University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

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Question 1

Calculate the metric g_{ij} and its inverse g^{ij} , the affine connection Γ^i_{jk}, and the Laplacian ∇^2 in two dimensions for a polar coordinate system with $\xi^1 = x$ and $\xi^2 = y$ being Cartesian coordinates and $x^1 = r$ and $x^2 = \theta$ being polar coordinates.

Answer

First, we have the transformation relations between Cartesian coordinates and polar coordinates:

$$x = r\cos\theta, \tag{1}$$
$$y = r\sin\theta. \tag{2}$$

The metric in Cartesian coordinates is given by:

$$\eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

To find the metric in polar coordinates, we use the transformation:

$$g_{ij} = \frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^l}{\partial x^j}\eta_{kl}. \tag{4}$$

Calculating the partial derivatives, we have:

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \tag{5}$$
$$\frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta. \tag{6}$$

Thus, the metric components in polar coordinates are:

$$g_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 = \cos^2\theta + \sin^2\theta = 1, \tag{7}$$
$$g_{r\theta} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} = \cos\theta(-r\sin\theta) + \sin\theta(r\cos\theta) = 0, \tag{8}$$
$$g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = (-r\sin\theta)^2 + (r\cos\theta)^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2. \tag{9}$$

Therefore, the metric in polar coordinates is:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{10}$$

The inverse metric is given by:

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{11}$$

Next, we calculate the affine connection components using the formula:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{12}$$

Now, we have:

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) = 0, \tag{13}$$

$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{rr}}{\partial \theta} - \frac{\partial g_{r\theta}}{\partial r} \right) = 0, \tag{14}$$

$$\Gamma^\theta_{rr} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta r}}{\partial r} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) = 0, \tag{15}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial \theta} \right) = 0. \tag{16}$$

The non-zero components of the affine connection are:

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{r\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right) = -r, \tag{17}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta r}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{r\theta}}{\partial \theta} \right) = \frac{1}{r}. \tag{18}$$

Finally, we calculate the Laplacian in polar coordinates:

$$\partial_r f = \frac{\partial f}{\partial r}, \quad \partial_\theta f = \frac{\partial f}{\partial \theta}, \tag{19}$$

$$\partial_r f = \partial_x f \frac{\partial x}{\partial r} + \partial_y f \frac{\partial y}{\partial r} = \cos\theta \partial_x f + \sin\theta \partial_y f, \tag{20}$$

$$\partial_\theta f = \partial_x f \frac{\partial x}{\partial \theta} + \partial_y f \frac{\partial y}{\partial \theta} = -r \sin\theta \partial_x f + r \cos\theta \partial_y f, \tag{21}$$

$$\partial_{rr} f = \partial_r(\partial_r f) = \cos\theta \partial_{xx} f \cos\theta + \sin\theta \partial_{yy} f \sin\theta + 2\cos\theta \sin\theta \partial_{xy} f, \tag{22}$$

$$\partial_{\theta\theta} f = \partial_\theta(\partial_\theta f) = r^2 \sin^2\theta \partial_{xx} f + r^2 \cos^2\theta \partial_{yy} f - 2r^2 \sin\theta \cos\theta \partial_{xy} f - r\cos\theta \partial_x f - r \sin\theta \partial_y f. \tag{23}$$

$$\partial_{rr} f = \cos^2\theta \partial_{xx} f + \sin^2\theta \partial_{yy} f + 2\cos\theta\sin\theta \partial_{xy} f, \tag{24}$$

$$\frac{1}{r}\partial_r f = \frac{1}{r}(\cos\theta \partial_x f + \sin\theta \partial_y f), \tag{25}$$

$$\frac{1}{r^2}\partial_{\theta\theta} f = \sin^2\theta \partial_{xx} f + \cos^2\theta \partial_{yy} f - 2\sin\theta\cos\theta \partial_{xy} f - \frac{1}{r}\cos\theta \partial_x f - \frac{1}{r}\sin\theta \partial_y f. \tag{26}$$

Thus, the Laplacian in polar coordinates is:

$$\begin{aligned} \nabla^2 f &= \partial_{xx} f + \partial_{yy} f && (27) \\ &= \partial_{rr} f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_{\theta\theta} f. && (28) \end{aligned}$$

□

Question 2

Calculate the compact expressions for the components of the affine connection when the metric g_{ij} is diagonal. See problem 3 in chapter 3 of Carroll's book.
Problem 3 in chapter 3 of Carroll's book: Imagine we have a diagonal metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = 0 \tag{29}$$

$$\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda g_{\mu\mu} \tag{30}$$

$$\Gamma^{\lambda}_{\mu\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right) \tag{31}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right). \tag{32}$$

In these expressions, $\mu \neq \nu \neq \lambda$, and repeated indices are not summed over.

Answer

By the definedion of the Christoffel symbols, we have

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{33}$$

Since the metric is diagonal, we have $g_{\mu\nu} = 0$ for $\mu \neq \nu$. Therefore, we can analyze the different cases:
1. For $\mu \neq \nu \neq \lambda$:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) = 0, \tag{34}$$

where we used the fact that all terms vanish because $g_{\mu\nu} = 0$ for $\mu \neq \nu$ when we sum over σ.
2. For $\mu = \nu \neq \lambda$:

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\mu\sigma} + \partial_\mu g_{\mu\sigma} - \partial_\sigma g_{\mu\mu}\right) = -\frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\mu\mu} = -\frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda g_{\mu\mu}, \tag{35}$$

where we used the fact that only the term with $\sigma = \lambda$ survives in the sum. When g is diagonal, $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}}$.
3. For $\mu \neq \lambda = \nu$:

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\lambda\sigma} + \partial_\lambda g_{\mu\sigma} - \partial_\sigma g_{\mu\lambda}\right) = \frac{1}{2}g^{\lambda\lambda}\partial_\mu g_{\lambda\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right), \tag{36}$$

where we used the fact that only the term with $\sigma = \lambda$ survives in the sum. When g is diagonal, $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}}$.
4. For $\mu = \nu = \lambda$:

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\lambda g_{\lambda\sigma} + \partial_\lambda g_{\lambda\sigma} - \partial_\sigma g_{\lambda\lambda}\right) = \frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right), \tag{37}$$

where we used the fact that only the term with $\sigma = \lambda$ survives in the sum. When g is diagonal, $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}}$. $\square$

Question 3

Prove that if the equation for a geodesic has the form

$$\frac{d^2x^\alpha}{dp_i^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_i}\frac{dx^\gamma}{dp_i} = 0, \tag{38}$$

for two different parameters p_1 and p_2 defined along the geodesic then the most general relation between them is $p_2 = Ap_1 + B$ where A and B are constants.

Answer

Let $x^\alpha(p_1)$ be a geodesic parameterized by p_1. We want to show that if we reparameterize the geodesic using a different parameter p_2, the new parameter must be a linear function of the old one, i.e., $p_2 = Ap_1 + B$ for some constants A and B.

Assume that $x^\alpha(p_2)$ is the same geodesic but parameterized by p_2. We can express p_2 as a function of p_1, i.e., $p_2 = f(p_1)$ for some function f. Then, we have:

$$\frac{dx^\alpha}{dp_2} = \frac{dx^\alpha}{dp_1}\frac{dp_1}{dp_2}, \tag{39}$$

$$\frac{d^2x^\alpha}{dp_2^2} = \frac{d}{dp_2}\left(\frac{dx^\alpha}{dp_1}\frac{dp_1}{dp_2}\right) = \frac{d^2x^\alpha}{dp_1^2}\left(\frac{dp_1}{dp_2}\right)^2 + \frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2}. \tag{40}$$

Substituting these into the geodesic equation parameterized by p_2, we get:

$$\frac{d^2x^\alpha}{dp_2^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_2}\frac{dx^\gamma}{dp_2} = 0. \tag{41}$$

Substituting the expressions for the derivatives, we have:

$$\left(\frac{d^2x^\alpha}{dp_1^2}\left(\frac{dp_1}{dp_2}\right)^2 + \frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2}\right) + \Gamma^\alpha_{\beta\gamma}\left(\frac{dx^\beta}{dp_1}\frac{dp_1}{dp_2}\right)\left(\frac{dx^\gamma}{dp_1}\frac{dp_1}{dp_2}\right) = 0. \tag{42}$$

Rearranging, we get:

$$\frac{d^2x^\alpha}{dp_1^2}\left(\frac{dp_1}{dp_2}\right)^2 + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_1}\frac{dx^\gamma}{dp_1}\left(\frac{dp_1}{dp_2}\right)^2 + \frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2} = 0. \tag{43}$$

Since $x^\alpha(p_1)$ satisfies the geodesic equation, we have:

$$\frac{d^2x^\alpha}{dp_1^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dp_1}\frac{dx^\gamma}{dp_1} = 0. \tag{44}$$

Thus, the first two terms cancel out, leaving us with:

$$\frac{dx^\alpha}{dp_1}\frac{d^2p_1}{dp_2^2} = 0. \tag{45}$$

Since $\frac{dx^\alpha}{dp_1}$ is not zero (as we are moving along the geodesic), we must have:

$$\frac{d^2p_1}{dp_2^2} = 0. \tag{46}$$

This implies that p_1 is a linear function of p_2, i.e.,

$$p_1 = Ap_2 + B, \tag{47}$$

for some constants A and B. We can rewrite this as:

$$p_2 = Cp_1 + D \tag{48}$$

where $C = \frac{1}{A}$ and $D = -\frac{B}{A}$. □

Remark: I am not sure A or C should be positive or negative. If we want p_1 and p_2 to have the same orientation, then A and C should be positive.

Remark: I am not sure if A or C could be 0, because if $A = 0$, then $p_1 = B$ is a constant, which means the parameterization is not valid. Similarly, if $C = 0$, then $p_2 = D$ is a constant, which also means the parameterization is not valid. In other words, I don't know how to make sure p_1 and p_2 can be non-constant functions.