University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Question 1

p-d reactions

Consider the reactions

$$p + d \to \pi^+ + {}^3\text{H}, \quad p + d \to \pi^0 + {}^3\text{He}. \tag{1}$$

Since the deuteron is in a ${}^3S_1$ state, it must be an isospin singlet. Therefore, the initial state $p + d$ is a pure $I = \frac{1}{2}$ state. Given that ^{3}H and ^{3}He form an isodoublet, write down the isospin decomposition of the final states, and from this, the ratio of the two cross sections.

Answer

First, we can use I_3 to decide the isospin for ^{3}H and ^{3}He. See the initial state $p+d$ has $I_3 = +\frac{1}{2}$, so the final state must also have $I_3 = +\frac{1}{2}$. Since π^+ has $I_3 = +1$ and π^0 has $I_3 = 0$, we can conclude that ^{3}H has $I_3 = -\frac{1}{2}$ and ^{3}He has $I_3 = +\frac{1}{2}$. Therefore, ^{3}H and ^{3}He form an isodoublet with $I = \frac{1}{2}$. Now we can write down the isospin decomposition of the final states. For the first reaction, we have

$$|\pi^+ + {}^3\text{H}\rangle = |\pi^+\rangle \otimes |{}^3\text{H}\rangle = |1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle. \tag{2}$$

Using the Clebsch-Gordan coefficients, we can decompose this into total isospin

$$|1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{3}$$

For the second reaction, we have

$$|\pi^0 + {}^3\text{He}\rangle = |\pi^0\rangle \otimes |{}^3\text{He}\rangle = |1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{4}$$

Using the Clebsch-Gordan coefficients, we can decompose this into total isospin

$$|1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}, \tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{5}$$

Now, since the initial state $p+d$ is a pure $I = \frac{1}{2}$ state, only the $I = \frac{1}{2}$ component of the final states will contribute to the cross sections. Therefore, we can write the amplitudes for the two reactions as

$$\mathcal{A}(p + d \to \pi^+ + {}^3\text{H}) \propto \sqrt{\tfrac{2}{3}}, \tag{6}$$

$$\mathcal{A}(p + d \to \pi^0 + {}^3\text{He}) \propto -\tfrac{1}{\sqrt{3}}. \tag{7}$$

The cross sections are proportional to the square of the amplitudes, so we have

$$\sigma(p + d \to \pi^+ + {}^3\text{H}) \propto \left|\sqrt{\tfrac{2}{3}}\right|^2 = \tfrac{2}{3}, \tag{8}$$

$$\sigma(p + d \to \pi^0 + {}^3\text{He}) \propto \left|-\tfrac{1}{\sqrt{3}}\right|^2 = \tfrac{1}{3}. \tag{9}$$

Finally, the ratio of the two cross sections is

$$\frac{\sigma(p + d \to \pi^+ + {}^3\text{H})}{\sigma(p + d \to \pi^0 + {}^3\text{He})} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2. \tag{10}$$

$\square$

Question 2

Particle production by strong interactions

Explain why the processes $\pi^- + p \to \pi^+ + \Sigma^-$, $\quad \pi^- + p \to K^0 + n$, $\quad \pi^- + p \to \Sigma^+ + K^-$ cannot be observed.

Answer

Before we analyze the processes, let's summarize the quantum numbers of the particles involved:

- π^-: $I = 1$, $I_3 = -1$, $S = 0$, $B = 0$
- π^+: $I = 1$, $I_3 = +1$, $S = 0$, $B = 0$
- p: $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$, $S = 0$, $B = 1$
- n: $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, $S = 0$, $B = 1$
- Σ^-: $I = 1$, $I_3 = -1$, $S = -1$, $B = 1$
- Σ^+: $I = 1$, $I_3 = +1$, $S = -1$, $B = 1$
- K^0: $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$, $S = +1$, $B = 0$
- K^-: $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, $S = -1$, $B = 0$

Now, let's analyze each process:

(a) $\pi^- + p \to \pi^+ + \Sigma^-$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$
- Final state: $I_3 = +1 - 1 = 0$, $S = 0 - 1 = -1$, $B = 0 + 1 = 1$

The strangeness S changes from 0 to -1, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to 0. Therefore, this process cannot be observed.

(b) $\pi^- + p \to K^0 + n$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$
- Final state: $I_3 = +\frac{1}{2} - \frac{1}{2} = 0$, $S = +1 + 0 = +1$, $B = 0 + 1 = 1$

The strangeness S changes from 0 to +1, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to 0. Therefore, this process cannot be observed.

(c) $\pi^- + p \to \Sigma^+ + K^-$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$

- Final state: $I_3 = +1 - \frac{1}{2} = +\frac{1}{2}$, $S = -1 - 1 = -2$, $B = 1 + 0 = 1$

The strangeness S changes from 0 to -2, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to $+\frac{1}{2}$. Therefore, this process cannot be observed.

□

Question 3

SU(2) invariants and pseudoreal representations

(a) Show that $\delta^a{}_b$ and ϵ_{ab} are invariant tensors under SU(2) transformations.

(b) The nucleon doublet $N^a = \binom{p}{n}, a = 1, 2$ transforms as the fundamental 2 of SU(2), while its conjugate $\bar{N}_a \equiv (N^a)^\dagger = (\bar{p}, \bar{n})$ transforms as $\overline{\mathbf{2}}$. Use $\delta^a{}_b$ to form an SU(2) invariant with $N, \bar{N}$ and write it explicitly in terms of the proton and neutron fields.

(c) Define $\tilde{N}^b = \epsilon^{bc}\bar{N}_c^T$ which maps the $\overline{\mathbf{2}}$ representation (lower index) into $\mathbf{2}$ (upper index). Construct an SU(2) invariant with $N, \tilde{N}$ using ϵ_{ab}, and write it in terms of the components. Verify that the result is identical to part (b), demonstrating that the $\mathbf{2}$ and $\overline{\mathbf{2}}$ representations are equivalent (or pseudoreal) in SU(2) and that any invariant constructed with $\delta^a{}_b$ can be rewritten using ϵ_{ab}.

(d) Consider SU(3), with the quark triplet $q^a (a = 1, 2, 3)$ transforming as $\mathbf{3}$ and its conjugate $\overline{q}_a \equiv (q^a)^\dagger$ transforming as $\overline{\mathbf{3}}$. Discuss why a similar mapping using the SU(3) invariant ϵ_{abc} does not make $\mathbf{3}$ and $\overline{\mathbf{3}}$ equivalent. Write down the possible SU(3) invariants involving $q, \overline{q}$.

Answer

(a)

$$\delta^a{}_b \to = \delta'^a{}_b = U^a{}_c\delta^c{}_d(U^\dagger)^d{}_b = U^a{}_c(U^\dagger)^c{}_b = \mathbf{1}^a{}_b = \delta^a{}_b, \tag{11}$$

$$\epsilon_{ab} \to \epsilon'_{ab} = (U^\dagger)^c{}_a(U^\dagger)^d{}_b\epsilon_{cd} = \det(U^\dagger)\epsilon_{ab} = \epsilon_{ab}. \tag{12}$$

(b)

Using $\delta^a{}_b$, we can form the invariant

$$\bar{N}_a N^a = \delta^a{}_b\bar{N}_a N^b = \bar{p}p + \bar{n}n. \tag{13}$$

We can verify that this is indeed invariant under SU(2) transformations:

$$\bar{N}_a N^a \to \bar{N}'_a N'^a = \bar{N}_b(U^\dagger)^b{}_a U^a{}_c N^c = \bar{N}_b\delta^b{}_c N^c = \bar{N}_a N^a. \tag{14}$$

(c)

Using ϵ_{ab}, we can form the invariant

$$\epsilon_{ab}N^a\tilde{N}^b = \epsilon_{ab}N^a\epsilon^{bc}\bar{N}_c^T = \delta_a{}^c N^a\bar{N}_c^T = N^a\bar{N}_a^T = \bar{N}_a N^a = \bar{p}p + \bar{n}n. \tag{15}$$

We can verify that this is indeed invariant under SU(2) transformations:

$$\epsilon_{ab}N^a\tilde{N}^b \to \epsilon_{ab}N'^a\tilde{N}'^b = \epsilon_{ab}U^a{}_cN^cU^b{}_d\tilde{N}^d = \det(U)\epsilon_{cd}N^c\tilde{N}^d = \epsilon_{cd}N^c\tilde{N}^d. \tag{16}$$

This demonstrates that the **2** and $\overline{\mathbf{2}}$ representations are equivalent (or pseudoreal) in SU(2) and that any invariant constructed with $\delta^a{}_b$ can be rewritten using ϵ_{ab}.

(d)

The possible SU(3) invariants involving q and $\overline{q}$ are:

$$\overline{q}_aq^a, \quad \epsilon_{abc}q^aq^bq^c, \quad \epsilon^{abc}\overline{q}_a\overline{q}_b\overline{q}_c. \tag{17}$$

We can start with the q^aq^b,

$$q^aq^b = \frac{1}{2}(q^aq^b + q^bq^a) + \frac{1}{2}(q^aq^b - q^bq^a) = S^{ab} + A^{ab}, \tag{18}$$

where S^{ab} is symmetric and A^{ab} is antisymmetric. Moreover, for the antisymmetric part, we can use ϵ_{abc} to lower an index and get

$$\theta_c = \epsilon_{abc}A^{ab} = \epsilon_{abc}\frac{1}{2}(q^aq^b - q^bq^a) = \epsilon_{abc}q^aq^b. \tag{19}$$

Now we can see that θ_c transforms as $\overline{\mathbf{3}}$. In order to see this, we can apply an SU(3) transformation:

$$\theta'_c = \epsilon_{abc}q'^aq'^b = \epsilon_{abc}U^a{}_{a'}U^b{}_{b'}q^{a'}q^{b'} = \epsilon_{abc'}\delta^{c'}{}_cU^a{}_{a'}U^b{}_{b'}q^{a'}q^{b'} \tag{20}$$

$$= \epsilon_{abc'}U^{c'}{}_k(U^\dagger)^k{}_cU^a{}_{a'}U^b{}_{b'}q^{a'}q^{b'} = \det(U)(U^\dagger)^k{}_c\epsilon_{a'b'k}q^{a'}q^{b'} \tag{21}$$

$$= (U^\dagger)^k{}_c\epsilon_{a'b'k}q^{a'}q^{b'} = (U^\dagger)^k{}_c\theta_k = (U^\dagger)^{c'}{}_c\theta_{c'}. \tag{22}$$

Therefore, q^aq^b can be decomposed into a symmetric part transforming as **6** and an antisymmetric part transforming as $\overline{\mathbf{3}}$. This shows that there is no way to map $\overline{\mathbf{3}}$ back to **3** using ϵ_{abc}, unlike the case in SU(2) where we could use ϵ_{ab} to map between **2** and $\overline{\mathbf{2}}$. Hence, the representations **3** and $\overline{\mathbf{3}}$ are not equivalent in SU(3). □

Question 4

Applications of U-spin

(a) Show that $U_\pm = t_6 \pm it_7$ and $U_3 = \left(\sqrt{3}t_8 - t_3\right)/2$ satisfy the SU(2) algebra

$$[U_3, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = 2U_3.$$

(b) Show that the charge operator $Q = t_3 + t_8/\sqrt{3}$ is a U-scalar i.e. it has U-spin $U = 0$ or $[Q, U_i] = 0$ for $i = \pm, 3$. Write the electromagnetic current operator in terms of quark fields.

(c) Show that for the meson octet, the ($U_3 = 0$) component of the U-triplet is $\pi_U^0 = \left(-\pi^0 + \sqrt{3}\eta\right)/2$, and the U-singlet is $\eta_U^0 = \left(\sqrt{3}\pi^0 + \eta\right)/2$. Since π_U^0 is a U -spin vector component it cannot couple to the electromagnetic current. Show that for the 2γ decay mode, $\langle \pi^0 \mid 2\gamma \rangle = \sqrt{3}\langle \eta \mid 2\gamma \rangle$. How does this U-spin prediction compare with the experimental decay widths?

Answer

(a)

Here we recap the Gell-Mann matrices t_3, t_6, t_7, t_8:

$$t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{23}$$

$$t_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{24}$$

Hence, we have

$$U_+ = t_6 + it_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_- = t_6 - it_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{25}$$

$$U_3 = \frac{\sqrt{3}t_8 - t_3}{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}. \tag{26}$$

Now we can verify the SU(2) algebra:

$$[U_3, U_+] = U_3U_+ - U_+U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = U_+, \tag{28}$$

$$[U_3, U_-] = U_3U_- - U_-U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{29}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -U_-, \tag{30}$$

$$[U_+, U_-] = U_+U_- - U_-U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{31}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 2U_3. \tag{32}$$

(b)

First, we write down the charge operator:

$$Q = t_3 + \frac{t_8}{\sqrt{3}} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{33}$$

Now we can verify that Q is a U-scalar:

$$[Q, U_+] = QU_+ - U_+Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{34}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix} = 0, \tag{35}$$

$$[Q, U_-] = QU_- - U_-Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix} = 0, \tag{37}$$

$$[Q, U_3] = QU_3 - U_3Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} = 0. \tag{39}$$

Thus, we have shown that $[Q, U_i] = 0$ for $i = \pm, 3$, confirming that Q is a U-scalar. In QFT, the electormagnetic current operator in terms of fermion fields is given by

$$J_\mu^{\text{em}} = \sum_f Q_f \overline{\psi}_f \gamma_\mu \psi_f, \tag{40}$$

where the sum runs over all fermion flavors f, Q_f is the electric charge of the fermion in units of the elementary charge, ψ_f is the fermion field, and γ_μ are the gamma matrices. For the quark fields u, d, s, the electromagnetic current operator can be explicitly written as

$$J_\mu^{\text{em}} = \frac{2}{3}\overline{u}\gamma_\mu u - \frac{1}{3}\overline{d}\gamma_\mu d - \frac{1}{3}\overline{s}\gamma_\mu s. \tag{41}$$

(c)

First, we can express π^0 and η in terms of quark content:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d}), \tag{42}$$

$$\eta = \frac{1}{\sqrt{6}}(u\overline{u} + d\overline{d} - 2s\overline{s}). \tag{43}$$

Now, we can construct the U-triplet and U-singlet components:

$$\pi_U^0 = \frac{-\pi^0 + \sqrt{3}\eta}{2} = \frac{-\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) + \sqrt{3}\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}{2} \tag{44}$$

$$= \frac{-\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) + \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d} - 2s\bar{s})}{2} = \frac{2d\bar{d} - 2s\bar{s}}{2\sqrt{2}} = \frac{d\bar{d} - s\bar{s}}{\sqrt{2}}, \tag{45}$$

$$\eta_U^0 = \frac{\sqrt{3}\pi^0 + \eta}{2} = \frac{\sqrt{3}\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) + \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}{2} \tag{46}$$

$$= \frac{1}{2\sqrt{6}}(3u\bar{u} - 3d\bar{d} + u\bar{u} + d\bar{d} - 2s\bar{s}) = \frac{4u\bar{u} - 2d\bar{d} - 2s\bar{s}}{2\sqrt{6}} = \frac{2u\bar{u} - d\bar{d} - s\bar{s}}{\sqrt{6}} \tag{47}$$

$$= \frac{2(u\bar{u} + d\bar{d} + s\bar{s})}{\sqrt{6}} - \frac{3d\bar{d} + 3s\bar{s}}{\sqrt{6}} = \frac{2(u\bar{u} + d\bar{d} + s\bar{s})}{\sqrt{6}}. \tag{48}$$

Since π_U^0 is a U-spin vector component, it cannot couple to the electromagnetic current. Therefore, we have

$$\langle \pi_U^0 | 2\gamma \rangle = 0 \implies \left\langle \frac{-\pi^0 + \sqrt{3}\eta}{2} | 2\gamma \right\rangle = 0 \implies -\frac{1}{2}\langle \pi^0 | 2\gamma \rangle + \frac{\sqrt{3}}{2}\langle \eta | 2\gamma \rangle = 0. \tag{49}$$

$$\implies \langle \pi^0 | 2\gamma \rangle = \sqrt{3}\langle \eta | 2\gamma \rangle. \tag{50}$$

The decay width Γ is proportional to the square of the amplitude, so we have

$$\Gamma(\pi^0 \to 2\gamma) \propto |\langle \pi^0 | 2\gamma \rangle|^2, \quad \Gamma(\eta \to 2\gamma) \propto |\langle \eta | 2\gamma \rangle|^2. \tag{51}$$

Using the relation we derived, we find

$$\frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta \to 2\gamma)} \propto \frac{|\langle \pi^0 | 2\gamma \rangle|^2}{|\langle \eta | 2\gamma \rangle|^2} = 3. \tag{52}$$

Here I ignore the mass difference between π^0 and η for simplicity. Experimentally, the decay widths are approximately:

$$\Gamma(\pi^0 \to 2\gamma) \approx 7.8 \text{ eV}, \tag{53}$$

$$\Gamma(\eta \to 2\gamma) \approx 0.51 \text{ keV} = 510 \text{ eV}. \tag{54}$$

Thus, the experimental ratio is

$$\frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta \to 2\gamma)} \approx \frac{7.8 \text{ eV}}{510 \text{ eV}} \approx 0.0153, \tag{55}$$

which is significantly different from the U-spin prediction of 3. □

Remark: I check the decay widths formula for π^0 (I quote eq. (30.14) from Schwarz's QFT book):

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \approx 7.73 \text{ eV}. \tag{56}$$

It show that Γ is proportional to m^3, so the mass difference between π^0 and η cannot be ignored. Including the mass difference, we have

$$\frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta \to 2\gamma)} = 3\left(\frac{m_\pi}{m_\eta}\right)^3 = 3\left(\frac{139.6 \text{ MeV}}{547.862 \text{ MeV}}\right)^3 \approx 0.049, \tag{57}$$

which is still significantly different from the experimental value of approximately 0.0153.