University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Question 1

Three-photon decay of a scalar particle

Consider the decay $X \to 3\gamma$, where X is a scalar particle of mass M. Assume the decay amplitude M_{fi} is approximately constant (i.e. independent of photon energies and angles) and can be written as $M_{fi} = A$. This is a good approximation for the decay of orthopositronium in its ground state.

(a) Derive the differential decay rate $\frac{d\Gamma}{dw}$ corresponding to the measured energy ω of a single photon in the rest frame of X.

(b) Express the total decay rate Γ in terms of the constant amplitude A.

Answer

We can start from the general expression for the decay rate of a particle decaying into three massless particles:

$$d\Gamma = \frac{1}{2M}\frac{1}{3!}|\mathcal{M}_{fi}|^2 d\tau_3, \tag{1}$$

where $d\tau_3$ is the three-body phase space element, M is the mass of the decaying particle, $1/3!$ accounts for the identical photons in the final state and $\mathcal{M}_{fi} = A$ is the invariant matrix element. The three-body phase space element for massless particles can be expressed as:

$$d\tau_3 = (2\pi)^4\delta^4(P - p_1 - p_2 - p_3)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}, \tag{2}$$

where P is the four-momentum of the decaying particle, and p_i and E_i are the momenta and energies of the final state photons, respectively. If we consider the rest frame of the decaying particle, we have $P = (M, 0, 0, 0)$. Besides, we can apply the splitting formula for three-body phase space:

$$d\tau_3 = d\tau_2(M \to p_1 + q)\frac{dq^2}{2\pi}d\tau_2(q \to p_2 + p_3), \tag{3}$$

where $q = p_2 + p_3$ is the combined four-momentum of photons 2 and 3. The two-body phase space elements can be expressed as:

$$d\tau_2(M \to p_1 + q) = (2\pi)^4\delta^4(P - p_1 - q)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3q}{(2\pi)^3 2E_q}, \tag{4}$$

$$d\tau_2(q \to p_2 + p_3) = (2\pi)^4\delta^4(q - p_2 - p_3)\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}. \tag{5}$$

To find the differential decay rate with respect to the energy of one photon, say $\omega = E_1$, we can integrate over the other variables. The total energy conservation gives us:

$$M = E_1 + E_2 + E_3. \tag{6}$$

Since the photons are massless, we have $E_i = |\vec{p}_i|$.

$$d\Gamma = \frac{1}{2M}\frac{1}{3!}|A|^2 d\tau_3 \tag{7}$$

$$= \frac{1}{2M}\frac{1}{3!}|A|^2 d\tau_2(M \to p_1 + q)\frac{dq^2}{2\pi} d\tau_2(q \to p_2 + p_3). \tag{8}$$

The two-body phase space elements can be evaluated in their respective rest frames. In this frame, $q = (\sqrt{q^2}, 0, 0, 0)$, and the energies of the photons are $E_2 = |\vec{p}_2|$ and $E_3 = |\vec{p}_3|$.

$$\int d\tau_2(q \to p_2 + p_3) = \int (2\pi)^4 \delta(\sqrt{q^2} - E_2 - E_3)\delta^3(\vec{0} - \vec{p}_2 - \vec{p}_3)\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3} \tag{9}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_2 - E_3)\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{1}{(2\pi)^3 2E_2} \tag{10}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_2)\frac{4\pi E_2^2 dE_2}{(2\pi)^3 2E_2}\frac{1}{(2\pi)^3 2E_2} \tag{11}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_2)\frac{4\pi}{4(2\pi)^6} dE_2 \tag{12}$$

$$= (2\pi)^4 \frac{\pi}{(2\pi)^6}\frac{1}{2} \tag{13}$$

$$= \frac{1}{8\pi}, \tag{14}$$

where we have used the delta function to perform the integral over E_2 to get extra factor of $1/2$. Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of M. In this frame, $P = (M, 0, 0, 0)$, and the energies are $p_1^\mu = (\omega, \vec{p}_1)$ and $q^\mu = (M - \omega, \vec{q}) = (E_q, \vec{q})$.

$$\int d\tau_2(M \to p_1 + q) = \int (2\pi)^4 \delta(M - E_1 - E_q)\delta^3(\vec{0} - \vec{p}_1 - \vec{q})\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3q}{(2\pi)^3 2E_q} \tag{15}$$

$$= \int (2\pi)^4 \delta(M - \omega - E_q)\frac{d^3p_1}{(2\pi)^3 2\omega}\frac{1}{(2\pi)^3 2E_q} \tag{16}$$

$$= \int (2\pi)^4 \delta(M - \omega - E_q)\frac{4\pi\omega^2 d\omega}{(2\pi)^3 2\omega}\frac{1}{(2\pi)^3 2E_q} \tag{17}$$

$$= \frac{4\pi}{4(2\pi)^2}\int \delta(M - \omega - E_q)\frac{\omega}{E_q} d\omega \tag{18}$$

$$= \frac{1}{4\pi}\int \delta(M - \omega - E_q)\frac{\omega}{E_q} d\omega. \tag{19}$$

Thus, we have

$$\int d\tau_2(M \to p_1 + q) = \frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega. \tag{20}$$

Now we can combine the results to get the differential decay rate:

$$d\Gamma = \frac{1}{2M} \frac{1}{3!} |A|^2 \left(\frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \right) \tag{21}$$

$$= \frac{|A|^2}{768\pi^3 M} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega dq^2. \tag{22}$$

We have to be careful when we use the delta function to perform the integral over ω since $E_q = \sqrt{(\vec{q})^2 + q^2} = \sqrt{\omega^2 + q^2}$

$$\frac{d}{dq^2}(\omega + E_q) = \frac{d}{dq^2}(\omega + \sqrt{\omega^2 + q^2}) = \frac{1}{2\sqrt{\omega^2 + q^2}} = \frac{1}{2E_q}. \tag{23}$$

Thus, we have

$$d\Gamma = \frac{|A|^2}{768\pi^3 M} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega dq^2 \tag{24}$$

$$= \frac{|A|^2}{768\pi^3 M} \frac{\omega}{E_q} 2E_q d\omega \tag{25}$$

$$= \frac{|A|^2}{384\pi^3 M} \omega d\omega. \tag{26}$$

However, we cannot disthinguish which photon we are measuring, so we have to multiply by a factor of 3. Therefore, the final expression for the differential decay rate is:

$$\frac{d\Gamma}{d\omega} = \frac{|A|^2}{128\pi^3 M} \omega, \quad 0 \leq \omega \leq \frac{M}{2}. \tag{27}$$

To find the total decay rate, we can integrate over the allowed range of ω:

$$\Gamma = \int_0^{M/2} \frac{d\Gamma}{d\omega} d\omega \tag{28}$$

$$= \int_0^{M/2} \frac{|A|^2}{128\pi^3 M} \omega d\omega \tag{29}$$

$$= \frac{|A|^2}{128\pi^3 M} \left[\frac{\omega^2}{2} \right]_0^{M/2} \tag{30}$$

$$= \frac{|A|^2}{128\pi^3 M} \frac{M^2}{8} \tag{31}$$

$$= \frac{|A|^2 M}{1024\pi^3}. \tag{32}$$

□

Question 2

Find the total Lorentz-invariant three-body phase space τ_3 for a final state containing one particle of mass m and two massless particles, produced from an initial particle of mass M. Express your final result in terms of the Mandelstam variable s and m.

Answer

From the definition of the three-body phase space, we have

$$\tau_3 = \int (2\pi)^4 \delta^4(P - p_1 - p_2 - p_3) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3}, \tag{33}$$

where P is the four-momentum of the initial particle, p_1 is the four-momentum of the massive particle with mass m, and p_2 and p_3 are the four-momenta of the two massless particles. We can use the splitting formula for three-body phase space:

$$\tau_3 = \int d\tau_2(M \to p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \to p_2 + p_3), \tag{34}$$

where $q = p_2 + p_3$ is the combined four-momentum of the two massless particles. The two-body phase space elements can be expressed as:

$$d\tau_2(M \to p_1 + q) = (2\pi)^4 \delta^4(P - p_1 - q) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3q}{(2\pi)^3 2E_q}, \tag{35}$$

$$d\tau_2(q \to p_2 + p_3) = (2\pi)^4 \delta^4(q - p_2 - p_3) \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3}. \tag{36}$$

For the second two-body phase space element, we can quote the result from the previous problem, since both particles are massless:

$$\int d\tau_2(q \to p_2 + p_3) = \frac{1}{8\pi}. \tag{37}$$

Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of M. In this frame, $P = (M, 0, 0, 0)$, and the energies are $E_1 = \sqrt{|\vec{p}_1|^2 + m^2}$ and $E_q = M - E_1 = \sqrt{|\vec{q}|^2 + q^2}$.

$$\int d\tau_2(M \to p_1 + q) = \int (2\pi)^4 \delta(M - E_1 - E_q)\delta^3(\vec{0} - \vec{p}_1 - \vec{q}) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3q}{(2\pi)^3 2E_q} \tag{38}$$

$$= \int (2\pi)^4 \delta(M - E_1 - E_q) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_q} \tag{39}$$

$$= \int (2\pi)^4 \delta(M - E_1 - E_q) \frac{4\pi|\vec{p}_1|^2 d|\vec{p}_1|}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_q} \tag{40}$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| \tag{41}$$

$$= \frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1|. \tag{42}$$

Thus, we have

$$\int d\tau_2(M \to p_1 + q) = \frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1|. \tag{43}$$

Now we can combine the results to get the total three-body phase space:

$$\tau_3 = \int d\tau_2(M \to p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \to p_2 + p_3) \tag{44}$$

$$= \int \left(\frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \right) \tag{45}$$

$$= \frac{1}{64\pi^3} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| dq^2. \tag{46}$$

To perform the integral over $|\vec{p}_1|$, we need to express E_q in terms of $|\vec{p}_1|$ and q^2:

$$E_q = \sqrt{|\vec{q}|^2 + q^2} = \sqrt{|\vec{p}_1|^2 + q^2} \tag{47}$$

$$E_1 = \sqrt{|\vec{p}_1|^2 + m^2}. \tag{48}$$

We also need to compute the derivative of $(E_1 + E_q)$ with respect to $|\vec{p}_1|$:

$$\frac{d}{d|\vec{p}_1|}(E_1 + E_q) = \frac{d}{d|\vec{p}_1|} \left(\sqrt{|\vec{p}_1|^2 + m^2} + \sqrt{|\vec{p}_1|^2 + q^2} \right) = \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + m^2}} + \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + q^2}} = \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_1|}{E_q} \tag{49}$$

$$= |\vec{p}_1| \left(\frac{E_1 + E_q}{E_1 E_q} \right) = |\vec{p}_1| \left(\frac{M}{E_1 E_q} \right). \tag{50}$$

Thus, we have

$$\tau_3 = \frac{1}{64\pi^3} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| dq^2 \tag{51}$$

$$= \frac{1}{64\pi^3} \int \frac{|\vec{p}_1|^2}{E_1 E_q} \frac{E_1 E_q}{M|\vec{p}_1|} dq^2 \tag{52}$$

$$= \frac{1}{64\pi^3 M} \int |\vec{p}_1| dq^2. \tag{53}$$

We can apply the relation between q^2 and $|\vec{p}_1|$ to change the integration variable in the rest frame of M, where $P^\mu = (M, 0, 0, 0)$ and $p_1^\mu = (E_1, \vec{p}_1)$, $q^\mu = P^\mu - p_1^\mu$:

$$q^2 = (P - p_1)^2 = M^2 + m^2 - 2ME_1 = M^2 + m^2 - 2M\sqrt{|\vec{p}_1|^2 + m^2}, \tag{54}$$

$$\frac{dq^2}{d|\vec{p}_1|} = -2M \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + m^2}} = -2M \frac{|\vec{p}_1|}{E_1}. \tag{55}$$

By $E_1^2 = |\vec{p}_1|^2 + m^2$, we have $dE_1 = \frac{|\vec{p}_1|}{E_1} d|\vec{p}_1|$. Thus, we have

$$\tau_3 = \frac{1}{64\pi^3 M} \int_{q_{min}}^{q_{max}} |\vec{p}_1| dq^2 \tag{56}$$

$$= \frac{1}{64\pi^3 M} \int_{|\vec{p}_1|_{max}}^{|\vec{p}_1|_{min}} |\vec{p}_1| \left(-2M \frac{|\vec{p}_1|}{E_1} \right) d|\vec{p}_1| \tag{57}$$

$$= \frac{1}{32\pi^3} \int_{|\vec{p}_1|_{min}}^{|\vec{p}_1|_{max}} \frac{|\vec{p}_1|^2}{E_1} d|\vec{p}_1| \tag{58}$$

$$= \frac{1}{32\pi^3} \int_{E_{1,min}}^{E_{1,max}} \sqrt{E_1^2 - m^2} dE_1. \tag{59}$$

The limits of integration for E_1 can be found from the kinematic constraints. The minimum energy occurs when the two massless particles are emitted back-to-back with maximum energy, and the maximum energy occurs when the massive particle is at rest:

$$E_{1,min} = m, \tag{60}$$

$$E_{1,max} = \frac{M^2 + m^2}{2M} = \frac{s + m^2}{2\sqrt{s}}, \text{ from } q_{min}^2 = 0 \Rightarrow M^2 + m^2 - 2ME_{1,max} = 0. \tag{61}$$

Thus, we have (by *Mathematica*)

$$\tau_3 = \frac{1}{32\pi^3} \int_m^{\frac{s+m^2}{2\sqrt{s}}} \sqrt{E_1^2 - m^2} dE_1 \tag{62}$$

$$= \frac{1}{32\pi^3} \frac{-m^4 + 2m^2 s \log\left(\frac{m^2}{s}\right) + s^2}{8s} \tag{63}$$

$$= \frac{-m^4 + 2m^2 s \log\left(\frac{m^2}{s}\right) + s^2}{256\pi^3 s}. \tag{64}$$

□

Question 3

Hadronic Transitions in Quarkonium

(a) The decay amplitude for the transition $\psi(2S) \to J/\psi(1S)\pi^+\pi^-$ can be approximated by

$$M_{fi} = a_\psi \sqrt{4m_{\psi(2S)}m_{J/\psi}}(q^2 - 4.5m_\pi^2), \tag{65}$$

where q is the total four-momentum of the emitted pion pair, and a_ψ is a dimensionful coupling constant. Using the experimental decay rate (performing a numerical phase-space integration if necessary), determine the absolute value $|a_\psi|$ in appropriate units of GeV.

(b) Perform the same analysis for the decay $\Upsilon(2S) \to \Upsilon(1S)\pi^+\pi^-$, for which the phenomenological amplitude is

$$M_{fi} = a_\Upsilon \sqrt{4m_{\Upsilon(2S)}m_{\Upsilon(1S)}}(q^2 - 3.2m_\pi^2). \tag{66}$$

Compare the extracted magnitudes of $|a_\psi|$ and $|a_\Upsilon|$. What can you infer about the relative spatial extent of the charmonium and bottomonium bound states?
Note: The constants a_ψ and a_Υ reflect overlap integrals between the $2S$ and $1S$ quarkonium wavefunctions and scale with the mean-square radius $\langle r^2 \rangle$ of the bound state.

Answer

(a)

The decay rate for the process $\psi(2S) \to J/\psi(1S)\pi^+\pi^-$ can be expressed as:

$$d\Gamma = \frac{1}{2m_{\psi(2S)}}|M_{fi}|^2 d\tau_3, \tag{67}$$

where $d\tau_3$ is the three-body phase space element for the final state particles. The three-body phase space element can be expressed as:

$$d\tau_3 = (2\pi)^4\delta^4(P - p_{J/\psi} - p_{\pi^+} - p_{\pi^-})\frac{d^3p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}}\frac{d^3p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}}\frac{d^3p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}}, \tag{68}$$

where P is the four-momentum of the initial $\psi(2S)$ particle, and $p_{J/\psi}$, p_{π^+}, and p_{π^-} are the four-momenta of the final state particles. In the rest frame of $\psi(2S)$, we have $P = (m_{\psi(2S)}, 0, 0, 0)$. The invariant matrix element is given by:

$$M_{fi} = a_\psi \sqrt{4m_{\psi(2S)}m_{J/\psi}}(q^2 - 4.5m_\pi^2), \tag{69}$$

where $q = p_{\pi^+} + p_{\pi^-}$ is the combined four-momentum of the pion pair. To find the total decay rate, we need to integrate over the three-body phase space:

$$\Gamma = \int d\Gamma = \frac{1}{2m_{\psi(2S)}} |a_\psi|^2 4m_{\psi(2S)} m_{J/\psi} \int (q^2 - 4.5m_\pi^2)^2 d\tau_3. \tag{70}$$

For $d\tau_3$, we can use the splitting formula for three-body phase space:

$$d\tau_3 = d\tau_2(m_{\psi(2S)} \to p_{J/\psi} + q) \frac{dq^2}{2\pi} d\tau_2(q \to p_{\pi^+} + p_{\pi^-}), \tag{71}$$

where $q = p_{\pi^+} + p_{\pi^-}$ is the combined four-momentum of the pion pair. The two-body phase space elements can be expressed as:

$$d\tau_2(m_{\psi(2S)} \to p_{J/\psi} + q) = (2\pi)^4 \delta^4(P - p_{J/\psi} - q) \frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}} \frac{d^3 q}{(2\pi)^3 2E_q}, \tag{72}$$

$$d\tau_2(q \to p_{\pi^+} + p_{\pi^-}) = (2\pi)^4 \delta^4(q - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}}. \tag{73}$$

Now we have to evaluate the two-body phase space elements. For the second two-body phase space element, and we can set $q = (\sqrt{q^2}, 0, 0, 0)$,

$$\int d\tau_2(q \to p_{\pi^+} + p_{\pi^-}) = \int (2\pi)^4 \delta^4(q - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}} \tag{74}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_{\pi^+} - E_{\pi^-}) \delta^3(\vec{0} - \vec{p}_{\pi^+} - \vec{p}_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}} \tag{75}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_{\pi^+} - E_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{1}{(2\pi)^3 2E_{\pi^-}}, \quad \text{where } E_{\pi^+} = E_{\pi^-} \tag{76}$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_{\pi^+}) \frac{4\pi |p|^2 d|p|}{(2\pi)^3 2E_{\pi^+}} \frac{1}{(2\pi)^3 2E_{\pi^+}}, \quad \text{where } |p| = |\vec{p}_{\pi^+}| \tag{77}$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} dp, \quad E = E_{\pi^+} \tag{78}$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} dp \tag{79}$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} \frac{E}{|p|} dE \quad \text{using } dE = \frac{|p|}{E} dp \tag{80}$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{\sqrt{E^2 - m_\pi^2}}{E} dE \tag{81}$$

$$= \frac{1}{4\pi} \frac{\sqrt{\frac{q^2}{4} - m_\pi^2}}{\frac{q^2}{2}} \frac{1}{2} \quad \text{using } E = \frac{\sqrt{q^2}}{2} \tag{82}$$

$$= \frac{1}{8\pi} \sqrt{1 - \frac{4m_\pi^2}{q^2}}. \tag{83}$$

Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of $\psi(2S)$. In this frame, $P = (m_{\psi(2S)}, 0, 0, 0)$, and the energies are $E_{J/\psi} = \sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}$ and $E_q = m_{\psi(2S)} - E_{J/\psi} = \sqrt{|\vec{q}|^2 + q^2}$.

$$\int d\tau_2(m_{\psi(2S)} \to p_{J/\psi} + q) = \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\delta^3(\vec{0} - \vec{p}_{J/\psi} - \vec{q})\frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}}\frac{d^3 q}{(2\pi)^3 2E_q} \tag{84}$$

$$\begin{aligned}
=&\int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}}\frac{1}{(2\pi)^3 2E_q} && (85)\\
=&\int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{4\pi|\vec{p}_{J/\psi}|^2 d|\vec{p}_{J/\psi}|}{(2\pi)^3 2E_{J/\psi}}\frac{1}{(2\pi)^3 2E_q} && (86)\\
=&\frac{4\pi}{4(2\pi)^2}\int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}d|\vec{p}_{J/\psi}| && (87)\\
=&\frac{1}{4\pi}\int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}d|\vec{p}_{J/\psi}|. && (88)
\end{aligned}$$

$E_{J/\psi}$ and E_q can be expressed in terms of $|\vec{p}_{J/\psi}|$ and q^2:

$$\begin{aligned}
E_q &= \sqrt{|\vec{q}|^2 + q^2} = \sqrt{|\vec{p}_{J/\psi}|^2 + q^2}, && (89)\\
E_{J/\psi} &= \sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}. && (90)
\end{aligned}$$

We also need to compute the derivative of $(E_{J/\psi} + E_q)$ with respect to $|\vec{p}_{J/\psi}|$:

$$\begin{aligned}
\frac{d}{d|\vec{p}_{J/\psi}|}(E_{J/\psi} + E_q) =&\frac{d}{d|\vec{p}_{J/\psi}|}\left(\sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2} + \sqrt{|\vec{p}_{J/\psi}|^2 + q^2}\right) && (91)\\
=&\frac{|\vec{p}_{J/\psi}|}{\sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}} + \frac{|\vec{p}_{J/\psi}|}{\sqrt{|\vec{p}_{J/\psi}|^2 + q^2}} = \frac{|\vec{p}_{J/\psi}|}{E_{J/\psi}} + \frac{|\vec{p}_{J/\psi}|}{E_q} && (92)\\
=&|\vec{p}_{J/\psi}|\left(\frac{E_{J/\psi} + E_q}{E_{J/\psi}E_q}\right) = |\vec{p}_{J/\psi}|\left(\frac{m_{\psi(2S)}}{E_{J/\psi}E_q}\right). && (93)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{4\pi}\int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q)\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}d|\vec{p}_{J/\psi}| && (94)\\
=&\frac{1}{4\pi}\frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi}E_q}\frac{E_{J/\psi}E_q}{m_{\psi(2S)}|\vec{p}_{J/\psi}|} && (95)\\
=&\frac{1}{4\pi}\frac{|\vec{p}_{J/\psi}|}{m_{\psi(2S)}}. && (96)
\end{aligned}$$

Now we can combine the results to get the total three-body phase space:

$$d\Gamma = \frac{1}{2m_{\psi(2S)}}|a_\psi|^2 4m_{\psi(2S)}m_{J/\psi}\int\left(\frac{1}{4\pi}\frac{|\vec{p}_{J/\psi}|}{m_{\psi(2S)}}\right)\frac{dq^2}{2\pi}\left(\frac{1}{8\pi}\sqrt{1-\frac{4m_\pi^2}{q^2}}\right)(q^2-4.5m_\pi^2)^2 \tag{97}$$

$$=\frac{|a_\psi|^2 m_{J/\psi}}{32\pi^3 m_{\psi(2S)}}\int|\vec{p}_{J/\psi}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-4.5m_\pi^2)^2 dq^2. \tag{98}$$

We can express $|\vec{p}_{J/\psi}|$ in terms of q^2:

$$m_{J/\psi}^2 = E_{J/\psi}^2 - |\vec{p}_{J/\psi}|^2 = (m_{\psi(2S)} - E_q)^2 - |\vec{p}_{J/\psi}|^2 = (m_{\psi(2S)} - \sqrt{|\vec{p}_{J/\psi}|^2 + q^2})^2 - |\vec{p}_{J/\psi}|^2, \tag{99}$$

which gives (by *Mathematica*):

$$|\vec{p}_{J/\psi}| = \frac{\sqrt{(M-(m-q))(M+(m-q))(M-(m+q))(M+(m+q))}}{2M} \tag{100}$$

$$=\frac{\sqrt{(M^2-(m+q)^2)(M^2-(m-q)^2)}}{2M}, \quad \text{where } M = m_{\psi(2S)}, m = m_{J/\psi}, q = \sqrt{q^2} \tag{101}$$

$$=\frac{\sqrt{(m_{\psi(2S)}^2-(m_{J/\psi}+\sqrt{q^2})^2)(m_{\psi(2S)}^2-(m_{J/\psi}-\sqrt{q^2})^2)}}{2m_{\psi(2S)}}. \tag{102}$$

We can discuss the limits of integration for q^2. $q^2 = (p_{\pi^+}+p_{\pi^-})^2 = (p_{\psi(2S)} - p_{J/\psi})^2$, which is the invariant mass squared of the pion pair. The minimum value of q^2 occurs when the two pions are produced at rest in their center-of-mass frame, which gives:

$$q_{min}^2 = (2m_\pi)^2 = 4m_\pi^2. \tag{103}$$

The maximum value of q^2 occurs when the J/ψ is produced at rest in the $\psi(2S)$ rest frame, which gives:

$$q_{max}^2 = (m_{\psi(2S)} - m_{J/\psi})^2. \tag{104}$$

Thus, we have with ($m_{J/\psi} = 3.096$ GeV, $m_{\psi(2S)} = 3.686$ GeV, $m_\pi = 0.13957$ GeV, and the experimental decay rate $\Gamma_{exp} = 101.64$ keV $= 1.01\times10^{-4}$ GeV):

$$\Gamma = \frac{|a_\psi|^2 m_{J/\psi}}{32\pi^3 m_{\psi(2S)}}\int_{4m_\pi^2}^{(m_{\psi(2S)}-m_{J/\psi})^2}|\vec{p}_{J/\psi}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-4.5m_\pi^2)^2 dq^2 \tag{105}$$

$$=|a_\psi|^2\times 8.82391\times10^{-7}\text{GeV}^5 = 1.01\times10^{-4}\text{ GeV} \tag{106}$$

$$\Rightarrow|a_\psi| = 10.6987\text{ GeV}^{-3}. \tag{107}$$

(b)

We can perform a similar analysis for the decay $\Upsilon(2S)\to\Upsilon(1S)\pi^+\pi^-$. The decay rate can be expressed as:

$$\Gamma = \frac{1}{2m_{\Upsilon(2S)}}|a_\Upsilon|^2 4m_{\Upsilon(2S)}m_{\Upsilon(1S)} \int \left(\frac{1}{4\pi}\frac{|\vec{p}_{\Upsilon(1S)}|}{m_{\Upsilon(2S)}}\right)\frac{dq^2}{2\pi}\left(\frac{1}{8\pi}\sqrt{1-\frac{4m_\pi^2}{q^2}}\right)(q^2-3.2m_\pi^2)^2 \tag{108}$$

$$=\frac{|a_\Upsilon|^2 m_{\Upsilon(1S)}}{32\pi^3 m_{\Upsilon(2S)}}\int |\vec{p}_{\Upsilon(1S)}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-3.2m_\pi^2)^2 dq^2. \tag{109}$$

We can express $|\vec{p}_{\Upsilon(1S)}|$ in terms of q^2:

$$|\vec{p}_{\Upsilon(1S)}| = \frac{\sqrt{(m_{\Upsilon(2S)}^2-(m_{\Upsilon(1S)}+\sqrt{q^2})^2)(m_{\Upsilon(2S)}^2-(m_{\Upsilon(1S)}-\sqrt{q^2})^2}}{2m_{\Upsilon(2S)}}. \tag{110}$$

The limits of integration for q^2 are:

$$q_{min}^2 = 4m_\pi^2, \tag{111}$$

$$q_{max}^2 = (m_{\Upsilon(2S)}-m_{\Upsilon(1S)})^2. \tag{112}$$

With ($m_{\Upsilon(1S)} = 9.460$ GeV, $m_{\Upsilon(2S)} = 10.023$ GeV, $m_\pi = 0.13957$ GeV, and the experimental decay rate $\Gamma_{exp} = 5.71$ keV $= 5.71 \times 10^{-6}$ GeV):

$$\Gamma = \frac{|a_\Upsilon|^2 m_{\Upsilon(1S)}}{32\pi^3 m_{\Upsilon(2S)}}\int_{4m_\pi^2}^{(m_{\Upsilon(2S)}-m_{\Upsilon(1S)})^2} |\vec{p}_{\Upsilon(1S)}|\sqrt{1-\frac{4m_\pi^2}{q^2}}(q^2-3.2m_\pi^2)^2 dq^2 \tag{113}$$

$$=|a_\Upsilon|^2 \times 9.36631 \times 10^-7\text{GeV}^5 = 5.71 \times 10^{-6}\ \text{GeV} \tag{114}$$

$$\Rightarrow |a_\Upsilon| = 2.46907\ \text{GeV}^{-3}. \tag{115}$$

Comparing the extracted magnitudes of $|a_\psi|$ and $|a_\Upsilon|$, we find that $|a_\psi|$ is significantly larger than $|a_\Upsilon|$. Since the constants a_ψ and a_Υ reflect overlap integrals between the $2S$ and $1S$ quarkonium wavefunctions and scale with the mean-square radius $\langle r^2\rangle$ of the bound state, we can infer that the charmonium bound state (associated with a_ψ) has a larger spatial extent compared to the bottomonium bound state (associated with a_Υ). This suggests that charmonium states are more loosely bound and have a larger size than bottomonium states, which are more tightly bound and compact. □