University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

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Assignment 3 due on Monday September 22th at 5PM

Question 1

Show explicitly that the 4-vector current density for a collection of point charges satisfies $\partial_\mu J^\mu = 0$

Answer

In the class, we defined the 4-vector current density for a collection of point charges as

$$J^0(t,\mathbf{x}) = \rho(t,\mathbf{x}) = \sum_a q_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{1}$$

$$\mathbf{J}(t,\mathbf{x}) = \sum_a q_a \mathbf{v}_a(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)), \quad \mathbf{v}_a(t) = \frac{d\mathbf{x}_a(t)}{dt}. \tag{2}$$

Then we have

$$\begin{aligned}
\partial_\mu J^\mu &= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} && (3)\\
&= \sum_a q_a \left[\frac{\partial}{\partial t}\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)))\right] && (4)\\
&= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)))\right] && (5)\\
&= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \mathbf{v}_a(t) \cdot \nabla\delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))\right] && (6)\\
&= 0. && (7)
\end{aligned}$$

Question 2

Prove that the electromagnetic energy density squared minus the square of the Poynting vector is a Lorentz invariant for an electromagnetic field by expressing this quantity in terms of tensors. You might consider using the dual field strength tensor defined by $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Answer

In the EM class, we defined the electromagnetic energy density and the Poynting vector as

$$u = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{8}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}. \tag{9}$$

Also, we have the following relations

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{10}$$

Besides, the EM field energy momentum strength tensor is given by (from Weinberg's GR book)

$$T^{\mu\nu}_{EM} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{11}$$

$$u = T^{00}_{EM} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad S^i = T^{i0}_{EM} = (\mathbf{E} \times \mathbf{B})^i, \tag{12}$$

where u is the EM energy density and $\mathbf{S}$ is the Poynting vector. Hence, we can define a 4-vector U^μ as

$$U^\mu = (u, \mathbf{S}) = (T^{00}_{EM}, T^{i0}_{EM}) = T^{\mu 0}_{EM}. \tag{13}$$

Then we have

$$T^{00}_{EM} = F^{0\alpha}F^{0}{}_{\alpha} - \frac{1}{4}\eta^{00}F_{\alpha\beta}F^{\alpha\beta} = F^{0i}F^{0}{}_{i} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{14}$$

$$= \mathbf{E}^2 - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{15}$$

$$= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = u \tag{16}$$

$$\tag{17}$$

and

$$T^{i0}_{EM} = F^{i\alpha}F^{0}{}_{\alpha} - \frac{1}{4}\eta^{i0}F_{\alpha\beta}F^{\alpha\beta} = F^{ij}F^{0}{}_{j} \tag{18}$$

$$= \epsilon_{ijk}B_k E_j \tag{19}$$

$$= (\mathbf{E}\times\mathbf{B})^i = S^i. \tag{20}$$

Therefore, we have

$$U^\mu U_\mu = -\eta_{\mu\nu}U^\mu U^\nu \tag{21}$$

$$= -\eta_{\mu\nu}T^{\mu 0}_{EM}T^{\nu 0}_{EM} \tag{22}$$

$$= T^{00}_{EM}T^{00}_{EM} - T^{i0}_{EM}T^{i0}_{EM} = u^2 - \mathbf{S}^2. \tag{23}$$

We can claim that $U^\mu U_\mu$ is a Lorentz invariant since it is the contraction of two tensors. Hence, we conclude that $u^2 - \mathbf{S}^2$ is a Lorentz invariant.

Remark: We can also prove this by using the dual field strength tensor $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. First, we have

$$\widetilde{F}^{0i} = \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}\epsilon_{jkl}B_l = B^i, \tag{24}$$

$$\widetilde{F}^{ij} = \frac{1}{2}\varepsilon^{ij0k}F_{0k} = \frac{1}{2}(\varepsilon^{ij0k} - \varepsilon^{ji0k})F_{0k} = \varepsilon^{ij0k}E_k = \epsilon^{ijk}E_k. \tag{25}$$

Then we can calculate the following two Lorentz invariants:

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2), \tag{26}$$

$$\widetilde{F}_{\mu\nu}F^{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}. \tag{27}$$

Now we can calculate

$$(F_{\mu\nu}F^{\mu\nu})^2 + (\widetilde{F}_{\mu\nu}F^{\mu\nu})^2 = 4(\mathbf{B}^2 - \mathbf{E}^2)^2 + 16(\mathbf{E}\cdot\mathbf{B})^2 \tag{28}$$

$$= 4[(\mathbf{B}^2 + \mathbf{E}^2)^2 - 4\mathbf{E}^2\mathbf{B}^2 + 4(\mathbf{E}\cdot\mathbf{B})^2] \tag{29}$$

$$= 4[(\mathbf{B}^2 + \mathbf{E}^2)^2 - 4(\mathbf{E}\times\mathbf{B})^2] \tag{30}$$

$$= 4(u^2 - \mathbf{S}^2). \tag{31}$$

This quantity is Lorentz invariant since all indices are contracted. Hence, we conclude that $u^2 - \mathbf{S}^2$ is a Lorentz invariant.

Question 3

Calculate the scalar $T^{\alpha}{}_{\alpha}$ associated with the electromagnetic stress tensor.

Answer

Consider the energy momentum tensor with the EM field:

$$\begin{aligned}
T^{\alpha\beta}_{total} = T^{\alpha\beta} + T^{\alpha\beta}_{EM} &= \sum_n p^{\alpha}_n(t)\frac{dx^{\beta}_n}{dt}\delta^{(3)}(\mathbf{x}-\mathbf{x}_n(t)) + F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} && (32)\\
&= \sum_n \frac{p^{\alpha}_n p^{\beta}_n}{E_n}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) + F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}, && (33)
\end{aligned}$$

We have

$$\begin{aligned}
T^{\alpha}{}_{\alpha} &= \eta_{\alpha\beta}T^{\alpha\beta}_{total} && (34)\\
&= \sum_n \frac{p^{\alpha}_n p_{n\alpha}}{E_n}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) + F^{\alpha\mu}F_{\alpha\mu} - \frac{1}{4}\eta_{\alpha\beta}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} && (35)\\
&= \sum_n \frac{m^2_n}{E_n}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) + 0, \quad \text{by } \eta^{\alpha\beta}\eta_{\alpha\beta} = 4 && (36)\\
&= \sum_n \frac{m^2_n}{E_n}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) && (37)\\
&= \sum_n \frac{m^2_n}{E_n}\delta^3(\mathbf{x}-\mathbf{x}_n(t)). && (38)
\end{aligned}$$

Remark: In class, we have shown that

$$\frac{\delta^3(\mathbf{x}-\mathbf{x}_n(t))}{E_n} \qquad (39)$$

is a Lorentz invariant. Hence, we can conclude that $T^{\alpha}{}_{\alpha}$ is a Lorentz invariant since m_n is also a Lorentz invariant. In other words, $T^{\alpha}{}_{\alpha}$ is a Lorentz scalar.