

University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I
Assignment Solution

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HW2 Due to October 7 11:59 PM

Question 1

Problem 5.1

Work out the LSZ reduction formula for the complex scalar field that was introduced in problem 3.5. Note that we must specify the type (*a* or *b*) of each incoming and outgoing particle.

Answer

We start with the mode expansion of the complex scalar field:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} [a(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}] \quad (1)$$

$$\varphi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} [b(\mathbf{k}) e^{ikx} + a^\dagger(\mathbf{k}) e^{-ikx}] \quad (2)$$

$$a(\mathbf{k}) = \int d^3x e^{-ikx} [i\partial_0 \varphi(x) + \omega \varphi(x)], \quad (3)$$

$$b(\mathbf{k}) = \int d^3x e^{-ikx} [\omega \varphi^\dagger(x) + i\partial_0 \varphi^\dagger(x)]. \quad (4)$$

First, we define the $|i\rangle$ and $|f\rangle$ states as

$$|i\rangle = \lim_{t \rightarrow -\infty} a_1^\dagger(t) a_2^\dagger(t) \cdots b_1^\dagger(t) b_2^\dagger(t) \cdots |0\rangle, \quad (5)$$

$$|f\rangle = \lim_{t \rightarrow +\infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t) \cdots b_{1'}^\dagger(t) b_{2'}^\dagger(t) \cdots |0\rangle. \quad (6)$$

And a_i and b_i are given by

$$a_i^\dagger = \int d^3k f_i(\mathbf{k}) a^\dagger(\mathbf{k}) \quad (7)$$

$$b_i^\dagger = \int d^3k g_i(\mathbf{k}) b^\dagger(\mathbf{k}), \quad (8)$$

where

$$f_i(\mathbf{k}), g_i(\mathbf{k}) \propto \exp(-(\mathbf{k} - \mathbf{k}_i)^2 / 4\sigma^2). \quad (9)$$

Now we can compute the difference between $a_1^\dagger(+\infty)$ and $a_1^\dagger(-\infty)$:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt \partial_0 a_1^\dagger(t) \quad (10)$$

$$= \int_{-\infty}^{+\infty} dt \int d^3k f_1(\mathbf{k}) \int d^3x e^{ikx} [\omega \varphi(x) - i \partial_0 \varphi(x)] \quad (11)$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial_\mu \partial^\mu + m^2) \varphi(x), \quad (12)$$

where I quote the equation in the textbook. Similarly, we can get

$$b_1^\dagger(+\infty) - b_1^\dagger(-\infty) = -i \int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x), \quad (13)$$

$$a_{1'}(+\infty) - a_{1'}(-\infty) = i \int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx} (-\partial_\mu \partial^\mu + m^2) \varphi(x), \quad (14)$$

$$b_{1'}(+\infty) - b_{1'}(-\infty) = i \int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x). \quad (15)$$

Now we can express the S-matrix element $\langle f|i\rangle$ as

$$\langle f|i\rangle = \langle 0 | \mathcal{T} b_{1'}(+\infty) b_{2'}(+\infty) \cdots a_{1'}(+\infty) a_{2'}(+\infty) \cdots a_1^\dagger(-\infty) a_2^\dagger(-\infty) \cdots b_1^\dagger(-\infty) b_2^\dagger(-\infty) \cdots | 0 \rangle \quad (16)$$

$$\begin{aligned} &= \langle 0 | \mathcal{T} [b_{1'}(-\infty) + i \int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x)] \cdots \\ &\quad \cdots [a_{1'}(-\infty) + i \int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx} (-\partial_\mu \partial^\mu + m^2) \varphi(x)] \cdots \\ &\quad \cdots [a_1^\dagger(+\infty) + i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial_\mu \partial^\mu + m^2) \varphi(x)] \cdots \\ &\quad \cdots [b_1^\dagger(+\infty) + i \int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x)] \cdots | 0 \rangle \end{aligned} \quad (17)$$

$$\begin{aligned} &= \langle 0 | \mathcal{T} [i \int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x)] \cdots \\ &\quad \cdots [i \int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx} (-\partial_\mu \partial^\mu + m^2) \varphi(x)] \cdots \\ &\quad \cdots [i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial_\mu \partial^\mu + m^2) \varphi(x)] \cdots \\ &\quad \cdots [i \int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x)] \cdots | 0 \rangle \end{aligned} \quad (18)$$

$$= (i)^{n+n'+m+m'} \langle 0 | \mathcal{T} \left[\prod_{j'}^{n'} \int d^4x e^{-ik_{j'}x} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x) \right] \left[\prod_{l'}^{m'} \int d^4x e^{-ik_{l'}x} (-\partial_\mu \partial^\mu + m^2) \varphi(x) \right] \quad (19)$$

$$\left[\prod_l^m \int d^4x e^{ik_lx} (-\partial_\mu \partial^\mu + m^2) \varphi(x) \right] \left[\prod_j^n \int d^4x e^{ik_jx} (-\partial_\mu \partial^\mu + m^2) \varphi^\dagger(x) \right] | 0 \rangle, \quad (20)$$

where we have used the fact that $a_i|0\rangle = b_i|0\rangle = 0$ and $\langle 0|a_i^\dagger = \langle 0|b_i^\dagger = 0$. Here n and m are the number of incoming a and b particles, while n' and m' are the number of outgoing a and b particles, respectively. We

also impose the $\sigma \rightarrow 0$ limit, so that $f_i(\mathbf{k})$ and $g_i(\mathbf{k})$ become delta functions. Finally, we can rewrite the S-matrix element as

$$\begin{aligned} \langle f | i \rangle = & (i)^{n+n'+m+m'} \int d^4x_1 e^{-ik_1 x_1} \dots \int d^4x_n e^{-ik_n x_n} \int d^4x_{1'} e^{ik_{1'} x_{1'}} \dots \int d^4x_{n'} e^{ik_{n'} x_{n'}} \\ & \int d^4y_1 e^{-ip_1 y_1} \dots \int d^4y_m e^{-ip_m y_m} \int d^4y_{1'} e^{ip_{1'} y_{1'}} \dots \int d^4y_{m'} e^{ip_{m'} y_{m'}} \\ & (-\partial_\mu \partial_{x_1}^\mu + m^2) \dots (-\partial_\mu \partial_{x_n}^\mu + m^2) (-\partial_\mu \partial_{x_{1'}}^\mu + m^2) \dots (-\partial_\mu \partial_{x_{n'}}^\mu + m^2) \\ & (-\partial_\mu \partial_{y_1}^\mu + m^2) \dots (-\partial_\mu \partial_{y_m}^\mu + m^2) (-\partial_\mu \partial_{y_{1'}}^\mu + m^2) \dots (-\partial_\mu \partial_{y_{m'}}^\mu + m^2) \\ & \langle 0 | \mathcal{T} \varphi^\dagger(y_{1'}) \dots \varphi^\dagger(y_{m'}) \varphi(x_{1'}) \dots \varphi(x_{n'}) \varphi(x_1) \dots \varphi(x_n) \varphi^\dagger(y_1) \dots \varphi^\dagger(y_m) | 0 \rangle. \end{aligned} \quad (21)$$

This is the LSZ reduction formula for the complex scalar field. \square

Question 2

Problem 6.1

- (a) Find an explicit formula for $\mathcal{D}q$ in eq. (6.9). Your formula should be of the form $\mathcal{D}q = C \prod_{j=1}^N dq_j$, where C is a constant that you should compute.
- (b) For the case of a free particle, $V(Q) = 0$, evaluate the path integral of eq. (6.9) explicitly. Hint: integrate over q_1 , then q_2 , etc, and look for a pattern. Express your final answer in terms of q', t', q'', t'' and m . Restore \hbar by dimensional analysis.
- (c) Compute the $\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')} | q' \rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare your result in part (b).

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j, \bar{q}_j)\delta t}, \quad (6.7)$$

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \exp \left[i \int_{t'}^{t''} dt L(\dot{q}(t), q(t)) \right]. \quad (6.9)$$

Answer

- (a) First, from eq. (6.7), we can see that

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j, \bar{q}_j)\delta t}, \quad \text{assuming } H(p, q) = \frac{1}{2m}p^2 + V(q) \quad (22)$$

$$= \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i(\frac{1}{2m}p_j^2 + V(\bar{q}_j))\delta t} \quad (23)$$

$$= \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{ip_j \delta t \dot{q}_j} e^{-i(\frac{1}{2m}p_j^2 + V(\bar{q}_j))\delta t}, \quad \text{where } \dot{q}_j = \frac{q_{j+1} - q_j}{\delta t} \quad (24)$$

$$= \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{-i\delta t(\frac{1}{2m}p_j^2 - p_j \dot{q}_j + V(\bar{q}_j))} \quad (25)$$

$$= \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{-i\delta t(\frac{1}{2m}(p_j - m\dot{q}_j)^2 - \frac{1}{2}m\dot{q}_j^2 + V(\bar{q}_j))} \quad (26)$$

$$= \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{-i\delta t \frac{1}{2m}(p_j - m\dot{q}_j)^2} e^{i\delta t(\frac{1}{2}m\dot{q}_j^2 - V(\bar{q}_j))} \quad (27)$$

$$= \int \prod_{k=1}^N dq_k \prod_{j=0}^N \frac{dp_j}{2\pi} e^{-i\delta t \frac{1}{2m}(p_j - m\dot{q}_j)^2} e^{i\delta t L(\dot{q}_j, \bar{q}_j)}, \quad \text{where } L(\dot{q}, q) = \frac{1}{2}m\dot{q}^2 - V(q) \quad (28)$$

$$= \int \prod_{k=1}^N dq_k \left[\prod_{j=0}^N \int \frac{dp_j}{2\pi} e^{-i\delta t \frac{1}{2m}(p_j - m\dot{q}_j)^2} \right] e^{i \sum_{j=0}^N \delta t L(\dot{q}_j, \bar{q}_j)}, \quad (29)$$

where we have used the definition of \dot{q}_j and $L(\dot{q}, q)$. Now we can compute the integral over p_j :

$$\int \frac{dp_j}{2\pi} e^{-i\delta t \frac{1}{2m}(p_j - mq_j)^2} = \int \frac{dp_j}{2\pi} e^{-i\frac{\delta t}{2m} p_j^2} \quad (\text{by shifting } p_j \rightarrow p_j + mq_j) \quad (30)$$

$$= \frac{1}{2\pi} \sqrt{\frac{2m\pi}{i\delta t}} \quad (\text{by Gaussian integral}) \quad (31)$$

$$= \sqrt{\frac{m}{2\pi i\delta t}}. \quad (32)$$

Thus, we have

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^N dq_k \left[\prod_{j=0}^N \sqrt{\frac{m}{2\pi i\delta t}} \right] e^{i \sum_{j=0}^N \delta t L(\dot{q}_j, \bar{q}_j)} \quad (33)$$

$$= \int \prod_{k=1}^N dq_k \left(\frac{m}{2\pi i\delta t} \right)^{\frac{N+1}{2}} e^{i \sum_{j=0}^N \delta t L(\dot{q}_j, \bar{q}_j)} \quad (34)$$

$$= \int \prod_{k=1}^N dq_k \left(\frac{m}{2\pi i\delta t} \right)^{\frac{N+1}{2}} e^{i \int_{t'}^{t''} dt L(\dot{q}(t), q(t))} \quad (\text{by definition of Riemann integral}). \quad (35)$$

Therefore, we can identify

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t} \right)^{\frac{N+1}{2}} \prod_{j=1}^N dq_j. \quad (36)$$

This is the explicit formula for $\mathcal{D}q$ in eq. (6.9).

(b) Now if we consider the case of a free particle, i.e. $V(Q) = 0$, then we have

$$L(\dot{q}, q) = \frac{1}{2} m \dot{q}^2, \quad (37)$$

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^N dq_k \left(\frac{m}{2\pi i\delta t} \right)^{\frac{N+1}{2}} e^{i \int_{t'}^{t''} dt \frac{1}{2} m \dot{q}^2} \quad (38)$$

$$= \int \prod_{k=1}^N dq_k \left(\frac{m}{2\pi i\delta t} \right)^{\frac{N+1}{2}} e^{i \sum_{j=0}^N \delta t \frac{1}{2} m \dot{q}_j^2}. \quad (39)$$

(40)

The terms in the exponent are given by:

$$\sum_{j=0}^N \delta t \frac{1}{2} m \dot{q}_j^2 = \sum_{j=0}^N \delta t \frac{1}{2} m \left(\frac{q_{j+1} - q_j}{\delta t} \right)^2 = \sum_{j=0}^N \frac{m}{2\delta t} (q_{j+1}^2 - 2q_{j+1}q_j + q_j^2). \quad (41)$$

Thus, we focus on the integral and compute it step by step:

$$\int dq_N \cdots \int dq_2 \exp \left(\sum_{j=2}^N \frac{im}{2\delta t} (q_{j+1}^2 - 2q_{j+1}q_j + q_j^2) \right) \int dq_1 \exp \left[\frac{im}{2\delta t} ((q_2^2 - 2q_2q_1 + q_1^2) + (q_1^2 - 2q_1q_0 + q_0^2)) \right] \quad (42)$$

$$= \int dq_N \cdots \int dq_2 \exp \left(\sum_{j=2}^N i\lambda (q_{j+1} - q_j)^2 \right) \int dq_1 \exp \left[i\lambda ((q_2 - q_1)^2 + (q_1 - q_0)^2) \right], \quad \text{where } \lambda = \frac{m}{2\delta t} \quad (43)$$

Before performing the integral over q_1 , we consider the following integral:

$$\int_{-\infty}^{+\infty} dx e^{i\alpha(x-\beta)^2} = \sqrt{\frac{i\pi}{\alpha}} \quad (44)$$

Then we also consider the more complicated integral:

$$\int_{-\infty}^{+\infty} dx e^{i\alpha(x-c_1)^2 + i\beta(x-c_2)^2} = \int_{-\infty}^{+\infty} dx e^{i(\alpha+\beta)x^2 - 2i(\alpha c_1 + \beta c_2)x + i(\alpha c_1^2 + \beta c_2^2)} \quad (45)$$

$$= e^{i\frac{\alpha\beta}{\alpha+\beta}(c_1-c_2)^2} \int_{-\infty}^{+\infty} dx e^{i(\alpha+\beta)(x - \frac{\alpha c_1 + \beta c_2}{\alpha+\beta})^2} \quad (46)$$

$$= e^{i\frac{\alpha\beta}{\alpha+\beta}(c_1-c_2)^2} \sqrt{\frac{i\pi}{\alpha+\beta}} = e^{i\frac{1}{\alpha+\beta}(c_1-c_2)^2} \sqrt{\frac{i\pi}{\alpha+\beta}}, \quad (47)$$

where I quoted the result from **Mathematica**. Now we can perform the integral over q_1 :

$$\int dq_1 \exp \left[i\lambda ((q_2 - q_1)^2 + (q_1 - q_0)^2) \right] \quad (48)$$

$$= \int dq_1 \exp \left[i\lambda (q_1 - q_2)^2 + i\lambda (q_1 - q_0)^2 \right] \quad (49)$$

$$= e^{i\frac{\lambda^2}{2\lambda}(q_2-q_0)^2} \sqrt{\frac{i\pi}{2\lambda}} \quad (50)$$

$$= e^{i\frac{\lambda}{2}(q_2-q_0)^2} \sqrt{\frac{i\pi}{2\lambda}}. \quad (51)$$

Thus, we have

$$\int dq_N \cdots \int dq_2 \exp \left(\sum_{j=2}^N i\lambda(q_{j+1} - q_j)^2 \right) \int dq_1 \exp \left[i\lambda \left((q_2 - q_1)^2 + (q_1 - q_0)^2 \right) \right] \quad (52)$$

$$= \sqrt{\frac{i\pi}{2\lambda}} \int dq_N \cdots \int dq_2 \exp \left(\sum_{j=2}^N i\lambda(q_{j+1} - q_j)^2 + i\frac{\lambda}{2}(q_2 - q_0)^2 \right) \quad (53)$$

$$= \sqrt{\frac{i\pi}{2\lambda}} \int dq_N \cdots \int dq_3 \exp \left(\sum_{j=3}^N i\lambda(q_{j+1} - q_j)^2 \right) \int dq_2 \exp \left(i\lambda(q_3 - q_2)^2 + i\frac{\lambda}{2}(q_2 - q_0)^2 \right) \quad (54)$$

$$= \sqrt{\frac{i\pi}{2\lambda}} \sqrt{\frac{i\pi}{\frac{3}{2}\lambda}} \int dq_N \cdots \int dq_3 \exp \left(\sum_{j=3}^N i\lambda(q_{j+1} - q_j)^2 + i\frac{\lambda}{3}(q_3 - q_0)^2 \right) \quad (55)$$

$$= \sqrt{\left(\frac{i\pi}{\lambda}\right)^2} \frac{1}{\sqrt{3}} \int dq_N \cdots \int dq_4 \exp \left(\sum_{j=4}^N i\lambda(q_{j+1} - q_j)^2 \right) \int dq_3 \exp \left(i\lambda(q_4 - q_3)^2 + i\frac{\lambda}{3}(q_3 - q_0)^2 \right) \quad (56)$$

$$= \cdots \quad (57)$$

$$= \sqrt{\left(\frac{i\pi}{\lambda}\right)^{N-1}} \frac{1}{\sqrt{N}} \int dq_N \exp \left(i\lambda(q_{N+1} - q_N)^2 + i\frac{\lambda}{N}(q_N - q_0)^2 \right) \quad (58)$$

$$= \sqrt{\left(\frac{i\pi}{\lambda}\right)^N} \frac{1}{\sqrt{N+1}} e^{i\frac{\lambda}{N+1}(q_{N+1} - q_0)^2}. \quad (59)$$

Combine with the prefactor, we have

$$\langle q'', t'' | q', t' \rangle = \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}} \sqrt{\left(\frac{i\pi}{\lambda} \right)^N} \frac{1}{\sqrt{N+1}} e^{i\frac{\lambda}{N+1}(q_{N+1} - q_0)^2} \quad (60)$$

$$= \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}} \sqrt{\left(\frac{i\pi}{\frac{m}{2\delta t}} \right)^N} \frac{1}{\sqrt{N+1}} e^{i\frac{m}{N+1}(q'' - q')^2} \quad (61)$$

$$= \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}} \left(\frac{2i\pi \delta t}{m} \right)^{\frac{N}{2}} \frac{1}{\sqrt{N+1}} e^{i\frac{m}{2(N+1)\delta t}(q'' - q')^2} \quad (62)$$

$$= \sqrt{\frac{m}{2\pi i (N+1)\delta t}} e^{i\frac{m}{2(N+1)\delta t}(q'' - q')^2} \quad (63)$$

$$= \sqrt{\frac{m}{2\pi i (t'' - t')}} e^{\frac{im}{2(t'' - t')}(q'' - q')^2}, \quad \text{where } t'' - t' = (N+1)\delta t. \quad (64)$$

Then restore \hbar by dimensional analysis, we have

$$\langle q'', t'' | q', t' \rangle = \sqrt{\frac{m}{2\pi i \hbar (t'' - t')}} e^{\frac{im}{2\hbar (t'' - t')}(q'' - q')^2}. \quad (65)$$

(c) We can also compute $\langle q'', t'' | q', t' \rangle$ by inserting a complete set of momentum eigenstates:

$$\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t'' - t')} | q' \rangle \quad (66)$$

$$= \int dp \langle q'' | e^{-iH(t'' - t')} | p \rangle \langle p | q' \rangle \quad (67)$$

$$= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}(t'' - t')} e^{ip(q'' - q')}, \quad (68)$$

where we have used $H = \frac{p^2}{2m}$ and $\langle p | q' \rangle = \frac{1}{\sqrt{2\pi}} e^{-ipq'}$. Now we can perform the integral over p :

$$\int_{-\infty}^{+\infty} dp e^{-i\frac{p^2}{2m}(t'' - t')} e^{ip(q'' - q')} = \int_{-\infty}^{+\infty} dp e^{-i\frac{t'' - t'}{2m} \left(p^2 - \frac{2m}{t'' - t'} (q'' - q') p \right)} \quad (69)$$

$$= \int_{-\infty}^{+\infty} dp e^{-i\frac{t'' - t'}{2m} \left(p - \frac{m}{t'' - t'} (q'' - q') \right)^2 + i\frac{m}{2(t'' - t')} (q'' - q')^2} \quad (70)$$

$$= e^{i\frac{m}{2(t'' - t')} (q'' - q')^2} \int_{-\infty}^{+\infty} dp e^{-i\frac{t'' - t'}{2m} \left(p - \frac{m}{t'' - t'} (q'' - q') \right)^2} \quad (71)$$

$$= e^{i\frac{m}{2(t'' - t')} (q'' - q')^2} \sqrt{\frac{2m\pi}{i(t'' - t')}}. \quad (72)$$

Thus, we have

$$\langle q'', t'' | q', t' \rangle = \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}(t'' - t')} e^{ip(q'' - q')} \quad (73)$$

$$= \frac{1}{2\pi} e^{i\frac{m}{2(t'' - t')} (q'' - q')^2} \sqrt{\frac{2m\pi}{i(t'' - t')}} \quad (74)$$

$$= \sqrt{\frac{m}{2\pi i(t'' - t')}} e^{\frac{im}{2(t'' - t')} (q'' - q')^2}. \quad (75)$$

This is exactly the same as the result we obtained in part (b). \square

Question 3

Problem 7.3

- (a) Use the Heisenberg equations of motion, $\dot{A} = i[H, A]$, to find explicit expressions for \dot{Q} and \dot{P} . Solve these to get the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of the Schrödinger-picture operators Q and P .
- (b) Write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and a^\dagger , where $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. Then, using your result from part (a), write the Heisenberg-picture operator $Q(t)$ and $P(t)$ in terms of a and a^\dagger .
- (c) Using your result from part (b), and $a|0\rangle = \langle 0|a^\dagger = 0$, verify eqs. (7.16) and (7.17).

Answer

- (a) First, we can compute \dot{Q} and \dot{P} using the Heisenberg equations of motion:

$$\dot{Q} = i[H, Q] = i\left[\frac{P^2}{2m} + \frac{1}{2}m\omega^2Q^2, Q\right] = i\frac{1}{2m}[P^2, Q] = \frac{P}{m}, \quad (76)$$

$$\dot{P} = i[H, P] = i\left[\frac{P^2}{2m} + \frac{1}{2}m\omega^2Q^2, P\right] = i\frac{1}{2}m\omega^2[Q^2, P] = -m\omega^2Q. \quad (77)$$

These are the equations of motion for a harmonic oscillator. Now we can solve these equations to get $Q(t)$ and $P(t)$:

$$\ddot{Q}(t) = \frac{\dot{P}}{m} = -\omega^2Q(t), \quad (78)$$

$$Q(t) = Q \cos \omega t + \frac{P}{m\omega} \sin \omega t, \quad (79)$$

$$P(t) = m\dot{Q}(t) = -m\omega Q \sin \omega t + P \cos \omega t. \quad (80)$$

Note that we have used the initial conditions $Q(0) = Q$ and $P(0) = P$ to determine the integration constants.

- (b) Next, we can write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and a^\dagger :

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad (81)$$

$$P = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger). \quad (82)$$

Then, using the result from part (a), we can write the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of a and a^\dagger :

$$Q(t) = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \cos \omega t + \frac{-i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger)}{m\omega} \sin \omega t \quad (83)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \cos \omega t - i\sqrt{\frac{\hbar}{2m\omega}}(a - a^\dagger) \sin \omega t \quad (84)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[a(\cos \omega t - i \sin \omega t) + a^\dagger(\cos \omega t + i \sin \omega t) \right] \quad (85)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[ae^{-i\omega t} + a^\dagger e^{i\omega t} \right], \quad (86)$$

$$P(t) = -m\omega \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \sin \omega t - i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) \cos \omega t \quad (87)$$

$$= -\sqrt{\frac{m\omega\hbar}{2}}(a + a^\dagger) \sin \omega t - i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) \cos \omega t \quad (88)$$

$$= -\sqrt{\frac{m\omega\hbar}{2}} \left[a(\sin \omega t + i \cos \omega t) + a^\dagger(\sin \omega t - i \cos \omega t) \right] \quad (89)$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}} \left[a(\cos \omega t - i \sin \omega t) - a^\dagger(\cos \omega t + i \sin \omega t) \right] \quad (90)$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}} \left[ae^{-i\omega t} - a^\dagger e^{i\omega t} \right]. \quad (91)$$

(c) Recall eqs. (7.14), (7.16) and (7.17):

$$\begin{aligned} G(t - t') &= \frac{i}{2\omega} \exp \left(-i\omega|t - t'| \right) \\ &= \frac{i}{2\omega} \left(\theta(t - t') e^{-i\omega(t-t')} + \theta(t' - t) e^{-i\omega(t'-t)} \right), \end{aligned} \quad (7.14)$$

$$\begin{aligned} \langle 0 | TQ(t_1) Q(t_2) | 0 \rangle &= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \frac{1}{i} \frac{\delta}{\delta f(t_2)} \langle 0 | 0 \rangle_f \Big|_{f=0} \\ &= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \left[\int_{-\infty}^{+\infty} dt' G(t_2 - t') f(t') \right] \langle 0 | 0 \rangle_f \Big|_{f=0} \\ &= \left[\frac{1}{i} G(t_2 - t_1) + (\text{term with } f's) \right] \langle 0 | 0 \rangle_f \Big|_{f=0} \\ &= \frac{1}{i} G(t_2 - t_1), \end{aligned} \quad (7.16)$$

$$\begin{aligned} \langle 0 | TQ(t_1) Q(t_2) Q(t_3) Q(t_4) | 0 \rangle &= \frac{1}{i^2} [G(t_1 - t_2) G(t_3 - t_4) \\ &\quad + G(t_1 - t_3) G(t_2 - t_4) \\ &\quad + G(t_1 - t_4) G(t_2 - t_3)]. \end{aligned} \quad (7.17)$$

Using the result from part (b), we can compute $\langle 0 | TQ(t_1) Q(t_2) | 0 \rangle$:

$$\langle 0 | TQ(t_1) Q(t_2) | 0 \rangle = \frac{\hbar}{2m\omega} \langle 0 | T \left[ae^{-i\omega t_1} + a^\dagger e^{i\omega t_1} \right] \left[ae^{-i\omega t_2} + a^\dagger e^{i\omega t_2} \right] | 0 \rangle \quad (92)$$

$$= \frac{\hbar}{2m\omega} \langle 0 | T \left[aae^{-i\omega(t_1+t_2)} + aa^\dagger e^{-i\omega t_1} e^{i\omega t_2} + a^\dagger a e^{i\omega t_1} e^{-i\omega t_2} + a^\dagger a^\dagger e^{i\omega(t_1+t_2)} \right] | 0 \rangle \quad (93)$$

$$= \frac{\hbar}{2m\omega} \langle 0 | T \left[aa^\dagger e^{-i\omega t_1} e^{i\omega t_2} \right] | 0 \rangle \quad (94)$$

$$= \frac{\hbar}{2m\omega} \langle 0 | T \left[(1 + a^\dagger a) e^{-i\omega t_1} e^{i\omega t_2} \right] | 0 \rangle \quad (95)$$

$$= \frac{\hbar}{2m\omega} \langle 0 | T e^{-i\omega t_1} e^{i\omega t_2} | 0 \rangle \quad (96)$$

$$= \frac{\hbar}{2m\omega} \left[\theta(t_1 - t_2) e^{-i\omega(t_1-t_2)} + \theta(t_2 - t_1) e^{-i\omega(t_2-t_1)} \right] \quad (97)$$

$$= \frac{1}{2\omega} \left[\theta(t_1 - t_2) e^{-i\omega(t_1-t_2)} + \theta(t_2 - t_1) e^{-i\omega(t_2-t_1)} \right], \quad \text{by setting } \hbar = m = 1, \quad (98)$$

$$= \frac{1}{2\omega} e^{-i\omega|t_1-t_2|} \quad (99)$$

$$= \frac{1}{i} G(t_2 - t_1) \quad (100)$$

where we have used $a|0\rangle = \langle 0|a^\dagger = 0$ and the definition of $G(t)$ in eq. (7.14). This verifies eq. (7.16). Next, we can compute $\langle 0 | TQ(t_1) Q(t_2) Q(t_3) Q(t_4) | 0 \rangle$:

$$\begin{aligned} & \langle 0 | TQ(t_1) Q(t_2) Q(t_3) Q(t_4) | 0 \rangle \\ &= \frac{\hbar^2}{4m^2\omega^2} \langle 0 | T \left[ae^{-i\omega t_1} + a^\dagger e^{i\omega t_1} \right] \left[ae^{-i\omega t_2} + a^\dagger e^{i\omega t_2} \right] \\ & \quad \left[ae^{-i\omega t_3} + a^\dagger e^{i\omega t_3} \right] \left[ae^{-i\omega t_4} + a^\dagger e^{i\omega t_4} \right] | 0 \rangle \end{aligned} \quad (101)$$

$$\begin{aligned} &= \frac{\hbar^2}{4m^2\omega^2} \langle 0 | T \left[\color{red}{aaaae^{-i\omega(t_1+t_2+t_3+t_4)}} + \color{red}{aaa^\dagger e^{-i\omega(t_1+t_2+t_3)} e^{i\omega t_4}} + \color{red}{aaa^\dagger a e^{-i\omega(t_1+t_2+t_4)} e^{i\omega t_3}} \right. \\ & \quad \left. + \color{blue}{aa^\dagger a^\dagger e^{-i\omega(t_1+t_2)} e^{i\omega(t_3+t_4)}} + \color{blue}{aa^\dagger a a e^{-i\omega(t_1+t_3+t_4)} e^{i\omega t_2}} + \color{blue}{aa^\dagger a a^\dagger e^{-i\omega(t_1+t_3)} e^{i\omega(t_2+t_4)}} \right. \\ & \quad \left. + \color{red}{aa^\dagger a^\dagger a e^{-i\omega(t_1+t_4)} e^{i\omega(t_2+t_3)}} + \color{blue}{aa^\dagger a^\dagger a^\dagger e^{-i\omega t_1} e^{i\omega(t_2+t_3+t_4)}} + \color{red}{a^\dagger a a a e^{-i\omega(t_2+t_3+t_4)} e^{i\omega t_1}} \right. \\ & \quad \left. + \color{red}{a^\dagger a a a^\dagger e^{-i\omega(t_2+t_3)} e^{i\omega(t_1+t_4)}} + \color{blue}{a^\dagger a a^\dagger a e^{-i\omega(t_2+t_4)} e^{i\omega(t_1+t_3)}} + \color{blue}{a^\dagger a a^\dagger a^\dagger e^{-i\omega t_2} e^{i\omega(t_1+t_3+t_4)}} \right. \\ & \quad \left. + \color{red}{a^\dagger a^\dagger a a e^{-i\omega(t_3+t_4)} e^{i\omega(t_1+t_2)}} + \color{blue}{a^\dagger a^\dagger a a^\dagger e^{-i\omega t_3} e^{i\omega(t_1+t_2+t_4)}} + \color{red}{a^\dagger a^\dagger a^\dagger a e^{-i\omega t_4} e^{i\omega(t_1+t_2+t_3)}} \right. \\ & \quad \left. + \color{red}{a^\dagger a^\dagger a^\dagger a^\dagger e^{i\omega(t_1+t_2+t_3+t_4)}} \right] | 0 \rangle, \quad \text{red terms vanish but blue terms can survive,} \end{aligned} \quad (102)$$

$$= \frac{\hbar^2}{4m^2\omega^2} \langle 0 | T \left[\color{blue}{aaa^\dagger a^\dagger e^{-i\omega(t_1+t_2)} e^{i\omega(t_3+t_4)}} + \color{blue}{aa^\dagger a a^\dagger e^{-i\omega(t_1+t_3)} e^{i\omega(t_2+t_4)}} \right] | 0 \rangle \quad (103)$$

$$\langle 0 | aaa^\dagger a^\dagger | 0 \rangle = \langle 0 | a(1 + a^\dagger a)a^\dagger | 0 \rangle = \langle 0 | aa^\dagger | 0 \rangle + \langle 0 | aa^\dagger aa^\dagger | 0 \rangle \quad (104)$$

$$= \langle 0 | (1 + a^\dagger a) | 0 \rangle + \langle 0 | (1 + a^\dagger a)(1 + a^\dagger a) | 0 \rangle = 2 \quad (105)$$

$$\langle 0 | aa^\dagger aa^\dagger | 0 \rangle = \langle 0 | (1 + a^\dagger a)(1 + a^\dagger a) | 0 \rangle = 1, \quad (106)$$

Hence, we have

$$\begin{aligned} & \langle 0 | T Q(t_1) Q(t_2) Q(t_3) Q(t_4) | 0 \rangle \\ &= \frac{\hbar^2}{4m^2\omega^2} \langle 0 | T \left[2e^{-i\omega(t_1+t_2)} e^{i\omega(t_3+t_4)} + e^{-i\omega(t_1+t_3)} e^{i\omega(t_2+t_4)} \right] | 0 \rangle \end{aligned} \quad (107)$$

$$= \frac{\hbar^2}{4m^2\omega^2} \left[\langle 0 | T e^{-i\omega(t_1+t_2)} e^{i\omega(t_3+t_4)} | 0 \rangle + \langle 0 | T e^{-i\omega(t_1+t_2)} e^{i\omega(t_3+t_4)} | 0 \rangle + \langle 0 | T e^{-i\omega(t_1+t_3)} e^{i\omega(t_2+t_4)} | 0 \rangle \right] \quad (108)$$

$$= \frac{\hbar^2}{4m^2\omega^2} \left[\langle 0 | T e^{-i\omega(t_1-t_3)} e^{-i\omega(t_2-t_4)} | 0 \rangle + \langle 0 | T e^{-i\omega(t_1-t_4)} e^{-i\omega(t_2-t_3)} | 0 \rangle + \langle 0 | T e^{-i\omega(t_1-t_2)} e^{-i\omega(t_3-t_4)} | 0 \rangle \right] \quad (109)$$

$$\begin{aligned} &= \frac{1}{4\omega^2} \left[\left(\theta(t_1 - t_3) e^{-i\omega(t_1-t_3)} + \theta(t_3 - t_1) e^{-i\omega(t_3-t_1)} \right) \left(\theta(t_2 - t_4) e^{-i\omega(t_2-t_4)} + \theta(t_4 - t_2) e^{-i\omega(t_4-t_2)} \right) \right. \\ &\quad + \left(\theta(t_1 - t_4) e^{-i\omega(t_1-t_4)} + \theta(t_4 - t_1) e^{-i\omega(t_4-t_1)} \right) \left(\theta(t_2 - t_3) e^{-i\omega(t_2-t_3)} + \theta(t_3 - t_2) e^{-i\omega(t_3-t_2)} \right) \\ &\quad \left. + \left(\theta(t_1 - t_2) e^{-i\omega(t_1-t_2)} + \theta(t_2 - t_1) e^{-i\omega(t_2-t_1)} \right) \left(\theta(t_3 - t_4) e^{-i\omega(t_3-t_4)} + \theta(t_4 - t_3) e^{-i\omega(t_4-t_3)} \right) \right], \end{aligned} \quad (110)$$

by setting $\hbar = m = 1$,

$$= \frac{1}{i^2} \left[G(t_1 - t_3) G(t_2 - t_4) + G(t_1 - t_4) G(t_2 - t_3) + G(t_1 - t_2) G(t_3 - t_4) \right] \quad (111)$$

where we have used the definition of $G(t)$ in eq. (7.14). This verifies eq. (7.17). \square

Question 4

Problem 7.4

Consider a harmonic oscillator in its ground state at $t = -\infty$. It is then subjected to an external force $f(t)$. Compute the probability $|\langle 0|0 \rangle_f|^2$ that the oscillator is still in its ground state at $t = +\infty$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\tilde{f}(E) = \int_{-\infty}^{+\infty} e^{iEt} f(t) dt$. Your answer should not involve any other unevaluated integrals.

Answer

Recall eqs. (7.10), (7.11) and (7.14):

$$\langle 0|0 \rangle_f = \exp \left[\frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right], \quad (7.10)$$

$$= \exp \left[\frac{i}{2} \int_{-\infty}^{+\infty} dt dt' f(t) G(t-t') f(t') \right] \quad (7.11)$$

$$= \exp \left[\frac{i}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{i}{2\omega} e^{-i\omega|t-t'|} f(t') \right], \quad \text{by eq. (7.14)}$$

$$|\langle 0|0 \rangle_f|^2 = \langle 0|0 \rangle_f \langle 0|0 \rangle_f^* \quad (112)$$

$$= \exp \left[\frac{i}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{i}{2\omega} e^{-i\omega|t-t'|} f(t') \right] \exp \left[- \frac{i}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{-i}{2\omega} e^{i\omega|t-t'|} f(t') \right] \quad (113)$$

$$= \exp \left[- \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} e^{-i\omega|t-t'|} f(t') - \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} e^{i\omega|t-t'|} f(t') \right] \quad (114)$$

$$= \exp \left[- \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} (e^{-i\omega|t-t'|} + e^{i\omega|t-t'|}) f(t') \right] \quad (115)$$

$$= \exp \left[- \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{\omega} \cos \omega|t-t'| f(t') \right] \quad (116)$$

$$= \exp \left[- \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} (e^{-i\omega(t-t')} + e^{i\omega(t-t')}) f(t') \right] \quad (117)$$

By the definition of Fourier transform, we have

$$\tilde{f}(E) = \int_{-\infty}^{+\infty} e^{iEt} f(t) dt. \quad (118)$$

Thus, we have

$$\int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} (e^{-i\omega(t-t')} + e^{i\omega(t-t')}) f(t') \quad (119)$$

$$= \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} e^{-i\omega(t-t')} f(t') + \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} e^{i\omega(t-t')} f(t') \quad (120)$$

$$= \frac{1}{2\omega} \int_{-\infty}^{+\infty} dt dt' f(t) e^{-i\omega t} e^{i\omega t'} f(t') + \frac{1}{2\omega} \int_{-\infty}^{+\infty} dt dt' f(t) e^{i\omega t} e^{-i\omega t'} f(t') \quad (121)$$

$$= \frac{1}{2\omega} \left(\int_{-\infty}^{+\infty} dt f(t) e^{-i\omega t} \right) \left(\int_{-\infty}^{+\infty} dt' f(t') e^{i\omega t'} \right) + \frac{1}{2\omega} \left(\int_{-\infty}^{+\infty} dt f(t) e^{i\omega t} \right) \left(\int_{-\infty}^{+\infty} dt' f(t') e^{-i\omega t'} \right) \quad (122)$$

$$= \frac{1}{2\omega} \tilde{f}(-\omega) \tilde{f}(\omega) + \frac{1}{2\omega} \tilde{f}(\omega) \tilde{f}(-\omega) \quad (123)$$

$$= \frac{1}{\omega} \tilde{f}(\omega) \tilde{f}(-\omega) \quad (124)$$

Therefore, we have

$$|\langle 0|0 \rangle_f|^2 = \exp \left[- \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' f(t) \frac{1}{2\omega} (e^{-i\omega(t-t')} + e^{i\omega(t-t')}) f(t') \right] \quad (125)$$

$$= \exp \left[- \frac{1}{2} \cdot \frac{1}{\omega} \tilde{f}(\omega) \tilde{f}(-\omega) \right] \quad (126)$$

$$= \exp \left[- \frac{1}{2\omega} |\tilde{f}(\omega)|^2 \right], \quad (127)$$

where we have used the fact that $\tilde{f}(-\omega)$ is the complex conjugate of $\tilde{f}(\omega)$. This is a manifestly real expression and does not involve any other unevaluated integrals. \square