

University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I
Assignment Solution

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HW1 Due to September 23 11:59 PM

Question 1

Problem 1.2

With the Hamiltonian of eq. (1.32), show that the state defined in eq. (1.33) obeys the abstract Schrodinger equation, eq. (1.1), if and only if the wave function obeys eq. (1.30). Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations of eq. (1.31), and to fermions, where the particle creation and annihilation operators obey the anti-commutation relations of eq. (1.38).

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = H |\psi, t\rangle \quad (1.1)$$

$$i\hbar \frac{\partial}{\partial t} \psi = \left[\sum_{j=1}^n \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) + \sum_{j=1}^n \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \right] \psi \quad (1.30)$$

$$\begin{aligned} [a(\mathbf{x}), a(\mathbf{x}')] &= 0 \\ [a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] &= 0 \\ [a(\mathbf{x}), a^\dagger(\mathbf{x}')] &= \delta^3(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (1.31)$$

$$\begin{aligned} H &= \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) \\ &\quad + \frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \end{aligned} \quad (1.32)$$

$$|\psi, t\rangle = \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (1.33)$$

Answer

We first consider boson case, and then we have

$$LHS = i\hbar \frac{\partial}{\partial t} |\psi, t\rangle \quad (1)$$

$$= i\hbar \frac{\partial}{\partial t} \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (2)$$

$$= \int d^3x_1 \dots d^3x_n i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (3)$$

$$= \int d^3x_1 \dots d^3x_n \left[\sum_{j=1}^n \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) + \sum_{j=1}^n \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \right] \psi a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (4)$$

$$RHS = H|\psi, t\rangle \quad (5)$$

$$= \left[\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) + \frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \right] |\psi, t\rangle \quad (6)$$

$$= \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (7)$$

$$+ \frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (8)$$

For the term in Equation 7, by considering $[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}')$, we have

$$a(\mathbf{x}) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) \quad (9)$$

$$= [a^\dagger(\mathbf{x}_1) a(\mathbf{x}) + \delta^3(\mathbf{x} - \mathbf{x}_1)] a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_n) \quad (10)$$

$$= a^\dagger(\mathbf{x}_1) a(\mathbf{x}) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_n) + \delta^3(\mathbf{x} - \mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_n) \quad (11)$$

$$= a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) a(\mathbf{x}) + \sum_{j=1}^n \delta^3(\mathbf{x} - \mathbf{x}_j) a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \quad (12)$$

$$= 0 + \sum_{j=1}^n \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \quad (13)$$

we can drop the first term in Equation 12 since this term will act on the $|0\rangle$, giving 0. Hence, we have

$$\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (14)$$

$$= \sum_{j=1}^n \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \quad (15)$$

$$= \sum_{j=1}^n \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \quad (16)$$

$$= \sum_{j=1}^n \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \mathcal{O}_j |0\rangle \quad (17)$$

$$= \sum_{j=1}^n \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \mathcal{O}_j |0\rangle \quad (18)$$

$$= \sum_{j=1}^n \int d^3x_1 \dots d^3x_n \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle \quad (19)$$

$$= \int d^3x_1 \dots d^3x_n \sum_{j=1}^n \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle, \quad (20)$$

where $a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) = a^\dagger(\mathbf{x}_j)\mathcal{O}_j$ since they (boson fields) commute. Now, we do the same thing for the term in Equation 8, we have

$$a(\mathbf{y})a(\mathbf{x})a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) \quad (21)$$

$$= a(\mathbf{y}) \sum_{j=1}^n \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \quad (22)$$

$$= \sum_{i \neq j} \sum_{j=1}^n \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij}, \quad \mathcal{T}_{ij} = \prod_{k \neq i,j}^n a^\dagger(\mathbf{x}_k). \quad (23)$$

Hence, we have

$$\frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (24)$$

$$= \frac{1}{2} \int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a(\mathbf{y}) a(\mathbf{x}) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (25)$$

$$= \frac{1}{2} \int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \sum_{i \neq j} \sum_{j=1}^n \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij} |0\rangle \quad (26)$$

$$= \sum_{i \neq j} \sum_{j=1}^n \frac{1}{2} \int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x} - \mathbf{y}) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \mathcal{T}_{ij} |0\rangle \quad (27)$$

$$= \sum_{i \neq j} \sum_{j=1}^n \frac{1}{2} \int d^3x_1 \dots d^3x_n V(\mathbf{x}_j - \mathbf{x}_i) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i) \mathcal{T}_{ij} |0\rangle \quad (28)$$

$$= \int d^3x_1 \dots d^3x_n \sum_{i \neq j} \sum_{j=1}^n \frac{1}{2} V(\mathbf{x}_j - \mathbf{x}_i) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (29)$$

$$= \int d^3x_1 \dots d^3x_n \sum_{j=1}^n \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle, \quad (30)$$

where $a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i) \mathcal{T}_{ij} = a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n)$ for the same reason. Hence, we have proved the $LHS = RHS$ and Equation 1.1 for the boson field case.

For fermion fields, the only difference is the anti-commutation relation. We start from Equation 7 again, by considering $\{a(\mathbf{x}), a^\dagger(\mathbf{x})\} = \delta^3(\mathbf{x} - \mathbf{x}')$, and we have

$$a(\mathbf{x})a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) \quad (31)$$

$$= [-a^\dagger(\mathbf{x}_1)a(\mathbf{x}) + \delta^3(\mathbf{x} - \mathbf{x}_1)] a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_n) \quad (32)$$

$$= (-1)^n a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) a(\mathbf{x}) + \sum_{j=1}^n (-1)^{j-1} \delta^3(\mathbf{x} - \mathbf{x}_j) a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \quad (33)$$

$$= 0 + \sum_{j=1}^n (-1)^{j-1} \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n), \quad (34)$$

where the 0 term comes from the same reason. Then the term in Equation 7 is given by

$$\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (35)$$

$$= \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a(\mathbf{x}) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (36)$$

$$= \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \sum_{j=1}^n (-1)^j \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \quad (37)$$

$$= \sum_{j=1}^n \int d^3x \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) (-1)^j \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j |0\rangle \quad (38)$$

$$= \sum_{j=1}^n \int d^3x_1 \dots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) (-1)^j \mathcal{O}_j |0\rangle \quad (39)$$

$$= \sum_{j=1}^n \int d^3x_1 \dots d^3x_n \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) (-1)^j a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle \quad (40)$$

$$= \int d^3x_1 \dots d^3x_n \sum_{j=1}^n \left(-\frac{\hbar^2}{2m} \nabla_j^2 + U(\mathbf{x}_j) \right) \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle, \quad (41)$$

where $a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) = (-1)^{j-1} a^\dagger(\mathbf{x}_j) \mathcal{O}_j$ by anti-commutation relation of fermion fields. Next, given the term in Equation 8, we have

$$a(\mathbf{y}) a(\mathbf{x}) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) \quad (42)$$

$$= a(\mathbf{y}) \sum_{j=1}^n (-1)^{j-1} \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \dots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \dots a^\dagger(\mathbf{x}_n) \quad (43)$$

$$= \sum_{i < j} \sum_{j=1}^n (-1)^{i-1} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij} + \sum_{i > j} \sum_{j=1}^n (-1)^{i-2} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij}, \quad (44)$$

where $\mathcal{T}_{ij} = \prod_{k \neq i,j}^n a^\dagger(\mathbf{x}_k)$. Now, we can simplify Equation 8 and it is given by

$$\frac{1}{2} \int d^3x d^3y V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (45)$$

$$= \frac{1}{2} \int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi a(\mathbf{y}) a(\mathbf{x}) a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (46)$$

$$= \frac{1}{2} \int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i < j}^n \sum_{j=1}^n (-1)^{i-1} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij} |0\rangle \\ + \frac{1}{2} \int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i > j}^n \sum_{j=1}^n (-1)^{i-2} (-1)^{j-1} \delta^3(\mathbf{y} - \mathbf{x}_i) \delta^3(\mathbf{x} - \mathbf{x}_j) \mathcal{T}_{ij} |0\rangle \quad (47)$$

$$= \frac{1}{2} \int d^3x_1 \dots d^3x_n \sum_{i < j}^n \sum_{j=1}^n (-1)^{i-1} (-1)^{j-1} V(\mathbf{x}_j - \mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i) \psi \mathcal{T}_{ij} |0\rangle \\ + \frac{1}{2} \int d^3x_1 \dots d^3x_n \sum_{i > j}^n \sum_{j=1}^n (-1)^{i-2} (-1)^{j-1} V(\mathbf{x}_j - \mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i) \psi \mathcal{T}_{ij} |0\rangle \quad (48)$$

$$= \frac{1}{2} \int d^3x_1 \dots d^3x_n \sum_{i \neq j}^n \sum_{j=1}^n (-1)^{i-1} (-1)^{j-1} V(\mathbf{x}_j - \mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i) \psi \mathcal{T}_{ij} |0\rangle \quad (49)$$

$$= \frac{1}{2} \int d^3x_1 \dots d^3x_n \sum_{i \neq j}^n \sum_{j=1}^n V(\mathbf{x}_j - \mathbf{x}_i) \psi a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle \quad (50)$$

$$= \int d^3x_1 \dots d^3x_n \sum_{j=1}^n \sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k) \psi a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n) |0\rangle, \quad (51)$$

where $(-1)^{i-1} (-1)^{j-1} a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i) \mathcal{T}_{ij} = a^\dagger(\mathbf{x}_1) \dots a^\dagger(\mathbf{x}_n)$ by anti-commutation relation. In summary, we have proved both cases for boson fields and fermion fields. \square

Question 2

Problem 2.3

Verify that eq. (2.16) follows from eq. (2.14).

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma M^{\rho\sigma} \quad (2.14)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma) \quad (2.16)$$

$$= i\hbar(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}) \quad (2.16)$$

Answer

Considering an infinitesimal transformation in $U(\Lambda) = 1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}$ and $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, now we get

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) \quad (52)$$

$$\Rightarrow \delta\omega_{\alpha\beta}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\alpha\beta}] = \delta\omega_{\rho\sigma}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\rho\sigma}] \quad (53)$$

$$RHS = (\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma)M^{\rho\sigma} \quad (54)$$

$$\rightarrow \delta^\mu{}_\rho\delta\omega^\nu{}_\sigma M^{\rho\sigma} + \delta^\nu{}_\sigma\delta\omega^\mu{}_\rho M^{\rho\sigma} = \delta\omega^\nu{}_\sigma M^{\mu\sigma} + \delta\omega^\mu{}_\rho M^{\rho\sigma} \quad (55)$$

$$= g^{\nu\rho}\delta\omega_{\rho\sigma}M^{\mu\sigma} + g^{\sigma\mu}\delta\omega_{\sigma\rho}M^{\rho\sigma} = \delta\omega_{\rho\sigma}(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\sigma}), \quad (56)$$

we only consider the linear term $\delta\omega$. We can further simplify it to

$$[M^{\mu\nu}, M^{\rho\sigma}] = \frac{2\hbar}{i}(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\sigma}) = 2i\hbar(g^{\sigma\mu}M^{\rho\nu} - g^{\nu\rho}M^{\mu\sigma}) = 2i\hbar(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \quad (57)$$

$$= -[M^{\nu\mu}, M^{\rho\sigma}] = 2i\hbar(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma}) \quad (58)$$

Finally, we have

$$[M^{\mu\nu}, M^{\rho\sigma}] \quad (59)$$

$$= \frac{1}{2}([M^{\mu\nu}, M^{\rho\sigma}] - [M^{\nu\mu}, M^{\rho\sigma}]) \quad (60)$$

$$= i\hbar(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}). \quad (61)$$

□

Question 3

Problem 2.8

- (a) Let $\Lambda = 1 + \delta\omega$ in eq.(2.26), and show that

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x),$$

where

$$\mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu).$$

- (b) Show that $[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x)$.

- (c) Prove the *Jacobi identity*, $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$. Hint: write out all the commutations.

- (d) Use your results from parts (b) and (c) to show that

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \quad (2.31)$$

- (e) Simplify the right-hand side of eq. (2.31) as much as possible.

- (f) Use your results from part (e) to verify eq. (2.16), up to the possibility of a term on the right-hand side that commutes with $\varphi(x)$ and its derivatives. (Such a term, called a *central charge*, in fact does not arise for the Lorentz algebra.)

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \quad (2.26)$$

Answer

- (a)

$$LHS = U^{-1}(\Lambda)\varphi(x)U(\Lambda) \quad (62)$$

$$= \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \quad (63)$$

$$\rightarrow \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\varphi(x), M^{\mu\nu}] \quad (64)$$

$$RHS = \varphi(\Lambda^{-1}x) = \varphi((\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)x^\nu) = \phi(x) - \delta\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) \quad (65)$$

$$\rightarrow -\delta\omega^\mu{}_\nu x^\nu \partial_\mu \varphi(x) = -\delta\omega_{\mu\nu}x^\nu \partial^\mu \varphi(x) = \delta\omega_{\mu\nu}\frac{1}{2}(x^\mu \partial^\nu - x^\nu \partial^\mu)\varphi(x), \quad (66)$$

we only focus on the linear term $\delta\omega$. Now we have

$$[\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu)\varphi(x) = \mathcal{L}^{\mu\nu}\varphi(x). \quad (67)$$

(b)

$$[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\varphi(x)M^{\rho\sigma} - M^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \quad (68)$$

$$= \mathcal{L}^{\mu\nu}[\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x). \quad (69)$$

(c)

$$[[A, B], C] + [[B, C], A] + [[C, A], B] \quad (70)$$

$$= (\textcolor{blue}{CAB} - \textcolor{red}{CBA} - \textcolor{brown}{ABC} + \textcolor{teal}{BAC}) + (\textcolor{brown}{ABC} - \textcolor{orange}{ACB} - \textcolor{red}{BCA} + \textcolor{red}{CBA}) + (\textcolor{red}{BCA} - \textcolor{teal}{BAC} - \textcolor{blue}{CAB} + \textcolor{brown}{ACB}) \quad (71)$$

$$= 0. \quad (72)$$

(d)

By the Jacobi identity, we have

$$0 = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\mu\nu}, M^{\rho\sigma}], \varphi(x)] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \quad (73)$$

$$\rightarrow [\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \quad (74)$$

$$= [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] - [[\varphi(x), M^{\rho\sigma}], M^{\mu\nu}] \quad (75)$$

$$= \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \quad (76)$$

$$= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \quad (77)$$

(e)

For the result in Equation 77, considering the relation $\partial^\mu x^\nu \varphi(x) = (g^{\mu\nu} + x^\nu \partial^\mu)\varphi(x)$, we can have

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \frac{\hbar}{i} \frac{\hbar}{i} (x^\mu \partial^\nu - x^\nu \partial^\mu)(x^\rho \partial^\sigma - x^\sigma \partial^\rho)\varphi(x) \quad (78)$$

$$= \left(\frac{\hbar}{i}\right)^2 [x^\mu(g^{\nu\rho} + x^\rho \partial^\nu)\partial^\sigma - x^\nu(g^{\mu\rho} + x^\rho \partial^\mu)\partial^\sigma - x^\mu(g^{\nu\sigma} + x^\sigma \partial^\nu)\partial^\rho + x^\nu(g^{\mu\sigma} + x^\sigma \partial^\mu)\partial^\rho]\varphi(x) \quad (79)$$

$$= \left(\frac{\hbar}{i}\right)^2 [g^{\nu\rho}x^\mu\partial^\sigma - g^{\mu\rho}x^\nu\partial^\sigma - g^{\nu\sigma}x^\mu\partial^\rho + g^{\mu\sigma}x^\nu\partial^\rho + x^\mu x^\rho \partial^\nu \partial^\sigma - x^\nu x^\rho \partial^\mu \partial^\sigma - x^\mu x^\sigma \partial^\nu \partial^\rho + x^\nu x^\sigma \partial^\mu \partial^\sigma]\varphi(x). \quad (80)$$

$$\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \quad (81)$$

$$= \left(\frac{\hbar}{i}\right)^2 [g^{\sigma\mu}x^\rho\partial^\nu - g^{\rho\mu}x^\sigma\partial^\nu - g^{\sigma\nu}x^\rho\partial^\mu + g^{\rho\nu}x^\sigma\partial^\mu + x^\rho x^\mu\partial^\sigma\partial^\nu - x^\rho x^\nu\partial^\sigma\partial^\mu - x^\sigma x^\mu\partial^\rho\partial^\nu + x^\sigma x^\nu\partial^\rho\partial^\mu] \varphi(x). \quad (82)$$

Using simpler forms to express:

$$\begin{aligned} \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) &= \left(\frac{\hbar}{i}\right)^2 \left[g^{\nu\rho}x^\mu\partial^\sigma - g^{\mu\rho}x^\nu\partial^\sigma - g^{\nu\sigma}x^\mu\partial^\rho + g^{\mu\sigma}x^\nu\partial^\rho \right. \\ &\quad \left. + x^\mu x^\rho\partial^\nu\partial^\sigma - x^\mu x^\sigma\partial^\nu\partial^\rho - x^\nu x^\rho\partial^\mu\partial^\sigma + x^\nu x^\sigma\partial^\mu\partial^\rho \right] \varphi(x), \end{aligned} \quad (83)$$

$$\begin{aligned} \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) &= \left(\frac{\hbar}{i}\right)^2 \left[g^{\sigma\mu}x^\rho\partial^\nu - g^{\rho\mu}x^\sigma\partial^\nu - g^{\sigma\nu}x^\rho\partial^\mu + g^{\rho\nu}x^\sigma\partial^\mu \right. \\ &\quad \left. + x^\rho x^\mu\partial^\sigma\partial^\nu - x^\sigma x^\mu\partial^\rho\partial^\nu - x^\rho x^\nu\partial^\sigma\partial^\mu + x^\sigma x^\nu\partial^\rho\partial^\mu \right] \varphi(x). \end{aligned} \quad (84)$$

Combining those two results together, it gives

$$\begin{aligned} (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) &= \left(\frac{\hbar}{i}\right)^2 \left[g^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) - g^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu) \right. \\ &\quad \left. - g^{\nu\sigma}(x^\mu\partial^\rho - x^\rho\partial^\mu) + g^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu) \right] \varphi(x) \end{aligned} \quad (85)$$

$$= \frac{\hbar}{i} \left(g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\sigma}\mathcal{L}^{\mu\rho} + g^{\mu\sigma}\mathcal{L}^{\nu\rho} \right) \varphi(x) \quad (86)$$

$$= i\hbar \left(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho} \right) \varphi(x). \quad (87)$$

Actually, it looks similar to

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= i\hbar (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \end{aligned} \quad (2.16)$$

(f)

Now we assume there is a non trivial term \mathcal{C} on $[M^{\mu\nu}, M^{\rho\sigma}]$, giving that

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C}), \quad (88)$$

where C can commutes with φ and its derivatives. Now, we have

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] \quad (2.31)$$

$$= i\hbar [\varphi(x), (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C})] \quad (89)$$

$$= i\hbar \left(g^{\nu\sigma}[\varphi(x), M^{\mu\rho}] + g^{\mu\rho}[\varphi(x), M^{\nu\sigma}] - g^{\mu\sigma}[\varphi, M^{\nu\rho}] - g^{\nu\rho}[\varphi(x), M^{\mu\sigma}] + [\varphi(x), \mathcal{C}] \right) \quad (90)$$

$$= i\hbar (g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho})\varphi(x) = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) \quad (91)$$

$$= [\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\varphi(x). \quad (92)$$

Hence, with the central charge \mathcal{C} , the relation still holds. \square

Question 4

Problem 2.9

Let us write

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau + \frac{i}{2\hbar} \delta\omega_{\mu\nu} (S_V^{\mu\nu})^\rho{}_\tau, \quad (2.32)$$

where

$$(S_V^{\mu\nu})^\rho{}_\tau \equiv \frac{\hbar}{i} (g^{\mu\rho} \delta^\nu{}_\tau - g^{\nu\rho} \delta^\mu{}_\tau) \quad (2.33)$$

are matrices which constitute the *vector representation* of the Lorentz generators.

(a) Let $\Lambda = 1 + \delta\omega$ in eq. (2.27), and show that

$$[\partial^\rho \varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu} \partial^\rho \varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau \partial^\tau \varphi(x) \quad (2.34)$$

(b) Show that the matrices $(S_V^{\mu\nu})$ must have the same commutation relations as the operators $M^{\mu\nu}$. Hint: see the previous problem.

(c) For a rotation by an angle θ about the z axis, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.35)$$

Show that

$$\Lambda = \exp(-i\theta S_V^{12}/\hbar). \quad (2.36)$$

(d) For a boost by *rapidity* η in the z direction, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh \eta & 0 & 0 & \sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix}. \quad (2.37)$$

Show that

$$\Lambda = \exp(+i\eta S_V^{30}/\hbar). \quad (2.38)$$

$$U^{-1}(\Lambda) \partial^\rho \varphi(x) U(\Lambda) = \Lambda^\rho_\mu \bar{\partial}^\mu \varphi(\Lambda^{-1}x), \quad \bar{x}^\mu = (\Lambda^{-1})^\mu_\nu x^\nu, \quad \bar{\partial}^\mu = (\Lambda^{-1})^\mu_\nu \partial^\nu \quad (2.27)$$

Answer

(a)

$$LHS = \left(1 - \frac{i}{2\hbar} \delta\omega_{\mu\nu} M^{\mu\nu}\right) \partial^\rho \varphi(x) \left(1 + \frac{i}{2\hbar} \delta\omega_{\mu\nu} M^{\mu\nu}\right) \quad (93)$$

$$= \partial^\rho \varphi(x) + \delta\omega_{\mu\nu} \frac{i}{2\hbar} [\partial^\rho \varphi(x), M^{\mu\nu}] \quad (94)$$

$$RHS = \Lambda^\rho_\mu \bar{\partial}^\mu \varphi(\Lambda^{-1}x) \quad (95)$$

$$= (\delta^\rho_\mu + \delta\omega^\rho_\mu) \bar{\partial}^\mu \varphi((\delta^\mu_\nu - \delta\omega^\mu_\nu) x^\nu) \quad (96)$$

$$= (\delta^\rho_\mu + \delta\omega^\rho_\mu) (\delta^\mu_\nu - \delta\omega^\mu_\nu) \partial^\nu (\varphi(x) - \delta\omega^\alpha_\beta x^\beta \partial_\alpha \varphi(x)) \quad (97)$$

$$= \delta^\rho_\nu (\partial^\nu \varphi(x) - \delta\omega^\alpha_\beta g^{\nu\beta} \partial_\alpha \varphi(x) - \delta\omega^\alpha_\beta x^\beta \partial^\nu \partial_\alpha \varphi(x)) \quad (98)$$

$$= \partial^\rho \varphi(x) - \delta\omega^\alpha_\beta g^{\rho\beta} \partial_\alpha \varphi(x) - \delta\omega^\alpha_\beta x^\beta \partial^\rho \partial_\alpha \varphi(x) \quad (99)$$

$$= \partial^\rho \varphi(x) - \delta\omega_{\alpha\beta} g^{\rho\beta} \partial^\alpha \varphi(x) - \delta\omega_{\alpha\beta} x^\beta \partial^\rho \partial^\alpha \varphi(x) \quad (100)$$

$$= \partial^\rho \varphi(x) - \delta\omega_{\mu\nu} g^{\rho\nu} \partial^\mu \varphi(x) - \delta\omega_{\mu\nu} x^\nu \partial^\rho \partial^\mu \varphi(x) \quad (101)$$

$$= \partial^\rho \varphi(x) + \delta\omega_{\mu\nu} (-g^{\rho\nu} \partial^\mu - x^\nu \partial^\rho \partial^\mu) \varphi(x) \quad (102)$$

$$= \partial^\rho \varphi(x) + \delta\omega_{\mu\nu} \frac{1}{2} (-g^{\rho\nu} \partial^\mu - x^\nu \partial^\rho \partial^\mu + g^{\rho\mu} \partial^\nu + x^\mu \partial^\rho \partial^\nu) \varphi(x) \quad (103)$$

Hence, we have

$$[\partial^\rho \varphi(x), M^{\mu\nu}] = \frac{\hbar}{i} (-g^{\rho\nu} \partial^\mu - x^\nu \partial^\rho \partial^\mu + g^{\rho\mu} \partial^\nu + x^\mu \partial^\rho \partial^\nu) \varphi(x) \quad (104)$$

$$= \frac{\hbar}{i} (x^\mu \partial^\nu - x^\nu \partial^\mu) \partial^\rho \varphi(x) + \frac{\hbar}{i} (g^{\rho\mu} \delta^\nu_\tau \partial^\tau - g^{\rho\nu} \delta^\mu_\tau \partial^\tau) \varphi(x) \quad (105)$$

$$= \frac{\hbar}{i} (x^\mu \partial^\nu - x^\nu \partial^\mu) \partial^\rho \varphi(x) + \frac{\hbar}{i} (g^{\mu\rho} \delta^\nu_\tau - g^{\nu\rho} \delta^\mu_\tau) \partial^\tau \varphi(x) \quad (106)$$

$$= \mathcal{L}^{\mu\nu} \partial^\rho \varphi(x) + (S_V^{\mu\nu})^\rho_\tau \partial^\tau \varphi(x), \quad (107)$$

which is exactly the result we want to prove.

(b)

$$[(S_V^{\mu\nu}), (S_V^{\rho\sigma})]^\alpha{}_\beta \quad (108)$$

$$= (S_V^{\mu\nu})^\alpha{}_\tau (S_V^{\rho\sigma})^\tau{}_\beta - (S_V^{\rho\sigma})^\alpha{}_\tau (S_V^{\mu\nu})^\tau{}_\beta \quad (109)$$

$$= \frac{\hbar}{i} (g^{\mu\alpha} \delta^\nu{}_\tau - g^{\nu\alpha} \delta^\mu{}_\tau) \frac{\hbar}{i} (g^{\rho\tau} \delta^\sigma{}_\beta - g^{\sigma\tau} \delta^\rho{}_\beta) - \frac{\hbar}{i} (g^{\rho\alpha} \delta^\sigma{}_\tau - g^{\sigma\alpha} \delta^\rho{}_\tau) \frac{\hbar}{i} (g^{\mu\tau} \delta^\nu{}_\beta - g^{\nu\tau} \delta^\mu{}_\beta) \quad (110)$$

$$= \left(\frac{\hbar}{i} \right)^2 \left[\color{red} g^{\mu\alpha} g^{\rho\nu} \delta^\sigma{}_\beta - g^{\mu\alpha} g^{\sigma\nu} \delta^\rho{}_\beta \color{black} - g^{\nu\alpha} g^{\rho\mu} \delta^\sigma{}_\beta + g^{\nu\alpha} g^{\sigma\mu} \delta^\rho{}_\beta \right. \quad (111)$$

$$\left. \color{red} - g^{\rho\alpha} g^{\mu\sigma} \delta^\nu{}_\beta \color{black} + g^{\rho\alpha} g^{\nu\sigma} \delta^\mu{}_\beta \color{blue} + g^{\sigma\alpha} g^{\mu\rho} \delta^\nu{}_\beta - g^{\sigma\alpha} g^{\nu\rho} \delta^\mu{}_\beta \right] \quad (112)$$

$$= i\hbar \left(\frac{\hbar}{i} \right) \left[\color{blue} g^{\mu\rho} (g^{\nu\alpha} \delta^\sigma{}_\beta - g^{\sigma\alpha} \delta^\nu{}_\beta) \color{black} - g^{\nu\rho} (g^{\mu\alpha} \delta^\sigma{}_\beta - g^{\sigma\alpha} \delta^\mu{}_\beta) - g^{\mu\sigma} (g^{\nu\alpha} \delta^\rho{}_\beta - g^{\rho\alpha} \delta^\nu{}_\beta) + g^{\nu\sigma} (g^{\mu\alpha} \delta^\rho{}_\beta - g^{\rho\alpha} \delta^\mu{}_\beta) \right] \quad (113)$$

$$= i\hbar \left(g^{\mu\rho} (S_V^{\nu\sigma})^\alpha{}_\beta - g^{\nu\rho} (S_V^{\mu\sigma})^\alpha{}_\beta - g^{\mu\sigma} (S_V^{\nu\rho})^\alpha{}_\beta + g^{\nu\sigma} (S_V^{\mu\rho})^\alpha{}_\beta \right) \quad (114)$$

$$= i\hbar (g^{\mu\rho} (S_V^{\nu\sigma}) - g^{\nu\rho} (S_V^{\mu\sigma}) - g^{\mu\sigma} (S_V^{\nu\rho}) + g^{\nu\sigma} (S_V^{\mu\rho}))^\alpha{}_\beta. \quad (115)$$

This looks exactly the same as

$$\begin{aligned} & [M^{\mu\nu}, M^{\rho\sigma}] \\ & = i\hbar (g^{\nu\sigma} M^{\mu\rho} + g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma}) \end{aligned} \quad (2.16)$$

(c)

$$\frac{i}{\hbar} (S_V^{12})^\mu{}_\nu = (g^{1\mu} \delta^2{}_\nu - g^{2\mu} \delta^1{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv \mathcal{R}, \quad (116)$$

$$\mathcal{R}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (117)$$

$$\mathcal{R}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\mathcal{R}, \quad (118)$$

$$\mathcal{R}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\mathcal{R}^2. \quad (119)$$

Hence, we have

$$\Lambda = \exp(-i\theta S_V^{12}/\hbar) = \exp(-\theta \mathcal{R}) \quad (120)$$

$$= \sum_{n=0}^{\infty} \frac{(-\theta \mathcal{R})^n}{n!} = \sum_{m=0}^{\infty} \frac{(-\theta \mathcal{R})^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(-\theta \mathcal{R})^{2m+1}}{(2m+1)!} \quad (121)$$

$$= I + \sum_{m=1}^{\infty} \frac{(-1)^m \theta^{2m} \mathcal{R}^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(-1)^{2m+1} \theta^{2m+1} \mathcal{R}^{2m+1}}{(2m+1)!} \quad (122)$$

$$= I + \sum_{m=1}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!} \mathcal{R}^2 - \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m+1}}{(2m+1)!} \mathcal{R} \quad (123)$$

$$= I + (\cos \theta - 1) \mathcal{R}^2 - \sin \theta \mathcal{R} \quad (124)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos \theta - 1 & 0 & 0 \\ 0 & 0 & \cos \theta - 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin \theta & 0 \\ 0 & \sin \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (125)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (126)$$

This is exactly the same as the result in Equation (2.35).

(d)

$$\frac{i}{\hbar}(S_V^{30})^\mu{}_\nu = (g^{3\mu}\delta^0{}_\nu - g^{0\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \equiv \mathcal{B}, \quad (127)$$

$$\mathcal{B}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (128)$$

$$\mathcal{B}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathcal{B}, \quad (129)$$

$$\mathcal{B}^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathcal{B}^2. \quad (130)$$

Hence, we have

$$\Lambda = \exp(+i\eta S_V^{30}/\hbar) = \exp(\eta \mathcal{B}) \quad (131)$$

$$= \sum_{n=0}^{\infty} \frac{(\eta \mathcal{B})^n}{n!} = \sum_{m=0}^{\infty} \frac{(\eta \mathcal{B})^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(\eta \mathcal{B})^{2m+1}}{(2m+1)!} \quad (132)$$

$$= I + \sum_{m=1}^{\infty} \frac{\eta^{2m} \mathcal{B}^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{\eta^{2m+1} \mathcal{B}^{2m+1}}{(2m+1)!} \quad (133)$$

$$= I + \sum_{m=1}^{\infty} \frac{\eta^{2m}}{(2m)!} \mathcal{B}^2 + \sum_{m=0}^{\infty} \frac{\eta^{2m+1}}{(2m+1)!} \mathcal{B} \quad (134)$$

$$= I + (\cosh \eta - 1) \mathcal{B}^2 + \sinh \eta \mathcal{B} \quad (135)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cosh \eta - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh \eta - 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \sinh \eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh \eta & 0 & 0 & 0 \end{pmatrix} \quad (136)$$

$$= \begin{pmatrix} \cosh \eta & 0 & 0 & \sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix}. \quad (137)$$

This is exactly the same as the result in Equation (2.37). \square

Question 5

Problem 3.1

Derive eq. (3.29) from eqs. (3.21), (3.24), and (3.28).

$$a(\mathbf{k}) = \int d^3x e^{-ikx} [i\partial_0\varphi(x) + \omega\varphi(x)] \quad (3.21)$$

$$\Pi(x) = \dot{\varphi}(x) = \partial_0\varphi(x) \quad (3.24)$$

$$[\varphi(\mathbf{x}, t), \varphi(\mathbf{x}', t)] = 0, \quad [\Pi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = 0, \quad [\varphi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (3.28)$$

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \quad [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0, \quad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (3.29)$$

Answer

Since $a(\mathbf{x})$ is independent of time, we can set $x^0 = y^0$, meaning all time variables are the same. Now, we compute

$$[a(\mathbf{k}), a(\mathbf{k}')] \quad (138)$$

$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} [i\partial_0\varphi(x) + \omega\varphi(x), i\partial_0\varphi(y) + \omega\varphi(y)] \quad (139)$$

$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} \left(-[\partial_0\varphi(x), \partial_0\varphi(y)] + i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)] + \omega^2[\varphi(x), \varphi(y)] \right) \quad (140)$$

$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} \left(i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)] \right) \quad (141)$$

$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} \left(i\omega(-i\delta^{(3)}(\mathbf{x} - \mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x} - \mathbf{y})) \right) \quad (142)$$

$$= 0, \quad (143)$$

where we have used the commutation relations in Equation (3.28). Similarly, we can show that

$$[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] \quad (144)$$

$$= \int d^3x d^3y e^{ikx} e^{ik'y} [i\partial_0\varphi(x) - \omega\varphi(x), i\partial_0\varphi(y) - \omega\varphi(y)] \quad (145)$$

$$= \int d^3x d^3y e^{ikx} e^{ik'y} \left(-[\partial_0\varphi(x), \partial_0\varphi(y)] - i\omega[\partial_0\varphi(x), \varphi(y)] - i\omega[\varphi(x), \partial_0\varphi(y)] + \omega^2[\varphi(x), \varphi(y)] \right) \quad (146)$$

$$= \int d^3x d^3y e^{ikx} e^{ik'y} \left(-i\omega[\partial_0\varphi(x), \varphi(y)] - i\omega[\varphi(x), \partial_0\varphi(y)] \right) \quad (147)$$

$$= \int d^3x d^3y e^{ikx} e^{ik'y} \left(-i\omega(-i\delta^{(3)}(\mathbf{x} - \mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x} - \mathbf{y})) \right) \quad (148)$$

$$= 0. \quad (149)$$

Now, we compute

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] \quad (150)$$

$$= \int d^3x d^3y e^{-ikx} e^{ik'y} [i\partial_0 \varphi(x) + \omega \varphi(x), i\partial_0 \varphi(y) - \omega \varphi(y)] \quad (151)$$

$$= \int d^3x d^3y e^{-ikx} e^{ik'y} \left(-[\partial_0 \varphi(x), \partial_0 \varphi(y)] - i\omega [\partial_0 \varphi(x), \varphi(y)] + i\omega [\varphi(x), \partial_0 \varphi(y)] - \omega^2 [\varphi(x), \varphi(y)] \right) \quad (152)$$

$$= \int d^3x d^3y e^{-ikx} e^{ik'y} \left(-i\omega [\partial_0 \varphi(x), \varphi(y)] + i\omega [\varphi(x), \partial_0 \varphi(y)] \right) \quad (153)$$

$$= \int d^3x d^3y e^{-ikx} e^{ik'y} \left(-i\omega (-i\delta^{(3)}(\mathbf{x} - \mathbf{y})) + i\omega (i\delta^{(3)}(\mathbf{x} - \mathbf{y})) \right) \quad (154)$$

$$= \int d^3x d^3y e^{-ikx} e^{ik'y} (2\omega \delta^{(3)}(\mathbf{x} - \mathbf{y})) \quad (155)$$

$$= \int d^3x e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}} (2\omega) \quad (156)$$

$$= (2\pi)^3 2\omega \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \quad (157)$$

□

Question 6

Problem 3.5

Consider a complex (that is, non-hermitian) scalar field φ with Lagrangian density

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0.$$

- (a) Show that φ obeys the Klein-Gordon equation.
- (b) Treat φ and φ^\dagger as independent fields, and find the conjugate momentum for each. Compute the Hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).
- (c) Write the mode expansion of φ as

$$\varphi(x) = \int \widetilde{dk} [a(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}].$$

Express $a(\mathbf{k})$ and $b(\mathbf{k})$ in terms of φ and φ^\dagger and their time derivatives.

- (d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates.
- (e) Express the Hamiltonian in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?

Answer

(a)

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi^\dagger, \quad (158)$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = -\partial^\mu \varphi^\dagger, \quad (159)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) = -\partial_\mu \partial^\mu \varphi^\dagger. \quad (160)$$

Hence, the Euler-Lagrange equation gives that

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) = 0 \quad (161)$$

$$\Rightarrow -m^2 \varphi^\dagger + \partial_\mu \partial^\mu \varphi^\dagger = 0 \quad (162)$$

$$\Rightarrow (\partial_\mu \partial^\mu - m^2) \varphi^\dagger = 0. \quad (163)$$

Similarly, we can show that

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} = -m^2 \varphi, \quad (164)$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^\dagger)} = -\partial^\mu \varphi, \quad (165)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^\dagger)} \right) = -\partial_\mu \partial^\mu \varphi. \quad (166)$$

Hence, the Euler-Lagrange equation gives that

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^\dagger)} \right) = 0 \quad (167)$$

$$\Rightarrow -m^2 \varphi + \partial_\mu \partial^\mu \varphi = 0 \quad (168)$$

$$\Rightarrow (\partial_\mu \partial^\mu - m^2) \varphi = 0. \quad (169)$$

Therefore, both φ and φ^\dagger obey the Klein-Gordon equation.

(b)

$$\Pi_\varphi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \varphi)} = -\partial^0 \varphi^\dagger = -\frac{-\partial}{\partial t} \varphi = +\dot{\varphi}^\dagger, \quad (170)$$

$$\Pi_{\varphi^\dagger} = \frac{\partial \mathcal{L}}{\partial (\partial_0 \varphi^\dagger)} = -\partial^0 \varphi = -\frac{-\partial}{\partial t} \varphi = +\dot{\varphi}. \quad (171)$$

The Hamiltonian density is given by

$$\mathcal{H} = \Pi_\varphi \dot{\varphi} + \Pi_{\varphi^\dagger} \dot{\varphi}^\dagger - \mathcal{L} \quad (172)$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger + \partial^\mu \varphi^\dagger \partial_\mu \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \quad (173)$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger - \dot{\varphi}^\dagger \dot{\varphi} + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \quad (174)$$

$$= \dot{\varphi} \dot{\varphi}^\dagger + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \quad (175)$$

$$= \Pi_\varphi \Pi_{\varphi^\dagger} + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0. \quad (176)$$

(c)

$$\varphi(x) = \int \widetilde{dk} [a(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}] \quad (177)$$

$$\Rightarrow \dot{\varphi}(x) = \int \widetilde{dk} [i\omega a(\mathbf{k}) e^{ikx} - i\omega b^\dagger(\mathbf{k}) e^{-ikx}] \quad (178)$$

$$\varphi^\dagger(x) = \int \widetilde{dk} [a^\dagger(\mathbf{k}) e^{-ikx} + b(\mathbf{k}) e^{ikx}] \quad (179)$$

$$\Rightarrow \dot{\varphi}^\dagger(x) = \int \widetilde{dk} [-i\omega a^\dagger(\mathbf{k}) e^{-ikx} + i\omega b(\mathbf{k}) e^{ikx}] \quad (180)$$

Hence, we have

$$a(\mathbf{k}) = \int d^3x e^{-ikx} [i\partial_0 \varphi(x) + \omega \varphi(x)], \quad (181)$$

$$b(\mathbf{k}) = \int d^3x e^{-ikx} [i\partial_0 \varphi^\dagger(x) + \omega \varphi^\dagger(x)]. \quad (182)$$

(d)

(e)