University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

Lecture Instructor: Professor Zhen Liu

Zong-En Chen
chen9613@umn.edu

November 18, 2025

HW5 Due to November 18 11:59 PM

Question 1

Problem 18.1

In any number d of spacetime dimensions, a *Dirac field* Ψ_α carries a spin index α, and has a kinetic term of the form $i\overline{\Psi}\gamma^\mu\partial_\mu\Psi$, where we have suppressed the spin indices; the *gamma matrices* γ^μ are dimensionless, and $\overline{\Psi} = \Psi^\dagger\gamma^0$.

(a) What is the mass dimension $[\Psi]$ of the field Ψ.

(b) Consider interaction of the form $g_n(\overline{\Psi}\Psi)^n$, where $n \geq 2$ is an integer. What is the mass dimension $[g_n]$ of g_n?

(c) Consider interaction of the form $g_{m,n}\varphi^m(\overline{\Psi}\Psi)^n$, where φ is a scalar field, and $m, n > 0$ are integers. What is the mass dimension $[g_{m,n}]$ of $g_{m,n}$?

(d) In $d = 4$ spacetime dimensions, which of these interactions are allowed in a renormalizable theory?

Answer

(a)

We have in d spacetime dimensions, the action is dimensionless, so the Lagrangian density has mass dimension $[\mathcal{L}] = d$. The kinetic term for the Dirac field is given by:

$$\mathcal{L}_{\text{kin}} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi. \tag{1}$$

The derivative ∂_μ has mass dimension 1, and the gamma matrices γ^μ are dimensionless. Therefore, we can write:

$$[\mathcal{L}_{\text{kin}}] = [\overline{\Psi}] + [\partial_\mu] + [\Psi] = 2[\Psi] + 1. \tag{2}$$

Setting this equal to the mass dimension of the Lagrangian density, we have:

$$2[\Psi] + 1 = d \implies [\Psi] = \frac{d-1}{2}. \tag{3}$$

(b)

The interaction term is given by:

$$\mathcal{L}_{\text{int}} = g_n(\overline{\Psi}\Psi)^n. \tag{4}$$

The mass dimension of this term is:

$$[\mathcal{L}_{\text{int}}] = [g_n] + n([\overline{\Psi}] + [\Psi]) = [g_n] + 2n[\Psi]. \tag{5}$$

Setting this equal to the mass dimension of the Lagrangian density, we have:

$$[g_n] + 2n[\Psi] = d \implies [g_n] = d - 2n[\Psi] = d - 2n\left(\frac{d-1}{2}\right) = d - n(d-1) = d(1-n) + n. \tag{6}$$

(c)

The interaction term is given by:

$$\mathcal{L}_{\text{int}} = g_{m,n}\varphi^m(\overline{\Psi}\Psi)^n. \tag{7}$$

The mass dimension of this term is:

$$[\mathcal{L}_{\text{int}}] = [g_{m,n}] + m[\varphi] + n([\overline{\Psi}] + [\Psi]) = [g_{m,n}] + m[\varphi] + 2n[\Psi]. \tag{8}$$

Setting this equal to the mass dimension of the Lagrangian density, we have:

$$[g_{m,n}] + m[\varphi] + 2n[\Psi] = d \implies [g_{m,n}] = d - m[\varphi] - 2n[\Psi]. \tag{9}$$

The mass dimension of the scalar field φ in d dimensions is given by:

$$[\varphi] = \frac{d-2}{2}. \tag{10}$$

Substituting this and the expression for $[\Psi]$ into the equation for $[g_{m,n}]$, we get:

$$[g_{m,n}] = d - m\left(\frac{d-2}{2}\right) - 2n\left(\frac{d-1}{2}\right) = d - \frac{m(d-2)}{2} - n(d-1). \tag{11}$$

(d)

In $d = 4$ spacetime dimensions, we have:

$$[\Psi] = \frac{4-1}{2} = \frac{3}{2}. \tag{12}$$

For the interaction $g_n(\overline{\Psi}\Psi)^n$, we have:

$$[g_n] = 4(1-n) + n = 4 - 3n. \tag{13}$$

For the interaction $g_{m,n}\varphi^m(\overline{\Psi}\Psi)^n$, we have:

$$[g_{m,n}] = 4 - m\left(\frac{4-2}{2}\right) - 2n\left(\frac{4-1}{2}\right) = 4 - m - 3n. \tag{14}$$

For a theory to be renormalizable, the coupling constants must have non-negative mass dimensions. Therefore:

- For g_n:

  $$4 - 3n \geq 0 \implies n \leq \frac{4}{3}. \tag{15}$$

  Since n is an integer and $n \geq 2$, there are no renormalizable interactions of this form.

- For $g_{m,n}$:

  $$4 - m - 3n \geq 0 \implies m + 3n \leq 4. \tag{16}$$

  The possible integer pairs (m, n) that satisfy this inequality with $m, n > 0$ are:

  - $(m, n) = (1, 1)$
  - $(m, n) = (2, 1)$
  - $(m, n) = (1, 2)$

$\square$

Question 2

Problem 20.2

Compute the $\mathcal{O}(\alpha)$ correction to the two-particle scattering amplitude at *threshold*, that is, for $s = 4m^2$ and $t = u = 0$, corresponding to zero three-momentum for both the incoming and outgoing particles.

Hint: for 20.2, do not ues Eq. (20.12)-(20.19) they are in different unit.

Answer

Starting with the equation (20.2) in the Srednicki's textbook, we have

$$i\mathcal{T}_{1-loop} = \frac{1}{i}\Big(i[\mathbf{V}_3(s)]^2\widetilde{\mathbf{\Delta}}(-s) + i[\mathbf{V}_3(t)]^2\widetilde{\mathbf{\Delta}}(-t) + i[\mathbf{V}_3(u)]^2\widetilde{\mathbf{\Delta}}(-u)\Big) + i\mathbf{V}_4(s,t,u), \tag{17}$$

where

$$\tilde{\mathbf{\Delta}}(-s) = \frac{1}{-s + m^2 - \Pi(-s)} \tag{18}$$

$$\Pi(-s) = \frac{1}{2}\alpha\int_0^1 dx D_2(s)\ln\left(D_2(s)/D_0\right) - \frac{1}{12}\alpha\left(-s+m^2\right) \tag{19}$$

$$\mathbf{V}_3(s)/g = 1 - \frac{1}{2}\alpha\int dF_3 \ln\left(D_3(s)/m^2\right) \tag{20}$$

$$\mathbf{V}_4(s,t,u) = \frac{1}{6}g^2\alpha\int dF_4\left[\frac{1}{D_4(s,t)} + \frac{1}{D_4(t,u)} + \frac{1}{D_4(u,s)}\right]. \tag{21}$$

Also, we have

$$D_2(s) = -x(1-x)s + m^2 \tag{22}$$

$$D_0 = +[1 - x(1-x)]m^2 \tag{23}$$

$$D_3(s) = -x_1x_2s + \left[1 - (x_1+x_2)\,x_3\right]m^2 \tag{24}$$

$$D_4(s,t) = -x_1x_2s - x_3x_4t + \left[1 - (x_1+x_2)\,(x_3+x_4)\right]m^2 \tag{25}$$

and the integration measures are given by

$$\int dF_3 = 2\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\delta\left(x_1+x_2+x_3-1\right) \tag{26}$$

$$\int dF_4 = 6\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\int_0^1 dx_4\delta\left(x_1+x_2+x_3+x_4-1\right). \tag{27}$$

By *Mathematica* calculation, we have the following results at threshold $s = 4m^2$ and $t = u = 0$:

$$\tilde{\mathbf{\Delta}}(-4m^2) = -\frac{12}{\left(\left(2\sqrt{3}\pi - 9\right)\alpha + 36\right)m^2}, \tag{28}$$

$$\mathbf{V}_3(4m^2) = g - g\alpha\Big(-\frac{4}{3} + \frac{5i\pi}{12} + \frac{11\pi}{4\sqrt{3}}\Big), \tag{29}$$

$$\tilde{\mathbf{\Delta}}(0) = \frac{12}{\left(\left(2\sqrt{3}\pi - 11\right)\alpha + 12\right)m^2}, \tag{30}$$

$$\mathbf{V}_3(0) = g - g\alpha\Big(\frac{\pi}{2\sqrt{3}} - 1\Big), \tag{31}$$

$$\mathbf{V}_4(4m^2, 0, 0) = g^2\alpha\left(\frac{-6 + 6i\pi + 13\sqrt{3}\pi}{18m^2}\right) \tag{32}$$

Thus, substituting these results into the expression for $i\mathcal{T}_{1-loop}$, we obtain:

$$i\mathcal{T}_{1-loop} = \frac{5g^2}{3m^2} + \frac{\left((525 - 36i) + \pi\left((-36 + 30i) - (40 - 78i)\sqrt{3}\right)\right)\alpha g^2}{108m^2} \tag{33}$$

$$= \frac{1.66667g^2}{m^2} + \frac{(1.79858 + 4.46923i)\alpha g^2}{m^2} \tag{34}$$

Second line is numerical result.

Remark: My detail calculation in Mathematica. □

Question 3

Problem 27.1
Suppose that we have a theory with

$$\beta(\alpha) = b_1\alpha^2 + \mathcal{O}(\alpha^3), \tag{35}$$

$$\gamma_m(\alpha) = c_1\alpha + \mathcal{O}(\alpha^2). \tag{36}$$

Neglecting the higher-order terms, show that

$$m(\mu_2) = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{c_1/b_1} m(\mu_1). \tag{37}$$

Answer

The gamma and beta function are given by:

$$\gamma_m(\alpha) = \frac{d}{d\ln\mu}\ln m(\mu) = c_1\alpha, \tag{38}$$

$$\beta(\alpha) = \frac{d}{d\ln\mu}\alpha = b_1\alpha^2. \tag{39}$$

We can rearrange the beta function to express $d\ln\mu$ in terms of $d\alpha$:

$$d\ln\mu = \frac{d\alpha}{\beta(\alpha)} = \frac{d\alpha}{b_1\alpha^2}. \tag{40}$$

Substituting this into the expression for $\gamma_m(\alpha)$, we have:

$$\frac{d}{d\ln\mu}\ln m(\mu) = c_1\alpha \implies d\ln m(\mu) = c_1\alpha\, d\ln\mu = c_1\alpha\cdot\frac{d\alpha}{b_1\alpha^2} = \frac{c_1}{b_1}\frac{d\alpha}{\alpha}. \tag{41}$$

Integrating both sides from μ_1 to μ_2, we get:

$$\int_{m(\mu_1)}^{m(\mu_2)} d\ln m(\mu) = \frac{c_1}{b_1}\int_{\alpha(\mu_1)}^{\alpha(\mu_2)}\frac{d\alpha}{\alpha}. \tag{42}$$

This gives:

$$\ln\left(\frac{m(\mu_2)}{m(\mu_1)}\right) = \frac{c_1}{b_1}\ln\left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right). \tag{43}$$

Exponentiating both sides, we obtain:

$$\frac{m(\mu_2)}{m(\mu_1)} = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{c_1/b_1}. \tag{44}$$

Thus, we have shown that:

$$m(\mu_2) = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{c_1/b_1} m(\mu_1). \tag{45}$$

□

Question 4

Problem 28.1

Consider φ^4 theory ,

$$\mathcal{L} = -Z_\varphi \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - Z_m \frac{1}{2} m^2 \varphi^2 - Z_\lambda \frac{\lambda \widetilde{\mu}^\epsilon}{4!} \varphi^4, \tag{46}$$

in $d = 4 - \epsilon$ spacetime dimensions. Compute the beta function to $\mathcal{O}(\lambda^2)$, the anomalous dimension of m to $\mathcal{O}(\lambda)$, and the anomalous dimension of φ to $\mathcal{O}(\lambda)$.

Answer

We first write down the Lagrangian for ϕ^4 theory in $d = 4 - \epsilon$ dimensions:

$$\mathcal{L} = -Z_\varphi \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - Z_m \frac{1}{2} m^2 \varphi^2 - Z_\lambda \frac{\lambda \widetilde{\mu}^\epsilon}{4!} \varphi^4, \tag{47}$$

where Z_φ, Z_m, and Z_λ are the renormalization constants for the field, mass, and coupling constant, respectively. We also write down the Lagrangian in terms of bare quantities:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \varphi_0)(\partial_\mu \varphi_0) - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{\lambda_0}{4!} \varphi_0^4, \tag{48}$$

where the bare quantities are related to the renormalized quantities by:

$$\varphi_0 = Z_\varphi^{1/2} \varphi, \tag{49}$$

$$m_0^2 = Z_m Z_\varphi^{-1} m^2, \tag{50}$$

$$\lambda_0 = Z_\lambda Z_\varphi^{-2} \lambda \widetilde{\mu}^\epsilon. \tag{51}$$

From our previous calculations in ϕ^4 theory, we have the following results for the renormalization constants to the required orders:

$$Z_\varphi = 1 + \mathcal{O}(\lambda^2) = 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n}, \tag{52}$$

$$Z_m = 1 + \frac{\lambda}{16\pi^2 \epsilon} + \mathcal{O}(\lambda^2) = 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n}, \tag{53}$$

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2 \epsilon} + \mathcal{O}(\lambda^2) = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n}. \tag{54}$$

We know $a_1(\lambda) = 0 + \mathcal{O}(\lambda^2)$, $b_1(\lambda) = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$, and $c_1(\lambda) = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$. Using these results, we can compute the beta function and anomalous dimensions.

$$\ln \lambda_0 = \ln(Z_\lambda Z_\varphi^{-2}) + \ln \lambda + \epsilon \ln \widetilde{\mu}, \tag{55}$$

$$0 = \frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\ln\mu} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\ln\mu} + \frac{1}{\lambda}\frac{d\lambda}{d\ln\mu} + \epsilon, \tag{56}$$

$$\beta(\lambda) \equiv \frac{d\lambda}{d\ln\mu} = \lambda\left(-\epsilon - \frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\ln\mu} + \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\ln\mu}\right). \tag{57}$$

To compute $\frac{dZ_\lambda}{d\ln\mu}$ and $\frac{dZ_\varphi}{d\ln\mu}$, we use the chain rule:

$$\frac{dZ_\lambda}{d\ln\mu} = \frac{dZ_\lambda}{d\lambda}\frac{d\lambda}{d\ln\mu} = \frac{dZ_\lambda}{d\lambda}\beta(\lambda), \tag{58}$$

$$\frac{dZ_\varphi}{d\ln\mu} = \frac{dZ_\varphi}{d\lambda}\frac{d\lambda}{d\ln\mu} = \frac{dZ_\varphi}{d\lambda}\beta(\lambda). \tag{59}$$

Substituting these into the expression for $\beta(\lambda)$, we have:

$$\beta(\lambda) = \lambda\left(-\epsilon - \frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda}\beta(\lambda) + \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda)\right) \tag{60}$$

$$= \lambda\left(-\epsilon - \beta(\lambda)\left(\frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right)\right). \tag{61}$$

We can solve for $\beta(\lambda)$:

$$\beta(\lambda)\left(1 + \lambda\left(\frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right)\right) = -\lambda\epsilon. \tag{62}$$

In the limit $\epsilon \to 0$, we have:

$$\beta(\lambda) = -\lambda\epsilon\left(1 + \lambda\left(\frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right)\right)^{-1} \tag{63}$$

$$= -\lambda\epsilon\left(1 - \lambda\left(\frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right) + \mathcal{O}(\lambda^2)\right) \tag{64}$$

$$= -\lambda\epsilon + \lambda^2\epsilon\left(\frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right) + \mathcal{O}(\lambda^3) \tag{65}$$

$$= -\lambda\epsilon + \lambda^2\epsilon\left(\frac{1}{Z_\lambda}\frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right) + \mathcal{O}(\lambda^3) \tag{66}$$

$$= -\lambda\epsilon + \lambda^2\epsilon\left(\frac{d}{d\lambda}\left(\frac{3\lambda}{16\pi^2\epsilon}\right) - 0\right) + \mathcal{O}(\lambda^3) \tag{67}$$

$$= -\lambda\epsilon + \lambda^2\epsilon\left(\frac{3}{16\pi^2\epsilon}\right) + \mathcal{O}(\lambda^3). \tag{68}$$

Expanding to $\mathcal{O}(\lambda^2)$, we find:

$$\beta(\lambda) = -\lambda\epsilon + \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3). \tag{69}$$

Next, we compute the anomalous dimension of the mass m:

$$\begin{aligned}
0 =& \frac{d}{d\ln\mu}\ln m_0 = \frac{d}{d\ln\mu}\ln(Z_m^{1/2} Z_\varphi^{-1/2} m) && (70)\\
=& \frac{1}{2Z_m}\frac{dZ_m}{d\ln\mu} - \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} && (71)\\
=& \frac{1}{2Z_m}\frac{dZ_m}{d\lambda}\beta(\lambda) - \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda) + \frac{1}{m}\frac{dm}{d\ln\mu}. && (72)
\end{aligned}$$

Solving for $\frac{dm}{d\ln\mu}$, we have:

$$\frac{dm}{d\ln\mu} = -m\beta(\lambda)\left(\frac{1}{2Z_m}\frac{dZ_m}{d\lambda} - \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right). \tag{73}$$

Substituting the expressions for Z_m and Z_φ, we find:

$$\begin{aligned}
\frac{dm}{d\ln\mu} =& -m\left(-\lambda\epsilon + \frac{3\lambda^2}{16\pi^2}\right)\left(\frac{1}{2}\frac{d}{d\lambda}\left(\frac{\lambda}{16\pi^2\epsilon}\right) - 0\right) + \mathcal{O}(\lambda^2) && (74)\\
=& -m\left(-\lambda\epsilon + \frac{3\lambda^2}{16\pi^2}\right)\left(\frac{1}{2}\cdot\frac{1}{16\pi^2\epsilon}\right) + \mathcal{O}(\lambda^2) && (75)\\
=& -m\left(-\frac{\lambda}{32\pi^2} + \frac{3\lambda^2}{32\pi^4\epsilon}\right) + \mathcal{O}(\lambda^2) && (76)\\
=& \frac{\lambda m}{32\pi^2} + \mathcal{O}(\lambda^2). && (77)
\end{aligned}$$

Thus, the anomalous dimension of the mass m to $\mathcal{O}(\lambda)$ is:

$$\gamma_m(\lambda) = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2). \tag{78}$$

Finally, we compute the anomalous dimension of the field φ:

$$\begin{aligned}
0 =& \frac{d}{d\ln\mu}\ln\varphi_0 = \frac{d}{d\ln\mu}\ln(Z_\varphi^{1/2}\varphi) && (79)\\
=& \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\ln\mu} + \frac{1}{\varphi}\frac{d\varphi}{d\ln\mu} && (80)\\
=& \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda) + \frac{1}{\varphi}\frac{d\varphi}{d\ln\mu}. && (81)
\end{aligned}$$

Solving for $\frac{d\varphi}{d\ln\mu}$ ($\varphi = Z_\varphi^{-1/2}\varphi_0$), we have:

$$\frac{d\varphi}{d\ln\mu} = -\varphi \cdot \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda). \tag{82}$$

Substituting the expressions for Z_φ and $\beta(\lambda)$, we find:

$$\begin{aligned} \frac{d\varphi}{d\ln\mu} &= -\varphi \cdot \frac{1}{2} \cdot 0 \cdot \left(-\lambda\epsilon + \frac{3\lambda^2}{16\pi^2}\right) + \mathcal{O}(\lambda^2) && (83) \\ &= 0 + \mathcal{O}(\lambda^2). && (84) \end{aligned}$$

Thus, the anomalous dimension of the field φ to $\mathcal{O}(\lambda)$ is:

$$\gamma_\varphi(\lambda) = \frac{1}{\varphi}\frac{d\varphi}{d\ln\mu} = 0 + \mathcal{O}(\lambda^2). \tag{85}$$

In summary, we have:

$$\begin{aligned} \beta(\lambda) &= -\lambda\epsilon + \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3), && (86) \\ \gamma_m(\lambda) &= \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2), && (87) \\ \gamma_\varphi(\lambda) &= 0 + \mathcal{O}(\lambda^2). && (88) \end{aligned}$$

□

Question 5

Extra Problem

Dirac equation: Use find solutions of the matrices α_i and β of Dirac equation,

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \left(i\hbar c(\alpha^i)_{ab}\partial_i + (\beta)_{ab}mc^2\right)\Psi_b, \tag{89}$$

which makes the wavefunction Ψ satisfy the Klein-Gordon equation. (This is how Dirac discovered his equation.)

(a) What are the lowest dimensional matrices α^i and β for $m = 0$? Derive the solutions.

(b) Is the above solution unique? f not, can you write down another one of the same dimension?

(c) What are the lowest dimensional matrices α^i and β for $m \neq 0$? Derive the solutions.

(d) Is the above solution unique? If not, can you write down another one of the same dimension?

Answer

Starting with the Dirac equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \left(i\hbar c(\alpha^i)_{ab}\partial_i + (\beta)_{ab}mc^2\right)\Psi_b, \tag{90}$$

we want to find matrices α^i and β such that the wavefunction Ψ satisfies the Klein-Gordon equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\Psi = 0. \tag{91}$$

(a)

For $m = 0$, the Dirac equation simplifies to:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = i\hbar c(\alpha^i)_{ab}\partial_i\Psi_b. \tag{92}$$

To ensure that Ψ satisfies the Klein-Gordon equation, we require that the matrices α^i satisfy the anticommutation relations:

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}I, \tag{93}$$

where I is the identity matrix. The lowest dimensional matrices that satisfy these relations are the Pauli matrices, which are 2×2 matrices. Therefore, we can choose:

$$\alpha^1 = \sigma_x, \quad \alpha^2 = \sigma_y, \quad \alpha^3 = \sigma_z, \tag{94}$$

where σ_x, σ_y, and σ_z are the Pauli matrices.

(b)

The solution is not unique. Another set of 2×2 matrices that satisfy the same anticommutation relations can be obtained by multiplying the Pauli matrices by a unitary transformation. For example, we can choose:

$$\alpha^1 = U\sigma_x U^\dagger, \quad \alpha^2 = U\sigma_y U^\dagger, \quad \alpha^3 = U\sigma_z U^\dagger, \tag{95}$$

where U is any 2×2 unitary matrix.

(c)

For $m \neq 0$, the Dirac equation is:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \left(i\hbar c(\alpha^i)_{ab}\partial_i + (\beta)_{ab}mc^2\right)\Psi_b. \tag{96}$$

To ensure that Ψ satisfies the Klein-Gordon equation, we require that the matrices α^i and β satisfy the following relations:

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}I, \quad \{\alpha^i, \beta\} = 0, \quad \beta^2 = I. \tag{97}$$

The lowest dimensional matrices that satisfy these relations are the 4×4 Dirac matrices. This is because we need to accommodate both the spin and particle-antiparticle degrees of freedom. A common choice is:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{98}$$

where σ^i are the Pauli matrices and I is the 2×2 identity matrix.

(d)

The solution is not unique. Another set of 4×4 matrices that satisfy the same relations can be obtained by multiplying the Dirac matrices by a unitary transformation. For example, we can choose:

$$\alpha^i = U\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}U^\dagger, \quad \beta = U\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}U^\dagger, \tag{99}$$

where U is any 4×4 unitary matrix.

Remark: This representation is known as the Dirac representation. Other representations, such as the Weyl or Majorana representations, can also be used to express the Dirac matrices. □