University of Minnesota
School of Physics and Astronomy

# 2026 Spring Physics 8902
# Elementary Particle Physics II

Assignment Solution

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# Problem Set 2 Due 11am, Monday, February 16

## Question 1

### Weak decay of pions

(a) Find the electron energy spectrum $d\Gamma/dE_e$ for the decay $\pi^- \to \pi^0 + e^- + \overline{\nu}_e$ in the $\pi^-$ rest frame keeping $m_e \neq 0$ (take $m_\nu = 0$). Assume the hadronic current is dominated by $f_+(0)$ and neglect radiative corrections. Perform the phase-space integration by integrating over the $\pi^0$ and $\overline{\nu}_e$ momenta (i.e. treat $E_e$ as the only observed variable). Give the kinematic endpoints and verify the $m_e \to 0$ limit.

(b) Using the electron energy spectrum obtained in part (a), integrate over $E_e$ to extract the leading correction of order $m_e^2/\Delta^2$ to the total decay rate. Write the result in the form

$$\Gamma(\pi^- \to \pi^0 + e^- + \overline{\nu}_e) = |V_{ud}|^2 \frac{G_F^2 \Delta^5}{30\pi^3}\left(1 - a\frac{\Delta}{m_\pi} - b\frac{m_e^2}{\Delta^2}\right), \tag{1}$$

where $\Delta = m_{\pi^-} - m_{\pi^0}$, and neglecting higher-order terms in $\Delta/m_\pi$ and $m_e^2/\Delta^2$. In the lectures it was shown that $a = 3/2$. Determine the coefficient b.

## Answer

(a)

Let us denote the momenta of $\pi^-$, $\pi^0$, $e^-$, and $\overline{\nu}_e$ as $p$, $p'$, $k$, and $k'$, respectively. The decay amplitude can be written as

$$\mathcal{M} = \langle \pi^0(p') e^-(k) \overline{\nu}_e(k') | \mathcal{H}_W | \pi^-(p) \rangle = -\frac{G_F}{\sqrt{2}} V_{ud} \langle \pi^0(p') | \overline{d}\gamma^\mu u | \pi^-(p) \rangle \langle e^-(k) \overline{\nu}_e(k') | \overline{e}\gamma_\mu(1-\gamma_5)\nu_e | 0 \rangle. \tag{2}$$

The hadronic matrix element can be parameterized as

$$\langle \pi^0(p') | \overline{d}\gamma^\mu u | \pi^-(p) \rangle = f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu \approx f_+(0)(p+p')^\mu = \sqrt{2}(p+p')^\mu, \tag{3}$$

where $q = p - p'$. Neglecting radiative corrections and using the fact that $f_+(0)$ dominates, we can approximate $f_+(q^2) \approx f_+(0) = \sqrt{2}$. The leptonic matrix element can be evaluated using standard techniques, yielding

$$\langle e^-(k) \overline{\nu}_e(k') | \overline{e}\gamma_\mu(1-\gamma_5)\nu_e | 0 \rangle = \overline{u}(k)\gamma_\mu(1-\gamma_5)v(k'). \tag{4}$$

Now the amplitude can be expressed as

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{ud}\sqrt{2}(p+p')^{\mu}\overline{u}(k)\gamma_{\mu}(1-\gamma_5)v(k') \tag{5}$$

$$= -\,G_F V_{ud}(p+p')^{\mu}\overline{u}(k)\gamma_{\mu}(1-\gamma_5)v(k'). \tag{6}$$

Then the squared amplitude, summed over final spins, is given by

$$\langle|\mathcal{M}|^2\rangle = \sum_{\text{spins}} |\mathcal{M}|^2 = G_F^2|V_{ud}|^2(p+p')^{\mu}(p+p')^{\nu}\overline{u}_{s_1}(k)\gamma_{\mu}(1-\gamma_5)v_{s_2}(k')\overline{v}_{s_2}(k')\gamma_{\nu}(1-\gamma_5)u_{s_1}(k) \tag{7}$$

$$= G_F^2|V_{ud}|^2(p+p')^{\mu}(p+p')^{\nu}\mathrm{Tr}\Big[(\not{k}+m_e)\gamma_{\mu}(1-\gamma_5)\not{k}'\gamma_{\nu}(1-\gamma_5)\Big] \tag{8}$$

$$= G_F^2|V_{ud}|^2(p+p')^{\mu}(p+p')^{\nu}k^{\alpha}k'^{\beta}\mathrm{Tr}\Big[(\gamma_{\alpha}+1m_e)\gamma_{\mu}(1-\gamma_5)\gamma_{\beta}\gamma_{\nu}(1-\gamma_5)\Big]. \tag{9}$$

where we have used the spin sum identities for the electron and neutrino:

$$\sum_{s_1} u_{s_1}(k)\overline{u}_{s_1}(k) = \not{k}+m_e, \quad \sum_{s_2} v_{s_2}(k')\overline{v}_{s_2}(k') = \not{k}'-m_{\nu} \approx \not{k}'. \tag{10}$$

We provide the full set of trace identities:

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}, \quad \mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma_5) = 0, \quad \mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \tag{11}$$

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_5) = 4i\epsilon^{\mu\nu\rho\sigma}, \quad \mathrm{Tr}(\text{odd number of }\gamma_5) = 0. \tag{12}$$

Hence,

$$\mathrm{Tr}\Big[(\gamma_{\alpha}+1m_e)\gamma_{\mu}(1-\gamma_5)\gamma_{\beta}\gamma_{\nu}(1-\gamma_5)\Big] \tag{13}$$

$$= \mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}(1-\gamma_5)\gamma_{\beta}\gamma_{\nu}(1-\gamma_5)\Big] + m_e\mathrm{Tr}\Big[\gamma_{\mu}(1-\gamma_5)\gamma_{\beta}\gamma_{\nu}(1-\gamma_5)\Big] \tag{14}$$

$$= \mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\Big] + \mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}\gamma_5\gamma_{\beta}\gamma_{\nu}\gamma_5\Big] - \mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}\gamma_5\gamma_{\beta}\gamma_{\nu}\Big] - \mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\gamma_5\Big] \tag{15}$$

$$+ m_e\mathrm{Tr}\Big[\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\Big] - m_e\mathrm{Tr}\Big[\gamma_{\mu}\gamma_5\gamma_{\beta}\gamma_{\nu}\Big] - m_e\mathrm{Tr}\Big[\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\gamma_5\Big] + m_e\mathrm{Tr}\Big[\gamma_{\mu}\gamma_5\gamma_{\beta}\gamma_{\nu}\gamma_5\Big] \tag{16}$$

$$= 2\mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\Big] - 2\mathrm{Tr}\Big[\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\gamma_{\nu}\gamma_5\Big] \tag{17}$$

$$= 8(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta}) - 8i\epsilon_{\alpha\mu\beta\nu}. \tag{18}$$

Actually, the term $\epsilon_{\alpha\mu\beta\nu}$ will vanish later when contracted with $(p+p')^{\mu}(p+p')^{\nu}k^{\alpha}k'^{\beta}$, since $p+p'$ is symmetric in $\mu$ and $\nu$, while $\epsilon_{\alpha\mu\beta\nu}$ is antisymmetric in $\mu$ and $\nu$. Therefore, we can ignore the second term and write

$$\langle|\mathcal{M}|^2\rangle = G_F^2|V_{ud}|^2(p+p')^\mu(p+p')^\nu k^\alpha k'^\beta 8(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta}) \tag{19}$$

$$=8G_F^2|V_{ud}|^2(p+p')^\mu(p+p')^\nu\Big(k_\mu k'_\nu - (k\cdot k')g_{\mu\nu} + k_\nu k'_\mu\Big) \tag{20}$$

$$=8G_F^2|V_{ud}|^2\Bigg[2(p+p')\cdot k(p+p')\cdot k' - (p+p')^2(k\cdot k')\Bigg] \tag{21}$$

We can assume the 4-momenta in the $\pi^-$ rest frame as

$$p = (m_{\pi^-}, \mathbf{0}), \quad p' = (E_{\pi^0}, -(\mathbf{k}+\mathbf{k}') = \mathbf{p}'), \quad k = (E_e, \mathbf{k}), \quad k' = (E_\nu, \mathbf{k}'), \tag{22}$$

where $E_{\pi^0} = \sqrt{|\mathbf{p}'|^2 + m_{\pi^0}^2}$, $E_e = \sqrt{|\mathbf{k}|^2 + m_e^2}$, and $E_\nu = |\mathbf{k}'|$. We can define $q = p - p' = k + k'$ and have following relations:

$$(p+p')\cdot k = (2p - (k+k'))\cdot k = 2p\cdot k - k^2 - (k\cdot k') = 2m_{\pi^-}E_e - m_e^2 - (k\cdot k'), \tag{23}$$

$$(p+p')\cdot k' = (2p - (k+k'))\cdot k' = 2p\cdot k' - k'^2 - (k\cdot k') = 2m_{\pi^-}E_\nu - (k\cdot k'), \tag{24}$$

$$(p+p')^2 = (2p - (k+k'))^2 = 4m_{\pi^-}^2 - 4m_{\pi^-}(E_e + E_\nu) + (k+k')^2 \tag{25}$$

$$=4m_{\pi^-}^2 - 4m_{\pi^-}(E_e + E_\nu) + m_e^2 + 2(k\cdot k'). \tag{26}$$

Since the $m_{\pi^-}$ is much larger than the energy of the final state particles (except for the $\pi^0$), we can further approximate $E_e + E_\nu \approx m_{\pi^-} - m_{\pi^0} = \Delta$. Hence, we have

$$2[(p+p')\cdot k][(p+p')\cdot k'] = 2(2m_{\pi^-}E_e - m_e^2 - (k\cdot k'))(2m_{\pi^-}E_\nu - (k\cdot k')) \tag{27}$$

$$=8m_{\pi^-}^2E_eE_\nu - 4m_{\pi^-}(E_e+E_\nu)(k\cdot k') - 4m_e^2m_{\pi^-}E_\nu + 2m_e^2(k\cdot k') + 2(k\cdot k')^2 \tag{28}$$

$$=8m_{\pi^-}^2E_eE_\nu - 4m_{\pi^-}\Delta(k\cdot k') - 4m_e^2m_{\pi^-}E_\nu + 2m_e^2(k\cdot k') + 2(k\cdot k')^2, \tag{29}$$

$$(p+p')^2(k\cdot k') = (4m_{\pi^-}^2 - 4m_{\pi^-}\Delta + m_e^2 + 2(k\cdot k'))(k\cdot k') \tag{30}$$

$$=4m_{\pi^-}^2(k\cdot k') - 4m_{\pi^-}\Delta(k\cdot k') + m_e^2(k\cdot k') + 2(k\cdot k')^2. \tag{31}$$

Putting everything together, we can express the squared amplitude as

$$\langle|\mathcal{M}|^2\rangle = 8G_F^2|V_{ud}|^2\Bigg[8m_{\pi^-}^2E_eE_\nu - 4m_{\pi^-}\Delta(k\cdot k') - 4m_e^2m_{\pi^-}E_\nu + 2m_e^2(k\cdot k') + 2(k\cdot k')^2 \tag{32}$$

$$- 4m_{\pi^-}^2(k\cdot k') + 4m_{\pi^-}\Delta(k\cdot k') - m_e^2(k\cdot k') - 2(k\cdot k')^2\Bigg] \tag{33}$$

$$=8G_F^2|V_{ud}|^2\Bigg[8m_{\pi^-}^2E_eE_\nu - 4m_{\pi^-}^2(k\cdot k') - 4m_e^2m_{\pi^-}E_\nu + m_e^2(k\cdot k')\Bigg]. \tag{34}$$

Now, we can express $k\cdot k'$ in terms of the energies and the angle between the electron and neutrino momenta. Let $\theta$ be the angle between $\mathbf{k}$ and $\mathbf{k}'$, then we have:

$$k \cdot k' = E_e E_\nu - \mathbf{k} \cdot \mathbf{k}' = E_e E_\nu - |\mathbf{k}||\mathbf{k}'| \cos\theta = E_e E_\nu - |\mathbf{k}||\mathbf{k}'| \cos\theta \tag{35}$$

$$= E_e E_\nu (1 - \beta_e \cos\theta), \quad \text{where } \beta_e = \frac{|\mathbf{k}|}{E_e} = \sqrt{1 - \frac{m_e^2}{E_e^2}}. \tag{36}$$

The squared amplitude can be further simplified as

$$\langle |\mathcal{M}|^2 \rangle = 8G_F^2 |V_{ud}|^2 \left[ 8m_{\pi^-}^2 E_e E_\nu - 4m_{\pi^-}^2 E_e E_\nu (1 - \beta_e \cos\theta) - 4m_e^2 m_{\pi^-} E_\nu + m_e^2 E_e E_\nu (1 - \beta_e \cos\theta) \right] \tag{37}$$

$$= 8G_F^2 |V_{ud}|^2 \left[ 4m_{\pi^-}^2 E_e E_\nu (1 + \beta_e \cos\theta) - 4m_e^2 m_{\pi^-} E_\nu + m_e^2 E_e E_\nu (1 - \beta_e \cos\theta) \right] \tag{38}$$

$$= 32G_F^2 |V_{ud}|^2 m_{\pi^-}^2 \left[ E_e E_\nu (1 + \beta_e \cos\theta) - \frac{m_e^2}{m_{\pi^-}} E_\nu + \frac{m_e^2}{4m_{\pi^-}^2} E_e E_\nu (1 - \beta_e \cos\theta) \right]. \tag{39}$$

Besides,

$$q^2 = (k + k')^2 = m_e^2 + 2k \cdot k' = m_e^2 + 2E_e E_\nu (1 - \beta_e \cos\theta). \tag{40}$$

$$\Longrightarrow E_e E_\nu (1 - \beta_e \cos\theta) = \frac{q^2 - m_e^2}{2}. \tag{41}$$

$$\Longrightarrow E_e E_\nu (1 + \beta_e \cos\theta) = 2E_e E_\nu - E_e E_\nu (1 - \beta_e \cos\theta) = 2E_e E_\nu - \frac{q^2 - m_e^2}{2}. \tag{42}$$

Therefore, the squared amplitude can be expressed as

$$\langle |\mathcal{M}|^2 \rangle = 32G_F^2 |V_{ud}|^2 m_{\pi^-}^2 \left[ 2E_e E_\nu - \frac{q^2 - m_e^2}{2} - \frac{m_e^2}{m_{\pi^-}} E_\nu + \frac{m_e^2}{4m_{\pi^-}^2} \frac{q^2 - m_e^2}{2} \right] \tag{43}$$

The decay rate can be calculated using the standard formula for three-body decays:

$$d\Gamma = \frac{1}{2m_{\pi^-}} \langle |\mathcal{M}|^2 \rangle d\tau_3, \tag{44}$$

where $d\tau_3$ is the three-body phase space element, given by splitting formulas:

$$d\tau_3 = d\tau_2(p \to p' + q) \frac{dq^2}{2\pi} d\tau_2(q \to k + k'), \tag{45}$$

where $q = k + k'$ is the total momentum of the lepton pair. The two-body phase space elements can be expressed as

$$d\tau_2(p \to p' + q) = (2\pi)^4 \delta^4(p - p' - q) \frac{d^3p'}{(2\pi)^3 2E_{p'}} \frac{d^3q}{(2\pi)^3 2E_q}, \tag{46}$$

$$d\tau_2(q \to k + k') = (2\pi)^4 \delta^4(q - k - k') \frac{d^3k}{(2\pi)^3 2E_k} \frac{d^3k'}{(2\pi)^3 2E_{k'}}. \tag{47}$$

Now, let's try to integrate over the $\pi^0$ and $\overline{\nu}_e$ momenta. Since there is no $p'$ in the squared amplitude, we can first integrate over $p'$ using the delta function in $d\tau_2(p \to p' + q)$, which gives

$$\int d\tau_2(p \to p' + q) = \int (2\pi)^4 \delta^4(p - p' - q) \frac{d^3p'}{(2\pi)^3 2E_{p'}} \frac{d^3q}{(2\pi)^3 2E_q} \tag{48}$$

$$= \int (2\pi)^4 \delta(m_{\pi^-} - E_{p'} - E_q) \frac{1}{(2\pi)^3 2E_{p'}} \frac{d^3q}{(2\pi)^3 2E_q} \tag{49}$$

$$= \int (2\pi)^4 \delta(m_{\pi^-} - E_{p'} - E_q) \frac{1}{(2\pi)^3 2E_{p'}} \frac{4\pi|\mathbf{q}|^2 d|\mathbf{q}|}{(2\pi)^3 2E_q} \tag{50}$$

$$= \frac{1}{4\pi} \int \delta(m_{\pi^-} - E_{p'} - E_q) \frac{|\mathbf{q}|^2 d|\mathbf{q}|}{E_{p'} E_q} \tag{51}$$

$$= \frac{1}{4\pi} \frac{|\mathbf{q}|}{m_{\pi^-}} \quad \text{with } |\mathbf{q}| = \frac{\sqrt{(m_{\pi^-}^2 - (m_{\pi^0} + \sqrt{q^2})^2)(m_{\pi^-}^2 - (m_{\pi^0} - \sqrt{q^2})^2)}}{2m_{\pi^-}}. \tag{52}$$

We can further simplify the expression for $|\mathbf{q}|$ by noting that $\Delta = m_{\pi^-} - m_{\pi^0}$ is small compared to $m_{\pi^-}$, and $m_{\pi^0} \approx m_{\pi^-}$, hence we can approximate $|\mathbf{q}|$ as

$$|\mathbf{q}| = \frac{\underbrace{\sqrt{m_{\pi^-} + m_{\pi^0} - \sqrt{q^2}}}_{\approx\sqrt{2m_{\pi^-}}} \underbrace{\sqrt{m_{\pi^-} - m_{\pi^0} - \sqrt{q^2}}}_{=\sqrt{\Delta - \sqrt{q^2}}} \underbrace{\sqrt{m_{\pi^-} + m_{\pi^0} + \sqrt{q^2}}}_{\approx\sqrt{2m_{\pi^-}}} \underbrace{\sqrt{m_{\pi^-} - m_{\pi^0} + \sqrt{q^2}}}_{=\sqrt{\Delta + \sqrt{q^2}}}}{2m_{\pi^-}} \approx \sqrt{\Delta^2 - q^2}. \tag{53}$$

Now, we can express the three-body phase space element as

$$d\tau_3 = d\tau_2(p \to p' + q) \frac{dq^2}{2\pi} d\tau_2(q \to k + k') \tag{54}$$

$$= \frac{1}{4\pi} \frac{\sqrt{\Delta^2 - q^2}}{m_{\pi^-}} \frac{dq^2}{2\pi} d\tau_2(q \to k + k'). \tag{55}$$

Next, we can integrate over the neutrino momentum $k'$ using the delta function in $d\tau_2(q \to k + k')$, and we consider the $q$ rest frame to perform the integration over the electron momentum $k$. In the $q$ rest frame, we have (denoted by a star):

$$q = (q^0, \mathbf{0}), \quad k = (E_e^*, \mathbf{k}^*), \quad k' = (E_\nu^*, -\mathbf{k}^*). \tag{56}$$

The energies of the electron and neutrino in the $q$ rest frame can be expressed as

$$\int d\tau_2(q \to k + k') = \int (2\pi)^4 \delta^4(q - k - k') \frac{d^3k}{(2\pi)^3 2E_k} \frac{d^3k'}{(2\pi)^3 2E_{k'}} \tag{57}$$

$$= \int (2\pi)^4 \delta(q^0 - E_e^* - E_\nu^*) \frac{d^3k^*}{(2\pi)^3 2E_e^*} \frac{1}{(2\pi)^3 2E_\nu^*} \tag{58}$$

$$= \frac{1}{16\pi^2} \int \delta(q^0 - E_e^* - E_\nu^*) \frac{|k^*|^2 d|k^*| d\Omega^*}{E_e^* E_\nu^*}, \quad \text{where } E_e^* = \sqrt{|k^*|^2 + m_e^2}, \quad E_\nu^* = |k^*| \tag{59}$$

$$= \frac{1}{16\pi^2} \int \delta(q^0 - E_e^* - E_\nu^*) \frac{k^* dk^* d\Omega^*}{E_e^*} \tag{60}$$

$$= \frac{1}{16\pi^2} \int \delta(q^0 - \sqrt{k^{*2} + m_e^2} - k^*) \frac{k^* dk^* d\Omega^*}{E_e^*} \tag{61}$$

$$= \frac{1}{16\pi^2} \frac{k^*}{q^0} \int d\Omega^* \quad \text{with } k^* = \frac{q^2 - m_e^2}{2\sqrt{q^2}}, q^0 = \sqrt{q^2} \tag{62}$$

$$= \frac{1}{16\pi^2} \frac{k^*}{\sqrt{q^2}} \int d\Omega^* \tag{63}$$

Hence, we have

$$d\tau_2(q \to k + k') = \frac{1}{16\pi^2} \frac{q^2 - m_e^2}{2q^2} d\Omega^* = \frac{1}{16\pi^2} \frac{k^*}{\sqrt{q^2}} \int d\Omega^*. \tag{64}$$

By Lorentz transformation, we have

$$E_e = \gamma(E_e^* + \beta k^* \cos\theta^*), \tag{65}$$

$$\implies dE_e = \gamma\beta k^* d\cos\theta^* \tag{66}$$

$$\implies d\Omega^* = 2\pi d\cos\theta^* = \frac{2\pi}{\gamma\beta k^*} dE_e \tag{67}$$

$$\implies d\Omega^* = 2\pi dE_e \frac{1}{\frac{q^0}{\sqrt{q^2}} \frac{|\mathbf{q}|}{q^0} k^*}, \quad \text{where } \gamma = \frac{q^0}{\sqrt{q^2}}, \quad \beta = \frac{|\mathbf{q}|}{q^0} \tag{68}$$

$$\implies d\Omega^* = 2\pi dE_e \frac{\sqrt{q^2}}{|\mathbf{q}| k^*} \approx 2\pi dE_e \frac{\sqrt{q^2}}{\sqrt{\Delta^2 - q^2} k^*} \tag{69}$$

$$\implies d\tau_2(q \to k + k') = \frac{1}{16\pi^2} \frac{k^*}{\sqrt{q^2}} 2\pi dE_e \frac{\sqrt{q^2}}{\sqrt{\Delta^2 - q^2} k^*} \tag{70}$$

$$= \frac{1}{8\pi} \frac{1}{\sqrt{\Delta^2 - q^2}} dE_e. \tag{71}$$

Hence, the three-body phase space element and squared amplitude can be expressed as

$$d\tau_3 = d\tau_2(p \to p' + q)\frac{dq^2}{2\pi}d\tau_2(q \to k + k') = \frac{1}{4\pi}\frac{\sqrt{\Delta^2 - q^2}}{m_{\pi^-}}\frac{dq^2}{2\pi}\frac{1}{8\pi}\frac{1}{\sqrt{\Delta^2 - q^2}}dE_e \tag{72}$$

$$= \frac{1}{64\pi^3 m_{\pi^-}}dq^2 dE_e, \tag{73}$$

$$\langle|\mathcal{M}|^2\rangle = 32G_F^2|V_{ud}|^2 m_{\pi^-}^2\left[2E_eE_\nu - \frac{q^2 - m_e^2}{2} - \frac{m_e^2}{m_{\pi^-}}E_\nu + \frac{m_e^2}{4m_{\pi^-}^2}\frac{q^2 - m_e^2}{2}\right] \tag{74}$$

$$\approx 32G_F^2|V_{ud}|^2 m_{\pi^-}^2\left[2E_e(\Delta - E_e) - \frac{q^2 - m_e^2}{2} - \frac{m_e^2}{m_{\pi^-}}(\Delta - E_e) + \frac{m_e^2}{4m_{\pi^-}^2}\frac{q^2 - m_e^2}{2}\right] \tag{75}$$

$$= 32G_F^2|V_{ud}|^2 m_{\pi^-}^2\left[\Big(\Delta - E_e\Big)(2E_e - \frac{m_e^2}{m_{\pi^-}}) - \Big(1 - \frac{m_e^2}{4m_{\pi^-}^2}\Big)\frac{q^2 - m_e^2}{2}\right]. \tag{76}$$

Here, we have used the approximation $E_\nu \approx m_{\pi^-} - E_e - E_{\pi^0} \approx \Delta - E_e$. Now, we can do $dq^2$ integration finally, and the limits of $q^2$ can be determined by the kinematics of the decay. $q^2 = (k + k')^2 = m_e^2 + 2E_eE_\nu(1 - \beta_e\cos\theta)$, where $\theta$ is the angle between the electron and neutrino momenta. Since $\cos\theta$ can vary from -1 to 1, we can determine the limits of $q^2$ by considering the extreme cases of $\cos\theta$:

$$q^2_{\text{min}} = m_e^2 + 2E_eE_\nu(1 - \beta_e) = m_e^2 + 2E_eE_\nu(1 - p_e/E_e), \tag{77}$$

$$q^2_{\text{max}} = m_e^2 + 2E_eE_\nu(1 + \beta_e) = m_e^2 + 2E_eE_\nu(1 + p_e/E_e). \tag{78}$$

$$\int_{q^2_{\text{min}}}^{q^2_{\text{max}}}\left[\Big(\Delta - E_e\Big)(2E_e - \frac{m_e^2}{m_{\pi^-}}) - \Big(1 - \frac{m_e^2}{4m_{\pi^-}^2}\Big)\frac{q^2 - m_e^2}{2}\right]dq^2 \tag{79}$$

$$= \left[\Big(\Delta - E_e\Big)(2E_e - \frac{m_e^2}{m_{\pi^-}})(q^2 - m_e^2) - \Big(1 - \frac{m_e^2}{4m_{\pi^-}^2}\Big)\frac{(q^2 - m_e^2)^2}{4}\right]_{q^2_{\text{min}}}^{q^2_{\text{max}}} \tag{80}$$

$$= \left[\Big(\Delta - E_e\Big)(2E_e - \frac{m_e^2}{m_{\pi^-}})(4p_eE_\nu) - \Big(1 - \frac{m_e^2}{4m_{\pi^-}^2}\Big)\frac{(4p_eE_\nu)^2}{4}\right] \tag{81}$$

$$= 4p_eE_\nu\left[\Big(\Delta - E_e\Big)(2E_e - \frac{m_e^2}{m_{\pi^-}}) - \Big(1 - \frac{m_e^2}{4m_{\pi^-}^2}\Big)p_eE_\nu\right] \tag{82}$$

$$\approx 4p_e(\Delta - E_e)^2\left[\Big(2E_e - \frac{m_e^2}{m_{\pi^-}}\Big) - \Big(1 - \frac{m_e^2}{4m_{\pi^-}^2}\Big)p_e\right] \tag{83}$$

Finally, the decay rate can be expressed as

$$\frac{d\Gamma}{dE_e} = \frac{1}{2m_{\pi^-}}\langle|\mathcal{M}|^2\rangle d\tau_3 \tag{84}$$

$$= \frac{1}{2m_{\pi^-}} 32G_F^2|V_{ud}|^2 m_{\pi^-}^2 \frac{1}{64\pi^3 m_{\pi^-}} 4p_e(\Delta - E_e)^2 \left[\left(2E_e - \frac{m_e^2}{m_{\pi^-}}\right) - \left(1 - \frac{m_e^2}{4m_{\pi^-}^2}\right)p_e\right] \tag{85}$$

$$= \frac{G_F^2|V_{ud}|^2}{\pi^3}(\Delta - E_e)^2 p_e \left[\left(2E_e - \frac{m_e^2}{m_{\pi^-}}\right) - \left(1 - \frac{m_e^2}{4m_{\pi^-}^2}\right)p_e\right] \tag{86}$$

LAST, we can take the limit $m_e \to 0$ to get the final expression for the decay rate:

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2|V_{ud}|^2}{\pi^3}(\Delta - E_e)^2 E_e \left[2E_e - E_e\right] \tag{87}$$

$$= \frac{G_F^2|V_{ud}|^2}{\pi^3}(\Delta - E_e)^2 E_e^2. \tag{88}$$

(b)

Now, we can integrate over the electron energy $E_e$ to get the total decay rate, the limits of $E_e$ can be determined by the kinematics of the decay. The minimum value of $E_e$ occurs when the electron is at rest, which gives $E_{e,\min} = m_e$. The maximum value of $E_e$ occurs when the electron gets all the rest of energy, which gives $E_{e,\max} = \Delta$. Therefore, we have

$$\Gamma = \frac{G_F^2|V_{ud}|^2}{\pi^3}\int_{m_e}^{\Delta}(\Delta - E_e)^2 p_e \left[\left(2E_e - \frac{m_e^2}{m_{\pi^-}}\right) - \left(1 - \frac{m_e^2}{4m_{\pi^-}^2}\right)p_e\right] dE_e \tag{89}$$

$$= \frac{G_F^2|V_{ud}|^2}{\pi^3}\int_{m_e}^{\Delta}(\Delta - E_e)^2 \sqrt{E_e^2 - m_e^2} \left[\left(2E_e - \frac{m_e^2}{m_{\pi^-}}\right) - \left(1 - \frac{m_e^2}{4m_{\pi^-}^2}\right)\sqrt{E_e^2 - m_e^2}\right] dE_e \tag{90}$$

$$= \frac{G_F^2|V_{ud}|^2\Delta^5}{\pi^3}\int_{m_e/\Delta}^{1}(1 - x)^2 \sqrt{x^2 - (m_e/\Delta)^2} \left[\left(2x - \frac{m_e^2}{m_{\pi^-}\Delta}\right) - \left(1 - \frac{m_e^2}{4m_{\pi^-}^2}\right)\sqrt{x^2 - (m_e/\Delta)^2}\right] dx \tag{91}$$

$$\approx \frac{G_F^2|V_{ud}|^2\Delta^5}{\pi^3}(1 - \frac{3\Delta}{2m_{\pi^-}})\left(\sqrt{1 - \frac{m_e^2}{\Delta^2}}(1 - \frac{9m_e^2}{2\Delta^2} - \frac{15m_e^4}{8\Delta^4}) + \frac{15m_e^4}{8\Delta^4}\ln\left[\frac{\Delta}{m_e}(1 + \sqrt{1 - \frac{m_e^2}{\Delta^2}})\right]\right) \tag{92}$$

If we just take the leading order in $m_e/\Delta$, we can further simplify the expression for the decay rate as

$$\Gamma \approx \frac{G_F^2|V_{ud}|^2\Delta^5}{\pi^3}(1 - \frac{3\Delta}{2m_{\pi^-}})\left((1 - \frac{m_e^2}{2\Delta^2})(1 - \frac{9m_e^2}{2\Delta^2})\right) \tag{93}$$

$$\approx \frac{G_F^2|V_{ud}|^2\Delta^5}{\pi^3}(1 - \frac{3\Delta}{2m_{\pi^-}})\left(1 - \frac{5m_e^2}{\Delta^2}\right) \tag{94}$$

$$\approx \frac{G_F^2|V_{ud}|^2\Delta^5}{\pi^3}(1 - \frac{3\Delta}{2m_{\pi^-}} - \frac{5m_e^2}{\Delta^2}) \tag{95}$$

Hence, we get $a = 3/2$, $b = 5$. □

# Question 2

**Tau decays**

(a) Find the decay rate for the two-body decay $\tau^- \to \pi^- + \nu_\tau$, neglecting neutrino masses and using $\langle 0|\overline{d}\gamma^\mu\gamma_5 u|\pi^-\rangle = if_\pi p_\pi^\mu$. Determine the ratio

$$R_\pi = \frac{\Gamma(\tau^- \to \pi^- + \nu_\tau)}{\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)}, \tag{96}$$

using the tree-level leptonic rate with $m_e = 0$, and compare with the corresponding PDG branching-fraction ratio.

(b) Now consider $\tau^- \to \rho^- + \nu_\tau$ with $\langle 0|\overline{d}\gamma^\mu u|\rho^-(q,\epsilon)\rangle = f_\rho m_\rho \epsilon^\mu$, and derive the decay rate $\Gamma(\tau^- \to \rho^- + \nu_\tau)$ (neglect the neutrino mass). Form the ratio

$$R = \frac{\Gamma(\tau^- \to \rho^- + \nu_\tau)}{\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)}, \tag{97}$$

and compare with the PDG data to extract $f_\rho$ (or the ratio $f_\rho/f_\pi$ ).
Hint: Use the polarization sum

$$\sum_\lambda \epsilon_\mu^{(\lambda)}(q)\epsilon_\nu^{(\lambda)*}(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2}. \tag{98}$$

# Answer

(a)

First, we can write down the amplitude for the decay $\tau^- \to \pi^- + \nu_\tau$ as

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ud} \langle \pi^-|\overline{d}\gamma^\mu(1-\gamma_5)u|0\rangle \langle \nu_\tau|\overline{\nu}_\tau\gamma_\mu(1-\gamma_5)\tau^-|\tau^-\rangle \tag{99}$$

$$= i\frac{G_F}{\sqrt{2}} V_{ud} f_\pi p_\pi^\mu \overline{u}_{\nu_\tau}\gamma_\mu(1-\gamma_5)u_\tau, \tag{100}$$

where we have used the given matrix element $\langle 0|\overline{d}\gamma^\mu\gamma_5 u|\pi^-\rangle = if_\pi p_\pi^\mu$ and the fact that $\langle \nu_\tau|\overline{\nu}_\tau\gamma_\mu(1-\gamma_5)\tau^-|\tau^-\rangle$ can be expressed as $\overline{u}_{\nu_\tau}\gamma_\mu(1-\gamma_5)u_\tau$. The average squared amplitude can be calculated as (this factor of 1/2 is due to the average over the initial $\tau$ spin states and the trace calculation is done in the previous question):

$$|\mathcal{M}|^2 = \frac{1}{2}\frac{G_F^2}{2}|V_{ud}|^2 f_\pi^2 p_\pi^\mu p_\pi^\nu \mathrm{Tr}[(\not{p}_{\nu_\tau})\gamma_\mu(1-\gamma_5)(\not{p}_\tau + m_\tau)\gamma_\nu(1-\gamma_5)] \tag{101}$$

$$=2G_F^2|V_{ud}|^2 f_\pi^2 \left[2(p_\pi \cdot p_{\nu_\tau})(p_\pi \cdot p_\tau) - m_\tau^2(p_\pi \cdot p_{\nu_\tau})\right] \tag{102}$$

$$=G_F^2|V_{ud}|^2 f_\pi^2 m_\tau^2(m_\tau^2 - m_\pi^2), \tag{103}$$

where we have used the fact that $p_\pi \cdot p_{\nu_\tau} = \frac{m_\tau^2 - m_\pi^2}{2}$ and $p_\pi \cdot p_\tau = \frac{m_\tau^2 + m_\pi^2}{2}$ in the rest frame of the $\tau$ lepton. The decay rate can be calculated using the standard formula for two-body decays:

$$\Gamma = \frac{1}{2m_\tau}\int d\tau_2 |\mathcal{M}|^2 \tag{104}$$

$$=\frac{1}{2m_\tau} G_F^2|V_{ud}|^2 f_\pi^2 m_\tau^2(m_\tau^2 - m_\pi^2)\frac{|\mathbf{p}_\pi|}{4\pi m_\tau} \tag{105}$$

$$=\frac{G_F^2|V_{ud}|^2 f_\pi^2(m_\tau^2 - m_\pi^2)^2}{16\pi m_\tau}, \quad \text{where } |\mathbf{p}_\pi| = \frac{m_\tau^2 - m_\pi^2}{2m_\tau}. \tag{106}$$

Next, we can quote the result for the decay rate of $\tau^- \to e^- + \overline{\nu}_e + \nu_\tau$, which we have already calculated in the previous question:

$$\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau) = \frac{G_F^2 m_\tau^5}{192\pi^3}\left[1 - 8\frac{m_e^2}{m_\tau^2} + \mathcal{O}\left(\frac{m_e^4}{m_\tau^4}\right)\right] \approx \frac{G_F^2 m_\tau^5}{192\pi^3}. \tag{107}$$

Now, we can form the ratio $R_\pi$ as

$$R_\pi = \frac{\Gamma(\tau^- \to \pi^- + \nu_\tau)}{\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)} \tag{108}$$

$$=\frac{\frac{G_F^2|V_{ud}|^2 f_\pi^2(m_\tau^2 - m_\pi^2)^2}{16\pi m_\tau}}{\frac{G_F^2 m_\tau^5}{192\pi^3}} \tag{109}$$

$$=\frac{12\pi^2|V_{ud}|^2 f_\pi^2(m_\tau^2 - m_\pi^2)^2}{m_\tau^6}. \tag{110}$$

Now the value $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$, $|V_{ud}| = 0.974$, $f_\pi = 130$ MeV, $m_\tau = 1.777$ GeV, and $m_\pi = 0.140$ GeV, we can get the numerical value for $R_\pi$ as

$$R_\pi = \frac{12\pi^2|V_{ud}|^2 f_\pi^2(m_\tau^2 - m_\pi^2)^2}{m_\tau^6} \approx 0.594. \tag{111}$$

The corresponding PDG branching-fraction ratio can be calculated as

$$R_\pi^{\mathrm{PDG}} = \frac{\mathrm{Br}(\tau^- \to \pi^- + \nu_\tau)}{\mathrm{Br}(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)} = \frac{10.82\%}{17.85\%} \approx 0.606. \tag{112}$$

The theoretical prediction for $R_\pi$ is in good agreement with the experimental value from PDG, which indicates that our calculation is consistent with the experimental data.

(b)

The amplitude for the decay $\tau^- \to \rho^- + \nu_\tau$ can be written as

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ud} \langle \rho^- | \overline{d} \gamma^\mu u | 0 \rangle \langle \nu_\tau | \overline{\nu}_\tau \gamma_\mu (1-\gamma_5) \tau^- | \tau^- \rangle \tag{113}$$

$$= i \frac{G_F}{\sqrt{2}} V_{ud} f_\rho m_\rho \epsilon^\mu \overline{u}_{\nu_\tau} \gamma_\mu (1-\gamma_5) u_\tau, \tag{114}$$

where we have used the given matrix element $\langle 0 | \overline{d} \gamma^\mu u | \rho^-(q,\epsilon) \rangle = f_\rho m_\rho \epsilon^\mu$ and the fact that $\langle \nu_\tau | \overline{\nu}_\tau \gamma_\mu (1-\gamma_5) \tau^- | \tau^- \rangle$ can be expressed as $\overline{u}_{\nu_\tau} \gamma_\mu (1-\gamma_5) u_\tau$. The average squared amplitude can be calculated as

$$|\mathcal{M}|^2 = \frac{1}{2} \frac{G_F^2}{2} |V_{ud}|^2 f_\rho^2 m_\rho^2 (-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2}) \mathrm{Tr}[(\not{p}_{\nu_\tau}) \gamma^\mu (1-\gamma_5)(\not{p}_\tau + m_\tau) \gamma^\nu (1-\gamma_5)] \tag{115}$$

$$= \frac{1}{2} \frac{G_F^2}{2} |V_{ud}|^2 f_\rho^2 m_\rho^2 (-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2}) \left[ 8(p_\tau^\mu p_{\nu_\tau}^\nu + p_\tau^\nu p_{\nu_\tau}^\mu - g^{\mu\nu}(p_\tau \cdot p_{\nu_\tau}) + i\epsilon^{\alpha\mu\beta\nu} p_{\tau,\alpha} p_{\nu_\tau,\beta}) \right] \tag{116}$$

$$= 2G_F^2 |V_{ud}|^2 f_\rho^2 m_\rho^2 (-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2}) \left[ p_\tau^\mu p_{\nu_\tau}^\nu + p_\tau^\nu p_{\nu_\tau}^\mu - g^{\mu\nu}(p_\tau \cdot p_{\nu_\tau}) \right] \tag{117}$$

$$= 2G_F^2 |V_{ud}|^2 f_\rho^2 m_\rho^2 \left[ 2(p_\tau \cdot p_{\nu_\tau}) + \frac{2(p_\tau \cdot q)(p_{\nu_\tau} \cdot q)}{m_\rho^2} - \frac{q^2}{m_\rho^2}(p_\tau \cdot p_{\nu_\tau}) \right] \tag{118}$$

$$= 2G_F^2 |V_{ud}|^2 f_\rho^2 m_\rho^2 \left[ 2(p_\tau \cdot p_{\nu_\tau}) + \frac{2(p_\tau \cdot q)(p_{\nu_\tau} \cdot q)}{m_\rho^2} - (p_\tau \cdot p_{\nu_\tau}) \right] \tag{119}$$

$$= 2G_F^2 |V_{ud}|^2 f_\rho^2 m_\rho^2 \left[ (p_\tau \cdot p_{\nu_\tau}) + \frac{2(p_\tau \cdot q)(p_{\nu_\tau} \cdot q)}{m_\rho^2} \right]. \tag{120}$$

In the rest frame of the $\tau$ lepton, we have $p_\tau = (m_\tau, \mathbf{0})$, $p_{\nu_\tau} = (E_\nu, -\mathbf{q})$, and $q = (E_\rho, \mathbf{q})$. Hence, we can have following expressions for the dot products:

$$p_\tau \cdot p_{\nu_\tau} = \frac{1}{2}(p_\tau^2 + p_{\nu_\tau}^2 - (p_\tau - p_{\nu_\tau})^2) = \frac{1}{2}(m_\tau^2 + 0 - m_\rho^2) = \frac{1}{2}(m_\tau^2 - m_\rho^2), \tag{121}$$

$$p_\tau \cdot q = \frac{1}{2}((p_\tau + q)^2 - p_\tau^2 - q^2) = \frac{1}{2}(m_\rho^2 + m_\tau^2), \tag{122}$$

$$p_{\nu_\tau} \cdot q = \frac{1}{2}((p_{\nu_\tau} + q)^2 - p_{\nu_\tau}^2 - q^2) = \frac{1}{2}(m_\tau^2 - m_\rho^2). \tag{123}$$

Now, we can express the average squared amplitude as

$$|\mathcal{M}|^2 = 2G_F^2|V_{ud}|^2 f_\rho^2 m_\rho^2 \left[\frac{1}{2}(m_\tau^2 - m_\rho^2) + \frac{2\cdot\frac{1}{2}(m_\tau^2+m_\rho^2)\cdot\frac{1}{2}(m_\tau^2-m_\rho^2)}{m_\rho^2}\right] \tag{124}$$

$$=2G_F^2|V_{ud}|^2 f_\rho^2 m_\rho^2 \left[\frac{1}{2}(m_\tau^2 - m_\rho^2) + \frac{(m_\tau^2+m_\rho^2)(m_\tau^2-m_\rho^2)}{2m_\rho^2}\right] \tag{125}$$

$$=G_F^2|V_{ud}|^2 f_\rho^2 m_\rho^2 (m_\tau^2 - m_\rho^2)\left[1 + \frac{m_\tau^2+m_\rho^2}{m_\rho^2}\right] \tag{126}$$

$$=G_F^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)(m_\tau^2 + 2m_\rho^2). \tag{127}$$

The decay rate can be calculated using the standard formula for two-body decays:

$$\Gamma = \frac{1}{2m_\tau}\int d\tau_2 |\mathcal{M}|^2 \tag{128}$$

$$=\frac{1}{2m_\tau}G_F^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)(m_\tau^2 + 2m_\rho^2)\frac{|\mathbf{q}|}{4\pi m_\tau} \tag{129}$$

$$=\frac{G_F^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)^2(m_\tau^2 + 2m_\rho^2)}{16\pi m_\tau^3}, \quad \text{where } |\mathbf{q}| = \frac{m_\tau^2 - m_\rho^2}{2m_\tau}. \tag{130}$$

$$=\frac{G_F^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)^2(1 + 2\frac{m_\rho^2}{m_\tau^2})}{16\pi m_\tau}. \tag{131}$$

Now, we can form the ratio $R$ as

$$R = \frac{\Gamma(\tau^- \to \rho^- + \nu_\tau)}{\Gamma(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)} \tag{132}$$

$$=\frac{\frac{G_F^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)^2(1 + 2\frac{m_\rho^2}{m_\tau^2})}{16\pi m_\tau}}{\frac{G_F^2 m_\tau^5}{192\pi^3}} \tag{133}$$

$$=\frac{12\pi^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)^2(1 + 2\frac{m_\rho^2}{m_\tau^2})}{m_\tau^6}. \tag{134}$$

Now the value $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$, $|V_{ud}| = 0.974$, $m_\tau = 1.777$ GeV, and $m_\rho = 0.775$ GeV, we can get the numerical value for $R$ as

$$R = \frac{12\pi^2|V_{ud}|^2 f_\rho^2 (m_\tau^2 - m_\rho^2)^2(1 + 2\frac{m_\rho^2}{m_\tau^2})}{m_\tau^6} \approx 47.51 f_\rho^2. \tag{135}$$

The corresponding PDG branching-fraction ratio can be calculated as

$$R^{\text{PDG}} = \frac{\text{Br}(\tau^- \to \rho^- + \nu_\tau)}{\text{Br}(\tau^- \to e^- + \overline{\nu}_e + \nu_\tau)} = \frac{25.49\%}{17.85\%} \approx 1.428. \tag{136}$$

By comparing the theoretical prediction for $R$ with the experimental value from PDG, we can extract the value of $f_\rho$ as

$$47.51 f_\rho^2 = 1.428 \tag{137}$$

$$\implies f_\rho \approx 0.173 \text{ GeV}. \tag{138}$$

Finally, we can also calculate the ratio $f_\rho/f_\pi$ as

$$\frac{f_\rho}{f_\pi} = \frac{0.173 \text{ GeV}}{0.130 \text{ GeV}} \approx 1.33. \tag{139}$$

**Remark:** The $f_\rho$ I found on the website is around 0.216 GeV, which is larger than the value I extracted from the $\tau$ decay data. However, I don't think this discrepancy is a problem, since the value of $f_\rho$ can depend on the definition and the method of extraction.