University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Question 1

Gell-Mann Okubo for the baryon octet

The generators $t_a (a = 1, \ldots, 8)$ of SU(3) are normalized as $t_a = \frac{\lambda_a}{2}$ with $\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab}$, where λ_a are the Gell-Mann matrices. They satisfy $[t_a, t_b] = i f_{abc} t_c$ and $\{t_a, t_b\} = \frac{1}{3}\delta_{ab}\mathbf{1} + d_{abc} t_c$, where f_{abc} are totally antisymmetric and d_{abc} are totally symmetric structure constants.

Let B and $\bar{B}$ be the baryon octet 3×3 traceless matrices, expanded in the generator basis as $B = B^i t_i$ and $\bar{B} = \bar{B}^i t_i$ where $B^i, \bar{B}^i$ are the adjoint components. Define the two bilinear combinations $O_A \equiv [\bar{B}, B] = \bar{B}B - B\bar{B}$ and $O_S \equiv \{\bar{B}, B\} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}B)$.

(a) Show that both O_A and O_S are traceless and therefore transform in the adjoint (octet) representation.

(b) Expand O_A and O_S in components using the generator basis and show that $O_A = i\left(\bar{B}^i B^j\right) f_{ijk} t_k$ and $O_S = \left(\bar{B}^i B^j\right) d_{ijk} t_k$, so that $(O_A)^k = i f_{ijk} \bar{B}^i B^j$ and $(O_S)^k = d_{ijk} \bar{B}^i B^j$.

(c) Introduce a flavor-breaking spurion $H_8 = H_8^i t_i$, with real components H_8^i. Construct the two independent SU(3)-invariant mass terms:

$$S_f = (O_A)_b^a (H_8)_a^b, \quad S_d = (O_S)_b^a (H_8)_a^b \tag{1}$$

Assuming H_8 points in the 8 -direction (i.e. $H_8^i \propto \delta_{i8}$), argue that S_f and S_d correspond to the f-type and d-type symmetry breaking terms in the baryon mass operator, respectively.

(d) Given that an adjoint operator O_8 acts on an octet state B as $O_8(B) = [O_8, B]$, show that the invariant scalars in (c) are equivalent to the matrix elements $S_f \propto \langle \bar{B}|t_8|B\rangle \equiv \mathrm{Tr}\left(\bar{B}\,[t_8, B]\right)$ and $S_d \propto \langle \bar{B}|d_{8ij} t_i t_j|B\rangle \equiv \mathrm{Tr}\left(\bar{B}\,[d_{8ij}\,[t_i, [t_j, B]]\right)$.

(e) Hence, argue that for each entry B_{ij} of the baryon octet matrix, $S_f \propto Y$ and $S_d \propto I(I+1) - Y^2/4$ where I, Y are the isospin and hypercharge of the baryon B, respectively, thereby reproducing the Gell-Mann-Okubo mass formula for the baryon octet.

(Hint: Verify, entrywise, that $\left[\frac{2}{\sqrt{3}} t_8, B\right] = YB$ and the normalized operator $\frac{2}{\sqrt{3}} d_{8ij}\,[t_i, [t_j, B]] + \frac{1}{3}\,[t_i, [t_i, B]] = (I(I+1) - Y^2/4)\,B$ acts diagonally on each baryon field. The $[t_i, [t_i, B]]$ term is the adjoint Casimir ($SU(3)$ singlet) which just shifts all octet components uniformly so that the Λ eigenvalue becomes 0 . It can be absorbed into the overall singlet part of the GMO formula. On the diagonal remember B_{11}, B_{22} and B_{33} mix Σ^0 and Λ, so $B_{\text{diag}} = \Sigma^0\,\mathrm{diag}(1, -1, 0)/\sqrt{2} + \Lambda\,\mathrm{diag}(1, 1, -2)/\sqrt{6}$).

Answer

(a)

To show that both O_A and O_S are traceless, we first understand their definitions:

$$O_A = [\bar{B}, B] = \bar{B}B - B\bar{B} = [\bar{B}, B] \tag{2}$$
$$= [\bar{B}^i t_i, B^j t_j] = \bar{B}^i B^j [t_i, t_j] = i\bar{B}^i B^j f_{ijk} t_k \tag{3}$$

Taking the trace of O_A:

$$\mathrm{Tr}(O_A) = \mathrm{Tr}(iBarB^i B^j f_{ijk} t_k) = i\bar{B}^i B^j f_{ijk} \,\mathrm{Tr}(t_k) = 0. \tag{4}$$

Similarly, for O_S:

$$O_S = \{\bar{B}, B\} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}B) = \bar{B}B + B\bar{B} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}B) \tag{5}$$
$$= (\bar{B}^i t_i)(B^j t_j) + (B^j t_j)(\bar{B}^i t_i) - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}^i t_i B^j t_j) \tag{6}$$
$$= \bar{B}^i B^j \{t_i, t_j\} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}^i B^j t_i t_j) \tag{7}$$
$$= \bar{B}^i B^j \left(\frac{1}{3}\delta_{ij}\mathbf{1} + d_{ijk} t_k\right) - \frac{2}{3}\mathbf{1}\left(\frac{1}{2}\bar{B}^i B^j \delta_{ij}\right) \tag{8}$$
$$= \bar{B}^i B^j d_{ijk} t_k \tag{9}$$

Taking the trace of O_S:

$$\mathrm{Tr}(O_S) = \mathrm{Tr}(\bar{B}^i B^j d_{ijk} t_k) = \bar{B}^i B^j d_{ijk} \,\mathrm{Tr}(t_k) = 0. \tag{10}$$

Thus, both O_A and O_S are traceless and transform in the adjoint (octet) representation.

(b)

Expanding O_A and O_S in components using the generator basis, we have already derived:

$$O_A = i\bar{B}^i B^j f_{ijk} t_k \tag{11}$$
$$O_S = \bar{B}^i B^j d_{ijk} t_k \tag{12}$$

Thus, the components are:

$$(O_A)^k = i f_{ijk} \bar{B}^i B^j \tag{13}$$
$$(O_S)^k = d_{ijk} \bar{B}^i B^j \tag{14}$$

(c)

Introducing a flavor-breaking spurion $H_8 = H_8^i t_i$, with real components H_8^i, we can construct the two independent SU(3)-invariant mass terms:

$$S_f = (O_A)^a_b (H_8)^b_a = \mathrm{Tr}(O_A H_8) \tag{15}$$
$$S_d = (O_S)^a_b (H_8)^b_a = \mathrm{Tr}(O_S H_8) \tag{16}$$

Assuming H_8 points in the 8-direction (i.e., $H_8^i \propto \delta_{i8}$), we can write:

$$H_8 = H_8^i t_i = H_8^8 t_8 \tag{17}$$

Substituting this into the expressions for S_f and S_d:

$$S_f = \mathrm{Tr}(O_A H_8) = \mathrm{Tr}(i f_{ijk} \bar{B}^i B^j t_k H_8^l t_l) = i H_8^8 f_{ij8} \bar{B}^i B^j \,\mathrm{Tr}(t_k t_8) = \frac{i}{2} H_8^8 f_{ij8} \bar{B}^i B^j \tag{18}$$
$$S_d = \mathrm{Tr}(O_S H_8) = \mathrm{Tr}(d_{ijk} \bar{B}^i B^j t_k H_8^l t_l) = H_8^8 d_{ij8} \bar{B}^i B^j \,\mathrm{Tr}(t_k t_8) = \frac{1}{2} H_8^8 d_{ij8} \bar{B}^i B^j \tag{19}$$

Since O_A and O_S are octet operators, S_f and S_d correspond to the f-type and d-type symmetry breaking terms in the baryon mass operator, respectively.

(d)

$$\langle \bar{B} | t_8 | B \rangle \equiv \mathrm{Tr}(\bar{B}[t_8, B]) \tag{20}$$
$$= \mathrm{Tr}(\bar{B}(t_8 B - B t_8)) = \mathrm{Tr}(\bar{B} t_8 B) - \mathrm{Tr}(\bar{B} B t_8) \tag{21}$$
$$= \mathrm{Tr}(B \bar{B} t_8) - \mathrm{Tr}(\bar{B} B t_8) \tag{22}$$
$$= \mathrm{Tr}((\bar{B} B - B \bar{B}) t_8) = \mathrm{Tr}(O_A t_8) \propto S_f \tag{23}$$

This is beccause $S_f = \mathrm{Tr}(O_A H_8) = \mathrm{Tr}(O_A H_8^8 t_8) = H_8^8 \,\mathrm{Tr}(O_A t_8)$. Similarly, for S_d:

$$\langle \bar{B} | d_{8ij} t_i t_j | B \rangle \equiv \mathrm{Tr}(\bar{B} d_{8ij} [t_i, [t_j, B]]) \tag{24}$$
$$= \mathrm{Tr}(\bar{B}^\alpha B^\beta t_\alpha d_{8ij} [t_i, [t_j, t_\beta]]) \tag{25}$$
$$= \bar{B}^\alpha B^\beta d_{8ij} \,\mathrm{Tr}(t_\alpha [t_i, [t_j, t_\beta]]) \tag{26}$$
$$= \bar{B}^\alpha B^\beta d_{8ij} \,\mathrm{Tr}(t_\alpha [t_i, i f_{j\beta\gamma} t_\gamma]) \tag{27}$$
$$= i \bar{B}^\alpha B^\beta d_{8ij} f_{j\beta\gamma} \,\mathrm{Tr}(t_\alpha [t_i, t_\gamma]) \tag{28}$$
$$= i \bar{B}^\alpha B^\beta d_{8ij} f_{j\beta\gamma} i f_{i\alpha\delta} \,\mathrm{Tr}(t_\alpha t_\delta) \tag{29}$$
$$= -\frac{1}{2} \bar{B}^\alpha B^\beta d_{8ij} f_{j\beta\gamma} f_{i\alpha\gamma} \tag{30}$$
$$= -\frac{1}{2} \bar{B}^a B^b d_{8ij} f_{jbc} f_{iac} \tag{31}$$

By `Mathematica`, we have

$$d_{8ij} f_{jbc} f_{iac} = 3/2 d_{8ab} \tag{32}$$

Thus,

$$\langle\bar{B}|d_{8ij}t_it_j|B\rangle = -\frac{3}{4}\bar{B}^aB^bd_{8ab} = -\frac{3}{2}\,\mathrm{Tr}(O_St_8) \propto S_d \tag{33}$$

This is because $S_d = \mathrm{Tr}(O_SH_8) = \mathrm{Tr}(O_SH_8^8t_8) = H_8^8\,\mathrm{Tr}(O_St_8)$.

(e)

First, we write down the explicit form of B:

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{pmatrix} \tag{34}$$

Next, we write down the hypercharge Y and isospin I and $I(I+1) - Y^2/4$ values for each baryon:

$$p : Y = 1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{35}$$
$$n : Y = 1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{36}$$
$$\Sigma^+ : Y = 0, I = 1, I(I+1) - Y^2/4 = 2 - 0 = 2 \tag{37}$$
$$\Sigma^0 : Y = 0, I = 1, I(I+1) - Y^2/4 = 2 - 0 = 2 \tag{38}$$
$$\Sigma^- : Y = 0, I = 1, I(I+1) - Y^2/4 = 2 - 0 = 2 \tag{39}$$
$$\Xi^0 : Y = -1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{40}$$
$$\Xi^- : Y = -1, I = 1/2, I(I+1) - Y^2/4 = 3/4 - 1/4 = 1/2 \tag{41}$$
$$\Lambda^0 : Y = 0, I = 0, I(I+1) - Y^2/4 = 0 - 0 = 0 \tag{42}$$

Now, we compute $\left[\frac{2}{\sqrt{3}}t_8, B\right]$, and it can be easily show that (see my *Mathematica* notebook for details):

$$[\frac{2}{\sqrt{3}}t_8, B] = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & n \\ -(-\Xi^-) & -\Xi^0 & 0 \end{pmatrix} = YB \tag{43}$$

where Y is the hypercharge of the baryon B. Next, we compute the normalized operator $\frac{2}{\sqrt{3}}d_{8ij}\,[t_i,[t_j,B]] + \frac{1}{3}\,[t_i,[t_i,B]]$:

$$\frac{2}{\sqrt{3}}d_{8ij}\,[t_i,[t_j,B]] + \frac{1}{3}\,[t_i,[t_i,B]] \tag{44}$$

$$= \begin{pmatrix} \sqrt{2}\Sigma^0 & 2\Sigma^+ & \frac{1}{2}p \\ 2\Sigma^- & -\sqrt{2}\Sigma^0 & \frac{1}{2}n \\ \frac{1}{2}(-\Xi^-) & \frac{1}{2}\Xi^0 & 0 \end{pmatrix} = \begin{pmatrix} 2\frac{\Sigma^0}{\sqrt{2}} & 2\Sigma^+ & \frac{1}{2}p \\ 2\Sigma^- & -2\frac{\Sigma^0}{\sqrt{2}} & \frac{1}{2}n \\ \frac{1}{2}(-\Xi^-) & \frac{1}{2}\Xi^0 & 0 \end{pmatrix} = \left(I(I+1) - \frac{Y^2}{4}\right)B \tag{45}$$

where $I(I+1) - \frac{Y^2}{4}$ is the value for each baryon B. Thus, we have shown that for each entry B_{ij} of the baryon octet matrix, $S_f \propto Y$ and $S_d \propto I(I+1) - Y^2/4$, thereby reproducing the Gell-Mann-Okubo mass formula for the baryon octet, meaning

$$M_B(Y, I) = M_0 + M_A Y + M_S \left(I(I+1) - \frac{Y^2}{4} \right), \tag{46}$$

where M_0, M_A, M_S are constants. $\square$

Question 2

ρ-ω **mixing**

The vector mesons $\rho(770)$ and $\omega(782)$ are very close in mass. For this reason the effects of isospin violation are somewhat enhanced in these mesons and can be parametrized in terms of ρ-ω mixing. Namely, the physical ρ^0 and ω mesons can be viewed as orthogonal mixed states of a pure isospin triplet and isospin singlet:

$$\rho^0 = \cos\theta \frac{(u\bar{u} - d\bar{d})}{\sqrt{2}} + \sin\theta \frac{(u\bar{u} + d\bar{d})}{\sqrt{2}},$$

$$\omega = -\sin\theta \frac{(u\bar{u} - d\bar{d})}{\sqrt{2}} + \cos\theta \frac{(u\bar{u} + d\bar{d})}{\sqrt{2}},$$

where θ is a (small) mixing angle.

(a) Determine θ (up to a sign) using experimental data on the decay $\omega \to \pi^+\pi^-$. Estimate the error in the value of the mixing angle.

(b) Using the value of θ predict the decay rates $\Gamma(\rho^0 \to e^+e^-)$ and $\Gamma(\omega \to e^+e^-)$, assuming the amplitude for a quark pair annihilation into an e^+e^- pair is proportional to the electric charge Q of the quark.

(c) Assume that the transition amplitude between different spin states of a $q\bar{q}$ quark pair with emission of a photon: $(q\bar{q}) \to (q\bar{q}) + \gamma$ is proportional to the quark electric charge Q. Use the value of the ρ-ω mixing angle θ to determine the ratios of the decay rates:

(i) $\Gamma(\rho^0 \to \pi^0\gamma)/\Gamma(\omega^0 \to \pi^0\gamma)$,

(ii) $\Gamma(\rho^0 \to \eta\gamma)/\Gamma(\omega^0 \to \eta\gamma)$.

Compare with the PDG experimental data. How does the inclusion of ρ-ω mixing improve the agreement with the data?

Answer

(a)

Question 3

Baryon magnetic moments

The octet of spin-$\frac{1}{2}$ baryons has magnetic moments μ. The operator that describes the magnetic moment is an $\mathrm{SU}(3)_f$ octet operator which is proportional to the quark charge Q. The charge

$$Q = t_3 + \frac{1}{\sqrt{3}} t_8$$

is traceless ($\operatorname{Tr} Q = 0$) and can be promoted to a purely $\mathrm{SU}(3)_f$ octet spurion $\mathbf{8}_Q$ (with no singlet piece, as in contrast to the GMO mass formula). Hence, when determining the baryon magnetic moment

$$\mu(B) = \langle \bar{B} | \mu | B \rangle \propto \mathbf{8}_{\bar{B}} \times \mathbf{8}_Q \times \mathbf{8}_B$$

there are two independent octet structures (the f- and d-type couplings, as for the baryon mass), given by

$$\mu(B) = c_f \operatorname{Tr}(B^\dagger [Q, B]) + c_d \operatorname{Tr}(B^\dagger \{Q, B\}) = \alpha_+ \operatorname{Tr}(B B^\dagger Q) + \alpha_- \operatorname{Tr}(B^\dagger B Q),$$

where $\alpha_+ \equiv c_d + c_f$, $\alpha_- \equiv c_d - c_f$ are arbitrary constants and

$$B = \begin{pmatrix} \frac{\Sigma_u^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma_u^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

Determine all the spin-$\frac{1}{2}$ baryon magnetic moments in terms of $\mu(p)$ and $\mu(n)$ (by eliminating $c_{f,d}$ or $\alpha_\pm$) and compare with the PDG experimental values. These predictions were first worked out by Coleman and Glashow in 1961. Note that imposing the full SU(6) spin-flavor symmetry further predicts $\mu(p)/\mu(n) = -\frac{3}{2}$, which you can ignore in this problem.

Answer

See my *Mathematica* notebook for detailed calculations. The final results are summarized in the table below:

- p: $c_f + \frac{1}{3} c_d$
- n: $-\frac{2}{3} c_d$
- Λ: $-\frac{1}{3} c_d$
- Σ^+: $c_f + \frac{1}{3} c_d$
- Σ^0: $\frac{1}{3} c_d$
- Σ^-: $-c_f + \frac{1}{3} c_d$

- Ξ^0: $-\frac{2}{3}c_d$
- Ξ^-: $-c_f + \frac{1}{3}c_d$

Baryon	Predicted Magnetic Moment (μ_N)	Experimental Magnetic Moment (μ_N)
p	$\mu(p)$	2.793
n	$\mu(n)$	-1.913
Λ	$\frac{1}{2}\mu(n)$	-0.613
Σ^+	$\mu(p)$	2.458
Σ^0	$-\frac{1}{2}\mu(n)$	≈ 0
Σ^-	$-(\mu(p)+\mu(n))$	-1.160
Ξ^0	$\mu(n)$	-1.250
Ξ^-	$-(\mu(p)+\mu(n))$	-0.651

The predictions are in good agreement with the experimental values, demonstrating the effectiveness of the SU(3)$_f$ symmetry approach in describing baryon magnetic moments except for Σ^0 where the experimental value is not well matched. $\square$