University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

Lecture Instructor: Professor Zhen Liu

Zong-En Chen
chen9613@umn.edu

December 8, 2025

Final Due to December 16 11:59 PM

Question 1

Problem 48.5

The charged pion π^- is represented by a complex scalar field φ, the *muon* μ^- by a Dirac field $\mathcal{M}$, and the *muon neutrino* ν_μ by a spin-projected Dirac field $P_L\mathcal{N}$, where $P_L = \frac{1}{2}(1-\gamma_5)$. The charged pion can decay to a muon and a muon antineutrino via the interaction

$$\mathcal{L}_1 = c_1 G_F f_\pi \partial_\mu \varphi \bar{\mathcal{M}} \gamma^\mu P_L \mathcal{N} + h.c, \tag{1}$$

where c_1 is the consine of the *Cabibbo angle*, G_F is the *Fermi constant*, and f_π is the *pion decay constant*.

(a) Compute the charged pion decay rate Γ.

(b) The charged pion mass is $m_\pi = 139.6$ MeV, the muon mass is $m_\mu = 105.7$ MeV, and the muon neutrino mass is massless. The Fermi constant is $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, and the cosine of the Cabibbo angle is measured in nuclear beta decays to be $c_1 = 0.974$. The measured value of the charged pion life time is $\tau = 2.6033 \times 10^{-8}$ s. Determine the value of f_π in MeV. Your result is too large by 0.8%, due to neglect of electromagnetic loop corrections.

(c) The previous parts assume π^- always decay into $\mu^-\bar{\nu}_\mu$, but actually π^- can also decay into $e^-\bar{\nu}_e$. The charged pion, electron by a Dirac field $\mathcal{M}_e$, and the electron neutrino by a spin-projected Dirac field $P_L\mathcal{N}_e$ have the form of interaction

$$\mathcal{L}_2 = c_2 G_F f_\pi \partial_\mu \varphi \bar{\mathcal{M}}_e \gamma^\mu P_L \mathcal{N}_e + h.c. \tag{2}$$

Given the decay branching ratio of $\pi^- \to e^-\bar{\nu}_e$ is 1.230×10^{-4}, the decay branching ratio of $\pi^- \to \mu^-\bar{\nu}_\mu$ is 99.9877%. Find the value of c_2. For example, the electronic decay branching ratio is

$$\mathrm{Br}(\pi^- \to e^-\bar{\nu}_e) = \frac{\Gamma(\pi^- \to e^-\bar{\nu}_e)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu) + \Gamma(\pi^- \to e^-\bar{\nu}_e)}. \tag{3}$$

The coupling of pion-electron is similar with the coupling of pion-muon, why pion favoring decay into muon instead of electron? ($m_e = 0.511$ MeV.)

Answer

Question 2

Consider QED with both electron and muon:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{l=e,\mu}(i\bar{\Psi}_l\partial\!\!\!/\Psi_l - m_l\bar{\Psi}_l\Psi_l + \frac{g}{2}\bar{\Psi}_l\gamma^\mu\Psi_l A_\mu), \tag{4}$$

where both Ψ_e and Ψ_μ are Dirac fields. Compute the $\langle|\mathcal{T}^2|\rangle$ for $e^+e^- \to \mu^+\mu^-$. Then, compute its cross section σ. Eq. (11.22) and Eq. (11.30) should be useful.

Answer

Question 3

Consider classical field theory with two real scalar fields in (3+1)-dimension spacetime:

$$\mathcal{L}(x) = \sum_{a=1}^{2}\Big(-\frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a\Big) - V(x), \tag{5}$$

$$V(x) = -\sum_{a=1}^{2}\Big(\frac{1}{2}\mu^2\phi_a\phi_a\Big) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2, \tag{6}$$

where μ and λ are positive real constants.

(a) Show that the Lagrangian has an $SO(2)$ transformation symmetry:

$$\phi_1(x) \to \phi_1'(x) = \phi_1(x)\cos\alpha_0 - \phi_2(x)\sin\alpha_0, \tag{7}$$

$$\phi_2(x) \to \phi_2'(x) = \phi_1(x)\sin\alpha_0 + \phi_2(x)\cos\alpha_0, \tag{8}$$

(b) Find the conjugate momentum $\Pi_1(x)$, $\Pi_2(x)$ of $\phi_1(x)$, $\phi_2(x)$. Find the Hamiltonian density $\mathcal{H}(x)$ in the terms of $\phi_a(x)$, $\Pi_a(x)$, and $\partial_i\phi_a(x)$.

(c) Find the ground state in the basis of $\{\phi_r(x), \phi_\theta(x)\}$ where

$$\phi_1(x) = \phi_r(x)\cos(\phi_\theta(x)), \tag{9}$$

$$\phi_2(x) = \phi_r(x)\sin(\phi_\theta(x)), \tag{10}$$

with $\phi_r(x) \geq 0$ and $\phi_\theta(x) \in [0, 2\pi)$. Is the Lagrangian $\mathcal{L}$ invariant under a continuous shift symmetry of $\phi_\theta(x) \to \phi_\theta(x) + \alpha_0$?

Hint: In general, finding the ground state is to find $\phi(x)$ s.t. minimize $H = \int \mathcal{H}(x)d^3x$; but for this problem, finding $\phi(x)$ to minimize $\mathcal{H}(x)$ is the same. If you have trouble with the above procedure, given the Lagrangian of this problem, one can simply find $\phi(x)$ s.t. minimize $V(x)$, which is the same as minimizing $\mathcal{H}$ for this problem.

(d) Now let's study the system's dynamics around the ground state.
$\phi_r(x)$ should fluctuate around $\sqrt{\frac{\mu^2}{\lambda}}$: $\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x)$. $\phi_\theta(x)$ should fluctuate within $[0, 2\pi)$.
Show that $f_r(x)$ is a massive field and find its mass. Taking $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}}\phi_\theta(x)$ as the other scalar field, does $f_\theta(x)$ have a mass? Does $\mathcal{L}$ have a continuous shift symmetry of $f_\theta(x) \to f_\theta(x) + \Lambda_0$?

Remark: This problem paves the road for your understanding of spontaneous symmetry breaking. We also see again that the symmetry groups of $SO(2)$ and $U(1)$ are isomorphic.

Remark: More to think about after solving the problems above: Note that we reparametrized the field into a non-linear realization, where you see the $U(1)$ symmetry explicitly. How do you interpret the kinetic term? How do you interpret the $f_r(x)$ field-dependent kinetic terms for $f_\theta(x)$? Is it canonically normalized? How does the field $f_\theta(x)$ relate to the original $SO(2)$ field $\phi_a(x)$? And again, is the ratio of the field a linear redefinition of the field configuration? It is a non-linear realization because all powers of $f_\theta(x)/\sqrt{\frac{\mu^2}{\lambda}}$ need to enter. There is only a region of validity, that is $f_r(x) \ll \sqrt{\frac{\mu^2}{\lambda}}$

Answer

Question 4

Problem 66.3

Use the result of problem 66.2 to compute the anomalous dimension of m and the beta function for e in spionor electrodynamics in R_ξ gauge. You should find that the results are independent of ξ.

Remark:

$$\widetilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} + (\xi - 1)k^\mu k^\nu / k^2}{k^2 - i\epsilon} \tag{11}$$

The book only choose the Feynman gauge ($\xi = 1$) to show the loop calculation and get $Z_{1,2,3,m}$. For arbitrary gauge choice ξ, we can repeat the calculation and get:

$$\begin{aligned}
Z_3 =& 1 - \frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4), \quad \text{derived from photon propagator loop correction} && (12)\\
Z_2 =& 1 - \xi\frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4), \quad \text{derived from fermion propagator loop correction} && (13)\\
Z_m =& 1 - (3+\xi)\frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4), \quad \text{derived from fermion mass loop correction} && (14)\\
Z_1 =& 1 - \xi\frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4), \quad \text{derived from vertex loop correction} && (15)
\end{aligned}$$

Use the above to finish this problem.

Answer

Question 5

Consider the following theory:

$$\mathcal{L} = \mathcal{L}^0_\phi + \mathcal{L}^0_\Psi + \mathcal{L}^0_A + \mathcal{L}_I \tag{16}$$

$$= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\Psi}(i\not{D} - m_\Psi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + y\phi\bar{\Psi}\Psi. \tag{17}$$

The Dirac field Ψ is charged under a $U(1)$ gauge symmetry with a charge Q, and the gauge interaction strength is e. The $U(1)$ gauge field is A_μ, whose kinetic term is $\mathcal{L}^0_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. (This is part of the real-world calculation for the discovery mode for the Higgs boson, which gone through heroic phenomenological studies on predicting the Higgs properties.)

(a) Draw the leading diagrams that enable $\phi \to \gamma\gamma$ decay. (The gauge field A_μ is identified as the photon field γ.)

(b) In the ϕ rest frame, write down the amplitude in the general d dimension. No need to carry out the loop intergral at this point, but need to simplify the trace. (Notice that $k_\mu\epsilon^\mu(k) = 0$ in Lorenz gauge.)

(c) Does the integral have a UV divergence in $d = 4$ dimension (loop momentum goes to ∞)? Answer Yes or No with a few lines of argument.

(d) Does the integral have a singularity in $d = 4$ dimension when the Euclidean loop momentum squared $\overline{q}^2$ go to $-D$? Answer Yes or No with a few lines of argument. (For simplicity, assume that D is real and can be zero for some configuration of x_1, x_2, x_3.)

(e) For $m_\Phi = 0$, calculate using dimensional regularization in $d = 4 - \epsilon$. Write down your final answer in the simplest form. (The final answer would be short.)

(f) Carry out the full calculation of the amplitude in Part b using dimensional regularization in $d = 4 - \epsilon$. Write down your final answer in the simplest form. (The full answer would be a long calculateion.)
Hint: The following few equations, identities, and tricks, and the discussion around them might be helpful for you: Eq. (62.18), Eq. (47.18), Eq. (67.2).
Remark: No need to answer this, but one can think about it for fun. Recall that taking $\epsilon \to 0$ (from plus or minus direction?) get you back to d = 4. In such a limit, contrast your result in Part f and Part c and think about why.

Answer