

University of Minnesota  
School of Physics and Astronomy

**2025 Fall Physics 8011**  
**Quantum Field Theory I**  
Assignment Solution

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## Question 1

Problem 48.5

The charged pion  $\pi^-$  is represented by a complex scalar field  $\varphi$ , the muon  $\mu^-$  by a Dirac field  $\mathcal{M}$ , and the muon neutrino  $\nu_\mu$  by a spin-projected Dirac field  $P_L\mathcal{N}$ , where  $P_L = \frac{1}{2}(1 - \gamma_5)$ . The charged pion can decay to a muon and a muon antineutrino via the interaction

$$\mathcal{L}_1 = 2c_1 G_F f_\pi \partial_\mu \varphi \overline{\mathcal{M}} \gamma^\mu P_L \mathcal{N} + h.c., \quad (1)$$

where  $c_1$  is the cosine of the *Cabibbo angle*,  $G_F$  is the *Fermi constant*, and  $f_\pi$  is the *pion decay constant*.

- (a) Compute the charged pion decay rate  $\Gamma$ .
- (b) The charged pion mass is  $m_\pi = 139.6$  MeV, the muon mass is  $m_\mu = 105.7$  MeV, and the muon neutrino mass is massless. The Fermi constant is  $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$ , and the cosine of the Cabibbo angle is measured in nuclear beta decays to be  $c_1 = 0.974$ . The measured value of the charged pion life time is  $\tau = 2.6033 \times 10^{-8} \text{ s}$ . Determine the value of  $f_\pi$  in MeV. Your result is too large by 0.8%, due to neglect of electromagnetic loop corrections.
- (c) The previous parts assume  $\pi^-$  always decay into  $\mu^- \bar{\nu}_\mu$ , but actually  $\pi^-$  can also decay into  $e^- \bar{\nu}_e$ . The charged pion, electron by a Dirac field  $\mathcal{M}_e$ , and the electron neutrino by a spin-projected Dirac field  $P_L\mathcal{N}_e$  have the form of interaction

$$\mathcal{L}_2 = 2c_2 G_F f_\pi \partial_\mu \varphi \overline{\mathcal{M}_e} \gamma^\mu P_L \mathcal{N}_e + h.c. \quad (2)$$

Given the decay branching ratio of  $\pi^- \rightarrow e^- \bar{\nu}_e$  is  $1.230 \times 10^{-4}$ , the decay branching ratio of  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$  is 99.9877%. Find the value of  $c_2$ . For example, the electronic decay branching ratio is

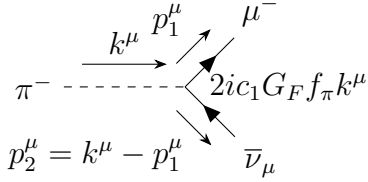
$$\text{Br}(\pi^- \rightarrow e^- \bar{\nu}_e) = \frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu) + \Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}. \quad (3)$$

The coupling of pion-electron is similar with the coupling of pion-muon, why pion favoring decay into muon instead of electron? ( $m_e = 0.511$  MeV.)

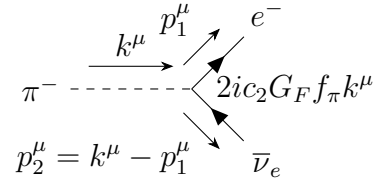
## Answer

(a)

First, we analyze the decay  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ . The Feynman diagram is shown in Fig. 1a. Now, we can write down



(a)  $\pi^-$  decay diagram.



(b)  $\pi^-$  decay diagram.

Figure 1: Feynman diagram for  $\pi^-$  decay into (a) muon and muon antineutrino; (b) electron and electron antineutrino.

the amplitude:

$$i\mathcal{T} = 2ic_1 G_F f_\pi k^\mu \bar{u}_{s_1}(p_1) \gamma_\mu P_L v_{s_2}(p_2) \quad (4)$$

$$= 2ic_1 G_F f_\pi \bar{u}_{s_1}(p_1) \not{k} P_L v_{s_2}(p_2) \quad (5)$$

$$= ic_1 G_F f_\pi \bar{u}_{s_1}(p_1) \not{k} (1 - \gamma_5) v_{s_2}(p_2), \quad (6)$$

where  $k^\mu$ ,  $p_1^\mu$ , and  $p_2^\mu$  are the four-momenta of  $\pi^-$ ,  $\mu^-$ , and  $\bar{\nu}_\mu$ , respectively.  $s_1$  and  $s_2$  are the spin indices of  $\mu^-$  and  $\bar{\nu}_\mu$ . We can write  $k^\mu = p_1^\mu + p_2^\mu$  due to momentum conservation. Thus, the amplitude can be further simplified as:

$$i\mathcal{T} = ic_1 G_F f_\pi \bar{u}_{s_1}(p_1) (\not{p}_1 + \not{p}_2) (1 - \gamma_5) v_{s_2}(p_2) \quad (7)$$

$$= ic_1 G_F f_\pi \left[ \bar{u}_{s_1}(p_1) \not{p}_1 (1 - \gamma_5) v_{s_2}(p_2) + \bar{u}_{s_1}(p_1) \not{p}_2 (1 - \gamma_5) v_{s_2}(p_2) \right] \quad (8)$$

$$= ic_1 G_F f_\pi \left[ (-m) \bar{u}_{s_1}(p_1) (1 - \gamma_5) v_{s_2}(p_2) + 0 \right] \quad (9)$$

where we have used the Dirac equation  $\bar{u}_{s_1}(p_1) (\not{p}_1 + m) = 0$  and the massless neutrino condition  $\not{p}_2 v_{s_2}(p_2) = 0$ . Therefore, the amplitude becomes:

$$i\mathcal{T} = -ic_1 G_F f_\pi m \bar{u}_{s_1}(p_1) (1 - \gamma_5) v_{s_2}(p_2). \quad (10)$$

Next, we can write down the Hermitian conjugate of the amplitude:

$$-i\mathcal{T}^* = ic_1 G_F f_\pi m \left[ \bar{u}_{s_1}(p_1) (1 - \gamma_5) v_{s_2}(p_2) \right]^\dagger \quad (11)$$

$$= ic_1 G_F f_\pi m \left[ v_{s_2}^\dagger(p_2) (1 - \gamma_5)^\dagger \bar{u}_{s_1}^\dagger(p_1) \right] \quad (12)$$

$$= ic_1 G_F f_\pi m \left[ v_{s_2}^\dagger(p_2) \gamma^0 (1 - \gamma_5)^\dagger \gamma^0 u_{s_1}(p_1) \right] \quad (13)$$

$$= ic_1 G_F f_\pi m \left[ \bar{v}_{s_2}(p_2) (1 + \gamma_5) u_{s_1}(p_1) \right], \quad (14)$$

where we have used the relation  $\bar{u} = u^\dagger \gamma^0$  and the Hermitian property of  $\gamma_5$  (that is,  $\gamma_5^\dagger = \gamma_5$ ) to get the last

line. Now, we can compute the squared amplitude averaged over initial spins and summed over final spins:

$$\langle |\mathcal{T}|^2 \rangle = \frac{1}{1} \sum_{s_1, s_2} \mathcal{T} \mathcal{T}^* \quad (15)$$

$$= c_1^2 G_F^2 f_\pi^2 m^2 \sum_{s_1, s_2} \left[ \bar{u}_{s_1}(p_1)(1 - \gamma_5)v_{s_2}(p_2) \right] \left[ \bar{v}_{s_2}(p_2)(1 + \gamma_5)u_{s_1}(p_1) \right] \quad (16)$$

$$= c_1^2 G_F^2 f_\pi^2 m^2 \sum_{s_1, s_2} \text{Tr} \left[ (1 + \gamma_5)u_{s_1}(p_1)\bar{u}_{s_1}(p_1)(1 - \gamma_5)v_{s_2}(p_2)\bar{v}_{s_2}(p_2) \right] \quad (17)$$

$$= c_1^2 G_F^2 f_\pi^2 m^2 \text{Tr} \left[ (1 + \gamma_5)(-\not{p}_1 + m)(1 - \gamma_5)(-\not{p}_2) \right] \quad (18)$$

where we have used the completeness relations for spinors and the trace properties of gamma matrices:

$$\sum_s u_s(p)\bar{u}_s(p) = -\not{p} + m, \quad (19)$$

$$\sum_s v_s(p)\bar{v}_s(p) = -\not{p} - m, \quad (20)$$

$$\text{Tr}(\not{a}\not{b}) = -4(a \cdot b), \quad (21)$$

$$\text{Tr}(\text{odd number of } \gamma\text{-matrices}) = 0, \quad (22)$$

$$\gamma_5^2 = 1. \quad (23)$$

We can expand the trace:

$$(1 + \gamma_5)(-\not{p}_1 + m)(1 - \gamma_5)(-\not{p}_2) = (1 + \gamma_5)(-\not{p}_1)(1 - \gamma_5)(-\not{p}_2) + (1 + \gamma_5)m(1 - \gamma_5)(-\not{p}_2) \quad (24)$$

$$= (1 + \gamma_5)(-\not{p}_1)(1 - \gamma_5)(-\not{p}_2) + 0 \quad (25)$$

$$= (1 + \gamma_5)(\not{p}_1)(1 - \gamma_5)(\not{p}_2) \quad (26)$$

$$= (\not{p}_1)(\not{p}_2) + (\not{p}_1)(-\gamma_5)(\not{p}_2) + \gamma_5(\not{p}_1)(\not{p}_2) + \gamma_5(\not{p}_1)(-\gamma_5)(\not{p}_2) \quad (27)$$

$$= 2(\not{p}_1)(\not{p}_2) + 2\gamma_5(\not{p}_1)(\not{p}_2) \quad (28)$$

where we have used the anticommutation relation  $\{\gamma_5, \gamma^\mu\} = 0$  to get the last line. Therefore, we have:

$$\langle |\mathcal{T}|^2 \rangle = c_1^2 G_F^2 f_\pi^2 m^2 \text{Tr} \left[ 2(\not{p}_1)(\not{p}_2) + 2\gamma_5(\not{p}_1)(\not{p}_2) \right] \quad (29)$$

$$= 2c_1^2 G_F^2 f_\pi^2 m^2 \left[ \text{Tr}(\not{p}_1 \not{p}_2) + \text{Tr}(\gamma_5 \not{p}_1 \not{p}_2) \right] \quad (30)$$

$$= 2c_1^2 G_F^2 f_\pi^2 m^2 \left[ -4(p_1 \cdot p_2) + 0 \right] \quad (31)$$

$$= -8c_1^2 G_F^2 f_\pi^2 m^2 (p_1 \cdot p_2) \quad (32)$$

where we have used the trace properties of gamma matrices again. In the rest frame of  $\pi^-$ , we have:

$$k^2 = -m_\pi^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2(p_1 \cdot p_2) = -m^2 + 0 + 2(p_1 \cdot p_2) \quad (33)$$

$$\Rightarrow -(p_1 \cdot p_2) = \frac{m_\pi^2 - m^2}{2} \quad (34)$$

Thus, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = 4c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2) \quad (35)$$

Note that the  $m$  is the muon mass. Finally, we can compute the decay rate:

$$\Gamma = \frac{1}{2m_\pi} \int d\Phi_2 \langle |\mathcal{T}|^2 \rangle, \quad (36)$$

where the two-body phase space integral is:

$$\int d\Phi_2 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(k - p_1 - p_2) \quad (37)$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \delta^3(\mathbf{0} - \mathbf{p}_1 - \mathbf{p}_2) \quad (38)$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \quad (39)$$

$$= \int \frac{4\pi p_1^2 dp_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \quad (40)$$

$$= \int \frac{4\pi p_1^2 dp_1}{(2\pi)^2 4E_1 E_2} \delta(m_\pi - E_1 - E_2) \quad (41)$$

where we have used the delta function to perform the  $\mathbf{p}_2$  integral. In the rest frame of  $\pi^-$ , we have  $\mathbf{p}_2 = -\mathbf{p}_1$  and  $E_2 = |\mathbf{p}_2| = |\mathbf{p}_1|$ . Thus, we can write  $E_1 + E_2 - m_\pi = \sqrt{p_1^2 + m^2} + p_1 - m_\pi$ . The root of the equation  $E_1 + E_2 - m_\pi = 0$  is:

$$p_1 = \frac{m_\pi^2 - m^2}{2m_\pi} \quad (42)$$

Also, we can compute the derivative:

$$\frac{d}{dp_1}(E_1 + E_2 - m_\pi) = \frac{p_1}{\sqrt{p_1^2 + m^2}} + 1 = \frac{E_1 + E_2}{E_1} = \frac{m_\pi}{E_1} \quad (43)$$

Therefore, the phase space integral becomes:

$$\int d\Phi_2 = \frac{4\pi p_1^2}{(2\pi)^2 4E_1 E_2} \frac{E_1}{m_\pi} \quad (44)$$

$$= \frac{p_1^2}{4\pi m_\pi E_2} = \frac{p_1}{4\pi m_\pi} \quad (45)$$

$$= \frac{m_\pi^2 - m^2}{8\pi m_\pi^2} \quad (46)$$

Finally, the decay rate is:

$$\Gamma_{\pi^- \rightarrow \mu \bar{\nu}_\mu} = \frac{1}{2m_\pi} \langle |\mathcal{T}|^2 \rangle \int d\Phi_2 \quad (47)$$

$$= \frac{1}{2m_\pi} \left[ 4c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2) \right] \left[ \frac{m_\pi^2 - m^2}{8\pi m_\pi^2} \right] \quad (48)$$

$$= \frac{c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2)^2}{4\pi m_\pi^3} \quad (49)$$

$$= \frac{c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{4\pi m_\pi^3} \quad (50)$$

(b)

The charged pion life time is related to the decay rate by  $\tau = 1/\Gamma$ . Note that  $2.6033 \times 10^{-8} \text{ s} \approx 3.955 \times 10^{16} \text{ MeV}^{-1}$ . Thus, we can solve for  $f_\pi$ :

$$f_\pi = \sqrt{\frac{4\pi m_\pi^3}{c_1^2 G_F^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2 \tau}} \quad (51)$$

$$= \sqrt{\frac{4\pi (139.6 \text{ MeV})^3}{(0.974)^2 (1.166 \times 10^{-5} \text{ GeV}^{-2})^2 (105.7 \text{ MeV})^2 ((139.6 \text{ MeV})^2 - (105.7 \text{ MeV})^2) (3.955 \times 10^{16} \text{ MeV}^{-1})}} \quad (52)$$

$$\approx 0.09314 \text{ GeV} = 93.14 \text{ MeV}. \quad (53)$$

(c)

Now we analyze the decay  $\pi^- \rightarrow e^- \bar{\nu}_e$ . The Feynman diagram is shown in Fig. 1b. By following the same procedure as in part (a), we can write down the decay rate:

$$\Gamma_{\pi^- \rightarrow e \bar{\nu}_e} = \frac{c_2^2 G_F^2 f_\pi^2 m_e^2 (m_\pi^2 - m_e^2)^2}{4\pi m_\pi^3}. \quad (54)$$

Given the branching ratio  $\text{Br}(\pi^- \rightarrow e^- \bar{\nu}_e) = 1.230 \times 10^{-4}$ , we have:

$$\text{Br}(\pi^- \rightarrow e^- \bar{\nu}_e) = \frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu) + \Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)} \quad (55)$$

$$\approx \frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu)} \quad (56)$$

$$= \frac{c_2^2 m_e^2 (m_\pi^2 - m_e^2)^2}{c_1^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2} \quad (57)$$

where we have used the fact that  $\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e) \ll \Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu)$  to get the second line. Therefore, we can solve for  $c_2$ :

$$c_2 = c_1 \sqrt{\text{Br}(\pi^- \rightarrow e^- \bar{\nu}_e) \frac{m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{m_e^2 (m_\pi^2 - m_e^2)^2}} \quad (58)$$

$$= 0.974 \sqrt{(1.230 \times 10^{-4}) \frac{(105.7 \text{ MeV})^2 ((139.6 \text{ MeV})^2 - (105.7 \text{ MeV})^2)^2}{(0.511 \text{ MeV})^2 ((139.6 \text{ MeV})^2 - (0.511 \text{ MeV})^2)^2}} \quad (59)$$

$$\approx 0.95345. \quad (60)$$

The most obvious reason that pion favoring decay into muon instead of electron is that the muon mass is much larger than the electron mass. Since the  $\Gamma \propto m_l^2$  ( $l = e, \mu$ ), the decay rate into muon is greatly enhanced compared to that into electron.

**Remark:** This phenomenon is known as *helicity suppression*. But I think the explanation above is sufficient for this problem. □

## Question 2

Consider QED with both electron and muon:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{l=e,\mu} (i\bar{\Psi}_l \not{\partial} \Psi_l - m_l \bar{\Psi}_l \Psi_l + \frac{g}{2} \bar{\Psi}_l \gamma^\mu \Psi_l A_\mu), \quad (61)$$

where both  $\Psi_e$  and  $\Psi_\mu$  are Dirac fields. Compute the  $\langle |\mathcal{T}^2| \rangle$  for  $e^+e^- \rightarrow \mu^+\mu^-$ . Then, compute its cross section  $\sigma$ . Eq. (11.22) and Eq. (11.30) should be useful.

## Answer

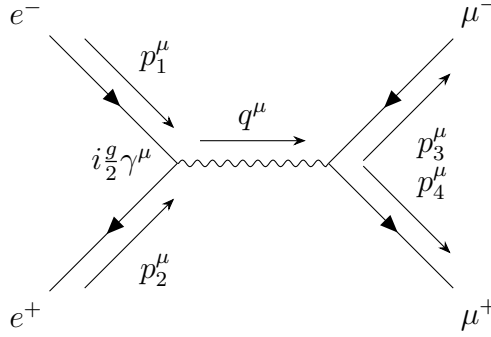


Figure 2: Feynman diagram for  $e^+e^- \rightarrow \mu^+\mu^-$ .

The Feynman diagram for  $e^+e^- \rightarrow \mu^+\mu^-$  is shown in Fig. 2. We can write down the amplitude:

$$i\mathcal{T} = \bar{v}_{s_2}(p_2) \left( i\frac{g}{2} \gamma^\mu \right) u_{s_1}(p_1) \frac{-ig_{\mu\nu}}{q^2} \bar{u}_{s_3}(p_3) \left( i\frac{g}{2} \gamma^\nu \right) v_{s_4}(p_4) \quad (62)$$

$$= i\frac{g^2}{4q^2} \left[ \bar{v}_{s_2}(p_2) \gamma^\mu u_{s_1}(p_1) \right] \left[ \bar{u}_{s_3}(p_3) \gamma_\mu v_{s_4}(p_4) \right], \quad (63)$$

where  $p_1^\mu$ ,  $p_2^\mu$ ,  $p_3^\mu$ , and  $p_4^\mu$  are the four-momenta of  $e^-$ ,  $e^+$ ,  $\mu^-$ , and  $\mu^+$ , respectively.  $s_1$ ,  $s_2$ ,  $s_3$ , and  $s_4$  are the spin indices of  $e^-$ ,  $e^+$ ,  $\mu^-$ , and  $\mu^+$ , respectively. Also, we have defined  $q^\mu = p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$ . Next, we can write down the Hermitian conjugate of the amplitude:

$$-i\mathcal{T}^* = -i\frac{g^2}{4q^2} \left[ \bar{v}_{s_2}(p_2) \gamma^\mu u_{s_1}(p_1) \right]^\dagger \left[ \bar{u}_{s_3}(p_3) \gamma_\mu v_{s_4}(p_4) \right]^\dagger \quad (64)$$

$$= -i\frac{g^2}{4q^2} \left[ u_{s_1}^\dagger(p_1) \gamma^{\mu\dagger} (\gamma^0)^\dagger v_{s_2}(p_2) \right] \left[ v_{s_4}^\dagger(p_4) \gamma_\mu^\dagger (\gamma^0)^\dagger u_{s_3}(p_3) \right] \quad (65)$$

$$= -i\frac{g^2}{4q^2} \left[ \bar{u}_{s_1}(p_1) \gamma^\mu v_{s_2}(p_2) \right] \left[ \bar{v}_{s_4}(p_4) \gamma_\mu u_{s_3}(p_3) \right]. \quad (66)$$



Therefore, we can compute the squared amplitude averaged over initial spins and summed over final spins:

$$\langle |\mathcal{T}|^2 \rangle = \frac{1}{4} \sum_{s_1, s_2, s_3, s_4} \mathcal{T} \mathcal{T}^* \quad (67)$$

$$= \frac{g^4}{64q^4} \sum_{s_1, s_2, s_3, s_4} \left[ \bar{v}_{s_2}(p_2) \gamma^\mu u_{s_1}(p_1) \right] \left[ \bar{u}_{s_3}(p_3) \gamma_\mu v_{s_4}(p_4) \right] \left[ \bar{u}_{s_1}(p_1) \gamma^\nu v_{s_2}(p_2) \right] \left[ \bar{v}_{s_4}(p_4) \gamma_\nu u_{s_3}(p_3) \right] \quad (68)$$

$$= \frac{g^4}{64q^4} \sum_{s_1, s_2, s_3, s_4} \text{Tr} \left[ \gamma^\mu u_{s_1}(p_1) \bar{u}_{s_1}(p_1) \gamma^\nu v_{s_2}(p_2) \bar{v}_{s_2}(p_2) \right] \text{Tr} \left[ \gamma_\mu u_{s_3}(p_3) \bar{u}_{s_3}(p_3) \gamma_\nu v_{s_4}(p_4) \bar{v}_{s_4}(p_4) \right] \quad (69)$$

$$= \frac{g^4}{64q^4} \text{Tr} \left[ \gamma^\mu (-\not{p}_1 + m_e) \gamma^\nu (-\not{p}_2 - m_e) \right] \text{Tr} \left[ \gamma_\mu (-\not{p}_3 + m_\mu) \gamma_\nu (-\not{p}_4 - m_\mu) \right], \quad (70)$$

where we have used the completeness relations for spinors and the trace properties of gamma matrices:

$$\sum_s u_s(p) \bar{u}_s(p) = -\not{p} + m, \quad (71)$$

$$\sum_s v_s(p) \bar{v}_s(p) = -\not{p} - m, \quad (72)$$

$$\text{Tr}(\not{a} \not{b}) = -4(a \cdot b), \quad (73)$$

$$\text{Tr}(\text{odd number of } \gamma\text{-matrices}) = 0 \quad (74)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = -4g^{\mu\nu} \quad (75)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \quad (76)$$

$$g_{\mu\nu} g^{\mu\nu} = 4 \quad (77)$$

We can expand the traces:

$$\text{Tr} \left[ \gamma^\mu (-\not{p}_1 + m_e) \gamma^\nu (-\not{p}_2 - m_e) \right] = \text{Tr} \left[ \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2 \right] - m_e^2 \text{Tr} \left[ \gamma^\mu \gamma^\nu \right] \quad (78)$$

$$= (p_1)_\alpha (p_2)_\beta \text{Tr} \left[ \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \right] + 4m_e^2 g^{\mu\nu} \quad (79)$$

$$= (p_1)_\alpha (p_2)_\beta 4(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha}) + 4m_e^2 g^{\mu\nu} \quad (80)$$

$$= 4 \left[ p_1^\mu p_2^\nu - g^{\mu\nu} (p_1 \cdot p_2) + p_1^\nu p_2^\mu + m_e^2 g^{\mu\nu} \right] \quad (81)$$

$$= 4 \left[ p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} (p_1 \cdot p_2 - m_e^2) \right], \quad (82)$$

and

$$\text{Tr} \left[ \gamma_\mu (-\not{p}_3 + m_\mu) \gamma_\nu (-\not{p}_4 - m_\mu) \right] = \text{Tr} \left[ \gamma_\mu \not{p}_3 \gamma_\nu \not{p}_4 \right] - m_\mu^2 \text{Tr} \left[ \gamma_\mu \gamma_\nu \right] \quad (83)$$

$$= (p_3)^\rho (p_4)^\sigma \text{Tr} \left[ \gamma_\mu \gamma_\rho \gamma_\nu \gamma_\sigma \right] + 4m_\mu^2 g_{\mu\nu} \quad (84)$$

$$= (p_3)^\rho (p_4)^\sigma 4(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho}) + 4m_\mu^2 g_{\mu\nu} \quad (85)$$

$$= 4 \left[ p_{3\mu} p_{4\nu} - g_{\mu\nu} (p_3 \cdot p_4) + p_{3\nu} p_{4\mu} + m_\mu^2 g_{\mu\nu} \right] \quad (86)$$

$$= 4 \left[ p_{3\mu} p_{4\nu} + p_{3\nu} p_{4\mu} - g_{\mu\nu} (p_3 \cdot p_4 - m_\mu^2) \right]. \quad (87)$$

Therefore, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{64q^4} 16 \left[ p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} (p_1 \cdot p_2 - m_e^2) \right] \left[ p_{3\mu} p_{4\nu} + p_{3\nu} p_{4\mu} - g_{\mu\nu} (p_3 \cdot p_4 - m_\mu^2) \right] \quad (88)$$

$$= \frac{g^4}{4q^4} \left[ (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4 - m_\mu^2) \right. \quad (89)$$

$$\left. + (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4 - m_\mu^2) \right. \quad (90)$$

$$\left. - (p_1 \cdot p_2 - m_e^2)(p_3 \cdot p_4) - (p_1 \cdot p_2 - m_e^2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2 - m_e^2)(p_3 \cdot p_4 - m_\mu^2) \right] \quad (91)$$

$$= \frac{g^4}{4q^4} \left[ (\textcolor{blue}{p}_1 p_3)(p_2 p_4) + (\textcolor{red}{p}_1 p_4)(p_2 p_3) - (\textcolor{red}{p}_1 p_2)(p_3 p_4) + (\textcolor{blue}{p}_1 p_2) m_\mu^2 \right. \quad (92)$$

$$\left. + (\textcolor{red}{p}_1 p_4)(p_2 p_3) + (\textcolor{blue}{p}_1 p_3)(p_2 p_4) - (\textcolor{red}{p}_1 p_2)(p_3 p_4) + (\textcolor{blue}{p}_1 p_2) m_\mu^2 \right. \quad (93)$$

$$\left. - (\textcolor{red}{p}_1 p_2)(p_3 p_4) + m_e^2 (p_3 p_4) - (\textcolor{red}{p}_1 p_2)(p_3 p_4) + m_e^2 (p_3 p_4) \right. \quad (94)$$

$$\left. + 4(\textcolor{red}{p}_1 p_2)(p_3 p_4) - 4m_\mu^2 (\textcolor{blue}{p}_1 p_2) - 4m_e^2 (p_3 p_4) + 4m_e^2 m_\mu^2 \right] \quad (95)$$

$$= \frac{g^4}{4q^4} \left[ 2(\textcolor{blue}{p}_1 p_3)(p_2 p_4) + 2(\textcolor{red}{p}_1 p_4)(p_2 p_3) - 2(\textcolor{blue}{p}_1 p_2) m_\mu^2 - 2m_e^2 (p_3 p_4) + 4m_e^2 m_\mu^2 \right] \quad (96)$$

Next, we can compute the cross section in the center-of-mass frame. In this frame, we have:

$$p_1^\mu = (E, 0, 0, p), \quad (97)$$

$$p_2^\mu = (E, 0, 0, -p), \quad (98)$$

$$p_3^\mu = (E, p' \sin \theta, 0, p' \cos \theta), \quad (99)$$

$$p_4^\mu = (E, -p' \sin \theta, 0, -p' \cos \theta), \quad (100)$$

where  $E = \sqrt{p^2 + m_e^2} = \sqrt{p'^2 + m_\mu^2} = \frac{\sqrt{s}}{2}$ . We can compute the dot products:

$$(p_1 \cdot p_2) = -E^2 + (-p^2) = m_e^2 - 2E^2 \quad (101)$$

$$(p_3 \cdot p_4) = -E^2 + (-p'^2) = m_\mu^2 - 2E^2 \quad (102)$$

$$(p_1 \cdot p_3) = -E^2 + pp' \cos \theta \quad (103)$$

$$(p_2 \cdot p_4) = -E^2 + pp' \cos \theta \quad (104)$$

$$(p_1 \cdot p_4) = -E^2 - pp' \cos \theta \quad (105)$$

$$(p_2 \cdot p_3) = -E^2 - pp' \cos \theta \quad (106)$$

Therefore, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = \frac{g^4}{4q^4} \left[ 2(-E^2 + pp' \cos \theta)^2 + 2(-E^2 - pp' \cos \theta)^2 - 2(m_e^2 - 2E^2)m_\mu^2 - 2m_e^2(m_\mu^2 - 2E^2) + 4m_e^2m_\mu^2 \right] \quad (107)$$

$$= \frac{g^4}{4q^4} \left[ 4(E^4 + p^2p'^2 \cos^2 \theta) + 4E^2(m_e^2 + m_\mu^2) \right] \quad (108)$$

$$= \frac{g^4}{q^4} \left[ E^4 + p^2p'^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2) \right]. \quad (109)$$

Next, we can compute the cross section (by eq. (11.31) in the textbook):

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \langle |\mathcal{T}|^2 \rangle \quad (110)$$

$$= \frac{1}{64\pi^2 s} \frac{p'}{p} \frac{g^4}{q^4} \left[ E^4 + p^2p'^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2) \right] \quad (111)$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[ E^4 + p^2p'^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2) \right], \quad (112)$$

where we have used the fact that  $q^2 = (p_1 + p_2)^2 = -s = -4E^2$ . Finally, we can integrate over the solid angle to get the total cross section:

$$\int d\Omega = 4\pi, \quad (113)$$

$$\int d\Omega \cos^2 \theta = 2\pi \int_{-1}^1 d\cos \theta \cos^2 \theta = \frac{4\pi}{3}, \quad (114)$$

thus,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega \quad (115)$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[ E^4 \int d\Omega + p^2 p'^2 \int d\Omega \cos^2 \theta + E^2 (m_e^2 + m_\mu^2) \int d\Omega \right] \quad (116)$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[ 4\pi E^4 + \frac{4\pi}{3} p^2 p'^2 + 4\pi E^2 (m_e^2 + m_\mu^2) \right] \quad (117)$$

$$= \frac{g^4}{16\pi s^3} \frac{p'}{p} \left[ E^4 + \frac{1}{3} p^2 p'^2 + E^2 (m_e^2 + m_\mu^2) \right]. \quad (118)$$

Now, we can express  $p$  and  $p'$  in terms of  $s$ :

$$p = \sqrt{E^2 - m_e^2} = \sqrt{\frac{s}{4} - m_e^2}, \quad (119)$$

$$p' = \sqrt{E^2 - m_\mu^2} = \sqrt{\frac{s}{4} - m_\mu^2}, \quad (120)$$

$$E = \frac{\sqrt{s}}{2}. \quad (121)$$

Therefore, the final expression for the cross section is:

$$\sigma = \frac{g^4}{16\pi s^3} \frac{\sqrt{\frac{s}{4} - m_\mu^2}}{\sqrt{\frac{s}{4} - m_e^2}} \left[ \left( \frac{s}{4} \right)^2 + \frac{1}{3} \left( \frac{s}{4} - m_e^2 \right) \left( \frac{s}{4} - m_\mu^2 \right) + \frac{s}{4} (m_e^2 + m_\mu^2) \right] \quad (122)$$

$$= \frac{g^4}{192\pi s^3} \frac{\sqrt{s - 4m_\mu^2}}{\sqrt{s - 4m_e^2}} \left[ (s + 2m_e^2)(s + 2m_\mu^2) \right], \quad \text{in terms of } s, \quad (123)$$

$$= \frac{g^4}{3072\pi E^6} \frac{\sqrt{E^2 - m_\mu^2}}{\sqrt{E^2 - m_e^2}} \left[ (2E^2 + m_e^2)(2E^2 + m_\mu^2) \right], \quad \text{in terms of } E. \quad (124)$$

□

### Question 3

Consider classical field theory with two real scalar fields in (3+1)-dimension spacetime:

$$\mathcal{L}(x) = \sum_{a=1}^2 \left( -\frac{1}{2} \partial^\mu \phi_a \partial_\mu \phi_a \right) - V(x), \quad (125)$$

$$V(x) = -\sum_{a=1}^2 \left( \frac{1}{2} \mu^2 \phi_a \phi_a \right) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2, \quad (126)$$

where  $\mu$  and  $\lambda$  are positive real constants.

(a) Show that the Lagrangian has an  $SO(2)$  transformation symmetry:

$$\phi_1(x) \rightarrow \phi'_1(x) = \phi_1(x) \cos \alpha_0 - \phi_2(x) \sin \alpha_0, \quad (127)$$

$$\phi_2(x) \rightarrow \phi'_2(x) = \phi_1(x) \sin \alpha_0 + \phi_2(x) \cos \alpha_0, \quad (128)$$

(b) Find the conjugate momentum  $\Pi_1(x)$ ,  $\Pi_2(x)$  of  $\phi_1(x)$ ,  $\phi_2(x)$ . Find the Hamiltonian density  $\mathcal{H}(x)$  in the terms of  $\phi_a(x)$ ,  $\Pi_a(x)$ , and  $\partial_i \phi_a(x)$ .

(c) Find the ground state in the basis of  $\{\phi_r(x), \phi_\theta(x)\}$  where

$$\phi_1(x) = \phi_r(x) \cos(\phi_\theta(x)), \quad (129)$$

$$\phi_2(x) = \phi_r(x) \sin(\phi_\theta(x)), \quad (130)$$

with  $\phi_r(x) \geq 0$  and  $\phi_\theta(x) \in [0, 2\pi)$ . Is the Lagrangian  $\mathcal{L}$  invariant under a continuous shift symmetry of  $\phi_\theta(x) \rightarrow \phi_\theta(x) + \alpha_0$ ?

**Hint:** In general, finding the ground state is to find  $\phi(x)$  s.t. minimize  $H = \int \mathcal{H}(x) d^3x$ ; but for this problem, finding  $\phi(x)$  to minimize  $\mathcal{H}(x)$  is the same. If you have trouble with the above procedure, given the Lagrangian of this problem, one can simply find  $\phi(x)$  s.t. minimize  $V(x)$ , which is the same as minimizing  $\mathcal{H}$  for this problem.

(d) Now let's study the system's dynamics around the ground state.

$\phi_r(x)$  should fluctuate around  $\sqrt{\frac{\mu^2}{\lambda}}$ :  $\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x)$ .  $\phi_\theta(x)$  should fluctuate within  $[0, 2\pi)$ .

Show that  $f_r(x)$  is a massive field and find its mass. Taking  $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}} \phi_\theta(x)$  as the other scalar field, does  $f_\theta(x)$  have a mass? Does  $\mathcal{L}$  have a continuous shift symmetry of  $f_\theta(x) \rightarrow f_\theta(x) + \Lambda_0$ ?

**Remark:** This problem paves the road for your understanding of spontaneous symmetry breaking. We also see again that the symmetry groups of  $SO(2)$  and  $U(1)$  are isomorphic.

**Remark:** More to think about after solving the problems above: Note that we reparametrized the field into a non-linear realization, where you see the  $U(1)$  symmetry explicitly. How do you interpret the kinetic term? How do you interpret the  $f_r(x)$  field-dependent kinetic terms for  $f_\theta(x)$ ? Is it canonically normalized? How does the field  $f_\theta(x)$  relate to the original  $SO(2)$  field  $\phi_a(x)$ ? And again, is the ratio of

the field a linear redefinition of the field configuration? It is a non-linear realization because all powers of  $f_\theta(x)/\sqrt{\frac{\mu^2}{\lambda}}$  need to enter. There is only a region of validity, that is  $f_r(x) \ll \sqrt{\frac{\mu^2}{\lambda}}$

## Answer

(a)

The Lagrangian can be separated into the kinetic term and the potential term:

$$\mathcal{L}(x) = \sum_{a=1}^2 \left( -\frac{1}{2} \partial^\mu \phi_a \partial_\mu \phi_a \right) - V(x) \quad (131)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu \phi_1 & \partial^\mu \phi_2 \end{pmatrix} \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix} - \left[ -\frac{\mu^2}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{\lambda}{4} \left( \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)^2 \right] \quad (132)$$

Under the  $SO(2)$  transformation, the fields transform as:

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha_0 & -\sin \alpha_0 \\ \sin \alpha_0 & \cos \alpha_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \equiv R(\alpha_0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (133)$$

where  $R(\alpha_0)$  is the rotation matrix. The kinetic term transforms as:

$$-\frac{1}{2} \begin{pmatrix} \partial^\mu \phi'_1 & \partial^\mu \phi'_2 \end{pmatrix} \begin{pmatrix} \partial_\mu \phi'_1 \\ \partial_\mu \phi'_2 \end{pmatrix} \quad (134)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu (R_{11}\phi_1 + R_{12}\phi_2) & \partial^\mu (R_{21}\phi_1 + R_{22}\phi_2) \end{pmatrix} \begin{pmatrix} \partial_\mu (R_{11}\phi_1 + R_{12}\phi_2) \\ \partial_\mu (R_{21}\phi_1 + R_{22}\phi_2) \end{pmatrix} \quad (135)$$

$$= -\frac{1}{2} \begin{pmatrix} R_{11}\partial^\mu \phi_1 + R_{12}\partial^\mu \phi_2 & R_{21}\partial^\mu \phi_1 + R_{22}\partial^\mu \phi_2 \end{pmatrix} \begin{pmatrix} R_{11}\partial_\mu \phi_1 + R_{12}\partial_\mu \phi_2 \\ R_{21}\partial_\mu \phi_1 + R_{22}\partial_\mu \phi_2 \end{pmatrix} \quad (136)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu \phi_1 & \partial^\mu \phi_2 \end{pmatrix} R^T(\alpha_0) R(\alpha_0) \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix} \quad (137)$$

$$= -\frac{1}{2} \begin{pmatrix} \partial^\mu \phi_1 & \partial^\mu \phi_2 \end{pmatrix} \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix}, \quad (138)$$

where we have used the orthogonality of the rotation matrix:  $R^T(\alpha_0)R(\alpha_0) = I$ . Similarly, the potential term

transforms as:

$$-\frac{\mu^2}{2} \begin{pmatrix} \phi'_1 & \phi'_2 \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} + \frac{\lambda}{4} \left( \begin{pmatrix} \phi'_1 & \phi'_2 \end{pmatrix} \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} \right)^2 \quad (139)$$

$$= -\frac{\mu^2}{2} \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 & R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 \\ R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \quad (140)$$

$$+ \frac{\lambda}{4} \left( \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 & R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \begin{pmatrix} R_{11}\phi_1 + R_{12}\phi_2 \\ R_{21}\phi_1 + R_{22}\phi_2 \end{pmatrix} \right)^2 \quad (141)$$

$$= -\frac{\mu^2}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} R^T(\alpha_0) R(\alpha_0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{\lambda}{4} \left( \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} R^T(\alpha_0) R(\alpha_0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)^2 \quad (142)$$

$$= -\frac{\mu^2}{2} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{\lambda}{4} \left( \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)^2 \quad (143)$$

Thus, the Lagrangian is invariant under the  $SO(2)$  transformation.

(b)

The conjugate momenta are given by:

$$\Pi_1(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_1)} = -\frac{1}{2} \cdot (-2) \cdot (\partial^0 \phi_1) = \partial^0 \phi_1, \quad (144)$$

$$\Pi_2(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_2)} = -\frac{1}{2} \cdot (-2) \cdot (\partial^0 \phi_2) = \partial^0 \phi_2. \quad (145)$$

The Hamiltonian density is given by:

$$\mathcal{H}(x) = \Pi_1(x) \partial_0 \phi_1 + \Pi_2(x) \partial_0 \phi_2 - \mathcal{L}(x) \quad (146)$$

$$= (\partial^0 \phi_1)(\partial_0 \phi_1) + (\partial^0 \phi_2)(\partial_0 \phi_2) - \left[ -\frac{1}{2}(\partial^\mu \phi_1 \partial_\mu \phi_1 + \partial^\mu \phi_2 \partial_\mu \phi_2) - V(x) \right] \quad (147)$$

$$= (\partial^0 \phi_1)^2 + (\partial^0 \phi_2)^2 + \frac{1}{2}(\partial^\mu \phi_1 \partial_\mu \phi_1 + \partial^\mu \phi_2 \partial_\mu \phi_2) + V(x) \quad (148)$$

$$= (\partial^0 \phi_1)^2 + (\partial^0 \phi_2)^2 + \frac{1}{2} \left( -(\partial^0 \phi_1)^2 + (\partial^i \phi_1)^2 - (\partial^0 \phi_2)^2 + (\partial^i \phi_2)^2 \right) + V(x) \quad (149)$$

$$= \frac{1}{2}(\partial^0 \phi_1)^2 + \frac{1}{2}(\partial^i \phi_1)^2 + \frac{1}{2}(\partial^0 \phi_2)^2 + \frac{1}{2}(\partial^i \phi_2)^2 + V(x) \quad (150)$$

$$= \frac{1}{2}\Pi_1^2 + \frac{1}{2}(\nabla \phi_1)^2 + \frac{1}{2}\Pi_2^2 + \frac{1}{2}(\nabla \phi_2)^2 + V(x). \quad (151)$$

(c)

To find the ground state, we need to minimize the potential  $V(x)$ :

$$V = -\frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \quad (152)$$

$$= -\frac{\mu^2}{2}\phi_r^2 + \frac{\lambda}{4}\phi_r^4, \quad (153)$$

where we have used the transformation:

$$\phi_1(x) = \phi_r(x) \cos(\phi_\theta(x)), \quad (154)$$

$$\phi_2(x) = \phi_r(x) \sin(\phi_\theta(x)). \quad (155)$$

To minimize  $V$ , we take the derivative with respect to  $\phi_r$  and set it to zero:

$$\frac{dV}{d\phi_r} = -\mu^2 \phi_r + \lambda \phi_r^3 = 0 \quad (156)$$

$$\Rightarrow \phi_r(\lambda \phi_r^2 - \mu^2) = 0. \quad (157)$$

The solutions are:

$$\phi_r = 0, \quad \text{or} \quad \phi_r = \sqrt{\frac{\mu^2}{\lambda}}. \quad (158)$$

To determine which solution corresponds to the ground state, we evaluate the second derivative of  $V$ :

$$\frac{d^2V}{d\phi_r^2} = -\mu^2 + 3\lambda \phi_r^2. \quad (159)$$

At  $\phi_r = 0$ :

$$\left. \frac{d^2V}{d\phi_r^2} \right|_{\phi_r=0} = -\mu^2 < 0, \quad (160)$$

indicating a local maximum. At  $\phi_r = \sqrt{\frac{\mu^2}{\lambda}}$ :

$$\left. \frac{d^2V}{d\phi_r^2} \right|_{\phi_r=\sqrt{\frac{\mu^2}{\lambda}}} = -\mu^2 + 3\lambda \left( \frac{\mu^2}{\lambda} \right) = 2\mu^2 > 0, \quad (161)$$

indicating a local minimum. Therefore, the ground state is at:

$$\phi_r = \sqrt{\frac{\mu^2}{\lambda}}, \quad \phi_\theta \text{ is arbitrary}. \quad (162)$$

Next, we check if the Lagrangian is invariant under the continuous shift symmetry  $\phi_\theta(x) \rightarrow \phi_\theta(x) + \alpha_0$ . We can rewrite the kinetic term in terms of  $\phi_r$  and  $\phi_\theta$ :

$$\partial^\mu \phi_1 \partial_\mu \phi_1 + \partial^\mu \phi_2 \partial_\mu \phi_2 = (\partial^\mu (\phi_r \cos \phi_\theta)) (\partial_\mu (\phi_r \cos \phi_\theta)) + (\partial^\mu (\phi_r \sin \phi_\theta)) (\partial_\mu (\phi_r \sin \phi_\theta)) \quad (163)$$

$$= (\partial^\mu \phi_r \cos \phi_\theta - \phi_r \sin \phi_\theta \partial^\mu \phi_\theta) (\partial_\mu \phi_r \cos \phi_\theta - \phi_r \sin \phi_\theta \partial_\mu \phi_\theta) \quad (164)$$

$$+ (\partial^\mu \phi_r \sin \phi_\theta + \phi_r \cos \phi_\theta \partial^\mu \phi_\theta) (\partial_\mu \phi_r \sin \phi_\theta + \phi_r \cos \phi_\theta \partial_\mu \phi_\theta) \quad (165)$$

$$= (\partial^\mu \phi_r) (\partial_\mu \phi_r) + \phi_r^2 (\partial^\mu \phi_\theta) (\partial_\mu \phi_\theta). \quad (166)$$



The potential term depends only on  $\phi_r$ :

$$V = -\frac{\mu^2}{2}\phi_r^2 + \frac{\lambda}{4}\phi_r^4. \quad (167)$$

Thus, the Lagrangian in terms of  $\phi_r$  and  $\phi_\theta$  is:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \phi_r)(\partial_\mu \phi_r) - \frac{1}{2}\phi_r^2(\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \frac{\mu^2}{2}\phi_r^2 - \frac{\lambda}{4}\phi_r^4. \quad (168)$$

This Lagrangian is invariant under the shift  $\phi_\theta(x) \rightarrow \phi_\theta(x) + \alpha_0$  since  $\phi_\theta$  appears only through its derivatives. Therefore, the Lagrangian has a continuous shift symmetry in  $\phi_\theta$ .

(d)

We expand  $\phi_r(x)$  around its vacuum expectation value:

$$\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x). \quad (169)$$

Substituting this into the Lagrangian, we have:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) - \frac{1}{2}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^2 (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \frac{\mu^2}{2}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^2 - \frac{\lambda}{4}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^4 \quad (170)$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) - \frac{1}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right) (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right) \quad (171)$$

$$- \frac{\lambda}{4}\left(\frac{\mu^4}{\lambda^2} + 4\frac{\mu^2}{\lambda}\sqrt{\frac{\mu^2}{\lambda}}f_r + 6\frac{\mu^2}{\lambda}f_r^2 + 4f_r^3\sqrt{\frac{\mu^2}{\lambda}} + f_r^4\right) \quad (172)$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) + \left(-\frac{\mu^2}{2\lambda} - \sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{1}{2}f_r^2\right) (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \left(\frac{\mu^4}{2\lambda} + \mu^2\sqrt{\frac{\mu^2}{\lambda}}f_r + \frac{\mu^2}{2}f_r^2\right) \quad (173)$$

$$+ \left(-\frac{\mu^4}{4\lambda} - \mu^2\sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{3\mu^2}{2}f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right) \quad (174)$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) + \left(-\frac{\mu^2}{2\lambda} - \sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{1}{2}f_r^2\right) (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \left(\frac{\mu^4}{4\lambda} - \mu^2 f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right). \quad (175)$$

The mass term for  $f_r$  can be identified from the potential part of the Lagrangian:

$$V(f_r) = -\left(\frac{\mu^4}{4\lambda} - \mu^2 f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right) \quad (176)$$

$$= -\frac{\mu^4}{4\lambda} + \mu^2 f_r^2 + \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} + \frac{\lambda}{4}f_r^4. \quad (177)$$

The mass term for  $f_r$  is given by the coefficient of the  $f_r^2$  term:

$$m_{f_r}^2 = 2\mu^2. \quad (178)$$

Thus,  $f_r(x)$  is a massive field with mass  $m_{f_r} = \sqrt{2}\mu$ . For the field  $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}}\phi_\theta(x)$ , we can rewrite the kinetic term involving  $\phi_\theta$  as:

$$-\frac{1}{2} \left( \frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2 \right) (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) = -\frac{1}{2} \left( 1 + \frac{2f_r}{\sqrt{\frac{\mu^2}{\lambda}}} + \frac{f_r^2}{\frac{\mu^2}{\lambda}} \right) (\partial^\mu f_\theta)(\partial_\mu f_\theta). \quad (179)$$

The field  $f_\theta(x)$  does not have a mass term, as there is no term proportional to  $f_\theta^2$  in the potential. Therefore,  $f_\theta(x)$  is a massless field. The Lagrangian remains invariant under the continuous shift symmetry  $f_\theta(x) \rightarrow f_\theta(x) + \Lambda_0$ , since  $f_\theta$  appears only through its derivatives. Thus, the shift symmetry is preserved.  $\square$

## Question 4

### Problem 66.3

Use the result of problem 66.2 to compute the anomalous dimension of  $m$  and the beta function for  $e$  in spinor electrodynamics in  $R_\xi$  gauge. You should find that the results are independent of  $\xi$ .

**Remark:**

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} + (\xi - 1)k^\mu k^\nu / k^2}{k^2 - i\epsilon} \quad (180)$$

The book only choose the Feynman gauge ( $\xi = 1$ ) to show the loop calculation and get  $Z_{1,2,3,m}$ . For arbitrary gauge choice  $\xi$ , we can repeat the calculation and get:

$$Z_3 = 1 - \frac{e^2}{6\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from photon propagator loop correction} \quad (181)$$

$$Z_2 = 1 - \xi \frac{e^2}{8\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from fermion propagator loop correction} \quad (182)$$

$$Z_m = 1 - (3 + \xi) \frac{e^2}{8\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from fermion mass loop correction} \quad (183)$$

$$Z_1 = 1 - \xi \frac{e^2}{8\pi^2} \left( \frac{1}{\epsilon} + \text{finite} \right) + \mathcal{O}(e^4), \quad \text{derived from vertex loop correction} \quad (184)$$

Use the above to finish this problem.

## Answer

## Question 5

Consider the following theory:

$$\mathcal{L} = \mathcal{L}_\phi^0 + \mathcal{L}_\Psi^0 + \mathcal{L}_A^0 + \mathcal{L}_I \quad (185)$$

$$= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\Psi}(i\not{D} - m_\Psi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + y\phi\bar{\Psi}\Psi. \quad (186)$$

The Dirac field  $\Psi$  is charged under a  $U(1)$  gauge symmetry with a charge  $Q$ , and the gauge interaction strength is  $e$ . The  $U(1)$  gauge field is  $A_\mu$ , whose kinetic term is  $\mathcal{L}_A^0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ . (This is part of the real-world calculation for the discovery mode for the Higgs boson, which gone through heroic phenomenological studies on predicting the Higgs properties.)

- Draw the leading diagrams that enable  $\phi \rightarrow \gamma\gamma$  decay. (The gauge field  $A_\mu$  is identified as the photon field  $\gamma$ .)
- In the  $\phi$  rest frame, write down the amplitude in the general  $d$  dimension. No need to carry out the loop intergral at this point, but need to simplify the trace. (Notice that  $k_\mu\epsilon^\mu(k) = 0$  in Lorenz gauge.)
- Does the integral have a UV divergence in  $d = 4$  dimension (loop momentum goes to  $\infty$ )? Answer Yes or No with a few lines of argument.
- Does the integral have a singularity in  $d = 4$  dimension when the Euclidean loop momentum squared  $\bar{q}^2$  go to  $-D$ ? Answer Yes or No with a few lines of argument. (For simplicity, assume that  $D$  is real and can be zero for some configuration of  $x_1, x_2, x_3$ .)
- For  $m_\phi = 0$ , calculate using dimensional regularization in  $d = 4 - \epsilon$ . Write down your final answer in the simplest form. (The final answer would be short.)
- Carry out the full calculation of the amplitude in Part b using dimensional regularization in  $d = 4 - \epsilon$ . Write down your final answer in the simplest form. (The full answer would be a long calculateion.)

**Hint:** The following few equations, identities, and tricks, and the discussion around them might be helpful for you: Eq. (62.18), Eq. (47.18), Eq. (67.2).

**Remark:** No need to answer this, but one can think about it for fun. Recall that taking  $\epsilon \rightarrow 0$  (from plus or minus direction?) get you back to  $d = 4$ . In such a limit, contrast your result in Part f and Part c and think about why.

## Answer