University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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HW4 Due to November 4 11:59 PM

Question 1

Problem 14.1

Derive a generalization of Feynman's formula,

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_i x_i^{\alpha_i - 1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}. \tag{1}$$

$$\int dF_n = (n-1)! \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \delta\left(\sum_{i=1}^n x_i - 1\right). \tag{2}$$

Hint: start with

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty dt\, t^{\alpha-1} e^{-tA}, \tag{3}$$

which defines the gamma function. Put an index on A, α and t, and take the product. Then multiply on the right-hand side by

$$1 = \int_0^\infty ds \delta(s - \sum_i t_i). \tag{4}$$

Make the change of variables $t_i = sx_i$ and carry out the integral over s.

Answer

By definesion of the gamma function, we have

$$\frac{1}{A_i^{\alpha_i}} = \frac{1}{\Gamma(\alpha_i)} \int_0^\infty dt_i\, t_i^{\alpha_i - 1} e^{-t_i A_i}. \tag{5}$$

Then we have

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} \int_0^\infty dt_i\, t_i^{\alpha_i - 1} e^{-t_i A_i} \tag{6}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_n \prod_{i=1}^n t_i^{\alpha_i - 1} e^{-t_i A_i} \tag{7}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_n \prod_{i=1}^n \left(t_i^{\alpha_i - 1} e^{-t_i A_i}\right) \int_0^\infty ds \delta(s - \sum_{i=1}^n t_i). \tag{8}$$

We make the change of variables $t_i = sx_i$, then we have

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n \left((sx_i)^{\alpha_i - 1} e^{-sx_i A_i}\right) \delta(s - s\sum_{i=1}^n x_i) s \tag{9}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds\, s^{\sum_{i=1}^n \alpha_i - 1} e^{-s\sum_{i=1}^n x_i A_i} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \tag{10}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \int_0^\infty ds\, s^{\sum_{i=1}^n \alpha_i - 1} e^{-s\sum_{i=1}^n x_i A_i} \tag{11}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{(\sum_{i=1}^n x_i A_i)^{\sum_{i=1}^n \alpha_i}} \tag{12}$$

$$= \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_{i=1}^n x_i^{\alpha_i - 1}}{(\sum_{i=1}^n x_i A_i)^{\sum_{i=1}^n \alpha_i}}. \tag{13}$$

Hence proved the formula. $\square$

Question 2

Problem 14.2
Verify eq. (14.23).

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{14}$$

Answer

We start with the Gaussian integral in d dimensions,

$$I_d = \int d^d x \, e^{-\mathbf{x}^2}. \tag{15}$$

In catrtesian coordinates, we have

$$I_d = \left(\int_{-\infty}^{\infty} dx \, e^{-x^2} \right)^d = (\sqrt{\pi})^d = \pi^{d/2}. \tag{16}$$

In spherical coordinates, we have

$$I_d = \int_0^\infty dr \, r^{d-1} e^{-r^2} \int d\Omega_d = \Omega_d \int_0^\infty dr \, r^{d-1} e^{-r^2}. \tag{17}$$

Make the change of variable $t = r^2$, then we have

$$I_d = \frac{\Omega_d}{2} \int_0^\infty dt \, t^{d/2-1} e^{-t} = \frac{\Omega_d}{2} \Gamma(d/2), \tag{18}$$

where we have used the definition of the gamma function

$$\Gamma(\alpha) = \int_0^\infty dt \, t^{\alpha-1} e^{-t}. \tag{19}$$

Equating the two expressions for I_d, we have

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{20}$$

□

Question 3

Problem 14.5

Compute the $O(\lambda)$ correction ot the propagator in φ^4 theory (see problem 9.2) in $d = 4 - \epsilon$ spacetime dimensions, and compute the $O(\lambda)$ terms in A and B.

Answer

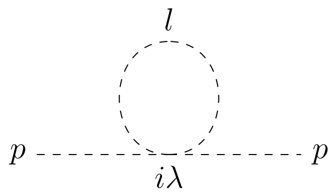

Figure 1: The Feynman diagram with the ϕ^4 propagator for 1-loop correction at $O(\lambda)$.

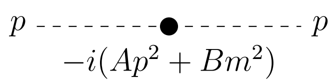

Figure 2: The Feynman diagram with the ϕ^4 propagator for 1-loop counter term at $O(\lambda)$.

First, we write down the Lagrangian for the φ^4 theory,

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{,} + \mathcal{L}_I, && (21)\\
\mathcal{L}_0 &= -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2, && (22)\\
\mathcal{L}_I &= -\frac{Z_\lambda}{4!}\lambda\varphi^4 + \mathcal{L}_{ct}, && (23)\\
\mathcal{L}_{ct} &= -\frac{1}{2}(Z_\varphi - 1)\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2. && (24)
\end{aligned}$$

For the $O(\lambda)$ correction to the propagator, the Feynman diagram is shown in Figure 1. The corresponding amplitude is given by

$$i\Sigma(p) = \frac{1}{2}(i\lambda)\frac{1}{i}\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2+m^2-i\epsilon} - i(Ap^2+Bm^2) + O(\lambda^2), \tag{25}$$

where the factor $\frac{1}{2}$ is the symmetry factor for this diagram. Consider the wick rotation to Euclidean space ($d^4l \to id^4l_E$ and $l^2 \to l_E^2$), we have

$$\Sigma(p) = \frac{\lambda}{2}\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{l_E^2+m^2} - (Ap^2+Bm^2) + O(\lambda^2), \tag{26}$$

where the $m = m - i\epsilon$ prescription is implied. Using the formula derived in Problem 14.1, we have

$$\begin{aligned}
\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 + m^2} &= \int \frac{d^d l_E}{(2\pi)^d} \int_0^\infty dx\, e^{-x(l_E^2+m^2)} && (27)\\
&= \int_0^\infty dx\, e^{-xm^2} \int \frac{d^d l_E}{(2\pi)^d} e^{-x l_E^2} && (28)\\
&= \int_0^\infty dx\, e^{-xm^2} \frac{1}{(2\pi)^d} \left(\frac{\pi}{x}\right)^{d/2} && (29)\\
&= \frac{1}{(4\pi)^{d/2}} \int_0^\infty dx\, x^{-d/2} e^{-xm^2} && (30)\\
&= \frac{1}{(4\pi)^{d/2}} (m^2)^{d/2-1} \Gamma(1-d/2). && (31)
\end{aligned}$$

Substituting $d = 4 - \epsilon$, we have

$$\begin{aligned}
\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 + m^2} &= \frac{1}{(4\pi)^{2-\epsilon/2}} (m^2)^{1-\epsilon/2} \Gamma(-1+\epsilon/2) && (32)\\
&= \frac{m^2}{16\pi^2} \left(\frac{4\pi}{m^2}\right)^{\epsilon/2} \Gamma(-1+\epsilon/2). && (33)
\end{aligned}$$

Also, when we consider dimension regularization, we need to introduce a mass scale μ to keep the coupling constant λ dimensionless. Thus, we have

$$\lambda \to \lambda \widetilde{\mu}^{\epsilon}. \tag{34}$$

Question 4

Problem 16.1
Compute the $O(\lambda^2)$ correction in $\mathbf{V}_4$ in φ^4 theory in $d = 4 - \epsilon$ spacetime dimensions. Take $\mathbf{V}_4 = \lambda$ when all four external momenta are on shell, and $s = 4m^2$. What is the $O(\lambda)$ contribution to C?

Answer