

University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I
Assignment Solution

Lecture Instructor: Professor Zhen Liu

Zong-En Chen
chen9613@umn.edu

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Question 1

Problem 48.5

The charged pion π^- is represented by a complex scalar field φ , the muon μ^- by a Dirac field \mathcal{M} , and the muon neutrino ν_μ by a spin-projected Dirac field $P_L \mathcal{N}$, where $P_L = \frac{1}{2}(1 - \gamma_5)$. The charged pion can decay to a muon and a muon antineutrino via the interaction

$$\mathcal{L}_1 = 2c_1 G_F f_\pi \partial_\mu \varphi \overline{\mathcal{M}} \gamma^\mu P_L \mathcal{N} + h.c., \quad (1)$$

where c_1 is the cosine of the *Cabibbo angle*, G_F is the *Fermi constant*, and f_π is the *pion decay constant*.

- (a) Compute the charged pion decay rate Γ .
- (b) The charged pion mass is $m_\pi = 139.6$ MeV, the muon mass is $m_\mu = 105.7$ MeV, and the muon neutrino mass is massless. The Fermi constant is $G_F = 1.166 \times 10^{-5}$ GeV $^{-2}$, and the cosine of the Cabibbo angle is measured in nuclear beta decays to be $c_1 = 0.974$. The measured value of the charged pion life time is $\tau = 2.6033 \times 10^{-8}$ s. Determine the value of f_π in MeV. Your result is too large by 0.8%, due to neglect of electromagnetic loop corrections.
- (c) The previous parts assume π^- always decay into $\mu^- \bar{\nu}_\mu$, but actually π^- can also decay into $e^- \bar{\nu}_e$. The charged pion, electron by a Dirac field \mathcal{M}_e , and the electron neutrino by a spin-projected Dirac field $P_L \mathcal{N}_e$ have the form of interaction

$$\mathcal{L}_2 = 2c_2 G_F f_\pi \partial_\mu \varphi \overline{\mathcal{M}}_e \gamma^\mu P_L \mathcal{N}_e + h.c. \quad (2)$$

Given the decay branching ratio of $\pi^- \rightarrow e^- \bar{\nu}_e$ is 1.230×10^{-4} , the decay branching ratio of $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ is 99.9877%. Find the value of c_2 . For example, the electronic decay branching ratio is

$$\text{Br}(\pi^- \rightarrow e^- \bar{\nu}_e) = \frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu) + \Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}. \quad (3)$$

The coupling of pion-electron is similar with the coupling of pion-muon, why pion favoring decay into muon instead of electron? ($m_e = 0.511$ MeV.)

Answer

- (a)

First, we analyze the decay $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$. The Feynman diagram is shown in Fig. 1a. Now, we can write down



Figure 1: Feynman diagram for π^- decay into (a) muon and muon antineutrino; (b) electron and electron antineutrino.

the amplitude:

$$i\mathcal{T} = 2ic_1 G_F f_\pi k^\mu \bar{u}_{s_1}(p_1) \gamma_\mu P_L v_{s_2}(p_2) \quad (4)$$

$$= 2ic_1 G_F f_\pi \bar{u}_{s_1}(p_1) \not{k} P_L v_{s_2}(p_2) \quad (5)$$

$$= ic_1 G_F f_\pi \bar{u}_{s_1}(p_1) \not{k} (1 - \gamma_5) v_{s_2}(p_2), \quad (6)$$

where k^μ , p_1^μ , and p_2^μ are the four-momenta of π^- , μ^- , and $\bar{\nu}_\mu$, respectively. s_1 and s_2 are the spin indices of μ^- and $\bar{\nu}_\mu$. We can write $k^\mu = p_1^\mu + p_2^\mu$ due to momentum conservation. Thus, the amplitude can be further simplified as:

$$i\mathcal{T} = ic_1 G_F f_\pi \bar{u}_{s_1}(p_1) (\not{p}_1 + \not{p}_2)(1 - \gamma_5) v_{s_2}(p_2) \quad (7)$$

$$= ic_1 G_F f_\pi \left[\bar{u}_{s_1}(p_1) \not{p}_1 (1 - \gamma_5) v_{s_2}(p_2) + \bar{u}_{s_1}(p_1) \not{p}_2 (1 - \gamma_5) v_{s_2}(p_2) \right] \quad (8)$$

$$= ic_1 G_F f_\pi \left[(-m) \bar{u}_{s_1}(p_1) (1 - \gamma_5) v_{s_2}(p_2) + 0 \right] \quad (9)$$

where we have used the Dirac equation $\bar{u}_{s_1}(p_1)(\not{p}_1 + m) = 0$ and the massless neutrino condition $\not{p}_2 v_{s_2}(p_2) = 0$. Therefore, the amplitude becomes:

$$i\mathcal{T} = -ic_1 G_F f_\pi m \bar{u}_{s_1}(p_1) (1 - \gamma_5) v_{s_2}(p_2). \quad (10)$$

Next, we can write down the Hermitian conjugate of the amplitude:

$$-i\mathcal{T}^* = ic_1 G_F f_\pi m \left[\bar{u}_{s_1}(p_1) (1 - \gamma_5) v_{s_2}(p_2) \right]^\dagger \quad (11)$$

$$= ic_1 G_F f_\pi m \left[v_{s_2}^\dagger(p_2) (1 - \gamma_5)^\dagger \bar{u}_{s_1}^\dagger(p_1) \right] \quad (12)$$

$$= ic_1 G_F f_\pi m \left[v_{s_2}^\dagger(p_2) \gamma^0 (1 - \gamma_5)^\dagger \gamma^0 u_{s_1}(p_1) \right] \quad (13)$$

$$= ic_1 G_F f_\pi m \left[\bar{v}_{s_2}(p_2) (1 + \gamma_5) u_{s_1}(p_1) \right], \quad (14)$$

where we have used the relation $\bar{u} = u^\dagger \gamma^0$ and the Hermitian property of γ_5 (that is, $\gamma_5^\dagger = \gamma_5$) to get the last

line. Now, we can compute the squared amplitude averaged over initial spins and summed over final spins:

$$\langle |\mathcal{T}|^2 \rangle = \frac{1}{1} \sum_{s_1, s_2} \mathcal{T} \mathcal{T}^* \quad (15)$$

$$= c_1^2 G_F^2 f_\pi^2 m^2 \sum_{s_1, s_2} \left[\bar{u}_{s_1}(p_1)(1 - \gamma_5)v_{s_2}(p_2) \right] \left[\bar{v}_{s_2}(p_2)(1 + \gamma_5)u_{s_1}(p_1) \right] \quad (16)$$

$$= c_1^2 G_F^2 f_\pi^2 m^2 \sum_{s_1, s_2} \text{Tr} \left[(1 + \gamma_5)u_{s_1}(p_1)\bar{u}_{s_1}(p_1)(1 - \gamma_5)v_{s_2}(p_2)\bar{v}_{s_2}(p_2) \right] \quad (17)$$

$$= c_1^2 G_F^2 f_\pi^2 m^2 \text{Tr} \left[(1 + \gamma_5)(-\not{p}_1 + m)(1 - \gamma_5)(-\not{p}_2) \right] \quad (18)$$

where we have used the completeness relations for spinors and the trace properties of gamma matrices:

$$\sum_s u_s(p)\bar{u}_s(p) = -\not{p} + m, \quad (19)$$

$$\sum_s v_s(p)\bar{v}_s(p) = -\not{p} - m, \quad (20)$$

$$\text{Tr}(\not{a}\not{b}) = -4(a \cdot b), \quad (21)$$

$$\text{Tr}(\text{odd number of } \gamma\text{-matrices}) = 0, \quad (22)$$

$$\gamma_5^2 = 1. \quad (23)$$

We can expand the trace:

$$(1 + \gamma_5)(-\not{p}_1 + m)(1 - \gamma_5)(-\not{p}_2) = (1 + \gamma_5)(-\not{p}_1)(1 - \gamma_5)(-\not{p}_2) + (1 + \gamma_5)m(1 - \gamma_5)(-\not{p}_2) \quad (24)$$

$$= (1 + \gamma_5)(-\not{p}_1)(1 - \gamma_5)(-\not{p}_2) + 0 \quad (25)$$

$$= (1 + \gamma_5)(\not{p}_1)(1 - \gamma_5)(\not{p}_2) \quad (26)$$

$$= (\not{p}_1)(\not{p}_2) + (\not{p}_1)(-\gamma_5)(\not{p}_2) + \gamma_5(\not{p}_1)(\not{p}_2) + \gamma_5(\not{p}_1)(-\gamma_5)(\not{p}_2) \quad (27)$$

$$= 2(\not{p}_1)(\not{p}_2) + 2\gamma_5(\not{p}_1)(\not{p}_2) \quad (28)$$

where we have used the anticommutation relation $\{\gamma_5, \gamma^\mu\} = 0$ to get the last line. Therefore, we have:

$$\langle |\mathcal{T}|^2 \rangle = c_1^2 G_F^2 f_\pi^2 m^2 \text{Tr} \left[2(\not{p}_1)(\not{p}_2) + 2\gamma_5(\not{p}_1)(\not{p}_2) \right] \quad (29)$$

$$= 2c_1^2 G_F^2 f_\pi^2 m^2 \left[\text{Tr}(\not{p}_1\not{p}_2) + \text{Tr}(\gamma_5\not{p}_1\not{p}_2) \right] \quad (30)$$

$$= 2c_1^2 G_F^2 f_\pi^2 m^2 \left[-4(p_1 \cdot p_2) + 0 \right] \quad (31)$$

$$= -8c_1^2 G_F^2 f_\pi^2 m^2 (p_1 \cdot p_2) \quad (32)$$

where we have used the trace properties of gamma matrices again. In the rest frame of π^- , we have:

$$k^2 = -m_\pi^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2(p_1 \cdot p_2) = -m^2 + 0 + 2(p_1 \cdot p_2) \quad (33)$$

$$\Rightarrow -(p_1 \cdot p_2) = \frac{m_\pi^2 - m^2}{2} \quad (34)$$

Thus, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = 4c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2) \quad (35)$$

Note that the m is the muon mass. Finally, we can compute the decay rate:

$$\Gamma = \frac{1}{2m_\pi} \int d\Phi_2 \langle |\mathcal{T}|^2 \rangle, \quad (36)$$

where the two-body phase space integral is:

$$\int d\Phi_2 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(k - p_1 - p_2) \quad (37)$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \delta^3(\mathbf{0} - \mathbf{p}_1 - \mathbf{p}_2) \quad (38)$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \quad (39)$$

$$= \int \frac{4\pi p_1^2 dp_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \quad (40)$$

$$= \int \frac{4\pi p_1^2 dp_1}{(2\pi)^2 4E_1 E_2} \delta(m_\pi - E_1 - E_2) \quad (41)$$

where we have used the delta function to perform the \mathbf{p}_2 integral. In the rest frame of π^- , we have $\mathbf{p}_2 = -\mathbf{p}_1$ and $E_2 = |\mathbf{p}_2| = |\mathbf{p}_1|$. Thus, we can write $E_1 + E_2 - m_\pi = \sqrt{p_1^2 + m^2} + p_1 - m_\pi$. The root of the equation $E_1 + E_2 - m_\pi = 0$ is:

$$p_1 = \frac{m_\pi^2 - m^2}{2m_\pi} \quad (42)$$

Also, we can compute the derivative:

$$\frac{d}{dp_1} (E_1 + E_2 - m_\pi) = \frac{p_1}{\sqrt{p_1^2 + m^2}} + 1 = \frac{E_1 + E_2}{E_1} = \frac{m_\pi}{E_1} \quad (43)$$

Therefore, the phase space integral becomes:

$$\int d\Phi_2 = \frac{4\pi p_1^2}{(2\pi)^2 4E_1 E_2} \frac{E_1}{m_\pi} \quad (44)$$

$$= \frac{p_1^2}{4\pi m_\pi E_2} = \frac{p_1}{4\pi m_\pi} \quad (45)$$

$$= \frac{m_\pi^2 - m^2}{8\pi m_\pi^2} \quad (46)$$

Finally, the decay rate is:

$$\Gamma_{\pi^- \rightarrow \mu \bar{\nu}_\mu} = \frac{1}{2m_\pi} \langle |\mathcal{T}|^2 \rangle \int d\Phi_2 \quad (47)$$

$$= \frac{1}{2m_\pi} \left[4c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2) \right] \left[\frac{m_\pi^2 - m^2}{8\pi m_\pi^2} \right] \quad (48)$$

$$= \frac{c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2)^2}{4\pi m_\pi^3} \quad (49)$$

$$= \frac{c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{4\pi m_\pi^3} \quad (50)$$

(b)

The charged pion life time is related to the decay rate by $\tau = 1/\Gamma$. Note that 2.6033×10^{-8} s $\approx 3.955 \times 10^{16}$ MeV $^{-1}$. Thus, we can solve for f_π :

$$f_\pi = \sqrt{\frac{4\pi m_\pi^3}{c_1^2 G_F^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2 \tau}} \quad (51)$$

$$= \sqrt{\frac{4\pi (139.6 \text{ MeV})^3}{(0.974)^2 (1.166 \times 10^{-5} \text{ GeV}^{-2})^2 (105.7 \text{ MeV})^2 ((139.6 \text{ MeV})^2 - (105.7 \text{ MeV})^2)^2 (3.955 \times 10^{16} \text{ MeV}^{-1})}} \quad (52)$$

$$\approx 0.09314 \text{ GeV} = 93.14 \text{ MeV}. \quad (53)$$

(c)

Now we analyze the decay $\pi^- \rightarrow e^- \bar{\nu}_e$. The Feynman diagram is shown in Fig. 1b. By following the same procedure as in part (a), we can write down the decay rate:

$$\Gamma_{\pi^- \rightarrow e \bar{\nu}_e} = \frac{c_2^2 G_F^2 f_\pi^2 m_e^2 (m_\pi^2 - m_e^2)^2}{4\pi m_\pi^3}. \quad (54)$$

Given the branching ratio $\text{Br}(\pi^- \rightarrow e^-\bar{\nu}_e) = 1.230 \times 10^{-4}$, we have:

$$\text{Br}(\pi^- \rightarrow e^-\bar{\nu}_e) = \frac{\Gamma(\pi^- \rightarrow e^-\bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^-\bar{\nu}_\mu) + \Gamma(\pi^- \rightarrow e^-\bar{\nu}_e)} \quad (55)$$

$$\approx \frac{\Gamma(\pi^- \rightarrow e^-\bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^-\bar{\nu}_\mu)} \quad (56)$$

$$= \frac{c_2^2 m_e^2 (m_\pi^2 - m_e^2)^2}{c_1^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2} \quad (57)$$

where we have used the fact that $\Gamma(\pi^- \rightarrow e^-\bar{\nu}_e) \ll \Gamma(\pi^- \rightarrow \mu^-\bar{\nu}_\mu)$ to get the second line. Therefore, we can solve for c_2 :

$$c_2 = c_1 \sqrt{\text{Br}(\pi^- \rightarrow e^-\bar{\nu}_e) \frac{m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{m_e^2 (m_\pi^2 - m_e^2)^2}} \quad (58)$$

$$= 0.974 \sqrt{(1.230 \times 10^{-4}) \frac{(105.7 \text{ MeV})^2 ((139.6 \text{ MeV})^2 - (105.7 \text{ MeV})^2)^2}{(0.511 \text{ MeV})^2 ((139.6 \text{ MeV})^2 - (0.511 \text{ MeV})^2)^2}} \quad (59)$$

$$\approx 0.95345. \quad (60)$$

The most obvious reason that pion favoring decay into muon instead of electron is that the muon mass is much larger than the electron mass. Since the $\Gamma \propto m_l^2$ ($l = e, \mu$), the decay rate into muon is greatly enhanced compared to that into electron.

Remark: This phenomenon is known as *helicity suppression*. But I think the explanation above is sufficient for this problem. \square

Question 2

Consider QED with both electron and muon:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{l=e,\mu} (i\bar{\Psi}_l \not{d}\Psi_l - m_l \bar{\Psi}_l \Psi_l + \frac{g}{2} \bar{\Psi}_l \gamma^\mu \Psi_l A_\mu), \quad (61)$$

where both Ψ_e and Ψ_μ are Dirac fields. Compute the $\langle |\mathcal{T}|^2 \rangle$ for $e^+e^- \rightarrow \mu^+\mu^-$. Then, compute its cross section σ . Eq. (11.22) and Eq. (11.30) should be useful.

Answer

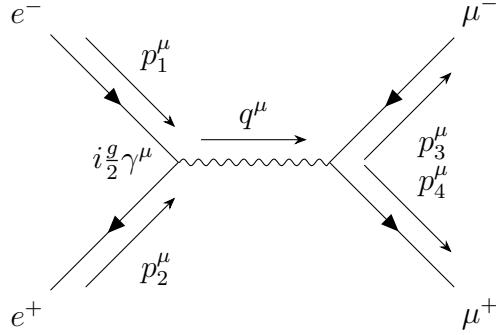


Figure 2: Feynman diagram for $e^+e^- \rightarrow \mu^+\mu^-$.

The Feynman diagram for $e^+e^- \rightarrow \mu^+\mu^-$ is shown in Fig. 2. We can write down the amplitude:

$$i\mathcal{T} = \bar{v}_{s_2}(p_2)(i\frac{g}{2}\gamma^\mu)u_{s_1}(p_1)\frac{-ig_{\mu\nu}}{q^2}\bar{u}_{s_3}(p_3)(i\frac{g}{2}\gamma^\nu)v_{s_4}(p_4) \quad (62)$$

$$= i\frac{g^2}{4q^2} \left[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1) \right] \left[\bar{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4) \right], \quad (63)$$

where p_1^μ , p_2^μ , p_3^μ , and p_4^μ are the four-momenta of e^- , e^+ , μ^- , and μ^+ , respectively. s_1 , s_2 , s_3 , and s_4 are the spin indices of e^- , e^+ , μ^- , and μ^+ , respectively. Also, we have defined $q^\mu = p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$. Next, we can write down the Hermitian conjugate of the amplitude:

$$-i\mathcal{T}^* = -i\frac{g^2}{4q^2} \left[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1) \right]^\dagger \left[\bar{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4) \right]^\dagger \quad (64)$$

$$= -i\frac{g^2}{4q^2} \left[u_{s_1}^\dagger(p_1)\gamma^\mu (\gamma^0)^\dagger v_{s_2}(p_2) \right] \left[v_{s_4}^\dagger(p_4)\gamma_\mu (\gamma^0)^\dagger u_{s_3}(p_3) \right] \quad (65)$$

$$= -i\frac{g^2}{4q^2} \left[\bar{u}_{s_1}(p_1)\gamma^\mu v_{s_2}(p_2) \right] \left[\bar{v}_{s_4}(p_4)\gamma_\mu u_{s_3}(p_3) \right]. \quad (66)$$

Therefore, we can compute the squared amplitude averaged over initial spins and summed over final spins:

$$\langle |\mathcal{T}|^2 \rangle = \frac{1}{4} \sum_{s_1, s_2, s_3, s_4} \mathcal{T} \mathcal{T}^* \quad (67)$$

$$= \frac{g^4}{64q^4} \sum_{s_1, s_2, s_3, s_4} \left[\bar{v}_{s_2}(p_2) \gamma^\mu u_{s_1}(p_1) \right] \left[\bar{u}_{s_3}(p_3) \gamma_\mu v_{s_4}(p_4) \right] \left[\bar{u}_{s_1}(p_1) \gamma^\nu v_{s_2}(p_2) \right] \left[\bar{v}_{s_4}(p_4) \gamma_\nu u_{s_3}(p_3) \right] \quad (68)$$

$$= \frac{g^4}{64q^4} \sum_{s_1, s_2, s_3, s_4} \text{Tr} \left[\gamma^\mu u_{s_1}(p_1) \bar{u}_{s_1}(p_1) \gamma^\nu v_{s_2}(p_2) \bar{v}_{s_2}(p_2) \right] \text{Tr} \left[\gamma_\mu u_{s_3}(p_3) \bar{u}_{s_3}(p_3) \gamma_\nu v_{s_4}(p_4) \bar{v}_{s_4}(p_4) \right] \quad (69)$$

$$= \frac{g^4}{64q^4} \text{Tr} \left[\gamma^\mu (-\not{p}_1 + m_e) \gamma^\nu (-\not{p}_2 - m_e) \right] \text{Tr} \left[\gamma_\mu (-\not{p}_3 + m_\mu) \gamma_\nu (-\not{p}_4 - m_\mu) \right], \quad (70)$$

where we have used the completeness relations for spinors and the trace properties of gamma matrices:

$$\sum_s u_s(p) \bar{u}_s(p) = -\not{p} + m, \quad (71)$$

$$\sum_s v_s(p) \bar{v}_s(p) = -\not{p} - m, \quad (72)$$

$$\text{Tr}(\not{a} \not{b}) = -4(a \cdot b), \quad (73)$$

$$\text{Tr}(\text{odd number of } \gamma\text{-matrices}) = 0 \quad (74)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = -4g^{\mu\nu} \quad (75)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \quad (76)$$

$$g_{\mu\nu} g^{\mu\nu} = 4 \quad (77)$$

We can expand the traces:

$$\text{Tr} \left[\gamma^\mu (-\not{p}_1 + m_e) \gamma^\nu (-\not{p}_2 - m_e) \right] = \text{Tr} \left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2 \right] - m_e^2 \text{Tr} \left[\gamma^\mu \gamma^\nu \right] \quad (78)$$

$$= (p_1)_\alpha (p_2)_\beta \text{Tr} \left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \right] + 4m_e^2 g^{\mu\nu} \quad (79)$$

$$= (p_1)_\alpha (p_2)_\beta 4(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha}) + 4m_e^2 g^{\mu\nu} \quad (80)$$

$$= 4 \left[p_1^\mu p_2^\nu - g^{\mu\nu} (p_1 \cdot p_2) + p_1^\nu p_2^\mu + m_e^2 g^{\mu\nu} \right] \quad (81)$$

$$= 4 \left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} (p_1 \cdot p_2 - m_e^2) \right], \quad (82)$$

and

$$\text{Tr} \left[\gamma_\mu (-\not{p}_3 + m_\mu) \gamma_\nu (-\not{p}_4 - m_\mu) \right] = \text{Tr} \left[\gamma_\mu \not{p}_3 \gamma_\nu \not{p}_4 \right] - m_\mu^2 \text{Tr} \left[\gamma_\mu \gamma_\nu \right] \quad (83)$$

$$= (p_3)^\rho (p_4)^\sigma \text{Tr} \left[\gamma_\mu \gamma_\rho \gamma_\nu \gamma_\sigma \right] + 4m_\mu^2 g_{\mu\nu} \quad (84)$$

$$= (p_3)^\rho (p_4)^\sigma 4(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho}) + 4m_\mu^2 g_{\mu\nu} \quad (85)$$

$$= 4 \left[p_{3\mu}p_{4\nu} - g_{\mu\nu}(p_3 \cdot p_4) + p_{3\nu}p_{4\mu} + m_\mu^2 g_{\mu\nu} \right] \quad (86)$$

$$= 4 \left[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - g_{\mu\nu}(p_3 \cdot p_4 - m_\mu^2) \right]. \quad (87)$$

Therefore, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{64q^4} 16 \left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1 \cdot p_2 - m_e^2) \right] \left[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - g_{\mu\nu}(p_3 \cdot p_4 - m_\mu^2) \right] \quad (88)$$

$$= \frac{g^4}{4q^4} \left[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4 - m_\mu^2) \right. \quad (89)$$

$$+ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4 - m_\mu^2) \quad (90)$$

$$- (p_1 \cdot p_2 - m_e^2)(p_3 \cdot p_4) - (p_1 \cdot p_2 - m_e^2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2 - m_e^2)(p_3 \cdot p_4 - m_\mu^2) \quad (91)$$

$$= \frac{g^4}{4q^4} \left[(\textcolor{blue}{p_1 p_3})(\textcolor{blue}{p_2 p_4}) + (\textcolor{red}{p_1 p_4})(\textcolor{blue}{p_2 p_3}) - (\textcolor{red}{p_1 p_2})(\textcolor{red}{p_3 p_4}) + (\textcolor{cyan}{p_1 p_2})m_\mu^2 \right. \quad (92)$$

$$+ (\textcolor{red}{p_1 p_4})(\textcolor{red}{p_2 p_3}) + (\textcolor{blue}{p_1 p_3})(\textcolor{blue}{p_2 p_4}) - (\textcolor{red}{p_1 p_2})(\textcolor{red}{p_3 p_4}) + (\textcolor{cyan}{p_1 p_2})m_\mu^2 \quad (93)$$

$$- (\textcolor{red}{p_1 p_2})(\textcolor{yellow}{p_3 p_4}) + m_e^2(\textcolor{yellow}{p_3 p_4}) - (\textcolor{red}{p_1 p_2})(\textcolor{red}{p_3 p_4}) + m_e^2(\textcolor{yellow}{p_3 p_4}) \quad (94)$$

$$+ 4(\textcolor{red}{p_1 p_2})(\textcolor{red}{p_3 p_4}) - 4m_\mu^2(\textcolor{cyan}{p_1 p_2}) - 4m_e^2(\textcolor{yellow}{p_3 p_4}) + 4m_e^2 m_\mu^2 \quad (95)$$

$$= \frac{g^4}{4q^4} \left[2(\textcolor{blue}{p_1 p_3})(\textcolor{blue}{p_2 p_4}) + 2(\textcolor{red}{p_1 p_4})(\textcolor{blue}{p_2 p_3}) - 2(\textcolor{cyan}{p_1 p_2})m_\mu^2 - 2m_e^2(\textcolor{yellow}{p_3 p_4}) + 4m_e^2 m_\mu^2 \right] \quad (96)$$

Next, we can compute the cross section in the center-of-mass frame. In this frame, we have:

$$p_1^\mu = (E, 0, 0, p), \quad (97)$$

$$p_2^\mu = (E, 0, 0, -p), \quad (98)$$

$$p_3^\mu = (E, p' \sin \theta, 0, p' \cos \theta), \quad (99)$$

$$p_4^\mu = (E, -p' \sin \theta, 0, -p' \cos \theta), \quad (100)$$

where $E = \sqrt{p^2 + m_e^2} = \sqrt{p'^2 + m_\mu^2} = \frac{\sqrt{s}}{2}$. We can compute the dot products:

$$(p_1 \cdot p_2) = -E^2 + (-p^2) = m_e^2 - 2E^2 \quad (101)$$

$$(p_3 \cdot p_4) = -E^2 + (-p'^2) = m_\mu^2 - 2E^2 \quad (102)$$

$$(p_1 \cdot p_3) = -E^2 + pp' \cos \theta \quad (103)$$

$$(p_2 \cdot p_4) = -E^2 + pp' \cos \theta \quad (104)$$

$$(p_1 \cdot p_4) = -E^2 - pp' \cos \theta \quad (105)$$

$$(p_2 \cdot p_3) = -E^2 - pp' \cos \theta \quad (106)$$

Therefore, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{4q^4} \left[2(-E^2 + pp' \cos \theta)^2 + 2(-E^2 - pp' \cos \theta)^2 - 2(m_e^2 - 2E^2)m_\mu^2 - 2m_e^2(m_\mu^2 - 2E^2) + 4m_e^2m_\mu^2 \right] \quad (107)$$

$$= \frac{g^4}{4q^4} \left[4(E^4 + p^2 p'^2 \cos^2 \theta) + 4E^2(m_e^2 + m_\mu^2) \right] \quad (108)$$

$$= \frac{g^4}{q^4} \left[E^4 + p^2 p'^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2) \right]. \quad (109)$$

Next, we can compute the cross section (by eq. (11.31) in the textbook):

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \langle |\mathcal{T}|^2 \rangle \quad (110)$$

$$= \frac{1}{64\pi^2 s} \frac{p'}{p} \frac{g^4}{q^4} \left[E^4 + p^2 p'^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2) \right] \quad (111)$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[E^4 + p^2 p'^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2) \right], \quad (112)$$

where we have used the fact that $q^2 = (p_1 + p_2)^2 = -s = -4E^2$. Finally, we can integrate over the solid angle to get the total cross section:

$$\int d\Omega = 4\pi, \quad (113)$$

$$\int d\Omega \cos^2 \theta = 2\pi \int_{-1}^1 d\cos \theta \cos^2 \theta = \frac{4\pi}{3}, \quad (114)$$

thus,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega \quad (115)$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[E^4 \int d\Omega + p^2 p'^2 \int d\Omega \cos^2 \theta + E^2 (m_e^2 + m_\mu^2) \int d\Omega \right] \quad (116)$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[4\pi E^4 + \frac{4\pi}{3} p^2 p'^2 + 4\pi E^2 (m_e^2 + m_\mu^2) \right] \quad (117)$$

$$= \frac{g^4}{16\pi s^3} \frac{p'}{p} \left[E^4 + \frac{1}{3} p^2 p'^2 + E^2 (m_e^2 + m_\mu^2) \right]. \quad (118)$$

Now, we can express p and p' in terms of s :

$$p = \sqrt{E^2 - m_e^2} = \sqrt{\frac{s}{4} - m_e^2}, \quad (119)$$

$$p' = \sqrt{E^2 - m_\mu^2} = \sqrt{\frac{s}{4} - m_\mu^2}, \quad (120)$$

$$E = \frac{\sqrt{s}}{2}. \quad (121)$$

Therefore, the final expression for the cross section is:

$$\sigma = \frac{g^4}{16\pi s^3} \frac{\sqrt{\frac{s}{4} - m_\mu^2}}{\sqrt{\frac{s}{4} - m_e^2}} \left[\left(\frac{s}{4}\right)^2 + \frac{1}{3} \left(\frac{s}{4} - m_e^2\right) \left(\frac{s}{4} - m_\mu^2\right) + \frac{s}{4} (m_e^2 + m_\mu^2) \right] \quad (122)$$

$$= \frac{g^4}{192\pi s^3} \frac{\sqrt{s - 4m_\mu^2}}{\sqrt{s - 4m_e^2}} \left[(s + 2m_e^2)(s + 2m_\mu^2) \right], \quad \text{in terms of } s, \quad (123)$$

$$= \frac{g^4}{3072\pi E^6} \frac{\sqrt{E^2 - m_\mu^2}}{\sqrt{E^2 - m_e^2}} \left[(2E^2 + m_e^2)(2E^2 + m_\mu^2) \right], \quad \text{in terms of } E. \quad (124)$$

□

Question 3

Consider classical field theory with two real scalar fields in (3+1)-dimension spacetime:

$$\mathcal{L}(x) = \sum_{a=1}^2 \left(-\frac{1}{2} \partial^\mu \phi_a \partial_\mu \phi_a \right) - V(x), \quad (125)$$

$$V(x) = - \sum_{a=1}^2 \left(\frac{1}{2} \mu^2 \phi_a \phi_a \right) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2, \quad (126)$$

where μ and λ are positive real constants.

- (a) Show that the Lagrangian has an $SO(2)$ transformation symmetry:

$$\phi_1(x) \rightarrow \phi'_1(x) = \phi_1(x) \cos \alpha_0 - \phi_2(x) \sin \alpha_0, \quad (127)$$

$$\phi_2(x) \rightarrow \phi'_2(x) = \phi_1(x) \sin \alpha_0 + \phi_2(x) \cos \alpha_0, \quad (128)$$

- (b) Find the conjugate momentum $\Pi_1(x)$, $\Pi_2(x)$ of $\phi_1(x)$, $\phi_2(x)$. Find the Hamiltonian density $\mathcal{H}(x)$ in the terms of $\phi_a(x)$, $\Pi_a(x)$, and $\partial_i \phi_a(x)$.

- (c) Find the ground state in the basis of $\{\phi_r(x), \phi_\theta(x)\}$ where

$$\phi_1(x) = \phi_r(x) \cos(\phi_\theta(x)), \quad (129)$$

$$\phi_2(x) = \phi_r(x) \sin(\phi_\theta(x)), \quad (130)$$

with $\phi_r(x) \geq 0$ and $\phi_\theta(x) \in [0, 2\pi]$. Is the Lagrangian \mathcal{L} invariant under a continuous shift symmetry of $\phi_\theta(x) \rightarrow \phi_\theta(x) + \alpha_0$?

Hint: In general, finding the ground state is to find $\phi(x)$ s.t. minimize $H = \int \mathcal{H}(x) d^3x$; but for this problem, finding $\phi(x)$ to minimize $\mathcal{H}(x)$ is the same. If you have trouble with the above procedure, given the Lagrangian of this problem, one can simply find $\phi(x)$ s.t. minimize $V(x)$, which is the same as minimizing \mathcal{H} for this problem.

- (d) Now let's study the system's dynamics around the ground state.

$\phi_r(x)$ should fluctuate around $\sqrt{\frac{\mu^2}{\lambda}}$: $\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x)$. $\phi_\theta(x)$ should fluctuate within $[0, 2\pi]$.

Show that $f_r(x)$ is a massive field and find its mass. Taking $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}} \phi_\theta(x)$ as the other scalar field, does $f_\theta(x)$ have a mass? Does \mathcal{L} have a continuous shift symmetry of $f_\theta(x) \rightarrow f_\theta(x) + \Lambda_0$?

Remark: This problem paves the road for your understanding of spontaneous symmetry breaking. We also see again that the symmetry groups of $SO(2)$ and $U(1)$ are isomorphic.

Remark: More to think about after solving the problems above: Note that we reparametrized the field into a non-linear realization, where you see the $U(1)$ symmetry explicitly. How do you interpret the kinetic term? How do you interpret the $f_r(x)$ field-dependent kinetic terms for $f_\theta(x)$? Is it canonically normalized? How does the field $f_\theta(x)$ relate to the original $SO(2)$ field $\phi_a(x)$? And again, is the ratio of

the field a linear redefinition of the field configuration? It is a non-linear realization because all powers of $f_\theta(x)/\sqrt{\frac{\mu^2}{\lambda}}$ need to enter. There is only a region of validity, that is $f_r(x) \ll \sqrt{\frac{\mu^2}{\lambda}}$

Answer

(a)

The Lagrangian can be separated into the kinetic term and the potential term:

$$\mathcal{L}(x) = \sum_{a=1}^2 \left(-\frac{1}{2} \partial^\mu \phi_a \partial_\mu \phi_a \right) - V(x) \quad (131)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu \phi_1 & \partial^\mu \phi_2 \end{pmatrix} \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix} - \left[-\frac{\mu^2}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{\lambda}{4} \left(\begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)^2 \right] \quad (132)$$

Under the $SO(2)$ transformation, the fields transform as:

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha_0 & -\sin \alpha_0 \\ \sin \alpha_0 & \cos \alpha_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \equiv R(\alpha_0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (133)$$

where $R(\alpha_0)$ is the rotation matrix. The kinetic term transforms as:

$$-\frac{1}{2} \begin{pmatrix} \partial^\mu \phi'_1 & \partial^\mu \phi'_2 \end{pmatrix} \begin{pmatrix} \partial_\mu \phi'_1 \\ \partial_\mu \phi'_2 \end{pmatrix} \quad (134)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu (R_{11}\phi_1 + R_{12}\phi_2) & \partial^\mu (R_{21}\phi_1 + R_{22}\phi_2) \end{pmatrix} \begin{pmatrix} \partial_\mu (R_{11}\phi_1 + R_{12}\phi_2) \\ \partial_\mu (R_{21}\phi_1 + R_{22}\phi_2) \end{pmatrix} \quad (135)$$

$$= -\frac{1}{2} \begin{pmatrix} R_{11}\partial^\mu \phi_1 + R_{12}\partial^\mu \phi_2 & R_{21}\partial^\mu \phi_1 + R_{22}\partial^\mu \phi_2 \end{pmatrix} \begin{pmatrix} R_{11}\partial_\mu \phi_1 + R_{12}\partial_\mu \phi_2 \\ R_{21}\partial_\mu \phi_1 + R_{22}\partial_\mu \phi_2 \end{pmatrix} \quad (136)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu \phi_1 & \partial^\mu \phi_2 \end{pmatrix} R^T(\alpha_0) R(\alpha_0) \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix} \quad (137)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu \phi_1 & \partial^\mu \phi_2 \end{pmatrix} \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix}, \quad (138)$$

where we have used the orthogonality of the rotation matrix: $R^T(\alpha_0) R(\alpha_0) = I$. Similarly, the potential term

transforms as:

$$-\frac{\mu^2}{2} \begin{pmatrix} \phi'_1 & \phi'_2 \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} + \frac{\lambda}{4} \left(\begin{pmatrix} \phi'_1 & \phi'_2 \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} \right)^2 \quad (139)$$

$$= -\frac{\mu^2}{2} \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 & R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 \\ R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \quad (140)$$

$$+ \frac{\lambda}{4} \left(\begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 & R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 \\ R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \right)^2 \quad (141)$$

$$= -\frac{\mu^2}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} R^T(\alpha_0) R(\alpha_0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{\lambda}{4} \left(\begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} R^T(\alpha_0) R(\alpha_0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)^2 \quad (142)$$

$$= -\frac{\mu^2}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{\lambda}{4} \left(\begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)^2 \quad (143)$$

Thus, the Lagrangian is invariant under the $SO(2)$ transformation.

(b)

The conjugate momenta are given by:

$$\Pi_1(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_1)} = -\frac{1}{2} \cdot (-2) \cdot (\partial^0\phi_1) = \partial^0\phi_1, \quad (144)$$

$$\Pi_2(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_2)} = -\frac{1}{2} \cdot (-2) \cdot (\partial^0\phi_2) = \partial^0\phi_2. \quad (145)$$

The Hamiltonian density is given by:

$$\mathcal{H}(x) = \Pi_1(x)\partial_0\phi_1 + \Pi_2(x)\partial_0\phi_2 - \mathcal{L}(x) \quad (146)$$

$$= (\partial^0\phi_1)(\partial_0\phi_1) + (\partial^0\phi_2)(\partial_0\phi_2) - \left[-\frac{1}{2}(\partial^\mu\phi_1\partial_\mu\phi_1 + \partial^\mu\phi_2\partial_\mu\phi_2) - V(x) \right] \quad (147)$$

$$= (\partial^0\phi_1)^2 + (\partial^0\phi_2)^2 + \frac{1}{2}(\partial^\mu\phi_1\partial_\mu\phi_1 + \partial^\mu\phi_2\partial_\mu\phi_2) + V(x) \quad (148)$$

$$= (\partial^0\phi_1)^2 + (\partial^0\phi_2)^2 + \frac{1}{2} \left(-(\partial^0\phi_1)^2 + (\partial^i\phi_1)^2 - (\partial^0\phi_2)^2 + (\partial^i\phi_2)^2 \right) + V(x) \quad (149)$$

$$= \frac{1}{2}(\partial^0\phi_1)^2 + \frac{1}{2}(\partial^i\phi_1)^2 + \frac{1}{2}(\partial^0\phi_2)^2 + \frac{1}{2}(\partial^i\phi_2)^2 + V(x) \quad (150)$$

$$= \frac{1}{2}\Pi_1^2 + \frac{1}{2}(\nabla\phi_1)^2 + \frac{1}{2}\Pi_2^2 + \frac{1}{2}(\nabla\phi_2)^2 + V(x). \quad (151)$$

(c)

To find the ground state, we need to minimize the potential $V(x)$:

$$V = -\frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \quad (152)$$

$$= -\frac{\mu^2}{2}\phi_r^2 + \frac{\lambda}{4}\phi_r^4, \quad (153)$$

where we have used the transformation:

$$\phi_1(x) = \phi_r(x) \cos(\phi_\theta(x)), \quad (154)$$

$$\phi_2(x) = \phi_r(x) \sin(\phi_\theta(x)). \quad (155)$$

To minimize V , we take the derivative with respect to ϕ_r and set it to zero:

$$\frac{dV}{d\phi_r} = -\mu^2 \phi_r + \lambda \phi_r^3 = 0 \quad (156)$$

$$\Rightarrow \phi_r (\lambda \phi_r^2 - \mu^2) = 0. \quad (157)$$

The solutions are:

$$\phi_r = 0, \quad \text{or} \quad \phi_r = \sqrt{\frac{\mu^2}{\lambda}}. \quad (158)$$

To determine which solution corresponds to the ground state, we evaluate the second derivative of V :

$$\frac{d^2V}{d\phi_r^2} = -\mu^2 + 3\lambda \phi_r^2. \quad (159)$$

At $\phi_r = 0$:

$$\left. \frac{d^2V}{d\phi_r^2} \right|_{\phi_r=0} = -\mu^2 < 0, \quad (160)$$

indicating a local maximum. At $\phi_r = \sqrt{\frac{\mu^2}{\lambda}}$:

$$\left. \frac{d^2V}{d\phi_r^2} \right|_{\phi_r=\sqrt{\frac{\mu^2}{\lambda}}} = -\mu^2 + 3\lambda \left(\frac{\mu^2}{\lambda} \right) = 2\mu^2 > 0, \quad (161)$$

indicating a local minimum. Therefore, the ground state is at:

$$\phi_r = \sqrt{\frac{\mu^2}{\lambda}}, \quad \phi_\theta \text{ is arbitrary.} \quad (162)$$

Next, we check if the Lagrangian is invariant under the continuous shift symmetry $\phi_\theta(x) \rightarrow \phi_\theta(x) + \alpha_0$. We can rewrite the kinetic term in terms of ϕ_r and ϕ_θ :

$$\partial^\mu \phi_1 \partial_\mu \phi_1 + \partial^\mu \phi_2 \partial_\mu \phi_2 = (\partial^\mu(\phi_r \cos \phi_\theta))(\partial_\mu(\phi_r \cos \phi_\theta)) + (\partial^\mu(\phi_r \sin \phi_\theta))(\partial_\mu(\phi_r \sin \phi_\theta)) \quad (163)$$

$$= (\partial^\mu \phi_r \cos \phi_\theta - \phi_r \sin \phi_\theta \partial^\mu \phi_\theta)(\partial_\mu \phi_r \cos \phi_\theta - \phi_r \sin \phi_\theta \partial_\mu \phi_\theta) \quad (164)$$

$$+ (\partial^\mu \phi_r \sin \phi_\theta + \phi_r \cos \phi_\theta \partial^\mu \phi_\theta)(\partial_\mu \phi_r \sin \phi_\theta + \phi_r \cos \phi_\theta \partial_\mu \phi_\theta) \quad (165)$$

$$= (\partial^\mu \phi_r)(\partial_\mu \phi_r) + \phi_r^2 (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta). \quad (166)$$

The potential term depends only on ϕ_r :

$$V = -\frac{\mu^2}{2}\phi_r^2 + \frac{\lambda}{4}\phi_r^4. \quad (167)$$

Thus, the Lagrangian in terms of ϕ_r and ϕ_θ is:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi_r)(\partial_\mu\phi_r) - \frac{1}{2}\phi_r^2(\partial^\mu\phi_\theta)(\partial_\mu\phi_\theta) + \frac{\mu^2}{2}\phi_r^2 - \frac{\lambda}{4}\phi_r^4. \quad (168)$$

This Lagrangian is invariant under the shift $\phi_\theta(x) \rightarrow \phi_\theta(x) + \alpha_0$ since ϕ_θ appears only through its derivatives. Therefore, the Lagrangian has a continuous shift symmetry in ϕ_θ .

(d)

We expand $\phi_r(x)$ around its vacuum expectation value:

$$\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x). \quad (169)$$

Substituting this into the Lagrangian, we have:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) - \frac{1}{2}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^2 (\partial^\mu\phi_\theta)(\partial_\mu\phi_\theta) + \frac{\mu^2}{2}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^2 - \frac{\lambda}{4}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^4 \quad (170)$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) - \frac{1}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right) (\partial^\mu\phi_\theta)(\partial_\mu\phi_\theta) + \frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right) \quad (171)$$

$$-\frac{\lambda}{4}\left(\frac{\mu^4}{\lambda^2} + 4\frac{\mu^2}{\lambda}\sqrt{\frac{\mu^2}{\lambda}}f_r + 6\frac{\mu^2}{\lambda}f_r^2 + 4f_r^3\sqrt{\frac{\mu^2}{\lambda}} + f_r^4\right) \quad (172)$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) + \left(-\frac{\mu^2}{2\lambda} - \sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{1}{2}f_r^2\right) (\partial^\mu\phi_\theta)(\partial_\mu\phi_\theta) + \left(\frac{\mu^4}{2\lambda} + \mu^2\sqrt{\frac{\mu^2}{\lambda}}f_r + \frac{\mu^2}{2}f_r^2\right) \quad (173)$$

$$+ \left(-\frac{\mu^4}{4\lambda} - \mu^2\sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{3\mu^2}{2}f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right) \quad (174)$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) + \left(-\frac{\mu^2}{2\lambda} - \sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{1}{2}f_r^2\right) (\partial^\mu\phi_\theta)(\partial_\mu\phi_\theta) + \left(\frac{\mu^4}{4\lambda} - \mu^2f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right). \quad (175)$$

The mass term for f_r can be identified from the potential part of the Lagrangian:

$$V(f_r) = -\left(\frac{\mu^4}{4\lambda} - \mu^2f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right) \quad (176)$$

$$= -\frac{\mu^4}{4\lambda} + \mu^2f_r^2 + \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} + \frac{\lambda}{4}f_r^4. \quad (177)$$

The mass term for f_r is given by the coefficient of the f_r^2 term:

$$m_{f_r}^2 = 2\mu^2. \quad (178)$$

Thus, $f_r(x)$ is a massive field with mass $m_{f_r} = \sqrt{2}\mu$. For the field $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}}\phi_\theta(x)$, we can rewrite the kinetic term involving ϕ_θ as:

$$-\frac{1}{2} \left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}} f_r + f_r^2 \right) (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) = -\frac{1}{2} \left(1 + \frac{2f_r}{\sqrt{\frac{\mu^2}{\lambda}}} + \frac{f_r^2}{\frac{\mu^2}{\lambda}} \right) (\partial^\mu f_\theta)(\partial_\mu f_\theta). \quad (179)$$

The field $f_\theta(x)$ does not have a mass term, as there is no term proportional to f_θ^2 in the potential. Therefore, $f_\theta(x)$ is a massless field. The Lagrangian remains invariant under the continuous shift symmetry $f_\theta(x) \rightarrow f_\theta(x) + \Lambda_0$, since f_θ appears only through its derivatives. Thus, the shift symmetry is preserved. \square

Question 4

Problem 66.3

Use the result of problem 66.2 to compute the anomalous dimension of m and the beta function for e in spinor electrodynamics in R_ξ gauge. You should find that the results are independent of ξ .

Remark:

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} + (\xi - 1)k^\mu k^\nu/k^2}{k^2 - i\epsilon} \quad (180)$$

The book only choose the Feynman gauge ($\xi = 1$) to show the loop calculation and get $Z_{1,2,3,m}$. For arbitrary gauge choice ξ , we can repeat the calculation and get:

$$Z_3 = 1 - \frac{e^2}{6\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from photon propagator loop correction} \quad (181)$$

$$Z_2 = 1 - \xi \frac{e^2}{8\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from fermion propagator loop correction} \quad (182)$$

$$Z_m = 1 - (3 + \xi) \frac{e^2}{8\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from fermion mass loop correction} \quad (183)$$

$$Z_1 = 1 - \xi \frac{e^2}{8\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from vertex loop correction} \quad (184)$$

Use the above to finish this problem.

Answer

Now, let's write down the bare Lagrangian and the renormalized Lagrangian:

$$\mathcal{L}_{bare} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \bar{\Psi}_0(i\cancel{D}_0 - m_0)\Psi_0 \quad (185)$$

$$= -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \bar{\Psi}_0(i\cancel{\partial} - e_0\cancel{A}_0 - m_0)\Psi_0, \quad (186)$$

$$= -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + i\bar{\Psi}_0\cancel{\partial}\Psi_0 - e_0\bar{\Psi}_0\cancel{A}_0\Psi_0 - m_0\bar{\Psi}_0\Psi_0, \quad (187)$$

and

$$\mathcal{L}_{re} = \mathcal{L}_0 + \mathcal{L}_1, \quad (188)$$

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\cancel{\partial}\Psi - m\bar{\Psi}\Psi \quad (189)$$

$$\mathcal{L}_1 = Z_1 e\bar{\Psi}\cancel{A}\Psi + \mathcal{L}_{ct}, \quad (190)$$

$$\mathcal{L}_{ct} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} + i(Z_2 - 1)\bar{\Psi}\cancel{\partial}\Psi - (Z_m - 1)m\bar{\Psi}\Psi, \quad (191)$$

Hence, we have

$$\mathcal{L}_{bare} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + i\bar{\Psi}_0\partial\Psi_0 - e_0\bar{\Psi}_0\mathcal{A}_0\Psi_0 - m_0\bar{\Psi}_0\Psi_0, \quad (192)$$

$$\mathcal{L}_{re} = -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\Psi}\partial\Psi + Z_1e\bar{\Psi}\mathcal{A}\Psi - Z_mm\bar{\Psi}\Psi. \quad (193)$$

From the above two equations, we can identify the relations between bare and renormalized quantities:

$$A_{0\mu} = \sqrt{Z_3}A_\mu, \quad (194)$$

$$\Psi_0 = \sqrt{Z_2}\Psi, \quad (195)$$

$$e_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}e\tilde{\mu}^{\epsilon/2}, \quad (196)$$

$$m_0 = \frac{Z_m}{Z_2}m. \quad (197)$$

Note that $\tilde{\mu}$ is the renormalization scale introduced in dimensional regularization to keep the coupling constant dimensionless in $d = 4 - \epsilon$ dimensions. We first compute the beta function for e :

$$0 = \frac{d \log e_0}{d \log \mu} = \frac{d}{d \log \mu} \left(\log Z_1 - \log Z_2 - \frac{1}{2} \log Z_3 + \log e + \frac{\epsilon}{2} \log \tilde{\mu} \right), \quad (198)$$

which gives

$$\beta(e) = \frac{de}{d \log \mu} = e \left(-\frac{d \log Z_1}{d \log \mu} + \frac{d \log Z_2}{d \log \mu} + \frac{1}{2} \frac{d \log Z_3}{d \log \mu} - \frac{\epsilon}{2} \right). \quad (199)$$

To compute the derivatives of the Z factors, we use the expressions given in the problem statement:

$$\frac{d \log Z_2}{d \log \mu} = \frac{1}{Z_2} \frac{dZ_2}{d \log \mu} = \frac{1}{Z_2} \frac{dZ_2}{de} \frac{de}{d \log \mu} = \frac{1}{Z_2} \frac{dZ_2}{de} \beta(e), \quad (200)$$

$$\frac{d \log Z_1}{d \log \mu} = \frac{1}{Z_1} \frac{dZ_1}{d \log \mu} = \frac{1}{Z_1} \frac{dZ_1}{de} \frac{de}{d \log \mu} = \frac{1}{Z_1} \frac{dZ_1}{de} \beta(e), \quad (201)$$

$$\frac{d \log Z_3}{d \log \mu} = \frac{1}{Z_3} \frac{dZ_3}{d \log \mu} = \frac{1}{Z_3} \frac{dZ_3}{de} \frac{de}{d \log \mu} = \frac{1}{Z_3} \frac{dZ_3}{de} \beta(e). \quad (202)$$

Substituting these into the expression for $\beta(e)$, we have:

$$\beta(e) = e \left(-\frac{1}{Z_1} \frac{dZ_1}{de} \beta(e) + \frac{1}{Z_2} \frac{dZ_2}{de} \beta(e) + \frac{1}{2} \frac{1}{Z_3} \frac{dZ_3}{de} \beta(e) - \frac{\epsilon}{2} \right) \quad (203)$$

$$\implies \beta(e) \left(1 + e \left(\frac{1}{Z_1} \frac{dZ_1}{de} - \frac{1}{Z_2} \frac{dZ_2}{de} - \frac{1}{2} \frac{1}{Z_3} \frac{dZ_3}{de} \right) \right) = -\frac{\epsilon}{2} e. \quad (204)$$

$$\implies \beta(e) = -\frac{\epsilon}{2} e \left(1 + e \left(\frac{1}{Z_1} \frac{dZ_1}{de} - \frac{1}{Z_2} \frac{dZ_2}{de} - \frac{1}{2} \frac{1}{Z_3} \frac{dZ_3}{de} \right) \right)^{-1} \quad (205)$$

$$= -\frac{\epsilon}{2} e \left(1 - e \left(\frac{1}{Z_1} \frac{dZ_1}{de} - \frac{1}{Z_2} \frac{dZ_2}{de} - \frac{1}{2} \frac{1}{Z_3} \frac{dZ_3}{de} \right) \right) + \mathcal{O}(e^4) \quad (206)$$

$$= -\frac{\epsilon}{2} e + \frac{\epsilon}{2} e^2 \left(\frac{1}{Z_1} \frac{dZ_1}{de} - \frac{1}{Z_2} \frac{dZ_2}{de} - \frac{1}{2} \frac{1}{Z_3} \frac{dZ_3}{de} \right) + \mathcal{O}(e^4). \quad (207)$$

Now we can apply the expressions for Z_1 , Z_2 , and Z_3 (also Z_m) given in the problem statement to compute the derivatives:

$$\frac{1}{Z_1} \frac{dZ_1}{de} = -\xi \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3), \quad (208)$$

$$\frac{1}{Z_2} \frac{dZ_2}{de} = -\xi \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3), \quad (209)$$

$$\frac{1}{Z_3} \frac{dZ_3}{de} = -\frac{e}{3\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3) \quad (210)$$

$$\frac{1}{Z_m} \frac{dZ_m}{de} = -(3 + \xi) \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3). \quad (211)$$

Substituting these into the expression for $\beta(e)$, we have:

$$\beta(e) = -\frac{\epsilon}{2} e + \frac{\epsilon}{2} e^2 \left(-\xi \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \xi \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \frac{1}{2} \cdot \frac{e}{3\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) \right) + \mathcal{O}(e^4) \quad (212)$$

$$= -\frac{\epsilon}{2} e + \frac{\epsilon}{2} e^2 \cdot \frac{e}{6\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4) \quad (213)$$

$$= -\frac{\epsilon}{2} e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4). \quad (214)$$

Now we can compute the anomalous dimension of m :

$$0 = \frac{d \log m_0}{d \log \mu} = \frac{d}{d \log \mu} (\log Z_m - \log Z_2 + \log m), \quad (215)$$

which gives

$$\gamma_m = \frac{d \log m}{d \log \mu} = -\frac{d \log Z_m}{d \log \mu} + \frac{d \log Z_2}{d \log \mu}. \quad (216)$$

Using the expressions for Z_m and Z_2 , we have:

$$\frac{d \log Z_m}{d \log \mu} = \frac{1}{Z_m} \frac{dZ_m}{d \log \mu} = \frac{1}{Z_m} \frac{dZ_m}{de} \frac{de}{d \log \mu} = \frac{1}{Z_m} \frac{dZ_m}{de} \beta(e), \quad (217)$$

$$\frac{d \log Z_2}{d \log \mu} = \frac{1}{Z_2} \frac{dZ_2}{d \log \mu} = \frac{1}{Z_2} \frac{dZ_2}{de} \frac{de}{d \log \mu} = \frac{1}{Z_2} \frac{dZ_2}{de} \beta(e). \quad (218)$$

Substituting these into the expression for γ_m , we have:

$$\gamma_m = -\frac{1}{Z_m} \frac{dZ_m}{de} \beta(e) + \frac{1}{Z_2} \frac{dZ_2}{de} \beta(e) \quad (219)$$

$$= \beta(e) \left(-\frac{1}{Z_m} \frac{dZ_m}{de} + \frac{1}{Z_2} \frac{dZ_2}{de} \right) \quad (220)$$

$$= \left(-\frac{\epsilon}{2} e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4) \right) \left(- \left(-(3+\xi) \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3) \right) + \left(-\xi \frac{e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3) \right) \right) \quad (221)$$

$$= \left(-\frac{\epsilon}{2} e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4) \right) \left(\frac{3e}{4\pi^2} \left(\frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^3) \right) \quad (222)$$

$$= -\frac{3e^2}{8\pi^2} + \mathcal{O}(e^4). \quad (223)$$

Thus, we have found that the beta function for e is:

$$\beta(e) = -\frac{\epsilon}{2} e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4), \quad (224)$$

and the anomalous dimension of m is:

$$\gamma_m = -\frac{3e^2}{8\pi^2} + \mathcal{O}(e^4). \quad (225)$$

Remark: Notice that both results are independent of the gauge parameter ξ . \square

Question 5

Consider the following theory:

$$\mathcal{L} = \mathcal{L}_\phi^0 + \mathcal{L}_\Psi^0 + \mathcal{L}_A^0 + \mathcal{L}_I \quad (226)$$

$$= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\Psi}(iD^\mu - m_\Psi)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + y\phi\bar{\Psi}\Psi. \quad (227)$$

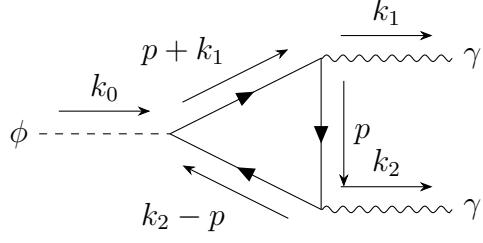
The Dirac field Ψ is charged under a $U(1)$ gauge symmetry with a charge Q , and the gauge interaction strength is e . The $U(1)$ gauge field is A_μ , whose kinetic term is $\mathcal{L}_A^0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. (This is part of the real-world calculation for the discovery mode for the Higgs boson, which gone through heroic phenomenological studies on predicting the Higgs properties.)

- (a) Draw the leading diagrams that enable $\phi \rightarrow \gamma\gamma$ decay. (The gauge field A_μ is identified as the photon field γ .)
 - (b) In the ϕ rest frame, write down the amplitude in the general d dimension. No need to carry out the loop integral at this point, but need to simplify the trace. (Notice that $k_\mu\epsilon^\mu(k) = 0$ in Lorenz gauge.)
 - (c) Does the integral have a UV divergence in $d = 4$ dimension (loop momentum goes to ∞)? Answer Yes or No with a few lines of argument.
 - (d) Does the integral have a singularity in $d = 4$ dimension when the Euclidean loop momentum squared \bar{q}^2 go to $-D$? Answer Yes or No with a few lines of argument. (For simplicity, assume that D is real and can be zero for some configuration of x_1, x_2, x_3 .)
 - (e) For $m_\Phi = 0$, calculate using dimensional regularization in $d = 4 - \epsilon$. Write down your final answer in the simplest form. (The final answer would be short.)
 - (f) Carry out the full calculation of the amplitude in Part b using dimensional regularization in $d = 4 - \epsilon$. Write down your final answer in the simplest form. (The full answer would be a long calculateion.)
- Hint:** The following few equations, identities, and tricks, and the discussion around them might be helpful for you: Eq. (62.18), Eq. (47.18), Eq. (67.2).
- Remark:** No need to answer this, but one can think about it for fun. Recall that taking $\epsilon \rightarrow 0$ (from plus or minus direction?) get you back to $d = 4$. In such a limit, contrast your result in Part f and Part c and think about why.

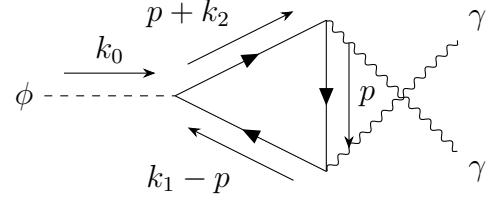
Answer

- (a)

We have two leading diagrams that contribute to the decay $\phi \rightarrow \gamma\gamma$, as shown in Fig. 3a and Fig. 3b. Both diagrams involve a fermion loop with two photon vertices and one scalar vertex.



(a) Leading diagram for $\phi \rightarrow \gamma\gamma$ decay.



(b) Leading diagram for $\phi \rightarrow \gamma\gamma$ decay.

Figure 3: Leading diagrams for $\phi \rightarrow \gamma\gamma$ decay.

(b)

The amplitude for the decay $\phi \rightarrow \gamma\gamma$ can be written as:

$$i\mathcal{M} = i\mathcal{M}_a + i\mathcal{M}_b, \quad (228)$$

where $i\mathcal{M}_a$ and $i\mathcal{M}_b$ are the contributions from the two diagrams. The contribution from the first diagram (Fig. 3a) is:

$$i\mathcal{M}_a = (-1)(iy) \int \frac{d^d p}{(2\pi)^d} \left[\frac{-i(-(\not{p} + \not{k}_1))}{(p + k_1)^2 + m_\Psi^2 - i\epsilon} (ieQ\gamma^{\mu_1}) \epsilon_{\mu_1}(k_1) \frac{-i(-\not{p})}{p^2 + m_\Psi^2 - i\epsilon} (ieQ\gamma^{\mu_2}) \epsilon_{\mu_2}(k_2) \frac{-i(-(\not{k}_2 - \not{p}))}{(k_2 - p)^2 + m_\Psi^2 - i\epsilon} \right]_{\text{Tr}} \quad (229)$$

$$= iy e^2 Q^2 \int \frac{d^d p}{(2\pi)^d} \frac{\text{Tr} [(\not{p} + \not{k}_1)\gamma^{\mu_1}\not{p}\gamma^{\mu_2}(\not{k}_2 - \not{p})] \epsilon_{\mu_1}(k_1) \epsilon_{\mu_2}(k_2)}{[(p + k_1)^2 + m_\Psi^2 - i\epsilon][p^2 + m_\Psi^2 - i\epsilon][(k_2 - p)^2 + m_\Psi^2 - i\epsilon]}. \quad (230)$$