University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

December 16, 2025

Assignment 1 due on Wednesday September 10th at 5PM

Question 1

Answer

We can use tensor notation to reduce the work. From the constraint

$$\eta_{\alpha\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta} = \eta_{\gamma\delta}, \tag{1}$$

and the relation $\gamma = \frac{1}{\sqrt{1-v^2}}$ (or $\gamma^2 - \gamma^2 v^2 = \gamma^2(1-v^2) = 1$), we can consider different γ, δ, meaning that

- $(\gamma, \delta) = (0, 0)$:

$$-(\Lambda^0{}_0)^2 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 = -\gamma^2 + \gamma^2 \sum_{i=1}^{3}(v_i)^2 = -\gamma^2 + \gamma^2 v^2 = -1 = \eta_{00} \tag{2}$$

- $(\gamma, \delta) = (0, i)$:

$$\begin{aligned} & -\Lambda^0{}_0\Lambda^0{}_i + \sum_{k=1}^{3}\Lambda^k{}_0\Lambda^k{}_i && (3) \\ =& -\gamma c v_i + \sum_{k=1}^{3}\gamma v_k(a\delta_{ki} + b v_k v_i) && (4) \\ =& -\gamma c v_i + \gamma a v_i + \gamma b v^2 v_i = \gamma v_i(-c + a + bv^2) = \eta_{0i} = 0 && (5) \end{aligned}$$

  Then we have $c = a + bv^2$.

- $(\gamma, \delta) = (i, 0)$

$$\begin{aligned} & -\Lambda^0{}_i\Lambda^0{}_0 + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_0 && (6) \\ =& \gamma v_i(-c + a + bv^2) && (7) \end{aligned}$$

- $(\gamma, \delta) = (i, i)$

$$\begin{aligned}
&-\Lambda^0{}_i\Lambda^0{}_i + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_i && (8)\\
=&-c^2v_i^2 + \sum_{k=1}^{3}(a\delta_{ki} + bv_kv_i)^2 && (9)\\
=&-c^2v_i^2 + \sum_{k=1}^{3}(a^2\delta_{ki} + 2ab\delta_{ki}v_kv_i + b^2v_i^2v_k^2) && (10)\\
=&-c^2v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 = \eta_{ii} = 1 && (11)
\end{aligned}$$

- $(\gamma, \delta) = (i, j), i \neq j$

$$\begin{aligned}
&-\Lambda^0{}_i\Lambda^0{}_j + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_j && (12)\\
=&-c^2v_iv_j + \sum_{k=1}^{3}(a\delta_{ki} + bv_kv_i)(a\delta_{kj} + bv_kv_j) && (13)\\
=&-c^2v_iv_j + \sum_{k=1}^{3}(a^2\delta_{ki}\delta_{kj} + ab(\delta_{ki}v_kv_j + \delta_{kj}v_kv_i) + b^2v_k^2v_iv_j) && (14)\\
=&-c^2v_iv_j + 2abv_iv_j + b^2v^2v_iv_j = \eta_{ij} = 0 && (15)
\end{aligned}$$

Then we have $c^2 = b^2v^2 + 2ab$

Combining the above information, we can have

$$\begin{aligned}
&c = a + bv^2 && (16)\\
&c^2 = a^2 + b^2v^4 + 2abv^2 = b^2v^2 + 2ab. && (17)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&-c^2v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 = 1 && (18)\\
\rightarrow 1 =& -(b^2v^2 + 2ab)v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 && (19)\\
&= a^2 && (20)
\end{aligned}$$

Hence, from Eq. 17 and Eq. 20, we have

$$\begin{aligned}
&c^2 = 1 + b^2v^4 + 2abv^2 = b^2v^2 + 2ab && (21)\\
\rightarrow &a = \pm 1 = \frac{1 + b^2v^4 - b^2v^2}{2b(1 - v^2)} = \frac{1 + b^2v^2(v^2 - 1)}{2b(1 - v^2)} = \frac{\gamma^2 - b^2v^2}{2b} && (22)\\
\rightarrow &b^2v^2 \pm 2b - \gamma^2 = 0 && (23)
\end{aligned}$$

For $a = +1$, we have

$$b = \frac{-1 \pm \sqrt{1 + \gamma^2 v^2}}{v^2} = \frac{-1 \pm \gamma}{v^2}, \tag{24}$$

and for $a = -1$, we have

$$b = \frac{1 \pm \gamma}{v^2}. \tag{25}$$

Hence, we choose $a = 1$ and $b = \frac{-1+\gamma}{v^2}$ by convention and have

$$c = a + bv^2 = 1 + (-1 + \gamma) = +\gamma \tag{26}$$

$$\tag{27}$$

Finally, we derive

$$\Lambda^i{}_j = a(v)\delta_{ij} + b(v)v_i v_j = \delta_{ij} + \frac{\gamma - 1}{v^2} v_i v_j \tag{28}$$

$$\Lambda^0{}_j = c(v)v_j = \gamma v_j, \tag{29}$$

which are mentioned in the Week 1 lecture. $\square$

Question 2

Answer

In the $\mathcal{O}$ frame, we set the condition for the right-hand side of the rod:

$$
\begin{aligned}
&x^{\mu}_{emit} = (0, L/2, D, 0), x^{\mu}_{receive} = (t, vt, 0, 0),\\
&\Delta x^{\mu} = x^{\mu}_{receive} - x^{\mu}_{emit} = (t, vt - L/2, -D, 0)\\
&\Delta x^{\mu}\Delta x_{\mu} = 0 = t^2 - (L/2 - vt)^2 - D^2.
\end{aligned}
$$

Also, with $dt = t, dx = vt - L/2, dy = -D$, we have

$$
\begin{aligned}
\Delta t' &= \gamma(t + v(vt - L/2))\\
\Delta x' &= \gamma(v + (vt - L/2))\\
\Delta y' &= -D
\end{aligned}
$$

We can substitute the solution t to solve the dx' and dt', and define the opening angle

$$
\tan\theta'/2 = \frac{\Delta x'}{\Delta y'}. \tag{30}
$$

Note that this is the calculation for right-hand side of the rod. Hence, it should be denoted as θ'_R. Now we can apply the procedure to the left-hand side of the rod. After plugging all information into `Mathematica` and using the equation, we have

$$
\tan\left(\theta'_R + \theta'_L\right) = \frac{\tan\theta'_R + \tan\theta'_L}{1 - \tan\theta'_R\tan\theta'_L} = \frac{4DL}{4D^2 - L^2} = \frac{2\frac{L}{2D}}{1 - \left(\frac{L}{2D}\right)^2} = \tan\theta. \tag{31}
$$

Hence, we prove that the opening angle keeps the same value. □

Assignment 2 due on Monday September 15th at 5PM

Question 1

Humans have finally been able to design and build a spaceship that can travel distances up to 100 light-years away. The propulsion system is capable of providing a constant acceleration $g = 9.8$ m/s in the rest frame of the spaceship; this simulates gravity so that people on board are as comfortable as on Earth. The spaceship leaves Earth from rest, accelerates towards its destination, and, halfway there, it reverses its engines so that it will come to rest when it reaches its destination. Find the position x and velocity v as functions of both Earth time t and proper time τ. How long does it take to reach a destination 10 light-years away according to a clock on Earth versus a clock in the spaceship? Repeat the calculation for a destination 100 light-years away. It is interesting to note that the characteristic time c/g is almost identically equal to one year.

Answer

Let's define the acceleration a^μ,

$$a^\mu = \frac{du^\mu}{d\tau}, a^\mu a_\mu = g^2, u^\mu = \frac{dx^\mu}{d\tau}, \tag{32}$$

where u^μ is the 4-velocity. We can re-parameterize to easily define the velocity: let y and z to be zero for convention,

$$u^\mu = (\cosh\eta, \sinh\eta, 0, 0), \quad \eta = \eta(\tau), \tag{33}$$

where η is the rapidity. Now we have

$$\frac{du^\mu}{d\tau} = \frac{d\eta}{d\tau}(\sinh\eta, \cosh\eta, 0, 0) \tag{34}$$

$$\frac{du^\mu}{d\tau}\frac{du^\mu}{d\tau} = -(\sinh^2\eta - \cosh^2\eta)\left(\frac{d\eta}{d\tau}\right)^2 = \left(\frac{d\eta}{d\tau}\right)^2 = g^2 \tag{35}$$

$$\rightarrow \left(\frac{d\eta}{d\tau}\right) = g \rightarrow \eta = g\tau + C \tag{36}$$

By initial condition, C should be 0, and then we have

$$u^\mu = (\cosh(g\tau), \sinh(g\tau), 0, 0), \tag{37}$$

$$x^\mu = \frac{1}{g}(\sinh(g\tau) + C_1, \cosh(g\tau) + C_2, 0, 0) \tag{38}$$

$$= \frac{1}{g}(\sinh(g\tau), \cosh(g\tau) - 1, 0, 0), \tag{39}$$

where C_1 and C_2 are determined by initial condition $t(\tau = 0) = 0, x(\tau = 0) = 0$. Finally, we have

$$t = x^0 = \frac{1}{g}\sinh{(g\tau)} = \frac{c}{g}\sinh{(\frac{g\tau}{c})} \tag{40}$$

$$x = x^1 = \frac{1}{g}\cosh{(g\tau)} - 1 = \frac{c^2}{g}\left(\cosh{(\frac{g\tau}{c})} - 1\right) \tag{41}$$

Considering the half of total traveling time $t_{1/2}$ when $x = L/2$, we have

$$t = \frac{1}{c}\sqrt{\left(x + \frac{c^2}{g}\right)^2 - \frac{c^4}{g^2}} \tag{42}$$

$$t_{1/2} = \frac{1}{c}\sqrt{\left(\frac{L}{2} + \frac{c^2}{g}\right)^2 - \frac{c^4}{g^2}} \tag{43}$$

$$\tag{44}$$

For $L/2 = 5$ light-years $= 5 \times \frac{c}{g} \times c = \frac{5c^2}{g}$, we have $t_{tatal} = 2t_{1/2} = 2\sqrt{35}c/g \approx 11.8$ years. For $L/2 = 50$ light-years $= 5 \times \frac{c}{g} \times c = \frac{50c^2}{g}$, we have $t_{tatal} = 2t_{1/2} = 20\sqrt{26}c/g \approx 102$ years. This is the time for the people on the earth (in $\mathcal{O}$ frame).

On the other hand, for people on the spaceship, the (proper) time would be τ, given by

$$\tau_{1/2} = \frac{c}{g}\sinh^{-1}(\frac{t_{1/2}}{c/g}) \tag{45}$$

Then for $L/2 = 5$ light-years $= 5 \times \frac{c}{g} \times c = \frac{5c^2}{g}$, we have $\tau_{total} = 2\tau_{1/2} = 4.96c/g \approx 4.96$ years. Also, for $L/2 = 50$ light-years, we have $\tau_{total} = 2\tau_{1/2} = 9.25c/g \approx 9.25$ years. $\square$

Question 2

Prove component by component that $\varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon^{\alpha\beta\gamma\delta}$, and evaluate the scalar $\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta}$.

Answer

$$\varepsilon_{\alpha\beta\gamma\delta} = \eta_{\alpha\rho}\eta_{\lambda\beta}\eta_{\gamma\kappa}\eta_{\zeta\delta}\varepsilon^{\rho\lambda\kappa\zeta}$$

If we choose $(\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)$ as an example, we will have

$$\varepsilon_{0123} = \eta_{0\rho}\eta_{1\beta}\eta_{2\kappa}\eta_{3\delta}\varepsilon^{\rho\lambda\kappa\zeta} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\varepsilon^{0123} = -1.$$

Also, the value is zero if any two of the variables α, β, γ, or δ are equal. Last, if the permutation of $(\alpha, \beta, \gamma, \delta)$ is even, the value is -1 due to the property of $\varepsilon^{\rho\lambda\kappa\zeta}$. Once the permutation of $(\alpha, \beta, \gamma, \delta)$ is odd, the value is $+1$ due to the property of $\varepsilon^{\rho\lambda\kappa\zeta}$. Hence, we prove the statement:

$$\varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon^{\alpha\beta\gamma\delta}.$$

Last,

$$\begin{aligned}
&\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta} \\
=& -\left(\varepsilon^{\alpha\beta\gamma\delta}\right)^2 \\
=& -1 \times (4!) = -24,
\end{aligned}$$

where 4! means the number of possible permutation leaving non-vanishing terms. $\square$

Assignment 3 due on Monday September 22th at 5PM

Question 1

Show explicitly that the 4-vector current density for a collection of point charges satisfies $\partial_\mu J^\mu = 0$

Answer

In the class, we defined the 4-vector current density for a collection of point charges as

$$J^0(t, \mathbf{x}) = \rho(t, \mathbf{x}) = \sum_a q_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{46}$$

$$\mathbf{J}(t, \mathbf{x}) = \sum_a q_a \mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)), \quad \mathbf{v}_a(t) = \frac{d\mathbf{x}_a(t)}{dt}. \tag{47}$$

Then we have

$$\begin{aligned}
\partial_\mu J^\mu &= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} && (48)\\
&= \sum_a q_a \left[\frac{\partial}{\partial t} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))) \right] && (49)\\
&= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))) \right] && (50)\\
&= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \right] && (51)\\
&= 0. && (52)
\end{aligned}$$

Question 2

Prove that the electromagnetic energy density squared minus the square of the Poynting vector is a Lorentz invariant for an electromagnetic field by expressing this quantity in terms of tensors. You might consider using the dual field strength tensor defined by $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Answer

In the EM class, we defined the electromagnetic energy density and the Poynting vector as

$$u = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{53}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}. \tag{54}$$

Also, we have the following relations

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{55}$$

Besides, the EM field energy momentum strength tensor is given by (from Weinberg's GR book)

$$T^{\mu\nu}_{EM} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{56}$$

$$u = T^{00}_{EM} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad S^i = T^{i0}_{EM} = (\mathbf{E} \times \mathbf{B})^i, \tag{57}$$

where u is the EM energy density and $\mathbf{S}$ is the Poynting vector. Hence, we can define a 4-vector U^μ as

$$U^\mu = (u, \mathbf{S}) = (T^{00}_{EM}, T^{i0}_{EM}) = T^{\mu 0}_{EM}. \tag{58}$$

Then we have

$$T^{00}_{EM} = F^{0\alpha}F^{0}{}_{\alpha} - \frac{1}{4}\eta^{00}F_{\alpha\beta}F^{\alpha\beta} = F^{0i}F^{0}{}_{i} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{59}$$

$$= \mathbf{E}^2 - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{60}$$

$$= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = u \tag{61}$$

$$\tag{62}$$

and

$$
\begin{aligned}
T^{i0}_{EM} &= F^{i\alpha}F^{0}{}_{\alpha} - \frac{1}{4}\eta^{i0}F_{\alpha\beta}F^{\alpha\beta} = F^{ij}F^{0}{}_{j} && (63)\\
&= \epsilon_{ijk}B_k E_j && (64)\\
&= (\mathbf{E}\times\mathbf{B})^i = S^i. && (65)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
U^\mu U_\mu &= -\eta_{\mu\nu}U^\mu U^\nu && (66)\\
&= -\eta_{\mu\nu}T^{\mu 0}_{EM}T^{\nu 0}_{EM} && (67)\\
&= T^{00}_{EM}T^{00}_{EM} - T^{i0}_{EM}T^{i0}_{EM} = u^2 - \mathbf{S}^2. && (68)
\end{aligned}
$$

We can claim that $U^\mu U_\mu$ is a Lorentz invariant since it is the contraction of two tensors. Hence, we conclude that $u^2 - \mathbf{S}^2$ is a Lorentz invariant.

Remark: We can also prove this by using the dual field strength tensor $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. First, we have

$$
\begin{aligned}
\widetilde{F}^{0i} &= \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}\epsilon_{jkl}B_l = B^i, && (69)\\
\widetilde{F}^{ij} &= \frac{1}{2}\varepsilon^{ij0k}F_{0k} = \frac{1}{2}(\varepsilon^{ij0k} - \varepsilon^{ji0k})F_{0k} = \varepsilon^{ij0k}E_k = \epsilon^{ijk}E_k. && (70)
\end{aligned}
$$

Then we can calculate the following two Lorentz invariants:

$$
\begin{aligned}
F_{\mu\nu}F^{\mu\nu} &= 2(\mathbf{B}^2 - \mathbf{E}^2), && (71)\\
\widetilde{F}_{\mu\nu}F^{\mu\nu} &= -4\mathbf{E}\cdot\mathbf{B}. && (72)
\end{aligned}
$$

Now we can calculate

$$
\begin{aligned}
(F_{\mu\nu}F^{\mu\nu})^2 + (\widetilde{F}_{\mu\nu}F^{\mu\nu})^2 &= 4(\mathbf{B}^2 - \mathbf{E}^2)^2 + 16(\mathbf{E}\cdot\mathbf{B})^2 && (73)\\
&= 4[(\mathbf{B}^2 + \mathbf{E}^2)^2 - 4\mathbf{E}^2\mathbf{B}^2 + 4(\mathbf{E}\cdot\mathbf{B})^2] && (74)\\
&= 4[(\mathbf{B}^2 + \mathbf{E}^2)^2 - 4(\mathbf{E}\times\mathbf{B})^2] && (75)\\
&= 4(u^2 - \mathbf{S}^2). && (76)
\end{aligned}
$$

This quantity is Lorentz invariant since all indices are contracted. Hence, we conclude that $u^2 - \mathbf{S}^2$ is a Lorentz invariant.

Question 3

Calculate the scalar $T^{\alpha}{}_{\alpha}$ associated with the electromagnetic stress tensor.

Answer

Consider the energy momentum tensor with the EM field:

$$T^{\alpha\beta}_{total} = T^{\alpha\beta} + T^{\alpha\beta}_{EM} = \sum_n p^{\alpha}_n(t)\frac{dx^{\beta}_n}{dt}\delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)) + F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \tag{77}$$

$$= \sum_n \frac{p^{\alpha}_n p^{\beta}_n}{E_n}\delta^3(\mathbf{x} - \mathbf{x}_n(t)) + F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}, \tag{78}$$

We have

$$T^{\alpha}{}_{\alpha} = \eta_{\alpha\beta}T^{\alpha\beta}_{total} \tag{79}$$

$$= \sum_n \frac{p^{\alpha}_n p_{n\alpha}}{E_n}\delta^3(\mathbf{x} - \mathbf{x}_n(t)) + F^{\alpha\mu}F_{\alpha\mu} - \frac{1}{4}\eta_{\alpha\beta}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \tag{80}$$

$$= \sum_n \frac{m^2_n}{E_n}\delta^3(\mathbf{x} - \mathbf{x}_n(t)) + 0, \quad \text{by } \eta^{\alpha\beta}\eta_{\alpha\beta} = 4 \tag{81}$$

$$= \sum_n \frac{m^2_n}{E_n}\delta^3(\mathbf{x} - \mathbf{x}_n(t)) \tag{82}$$

$$= \sum_n \frac{m^2_n}{E_n}\delta^3(\mathbf{x} - \mathbf{x}_n(t)). \tag{83}$$

Remark: In class, we have shown that

$$\frac{\delta^3(\mathbf{x} - \mathbf{x}_n(t))}{E_n} \tag{84}$$

is a Lorentz invariant. Hence, we can conclude that $T^{\alpha}{}_{\alpha}$ is a Lorentz invariant since m_n is also a Lorentz invariant. In other words, $T^{\alpha}{}_{\alpha}$ is a Lorentz scalar.

Remark 2 (after deadline): Actually, I don't know whetehr should I only consider the EM part or the total energy momentum tensor. If I only consider the EM part, then we have 0. *Seth* and I discussed this question and we think that the question might be ambiguous. So I just write both of them here.

Assignment 4 due on Monday September 29th at 5PM

Question 1

Calculate the metric g_{ij} and its inverse g^{ij} , the affine connection Γ^i_{jk}, and the Laplacian ∇^2 in two dimensions for a polar coordinate system with $\xi^1 = x$ and $\xi^2 = y$ being Cartesian coordinates and $x^1 = r$ and $x^2 = \theta$ being polar coordinates.

Answer

First, we have the transformation relations between Cartesian coordinates and polar coordinates:

$$x = r\cos\theta, \tag{85}$$
$$y = r\sin\theta. \tag{86}$$

The metric in Cartesian coordinates is given by:

$$\eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{87}$$

To find the metric in polar coordinates, we use the transformation:

$$g_{ij} = \frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^l}{\partial x^j}\eta_{kl}. \tag{88}$$

Calculating the partial derivatives, we have:

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \tag{89}$$
$$\frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta. \tag{90}$$

Thus, the metric components in polar coordinates are:

$$g_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 = \cos^2\theta + \sin^2\theta = 1, \tag{91}$$
$$g_{r\theta} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} = \cos\theta(-r\sin\theta) + \sin\theta(r\cos\theta) = 0, \tag{92}$$
$$g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = (-r\sin\theta)^2 + (r\cos\theta)^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2. \tag{93}$$

Therefore, the metric in polar coordinates is:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{94}$$

The inverse metric is given by:

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{95}$$

Next, we calculate the affine connection components using the formula:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{96}$$

Now, we have:

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) = 0, \tag{97}$$

$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{rr}}{\partial \theta} - \frac{\partial g_{r\theta}}{\partial r} \right) = 0, \tag{98}$$

$$\Gamma^\theta_{rr} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta r}}{\partial r} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) = 0, \tag{99}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial \theta} \right) = 0. \tag{100}$$

The non-zero components of the affine connection are:

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{r\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right) = -r, \tag{101}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta r}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{r\theta}}{\partial \theta} \right) = \frac{1}{r}. \tag{102}$$

Finally, we calculate the Laplacian in polar coordinates:

$$\partial_r f = \frac{\partial f}{\partial r}, \quad \partial_\theta f = \frac{\partial f}{\partial \theta}, \tag{103}$$

$$\partial_r f = \partial_x f \frac{\partial x}{\partial r} + \partial_y f \frac{\partial y}{\partial r} = \cos\theta \partial_x f + \sin\theta \partial_y f, \tag{104}$$

$$\partial_\theta f = \partial_x f \frac{\partial x}{\partial \theta} + \partial_y f \frac{\partial y}{\partial \theta} = -r\sin\theta \partial_x f + r\cos\theta \partial_y f, \tag{105}$$

$$\partial_{rr} f = \partial_r(\partial_r f) = \cos\theta \partial_{xx} f \cos\theta + \sin\theta \partial_{yy} f \sin\theta + 2\cos\theta \sin\theta \partial_{xy} f, \tag{106}$$

$$\partial_{\theta\theta} f = \partial_\theta(\partial_\theta f) = r^2 \sin^2\theta \partial_{xx} f + r^2\cos^2\theta \partial_{yy} f - 2r^2 \sin\theta\cos\theta \partial_{xy} f - r\cos\theta \partial_x f - r\sin\theta \partial_y f. \tag{107}$$

$$\partial_{rr} f = \cos^2\theta \partial_{xx} f + \sin^2\theta \partial_{yy} f + 2\cos\theta\sin\theta \partial_{xy} f, \tag{108}$$

$$\frac{1}{r}\partial_r f = \frac{1}{r}(\cos\theta \partial_x f + \sin\theta \partial_y f), \tag{109}$$

$$\frac{1}{r^2}\partial_{\theta\theta} f = \sin^2\theta \partial_{xx} f + \cos^2\theta \partial_{yy} f - 2\sin\theta\cos\theta \partial_{xy} f - \frac{1}{r}\cos\theta \partial_x f - \frac{1}{r}\sin\theta \partial_y f. \tag{110}$$

Thus, the Laplacian in polar coordinates is:

$$\nabla^2 f = \partial_{xx} f + \partial_{yy} f \tag{111}$$

$$= \partial_{rr} f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_{\theta\theta} f. \tag{112}$$

□

Question 2

Calculate the compact expressions for the components of the affine connection when the metric g_{ij} is diagonal. See problem 3 in chapter 3 of Carroll's book.
Problem 3 in chapter 3 of Carroll's book: Imagine we have a diagonal metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = 0 \tag{113}$$

$$\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda g_{\mu\mu} \tag{114}$$

$$\Gamma^{\lambda}_{\mu\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right) \tag{115}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right). \tag{116}$$

In these expressions, $\mu \neq \nu \neq \lambda$, and repeated indices are not summed over.

Answer

By the definedion of the Christoffel symbols, we have

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{117}$$

Since the metric is diagonal, we have $g_{\mu\nu} = 0$ for $\mu \neq \nu$. Therefore, we can analyze the different cases:
1. For $\mu \neq \nu \neq \lambda$:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) = 0, \tag{118}$$

where we used the fact that all terms vanish because $g_{\mu\nu} = 0$ for $\mu \neq \nu$ when we sum over σ.
2. For $\mu = \nu \neq \lambda$:

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\mu\sigma} + \partial_\mu g_{\mu\sigma} - \partial_\sigma g_{\mu\mu}\right) = -\frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\mu\mu} = -\frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda g_{\mu\mu}, \tag{119}$$

where we used the fact that only the term with $\sigma = \lambda$ survives in the sum. When g is diagonal, $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}}$.
3. For $\mu \neq \lambda = \nu$:

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\lambda\sigma} + \partial_\lambda g_{\mu\sigma} - \partial_\sigma g_{\mu\lambda}\right) = \frac{1}{2}g^{\lambda\lambda}\partial_\mu g_{\lambda\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right), \tag{120}$$

where we used the fact that only the term with $\sigma = \lambda$ survives in the sum. When g is diagonal, $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}}$.
4. For $\mu = \nu = \lambda$:

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\lambda g_{\lambda\sigma} + \partial_\lambda g_{\lambda\sigma} - \partial_\sigma g_{\lambda\lambda}\right) = \frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right), \tag{121}$$

where we used the fact that only the term with $\sigma = \lambda$ survives in the sum. When g is diagonal, $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}}$. $\square$

Question 3

Prove that if the equation for a geodesic has the form

$$\frac{d^2x^\alpha}{dp_i^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_i}\frac{dx^\gamma}{dp_i} = 0, \tag{122}$$

for two different parameters p_1 and p_2 defined along the geodesic then the most general relation between them is $p_2 = Ap_1 + B$ where A and B are constants.

Answer

Let $x^\alpha(p_1)$ be a geodesic parameterized by p_1. We want to show that if we reparameterize the geodesic using a different parameter p_2, the new parameter must be a linear function of the old one, i.e., $p_2 = Ap_1 + B$ for some constants A and B.

Assume that $x^\alpha(p_2)$ is the same geodesic but parameterized by p_2. We can express p_2 as a function of p_1, i.e., $p_2 = f(p_1)$ for some function f. Then, we have:

$$\frac{dx^\alpha}{dp_2} = \frac{dx^\alpha}{dp_1}\frac{dp_1}{dp_2}, \tag{123}$$

$$\frac{d^2x^\alpha}{dp_2^2} = \frac{d}{dp_2}\left(\frac{dx^\alpha}{dp_1}\frac{dp_1}{dp_2}\right) = \frac{d^2x^\alpha}{dp_1^2}\left(\frac{dp_1}{dp_2}\right)^2 + \frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2}. \tag{124}$$

Substituting these into the geodesic equation parameterized by p_2, we get:

$$\frac{d^2x^\alpha}{dp_2^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_2}\frac{dx^\gamma}{dp_2} = 0. \tag{125}$$

Substituting the expressions for the derivatives, we have:

$$\left(\frac{d^2x^\alpha}{dp_1^2}\left(\frac{dp_1}{dp_2}\right)^2 + \frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2}\right) + \Gamma^\alpha_{\beta\gamma}\left(\frac{dx^\beta}{dp_1}\frac{dp_1}{dp_2}\right)\left(\frac{dx^\gamma}{dp_1}\frac{dp_1}{dp_2}\right) = 0. \tag{126}$$

Rearranging, we get:

$$\frac{d^2x^\alpha}{dp_1^2}\left(\frac{dp_1}{dp_2}\right)^2 + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_1}\frac{dx^\gamma}{dp_1}\left(\frac{dp_1}{dp_2}\right)^2 + \frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2} = 0. \tag{127}$$

Since $x^\alpha(p_1)$ satisfies the geodesic equation, we have:

$$\frac{d^2x^\alpha}{dp_1^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_1}\frac{dx^\gamma}{dp_1} = 0. \tag{128}$$

Thus, the first two terms cancel out, leaving us with:

$$\frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2} = 0. \tag{129}$$

Since $\frac{dx^\alpha}{dp_1}$ is not zero (as we are moving along the geodesic), we must have:

$$\frac{d^2p_1}{dp_2^2} = 0. \tag{130}$$

This implies that p_1 is a linear function of p_2, i.e.,

$$p_1 = Ap_2 + B, \tag{131}$$

for some constants A and B. We can rewrite this as:

$$p_2 = Cp_1 + D \tag{132}$$

where $C = \frac{1}{A}$ and $D = -\frac{B}{A}$. □

Remark: I am not sure A or C should be positive or negative. If we want p_1 and p_2 to have the same orientation, then A and C should be positive.

Remark: I am not sure if A or C could be 0, because if $A = 0$, then $p_1 = B$ is a constant, which means the parameterization is not valid. Similarly, if $C = 0$, then $p_2 = D$ is a constant, which also means the parameterization is not valid. In other words, I don't know how to make sure p_1 and p_2 can be non-constant functions.

Assignment 5 due on Monday October 6th at 5PM

Question 1

In lecture we studied the time dilation of a slowly moving object in a weak stationary gravitational field. We found that the frequency difference of identical clocks located at points $\mathbf{x}_1$ and $\mathbf{x}_2$ and at rest in the gravitational field is

$$\frac{\nu_2 - \nu_1}{\nu_0} = \frac{\Delta\nu}{\nu_0} = \phi_2 - \phi_1, \tag{133}$$

where ϕ is the gravitational potential. Generalize this formula to the case when they are moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which are small compared to the speed of light. This results in contributions from both gravity and special relativity.

Answer

In a weak stationary gravitational field, the metric can be approximated as

$$\begin{aligned} d\tau^2 =& -\left(g_{\mu\nu} dx^\mu dx^\nu\right) & (134) \\ =&(1+2\phi)dt^2 - (1-2\phi)(dx^2 + dy^2 + dz^2), & (135) \end{aligned}$$

where ϕ is the gravitational potential and $|\phi| \ll 1$. For a clock moving with velocity $\mathbf{v} = (v_x, v_y, v_z)$, we have $dx = v_x dt$, $dy = v_y dt$, and $dz = v_z dt$. Therefore, the line element becomes

$$d\tau^2 = (1+2\phi)dt^2 - (1-2\phi)(v_x^2 + v_y^2 + v_z^2)dt^2. \tag{136}$$

This simplifies to

$$d\tau^2 = \left[(1+2\phi) - (1-2\phi)v^2\right] dt^2 \tag{137}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. Hence, if we consider the $dt/d\tau$, we will have

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{1}{\sqrt{(1+2\phi) - (1-2\phi)v^2}} = \frac{1}{\sqrt{1+2\phi - v^2 + 2\phi v^2}} \approx \frac{1}{\sqrt{1+2\phi - v^2}} & (138) \\ &= (1+2\phi - v^2)^{-\frac{1}{2}}, & (139) \end{aligned}$$

where we have used the binomial expansion. Therefore, the frequency of the clock moving with velocity $\mathbf{v}$ in a weak stationary gravitational field is

$$\nu_i = \frac{1}{dt_i} = \frac{1}{d\tau}\frac{d\tau}{dt_i} = \nu_0\sqrt{1+2\phi_i - v_i^2} \approx \nu_0\left(1+\phi_i - \frac{v_i^2}{2}\right), \quad i = 1, 2. \tag{140}$$

where $\nu_0 = 1/d\tau$ is the frequency of the clock at rest in the absence of gravitational field. Thus, we have

$$\frac{\Delta\nu}{\nu_0} = \frac{\nu_2 - \nu_1}{\nu_0} = (\phi_2 - \phi_1) - \frac{1}{2}(v_2^2 - v_1^2). \tag{141}$$

□

Question 2

Prove that $V_{\mu;\nu} \equiv \partial V_\mu / \partial x^\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$ transforms as a second rank tensor, similar to how we proved it for ${V^\mu}_{;\nu}$ in class, assuming that V transforms as a 4 -vector.

Answer

First, by the definition, we have

$$
\begin{aligned}
{V'}_\mu &= \frac{\partial x^\nu}{\partial x'^\mu} V_\nu, && (142)\\
\frac{\partial {V'}_\mu}{\partial x'^\nu} &= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x'^\mu} V_\beta\right) = \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial^2 x^\beta}{\partial x^\alpha \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} && (143)\\
&= \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} && (144)\\
{\Gamma'}^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} && (145)\\
&= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}, \quad \text{by } \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu}\right) = 0, && (146)\\
{\Gamma'}^\lambda_{\mu\nu}{V'}_\lambda &= \left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right)\frac{\partial x^\beta}{\partial x'^\lambda} V_\beta && (147)\\
&= \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma}\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^\lambda} V_\beta + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^\lambda} V_\beta && (148)\\
&= \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma}{\delta^\beta}_\rho V_\beta + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}{\delta^\beta}_\rho V_\beta && (149)\\
&= \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} V_\rho + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho. && (150)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
{V'}_{\mu;\nu} &= \frac{\partial {V'}_\mu}{\partial x'^\nu} - {\Gamma'}^\lambda_{\mu\nu}{V'}_\lambda && (151)\\
&= \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} - \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} V_\rho - \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho && (152)\\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} - \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} V_\rho && (153)\\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\left(\frac{\partial V_\beta}{\partial x^\alpha} - \Gamma^\rho_{\alpha\beta} V_\rho\right) = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\rho_{\alpha\beta} V_\rho\right) && (154)\\
&= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} V_{\alpha;\beta}. && (155)
\end{aligned}
$$

This shows that $V_{\mu;\nu}$ transforms as a second rank tensor. $\square$

Question 3

Show that

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = A_{\mu\nu,\lambda} + A_{\lambda\mu,\nu} + A_{\nu\lambda,\mu}, \tag{156}$$

when $A_{\mu\nu}$ is an anti-symmetric tensor.

Answer

By the definition of covariant derivative, we have

$$A_{\mu\nu;\lambda} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho}, \tag{157}$$

$$A_{\lambda\mu;\nu} = \frac{\partial A_{\lambda\mu}}{\partial x^\nu} - \Gamma^\rho_{\lambda\nu} A_{\rho\mu} - \Gamma^\rho_{\mu\nu} A_{\lambda\rho}, \tag{158}$$

$$A_{\nu\lambda;\mu} = \frac{\partial A_{\nu\lambda}}{\partial x^\mu} - \Gamma^\rho_{\nu\mu} A_{\rho\lambda} - \Gamma^\rho_{\lambda\mu} A_{\nu\rho}. \tag{159}$$

Adding them together, we have

$$\begin{aligned} A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = & \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} \\ & - \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho} - \Gamma^\rho_{\lambda\nu} A_{\rho\mu} - \Gamma^\rho_{\mu\nu} A_{\lambda\rho} - \Gamma^\rho_{\nu\mu} A_{\rho\lambda} - \Gamma^\rho_{\lambda\mu} A_{\nu\rho}. \end{aligned} \tag{160}$$

Since $A_{\mu\nu}$ is an anti-symmetric tensor, we have $A_{\mu\nu} = -A_{\nu\mu}$. Therefore, we have

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} \tag{161}$$

$$= A_{\mu\nu,\lambda} + A_{\lambda\mu,\nu} + A_{\nu\lambda,\mu}. \tag{162}$$

$\square$

Assignment 6 due on Monday October 13th at 10PM

Question 1

Show that

$$g_{\mu\nu,\gamma} = \Gamma_{\mu\nu\gamma} + \Gamma_{\nu\mu\gamma}, \tag{163}$$

where

$$\Gamma_{\mu\nu\lambda} \equiv g_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda}, \tag{164}$$

and that

$$g_{,\gamma} = g g^{\mu\nu} g_{\mu\nu,\gamma}. \tag{165}$$

Answer

By definition of the covariant derivative, we have

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}_{\nu\lambda} g_{\mu\sigma} = 0. \tag{166}$$

Rearranging gives

$$g_{\mu\nu,\lambda} = \Gamma^{\sigma}_{\nu\lambda} g_{\mu\sigma} + \Gamma^{\sigma}_{\mu\lambda} g_{\sigma\nu}. \tag{167}$$

Using the definition of $\Gamma_{\mu\nu\lambda}$, we can rewrite this as

$$g_{\mu\nu,\lambda} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}. \tag{168}$$

This proves the first part.

Second, we have

$$g = -\det(g_{\mu\nu}). \tag{169}$$

In the lecture notes, we showed that for any invertible matrix M, the variation of its determinant is given by

$$\frac{\partial}{\partial x^\lambda} \ln\det(M) = \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial x^\lambda}\right). \tag{170}$$

Here we can identify M with $g_{\mu\nu}$, and thus M^{-1} with $g^{\mu\nu}$. Therefore, we have

$$\frac{\partial}{\partial x^\lambda} \ln(g) = \frac{1}{g} g_{,\lambda} = g^{\mu\nu} g_{\mu\nu,\lambda}. \tag{171}$$

Rearranging gives

$$g_{,\lambda} = g g^{\mu\nu} g_{\mu\nu,\lambda}. \tag{172}$$

This proves the second part. $\square$

Question 2

The metric for the surface of a sphere of radius a is determined by

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{173}$$

(i) Calculate the components of the affine connection. It is conventional to denote the components as $\Gamma^{\phi}_{\theta\phi}$ and similarly for the other ones. In other words, $x^1 = \theta$ and $x^2 = \phi$.

(ii) Referring to figure 3.2 of the textbook, parallel transport a vector along two different paths, starting from a point on the equator $\theta = \pi/2, \phi = 0$, and ending at the north pole. Path A is along a line of fixed longitude $\phi = 0$. Path B first goes along the equator to $\phi = \pi/2$ and then goes north along a line of fixed longitude. Initially the vector you are parallel transporting points in the $-\hat{\theta}$ direction. What is the angle between the two vectors at the north pole? (You must solve the parallel transport equation, just drawing a picture is not sufficient.)

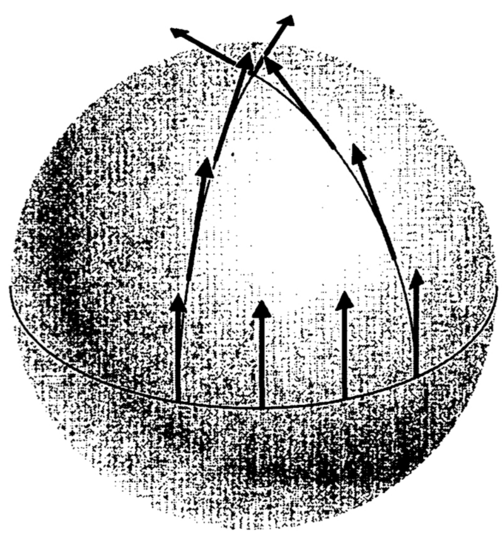

FIGURE 3.2 **Parallel transport on a two-sphere. On a curved manifold, the result of parallel transport can depend on the path taken.**

Figure 1: From Sean Carroll's textbook *Spacetime and Geometry*, figure 3.2.

Answer

(i)

The metric components are given by

$$g_{\theta\theta} = a^2, \quad g_{\phi\phi} = a^2 \sin^2\theta, \quad g_{\theta\phi} = g_{\phi\theta} = 0. \tag{174}$$

The inverse metric components are

$$g^{\theta\theta} = \frac{1}{a^2}, \quad g^{\phi\phi} = \frac{1}{a^2 \sin^2\theta}, \quad g^{\theta\phi} = g^{\phi\theta} = 0. \tag{175}$$

Using the formula for the Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right), \tag{176}$$

we can compute the non-zero components of the affine connection:

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta} \left(g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta}\right) = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\sin\theta\cos\theta, \tag{177}$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \frac{1}{2} g^{\phi\phi} \left(g_{\phi\theta,\phi} + g_{\phi\phi,\theta} - g_{\theta\phi,\phi}\right) = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\theta} = \cot\theta, \tag{178}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}\right) = 0, \tag{179}$$

$$\Gamma^\phi_{\phi\phi} = \frac{1}{2} g^{\phi\phi} \left(g_{\phi\phi,\phi} + g_{\phi\phi,\phi} - g_{\phi\phi,\phi}\right) = 0, \tag{180}$$

$$\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \frac{1}{2} g^{\theta\theta} \left(g_{\theta\theta,\phi} + g_{\theta\phi,\theta} - g_{\phi\theta,\theta}\right) = 0, \tag{181}$$

$$\Gamma^\phi_{\theta\theta} = \frac{1}{2} g^{\phi\phi} \left(g_{\phi\theta,\theta} + g_{\phi\theta,\theta} - g_{\theta\theta,\phi}\right) = 0. \tag{182}$$

(ii)

The parallel transport equation is given by

$$0 = \frac{DV^\mu}{D\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} V^\rho, \tag{183}$$

where τ is a parameter along the curve. We will solve this equation for both paths A and B. Initially, the vector points in the $-\hat{\theta}$ direction, so we have

$$V^\theta(\tau = 0) = -1, \quad V^\phi(\tau = 0) = 0. \tag{184}$$

Path A: Along path A, we have $\phi = 0$ and θ varies from $\pi/2$ to 0. We can choose $\tau = \theta$, so that $d\theta/d\tau = 1$ and $d\phi/d\tau = 0$. The parallel transport equations for the components of V^μ are then

$$\begin{aligned} 0 &= \frac{dV^\theta}{d\tau} + \Gamma^\theta_{\theta\theta}\frac{d\theta}{d\tau}V^\theta + \Gamma^\theta_{\phi\theta}\frac{d\phi}{d\tau}V^\theta + \Gamma^\theta_{\theta\phi}\frac{d\theta}{d\tau}V^\phi + \Gamma^\theta_{\phi\phi}\frac{d\phi}{d\tau}V^\phi, && (185)\\ &= \frac{dV^\theta}{d\tau} + 0 + 0 + 0 + 0, && (186)\\ &= \frac{dV^\theta}{d\tau}, && (187)\end{aligned}$$

and

$$0 = \frac{dV^\phi}{d\tau} + \Gamma^\phi_{\theta\theta}\frac{d\theta}{d\tau}V^\theta + \Gamma^\phi_{\phi\theta}\frac{d\phi}{d\tau}V^\theta + \Gamma^\phi_{\theta\phi}\frac{d\theta}{d\tau}V^\phi + \Gamma^\phi_{\phi\phi}\frac{d\phi}{d\tau}V^\phi, \tag{188}$$

$$= \frac{dV^\phi}{d\tau} + 0 + 0 + \cot\theta V^\phi + 0, \tag{189}$$

$$= \frac{dV^\phi}{d\tau} + \cot\theta V^\phi. \tag{190}$$

The first equation implies that V^θ is constant along path A. Since it starts at -1, we have

$$V^\theta(\theta) = V^\theta = -1. \tag{191}$$

The second equation is a first-order linear ordinary differential equation for V^ϕ. We can solve it using an integrating factor:

$$\frac{dV^\phi}{d\tau} + \cot\theta V^\phi = \frac{dV^\phi}{d\theta} + \cot\theta V^\phi = 0. \tag{192}$$

By `Mathematica`, we find that

$$V^\phi(\theta) = \frac{C}{\sin\theta}, \tag{193}$$

where C is an integration constant. Since $V^\phi(\theta = \pi/2) = 0$, we have $C = 0$. Therefore, along path A, we have

$$V^\theta = -1, \quad V^\phi = 0. \tag{194}$$

Path B: Along path B, we first move along the equator from $\phi = 0$ to $\phi = \pi/2$ at fixed $\theta = \pi/2$, and then move north along a line of fixed longitude from $\theta = \pi/2$ to $\theta = 0$ at fixed $\phi = \pi/2$. We will solve the parallel transport equation in two segments.

Segment 1: Along the equator, we have $\theta = \pi/2$ and ϕ varies from 0 to $\pi/2$. We can choose $\tau = \phi$, so that $d\phi/d\tau = 1$ and $d\theta/d\tau = 0$. The parallel transport equations for the components of V^μ are then

$$0 = \frac{dV^\theta}{d\tau} + \Gamma^\theta_{\theta\theta}\frac{d\theta}{d\tau}V^\theta + \Gamma^\theta_{\phi\theta}\frac{d\phi}{d\tau}V^\theta + \Gamma^\theta_{\theta\phi}\frac{d\theta}{d\tau}V^\phi + \Gamma^\theta_{\phi\phi}\frac{d\phi}{d\tau}V^\phi, \tag{195}$$

$$= \frac{dV^\theta}{d\tau} + 0 + 0 + 0 + 0, \tag{196}$$

$$= \frac{dV^\theta}{d\tau}, \tag{197}$$

and

$$0 = \frac{dV^\phi}{d\tau} + \Gamma^\phi_{\theta\theta}\frac{d\theta}{d\tau}V^\theta + \Gamma^\phi_{\phi\theta}\frac{d\phi}{d\tau}V^\theta + \Gamma^\phi_{\theta\phi}\frac{d\theta}{d\tau}V^\phi + \Gamma^\phi_{\phi\phi}\frac{d\phi}{d\tau}V^\phi, \tag{198}$$

$$= \frac{dV^\phi}{d\tau} + 0 + \cot(\pi/2)\cdot 1 \cdot V^\theta + 0 + 0, \tag{199}$$

$$= \frac{dV^\phi}{d\tau}. \tag{200}$$

The first equation implies that V^θ is constant along segment 1. Since it starts at -1, we have

$$V^\theta(\phi) = V^\theta = -1. \tag{201}$$

The second equation implies that V^ϕ is also constant along segment 1. Since it starts at 0, we have

$$V^\phi(\phi) = V^\phi = 0. \tag{202}$$

At the end of segment 1, we have

$$V^\theta = -1, \quad V^\phi = 0. \tag{203}$$

Segment 2: similarly, we can apply the same procedure as in path A, and we find that at the end of segment 2, we have

$$V^\theta = -1, \quad V^\phi = 0. \tag{204}$$

Conclusion: Both paths A and B yield the same vector along their paths direction at the north pole:

$$V^\theta = -1, \quad V^\phi = 0. \tag{205}$$

However, the basis vectors at the north pole are not well-defined in the ϕ direction, since all lines of longitude converge there. To find the angle between the two vectors at the north pole, we can consider their projections onto the tangent plane at the north pole in xyz-coordinates. The north pole corresponds to the point $(0,0,a)$ in Cartesian coordinates. The tangent plane at this point is spanned by the vectors $\hat{x}$ and $\hat{y}$. The vector V in spherical coordinates can be expressed in Cartesian coordinates as

$$\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}, \tag{206}$$

where we have (with $r = a = 1$ for simplicity)

$$V_x = V^\theta \frac{\partial x}{\partial \theta} + V^\phi \frac{\partial x}{\partial \phi} = V^\theta \cos\theta \cos\phi - V^\phi \sin\theta \sin\phi, \tag{207}$$

$$V_y = V^\theta \frac{\partial y}{\partial \theta} + V^\phi \frac{\partial y}{\partial \phi} = V^\theta \cos\theta \sin\phi + V^\phi \sin\theta \cos\phi, \tag{208}$$

$$V_z = V^\theta \frac{\partial z}{\partial \theta} + V^\phi \frac{\partial z}{\partial \phi} = -V^\theta \sin\theta + V^\phi \cdot 0. \tag{209}$$

For the path A, at the north pole, $\theta = 0, \phi = 0$, so we have

$$V_x^A = -1 \cdot 1 \cdot 1 - 0 \cdot 0 \cdot 0 = -1, \tag{210}$$

$$V_y^A = -1 \cdot 1 \cdot 0 + 0 \cdot 0 \cdot 1 = 0, \tag{211}$$

$$V_z^A = -(-1) \cdot 0 + 0 = 0. \tag{212}$$

For the path B, at the north pole, $\theta = 0, \phi = \pi/2$, so we have

$$V_x^B = -1 \cdot 1 \cdot 0 - 0 \cdot 0 \cdot 1 = 0, \tag{213}$$

$$V_y^B = -1 \cdot 1 \cdot 1 + 0 \cdot 0 \cdot 0 = -1, \tag{214}$$

$$V_z^B = -(-1) \cdot 0 + 0 = 0. \tag{215}$$

Finally, we can compute the angle α between the two vectors using the dot product. It is straightforward to see that

$$\vec{V}^A \cdot \vec{V}^B = V_x^A V_x^B + V_y^A V_y^B + V_z^A V_z^B = 0 + 0 + 0 = 0. \tag{216}$$

Meaning that the two vectors are orthogonal to each other. Therefore, the angle between the two vectors at the north pole is $\alpha = 90°$. □

Remark: I am not sure if this is the expected answer, since in the textbook it is mentioned that the angle should be 90°. Also, if I do not convert to Cartesian coordinates, I get the same vector components for both paths at the north pole, which does not make sense since the basis vectors are not well-defined there. However, I still do not know how to get the expected answer of 90° for original spherical coordinates. If you have any suggestions, please let me know.

Question 3

Starting with the energy-momentum tensor of a massless scalar field ϕ

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}, \tag{217}$$

derive the equation of motion for ϕ in the presence of a gravitational field.

Answer

First, notice that the normal derivative can be replaced by the covariant derivative for scalar field ϕ, so this energy-momentum has considered in the presence of a gravitational field. But for clarity, we keep the comma notation for partial derivatives. Next, we can use $g^{\alpha\beta}$ to raise the indices of $T_{\mu\nu}$:

$$T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta} = \phi^{,\mu}\phi^{,\nu} - \frac{1}{2}g^{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}. \tag{218}$$

The equation of motion for ϕ can be derived from the conservation of the energy-momentum tensor, which states that

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{219}$$

Expanding the covariant derivative, we have

$$\begin{aligned}
T^{\mu\nu}{}_{;\nu} &= (\phi^{,\mu}\phi^{,\nu})_{;\nu} - \frac{1}{2}(g^{\mu\nu}\phi_{,\sigma}\phi^{,\sigma})_{;\nu} && (220)\\
&= (\phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu}) - \frac{1}{2}(g^{\mu\nu}{}_{;\nu}\phi_{,\sigma}\phi^{,\sigma} + g^{\mu\nu}(\phi_{,\sigma}\phi^{,\sigma})_{;\nu}) && (221)\\
&= \phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - \frac{1}{2}g^{\mu\nu}(\phi_{,\sigma}\phi^{,\sigma})_{;\nu} && (222)\\
&= \phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - \frac{1}{2}g^{\mu\nu}(g_{\alpha\beta}\phi^{,\alpha}\phi^{,\beta})_{;\nu} && (223)\\
&= \phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - \frac{1}{2}g^{\mu\nu}(g_{\alpha\beta;\nu}\phi^{,\alpha}\phi^{,\beta} + g_{\alpha\beta}\phi^{,\alpha}{}_{;\nu}\phi^{,\beta} + g_{\alpha\beta}\phi^{,\alpha}\phi^{,\beta}{}_{;\nu}) && (224)\\
&= \phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - \frac{1}{2}g^{\mu\nu}(0 + \phi^{,\alpha}{}_{;\nu}\phi_{,\alpha} + \phi^{,\alpha}\phi_{,\alpha;\nu}) && (225)\\
&= \phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - \frac{1}{2}g^{\mu\nu}(2\phi^{,\alpha}\phi_{,\alpha;\nu}) && (226)\\
&= \phi^{,\mu}{}_{;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha;\nu} && (227)\\
&= g^{\alpha\mu}\phi_{,\alpha;\nu}\phi^{,\nu} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha;\nu} && (228)\\
&= g^{\nu\mu}\phi_{,\nu;\alpha}\phi^{,\alpha} + \phi^{,\mu}\phi^{,\nu}{}_{;\nu} - g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha;\nu}, && (229)
\end{aligned}$$

where we rewrite the first term in the last step by swapping the dummy indices α and ν. We have to show that the first and the last terms cancel each other. Indeed, we have

$$
\begin{aligned}
\phi_{,\nu;\alpha} &= \phi_{,\nu,\alpha} - \Gamma^{\sigma}_{\nu\alpha}\phi_{,\sigma} && (230)\\
&= \phi_{,\alpha,\nu} - \Gamma^{\sigma}_{\alpha\nu}\phi_{,\sigma} && (231)\\
&= \phi_{,\alpha;\nu}, && (232)
\end{aligned}
$$

Hence, we have

$$g^{\nu\mu}\phi_{,\nu;\alpha}\phi^{,\alpha} = g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha;\nu}. \tag{233}$$

In the end, we have

$$T^{\mu\nu}{}_{;\nu} = \phi^{,\mu}\phi^{,\nu}{}_{;\nu} = \phi^{,\mu}\phi^{;\nu}{}_{;\nu} = 0. \tag{234}$$

Since $\phi^{,\mu}$ is not necessarily zero (otherwise ϕ is a trivial constant function), we must have

$$\phi^{;\nu}{}_{;\nu} = 0. \tag{235}$$

Remark: This is the covariant form of the d'Alembertian operator acting on ϕ, often denoted as $\Box\phi = 0$. In flat spacetime, this reduces to the Kelin-Gordon equation for a massless scalar field. $\Box$

Assignment 7 due on Monday October 20th at 10PM

Question 1

The metric for the surface of a sphere of radius a is given by

$$g_{\theta\theta} = a^2, \quad g_{\phi\phi} = a^2 \sin^2\theta, \quad g_{\theta\phi} = g_{\phi\theta} = 0. \tag{236}$$

Calculate the Gaussian curvature $K = -\frac{1}{2}R$ for this space. Then calculate the Gaussian curvature for a space with a metric given by

$$g_{xx} = \frac{a^2(1-y^2)}{(1-x^2-y^2)^2}, \quad g_{yy} = \frac{a^2(1-x^2)}{(1-x^2-y^2)^2}, \quad g_{xy} = g_{yx} = \frac{a^2 xy}{(1-x^2-y^2)^2}, \tag{237}$$

where $x^2 + y^2 < 1$. Notice that the distance between two points is unbounded because of the denominators. Can you imagine such a space?

Answer

First, we calculate the Christoffel symbols for the sphere metric by last Assignment's formula:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \tag{238}$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta. \tag{239}$$

Then we can calculate the Riemann tensor component:

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi} & (240)\\ &= \partial_\theta(-\sin\theta\cos\theta) - 0 + 0 - (-\sin\theta\cos\theta)(\cot\theta) & (241)\\ &= -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta. & (242) \end{aligned}$$

Raising the first index, we have

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda} R^{\lambda}{}_{\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta. \tag{243}$$

Next, we calculate the Ricci tensor component:

$$R_{\phi\phi} = R^{\theta}{}_{\phi\theta\phi} + R^{\phi}{}_{\phi\phi\phi} = R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta, \tag{244}$$

$$R_{\theta\theta} = R^{\theta}{}_{\theta\theta\theta} + R^{\phi}{}_{\theta\phi\theta} = R^{\phi}{}_{\theta\phi\theta} = g^{\phi\lambda} R_{\lambda\theta\phi\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = \frac{1}{a^2\sin^2\theta} a^2\sin^2\theta = 1. \tag{245}$$

The Ricci scalar is then

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{1}{a^2} + \frac{1}{a^2 \sin^2\theta} \sin^2\theta = \frac{2}{a^2}. \tag{246}$$

Thus, the Gaussian curvature is

$$K = -\frac{1}{2} R = -\frac{1}{a^2}. \tag{247}$$

By the definition of Christoffel symbols, we have

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}). \tag{248}$$

Next, we calculate the Christoffel symbols for the second metric:

$$\Gamma^x_{xx} = g^{xx} \partial_x g_{xx} + g^{xy} \left(\partial_x g_{xy} - \frac{1}{2} \partial_y g_{xx} \right) = \frac{2x}{1 - x^2 - y^2}, \tag{249}$$

$$\Gamma^x_{yy} = g^{xx} \left(-\frac{1}{2} \partial_x g_{yy} \right) + g^{xy} \left(\partial_y g_{yy} - \frac{1}{2} \partial_y g_{yy} \right) = -\frac{2x}{1 - x^2 - y^2}, \tag{250}$$

$$\Gamma^x_{xy} = g^{xx} \left(\frac{1}{2} \partial_y g_{xx} \right) + g^{xy} \left(\frac{1}{2} \partial_x g_{yy} \right) = \frac{2y}{1 - x^2 - y^2}, \tag{251}$$

$$\Gamma^y_{yy} = g^{yy} \partial_y g_{yy} + g^{yx} \left(\partial_y g_{yx} - \frac{1}{2} \partial_x g_{yy} \right) = \frac{2y}{1 - x^2 - y^2}, \tag{252}$$

$$\Gamma^y_{xx} = g^{yy} \left(-\frac{1}{2} \partial_y g_{xx} \right) + g^{yx} \left(\partial_x g_{xx} - \frac{1}{2} \partial_x g_{xx} \right) = -\frac{2y}{1 - x^2 - y^2}, \tag{253}$$

$$\Gamma^y_{xy} = g^{yy} \left(\frac{1}{2} \partial_x g_{yy} \right) + g^{yx} \left(\frac{1}{2} \partial_y g_{xx} \right) = \frac{2x}{1 - x^2 - y^2}. \tag{254}$$

Using Mathematica to calculate the Riemann tensor, Ricci tensor and Ricci scalar, we find that the Ricci scalar is

$$R = -\frac{2}{a^2}. \tag{255}$$

Thus, the Gaussian curvature is

$$K = -\frac{1}{2} R = \frac{1}{a^2}. \tag{256}$$

This space is known as the Poincaré disk model of hyperbolic geometry. In this model, the entire hyperbolic plane is represented within the unit disk, and distances increase rapidly as one approaches the boundary of the disk. Although the disk appears finite, the geometry within it is infinite, allowing for unbounded distances between points.

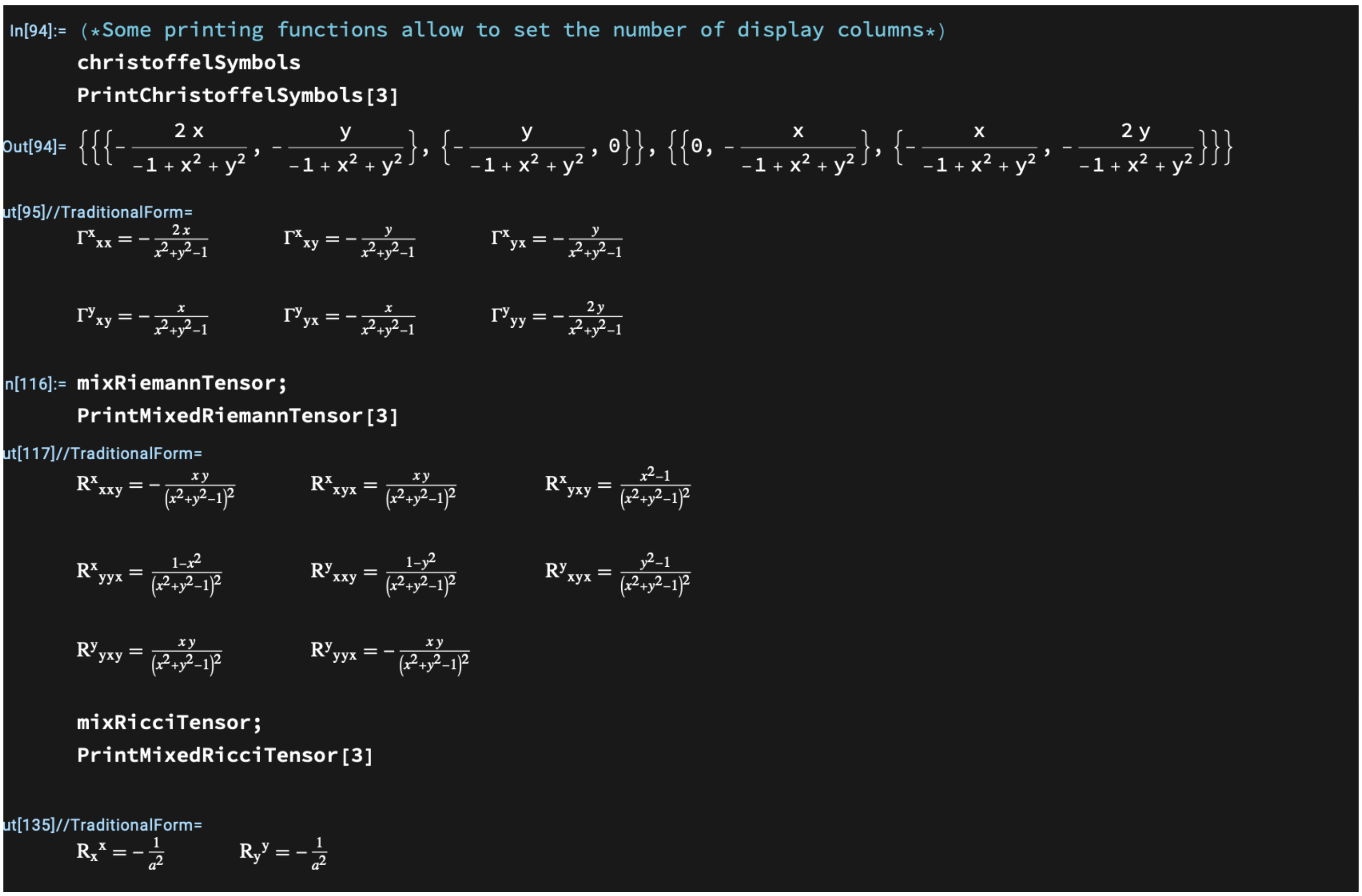

Figure 2: Calculation of Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar in Mathematica

Question 2

In class we were to led to Einstein's field equation with the inclusion of a cosmological constant.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{257}$$

Find the nonrelativistic, static, weak field limit of this equation. The constant Λ has dimension of $1/\text{length}^2$. Calculate the numerical value of Λ based on the numerical value of the dark energy inferred from cosmological observations.

Answer

In the nonrelativistic, static, weak field limit, we can approximate the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{258}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small perturbation. In this limit, the energy-momentum tensor $T_{\mu\nu}$ is dominated by the energy density ρ, so we have

$$T_{00} \approx \rho, \quad T_{0i} \approx 0, \quad T_{ij} \approx 0. \tag{259}$$

The Ricci tensor and Ricci scalar can be approximated as

$$R_{00} \approx -\frac{1}{2}\nabla^2 h_{00}, \tag{260}$$

$$R \approx -\nabla^2 h_{00}. \tag{261}$$

Substituting these approximations into Einstein's field equation, we get

$$-\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}\eta_{00}(-\nabla^2 h_{00}) + \Lambda\eta_{00} = -8\pi G\rho. \tag{262}$$

Simplifying this equation, we find

$$\nabla^2 h_{00} = 16\pi G\rho - 2\Lambda. \tag{263}$$

In the Newtonian limit, the gravitational potential Φ is related to h_{00} by

$$h_{00} = -2\Phi. \tag{264}$$

Thus, we have

$$\nabla^2\Phi = 4\pi G\rho - \Lambda. \tag{265}$$

This is the modified Poisson equation in the presence of a cosmological constant. For the numerical value of Λ, we can use the observed value of dark energy density $\rho_\Lambda \approx 6.91 \times 10^{-30}$ g/cm^3. The relationship between Λ and ρ_Λ is given by (see equation (14) in this Paper by Sean Carroll):

$$\Lambda = 8\pi G\rho_\Lambda. \tag{266}$$

Using $G \approx 6.674 \times 10^{-8}$ cm^3 g^{-1} s^{-2}, we find

$$\Lambda \approx 8\pi(6.674 \times 10^{-8})(6.91 \times 10^{-30}) \text{ cm}^{-2} \approx 1.19 \times 10^{-56} \text{ cm}^{-2}. \tag{267}$$

Thus, the numerical value of the cosmological constant Λ based on the observed dark energy density is approximately 1.19×10^{-56} cm^{-2}.

Assignment 8 due on Monday November 3 at 10PM

Question 1

In lecture we showed that $P^0 = M$ for the Schwarzschild solution in standard coordinates. Calculate P^z and show that it is zero. Since the metric is isotropic this shows that all components of the 3-momentum are zero.

Answer

$$\begin{aligned}
P^j =& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij} + \partial_t h_{ij} + \partial_k h_{0k}\delta_{ij} - \partial_i h_{0j}\Bigg)n^i r^2 d\Omega && (268)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij} + \partial_t h_{ij} + \partial_k h_{0k}\delta_{ij} - \partial_i h_{0j}\Bigg)\frac{x^i}{r} r^2 d\Omega && (269)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij} + \partial_t h_{ij} + \partial_k h_{0k}\delta_{ij} - \partial_i h_{0j}\Bigg)x^i r d\Omega && (270)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}\delta_{ij}x^i + \partial_t h_{ij}x^i + \partial_k h_{0k}\delta_{ij}x^i - \partial_i h_{0j}x^i\Bigg) r d\Omega && (271)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}x^j + \partial_t h_{ij}x^i + \partial_k h_{0k}x^j - \partial_i h_{0j}x^i\Bigg) r d\Omega && (272)\\
=& \frac{-1}{16\pi G}\int\Bigg(-\partial_t h_{kk}x^j + \partial_t h_{ij}x^i\Bigg) r d\Omega \quad \text{(since } h_{0j} = h_{0k} = 0 \text{ for Schwarzschild metric)} && (273)\\
=& 0 \quad \text{(since } h_{\mu\nu} \text{ is time-independent for Schwarzschild metric)} \quad \square && (274)
\end{aligned}$$

Question 2

In lecture we were given the components of the affine connection $\Gamma^\lambda_{\mu\nu}$ for the Scharzschild solution in standard coordinates. Using these, calculate one component of the Riemann-Christoffel curvature tensor, namely R^t_{rtr}. This is nonvanishing everywhere and goes to zero as $r \to \infty$. This shows that space is curved even though both the Ricci tensor and curvature scalar vanish.

Answer

By the definition of the Riemann-Christoffel curvature tensor, we have

$$R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} - \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\mu\nu}. \tag{275}$$

Thus, we have

$$R^t{}_{rtr} = \partial_t \Gamma^t_{rr} - \partial_r \Gamma^t_{rt} + \Gamma^t_{t\sigma}\Gamma^\sigma_{rr} - \Gamma^t_{r\sigma}\Gamma^\sigma_{rt}. \tag{276}$$

From the lecture notes, we have

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}, \quad \Gamma^t_{rr} = 0. \tag{277}$$

Thus, we have (by Mathematica)

$$R^t_{rtr} = -\partial_r \Gamma^t_{rt} + \Gamma^t_{tt}\Gamma^t_{rr} + \Gamma^t_{tr}\Gamma^r_{rr} - \Gamma^t_{rt}\Gamma^t_{rt} - \Gamma^t_{rr}\Gamma^r_{rt} \tag{278}$$

$$= \frac{2M}{r^2(r-2M)} \tag{279}$$

We can see that R^t_{rtr} is nonvanishing everywhere and it goes to zero as $r \to \infty$. $\square$

Question 3

A photon moves in the Schwarzschild metric in the equatorial plane $\theta = \pi/2$. Using standard coordinates, show that the shape of the orbit is given by the solution to the differential equation

$$\frac{d^2w}{d\phi^2} + w = 3w^2, \tag{280}$$

where $w = GM/r$. Assuming that $|w| \ll 1$, solve this equation iteratively to find the deflection angle $\Delta\phi$, and show that it agrees with the answer obtained in class by other means.

Answer

For a photon moving in the Schwarzschild metric in the equatorial plane $\theta = \pi/2$, we have the following equations of motion:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2GM}{r}\right)\frac{L^2}{r^2}, \tag{281}$$

$$\frac{d\phi}{d\lambda} = \frac{L}{r^2}, \tag{282}$$

where λ is an affine parameter along the photon's trajectory, E is the energy per unit mass, and L is the angular momentum per unit mass. Dividing the first equation by the square of the second equation, we have

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\left(E^2 - \left(1 - \frac{2GM}{r}\right)\frac{L^2}{r^2}\right). \tag{283}$$

Rearranging this equation, we have

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4E^2}{L^2} - r^2 + 2GMr. \tag{284}$$

Next, we introduce the variable $w = \frac{GM}{r}$. Thus, we have $r = \frac{GM}{w}$ and $\frac{dr}{d\phi} = -\frac{GM}{w^2}\frac{dw}{d\phi}$. Substituting these into the previous equation, we have

$$\left(-\frac{GM}{w^2}\frac{dw}{d\phi}\right)^2 = \frac{\left(\frac{GM}{w}\right)^4 E^2}{L^2} - \left(\frac{GM}{w}\right)^2 + 2GM\left(\frac{GM}{w}\right). \tag{285}$$

Simplifying this equation, we have

$$\left(\frac{dw}{d\phi}\right)^2 = \frac{E^2(GM)^2}{L^2} - w^2 + 2w^3. \tag{286}$$

Taking the derivative of both sides with respect to ϕ, we have

$$2\frac{dw}{d\phi}\frac{d^2w}{d\phi^2} = -2w\frac{dw}{d\phi} + 6w^2\frac{dw}{d\phi}. \tag{287}$$

Dividing both sides by $2\frac{dw}{d\phi}$, we have

$$\frac{d^2w}{d\phi^2} = -w + 3w^2. \tag{288}$$

Rearranging this equation, we have

$$\frac{d^2w}{d\phi^2} + w = 3w^2. \tag{289}$$

To solve this equation iteratively, we first solve the homogeneous equation:

$$\frac{d^2w_0}{d\phi^2} + w_0 = 0. \tag{290}$$

The general solution to this equation is

$$w_0(\phi) = A\cos\phi + B\sin\phi, \tag{291}$$

where A and B are constants determined by initial conditions. By considering a photon coming from infinity, we set $A = \frac{GM}{b}$ and $B = 0$, where b is the impact parameter. In other words,

$$w_0(0) = \frac{GM}{b}, \quad \left.\frac{dw_0}{d\phi}\right|_{\phi=0} = 0. \tag{292}$$

Thus, we have

$$w_0(\phi) = \frac{GM}{b}\cos\phi. \tag{293}$$

Next, we substitute w_0 into the right-hand side of the original equation to find the first-order correction w_1:

$$\frac{d^2w_1}{d\phi^2} + w_1 = 3w_0^2 = 3\left(\frac{GM}{b}\cos\phi\right)^2 = 3\left(\frac{GM}{b}\right)^2\cos^2\phi. \tag{294}$$

Using the identity $\cos^2\phi = \frac{1+\cos 2\phi}{2}$, we have

$$\frac{d^2w_1}{d\phi^2} + w_1 = \frac{3}{2}\left(\frac{GM}{b}\right)^2(1+\cos 2\phi). \tag{295}$$

By considering a photon coming from infinity, we have

$$w_1(0) = 0, \quad \left.\frac{dw_1}{d\phi}\right|_{\phi=0} = 0. \tag{296}$$

Thus, we have (by mathematica)

$$\begin{aligned} w_1(\phi) &= \frac{2G^2M^2\sin^2\left(\frac{\phi}{2}\right)(\cos(\phi)+2)}{b^2} && (297)\\ &= \frac{G^2M^2}{2b^2}(3 - 2\cos\phi - \cos 2\phi). && (298)\end{aligned}$$

Hence the approximate solution up to first order is

$$w(\phi) = w_0(\phi) + w_1(\phi) = \frac{GM}{b}\cos\phi + \frac{G^2M^2}{2b^2}(3 - 2\cos\phi - \cos 2\phi). \tag{299}$$

To find the deflection angle $\Delta\phi$, we set $w(\phi) = 0$ and solve for ϕ when the photon is far away from the mass. Thus, we have

$$0 = \frac{GM}{b}\cos\phi + \frac{G^2M^2}{2b^2}(3 - 2\cos\phi - \cos 2\phi). \tag{300}$$

Expanding $\cos 2\phi = 2\cos^2\phi - 1$, we have

$$\begin{aligned} 0 =& \frac{GM}{b}\cos\phi + \frac{G^2M^2}{2b^2}(4 - 4\cos\phi + 2\cos^2\phi) && (301)\\ =& \frac{G^2M^2}{b^2}\cos^2\phi + \left(\frac{GM}{b} - \frac{2G^2M^2}{b^2}\right)\cos\phi + \frac{2G^2M^2}{b^2} && (302)\\ \approx& \frac{G^2M^2}{b^2}\cos^2\phi + \frac{GM}{b}\cos\phi + \frac{2G^2M^2}{b^2}. && (303)\end{aligned}$$

Solving this quadratic equation for $\cos\phi$, we have

$$\cos\phi = \frac{-\frac{GM}{b} \pm \sqrt{\left(\frac{GM}{b}\right)^2 - 8\left(\frac{GM}{b}\right)^4}}{2\frac{G^2M^2}{b^2}} \approx -2\frac{GM}{b} \tag{304}$$

Thus, we have

$$\phi \approx \frac{\pi}{2} + 2\frac{GM}{b}. \tag{305}$$

Since the photon comes from infinity and goes back to infinity, the total deflection angle is

$$\Delta\phi = 2\left(\phi - \frac{\pi}{2}\right) = \frac{4GM}{b}. \tag{306}$$

This agrees with the answer obtained in class by other means. □

Question 4

The deflection of light by a spherical static body whose physical radius is smaller than its Schwarzschild radius should produce comet-like orbits suffering substantial deflection before returning to infinite radius if the distance of closest approach r_0 becomes comparable to the Schwarzschild radius. Using the differential equation in problem 1 show that there is a critical orbit $w_c(\phi)$, corresponding to a critical radius r_c. Explain what happens when $r_0 > r_c, r_0 = r_c$, and $r_0 < r_c$. Draw some orbits for illustration. This phenomenon was observed by the "Event Horizon Telescope".

Answer

From problem 3, we have the differential equation

$$\frac{d^2w}{d\phi^2} + w = 3w^2, \tag{307}$$

where $w = \frac{GM}{r}$. To find the critical orbit $w_c(\phi)$, we look for a circular orbit where w is constant. Thus, we set $\frac{d^2w}{d\phi^2} = 0$. Thus, we have

$$w_c = 3w_c^2. \tag{308}$$

Solving this equation, we have

$$w_c = 0 \quad \text{or} \quad w_c = \frac{1}{3}. \tag{309}$$

The solution $w_c = 0$ corresponds to an orbit at infinite radius, which is not physically interesting. The solution $w_c = \frac{1}{3}$ corresponds to a critical radius

$$r_c = 3GM. \tag{310}$$

When the distance of closest approach r_0 is greater than the critical radius r_c ($r_0 > r_c$), the photon will be deflected but will eventually escape to infinity, resulting in a hyperbolic-like orbit. When r_0 is equal to the critical radius r_c ($r_0 = r_c$), the photon will spiral around the mass at the critical radius, resulting in a circular orbit. When r_0 is less than the critical radius r_c ($r_0 < r_c$), the photon will be captured by the mass and will spiral inward, eventually falling into the black hole. This results in a trajectory that does not return to infinity.

□

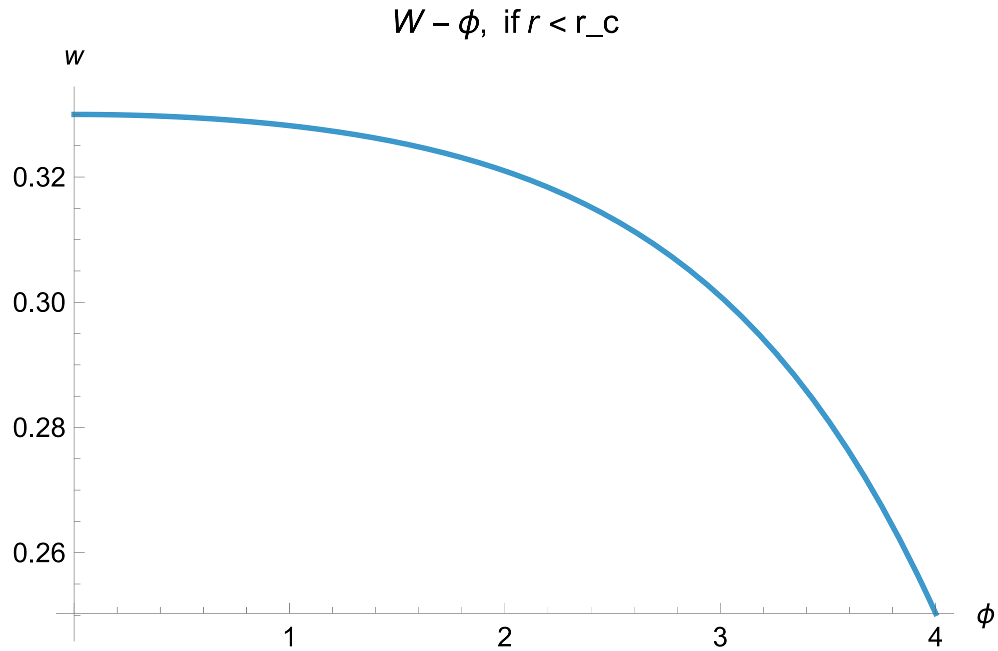

Figure 3: Orbit for $r_0 < r_c$, photon spirals inward and falls into the black hole.

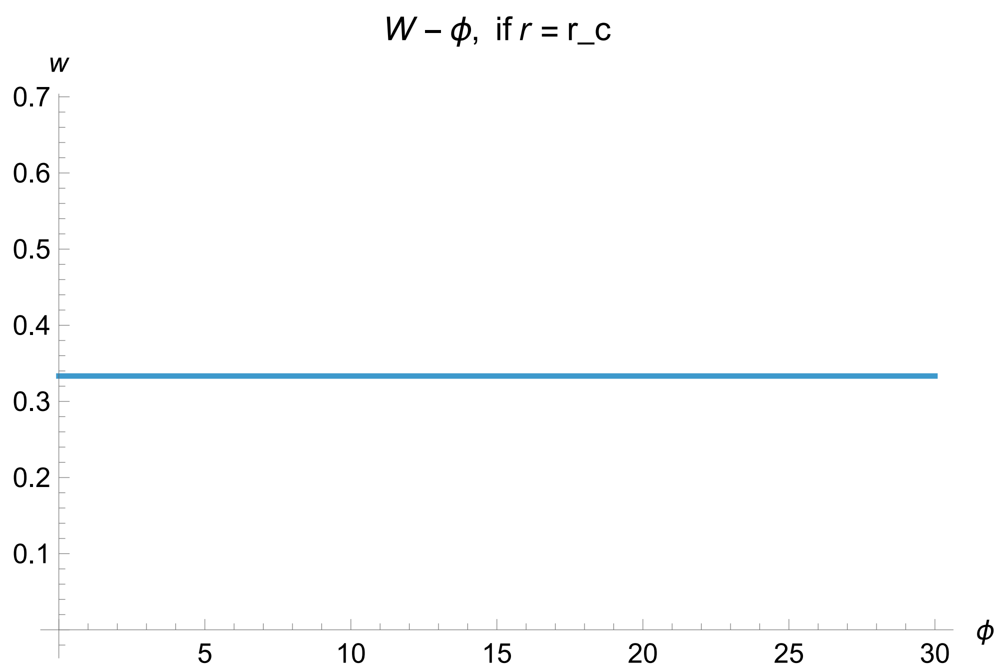

Figure 4: Orbit for $r_0 = r_c$, photon spirals around the mass at the critical radius.

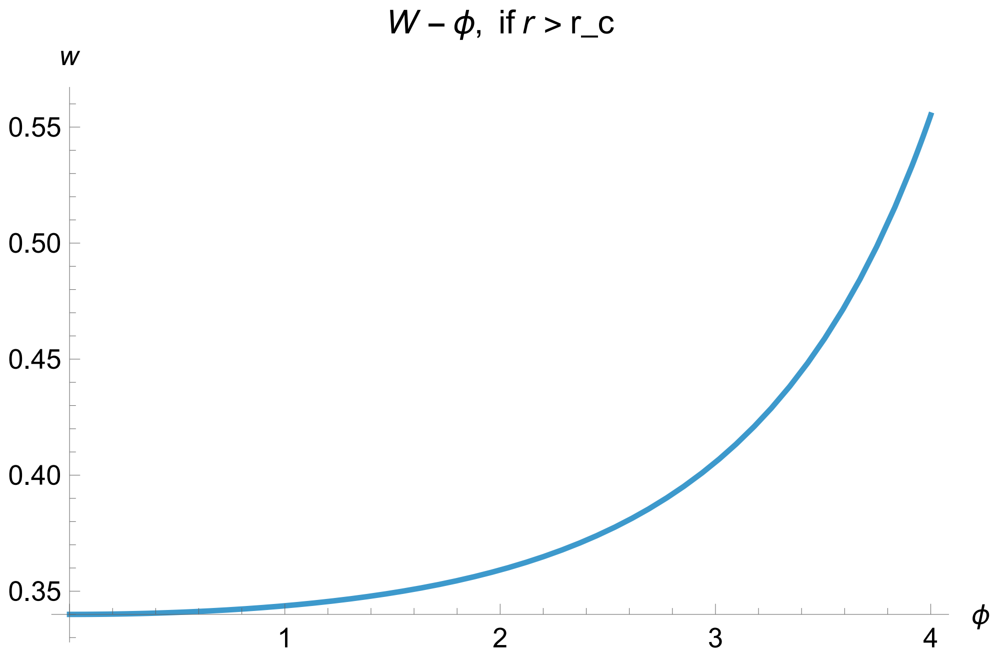

Figure 5: Orbit for $r_0 > r_c$, photon is deflected but escapes to infinity.

Assignment 9 due on Monday November 10 at 10PM

Question 1

Find the metric for the Scharzschild solution in terms of harmonic coordinates t and

$$X_1 = R(r)\cos\phi\sin\theta, \quad X_2 = R(r)\sin\phi\sin\theta, \quad X_3 = R(r)\cos\theta, \tag{311}$$

by solving for the function $R(r)$. The metric should be in the form

$$d\tau^2 = P(R)dt^2 - Q(R)d\mathbf{X}^2 - S(R)(\mathbf{X}\cdot d\mathbf{X})^2. \tag{312}$$

Answer

First we write the Schwarzschild metric in the usual coordinates:

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{313}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

To find $R(r)$, we impose the harmonic coordinate condition $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$. Calculating $\sqrt{-g} = r^2\sin\theta$ and $g^{\mu\nu}$ from the metric, we find that the harmonic condition leads to the differential equation:

$$\frac{d}{dr}\left(r^2\left(1 - \frac{2GM}{r}\right)\frac{dR}{dr}\right) - 2R = 0. \tag{314}$$

This also comes from the equation 8.1.15 in Weinberg's Gravitation and Cosmology book:

$$\frac{d}{dr}\left(r^2 B^{1/2}A^{-1/2}\frac{dR}{dr}\right) - 2A^{1/2}B^{1/2}R = 0, \tag{315}$$

where $A = 1 - \frac{2GM}{r}$ and $B = \left(1 - \frac{2GM}{r}\right)^{-1}$. By Mathematica, the solution to this differential equation is:

$$R(r) = \frac{2c_2(r - GM)\coth^{-1}\left(\frac{GM}{r-GM}\right) - 2(c_1 + c_2)GM + 2c_1 r}{2GM} \tag{316}$$

where c_1 and c_2 are integration constants. To ensure that $R(r) \to r$ as $r \to \infty$, we set $c_1 = GM$ and $c_2 = 0$. Thus, the final solution is:

$$R(r) = r - GM \implies r = R + GM. \tag{317}$$

Substituting this back into the Schwarzschild metric, we get:

$$
\begin{aligned}
d\tau^2 =& \left(1-\frac{2GM}{R+GM}\right)dt^2 - \left(1-\frac{2GM}{R+GM}\right)^{-1}dR^2 - (R+GM)^2 d\Omega^2 && (318)\\
=& \frac{R-GM}{R+GM}dt^2 - \frac{R+GM}{R-GM}dR^2 - (R+GM)^2 d\Omega^2 && (319)\\
=& P(R)dt^2 - Q(R)d\mathbf{X}^2 - S(R)(\mathbf{X}\cdot d\mathbf{X})^2 && (320)
\end{aligned}
$$

Next, we express $dR^2 + R^2 d\Omega^2$ in terms of $d\mathbf{X}^2$ and $(\mathbf{X}\cdot d\mathbf{X})^2$:

$$
\begin{aligned}
d\mathbf{X}^2 &= dX_1^2 + dX_2^2 + dX_3^2 = dR^2 + R^2 d\Omega^2, && (321)\\
(\mathbf{X}\cdot d\mathbf{X})^2 &= R^2 dR^2. && (322)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d\tau^2 =& P(R)dt^2 - Q(R)d\mathbf{X}^2 - S(R)(\mathbf{X}\cdot d\mathbf{X})^2 && (323)\\
=& P(R)dt^2 - Q(R)(dR^2 + R^2 d\Omega^2) - S(R)R^2 dR^2 && (324)\\
=& P(R)dt^2 - \left(Q(R) + S(R)R^2\right)dR^2 - Q(R)R^2 d\Omega^2. && (325)
\end{aligned}
$$

Comparing coefficients, we find:

$$
\begin{aligned}
P(R) &= \frac{R-GM}{R+GM}, && (326)\\
Q(R) + S(R)R^2 &= \frac{R+GM}{R-GM}, && (327)\\
Q(R)R^2 &= (R+GM)^2. && (328)
\end{aligned}
$$

Solving these equations, we obtain:

$$
\begin{aligned}
Q(R) &= \frac{(R+GM)^2}{R^2} = (1+\frac{GM}{R})^2, && (329)\\
S(R) &= (G^2M^2(GM+R))/(R^4(-GM+R)) = \frac{G^2M^2(R+GM)}{R^4(R-GM)}. && (330)
\end{aligned}
$$

Hence, the Schwarzschild metric in harmonic coordinates is:

$$
d\tau^2 = \frac{R-GM}{R+GM}dt^2 - \left(1+\frac{GM}{R}\right)^2 d\mathbf{X}^2 - \frac{G^2M^2(R+GM)}{R^4(R-GM)}(\mathbf{X}\cdot d\mathbf{X})^2. \quad (331)
$$

This is identical to equation 8.2.15 in Weinberg's Gravitation and Cosmology book. □

Question 2

Consider a gravitational action of the form

$$I_G = -\frac{1}{16\pi G}\int d^4x\sqrt{g}f(R), \tag{332}$$

where $f(R)$ is a smooth differentiable fucntion of the scalar curvature R. In the limit $R \to 0$, we should require $f(R) \to R$ so as to recover the well-tested weak field limit of Einstein's gravity. Find the equation of motion that follows from this action, which is a generalization of

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{333}$$

Answer

We have known that the variation of the matter action gives (see equation 12.2.2 in Weinberg's Gravitation and Cosmology book):

$$\delta I_M = \frac{1}{2}\int d^4x\sqrt{g}T^{\mu\nu}\delta g_{\mu\nu}. \tag{334}$$

Next, we calculate the variation of the gravitational action:

$$\begin{aligned}
\delta I_G =& -\frac{1}{16\pi G}\int d^4x\left(\delta\sqrt{g}f(R) + \sqrt{g}f'(R)\delta R\right) & (335)\\
=& -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)\delta R\right) & (336)\\
=& -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}f(R)\delta g_{\mu\nu} + f'(R)\delta R\right) & (337)
\end{aligned}$$

where $f'(R) = \frac{df}{dR}$ and $\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}$. To find δR, we use the relation:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{338}$$

Then we have

$$\begin{aligned}
\delta I_G =& -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}f(R)\delta g_{\mu\nu} - f'(R)R^{\mu\nu}\delta g_{\mu\nu} + f'(R)g^{\mu\nu}\delta R_{\mu\nu}\right) & (339)\\
=& -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\left(\frac{1}{2}g^{\mu\nu}f(R) - f'(R)R^{\mu\nu}\right)\delta g_{\mu\nu} + f'(R)g^{\mu\nu}\delta R_{\mu\nu}\right) & (340)
\end{aligned}$$

The term $\delta R_{\mu\nu}$ can be expressed as (see *Palatini identity* equation 10.9.3 in Weinberg's Gravitation and Cosmology book):

$$\delta R_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \Big(\delta g_{\lambda\sigma;\mu;\nu} + \delta g_{\mu\nu;\lambda;\sigma} - \delta g_{\mu\lambda;\nu;\sigma} - \delta g_{\nu\lambda;\mu;\sigma} \Big) \tag{341}$$

Substituting this into the expression for δI_G, we get:

$$\delta I_G = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \Bigg(\Big(\frac{1}{2} g^{\mu\nu} f(R) - f'(R) R^{\mu\nu} \Big) \delta g_{\mu\nu} \tag{342}$$

$$+ \frac{1}{2} f'(R) g^{\mu\nu} g^{\lambda\sigma} \Big(\delta g_{\lambda\sigma;\mu;\nu} + \delta g_{\mu\nu;\lambda;\sigma} - \delta g_{\mu\lambda;\nu;\sigma} - \delta g_{\nu\lambda;\mu;\sigma} \Big) \Bigg) \tag{343}$$

The second line can be simplified using integration by parts and the fact that the covariant derivative of the metric tensor vanishes. That means:

$$\delta I_G = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \frac{1}{2} f'(R) g^{\mu\nu} g^{\lambda\sigma} \Big(\delta g_{\lambda\sigma;\mu;\nu} + \delta g_{\mu\nu;\lambda;\sigma} - \delta g_{\mu\lambda;\nu;\sigma} - \delta g_{\nu\lambda;\mu;\sigma} \Big) \tag{344}$$

$$= -\frac{1}{16\pi G} \int d^4x \sqrt{g} \frac{1}{2} f'(R) \Big(g^{\lambda\sigma} g^{\mu\nu} \delta g_{\lambda\sigma;\mu;\nu} + g^{\mu\nu} g^{\lambda\sigma} \delta g_{\mu\nu;\lambda;\sigma} - g^{\mu\lambda} g^{\nu\sigma} \delta g_{\mu\lambda;\nu;\sigma} - g^{\nu\lambda} g^{\mu\sigma} \delta g_{\nu\lambda;\mu;\sigma} \Big) \tag{345}$$

$$= -\frac{1}{16\pi G} \int d^4x \sqrt{g} \frac{1}{2} f'(R) \Big(g^{\lambda\sigma} g^{\mu\nu} \delta g_{\lambda\sigma;\mu;\nu} + g^{\mu\nu} g^{\lambda\sigma} \delta g_{\mu\nu;\lambda;\sigma} - 2 g^{\mu\lambda} g^{\nu\sigma} \delta g_{\mu\lambda;\nu;\sigma} \Big) \tag{346}$$

$$= -\frac{1}{16\pi G} \int d^4x \sqrt{g} \frac{1}{2} f'(R) \Big(\Box (g^{\lambda\sigma} \delta g_{\lambda\sigma}) + \Box (g^{\mu\nu} \delta g_{\mu\nu}) - 2 (g^{\mu\lambda} g^{\nu\sigma} \delta g_{\mu\lambda})_{;\nu;\sigma} \Big) \tag{347}$$

$$= -\frac{1}{16\pi G} \int d^4x \sqrt{g} \Big(f'(R) \Box (g^{\mu\nu} \delta g_{\mu\nu}) - f'(R) (g^{\mu\lambda} g^{\nu\sigma} \delta g_{\mu\lambda})_{;\nu;\sigma} \Big) \tag{348}$$

$$= -\frac{1}{16\pi G} \int d^4x \sqrt{g} \Big(g^{\mu\nu} \Box f'(R) \delta g_{\mu\nu} - f'(R)^{;\mu;\nu} \delta g_{\mu\nu} \Big), \tag{349}$$

where we have used integration by parts and dropped the boundary terms. Hence, we have:

$$\delta I_G = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \Bigg(\Big(\frac{1}{2} g^{\mu\nu} f(R) - f'(R) R^{\mu\nu} \Big) \delta g_{\mu\nu} \tag{350}$$

$$+ \Big(g^{\mu\nu} \Box f'(R) - f'(R)^{;\mu;\nu} \Big) \delta g_{\mu\nu} \Bigg) \tag{351}$$

Combining terms, we have:

$$\delta I_G = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \Bigg(\Big(\frac{1}{2} g^{\mu\nu} f(R) - f'(R) R^{\mu\nu} + g^{\mu\nu} \Box f'(R) - f'(R)^{;\mu;\nu} \Big) \delta g_{\mu\nu} \Bigg) \tag{352}$$

Setting the total variation $\delta I_G + \delta I_M = 0$ for arbitrary $\delta g_{\mu\nu}$, we obtain the equation of motion:

$$f'(R)R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}f(R) - g^{\mu\nu}\Box f'(R) + f'(R)^{;\mu;\nu} = -8\pi G T^{\mu\nu} \tag{353}$$

or equivalently,

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - g_{\mu\nu}\Box f'(R) + f'(R)_{;\mu;\nu} = -8\pi G T_{\mu\nu} \tag{354}$$

This is the generalized Einstein equation for the action with $f(R)$ gravity. $\square$

Assignment 10 due on Monday November 17 at 10PM

Question 1

The Lagrangian density for a scalar field ϕ is givesn by

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - U(\phi), \tag{355}$$

where $U(\phi)$ is a potential, typically $\lambda\phi^4$, and the metric is given. The action is

$$I_\phi = \int d^4x \sqrt{g} \mathcal{L}. \tag{356}$$

There are two ways to calculate the eneergy-momentum tensor.

(a) In field theory, it is usually calculated by varying the cation with respect to the field, yielding the formula

$$T^{\mu\nu} = g^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\kappa \phi g^{\kappa\nu}. \tag{357}$$

similar to classical mechanics. Calculate it this way.

(b) As discussed in lecture, it can also be calculated by varying the action with respect to the metric via the formula

$$\delta I_\phi = \frac{1}{2} \int d^4x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu}. \tag{358}$$

Calculate it this way. Does it agree with part (a)?

Answer

(a)

We have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\frac{1}{2} g^{\alpha\beta} \left(\delta^\mu_\alpha \partial_\beta \phi + \partial_\alpha \phi \delta^\mu_\beta \right) = -g^{\mu\beta} \partial_\beta \phi. \tag{359}$$

Hence,

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\kappa\phi g^{\kappa\nu} \tag{360}$$
$$= g^{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2 - U(\phi)\right) + g^{\mu\beta}\partial_\beta\phi\partial_\kappa\phi g^{\kappa\nu} \tag{361}$$
$$= g^{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2 - U(\phi)\right) + \partial^\mu\phi\partial^\nu\phi. \tag{362}$$

(b)

We have

$$\delta I_\phi = \int d^4x\,(\delta\sqrt{g}\mathcal{L} + \sqrt{g}\delta\mathcal{L}) \tag{363}$$
$$= \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\mathcal{L}\delta g_{\mu\nu} + \delta\mathcal{L}\right) \tag{364}$$

where we have used $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}$ (equation on page 364 in Weinberg's Gravitation and Cosmology). Next, we calculate $\delta\mathcal{L}$:

$$\delta\mathcal{L} = -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g^{\mu\nu}\delta(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}m^2\delta(\phi^2) - \delta U(\phi) \tag{365}$$
$$= -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{366}$$
$$= \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\partial_\mu\phi\partial_\nu\phi \tag{367}$$
$$= \frac{1}{2}\delta g_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi \tag{368}$$
$$= \frac{1}{2}\delta g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi, \tag{369}$$

since the other terms do not depend on the metric. Therefore, we have

$$\delta I_\phi = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\mathcal{L}\delta g_{\mu\nu} + \frac{1}{2}\delta g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\right) \tag{370}$$
$$= \frac{1}{2}\int d^4x\sqrt{g}\left(g^{\mu\nu}\mathcal{L} + \partial^\mu\phi\partial^\nu\phi\right)\delta g_{\mu\nu}. \tag{371}$$

Comparing this with

$$\delta I_\phi = \frac{1}{2}\int d^4x\sqrt{g}T^{\mu\nu}\delta g_{\mu\nu}, \tag{372}$$

we find

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} + \partial^\mu\phi\partial^\nu\phi, \tag{373}$$

which agrees with the result from part (a). $\square$

Question 2

In Newtonian mechanics a space probe in a circular orbit of radius r about the sun (for example) with mass M has a period given by

$$t_N = 2\pi\sqrt{\frac{r^3}{GM}}. \tag{374}$$

Consider a space probe in a circular orbit of radius $r > \frac{3}{2}GM$ in the plane $\theta = \frac{\pi}{2}$ in standard coordinates about Schwarzschild black hole of mass M . We need to solve the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{375}$$

for $\mu = t, r, \theta, \phi$. Yes, all four! The affine connection/Christoffel symbol can be found in various sources, but be sure their conventions align with ours. When solving these equations use the initial condition $t = 0$ and $\phi = 0$ at $\tau = 0$. Note that in this situation

$$d\tau^2 = \left(1 - \frac{R_s}{r}\right)dt^2 - r^2 d\phi^2. \tag{376}$$

(a) What is the period τ_p as measured by a clock in the space probe? How does it relate to t_N ? What happens as $r \to \frac{3}{2}R_s$?

(b) What is the period t_p in standard coordinates? How does it relate to t_N?

Answer

(a)

We first write down the non-zero Christoffel symbols in Schwarzschild coordinates:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{GM}{r(r-2GM)}, \tag{377}$$

$$\Gamma^r_{tt} = \frac{GM(r-2GM)}{r^3}, \tag{378}$$

$$\Gamma^r_{rr} = -\frac{GM}{r(r-2GM)}, \tag{379}$$

$$\Gamma^r_{\theta\theta} = -(r-2GM), \tag{380}$$

$$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\theta, \tag{381}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \tag{382}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \tag{383}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \tag{384}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{385}$$

For a circular orbit, we have $r =$ constant and $\theta = \frac{\pi}{2}$. Therefore, the geodesic equations for $\mu = t, r, \theta, \phi$ reduce to

$$\frac{d^2t}{d\tau^2} + 2\Gamma^t_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0, \qquad \text{t component}, \tag{386}$$

$$\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = 0, \qquad \text{r component}, \tag{387}$$

$$\Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = 0, \qquad \theta \text{ component}, \tag{388}$$

$$\frac{d^2\phi}{d\tau^2} + 2\Gamma^\phi_{r\phi}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0, \qquad \phi \text{ component}. \tag{389}$$

The t and ϕ components are automatically satisfied since $dr/d\tau = 0$. The solution for t is straightforward:

$$\frac{d^2t}{d\tau^2} = 0 \implies \frac{dt}{d\tau} = \text{constant}. \tag{390}$$

The solution for ϕ is also straightforward:

$$\frac{d^2\phi}{d\tau^2} = 0 \implies \frac{d\phi}{d\tau} = \text{constant}. \tag{391}$$

The θ component is automatically satisfied since $\Gamma^\theta_{\phi\phi} = 0$ at $\theta = \frac{\pi}{2}$. The r component gives

$$\frac{GM(r-2GM)}{r^3}\left(\frac{dt}{d\tau}\right)^2 - (r-2GM)\left(\frac{d\phi}{d\tau}\right)^2 = 0. \tag{392}$$

Rearranging, we find

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^3}\left(\frac{dt}{d\tau}\right)^2. \tag{393}$$

Next, we use the relation

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - r^2 d\phi^2, \tag{394}$$

to express $d\tau$ in terms of dt and $d\phi$. Dividing both sides by $d\tau^2$, we get

$$1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2. \tag{395}$$

Substituting the expression for $\left(\frac{d\phi}{d\tau}\right)^2$, we have

$$1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - r^2 \cdot \frac{GM}{r^3}\left(\frac{dt}{d\tau}\right)^2. \tag{396}$$

Simplifying, we find

$$1 = \left(1 - \frac{3GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2. \tag{397}$$

Solving for $\frac{dt}{d\tau}$, we get

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{3GM}{r}}} = \frac{1}{\sqrt{1 - \frac{3R_s/2}{r}}}, \tag{398}$$

$$\implies \frac{d\phi}{d\tau} = \sqrt{\frac{GM}{r^3}} \cdot \frac{1}{\sqrt{1 - \frac{3R_s/2}{r}}}. \tag{399}$$

The period τ_p as measured by a clock in the space probe is given by

$$\tau_p = \frac{2\pi}{\frac{d\phi}{d\tau}}. \tag{400}$$

Using the relation between $\frac{d\phi}{d\tau}$ and $\frac{dt}{d\tau}$, we find

$$\tau_p = 2\pi\sqrt{\frac{r^3}{GM}}\sqrt{1 - \frac{3R_s/2}{r}} = t_N\sqrt{1 - \frac{3R_s/2}{r}}. \tag{401}$$

As $r \to \frac{3}{2}R_s$, we see that $\tau_p \to 0$. As $r \gg R_s$, we have $\tau_p \approx t_N$.

(b)

The period t_p in standard coordinates is given by

$$t_p = \frac{2\pi}{\frac{d\phi}{dt}}. \tag{402}$$

Using the chain rule, we have

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau} \cdot \frac{d\tau}{dt} = \frac{d\phi}{d\tau} \cdot \frac{1}{\frac{dt}{d\tau}}. \tag{403}$$

Therefore,

$$\frac{d\phi}{dt} = \sqrt{\frac{GM}{r^3}} \cdot \frac{1}{\sqrt{1 - \frac{3R_s/2}{r}}} \cdot \sqrt{1 - \frac{3R_s/2}{r}} \tag{404}$$

$$= \sqrt{\frac{GM}{r^3}}. \tag{405}$$

Thus, we find

$$t_p = 2\pi\sqrt{\frac{r^3}{GM}} = t_N. \tag{406}$$

□

Assignment 11 due on Monday November 24 at 10PM

Question 1

Calculate numerically the Schwarzschild radius R_s, the characteristic collapse time t_{collapse}, the characteristic radial redshift time $2R_s/c$, the characteristic radial flux time $R_s/2c$, and the characteristic total luminosity time $3\sqrt{3}R_s/2c$ in the appropriate units (seconds, days, years, kilometers, light-years, etc) using the dust cloud model for the following initial conditions:

(i) An object of three solar masses with an initial radius of 1 AU.

(ii) An object of 108 solar masses with an initial radius of 100 light-years.

Answer

We have

$$R_s = \frac{2GM}{c^2}, \tag{407}$$

$$t_{\text{collapse}} = \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}}, \tag{408}$$

$$t_{\text{redshift}} = \frac{2R_s}{c}, \tag{409}$$

$$t_{\text{flux}} = \frac{R_s}{2c}, \tag{410}$$

$$t_{\text{luminosity}} = \frac{3\sqrt{3}R_s}{2c}. \tag{411}$$

(i) For an object of three solar masses with an initial radius of 1 AU:

Using $M = 3M_\odot = 3 \times 1.989 \times 10^{30}$ kg, $R = 1$ AU $= 1.496 \times 10^{11}$ m, $G = 6.67430 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, and $c = 3 \times 10^8$ m/s, we find (see *mathematica* code for calculation):

$$R_s \approx 8844.42 \text{ m} = 8.84442 \text{ km}, \tag{412}$$

$$t_{\text{collapse}} \approx 3.22152 \times 10^6 \text{ sec} = 37.2862 \text{ days}, \tag{413}$$

$$t_{\text{redshift}} \approx 5.89628 \times 10^{-5} \text{ seconds} = 58.9628 \text{ microseconds}, \tag{414}$$

$$t_{\text{flux}} \approx 1.47407 \times 10^{-5} \text{ seconds} = 14.7407 \text{ microseconds}, \tag{415}$$

$$t_{\text{luminosity}} \approx 7.65949 \times 10^{-5} \text{ seconds} = 76.5949 \text{ microseconds}. \tag{416}$$

(ii) For an object of 10^8 solar masses with an initial radius of 100 light-years:

Using $M = 10^8 M_\odot = 10^8 \times 1.989 \times 10^{30}$ kg, $R = 100$ light-years $= 9.461 \times 10^{17}$ m, $G = 6.67430 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, and $c = 3 \times 10^8$ m/s, we find (see *mathematica* code for calculation):

$$R_s \approx 2.94814 \times 10^{11} \text{ m} = 1.971 \text{ AU}, \tag{417}$$

$$t_{\text{collapse}} \approx 8.87422 \times 10^{12} \text{ sec} = 281400 \text{ years}, \tag{418}$$

$$t_{\text{redshift}} \approx 1965.43 \text{ seconds} = 32.7572 \text{ minutes}, \tag{419}$$

$$t_{\text{flux}} \approx 491.357 \text{ seconds} = 8.18928 \text{ minutes}, \tag{420}$$

$$t_{\text{luminosity}} \approx 2553.16 \text{ seconds} = 42.5527 \text{ minutes}. \tag{421}$$

□

Question 2

In the 1960's it was shown that the light received from a collapsing, luminous cloud of dust is dominated by photons deposited near the unstable orbit as the surface of the cloud crosses the radius $r = 3/2R_s$. Calculate the redshift z for photons emitted *radially* near this surface to at least 2 significant digits. (A more elaborate calculation shows that most photons are emitted with nonzero angular momentum. These photons escape after orbiting the dust cloud many times, resulting in a redshift $z = 2$.)

Answer

The redshift z for photons emitted radially from a radius r in the Schwarzschild metric is given by

$$1 + z = \frac{1}{\sqrt{1 - \frac{R_s}{r}}}. \tag{422}$$

Hence, for $r = \frac{3}{2}R_s$, we have

$$1 + z = \frac{1}{\sqrt{1 - \frac{R_s}{\frac{3}{2}R_s}}} = \frac{1}{\sqrt{1 - \frac{2}{3}}} = \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{3}. \tag{423}$$

Thus, the redshift is

$$z = \sqrt{3} - 1 \approx 0.732. \tag{424}$$

But this is the redshift effect only due to the gravitational field. We can also consider the Doppler effect due to the motion of the collapsing surface. The total redshift considering both effects is given by

$$1 + z_{total} = (1 + z_{\text{gravitational}})(1 + z_{\text{Doppler}}). \tag{425}$$

The Doppler redshift for a radially infalling object is given by

$$1 + z_{\text{Doppler}} = \sqrt{\frac{1 + v/c}{1 - v/c}}, \tag{426}$$

where v is the infall velocity at radius r. The infall velocity can be found using energy conservation in the Schwarzschild metric:

$$v = c\sqrt{\frac{R_s}{r}} = \sqrt{\frac{R_s}{r}}. \tag{427}$$

For $r = \frac{3}{2}R_s$, we have

$$v = \sqrt{\frac{R_s}{\frac{3}{2}R_s}} = \sqrt{\frac{2}{3}}. \tag{428}$$

Thus, the Doppler redshift is

$$1 + z_{\text{Doppler}} = \sqrt{\frac{1 + \sqrt{\frac{2}{3}}}{1 - \sqrt{\frac{2}{3}}}}. \tag{429}$$

Combining both effects, we have

$$1 + z_{total} = \sqrt{3} \cdot \sqrt{\frac{1 + \sqrt{\frac{2}{3}}}{1 - \sqrt{\frac{2}{3}}}}. \tag{430}$$

Calculating this numerically, we find

$$z_{total} \approx 4.45. \tag{431}$$

□

Assignment 12 due on Monday December 1 at 10PM

Question 1

Verify the Reissner-Nordstrom solution as presented in class and in the textbook. That is, show in detail that both Maxwell's equations

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{\mu\nu}\right) = 0, \tag{432}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0, \tag{433}$$

and Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{434}$$

are satisfied with the metric:

$$d\tau^2 = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 - \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}dr^2 - r^2 d\Omega^2, \tag{435}$$

and with a radial electric field $E_r = \frac{Q}{r^2}$. Note that this is in Gaussian units. Weinberg uses Heaviside-Lorentz units.

Answer

First, we write down the field strength tensor $F_{\mu\nu}$ for a radial electric field $E_r = \frac{Q}{r^2}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{Q}{r^2} & 0 & 0 \\ \frac{Q}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{436}$$

with $g^{tt} = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}$ and $g^{rr} = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)$. Raising the indices, we have

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{Q}{r^2} & 0 & 0 \\ -\frac{Q}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{437}$$

Next, we calculate $\sqrt{g}$:

$$\sqrt{g} = r^2 \sin\theta. \tag{438}$$

Now, we can verify Maxwell's equations. For $\mu = 0$,

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{0\nu}\right) = \frac{\partial}{\partial r}\left(r^2 \sin\theta \cdot \frac{Q}{r^2}\right) = 0. \tag{439}$$

For $\mu = 1$,

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{1\nu}\right) = \frac{\partial}{\partial t}\left(r^2 \sin\theta \cdot -\frac{Q}{r^2}\right) = 0. \tag{440}$$

For $\mu = 2$ and $\mu = 3$, the equations are trivially satisfied since $F^{2\nu} = F^{3\nu} = 0$. Thus, $\frac{\partial}{\partial x^\nu}\left(\sqrt{g}F^{\mu\nu}\right) = 0$ holds for all μ. Now we check the second Maxwell equation:

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0. \tag{441}$$

Since $F_{\mu\nu}$ only has non-zero components for $\mu = 0$ and $\nu = 1$ (or vice versa), we only need to check the case when $(\mu, \nu, \lambda) = (0, 1, r)$:

$$F_{01,r} + F_{1r,0} + F_{r0,1} = \frac{\partial}{\partial r}\left(-\frac{Q}{r^2}\right) + 0 + \frac{\partial}{\partial r}\left(\frac{Q}{r^2}\right) = 0. \tag{442}$$

All other combinations yield zero trivially. Thus, the second Maxwell equation is also satisfied. For Einstein's equations, please check the attached file for the detailed calculations of the Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and energy-momentum tensor $T_{\mu\nu}$. After computing these quantities, we find that Einstein's equations are satisfied with the given metric and electric field configuration. $\square$

Question 2

Read pages 259-261 of the textbook by Carroll and then solve exercise 1 on page 272. You can use any information provided in class. As Carroll writes in his book, this is an amazing exact solution.
Carroll: Show that the coupled Einstein-Maxwell equations can be simultaneously solved by the metric (6.62) and the electrostatic potential (6.67) if H (i) obeys Laplace's equation,

$$\nabla^2 H = 0. \tag{443}$$

$$ds = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2), \tag{444}$$

where where $H = H(\vec{x})$. The electrostatic potential is given by

$$A_\mu = \left(\frac{1}{\sqrt{G}}\frac{1}{H} - 1, 0, 0, 0\right). \tag{445}$$

Answer

To verify that the coupled Einstein-Maxwell equations are satisfied by the given metric and electrostatic potential, we start by writing down the metric:

$$ds^2 = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2), \tag{446}$$

where $H = H(\vec{x})$. The electrostatic potential is given by

$$A_\mu = \left(\frac{1}{\sqrt{G}}\frac{1}{H} - 1, 0, 0, 0\right). \tag{447}$$

See the attached file for the full solution.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\pi G T_{\mu\nu} = \begin{pmatrix} \frac{2\left(H^{(0,0,2)}(x,y,z)+H^{(0,2,0)}(x,y,z)+H^{(2,0,0)}(x,y,z)\right)}{H(x,y,z)^5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{448}$$

Thus, if H satisfies Laplace's equation $\nabla^2 H = 0$, then the coupled Einstein-Maxwell equations are satisfied. I also verified Maxwell's equations in the attached file. $\square$.

Assignment 13 due on Monday December 8 at 10PM

The metric for an electrically charged and rotating black hole is called the Kerr-Newman metric. It is a simple extension of the Kerr metric but with the addition of the electromagnetic field. Read pp. 261-267 of Carroll before starting this problem.

Question 1

In class we derived the thermodynamic-like relationship

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J, \tag{449}$$

for the Kerr solution. For the Kerr-Newman solution, there should be another term on the right hand sides of the form $\mu\delta Q$. In this case, what are κ, Ω_H, and μ?

Answer

We start from the Kerr-Newman metric in Boyer-Lindquist coordinates:

$$ds^2 = -\left(1 - \frac{2GMr - GQ^2}{\rho^2}\right)dt^2 - \frac{2a(2GMr - GQ^2)\sin^2\theta}{\rho^2}dtd\phi \tag{450}$$

$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{(2GMr - GQ^2)a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \tag{451}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2GMr + a^2 + GQ^2$, and $a = \frac{J}{M}$ is the specific angular momentum. The event horizon is located at $r_+ = GM + \sqrt{(GM)^2 - a^2 - GQ^2}$. The surface gravity κ is given by

$$\kappa = \frac{r_+ - GM}{r_+^2 + a^2} = \frac{\sqrt{(GM)^2 - a^2 - GQ^2}}{r_+^2 + a^2}. \tag{452}$$

The angular velocity of the horizon Ω_H is

$$\Omega_H = \frac{a}{r_+^2 + a^2}. \tag{453}$$

The electrostatic potential μ at the horizon is given by

$$\mu = \frac{Qr_+}{r_+^2 + a^2}. \tag{454}$$

Thus, the first law of black hole mechanics for the Kerr-Newman black hole is

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J + \mu\delta Q. \tag{455}$$

Assignment 14 due on Monday December 15 at 10PM

Question 1

Solve problem 6 of chapter 6 of Carroll's book. It is worthwhile to remind yourself of time dilation, the normal Doppler effect, and the relativistic Doppler effect in special relativity as a warmup.

Carroll problem 6 of chapter 6: Consider a Kerr black hole with an accretion disk of negligible mass in the equatorial plane. Assume that particles in the disk follow geodesics (that is, ignore any pressure support). Now, suppose the disk contains some iron atoms that are being excited by a source of radiation. When the iron atoms de-excite, they emit radiation with a known frequency ν_0, as measured in their rest frame. Suppose we detect this radiation far from the black hole (we also lie in the equatorial plane). What is the observed frequency of photons emitted from either edge of the disk, and from the center of the disk? Consider cases where the disk and the black hole are rotating in the same and opposite directions. Can we use these measurements to determine the mass and angular momentum of the black hole?

Answer

We start with the Kerr metric in Boyer-Lindquist coordinates:

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{4aGMr\sin^2\theta}{\rho^2} dtd\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - a^2\ \Delta \sin^2\theta\right)\sin^2\theta d\phi^2, \tag{456}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2GMr + a^2$. For the equatorial plane, $\theta = \frac{\pi}{2}$, so $\rho^2 = r^2$ and $\sin\theta = 1$. The metric simplifies to:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GMr}{r^2}\right) dt^2 - \frac{4aGMr}{r^2} dtd\phi + \frac{r^2}{r^2 - 2GMr + a^2} dr^2 && (457)\\ &\quad + r^2 d\theta^2 + \frac{1}{r^2}\left((r^2+a^2)^2 - a^2(r^2 - 2GMr + a^2)\right) d\phi^2 && (458)\\ &= -\left(1 - \frac{2GM}{r}\right) dt^2 - \frac{4aGM}{r} dtd\phi + \frac{r^2}{r^2 - 2Mr + a^2} dr^2 + r^2 d\theta^2 + \frac{(r^2+a^2)^2 - a^2(r^2 - 2Mr + a^2)}{r^2} d\phi^2. && (459) \end{aligned}$$

We can solve the geodesic equations for the disk particles, but for simplicity, we assume that the particles are moving in circular orbits at a fixed radius r. The geodesic equations with affine connection coefficients $\Gamma^\mu_{\nu\lambda}$ are:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = 0. \tag{460}$$

For circular orbits, the four velocity components are $u^t = \frac{dt}{d\tau}$, $u^r = 0$, $u^\theta = 0$, and $u^\phi = \frac{d\phi}{d\tau}$. Also, the angular velocity it $\Omega = \frac{d\phi}{dt}$. The geodesic equations for r is:

$$\frac{d^2r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + 2\Gamma^r_{t\phi}\frac{dt}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{461}$$

$$\implies \frac{d^2r}{d\tau^2} + \left(\frac{dt}{d\tau}\right)^2\Gamma^r_{tt} + \left(\frac{d\phi}{d\tau}\right)^2\Gamma^r_{\phi\phi} + 2\frac{dt}{d\tau}\frac{d\phi}{d\tau}\Gamma^r_{t\phi} = 0. \tag{462}$$

The non-zero connection coefficients for r are:

$$\Gamma^r_{tt} = \frac{GM\left(a^2 + r(r-2GM)\right)}{r^4}, \tag{463}$$

$$\Gamma^r_{t\phi} = -\frac{\left(a^2 + r(r-2GM)\right)\left(-a^4 + a^2\left(a^2 - GMr\right) + r^4\right)}{r^5}, \tag{464}$$

$$\Gamma^r_{\phi\phi} = -\frac{aGM\left(a^2 + r(r-2GM)\right)}{r^4}. \tag{465}$$

Substituting these into the geodesic equation, we get:

$$\Omega^2\left[-\frac{aGM\left(a^2 + r(r-2GM)\right)}{r^4}\right] \tag{466}$$

$$+2\Omega\left[-\frac{\left(a^2 + r(r-2GM)\right)\left(-a^4 + a^2\left(a^2 - GMr\right) + r^4\right)}{r^5}\right] \tag{467}$$

$$+\left[\frac{GM\left(a^2 + r(r-2GM)\right)}{r^4}\right] = 0, \tag{468}$$

where $\Omega = \frac{d\phi}{dt}$. Solving for Ω, we get:

$$\Omega_\pm = \frac{a^3GM + aGMr(r-2GM) \pm r^{3/2}\sqrt{GM(a^2 + r(r-2GM))^2}}{a^4GM - a^2r(2G^2M^2 - GMr + r^2) + r^4(2GM - r)} \tag{469}$$

$$= \frac{aGM\Delta \pm r^{3/2}\sqrt{GM\Delta^2}}{(a^2GM - r^3)\Delta} = \frac{aGM \pm r^{3/2}\sqrt{GM}}{(a^2GM - r^3)} \tag{470}$$

$$= \pm\frac{\sqrt{GM}}{a\sqrt{GM} \pm r^{3/2}}, \tag{471}$$

where $\Delta = r^2 - 2GM + a^2$. The positive and negative signs correspond to the disk rotating in the same and opposite directions as the black hole, respectively. Next, the red shift factor is given by:

$$1 + z = \frac{\nu_{obs}}{\nu_0} = \frac{(p_\mu u^\mu)_{detector}}{(p_\mu u^\mu)_{emitter}} = \frac{-p_t}{-p_\mu u^\mu} = \frac{-p_t}{-p_t u^t - p_\phi u^\phi}. \tag{472}$$

Now we need to find the components of the four-velocity u_μ in emitter's frame. In the emitter's frame, the four-velocity is:

$$u_\mu^{emitter}(u^\mu)^{emitter} = -1 \implies u^t = \frac{1}{\sqrt{-(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)}}. \tag{473}$$

The observed frequency is then:

$$1 + z = \frac{1}{u^t(1 + \Omega p_\phi / p_t)}. \tag{474}$$

If the light comes radially from the disk, then $p_\phi = 0$, and the observed frequency is:

$$1 + z = \frac{1}{u^t} = \sqrt{-(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)}. \tag{475}$$

We can plug the values of g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$ from the Kerr metric to get the observed frequency and the rotation of the disk. Next, if the light is emitted from the edge of the disk, we need to consider $p_\phi \neq 0$. The observed frequency is:

$$1 + z = \frac{1}{u^t(1 + \Omega p_\phi / p_t)} = \frac{\sqrt{-(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)}}{1 + \Omega p_\phi / p_t}. \tag{476}$$

The observed frequency depends on the radius r of the disk, the angular velocity Ω, and the direction of rotation of the disk relative to the black hole. By measuring the observed frequency at different radii, we can determine the mass and angular momentum of the black hole. The observed frequency will be higher for a faster-rotating black hole, and lower for a slower-rotating black hole. The observed frequency will also be higher for a disk rotating in the same direction as the black hole, and lower for a disk rotating in the opposite direction. Therefore, by measuring the observed frequency at different radii, we can determine the mass and angular momentum of the black hole. **Hence, we can use these measurements to determine the mass and angular momentum of the black hole.** $\square$