University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

December 1, 2025

Assignment 11 due on Monday December 1 at 10PM

Question 1

Verify the Reissner-Nordstrom solution as presented in class and in the textbook. That is, show in detail that both Maxwell's equations

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{\mu\nu}\right) = 0, \tag{1}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0, \tag{2}$$

and Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{3}$$

are satisfied with the metric:

$$d\tau^2 = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 - \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}dr^2 - r^2 d\Omega^2, \tag{4}$$

and with a radial electric field $E_r = \frac{Q}{r^2}$. Note that this is in Gaussian units. Weinberg uses Heaviside-Lorentz units.

Answer

First, we write down the field strength tensor $F_{\mu\nu}$ for a radial electric field $E_r = \frac{Q}{r^2}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{Q}{r^2} & 0 & 0 \\ \frac{Q}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

with $g^{tt} = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}$ and $g^{rr} = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)$. Raising the indices, we have

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{Q}{r^2} & 0 & 0 \\ -\frac{Q}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Next, we calculate $\sqrt{g}$:

$$\sqrt{g} = r^2 \sin\theta. \tag{7}$$

Now, we can verify Maxwell's equations. For $\mu = 0$,

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{0\nu}\right) = \frac{\partial}{\partial r}\left(r^2 \sin\theta \cdot \frac{Q}{r^2}\right) = 0. \tag{8}$$

For $\mu = 1$,

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{1\nu}\right) = \frac{\partial}{\partial t}\left(r^2 \sin\theta \cdot -\frac{Q}{r^2}\right) = 0. \tag{9}$$

For $\mu = 2$ and $\mu = 3$, the equations are trivially satisfied since $F^{2\nu} = F^{3\nu} = 0$. Thus, $\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{\mu\nu}\right) = 0$ holds for all μ. Now we check the second Maxwell equation:

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0. \tag{10}$$

Since $F_{\mu\nu}$ only has non-zero components for $\mu = 0$ and $\nu = 1$ (or vice versa), we only need to check the case when $(\mu, \nu, \lambda) = (0, 1, r)$:

$$F_{01,r} + F_{1r,0} + F_{r0,1} = \frac{\partial}{\partial r}\left(-\frac{Q}{r^2}\right) + 0 + \frac{\partial}{\partial r}\left(\frac{Q}{r^2}\right) = 0. \tag{11}$$

All other combinations yield zero trivially. Thus, the second Maxwell equation is also satisfied. For Einstein's equations, please check the attached file for the detailed calculations of the Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and energy-momentum tensor $T_{\mu\nu}$. After computing these quantities, we find that Einstein's equations are satisfied with the given metric and electric field configuration. $\square$

Question 2

Read pages 259-261 of the textbook by Carroll and then solve exercise 1 on page 272. You can use any information provided in class. As Carroll writes in his book, this is an amazing exact solution.
Carroll: Show that the coupled Einstein-Maxwell equations can be simultaneously solved by the metric (6.62) and the electrostatic potential (6.67) if H (i) obeys Laplace's equation,

$$\nabla^2 H = 0. \tag{12}$$

$$ds = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2), \tag{13}$$

where where $H = H(\vec{x})$. The electrostatic potential is given by

$$A_\mu = \left(\frac{1}{\sqrt{G}}\frac{1}{H} - 1, 0, 0, 0\right). \tag{14}$$

Answer

To verify that the coupled Einstein-Maxwell equations are satisfied by the given metric and electrostatic potential, we start by writing down the metric:

$$ds^2 = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2), \tag{15}$$

where $H = H(\vec{x})$. The electrostatic potential is given by

$$A_\mu = \left(\frac{1}{\sqrt{G}}\frac{1}{H} - 1, 0, 0, 0\right). \tag{16}$$

See the attached file for the full solution.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\pi G T_{\mu\nu} = \begin{pmatrix} \frac{2\left(H^{(0,0,2)}(x,y,z)+H^{(0,2,0)}(x,y,z)+H^{(2,0,0)}(x,y,z)\right)}{H(x,y,z)^5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Thus, if H satisfies Laplace's equation $\nabla^2 H = 0$, then the coupled Einstein-Maxwell equations are satisfied. I also verified Maxwell's equations in the attached file. $\square$.