

University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501

General Relativity I

Assignment Solution

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Assignment 3 due on Monday September 22th at 5PM

Question 1

Show explicitly that the 4-vector current density for a collection of point charges satisfies $\partial_\mu J^\mu = 0$

Answer

In the class, we defined the 4-vector current density for a collection of point charges as

$$J^0(t, \mathbf{x}) = \rho(t, \mathbf{x}) = \sum_a q_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \quad (1)$$

$$\mathbf{J}(t, \mathbf{x}) = \sum_a q_a \mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)), \quad \mathbf{v}_a(t) = \frac{d\mathbf{x}_a(t)}{dt}. \quad (2)$$

Then we have

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \quad (3)$$

$$= \sum_a q_a \left[\frac{\partial}{\partial t} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))) \right] \quad (4)$$

$$= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \nabla \cdot (\mathbf{v}_a(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t))) \right] \quad (5)$$

$$= \sum_a q_a \left[-\mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) + \mathbf{v}_a(t) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \right] \quad (6)$$

$$= 0. \quad (7)$$

Question 2

Prove that the electromagnetic energy density squared minus the square of the Poynting vector is a Lorentz invariant for an electromagnetic field by expressing this quantity in terms of tensors. You might consider using the dual field strength tensor defined by $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Answer

In the EM class, we defined the electromagnetic energy density and the Poynting vector as

$$u = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \quad (8)$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}. \quad (9)$$

Also, we have the following relations

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad (10)$$

Besides, the EM field energy momentum strength tensor is given by

$$T_{EM}^{\mu\nu} = F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \quad (11)$$

$$u = T_{EM}^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad S^i = T_{EM}^{i0} = (\mathbf{E} \times \mathbf{B})^i. \quad (12)$$

Question 3

Calculate the scalar T^α_α associated with the electromagnetic stress tensor.

Answer