University of Minnesota

School of Physics and Astronomy

2026 Spring Physics 8902
Elementary Particle Physics II

Assignment Solution

Lecture Instructor: Professor Tony Gherghetta

Zong-En Chen

chen9613@umn.edu

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Problem Set 1 Due 11am, Monday, February 2

Question 1

Show that the charge-lowering weak current of the form

$$J^\mu = \bar{e}\gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_e \tag{1}$$

involves only left-handed electrons (or right-handed positrons). In the relativistic limit ($v \approx c$) show that the electrons have negative helicity.

Answer

We start with the Dirac spinor:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2}$$

where ψ_L and ψ_R are the left-handed and right-handed components, respectively. To be more specific, here the left-handed and right-handed are referring to chirality. Besides, this notation is called Weyl or chiral representation. In this representation, the gamma matrices are expressed as:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{3}$$

where I is the 2×2 identity matrix and σ^i are the Pauli matrices. The projection operators for left-handed and right-handed components are given by:

$$P_L = \frac{1}{2}(1-\gamma^5), \quad P_R = \frac{1}{2}(1+\gamma^5). \tag{4}$$

Then the projection operator P_L can be explicitly written as:

$$P_L = \frac{1}{2}\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

Applying this operator to the Dirac spinor ψ, we have:

$$P_L\psi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}. \tag{6}$$

Besides, we have this relation:

$$\overline{\psi}\gamma^\mu P_L \psi = \overline{\psi} P_R \gamma^\mu \psi, \tag{7}$$

where we used the fact that $\gamma^\mu P_L = P_R \gamma^\mu$. The $\overline{\psi}$ is defined as $\overline{\psi} = \psi^\dagger \gamma^0$. Therefore, we have:

$$\begin{aligned}
J^\mu &= \bar{e}\gamma^\mu P_L \nu_e = \bar{e}\gamma^\mu P_L P_L \nu_e && (8)\\
&= \bar{e} P_R \gamma^\mu P_L \nu_e = e^\dagger \gamma^0 P_R \gamma^\mu P_L \nu_e && (9)\\
&= e^\dagger P_L \gamma^0 \gamma^\mu P_L \nu_e && (10)\\
&= (P_L e)^\dagger \gamma^0 \gamma^\mu (P_L \nu_e) && (11)\\
&= \overline{e_L} \gamma^\mu \nu_{eL}. && (12)
\end{aligned}$$

This shows that the weak current J^μ only involves left-handed electrons (or right-handed positrons). Besides, it also involves left-handed neutrinos.

Next, we start from the Dirac equation, the solution for Dirac spinor $u(p)$ with momentum p and mass m can be expressed as:

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi_s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi_s \end{pmatrix}, \tag{13}$$

where $\sigma^\mu = (I, \sigma^i)$, $\bar{\sigma}^\mu = (I, -\sigma^i)$, and ξ_s is a two-component spinor representing the spin state:

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{14}$$

For simplicity, we consider the electron moving along the z-axis, so the four-momentum can be written as:

$$p^\mu = (E, 0, 0, p_z), \tag{15}$$

where $E = \sqrt{p_z^2 + m^2}$. Hence, we have:

$$u_+(p) = \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \\ 0 \end{pmatrix}, \quad u_-(p) = \begin{pmatrix} 0 \\ \sqrt{E + p_z} \\ 0 \\ \sqrt{E - p_z} \end{pmatrix}. \tag{16}$$

In particular, the left-handed component of the Dirac spinor can be obtained by applying the projection operator P_L:

$$P_L u_+(p) = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_L u_-(p) = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ 0 \end{pmatrix}. \tag{17}$$

Now the helicity operator is defined as:

$$h = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = \Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \frac{p_z}{|p_z|} \tag{18}$$

Now we can apply this operator to act on the solution with left-handed spinor, we have:

$$hP_L u_-(p) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \frac{p_z}{|p_z|} \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{p_z}{|p_z|} \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{19}$$

and

$$hP_L u_-(p) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \frac{p_z}{|p_z|} \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ 0 \end{pmatrix} = -\frac{p_z}{|p_z|} \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

In the relativistic limit, we have $p_z = \pm E$. For $p_z = +E$, only $u_-(p)$ survives after the projection, and we have:

$$hP_L u_-(p) = -P_L u_-(p), \tag{21}$$

which means the helicity is negative. For $p_z = -E$, only $u_+(p)$ survives after the projection, and we have:

$$hP_L u_+(p) = -P_L u_+(p), \tag{22}$$

which also means the helicity is negative. **Therefore, in the relativistic limit, the left-handed electrons have negative helicity.** $\square$

Question 2

Weak decays of leptons

(a) Calculate the muon total decay width, $\Gamma(\mu^- \to e^-\overline{\nu}_e\nu_\mu)$, accounting for the finite mass of the electron.

(b) Use this result to determine the numerical value of the ratio

$$R = \frac{\Gamma(\tau^- \to \mu^-\overline{\nu}_\mu\nu_\tau)}{\Gamma(\tau^- \to e^-\overline{\nu}_e\nu_\tau)}. \tag{23}$$

Compare the theoretical prediction for this ratio with the experimental value.

Answer

(a) We can start from the differential decay width for the muon decay process $\mu^- \to e^-\overline{\nu}_e\nu_\mu$, which is given by (this is allowed in the lecture notes):

$$d\Gamma = \frac{1}{12\pi}G_F^2\left[3(m_\mu^2+m_e^2)E - 4m_\mu^2E - 2m_\mu m_e^2\right]|\vec{p}|dE, \tag{24}$$

where G_F is the Fermi coupling constant, m_μ and m_e are the masses of muon and electron, respectively. Here E and $\vec{p}$ are the energy and momentum of the electron in the final state. Hence, we can integrate over the electron energy E to get the total decay width. The range of E is from m_e to $\frac{m_\mu^2+m_e^2}{2m_\mu}$. The upper limit can be derived from the energy-momentum conservation in the muon rest frame:

$$m_\mu = E + E_{\overline{\nu}_e} + E_{\nu_\mu} \geq E + |\vec{p}_{\overline{\nu}_e}| + |\vec{p}_{\nu_\mu}| \geq E + |\vec{p}_{\overline{\nu}_e} + \vec{p}_{\nu_\mu}| = E + |\vec{p}| = E + \sqrt{E^2 - m_e^2} \tag{25}$$

$$\implies E \leq \frac{m_\mu^2+m_e^2}{2m_\mu}. \tag{26}$$

Therefore, we have:

$$\Gamma = \frac{1}{12\pi}G_F^2\int_{m_e}^{\frac{m_\mu^2+m_e^2}{2m_\mu}}\left[3(m_\mu^2+m_e^2)E - 4m_\mu^2E - 2m_\mu, m_e^2\right]\sqrt{E^2-m_e^2}dE \tag{27}$$

$$= \frac{1}{12\pi}G_F^2\frac{8m_e^6m_\mu^2 - 8m_e^2m_\mu^6 + 24m_e^4m_\mu^4\log\left(\frac{m_\mu}{m_e}\right) - m_e^8 + m_\mu^8}{16m_\mu^3}, \quad \text{by using Mathematica.} \tag{28}$$

$$\approx \frac{G_F^2m_\mu^5}{192\pi^3}\left[1 - 8\frac{m_e^2}{m_\mu^2} + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\right)\right]. \tag{29}$$

This is the total decay width of muon accounting for the finite mass of the electron.

(b) Now we can use the result from part (a) to calculate the ratio:

$$R_{\text{th}} = \frac{\Gamma(\tau^- \to \mu^- \overline{\nu}_\mu \nu_\tau)}{\Gamma(\tau^- \to e^- \overline{\nu}_e \nu_\tau)} \tag{30}$$

$$= \frac{1 - 8\frac{m_\mu^2}{m_\tau^2} + \mathcal{O}\left(\frac{m_\mu^4}{m_\tau^4}\right)}{1 - 8\frac{m_e^2}{m_\tau^2} + \mathcal{O}\left(\frac{m_e^4}{m_\tau^4}\right)} \tag{31}$$

$$\approx (1 - 8\frac{m_\mu^2}{m_\tau^2})(1 + 8\frac{m_e^2}{m_\tau^2}) \tag{32}$$

$$\approx 1 - 8\frac{m_\mu^2 - m_e^2}{m_\tau^2}. \tag{33}$$

Using the known values of the masses: $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, and $m_\tau = 1776.86$ MeV, we can calculate the numerical value of $R_{\text{th}} = 0.971712$. The experimental value is given by

$$R_{\text{exp}} = \frac{B(\tau^- \to \mu^- \overline{\nu}_\mu \nu_\tau)}{B(\tau^- \to e^- \overline{\nu}_e \nu_\tau)} = \frac{0.1739}{0.1782} = 0.97587. \tag{34}$$

The theoretical prediction is in good agreement with the experimental value, with a small difference that can be attributed to higher-order corrections and experimental uncertainties. □