University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

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HW1 Due to September 23 11:59 PM

Question 1

Problem 1.2

With the Hamiltonian of eq. (1.32), show that the state defined in eq. (1.33) obeys the abstract Schrodinger equation, eq. (1.1), if and only if the wave function obeys eq. (1.30). Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations of eq. (1.31), and to fermions, where the particle creation and annihilation operators obey the anti-commutation relations of eq. (1.38).

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle \tag{1.1}$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U\left(\mathbf{x}_j\right)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1} V\left(\mathbf{x}_j - \mathbf{x}_k\right)\right]\psi \tag{1.30}$$

$$\begin{aligned} [a(\mathbf{x}), a\left(\mathbf{x}'\right)] &= 0 \\ \left[a^\dagger(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] &= 0 \\ \left[a(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] &= \delta^3\left(\mathbf{x}-\mathbf{x}'\right) \end{aligned} \tag{1.31}$$

$$\begin{aligned} H = &\int d^3x a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right)a(\mathbf{x}) \\ &+ \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \end{aligned} \tag{1.32}$$

$$|\psi,t\rangle = \int d^3x_1\ldots d^3x_n \psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right) a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{1.33}$$

Answer

We first consider boson case, and then we have

$$\begin{aligned} LHS &= i\hbar\frac{\partial}{\partial t}|\psi,t\rangle && (1) \\ =& i\hbar\frac{\partial}{\partial t}\int d^3x_1\ldots d^3x_n\psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (2) \\ =& \int d^3x_1\ldots d^3x_n i\hbar\frac{\partial}{\partial t}\psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (3) \\ =& \int d^3x_1\ldots d^3x_n\left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U\left(\mathbf{x}_j\right)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V\left(\mathbf{x}_j-\mathbf{x}_k\right)\right]\psi a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (4) \end{aligned}$$

$$
\begin{aligned}
&RHS = H|\psi, t\rangle && (5)\\
&= \left[\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) + \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x})\right] |\psi, t\rangle && (6)\\
&= \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle && (7)\\
&+\frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle && (8)
\end{aligned}
$$

For the term in Equation 7, by considering $[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}')$, we have

$$
\begin{aligned}
&a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) && (9)\\
=&[a^\dagger(\mathbf{x}_1) a(\mathbf{x}) + \delta^3(\mathbf{x}-\mathbf{x}_1)] a^\dagger\left(\mathbf{x}_2\right) \dots a^\dagger\left(\mathbf{x}_n\right) && (10)\\
=&a^\dagger(\mathbf{x}_1) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_2\right) \dots a^\dagger\left(\mathbf{x}_n\right) + \delta^3(\mathbf{x}-\mathbf{x}_1) a^\dagger\left(\mathbf{x}_2\right) \dots a^\dagger\left(\mathbf{x}_n\right) && (11)\\
=&a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) a(\mathbf{x}) + \sum_{j=1}^n \delta^3(\mathbf{x}-\mathbf{x}_j) a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) && (12)\\
=&0 + \sum_{j=1}^n \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) && (13)
\end{aligned}
$$

we can drop the first term in Equation 12 since this term will act on the $|0\rangle$, giving 0. Hence, we have

$$
\begin{aligned}
&\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle && (14)\\
=&\sum_{j=1}^n \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j |0\rangle && (15)\\
=&\sum_{j=1}^n \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j |0\rangle && (16)\\
=&\sum_{j=1}^n \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \mathcal{O}_j |0\rangle && (17)\\
=&\sum_{j=1}^n \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \mathcal{O}_j |0\rangle && (18)\\
=&\sum_{j=1}^n \int d^3x_1 \dots d^3x_n \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle && (19)\\
=&\int d^3x_1 \dots d^3x_n \sum_{j=1}^n \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) |0\rangle, && (20)
\end{aligned}
$$

where $a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n) = a^\dagger(\mathbf{x}_j)\mathcal{O}_j$ since they (boson fields) commute. Now, we do the same thing for the term in Equation 8, we have

$$
\begin{aligned}
&a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n) &(21)\\
=&a(\mathbf{y})\sum_{j=1}^{n}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n) &(22)\\
=&\sum_{i\neq j}^{n}\sum_{j=1}^{n}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}, \quad \mathcal{T}_{ij} = \prod_{k\neq i,j}^{n} a^\dagger(\mathbf{x}_k). &(23)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\int d^3x_1\dots d^3x_n\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle &(24)\\
=&\frac{1}{2}\int d^3x d^3y\int d^3x_1\dots d^3x_n V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle &(25)\\
=&\frac{1}{2}\int d^3x d^3y\int d^3x_1\dots d^3x_n V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\sum_{i\neq j}^{n}\sum_{j=1}^{n}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle &(26)\\
=&\sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}\int d^3x d^3y\int d^3x_1\dots d^3x_n V(\mathbf{x}-\mathbf{y})\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})\mathcal{T}_{ij}|0\rangle &(27)\\
=&\sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}\int d^3x_1\dots d^3x_n V(\mathbf{x}_j-\mathbf{x}_i)\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij}|0\rangle &(28)\\
=&\int d^3x_1\dots d^3x_n\sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}V(\mathbf{x}_j-\mathbf{x}_i)\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle &(29)\\
=&\int d^3x_1\dots d^3x_n\sum_{j=1}^{n}\sum_{k=1}^{j-1}V(\mathbf{x}_j-\mathbf{x}_k)\,\psi(\mathbf{x}_1,\dots,\mathbf{x}_n;t)\,a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)|0\rangle, &(30)
\end{aligned}
$$

where $a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij} = a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)$ for the same reason. Hence, we have proved the $LHS = RHS$ and Equation 1.1 for the boson field case.

For fermion fields, the only difference is the anti-commutation relation. We start from Equation 7 again, by considering$\{a(\mathbf{x}), a^\dagger(\mathbf{x})\} = \delta^3(\mathbf{x}-\mathbf{x}')$, and we have

$$
\begin{aligned}
&a(\mathbf{x})a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n) &(31)\\
=&[-a^\dagger(\mathbf{x}_1)a(\mathbf{x})+\delta^3(\mathbf{x}-\mathbf{x}_1)]a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_n) &(32)\\
=&(-1)^n a^\dagger(\mathbf{x}_1)\dots a^\dagger(\mathbf{x}_n)\,a(\mathbf{x})+\sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n) &(33)\\
=&0+\sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n), &(34)
\end{aligned}
$$

where the 0 term comes from the same reason. Then the term in Equation 7 is given by

$$\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{35}$$

$$= \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{36}$$

$$= \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) \sum_{j=1}^{n} (-1)^j \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{37}$$

$$= \sum_{j=1}^{n} \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) (-1)^j \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{38}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) (-1)^j \mathcal{O}_j |0\rangle \tag{39}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \dots d^3x_n \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) (-1)^j a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{40}$$

$$= \int d^3x_1 \dots d^3x_n \sum_{j=1}^{n} \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \dots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle, \tag{41}$$

where $a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) = (-1)^{j-1} a^\dagger(\mathbf{x}_j) \mathcal{O}_j$ by anti-commutation relation of fermion fields. Next, given the term in Equation 8, we have

$$a(\mathbf{y}) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \dots a^\dagger\left(\mathbf{x}_n\right) \tag{42}$$

$$= a(\mathbf{y}) \sum_{j=1}^{n} (-1)^{j-1} \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \tag{43}$$

$$= \sum_{i<j}^{n} \sum_{j=1}^{n} (-1)^{i-1} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij} + \sum_{i>j}^{n} \sum_{j=1}^{n} (-1)^{i-2} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij}, \tag{44}$$

where $\mathcal{T}_{ij} = \prod_{k\neq i,j}^{n} a^\dagger(\mathbf{x}_k)$. Now, we can simplify Equation 8 and it is given by

$$
\begin{aligned}
&\frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right) a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (45)\\
=&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi a(\mathbf{y}) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (46)\\
=&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i<j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle \\
+&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i>j}^{n}\sum_{j=1}^{n}(-1)^{i-2}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle && (47)\\
=&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i<j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1} V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle \\
+&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i>j}^{n}\sum_{j=1}^{n}(-1)^{i-2}(-1)^{j-1} V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle && (48)\\
=&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i\neq j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1} V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle && (49)\\
=&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i\neq j}^{n}\sum_{j=1}^{n} V(\mathbf{x}_j-\mathbf{x}_i)\psi a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (50)\\
=&\int d^3x_1 \dots d^3x_n \sum_{j=1}^{n}\sum_{k=1}^{j-1} V\left(\mathbf{x}_j-\mathbf{x}_k\right)\psi a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle, && (51)
\end{aligned}
$$

where $(-1)^{i-1}(-1)^{j-1}a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij} = a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)$ by anti-commutation relation. In summary, we have proved both cases for boson fields and fermion fields. □

Question 2

Problem 2.3

Verify that eq. (2.16) follows from eq. (2.14).

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \tag{2.14}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma) \tag{2.16}$$

$$= i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{2.16}$$

Answer

Considering an infinitesimal transformation in $U(\Lambda) = 1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}$ and $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$, now we get

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) M^{\mu\nu} \left(1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) \tag{52}$$

$$\rightarrow \delta\omega_{\alpha\beta}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\alpha\beta}] = \delta\omega_{\rho\sigma}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\rho\sigma}] \tag{53}$$

$$RHS = (\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma})M^{\rho\sigma} \tag{54}$$

$$\rightarrow \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} = \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} \tag{55}$$

$$= g^{\nu\rho}\delta\omega_{\rho\sigma}M^{\mu\sigma} + g^{\sigma\mu}\delta\omega_{\sigma\rho}M^{\rho\nu} = \delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\nu}\right), \tag{56}$$

we only consider the linear term $\delta\omega$. We can further simplify it to

$$[M^{\mu\nu}, M^{\rho\sigma}] = \frac{2\hbar}{i}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\nu}\right) = 2i\hbar\left(g^{\sigma\mu}M^{\rho\nu} - g^{\nu\rho}M^{\mu\sigma}\right) = 2i\hbar\left(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right) \tag{57}$$

$$= -[M^{\nu\mu}, M^{\rho\sigma}] = 2i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma}\right) \tag{58}$$

Finally, we have

$$[M^{\mu\nu}, M^{\rho\sigma}] \tag{59}$$

$$= \frac{1}{2}\left([M^{\mu\nu}, M^{\rho\sigma}] - [M^{\nu\mu}, M^{\rho\sigma}]\right) \tag{60}$$

$$= i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right). \tag{61}$$

□

Question 3

Problem 2.8

(a) Let $\Lambda = 1 + \delta\omega$ in eq.(2.26), and show that

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x),$$

where

$$\mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

(b) Show that $[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x)$.

(c) Prove the *Jacobi identity*, $[[A, B], C]+[[B, C], A]+[[C, A], B] = 0$. Hint: write out all the commutations.

(d) Use your results from parts (b) and (c) to show that

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \tag{2.31}$$

(e) Simplify the right-hand side of eq. (2.31) as much as possible.

(f) Use your results from part (e) to verify eq. (2.16), up to the possibility of a term on the right-hand side that commutes with $\varphi(x)$ and its derivatives. (Such a term, called a *central charge*, in fact does not arise for the Lorentz algebra.)

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \tag{2.26}$$

Answer

(a)

$$LHS = U^{-1}(\Lambda)\varphi(x)U(\Lambda) \tag{62}$$

$$= \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \tag{63}$$

$$\rightarrow \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\varphi(x), M^{\mu\nu}] \tag{64}$$

$$RHS = \varphi(\Lambda^{-1}x) = \varphi((\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)x^\nu) = \phi(x) - \delta\omega^\mu{}_\nu x^\nu\partial_\mu\phi(x) \tag{65}$$

$$\rightarrow -\delta\omega^\mu{}_\nu x^\nu\partial_\mu\varphi(x) = -\delta\omega_{\mu\nu}x^\nu\partial^\mu\varphi(x) = \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\varphi(x), \tag{66}$$

we only focus on the linear term $\delta\omega$. Now we have

$$[\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi(x) = \mathcal{L}^{\mu\nu}\varphi(x). \tag{67}$$

(b)

$$[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\varphi(x)M^{\rho\sigma} - M^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{68}$$

$$=\mathcal{L}^{\mu\nu}[\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x). \tag{69}$$

(c)

$$[[A,B],C] + [[B,C],A] + [[C,A],B] \tag{70}$$

$$=(CAB - CBA - ABC + BAC) + (ABC - ACB - BCA + CBA) + (BCA - BAC - CAB + ACB) \tag{71}$$

$$=0. \tag{72}$$

(d)

By the Jacobi identity, we have

$$0 = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\mu\nu}, M^{\rho\sigma}], \varphi(x)] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \tag{73}$$

$$\rightarrow[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \tag{74}$$

$$=[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] - [[\varphi(x), M^{\rho\sigma}], M^{\mu\nu}] \tag{75}$$

$$=\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{76}$$

$$=(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \tag{77}$$

(e)

For the result in Equation 77, considering the relation $\partial^\mu x^\nu\varphi(x) = (g^{\mu\nu} + x^\nu\partial^\mu)\varphi(x)$, we can have

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \frac{\hbar}{i}\frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)(x^\rho\partial^\sigma - x^\sigma\partial^\rho)\varphi(x) \tag{78}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[x^\mu(g^{\nu\rho} + x^\rho\partial^\nu)\partial^\sigma - x^\nu(g^{\mu\rho} + x^\rho\partial^\mu)\partial^\sigma - x^\mu(g^{\nu\sigma} + x^\sigma\partial^\nu)\partial^\rho + x^\nu(g^{\mu\sigma} + x^\sigma\partial^\mu)\partial^\rho\right]\varphi(x) \tag{79}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[g^{\nu\rho}x^\mu\partial^\sigma - g^{\mu\rho}x^\nu\partial^\sigma - g^{\nu\sigma}x^\mu\partial^\rho + g^{\mu\sigma}x^\nu\partial^\rho + x^\mu x^\rho\partial^\nu\partial^\sigma - x^\nu x^\rho\partial^\mu\partial^\sigma - x^\mu x^\sigma\partial^\nu\partial^\rho + x^\nu x^\sigma\partial^\mu\partial^\sigma\right]\varphi(x). \tag{80}$$

$$\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{81}$$

$$= \left(\frac{\hbar}{i}\right)^2 [g^{\sigma\mu}\, x^\rho\partial^\nu - g^{\rho\mu}\, x^\sigma\partial^\nu - g^{\sigma\nu}\, x^\rho\partial^\mu + g^{\rho\nu}\, x^\sigma\partial^\mu +\, x^\rho x^\mu\, \partial^\sigma\partial^\nu - x^\rho x^\nu\, \partial^\sigma\partial^\mu - x^\sigma x^\mu\, \partial^\rho\partial^\nu + x^\sigma x^\nu\, \partial^\rho\partial^\mu]\, \varphi(x). \tag{82}$$

Using simpler forms to express:

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[g^{\nu\rho}\, x^\mu\partial^\sigma - g^{\mu\rho}\, x^\nu\partial^\sigma - g^{\nu\sigma}\, x^\mu\partial^\rho + g^{\mu\sigma}\, x^\nu\partial^\rho$$
$$+\, x^\mu x^\rho\, \partial^\nu\partial^\sigma - x^\mu x^\sigma\, \partial^\nu\partial^\rho - x^\nu x^\rho\, \partial^\mu\partial^\sigma + x^\nu x^\sigma\, \partial^\mu\partial^\rho\Big]\varphi(x), \tag{83}$$

$$\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[g^{\sigma\mu}\, x^\rho\partial^\nu - g^{\rho\mu}\, x^\sigma\partial^\nu - g^{\sigma\nu}\, x^\rho\partial^\mu + g^{\rho\nu}\, x^\sigma\partial^\mu$$
$$+\, x^\rho x^\mu\, \partial^\sigma\partial^\nu - x^\sigma x^\mu\, \partial^\rho\partial^\nu - x^\rho x^\nu\, \partial^\sigma\partial^\mu + x^\sigma x^\nu\, \partial^\rho\partial^\mu\Big]\varphi(x). \tag{84}$$

Combining those two results together, it gives

$$(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[g^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) - g^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu)$$
$$- g^{\nu\sigma}(x^\mu\partial^\rho - x^\rho\partial^\mu) + g^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu)\Big]\varphi(x) \tag{85}$$

$$= \frac{\hbar}{i}\Big(g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\sigma}\mathcal{L}^{\mu\rho} + g^{\mu\sigma}\mathcal{L}^{\nu\rho}\Big)\varphi(x) \tag{86}$$

$$= i\hbar\Big(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\Big)\varphi(x). \tag{87}$$

Actually, it looks similar to

$$[M^{\mu\nu}, M^{\rho\sigma}]$$
$$= i\hbar\, (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \tag{2.16}$$

(f)

Now we assume there is a non trivial term $\mathcal{C}$ on $[M^{\mu\nu}, M^{\rho\sigma}]$, giving that

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\, (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C})\,, \tag{88}$$

where C can commutes with φ and its derivatives. Now, we have

$$
\begin{aligned}
&[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] && (2.31)\\
=&i\hbar[\varphi(x), (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C})] && (89)\\
=&i\hbar\bigg(g^{\nu\sigma}[\varphi(x), M^{\mu\rho}] + g^{\mu\rho}[\varphi(x), M^{\nu\sigma}] - g^{\mu\sigma}[\varphi, M^{\nu\rho}] - g^{\nu\rho}[\varphi(x), M^{\mu\sigma}] + [\varphi(x), \mathcal{C}]\bigg) && (90)\\
=&i\hbar\big(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\big)\varphi(x) = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) && (91)\\
=&[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\varphi(x). && (92)
\end{aligned}
$$

Hence, with the central charge $\mathcal{C}$, the relation still holds. $\square$

Question 4

Problem 2.9

Let us write

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau + \frac{i}{2\hbar}\delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\tau, \tag{2.32}$$

where

$$(S_V^{\mu\nu})^\rho{}_\tau \equiv \frac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau) \tag{2.33}$$

are matrices which constitute the *vector representation* of the Lorentz generators.

(a) Let $\Lambda = 1 + \delta\omega$ in eq. (2.27), and show that

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x) \tag{2.34}$$

(b) Show that the matrices $(S_V^{\mu\nu})$ must have the same commutation relations as the operators $M^{\mu\nu}$. Hint: see the previous problem.

(c) For a rotation by an angle θ about the z axis, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Show that

$$\Lambda = \exp\left(-i\theta S_V^{12}/\hbar\right). \tag{2.36}$$

(d) For a boost by *rapidity* η in the z direction, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}. \tag{2.37}$$

Show that

$$\Lambda = \exp\left(+i\eta S_V^{30}/\hbar\right). \tag{2.38}$$

$$U^{-1}(\Lambda)\partial^\mu\varphi(x)U(\Lambda) = \Lambda^\mu{}_\rho\overline{\partial}^\rho\varphi(\Lambda^{-1}x), \quad \overline{x}^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu, \quad \overline{\partial}^\mu = (\Lambda^{-1})^\mu{}_\nu\partial^\nu \tag{2.27}$$

Answer

(a)

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \tag{93}$$

$$\rightarrow \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\partial^\rho\varphi(x), M^{\mu\nu}] \tag{94}$$

$$RHS = \Lambda^\mu{}_\rho\overline{\partial}^\rho\varphi(\Lambda^{-1}x) \tag{95}$$

$$= \Lambda^\mu{}_\rho(\Lambda^{-1})^\rho{}_\tau\partial^\tau\varphi((\Lambda^{-1})^\alpha{}_\nu x^\nu) \tag{96}$$

$$= (\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\rho{}_\tau - \delta\omega^\rho{}_\tau)\partial^\tau\varphi((\delta^\alpha{}_\nu - \delta\omega^\alpha{}_\nu)x^\nu) \tag{97}$$

$$= (\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\rho{}_\tau - \delta\omega^\rho{}_\tau)\partial^\tau\left[\varphi(x) - \delta\omega^\alpha{}_\nu x^\nu\partial_\alpha\varphi(x)\right] \tag{98}$$

$$= (\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\rho{}_\tau - \delta\omega^\rho{}_\tau)\left[\partial^\tau\varphi(x) - \delta\omega^\alpha{}_\nu(\delta^\nu_\tau\partial_\alpha + x^\nu\partial_\alpha\partial_\tau)\varphi(x)\right] \tag{99}$$

$$= \partial^\mu\varphi(x) + \delta\omega^\mu{}_\rho\partial^\rho\varphi(x) - \delta\omega^\rho{}_\tau\partial^\tau\varphi(x) - \delta\omega^\alpha{}_\nu(\delta^\nu_\mu\partial_\alpha + x^\nu\partial_\alpha\partial_\mu)\varphi(x) \tag{100}$$

$$= \partial^\mu\varphi(x) + \delta\omega_{\mu\rho}\partial^\rho\varphi(x) - \delta\omega_{\rho\tau}\partial^\tau\varphi(x) - \delta\omega_{\nu\alpha}(\delta^\nu_\mu\partial_\alpha + x^\nu\partial_\alpha\partial_\mu)\varphi(x) \tag{101}$$

If someting is antisymmetric, then we have $\delta\omega_{\mu\rho} = -\delta\omega_{\rho\mu}$. Now we get.

(b)

(c)

(d)

Question 5

Problem 3.1

Derive eq. (3.29) from eqs. (3.21), (3.24), and (3.28).

Question 6

Problem 3.5

Consider a complex (that is, non-hermitian) scalar field φ with Lagrangian density

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0.$$

(a) Show that φ obeys the Klein-Gordon equation.

(b) Treat φ and $\varphi^\dagger$ as independent fields, and find the conjugate momentum for each. Compute the Hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).

(c) Write the mode expansion of φ as

$$\varphi(x) = \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}].$$

Express $a(\mathbf{k})$ and $b(\mathbf{k})$ in terms of φ and $\varphi^\dagger$ and their time derivatives.

(d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates.

(e) Express the Hamiltonian in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?