula for the baryon octet.

(Hint: Verify, entrywise, that $\left[\frac{2}{\sqrt{3}}t_8, B\right] = YB$ and the normalized operator $\frac{2}{\sqrt{3}}d_{8ij}\,[t_i, [t_j, B]] + \frac{1}{3}\,[t_i, [t_i, B]] = (I(I+1) - Y^2/4)\,B$ acts diagonally on each baryon field. The $[t_i, [t_i, B]]$ term is the adjoint Casimir ($SU(3)$ singlet) which just shifts all octet components uniformly so that the Λ eigenvalue becomes 0 . It can be absorbed into the overall singlet part of the GMO formula. On the diagonal remember B_{11}, B_{22} and B_{33} mix Σ^0 and Λ, so $B_{\mathrm{diag}} = \Sigma^0 \mathrm{diag}(1, -1, 0)/\sqrt{2} + \Lambda\,\mathrm{diag}(1, 1, -2)/\sqrt{6}$.

Answer

(a)

To show that both O_A and O_S are traceless, we first understand their definitions:

$$O_A = [\bar{B}, B] = \bar{B}B - B\bar{B} = [\bar{B}, B] \tag{110}$$

$$= [\bar{B}^i t_i, B^j t_j] = \bar{B}^i B^j [t_i, t_j] = i\bar{B}^i B^j f_{ijk} t_k \tag{111}$$

Taking the trace of O_A:

$$\text{Tr}(O_A) = \text{Tr}(iBar B^i B^j f_{ijk} t_k) = i\bar{B}^i B^j f_{ijk} \text{Tr}(t_k) = 0. \tag{112}$$

Similarly, for O_S:

$$O_S = \{\bar{B}, B\} - \frac{2}{3}\mathbf{1}\,\text{Tr}(\bar{B}B) = \bar{B}B + B\bar{B} - \frac{2}{3}\mathbf{1}\,\text{Tr}(\bar{B}B) \tag{113}$$

$$= (\bar{B}^i t_i)(B^j t_j) + (B^j t_j)(\bar{B}^i t_i) - \frac{2}{3}\mathbf{1}\,\text{Tr}(\bar{B}^i t_i B^j t_j) \tag{114}$$

$$= \bar{B}^i B^j \{t_i, t_j\} - \frac{2}{3}\mathbf{1}\,\text{Tr}(\bar{B}^i B^j t_i t_j) \tag{115}$$

$$= \bar{B}^i B^j \left(\frac{1}{3}\delta_{ij}\mathbf{1} + d_{ijk} t_k\right) - \frac{2}{3}\mathbf{1}\left(\frac{1}{2}\bar{B}^i B^j \delta_{ij}\right) \tag{116}$$

$$= \bar{B}^i B^j d_{ijk} t_k \tag{117}$$

Taking the trace of O_S:

$$\text{Tr}(O_S) = \text{Tr}(\bar{B}^i B^j d_{ijk} t_k) = \bar{B}^i B^j d_{ijk} \text{Tr}(t_k) = 0. \tag{118}$$

Thus, both O_A and O_S are traceless and transform in the adjoint (octet) representation.

(b)

Expanding O_A and O_S in components using the generator basis, we have already derived:

$$O_A = i\bar{B}^i B^j f_{ijk} t_k \tag{119}$$

$$O_S = \bar{B}^i B^j d_{ijk} t_k \tag{120}$$

Thus, the components are:

$$(O_A)^k = if_{ijk}\bar{B}^i B^j \tag{121}$$

$$(O_S)^k = d_{ijk}\bar{B}^i B^j \tag{122}$$

(c)

Introducing a flavor-breaking spurion $H_8 = H_8^i t_i$, with real components H_8^i, we can construct the two

independent SU(3)-invariant mass terms:

$$S_f = (O_A)^a_b (H_8)^b_a = \text{Tr}(O_A H_8) \tag{123}$$

$$S_d = (O_S)^a_b (H_8)^b_a = \text{Tr}(O_S H_8) \tag{124}$$

Assuming H_8 points in the 8-direction (i.e., $H_8^i \propto \delta_{i8}$), we can write:

$$H_8 = H_8^i t_i = H_8^8 t_8 \tag{125}$$

Substituting this into the expressions for S_f and S_d:

$$S_f = \text{Tr}(O_A H_8) = \text{Tr}(i f_{ijk} \bar{B}^i B^j t_k H_8^l t_l) = i H_8^8 f_{ij8} \bar{B}^i B^j \, \text{Tr}(t_k t_8) = \frac{i}{2} H_8^8 f_{ij8} \bar{B}^i B^j \tag{126}$$

$$S_d = \text{Tr}(O_S H_8) = \text{Tr}(d_{ijk} \bar{B}^i B^j t_k H_8^l t_l) = H_8^8 d_{ij8} \bar{B}^i B^j \, \text{Tr}(t_k t_8) = \frac{1}{2} H_8^8 d_{ij8} \bar{B}^i B^j \tag{127}$$

Since O_A and O_S are octet operators, S_f and S_d correspond to the f-type and d-type symmetry breaking terms in the baryon mass operator, respectively.

(d)

$$\langle \bar{B} | t_8 | B \rangle \equiv \text{Tr}(\bar{B}[t_8, B]) \tag{128}$$

$$= \text{Tr}(\bar{B}(t_8 B - B t_8)) = \text{Tr}(\bar{B} t_8 B) - \text{Tr}(\bar{B} B t_8) \tag{129}$$

$$= \text{Tr}(B \bar{B} t_8) - \text{Tr}(\bar{B} B t_8) \tag{130}$$

$$= \text{Tr}((\bar{B} B - B \bar{B}) t_8) = \text{Tr}(O_A t_8) \propto S_f \tag{131}$$

This is beccause $S_f = \text{Tr}(O_A H_8) = \text{Tr}(O_A H_8^8 t_8) = H_8^8 \, \text{Tr}(O_A t_8)$. Similarly, for S_d:

$$\langle \bar{B} | d_{8ij} t_i t_j | B \rangle \equiv \text{Tr}(\bar{B} d_{8ij}[t_i, [t_j, B]]) \tag{132}$$

$$= \text{Tr}(\bar{B}^\alpha B^\beta t_\alpha d_{8ij}[t_i, [t_j, t_\beta]]) \tag{133}$$

$$= \bar{B}^\alpha B^\beta d_{8ij} \, \text{Tr}(t_\alpha [t_i, [t_j, t_\beta]]) \tag{134}$$

$$= \bar{B}^\alpha B^\beta d_{8ij} \, \text{Tr}(t_\alpha [t_i, i f_{j\beta\gamma} t_\gamma]) \tag{135}$$

$$= i \bar{B}^\alpha B^\beta d_{8ij} f_{j\beta\gamma} \, \text{Tr}(t_\alpha [t_i, t_\gamma]) \tag{136}$$

$$= i \bar{B}^\alpha B^\beta d_{8ij} f_{j\beta\gamma} i f_{i\alpha\delta} \, \text{Tr}(t_\alpha t_\delta) \tag{137}$$

$$= -\frac{1}{2} \bar{B}^\alpha B^\beta d_{8ij} f_{j\beta\gamma} f_{i\alpha\gamma} \tag{138}$$

$$= -\frac{1}{2} \bar{B}^a B^b d_{8ij} f_{jbc} f_{iac} \tag{139}$$

By `Mathematica`, we have

$$d_{8ij} f_{jbc} f_{iac} = 3/2 d_{8ab} \tag{140}$$

Thus,

$$\langle \bar{B}|d_{8ij}t_it_j|B\rangle = -\frac{3}{4}\bar{B}^a B^b d_{8ab} = -\frac{3}{2}\mathrm{Tr}(O_S t_8) \propto S_d \tag{141}$$

This is because $S_d = \mathrm{Tr}(O_S H_8) = \mathrm{Tr}(O_S H_8^8 t_8) = H_8^8 \mathrm{Tr}(O_S t_8)$.

(e)

First, we write down the explicit form of B:

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{pmatrix} \tag{142}$$

Next, we write down the hypercharge Y and isospin I and $I(I+1) - Y^2/4$ values for each baryon:

$$p : Y = 1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{143}$$
$$n : Y = 1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{144}$$
$$\Sigma^+ : Y = 0, I = 1, I(I+1) - Y^2/4 = 2 - 0 = 2 \tag{145}$$
$$\Sigma^0 : Y = 0, I = 1, I(I+1) - Y^2/4 = 2 - 0 = 2 \tag{146}$$
$$\Sigma^- : Y = 0, I = 1, I(I+1) - Y^2/4 = 2 - 0 = 2 \tag{147}$$
$$\Xi^0 : Y = -1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{148}$$
$$\Xi^- : Y = -1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{149}$$
$$\Lambda^0 : Y = 0, I = 0, I(I+1) - Y^2/4 = 0 - 0 = 0 \tag{150}$$

Now, we compute $\left[\frac{2}{\sqrt{3}}t_8, B\right]$, and it can be easily show that (see my *Mathematica* notebook for details):

$$[\frac{2}{\sqrt{3}}t_8, B] = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & n \\ -(-\Xi^-) & -\Xi^0 & 0 \end{pmatrix} = YB \tag{151}$$

where Y is the hypercharge of the baryon B. Next, we compute the normalized operator $\frac{2}{\sqrt{3}}d_{8ij}[t_i, [t_j, B]] + \frac{1}{3}[t_i, [t_i, B]]$:

$$\frac{2}{\sqrt{3}}d_{8ij}[t_i, [t_j, B]] + \frac{1}{3}[t_i, [t_i, B]] \tag{152}$$

$$= \begin{pmatrix} \sqrt{2}\Sigma^0 & 2\Sigma^+ & \frac{1}{2}p \\ 2\Sigma^- & -\sqrt{2}\Sigma^0 & \frac{1}{2}n \\ \frac{1}{2}(-\Xi^-) & \frac{1}{2}\Xi^0 & 0 \end{pmatrix} = \begin{pmatrix} 2\frac{\Sigma^0}{\sqrt{2}} & 2\Sigma^+ & \frac{1}{2}p \\ 2\Sigma^- & -2\frac{\Sigma^0}{\sqrt{2}} & \frac{1}{2}n \\ \frac{1}{2}(-\Xi^-) & \frac{1}{2}\Xi^0 & 0 \end{pmatrix} = \left(I(I+1) - \frac{Y^2}{4}\right)B \tag{153}$$

where $I(I+1) - \frac{Y^2}{4}$ is the value for each baryon B. Thus, we have shown that for each entry B_{ij} of

the baryon octet matrix, $S_f \propto Y$ and $S_d \propto I(I+1) - Y^2/4$, thereby reproducing the Gell-Mann-Okubo mass formula for the baryon octet, meaning

$$M_B(Y, I) = M_0 + M_A Y + M_S \left(I(I+1) - \frac{Y^2}{4} \right), \tag{154}$$

where M_0, M_A, M_S are constants. $\qquad\square$

Question 2

ρ-ω mixing

The vector mesons $\rho(770)$ and $\omega(782)$ are very close in mass. For this reason the effects of isospin violation are somewhat enhanced in these mesons and can be parametrized in terms of ρ-ω mixing. Namely, the physical ρ^0 and ω mesons can be viewed as orthogonal mixed states of a pure isospin triplet and isospin singlet:

$$\rho^0 = \cos\theta\frac{(u\bar{u} - d\bar{d})}{\sqrt{2}} + \sin\theta\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}},$$

$$\omega = -\sin\theta\frac{(u\bar{u} - d\bar{d})}{\sqrt{2}} + \cos\theta\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}},$$

where θ is a (small) mixing angle.

(a) Determine θ (up to a sign) using experimental data on the decay $\omega \to \pi^+\pi^-$. Estimate the error in the value of the mixing angle.

(b) Using the value of θ predict the decay rates $\Gamma(\rho^0 \to e^+e^-)$ and $\Gamma(\omega \to e^+e^-)$, assuming the amplitude for a quark pair annihilation into an e^+e^- pair is proportional to the electric charge Q of the quark.

(c) Assume that the transition amplitude between different spin states of a $q\bar{q}$ quark pair with emission of a photon: $(q\bar{q}) \to (q\bar{q}) + \gamma$ is proportional to the quark electric charge Q. Use the value of the ρ-ω mixing angle θ to determine the ratios of the decay rates:

   (i) $\Gamma(\rho^0 \to \pi^0\gamma)/\Gamma(\omega^0 \to \pi^0\gamma)$,

   (ii) $\Gamma(\rho^0 \to \eta\gamma)/\Gamma(\omega^0 \to \eta\gamma)$.

   Compare with the PDG experimental data. How does the inclusion of ρ-ω mixing improve the agreement with the data?

Answer

Before we start, we declare the notation for physical states and pure isospin states as follows:

$$|\rho^0\rangle = \cos\theta|\rho_I^0\rangle + \sin\theta|\omega_I\rangle, \tag{155}$$

$$|\omega\rangle = -\sin\theta|\rho_I^0\rangle + \cos\theta|\omega_I\rangle, \tag{156}$$

$$|\rho_I^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) = |I = 1, I_3 = 0\rangle, \tag{157}$$

$$|\omega_I\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = |I = 0, I_3 = 0\rangle. \tag{158}$$

(a)

The decay amplitude for $\omega \to \pi^+\pi^-$ and $\rho^0 \to \pi^+\pi^-$ can be written as:

$$\mathcal{M}(\omega \to \pi^+\pi^-) = \langle \pi^+\pi^- | H | \omega \rangle \tag{159}$$

$$= \langle \pi^+\pi^- | H | -\sin\theta | \rho_I^0 \rangle + \cos\theta | \omega_I \rangle \tag{160}$$

$$= -\sin\theta \langle \pi^+\pi^- | H | \rho_I^0 \rangle + \cos\theta \langle \pi^+\pi^- | H | \omega_I \rangle \tag{161}$$

$$\mathcal{M}(\rho^0 \to \pi^+\pi^-) = \langle \pi^+\pi^- | H | \rho^0 \rangle \tag{162}$$

$$= \langle \pi^+\pi^- | H | \cos\theta | \rho_I^0 \rangle + \sin\theta | \omega_I \rangle \tag{163}$$

$$= \cos\theta \langle \pi^+\pi^- | H | \rho_I^0 \rangle + \sin\theta \langle \pi^+\pi^- | H | \omega_I \rangle. \tag{164}$$

Actually, since both of ρ^0 and ω are vector mesons with $J^{PC} = 1^{--}$ and $\pi^+\pi^-$ is a pseudoscalar meson pair with $J^{PC} = 0^{-+}$, the decay must proceed via a P-wave to conserve angular momentum and parity due to strong interaction. Thus the spatial wave function of $\pi^+\pi^-$ must be antisymmetric under exchange of the two pions. Since pions are bosons, the total wave function must be symmetric under exchange of the two pions. Therefore, the isospin wave function of $\pi^+\pi^-$ must also be antisymmetric under exchange of the two pions. This means that the $\pi^+\pi^-$ state can only be in an isospin $I = 1$ state, since the $I = 0$ and $I = 2$ states are symmetric under exchange of the two pions. Thus, by Clebsch-Gordan decomposition, we only have:

$$|\pi^+\pi^-\rangle_{\text{anti-sym}} = \frac{1}{\sqrt{2}} \left(|\pi^+\rangle|\pi^-\rangle - |\pi^-\rangle|\pi^+\rangle \right) \tag{165}$$

$$= |I = 1, I_3 = 0\rangle. \tag{166}$$

Using this, we can evaluate the matrix elements:

$$\langle \pi^+\pi^- | H | \rho_I^0 \rangle \propto \langle I = 1, I_3 = 0 | H | I = 1, I_3 = 0 \rangle = A, \tag{167}$$

$$\langle \pi^+\pi^- | H | \omega_I \rangle \propto \langle I = 1, I_3 = 0 | H | I = 0, I_3 = 0 \rangle = 0, \tag{168}$$

where A is the amplitude for the isospin-conserving decay. Thus, the decay amplitudes become:

$$\mathcal{M}(\omega \to \pi^+\pi^-) = -\sin\theta A, \tag{169}$$

$$\mathcal{M}(\rho^0 \to \pi^+\pi^-) = \cos\theta A. \tag{170}$$

The decay rates are proportional to the square of the amplitude:

$$\Gamma(\omega \to \pi^+\pi^-) \propto |\mathcal{M}(\omega \to \pi^+\pi^-)|^2 = \sin^2\theta |A|^2, \tag{171}$$

$$\Gamma(\rho^0 \to \pi^+\pi^-) \propto |\mathcal{M}(\rho^0 \to \pi^+\pi^-)|^2 = \cos^2\theta |A|^2. \tag{172}$$

Note that we ignore the phase space factors since $m_\omega \approx m_{\rho^0}$. Taking the ratio of the decay rates, we have:

$$\frac{\Gamma(\omega \to \pi^+\pi^-)}{\Gamma(\rho^0 \to \pi^+\pi^-)} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta. \tag{173}$$

Using the experimental values from PDG:

$$\Gamma(\omega \to \pi^+\pi^-) \approx 0.133 \text{ MeV}, \tag{174}$$

$$\Gamma(\rho^0 \to \pi^+\pi^-) \approx 147.1 \text{ MeV}, \tag{175}$$

we can solve for θ:

$$\tan^2\theta = \frac{0.133}{147.1}, \tag{176}$$

$$\theta \approx 0.0301 \text{ radians} \approx 1.72°. \tag{177}$$

The error in the value of the mixing angle can be estimated by propagating the uncertainties in the decay rates. However, since the uncertainties in the decay rates are relatively small compared to their values, the error in θ will also be small. For simplicity, we can estimate the error as:

$$\Delta\theta \approx \frac{1}{2}\frac{\Delta\Gamma(\omega \to \pi^+\pi^-)}{\Gamma(\rho^0 \to \pi^+\pi^-)}\frac{1}{\tan\theta}. \tag{178}$$

Using the uncertainties from PDG, $\Delta\Gamma(\omega \to \pi^+\pi^-)$ is about 0.01 MeV, we find:

$$\Delta\theta \approx \frac{1}{2}\frac{0.01}{147.1}\frac{1}{\tan(0.0301)} \approx 0.00113041 \text{ radians} \approx 0.065°. \tag{179}$$

(b)
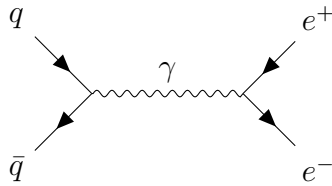

Figure 5: The Feynman diagram for $q\bar{q} \to e^+e^-$.

We can draw the Feynman diagram in Figure 5 for the decay of a vector meson into an electron-positron pair via a virtual photon. The amplitude for the decay can be written as:

$$\mathcal{M}(V \to e^+e^-) \propto \langle e^+e^-|H|V\rangle, \tag{180}$$

where V represents either ρ^0 or ω. The quark content of the vector mesons is:

$$|\rho_I^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \tag{181}$$

$$|\omega_I\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle). \tag{182}$$

The amplitude for the decay can be expressed in terms of the quark charges:

$$\mathcal{M}(\rho_I^0 \to e^+e^-) \propto \frac{1}{\sqrt{2}}(Q_u - Q_d) = \frac{1}{\sqrt{2}}\left(\frac{2}{3} - \left(-\frac{1}{3}\right)\right) = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}, \tag{183}$$

$$\mathcal{M}(\omega_I \to e^+e^-) \propto \frac{1}{\sqrt{2}}(Q_u + Q_d) = \frac{1}{\sqrt{2}}\left(\frac{2}{3} + \left(-\frac{1}{3}\right)\right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{3} = \frac{1}{3\sqrt{2}}. \tag{184}$$

Using the mixing relations, we can write the amplitudes for the physical states:

$$\mathcal{M}(\rho^0 \to e^+e^-) = \cos\theta \mathcal{M}(\rho_I^0 \to e^+e^-) + \sin\theta \mathcal{M}(\omega_I \to e^+e^-) \tag{185}$$

$$= \cos\theta \cdot \frac{1}{\sqrt{2}} + \sin\theta \cdot \frac{1}{3\sqrt{2}}, \tag{186}$$

$$\mathcal{M}(\omega \to e^+e^-) = -\sin\theta \mathcal{M}(\rho_I^0 \to e^+e^-) + \cos\theta \mathcal{M}(\omega_I \to e^+e^-) \tag{187}$$

$$= -\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{3\sqrt{2}}. \tag{188}$$

The decay rates are proportional to the square of the amplitudes:

$$\Gamma(\rho^0 \to e^+e^-) \propto \left|\cos\theta \cdot \frac{1}{\sqrt{2}} + \sin\theta \cdot \frac{1}{3\sqrt{2}}\right|^2, \tag{189}$$

$$\Gamma(\omega \to e^+e^-) \propto \left|-\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{3\sqrt{2}}\right|^2. \tag{190}$$

Substituting the value of $\theta \approx 0.0301$ radians, we can calculate the decay rates:

$$\Gamma(\rho^0 \to e^+e^-) \propto \left|\cos(0.0301) \cdot \frac{1}{\sqrt{2}} + \sin(0.0301) \cdot \frac{1}{3\sqrt{2}}\right|^2 \approx 0.5009, \tag{191}$$

$$\Gamma(\omega \to e^+e^-) \propto \left|-\sin(0.0301) \cdot \frac{1}{\sqrt{2}} + \cos(0.0301) \cdot \frac{1}{3\sqrt{2}}\right|^2 \approx 0.046. \tag{192}$$

Taking the ratio of the decay rates, we have:

$$\frac{\Gamma(\rho^0 \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} \approx \frac{0.5009}{0.046} \approx 11.09. \tag{193}$$

Using the experimental values from PDG:

$$\frac{\Gamma(\rho^0 \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} \approx \frac{6.99 \text{ keV}}{0.64 \text{ keV}} \approx 10.86, \tag{194}$$

we see that our prediction is in good agreement with the experimental data.

(c)

The decay amplitude for the transition $(q\bar{q}) \to (q\bar{q}) + \gamma$ can be written as:

$$\mathcal{M}(V \to P + \gamma) \propto \langle P\gamma|H|V\rangle, \tag{195}$$

where V represents either ρ^0 or ω, and P represents either π^0 or η. The quark content of the pseudoscalar mesons is:

$$|\rho_I^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \tag{196}$$

$$|\omega_I\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle), \tag{197}$$

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle), \tag{198}$$

$$|\eta\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle). \tag{199}$$

The amplitude for the decay can be expressed in terms of the quark charges:

$$\mathcal{M}(\rho_I^0 \to \pi^0 + \gamma) \propto Q_u\langle u\bar{u}|u\bar{u}\rangle + Q_d\langle d\bar{d}|d\bar{d}\rangle = \frac{2}{3} + \left(-\frac{1}{3}\right) = \frac{1}{3}, \tag{200}$$

$$\mathcal{M}(\omega_I \to \pi^0 + \gamma) \propto Q_u\langle u\bar{u}|u\bar{u}\rangle - Q_d\langle d\bar{d}|d\bar{d}\rangle = \frac{2}{3} - \left(-\frac{1}{3}\right) = 1, \tag{201}$$

$$\mathcal{M}(\rho_I^0 \to \eta + \gamma) \propto Q_u\langle u\bar{u}|u\bar{u}\rangle - Q_d\langle d\bar{d}|d\bar{d}\rangle = \frac{2}{3} - \left(-\frac{1}{3}\right) = 1, \tag{202}$$

$$\mathcal{M}(\omega_I \to \eta + \gamma) \propto Q_u\langle u\bar{u}|u\bar{u}\rangle + Q_d\langle d\bar{d}|d\bar{d}\rangle = \frac{2}{3} + \left(-\frac{1}{3}\right) = \frac{1}{3}. \tag{203}$$

Using the mixing relations, we can write the amplitudes for the physical states:

$$\mathcal{M}(\rho^0 \to \pi^0 + \gamma) = \cos\theta \mathcal{M}(\rho_I^0 \to \pi^0 + \gamma) + \sin\theta \mathcal{M}(\omega_I \to \pi^0 + \gamma) \tag{204}$$

$$= \cos\theta \cdot \frac{1}{3} + \sin\theta \cdot 1, \tag{205}$$

$$\mathcal{M}(\omega \to \pi^0 + \gamma) = -\sin\theta \mathcal{M}(\rho_\mathcal{I}' \to \pi' + \gamma) + \cos\theta \mathcal{M}(\omega_\mathcal{I} \to \pi' + \gamma) \tag{206}$$

$$= -\sin\theta \cdot \frac{1}{3} + \cos\theta \cdot 1, \tag{207}$$

$$\mathcal{M}(\rho^0 \to \eta + \gamma) = \cos\theta \mathcal{M}(\rho_\mathcal{I}' \to \eta + \gamma) + \sin\theta \mathcal{M}(\omega_\mathcal{I} \to \eta + \gamma) \tag{208}$$

$$= \cos\theta \cdot 1 + \sin\theta \cdot \frac{1}{3}, \tag{209}$$

$$\mathcal{M}(\omega \to \eta + \gamma) = -\sin\theta \mathcal{M}(\rho_\mathcal{I}' \to \eta + \gamma) + \cos\theta \mathcal{M}(\omega_\mathcal{I} \to \eta + \gamma) \tag{210}$$

$$= -\sin\theta \cdot 1 + \cos\theta \cdot \frac{1}{3}. \tag{211}$$

The decay rates are proportional to the square of the amplitudes:

$$\Gamma(\rho^0 \to \pi^0 + \gamma) \propto \left| \cos\theta \cdot \frac{1}{3} + \sin\theta \cdot 1 \right|^2, \tag{212}$$

$$\Gamma(\omega \to \pi^0 + \gamma) \propto \left| -\sin\theta \cdot \frac{1}{3} + \cos\theta \cdot 1 \right|^2, \tag{213}$$

$$\Gamma(\rho^0 \to \eta + \gamma) \propto \left| \cos\theta \cdot 1 + \sin\theta \cdot \frac{1}{3} \right|^2, \tag{214}$$

$$\Gamma(\omega \to \eta + \gamma) \propto \left| -\sin\theta \cdot 1 + \cos\theta \cdot \frac{1}{3} \right|^2. \tag{215}$$

Hence, the ratios of the decay rates are:

$$\frac{\Gamma(\rho^0 \to \pi^0 + \gamma)}{\Gamma(\omega \to \pi^0 + \gamma)} \approx \left(\frac{\cos\theta + 3\sin\theta}{3\cos\theta - \sin\theta} \right)^2 \tag{216}$$

$$\frac{\Gamma(\rho^0 \to \eta + \gamma)}{\Gamma(\omega \to \eta + \gamma)} \approx \left(\frac{3\cos\theta + \sin\theta}{\cos\theta - 3\sin\theta} \right)^2. \tag{217}$$

Hence, we can have

$$\left. \frac{\Gamma(\rho^0 \to \pi^0 + \gamma)}{\Gamma(\omega \to \pi^0 + \gamma)} \right|_{\theta=0} = \left(\frac{1+0}{3-0} \right)^2 = \frac{1}{9} \approx 0.111, \tag{218}$$

$$\left. \frac{\Gamma(\rho^0 \to \pi^0 + \gamma)}{\Gamma(\omega \to \pi^0 + \gamma)} \right|_{\theta=0.0301} \approx 0.135, \tag{219}$$

$$\left. \frac{\Gamma(\rho^0 \to \eta + \gamma)}{\Gamma(\omega \to \eta + \gamma)} \right|_{\theta=0} = \left(\frac{3+0}{1-0} \right)^2 = 9, \tag{220}$$

$$\left. \frac{\Gamma(\rho^0 \to \eta + \gamma)}{\Gamma(\omega \to \eta + \gamma)} \right|_{\theta=0.0301} \approx 11.1. \tag{221}$$

Compare with the experimental data from PDG:

$$\frac{\Gamma(\rho^0 \to \pi^0 + \gamma)}{\Gamma(\omega \to \pi^0 + \gamma)} \approx \frac{69.3 \text{ keV}}{723 \text{ keV}} \approx 0.096, \tag{222}$$

$$\frac{\Gamma(\rho^0 \to \eta + \gamma)}{\Gamma(\omega \to \eta + \gamma)} \approx \frac{44.22 \text{ keV}}{3.9 \text{ keV}} \approx 11.32. \tag{223}$$

Hence, we see that the inclusion of ρ-ω mixing improves the agreement with the experimental data for the $\rho^0 \to \eta + \gamma$ decay, while it worsens the agreement for the $\rho^0 \to \pi^0 + \gamma$ decay. $\qquad\square$

Question 3

Baryon magnetic moments

The octet of spin-$\frac{1}{2}$ baryons has magnetic moments μ. The operator that describes the magnetic moment is an $SU(3)_f$ octet operator which is proportional to the quark charge Q. The charge

$$Q = t_3 + \frac{1}{\sqrt{3}} t_8$$

is traceless ($\operatorname{Tr} Q = 0$) and can be promoted to a purely $SU(3)_f$ octet spurion $\mathbf{8}_Q$ (with no singlet piece, as in contrast to the GMO mass formula). Hence, when determining the baryon magnetic moment

$$\mu(B) = \langle \bar{B} | \mu | B \rangle \propto \mathbf{8}_{\bar{B}} \times \mathbf{8}_Q \times \mathbf{8}_B$$

there are two independent octet structures (the f- and d-type couplings, as for the baryon mass), given by

$$\mu(B) = c_f \operatorname{Tr}(B^\dagger [Q, B]) + c_d \operatorname{Tr}(B^\dagger \{Q, B\}) = \alpha_+ \operatorname{Tr}(BB^\dagger Q) + \alpha_- \operatorname{Tr}(B^\dagger BQ),$$

where $\alpha_+ \equiv c_d + c_f$, $\alpha_- \equiv c_d - c_f$ are arbitrary constants and

$$B = \begin{pmatrix} \frac{\Sigma_u^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma_u^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

Determine all the spin-$\frac{1}{2}$ baryon magnetic moments in terms of $\mu(p)$ and $\mu(n)$ (by eliminating $c_{f,d}$ or $\alpha_\pm$) and compare with the PDG experimental values. These predictions were first worked out by Coleman and Glashow in 1961. Note that imposing the full $SU(6)$ spin-flavor symmetry further predicts $\mu(p)/\mu(n) = -\frac{3}{2}$, which you can ignore in this problem.

Answer

See my *Mathematica* notebook for detailed calculations. I also impose the normalized condition to simplify the result. The final results are summarized in the table below:

- p: $c_f + \frac{1}{3} c_d$

- n: $-\frac{2}{3} c_d$

- Λ: $-\frac{1}{3} c_d$

- Σ^+: $c_f + \frac{1}{3} c_d$

- Σ^0: $\frac{1}{3} c_d$

- Σ^-: $-c_f + \frac{1}{3} c_d$

- Ξ^0: $-\frac{2}{3}c_d$

- Ξ^-: $-c_f + \frac{1}{3}c_d$

Baryon	Predicted Magnetic Moment (μ_N)	Experimental Magnetic Moment (μ_N)
p	$\mu(p)$	2.793
n	$\mu(n)$	-1.913
Λ	$\frac{1}{2}\mu(n)$	-0.613
Σ^+	$\mu(p)$	2.458
Σ^0	$-\frac{1}{2}\mu(n)$	≈ 0
Σ^-	$-(\mu(p) + \mu(n))$	-1.160
Ξ^0	$\mu(n)$	-1.250
Ξ^-	$-(\mu(p) + \mu(n))$	-0.651

The predictions are in good agreement with the experimental values, demonstrating the effectiveness of the $\mathrm{SU}(3)_f$ symmetry approach in describing baryon magnetic moments except for Σ^0 where the experimental value is not well matched. $\qquad\square$

Problem Set 5 due 9:30 AM, Monday, November 10

Question 1

Use the nonrelativistic hydrogen atom result to determine the energy levels for two particles of equal mass m interacting via an attractive potential $-\alpha/r$, where α is the coupling constant. The $2^3S - 1^3S$ separation is ~ 600 MeV for charmonium and ~ 5 eV for positronium. Justify this factor 10^8 in energy scale in terms of the constituent masses and couplings in the two cases. For charmonium assume a color-Coulomb potential $V(r) = -C_F \frac{\alpha_s}{r}$ with $C_F = 4/3$. Estimate the strong force coupling α_S, needed to reproduce the observed splitting.

Answer

For a two-body system with equal masses m, we can reduce it to a one-body problem with reduced mass $\mu = m/2$. The energy levels for a hydrogen-like atom are given by:

$$E_n = -\frac{\mu\alpha^2}{2n^2}, \tag{224}$$

where n is the principal quantum number. We first derive this equation for our system. We can apply classical mechanics to find the energy levels. The circular orbit condition and quantization of angular momentum give:

$$\frac{\mu v^2}{r} = \frac{\alpha}{r^2} \implies \mu r v^2 = \alpha, \tag{225}$$

$$\mu v r = n\hbar = n. \tag{226}$$

We can have $v = \alpha/n$, then the total energy is:

$$E_n = -\frac{1}{2}\mu v^2 = -\frac{\mu\alpha^2}{2n^2}. \tag{227}$$

For positronium, the coupling constant is the fine-structure constant $\alpha \approx 1/137$. The energy difference between the 2^3S and 1^3S states is:

$$\Delta E_{pos} = E_2 - E_1 = -\frac{\mu\alpha^2}{8} + \frac{\mu\alpha^2}{2} = \frac{3\mu\alpha^2}{8}. \tag{228}$$

Substituting $\mu = m_e/2$ (where $m_e = 511$ keV is the electron mass) and $\alpha \approx 1/137$, we find:

$$\Delta E_{pos} \approx \frac{3 \cdot (511 \text{ keV}/2) \cdot (1/137)^2}{8} \approx 5 \text{ eV}. \tag{229}$$

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For charmonium, we use the color-Coulomb potential with $C_F = 4/3$ and the strong coupling constant α_S. The energy difference between the 2^3S and 1^3S states is:

$$\Delta E_{charm} = \frac{3\mu(C_F\alpha_S)^2}{8}. \tag{230}$$

Given that $\Delta E_{charm} \approx 600$ MeV, we can solve for α_S:

$$600 \text{ MeV} = \frac{3 \cdot (m_c/2) \cdot (4/3\alpha_S)^2}{8}, \tag{231}$$

where $m_c \approx 1.27$ GeV is the charm quark mass. Rearranging gives $\alpha_S \approx 1.2$. $\qquad\square$

Remark: The typical value of the strong coupling constant α_S at the charmonium scale is around 0.3 to 0.4, which is significantly lower than the estimated value of 1.2 obtained from this simple Coulombic model. This is not a physically reasonable value for α_S at the charmonium scale, indicating that the simple Coulombic model is insufficient to describe the charmonium system accurately. More sophisticated models that include confinement and relativistic effects are necessary for a better description.

Question 2

Quarkonia

(a) Meson such as the $\phi(s\bar{s})$, $J/\psi(c\bar{c})$ and $\Upsilon(b\bar{b})$ are comparatively narrow hadronic resonances, even though they are strongly interacting. Explain, using qualitative arguments, why these quarkonium state do *not* readily decay into lighter flavor mesons such as $Q\bar{q}$ and $\bar{Q}q$ (where $Q = c, b$ and $q = u, d, s$). Discuss how the mass of each resonance with respect to the lowest threshold (e.g. $K\bar{K}, D\bar{D}, B\bar{B}$) controls whether strong decays are allowed.

Hint: Model a $Q\bar{Q}$ ground state with the Cornell potential and include a kinetic term via the uncertainties principle $p \sim 1/r$ to obtain:

$$E(r) = \frac{1}{2\mu r^2} - \frac{A}{r} + kr, \tag{232}$$

where $\mu = m_Q/2$ and $A = 4\alpha_S/3$. Minimize $E(r)$ with respect to r to obtain an estimate for the ground state size r_*. Use your r_* to estimate each piece in $E(r)$ and show how this changes in going from $s\bar{s}$ to $c\bar{c}$ to $b\bar{b}$. When estimating the bound state energies, use $\alpha_S \sim 1(0.4)$ for the $s\bar{s}(c\bar{c}, b\bar{b})$ systems, and the "constituent" quark masses, $m_s \sim 450$ MeV, $m_c \sim 1.5$ GeV, and $m_b \sim 4.8$ GeV, which include the gluon self-energy corrections.

(b) The vector quarkonia $\phi(s\bar{s})$, $J/\psi(c\bar{c})$ and $\Upsilon(b\bar{b})$ follow an approximate factor-of-three scaling in mass from one flavor to the next. Before the discovery of the top quark in 1995, this scaling was used to estimate the "toponium" mass. Determine this mass and estimate whether this fictional top-antitop ($t\bar{t}$) pair would have had time to form a bound state before decaying? Compare with the real top quark which has a mass $m_t \approx 173$ GeV.

Answer

(a)

By Mathematica, we can minimize $E(r)$ with respect to r and obtain the ground state size r_* for each quarkonium system:

$$r_{s\bar{s}} \approx 2.05 \text{ GeV}^{-1} = 0.40 \text{ fm}, E(r_{s\bar{s}}) \approx 0.288 \text{ GeV}, \tag{233}$$

$$r_{c\bar{c}} \approx 1.42 \text{ GeV}^{-1} = 0.28 \text{ fm}, E(r_{c\bar{c}}) \approx 0.239 \text{ GeV}, \tag{234}$$

$$r_{b\bar{b}} \approx 0.67 \text{ GeV}^{-1} = 0.13 \text{ fm}, E(r_{b\bar{b}}) \approx -0.198 \text{ GeV}. \tag{235}$$

Hence, we can estimate each piece in $E(r)$:

$$E_{s\bar{s}} : \frac{1}{2\mu r_*^2} \approx 0.53 \text{ GeV}, -\frac{A}{r_*} \approx -0.65 \text{ GeV}, kr_* \approx 0.41 \text{ GeV}, \qquad (236)$$

$$E_{c\bar{c}} : \frac{1}{2\mu r_*^2} \approx 0.32 \text{ GeV}, -\frac{A}{r_*} \approx -0.38 \text{ GeV}, kr_* \approx 0.28 \text{ GeV}, \qquad (237)$$

$$E_{b\bar{b}} : \frac{1}{2\mu r_*^2} \approx 0.47 \text{ GeV}, -\frac{A}{r_*} \approx -0.80 \text{ GeV}, kr_* \approx 0.13 \text{ GeV}. \qquad (238)$$

By comparing $c\bar{c}$ with $b\bar{b}$, we can see that the kinetic term, while the Coulomb term decreases in magnitude. This is because as the quark mass increases, the quarkonium system becomes more tightly bound, leading to a smaller size r_*. A smaller size results in a higher kinetic energy due to the uncertainty principle, and a stronger Coulomb attraction due to the reduced distance between the quarks. The linear confinement term also decreases as the size decreases. It's hard to compare $s\bar{s}$ with the other two systems since its coupling constant is much larger, but we can still see that the kinetic term is the largest among the three systems, indicating a relatively larger size.

With the bounding energies estimated above, we can estimate the total masses of each quarkonium system:

$$M_{s\bar{s}} \approx 2m_s + E(r_{s\bar{s}}) \approx 1.188 \text{ GeV}, \qquad (239)$$

$$M_{c\bar{c}} \approx 2m_c + E(r_{c\bar{c}}) \approx 3.239 \text{ GeV}, \qquad (240)$$

$$M_{b\bar{b}} \approx 2m_b + E(r_{b\bar{b}}) \approx 9.402 \text{ GeV}. \qquad (241)$$

Compared to the actual masses of $\phi(1.019)$, $J/\psi(3.097)$ and $\Upsilon(9.460)$, our estimates are reasonably close.

Next, we consider the decay of these quarkonium states into lighter flavor mesons. For strong decays to occur, the mass of the quarkonium state must be greater than the sum of the masses of the decay products. The relevant thresholds are:

$$K\bar{K} \approx 0.494 \text{ GeV} \times 2 = 0.988 \text{ GeV}, \qquad (242)$$

$$D\bar{D} \approx 1.865 \text{ GeV} \times 2 = 3.730 \text{ GeV}, \qquad (243)$$

$$B\bar{B} \approx 5.280 \text{ GeV} \times 2 = 10.560 \text{ GeV}. \qquad (244)$$

Comparing these thresholds with the estimated masses:

$$M_{s\bar{s}} \approx 1.188 \text{ GeV} > K\bar{K} \text{ (allowed)}, \qquad (245)$$

$$M_{c\bar{c}} \approx 3.239 \text{ GeV} < D\bar{D} \text{ (not allowed)}, \qquad (246)$$

$$M_{b\bar{b}} \approx 9.402 \text{ GeV} < B\bar{B} \text{ (not allowed)}. \qquad (247)$$

Thus, the $\phi(s\bar{s})$ can decay into $K\bar{K}$, while the $J/\psi(c\bar{c})$ and $\Upsilon(b\bar{b})$ cannot decay into $D\bar{D}$ and $B\bar{B}$ respectively. This explains why J/ψ and Υ are comparatively narrow resonances, as they do not have

strong decay channels available.

(b)

Using the approximate factor-of-three scaling in mass, we can estimate the mass of the fictional toponium $(t\bar{t})$ state:

$$M_{t\bar{t}} \approx 3 \times M_{b\bar{b}} \approx 3 \times 9.460 \text{ GeV} \approx 28.380 \text{ GeV}. \tag{248}$$

We can define the constituent mass of the top quark as half of the toponium mass:

$$m_t \approx \frac{M_{t\bar{t}}}{2} \approx \frac{28.380 \text{ GeV}}{2} \approx 14.190 \text{ GeV}. \tag{249}$$

To determine whether the toponium state would have time to form a bound state before decaying, we need to compare the lifetime of the top quark with the timescale for bound state formation. The typical time to form a bound state is on the order of the inverse of the binding energy, which can be estimated from the strong interaction scale, roughly $\Lambda_{QCD} \sim 200$ MeV. Thus, the timescale for bound state formation is approximately:

$$\tau_{form} \sim \frac{1}{\Lambda_{QCD}} \sim \frac{1}{250 \text{ MeV}} \approx 2.6 \times 10^{-24} \text{ s}. \tag{250}$$

Here we try to estimate the lifetime of the top quark. Using the equation provided in the lecture notes:

$$\Gamma_t \approx G_F m_t^3 = 1.166 \times 10^{-5} \text{ GeV}^{-2} \times (14.190 \text{ GeV})^3 \approx 33.3 \text{ MeV}. \tag{251}$$

Thus, the lifetime of the toponium state is:

$$\tau_{t,fictional} \sim \frac{1}{\Gamma_t} \sim \frac{1}{33.3 \text{ MeV}} \approx 1.98 \times 10^{-23} \text{ s}. \tag{252}$$

Comparing the two timescales, we find that $\tau_{t,fictional} \gg \tau_{form}$, indicating that the toponium state would have had time to form a bound state before decaying. Now we can compare this with the real top quark mass $m_t \approx 173$ GeV:

$$\Gamma_t \approx G_F m_t^3 = 1.166 \times 10^{-5} \text{ GeV}^{-2} \times (173 \text{ GeV})^3 \approx 60.4 \text{ GeV}. \tag{253}$$

Thus, the lifetime of the real top quark is:

$$\tau_{t,real} \sim \frac{1}{\Gamma_t} \sim \frac{1}{60.4 \text{ GeV}} \approx 1.09 \times 10^{-26} \text{ s}. \tag{254}$$

Comparing this with the bound state formation timescale, we find that $\tau_{t,real} \ll \tau_{form}$, indicating that the real top quark decays too quickly to form a bound state. $\square$

Question 3

Regge trajectories

In the string model of hadrons a meson (quark-antiquark pair) can be thought of as a string of length $2r_0$ with string tension k and whose ends rotate at a speed $v = c$.

(a) Show that the total mass of the rotating string is $M = \pi k r_0$ and the orbital angular momentum is $J = \frac{1}{2}\pi k r_0^2$. Hence, obtain the relation between J and M^2.

(b) Draw a Chew-Frautschi plot (i.e. J vs. M^2) of the following mesons $\rho(770)$, $f_2(1270)$, $\rho_3(1690)$, $f_4(2050)$, $\rho_5(2350)$ and $f_6(2510)$ and baryons $\Delta(1232)$, $\Delta(1950)$, $\Delta(2420)$, $\Delta(2950)$. From your plot, determine the experimental value of the string tension k. Compare the meson and baryon values and discuss whether a universal string tension describes both within errors.

Answer

(a)

Consider a rotating string of length $2r_0$ with string tension k. The mass element dm of the string at a distance r from the center and moving with velocity v can be expressed as:

$$dm = k\, dr, \tag{255}$$

$$v = \omega r = \frac{c}{r_0} r, \tag{256}$$

where $\omega = c/r_0$ is the angular velocity. The relativistic factor γ is given by:

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} = \frac{1}{\sqrt{1 - (r/r_0)^2}}. \tag{257}$$

The total mass M of the string can be calculated by integrating the mass element over the length of the string:

$$E = Mc^2 = \int dE = \int \gamma c^2 dm = \int_{-r_0}^{r_0} \frac{kc^2}{\sqrt{1 - (r/r_0)^2}} dr = \pi k c^2 r_0. \tag{258}$$

Thus, the total mass of the rotating string is:

$$M = \pi k r_0. \tag{259}$$

Next, we calculate the orbital angular momentum J of the string. The angular momentum element dJ at a distance r from the center is given by:

$$dJ = \gamma r v\, dm = \frac{1}{\sqrt{1 - (r/r_0)^2}} r \frac{c}{r_0} r k\, dr = \frac{kc}{r_0} \frac{r^2}{\sqrt{1 - (r/r_0)^2}} dr. \tag{260}$$

Integrating this over the length of the string gives:

$$J = \int dJ = \int_{-r_0}^{r_0} \frac{kc}{r_0} \frac{r^2}{\sqrt{1 - (r/r_0)^2}} dr = \frac{\pi}{2} ckr_0^2 = \frac{\pi}{2} kr_0^2, \tag{261}$$

where we set $c = 1$ in the last step. Hence, the orbital angular momentum of the rotating string is:

$$J = \frac{1}{2}\pi kr_0^2. \tag{262}$$

Now, we can eliminate r_0 to find the relation between J and M^2:

$$r_0 = \frac{M}{\pi k} \implies J = \frac{1}{2}\pi k \left(\frac{M}{\pi k}\right)^2 = \frac{M^2}{2\pi k}. \tag{263}$$

(b)

Using the relation derived in part (a), we can plot J vs. M^2 for the given mesons and baryons. The data points are as follows:

- Mesons:

$$\rho(770) : J = 1, M^2 = (0.770 \text{ GeV})^2 = 0.593 \text{ GeV}^2,$$
$$f_2(1270) : J = 2, M^2 = (1.275 \text{ GeV})^2 = 1.626 \text{ GeV}^2,$$
$$\rho_3(1690) : J = 3, M^2 = (1.689 \text{ GeV})^2 = 2.852 \text{ GeV}^2,$$
$$f_4(2050) : J = 4, M^2 = (2.018 \text{ GeV})^2 = 4.072 \text{ GeV}^2,$$
$$\rho_5(2350) : J = 5, M^2 = (2.330 \text{ GeV})^2 = 5.4289 \text{ GeV}^2,$$
$$f_6(2510) : J = 6, M^2 = (2.470 \text{ GeV})^2 = 6.1009 \text{ GeV}^2.$$

- Baryons:

$$\Delta(1232) : J = 3/2, M^2 = (1.232 \text{ GeV})^2 = 1.518 \text{ GeV}^2,$$
$$\Delta(1950) : J = 7/2, M^2 = (1.930 \text{ GeV})^2 = 3.7249 \text{ GeV}^2,$$
$$\Delta(2420) : J = 11/2, M^2 = (2.450 \text{ GeV})^2 = 6.0025 \text{ GeV}^2,$$
$$\Delta(2950) : J = 15/2, M^2 = (2.990 \text{ GeV})^2 = 8.9401 \text{ GeV}^2.$$

Figure 6 shows the Chew-Frautschi plot of J vs. M^2 for the mesons and baryons. From the linear fit of the data points, we can determine the slope of the line, which is given by:

$$\text{slope} = \frac{1}{2\pi k}. \tag{264}$$

From the linear fit of the meson data, we find the slope to be approximately 0.8 GeV^{-2}, leading to:

$$k_{meson} \approx 0.184 \text{ GeV}^2 \tag{265}$$

$$k_{baryon} \approx 0.196 \text{ GeV}^2. \tag{266}$$

The values of the string tension k for mesons and baryons are quite close, suggesting that a universal string tension can describe both systems within experimental errors. This supports the idea that the underlying dynamics of quark confinement in hadrons can be effectively modeled using a common string tension parameter. □
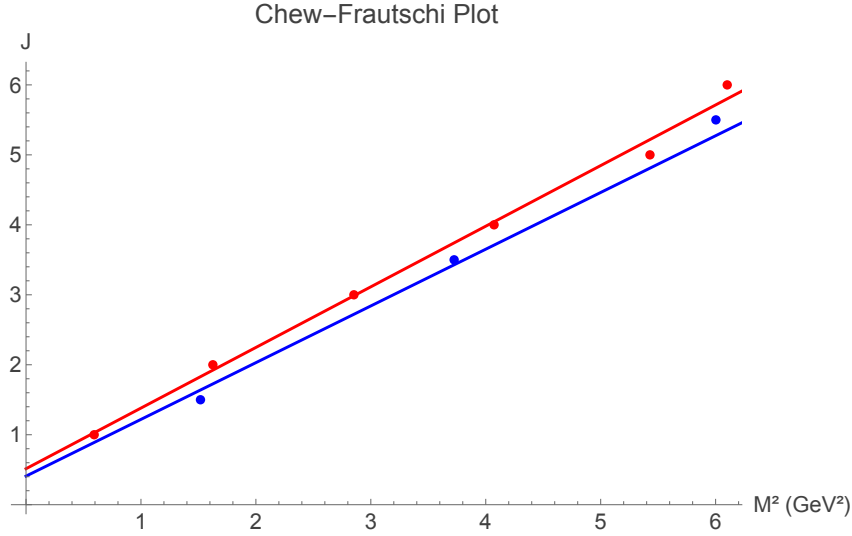

Figure 6: Chew-Frautschi plot of mesons and baryons.

Problem Set 6 due 9:30 AM, Monday, November 24

Question 1

Three-photon decay of a scalar particle

Consider the decay $X \to 3\gamma$, where X is a scalar particle of mass M. Assume the decay amplitude M_{fi} is approximately constant (i.e. independent of photon energies and angles) and can be written as $M_{fi} = A$. This is a good approximation for the decay of orthopositronium in its ground state.

(a) Derive the differential decay rate $\frac{d\Gamma}{d\omega}$ corresponding to the measured energy ω of a single photon in the rest frame of X.

(b) Express the total decay rate Γ in terms of the constant amplitude A.

Answer

We can start from the general expression for the decay rate of a particle decaying into three massless particles:

$$d\Gamma = \frac{1}{2M}\frac{1}{3!}|M_{fi}|^2 d\tau_3, \tag{267}$$

where $d\tau_3$ is the three-body phase space element, M is the mass of the decaying particle, $1/3!$ accounts for the identical photons in the final state and $M_{fi} = A$ is the invariant matrix element. The three-body phase space element for massless particles can be expressed as:

$$d\tau_3 = (2\pi)^4 \delta^4(P - p_1 - p_2 - p_3)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}, \tag{268}$$

where P is the four-momentum of the decaying particle, and p_i and E_i are the momenta and energies of the final state photons, respectively. If we consider the rest frame of the decaying particle, we have $P = (M, 0, 0, 0)$. Besides, we can apply the splitting formula for three-body phase space:

$$d\tau_3 = d\tau_2(M \to p_1 + q)\frac{dq^2}{2\pi}d\tau_2(q \to p_2 + p_3), \tag{269}$$

where $q = p_2 + p_3$ is the combined four-momentum of photons 2 and 3. The two-body phase space elements can be expressed as:

$$d\tau_2(M \to p_1 + q) = (2\pi)^4 \delta^4(P - p_1 - q)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3q}{(2\pi)^3 2E_q}, \tag{270}$$

$$d\tau_2(q \to p_2 + p_3) = (2\pi)^4 \delta^4(q - p_2 - p_3)\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}. \tag{271}$$

To find the differential decay rate with respect to the energy of one photon, say $\omega = E_1$, we can integrate over the other variables. The total energy conservation gives us:

$$M = E_1 + E_2 + E_3. \tag{272}$$

Since the photons are massless, we have $E_i = |\vec{p}_i|$.

$$d\Gamma = \frac{1}{2M}\frac{1}{3!}|A|^2 d\tau_3 \tag{273}$$

$$= \frac{1}{2M}\frac{1}{3!}|A|^2 d\tau_2(M \to p_1 + q)\frac{dq^2}{2\pi}d\tau_2(q \to p_2 + p_3). \tag{274}$$

The two-body phase space elements can be evaluated in their respective rest frames. In this frame, $q = (\sqrt{q^2}, 0, 0, 0)$, and the energies of the photons are $E_2 = |\vec{p}_2|$ and $E_3 = |\vec{p}_3|$.

$$\int d\tau_2(q \to p_2 + p_3) = \int (2\pi)^4 \delta(\sqrt{q^2} - E_2 - E_3)\delta^3(\vec{0} - \vec{p}_2 - \vec{p}_3)\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3} \tag{275}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_2 - E_3)\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{1}{(2\pi)^3 2E_2} \tag{276}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_2)\frac{4\pi E_2^2 dE_2}{(2\pi)^3 2E_2}\frac{1}{(2\pi)^3 2E_2} \tag{277}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_2)\frac{4\pi}{4(2\pi)^6}dE_2 \tag{278}$$

$$= (2\pi)^4 \frac{\pi}{(2\pi)^6}\frac{1}{2} \tag{279}$$

$$= \frac{1}{8\pi}, \tag{280}$$

where we have used the delta function to perform the integral over E_2 to get extra factor of $1/2$. Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of M. In this frame, $P = (M, 0, 0, 0)$, and the energies are $p_1^\mu = (\omega, \vec{p}_1)$ and $q^\mu = (M - \omega, \vec{q}) = (E_q, \vec{q})$.

$$\int d\tau_2(M \to p_1 + q) = \int (2\pi)^4 \delta(M - E_1 - E_q)\delta^3(\vec{0} - \vec{p}_1 - \vec{q})\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3q}{(2\pi)^3 2E_q} \tag{281}$$

$$= \int (2\pi)^4 \delta(M - \omega - E_q)\frac{d^3p_1}{(2\pi)^3 2\omega}\frac{1}{(2\pi)^3 2E_q} \tag{282}$$

$$= \int (2\pi)^4 \delta(M - \omega - E_q)\frac{4\pi\omega^2 d\omega}{(2\pi)^3 2\omega}\frac{1}{(2\pi)^3 2E_q} \tag{283}$$

$$= \frac{4\pi}{4(2\pi)^2}\int \delta(M - \omega - E_q)\frac{\omega}{E_q}d\omega \tag{284}$$

$$= \frac{1}{4\pi}\int \delta(M - \omega - E_q)\frac{\omega}{E_q}d\omega. \tag{285}$$

Thus, we have

$$\int d\tau_2(M \to p_1 + q) = \frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega. \tag{286}$$

Now we can combine the results to get the differential decay rate:

$$d\Gamma = \frac{1}{2M} \frac{1}{3!} |A|^2 \left(\frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \right) \tag{287}$$

$$= \frac{|A|^2}{768\pi^3 M} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega dq^2. \tag{288}$$

We have to be careful when we use the delta function to perform the integral over ω since $E_q = \sqrt{(\vec{q})^2 + q^2} = \sqrt{\omega^2 + q^2}$

$$\frac{d}{dq^2}(\omega + E_q) = \frac{d}{dq^2}(\omega + \sqrt{\omega^2 + q^2}) = \frac{1}{2\sqrt{\omega^2 + q^2}} = \frac{1}{2E_q}. \tag{289}$$

Thus, we have

$$d\Gamma = \frac{|A|^2}{768\pi^3 M} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega dq^2 \tag{290}$$

$$= \frac{|A|^2}{768\pi^3 M} \frac{\omega}{E_q} 2E_q d\omega \tag{291}$$

$$= \frac{|A|^2}{384\pi^3 M} \omega d\omega. \tag{292}$$

However, we cannot disthinguish which photon we are measuring, so we have to multiply by a factor of 3. Therefore, the final expression for the differential decay rate is:

$$\frac{d\Gamma}{d\omega} = \frac{|A|^2}{128\pi^3 M} \omega, \quad 0 \le \omega \le \frac{M}{2}. \tag{293}$$

To find the total decay rate, we can integrate over the allowed range of ω:

$$\Gamma = \int_0^{M/2} \frac{d\Gamma}{d\omega} d\omega \tag{294}$$

$$= \int_0^{M/2} \frac{|A|^2}{128\pi^3 M} \omega d\omega \tag{295}$$

$$= \frac{|A|^2}{128\pi^3 M} \left[\frac{\omega^2}{2} \right]_0^{M/2} \tag{296}$$

$$= \frac{|A|^2}{128\pi^3 M} \frac{M^2}{8} \tag{297}$$

$$= \frac{|A|^2 M}{1024\pi^3}. \tag{298}$$

$$\square$$

Question 2

Find the total Lorentz-invariant three-body phase space τ_3 for a final state containing one particle of mass m and two massless particles, produced from an initial particle of mass M. Express your final result in terms of the Mandelstam variable s and m.

Answer

From the definition of the three-body phase space, we have

$$\tau_3 = \int (2\pi)^4 \delta^4(P - p_1 - p_2 - p_3) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3}, \tag{299}$$

where P is the four-momentum of the initial particle, p_1 is the four-momentum of the massive particle with mass m, and p_2 and p_3 are the four-momenta of the two massless particles. We can use the splitting formula for three-body phase space:

$$\tau_3 = \int d\tau_2(M \to p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \to p_2 + p_3), \tag{300}$$

where $q = p_2 + p_3$ is the combined four-momentum of the two massless particles. The two-body phase space elements can be expressed as:

$$d\tau_2(M \to p_1 + q) = (2\pi)^4 \delta^4(P - p_1 - q) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3q}{(2\pi)^3 2E_q}, \tag{301}$$

$$d\tau_2(q \to p_2 + p_3) = (2\pi)^4 \delta^4(q - p_2 - p_3) \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3}. \tag{302}$$

For the second two-body phase space element, we can quote the result from the previous problem, since both particles are massless:

$$\int d\tau_2(q \to p_2 + p_3) = \frac{1}{8\pi}. \tag{303}$$

Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of M. In this frame, $P = (M, 0, 0, 0)$, and the energies are $E_1 = \sqrt{|\vec{p}_1|^2 + m^2}$ and $E_q = M - E_1 = \sqrt{|\vec{q}|^2 + q^2}$.

$$\int d\tau_2(M \to p_1 + q) = \int (2\pi)^4 \delta(M - E_1 - E_q) \delta^3(\vec{0} - \vec{p}_1 - \vec{q}) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 q}{(2\pi)^3 2E_q} \tag{304}$$

$$= \int (2\pi)^4 \delta(M - E_1 - E_q) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_q} \tag{305}$$

$$= \int (2\pi)^4 \delta(M - E_1 - E_q) \frac{4\pi |\vec{p}_1|^2 d|\vec{p}_1|}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_q} \tag{306}$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| \tag{307}$$

$$= \frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1|. \tag{308}$$

Thus, we have

$$\int d\tau_2(M \to p_1 + q) = \frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1|. \tag{309}$$

Now we can combine the results to get the total three-body phase space:

$$\tau_3 = \int d\tau_2(M \to p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \to p_2 + p_3) \tag{310}$$

$$= \int \left(\frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \right) \tag{311}$$

$$= \frac{1}{64\pi^3} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| dq^2. \tag{312}$$

To perform the integral over $|\vec{p}_1|$, we need to express E_q in terms of $|\vec{p}_1|$ and q^2:

$$E_q = \sqrt{|\vec{q}|^2 + q^2} = \sqrt{|\vec{p}_1|^2 + q^2} \tag{313}$$

$$E_1 = \sqrt{|\vec{p}_1|^2 + m^2}. \tag{314}$$

We also need to compute the derivative of $(E_1 + E_q)$ with respect to $|\vec{p}_1|$:

$$\frac{d}{d|\vec{p}_1|}(E_1 + E_q) = \frac{d}{d|\vec{p}_1|} \left(\sqrt{|\vec{p}_1|^2 + m^2} + \sqrt{|\vec{p}_1|^2 + q^2} \right) = \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + m^2}} + \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + q^2}} = \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_1|}{E_q}$$

$$\tag{315}$$

$$= |\vec{p}_1| \left(\frac{E_1 + E_q}{E_1 E_q} \right) = |\vec{p}_1| \left(\frac{M}{E_1 E_q} \right). \tag{316}$$

Thus, we have

$$\tau_3 = \frac{1}{64\pi^3} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| dq^2 \tag{317}$$

$$= \frac{1}{64\pi^3} \int \frac{|\vec{p}_1|^2}{E_1 E_q} \frac{E_1 E_q}{M|\vec{p}_1|} dq^2 \tag{318}$$

$$= \frac{1}{64\pi^3 M} \int |\vec{p}_1| dq^2. \tag{319}$$

We can apply the relation between q^2 and $|\vec{p}_1|$ to change the integration variable in the rest frame of M, where $P^\mu = (M, 0, 0, 0)$ and $p_1^\mu = (E_1, \vec{p}_1)$, $q^\mu = P^\mu - p_1^\mu$:

$$q^2 = (P - p_1)^2 = M^2 + m^2 - 2ME_1 = M^2 + m^2 - 2M\sqrt{|\vec{p}_1|^2 + m^2}, \tag{320}$$

$$\frac{dq^2}{d|\vec{p}_1|} = -2M \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + m^2}} = -2M \frac{|\vec{p}_1|}{E_1}. \tag{321}$$

By $E_1^2 = |\vec{p}_1|^2 + m^2$, we have $dE_1 = \frac{|\vec{p}_1|}{E_1} d|\vec{p}_1|$. Thus, we have

$$\tau_3 = \frac{1}{64\pi^3 M} \int_{q_{min}}^{q_{max}} |\vec{p}_1| dq^2 \tag{322}$$

$$= \frac{1}{64\pi^3 M} \int_{|\vec{p}_1|_{max}}^{|\vec{p}_1|_{min}} |\vec{p}_1| \left(-2M \frac{|\vec{p}_1|}{E_1} \right) d|\vec{p}_1| \tag{323}$$

$$= \frac{1}{32\pi^3} \int_{|\vec{p}_1|_{min}}^{|\vec{p}_1|_{max}} \frac{|\vec{p}_1|^2}{E_1} d|\vec{p}_1| \tag{324}$$

$$= \frac{1}{32\pi^3} \int_{E_{1,min}}^{E_{1,max}} \sqrt{E_1^2 - m^2} dE_1. \tag{325}$$

The limits of integration for E_1 can be found from the kinematic constraints. The minimum energy occurs when the two massless particles are emitted back-to-back with maximum energy, and the maximum energy occurs when the massive particle is at rest:

$$E_{1,min} = m, \tag{326}$$

$$E_{1,max} = \frac{M^2 + m^2}{2M} = \frac{s + m^2}{2\sqrt{s}}, \text{ from } q_{min}^2 = 0 \Rightarrow M^2 + m^2 - 2ME_{1,max} = 0. \tag{327}$$

Thus, we have (by *Mathematica*)

$$\tau_3 = \frac{1}{32\pi^3} \int_m^{\frac{s+m^2}{2\sqrt{s}}} \sqrt{E_1^2 - m^2} \, dE_1 \tag{328}$$

$$= \frac{1}{32\pi^3} \frac{-m^4 + 2m^2 s \log\left(\frac{m^2}{s}\right) + s^2}{8s} \tag{329}$$

$$= \frac{-m^4 + 2m^2 s \log\left(\frac{m^2}{s}\right) + s^2}{256\pi^3 s}. \tag{330}$$

$\square$

Question 3

Hadronic Transitions in Quarkonium

(a) The decay amplitude for the transition $\psi(2S) \to J/\psi(1S)\pi^+\pi^-$ can be approximated by

$$M_{fi} = a_\psi \sqrt{4m_{\psi(2S)}m_{J/\psi}}(q^2 - 4.5m_\pi^2), \tag{331}$$

where q is the total four-momentum of the emitted pion pair, and a_ψ is a dimensionful coupling constant. Using the experimental decay rate (performing a numerical phase-space integration if necessary), determine the absolute value $|a_\psi|$ in appropriate units of GeV.

(b) Perform the same analysis for the decay $\Upsilon(2S) \to \Upsilon(1S)\pi^+\pi^-$, for which the phenomenological amplitude is

$$M_{fi} = a_\Upsilon \sqrt{4m_{\Upsilon(2S)}m_{\Upsilon(1S)}}(q^2 - 3.2m_\pi^2). \tag{332}$$

Compare the extracted magnitudes of $|a_\psi|$ and $|a_\Upsilon|$. What can you infer about the relative spatial extent of the charmonium and bottomonium bound states?

Note: The constants a_ψ and a_Υ reflect overlap integrals between the $2S$ and $1S$ quarkonium wavefunctions and scale with the mean-square radius $\langle r^2 \rangle$ of the bound state.

Answer

(a)

The decay rate for the process $\psi(2S) \to J/\psi(1S)\pi^+\pi^-$ can be expressed as:

$$d\Gamma = \frac{1}{2m_{\psi(2S)}}|M_{fi}|^2 d\tau_3, \tag{333}$$

where $d\tau_3$ is the three-body phase space element for the final state particles. The three-body phase space element can be expressed as:

$$d\tau_3 = (2\pi)^4\delta^4(P - p_{J/\psi} - p_{\pi^+} - p_{\pi^-})\frac{d^3p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}}\frac{d^3p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}}\frac{d^3p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}}, \tag{334}$$

where P is the four-momentum of the initial $\psi(2S)$ particle, and $p_{J/\psi}$, p_{π^+}, and p_{π^-} are the four-momenta of the final state particles. In the rest frame of $\psi(2S)$, we have $P = (m_{\psi(2S)}, 0, 0, 0)$. The invariant matrix element is given by:

$$M_{fi} = a_\psi \sqrt{4m_{\psi(2S)}m_{J/\psi}}(q^2 - 4.5m_\pi^2), \tag{335}$$

where $q = p_{\pi^+} + p_{\pi^-}$ is the combined four-momentum of the pion pair. To find the total decay rate, we need to integrate over the three-body phase space:

$$\Gamma = \int d\Gamma = \frac{1}{2m_{\psi(2S)}} |a_\psi|^2 4 m_{\psi(2S)} m_{J/\psi} \int (q^2 - 4.5 m_\pi^2)^2 d\tau_3. \tag{336}$$

For $d\tau_3$, we can use the splitting formula for three-body phase space:

$$d\tau_3 = d\tau_2(m_{\psi(2S)} \to p_{J/\psi} + q) \frac{dq^2}{2\pi} d\tau_2(q \to p_{\pi^+} + p_{\pi^-}), \tag{337}$$

where $q = p_{\pi^+} + p_{\pi^-}$ is the combined four-momentum of the pion pair. The two-body phase space elements can be expressed as:

$$d\tau_2(m_{\psi(2S)} \to p_{J/\psi} + q) = (2\pi)^4 \delta^4(P - p_{J/\psi} - q) \frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}} \frac{d^3 q}{(2\pi)^3 2E_q}, \tag{338}$$

$$d\tau_2(q \to p_{\pi^+} + p_{\pi^-}) = (2\pi)^4 \delta^4(q - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}}. \tag{339}$$

Now we have to evaluate the two-body phase space elements. For the second two-body phase space element, and we can set $q = (\sqrt{q^2}, 0, 0, 0)$,

$$\int d\tau_2(q \to p_{\pi^+} + p_{\pi^-}) = \int (2\pi)^4 \delta^4(q - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}} \tag{340}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_{\pi^+} - E_{\pi^-}) \delta^3(\vec{0} - \vec{p}_{\pi^+} - \vec{p}_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}} \tag{341}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_{\pi^+} - E_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{1}{(2\pi)^3 2E_{\pi^-}}, \quad \text{where } E_{\pi^+} = E_{\pi^-} \tag{342}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_{\pi^+}) \frac{4\pi |p|^2 d|p|}{(2\pi)^3 2E_{\pi^+}} \frac{1}{(2\pi)^3 2E_{\pi^+}}, \quad \text{where } |p| = |\vec{p}_{\pi^+}| \tag{343}$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} dp, \quad E = E_{\pi^+} \tag{344}$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} dp \tag{345}$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} \frac{E}{|p|} dE \quad \text{using } dE = \frac{|p|}{E} dp \tag{346}$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{\sqrt{E^2 - m_\pi^2}}{E} dE \tag{347}$$

$$= \frac{1}{4\pi} \frac{\sqrt{\frac{q^2}{4} - m_\pi^2}}{\frac{q^2}{2}} \frac{1}{2} \quad \text{using } E = \frac{\sqrt{q^2}}{2} \tag{348}$$

$$= \frac{1}{8\pi} \sqrt{1 - \frac{4m_\pi^2}{q^2}}. \tag{349}$$

Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame

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of $\psi(2S)$. In this frame, $P = (m_{\psi(2S)}, 0, 0, 0)$, and the energies are $E_{J/\psi} = \sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}$ and $E_q = m_{\psi(2S)} - E_{J/\psi} = \sqrt{|\vec{q}|^2 + q^2}$.

$$\int d\tau_2(m_{\psi(2S)} \to p_{J/\psi} + q) = \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\delta^3(\vec{0} - \vec{p}_{J/\psi} - \vec{q})\frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}}\frac{d^3 q}{(2\pi)^3 2E_q} \tag{350}$$

$$= \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}}\frac{1}{(2\pi)^3 2E_q} \tag{351}$$

$$= \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{4\pi|\vec{p}_{J/\psi}|^2 d|\vec{p}_{J/\psi}|}{(2\pi)^3 2E_{J/\psi}}\frac{1}{(2\pi)^3 2E_q} \tag{352}$$

$$= \frac{4\pi}{4(2\pi)^2}\int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}d|\vec{p}_{J/\psi}| \tag{353}$$

$$= \frac{1}{4\pi}\int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}d|\vec{p}_{J/\psi}|. \tag{354}$$

$E_{J/\psi}$ and E_q can be expressed in terms of $|\vec{p}_{J/\psi}|$ and q^2:

$$E_q = \sqrt{|\vec{q}|^2 + q^2} = \sqrt{|\vec{p}_{J/\psi}|^2 + q^2}, \tag{355}$$

$$E_{J/\psi} = \sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}. \tag{356}$$

We also need to compute the derivative of $(E_{J/\psi} + E_q)$ with respect to $|\vec{p}_{J/\psi}|$:

$$\frac{d}{d|\vec{p}_{J/\psi}|}(E_{J/\psi} + E_q) = \frac{d}{d|\vec{p}_{J/\psi}|}\left(\sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2} + \sqrt{|\vec{p}_{J/\psi}|^2 + q^2}\right) \tag{357}$$

$$= \frac{|\vec{p}_{J/\psi}|}{\sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}} + \frac{|\vec{p}_{J/\psi}|}{\sqrt{|\vec{p}_{J/\psi}|^2 + q^2}} = \frac{|\vec{p}_{J/\psi}|}{E_{J/\psi}} + \frac{|\vec{p}_{J/\psi}|}{E_q} \tag{358}$$

$$= |\vec{p}_{J/\psi}|\left(\frac{E_{J/\psi} + E_q}{E_{J/\psi}E_q}\right) = |\vec{p}_{J/\psi}|\left(\frac{m_{\psi(2S)}}{E_{J/\psi}E_q}\right). \tag{359}$$

Thus, we have

$$\frac{1}{4\pi}\int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}d|\vec{p}_{J/\psi}| \tag{360}$$

$$= \frac{1}{4\pi}\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}\frac{E_{J/\psi}E_q}{m_{\psi(2S)}|\vec{p}_{J/\psi}|} \tag{361}$$

$$= \frac{1}{4\pi}\frac{|\vec{p}_{J/\psi}|}{m_{\psi(2S)}}. \tag{362}$$

Now we can combine the results to get the total three-body phase space:

$$d\Gamma = \frac{1}{2m_{\psi(2S)}}|a_\psi|^2 4m_{\psi(2S)}m_{J/\psi}\int\left(\frac{1}{4\pi}\frac{|\vec{p}_{J/\psi}|}{m_{\psi(2S)}}\right)\frac{dq^2}{2\pi}\left(\frac{1}{8\pi}\sqrt{1-\frac{4m_\pi^2}{q^2}}\right)(q^2-4.5m_\pi^2)^2 \tag{363}$$

$$=\frac{|a_\psi|^2 m_{J/\psi}}{32\pi^3 m_{\psi(2S)}}\int|\vec{p}_{J/\psi}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-4.5m_\pi^2)^2 dq^2. \tag{364}$$

We can express $|\vec{p}_{J/\psi}|$ in terms of q^2:

$$m_{J/\psi}^2 = E_{J/\psi}^2 - |\vec{p}_{J/\psi}|^2 = (m_{\psi(2S)} - E_q)^2 - |\vec{p}_{J/\psi}|^2 = (m_{\psi(2S)} - \sqrt{|\vec{p}_{J/\psi}|^2 + q^2})^2 - |\vec{p}_{J/\psi}|^2, \tag{365}$$

which gives (by *Mathematica*):

$$|\vec{p}_{J/\psi}| = \frac{\sqrt{(M-(m-q))(M+(m-q))(M-(m+q))(M+(m+q))}}{2M} \tag{366}$$

$$=\frac{\sqrt{(M^2-(m+q)^2)(M^2-(m-q)^2)}}{2M}, \quad \text{where } M = m_{\psi(2S)}, m = m_{J/\psi}, q = \sqrt{q^2} \tag{367}$$

$$=\frac{\sqrt{(m_{\psi(2S)}^2 - (m_{J/\psi} + \sqrt{q^2})^2)(m_{\psi(2S)}^2 - (m_{J/\psi} - \sqrt{q^2})^2)}}{2m_{\psi(2S)}}. \tag{368}$$

We can discuss the limits of integration for q^2. $q^2 = (p_{\pi^+} + p_{\pi^-})^2 = (p_{\psi(2S)} - p_{J/\psi})^2$, which is the invariant mass squared of the pion pair. The minimum value of q^2 occurs when the two pions are produced at rest in their center-of-mass frame, which gives:

$$q_{min}^2 = (2m_\pi)^2 = 4m_\pi^2. \tag{369}$$

The maximum value of q^2 occurs when the J/ψ is produced at rest in the $\psi(2S)$ rest frame, which gives:

$$q_{max}^2 = (m_{\psi(2S)} - m_{J/\psi})^2. \tag{370}$$

Thus, we have with ($m_{J/\psi} = 3.096$ GeV, $m_{\psi(2S)} = 3.686$ GeV, $m_\pi = 0.13957$ GeV, and the experimental decay rate $\Gamma_{exp} = 101.64$ keV $= 1.01 \times 10^{-4}$ GeV):

$$\Gamma = \frac{|a_\psi|^2 m_{J/\psi}}{32\pi^3 m_{\psi(2S)}}\int_{4m_\pi^2}^{(m_{\psi(2S)}-m_{J/\psi})^2}|\vec{p}_{J/\psi}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-4.5m_\pi^2)^2 dq^2 \tag{371}$$

$$=|a_\psi|^2 \times 8.82391 \times 10^{-7}\text{GeV}^5 = 1.01 \times 10^{-4} \text{ GeV} \tag{372}$$

$$\Rightarrow |a_\psi| = 10.6987 \text{ GeV}^{-3}. \tag{373}$$

(b)

We can perform a similar analysis for the decay $\Upsilon(2S) \to \Upsilon(1S)\pi^+\pi^-$. The decay rate can be expressed

as:

$$\Gamma = \frac{1}{2m_{\Upsilon(2S)}}|a_\Upsilon|^2 4m_{\Upsilon(2S)}m_{\Upsilon(1S)} \int \left(\frac{1}{4\pi}\frac{|\vec{p}_{\Upsilon(1S)}|}{m_{\Upsilon(2S)}}\right)\frac{dq^2}{2\pi}\left(\frac{1}{8\pi}\sqrt{1-\frac{4m_\pi^2}{q^2}}\right)(q^2-3.2m_\pi^2)^2 \tag{374}$$

$$= \frac{|a_\Upsilon|^2 m_{\Upsilon(1S)}}{32\pi^3 m_{\Upsilon(2S)}}\int|\vec{p}_{\Upsilon(1S)}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-3.2m_\pi^2)^2 dq^2. \tag{375}$$

We can express $|\vec{p}_{\Upsilon(1S)}|$ in terms of q^2:

$$|\vec{p}_{\Upsilon(1S)}| = \frac{\sqrt{(m_{\Upsilon(2S)}^2 - (m_{\Upsilon(1S)} + \sqrt{q^2})^2)(m_{\Upsilon(2S)}^2 - (m_{\Upsilon(1S)} - \sqrt{q^2})^2}}{2m_{\Upsilon(2S)}}. \tag{376}$$

The limits of integration for q^2 are:

$$q_{min}^2 = 4m_\pi^2, \tag{377}$$

$$q_{max}^2 = (m_{\Upsilon(2S)} - m_{\Upsilon(1S)})^2. \tag{378}$$

With ($m_{\Upsilon(1S)} = 9.460$ GeV, $m_{\Upsilon(2S)} = 10.023$ GeV, $m_\pi = 0.13957$ GeV, and the experimental decay rate $\Gamma_{exp} = 5.71$ keV $= 5.71 \times 10^{-6}$ GeV):

$$\Gamma = \frac{|a_\Upsilon|^2 m_{\Upsilon(1S)}}{32\pi^3 m_{\Upsilon(2S)}}\int_{4m_\pi^2}^{(m_{\Upsilon(2S)}-m_{\Upsilon(1S)})^2}|\vec{p}_{\Upsilon(1S)}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-3.2m_\pi^2)^2 dq^2 \tag{379}$$

$$= |a_\Upsilon|^2 \times 9.36631 \times 10^{-7}\text{GeV}^5 = 5.71 \times 10^{-6} \text{ GeV} \tag{380}$$

$$\Rightarrow |a_\Upsilon| = 2.46907 \text{ GeV}^{-3}. \tag{381}$$

Comparing the extracted magnitudes of $|a_\psi|$ and $|a_\Upsilon|$, we find that $|a_\psi|$ is significantly larger than $|a_\Upsilon|$. Since the constants a_ψ and a_Υ reflect overlap integrals between the $2S$ and $1S$ quarkonium wavefunctions and scale with the mean-square radius $\langle r^2\rangle$ of the bound state, we can infer that the charmonium bound state (associated with a_ψ) has a larger spatial extent compared to the bottomonium bound state (associated with a_Υ). This suggests that charmonium states are more loosely bound and have a larger size than bottomonium states, which are more tightly bound and compact. $\qquad\square$

Problem Set 7 due 9:30 AM, Wednesday, December 10

Question 1

Color Structure and One-Gluon Exchange

(a) The color force between quarks is mediated by a color-anticolor octet of vector gauge bosons called gluons. Denoting the three color charges by R (red), G (green) and B (blue) write down the color combinations of the gluon octet states in analogy with the meson flavor-antiflavor octet of $SU(3)_f$.

(b) A quark-antiquark meson is a color singlet with wavefunction $Q\overline{Q} = \frac{1}{\sqrt{3}}(R\overline{R} + G\overline{G} + B\overline{B})$. At short distances, the potential between quarks is approximately Coulombic $V(r) = \xi \frac{g^2}{r}$, arising from one-gluon exchange with coupling g between the gluon and the quark pair. Determine the color factor ξ for the color-singlet meson state. Is the potential attractive or repulsive?

(c) For two quarks, the color states combines as $\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$. Write down the color wavefunction for the antisymmetric $\overline{\mathbf{3}}$ and symmetric $\mathbf{6}$ states using the color basis $\{R, G, B\}$. Use one-gluon exchange arguments to determine which color configuration corresponding ot an attractive potential. Explain qualitatively how this leads to stable color-singlet baryons: two quarks attract in the $\overline{\mathbf{3}}$ channel, which then combines with the third quark ($\mathbf{3}$) to form a color singlet (using the result from (b)).

Hint: You can obtain the color diquark wavefunctions by recalling that $\overline{\mathbf{3}}_i \propto \epsilon_{ijk}Q_jQ_k$ and $\mathbf{6} \propto Q_iQ_j + Q_jQ_i$.

Answer

(a)

The eight gluon color states can be written as $(u \rightarrow R,\ d \rightarrow G,\ s \rightarrow B)$:

$$g_1 = \frac{1}{\sqrt{2}}(R\overline{G} + G\overline{R}), \quad g_2 = \frac{-i}{\sqrt{2}}(R\overline{G} - G\overline{R}), \quad g_3 = \frac{1}{\sqrt{2}}(R\overline{R} - G\overline{G}), \tag{382}$$

$$g_4 = \frac{1}{\sqrt{2}}(R\overline{B} + B\overline{R}), \quad g_5 = \frac{-i}{\sqrt{2}}(R\overline{B} - B\overline{R}), \quad g_6 = \frac{1}{\sqrt{2}}(G\overline{B} + B\overline{G}), \tag{383}$$

$$g_7 = \frac{-i}{\sqrt{2}}(G\overline{B} - B\overline{G}), \quad g_8 = \frac{1}{\sqrt{6}}(R\overline{R} + G\overline{G} - 2B\overline{B}). \tag{384}$$

(b)

We start from the QCD interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = g\overline{\psi}\gamma^{\mu}T^a\psi A^a_{\mu}, \tag{385}$$

where T^a are the generators of the $SU(3)$ color group in the adjoint representation, and A_μ^a are the gluon fields. We can write the transition amplitude for one-gluon exchange between a quark and an antiquark as:

$$\mathcal{M} \propto g^2 \left(\overline{u}_\alpha(p_f)\gamma^\mu T_{\alpha\beta}^a u_\beta(p_i)\right) \frac{-ig_{\mu\nu}}{q^2} \left(\overline{v}_\gamma(k_i)\gamma^\nu \overline{T}_{\gamma\delta}^a v_\delta(k_f)\right), \tag{386}$$

where $\alpha, \beta, \gamma, \delta$ are color indices, q is the four-momentum transfer, u and v are the quark and antiquark spinors, respectively, $\overline{T}$ is the antiquark representation of the color generators, and p_i, p_f, k_i, k_f are the initial and final momenta of the quark and antiquark, respectively. The color factor ξ arises from the contraction of the color indices:

$$\xi = \langle\psi|\mathcal{C}|\psi\rangle = T_{\alpha\beta}^a \overline{T}_{\gamma\delta}^a \langle Q_\alpha \overline{Q}_\gamma | Q_\beta \overline{Q}_\delta\rangle, \tag{387}$$

where $|\psi\rangle = |Q\overline{Q}\rangle$ is the color-singlet meson state, and $\mathcal{C} = T^a \otimes \overline{T}^a$ is the color operator for one-gluon exchange. Applying the relation $\overline{T}^a = -(T^a)^T$, we have:

$$\xi = -T_{\alpha\beta}^a T_{\delta\gamma}^a \langle Q_\alpha \overline{Q}_\gamma | Q_\beta \overline{Q}_\delta\rangle. \tag{388}$$

For the color-singlet meson state, we have:

$$|Q\overline{Q}\rangle = \frac{1}{\sqrt{3}}(R\overline{R} + G\overline{G} + B\overline{B}). \tag{389}$$

Applying $\langle Q_\alpha \overline{Q}_\gamma | Q_\beta \overline{Q}_\delta\rangle = \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta}$, we find:

$$\xi = -\frac{1}{3}T_{\alpha\beta}^a T_{\beta\alpha}^a \tag{390}$$

$$= -\frac{1}{3}\text{Tr}(T^a T^a) \tag{391}$$

$$= -\frac{1}{3}\cdot\frac{1}{2}\cdot 8 \quad \left(\text{since } \text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \text{ and } a = 1,\ldots,8\right) \tag{392}$$

$$= -\frac{4}{3}. \tag{393}$$

Thus, the color factor for the color-singlet meson state is $\xi = -\frac{4}{3}$, indicating that the potential is attractive.

(c)

Now we can write down:

$$\xi = \langle\psi_{\alpha\gamma}|\mathcal{C}_{\alpha\gamma,\beta\delta}|\psi_{\beta\delta}\rangle = T_{\alpha\beta}^a T_{\delta\gamma}^a \langle\psi_{\alpha\gamma}|\psi_{\beta\delta}\rangle, \tag{394}$$

where $\alpha, \beta, \gamma, \delta$ are color indices. We can apply Fierz identity for the generators of $SU(3)$:

$$T^a_{\alpha\beta}T^a_{\delta\gamma} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{3}\delta_{\alpha\beta}\delta_{\delta\gamma}\right). \tag{395}$$

Now we can have:

$$\xi = \frac{1}{2}\left(\langle\psi_{\alpha\gamma}|\psi_{\gamma\alpha}\rangle - \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle\right). \tag{396}$$

First, if $|\psi\rangle$ is in the antisymmetric $\mathbf{\bar{3}}$ representation, we have:

$$|\psi_{\alpha\gamma}\rangle = -|\psi_{\gamma\alpha}\rangle, \tag{397}$$

which leads to:

$$\xi_{\bar{3}} = \frac{1}{2}\left(-\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle - \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle\right) \tag{398}$$

$$= -\frac{2}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle = -\frac{2}{3}. \tag{399}$$

Next, if $|\psi\rangle$ is in the symmetric $\mathbf{6}$ representation, we have:

$$|\psi_{\alpha\gamma}\rangle = |\psi_{\gamma\alpha}\rangle, \tag{400}$$

which leads to:

$$\xi_6 = \frac{1}{2}\left(\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle - \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle\right) \tag{401}$$

$$= \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle = \frac{1}{3}. \tag{402}$$

Thus, the color factor for the antisymmetric $\mathbf{\bar{3}}$ state is $\xi_{\bar{3}} = -\frac{2}{3}$, indicating an attractive potential, while the color factor for the symmetric $\mathbf{6}$ state is $\xi_6 = \frac{1}{3}$, indicating a repulsive potential. This attraction in the $\mathbf{\bar{3}}$ channel allows two quarks to form a stable diquark state, which can then combine with a third quark in the $\mathbf{3}$ representation to form a color-singlet baryon, as the combination $\mathbf{\bar{3}} \otimes \mathbf{3}$ contains a singlet representation. $\qquad\square$

Question 2

Inverse Fourier Transform of Form Factors

Hadronic form factors $F(Q^2)$ are measured in elastic scattering, where the exchanged momentum is off-shell. In this regime no real particle is produced by the probe, and the form factor encodes information about the spatial distribution of charge and current within the target. It is conventional to define $Q^2 = -q^2$.

(a) In the nonrelativistic (static) limit $(v \ll c)$, explain why the energy transfer q^0 can be neglected relative to the spatial momentum $|\mathbf{q}|$. Show that this implies $Q^2 = |\mathbf{q}|^2 > 0$, corresponding to spacelike momentum exchange.

(b) In this limit the **spatial charge density** $\rho(r)$ is obtain as the inverse three-dimensionful Fourier transform of the form factor:

$$\rho(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} F(Q^2). \tag{403}$$

Evaluate $\rho(r)$ for the following model form factors:

$$F(Q^2) = \frac{1}{1 + Q^2/\Lambda^2} \quad \text{and} \quad F(Q^2) = e^{-Q^2/\Lambda^2}, \tag{404}$$

and show that they yield, respectively a Yukawa form $\propto e^{-\Lambda r}/r$ and a Gaussian form $\propto e^{-\Lambda^2 r^2/4}$.

(c) Compare the spatial falloff of these two distributions. What does each imply about the effective range and shape of the hadronic charge density?

Answer

(a)

We can define $q^\mu = (q^0, \mathbf{q}) = p_f^\mu - p_i^\mu$, where p_i^μ and p_f^μ are the initial and final four-momenta of the target hadron, respectively. In the nonrelativistic limit $(v \ll c)$, the kinetic energy of the hadron is much smaller than its rest mass energy, so we can approximate:

$$q^0 = E_f - E_i \approx \frac{\mathbf{p}_f^2}{2m} - \frac{\mathbf{p}_i^2}{2m} \ll |\mathbf{q}| = |\mathbf{p}_f - \mathbf{p}_i|. \tag{405}$$

Thus, we can neglect q^0 relative to $|\mathbf{q}|$, leading to:

$$Q^2 = -q^2 = -(q^0)^2 + |\mathbf{q}|^2 \approx |\mathbf{q}|^2 > 0, \tag{406}$$

which corresponds to spacelike momentum exchange.

(b)

To evaluate the spatial charge density $\rho(r)$ for the given form factors, we start with the integral:

$$\rho(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} F(Q^2). \tag{407}$$

For the first form factor $F(Q^2) = \frac{1}{1+Q^2/\Lambda^2}$, we have:

$$\rho(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{1+|\mathbf{q}|^2/\Lambda^2} \tag{408}$$

$$= \int \frac{d\phi\, d\cos\theta\, dq}{(2\pi)^3} q^2 e^{iqr\cos\theta} \frac{1}{1+q^2/\Lambda^2} \tag{409}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dq \frac{q^2}{1+q^2/\Lambda^2} \int_{-1}^1 d\cos\theta\, e^{iqr\cos\theta}, \quad \text{after integrating over } \phi \tag{410}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dq \frac{q^2}{1+q^2/\Lambda^2} \left(\frac{2\sin(qr)}{qr}\right), \quad \text{after integrating over } \cos\theta \tag{411}$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty dq \frac{q\sin(qr)}{1+q^2/\Lambda^2} \tag{412}$$

$$= \frac{1}{2\pi^2 r} \frac{\pi\Lambda^2 r e^{-\sqrt{\frac{r^2}{\frac{1}{\Lambda^2}}}}}{2\sqrt{r^2}}, \quad \text{by Mathematica} \tag{413}$$

$$= \frac{\Lambda^2}{4\pi} \frac{e^{-\Lambda r}}{r}. \tag{414}$$

For the second form factor $F(Q^2) = e^{-Q^2/\Lambda^2}$, we have:

$$\rho(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} e^{-|\mathbf{q}|^2/\Lambda^2} \tag{415}$$

$$= \int \frac{d\phi\, d\cos\theta\, dq}{(2\pi)^3} q^2 e^{iqr\cos\theta} e^{-q^2/\Lambda^2} \tag{416}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dq\, q^2 e^{-q^2/\Lambda^2} \int_{-1}^1 d\cos\theta\, e^{iqr\cos\theta}, \quad \text{after integrating over } \phi \tag{417}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dq\, q^2 e^{-q^2/\Lambda^2} \left(\frac{2\sin(qr)}{qr}\right), \quad \text{after integrating over } \cos\theta \tag{418}$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty dq\, q e^{-q^2/\Lambda^2} \sin(qr) \tag{419}$$

$$= \frac{1}{2\pi^2 r} \frac{\sqrt{\pi} r e^{-\frac{1}{4}\Lambda^2 r^2}}{4\left(\frac{1}{\Lambda^2}\right)^{3/2}}, \quad \text{by Mathematica} \tag{420}$$

$$= \frac{\Lambda^3}{8\pi^{3/2}} e^{-\frac{\Lambda^2 r^2}{4}}. \tag{421}$$

(c)

The spatial falloff of the two distributions can be compared as follows:

- The Yukawa form $\rho(r) \propto \frac{e^{-\Lambda r}}{r}$ indicates a long-range interaction that decays exponentially with distance r. The presence of the $1/r$ factor suggests that the charge density has a significant

contribution even at larger distances, although it decreases rapidly due to the exponential term. This form is characteristic of interactions mediated by massive particles, where Λ can be interpreted as the mass scale of the exchanged particle.

- The Gaussian form $\rho(r) \propto e^{-\frac{\Lambda^2 r^2}{4}}$ indicates a short-range interaction that decays very rapidly with distance r. The Gaussian decay implies that the charge density is highly localized around the origin, with negligible contributions at larger distances. This form is characteristic of interactions where the charge distribution is tightly confined, leading to a rapid falloff.

In summary, the Yukawa form suggests a more extended charge distribution with a longer effective range, while the Gaussian form indicates a highly localized charge distribution with a very short effective range. $\qquad\square$

Question 3

Deep Inelastic Structure Functions and the Gottfried Sum Rule

In deep inelastic electron–nucleon scattering, the nucleon structure functions $F_2^p(x)$ and $F_2^n(x)$ describe the momentum distributions of quarks carrying a fraction x of the nucleon's momentum.

(a) Using the quark–parton model and the fact that quark distributions are positive definite, verify that the structure functions satisfy

$$\frac{1}{4} \leq \frac{F_2^n(x)}{F_2^p(x)} \leq 4. \tag{422}$$

(b) In the limit $x \to 0$, the sea quarks dominate and may be taken as $SU(2)$-flavor symmetric. What limit do you expect for the ratio $F_2^n(x)/F_2^p(x)$ in this case?

(c) The *Gottfried sum rule* is defined through the integral

$$I_G(x) = \int_x^1 \frac{F_2^p(x') - F_2^n(x')}{x'} dx'. \tag{423}$$

Assuming an $SU(2)$ flavor-symmetric sea, what value do you predict for $I_G(0)$? Compare your result with the experimental measurement $I_G(0) = 0.235 \pm 0.026$ at $Q^2 = 4$ GeV^2, first reported by the NMC collaboration at CERN in 1991. Is this surprising?

Answer

(a)

In the quark-parton model, the structure functions for the proton and neutron can be expressed in terms of the quark distribution functions as follows:

$$F_2^p(x) = x \left[\frac{4}{9}(u(x) + \bar{u}(x)) + \frac{1}{9}(d(x) + \bar{d}(x)) + \frac{1}{9}(s(x) + \bar{s}(x)) \right], \tag{424}$$

$$F_2^n(x) = x \left[\frac{4}{9}(d(x) + \bar{d}(x)) + \frac{1}{9}(u(x) + \bar{u}(x)) + \frac{1}{9}(s(x) + \bar{s}(x)) \right]. \tag{425}$$

To find the ratio $\frac{F_2^n(x)}{F_2^p(x)}$, we can write:

$$\frac{F_2^n(x)}{F_2^p(x)} = \frac{\frac{4}{9}(d(x) + \bar{d}(x)) + \frac{1}{9}(u(x) + \bar{u}(x)) + \frac{1}{9}(s(x) + \bar{s}(x))}{\frac{4}{9}(u(x) + \bar{u}(x)) + \frac{1}{9}(d(x) + \bar{d}(x)) + \frac{1}{9}(s(x) + \bar{s}(x))}. \tag{426}$$

Since the quark distribution functions are positive definite, we can analyze the extremes of this ratio.

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The minimum value occurs when $d(x) + \bar{d}(x)$ is minimized and $u(x) + \bar{u}(x)$ is maximized, leading to:

$$\frac{F_2^n(x)}{F_2^p(x)} \geq \frac{\frac{1}{9}(u(x) + \bar{u}(x))}{\frac{4}{9}(u(x) + \bar{u}(x))} = \frac{1}{4}. \tag{427}$$

The maximum value occurs when $d(x) + \bar{d}(x)$ is maximized and $u(x) + \bar{u}(x)$ is minimized, leading to:

$$\frac{F_2^n(x)}{F_2^p(x)} \leq \frac{\frac{4}{9}(d(x) + \bar{d}(x))}{\frac{1}{9}(d(x) + \bar{d}(x))} = 4. \tag{428}$$

Thus, we have verified that:

$$\frac{1}{4} \leq \frac{F_2^n(x)}{F_2^p(x)} \leq 4. \tag{429}$$

(b)

In the limit $x \to 0$, the sea quarks dominate the structure functions, and we can assume $SU(2)$-flavor symmetry, which implies:

$$\bar{u}(x) \approx \bar{d}(x). \tag{430}$$

Under this assumption, the structure functions simplify to:

$$F_2^p(x) \approx x \left[\frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x) + \frac{1}{9}(s(x) + \bar{s}(x)) \right] = x \left[\frac{5}{9}\bar{u}(x) + \frac{1}{9}(s(x) + \bar{s}(x)) \right], \tag{431}$$

$$F_2^n(x) \approx x \left[\frac{4}{9}\bar{d}(x) + \frac{1}{9}\bar{u}(x) + \frac{1}{9}(s(x) + \bar{s}(x)) \right] = x \left[\frac{5}{9}\bar{u}(x) + \frac{1}{9}(s(x) + \bar{s}(x)) \right]. \tag{432}$$

Thus, in this limit, we find:

$$\frac{F_2^n(x)}{F_2^p(x)} \approx 1. \tag{433}$$

(c)

The Gottfried sum rule is given by:

$$I_G(0) = \int_0^1 \frac{F_2^p(x') - F_2^n(x')}{x'} dx'. \tag{434}$$

We can express all quark distributions in terms of valence and sea components:

$$u(x) = u_v(x) + u_s(x), \quad d(x) = d_v(x) + d_s(x), \tag{435}$$

$$\bar{u}(x) = \bar{u}_s(x), \quad \bar{d}(x) = \bar{d}_s(x), \quad s(x) = s_s(x), \quad \bar{s}(x) = \bar{s}_s(x). \tag{436}$$

Assuming an $SU(2)$ flavor-symmetric sea, giving $\bar{u}_s(x) = \bar{d}_s(x) = u_s(x) = d_s(x)$, we can simplify the

difference between the proton and neutron structure functions:

$$F_2^p(x) - F_2^n(x) = x \left[\frac{4}{9}(u_v(x) - d_v(x)) + \frac{1}{9}(d_v(x) - u_v(x)) \right] \tag{437}$$

$$= x \left[\frac{1}{3}(u_v(x) - d_v(x)) \right] \tag{438}$$

Substituting this into the Gottfried sum rule, we get:

$$I_G(0) = \int_0^1 \frac{x \left[\frac{1}{3}(u_v(x) - d_v(x)) \right]}{x} dx' \tag{439}$$

$$= \frac{1}{3} \int_0^1 (u_v(x) - d_v(x)) dx'. \tag{440}$$

The integrals of the valence quark distributions over the range $[0, 1]$ give the total number of valence quarks in the proton:

$$\int_0^1 u_v(x) dx' = 2, \quad \int_0^1 d_v(x) dx' = 1. \tag{441}$$

Thus, we find:

$$I_G(0) = \frac{1}{3}(2 - 1) = \frac{1}{3}. \tag{442}$$

Comparing this result with the experimental measurement $I_G(0) = 0.235 \pm 0.026$, we see that the experimental value is significantly lower than the predicted value of $\frac{1}{3} \approx 0.333$. This discrepancy suggests that the assumption of an $SU(2)$ flavor-symmetric sea may not hold true in reality. The lower experimental value indicates an asymmetry in the sea quark distributions, specifically that there are more $\bar{d}$ quarks than $\bar{u}$ quarks in the proton. $\qquad \square$