University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

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Question 1

Answer

We can use tensor notation to reduce the work. From the constraint

$$\eta_{\alpha\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta} = \eta_{\gamma\delta}, \tag{1}$$

and the relation $\gamma = \frac{1}{\sqrt{1-v^2}}$ (or $\gamma^2 - \gamma^2 v^2 = \gamma^2(1-v^2) = 1$), we can consider different γ, δ, meaning that

- $(\gamma, \delta) = (0, 0)$:

$$-(\Lambda^0{}_0)^2 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 = -\gamma^2 + \gamma^2\sum_{i=1}^{3}(v_i)^2 = -\gamma^2 + \gamma^2 v^2 = -1 = \eta_{00} \tag{2}$$

- $(\gamma, \delta) = (0, i)$:

$$\begin{aligned} &-\Lambda^0{}_0\Lambda^0{}_i + \sum_{k=1}^{3}\Lambda^k{}_0\Lambda^k{}_i &(3)\\ =&-\gamma c v_i + \sum_{k=1}^{3}\gamma v_k(a\delta_{ki} + b v_k v_i) &(4)\\ =&-\gamma c v_i + \gamma a v_i + \gamma b v^2 v_i = \gamma v_i(-c + a + bv^2) = \eta_{0i} = 0 &(5)\end{aligned}$$

  Then we have $c = a + bv^2$.

- $(\gamma, \delta) = (i, 0)$

$$\begin{aligned} &-\Lambda^0{}_i\Lambda^0{}_0 + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_0 &(6)\\ =&\gamma v_i(-c + a + bv^2) &(7)\end{aligned}$$

- $(\gamma, \delta) = (i, i)$

$$
\begin{aligned}
& -\Lambda^0{}_i\Lambda^0{}_i + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_i && (8) \\
=& -c^2v_i^2 + \sum_{k=1}^{3}(a\delta_{ki} + bv_kv_i)^2 && (9) \\
=& -c^2v_i^2 + \sum_{k=1}^{3}(a^2\delta_{ki} + 2ab\delta_{ki}v_kv_i + b^2v_i^2v_k^2) && (10) \\
=& -c^2v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 = \eta_{ii} = 1 && (11)
\end{aligned}
$$

- $(\gamma, \delta) = (i, j), i \neq j$

$$
\begin{aligned}
& -\Lambda^0{}_i\Lambda^0{}_j + \sum_{k=1}^{3}\Lambda^k{}_i\Lambda^k{}_j && (12) \\
=& -c^2v_iv_j + \sum_{k=1}^{3}(a\delta_{ki} + bv_kv_i)(a\delta_{kj} + bv_kv_j) && (13) \\
=& -c^2v_iv_j + \sum_{k=1}^{3}(a^2\delta_{ki}\delta_{kj} + ab(\delta_{ki}v_kv_j + \delta_{kj}v_kv_i) + b^2v_k^2v_iv_j) && (14) \\
=& -c^2v_iv_j + 2abv_iv_j + b^2v^2v_iv_j = \eta_{ij} = 0 && (15)
\end{aligned}
$$

Then we have $c^2 = b^2v^2 + 2ab$

Combining the above information, we can have

$$
\begin{aligned}
& c = a + bv^2 && (16) \\
& c^2 = a^2 + b^2v^4 + 2abv^2 = b^2v^2 + 2ab. && (17)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
& -c^2v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 = 1 && (18) \\
\rightarrow 1 =& -(b^2v^2 + 2ab)v_i^2 + a^2 + 2abv_i^2 + b^2v_i^2v^2 && (19) \\
=& a^2 && (20)
\end{aligned}
$$

Hence, from Eq. 17 and Eq. 20, we have

$$
\begin{aligned}
& c^2 = 1 + b^2v^4 + 2abv^2 = b^2v^2 + 2ab && (21) \\
\rightarrow & a = \pm 1 = \frac{1 + b^2v^4 - b^2v^2}{2b(1 - v^2)} = \frac{1 + b^2v^2(v^2 - 1)}{2b(1 - v^2)} = \frac{\gamma^2 - b^2v^2}{2b} && (22) \\
\rightarrow & b^2v^2 \pm 2b - \gamma^2 = 0 && (23)
\end{aligned}
$$

For $a = +1$, we have

$$b = \frac{-1 \pm \sqrt{1 + \gamma^2 v^2}}{v^2} = \frac{-1 \pm \gamma}{v^2}, \tag{24}$$

and for $a = -1$, we have

$$b = \frac{1 \pm \gamma}{v^2}. \tag{25}$$

Hence, we choose $a = 1$ and $b = \frac{-1+\gamma}{v^2}$ by convention and have

$$c = a + bv^2 = 1 + (-1 + \gamma) = +\gamma \tag{26}$$

$$\tag{27}$$

Finally, we derive

$$\Lambda^i{}_j = a(v)\delta_{ij} + b(v)v_i v_j = \delta_{ij} + \frac{\gamma - 1}{v^2} v_i v_j \tag{28}$$

$$\Lambda^0{}_j = c(v)v_j = \gamma v_j, \tag{29}$$

which are mentioned in the Week 1 lecture. $\square$

Question 2

Answer

In the $\mathcal{O}$ frame, we set the condition for the right-hand side of the rod:

$$\begin{aligned}
&x^{\mu}_{emit} = (0, L/2, D, 0), x^{\mu}_{receive} = (t, vt, 0, 0),\\
&\Delta x^{\mu} = x^{\mu}_{receive} - x^{\mu}_{emit} = (t, vt - L/2, -D, 0)\\
&\Delta x^{\mu}\Delta x_{\mu} = 0 = t^2 - (L/2 - vt)^2 - D^2.
\end{aligned}$$

Also, with $dt = t, dx = vt - L/2, dy = -D$, we have

$$\begin{aligned}
\Delta t' &= \gamma(t + v(vt - L/2))\\
\Delta x' &= \gamma(v + (vt - L/2))\\
\Delta y' &= -D
\end{aligned}$$

We can substitute the solution t to solve the dx' and dt', and define the opening angle

$$\tan\theta'/2 = \frac{\Delta x'}{\Delta y'}. \tag{30}$$

Note that this is the calculation for right-hand side of the rod. Hence, it should be denoted as θ'_R. Now we can apply the procedure to the left-hand side of the rod. After plugging all information into `Mathematica` and using the equation, we have

$$\tan(\theta'_R + \theta'_L) = \frac{\tan\theta'_R + \tan\theta'_L}{1 - \tan\theta'_R \tan\theta'_L} = \frac{4DL}{4D^2 - L^2} = \frac{2\frac{L}{2D}}{1 - \left(\frac{L}{2D}\right)^2} = \tan\theta. \tag{31}$$

Hence, we prove that the opening angle keeps the same value. □