

University of Minnesota  
School of Physics and Astronomy

**2026 Spring Physics 8502**  
**General Relativity II**  
Assignment Solution

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# Problem Set 2 due on Due Feb 18 at 11:59pm

## Question 1

Consider 2 objects of mass  $m_1$  and  $m_2$  with separation  $a$ , orbiting about a common center of mass. Find the change in the period ( $\dot{\tau}/\tau$ ) due to gravitational radiation. Assume the above result is valid as  $a \rightarrow 0$ , find the time to go from  $a = a_0$  to  $a = 0$ .

## Answer

We first write down the radiation power of the system:

$$P = \frac{dE}{dt} = -\frac{G}{5c^5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (1)$$

$$= -\frac{G}{5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (2)$$

where  $I_{ij}$  is the quadrupole moment of the system. For a binary system, we can express  $I_{ij}$  in terms of the reduced mass  $\mu = \frac{m_1 m_2}{m_1 + m_2}$  and the separation vector  $\mathbf{r}$  between the two masses:

$$I_{ij} = \mu(r_i r_j - \frac{1}{3} r^2 \delta_{ij}) \quad (3)$$

For a circular orbit, the separation vector can be expressed as  $\mathbf{r}(t) = a(\cos(\omega t), \sin(\omega t), 0)$ , where  $\omega$  is the angular frequency of the orbit. The third time derivative of  $I_{ij}$  can be calculated as follows:

$$\ddot{I}_{ij} = \mu \left( \frac{d^3}{dt^3}(r_i r_j) - \frac{1}{3} \frac{d^3}{dt^3}(r^2 \delta_{ij}) \right) \quad (4)$$

Calculating the third time derivative of  $r_i r_j$  and  $r^2 \delta_{ij}$ , we find:

$$\frac{d^3}{dt^3}(r_i r_j) = \frac{d^3}{dt^3} \begin{pmatrix} a^2 \cos^2(\omega t) & a^2 \cos(\omega t) \sin(\omega t) & 0 \\ a^2 \cos(\omega t) \sin(\omega t) & a^2 \sin^2(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5)$$

$$= \frac{d^3}{dt^3} \begin{pmatrix} \frac{a^2}{2}(1 + \cos(2\omega t)) & \frac{a^2}{2} \sin(2\omega t) & 0 \\ \frac{a^2}{2} \sin(2\omega t) & \frac{a^2}{2}(1 - \cos(2\omega t)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6)$$

$$= 4a^2 \omega^3 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7)$$

$$\frac{d^3}{dt^3}(r^2 \delta_{ij}) = \frac{d^3}{dt^3}(a^2 \delta_{ij}) = 0 \quad (8)$$

Substituting these results back into the expression for  $\ddot{I}_{ij}$ , we get:

$$\ddot{I}_{ij} = 4\mu a^2 \omega^3 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9)$$

Now we can calculate the power radiated by the system:

$$P = \frac{dE}{dt} = -\frac{G}{5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (10)$$

$$= -\frac{G}{5} \langle 16\mu^2 a^4 \omega^6 (2\sin^2(2\omega t) + 2\cos^2(2\omega t)) \rangle \quad (11)$$

$$= -\frac{32G}{5} \mu^2 a^4 \omega^6. \quad (12)$$

By Kepler's third law, we have  $\omega^2 = \frac{G(m_1+m_2)}{a^3}$ ,  $\mu = \frac{m_1 m_2}{m_1+m_2}$ , which allows us to express the power in terms of the separation  $a$ :

$$P = -\frac{32G^4}{5} \frac{\mu^2 (m_1 + m_2)^3}{a^5} \quad (13)$$

$$= -\frac{32G^4}{5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5} \quad (14)$$

The energy of the system is given by the sum of the kinetic and potential energy:

$$E = -\frac{Gm_1 m_2}{2a} \quad (15)$$

The rate of change of the energy is equal to the power radiated:

$$\frac{dE}{dt} = \frac{Gm_1 m_2}{2a^2} \frac{da}{dt} = -\frac{32G^4}{5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5} \quad (16)$$

Solving for  $\frac{da}{dt}$ , we find:

$$\frac{da}{dt} = -\frac{64G^3}{5} \frac{m_1 m_2 (m_1 + m_2)}{a^3} \quad (17)$$

Now, we can write down the period of the orbit  $\tau = \frac{2\pi}{\omega}$ :

$$\tau = 2\pi \sqrt{\frac{a^3}{G(m_1 + m_2)}} = \frac{2\pi}{\sqrt{G(m_1 + m_2)}} a^{3/2} \quad (18)$$

Taking the time derivative of the period, we get:

$$\frac{d\tau}{dt} = \frac{3\pi}{\sqrt{G(m_1 + m_2)}} a^{1/2} \frac{da}{dt} = -\frac{192\pi G^{5/2}}{5} \frac{m_1 m_2 (m_1 + m_2)^{1/2}}{a^{5/2}} \quad (19)$$

Finally, we can express the change in the period as:

$$\frac{\dot{\tau}}{\tau} = \frac{d\tau/dt}{\tau} = -\frac{96G^3}{5} \frac{m_1 m_2}{a^4} \quad (20)$$

To find the time it takes for the separation to go from  $a = a_0$  to  $a = 0$ , we can integrate the expression for  $\frac{da}{dt}$ :

$$t = \int dt = \int_{a_0}^0 \frac{da}{\frac{da}{dt}} = \int_{a_0}^0 -\frac{5}{64G^3} \frac{a^3}{m_1 m_2 (m_1 + m_2)} da = \frac{5a_0^4}{256G^3 m_1 m_2 (m_1 + m_2)}. \quad (21)$$

□

## Question 2

Problem 7.6 in Carroll. Two objects of mass  $M$  have a head-on collision at  $(0, 0, 0, 0)$ . In the distant past,  $t \rightarrow -\infty$ , the mass started at  $x \rightarrow \pm\infty$  with zero velocity.

- (a) Using Newtonian theory show that  $x(t) = \pm(\frac{9}{8}GMt^2)^{1/3}$ .
- (b) For what separations is the Newtonian approximation reasonable?
- (c) Calculate  $h_{xx}^{TT}$  at  $(0, R, 0)$ .
- (d) For the same problem, calculate the total energy radiated in the collision.

## Answer

(a)

Start with energy conservation:

$$2\frac{1}{2}M\dot{x}^2 - \frac{GM^2}{2x} = 0 \quad (22)$$

$$\implies \dot{x} = \sqrt{\frac{GM}{2x}} \quad (23)$$

Separating variables and integrating, we get:

$$\int x^{1/2} dx = \sqrt{\frac{GM}{2}} \int dt \quad (24)$$

$$\implies \frac{2}{3}x^{3/2} = \sqrt{\frac{GM}{2}}t + C, \quad \text{where } C = 0 \text{ since } x = 0 \text{ at } t = 0 \quad (25)$$

$$\implies x^{3/2} = \frac{3}{2}\sqrt{\frac{GM}{2}}t = \sqrt{\frac{9}{8}GMt} \quad (26)$$

$$\implies x(t) = \pm \left( \frac{9}{8}GMt^2 \right)^{1/3} \quad (27)$$

(b)

The Newtonian approximation is reasonable when the gravitational field is weak and the velocities are much less than the speed of light. For the speed, we can calculate  $\dot{x}$  from the expression we derived in

part (a):

$$\dot{x} = \frac{d}{dt} \left( \frac{9}{8} GM t^2 \right)^{1/3} = \frac{2}{3} \left( \frac{9}{8} GM \right)^{1/3} t^{-1/3} \ll c = 1 \quad (28)$$

$$\implies t \gg \left( \frac{2}{3} \left( \frac{9}{8} GM \right)^{1/3} \right)^3 = \frac{8}{27} GM \quad (29)$$

$$\implies x(t) \gg \left( \frac{9}{8} GM \left( \frac{8}{27} GM \right)^2 \right)^{1/3} = \frac{2}{3} GM \quad (30)$$

For the gravitational field, we can calculate the gravitational potential  $\Phi$  at the position of one of the masses:

$$\Phi = -\frac{GM}{2x} \quad (31)$$

The Newtonian approximation is reasonable when  $|\Phi| \ll 1$ , which implies:

$$x \gg \frac{GM}{2} \quad (32)$$

(c)

By equation (7.140) in Carroll, the transverse-traceless part of the metric perturbation is given by:

$$h_{ij}^{TT} = \frac{2G}{R} \frac{d^2}{dt^2} I_{ij}(t_r), \quad (33)$$

where  $I_{ij}$  is the quadrupole moment of the system and  $t_r = t - R$  is the retarded time. The quadrupole moment can be calculated as:

$$I_{ij} = \sum_a m_a (x_a^i x_a^j - \frac{1}{3} r_a^2 \delta_{ij}). \quad (34)$$

Since we want to calculate  $h_{xx}^{TT}$ , we need to find  $I_{xx}$ . For the two masses, we have:

$$I_{xx} = M(x^2 - \frac{1}{3} r^2) + M((-x)^2 - \frac{1}{3} r^2) = \frac{4}{3} M x^2 = \frac{4}{3} M \left( \frac{9}{8} GM t^2 \right)^{2/3} \quad (35)$$

Taking the second time derivative, we get:

$$\frac{d^2}{dt^2} I_{xx} = \frac{4}{3} M \frac{d^2}{dt^2} \left( \frac{9}{8} GM t^2 \right)^{2/3} = \frac{4}{3} M \left( \frac{9}{8} GM \right)^{2/3} \frac{d^2}{dt^2} t^{4/3} \quad (36)$$

$$= \frac{4}{3} M \left( \frac{9}{8} GM \right)^{2/3} \frac{4}{9} t^{-2/3} = \frac{16}{27} M \left( \frac{9}{8} GM \right)^{2/3} t^{-2/3} \quad (37)$$

Substituting this back into the expression for  $h_{xx}^{TT}$ , we get:

$$h_{xx}^{TT} = \frac{2G}{R} \frac{16}{27} M \left( \frac{9}{8} GM \right)^{2/3} t^{-2/3} = \frac{32GM}{27R} \left( \frac{9}{8} GM \right)^{2/3} (t - R)^{-2/3}, \quad (38)$$

where we have replaced  $t$  with the retarded time  $t_r = t - R$ .

(d)

The total energy radiated in the collision can be calculated using the quadrupole formula for gravitational radiation:

$$E = \frac{G}{5} \int_{-\infty}^{\infty} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle dt \quad (39)$$

Since we have already calculated  $I_{xx}$ , we can find  $\ddot{I}_{xx}$  by taking the third time derivative:

$$\ddot{I}_{xx} = \frac{d}{dt} \frac{d^2}{dt^2} I_{xx} = \frac{d}{dt} \left( \frac{16}{27} M \left( \frac{9}{8} GM \right)^{2/3} t^{-2/3} \right) = -\frac{32}{81} M \left( \frac{9}{8} GM \right)^{2/3} t^{-5/3} \quad (40)$$

Since the quadrupole moment is traceless, we have  $\ddot{I}_{kk} = 0$ . Therefore, the energy radiated can be expressed as:

$$I_{yy} = I_{zz} = -\frac{1}{2} I_{xx}, \quad \ddot{I}_{yy} = \ddot{I}_{zz} = -\frac{1}{2} \ddot{I}_{xx} \quad (41)$$

Back to the expression for the energy radiated, we have:

$$E = \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \ddot{I}_{xx}^2 + 2 \ddot{I}_{yy}^2 - \frac{1}{3} (\ddot{I}_{xx} + 2 \ddot{I}_{yy})^2 \right\rangle dt \quad (42)$$

$$= \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \ddot{I}_{xx}^2 + 2 \left( -\frac{1}{2} \ddot{I}_{xx} \right)^2 - \frac{1}{3} (\ddot{I}_{xx} - \ddot{I}_{xx})^2 \right\rangle dt \quad (43)$$

$$= \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \ddot{I}_{xx}^2 + 2 \cdot \frac{1}{4} \ddot{I}_{xx}^2 - 0 \right\rangle dt \quad (44)$$

$$= \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \frac{3}{2} \ddot{I}_{xx}^2 \right\rangle dt = \frac{3G}{10} \int_{-\infty}^{t_0} \left( -\frac{32}{81} M \left( \frac{9}{8} GM \right)^{2/3} t^{-5/3} \right)^2 dt \quad (45)$$

$$= \frac{3G}{10} \frac{1024}{6561} M^2 \left( \frac{9}{8} GM \right)^{4/3} \int_{-\infty}^{t_0} t^{-10/3} dt = \frac{3G}{10} \frac{1024}{6561} M^2 \left( \frac{9}{8} GM \right)^{4/3} \cdot \frac{3}{7} t_0^{-7/3} \quad (46)$$

$$= \frac{512}{10935} GM^2 \left( \frac{9}{8} GM \right)^{4/3} t_0^{-7/3}, \quad (47)$$

where  $t_0$  is the time at which the collision occurs. Since the collision occurs at  $t = 0$ , we can take the limit as  $t_0 \rightarrow 0$  to find the total energy radiated:

$$E = \lim_{t_0 \rightarrow 0} \frac{512}{10935} GM^2 \left( \frac{9}{8} GM \right)^{4/3} t_0^{-7/3} = \infty \quad (48)$$

This result indicates that an infinite amount of energy is radiated in the collision, which is a consequence

of the idealized nature of the problem. In reality, the energy radiated would be finite due to various factors such as the **finite size** of the masses and the presence of other forces that would come into play during the collision. □

## Question 3

A ball of mass  $m = 100$  g is thrown into the vacuum above the earth (ie. neglect all effects of air resistance), which produces a UNIFORM gravitational field ( $g = 10^3$  cm s $^{-2}$ ) with a velocity  $v_0 = 10^3$  cm/s. Normally, this ball would rise to a height of  $h = v_0^2/2g = 500$  cm. However, the ball will be a source of gravitational radiation and won't go quite so high. Find  $\Delta h$ . How does  $\Delta h/h$  depend on  $v_0$ ?

Now suppose the ball will fall back and elastically bounce. Left alone, the ball will eventually come to rest. How long will it take.

## Answer

The power radiated by the ball due to gravitational radiation can be calculated using the quadrupole formula:

$$P = \frac{G}{5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (49)$$

For a ball moving in a uniform gravitational field, the quadrupole moment can be expressed as:

$$I_{ij} = m(x_i x_j - \frac{1}{3} r^2 \delta_{ij}) \quad (50)$$

Since the ball is moving vertically, we can express its position as  $x(t) = (0, 0, z(t))$ , where  $z(t)$  is the height of the ball at time  $t$ . The quadrupole moment can then be simplified to:

$$I_{zz} = m(z^2 - \frac{1}{3} z^2) = \frac{2}{3} m z^2 \quad (51)$$

Since the quadrupole moment is traceless, we have  $I_{xx} = I_{yy} = -\frac{1}{2} I_{zz}$ . Therefore, the second time derivative of the quadrupole moment can be calculated as:

$$\ddot{I}_{zz} = \frac{d^3}{dt^3} I_{zz} = \frac{d^3}{dt^3} \left( \frac{2}{3} m z^2 \right) \quad (52)$$

$$= \frac{2}{3} m \frac{d^2}{dt^2} (2z\dot{z}) = \frac{4}{3} m \frac{d}{dt} (z\ddot{z} + \dot{z}^2) \quad (53)$$

$$= \frac{4m}{3} \frac{d}{dt} (z(-g) + \dot{z}^2) = \frac{4m}{3} (-g\dot{z} + 2\dot{z}\ddot{z}) = \frac{4m}{3} (-g\dot{z} + 2\dot{z}(-g)) = -4mg\dot{z} = -4mgv \quad (54)$$

Since  $I_{xx} = I_{yy} = -\frac{1}{2} I_{zz}$ , we have  $\ddot{I}_{xx} = \ddot{I}_{yy} = -\frac{1}{2} \ddot{I}_{zz}$ . Substituting these results back into the expression for the power, we get:

$$P = \frac{G}{5} \left\langle \ddot{I}_{zz}^2 + 2 \ddot{I}_{xx}^2 - \frac{1}{3} (\ddot{I}_{zz} + 2\ddot{I}_{xx})^2 \right\rangle = \frac{G}{5} \left\langle \ddot{I}_{zz}^2 + 2 \cdot \frac{1}{4} \ddot{I}_{zz}^2 - 0 \right\rangle = \frac{3G}{10} \left\langle \ddot{I}_{zz}^2 \right\rangle \quad (55)$$

$$= \frac{3G}{10} \langle (-4mgv)^2 \rangle = \frac{24Gm^2g^2}{5} v^2 \quad (56)$$

The energy radiated by the ball can be calculated by integrating the power over time, and we also restore the  $c^5$  in the denominator:

$$\Delta E = \int P dt = \int \frac{24Gm^2g^2}{5c^5} v^2 dt = \frac{24Gm^2g^2}{5c^5} \int v^2 dt \quad (57)$$

$$= \frac{24Gm^2g^2}{5c^5} \int v^2 \frac{dv}{-g} = -\frac{24Gm^2g}{5c^5} \int v^2 dv = -\frac{24Gm^2g}{5c^5} \cdot \frac{v_0^3}{3} = -\frac{8Gm^2gv_0^3}{5c^5} \quad (58)$$

The change in height  $\Delta h$  can be calculated by equating the energy radiated to the change in potential energy:

$$\Delta E = mg\Delta h \implies \Delta h = \frac{\Delta E}{mg} = -\frac{8Gm^2gv_0^3}{5c^5} \quad (59)$$

The ratio  $\Delta h/h$  can be expressed as:

$$\frac{\Delta h}{h} = \frac{-\frac{8Gm^2gv_0^3}{5c^5}}{\frac{v_0^2}{2g}} = -\frac{16Gm^2v_0}{5c^5} \quad (60)$$

This shows that the ratio  $\Delta h/h$  is proportional to the initial velocity  $v_0$  of the ball.

Now, we know the due to the radiation, we have  $\Delta E = -\frac{8Gm^2gv_0^3}{5c^5}$ , which means the ball loses energy at a rate of  $\frac{dE}{dt}$ :

$$\frac{dE}{dt} = -\frac{8Gm^2gv_0^3}{5c^5} \cdot \frac{1}{t_{\text{total}}} = -\frac{8Gm^2gv_0^3}{5c^5} \cdot \frac{1}{2t_{\text{up}}}, \quad (61)$$

where  $t_{\text{total}}$  is the total time for the ball to go up and come back down, and  $t_{\text{up}}$  is the time it takes for the ball to reach its maximum height. Since the ball is thrown upwards with an initial velocity of  $v_0$ , we can calculate  $t_{\text{up}}$  using the equation of motion:

$$v = v_0 - gt_{\text{up}} = 0 \implies t_{\text{up}} = \frac{v_0}{g} \quad (62)$$

Substituting this back into the expression for  $\frac{dE}{dt}$ ,

$$\frac{dE}{dt} = -\frac{8Gm^2gv_0^3}{5c^5} \cdot \frac{1}{2 \cdot \frac{v_0}{g}} = -\frac{4Gm^2g^2v_0^2}{5c^5} \quad (63)$$

Note that the energy of the ball at its maximum height is given by:

$$E = mgh = mg \frac{v_0^2}{2g} = \frac{1}{2}mv_0^2 \implies v_0^2 = \frac{2E}{m}, \quad (64)$$

Hence, we can express  $\frac{dE}{dt}$  in terms of the energy  $E$ :

$$\frac{dE}{dt} = -\frac{4Gm^2g^2}{5c^5} \cdot \frac{2E}{m} = -\frac{8Gmg^2}{5c^5}E \quad (65)$$

This is a first-order linear differential equation, the solution to which is given by:

$$E(t) = E_0 e^{-\frac{8Gmg^2}{5c^5}t} \quad (66)$$

where  $E_0$  is the initial energy of the ball at its maximum height. The ball will come to rest when its energy approaches zero, which occurs as  $t \rightarrow \infty$ .

□