University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I

Assignment Solution

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Question 1

Gell-Mann Okubo for the baryon octet

The generators $t_a(a=1,\ldots,8)$ of SU(3) are normalized as $t_a=\frac{\lambda_a}{2}$ with $\operatorname{Tr}(t_a t_b)=\frac{1}{2}\delta_{ab}$, where λ_a are the Gell-Mann matrices. They satisfy $[t_a,t_b]=if_{abc}t_c$ and $\{t_a,t_b\}=\frac{1}{3}\delta_{ab}\mathbf{1}+d_{abc}t_c$, where f_{abc} are totally antisymmetric and d_{abc} are totally symmetric structure constants.

Let B and $\bar{B}$ be the baryon octet 3×3 traceless matrices, expanded in the generator basis as $B=B^i t_i$ and $\bar{B}=\bar{B}^i t_i$ where $B^i,\bar{B}^i$ are the adjoint components. Define the two bilinear combinations $O_A\equiv[\bar{B},B]=\bar{B}B-B\bar{B}$ and $O_S\equiv\{\bar{B},B\}-\frac{2}{3}\mathbf{1}\operatorname{Tr}(\bar{B}B)$.

(a) Show that both O_A and O_S are traceless and therefore transform in the adjoint (octet) representation.

(b) Expand O_A and O_S in components using the generator basis and show that $O_A=i\left(\bar{B}^i B^j\right)f_{ijk}t_k$ and $O_S=\left(\bar{B}^i B^j\right)d_{ijk}t_k$, so that $\left(O_A\right)^k=if_{ijk}\bar{B}^iB^j$ and $\left(O_S\right)^k=d_{ijk}\bar{B}^iB^j$.

(c) Introduce a flavor-breaking spurion $H_8=H_8^i t_i$, with real components H_8^i. Construct the two independent SU(3)-invariant mass terms:

$$S_f=\left(O_A\right)_b^a\left(H_8\right)_a^b,\quad S_d=\left(O_S\right)_b^a\left(H_8\right)_a^b \tag{1}$$

Assuming H_8 points in the 8 -direction (i.e. $H_8^i\propto\delta_{i8}$), argue that S_f and S_d correspond to the f-type and d-type symmetry breaking terms in the baryon mass operator, respectively.

(d) Given that an adjoint operator O_8 acts on an octet state B as $O_8(B)=[O_8,B]$, show that the invariant scalars in (c) are equivalent to the matrix elements $S_f\propto\langle\bar{B}|t_8|B\rangle\equiv\operatorname{Tr}\left(\bar{B}\left[t_8,B\right]\right)$ and $S_d\propto\langle\bar{B}|d_{8ij}t_it_j|B\rangle\equiv\operatorname{Tr}\left(\bar{B}\left[d_{8ij}\left[t_i,\left[t_j,B\right]\right]\right)\right.$.

(e) Hence, argue that for each entry B_{ij} of the baryon octet matrix, $S_f\propto Y$ and $S_d\propto I(I+1)-Y^2/4$ where I,Y are the isospin and hypercharge of the baryon B, respectively, thereby reproducing the Gell-Mann-Okubo mass formula for the baryon octet.

(Hint: Verify, entrywise, that $\left[\frac{2}{\sqrt{3}}t_8,B\right]=YB$ and the normalized operator $\frac{2}{\sqrt{3}}d_{8ij}\left[t_i,\left[t_j,B\right]\right]+\frac{1}{3}\left[t_i,\left[t_i,B\right]\right]=\left(I(I+1)-Y^2/4\right)B$ acts diagonally on each baryon field. The $\left[t_i,\left[t_i,B\right]\right]$ term is the adjoint Casimir ($SU(3)$ singlet) which just shifts all octet components uniformly so that the Λ eigenvalue becomes 0 . It can be absorbed into the overall singlet part of the GMO formula. On the diagonal remember B_{11},B_{22} and B_{33} mix Σ^0 and Λ, so $B_{\text{diag}}=\Sigma^0\operatorname{diag}(1,-1,0)/\sqrt{2}+\Lambda\operatorname{diag}(1,1,-2)/\sqrt{6})$.

Answer

(a)

To show that both O_A and O_S are traceless, we first understand their definitions:

$$O_A = [\bar{B}, B] = \bar{B}B - B\bar{B} = [\bar{B}, B] \tag{2}$$
$$= [\bar{B}^i t_i, B^j t_j] = \bar{B}^i B^j [t_i, t_j] = i\bar{B}^i B^j f_{ijk} t_k \tag{3}$$

Taking the trace of O_A:

$$\mathrm{Tr}(O_A) = \mathrm{Tr}(iBarB^i B^j f_{ijk} t_k) = i\bar{B}^i B^j f_{ijk}\,\mathrm{Tr}(t_k) = 0. \tag{4}$$

Similarly, for O_S:

$$O_S = \{\bar{B}, B\} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}B) = \bar{B}B + B\bar{B} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}B) \tag{5}$$
$$= (\bar{B}^i t_i)(B^j t_j) + (B^j t_j)(\bar{B}^i t_i) - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}^i t_i B^j t_j) \tag{6}$$
$$= \bar{B}^i B^j \{t_i, t_j\} - \frac{2}{3}\mathbf{1}\,\mathrm{Tr}(\bar{B}^i B^j t_i t_j) \tag{7}$$
$$= \bar{B}^i B^j \left(\frac{1}{3}\delta_{ij}\mathbf{1} + d_{ijk} t_k\right) - \frac{2}{3}\mathbf{1}\left(\frac{1}{2}\bar{B}^i B^j \delta_{ij}\right) \tag{8}$$
$$= \bar{B}^i B^j d_{ijk} t_k \tag{9}$$

Question 2

ρ-ω mixing

The vector mesons $\rho(770)$ and $\omega(782)$ are very close in mass. For this reason the effects of isospin violation are somewhat enhanced in these mesons and can be parametrized in terms of ρ-ω mixing. Namely, the physical ρ^0 and ω mesons can be viewed as orthogonal mixed states of a pure isospin triplet and isospin singlet:

$$\rho^0 = \cos\theta\frac{(u\bar{u} - d\bar{d})}{\sqrt{2}} + \sin\theta\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}},$$

$$\omega = -\sin\theta\frac{(u\bar{u} - d\bar{d})}{\sqrt{2}} + \cos\theta\frac{(u\bar{u} + d\bar{d})}{\sqrt{2}},$$

where θ is a (small) mixing angle.

(a) Determine θ (up to a sign) using experimental data on the decay $\omega \to \pi^+\pi^-$. Estimate the error in the value of the mixing angle.

(b) Using the value of θ predict the decay rates $\Gamma(\rho^0 \to e^+e^-)$ and $\Gamma(\omega \to e^+e^-)$, assuming the amplitude for a quark pair annihilation into an e^+e^- pair is proportional to the electric charge Q of the quark.

(c) Assume that the transition amplitude between different spin states of a $q\bar{q}$ quark pair with emission of a photon: $(q\bar{q}) \to (q\bar{q}) + \gamma$ is proportional to the quark electric charge Q. Use the value of the ρ-ω mixing angle θ to determine the ratios of the decay rates:

(i) $\Gamma(\rho^0 \to \pi^0\gamma)/\Gamma(\omega^0 \to \pi^0\gamma)$,

(ii) $\Gamma(\rho^0 \to \eta\gamma)/\Gamma(\omega^0 \to \eta\gamma)$.

Compare with the PDG experimental data. How does the inclusion of ρ-ω mixing improve the agreement with the data?

Answer

Question 3

Baryon magnetic moments

The octet of spin-$\frac{1}{2}$ baryons has magnetic moments μ. The operator that describes the magnetic moment is an $\mathrm{SU}(3)_f$ octet operator which is proportional to the quark charge Q. The charge

$$Q = t_3 + \frac{1}{\sqrt{3}} t_8$$

is traceless ($\mathrm{Tr}\, Q = 0$) and can be promoted to a purely $\mathrm{SU}(3)_f$ octet spurion $\mathbf{8}_Q$ (with no singlet piece, as in contrast to the GMO mass formula). Hence, when determining the baryon magnetic moment

$$\mu(B) = \langle \bar{B} | \mu | B \rangle \propto \mathbf{8}_{\bar{B}} \times \mathbf{8}_Q \times \mathbf{8}_B$$

there are two independent octet structures (the f- and d-type couplings, as for the baryon mass), given by

$$\mu(B) = c_f \,\mathrm{Tr}(B^\dagger [Q, B]) + c_d \,\mathrm{Tr}(B^\dagger \{Q, B\}) = \alpha_+ \,\mathrm{Tr}(B B^\dagger Q) + \alpha_- \,\mathrm{Tr}(B^\dagger B Q),$$

where $\alpha_+ \equiv c_d + c_f$, $\alpha_- \equiv c_d - c_f$ are arbitrary constants and

$$B = \begin{pmatrix} \frac{\Sigma_u^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma_u^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

Determine all the spin-$\frac{1}{2}$ baryon magnetic moments in terms of $\mu(p)$ and $\mu(n)$ (by eliminating $c_{f,d}$ or $\alpha_\pm$) and compare with the PDG experimental values. These predictions were first worked out by Coleman and Glashow in 1961. Note that imposing the full SU(6) spin-flavor symmetry further predicts $\mu(p)/\mu(n) = -\frac{3}{2}$, which you can ignore in this problem.

Answer