University of Minnesota
School of Physics and Astronomy

2026 Spring Physics 8502
General Relativity II

Assignment Solution

Lecture Instructor: Professor Keith Olive

Zong-En Chen
chen9613@umn.edu

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Problem Set 1 due on Due Feb 9 at 11:59pm

Question 1

(a) Consider the outer surface of the ergosphere for a Kerr black hole. Construct the normal vector to this surface and show that it is a 2-way surface.

(b) Using the condition $n_\alpha n^\alpha = 0$ for a null surface, derive the equation equation for the critical 1-way surface for a time-independent axial symmetric surface. That is start with some $u(r,\theta)$ to derive the equation defining this surface. What is its physical interpretation.

Answer

(a)

Starting with the metric for a Kerr black hole in Boyer-Lindquist coordinates,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \tag{1}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. The outer surface of the ergosphere is defined by the condition $g_{tt} = 0$, which gives

$$1 - \frac{2Mr}{\Sigma} = 0 \implies r^2 - 2Mr + a^2\cos^2\theta = 0. \tag{2}$$

We can define a surface function $f(r,\theta) = r^2 - 2Mr + a^2\cos^2\theta = 0$. The normal vector to this surface is given by the gradient of f:

$$n_\mu = \partial_\mu f = \left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}\right) = (0, 2r - 2M, -2a^2\cos\theta\sin\theta, 0). \tag{3}$$

To show that this is a 2-way surface, we need to compute the norm of the normal vector:

$$n_\mu n^\mu = g^{\mu\nu} n_\mu n_\nu. \tag{4}$$

We try to write out the matrix form of $g_{\mu\nu}$:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2Mr}{\Sigma}\right) & 0 & 0 & -\frac{2Mar\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2Mar\sin^2\theta}{\Sigma} & 0 & 0 & \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix}. \tag{5}$$

Since we are only interested in the rr and $\theta\theta$ components for the norm calculation, we can focus on those:

$$n_\mu n^\mu = g^{rr} n_r n_r + g^{\theta\theta} n_\theta n_\theta = g^{rr}(2r - 2M)^2 + g^{\theta\theta}(-2a^2 \cos\theta \sin\theta)^2. \tag{6}$$

Calculating g^{rr} and $g^{\theta\theta}$ from the inverse metric, we find:

$$g^{rr} = \frac{\Delta}{\Sigma}, \quad g^{\theta\theta} = \frac{1}{\Sigma}. \tag{7}$$

Substituting these back into the norm expression:

$$\begin{aligned} n_\mu n^\mu =& \frac{\Delta}{\Sigma}(2r - 2M)^2 + \frac{1}{\Sigma}(-2a^2 \cos\theta \sin\theta)^2 & (8)\\ =& \frac{4}{\Sigma}\left[\Delta(r - M)^2 + a^4 \cos^2\theta \sin^2\theta\right] & (9)\\ =& \frac{4}{\Sigma}\left[(r^2 - 2Mr + a^2)(r - M)^2 + a^4 \cos^2\theta \sin^2\theta\right]. & (10) \end{aligned}$$

We have $r^2 - 2Mr = -a^2\cos^2\theta$, and $(r - M)^2 = r^2 - 2Mr + M^2 = -a^2\cos^2\theta + M^2$ on the ergosphere, so substituting this in:

$$n_\mu n^\mu = \frac{4}{\Sigma}\left[a^2 \sin^2\theta (r - M)^2 + a^4 \cos^2\theta \sin^2\theta\right] = \frac{4a^2 \sin^2\theta}{\Sigma}\left[(r - M)^2 + a^2 \cos^2\theta\right]. \tag{11}$$

Since for a Kerr black hole $a \neq 0$ and $\sin\theta \neq 0$ except at the poles, we have $n_\mu n^\mu > 0$ almost everywhere on the ergosphere, indicating that the normal vector is spacelike. Therefore, the ergosphere is a 2-way surface.

(b)

For a time-independent axially symmetric surface, we can define the surface function as $u(r, \theta) = 0$. The normal vector to this surface is given by:

$$n_\mu = \partial_\mu u = \left(0, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, 0\right). \tag{12}$$

Using the condition for a null surface, $n_\mu n^\mu = 0$, we have:

$$n_\mu n^\mu = g^{rr}\left(\frac{\partial u}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial u}{\partial \theta}\right)^2 = 0. \tag{13}$$

If we plug the $g^{rr} = \frac{\Delta}{\Sigma}$ and $g^{\theta\theta} = \frac{1}{\Sigma}$ from the Kerr metric, we get:

$$\begin{aligned} \frac{\Delta}{\Sigma}\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{\Sigma}\left(\frac{\partial u}{\partial \theta}\right)^2 = 0 & \quad (14)\\ \implies \Delta\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 = 0. & \quad (15) \end{aligned}$$

If we want this equation to hold, we can have following condition:

1. $\Delta = 0$ and $\frac{\partial u}{\partial \theta} = 0$. This corresponds to the event horizon of the Kerr black hole, which is a 1-way surface.

2. $\frac{\partial u}{\partial r} = 0$ and $\frac{\partial u}{\partial \theta} = 0$. This would imply that u is constant, which is not a valid surface definition.

Thus, the critical 1-way surface is defined by $\Delta = 0$ and $\frac{\partial u}{\partial \theta} = 0$. That is, $u(r, \theta) = u(r)$ only, and the surface is located at the event horizon of the Kerr black hole. The condition $\Delta = 0$ corresponds to the event horizon, which is a null surface that allows one-way passage of information. $\square$

Question 2

Show that

$$(L_X T)^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} = (\nabla_X T)^{a_1 a_2 \ldots a_r}_{b_1 b_2 \ldots b_s} - \sum_i T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^{a_i}_{;j} + \sum_i T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^j_{;b_i} \tag{16}$$

Answer

In the course, we have already know the definition of the Lie derivative of a tensor field T along a vector field X:

$$(L_X T)^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} = X^c \partial_c T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} - \sum_i T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^{a_i}_{,j} + \sum_i T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^j_{,b_i}. \tag{17}$$

On the other hand, the covariant derivative of T along X is given by:

$$(\nabla_X T)^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} = X^c \partial_c T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} + \sum_i \Gamma^{a_i}_{cj} T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^c - \sum_i \Gamma^j_{cb_i} T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^c. \tag{18}$$

Subtracting the two expressions, we get:

$$(L_X T)^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} - (\nabla_X T)^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} = -\sum_i T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^{a_i}_{,j} + \sum_i T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^j_{,b_i} \tag{19}$$

$$- \sum_i \Gamma^{a_i}_{cj} T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^c + \sum_i \Gamma^j_{cb_i} T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^c \tag{20}$$

$$= -\sum_i T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^{a_i}_{;j} + \sum_i T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^j_{;b_i}. \tag{21}$$

We have used the definition of the covariant derivative of a vector field, $X^a_{;b} = X^a_{,b} + \Gamma^a_{cb} X^c$, to rewrite the terms involving the Christoffel symbols. Thus, we have shown that:

$$(L_X T)^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} = (\nabla_X T)^{a_1 a_2 \ldots a_r}_{b_1 b_2 \ldots b_s} - \sum_i T^{a_1 \ldots j \ldots a_r}_{b_1 b_2 \ldots b_s} X^{a_i}_{;j} + \sum_i T^{a_1 a_2 \ldots a_r}_{b_1 \ldots j \ldots b_s} X^j_{;b_i}. \tag{22}$$

□

Question 3

Consider an arbitrary unit vector, X transported along a latitude line on the surface of the sphere. Use $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$, and recall that there are only 2 distinct Γ's.

(a) Show the behavior of the angle between that vector and the tangent vector, T , which generates the isometry (latitude line) as that vector is parallel transported along the latitude line.

(b) Do the same assuming the vector is Lie transported. That is instead of $\nabla_T X = 0$, assume that $L_T X = 0$.

Answer

Before we start, let's write down the non-zero Christoffel symbols for the metric on the surface of the sphere:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta. \tag{23}$$

(a)

The metric on the surface of the sphere is given by $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. The tangent vector to the latitude line is $T = \frac{\partial}{\partial\phi}$, which generates the isometry. The unit vector X can be expressed in terms of the coordinate basis as $X = X^\theta \frac{\partial}{\partial\theta} + X^\phi \frac{\partial}{\partial\phi}$. The angle α between X and T is given by:

$$\cos\alpha = \frac{g(X,T)}{\sqrt{g(X,X)g(T,T)}}. \tag{24}$$

Since $T = \frac{\partial}{\partial\phi}$, we have $g(T,T) = g_{\phi\phi} = \sin^2\theta$. The inner product $g(X,T)$ is given by:

$$g(X,T) = g_{\phi\phi}X^\phi = \sin^2\theta X^\phi. \tag{25}$$

The norm of X is given by:

$$g(X,X) = g_{\theta\theta}(X^\theta)^2 + g_{\phi\phi}(X^\phi)^2 = (X^\theta)^2 + \sin^2\theta(X^\phi)^2. \tag{26}$$

For convenience, we set the length of X to be 1,Thus, the cosine of the angle is:

$$\cos\alpha = \frac{\sin^2\theta X^\phi}{\sqrt{((X^\theta)^2 + \sin^2\theta(X^\phi)^2)\sin^2\theta}} = \frac{\sin\theta X^\phi}{\sqrt{(X^\theta)^2 + \sin^2\theta(X^\phi)^2}} \tag{27}$$

$$= \frac{\sin\theta X^\phi}{\sqrt{1}} = \sin\theta X^\phi. \tag{28}$$

To find how the angle changes as X is parallel transported along the latitude line, we need to solve the parallel transport equation $\nabla_T X = 0$. This gives us:

$$\nabla_T X^\theta = T^\phi \partial_\phi X^\theta + \Gamma^\theta_{\phi\phi} T^\phi X^\phi = 0, \tag{29}$$

$$\nabla_T X^\phi = T^\phi \partial_\phi X^\phi + \Gamma^\phi_{\phi\theta} T^\phi X^\theta + \Gamma^\phi_{\phi\phi} T^\phi X^\phi = 0. \tag{30}$$

Substituting $T^\phi = 1$ and the Christoffel symbols, we get:

$$\partial_\phi X^\theta - \sin\theta\cos\theta X^\phi = 0, \tag{31}$$

$$\partial_\phi X^\phi + \cot\theta X^\theta = 0. \tag{32}$$

Now we can try to solve these coupled differential equations. From the first equation, we can express X^ϕ in terms of X^θ:

$$X^\phi = \frac{1}{\sin\theta\cos\theta} \partial_\phi X^\theta. \tag{33}$$

Substituting this into the second equation gives us a second-order differential equation for X^θ:

$$\partial_\phi \left(\frac{1}{\sin\theta\cos\theta} \partial_\phi X^\theta \right) + \cot\theta X^\theta = 0 \tag{34}$$

$$\implies \frac{1}{\sin\theta\cos\theta} \partial_\phi^2 X^\theta + \cot\theta X^\theta = 0 \tag{35}$$

$$\implies \partial_\phi^2 X^\theta + \cot\theta \sin\theta\cos\theta X^\theta = 0 \tag{36}$$

$$\implies \partial_\phi^2 X^\theta + \cos^2\theta X^\theta = 0. \tag{37}$$

The general solution to this equation is:

$$X^\theta = A\cos((\cos\theta)\phi) + B\sin((\cos\theta)\phi) = C\sin((\cos\theta)\phi + \delta), \tag{38}$$

where A, B, C, and δ are constants determined by the initial conditions. Substituting this back into the expression for X^ϕ, we get:

$$X^\phi = \frac{1}{\sin\theta\cos\theta} \partial_\phi X^\theta = \frac{C}{\sin\theta} \cos((\cos\theta)\phi + \delta). \tag{39}$$

Now we can compute the angle α as a function of ϕ:

$$\cos\alpha(\phi) = \sin\theta X^\phi = C\cos((\cos\theta)\phi + \delta). \tag{40}$$

Thus, the angle between the vector X and the tangent vector T oscillates as X is parallel transported along the latitude line, with a frequency determined by $\cos\theta$.

(b)

Now we assume that the vector X is Lie transported along the latitude line, which means that $L_T X = 0$. The Lie derivative of X along T is given by:

$$(L_T X)^\theta = T^\phi \partial_\phi X^\theta - X^\phi \partial_\phi T^\theta = \partial_\phi X^\theta, \tag{41}$$

$$(L_T X)^\phi = T^\phi \partial_\phi X^\phi - X^\theta \partial_\phi T^\phi = \partial_\phi X^\phi. \tag{42}$$

Setting $L_T X = 0$ gives us:

$$\partial_\phi X^\theta = 0, \tag{43}$$

$$\partial_\phi X^\phi = 0. \tag{44}$$

This means that both X^θ and X^ϕ are constants along the latitude line. Therefore, the angle α between X and T is also constant:

$$\cos\alpha = \sin\theta X^\phi = \text{constant}. \tag{45}$$

Thus, when the vector X is Lie transported along the latitude line, the angle between X and the tangent vector T remains constant, in contrast to the oscillatory behavior observed in the case of parallel transport. □