University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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Question 1

Problem 36.3

(a) Prove the *Fierz identities*

$$(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{36.58}$$

$$(\chi_1^\dagger\overline{\sigma^\mu}\chi_2)(\chi_3^\dagger\overline{\sigma_\mu}\chi_4) = (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) \tag{36.59}$$

(b) Define the Dirac fields

$$\Psi_1 \equiv \begin{pmatrix}\chi_i\\ \xi_i^\dagger\end{pmatrix}, \quad \Psi_i^C \equiv \begin{pmatrix}\xi_i\\ \chi_i^\dagger\end{pmatrix} \tag{36.60}$$

Use eqs. (36.58) and (36.59) to prove the Dirac form of the Fierz identities,

$$(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) = -2(\overline{\Psi}_1 P_R\Psi_3^C)(\overline{\Psi}_4^C P_L\Psi_2) \tag{36.61}$$

$$(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) = (\overline{\Psi}_1\gamma^\mu P_L\Psi_4)(\overline{\Psi}_3\gamma_\mu P_L\Psi_2) \tag{36.62}$$

(c) By writing both sides out in terms of Weyl fields, show that

$$\overline{\Psi}_1\gamma^\mu P_R\Psi_2 = -\overline{\Psi}_2^C\gamma^\mu P_L\Psi_1^C \tag{36.63}$$

$$\overline{\Psi}_1 P_L\Psi_2 = \overline{\Psi}_2^C P_L\Psi_1^C \tag{36.64}$$

$$\overline{\Psi}_1 P_R\Psi_2 = \overline{\Psi}_2^C P_R\Psi_1^C. \tag{36.65}$$

Combining equations (36.63–36.65) with equations (36.61–36.62) yields more useful forms of the Fierz identities.

Answer

(a)

We start from the left-hand side of equation (36.58):

$$\begin{aligned}(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) &= (\chi_1^\dagger)_{\dot a}(\overline{\sigma}^\mu)^{\dot a b}(\chi_2)_b(\chi_3^\dagger)_{\dot c}(\overline{\sigma}_\mu)^{\dot c d}(\chi_4)_d & (1)\\ &= (\chi_1^\dagger)_{\dot a}(\chi_2)_b(\chi_3^\dagger)_{\dot c}(\chi_4)_d(\overline{\sigma}^\mu)^{\dot a b}(\overline{\sigma}_\mu)^{\dot c d} & (2)\end{aligned}$$

Using the identity in equations (35.4),(35.19)

$$(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} = -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \tag{35.4}$$
$$(\overline{\sigma}^\mu)^{\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}(\sigma_\mu)_{b\dot{b}} \tag{35.19}$$

we have

$$\begin{aligned}
(\overline{\sigma}^\mu)^{\dot{a}b}(\overline{\sigma}_\mu)^{\dot{c}d} &= \epsilon^{be}\epsilon^{\dot{a}\dot{f}}(\sigma^\mu)_{e\dot{f}}\epsilon^{dg}\epsilon^{\dot{c}\dot{h}}(\sigma_\mu)_{g\dot{h}} &(3)\\
&= \epsilon^{be}\epsilon^{dg}\epsilon^{\dot{a}\dot{f}}\epsilon^{\dot{c}\dot{h}}(\sigma^\mu)_{e\dot{f}}(\sigma_\mu)_{g\dot{h}} &(4)\\
&= -2\epsilon^{be}\epsilon^{dg}\epsilon^{\dot{a}\dot{f}}\epsilon^{\dot{c}\dot{h}}\epsilon_{eg}\epsilon_{\dot{f}\dot{h}} &(5)\\
&= -2\epsilon^{be}\delta^d_e\epsilon^{\dot{a}\dot{f}}\delta^{\dot{c}}_{\dot{f}} &(6)\\
&= -2\epsilon^{bd}\epsilon^{\dot{a}\dot{c}} &(7)
\end{aligned}$$

Substituting this back, we get

$$\begin{aligned}
(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) &= -2(\chi_1^\dagger)_{\dot{a}}(\chi_2)_b(\chi_3^\dagger)_{\dot{c}}(\chi_4)_d\epsilon^{bd}\epsilon^{\dot{a}\dot{c}} &(8)\\
&= -2(\chi_1^\dagger)_{\dot{a}}(\chi_3^\dagger)_{\dot{c}}\epsilon^{\dot{a}\dot{c}}(\chi_2)_b(\chi_4)_d\epsilon^{bd} &(9)\\
&= -2(\chi_1^\dagger)^{\dot{c}}(\chi_3^\dagger)_{\dot{c}}(\chi_2)_d(\chi_4)^d &(10)\\
&= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4)\times(-1)(-1) &(11)\\
&= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) &(12)
\end{aligned}$$

This proves equation (36.58). Similarly, we can prove equation (36.59):

$$\begin{aligned}
(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) &= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) &(13)\\
&= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_4\chi_2) &(14)\\
&= (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) &(15)
\end{aligned}$$

This proves equation (36.59).

(b)

First $\Psi = \begin{pmatrix}\chi_a\\(\xi^\dagger)^{\dot{a}}\end{pmatrix}$, so $\overline{\Psi} = (\xi^a, (\chi^\dagger)_{\dot{a}})$. Also, $P_L\Psi = \begin{pmatrix}\chi_a\\0\end{pmatrix}$ and $P_R\Psi = \begin{pmatrix}0\\(\xi^\dagger)^{\dot{a}}\end{pmatrix}$. Thus,

$$\begin{aligned}
\overline{\Psi}_1\gamma^\mu P_L\Psi_2 &= (\xi_1^a, (\chi_1^\dagger)_{\dot{a}})\begin{pmatrix}0 & \sigma^\mu_{a\dot{b}}\\ \overline{\sigma}^{\mu\dot{a}b} & 0\end{pmatrix}\begin{pmatrix}(\chi_2)_b\\0\end{pmatrix} &(16)\\
&= \xi_1^a\sigma^\mu_{a\dot{b}}(\chi_2)_b &(17)\\
&= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2) &(18)
\end{aligned}$$

Similarly,

$$\overline{\Psi}_3\gamma_\mu P_L\Psi_4 = (\chi_3^\dagger\overline{\sigma}_\mu\chi_4) \tag{19}$$

Therefore,

$$\begin{aligned}(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) && (20)\\ &= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) && \text{(from (a))}\end{aligned}$$

Now, for the right-hand side of equation (36.61):

$$\begin{aligned}\overline{\Psi}_1 P_R\Psi_3^C &= (\xi_1^a, (\chi_1^\dagger)_{\dot a})\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}(\xi_3)_b\\ (\chi_3^\dagger)^{\dot b}\end{pmatrix} && (21)\\ &= (\chi_1^\dagger)_{\dot a}(\chi_3^\dagger)^{\dot a} && (22)\\ &= (\chi_1^\dagger\chi_3^\dagger) && (23)\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\Psi}_4^C P_L\Psi_2 &= ((\xi_4^a, (\chi_4^\dagger)_{\dot a})\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}(\chi_2)_b\\ (\xi_2^\dagger)^{\dot b}\end{pmatrix} && (24)\\ &= (\xi_4^a)(\chi_2)_a && (25)\\ &= (\chi_4\chi_2) = (\chi_2\chi_4) && (26)\end{aligned}$$

Thus,

$$-2(\overline{\Psi}_1 P_R\Psi_3^C)(\overline{\Psi}_4^C P_L\Psi_2) = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{27}$$

This proves equation (36.61). Similarly, we can prove equation (36.62):

$$\begin{aligned}(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) && (28)\\ &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) && \text{(from (a))}\\ &= (\overline{\Psi}_1\gamma^\mu P_L\Psi_4)(\overline{\Psi}_3\gamma_\mu P_L\Psi_2) && (29)\end{aligned}$$

This proves equation (36.62).

(c)

First, we compute the left-hand side of equation (36.63):

$$\begin{aligned}\overline{\Psi}_1\gamma^\mu P_R\Psi_2 &= (\xi_1^a, (\chi_1^\dagger)_{\dot a})\begin{pmatrix}0 & \sigma^\mu_{a\dot b}\\ \overline{\sigma}^{\mu\dot a b} & 0\end{pmatrix}\begin{pmatrix}0\\ (\xi_2^\dagger)^{\dot b}\end{pmatrix} && (30)\\ &= \xi_1^a\sigma^\mu_{a\dot b}(\xi_2^\dagger)^{\dot b} && (31)\end{aligned}$$

Next, we compute the right-hand side of equation (36.63):

$$\overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \overline{\sigma}^{\mu\dot{a}b} & 0 \end{pmatrix} \begin{pmatrix} (\xi_1)_b \\ 0 \end{pmatrix} \tag{32}$$

$$= (\xi_2^\dagger)_{\dot{a}} \overline{\sigma}^{\mu\dot{a}b} (\xi_1)_b \tag{33}$$

Using the identity

$$(\overline{\sigma}^\mu)^{\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}(\sigma_\mu)_{b\dot{b}} \tag{35.19}$$

we have

$$(\xi_2^\dagger)_{\dot{a}} \overline{\sigma}^{\mu\dot{a}b} (\xi_1)_b = (\xi_2^\dagger)_{\dot{a}} \epsilon^{bc}\epsilon^{\dot{a}\dot{b}} (\sigma_\mu)_{c\dot{b}} (\xi_1)_b \tag{34}$$

$$= -\xi_1^c \sigma^\mu_{c\dot{b}} (\xi_2^\dagger)^{\dot{b}} \tag{35}$$

$$= -\xi_1^a \sigma^\mu_{a\dot{b}} (\xi_2^\dagger)^{\dot{b}} \tag{36}$$

$$= \overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C = -\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 \tag{37}$$

This proves equation (36.63). Similarly, we can prove equations (36.64) and (36.65):

$$LHS = \overline{\Psi}_1 P_L \Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\chi_2)_a \\ (\xi_2^\dagger)^{\dot{a}} \end{pmatrix} \tag{38}$$

$$= \xi_1^a (\chi_2)_a = (\xi_1 \chi_2) \tag{39}$$

$$RHS = \overline{\Psi}_2^C P_L \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\xi_1)_a \\ (\chi_1^\dagger)^{\dot{a}} \end{pmatrix} \tag{40}$$

$$= (\chi_2)_a \xi_1^a = (\chi_2 \xi_1) = (\xi_1 \chi_2) \tag{41}$$

This proves equation (36.64).

$$LHS = \overline{\Psi}_1 P_R \Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\chi_2)_a \\ (\xi_2^\dagger)^{\dot{a}} \end{pmatrix} \tag{42}$$

$$= (\chi_1^\dagger)_{\dot{a}} (\xi_2^\dagger)^{\dot{a}} = (\chi_1^\dagger \xi_2^\dagger) \tag{43}$$

$$RHS = \overline{\Psi}_2^C P_R \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\xi_1)_a \\ (\chi_1^\dagger)^{\dot{a}} \end{pmatrix} \tag{44}$$

$$= (\xi_2^\dagger)_{\dot{a}} (\chi_1^\dagger)^{\dot{a}} = (\xi_2^\dagger \chi_1^\dagger) = (\chi_1^\dagger \xi_2^\dagger) \tag{45}$$

This proves equation (36.65). $\square$

Question 2

38.1

Use equation (38.12) to compute $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ explicitly. Hint: Show that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1, and that, for any matrix A with eigenvalues ± 1, $\exp(cA) = \cosh(c) + A\sinh(c)$, where c is an arbitrary complex number.

Extra question: What is the expression in the large energy limit $E_{\mathbf{p}} \gg m$? Please write down the result.

$$u_s(\mathbf{p}) = \exp{(i\eta\hat{\mathbf{p}}\cdot\mathbf{K})}u_s(\mathbf{0}), \quad v_s(\mathbf{p}) = \exp{(i\eta\hat{\mathbf{p}}\cdot\mathbf{K})}v_s(\mathbf{0}) \tag{38.12}$$

Answer

We start by showing that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1. The boost generators $\mathbf{K}$ in the Dirac representation are given by

$$K^j = \frac{i}{2}\gamma^j\gamma^0 = \frac{i}{2}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \tag{46}$$

Thus,

$$2i\hat{\mathbf{p}}\cdot\mathbf{K} = 2i\sum_{j=1}^{3}\hat{p}_j K^j = 2i\sum_{j=1}^{3}\hat{p}_j\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} = -\begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & -\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}, \tag{47}$$

where $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \hat{p}_3 & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 \end{pmatrix}$. Now we want to prove $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ has eigenvalues ± 1, since $\hat{\mathbf{p}}$ is a unit vector. The characteristic polynomial of $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ is given by

$$\det(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} - \lambda I) = \det\begin{pmatrix} \hat{p}_3 - \lambda & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 - \lambda \end{pmatrix} = (\hat{p}_3 - \lambda)(-\hat{p}_3 - \lambda) - (\hat{p}_1 - i\hat{p}_2)(\hat{p}_1 + i\hat{p}_2) \tag{48}$$

Simplifying this, we get

$$\det(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} - \lambda I) = \lambda^2 - (\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) = \lambda^2 - 1 \tag{49}$$

Setting the determinant to zero, we find the eigenvalues:

$$\lambda^2 - 1 = 0 \implies \lambda^2 = 1 \implies \lambda = \pm 1 \tag{50}$$

Thus, $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ has eigenvalues ± 1. Consequently, the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1 as well. Now we can use the identity for any matrix A with eigenvalues ± 1:

$$\exp(cA) = \cosh(c) + A\sinh(c) \tag{51}$$

where c is an arbitrary complex number. Applying this to our case with $A = 2i\hat{\mathbf{p}}\cdot\mathbf{K}$ and $c = \eta/2$, we have

$$\begin{aligned}
\exp(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}) &= \cosh\left(\frac{\eta}{2}\right) + (2i\hat{\mathbf{p}}\cdot\mathbf{K})\sinh\left(\frac{\eta}{2}\right) && (52)\\
&= \cosh\left(\frac{\eta}{2}\right) - \begin{pmatrix}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & 0\\ 0 & -\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\end{pmatrix}\sinh\left(\frac{\eta}{2}\right) && (53)\\
&= \begin{pmatrix}\cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right) & 0\\ 0 & \cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\end{pmatrix} && (54)
\end{aligned}$$

Now, we can compute $u_s(\mathbf{p})$,

$$\begin{aligned}
u_s ame(\mathbf{p}) &= \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)u_s(\mathbf{0}) && (55)\\
&= \begin{pmatrix}\cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right) & 0\\ 0 & \cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\end{pmatrix}\begin{pmatrix}\sqrt{m}\,\xi_s\\ \sqrt{m}\,\xi_s\end{pmatrix} && (56)\\
&= \sqrt{m}\begin{pmatrix}\left(\cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\right)\xi_s\\ \left(\cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\right)\xi_s\end{pmatrix}, && (57)
\end{aligned}$$

where $\xi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ and $\xi_- = \begin{pmatrix}0\\1\end{pmatrix}$. Hence, we have

$$\begin{aligned}
u_+(\mathbf{p}) &= \sqrt{m}\begin{pmatrix}\cosh\left(\frac{\eta}{2}\right) - \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\\ -(\hat{p}_1 + i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right)\\ \cosh\left(\frac{\eta}{2}\right) + \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\\ (\hat{p}_1 + i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right)\end{pmatrix} && (58)\\
u_-(\mathbf{p}) &= \sqrt{m}\begin{pmatrix}-(\hat{p}_1 - i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right)\\ \cosh\left(\frac{\eta}{2}\right) + \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\\ (\hat{p}_1 - i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right)\\ \cosh\left(\frac{\eta}{2}\right) - \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\end{pmatrix} && (59)
\end{aligned}$$

Similarly, we can compute $v_s(\mathbf{p})$,

$$
\begin{aligned}
v_s(\mathbf{p}) =& \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)v_s(\mathbf{0}) && (60)\\
=& \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right) & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right) \end{pmatrix} \begin{pmatrix} \sqrt{m}\,\xi_s \\ -\sqrt{m}\,\xi_s \end{pmatrix} && (61)\\
=& \sqrt{m}\begin{pmatrix} \left(\cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\right)\xi_s \\ -\left(\cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\right)\xi_s \end{pmatrix}, && (62)
\end{aligned}
$$

where $\xi_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi_- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Hence, we have

$$
v_+(\mathbf{p}) = \sqrt{m}\begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \\ \cosh\left(\frac{\eta}{2}\right) + \hat{p}_3\sinh\left(\frac{\eta}{2}\right) \\ (\hat{p}_1 - i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \\ -\left(\cosh\left(\frac{\eta}{2}\right) - \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\right) \end{pmatrix} \tag{63}
$$

$$
v_-(\mathbf{p}) = \sqrt{m}\begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{p}_3\sinh\left(\frac{\eta}{2}\right) \\ -(\hat{p}_1 + i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \\ -\left(\cosh\left(\frac{\eta}{2}\right) + \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\right) \\ -(\hat{p}_1 + i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \end{pmatrix} \tag{64}
$$

Now, we can express $\cosh\left(\frac{\eta}{2}\right)$ and $\sinh\left(\frac{\eta}{2}\right)$ in terms of energy $E_{\mathbf{p}}$ and mass m. We know that

$$
\cosh(\eta) = \frac{E_{\mathbf{p}}}{m}, \quad \sinh(\eta) = \frac{|\mathbf{p}|}{m} \tag{65}
$$

Using the half-angle formulas for hyperbolic functions, we have

$$
\cosh\left(\frac{\eta}{2}\right) = \sqrt{\frac{E_{\mathbf{p}} + m}{2m}}, \quad \sinh\left(\frac{\eta}{2}\right) = \sqrt{\frac{E_{\mathbf{p}} - m}{2m}} \tag{66}
$$

Substituting these back into the expressions for $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$, we get

$$u_+(\mathbf{p}) = \begin{pmatrix} \sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -(\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ (\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \end{pmatrix} \tag{67}$$

$$u_-(\mathbf{p}) = \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ (\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \end{pmatrix} \tag{68}$$

$$v_+(\mathbf{p}) = \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ (\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -\left(\sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}}\right) \end{pmatrix} \tag{69}$$

$$v_-(\mathbf{p}) = \begin{pmatrix} \sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -(\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -\left(\sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}}\right) \\ -(\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \end{pmatrix} \tag{70}$$

In the large energy limit $E_\mathbf{p} \gg m$, we have

$$\sqrt{\frac{E_\mathbf{p}+m}{2}} \approx \sqrt{\frac{E_\mathbf{p}}{2}}, \quad \sqrt{\frac{E_\mathbf{p}-m}{2}} \approx \sqrt{\frac{E_\mathbf{p}}{2}} \tag{71}$$

Thus, the expressions for $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ simplify to

$$u_+(\mathbf{p}) \approx \sqrt{\frac{E_\mathbf{p}}{2}} \begin{pmatrix} 1 - \hat{p}_3 \\ -(\hat{p}_1 + i\hat{p}_2) \\ 1 + \hat{p}_3 \\ (\hat{p}_1 + i\hat{p}_2) \end{pmatrix} \tag{72}$$

$$u_-(\mathbf{p}) \approx \sqrt{\frac{E_\mathbf{p}}{2}} \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2) \\ 1 + \hat{p}_3 \\ (\hat{p}_1 - i\hat{p}_2) \\ 1 - \hat{p}_3 \end{pmatrix} \tag{73}$$

$$v_+(\mathbf{p}) \approx \sqrt{\frac{E_\mathbf{p}}{2}} \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2) \\ 1 + \hat{p}_3 \\ (\hat{p}_1 - i\hat{p}_2) \\ -(1 - \hat{p}_3) \end{pmatrix} \tag{74}$$

$$v_-(\mathbf{p}) \approx \sqrt{\frac{E_\mathbf{p}}{2}} \begin{pmatrix} 1 - \hat{p}_3 \\ -(\hat{p}_1 + i\hat{p}_2) \\ -(1 + \hat{p}_3) \\ -(\hat{p}_1 + i\hat{p}_2) \end{pmatrix} \tag{75}$$

□

Question 3

45.2

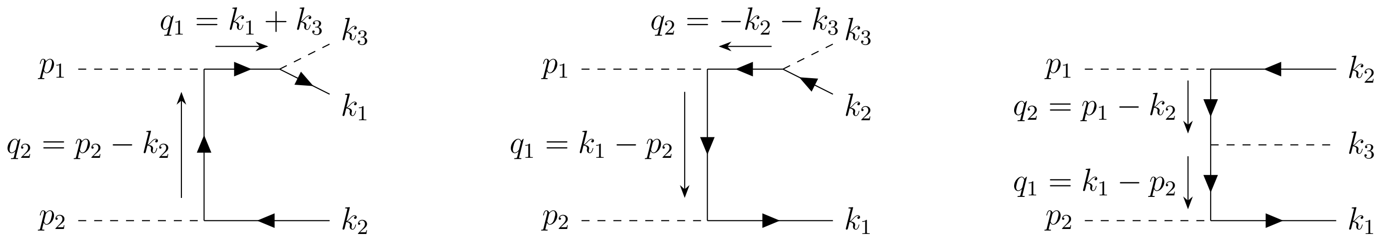
Use the Feynman rules to write down (at tree level) $i\mathcal{T}$ for the processes: $e^+e^+ \to e^+e^+$ and $\varphi\varphi \to e^+e^-\varphi$.
Remark: Do not write $\varphi\varphi \to e^+e^-$. Also, please draw Feynman diagrams when doing this problem. Remember the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + g\varphi\overline{\Psi}\Psi. \tag{76}$$

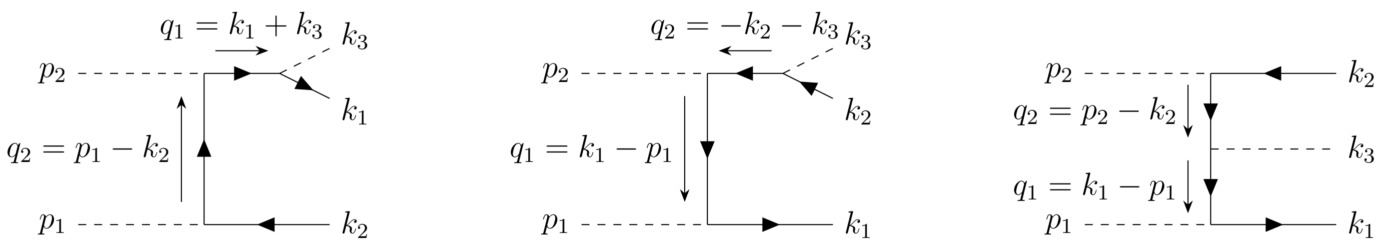
Answer

(a) t-channel diagram. (b) u-channel diagram.

Figure 1: Feynman diagrams for $e^+e^+ \to e^+e^+$ at tree level (t and u channels).



(a) Diagram 1 for $\varphi\varphi \to e^+e^-\varphi$. (b) Diagram 2 for $\varphi\varphi \to e^+e^-\varphi$. (c) Diagram 3 for $\varphi\varphi \to e^+e^-\varphi$.



(d) Diagram 1 for $\varphi\varphi \to e^+e^-\varphi$. (e) Diagram 2 for $\varphi\varphi \to e^+e^-\varphi$. (f) Diagram 3 for $\varphi\varphi \to e^+e^-\varphi$.

Figure 2: Two Feynman diagrams for $\varphi\varphi \to e^+e^-\varphi$ at tree level.

- **For the process** $e^+e^+ \to e^+e^+$**:**

The tree-level amplitude for the process $e^+e^+ \to e^+e^+$ consists of two diagrams: the t-channel and u-channel exchanges of a scalar particle. The total amplitude is given by the sum of the contributions from both channels. The amplitude for the t-channel diagram (Figure 1a) is

$$i\mathcal{T}_t = (ig)^2 \left[\overline{v}(p_2)v(k_2)\right] \frac{-i}{(p_1-k_1)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_1)v(k_1)\right]. \tag{77}$$

Similarly, the amplitude for the u-channel diagram (Figure 1b) is

$$i\mathcal{T}_u = (ig)^2 \left[\overline{v}(p_1)v(k_2)\right] \frac{-i}{(p_1-k_2)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_2)v(k_1)\right]. \tag{78}$$

Thus, the total amplitude for the process $e^+e^+ \to e^+e^+$ is

$$i\mathcal{T} = i\mathcal{T}_t - i\mathcal{T}_u \tag{79}$$

$$=(ig^2) \left[\overline{v}(p_2)v(k_2)\right] \frac{-i}{(p_1-k_1)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_1)v(k_1)\right] \tag{80}$$

$$-(ig^2) \left[\overline{v}(p_1)v(k_2)\right] \frac{-i}{(p_1-k_2)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_2)v(k_1)\right]. \tag{81}$$

The minus sign arises due to the antisymmetry of the fermionic wavefunctions under exchange.

- **For the process $\varphi\varphi \to e^+e^-\varphi$:**

The tree-level amplitude for the process $\varphi\varphi \to e^+e^-\varphi$ consists of three diagrams, as shown in Figure 2. The total amplitude is given by the sum of the contributions from all three diagrams. The amplitude for Diagram 1 (Figure 2a) is

$$i\mathcal{T}_1 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{82}$$

where $q_1 = k_1 + k_3$ and $q_2 = p_2 - k_2$. The amplitude for Diagram 2 (Figure 2b) is

$$i\mathcal{T}_2 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{83}$$

where $q_1 = k_1 - p_2$ and $q_2 = -k_2 - k_3$. The amplitude for Diagram 3 (Figure 2c) is

$$i\mathcal{T}_3 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{84}$$

where $q_1 = k_1 - p_2$ and $q_2 = p_1 - k_2$. The amplitude for Diagram 4 (Figure 2d) is

$$i\mathcal{T}_4 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{85}$$

where $q_1 = k_1 + k_3$ and $q_2 = p_1 - k_2$. The amplitude for Diagram 5 (Figure 2e) is

$$i\mathcal{T}_5 = (ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{q}_1 + m)}{q_1^2 + m^2 - i\epsilon} \frac{-i(-\not{q}_2 + m)}{q_2^2 + m^2 - i\epsilon} v(k_2)\right], \tag{86}$$

where $q_1 = k_1 - p_1$ and $q_2 = -k_2 - k_3$. The amplitude for Diagram 6 (Figure 2f) is

$$i\mathcal{T}_6 = (ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{q}_1 + m)}{q_1^2 + m^2 - i\epsilon} \frac{-i(-\not{q}_2 + m)}{q_2^2 + m^2 - i\epsilon} v(k_2)\right], \tag{87}$$

where $q_1 = k_1 - p_1$ and $q_2 = p_2 - k_2$. Thus, the total amplitude for the process $\varphi\varphi \to e^+e^-\varphi$ is

$$\begin{aligned}
i\mathcal{T} &= i\mathcal{T}_1 + i\mathcal{T}_2 + i\mathcal{T}_3 + i\mathcal{T}_4 + i\mathcal{T}_5 + i\mathcal{T}_6 && (88)\\
&=(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 - \not{k}_3 + m)}{(k_1 + k_3)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_2 + \not{k}_2 + m)}{(p_2 - k_2)^2 + m^2 - i\epsilon} v(k_2)\right] && (89)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_2 + m)}{(k_1 - p_2)^2 + m^2 - i\epsilon} \frac{-i(\not{k}_2 + \not{k}_3 + m)}{(-k_2 - k_3)^2 + m^2 - i\epsilon} v(k_2)\right] && (90)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_2 + m)}{(k_1 - p_2)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_1 + \not{k}_2 + m)}{(p_1 - k_2)^2 + m^2 - i\epsilon} v(k_2)\right] && (91)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 - \not{k}_3 + m)}{(k_1 + k_3)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_1 + \not{k}_2 + m)}{(p_1 - k_2)^2 + m^2 - i\epsilon} v(k_2)\right] && (92)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_1 + m)}{(k_1 - p_1)^2 + m^2 - i\epsilon} \frac{-i(\not{k}_2 + \not{k}_3 + m)}{(-k_2 - k_3)^2 + m^2 - i\epsilon} v(k_2)\right] && (93)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_1 + m)}{(k_1 - p_1)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_2 + \not{k}_2 + m)}{(p_2 - k_2)^2 + m^2 - i\epsilon} v(k_2)\right]. && (94)
\end{aligned}$$

□

Question 4

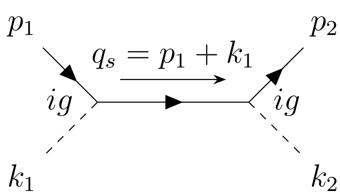
48.2

Compute $\langle|\mathcal{T}|^2\rangle$ for $e^+e^- \to \varphi\varphi$. You should find that your result is the same as that for $e^-\varphi \to e^-\varphi$, but with $s \leftrightarrow t$, and an extra overall minus sign. This relationship is known as *crossing symmetry*. There is an overall minus sign for each fermion that is moved from the initial to the final state.

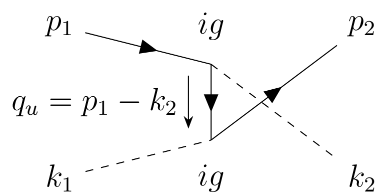
Remark: Please compute for $e^-\varphi \to e^-\varphi$, do not compute for $e^+e^- \to \varphi\varphi$. Please also draw Feynman diagrams. Remember the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + g\varphi\overline{\Psi}\Psi. \tag{95}$$

Answer



(a) s-channel diagram.



(b) u-channel diagram.

Figure 3: Feynman diagrams for $e^-\varphi \to e^-\varphi$ at tree level (s and u channels).

The tree-level amplitude for the process $e^-\varphi \to e^-\varphi$ consists of two diagrams: the s-channel and u-channel exchanges of a fermion. The total amplitude is given by the sum of the contributions from both channels. The amplitude for the s-channel diagram (Figure 3a) is

$$i\mathcal{T}_s = (ig)^2\left[\overline{u}(p_2)\frac{-i(-\not{q}_s + m)}{q_s^2 + m^2 - i\epsilon}u(p_1)\right], \tag{96}$$

where $q_s = p_1 + k_1$. Similarly, the amplitude for the u-channel diagram (Figure 3b) is

$$i\mathcal{T}_u = (ig)^2\left[\overline{u}(p_2)\frac{-i(-\not{q}_u + m)}{q_u^2 + m^2 - i\epsilon}u(p_1)\right], \tag{97}$$

where $q_u = p_1 - k_2$. Thus, the total amplitude for the process $e^-\varphi \to e^-\varphi$ is

$$\begin{aligned} i\mathcal{T} &= i\mathcal{T}_s + i\mathcal{T}_u & (98)\\ &=(ig^2)\left[\overline{u}(p_2)\frac{-i(-\not{q}_s + m)}{q_s^2 + m^2 - i\epsilon}u(p_1)\right] & (99)\\ &+(ig^2)\left[\overline{u}(p_2)\frac{-i(-\not{q}_u + m)}{q_u^2 + m^2 - i\epsilon}u(p_1)\right]. & (100)\end{aligned}$$

To compute $\langle|\mathcal{T}|^2\rangle$, we average over initial spins and sum over final spins:

$$\langle|\mathcal{T}|^2\rangle = \frac{1}{2}\sum_{\text{spins}}|\mathcal{T}|^2. \tag{101}$$

We define $\mathcal{A}$ as

$$\begin{aligned}
\mathcal{A} &= \left[\overline{u}_{s_2}(p_2)\frac{(-\not{q}_s+m)}{q_s^2+m^2}u_{s_1}(p_1)\right] + \left[\overline{u}_{s_2}(p_2)\frac{(-\not{q}_u+m)}{q_u^2+m^2}u_{s_1}(p_1)\right] && (102)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{q}_s+m}{q_s^2+m^2}+\frac{-\not{q}_u+m}{q_u^2+m^2}\right)u_{s_1}(p_1)\right] && (103)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-(\not{p}_1+\not{k}_1)+m}{(p_1+k_1)^2+m^2}+\frac{-(\not{p}_1-\not{k}_2)+m}{(p_1-k_2)^2+m^2}\right)u_{s_1}(p_1)\right] && (104)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{p}_1-\not{k}_1+m}{-s+m^2}+\frac{-\not{p}_1+\not{k}_2+m}{-u+m^2}\right)u_{s_1}(p_1)\right] && (105)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_1}(p_1)\right] && (106)
\end{aligned}$$

and its Hermitian conjugate $\mathcal{A}^\dagger$ as

$$\mathcal{A}^\dagger = \left[\overline{u}_{s_1}(p_1)\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}u_{s_2}(p_2)\right] \tag{107}$$

where s_1 and s_2 are the spin indices for the initial and final electrons, respectively, and we have used the Mandelstam variables:

$$s = -(p_1+k_1)^2, \quad u = -(p_1-k_2)^2. \tag{108}$$

Also, we apply the identity:

$$(\not{p}+m)u_s(p) = 0 \implies -\not{p}u_s(p) = +mu_s(p). \tag{109}$$

Then, we have

$$\langle|\mathcal{T}|^2\rangle = \frac{g^4}{2}\sum_{s_1,s_2}AA^\dagger. \tag{110}$$

Hence,

$$\sum_{s_1,s_2} \mathcal{A}\mathcal{A}^\dagger = \sum_{s_1,s_2} \left[\overline{u}_{s_2}(p_2) \left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) u_{s_1}(p_1) \right] \tag{111}$$

$$\times \left[\overline{u}_{s_1}(p_1) \left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) u_{s_2}(p_2) \right] \tag{112}$$

$$= \sum_{s_1,s_2} \left[\overline{u}_{s_2}(p_2) \left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) u_{s_1}(p_1)\overline{u}_{s_1}(p_1) \right. \tag{113}$$

$$\left. \times \left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) u_{s_2}(p_2) \right] \tag{114}$$

$$= \mathrm{Tr}\left[\left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) u_{s_1}(p_1)\overline{u}_{s_1}(p_1) \right. \tag{115}$$

$$\left. \times \left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) u_{s_2}(p_2)\overline{u}_{s_2}(p_2) \right] \tag{116}$$

$$= \mathrm{Tr}\left[\left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) (-\not{p}_1 + m) \right. \tag{117}$$

$$\left. \times \left(\frac{-\not{k}_1 + 2m}{-s+m^2} + \frac{+\not{k}_2 + 2m}{-u+m^2} \right) (-\not{p}_2 + m) \right] \tag{118}$$

where we have used the completeness relation for spinors:

$$\sum_s u_s(p)\overline{u}_s(p) = -\not{p} + m. \tag{119}$$

Since only even numbers of gamma matrices contribute to the trace, we find

$$\sum_{s_1,s_2} \mathcal{A}\mathcal{A}^\dagger = \frac{1}{(-s+m^2)^2} \mathrm{Tr}\Big[(-\not{k}_1 + 2m)(-\not{p}_1 + m)(-\not{k}_1 + 2m)(-\not{p}_2 + m) \Big] \tag{120}$$

$$+ \frac{1}{(-u+m^2)^2} \mathrm{Tr}\Big[(+\not{k}_2 + 2m)(-\not{p}_1 + m)(+\not{k}_2 + 2m)(-\not{p}_2 + m) \Big] \tag{121}$$

$$+ \frac{1}{(-s+m^2)(-u+m^2)} \mathrm{Tr}\Big[(-\not{k}_1 + 2m)(-\not{p}_1 + m)(+\not{k}_2 + 2m)(-\not{p}_2 + m) \Big] \tag{122}$$

$$+ \frac{1}{(-u+m^2)(-s+m^2)} \mathrm{Tr}\Big[(+\not{k}_2 + 2m)(-\not{p}_1 + m)(-\not{k}_1 + 2m)(-\not{p}_2 + m) \Big] \tag{123}$$

$$= \frac{1}{(s-m^2)^2} \mathrm{Tr}\Big[(\not{k}_1 - 2m)(\not{p}_1 - m)(\not{k}_1 - 2m)(\not{p}_2 - m) \Big] \tag{124}$$

$$+ \frac{1}{(u-m^2)^2} \mathrm{Tr}\Big[(\not{k}_2 + 2m)(\not{p}_1 - m)(\not{k}_2 + 2m)(\not{p}_2 - m) \Big] \tag{125}$$

$$+ \frac{-1}{(s-m^2)(u-m^2)} \mathrm{Tr}\Big[(\not{k}_1 - 2m)(\not{p}_1 - m)(+\not{k}_2 + 2m)(\not{p}_2 - m) \Big] \tag{126}$$

$$+ \frac{-1}{(u-m^2)(s-m^2)} \mathrm{Tr}\Big[(\not{k}_2 + 2m)(\not{p}_1 - m)(\not{k}_1 - 2m)(\not{p}_2 - m) \Big]. \tag{127}$$

Now, we compute each trace term separately:

$$\text{First term: } \mathrm{Tr}\Big[(\not{k}_1 - 2m)(\not{p}_1 - m)(\not{k}_1 - 2m)(\not{p}_2 - m)\Big] \tag{128}$$
$$= \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_1\not{k}_1 + 4m^2\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{k}_1 + 4m^4\Big] \tag{129}$$
$$\text{Second term: } \mathrm{Tr}\Big[(\not{k}_2 + 2m)(\not{p}_1 - m)(\not{k}_2 + 2m)(\not{p}_2 - m)\Big] \tag{130}$$
$$= \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_2\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_2\not{k}_2 - 4m^2\not{k}_2\not{p}_2 - 4m^2\not{p}_1\not{k}_2 + 4m^4\Big] \tag{131}$$
$$\text{Third term: } \mathrm{Tr}\Big[(\not{k}_1 - 2m)(\not{p}_1 - m)(\not{k}_2 + 2m)(\not{p}_2 - m)\Big] \tag{132}$$
$$= \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_2\not{p}_2 + m^2\not{k}_1\not{k}_2 - 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) - 4m^2\not{p}_1\not{p}_2 - 4m^4\Big] \tag{133}$$
$$\text{Fourth term: } \mathrm{Tr}\Big[(\not{k}_2 + 2m)(\not{p}_1 - m)(\not{k}_1 - 2m)(\not{p}_2 - m)\Big] \tag{134}$$
$$= \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_1\not{p}_2 + m^2\not{k}_2\not{k}_1 - 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) - 4m^2\not{p}_1\not{p}_2 - 4m^4\Big]. \tag{135}$$

Using the trace identities:

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(ad)(bc) - (ac)(bd) + (ab)(cd)], \quad \mathrm{Tr}[\not{a}\not{b}] = -4(ab), \quad \mathrm{Tr}[\mathcal{I}] = 4, \tag{136}$$

and the on-shell conditions $p_1^2 = p_2^2 = -m^2$ and $k_1^2 = k_2^2 = -m_\varphi^2$, we find

$$\text{First term: } \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_1\not{k}_1 + 4m^2\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{k}_1 + 4m^4\Big] \tag{137}$$

$$=4\Big[(k_1p_2)(p_1k_1) - (k_1k_1)(p_1p_2) + (k_1p_1)(k_1p_2) - 4m^2(p_1p_2) \tag{138}$$

$$-m^2(k_1k_1) - 4m^2(k_1p_2) - 4m^2(p_1k_1) + 4m^2\Big] \tag{139}$$

$$=4\Big[2(k_1p_2)(p_1k_1) + (m_\varphi^2 - 4m^2)(p_1p_2) + m^2m_\varphi^2 - 4m^2(k_1p_2) - 4m^2(p_1k_1) + 4m^4\Big] \tag{140}$$

$$\text{Second term: } \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_2\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_2\not{k}_2 - 4m^2\not{k}_2\not{p}_2 - 4m^2\not{p}_1\not{k}_2 + 4m^4\Big] \tag{141}$$

$$=4\Big[(k_2p_2)(p_1k_2) - (k_2k_2)(p_1p_2) + (k_2p_1)(k_2p_2) - 4m^2(p_1p_2) \tag{142}$$

$$-m^2(k_2k_2) + 4m^2(k_2p_2) + 4m^2(p_1k_2) + 4m^2\Big] \tag{143}$$

$$=4\Big[2(k_2p_2)(p_1k_2) + (m_\varphi^2 - 4m^2)(p_1p_2) + m^2m_\varphi^2 + 4m^2(k_2p_2) + 4m^2(p_1k_2) + 4m^4\Big] \tag{144}$$

$$\text{Third term: } \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_2\not{p}_2 + m^2\not{k}_1\not{k}_2 - 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) - 4m^2\not{p}_1\not{p}_2 - 4m^4\Big] \tag{145}$$

$$=4\Big[(k_1p_2)(p_1k_2) - (k_1k_2)(p_1p_2) + (k_1p_1)(k_2p_2) - m^2(k_1k_2) + 2m^2(k_1(p_1+p_2)) \tag{146}$$

$$-2m^2(k_2(p_1+p_2)) + 4m^2(p_1p_2) - 4m^4\Big] \tag{147}$$

$$=4\Big[(k_1p_2)(p_1k_2) - (k_1k_2)(p_1p_2) + (k_1p_1)(k_2p_2) - m^2(k_1k_2) \tag{148}$$

$$+2m^2(k_1p_1 + k_1p_2 - k_2p_1 - k_2p_2) - 4m^2(p_1p_2) - 4m^4\Big] \tag{149}$$

$$\text{Fourth term: } \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_1\not{p}_2 + m^2\not{k}_2\not{k}_1 - 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) + 4m^2\not{p}_1\not{p}_2 - 4m^4\Big] \tag{150}$$

$$=4\Big[(k_2p_2)(p_1k_1) - (k_2k_1)(p_1p_2) + (k_2p_1)(k_1p_2) - m^2(k_2k_1) + 2m^2(k_2(p_1+p_2)) \tag{151}$$

$$-2m^2(k_1(p_1+p_2)) + 4m^2(p_1p_2) - 4m^4\Big] \tag{152}$$

$$=4\Big[(k_2p_2)(p_1k_1) - (k_2k_1)(p_1p_2) + (k_2p_1)(k_1p_2) - m^2(k_2k_1) \tag{153}$$

$$+2m^2(k_2p_1 + k_2p_2 - k_1p_1 - k_1p_2) + 4m^2(p_1p_2) - 4m^4\Big]. \tag{154}$$

We can express the dot products in terms of the Mandelstam variables:

$$s = -(p_1+k_1)^2 = -(p_2+k_2)^2 = m^2 + m_\varphi^2 - 2(p_1k_1) = m^2 + m_\varphi^2 - 2(p_2k_2), \tag{155}$$

$$u = -(p_1-k_2)^2 = -(p_2-k_1)^2 = m^2 + m_\varphi^2 + 2(k_2p_1) = m^2 + m_\varphi^2 + 2(k_1p_2), \tag{156}$$

$$t = -(k_1-k_2)^2 = -(p_1-p_2)^2 = 2m^2 + 2(p_1p_2) = 2m_\varphi^2 + 2(k_1k_2). \tag{157}$$

Thus, by $s+t+u=2m^2+2m_\varphi^2$, we have

$$(p_1k_1) = \frac{m^2+m_\varphi^2-s}{2}, \tag{158}$$
$$(p_2k_2) = \frac{m^2+m_\varphi^2-s}{2}, \tag{159}$$
$$(k_2p_1) = \frac{u-m^2-m_\varphi^2}{2}, \tag{160}$$
$$(k_1p_2) = \frac{u-m^2-m_\varphi^2}{2}, \tag{161}$$
$$(p_1p_2) = \frac{t-2m^2}{2} = \frac{-(s+u)+2m_\varphi^2}{2}, \tag{162}$$
$$(k_1k_2) = \frac{t-2m_\varphi^2}{2} = \frac{-(s+u)+2m^2}{2}. \tag{163}$$

Substituting these expressions back into the sum, we find (by Mathematica):

$$\sum_{s_1,s_2}\mathcal{A}\mathcal{A}^\dagger = 2\times\left[\frac{7m^4+m^2\left(-8m_\varphi^2+9s+u\right)+m_\varphi^4-su}{\left(m^2-s\right)^2}\right. \tag{164}$$
$$+\frac{7m^4+m^2\left(-8m_\varphi^2+s+9u\right)+m_\varphi^4-su}{\left(m^2-u\right)^2} \tag{165}$$
$$\left.+\frac{2\left(9m^4+m^2\left(3(s+u)-8m_\varphi^2\right)-m_\varphi^4+su\right)}{\left(m^2-s\right)\left(m^2-u\right)}\right]. \tag{166}$$

Therefore, the final result for $\langle|\mathcal{T}|^2\rangle$ is

$$\langle|\mathcal{T}|^2\rangle = \frac{g^4}{2}\sum_{s_1,s_2}\mathcal{A}\mathcal{A}^\dagger \tag{167}$$
$$= g^4\left[\frac{7m^4+m^2\left(-8m_\varphi^2+9s+u\right)+m_\varphi^4-su}{\left(m^2-s\right)^2}\right. \tag{168}$$
$$+\frac{7m^4+m^2\left(-8m_\varphi^2+s+9u\right)+m_\varphi^4-su}{\left(m^2-u\right)^2} \tag{169}$$
$$\left.+\frac{2\left(9m^4+m^2\left(3(s+u)-8m_\varphi^2\right)-m_\varphi^4+su\right)}{\left(m^2-s\right)\left(m^2-u\right)}\right]. \tag{170}$$

□