University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Question 1

p-d reactions

Consider the reactions

$$p + d \to \pi^+ + {}^3\text{H}, \quad p + d \to \pi^0 + {}^3\text{He}. \tag{1}$$

Since the deuteron is in a ${}^3S_1$ state, it must be an isospin singlet. Therefore, the initial state $p + d$ is a pure $I = \frac{1}{2}$ state. Given that ^{3}H and ^{3}He form an isodoublet, write down the isospin decomposition of the final states, and from this, the ratio of the two cross sections.

Answer

First, we can use I_3 to decide the isospin for ^{3}H and ^{3}He. See the initial state $p+d$ has $I_3 = +\frac{1}{2}$, so the final state must also have $I_3 = +\frac{1}{2}$. Since π^+ has $I_3 = +1$ and π^0 has $I_3 = 0$, we can conclude that ^{3}H has $I_3 = -\frac{1}{2}$ and ^{3}He has $I_3 = +\frac{1}{2}$. Therefore, ^{3}H and ^{3}He form an isodoublet with $I = \frac{1}{2}$. Now we can write down the isospin decomposition of the final states. For the first reaction, we have

$$|\pi^+ + {}^3\text{H}\rangle = |\pi^+\rangle \otimes |{}^3\text{H}\rangle = |1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle. \tag{2}$$

Using the Clebsch-Gordan coefficients, we can decompose this into total isospin

$$|1, 1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{3}$$

For the second reaction, we have

$$|\pi^0 + {}^3\text{He}\rangle = |\pi^0\rangle \otimes |{}^3\text{He}\rangle = |1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{4}$$

Using the Clebsch-Gordan coefficients, we can decompose this into total isospin

$$|1, 0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}, \tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{5}$$

Now, since the initial state $p + d$ is a pure $I = \frac{1}{2}$ state, only the $I = \frac{1}{2}$ component of the final states will contribute to the cross sections. Therefore, we can write the amplitudes for the two reactions as

$$\mathcal{A}(p + d \to \pi^+ + {}^3\text{H}) \propto \sqrt{\tfrac{2}{3}}, \tag{6}$$

$$\mathcal{A}(p + d \to \pi^0 + {}^3\text{He}) \propto -\tfrac{1}{\sqrt{3}}. \tag{7}$$

The cross sections are proportional to the square of the amplitudes, so we have

$$\sigma(p + d \to \pi^+ + {}^3\mathrm{H}) \propto \left|\sqrt{\tfrac{2}{3}}\right|^2 = \tfrac{2}{3}, \tag{8}$$

$$\sigma(p + d \to \pi^0 + {}^3\mathrm{He}) \propto \left|-\tfrac{1}{\sqrt{3}}\right|^2 = \tfrac{1}{3}. \tag{9}$$

Finally, the ratio of the two cross sections is

$$\frac{\sigma(p + d \to \pi^+ + {}^3\mathrm{H})}{\sigma(p + d \to \pi^0 + {}^3\mathrm{He})} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2. \tag{10}$$

$\square$

Question 2

Particle production by strong interactions

Explain why the processes $\pi^- + p \to \pi^+ + \Sigma^-$, $\pi^- + p \to K^0 + n$, $\pi^- + p \to \Sigma^+ + K^-$ cannot be observed.

Answer

Before we analyze the processes, let's summarize the quantum numbers of the particles involved:

- π^-: $I = 1$, $I_3 = -1$, $S = 0$, $B = 0$
- π^+: $I = 1$, $I_3 = +1$, $S = 0$, $B = 0$
- p: $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$, $S = 0$, $B = 1$
- n: $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, $S = 0$, $B = 1$
- Σ^-: $I = 1$, $I_3 = -1$, $S = -1$, $B = 1$
- Σ^+: $I = 1$, $I_3 = +1$, $S = -1$, $B = 1$
- K^0: $I = \frac{1}{2}$, $I_3 = +\frac{1}{2}$, $S = +1$, $B = 0$
- K^-: $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, $S = -1$, $B = 0$

Now, let's analyze each process:

(a) $\pi^- + p \to \pi^+ + \Sigma^-$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$
- Final state: $I_3 = +1 - 1 = 0$, $S = 0 - 1 = -1$, $B = 0 + 1 = 1$

The strangeness S changes from 0 to -1, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to 0. Therefore, this process cannot be observed.

(b) $\pi^- + p \to K^0 + n$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$
- Final state: $I_3 = +\frac{1}{2} - \frac{1}{2} = 0$, $S = +1 + 0 = +1$, $B = 0 + 1 = 1$

The strangeness S changes from 0 to +1, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to 0. Therefore, this process cannot be observed.

(c) $\pi^- + p \to \Sigma^+ + K^-$:

- Initial state: $I_3 = -1 + \frac{1}{2} = -\frac{1}{2}$, $S = 0 + 0 = 0$, $B = 0 + 1 = 1$

- Final state: $I_3 = +1 - \frac{1}{2} = +\frac{1}{2}$, $S = -1 - 1 = -2$, $B = 1 + 0 = 1$

The strangeness S changes from 0 to -2, which is not allowed in strong interactions. The isospin I_3 also changes from $-\frac{1}{2}$ to $+\frac{1}{2}$. Therefore, this process cannot be observed.

$\square$

Question 3

SU(2) invariants and pseudoreal representations

(a) Show that $\delta^a{}_b$ and ϵ_{ab} are invariant tensors under SU(2) transformations.

(b) The nucleon doublet $N^a = \binom{p}{n}, a = 1, 2$ transforms as the fundamental 2 of SU(2), while its conjugate $\bar{N}_a \equiv (N^a)^\dagger = (\bar{p}, \bar{n})$ transforms as $\overline{\mathbf{2}}$. Use $\delta^a{}_b$ to form an SU(2) invariant with $N, \bar{N}$ and write it explicitly in terms of the proton and neutron fields.

(c) Define $\tilde{N}^b = \epsilon^{bc}\bar{N}_c^T$ which maps the $\overline{\mathbf{2}}$ representation (lower index) into $\mathbf{2}$ (upper index). Construct an SU(2) invariant with $N, \tilde{N}$ using ϵ_{ab}, and write it in terms of the components. Verify that the result is identical to part (b), demonstrating that the $\mathbf{2}$ and $\overline{\mathbf{2}}$ representations are equivalent (or pseudoreal) in SU(2) and that any invariant constructed with $\delta^a{}_b$ can be rewritten using ϵ_{ab}.

(d) Consider SU(3), with the quark triplet $q^a (a = 1, 2, 3)$ transforming as $\mathbf{3}$ and its conjugate $\overline{q}_a \equiv (q^a)^\dagger$ transforming as $\overline{\mathbf{3}}$. Discuss why a similar mapping using the SU(3) invariant ϵ_{abc} does not make $\mathbf{3}$ and $\overline{\mathbf{3}}$ equivalent. Write down the possible SU(3) invariants involving $q, \overline{q}$.

Answer

(a)

$$\delta^a{}_b \to= \delta'^a{}_b = U^a{}_c\delta^c{}_d(U^\dagger)^d{}_b = U^a{}_c(U^\dagger)^c{}_b = \mathbf{1}^a{}_b = \delta^a{}_b, \tag{11}$$

$$\epsilon_{ab} \to \epsilon'_{ab} = (U^\dagger)^c{}_a(U^\dagger)^d{}_b\epsilon_{cd} = \det(U^\dagger)\epsilon_{ab} = \epsilon_{ab}. \tag{12}$$

(b)

Using $\delta^a{}_b$, we can form the invariant

$$\bar{N}_a N^a = \delta^a{}_b\bar{N}_a N^b = \bar{p}p + \bar{n}n. \tag{13}$$

We can verify that this is indeed invariant under SU(2) transformations:

$$\bar{N}_a N^a \to \bar{N}'_a N'^a = \bar{N}_b(U^\dagger)^b{}_a U^a{}_c N^c = \bar{N}_b\delta^b{}_c N^c = \bar{N}_a N^a. \tag{14}$$

(c)

Using ϵ_{ab}, we can form the invariant

$$\epsilon_{ab}N^a\tilde{N}^b = \epsilon_{ab}N^a\epsilon^{bc}\bar{N}_c^T = \delta_a{}^c N^a\bar{N}_c^T = N^a\bar{N}_a^T = \bar{N}_a N^a = \bar{p}p + \bar{n}n. \tag{15}$$

We can verify that this is indeed invariant under SU(2) transformations:

$$\epsilon_{ab} N^a \tilde{N}^b \to \epsilon_{ab} N'^a \tilde{N}'^b = \epsilon_{ab} U^a{}_c N^c U^b{}_d \tilde{N}^d = \det(U) \epsilon_{cd} N^c \tilde{N}^d = \epsilon_{cd} N^c \tilde{N}^d. \tag{16}$$

This demonstrates that the **2** and $\overline{\mathbf{2}}$ representations are equivalent (or pseudoreal) in SU(2) and that any invariant constructed with $\delta^a{}_b$ can be rewritten using ϵ_{ab}.

(d)

The possible SU(3) invariants involving q and $\overline{q}$ are:

$$\overline{q}_a q^a, \quad \epsilon_{abc} q^a q^b q^c, \quad \epsilon^{abc} \overline{q}_a \overline{q}_b \overline{q}_c. \tag{17}$$

In other words, $\overline{q}_a$ and $\epsilon_{abc} q^a q^b$ can transforming as $\overline{\mathbf{3}}$ representation, but $\epsilon_{abc} q^a q^b$ is a composite object made of two quarks, not a single quark. That means

Question 4

Applications of U-spin

(a) Show that $U_\pm = t_6 \pm it_7$ and $U_3 = \left(\sqrt{3}t_8 - t_3\right)/2$ satisfy the SU(2) algebra

$$[U_3, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = 2U_3.$$

(b) Show that the charge operator $Q = t_3 + t_8/\sqrt{3}$ is a U-scalar i.e. it has U-spin $U = 0$ or $[Q, U_i] = 0$ for $i = \pm 3$. Write the electromagnetic current operator in terms of quark fields.

(c) Show that for the meson octet, the ($U_3 = 0$) component of the U-triplet is $\pi_U^0 = \left(-\pi^0 + \sqrt{3}\eta\right)/2$, and the U-singlet is $\eta_U^0 = \left(\sqrt{3}\pi^0 + \eta\right)/2$. Since π_U^0 is a U -spin vector component it cannot couple to the electromagnetic current. Show that for the 2γ decay mode, $\langle \pi^0 \mid 2\gamma \rangle = \sqrt{3}\langle \eta \mid 2\gamma \rangle$. How does this U-spin prediction compare with the experimental decay widths?

Answer

(a)

We start with the commutation relations of the t_i generators of SU(3):

$$[t_i, t_j] = if_{ijk}t_k, \tag{18}$$