

University of Minnesota  
School of Physics and Astronomy

**2026 Spring Physics 8502**  
**General Relativity II**  
Assignment Solution

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# Assignment 1 due on Wednesday September 10th at 5PM

## Question 1

### Answer

We can use tensor notation to reduce the work. From the constraint

$$\eta_{\alpha\beta}\Lambda^\alpha_\gamma\Lambda^\beta_\delta = \eta_{\gamma\delta}, \quad (1)$$

and the relation  $\gamma = \frac{1}{\sqrt{1-v^2}}$  (or  $\gamma^2 - \gamma^2 v^2 = \gamma^2(1 - v^2) = 1$ ), we can consider different  $\gamma, \delta$ , meaning that

- $(\gamma, \delta) = (0, 0)$ :

$$-(\Lambda^0_0)^2 + \sum_{i=1}^3 (\Lambda^i_0)^2 = -\gamma^2 + \gamma^2 \sum_{i=1}^3 (v_i)^2 = -\gamma^2 + \gamma^2 v^2 = -1 = \eta_{00} \quad (2)$$

- $(\gamma, \delta) = (0, i)$ :

$$-\Lambda^0_0\Lambda^0_i + \sum_{k=1}^3 \Lambda^k_0\Lambda^k_i \quad (3)$$

$$= -\gamma c v_i + \sum_{k=1}^3 \gamma v_k (a \delta_{ki} + b v_k v_i) \quad (4)$$

$$= -\gamma c v_i + \gamma a v_i + \gamma b v^2 v_i = \gamma v_i (-c + a + b v^2) = \eta_{0i} = 0 \quad (5)$$

Then we have  $c = a + b v^2$ .

- $(\gamma, \delta) = (i, 0)$

$$-\Lambda^0_i\Lambda^0_0 + \sum_{k=1}^3 \Lambda^k_i\Lambda^k_0 \quad (6)$$

$$= \gamma v_i (-c + a + b v^2) \quad (7)$$

- $(\gamma, \delta) = (i, i)$

$$- \Lambda^0_i \Lambda^0_i + \sum_{k=1}^3 \Lambda^k_i \Lambda^k_i \quad (8)$$

$$= -c^2 v_i^2 + \sum_{k=1}^3 (a\delta_{ki} + bv_k v_i)^2 \quad (9)$$

$$= -c^2 v_i^2 + \sum_{k=1}^3 (a^2 \delta_{ki} + 2ab\delta_{ki} v_k v_i + b^2 v_i^2 v_k^2) \quad (10)$$

$$= -c^2 v_i^2 + a^2 + 2abv_i^2 + b^2 v_i^2 v^2 = \eta_{ii} = 1 \quad (11)$$

- $(\gamma, \delta) = (i, j), i \neq j$

$$- \Lambda^0_i \Lambda^0_j + \sum_{k=1}^3 \Lambda^k_i \Lambda^k_j \quad (12)$$

$$= -c^2 v_i v_j + \sum_{k=1}^3 (a\delta_{ki} + bv_k v_i)(a\delta_{kj} + bv_k v_j) \quad (13)$$

$$= -c^2 v_i v_j + \sum_{k=1}^3 (a^2 \delta_{ki} \delta_{kj} + ab(\delta_{ki} v_k v_j + \delta_{kj} v_k v_i) + b^2 v_k^2 v_i v_j) \quad (14)$$

$$= -c^2 v_i v_j + 2abv_i v_j + b^2 v^2 v_i v_j = \eta_{ij} = 0 \quad (15)$$

Then we have  $c^2 = b^2 v^2 + 2ab$

Combining the above information, we can have

$$c = a + bv^2 \quad (16)$$

$$c^2 = a^2 + b^2 v^4 + 2abv^2 = b^2 v^2 + 2ab. \quad (17)$$

Also, we have

$$-c^2 v_i^2 + a^2 + 2abv_i^2 + b^2 v_i^2 v^2 = 1 \quad (18)$$

$$\rightarrow 1 = -(b^2 v^2 + 2ab)v_i^2 + a^2 + 2abv_i^2 + b^2 v_i^2 v^2 \quad (19)$$

$$= a^2 \quad (20)$$

Hence, from Eq. 17 and Eq. 20, we have

$$c^2 = 1 + b^2 v^4 + 2abv^2 = b^2 v^2 + 2ab \quad (21)$$

$$\rightarrow a = \pm 1 = \frac{1 + b^2 v^4 - b^2 v^2}{2b(1 - v^2)} = \frac{1 + b^2 v^2(v^2 - 1)}{2b(1 - v^2)} = \frac{\gamma^2 - b^2 v^2}{2b} \quad (22)$$

$$\rightarrow b^2 v^2 \pm 2b - \gamma^2 = 0 \quad (23)$$

For  $a = +1$ , we have

$$b = \frac{-1 \pm \sqrt{1 + \gamma^2 v^2}}{v^2} = \frac{-1 \pm \gamma}{v^2}, \quad (24)$$

and for  $a = -1$ , we have

$$b = \frac{1 \pm \gamma}{v^2}. \quad (25)$$

Hence, we choose  $a = 1$  and  $b = \frac{-1+\gamma}{v^2}$  by convention and have

$$c = a + bv^2 = 1 + (-1 + \gamma) = +\gamma \quad (26)$$

$$(27)$$

Finally, we derive

$$\Lambda^i_j = a(v)\delta_{ij} + b(v)v_i v_j = \delta_{ij} + \frac{\gamma - 1}{v^2} v_i v_j \quad (28)$$

$$\Lambda^0_j = c(v)v_j = \gamma v_j, \quad (29)$$

which are mentioned in the Week 1 lecture. □

## Question 2

### Answer

In the  $\mathcal{O}$  frame, we set the condition for the right-hand side of the rod:

$$\begin{aligned}x_{emit}^\mu &= (0, L/2, D, 0), x_{receive}^\mu = (t, vt, 0, 0), \\ \Delta x^\mu &= x_{receive}^\mu - x_{emit}^\mu = (t, vt - L/2, -D, 0) \\ \Delta x^\mu \Delta x_\mu &= 0 = t^2 - (L/2 - vt)^2 - D^2.\end{aligned}$$

Also, with  $dt = t, dx = vt - L/2, dy = -D$ , we have

$$\begin{aligned}\Delta t' &= \gamma(t + v(vt - L/2)) \\ \Delta x' &= \gamma(v + (vt - L/2)) \\ \Delta y' &= -D\end{aligned}$$

We can substitute the solution  $t$  to solve the  $dx'$  and  $dt'$ , and define the opening angle

$$\tan \theta'/2 = \frac{\Delta x'}{\Delta y'}. \quad (30)$$

Note that this is the calculation for right-hand side of the rod. Hence, it should be denoted as  $\theta'_R$ . Now we can apply the procedure to the left-hand side of the rod. After plugging all information into **Mathematica** and using the equation, we have

$$\tan(\theta'_R + \theta'_L) = \frac{\tan \theta'_R + \tan \theta'_L}{1 - \tan \theta'_R \tan \theta'_L} = \frac{4DL}{4D^2 - L^2} = \frac{2\frac{L}{2D}}{1 - \left(\frac{L}{2D}\right)^2} = \tan \theta. \quad (31)$$

Hence, we prove that the opening angle keeps the same value. □