University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

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Assignment 5 due on Monday October 6th at 5PM

Question 1

In lecture we studied the time dilation of a slowly moving object in a weak stationary gravitational field. We found that the frequency difference of identical clocks located at points $\mathbf{x}_1$ and $\mathbf{x}_2$ and at rest in the gravitational field is

$$\frac{\nu_2 - \nu_1}{\nu_0} = \frac{\Delta\nu}{\nu_0} = \phi_2 - \phi_1, \tag{1}$$

where ϕ is the gravitational potential. Generalize this formula to the case when they are moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which are small compared to the speed of light. This results in contributions from both gravity and special relativity.

Answer

Question 2

Prove that $V_{\mu;\nu} \equiv \partial V_\mu / \partial x^\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$ transforms as a second rank tensor, similar to how we proved it for ${V^\mu}_{;\nu}$ in class, assuming that V transforms as a 4 -vector.

Answer

First, by the definition, we have

$$\begin{aligned}
{V'}_\mu &= \frac{\partial x^\nu}{\partial x'^\mu} V_\nu, && (2)\\
\frac{\partial {V'}_\mu}{\partial x'^\nu} &= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x'^\mu} V_\beta\right) = \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial^2 x^\beta}{\partial x^\alpha \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} && (3)\\
&= \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} && (4)\\
{\Gamma'}^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} && (5)\\
&= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}, \quad \text{by } \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu}\right) = 0, && (6)\\
{\Gamma'}^\lambda_{\mu\nu}{V'}_\lambda &= \left(\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right)\frac{\partial x^\beta}{\partial x'^\lambda} V_\beta && (7)\\
&= \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma}\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^\lambda} V_\beta + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^\lambda} V_\beta && (8)\\
&= \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma}{\delta^\beta}_\rho V_\beta + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}{\delta^\beta}_\rho V_\beta && (9)\\
&= \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} V_\rho + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho. && (10)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
{V'}_{\mu;\nu} &= \frac{\partial {V'}_\mu}{\partial x'^\nu} - {\Gamma'}^\lambda_{\mu\nu}{V'}_\lambda && (11)\\
&= \frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} V_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} - \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} V_\rho - \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho && (12)\\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial V_\beta}{\partial x^\alpha} - \frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} V_\rho && (13)\\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\left(\frac{\partial V_\beta}{\partial x^\alpha} - \Gamma^\rho_{\alpha\beta} V_\rho\right) = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\left(\frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\rho_{\alpha\beta} V_\rho\right) && (14)\\
&= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} V_{\alpha;\beta}. && (15)
\end{aligned}$$

This shows that $V_{\mu;\nu}$ transforms as a second rank tensor. $\square$

Question 3

Show that

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = A_{\mu\nu,\lambda} + A_{\lambda\mu,\nu} + A_{\nu\lambda,\mu}, \tag{16}$$

when $A_{\mu\nu}$ is an anti-symmetric tensor.

Answer

By the definition of covariant derivative, we have

$$A_{\mu\nu;\lambda} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho}, \tag{17}$$

$$A_{\lambda\mu;\nu} = \frac{\partial A_{\lambda\mu}}{\partial x^\nu} - \Gamma^\rho_{\lambda\nu} A_{\rho\mu} - \Gamma^\rho_{\mu\nu} A_{\lambda\rho}, \tag{18}$$

$$A_{\nu\lambda;\mu} = \frac{\partial A_{\nu\lambda}}{\partial x^\mu} - \Gamma^\rho_{\nu\mu} A_{\rho\lambda} - \Gamma^\rho_{\lambda\mu} A_{\nu\rho}. \tag{19}$$

Adding them together, we have

$$\begin{aligned} A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = &\frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} \\ &- \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho} - \Gamma^\rho_{\lambda\nu} A_{\rho\mu} - \Gamma^\rho_{\mu\nu} A_{\lambda\rho} - \Gamma^\rho_{\nu\mu} A_{\rho\lambda} - \Gamma^\rho_{\lambda\mu} A_{\nu\rho}. \end{aligned} \tag{20}$$

Since $A_{\mu\nu}$ is an anti-symmetric tensor, we have $A_{\mu\nu} = -A_{\nu\mu}$. Therefore, we have

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} \tag{21}$$

$$= A_{\mu\nu,\lambda} + A_{\lambda\mu,\nu} + A_{\nu\lambda,\mu}. \tag{22}$$

□