University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

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Question 1

Calculate numerically the Schwarzschild radius R_s, the characteristic collapse time t_{collapse}, the characteristic radial redshift time $2R_s/c$, the characteristic radial flux time $R_s/2c$, and the characteristic total luminosity time $3\sqrt{3}R_s/2c$ in the appropriate units (seconds, days, years, kilometers, light-years, etc) using the dust cloud model for the following initial conditions:

(i) An object of three solar masses with an initial radius of 1 AU.

(ii) An object of 108 solar masses with an initial radius of 100 light-years.

Answer

We have

$$R_s = \frac{2GM}{c^2}, \tag{1}$$

$$t_{\text{collapse}} = \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}}, \tag{2}$$

$$t_{\text{redshift}} = \frac{2R_s}{c}, \tag{3}$$

$$t_{\text{flux}} = \frac{R_s}{2c}, \tag{4}$$

$$t_{\text{luminosity}} = \frac{3\sqrt{3}R_s}{2c}. \tag{5}$$

(i) For an object of three solar masses with an initial radius of 1 AU:

Using $M = 3M_\odot = 3 \times 1.989 \times 10^{30}$ kg, $R = 1$ AU $= 1.496 \times 10^{11}$ m, $G = 6.67430 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, and $c = 3 \times 10^8$ m/s, we find (see *mathematica* code for calculation):

$$R_s \approx 8844.42 \text{ m} = 8.84442 \text{ km}, \tag{6}$$

$$t_{\text{collapse}} \approx 3.22152 \times 10^6 \text{ sec} = 37.2862 \text{ days}, \tag{7}$$

$$t_{\text{redshift}} \approx 5.89628 \times 10^{-5} \text{ seconds} = 58.9628 \text{ microseconds}, \tag{8}$$

$$t_{\text{flux}} \approx 1.47407 \times 10^{-5} \text{ seconds} = 14.7407 \text{ microseconds}, \tag{9}$$

$$t_{\text{luminosity}} \approx 7.65949 \times 10^{-5} \text{ seconds} = 76.5949 \text{ microseconds}. \tag{10}$$

(ii) For an object of 10^8 solar masses with an initial radius of 100 light-years:

Using $M = 10^8 M_\odot = 10^8 \times 1.989 \times 10^{30}$ kg, $R = 100$ light-years $= 9.461 \times 10^{17}$ m, $G = 6.67430 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, and $c = 3 \times 10^8$ m/s, we find (see *mathematica* code for calculation):

$$
\begin{aligned}
R_s \approx& 2.94814 \times 10^{11} \text{ m} = 1.971 \text{ AU}, && (11)\\
t_{\text{collapse}} \approx& 8.87422 \times 10^{12} \text{ sec} = 281400 \text{ years}, && (12)\\
t_{\text{redshift}} \approx& 1965.43 \text{ seconds} = 32.7572 \text{ minutes}, && (13)\\
t_{\text{flux}} \approx& 491.357 \text{ seconds} = 8.18928 \text{ minutes}, && (14)\\
t_{\text{luminosity}} \approx& 2553.16 \text{ seconds} = 42.5527 \text{ minutes}. && (15)
\end{aligned}
$$

$\square$

Question 2

In the 1960's it was shown that the light received from a collapsing, luminous cloud of dust is dominated by photons deposited near the unstable orbit as the surface of the cloud crosses the radius $r = 3/2R_s$. Calculate the redshift z for photons emitted *radially* near this surface to at least 2 significant digits. (A more elaborate calculation shows that most photons are emitted with nonzero angular momentum. These photons escape after orbiting the dust cloud many times, resulting in a redshift $z = 2$.)

Answer

The redshift z for photons emitted radially from a radius r in the Schwarzschild metric is given by

$$1 + z = \frac{1}{\sqrt{1 - \frac{R_s}{r}}}. \tag{16}$$

Hence, for $r = \frac{3}{2}R_s$, we have

$$1 + z = \frac{1}{\sqrt{1 - \frac{R_s}{\frac{3}{2}R_s}}} = \frac{1}{\sqrt{1 - \frac{2}{3}}} = \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{3}. \tag{17}$$

Thus, the redshift is

$$z = \sqrt{3} - 1 \approx 0.732. \tag{18}$$

But this is the redshift effect only due to the gravitational field. We can also consider the Doppler effect due to the motion of the collapsing surface. The total redshift considering both effects is given by

$$1 + z_{total} = (1 + z_{\text{gravitational}})(1 + z_{\text{Doppler}}). \tag{19}$$

The Doppler redshift for a radially infalling object is given by

$$1 + z_{\text{Doppler}} = \sqrt{\frac{1 + v/c}{1 - v/c}}, \tag{20}$$

where v is the infall velocity at radius r. The infall velocity can be found using energy conservation in the Schwarzschild metric:

$$v = c\sqrt{\frac{R_s}{r}} = \sqrt{\frac{R_s}{r}}. \tag{21}$$

For $r = \frac{3}{2}R_s$, we have

$$v = \sqrt{\frac{R_s}{\frac{3}{2}R_s}} = \sqrt{\frac{2}{3}}. \tag{22}$$

Thus, the Doppler redshift is

$$1 + z_{\text{Doppler}} = \sqrt{\frac{1 + \sqrt{\frac{2}{3}}}{1 - \sqrt{\frac{2}{3}}}}. \tag{23}$$

Combining both effects, we have

$$1 + z_{total} = \sqrt{3} \cdot \sqrt{\frac{1 + \sqrt{\frac{2}{3}}}{1 - \sqrt{\frac{2}{3}}}}. \tag{24}$$

Calculating this numerically, we find

$$z_{total} \approx 4.45. \tag{25}$$

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