University of Minnesota

School of Physics and Astronomy

2026 Spring Physics 8902
Elementary Particle Physics II

Assignment Solution

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Problem Set 2 Due 11am, Monday, February 16

Question 1

Weak decay of pions

(a) Find the electron energy spectrum $d\Gamma/dE_e$ for the decay $\pi^- \to \pi^0 + e^- + \overline{\nu}_e$ in the π^- rest frame keeping $m_e \neq 0$ (take $m_\nu = 0$). Assume the hadronic current is dominated by $f_+(0)$ and neglect radiative corrections. Perform the phase-space integration by integrating over the π^0 and $\overline{\nu}_e$ momenta (i.e. treat E_e as the only observed variable). Give the kinematic endpoints and verify the $m_e \to 0$ limit.

(b) Using the electron energy spectrum obtained in part (a), integrate over E_e to extract the leading correction of order m_e^2/Δ^2 to the total decay rate. Write the result in the form

$$\Gamma(\pi^- \to \pi^0 + e^- + \overline{\nu}_e) = |V_{ud}|^2 \frac{G_F^2 \Delta^5}{30\pi^3}\left(1 - a\frac{\Delta}{m_\pi} - b\frac{m_e^2}{\Delta^2}\right), \tag{1}$$

where $\Delta = m_{\pi^-} - m_{\pi^0}$, and neglecting higher-order terms in Δ/m_π and m_e^2/Δ^2. In the lectures it was shown that $a = 3/2$. Determine the coefficient b.

Answer

(a)

Let us denote the momenta of π^-, π^0, e^-, and $\overline{\nu}_e$ as p, p', k, and k', respectively. The decay amplitude can be written as

$$\mathcal{M} = \langle \pi^0(p') e^-(k) \overline{\nu}_e(k') | \mathcal{H}_W | \pi^-(p) \rangle = -\frac{G_F}{\sqrt{2}} V_{ud} \langle \pi^0(p') | \overline{d}\gamma^\mu u | \pi^-(p) \rangle \langle e^-(k) \overline{\nu}_e(k') | \overline{e}\gamma_\mu(1-\gamma^5)\nu_e | 0 \rangle. \tag{2}$$

The hadronic matrix element can be parameterized as

$$\langle \pi^0(p') | \overline{d}\gamma^\mu u | \pi^-(p) \rangle = f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu \approx f_+(0)(p+p')^\mu = \sqrt{2}(p+p')^\mu, \tag{3}$$

where $q = p - p'$. Neglecting radiative corrections and using the fact that $f_+(0)$ dominates, we can approximate $f_+(q^2) \approx f_+(0) = \sqrt{2}$. The leptonic matrix element can be evaluated using standard techniques, yielding

$$\langle e^-(k) \overline{\nu}_e(k') | \overline{e}\gamma_\mu(1-\gamma^5)\nu_e | 0 \rangle = \overline{u}(k)\gamma_\mu(1-\gamma^5)v(k'). \tag{4}$$

Now the amplitude can be expressed as

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{ud}\sqrt{2}(p+p')^{\mu}\overline{u}(k)\gamma_{\mu}(1-\gamma^5)v(k') \tag{5}$$

$$= -\, G_F V_{ud}(p+p')^{\mu}\overline{u}(k)\gamma_{\mu}(1-\gamma^5)v(k'). \tag{6}$$

Then the squared amplitude, summed over final spins, is given by

$$\langle|\mathcal{M}|^2\rangle = \sum_{\text{spins}} |\mathcal{M}|^2 = G_F^2|V_{ud}|^2(p+p')^{\mu}(p+p')^{\nu}\overline{u}_{s_1}(k)\gamma_{\mu}(1-\gamma^5)v_{s_2}(k')\overline{v}_{s_2}(k')\gamma_{\nu}(1-\gamma^5)u_{s_1}(k) \tag{7}$$

$$= G_F^2|V_{ud}|^2(p+p')^{\mu}(p+p')^{\nu}\mathrm{Tr}\Big[(\not{k}+m_e)\gamma_{\mu}(1-\gamma^5)\not{k}'\gamma_{\nu}(1-\gamma^5)\Big] \tag{8}$$

$$= G_F^2|V_{ud}|^2(p+p')^{\mu}(p+p')^{\nu}k^{\alpha}k'^{\beta}\mathrm{Tr}\Big[(\gamma_{\alpha}+1m_e)\gamma_{\mu}(1-\gamma^5)\gamma_{\beta}\gamma_{\nu}(1-\gamma^5)\Big]. \tag{9}$$

where we have used the spin sum identities for the electron and neutrino:

$$\sum_{s_1} u_{s_1}(k)\overline{u}_{s_1}(k) = \not{k}+m_e, \quad \sum_{s_2} v_{s_2}(k')\overline{v}_{s_2}(k') = \not{k}' - m_{\nu} \approx \not{k}'. \tag{10}$$

We provide the full set of trace identities:

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}, \quad \mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^5) = 0, \quad \mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \tag{11}$$

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^5) = 4i\epsilon^{\mu\nu\rho\sigma}, \quad \mathrm{Tr}(\text{odd number of } \gamma^5) = 0. \tag{12}$$

Hence,

$$\mathrm{Tr}\Big[(\gamma_{\alpha}+1m_e)\gamma_{\mu}(1-\gamma^5)\gamma_{\beta}\gamma_{\nu}(1-\gamma^5)\Big] \tag{13}$$

$$= \mathrm{Tr}\Big[(\gamma_{\alpha}+1m_e)\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}(1-\gamma^5)\Big] - \mathrm{Tr}\Big[(\gamma_{\alpha}+1m_e)\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\gamma^5(1-\gamma^5)\Big] \tag{14}$$

Question 2

Tau decays

(a) Find the decay rate for the two-body decay $\tau^- \to \pi^- + \nu_\tau$, neglecting neutrino masses and using $\langle 0|\overline{d}\gamma^\mu\gamma^5 u|\pi^-\rangle = if_\pi p_\pi^\mu$. Determine the ratio

$$R_\pi = \frac{\Gamma(\tau^- \to \pi^- + \nu_\tau)}{\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)}, \tag{15}$$

using the tree-level leptonic rate with $m_e = 0$, and compare with the corresponding PDG branching-fraction ratio.

(b) Now consider $\tau^- \to \rho^- + \nu_\tau$ with $\langle 0|\overline{d}\gamma^\mu u|\rho^-(q,\epsilon)\rangle = f_\rho m_\rho \epsilon^\mu$, and derive the decay rate $\Gamma(\tau^- \to \rho^- + \nu_\tau)$ (neglect the neutrino mass). Form the ratio

$$R = \frac{\Gamma(\tau^- \to \rho^- + \nu_\tau)}{\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)}, \tag{16}$$

and nd compare with the PDG data to extract f_ρ (or the ratio f_ρ/f_π).
Hint: Use the polarization sum

$$\sum_\lambda \epsilon_\mu^{(\lambda)}(q)\epsilon_\nu^{(\lambda)*}(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2}. \tag{17}$$