

University of Minnesota
School of Physics and Astronomy

2026 Spring Physics 8502
General Relativity II
Assignment Solution

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Problem Set 2 due on Due Feb 18 at 11:59pm

Question 1

Consider 2 objects of mass m_1 and m_2 with separation a , orbiting about a common center of mass. Find the change in the period ($\dot{\tau}/\tau$) due to gravitational radiation. Assume the above result is valid as $a \rightarrow 0$, find the time to go from $a = a_0$ to $a = 0$.

Answer

We first write down the radiation power of the system:

$$P = \frac{dE}{dt} = -\frac{G}{5c^5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (1)$$

$$= -\frac{G}{5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (2)$$

where I_{ij} is the quadrupole moment of the system. For a binary system, we can express I_{ij} in terms of the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ and the separation vector \mathbf{r} between the two masses:

$$I_{ij} = \mu(r_i r_j - \frac{1}{3} r^2 \delta_{ij}) \quad (3)$$

For a circular orbit, the separation vector can be expressed as $\mathbf{r}(t) = a(\cos(\omega t), \sin(\omega t), 0)$, where ω is the angular frequency of the orbit. The third time derivative of I_{ij} can be calculated as follows:

$$\ddot{I}_{ij} = \mu \left(\frac{d^3}{dt^3} (r_i r_j) - \frac{1}{3} \frac{d^3}{dt^3} (r^2 \delta_{ij}) \right) \quad (4)$$

Calculating the third time derivative of $r_i r_j$ and $r^2 \delta_{ij}$, we find:

$$\frac{d^3}{dt^3} (r_i r_j) = \frac{d^3}{dt^3} \begin{pmatrix} a^2 \cos^2(\omega t) & a^2 \cos(\omega t) \sin(\omega t) & 0 \\ a^2 \cos(\omega t) \sin(\omega t) & a^2 \sin^2(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5)$$

$$= \frac{d^3}{dt^3} \begin{pmatrix} \frac{a^2}{2} (1 + \cos(2\omega t)) & \frac{a^2}{2} \sin(2\omega t) & 0 \\ \frac{a^2}{2} \sin(2\omega t) & \frac{a^2}{2} (1 - \cos(2\omega t)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6)$$

$$= 4a^2 \omega^3 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7)$$

$$\frac{d^3}{dt^3} (r^2 \delta_{ij}) = \frac{d^3}{dt^3} (a^2 \delta_{ij}) = 0 \quad (8)$$

Substituting these results back into the expression for \ddot{I}_{ij} , we get:

$$\ddot{I}_{ij} = 4\mu a^2 \omega^3 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9)$$

Now we can calculate the power radiated by the system:

$$P = \frac{dE}{dt} = -\frac{G}{5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (10)$$

$$= -\frac{G}{5} \langle 16\mu^2 a^4 \omega^6 (2\sin^2(2\omega t) + 2\cos^2(2\omega t)) \rangle \quad (11)$$

$$= -\frac{32G}{5} \mu^2 a^4 \omega^6. \quad (12)$$

By Kepler's third law, we have $\omega^2 = \frac{G(m_1+m_2)}{a^3}$, $\mu = \frac{m_1 m_2}{m_1+m_2}$, which allows us to express the power in terms of the separation a :

$$P = -\frac{32G^4}{5} \frac{\mu^2 (m_1 + m_2)^3}{a^5} \quad (13)$$

$$= -\frac{32G^4}{5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5} \quad (14)$$

The energy of the system is given by the sum of the kinetic and potential energy:

$$E = -\frac{Gm_1 m_2}{2a} \quad (15)$$

The rate of change of the energy is equal to the power radiated:

$$\frac{dE}{dt} = \frac{Gm_1 m_2}{2a^2} \frac{da}{dt} = -\frac{32G^4}{5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5} \quad (16)$$

Solving for $\frac{da}{dt}$, we find:

$$\frac{da}{dt} = -\frac{64G^3}{5} \frac{m_1 m_2 (m_1 + m_2)}{a^3} \quad (17)$$

Now, we can write down the period of the orbit $\tau = \frac{2\pi}{\omega}$:

$$\tau = 2\pi \sqrt{\frac{a^3}{G(m_1 + m_2)}} = \frac{2\pi}{\sqrt{G(m_1 + m_2)}} a^{3/2} \quad (18)$$

Taking the time derivative of the period, we get:

$$\frac{d\tau}{dt} = \frac{3\pi}{\sqrt{G(m_1 + m_2)}} a^{1/2} \frac{da}{dt} = -\frac{192\pi G^{5/2}}{5} \frac{m_1 m_2 (m_1 + m_2)^{1/2}}{a^{5/2}} \quad (19)$$

Finally, we can express the change in the period as:

$$\frac{\dot{\tau}}{\tau} = \frac{d\tau/dt}{\tau} = -\frac{96G^3}{5} \frac{m_1 m_2}{a^4} \quad (20)$$

To find the time it takes for the separation to go from $a = a_0$ to $a = 0$, we can integrate the expression for $\frac{da}{dt}$:

$$t = \int dt = \int_{a_0}^0 \frac{da}{\frac{da}{dt}} = \int_{a_0}^0 -\frac{5}{64G^3} \frac{a^3}{m_1 m_2 (m_1 + m_2)} da = \frac{5a_0^4}{256G^3 m_1 m_2 (m_1 + m_2)}. \quad (21)$$

□

Question 2

Problem 7.6 in Carroll. Two object of mass M have a head on collision at $(0,0,0,0)$. In the distant past, $t \rightarrow -\infty$, the mass mass started at $x \rightarrow \pm\infty$ with zero velocity.

- (a) Using Newtonian theory show that $x(t) = \pm(\frac{9}{8}GMt^2)^{1/3}$.
- (b) For what separations is the Newtonian approximation reasonable?
- (c) Calculate h_{xx}^{TT} at $(0, R, 0)$.
- (d) For the same problem, calculate the total energy radiated in the collision.

Answer

(a)

Start with energy conservation:

$$2\frac{1}{2}M\dot{x}^2 - \frac{GM^2}{2x} = 0 \quad (22)$$

$$\implies \dot{x} = \sqrt{\frac{GM}{2x}} \quad (23)$$

Separating variables and integrating, we get:

$$\int x^{1/2} dx = \sqrt{\frac{GM}{2}} \int dt \quad (24)$$

$$\implies \frac{2}{3}x^{3/2} = \sqrt{\frac{GM}{2}}t + C, \quad \text{where } C = 0 \text{ since } x = 0 \text{ at } t = 0 \quad (25)$$

$$\implies x^{3/2} = \frac{3}{2}\sqrt{\frac{GM}{2}}t = \sqrt{\frac{9}{8}GMt} \quad (26)$$

$$\implies x(t) = \pm \left(\frac{9}{8}GMt^2\right)^{1/3} \quad (27)$$

(b)

The Newtonian approximation is reasonable when the gravitational field is weak and the velocities are much less than the speed of light. For the speed, we can calculate \dot{x} from the expression we derived in

part (a):

$$\dot{x} = \frac{d}{dt} \left(\frac{9}{8} GM t^2 \right)^{1/3} = \frac{2}{3} \left(\frac{9}{8} GM \right)^{1/3} t^{-1/3} \ll c = 1 \quad (28)$$

$$\Rightarrow t \gg \left(\frac{2}{3} \left(\frac{9}{8} GM \right)^{1/3} \right)^3 = \frac{8}{27} GM \quad (29)$$

$$\Rightarrow x(t) \gg \left(\frac{9}{8} GM \left(\frac{8}{27} GM \right)^2 \right)^{1/3} = \frac{2}{3} GM \quad (30)$$

For the gravitational field, we can calculate the gravitational potential Φ at the position of one of the masses:

$$\Phi = -\frac{GM}{2x} \quad (31)$$

The Newtonian approximation is reasonable when $|\Phi| \ll 1$, which implies:

$$x \gg \frac{GM}{2} \quad (32)$$

(c)

By equation (7.140) in Carroll, the transverse-traceless part of the metric perturbation is given by:

$$h_{ij}^{TT} = \frac{2G}{R} \frac{d^2}{dt^2} I_{ij}(t_r), \quad (33)$$

where I_{ij} is the quadrupole moment of the system and $t_r = t - R$ is the retarded time. The quadrupole moment can be calculated as:

$$I_{ij} = \sum_a m_a (x_a^i x_a^j - \frac{1}{3} r_a^2 \delta_{ij}). \quad (34)$$

Since we want to calculate h_{xx}^{TT} , we need to find I_{xx} . For the two masses, we have:

$$I_{xx} = M(x^2 - \frac{1}{3}r^2) + M((-x)^2 - \frac{1}{3}r^2) = \frac{4}{3}Mx^2 = \frac{4}{3}M \left(\frac{9}{8} GM t^2 \right)^{2/3} \quad (35)$$

Taking the second time derivative, we get:

$$\frac{d^2}{dt^2} I_{xx} = \frac{4}{3} M \frac{d^2}{dt^2} \left(\frac{9}{8} GM t^2 \right)^{2/3} = \frac{4}{3} M \left(\frac{9}{8} GM \right)^{2/3} \frac{d^2}{dt^2} t^{4/3} \quad (36)$$

$$= \frac{4}{3} M \left(\frac{9}{8} GM \right)^{2/3} \frac{4}{9} t^{-2/3} = \frac{16}{27} M \left(\frac{9}{8} GM \right)^{2/3} t^{-2/3} \quad (37)$$

Substituting this back into the expression for h_{xx}^{TT} , we get:

$$h_{xx}^{TT} = \frac{2G}{R} \frac{16}{27} M \left(\frac{9}{8} GM \right)^{2/3} t^{-2/3} = \frac{32GM}{27R} \left(\frac{9}{8} GM \right)^{2/3} (t - R)^{-2/3}, \quad (38)$$

where we have replaced t with the retarded time $t_r = t - R$.

(d)

The total energy radiated in the collision can be calculated using the quadrupole formula for gravitational radiation:

$$E = \frac{G}{5} \int_{-\infty}^{\infty} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle dt \quad (39)$$

Since we have already calculated I_{xx} , we can find \ddot{I}_{xx} by taking the third time derivative:

$$\ddot{I}_{xx} = \frac{d}{dt} \frac{d^2}{dt^2} I_{xx} = \frac{d}{dt} \left(\frac{16}{27} M \left(\frac{9}{8} GM \right)^{2/3} t^{-2/3} \right) = -\frac{32}{81} M \left(\frac{9}{8} GM \right)^{2/3} t^{-5/3} \quad (40)$$

Since the quadrupole moment is traceless, we have $\ddot{I}_{kk} = 0$. Therefore, the energy radiated can be expressed as:

$$I_{yy} = I_{zz} = -\frac{1}{2} I_{xx}, \quad \ddot{I}_{yy} = \ddot{I}_{zz} = -\frac{1}{2} \ddot{I}_{xx} \quad (41)$$

Back to the expression for the energy radiated, we have:

$$E = \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \ddot{I}_{xx}^2 + 2 \ddot{I}_{yy}^2 - \frac{1}{3} (\ddot{I}_{xx} + 2 \ddot{I}_{yy})^2 \right\rangle dt \quad (42)$$

$$= \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \ddot{I}_{xx}^2 + 2 \left(-\frac{1}{2} \ddot{I}_{xx} \right)^2 - \frac{1}{3} (\ddot{I}_{xx} - \ddot{I}_{xx})^2 \right\rangle dt \quad (43)$$

$$= \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \ddot{I}_{xx}^2 + 2 \cdot \frac{1}{4} \ddot{I}_{xx}^2 - 0 \right\rangle dt \quad (44)$$

$$= \frac{G}{5} \int_{-\infty}^{t_0} \left\langle \frac{3}{2} \ddot{I}_{xx}^2 \right\rangle dt = \frac{3G}{10} \int_{-\infty}^{t_0} \left(-\frac{32}{81} M \left(\frac{9}{8} GM \right)^{2/3} t^{-5/3} \right)^2 dt \quad (45)$$

$$= \frac{3G}{10} \frac{1024}{6561} M^2 \left(\frac{9}{8} GM \right)^{4/3} \int_{-\infty}^{t_0} t^{-10/3} dt = \frac{3G}{10} \frac{1024}{6561} M^2 \left(\frac{9}{8} GM \right)^{4/3} \cdot \frac{3}{7} t_0^{-7/3} \quad (46)$$

$$= \frac{512}{10935} GM^2 \left(\frac{9}{8} GM \right)^{4/3} t_0^{-7/3}, \quad (47)$$

where t_0 is the time at which the collision occurs. Since the collision occurs at $t = 0$, we can take the limit as $t_0 \rightarrow 0$ to find the total energy radiated:

$$E = \lim_{t_0 \rightarrow 0} \frac{512}{10935} GM^2 \left(\frac{9}{8} GM \right)^{4/3} t_0^{-7/3} = \infty \quad (48)$$

This result indicates that an infinite amount of energy is radiated in the collision, which is a consequence

of the idealized nature of the problem. In reality, the energy radiated would be finite due to various factors such as the **finite size** of the masses and the presence of other forces that would come into play during the collision. \square

Question 3

A ball of mass $m = 100$ g is thrown into the vacuum above the earth (ie. neglect all effects of air resistance), which produces a UNIFORM gravitational field ($g = 10^3$ cm s²) with a velocity $v_0 = 10^3$ cm/s. Normally, this ball would rise to a height of $h = v_0^2/2g = 500$ cm. However, the ball will be a source of gravitational radiation and won't go quite so high. Find Δh . How does $\Delta h/h$ depend on v_0 ?

Now suppose the ball will fall back and elastically bounce. Left alone, the ball will eventually come to rest. How long will it take.

Answer

The power radiated by the ball due to gravitational radiation can be calculated using the quadrupole formula:

$$P = \frac{G}{5} \left\langle \ddot{I}_{ij} \ddot{I}_{ij} - \frac{1}{3} (\ddot{I}_{kk})^2 \right\rangle \quad (49)$$

For a ball moving in a uniform gravitational field, the quadrupole moment can be expressed as:

$$I_{ij} = m(x_i x_j - \frac{1}{3} r^2 \delta_{ij}) \quad (50)$$

Since the ball is moving vertically, we can express its position as $x(t) = (0, 0, z(t))$, where $z(t)$ is the height of the ball at time t . The quadrupole moment can then be simplified to:

$$I_{zz} = m(z^2 - \frac{1}{3} z^2) = \frac{2}{3} m z^2 \quad (51)$$

Since the quadrupole moment is traceless, we have $I_{xx} = I_{yy} = -\frac{1}{2} I_{zz}$. Therefore, the second time derivative of the quadrupole moment can be calculated as:

$$\ddot{I}_{zz} = \frac{d^3}{dt^3} I_{zz} = \frac{d^3}{dt^3} \left(\frac{2}{3} m z^2 \right) \quad (52)$$

$$= \frac{2}{3} m \frac{d^2}{dt^2} (2z\dot{z}) = \frac{4}{3} m \frac{d}{dt} (z\ddot{z} + \dot{z}^2) \quad (53)$$

$$= \frac{4m}{3} \frac{d}{dt} (z(-g) + \dot{z}^2) = \frac{4m}{3} (-g\dot{z} + 2\dot{z}\ddot{z}) = \frac{4m}{3} (-g\dot{z} + 2\dot{z}(-g)) = -4mg\dot{z} = -4mgv \quad (54)$$

Since $I_{xx} = I_{yy} = -\frac{1}{2} I_{zz}$, we have $\ddot{I}_{xx} = \ddot{I}_{yy} = -\frac{1}{2} \ddot{I}_{zz}$. Substituting these results back into the expression for the power, we get:

$$P = \frac{G}{5} \left\langle \ddot{I}_{zz}^2 + 2\ddot{I}_{xx}^2 - \frac{1}{3} (\ddot{I}_{zz} + 2\ddot{I}_{xx})^2 \right\rangle = \frac{G}{5} \left\langle \ddot{I}_{zz}^2 + 2 \cdot \frac{1}{4} \ddot{I}_{zz}^2 - 0 \right\rangle = \frac{3G}{10} \left\langle \ddot{I}_{zz}^2 \right\rangle \quad (55)$$

$$= \frac{3G}{10} \langle (-4mgv)^2 \rangle = \frac{24Gm^2g^2}{5} v^2 \quad (56)$$

The energy radiated by the ball can be calculated by integrating the power over time, and we also restore the c^5 in the denominator:

$$\Delta E = \int P dt = \int \frac{24Gm^2g^2}{5c^5} v^2 dt = \frac{24Gm^2g^2}{5c^5} \int v^2 dt \quad (57)$$

$$= \frac{24Gm^2g^2}{5c^5} \int v^2 \frac{dv}{-g} = -\frac{24Gm^2g}{5c^5} \int v^2 dv = -\frac{24Gm^2g}{5c^5} \cdot \frac{v_0^3}{3} = -\frac{8Gm^2gv_0^3}{5c^5} \quad (58)$$

The change in height Δh can be calculated by equating the energy radiated to the change in potential energy:

$$\Delta E = mg\Delta h \implies \Delta h = \frac{\Delta E}{mg} = -\frac{8Gm^2gv_0^3}{5c^5} \quad (59)$$

The ratio $\Delta h/h$ can be expressed as:

$$\frac{\Delta h}{h} = \frac{-\frac{8Gm^2gv_0^3}{5c^5}}{\frac{v_0^2}{2g}} = -\frac{16Gm^2g^2v_0}{5c^5} \quad (60)$$

This shows that the ratio $\Delta h/h$ is proportional to the initial velocity v_0 of the ball.

Now, we know the due to the radiation, we have $\Delta E = -\frac{8Gm^2gv_0^3}{5c^5}$, which means the ball loses energy at a rate of $\frac{dE}{dt}$:

$$\frac{dE}{dt} = -\frac{8Gm^2gv_0^3}{5c^5} \cdot \frac{1}{t_{\text{total}}} = -\frac{8Gm^2gv_0^3}{5c^5} \cdot \frac{1}{2t_{\text{up}}}, \quad (61)$$

where t_{total} is the total time for the ball to go up and come back down, and t_{up} is the time it takes for the ball to reach its maximum height. Since the ball is thrown upwards with an initial velocity of v_0 , we can calculate t_{up} using the equation of motion:

$$v = v_0 - gt_{\text{up}} = 0 \implies t_{\text{up}} = \frac{v_0}{g} \quad (62)$$

Substituting this back into the expression for $\frac{dE}{dt}$,

$$\frac{dE}{dt} = -\frac{8Gm^2gv_0^3}{5c^5} \cdot \frac{1}{2 \cdot \frac{v_0}{g}} = -\frac{4Gm^2g^2v_0^2}{5c^5} \quad (63)$$

Note that the energy of the ball at its maximum height is given by:

$$E = mgh = mg \frac{v_0^2}{2g} = \frac{1}{2}mv_0^2 \implies v_0^2 = \frac{2E}{m}, \quad (64)$$

Hence, we can express $\frac{dE}{dt}$ in terms of the energy E :

$$\frac{dE}{dt} = -\frac{4Gm^2g^2}{5c^5} \cdot \frac{2E}{m} = -\frac{8Gmg^2}{5c^5}E \quad (65)$$

This is a first-order linear differential equation, the solution to which is given by:

$$E(t) = E_0 e^{-\frac{8Gmg^2}{5c^5}t} \quad (66)$$

where E_0 is the initial energy of the ball at its maximum height. The ball will come to rest when its energy approaches zero, which occurs as $t \rightarrow \infty$.

□