University of Minnesota
School of Physics and Astronomy

# 2025 Fall Physics 8501
# General Relativity I

Assignment Solution

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# Assignment 8 due on Monday November 3 at 10PM

## Question 1

In lecture we showed that $P^0 = M$ for the Schwarzschild solution in standard coordinates. Calculate $P^z$ and show that it is zero. Since the metric is isotropic this shows that all components of the 3-momentum are zero.

## Answer

$$\begin{aligned}
P^j =& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij} + \partial_t h_{ij} + \partial_k h_{0k}\delta_{ij} - \partial_i h_{0j}\Bigg) n^i r^2 d\Omega && (1)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij} + \partial_t h_{ij} + \partial_k h_{0k}\delta_{ij} - \partial_i h_{0j}\Bigg) \frac{x^i}{r} r^2 d\Omega && (2)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij} + \partial_t h_{ij} + \partial_k h_{0k}\delta_{ij} - \partial_i h_{0j}\Bigg) x^i r d\Omega && (3)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij}x^i + \partial_t h_{ij}x^i + \partial_k h_{0k}\delta_{ij}x^i - \partial_i h_{0j}x^i\Bigg) r d\Omega && (4)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}x^j + \partial_t h_{ij}x^i + \partial_k h_{0k}x^j - \partial_i h_{0j}x^i\Bigg) r d\Omega && (5)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}x^j + \partial_t h_{ij}x^i\Bigg) r d\Omega \quad \text{(since } h_{0j} = h_{0k} = 0 \text{ for Schwarzschild metric)} && (6)\\
=& 0 \quad \text{(since } h_{\mu\nu} \text{ is time-independent for Schwarzschild metric)} \quad \square && (7)
\end{aligned}$$

# Question 2

In lecture we were given the components of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ for the Scharzschild solution in standard coordinates. Using these, calculate one component of the Riemann-Christoffel curvature tensor, namely $R^{t}_{rtr}$. This is nonvanishing everywhere and goes to zero as $r \to \infty$. This shows that space is curved even though both the Ricci tensor and curvature scalar vanish.

# Answer

By the definition of the Riemann-Christoffel curvature tensor, we have

$$R^{\lambda}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\lambda}_{\mu\rho} - \partial_\rho \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu}. \tag{8}$$

Thus, we have

$$R^{t}{}_{rtr} = \partial_t \Gamma^{t}_{rr} - \partial_r \Gamma^{t}_{rt} + \Gamma^{t}_{t\sigma}\Gamma^{\sigma}_{rr} - \Gamma^{t}_{r\sigma}\Gamma^{\sigma}_{rt}. \tag{9}$$

From the lecture notes, we have

$$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}, \quad \Gamma^{t}_{rr} = 0. \tag{10}$$

Thus, we have (by Mathematica)

$$\begin{aligned} R^{t}_{rtr} &= -\partial_r \Gamma^{t}_{rt} + \Gamma^{t}_{tt}\Gamma^{t}_{rr} + \Gamma^{t}_{tr}\Gamma^{r}_{rr} - \Gamma^{t}_{rt}\Gamma^{t}_{rt} - \Gamma^{t}_{rr}\Gamma^{r}_{rt} && (11)\\ &= \frac{2M}{r^2(r-2M)} && (12) \end{aligned}$$

We can see that $R^{t}_{rtr}$ is nonvanishing everywhere and it goes to zero as $r \to \infty$. $\square$

# Question 3

A photon moves in the Schwarzschild metric in the equatorial plane $\theta = \pi/2$. Using standard coordinates, show that the shape of the orbit is given by the solution to the differential equation

$$\frac{d^2w}{d\phi^2} + w = 3w^2, \tag{13}$$

where $w = GM/r$. Assuming that $|w| \ll 1$, solve this equation iteratively to find the deflection angle $\Delta\phi$, and show that it agrees with the answer obtained in class by other means.

# Answer

For a photon moving in the Schwarzschild metric in the equatorial plane $\theta = \pi/2$, we have the following equations of motion:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2GM}{r}\right)\frac{L^2}{r^2}, \tag{14}$$

$$\frac{d\phi}{d\lambda} = \frac{L}{r^2}, \tag{15}$$

where $\lambda$ is an affine parameter along the photon's trajectory, $E$ is the energy per unit mass, and $L$ is the angular momentum per unit mass. Dividing the first equation by the square of the second equation, we have

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\left(E^2 - \left(1 - \frac{2GM}{r}\right)\frac{L^2}{r^2}\right). \tag{16}$$

Rearranging this equation, we have

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4E^2}{L^2} - r^2 + 2GMr. \tag{17}$$

Next, we introduce the variable $w = \frac{GM}{r}$. Thus, we have $r = \frac{GM}{w}$ and $\frac{dr}{d\phi} = -\frac{GM}{w^2}\frac{dw}{d\phi}$. Substituting these into the previous equation, we have

$$\left(-\frac{GM}{w^2}\frac{dw}{d\phi}\right)^2 = \frac{\left(\frac{GM}{w}\right)^4 E^2}{L^2} - \left(\frac{GM}{w}\right)^2 + 2GM\left(\frac{GM}{w}\right). \tag{18}$$

Simplifying this equation, we have

$$\left(\frac{dw}{d\phi}\right)^2 = \frac{E^2(GM)^2}{L^2} - w^2 + 2w^3. \tag{19}$$

Taking the derivative of both sides with respect to $\phi$, we have

$$2\frac{dw}{d\phi}\frac{d^2w}{d\phi^2} = -2w\frac{dw}{d\phi} + 6w^2\frac{dw}{d\phi}. \tag{20}$$

Dividing both sides by $2\frac{dw}{d\phi}$, we have

$$\frac{d^2w}{d\phi^2} = -w + 3w^2. \tag{21}$$

Rearranging this equation, we have

$$\frac{d^2w}{d\phi^2} + w = 3w^2. \tag{22}$$

To solve this equation iteratively, we first solve the homogeneous equation:

$$\frac{d^2w_0}{d\phi^2} + w_0 = 0. \tag{23}$$

The general solution to this equation is

$$w_0(\phi) = A\cos\phi + B\sin\phi, \tag{24}$$

where $A$ and $B$ are constants determined by initial conditions. By considering a photon coming from infinity, we set $A = \frac{GM}{b}$ and $B = 0$, where $b$ is the impact parameter. In other words,

$$w_0(0) = \frac{GM}{b}, \quad \left.\frac{dw_0}{d\phi}\right|_{\phi=0} = 0. \tag{25}$$

Thus, we have

$$w_0(\phi) = \frac{GM}{b}\cos\phi. \tag{26}$$

Next, we substitute $w_0$ into the right-hand side of the original equation to find the first-order correction $w_1$:

$$\frac{d^2w_1}{d\phi^2} + w_1 = 3w_0^2 = 3\left(\frac{GM}{b}\cos\phi\right)^2 = 3\left(\frac{GM}{b}\right)^2\cos^2\phi. \tag{27}$$

Using the identity $\cos^2\phi = \frac{1+\cos 2\phi}{2}$, we have

$$\frac{d^2w_1}{d\phi^2} + w_1 = \frac{3}{2}\left(\frac{GM}{b}\right)^2(1+\cos 2\phi). \tag{28}$$

By considering a photon coming from infinity, we have

$$w_1(0) = 0, \quad \left.\frac{dw_1}{d\phi}\right|_{\phi=0} = 0. \tag{29}$$

Thus, we have (by mathematica)

$$\begin{align} w_1(\phi) &= \frac{2G^2M^2 \sin^2\left(\frac{\phi}{2}\right)(\cos(\phi)+2)}{b^2} \tag{30} \\ &= \frac{G^2M^2}{2b^2}(3 - 2\cos\phi - \cos 2\phi). \tag{31} \end{align}$$

Hence the approximate solution up to first order is

$$w(\phi) = w_0(\phi) + w_1(\phi) = \frac{GM}{b}\cos\phi + \frac{G^2M^2}{2b^2}(3 - 2\cos\phi - \cos 2\phi). \tag{32}$$

To find the deflection angle $\Delta\phi$, we set $w(\phi) = 0$ and solve for $\phi$ when the photon is far away from the mass. Thus, we have

$$0 = \frac{GM}{b}\cos\phi + \frac{G^2M^2}{2b^2}(3 - 2\cos\phi - \cos 2\phi). \tag{33}$$

Expanding $\cos 2\phi = 2\cos^2\phi - 1$, we have

$$\begin{align} 0 =& \frac{GM}{b}\cos\phi + \frac{G^2M^2}{2b^2}(4 - 4\cos\phi + 2\cos^2\phi) \tag{34} \\ =& \frac{G^2M^2}{b^2}\cos^2\phi + \left(\frac{GM}{b} - \frac{2G^2M^2}{b^2}\right)\cos\phi + \frac{2G^2M^2}{b^2} \tag{35} \\ \approx& \frac{G^2M^2}{b^2}\cos^2\phi + \frac{GM}{b}\cos\phi + \frac{2G^2M^2}{b^2}. \tag{36} \end{align}$$

Solving this quadratic equation for $\cos\phi$, we have

$$\cos\phi = \frac{-\frac{GM}{b} \pm \sqrt{\left(\frac{GM}{b}\right)^2 - 8\left(\frac{GM}{b}\right)^4}}{2\frac{G^2M^2}{b^2}} \approx -2\frac{GM}{b} \tag{37}$$

Thus, we have

$$\phi \approx \frac{\pi}{2} + 2\frac{GM}{b}. \tag{38}$$

Since the photon comes from infinity and goes back to infinity, the total deflection angle is

$$\Delta\phi = 2\left(\phi - \frac{\pi}{2}\right) = \frac{4GM}{b}. \tag{39}$$

This agrees with the answer obtained in class by other means. $\square$

# Question 4

The deflection of light by a spherical static body whose physical radius is smaller than its Schwarzschild radius should produce comet-like orbits suffering substantial deflection before returning to infinite radius if the distance of closest approach $r_0$ becomes comparable to the Schwarzschild radius. Using the differential equation in problem 1 show that there is a critical orbit $w_c(\phi)$, corresponding to a critical radius $r_c$. Explain what happens when $r_0 > r_c, r_0 = r_c$, and $r_0 < r_c$. Draw some orbits for illustration. This phenomenon was observed by the "Event Horizon Telescope".

# Answer

From problem 3, we have the differential equation

$$\frac{d^2w}{d\phi^2} + w = 3w^2, \tag{40}$$

where $w = \frac{GM}{r}$. To find the critical orbit $w_c(\phi)$, we look for a circular orbit where $w$ is constant. Thus, we set $\frac{d^2w}{d\phi^2} = 0$. Thus, we have

$$w_c = 3w_c^2. \tag{41}$$

Solving this equation, we have

$$w_c = 0 \quad \text{or} \quad w_c = \frac{1}{3}. \tag{42}$$

The solution $w_c = 0$ corresponds to an orbit at infinite radius, which is not physically interesting. The solution $w_c = \frac{1}{3}$ corresponds to a critical radius

$$r_c = 3GM. \tag{43}$$

When the distance of closest approach $r_0$ is greater than the critical radius $r_c$ ($r_0 > r_c$), the photon will be deflected but will eventually escape to infinity, resulting in a hyperbolic-like orbit. When $r_0$ is equal to the critical radius $r_c$ ($r_0 = r_c$), the photon will spiral around the mass at the critical radius, resulting in a circular orbit. When $r_0$ is less than the critical radius $r_c$ ($r_0 < r_c$), the photon will be captured by the mass and will spiral inward, eventually falling into the black hole. This results in a trajectory that does not return to infinity.

□

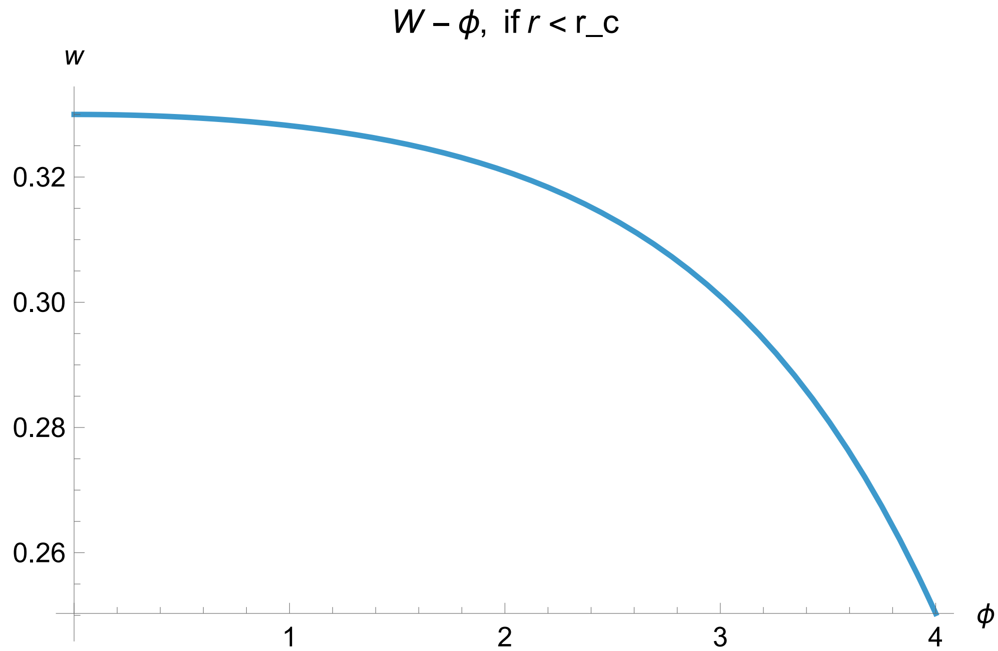


Figure 1: Orbit for $r_0 < r_c$, photon spirals inward and falls into the black hole.

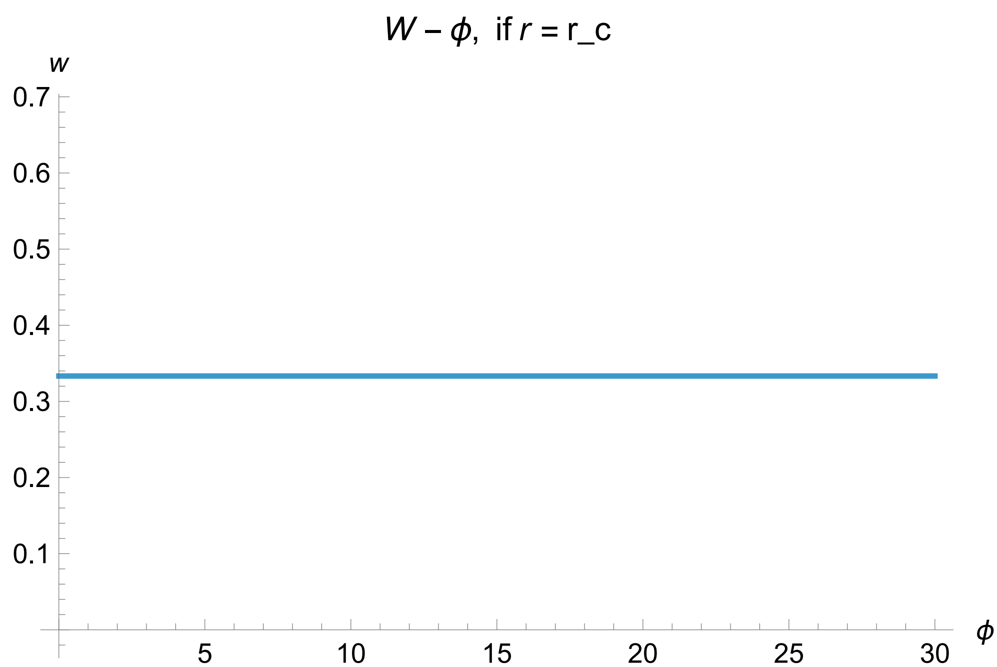


Figure 2: Orbit for $r_0 = r_c$, photon spirals around the mass at the critical radius.

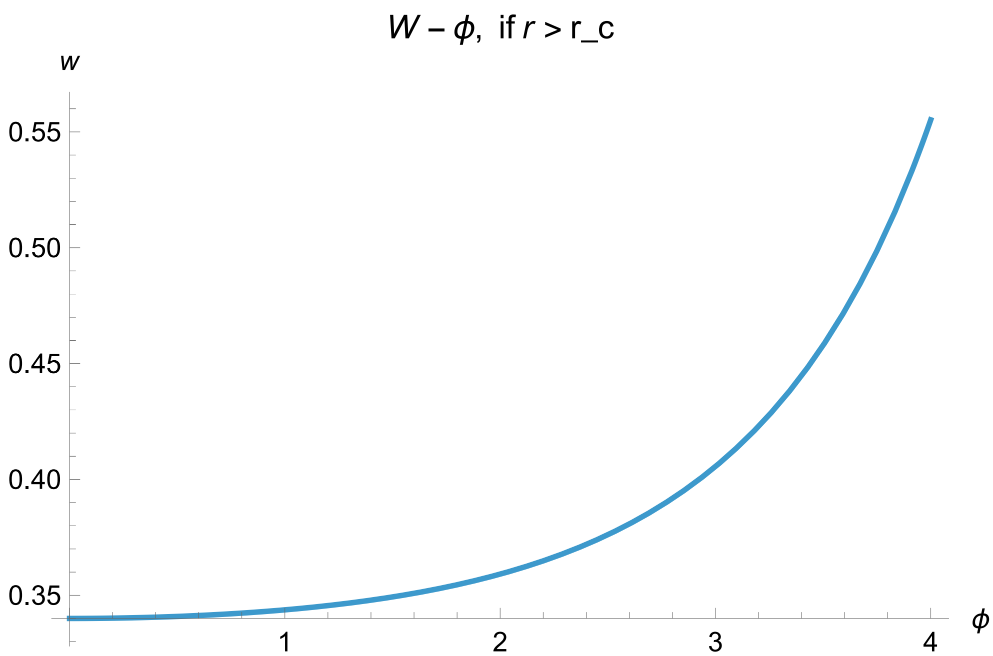


Figure 3: Orbit for $r_0 > r_c$, photon is deflected but escapes to infinity.