University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Question 1

Color Structure and One-Gluon Exchange

(a) The color force between quarks is mediated by a color-anticolor octet of vector gauge bosons called gluons. Denoting the three color charges by R (red), G (green) and B (blue) write down the color combinations of the gluon octet states in analogy with the meson flavor-antiflavor octet of $SU(3)_f$.

(b) A quark-antiquark meson is a color singlet with wavefunction $Q\overline{Q} = \frac{1}{\sqrt{3}}(R\overline{R} + G\overline{G} + B\overline{B})$. At short distances, the potential between quarks is approximately Coulombic $V(r) = \xi \frac{g^2}{r}$, arising from one-gluon exchange with coupling g between the gluon and the quark pair. Determine the color factor ξ for the color-singlet meson state. Is the potential attractive or repulsive?

(c) For two quarks, the color states combines as $\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$. Write down the color wavefunction for the antisymmetric $\overline{\mathbf{3}}$ and symmetric $\mathbf{6}$ states using the color basis $\{R, G, B\}$. Use one-gluon exchange arguments to determine which color configuration corresponding ot an attractive potential. Explain qualitatively how this leads to stable color-singlet baryons: two quarks attract in the $\overline{\mathbf{3}}$ channel, which then combines with the third quark ($\mathbf{3}$) to form a color singlet (using the result from (b)).
Hint: You can obtain the color diquark wavefunctions by recalling that $\overline{\mathbf{3}_i} \propto \epsilon_{ijk} Q_j Q_k$ and $\mathbf{6} \propto Q_i Q_j + Q_j Q_i$.

Answer

(a)

The eight gluon color states can be written as ($u \to R$, $d \to G$, $s \to B$):

$$g_1 = \frac{1}{\sqrt{2}}(R\overline{G} + G\overline{R}), \quad g_2 = \frac{-i}{\sqrt{2}}(R\overline{G} - G\overline{R}), \quad g_3 = \frac{1}{\sqrt{2}}(R\overline{R} - G\overline{G}), \tag{1}$$

$$g_4 = \frac{1}{\sqrt{2}}(R\overline{B} + B\overline{R}), \quad g_5 = \frac{-i}{\sqrt{2}}(R\overline{B} - B\overline{R}), \quad g_6 = \frac{1}{\sqrt{2}}(G\overline{B} + B\overline{G}), \tag{2}$$

$$g_7 = \frac{-i}{\sqrt{2}}(G\overline{B} - B\overline{G}), \quad g_8 = \frac{1}{\sqrt{6}}(R\overline{R} + G\overline{G} - 2B\overline{B}). \tag{3}$$

(b)

We start from the QCD interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = g\overline{\psi}\gamma^\mu T^a \psi A^a_\mu, \tag{4}$$

where T^a are the generators of the $SU(3)$ color group in the adjoint representation, and A^a_μ are the gluon fields. We can write the transition amplitude for one-gluon exchange between a quark and an antiquark as:

$$\mathcal{M} \propto g^2 \left(\overline{u}_\alpha(p_f)\gamma^\mu T^a_{\alpha\beta} u_\beta(p_i)\right) \frac{-ig_{\mu\nu}}{q^2} \left(\overline{v}_\gamma(k_i)\gamma^\nu \overline{T}^a_{\gamma\delta} v_\delta(k_f)\right), \tag{5}$$

where $\alpha, \beta, \gamma, \delta$ are color indices, q is the four-momentum transfer, u and v are the quark and antiquark spinors, respectively, $\overline{T}$ is the antiquark representation of the color generators, and p_i, p_f, k_i, k_f are the initial and final momenta of the quark and antiquark, respectively. The color factor ξ arises from the contraction of the color indices:

$$\xi = \langle\psi|\mathcal{C}|\psi\rangle = T^a_{\alpha\beta}\overline{T}^a_{\gamma\delta}\langle Q_\alpha\overline{Q}_\gamma|Q_\beta\overline{Q}_\delta\rangle, \tag{6}$$

where $|\psi\rangle = |Q\overline{Q}\rangle$ is the color-singlet meson state, and $\mathcal{C} = T^a \otimes \overline{T}^a$ is the color operator for one-gluon exchange. Applying the relation $\overline{T}^a = -(T^a)^T$, we have:

$$\xi = -T^a_{\alpha\beta}T^a_{\delta\gamma}\langle Q_\alpha\overline{Q}_\gamma|Q_\beta\overline{Q}_\delta\rangle. \tag{7}$$

For the color-singlet meson state, we have:

$$|Q\overline{Q}\rangle = \frac{1}{\sqrt{3}}(R\overline{R} + G\overline{G} + B\overline{B}). \tag{8}$$

Applying $\langle Q_\alpha\overline{Q}_\gamma|Q_\beta\overline{Q}_\delta\rangle = \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta}$, we find:

$$\begin{aligned}
\xi &= -\frac{1}{3}T^a_{\alpha\beta}T^a_{\beta\alpha} && (9)\\
&= -\frac{1}{3}\mathrm{Tr}(T^aT^a) && (10)\\
&= -\frac{1}{3}\cdot\frac{1}{2}\cdot 8 \quad \left(\text{since } \mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab} \text{ and } a = 1,\ldots,8\right) && (11)\\
&= -\frac{4}{3}. && (12)
\end{aligned}$$

Thus, the color factor for the color-singlet meson state is $\xi = -\frac{4}{3}$, indicating that the potential is attractive.

(c)

Now we can write down:

$$\xi = \langle\psi_{\alpha\gamma}|\mathcal{C}_{\alpha\gamma,\beta\delta}|\psi_{\beta\delta}\rangle = T^a_{\alpha\beta}T^a_{\delta\gamma}\langle\psi_{\alpha\gamma}|\psi_{\beta\delta}\rangle, \tag{13}$$

where $\alpha, \beta, \gamma, \delta$ are color indices. We can apply Fierz identity for the generators of $SU(3)$:

$$T^a_{\alpha\beta} T^a_{\delta\gamma} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{3}\delta_{\alpha\beta}\delta_{\delta\gamma}\right). \tag{14}$$

Now we can have:

$$\xi = \frac{1}{2}\left(\langle\psi_{\alpha\gamma}|\psi_{\gamma\alpha}\rangle - \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle\right). \tag{15}$$

First, if $|\psi\rangle$ is in the antisymmetric $\overline{\mathbf{3}}$ representation, we have:

$$|\psi_{\alpha\gamma}\rangle = -|\psi_{\gamma\alpha}\rangle, \tag{16}$$

which leads to:

$$\xi_{\overline{3}} = \frac{1}{2}\left(-\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle - \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle\right) \tag{17}$$

$$= -\frac{2}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle = -\frac{2}{3}. \tag{18}$$

Next, if $|\psi\rangle$ is in the symmetric $\mathbf{6}$ representation, we have:

$$|\psi_{\alpha\gamma}\rangle = |\psi_{\gamma\alpha}\rangle, \tag{19}$$

which leads to:

$$\xi_{6} = \frac{1}{2}\left(\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle - \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle\right) \tag{20}$$

$$= \frac{1}{3}\langle\psi_{\alpha\gamma}|\psi_{\alpha\gamma}\rangle = \frac{1}{3}. \tag{21}$$

Thus, the color factor for the antisymmetric $\overline{\mathbf{3}}$ state is $\xi_{\overline{3}} = -\frac{2}{3}$, indicating an attractive potential, while the color factor for the symmetric $\mathbf{6}$ state is $\xi_6 = \frac{1}{3}$, indicating a repulsive potential. This attraction in the $\overline{\mathbf{3}}$ channel allows two quarks to form a stable diquark state, which can then combine with a third quark in the $\mathbf{3}$ representation to form a color-singlet baryon, as the combination $\overline{\mathbf{3}} \otimes \mathbf{3}$ contains a singlet representation. $\square$

Question 2

Inverse Fourier Transform of Form Factors

Hadronic form factors $F(Q^2)$ are measured in elastic scattering, where the exchanged momentum is off-shell. In this regime no real particle is produced by the probe, and the form factor encodes information about the spatial distribution of charge and current within the target. It is conventional to define $Q^2 = -q^2$.

(a) In the nonrelativistic (static) limit ($v \ll c$), explain why the energy transfer q^0 can be neglected relative to the spatial momentum $|\mathbf{q}|$. Show that this implies $Q^2 = |\mathbf{q}|^2 > 0$, corresponding to spacelike momentum exchange.

(b) In this limit the **spatial charge density** $\rho(r)$ is obtain as the inverse three-dimensionful Fourier transform of the form factor:

$$\rho(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} F(Q^2). \tag{22}$$

Evaluate $\rho(r)$ for the following model form factors:

$$F(Q^2) = \frac{1}{1 + Q^2/\Lambda^2} \quad \text{and} \quad F(Q^2) = e^{-Q^2/\Lambda^2}, \tag{23}$$

and show that they yield, respectively a Yukawa form $\propto e^{-\Lambda r}/r$ and a Gaussian form $\propto e^{-\Lambda^2 r^2/4}$.

(c) Compare the spatial falloff of these two distributions. What does each imply about the effective range and shape of the hadronic charge density?

Answer

(a)

We can define $q^\mu = (q^0, \mathbf{q}) = p_f^\mu - p_i^\mu$, where p_i^μ and p_f^μ are the initial and final four-momenta of the target hadron, respectively. In the nonrelativistic limit ($v \ll c$), the kinetic energy of the hadron is much smaller than its rest mass energy, so we can approximate:

$$q^0 = E_f - E_i \approx \frac{\mathbf{p}_f^2}{2m} - \frac{\mathbf{p}_i^2}{2m} \ll |\mathbf{q}| = |\mathbf{p}_f - \mathbf{p}_i|. \tag{24}$$

Thus, we can neglect q^0 relative to $|\mathbf{q}|$, leading to:

$$Q^2 = -q^2 = -(q^0)^2 + |\mathbf{q}|^2 \approx |\mathbf{q}|^2 > 0, \tag{25}$$

which corresponds to spacelike momentum exchange.

(b)

To evaluate the spatial charge density $\rho(r)$ for the given form factors, we start with the integral:

$$\rho(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} F(Q^2). \tag{26}$$

For the first form factor $F(Q^2) = \frac{1}{1+Q^2/\Lambda^2}$, we have:

$$\begin{aligned}
\rho(r) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{1+|\mathbf{q}|^2/\Lambda^2} && (27)\\
&= \int \frac{d\phi d\cos\theta dq}{(2\pi)^3} q^2 e^{iqr\cos\theta} \frac{1}{1+q^2/\Lambda^2} && (28)\\
&= \frac{2\pi}{(2\pi)^3} \int_0^\infty dq \frac{q^2}{1+q^2/\Lambda^2} \int_{-1}^1 d\cos\theta e^{iqr\cos\theta}, \quad \text{after integrating over } \phi && (29)\\
&= \frac{1}{(2\pi)^2} \int_0^\infty dq \frac{q^2}{1+q^2/\Lambda^2} \left(\frac{2\sin(qr)}{qr}\right), \quad \text{after integrating over } \cos\theta && (30)\\
&= \frac{1}{2\pi^2 r} \int_0^\infty dq \frac{q\sin(qr)}{1+q^2/\Lambda^2} && (31)\\
&= \frac{1}{2\pi^2 r} \frac{\pi\Lambda^2 r e^{-\frac{\sqrt{r^2}}{\sqrt{\frac{1}{\Lambda^2}}}}}{2\sqrt{r^2}}, \quad \text{by Mathematica} && (32)\\
&= \frac{\Lambda^2}{4\pi} \frac{e^{-\Lambda r}}{r}. && (33)
\end{aligned}$$

For the second form factor $F(Q^2) = e^{-Q^2/\Lambda^2}$, we have:

$$\begin{aligned}
\rho(r) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} e^{-|\mathbf{q}|^2/\Lambda^2} && (34)\\
&= \int \frac{d\phi d\cos\theta dq}{(2\pi)^3} q^2 e^{iqr\cos\theta} e^{-q^2/\Lambda^2} && (35)\\
&= \frac{2\pi}{(2\pi)^3} \int_0^\infty dq q^2 e^{-q^2/\Lambda^2} \int_{-1}^1 d\cos\theta e^{iqr\cos\theta}, \quad \text{after integrating over } \phi && (36)\\
&= \frac{1}{(2\pi)^2} \int_0^\infty dq q^2 e^{-q^2/\Lambda^2} \left(\frac{2\sin(qr)}{qr}\right), \quad \text{after integrating over } \cos\theta && (37)\\
&= \frac{1}{2\pi^2 r} \int_0^\infty dq q e^{-q^2/\Lambda^2} \sin(qr) && (38)\\
&= \frac{1}{2\pi^2 r} \frac{\sqrt{\pi} r e^{-\frac{1}{4}\Lambda^2 r^2}}{4\left(\frac{1}{\Lambda^2}\right)^{3/2}}, \quad \text{by Mathematica} && (39)\\
&= \frac{\Lambda^3}{8\pi^{3/2}} e^{-\frac{\Lambda^2 r^2}{4}}. && (40)
\end{aligned}$$

(c)

The spatial falloff of the two distributions can be compared as follows:

- The Yukawa form $\rho(r) \propto \frac{e^{-\Lambda r}}{r}$ indicates a long-range interaction that decays exponentially with distance r. The presence of the $1/r$ factor suggests that the charge density has a significant contribution even at larger distances, although it decreases rapidly due to the exponential term. This form is characteristic of interactions mediated by massive particles, where Λ can be interpreted as the mass scale of the exchanged particle.

- The Gaussian form $\rho(r) \propto e^{-\frac{\Lambda^2 r^2}{4}}$ indicates a short-range interaction that decays very rapidly with distance r. The Gaussian decay implies that the charge density is highly localized around the origin, with negligible contributions at larger distances. This form is characteristic of interactions where the charge distribution is tightly confined, leading to a rapid falloff.

In summary, the Yukawa form suggests a more extended charge distribution with a longer effective range, while the Gaussian form indicates a highly localized charge distribution with a very short effective range. $\square$

Question 3

Deep Inelastic Structure Functions and the Gottfried Sum Rule

In deep inelastic electron–nucleon scattering, the nucleon structure functions $F_2^p(x)$ and $F_2^n(x)$ describe the momentum distributions of quarks carrying a fraction x of the nucleon's momentum.

(a) Using the quark–parton model and the fact that quark distributions are positive definite, verify that the structure functions satisfy

$$\frac{1}{4} \leq \frac{F_2^n(x)}{F_2^p(x)} \leq 4. \tag{41}$$

(b) In the limit $x \to 0$, the sea quarks dominate and may be taken as $SU(2)$-flavor symmetric. What limit do you expect for the ratio $F_2^n(x)/F_2^p(x)$ in this case?

(c) The *Gottfried sum rule* is defined through the integral

$$I_G(x) = \int_x^1 \frac{F_2^p(x') - F_2^n(x')}{x'} dx'. \tag{42}$$

Assuming an $SU(2)$ flavor-symmetric sea, what value do you predict for $I_G(0)$? Compare your result with the experimental measurement $I_G(0) = 0.235 \pm 0.026$ at $Q^2 = 4$ GeV^2, first reported by the NMC collaboration at CERN in 1991. Is this surprising?

Answer

(a)

In the quark-parton model, the structure functions for the proton and neutron can be expressed in terms of the quark distribution functions as follows:

$$F_2^p(x) = x\left[\frac{4}{9}(u(x) + \overline{u}(x)) + \frac{1}{9}(d(x) + \overline{d}(x)) + \frac{1}{9}(s(x) + \overline{s}(x))\right], \tag{43}$$

$$F_2^n(x) = x\left[\frac{4}{9}(d(x) + \overline{d}(x)) + \frac{1}{9}(u(x) + \overline{u}(x)) + \frac{1}{9}(s(x) + \overline{s}(x))\right]. \tag{44}$$

To find the ratio $\frac{F_2^n(x)}{F_2^p(x)}$, we can write:

$$\frac{F_2^n(x)}{F_2^p(x)} = \frac{\frac{4}{9}(d(x) + \overline{d}(x)) + \frac{1}{9}(u(x) + \overline{u}(x)) + \frac{1}{9}(s(x) + \overline{s}(x))}{\frac{4}{9}(u(x) + \overline{u}(x)) + \frac{1}{9}(d(x) + \overline{d}(x)) + \frac{1}{9}(s(x) + \overline{s}(x))}. \tag{45}$$

Since the quark distribution functions are positive definite, we can analyze the extremes of this ratio.

The minimum value occurs when $d(x)+\overline{d}(x)$ is minimized and $u(x)+\overline{u}(x)$ is maximized, leading to:

$$\frac{F_2^n(x)}{F_2^p(x)} \geq \frac{\frac{1}{9}(u(x)+\overline{u}(x))}{\frac{4}{9}(u(x)+\overline{u}(x))} = \frac{1}{4}. \tag{46}$$

The maximum value occurs when $d(x)+\overline{d}(x)$ is maximized and $u(x)+\overline{u}(x)$ is minimized, leading to:

$$\frac{F_2^n(x)}{F_2^p(x)} \leq \frac{\frac{4}{9}(d(x)+\overline{d}(x))}{\frac{1}{9}(d(x)+\overline{d}(x))} = 4. \tag{47}$$

Thus, we have verified that:

$$\frac{1}{4} \leq \frac{F_2^n(x)}{F_2^p(x)} \leq 4. \tag{48}$$

(b)

In the limit $x \to 0$, the sea quarks dominate the structure functions, and we can assume $SU(2)$-flavor symmetry, which implies:

$$\overline{u}(x) \approx \overline{d}(x). \tag{49}$$

Under this assumption, the structure functions simplify to:

$$F_2^p(x) \approx x\left[\frac{4}{9}\overline{u}(x)+\frac{1}{9}\overline{d}(x)+\frac{1}{9}(s(x)+\overline{s}(x))\right] = x\left[\frac{5}{9}\overline{u}(x)+\frac{1}{9}(s(x)+\overline{s}(x))\right], \tag{50}$$

$$F_2^n(x) \approx x\left[\frac{4}{9}\overline{d}(x)+\frac{1}{9}\overline{u}(x)+\frac{1}{9}(s(x)+\overline{s}(x))\right] = x\left[\frac{5}{9}\overline{u}(x)+\frac{1}{9}(s(x)+\overline{s}(x))\right]. \tag{51}$$

Thus, in this limit, we find:

$$\frac{F_2^n(x)}{F_2^p(x)} \approx 1. \tag{52}$$

(c)

The Gottfried sum rule is given by:

$$I_G(0) = \int_0^1 \frac{F_2^p(x')-F_2^n(x')}{x'}dx'. \tag{53}$$

We can express all quark distributions in terms of valence and sea components:

$$u(x) = u_v(x)+u_s(x), \quad d(x) = d_v(x)+d_s(x), \tag{54}$$

$$\overline{u}(x) = \overline{u}_s(x), \quad \overline{d}(x) = \overline{d}_s(x), \quad s(x) = s_s(x), \quad \overline{s}(x) = \overline{s}_s(x). \tag{55}$$

Assuming an $SU(2)$ flavor-symmetric sea, giving $\overline{u}_s(x) = \overline{d}_s(x) = u_s(x) = d_s(x)$, we can simplify the difference between the proton and neutron structure functions:

$$F_2^p(x) - F_2^n(x) = x\left[\frac{4}{9}(u_v(x) - d_v(x)) + \frac{1}{9}(d_v(x) - u_v(x))\right] \tag{56}$$

$$= x\left[\frac{1}{3}(u_v(x) - d_v(x))\right] \tag{57}$$

Substituting this into the Gottfried sum rule, we get:

$$I_G(0) = \int_0^1 \frac{x\left[\frac{1}{3}(u_v(x) - d_v(x))\right]}{x} dx' \tag{58}$$

$$= \frac{1}{3}\int_0^1 (u_v(x) - d_v(x))dx'. \tag{59}$$

The integrals of the valence quark distributions over the range $[0, 1]$ give the total number of valence quarks in the proton:

$$\int_0^1 u_v(x)dx' = 2, \quad \int_0^1 d_v(x)dx' = 1. \tag{60}$$

Thus, we find:

$$I_G(0) = \frac{1}{3}(2 - 1) = \frac{1}{3}. \tag{61}$$

Comparing this result with the experimental measurement $I_G(0) = 0.235 \pm 0.026$, we see that the experimental value is significantly lower than the predicted value of $\frac{1}{3} \approx 0.333$. This discrepancy suggests that the assumption of an $SU(2)$ flavor-symmetric sea may not hold true in reality. The lower experimental value indicates an asymmetry in the sea quark distributions, specifically that there are more $\overline{d}$ quarks than $\overline{u}$ quarks in the proton. $\square$