University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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October 2, 2025

HW2 Due to October 7 11:59 PM

Question 1

Problem 5.1
Work out the LSZ reduction formula for the complex scalar field that was introduced in problem 3.5. Note that we must specify the type (a or b) of each incoming and outgoing particle.

Answer

We start with the mode expansion of the complex scalar field:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} [a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] \tag{1}$$

$$\varphi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} [b(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}] \tag{2}$$

$$a(\mathbf{k}) = \int d^3x e^{-ikx} \left[i\partial_0\varphi(x) + \omega\varphi(x)\right], \tag{3}$$

$$b(\mathbf{k}) = \int d^3x e^{-ikx} [\omega\varphi^\dagger(x) + i\partial_0\varphi^\dagger(x)]. \tag{4}$$

First, we define the $|i\rangle$ and $|f\rangle$ states as

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\cdots b_1^\dagger(t)b_2^\dagger(t)\cdots|0\rangle, \tag{5}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\cdots b_{1'}^\dagger(t)b_{2'}^\dagger(t)\cdots|0\rangle. \tag{6}$$

And a_i and b_i are given by

$$a_i^\dagger = \int d^3k f_i(\mathbf{k})a^\dagger(\mathbf{k}) \tag{7}$$

$$b_i^\dagger = \int d^3k g_i(\mathbf{k})b^\dagger(\mathbf{k}), \tag{8}$$

where

$$f_i(\mathbf{k}), g_i(\mathbf{k}) \propto \exp(-(\mathbf{k}-\mathbf{k}_i)^2/4\sigma^2). \tag{9}$$

Now we can compute the difference between $a_1^\dagger(+\infty)$ and $a_1^\dagger(-\infty)$:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt \partial_0 a_1^\dagger(t) \tag{10}$$

$$= \int_{-\infty}^{+\infty} dt \int d^3k f_1(\mathbf{k}) \int d^3x e^{ikx}[\omega\varphi(x) - i\partial_0\varphi(x)] \tag{11}$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x), \tag{12}$$

where I quote the equation in the textbook. Similarly, we can get

$$b_1^\dagger(+\infty) - b_1^\dagger(-\infty) = -i \int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x), \tag{13}$$

$$a_{1'}(+\infty) - a_{1'}(-\infty) = i \int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x), \tag{14}$$

$$b_{1'}(+\infty) - b_{1'}(-\infty) = i \int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x). \tag{15}$$

Now we can express the S-matrix element $\langle f|i\rangle$ as

$$\langle f|i\rangle = \langle 0|\mathcal{T} b_{1'}(+\infty)b_{2'}(+\infty)\cdots a_{1'}(+\infty)a_{2'}(+\infty)\cdots a_1^\dagger(-\infty)a_2^\dagger(-\infty)\cdots b_1^\dagger(-\infty)b_2^\dagger(-\infty)\cdots|0\rangle \tag{16}$$

$$\begin{aligned}
=&\langle 0|\mathcal{T}[b_{1'}(-\infty) + i\int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots \\
&\cdots[a_{1'}(-\infty) + i\int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[a_1^\dagger(+\infty) + i\int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[b_1^\dagger(+\infty) + i\int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots|0\rangle
\end{aligned} \tag{17}$$

$$\begin{aligned}
=&\langle 0|\mathcal{T}[i\int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots \\
&\cdots[i\int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[i\int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[i\int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots|0\rangle
\end{aligned} \tag{18}$$

$$=(i)^{n+n'+m+m'}\langle 0|\mathcal{T}[\prod_{j'}^{n'}\int d^4x e^{-ik_{j'}x}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)][\prod_{l'}^{n'}\int d^4x e^{-ik_{l'}x}(-\partial_\mu\partial^\mu + m^2)\varphi(x)] \tag{19}$$

$$[\prod_{l}^{m}\int d^4x e^{ik_l x}(-\partial_\mu\partial^\mu + m^2)\varphi(x)][\prod_{j}^{n}\int d^4x e^{ik_j x}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]|0\rangle, \tag{20}$$

where we have used the fact that $a_i|0\rangle = b_i|0\rangle = 0$ and $\langle 0|a_i^\dagger = \langle 0|b_i^\dagger = 0$. Here n and m are the number of incoming a and b particles, while n' and m' are the number of outgoing a and b particles, respectively. We also impose the $\sigma \to 0$ limit, so that $f_i(\mathbf{k})$ and $g_i(\mathbf{k})$ become delta functions. Finally, we can rewrite the S-matrix element as

$$\begin{aligned}\langle f|i\rangle =&(i)^{n+n'+m+m'} \int d^4x_1 e^{-ik_1x_1} \cdots \int d^4x_n e^{-ik_nx_n} \int d^4x_{1'} e^{ik_{1'}x_{1'}} \cdots \int d^4x_{n'} e^{ik_{n'}x_{n'}} \\ &\int d^4y_1 e^{-ip_1y_1} \cdots \int d^4y_m e^{-ip_my_m} \int d^4y_{1'} e^{ip_{1'}y_{1'}} \cdots \int d^4y_{m'} e^{ip_{m'}y_{m'}} \\ &(-\partial_\mu \partial^\mu_{x_1} + m^2) \cdots (-\partial_\mu \partial^\mu_{x_n} + m^2)(-\partial_\mu \partial^\mu_{x_{1'}} + m^2) \cdots (-\partial_\mu \partial^\mu_{x_{n'}} + m^2) \\ &(-\partial_\mu \partial^\mu_{y_1} + m^2) \cdots (-\partial_\mu \partial^\mu_{y_m} + m^2)(-\partial_\mu \partial^\mu_{y_{1'}} + m^2) \cdots (-\partial_\mu \partial^\mu_{y_{m'}} + m^2) \\ &\langle 0|\mathcal{T}\varphi^\dagger(y_{1'}) \cdots \varphi^\dagger(y_{m'})\varphi(x_{1'}) \cdots \varphi(x_{n'})\varphi(x_1) \cdots \varphi(x_n)\varphi^\dagger(y_1) \cdots \varphi^\dagger(y_m)|0\rangle. \end{aligned} \tag{21}$$

This is the LSZ reduction formula for the complex scalar field. $\square$

Question 2

Problem 6.1

(a) Find an explicit formula for $\mathcal{D}q$ in eq. (6.9). Your formula should be of the form $\mathcal{D}q = C\prod_{j=1}^{N} dq_j$, where C is a constant that you should compute.

(b) For the case of a free particle, $V(Q) = 0$, evaluate the path integral of eq. (6.9) explicitly. Hint: integrate over q_1, then q_2, etc, and look for a pattern. Express you final answer in terms of q', t', q'', t'' and m. Restore $\hbar$ by dimensional analysis.

(c) Compute the $\langle q'', t''|q', t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare your result in part (b).

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\overline{q}_j)\delta t}, \tag{6.7}$$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q \exp\left[i\int_{t'}^{t''} dt L(\dot{q}(t), q(t))\right]. \tag{6.9}$$

Answer

(a) First, from eq. (6.7), we can see that

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\overline{q}_j)\delta t}, \quad \text{assuming } H(p,q) = \frac{1}{2m}p^2 + V(q) && (22)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i(\frac{1}{2m}p_j^2+V(\overline{q}_j))\delta t} && (23)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j\delta t\dot{q}_j} e^{-i(\frac{1}{2m}p_j^2+V(\overline{q}_j))\delta t}, \quad \text{where } \dot{q}_j = \frac{q_{j+1}-q_j}{\delta t} && (24)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t(\frac{1}{2m}p_j^2 - p_j\dot{q}_j + V(\overline{q}_j))} && (25)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t(\frac{1}{2m}(p_j-m\dot{q}_j)^2 - \frac{1}{2}m\dot{q}_j{}^2 + V(\overline{q}_j))} && (26)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t\frac{1}{2m}(p_j-m\dot{q}_j)^2} e^{i\delta t(\frac{1}{2}m\dot{q}_j{}^2 - V(\overline{q}_j))} && (27)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t\frac{1}{2m}(p_j-m\dot{q}_j)^2} e^{i\delta t L(\dot{q}_j,\overline{q}_j)}, \quad \text{where } L(\dot{q},q) = \frac{1}{2}m\dot{q}^2 - V(q) && (28)\\
&= \int \prod_{k=1}^{N} dq_k \left[\prod_{j=0}^{N} \int \frac{dp_j}{2\pi} e^{-i\delta t\frac{1}{2m}(p_j-m\dot{q}_j)^2}\right] e^{i\sum_{j=0}^{N}\delta t L(\dot{q}_j,\overline{q}_j)}, && (29)
\end{aligned}$$

where we have used the definition of $\dot{q}_j$ and $L(\dot{q}, q)$. Now we can compute the integral over p_j:

$$\begin{aligned}
\int \frac{dp_j}{2\pi} e^{-i\delta t \frac{1}{2m}(p_j - m\dot{q}_j)^2} &= \int \frac{dp_j}{2\pi} e^{-i\frac{\delta t}{2m} p_j^2} \quad \text{(by shifting } p_j \to p_j + m\dot{q}_j) && (30)\\
&= \frac{1}{2\pi}\sqrt{\frac{2m\pi}{i\delta t}} \quad \text{(by Gaussian integral)} && (31)\\
&= \sqrt{\frac{m}{2\pi i \delta t}}. && (32)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \left[\prod_{j=0}^{N} \sqrt{\frac{m}{2\pi i\delta t}}\right] e^{i\sum_{j=0}^{N} \delta t L(\dot{q}_j, \overline{q}_j)} && (33)\\
&= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\sum_{j=0}^{N} \delta t L(\dot{q}_j, \overline{q}_j)} && (34)\\
&= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\int_{t'}^{t''} dt L(\dot{q}(t), q(t))} \quad \text{(by definition of Riemann integral).} && (35)
\end{aligned}$$

Therefore, we can identify

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} dq_j. \qquad (36)$$

This is the explicit formula for $\mathcal{D}q$ in eq. (6.9).

(b) Now if we consider the case of a free particle, i.e. $V(Q) = 0$, then we have

$$\begin{aligned}
L(\dot{q}, q) &= \frac{1}{2} m\dot{q}^2, && (37)\\
\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\int_{t'}^{t''} dt \frac{1}{2} m\dot{q}^2} && (38)\\
&= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\sum_{j=0}^{N} \delta t \frac{1}{2} m\dot{q_j}^2}. && (39)\\
& && (40)
\end{aligned}$$

The terms in the exponent are given by:

$$\sum_{j=0}^{N} \delta t \frac{1}{2} m\dot{q_j}^2 = \sum_{j=0}^{N} \delta t \frac{1}{2} m\Big(\frac{q_{j+1} - q_j}{\delta t}\Big)^2 = \sum_{j=0}^{N} \frac{m}{2\delta t}(q_{j+1}^2 - 2q_{j+1}q_j + q_j^2). \qquad (41)$$

Thus, we focus on the integral and compute it step by step:

$$\int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} \frac{im}{2\delta t}(q_{j+1}^2 - 2q_{j+1}q_j + q_j^2)\Big) \int dq_1 \exp\Big[\frac{im}{2\delta t}\Big((q_2^2 - 2q_2q_1 + q_1^2) + (q_1^2 - 2q_1q_0 + q_0^2)\Big)\Big] \tag{42}$$

$$\begin{aligned} &= \int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} i\lambda(q_{j+1} - q_j)^2\Big) \\ &\quad \int dq_1 \exp\Big[i\lambda\Big((q_2 - q_1)^2 + (q_1 - q_0)^2\Big)\Big], \quad \text{where } \lambda = \frac{m}{2\delta t} \end{aligned} \tag{43}$$

Before performing the integral over q_1, we consider the following integral:

$$\int_{-\infty}^{+\infty} dx e^{i\alpha(x-\beta)^2} = \sqrt{\frac{i\pi}{\alpha}} \tag{44}$$

Then we also consider the more complicated integral:

$$\int_{-\infty}^{+\infty} dx e^{i\alpha(x-c_1)^2 + i\beta(x-c_2)^2} = \int_{-\infty}^{+\infty} dx e^{i(\alpha+\beta)x^2 - 2i(\alpha c_1 + \beta c_2)x + i(\alpha c_1^2 + \beta c_2^2)} \tag{45}$$

$$= e^{i\frac{\alpha\beta}{\alpha+\beta}(c_1-c_2)^2} \int_{-\infty}^{+\infty} dx e^{i(\alpha+\beta)(x - \frac{\alpha c_1 + \beta c_2}{\alpha+\beta})^2} \tag{46}$$

$$= e^{i\frac{\alpha\beta}{\alpha+\beta}(c_1-c_2)^2} \sqrt{\frac{i\pi}{\alpha+\beta}} = e^{i\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}(c_1-c_2)^2} \sqrt{\frac{i\pi}{\alpha+\beta}}, \tag{47}$$

where I quoted the result from `Mathematica`. Now we can perform the integral over q_1:

$$\int dq_1 \exp\Big[i\lambda\Big((q_2 - q_1)^2 + (q_1 - q_0)^2\Big)\Big] \tag{48}$$

$$= \int dq_1 \exp\Big[i\lambda(q_1 - q_2)^2 + i\lambda(q_1 - q_0)^2\Big] \tag{49}$$

$$= e^{i\frac{\lambda^2}{2\lambda}(q_2-q_0)^2} \sqrt{\frac{i\pi}{2\lambda}} \tag{50}$$

$$= e^{i\frac{\lambda}{2}(q_2-q_0)^2} \sqrt{\frac{i\pi}{2\lambda}}. \tag{51}$$

Thus, we have

$$\int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} i\lambda(q_{j+1}-q_j)^2\Big)\int dq_1 \exp\Big[i\lambda\Big((q_2-q_1)^2+(q_1-q_0)^2\Big)\Big] \tag{52}$$

$$=\sqrt{\frac{i\pi}{2\lambda}}\int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} i\lambda(q_{j+1}-q_j)^2 + i\frac{\lambda}{2}(q_2-q_0)^2\Big) \tag{53}$$

$$=\sqrt{\frac{i\pi}{2\lambda}}\int dq_N \cdots \int dq_3 \exp\Big(\sum_{j=3}^{N} i\lambda(q_{j+1}-q_j)^2\Big)\int dq_2 \exp\Big(i\lambda(q_3-q_2)^2 + i\frac{\lambda}{2}(q_2-q_0)^2\Big) \tag{54}$$

$$=\sqrt{\frac{i\pi}{2\lambda}}\sqrt{\frac{i\pi}{\frac{3}{2}\lambda}}\int dq_N \cdots \int dq_3 \exp\Big(\sum_{j=3}^{N} i\lambda(q_{j+1}-q_j)^2 + i\frac{\lambda}{3}(q_3-q_0)^2\Big) \tag{55}$$

$$=\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^2}\frac{1}{\sqrt{3}}\int dq_N \cdots \int dq_4 \exp\Big(\sum_{j=4}^{N} i\lambda(q_{j+1}-q_j)^2\Big)\int dq_3 \exp\Big(i\lambda(q_4-q_3)^2 + i\frac{\lambda}{3}(q_3-q_0)^2\Big) \tag{56}$$

$$=\cdots \tag{57}$$

$$=\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^{N-1}}\frac{1}{\sqrt{N}}\int dq_N \exp\Big(i\lambda(q_{N+1}-q_N)^2 + i\frac{\lambda}{N}(q_N-q_0)^2\Big) \tag{58}$$

$$=\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^{N}}\frac{1}{\sqrt{N+1}}e^{i\frac{\lambda}{N+1}(q_{N+1}-q_0)^2}. \tag{59}$$

Combine with the prefactor, we have

$$\langle q'',t''|q',t'\rangle = \Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}}\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^{N}}\frac{1}{\sqrt{N+1}}e^{i\frac{\lambda}{N+1}(q_{N+1}-q_0)^2} \tag{60}$$

$$= \Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}}\sqrt{\Big(\frac{i\pi}{\frac{m}{2\delta t}}\Big)^{N}}\frac{1}{\sqrt{N+1}}e^{i\frac{\frac{m}{2\delta t}}{N+1}(q''-q')^2} \tag{61}$$

$$= \Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}}\Big(\frac{2i\pi\delta t}{m}\Big)^{\frac{N}{2}}\frac{1}{\sqrt{N+1}}e^{i\frac{m}{2(N+1)\delta t}(q''-q')^2} \tag{62}$$

$$=\sqrt{\frac{m}{2\pi i(N+1)\delta t}}e^{i\frac{m}{2(N+1)\delta t}(q''-q')^2} \tag{63}$$

$$=\sqrt{\frac{m}{2\pi i(t''-t')}}e^{\frac{im}{2(t''-t')}(q''-q')^2}, \quad \text{where } t''-t'=(N+1)\delta t. \tag{64}$$

Then restore $\hbar$ by dimensional analysis, we have

$$\langle q'',t''|q',t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}}e^{\frac{im}{2\hbar(t''-t')}(q''-q')^2}. \tag{65}$$

(c) We can also compute $\langle q'',t''|q',t'\rangle$ by inserting a complete set of momentum eigenstates:

$$\begin{aligned}
\langle q'', t'' | q', t' \rangle =& \langle q'' | e^{-iH(t''-t')} | q' \rangle && (66)\\
=& \int dp \langle q'' | e^{-iH(t''-t')} | p \rangle \langle p | q' \rangle && (67)\\
=& \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}(t''-t')} e^{ip(q''-q')}, && (68)
\end{aligned}$$

where we have used $H = \frac{p^2}{2m}$ and $\langle p|q'\rangle = \frac{1}{\sqrt{2\pi}} e^{-ipq'}$. Now we can perform the integral over p:

$$\begin{aligned}
\int_{-\infty}^{+\infty} dp e^{-i\frac{p^2}{2m}(t''-t')} e^{ip(q''-q')} =& \int_{-\infty}^{+\infty} dp e^{-i\frac{t''-t'}{2m}\left(p^2 - \frac{2m}{t''-t'}(q''-q')p\right)} && (69)\\
=& \int_{-\infty}^{+\infty} dp e^{-i\frac{t''-t'}{2m}\left(p - \frac{m}{t''-t'}(q''-q')\right)^2 + i\frac{m}{2(t''-t')}(q''-q')^2} && (70)\\
=& e^{i\frac{m}{2(t''-t')}(q''-q')^2} \int_{-\infty}^{+\infty} dp e^{-i\frac{t''-t'}{2m}\left(p - \frac{m}{t''-t'}(q''-q')\right)^2} && (71)\\
=& e^{i\frac{m}{2(t''-t')}(q''-q')^2} \sqrt{\frac{2m\pi}{i(t''-t')}}. && (72)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\langle q'', t'' | q', t' \rangle =& \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}(t''-t')} e^{ip(q''-q')} && (73)\\
=& \frac{1}{2\pi} e^{i\frac{m}{2(t''-t')}(q''-q')^2} \sqrt{\frac{2m\pi}{i(t''-t')}} && (74)\\
=& \sqrt{\frac{m}{2\pi i(t''-t')}} e^{\frac{im}{2(t''-t')}(q''-q')^2}. && (75)
\end{aligned}$$

This is exactly the same as the result we obtained in part (b). $\square$

Question 3

Problem 7.3

(a) Use the Heisenberg equations of motion, $\dot{A} = i[H, A]$, to find explicit expressions for $\dot{Q}$ and $\dot{P}$. Solve these to get the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of the Schrödinger-picture operators Q and P.

(b) Write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$, where $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. Then, using your result from part (a), write the Heisenberg-picture operator $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$.

(c) Using your result from part (b), and $a|0\rangle = \langle 0|a^\dagger = 0$,verify eqs. (7.16) and (7.17).

Answer

(a) First, we can compute $\dot{Q}$ and $\dot{P}$ using the Heisenberg equations of motion:

$$\dot{Q} = i[H, Q] = i\left[\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2, Q\right] = i\frac{1}{2m}[P^2, Q] = \frac{P}{m}, \tag{76}$$

$$\dot{P} = i[H, P] = i\left[\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2, P\right] = i\frac{1}{2}m\omega^2[Q^2, P] = -m\omega^2 Q. \tag{77}$$

These are the equations of motion for a harmonic oscillator. Now we can solve these equations to get $Q(t)$ and $P(t)$:

$$\ddot{Q}(t) = \frac{\dot{P}}{m} = -\omega^2 Q(t), \tag{78}$$

$$Q(t) = Q\cos\omega t + \frac{P}{m\omega}\sin\omega t, \tag{79}$$

$$P(t) = m\dot{Q}(t) = -m\omega Q\sin\omega t + P\cos\omega t. \tag{80}$$

Note that we have used the initial conditions $Q(0) = Q$ and $P(0) = P$ to determine the integration constants.

(b) Next, we can write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$:

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \tag{81}$$

$$P = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger). \tag{82}$$

Then, using the result from part (a), we can write the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$:

$$Q(t) =\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\cos\omega t + \frac{-i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger)}{m\omega}\sin\omega t \tag{83}$$

$$=\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\cos\omega t - i\sqrt{\frac{\hbar}{2m\omega}}(a-a^\dagger)\sin\omega t \tag{84}$$

$$=\sqrt{\frac{\hbar}{2m\omega}}\Big[a(\cos\omega t - i\sin\omega t) + a^\dagger(\cos\omega t + i\sin\omega t)\Big] \tag{85}$$

$$=\sqrt{\frac{\hbar}{2m\omega}}\Big[ae^{-i\omega t} + a^\dagger e^{i\omega t}\Big], \tag{86}$$

$$P(t) =-m\omega\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\sin\omega t - i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger)\cos\omega t \tag{87}$$

$$=-\sqrt{\frac{m\omega\hbar}{2}}(a+a^\dagger)\sin\omega t - i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger)\cos\omega t \tag{88}$$

$$=-\sqrt{\frac{m\omega\hbar}{2}}\Big[a(\sin\omega t + i\cos\omega t) + a^\dagger(\sin\omega t - i\cos\omega t)\Big] \tag{89}$$

$$=-i\sqrt{\frac{m\omega\hbar}{2}}\Big[a(\cos\omega t - i\sin\omega t) - a^\dagger(\cos\omega t + i\sin\omega t)\Big] \tag{90}$$

$$=-i\sqrt{\frac{m\omega\hbar}{2}}\Big[ae^{-i\omega t} - a^\dagger e^{i\omega t}\Big]. \tag{91}$$

(c) Recall eqs. (7.14), (7.16) and (7.17):

$$\begin{aligned} G(t-t') &=\frac{i}{2\omega}\exp\Big(-i\omega|t-t'|\Big) \\ &=\frac{i}{2\omega}\Big(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{-i\omega(t'-t)}\Big), \end{aligned} \tag{7.14}$$

$$\begin{aligned} \langle 0|\mathrm{T}Q(t_1)\,Q(t_2)\,|0\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0\mid 0\rangle_f\Big|_{f=0} \\ &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty} dt' G(t_2-t')\,f(t')\right]\langle 0\mid 0\rangle_f\Big|_{f=0} \\ &= \left[\frac{1}{i}G(t_2-t_1) + (\text{ term with } f'\text{s})\right]\langle 0\mid 0\rangle_f\Big|_{f=0} \\ &= \frac{1}{i}G(t_2-t_1), \end{aligned} \tag{7.16}$$

$$\begin{aligned} \langle 0|\mathrm{T}Q(t_1)\,Q(t_2)\,Q(t_3)\,Q(t_4)\,|0\rangle =&\frac{1}{i^2}\left[G(t_1-t_2)\,G(t_3-t_4)\right. \\ &+G(t_1-t_3)\,G(t_2-t_4) \\ &\left.+G(t_1-t_4)\,G(t_2-t_3)\right]. \end{aligned} \tag{7.17}$$

Using the result from part (b), we can compute $\langle 0|\mathrm{T}Q(t_1)\,Q(t_2)\,|0\rangle$:

$$
\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)\,Q(t_2)\,|0\rangle =&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[ae^{-i\omega t_1}+a^\dagger e^{i\omega t_1}\Big]\Big[ae^{-i\omega t_2}+a^\dagger e^{i\omega t_2}\Big]|0\rangle && (92)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[aae^{-i\omega(t_1+t_2)}+aa^\dagger e^{-i\omega t_1}e^{i\omega t_2}+a^\dagger ae^{i\omega t_1}e^{-i\omega t_2}+a^\dagger a^\dagger e^{i\omega(t_1+t_2)}\Big]|0\rangle && (93)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[aa^\dagger e^{-i\omega t_1}e^{i\omega t_2}\Big]|0\rangle && (94)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[(1+a^\dagger a)e^{-i\omega t_1}e^{i\omega t_2}\Big]|0\rangle && (95)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}e^{-i\omega t_1}e^{i\omega t_2}|0\rangle && (96)\\
=&\frac{\hbar}{2m\omega}\Big[\theta(t_1-t_2)e^{-i\omega(t_1-t_2)}+\theta(t_2-t_1)e^{-i\omega(t_2-t_1)}\Big] && (97)\\
=&\frac{1}{2\omega}\Big[\theta(t_1-t_2)e^{-i\omega(t_1-t_2)}+\theta(t_2-t_1)e^{-i\omega(t_2-t_1)}\Big],\quad \text{by setting } \hbar=m=1, && (98)\\
=&\frac{1}{2\omega}e^{-i\omega|t_1-t_2|} && (99)\\
=&\frac{1}{i}G(t_2-t_1) && (100)
\end{aligned}
$$

where we have used $a|0\rangle=\langle 0|a^\dagger=0$ and the definition of $G(t)$ in eq. (7.14). This verifies eq. (7.16). Next, we can compute $\langle 0|\mathrm{T}Q(t_1)\,Q(t_2)\,Q(t_3)\,Q(t_4)\,|0\rangle$:

$$
\begin{aligned}
&\langle 0|\mathrm{T}Q(t_1)\,Q(t_2)\,Q(t_3)\,Q(t_4)\,|0\rangle\\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[ae^{-i\omega t_1}+a^\dagger e^{i\omega t_1}\Big]\Big[ae^{-i\omega t_2}+a^\dagger e^{i\omega t_2}\Big]\\
&\Big[ae^{-i\omega t_3}+a^\dagger e^{i\omega t_3}\Big]\Big[ae^{-i\omega t_4}+a^\dagger e^{i\omega t_4}\Big]|0\rangle && (101)\\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[aaaae^{-i\omega(t_1+t_2+t_3+t_4)}+aaaa^\dagger e^{-i\omega(t_1+t_2+t_3)}e^{i\omega t_4}+aaa^\dagger ae^{-i\omega(t_1+t_2+t_4)}e^{i\omega t_3}\\
&+aaa^\dagger a^\dagger e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}+aa^\dagger aae^{-i\omega(t_1+t_3+t_4)}e^{i\omega t_2}+aa^\dagger aa^\dagger e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}\\
&+aa^\dagger a^\dagger ae^{-i\omega(t_1+t_4)}e^{i\omega(t_2+t_3)}+aa^\dagger a^\dagger a^\dagger e^{-i\omega t_1}e^{i\omega(t_2+t_3+t_4)}+a^\dagger aaae^{-i\omega(t_2+t_3+t_4)}e^{i\omega t_1}\\
&+a^\dagger aaa^\dagger e^{-i\omega(t_2+t_3)}e^{i\omega(t_1+t_4)}+a^\dagger aa^\dagger ae^{-i\omega(t_2+t_4)}e^{i\omega(t_1+t_3)}+a^\dagger aa^\dagger a^\dagger e^{-i\omega t_2}e^{i\omega(t_1+t_3+t_4)}\\
&+a^\dagger a^\dagger aae^{-i\omega(t_3+t_4)}e^{i\omega(t_1+t_2)}+a^\dagger a^\dagger aa^\dagger e^{-i\omega t_3}e^{i\omega(t_1+t_2+t_4)}+a^\dagger a^\dagger a^\dagger ae^{-i\omega t_4}e^{i\omega(t_1+t_2+t_3)}\\
&+a^\dagger a^\dagger a^\dagger a^\dagger e^{i\omega(t_1+t_2+t_3+t_4)}\Big]|0\rangle,\quad \text{red terms vanish but blue terms can survive,} && (102)\\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[aaa^\dagger a^\dagger e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}+aa^\dagger aa^\dagger e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}\Big]|0\rangle && (103)
\end{aligned}
$$

$$
\begin{aligned}
\langle 0|aaa^\dagger a^\dagger|0\rangle =& \langle 0|a(1+a^\dagger a)a^\dagger|0\rangle = \langle 0|aa^\dagger|0\rangle + \langle 0|aa^\dagger aa^\dagger|0\rangle && (104)\\
=&\langle 0|(1+a^\dagger a)|0\rangle + \langle 0|(1+a^\dagger a)(1+a^\dagger a)|0\rangle = 2 && (105)\\
\langle 0|aa^\dagger aa^\dagger|0\rangle =& \langle 0|(1+a^\dagger a)(1+a^\dagger a)|0\rangle = 1, && (106)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\langle 0|\mathrm{T}Q\left(t_1\right)Q\left(t_2\right)Q\left(t_3\right)Q\left(t_4\right)|0\rangle \\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\left[2e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}+e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}\right]|0\rangle && (107)\\
=&\frac{\hbar^2}{4m^2\omega^2}\left[\langle 0|\mathrm{T}e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}|0\rangle\right] && (108)\\
=&\frac{\hbar^2}{4m^2\omega^2}\left[\langle 0|\mathrm{T}e^{-i\omega(t_1-t_3)}e^{-i\omega(t_2-t_4)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1-t_4)}e^{-i\omega(t_2-t_3)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1-t_2)}e^{-i\omega(t_3-t_4)}|0\rangle\right] && (109)\\
=&\frac{1}{4\omega^2}\Big[\Big(\theta(t_1-t_3)e^{-i\omega(t_1-t_3)}+\theta(t_3-t_1)e^{-i\omega(t_3-t_1)}\Big)\Big(\theta(t_2-t_4)e^{-i\omega(t_2-t_4)}+\theta(t_4-t_2)e^{-i\omega(t_4-t_2)}\Big)\\
&+\Big(\theta(t_1-t_4)e^{-i\omega(t_1-t_4)}+\theta(t_4-t_1)e^{-i\omega(t_4-t_1)}\Big)\Big(\theta(t_2-t_3)e^{-i\omega(t_2-t_3)}+\theta(t_3-t_2)e^{-i\omega(t_3-t_2)}\Big)\\
&+\Big(\theta(t_1-t_2)e^{-i\omega(t_1-t_2)}+\theta(t_2-t_1)e^{-i\omega(t_2-t_1)}\Big)\Big(\theta(t_3-t_4)e^{-i\omega(t_3-t_4)}+\theta(t_4-t_3)e^{-i\omega(t_4-t_3)}\Big)\Big], && (110)\\
&\quad\text{by setting } \hbar=m=1,\\
=&\frac{1}{i^2}\Big[G(t_1-t_3)G(t_2-t_4)+G(t_1-t_4)G(t_2-t_3)+G(t_1-t_2)G(t_3-t_4)\Big] && (111)
\end{aligned}
$$

where we have used the definition of $G(t)$ in eq. (7.14). This verifies eq. (7.17). □

Question 4

Problem 7.4

Consider a harmonic oscillator in its ground state at $t = -\infty$. It is then subjected to an external force $f(t)$. Compute the probability $|\langle 0|0\rangle_f|^2$ that the oscillator is still in its ground state at $t = +\infty$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\tilde{f}(E) = \int_{-\infty}^{+\infty} e^{iEt} f(t)$. Your answer should not involve any other unevaluated integrals.

Answer