University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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HW2 Due to October 7 11:59 PM

Question 1

Problem 5.1
Work out the LSZ reduction formula for the complex scalar field that was introduced in problem 3.5. Note that we must specify the type (a or b) of each incoming and outgoing particle.

Answer

We start with the mode expansion of the complex scalar field:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] \tag{1}$$

$$\varphi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}[b(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}] \tag{2}$$

$$a(\mathbf{k}) = \int d^3x e^{-ikx}\left[i\partial_0\varphi(x) + \omega\varphi(x)\right], \tag{3}$$

$$b(\mathbf{k}) = \int d^3x e^{-ikx}[\omega\varphi^\dagger(x) + i\partial_0\varphi^\dagger(x)]. \tag{4}$$

First, we define the $|i\rangle$ and $|f\rangle$ states as

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\cdots b_1^\dagger(t)b_2^\dagger(t)\cdots|0\rangle, \tag{5}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\cdots b_{1'}^\dagger(t)b_{2'}^\dagger(t)\cdots|0\rangle. \tag{6}$$

And a_i and b_i are given by

$$a_i^\dagger = \int d^3k f_i(\mathbf{k})a^\dagger(\mathbf{k}) \tag{7}$$

$$b_i^\dagger = \int d^3k g_i(\mathbf{k})b^\dagger(\mathbf{k}), \tag{8}$$

where

$$f_i(\mathbf{k}), g_i(\mathbf{k}) \propto \exp(-(\mathbf{k}-\mathbf{k}_i)^2/4\sigma^2). \tag{9}$$

Now we can compute the difference between $a_1^\dagger(+\infty)$ and $a_1^\dagger(-\infty)$:

$$
\begin{aligned}
a_1^\dagger(+\infty) - a_1^\dagger(-\infty) =& \int_{-\infty}^{+\infty} dt \partial_0 a_1^\dagger(t) && (10)\\
=& \int_{-\infty}^{+\infty} dt \int d^3k f_1(\mathbf{k}) \int d^3x e^{ikx}[\omega\varphi(x) - i\partial_0\varphi(x)] && (11)\\
=& -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x), && (12)
\end{aligned}
$$

where I quote the equation in the textbook. Similarly, we can get

$$
\begin{aligned}
b_1^\dagger(+\infty) - b_1^\dagger(-\infty) =& -i \int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x), && (13)\\
a_{1'}(+\infty) - a_{1'}(-\infty) =& i \int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x), && (14)\\
b_{1'}(+\infty) - b_{1'}(-\infty) =& i \int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x). && (15)
\end{aligned}
$$

Now we can express the S-matrix element $\langle f|i\rangle$ as

$$
\begin{aligned}
\langle f|i\rangle =& \langle 0|\mathcal{T} b_{1'}(+\infty) b_{2'}(+\infty)\cdots a_{1'}(+\infty) a_{2'}(+\infty)\cdots a_1^\dagger(-\infty) a_2^\dagger(-\infty)\cdots b_1^\dagger(-\infty) b_2^\dagger(-\infty)\cdots|0\rangle && (16)\\
=& \langle 0|\mathcal{T}[b_{1'}(-\infty) + i\int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots\\
& \cdots [a_{1'}(-\infty) + i\int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots\\
& \cdots [a_1^\dagger(+\infty) + i\int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots\\
& \cdots [b_1^\dagger(+\infty) + i\int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots|0\rangle && (17)\\
=& \langle 0|\mathcal{T}[i\int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots\\
& \cdots [i\int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots\\
& \cdots [i\int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots\\
& \cdots [i\int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots|0\rangle && (18)\\
=& (i)^{n+n'+m+m'}\langle 0|\mathcal{T}[\prod_{j'}^{n'}\int d^4x e^{-ik_{j'}x}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)][\prod_{l'}^{n'}\int d^4x e^{-ik_{l'}x}(-\partial_\mu\partial^\mu + m^2)\varphi(x)] && (19)\\
& [\prod_{l}^{m}\int d^4x e^{ik_l x}(-\partial_\mu\partial^\mu + m^2)\varphi(x)][\prod_{j}^{n}\int d^4x e^{ik_j x}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]|0\rangle, && (20)
\end{aligned}
$$

where we have used the fact that $a_i|0\rangle = b_i|0\rangle = 0$ and $\langle 0|a_i^\dagger = \langle 0|b_i^\dagger = 0$. Here n and m are the number of incoming a and b particles, while n' and m' are the number of outgoing a and b particles, respectively. We also impose the $\sigma \to 0$ limit, so that $f_i(\mathbf{k})$ and $g_i(\mathbf{k})$ become delta functions. Finally, we can rewrite the S-matrix element as

$$\begin{aligned}\langle f|i\rangle =&(i)^{n+n'+m+m'} \int d^4x_1 e^{-ik_1x_1} \cdots \int d^4x_n e^{-ik_nx_n} \int d^4x_{1'} e^{ik_{1'}x_{1'}} \cdots \int d^4x_{n'} e^{ik_{n'}x_{n'}} \\ &\int d^4y_1 e^{-ip_1y_1} \cdots \int d^4y_m e^{-ip_my_m} \int d^4y_{1'} e^{ip_{1'}y_{1'}} \cdots \int d^4y_{m'} e^{ip_{m'}y_{m'}} \\ &(-\partial_\mu\partial^\mu_{x_1} + m^2) \cdots (-\partial_\mu\partial^\mu_{x_n} + m^2)(-\partial_\mu\partial^\mu_{x_{1'}} + m^2) \cdots (-\partial_\mu\partial^\mu_{x_{n'}} + m^2) \\ &(-\partial_\mu\partial^\mu_{y_1} + m^2) \cdots (-\partial_\mu\partial^\mu_{y_m} + m^2)(-\partial_\mu\partial^\mu_{y_{1'}} + m^2) \cdots (-\partial_\mu\partial^\mu_{y_{m'}} + m^2) \\ &\langle 0|\mathcal{T}\varphi^\dagger(y_{1'}) \cdots \varphi^\dagger(y_{m'})\varphi(x_{1'}) \cdots \varphi(x_{n'})\varphi(x_1) \cdots \varphi(x_n)\varphi^\dagger(y_1) \cdots \varphi^\dagger(y_m)|0\rangle. \end{aligned} \tag{21}$$

This is the LSZ reduction formula for the complex scalar field. □

Question 2

Problem 6.1

(a) Find an explicit formula for $\mathcal{D}q$ in eq. (6.9). Your formula should be of the form $\mathcal{D}q = C\prod_{j=1}^{N} dq_j$, where C is a constant that you should compute.

(b) For the case of a free particle, $V(Q) = 0$, evaluate the path integral of eq. (7.9) explicitly. Hint: integrate over q_1, then q_2, etc, and look for a pattern. Express you final answer in terms of q', t', q'', t'' and m. Restore $\hbar$ by dimensional analysis.

(c) Compute the $\langle q'', t''|q', t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare your result in part (b).

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}, \tag{6.7}$$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q \exp\left[i\int_{t'}^{t''} dt L(\dot{q}(t), q(t))\right]. \tag{6.9}$$

Answer

First, from eq. (6.7), we can see that

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}, \quad \text{assuming } H(p,q) = \frac{1}{2m}p^2 + V(q) && (22)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i(\frac{1}{2m}p_j^2+V(\bar{q}_j))\delta t} && (23)\\
&= && (24)
\end{aligned}$$

Question 3

Problem 7.3

(a) Use the Heisenberg equations of motion, $\dot{A} = i[H, A]$, to find explicit expressions for $\dot{Q}$ and $\dot{P}$. Solve these to get the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of the Schrödinger-picture operators Q and P.

(b) Write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$, where $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. Then, using your result from part (a), write the Heisenberg-picture operator $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$.

(c) Using your result from part (b), and $a|0\rangle = \langle 0|a^\dagger = 0$,verify eqs. (7.16) and (7.17).

Answer

Question 4

Problem 7.4

Consider a harmonic oscillator in its ground state at $t = -\infty$. It is then subjected to an external force $f(t)$. Compute the probability $|\langle 0|0\rangle_f|^2$ that the oscillator is still in its ground state at $t = +\infty$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\tilde{f}(E) = \int_{-\infty}^{+\infty} e^{iEt} f(t)$. Your answer should not involve any other unevaluated integrals.

Answer