University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8011
Quantum Field Theory I

Assignment Solution

Lecture Instructor: Professor Zhen Liu

Zong-En Chen
chen9613@umn.edu

January 20, 2026

HW1 Due to September 23 11:59 PM

Question 1

Problem 1.2

With the Hamiltonian of eq. (1.32), show that the state defined in eq. (1.33) obeys the abstract Schrodinger equation, eq. (1.1), if and only if the wave function obeys eq. (1.30). Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations of eq. (1.31), and to fermions, where the particle creation and annihilation operators obey the anti-commutation relations of eq. (1.38).

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle \tag{1.1}$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U\left(\mathbf{x}_j\right)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V\left(\mathbf{x}_j - \mathbf{x}_k\right)\right]\psi \tag{1.30}$$

$$\begin{aligned} \left[a(\mathbf{x}), a\left(\mathbf{x}'\right)\right] &= 0 \\ \left[a^\dagger(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] &= 0 \\ \left[a(\mathbf{x}), a^\dagger\left(\mathbf{x}'\right)\right] &= \delta^3\left(\mathbf{x}-\mathbf{x}'\right) \end{aligned} \tag{1.31}$$

$$\begin{aligned} H = &\int d^3x a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right)a(\mathbf{x}) \\ &+ \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x}) \end{aligned} \tag{1.32}$$

$$|\psi,t\rangle = \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{1.33}$$

Answer

We first consider boson case, and then we have

$$LHS = i\hbar\frac{\partial}{\partial t}|\psi,t\rangle \tag{1}$$

$$= i\hbar\frac{\partial}{\partial t}\int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{2}$$

$$= \int d^3x_1 \ldots d^3x_n i\hbar\frac{\partial}{\partial t}\psi\left(\mathbf{x}_1,\ldots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{3}$$

$$= \int d^3x_1 \ldots d^3x_n \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U\left(\mathbf{x}_j\right)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V\left(\mathbf{x}_j - \mathbf{x}_k\right)\right]\psi a^\dagger\left(\mathbf{x}_1\right)\ldots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{4}$$

$$RHS = H|\psi, t\rangle \tag{5}$$

$$= \left[\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) + \frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x})\right] |\psi, t\rangle \tag{6}$$

$$= \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle \tag{7}$$

$$+\frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle \tag{8}$$

For the term in Equation 7, by considering $[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}')$, we have

$$a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) \tag{9}$$

$$=[a^\dagger(\mathbf{x}_1) a(\mathbf{x}) + \delta^3(\mathbf{x}-\mathbf{x}_1)] a^\dagger\left(\mathbf{x}_2\right) \ldots a^\dagger\left(\mathbf{x}_n\right) \tag{10}$$

$$=a^\dagger(\mathbf{x}_1) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_2\right) \ldots a^\dagger\left(\mathbf{x}_n\right) + \delta^3(\mathbf{x}-\mathbf{x}_1) a^\dagger\left(\mathbf{x}_2\right) \ldots a^\dagger\left(\mathbf{x}_n\right) \tag{11}$$

$$=a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) a(\mathbf{x}) + \sum_{j=1}^{n} \delta^3(\mathbf{x}-\mathbf{x}_j) a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \ldots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \ldots a^\dagger(\mathbf{x}_n) \tag{12}$$

$$=0 + \sum_{j=1}^{n} \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j, \quad \mathcal{O}_j = a^\dagger(\mathbf{x}_1) a^\dagger(\mathbf{x}_2) \ldots a^\dagger(\mathbf{x}_{j-1}) a^\dagger(\mathbf{x}_{j+1}) \ldots a^\dagger(\mathbf{x}_n) \tag{13}$$

we can drop the first term in Equation 12 since this term will act on the $|0\rangle$, giving 0. Hence, we have

$$\int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) a(\mathbf{x}) \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle \tag{14}$$

$$= \sum_{j=1}^{n} \int d^3x a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \int d^3x_1 \ldots d^3x_n \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{15}$$

$$= \sum_{j=1}^{n} \int d^3x \int d^3x_1 \ldots d^3x_n a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) \delta^3(\mathbf{x}-\mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{16}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \ldots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) \mathcal{O}_j |0\rangle \tag{17}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \ldots d^3x_n a^\dagger(\mathbf{x}_j) \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) \mathcal{O}_j |0\rangle \tag{18}$$

$$= \sum_{j=1}^{n} \int d^3x_1 \ldots d^3x_n \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger(\mathbf{x}_j) \mathcal{O}_j |0\rangle \tag{19}$$

$$= \int d^3x_1 \ldots d^3x_n \sum_{j=1}^{n} \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) \psi\left(\mathbf{x}_1, \ldots, \mathbf{x}_n; t\right) a^\dagger\left(\mathbf{x}_1\right) \ldots a^\dagger\left(\mathbf{x}_n\right) |0\rangle, \tag{20}$$

where $a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n) = a^{\dagger}(\mathbf{x}_j)\mathcal{O}_j$ since they (boson fields) commute. Now, we do the same thing for the term in Equation 8, we have

$$
\begin{aligned}
& a(\mathbf{y})a(\mathbf{x})a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n) && (21)\\
=& a(\mathbf{y})\sum_{j=1}^{n}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j, \quad \mathcal{O}_j = a^{\dagger}(\mathbf{x}_1)a^{\dagger}(\mathbf{x}_2)\ldots a^{\dagger}(\mathbf{x}_{j-1})a^{\dagger}(\mathbf{x}_{j+1})\ldots a^{\dagger}(\mathbf{x}_n) && (22)\\
=& \sum_{i\neq j}^{n}\sum_{j=1}^{n}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}, \quad \mathcal{T}_{ij} = \prod_{k\neq i,j}^{n} a^{\dagger}(\mathbf{x}_k). && (23)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& \frac{1}{2}\int d^3xd^3yV(\mathbf{x}-\mathbf{y})a^{\dagger}(\mathbf{x})a^{\dagger}(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\int d^3x_1\ldots d^3x_n\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n)|0\rangle && (24)\\
=& \frac{1}{2}\int d^3xd^3y\int d^3x_1\ldots d^3x_nV(\mathbf{x}-\mathbf{y})a^{\dagger}(\mathbf{x})a^{\dagger}(\mathbf{y})\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,a(\mathbf{y})a(\mathbf{x})a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n)|0\rangle && (25)\\
=& \frac{1}{2}\int d^3xd^3y\int d^3x_1\ldots d^3x_nV(\mathbf{x}-\mathbf{y})a^{\dagger}(\mathbf{x})a^{\dagger}(\mathbf{y})\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\sum_{i\neq j}^{n}\sum_{j=1}^{n}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle && (26)\\
=& \sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}\int d^3xd^3y\int d^3x_1\ldots d^3x_nV(\mathbf{x}-\mathbf{y})\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)a^{\dagger}(\mathbf{x})a^{\dagger}(\mathbf{y})\mathcal{T}_{ij}|0\rangle && (27)\\
=& \sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}\int d^3x_1\ldots d^3x_nV(\mathbf{x}_j-\mathbf{x}_i)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,a^{\dagger}(\mathbf{x}_j)a^{\dagger}(\mathbf{x}_i)\mathcal{T}_{ij}|0\rangle && (28)\\
=& \int d^3x_1\ldots d^3x_n\sum_{i\neq j}^{n}\sum_{j=1}^{n}\frac{1}{2}V(\mathbf{x}_j-\mathbf{x}_i)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n)|0\rangle && (29)\\
=& \int d^3x_1\ldots d^3x_n\sum_{j=1}^{n}\sum_{k=1}^{j-1}V(\mathbf{x}_j-\mathbf{x}_k)\,\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\,a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n)|0\rangle, && (30)
\end{aligned}
$$

where $a^{\dagger}(\mathbf{x}_j)a^{\dagger}(\mathbf{x}_i)\mathcal{T}_{ij} = a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n)$ for the same reason. Hence, we have proved the $LHS = RHS$ and Equation 1.1 for the boson field case.

For fermion fields, the only difference is the anti-commutation relation. We start from Equation 7 again, by considering$\{a(\mathbf{x}), a^{\dagger}(\mathbf{x})\} = \delta^3(\mathbf{x}-\mathbf{x}')$, and we have

$$
\begin{aligned}
& a(\mathbf{x})a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n) && (31)\\
=& [-a^{\dagger}(\mathbf{x}_1)a(\mathbf{x}) + \delta^3(\mathbf{x}-\mathbf{x}_1)]a^{\dagger}(\mathbf{x}_2)\ldots a^{\dagger}(\mathbf{x}_n) && (32)\\
=& (-1)^n a^{\dagger}(\mathbf{x}_1)\ldots a^{\dagger}(\mathbf{x}_n)\,a(\mathbf{x}) + \sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)a^{\dagger}(\mathbf{x}_1)a^{\dagger}(\mathbf{x}_2)\ldots a^{\dagger}(\mathbf{x}_{j-1})a^{\dagger}(\mathbf{x}_{j+1})\ldots a^{\dagger}(\mathbf{x}_n) && (33)\\
=& 0 + \sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j, \quad \mathcal{O}_j = a^{\dagger}(\mathbf{x}_1)a^{\dagger}(\mathbf{x}_2)\ldots a^{\dagger}(\mathbf{x}_{j-1})a^{\dagger}(\mathbf{x}_{j+1})\ldots a^{\dagger}(\mathbf{x}_n), && (34)
\end{aligned}
$$

where the 0 term comes from the same reason. Then the term in Equation 7 is given by

$$\int d^3x a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2+U(\mathbf{x})\right)a(\mathbf{x})\int d^3x_1\dots d^3x_n\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{35}$$

$$=\int d^3x\int d^3x_1\dots d^3x_n a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2+U(\mathbf{x})\right)\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)a(\mathbf{x})a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle \tag{36}$$

$$=\int d^3x\int d^3x_1\dots d^3x_n a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2+U(\mathbf{x})\right)\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)\sum_{j=1}^{n}(-1)^j\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j|0\rangle \tag{37}$$

$$=\sum_{j=1}^{n}\int d^3x\int d^3x_1\dots d^3x_n a^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2+U(\mathbf{x})\right)\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)(-1)^j\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j|0\rangle \tag{38}$$

$$=\sum_{j=1}^{n}\int d^3x_1\dots d^3x_n a^\dagger(\mathbf{x}_j)\left(-\frac{\hbar^2}{2m}\nabla_j^2+U(\mathbf{x}_j)\right)\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)(-1)^j\mathcal{O}_j|0\rangle \tag{39}$$

$$=\sum_{j=1}^{n}\int d^3x_1\dots d^3x_n\left(-\frac{\hbar^2}{2m}\nabla_j^2+U(\mathbf{x}_j)\right)\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)(-1)^j a^\dagger(\mathbf{x}_j)\mathcal{O}_j|0\rangle \tag{40}$$

$$=\int d^3x_1\dots d^3x_n\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2+U(\mathbf{x}_j)\right)\psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right)a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle, \tag{41}$$

where $a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)=(-1)^{j-1}a^\dagger(\mathbf{x}_j)\mathcal{O}_j$ by anti-commutation relation of fermion fields. Next, given the term in Equation 8, we have

$$a(\mathbf{y})a(\mathbf{x})a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right) \tag{42}$$

$$=a(\mathbf{y})\sum_{j=1}^{n}(-1)^{j-1}\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{O}_j,\quad \mathcal{O}_j=a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)\dots a^\dagger(\mathbf{x}_{j-1})a^\dagger(\mathbf{x}_{j+1})\dots a^\dagger(\mathbf{x}_n) \tag{43}$$

$$=\sum_{i<j}^{n}\sum_{j=1}^{n}(-1)^{i-1}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}+\sum_{i>j}^{n}\sum_{j=1}^{n}(-1)^{i-2}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}, \tag{44}$$

where $\mathcal{T}_{ij} = \prod_{k\neq i,j}^{n} a^\dagger(\mathbf{x}_k)$. Now, we can simplify Equation 8 and it is given by

$$
\begin{aligned}
&\frac{1}{2}\int d^3x d^3y V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) \int d^3x_1 \dots d^3x_n \psi\left(\mathbf{x}_1,\dots,\mathbf{x}_n;t\right) a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (45)\\
=&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi a(\mathbf{y}) a(\mathbf{x}) a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (46)\\
=&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i<j}^{n}\sum_{j=1}^{n} (-1)^{i-1}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle \\
+&\frac{1}{2}\int d^3x d^3y \int d^3x_1 \dots d^3x_n V(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) \psi \sum_{i>j}^{n}\sum_{j=1}^{n} (-1)^{i-2}(-1)^{j-1}\delta^3(\mathbf{y}-\mathbf{x}_i)\delta^3(\mathbf{x}-\mathbf{x}_j)\mathcal{T}_{ij}|0\rangle && (47)\\
=&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i<j}^{n}\sum_{j=1}^{n} (-1)^{i-1}(-1)^{j-1} V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle \\
+&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i>j}^{n}\sum_{j=1}^{n} (-1)^{i-2}(-1)^{j-1} V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle && (48)\\
=&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i\neq j}^{n}\sum_{j=1}^{n} (-1)^{i-1}(-1)^{j-1} V(\mathbf{x}_j-\mathbf{x}_i) a^\dagger(\mathbf{x}_j) a^\dagger(\mathbf{x}_i)\psi\mathcal{T}_{ij}|0\rangle && (49)\\
=&\frac{1}{2}\int d^3x_1 \dots d^3x_n \sum_{i\neq j}^{n}\sum_{j=1}^{n} V(\mathbf{x}_j-\mathbf{x}_i)\psi a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle && (50)\\
=&\int d^3x_1 \dots d^3x_n \sum_{j=1}^{n}\sum_{k=1}^{j-1} V\left(\mathbf{x}_j-\mathbf{x}_k\right)\psi a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)|0\rangle, && (51)
\end{aligned}
$$

where $(-1)^{i-1}(-1)^{j-1}a^\dagger(\mathbf{x}_j)a^\dagger(\mathbf{x}_i)\mathcal{T}_{ij} = a^\dagger\left(\mathbf{x}_1\right)\dots a^\dagger\left(\mathbf{x}_n\right)$ by anti-commutation relation. In summary, we have proved both cases for boson fields and fermion fields. □

Question 2

Problem 2.3

Verify that eq. (2.16) follows from eq. (2.14).

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \tag{2.14}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma) \tag{2.16}$$

$$= i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{2.16}$$

Answer

Considering an infinitesimal transformation in $U(\Lambda) = 1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}$ and $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$, now we get

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) M^{\mu\nu} \left(1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) \tag{52}$$

$$\Rightarrow \delta\omega_{\alpha\beta}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\alpha\beta}] = \delta\omega_{\rho\sigma}\frac{i}{2\hbar}[M^{\mu\nu}, M^{\rho\sigma}] \tag{53}$$

$$RHS = (\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma})M^{\rho\sigma} \tag{54}$$

$$\rightarrow \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} = \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} \tag{55}$$

$$= g^{\nu\rho}\delta\omega_{\rho\sigma}M^{\mu\sigma} + g^{\sigma\mu}\delta\omega_{\sigma\rho}M^{\rho\nu} = \delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\nu}\right), \tag{56}$$

we only consider the linear term $\delta\omega$. We can further simplify it to

$$[M^{\mu\nu}, M^{\rho\sigma}] = \frac{2\hbar}{i}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\sigma\mu}M^{\rho\nu}\right) = 2i\hbar\left(g^{\sigma\mu}M^{\rho\nu} - g^{\nu\rho}M^{\mu\sigma}\right) = 2i\hbar\left(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right) \tag{57}$$

$$= -[M^{\nu\mu}, M^{\rho\sigma}] = 2i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma}\right) \tag{58}$$

Finally, we have

$$[M^{\mu\nu}, M^{\rho\sigma}] \tag{59}$$

$$= \frac{1}{2}\left([M^{\mu\nu}, M^{\rho\sigma}] - [M^{\nu\mu}, M^{\rho\sigma}]\right) \tag{60}$$

$$= i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right). \tag{61}$$

□

Question 3

Problem 2.8

(a) Let $\Lambda = 1 + \delta\omega$ in eq.(2.26), and show that

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x),$$

where

$$\mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

(b) Show that $[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x)$.

(c) Prove the *Jacobi identity*, $[[A,B],C]+[[B,C],A]+[[C,A],B] = 0$. Hint: write out all the commutations.

(d) Use your results from parts (b) and (c) to show that

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \tag{2.31}$$

(e) Simplify the right-hand side of eq. (2.31) as much as possible.

(f) Use your results from part (e) to verify eq. (2.16), up to the possibility of a term on the right-hand side that commutes with $\varphi(x)$ and its derivatives. (Such a term, called a *central charge*, in fact does not arise for the Lorentz algebra.)

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \tag{2.26}$$

Answer

(a)

$$LHS = U^{-1}(\Lambda)\varphi(x)U(\Lambda) \tag{62}$$

$$= \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \tag{63}$$

$$\to \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\varphi(x), M^{\mu\nu}] \tag{64}$$

$$RHS = \varphi(\Lambda^{-1}x) = \varphi((\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)x^\nu) = \phi(x) - \delta\omega^\mu{}_\nu x^\nu\partial_\mu\phi(x) \tag{65}$$

$$\to -\delta\omega^\mu{}_\nu x^\nu\partial_\mu\varphi(x) = -\delta\omega_{\mu\nu}x^\nu\partial^\mu\varphi(x) = \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\varphi(x), \tag{66}$$

we only focus on the linear term $\delta\omega$. Now we have

$$[\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi(x) = \mathcal{L}^{\mu\nu}\varphi(x). \tag{67}$$

(b)

$$[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\varphi(x)M^{\rho\sigma} - M^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{68}$$

$$=\mathcal{L}^{\mu\nu}[\varphi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x). \tag{69}$$

(c)

$$[[A, B], C] + [[B, C], A] + [[C, A], B] \tag{70}$$

$$=(CAB - CBA - ABC + BAC) + (ABC - ACB - BCA + CBA) + (BCA - BAC - CAB + ACB) \tag{71}$$

$$=0. \tag{72}$$

(d)

By the Jacobi identity, we have

$$0 = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\mu\nu}, M^{\rho\sigma}], \varphi(x)] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \tag{73}$$

$$\rightarrow[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] + [[M^{\rho\sigma}, \varphi(x)], M^{\mu\nu}] \tag{74}$$

$$=[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] - [[\varphi(x), M^{\rho\sigma}], M^{\mu\nu}] \tag{75}$$

$$=\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{76}$$

$$=(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x). \tag{77}$$

(e)

For the result in Equation 77, considering the relation $\partial^\mu x^\nu\varphi(x) = (g^{\mu\nu} + x^\nu\partial^\mu)\varphi(x)$, we can have

$$\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \frac{\hbar}{i}\frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)(x^\rho\partial^\sigma - x^\sigma\partial^\rho)\varphi(x) \tag{78}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[x^\mu(g^{\nu\rho} + x^\rho\partial^\nu)\partial^\sigma - x^\nu(g^{\mu\rho} + x^\rho\partial^\mu)\partial^\sigma - x^\mu(g^{\nu\sigma} + x^\sigma\partial^\nu)\partial^\rho + x^\nu(g^{\mu\sigma} + x^\sigma\partial^\mu)\partial^\rho\right]\varphi(x) \tag{79}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[g^{\nu\rho}x^\mu\partial^\sigma - g^{\mu\rho}x^\nu\partial^\sigma - g^{\nu\sigma}x^\mu\partial^\rho + g^{\mu\sigma}x^\nu\partial^\rho + x^\mu x^\rho\partial^\nu\partial^\sigma - x^\nu x^\rho\partial^\mu\partial^\sigma - x^\mu x^\sigma\partial^\nu\partial^\rho + x^\nu x^\sigma\partial^\mu\partial^\sigma\right]\varphi(x). \tag{80}$$

$$\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) \tag{81}$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[g^{\sigma\mu}\, x^\rho\partial^\nu - g^{\rho\mu}\, x^\sigma\partial^\nu - g^{\sigma\nu}\, x^\rho\partial^\mu + g^{\rho\nu}\, x^\sigma\partial^\mu + x^\rho x^\mu\, \partial^\sigma\partial^\nu - x^\rho x^\nu\, \partial^\sigma\partial^\mu - x^\sigma x^\mu\, \partial^\rho\partial^\nu + x^\sigma x^\nu\, \partial^\rho\partial^\mu\right]\varphi(x). \tag{82}$$

Using simpler forms to express:

$$\begin{aligned}\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[& g^{\nu\rho}\, x^\mu\partial^\sigma - g^{\mu\rho}\, x^\nu\partial^\sigma - g^{\nu\sigma}\, x^\mu\partial^\rho + g^{\mu\sigma}\, x^\nu\partial^\rho \\ &+ x^\mu x^\rho\, \partial^\nu\partial^\sigma - x^\mu x^\sigma\, \partial^\nu\partial^\rho - x^\nu x^\rho\, \partial^\mu\partial^\sigma + x^\nu x^\sigma\, \partial^\mu\partial^\rho\Big]\varphi(x), \end{aligned} \tag{83}$$

$$\begin{aligned}\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[& g^{\sigma\mu}\, x^\rho\partial^\nu - g^{\rho\mu}\, x^\sigma\partial^\nu - g^{\sigma\nu}\, x^\rho\partial^\mu + g^{\rho\nu}\, x^\sigma\partial^\mu \\ &+ x^\rho x^\mu\, \partial^\sigma\partial^\nu - x^\sigma x^\mu\, \partial^\rho\partial^\nu - x^\rho x^\nu\, \partial^\sigma\partial^\mu + x^\sigma x^\nu\, \partial^\rho\partial^\mu\Big]\varphi(x). \end{aligned} \tag{84}$$

Combining those two results together, it gives

$$\begin{aligned}(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) = \left(\frac{\hbar}{i}\right)^2 \Big[& g^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) - g^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu) \\ &- g^{\nu\sigma}(x^\mu\partial^\rho - x^\rho\partial^\mu) + g^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu)\Big]\varphi(x) \end{aligned} \tag{85}$$

$$= \frac{\hbar}{i}\Big(g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\rho}\mathcal{L}^{\nu\sigma} - g^{\nu\sigma}\mathcal{L}^{\mu\rho} + g^{\mu\sigma}\mathcal{L}^{\nu\rho}\Big)\varphi(x) \tag{86}$$

$$= i\hbar\Big(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\Big)\varphi(x). \tag{87}$$

Actually, it looks similar to

$$\begin{aligned}&[M^{\mu\nu}, M^{\rho\sigma}] \\ =& i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right)\end{aligned} \tag{2.16}$$

(f)

Now we assume there is a non trivial term $\mathcal{C}$ on $[M^{\mu\nu}, M^{\rho\sigma}]$, giving that

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C}\right), \tag{88}$$

where C can commutes with φ and its derivatives. Now, we have

$$
\begin{aligned}
&[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] && (2.31)\\
=&i\hbar[\varphi(x), (g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + \mathcal{C})] && (89)\\
=&i\hbar\Big(g^{\nu\sigma}[\varphi(x), M^{\mu\rho}] + g^{\mu\rho}[\varphi(x), M^{\nu\sigma}] - g^{\mu\sigma}[\varphi, M^{\nu\rho}] - g^{\nu\rho}[\varphi(x), M^{\mu\sigma}] + [\varphi(x), \mathcal{C}]\Big) && (90)\\
=&i\hbar\big(g^{\mu\rho}\mathcal{L}^{\nu\sigma} + g^{\nu\sigma}\mathcal{L}^{\mu\rho} - g^{\nu\rho}\mathcal{L}^{\mu\sigma} - g^{\mu\sigma}\mathcal{L}^{\nu\rho}\big)\varphi(x) = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) && (91)\\
=&[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\varphi(x). && (92)
\end{aligned}
$$

Hence, with the central charge $\mathcal{C}$, the relation still holds. □

Question 4

Problem 2.9

Let us write

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau + \frac{i}{2\hbar}\delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\tau, \tag{2.32}$$

where

$$(S_V^{\mu\nu})^\rho{}_\tau \equiv \frac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau) \tag{2.33}$$

are matrices which constitute the *vector representation* of the Lorentz generators.

(a) Let $\Lambda = 1 + \delta\omega$ in eq. (2.27), and show that

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x) \tag{2.34}$$

(b) Show that the matrices $(S_V^{\mu\nu})$ must have the same commutation relations as the operators $M^{\mu\nu}$. Hint: see the previous problem.

(c) For a rotation by an angle θ about the z axis, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Show that

$$\Lambda = \exp(-i\theta S_V^{12}/\hbar). \tag{2.36}$$

(d) For a boost by *rapidity* η in the z direction, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}. \tag{2.37}$$

Show that

$$\Lambda = \exp(+i\eta S_V^{30}/\hbar). \tag{2.38}$$

$$U^{-1}(\Lambda)\partial^\rho\varphi(x)U(\Lambda) = \Lambda^\rho{}_\mu\overline{\partial}^\mu\varphi(\Lambda^{-1}x), \quad \overline{x}^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu, \quad \overline{\partial}^\mu = (\Lambda^{-1})^\mu{}_\nu\partial^\nu \tag{2.27}$$

Answer

(a)

$$LHS = \left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) \tag{93}$$

$$=\partial^\rho\varphi(x) + \delta\omega_{\mu\nu}\frac{i}{2\hbar}[\partial^\rho\varphi(x), M^{\mu\nu}] \tag{94}$$

$$RHS = \Lambda^\rho{}_\mu\overline{\partial}^\mu\varphi(\Lambda^{-1}x) \tag{95}$$

$$=(\delta^\rho{}_\mu + \delta\omega^\rho{}_\mu)\overline{\partial}^\mu\varphi((\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)x^\nu) \tag{96}$$

$$=(\delta^\rho{}_\mu + \delta\omega^\rho{}_\mu)(\delta^\mu{}_\nu - \delta\omega^\mu{}_\nu)\partial^\nu(\varphi(x) - \delta\omega^\alpha{}_\beta x^\beta\partial_\alpha\varphi(x)) \tag{97}$$

$$=\delta^\rho{}_\nu(\partial^\nu\varphi(x) - \delta\omega^\alpha{}_\beta g^{\nu\beta}\partial_\alpha\varphi(x) - \delta\omega^\alpha{}_\beta x^\beta\partial^\nu\partial_\alpha\varphi(x)) \tag{98}$$

$$=\partial^\rho\varphi(x) - \delta\omega^\alpha{}_\beta g^{\rho\beta}\partial_\alpha\varphi(x) - \delta\omega^\alpha{}_\beta x^\beta\partial^\rho\partial_\alpha\varphi(x) \tag{99}$$

$$=\partial^\rho\varphi(x) - \delta\omega_{\alpha\beta}g^{\rho\beta}\partial^\alpha\varphi(x) - \delta\omega_{\alpha\beta}x^\beta\partial^\rho\partial^\alpha\varphi(x) \tag{100}$$

$$=\partial^\rho\varphi(x) - \delta\omega_{\mu\nu}g^{\rho\nu}\partial^\mu\varphi(x) - \delta\omega_{\mu\nu}x^\nu\partial^\rho\partial^\mu\varphi(x) \tag{101}$$

$$=\partial^\rho\varphi(x) + \delta\omega_{\mu\nu}\left(-g^{\rho\nu}\partial^\mu - x^\nu\partial^\rho\partial^\mu\right)\varphi(x) \tag{102}$$

$$=\partial^\rho\varphi(x) + \delta\omega_{\mu\nu}\frac{1}{2}\left(-g^{\rho\nu}\partial^\mu - x^\nu\partial^\rho\partial^\mu + g^{\rho\mu}\partial^\nu + x^\mu\partial^\rho\partial^\nu\right)\varphi(x) \tag{103}$$

Hence, we have

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i}\left(-g^{\rho\nu}\partial^\mu - x^\nu\partial^\rho\partial^\mu + g^{\rho\mu}\partial^\nu + x^\mu\partial^\rho\partial^\nu\right)\varphi(x) \tag{104}$$

$$=\frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\partial^\rho\varphi(x) + \frac{\hbar}{i}(g^{\rho\mu}\delta^\nu{}_\tau\partial^\tau - g^{\rho\nu}\delta^\mu{}_\tau\partial^\tau)\varphi(x) \tag{105}$$

$$=\frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\partial^\rho\varphi(x) + \frac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)\partial^\tau\varphi(x) \tag{106}$$

$$=\mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x), \tag{107}$$

which is exactly the result we want to prove.

(b)

$$\begin{aligned}
&[(S_V^{\mu\nu}),(S_V^{\rho\sigma})]^\alpha{}_\beta &(108)\\
=&(S_V^{\mu\nu})^\alpha{}_\tau(S_V^{\rho\sigma})^\tau{}_\beta-(S_V^{\rho\sigma})^\alpha{}_\tau(S_V^{\mu\nu})^\tau{}_\beta &(109)\\
=&\frac{\hbar}{i}(g^{\mu\alpha}\delta^\nu{}_\tau-g^{\nu\alpha}\delta^\mu{}_\tau)\frac{\hbar}{i}(g^{\rho\tau}\delta^\sigma{}_\beta-g^{\sigma\tau}\delta^\rho{}_\beta)-\frac{\hbar}{i}(g^{\rho\alpha}\delta^\sigma{}_\tau-g^{\sigma\alpha}\delta^\rho{}_\tau)\frac{\hbar}{i}(g^{\mu\tau}\delta^\nu{}_\beta-g^{\nu\tau}\delta^\mu{}_\beta) &(110)\\
=&\left(\frac{\hbar}{i}\right)^2\Big[g^{\mu\alpha}g^{\rho\nu}\delta^\sigma{}_\beta-g^{\mu\alpha}g^{\sigma\nu}\delta^\rho{}_\beta-g^{\nu\alpha}g^{\rho\mu}\delta^\sigma{}_\beta+g^{\nu\alpha}g^{\sigma\mu}\delta^\rho{}_\beta &(111)\\
&\quad-g^{\rho\alpha}g^{\mu\sigma}\delta^\nu{}_\beta+g^{\rho\alpha}g^{\nu\sigma}\delta^\mu{}_\beta+g^{\sigma\alpha}g^{\mu\rho}\delta^\nu{}_\beta-g^{\sigma\alpha}g^{\nu\rho}\delta^\mu{}_\beta\Big] &(112)\\
=&i\hbar\left(\frac{\hbar}{i}\right)\Big[g^{\mu\rho}(g^{\nu\alpha}\delta^\sigma{}_\beta-g^{\sigma\alpha}\delta^\nu{}_\beta)-g^{\nu\rho}(g^{\mu\alpha}\delta^\sigma{}_\beta-g^{\sigma\alpha}\delta^\mu{}_\beta)-g^{\mu\sigma}(g^{\nu\alpha}\delta^\rho{}_\beta-g^{\rho\alpha}\delta^\nu{}_\beta)+g^{\nu\sigma}(g^{\mu\alpha}\delta^\rho{}_\beta-g^{\rho\alpha}\delta^\mu{}_\beta)\Big] &(113)\\
=&i\hbar\left(g^{\mu\rho}(S_V^{\nu\sigma})^\alpha{}_\beta-g^{\nu\rho}(S_V^{\mu\sigma})^\alpha{}_\beta-g^{\mu\sigma}(S_V^{\nu\rho})^\alpha{}_\beta+g^{\nu\sigma}(S_V^{\mu\rho})^\alpha{}_\beta\right) &(114)\\
=&i\hbar\Big(g^{\mu\rho}(S_V^{\nu\sigma})-g^{\nu\rho}(S_V^{\mu\sigma})-g^{\mu\sigma}(S_V^{\nu\rho})+g^{\nu\sigma}(S_V^{\mu\rho})\Big)^\alpha{}_\beta. &(115)
\end{aligned}$$

This looks exactly the same as

$$\begin{aligned}
&[M^{\mu\nu},M^{\rho\sigma}]\\
=&i\hbar\left(g^{\nu\sigma}M^{\mu\rho}+g^{\mu\rho}M^{\nu\sigma}-g^{\mu\sigma}M^{\nu\rho}-g^{\nu\rho}M^{\mu\sigma}\right) \qquad (2.16)
\end{aligned}$$

(c)

$$\frac{i}{\hbar}(S_V^{12})^\mu{}_\nu=(g^{1\mu}\delta^2{}_\nu-g^{2\mu}\delta^1{}_\nu)=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\equiv\mathcal{R}, \tag{116}$$

$$\mathcal{R}^2=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}, \tag{117}$$

$$\mathcal{R}^3=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}=-\mathcal{R}, \tag{118}$$

$$\mathcal{R}^4=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}=-\mathcal{R}^2. \tag{119}$$

Hence, we have

$$
\begin{aligned}
\Lambda =& \exp\left(-i\theta S_V^{12}/\hbar\right) = \exp\left(-\theta\mathcal{R}\right) && (120)\\
=& \sum_{n=0}^{\infty}\frac{(-\theta\mathcal{R})^n}{n!} = \sum_{m=0}^{\infty}\frac{(-\theta\mathcal{R})^{2m}}{(2m)!} + \sum_{m=0}^{\infty}\frac{(-\theta\mathcal{R})^{2m+1}}{(2m+1)!} && (121)\\
=& I + \sum_{m=1}^{\infty}\frac{(-1)^m\theta^{2m}\mathcal{R}^{2m}}{(2m)!} + \sum_{m=0}^{\infty}\frac{(-1)\theta^{2m+1}\mathcal{R}^{2m+1}}{(2m+1)!} && (122)\\
=& I + \sum_{m=1}^{\infty}\frac{(-1)^m\theta^{2m}}{(2m)!}\mathcal{R}^2 - \sum_{m=0}^{\infty}\frac{(-1)^m\theta^{2m+1}}{(2m+1)!}\mathcal{R} && (123)\\
=& I + (\cos\theta - 1)\mathcal{R}^2 - \sin\theta\mathcal{R} && (124)\\
=& \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \cos\theta - 1 & 0 & 0\\ 0 & 0 & \cos\theta - 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -\sin\theta & 0\\ 0 & \sin\theta & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} && (125)\\
=& \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. && (126)
\end{aligned}
$$

This is exactly the same as the result in Equation (2.35).

(d)

$$\frac{i}{\hbar}(S_V^{30})^\mu{}_\nu = (g^{3\mu}\delta^0{}_\nu - g^{0\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \equiv \mathcal{B}, \tag{127}$$

$$\mathcal{B}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{128}$$

$$\mathcal{B}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathcal{B}, \tag{129}$$

$$\mathcal{B}^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathcal{B}^2. \tag{130}$$

Hence, we have

$$\Lambda = \exp\left(+i\eta S_V^{30}/\hbar\right) = \exp\left(\eta\mathcal{B}\right) \tag{131}$$

$$= \sum_{n=0}^{\infty} \frac{(\eta\mathcal{B})^n}{n!} = \sum_{m=0}^{\infty} \frac{(\eta\mathcal{B})^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(\eta\mathcal{B})^{2m+1}}{(2m+1)!} \tag{132}$$

$$= I + \sum_{m=1}^{\infty} \frac{\eta^{2m}\mathcal{B}^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{\eta^{2m+1}\mathcal{B}^{2m+1}}{(2m+1)!} \tag{133}$$

$$= I + \sum_{m=1}^{\infty} \frac{\eta^{2m}}{(2m)!}\mathcal{B}^2 + \sum_{m=0}^{\infty} \frac{\eta^{2m+1}}{(2m+1)!}\mathcal{B} \tag{134}$$

$$= I + (\cosh\eta - 1)\mathcal{B}^2 + \sinh\eta\,\mathcal{B} \tag{135}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cosh\eta - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh\eta - 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \sinh\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\eta & 0 & 0 & 0 \end{pmatrix} \tag{136}$$

$$= \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}. \tag{137}$$

This is exactly the same as the result in Equation (2.37). □

Question 5

Problem 3.1

Derive eq. (3.29) from eqs. (3.21), (3.24), and (3.28).

$$a(\mathbf{k}) = \int d^3x e^{-ikx} \left[i\partial_0\varphi(x) + \omega\varphi(x) \right] \tag{3.21}$$

$$\Pi(x) = \dot{\varphi}(x) = \partial_0\varphi(x) \tag{3.24}$$

$$[\varphi(\mathbf{x},t), \varphi(\mathbf{x}',t)] = 0, \quad [\Pi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = 0, \quad [\varphi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{3.28}$$

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \quad [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0, \quad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{3.29}$$

Answer

Since $a(\mathbf{x})$ is independent of time, we can set $x^0 = y^0$, meaning all time variables are the same. Now, we compute

$$\begin{aligned}
&[a(\mathbf{k}), a(\mathbf{k}')] && (138)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y} [i\partial_0\varphi(x) + \omega\varphi(x), i\partial_0\varphi(y) + \omega\varphi(y)] && (139)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y} \Big(- [\partial_0\varphi(x), \partial_0\varphi(y)] + i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)] + \omega^2[\varphi(x), \varphi(y)] \Big) && (140)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y} \Big(i\omega[\partial_0\varphi(x), \varphi(y)] + i\omega[\varphi(x), \partial_0\varphi(y)] \Big) && (141)\\
=&\int d^3x d^3y e^{-ikx} e^{-ik'y} \Big(i\omega(-i\delta^{(3)}(\mathbf{x}-\mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y})) \Big) && (142)\\
=&0, && (143)
\end{aligned}$$

where we have used the commutation relations in Equation (3.28). Similarly, we can show that

$$\begin{aligned}
&[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] && (144)\\
=&\int d^3x d^3y e^{ikx} e^{ik'y} [i\partial_0\varphi(x) - \omega\varphi(x), i\partial_0\varphi(y) - \omega\varphi(y)] && (145)\\
=&\int d^3x d^3y e^{ikx} e^{ik'y} \Big(- [\partial_0\varphi(x), \partial_0\varphi(y)] - i\omega[\partial_0\varphi(x), \varphi(y)] - i\omega[\varphi(x), \partial_0\varphi(y)] + \omega^2[\varphi(x), \varphi(y)] \Big) && (146)\\
=&\int d^3x d^3y e^{ikx} e^{ik'y} \Big(- i\omega[\partial_0\varphi(x), \varphi(y)] - i\omega[\varphi(x), \partial_0\varphi(y)] \Big) && (147)\\
=&\int d^3x d^3y e^{ikx} e^{ik'y} \Big(- i\omega(-i\delta^{(3)}(\mathbf{x}-\mathbf{y})) + i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y})) \Big) && (148)\\
=&0. && (149)
\end{aligned}$$

Now, we compute

$$
\begin{aligned}
&[a(\mathbf{k}), a^\dagger(\mathbf{k}')] && (150)\\
=&\int d^3xd^3ye^{-ikx}e^{ik'y}[i\partial_0\varphi(x)+\omega\varphi(x), i\partial_0\varphi(y)-\omega\varphi(y)] && (151)\\
=&\int d^3xd^3ye^{-ikx}e^{ik'y}\Big(-[\partial_0\varphi(x),\partial_0\varphi(y)]-i\omega[\partial_0\varphi(x),\varphi(y)]+i\omega[\varphi(x),\partial_0\varphi(y)]-\omega^2[\varphi(x),\varphi(y)]\Big) && (152)\\
=&\int d^3xd^3ye^{-ikx}e^{ik'y}\Big(-i\omega[\partial_0\varphi(x),\varphi(y)]+i\omega[\varphi(x),\partial_0\varphi(y)]\Big) && (153)\\
=&\int d^3xd^3ye^{-ikx}e^{ik'y}\Big(-i\omega(-i\delta^{(3)}(\mathbf{x}-\mathbf{y}))+i\omega(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\Big) && (154)\\
=&\int d^3xd^3ye^{-ikx}e^{ik'y}(2\omega\delta^{(3)}(\mathbf{x}-\mathbf{y})) && (155)\\
=&\int d^3xe^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}(2\omega) && (156)\\
=&(2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}'). && (157)
\end{aligned}
$$

□

Question 6

Problem 3.5

Consider a complex (that is, non-hermitian) scalar field φ with Lagrangian density

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0.$$

(a) Show that φ obeys the Klein-Gordon equation.

(b) Treat φ and $\varphi^\dagger$ as independent fields, and find the conjugate momentum for each. Compute the Hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).

(c) Write the mode expansion of φ as

$$\varphi(x) = \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}].$$

Express $a(\mathbf{k})$ and $b(\mathbf{k})$ in terms of φ and $\varphi^\dagger$ and their time derivatives.

(d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates.

(e) Express the Hamiltonian in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$ and their Hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?

$$\widetilde{dk} = \frac{d^3k}{(2\pi)^3 2\omega}, \quad \omega = \sqrt{\mathbf{k}^2 + m^2} \tag{3.11}$$

Answer

(a)

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\, m^2 \varphi^\dagger, \tag{158}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = -\, \partial^\mu \varphi^\dagger, \tag{159}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = -\, \partial_\mu \partial^\mu \varphi^\dagger. \tag{160}$$

Hence, the Euler-Lagrange equation gives that

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0 \tag{161}$$

$$\Rightarrow - m^2 \varphi^\dagger + \partial_\mu \partial^\mu \varphi^\dagger = 0 \tag{162}$$

$$\Rightarrow (\partial_\mu \partial^\mu - m^2) \varphi^\dagger = 0. \tag{163}$$

Similarly, we can show that

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} = -m^2 \varphi, \tag{164}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)} = -\partial^\mu \varphi, \tag{165}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)} \right) = -\partial_\mu \partial^\mu \varphi. \tag{166}$$

Hence, the Euler-Lagrange equation gives that

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)} \right) = 0 \tag{167}$$

$$\Rightarrow - m^2 \varphi + \partial_\mu \partial^\mu \varphi = 0 \tag{168}$$

$$\Rightarrow (\partial_\mu \partial^\mu - m^2) \varphi = 0. \tag{169}$$

Therefore, both φ and $\varphi^\dagger$ obey the Klein-Gordon equation.

(b)

$$\Pi_\varphi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = -\partial^0 \varphi^\dagger = -\frac{-\partial}{\partial t} \varphi = +\dot{\varphi}^\dagger, \tag{170}$$

$$\Pi_{\varphi^\dagger} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^\dagger)} = -\partial^0 \varphi = -\frac{-\partial}{\partial t} \varphi = +\dot{\varphi}. \tag{171}$$

The Hamiltonian density is given by

$$\mathcal{H} = \Pi_\varphi \dot{\varphi} + \Pi_{\varphi^\dagger} \dot{\varphi}^\dagger - \mathcal{L} \tag{172}$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger + \partial^\mu \varphi^\dagger \partial_\mu \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{173}$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi} \dot{\varphi}^\dagger - \dot{\varphi}^\dagger \dot{\varphi} + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{174}$$

$$= \dot{\varphi} \dot{\varphi}^\dagger + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{175}$$

$$= \Pi_{\varphi^\dagger} \Pi_\varphi + \nabla \varphi^\dagger \cdot \nabla \varphi + m^2 \varphi^\dagger \varphi - \Omega_0. \tag{176}$$

(c)

$$\varphi(x) = \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] \tag{177}$$

$$\Rightarrow \dot{\varphi}(x) = \int \widetilde{dk}[i\omega a(\mathbf{k})e^{ikx} - i\omega b^\dagger(\mathbf{k})e^{-ikx}] \tag{178}$$

$$\varphi^\dagger(x) = \int \widetilde{dk}[a^\dagger(\mathbf{k})e^{-ikx} + b(\mathbf{k})e^{ikx}] \tag{179}$$

$$\Rightarrow \dot{\varphi}^\dagger(x) = \int \widetilde{dk}[-i\omega a^\dagger(\mathbf{k})e^{-ikx} + i\omega b(\mathbf{k})e^{ikx}] \tag{180}$$

Hence, we have

$$\int d^3x e^{-ikx}\varphi(x) \tag{181}$$

$$= \int d^3x e^{-ikx} \int \widetilde{dk'}[a(\mathbf{k'})e^{ik'x} + b^\dagger(\mathbf{k'})e^{-ik'x}] \tag{182}$$

$$= \int \widetilde{dk'} \int d^3x[a(\mathbf{k'})e^{i(k'-k)x} + b^\dagger(\mathbf{k'})e^{-i(k'+k)x}] \tag{183}$$

$$= \int \widetilde{dk'} \left[a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})e^{-i(\omega'-\omega)t} + b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}+\mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{184}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})e^{-i(\omega'-\omega)t} + b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}+\mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{185}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})e^{-i(\omega'-\omega)t} + b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}+\mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{186}$$

$$= \frac{1}{2\omega}a(\mathbf{k})e^{-i(\omega-\omega)t} + \frac{1}{2\omega}b^\dagger(-\mathbf{k})e^{i(\omega+\omega)t} \tag{187}$$

$$= \frac{1}{2\omega}a(\mathbf{k}) + \frac{1}{2\omega}b^\dagger(-\mathbf{k})e^{i2\omega t}. \tag{188}$$

Next, we compute

$$\int d^3x e^{-ikx}\dot{\varphi}(x) \tag{189}$$

$$= \int d^3x e^{-ikx} \int \widetilde{dk'}[-i\omega' a(\mathbf{k'})e^{ik'x} + i\omega' b^\dagger(\mathbf{k'})e^{-ik'x}] \tag{190}$$

$$= \int \widetilde{dk'} \int d^3x[-i\omega' a(\mathbf{k'})e^{i(k'-k)x} + i\omega' b^\dagger(\mathbf{k'})e^{-i(k'+k)x}] \tag{191}$$

$$= \int \widetilde{dk'} \left[-i\omega' a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})e^{-i(\omega'-\omega)t} + i\omega' b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}+\mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{192}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[-i\omega' a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})e^{-i(\omega'-\omega)t} + i\omega' b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}+\mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{193}$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \left[-i\omega' a(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})e^{-i(\omega'-\omega)t} + i\omega' b^\dagger(\mathbf{k'})(2\pi)^3\delta^{(3)}(\mathbf{k'}+\mathbf{k})e^{i(\omega'+\omega)t}\right] \tag{194}$$

$$= \frac{-i\omega}{2\omega}a(\mathbf{k})e^{-i(\omega-\omega)t} + \frac{i\omega}{2\omega}b^\dagger(-\mathbf{k})e^{i(\omega+\omega)t} \tag{195}$$

$$= \frac{-i}{2}a(\mathbf{k}) + \frac{i}{2}b^\dagger(-\mathbf{k})e^{i2\omega t}. \tag{196}$$

Therefore, we have

$$\int d^3xe^{-ikx}[i\partial_0\varphi(x)+\omega\varphi(x)] = \int d^3xe^{-ikx}[i\dot{\varphi}(x)+\omega\varphi(x)] \tag{197}$$

$$= \left[i\frac{-i}{2}a(\mathbf{k})+i\frac{i}{2}b^\dagger(-\mathbf{k})e^{i2\omega t}\right]+\left[\frac{\omega}{2\omega}a(\mathbf{k})+\frac{\omega}{2\omega}b^\dagger(-\mathbf{k})e^{i2\omega t}\right] \tag{198}$$

$$= a(\mathbf{k}). \tag{199}$$

Similarly, we can show that

$$\varphi^\dagger(x) = \int \widetilde{dk}[a^\dagger(\mathbf{k})e^{-ikx}+b(\mathbf{k})e^{ikx}] \tag{200}$$

$$\Rightarrow b(\mathbf{x}) = \int d^3xe^{-ikx}[\omega\varphi^\dagger(x)+i\partial_0\varphi^\dagger(x)], \tag{201}$$

by exchanging $a(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ with $a^\dagger(\mathbf{k})$ and $b(\mathbf{k})$, respectively.

(d)

Again, since $a(\mathbf{x})$ and $b(\mathbf{x})$ are independent of time, we can set $x^0 = y^0$, meaning all time variables are the same. Also, we rewrite the $a(\mathbf{k})$ and $b(\mathbf{k})$ for convention.

$$a(\mathbf{k}) = \int d^3xe^{-ikx}\left[i\partial_0\varphi(x)+\omega\varphi(x)\right] = \int d^3xe^{-ikx}\left[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x)\right], \tag{202}$$

$$b(\mathbf{k}) = \int d^3xe^{-ikx}[\omega\varphi^\dagger(x)+i\partial_0\varphi^\dagger(x)] = \int d^3xe^{-ikx}\left[i\Pi_\varphi(x)+\omega\varphi^\dagger(x)\right]. \tag{203}$$

Now, we compute

$$[a(\mathbf{k}),a(\mathbf{k}')] \tag{204}$$

$$= \int d^3xd^3ye^{-ikx}e^{-ik'y}[i\Pi_{\varphi^\dagger}(x)+\omega\varphi(x), i\Pi_{\varphi^\dagger}(y)+\omega\varphi(y)] \tag{205}$$

$$= \int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(-[\Pi_{\varphi^\dagger}(x),\Pi_{\varphi^\dagger}(y)]+i\omega[\Pi_{\varphi^\dagger}(x),\varphi(y)]+i\omega[\varphi(x),\Pi_{\varphi^\dagger}(y)]+\omega^2[\varphi(x),\varphi(y)]\Big) \tag{206}$$

$$= \int d^3xd^3ye^{-ikx}e^{-ik'y}\Big(0\Big) \tag{207}$$

$$=0, \tag{208}$$

where we have used the canonical commutation relations and the fact that φ and $\varphi^\dagger$ are independent fields. Similarly, we can show that

$$[b(\mathbf{k}),b(\mathbf{k}')] = 0, \tag{209}$$

by exchanging $a(\mathbf{k})$, φ and $\Pi_{\varphi^\dagger}$ with $b(\mathbf{k})$, $\varphi^\dagger$ and Π_φ, respectively. Now, we compute

$$[a(\mathbf{k}), b(\mathbf{k}')] \tag{210}$$
$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} [i\Pi_{\varphi^\dagger}(x) + \omega\varphi(x), i\Pi_\varphi(y) + \omega\varphi^\dagger(y)] \tag{211}$$
$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} \Big(- [\Pi_{\varphi^\dagger}(x), \Pi_\varphi(y)] + i\omega[\Pi_{\varphi^\dagger}(x), \varphi^\dagger(y)] + i\omega[\varphi(x), \Pi_\varphi(y)] - \omega^2[\varphi(x), \varphi^\dagger(y)]\Big) \tag{212}$$
$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} \Big(i\omega[\varphi(x), \Pi_\varphi(y)] - i\omega[\varphi^\dagger(y), \Pi_{\varphi^\dagger}(x),]\Big) \tag{213}$$
$$= \int d^3x d^3y e^{-ikx} e^{-ik'y} \Big(i\omega(i\delta^{(3)}(\mathbf{x} - \mathbf{y})) - i\omega(i\delta^{(3)}(\mathbf{x} - \mathbf{y}))\Big) \tag{214}$$
$$=0. \tag{215}$$

Next, we compute

$$[a(\mathbf{k}), b^\dagger(\mathbf{k}')] \tag{216}$$
$$= \int d^3x d^3y e^{-ikx} e^{ik'y} [i\Pi_{\varphi^\dagger}(x) + \omega\varphi(x), -i\Pi_{\varphi^\dagger}(y) + \omega\varphi(y)] \tag{217}$$
$$= \int d^3x d^3y e^{-ikx} e^{ik'y} \Big([\Pi_{\varphi^\dagger}(x), \Pi_{\varphi^\dagger}(y)] + i\omega[\Pi_{\varphi^\dagger}(x), \varphi(y)] - i\omega[\varphi(x), \Pi_{\varphi^\dagger}(y)] + \omega^2[\varphi(x), \varphi(y)]\Big) \tag{218}$$
$$= 0, \tag{219}$$

by the same reason as $[a(\mathbf{k}), a(\mathbf{k}')]$. Also, the result leads to

$$[a^\dagger(\mathbf{k}), b(\mathbf{k}')] = 0. \tag{220}$$

Finally, we compute

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] \tag{221}$$
$$= \int d^3x d^3y e^{-ikx} e^{ik'y} [i\Pi_{\varphi^\dagger}(x) + \omega\varphi(x), -i\Pi_\varphi(y) + \omega\varphi^\dagger(y)] \tag{222}$$
$$= \int d^3x d^3y e^{-ikx} e^{ik'y} \Big([\Pi_{\varphi^\dagger}(x), \Pi_\varphi(y)] + i\omega[\Pi_{\varphi^\dagger}(x), \varphi^\dagger(y)] - i\omega[\varphi(x), \Pi_\varphi(y)] + \omega^2[\varphi(x), \varphi^\dagger(y)]\Big) \tag{223}$$
$$= - i \int d^3x d^3y e^{-ikx} e^{ik'y} \Big(\omega[\varphi(x), \Pi_\varphi(y)] + \omega[\varphi^\dagger(y), \Pi_{\varphi^\dagger}(x)]\Big) \tag{224}$$
$$= - i \int d^3x d^3y e^{-ikx} e^{ik'y} \Big(\omega(i\delta^{(3)}(\mathbf{x} - \mathbf{y})) + \omega(i\delta^{(3)}(\mathbf{x} - \mathbf{y}))\Big) \tag{225}$$
$$= \int d^3x d^3y e^{-ikx} e^{ik'y} (2\omega\delta^{(3)}(\mathbf{x} - \mathbf{y})) \tag{226}$$
$$= \int d^3x e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} (2\omega) \tag{227}$$
$$=(2\pi)^3 2\omega\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{228}$$

Similarly, we can show that

$$[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{229}$$

by exchanging $a(\mathbf{k})$,$a^\dagger(\mathbf{k})$, φ and $\Pi_{\varphi^\dagger}$ with $b(\mathbf{k})$,$b^\dagger(\mathbf{k})$, $\varphi^\dagger$ and Π_φ, respectively.

(e)

First, we know that

$$H = \int d^3x \mathcal{H} = \int d^3x \left(\Pi_{\varphi^\dagger}\Pi_\varphi + \nabla\varphi^\dagger \cdot \nabla\varphi + m^2\varphi^\dagger\varphi - \Omega_0\right). \tag{230}$$

Before we proceed, we remind each term in the Hamiltonian density.

$$\varphi = \int \widetilde{dk}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] = \int \widetilde{dk}[a_\mathbf{k}e^{ikx} + b^\dagger_\mathbf{k}e^{-ikx}], \tag{231}$$

$$\varphi^\dagger = \int \widetilde{dk}[a^\dagger(\mathbf{k})e^{-ikx} + b(\mathbf{k})e^{ikx}] = \int \widetilde{dk}[a^\dagger_\mathbf{k}e^{-ikx} + b_\mathbf{k}e^{ikx}], \tag{232}$$

$$\Pi_\varphi = \dot{\varphi}^\dagger = \int \widetilde{dk}[i\omega a^\dagger_\mathbf{k}e^{-ikx} - i\omega b_\mathbf{k}e^{ikx}], \tag{233}$$

$$\Pi_{\varphi^\dagger} = \dot{\varphi} = \int \widetilde{dk}[-i\omega a_\mathbf{k}e^{ikx} + i\omega b^\dagger_\mathbf{k}e^{-ikx}], \tag{234}$$

$$\nabla\varphi = \int \widetilde{dk}[i\mathbf{k}a_\mathbf{k}e^{ikx} - i\mathbf{k}b^\dagger_\mathbf{k}e^{-ikx}], \tag{235}$$

$$\nabla\varphi^\dagger = \int \widetilde{dk}[-i\mathbf{k}a^\dagger_\mathbf{k}e^{-ikx} + i\mathbf{k}b_\mathbf{k}e^{ikx}]. \tag{236}$$

$$\tag{237}$$

Now, we compute each term in the Hamiltonian.

$$\int d^3x \Pi_{\varphi^\dagger}\Pi_\varphi \tag{238}$$

$$= \int d^3x \int \widetilde{dk}[-i\omega a_{\mathbf{k}} e^{ikx} + i\omega b^\dagger_{\mathbf{k}} e^{-ikx}] \int \widetilde{dk'}[i\omega' a^\dagger_{\mathbf{k}'} e^{-ik'x} - i\omega' b_{\mathbf{k}'} e^{ik'x}] \tag{239}$$

$$= \int \widetilde{dk}\widetilde{dk'} \int d^3x \left(\omega\omega' a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} e^{i(k-k')x} + \omega\omega' b^\dagger_{\mathbf{k}} b_{\mathbf{k}'} e^{-i(k-k')x} - \omega\omega' a_{\mathbf{k}} b_{\mathbf{k}'} e^{i(k+k')x} - \omega\omega' b^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{k}'} e^{-i(k+k')x}\right) \tag{240}$$

$$= \int \widetilde{dk}\widetilde{dk'} \Big(\omega\omega' a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{-i(\omega-\omega')t} + \omega\omega' b^\dagger_{\mathbf{k}} b_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{i(\omega-\omega')t} \tag{241}$$

$$- \omega\omega' a_{\mathbf{k}} b_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{-i(\omega+\omega')t} - \omega\omega' b^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{i(\omega+\omega')t}\Big) \tag{242}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(\omega\omega' a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{-i(\omega-\omega')t} + \omega\omega' b^\dagger_{\mathbf{k}} b_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{i(\omega-\omega')t} \tag{243}$$

$$- \omega\omega' a_{\mathbf{k}} b_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{-i(\omega+\omega')t} - \omega\omega' b^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{i(\omega+\omega')t}\Big) \tag{244}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{1}{2\omega} \left(\omega^2 a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + \omega^2 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \omega^2 a_{\mathbf{k}} b_{-\mathbf{k}} e^{-i2\omega t} - \omega^2 b^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} e^{i2\omega t}\right) \tag{245}$$

$$= \int \widetilde{dk} \frac{1}{2\omega} \left(\omega^2 a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + \omega^2 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \omega^2 a_{\mathbf{k}} b_{-\mathbf{k}} e^{-i2\omega t} - \omega^2 b^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} e^{i2\omega t}\right) \tag{246}$$

$$\int d^3x \nabla\varphi^\dagger \cdot \nabla\varphi \tag{247}$$

$$= \int d^3x \int \widetilde{dk}[-i\mathbf{k} a^\dagger_{\mathbf{k}} e^{-ikx} + i\mathbf{k} b_{\mathbf{k}} e^{ikx}] \cdot \int \widetilde{dk'}[i\mathbf{k}' a_{\mathbf{k}'} e^{ik'x} - i\mathbf{k}' b^\dagger_{\mathbf{k}'} e^{-ik'x}] \tag{248}$$

$$= \int \widetilde{dk}\widetilde{dk'} \int d^3x \left(\mathbf{k}\cdot\mathbf{k}' a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} e^{i(k'-k)x} + \mathbf{k}\cdot\mathbf{k}' b_{\mathbf{k}} b^\dagger_{\mathbf{k}'} e^{-i(k'-k)x} - \mathbf{k}\cdot\mathbf{k}' a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}'} e^{-i(k+k')x} - \mathbf{k}\cdot\mathbf{k}' b_{\mathbf{k}} a_{\mathbf{k}'} e^{i(k+k')x}\right) \tag{249}$$

$$= \int \widetilde{dk}\widetilde{dk'} \Big(\mathbf{k}\cdot\mathbf{k}' a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{-i(\omega'-\omega)t} + \mathbf{k}\cdot\mathbf{k}' b_{\mathbf{k}} b^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{i(\omega'-\omega)t} \tag{250}$$

$$- \mathbf{k}\cdot\mathbf{k}' a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{i(\omega+\omega')t} - \mathbf{k}\cdot\mathbf{k}' b_{\mathbf{k}} a_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{-i(\omega+\omega')t}\Big) \tag{251}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} \Big(\mathbf{k}\cdot\mathbf{k}' a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{-i(\omega'-\omega)t} + \mathbf{k}\cdot\mathbf{k}' b_{\mathbf{k}} b^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{i(\omega'-\omega)t} \tag{252}$$

$$- \mathbf{k}\cdot\mathbf{k}' a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{i(\omega+\omega')t} - \mathbf{k}\cdot\mathbf{k}' b_{\mathbf{k}} a_{\mathbf{k}'} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') e^{-i(\omega+\omega')t}\Big) \tag{253}$$

$$= \int \widetilde{dk} \frac{1}{2\omega} \left(\mathbf{k}^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \mathbf{k}^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right), \quad \text{by } \mathbf{k}' = -\mathbf{k}. \tag{254}$$

$$\int d^3x m^2 \varphi^\dagger \varphi \tag{255}$$

$$= \int d^3x m^2 \int \widetilde{dk}[a^\dagger_{\mathbf{k}} e^{-ikx} + b_{\mathbf{k}} e^{ikx}] \int \widetilde{dk'}[a_{\mathbf{k'}} e^{ik'x} + b^\dagger_{\mathbf{k'}} e^{-ik'x}] \tag{256}$$

$$= \int \widetilde{dk}\widetilde{dk'} \int d^3x m^2 \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k'}} e^{i(k'-k)x} + b_{\mathbf{k}} b^\dagger_{\mathbf{k'}} e^{-i(k'-k)x} + a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k'}} e^{-i(k+k')x} + b_{\mathbf{k}} a_{\mathbf{k'}} e^{i(k+k')x} \right) \tag{257}$$

$$= \int \widetilde{dk}\widetilde{dk'} m^2 \Big(a^\dagger_{\mathbf{k}} a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{-i(\omega'-\omega)t} + b_{\mathbf{k}} b^\dagger_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{i(\omega'-\omega)t} \tag{258}$$

$$+ a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{i(\omega+\omega')t} + b_{\mathbf{k}} a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{-i(\omega+\omega')t} \Big) \tag{259}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} m^2 \Big(a^\dagger_{\mathbf{k}} a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{-i(\omega'-\omega)t} + b_{\mathbf{k}} b^\dagger_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k'}) e^{i(\omega'-\omega)t} \tag{260}$$

$$+ a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{i(\omega+\omega')t} + b_{\mathbf{k}} a_{\mathbf{k'}} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k'}) e^{-i(\omega+\omega')t} \Big) \tag{261}$$

$$= \int \widetilde{dk} m^2 \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t} \right) \tag{262}$$

Therefore, the Hamiltonian is given by

$$H = \int \widetilde{dk} \frac{1}{2\omega} \left(\omega^2 a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + \omega^2 b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \omega^2 b^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} e^{i2\omega t} - \omega^2 a_{\mathbf{k}} b_{-\mathbf{k}} e^{-i2\omega t} \right) \tag{263}$$

$$+ \int \widetilde{dk} \frac{1}{2\omega} \left(\mathbf{k}^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \mathbf{k}^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t} \right) \tag{264}$$

$$+ \int \widetilde{dk} \frac{1}{2\omega} \left(m^2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + m^2 b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + m^2 a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} e^{i2\omega t} + m^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t} \right) - \int d^3x \Omega_0 \tag{265}$$

Also, we can change varible $\mathbf{k} \to \mathbf{k'} = -\mathbf{k}, d\mathbf{k} = d\mathbf{k'}$, but $\omega \to \omega' = \omega$. Then, the third and fourth terms in the first integral become $b_{\mathbf{k}}^\dagger \to b_{-\mathbf{k}}^\dagger$ and so on. Hence, we have

$$\begin{aligned}
H = & \int \widetilde{dk} \frac{1}{2\omega}\left(\omega^2 a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + \omega^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \omega^2 b_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{i2\omega t} - \omega^2 a_{-\mathbf{k}} b_{\mathbf{k}} e^{-i2\omega t}\right) && (266)\\
& + \int \widetilde{dk} \frac{1}{2\omega}\left(\mathbf{k}^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \mathbf{k}^2 a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) && (267)\\
& + \int \widetilde{dk} \frac{1}{2\omega}\left(m^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + m^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + m^2 a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger e^{i2\omega t} + m^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) - \int d^3x \Omega_0 && (268)\\
= & \int \widetilde{dk} \frac{1}{2\omega}\left(\omega^2 a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + \omega^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \omega^2 a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger e^{i2\omega t} - \omega^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) && (269)\\
& + \int \widetilde{dk} \frac{1}{2\omega}\left(\mathbf{k}^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \mathbf{k}^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \mathbf{k}^2 a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger e^{i2\omega t} + \mathbf{k}^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) && (270)\\
& + \int \widetilde{dk} \frac{1}{2\omega}\left(m^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + m^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + m^2 a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger e^{i2\omega t} + m^2 b_{\mathbf{k}} a_{-\mathbf{k}} e^{-i2\omega t}\right) - \int d^3x \Omega_0 && (271)\\
= & \int \widetilde{dk} \frac{1}{2\omega}\omega^2\left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\right) - \int d^3x \Omega_0, \quad \text{by } \omega^2 = \mathbf{k}^2 + m^2 && (272)\\
= & \int \widetilde{dk} \frac{\omega}{2}\left(2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + 2b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + 2(2\pi)^3 2\omega \delta^{(3)}(0)\right) - \int d^3x \Omega_0, \quad \text{by commutation relation} && (273)\\
= & \int \widetilde{dk}\omega\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) + \int \frac{d^3k}{(2\pi)^3 2\omega}\omega(2\pi)^3 2\omega\delta^{(3)}(0) - \int d^3x \Omega_0 && (274)\\
= & \int \widetilde{dk}\omega\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) + 2\frac{1}{2(2\pi)^3}\int d^3k\omega(2\pi)^3\delta^{(3)}(0) - \Omega_0 V, \quad \text{by } \int d^3x = V && (275)\\
= & \int \widetilde{dk}\omega\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) + 2\mathcal{E}_0 V - \Omega_0 V, \quad \text{by } \mathcal{E}_0 = \frac{1}{2(2\pi)^3}\int d^3k\omega, V = (2\pi)^3\delta^{(3)}(0). && (276)
\end{aligned}$$

In order to make the vacuum energy zero, we set $\Omega_0 = 2\mathcal{E}_0$. Hence, the Hamiltonian is given by

$$H = \int \widetilde{dk}\omega\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right). \tag{277}$$

□

HW2 Due to October 7 11:59 PM

Question 1

Problem 5.1
Work out the LSZ reduction formula for the complex scalar field that was introduced in problem 3.5. Note that we must specify the type (a or b) of each incoming and outgoing particle.

Answer

We start with the mode expansion of the complex scalar field:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}] \tag{278}$$

$$\varphi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}[b(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}] \tag{279}$$

$$a(\mathbf{k}) = \int d^3x e^{-ikx}\left[i\partial_0\varphi(x) + \omega\varphi(x)\right], \tag{280}$$

$$b(\mathbf{k}) = \int d^3x e^{-ikx}[\omega\varphi^\dagger(x) + i\partial_0\varphi^\dagger(x)]. \tag{281}$$

First, we define the $|i\rangle$ and $|f\rangle$ states as

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\cdots b_1^\dagger(t)b_2^\dagger(t)\cdots|0\rangle, \tag{282}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\cdots b_{1'}^\dagger(t)b_{2'}^\dagger(t)\cdots|0\rangle. \tag{283}$$

And a_i and b_i are given by

$$a_i^\dagger = \int d^3k f_i(\mathbf{k})a^\dagger(\mathbf{k}) \tag{284}$$

$$b_i^\dagger = \int d^3k g_i(\mathbf{k})b^\dagger(\mathbf{k}), \tag{285}$$

where

$$f_i(\mathbf{k}), g_i(\mathbf{k}) \propto \exp(-(\mathbf{k}-\mathbf{k}_i)^2/4\sigma^2). \tag{286}$$

Now we can compute the difference between $a_1^\dagger(+\infty)$ and $a_1^\dagger(-\infty)$:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt \partial_0 a_1^\dagger(t) \tag{287}$$

$$= \int_{-\infty}^{+\infty} dt \int d^3k f_1(\mathbf{k}) \int d^3x e^{ikx}[\omega\varphi(x) - i\partial_0\varphi(x)] \tag{288}$$

$$= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x), \tag{289}$$

where I quote the equation in the textbook. Similarly, we can get

$$b_1^\dagger(+\infty) - b_1^\dagger(-\infty) = -i \int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x), \tag{290}$$

$$a_{1'}(+\infty) - a_{1'}(-\infty) = i \int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x), \tag{291}$$

$$b_{1'}(+\infty) - b_{1'}(-\infty) = i \int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x). \tag{292}$$

Now we can express the S-matrix element $\langle f|i\rangle$ as

$$\langle f|i\rangle = \langle 0|\mathcal{T} b_{1'}(+\infty)b_{2'}(+\infty)\cdots a_{1'}(+\infty)a_{2'}(+\infty)\cdots a_1^\dagger(-\infty)a_2^\dagger(-\infty)\cdots b_1^\dagger(-\infty)b_2^\dagger(-\infty)\cdots|0\rangle \tag{293}$$

$$\begin{aligned}
=&\langle 0|\mathcal{T}[b_{1'}(-\infty) + i\int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots \\
&\cdots[a_{1'}(-\infty) + i\int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[a_1^\dagger(+\infty) + i\int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[b_1^\dagger(+\infty) + i\int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots|0\rangle
\end{aligned} \tag{294}$$

$$\begin{aligned}
=&\langle 0|\mathcal{T}[i\int d^3k g_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots \\
&\cdots[i\int d^3k f_{1'}^*(\mathbf{k}) \int d^4x e^{-ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[i\int d^3k f_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi(x)]\cdots \\
&\cdots[i\int d^3k g_1(\mathbf{k}) \int d^4x e^{ikx}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]\cdots|0\rangle
\end{aligned} \tag{295}$$

$$=(i)^{n+n'+m+m'}\langle 0|\mathcal{T}[\prod_{j'}^{n'}\int d^4x e^{-ik_{j'}x}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)][\prod_{l'}^{n'}\int d^4x e^{-ik_{l'}x}(-\partial_\mu\partial^\mu + m^2)\varphi(x)] \tag{296}$$

$$[\prod_{l}^{m}\int d^4x e^{ik_l x}(-\partial_\mu\partial^\mu + m^2)\varphi(x)][\prod_{j}^{n}\int d^4x e^{ik_j x}(-\partial_\mu\partial^\mu + m^2)\varphi^\dagger(x)]|0\rangle, \tag{297}$$

where we have used the fact that $a_i|0\rangle = b_i|0\rangle = 0$ and $\langle 0|a_i^\dagger = \langle 0|b_i^\dagger = 0$. Here n and m are the number of incoming a and b particles, while n' and m' are the number of outgoing a and b particles, respectively. We also impose the $\sigma \to 0$ limit, so that $f_i(\mathbf{k})$ and $g_i(\mathbf{k})$ become delta functions. Finally, we can rewrite the S-matrix element as

$$\begin{aligned}
\langle f|i\rangle =&(i)^{n+n'+m+m'} \int d^4x_1 e^{-ik_1x_1}\cdots\int d^4x_n e^{-ik_nx_n}\int d^4x_{1'}e^{ik_{1'}x_{1'}}\cdots\int d^4x_{n'}e^{ik_{n'}x_{n'}}\\
&\int d^4y_1 e^{-ip_1y_1}\cdots\int d^4y_m e^{-ip_my_m}\int d^4y_{1'}e^{ip_{1'}y_{1'}}\cdots\int d^4y_{m'}e^{ip_{m'}y_{m'}}\\
&(-\partial_\mu\partial^\mu_{x_1}+m^2)\cdots(-\partial_\mu\partial^\mu_{x_n}+m^2)(-\partial_\mu\partial^\mu_{x_{1'}}+m^2)\cdots(-\partial_\mu\partial^\mu_{x_{n'}}+m^2)\\
&(-\partial_\mu\partial^\mu_{y_1}+m^2)\cdots(-\partial_\mu\partial^\mu_{y_m}+m^2)(-\partial_\mu\partial^\mu_{y_{1'}}+m^2)\cdots(-\partial_\mu\partial^\mu_{y_{m'}}+m^2)\\
&\langle 0|\mathcal{T}\varphi^\dagger(y_{1'})\cdots\varphi^\dagger(y_{m'})\varphi(x_{1'})\cdots\varphi(x_{n'})\varphi(x_1)\cdots\varphi(x_n)\varphi^\dagger(y_1)\cdots\varphi^\dagger(y_m)|0\rangle.
\end{aligned}\tag{298}$$

This is the LSZ reduction formula for the complex scalar field. □

Question 2

Problem 6.1

(a) Find an explicit formula for $\mathcal{D}q$ in eq. (6.9). Your formula should be of the form $\mathcal{D}q = C\prod_{j=1}^{N} dq_j$, where C is a constant that you should compute.

(b) For the case of a free particle, $V(Q) = 0$, evaluate the path integral of eq. (6.9) explicitly. Hint: integrate over q_1, then q_2, etc, and look for a pattern. Express you final answer in terms of q', t', q'', t'' and m. Restore $\hbar$ by dimensional analysis.

(c) Compute the $\langle q'', t''|q', t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare your result in part (b).

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\overline{q}_j)\delta t}, \tag{6.7}$$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q \exp\left[i\int_{t'}^{t''} dt L(\dot{q}(t), q(t))\right]. \tag{6.9}$$

Answer

(a) First, from eq. (6.7), we can see that

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\overline{q}_j)\delta t}, \quad \text{assuming } H(p,q) = \frac{1}{2m}p^2 + V(q) && (299)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i(\frac{1}{2m}p_j^2 + V(\overline{q}_j))\delta t} && (300)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j\delta t\dot{q}_j} e^{-i(\frac{1}{2m}p_j^2 + V(\overline{q}_j))\delta t}, \quad \text{where } \dot{q}_j = \frac{q_{j+1}-q_j}{\delta t} && (301)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t(\frac{1}{2m}p_j^2 - p_j\dot{q}_j + V(\overline{q}_j))} && (302)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t(\frac{1}{2m}(p_j - m\dot{q}_j)^2 - \frac{1}{2}m\dot{q}_j{}^2 + V(\overline{q}_j))} && (303)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t\frac{1}{2m}(p_j - m\dot{q}_j)^2} e^{i\delta t(\frac{1}{2}m\dot{q}_j{}^2 - V(\overline{q}_j))} && (304)\\
&= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\delta t\frac{1}{2m}(p_j - m\dot{q}_j)^2} e^{i\delta t L(\dot{q}_j,\overline{q}_j)}, \quad \text{where } L(\dot{q}, q) = \frac{1}{2}m\dot{q}^2 - V(q) && (305)\\
&= \int \prod_{k=1}^{N} dq_k \left[\prod_{j=0}^{N} \int \frac{dp_j}{2\pi} e^{-i\delta t\frac{1}{2m}(p_j - m\dot{q}_j)^2}\right] e^{i\sum_{j=0}^{N}\delta t L(\dot{q}_j,\overline{q}_j)}, && (306)
\end{aligned}$$

where we have used the definition of $\dot{q}_j$ and $L(\dot{q}, q)$. Now we can compute the integral over p_j:

$$\begin{aligned}\int \frac{dp_j}{2\pi} e^{-i\delta t \frac{1}{2m}(p_j - m\dot{q}_j)^2} &= \int \frac{dp_j}{2\pi} e^{-i\frac{\delta t}{2m}p_j^2} \quad \text{(by shifting } p_j \to p_j + m\dot{q}_j\text{)} &(307)\\ &= \frac{1}{2\pi}\sqrt{\frac{2m\pi}{i\delta t}} \quad \text{(by Gaussian integral)} &(308)\\ &= \sqrt{\frac{m}{2\pi i \delta t}}. &(309)\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \left[\prod_{j=0}^{N}\sqrt{\frac{m}{2\pi i\delta t}}\right] e^{i\sum_{j=0}^{N}\delta t L(\dot{q}_j, \overline{q}_j)} &(310)\\ &= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\sum_{j=0}^{N}\delta t L(\dot{q}_j, \overline{q}_j)} &(311)\\ &= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\int_{t'}^{t''} dt L(\dot{q}(t), q(t))} \quad \text{(by definition of Riemann integral).} &(312)\end{aligned}$$

Therefore, we can identify

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} dq_j. \tag{313}$$

This is the explicit formula for $\mathcal{D}q$ in eq. (6.9).

(b) Now if we consider the case of a free particle, i.e. $V(Q) = 0$, then we have

$$\begin{aligned}L(\dot{q}, q) &= \frac{1}{2}m\dot{q}^2, &(314)\\ \langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\int_{t'}^{t''} dt \frac{1}{2}m\dot{q}^2} &(315)\\ &= \int \prod_{k=1}^{N} dq_k \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} e^{i\sum_{j=0}^{N}\delta t \frac{1}{2}m\dot{q}_j{}^2}. &(316)\\ & &(317)\end{aligned}$$

The terms in the exponent are given by:

$$\sum_{j=0}^{N}\delta t \frac{1}{2}m\dot{q}_j{}^2 = \sum_{j=0}^{N}\delta t \frac{1}{2}m\left(\frac{q_{j+1} - q_j}{\delta t}\right)^2 = \sum_{j=0}^{N}\frac{m}{2\delta t}(q_{j+1}^2 - 2q_{j+1}q_j + q_j^2). \tag{318}$$

Thus, we focus on the integral and compute it step by step:

$$\int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} \frac{im}{2\delta t}(q_{j+1}^2 - 2q_{j+1}q_j + q_j^2)\Big) \int dq_1 \exp\Big[\frac{im}{2\delta t}\Big((q_2^2 - 2q_2q_1 + q_1^2) + (q_1^2 - 2q_1q_0 + q_0^2)\Big)\Big] \tag{319}$$

$$= \int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} i\lambda(q_{j+1} - q_j)^2\Big)$$

$$\int dq_1 \exp\Big[i\lambda\Big((q_2 - q_1)^2 + (q_1 - q_0)^2\Big)\Big], \quad \text{where } \lambda = \frac{m}{2\delta t} \tag{320}$$

Before performing the integral over q_1, we consider the following integral:

$$\int_{-\infty}^{+\infty} dx e^{i\alpha(x-\beta)^2} = \sqrt{\frac{i\pi}{\alpha}} \tag{321}$$

Then we also consider the more complicated integral:

$$\int_{-\infty}^{+\infty} dx e^{i\alpha(x-c_1)^2 + i\beta(x-c_2)^2} = \int_{-\infty}^{+\infty} dx e^{i(\alpha+\beta)x^2 - 2i(\alpha c_1 + \beta c_2)x + i(\alpha c_1^2 + \beta c_2^2)} \tag{322}$$

$$= e^{i\frac{\alpha\beta}{\alpha+\beta}(c_1-c_2)^2} \int_{-\infty}^{+\infty} dx e^{i(\alpha+\beta)(x - \frac{\alpha c_1 + \beta c_2}{\alpha+\beta})^2} \tag{323}$$

$$= e^{i\frac{\alpha\beta}{\alpha+\beta}(c_1-c_2)^2} \sqrt{\frac{i\pi}{\alpha+\beta}} = e^{i\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}(c_1-c_2)^2} \sqrt{\frac{i\pi}{\alpha+\beta}}, \tag{324}$$

where I quoted the result from `Mathematica`. Now we can perform the integral over q_1:

$$\int dq_1 \exp\Big[i\lambda\Big((q_2 - q_1)^2 + (q_1 - q_0)^2\Big)\Big] \tag{325}$$

$$= \int dq_1 \exp\Big[i\lambda(q_1 - q_2)^2 + i\lambda(q_1 - q_0)^2\Big] \tag{326}$$

$$= e^{i\frac{\lambda^2}{2\lambda}(q_2-q_0)^2} \sqrt{\frac{i\pi}{2\lambda}} \tag{327}$$

$$= e^{i\frac{\lambda}{2}(q_2-q_0)^2} \sqrt{\frac{i\pi}{2\lambda}}. \tag{328}$$

Thus, we have

$$\int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} i\lambda(q_{j+1}-q_j)^2\Big) \int dq_1 \exp\Big[i\lambda\Big((q_2-q_1)^2+(q_1-q_0)^2\Big)\Big] \tag{329}$$

$$=\sqrt{\frac{i\pi}{2\lambda}} \int dq_N \cdots \int dq_2 \exp\Big(\sum_{j=2}^{N} i\lambda(q_{j+1}-q_j)^2 + i\frac{\lambda}{2}(q_2-q_0)^2\Big) \tag{330}$$

$$=\sqrt{\frac{i\pi}{2\lambda}} \int dq_N \cdots \int dq_3 \exp\Big(\sum_{j=3}^{N} i\lambda(q_{j+1}-q_j)^2\Big) \int dq_2 \exp\Big(i\lambda(q_3-q_2)^2 + i\frac{\lambda}{2}(q_2-q_0)^2\Big) \tag{331}$$

$$=\sqrt{\frac{i\pi}{2\lambda}}\sqrt{\frac{i\pi}{\frac{3}{2}\lambda}} \int dq_N \cdots \int dq_3 \exp\Big(\sum_{j=3}^{N} i\lambda(q_{j+1}-q_j)^2 + i\frac{\lambda}{3}(q_3-q_0)^2\Big) \tag{332}$$

$$=\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^2}\frac{1}{\sqrt{3}} \int dq_N \cdots \int dq_4 \exp\Big(\sum_{j=4}^{N} i\lambda(q_{j+1}-q_j)^2\Big) \int dq_3 \exp\Big(i\lambda(q_4-q_3)^2 + i\frac{\lambda}{3}(q_3-q_0)^2\Big) \tag{333}$$

$$=\cdots \tag{334}$$

$$=\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^{N-1}}\frac{1}{\sqrt{N}} \int dq_N \exp\Big(i\lambda(q_{N+1}-q_N)^2 + i\frac{\lambda}{N}(q_N-q_0)^2\Big) \tag{335}$$

$$=\sqrt{\Big(\frac{i\pi}{\lambda}\Big)^{N}}\frac{1}{\sqrt{N+1}} e^{i\frac{\lambda}{N+1}(q_{N+1}-q_0)^2}. \tag{336}$$

Combine with the prefactor, we have

$$\langle q'', t''|q', t'\rangle = \Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}} \sqrt{\Big(\frac{i\pi}{\lambda}\Big)^{N}}\frac{1}{\sqrt{N+1}} e^{i\frac{\lambda}{N+1}(q_{N+1}-q_0)^2} \tag{337}$$

$$= \Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}} \sqrt{\Big(\frac{i\pi}{\frac{m}{2\delta t}}\Big)^{N}}\frac{1}{\sqrt{N+1}} e^{i\frac{\frac{m}{2\delta t}}{N+1}(q''-q')^2} \tag{338}$$

$$= \Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}} \Big(\frac{2i\pi\delta t}{m}\Big)^{\frac{N}{2}}\frac{1}{\sqrt{N+1}} e^{i\frac{m}{2(N+1)\delta t}(q''-q')^2} \tag{339}$$

$$= \sqrt{\frac{m}{2\pi i(N+1)\delta t}} e^{i\frac{m}{2(N+1)\delta t}(q''-q')^2} \tag{340}$$

$$= \sqrt{\frac{m}{2\pi i(t''-t')}} e^{\frac{im}{2(t''-t')}(q''-q')^2}, \quad \text{where } t''-t' = (N+1)\delta t. \tag{341}$$

Then restore $\hbar$ by dimensional analysis, we have

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}} e^{\frac{im}{2\hbar(t''-t')}(q''-q')^2}. \tag{342}$$

(c) We can also compute $\langle q'', t''|q', t'\rangle$ by inserting a complete set of momentum eigenstates:

$$\begin{aligned}\langle q'',t''|q',t'\rangle =&\langle q''|e^{-iH(t''-t')}|q'\rangle & (343)\\ =&\int dp\langle q''|e^{-iH(t''-t')}|p\rangle\langle p|q'\rangle & (344)\\ =&\int \frac{dp}{2\pi}e^{-i\frac{p^2}{2m}(t''-t')}e^{ip(q''-q')}, & (345)\end{aligned}$$

where we have used $H=\frac{p^2}{2m}$ and $\langle p|q'\rangle=\frac{1}{\sqrt{2\pi}}e^{-ipq'}$. Now we can perform the integral over p:

$$\begin{aligned}\int_{-\infty}^{+\infty}dpe^{-i\frac{p^2}{2m}(t''-t')}e^{ip(q''-q')} =&\int_{-\infty}^{+\infty}dpe^{-i\frac{t''-t'}{2m}\left(p^2-\frac{2m}{t''-t'}(q''-q')p\right)} & (346)\\ =&\int_{-\infty}^{+\infty}dpe^{-i\frac{t''-t'}{2m}\left(p-\frac{m}{t''-t'}(q''-q')\right)^2+i\frac{m}{2(t''-t')}(q''-q')^2} & (347)\\ =&e^{i\frac{m}{2(t''-t')}(q''-q')^2}\int_{-\infty}^{+\infty}dpe^{-i\frac{t''-t'}{2m}\left(p-\frac{m}{t''-t'}(q''-q')\right)^2} & (348)\\ =&e^{i\frac{m}{2(t''-t')}(q''-q')^2}\sqrt{\frac{2m\pi}{i(t''-t')}}. & (349)\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle q'',t''|q',t'\rangle =&\int\frac{dp}{2\pi}e^{-i\frac{p^2}{2m}(t''-t')}e^{ip(q''-q')} & (350)\\ =&\frac{1}{2\pi}e^{i\frac{m}{2(t''-t')}(q''-q')^2}\sqrt{\frac{2m\pi}{i(t''-t')}} & (351)\\ =&\sqrt{\frac{m}{2\pi i(t''-t')}}e^{\frac{im}{2(t''-t')}(q''-q')^2}. & (352)\end{aligned}$$

This is exactly the same as the result we obtained in part (b). □

Question 3

Problem 7.3

(a) Use the Heisenberg equations of motion, $\dot{A} = i[H, A]$, to find explicit expressions for $\dot{Q}$ and $\dot{P}$. Solve these to get the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of the Schrödinger-picture operators Q and P.

(b) Write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$, where $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. Then, using your result from part (a), write the Heisenberg-picture operator $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$.

(c) Using your result from part (b), and $a|0\rangle = \langle 0|a^\dagger = 0$,verify eqs. (7.16) and (7.17).

Answer

(a) First, we can compute $\dot{Q}$ and $\dot{P}$ using the Heisenberg equations of motion:

$$\dot{Q} = i[H, Q] = i\left[\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2, Q\right] = i\frac{1}{2m}[P^2, Q] = \frac{P}{m}, \tag{353}$$

$$\dot{P} = i[H, P] = i\left[\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2, P\right] = i\frac{1}{2}m\omega^2[Q^2, P] = -m\omega^2 Q. \tag{354}$$

These are the equations of motion for a harmonic oscillator. Now we can solve these equations to get $Q(t)$ and $P(t)$:

$$\ddot{Q}(t) = \frac{\dot{P}}{m} = -\omega^2 Q(t), \tag{355}$$

$$Q(t) = Q\cos\omega t + \frac{P}{m\omega}\sin\omega t, \tag{356}$$

$$P(t) = m\dot{Q}(t) = -m\omega Q\sin\omega t + P\cos\omega t. \tag{357}$$

Note that we have used the initial conditions $Q(0) = Q$ and $P(0) = P$ to determine the integration constants.

(b) Next, we can write the Schrödinger-picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$:

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \tag{358}$$

$$P = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger). \tag{359}$$

Then, using the result from part (a), we can write the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$:

$$Q(t) = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\cos\omega t + \frac{-i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger)}{m\omega}\sin\omega t \tag{360}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\cos\omega t - i\sqrt{\frac{\hbar}{2m\omega}}(a-a^\dagger)\sin\omega t \tag{361}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\Big[a(\cos\omega t - i\sin\omega t) + a^\dagger(\cos\omega t + i\sin\omega t)\Big] \tag{362}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\Big[ae^{-i\omega t} + a^\dagger e^{i\omega t}\Big], \tag{363}$$

$$P(t) = -m\omega\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\sin\omega t - i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger)\cos\omega t \tag{364}$$

$$= -\sqrt{\frac{m\omega\hbar}{2}}(a+a^\dagger)\sin\omega t - i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger)\cos\omega t \tag{365}$$

$$= -\sqrt{\frac{m\omega\hbar}{2}}\Big[a(\sin\omega t + i\cos\omega t) + a^\dagger(\sin\omega t - i\cos\omega t)\Big] \tag{366}$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\Big[a(\cos\omega t - i\sin\omega t) - a^\dagger(\cos\omega t + i\sin\omega t)\Big] \tag{367}$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\Big[ae^{-i\omega t} - a^\dagger e^{i\omega t}\Big]. \tag{368}$$

(c) Recall eqs. (7.14), (7.16) and (7.17):

$$\begin{aligned} G(t-t') &= \frac{i}{2\omega}\exp\Big(-i\omega|t-t'|\Big) \\ &= \frac{i}{2\omega}\Big(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{-i\omega(t'-t)}\Big), \end{aligned} \tag{7.14}$$

$$\begin{aligned} \langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0 \mid 0\rangle_f\Big|_{f=0} \\ &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty} dt' G(t_2-t')f(t')\right]\langle 0 \mid 0\rangle_f\Big|_{f=0} \\ &= \left[\frac{1}{i}G(t_2-t_1) + (\text{ term with } f'\text{s})\right]\langle 0 \mid 0\rangle_f\Big|_{f=0} \\ &= \frac{1}{i}G(t_2-t_1), \end{aligned} \tag{7.16}$$

$$\begin{aligned} \langle 0|\mathrm{T}Q(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = &\frac{1}{i^2}[G(t_1-t_2)G(t_3-t_4) \\ &+ G(t_1-t_3)G(t_2-t_4) \\ &+ G(t_1-t_4)G(t_2-t_3)]. \end{aligned} \tag{7.17}$$

Using the result from part (b), we can compute $\langle 0|\mathrm{T}Q\left(t_1\right)Q\left(t_2\right)|0\rangle$:

$$
\begin{aligned}
\langle 0|\mathrm{T}Q\left(t_1\right)Q\left(t_2\right)|0\rangle =&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[ae^{-i\omega t_1}+a^\dagger e^{i\omega t_1}\Big]\Big[ae^{-i\omega t_2}+a^\dagger e^{i\omega t_2}\Big]|0\rangle && (369)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[aae^{-i\omega(t_1+t_2)}+aa^\dagger e^{-i\omega t_1}e^{i\omega t_2}+a^\dagger ae^{i\omega t_1}e^{-i\omega t_2}+a^\dagger a^\dagger e^{i\omega(t_1+t_2)}\Big]|0\rangle && (370)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[aa^\dagger e^{-i\omega t_1}e^{i\omega t_2}\Big]|0\rangle && (371)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}\Big[(1+a^\dagger a)e^{-i\omega t_1}e^{i\omega t_2}\Big]|0\rangle && (372)\\
=&\frac{\hbar}{2m\omega}\langle 0|\mathrm{T}e^{-i\omega t_1}e^{i\omega t_2}|0\rangle && (373)\\
=&\frac{\hbar}{2m\omega}\Big[\theta(t_1-t_2)e^{-i\omega(t_1-t_2)}+\theta(t_2-t_1)e^{-i\omega(t_2-t_1)}\Big] && (374)\\
=&\frac{1}{2\omega}\Big[\theta(t_1-t_2)e^{-i\omega(t_1-t_2)}+\theta(t_2-t_1)e^{-i\omega(t_2-t_1)}\Big],\quad \text{by setting } \hbar=m=1, && (375)\\
=&\frac{1}{2\omega}e^{-i\omega|t_1-t_2|} && (376)\\
=&\frac{1}{i}G(t_2-t_1) && (377)
\end{aligned}
$$

where we have used $a|0\rangle=\langle 0|a^\dagger=0$ and the definition of $G(t)$ in eq. (7.14). This verifies eq. (7.16). Next, we can compute $\langle 0|\mathrm{T}Q\left(t_1\right)Q\left(t_2\right)Q\left(t_3\right)Q\left(t_4\right)|0\rangle$:

$$
\begin{aligned}
&\langle 0|\mathrm{T}Q\left(t_1\right)Q\left(t_2\right)Q\left(t_3\right)Q\left(t_4\right)|0\rangle\\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[ae^{-i\omega t_1}+a^\dagger e^{i\omega t_1}\Big]\Big[ae^{-i\omega t_2}+a^\dagger e^{i\omega t_2}\Big]\\
&\Big[ae^{-i\omega t_3}+a^\dagger e^{i\omega t_3}\Big]\Big[ae^{-i\omega t_4}+a^\dagger e^{i\omega t_4}\Big]|0\rangle && (378)\\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[aaaae^{-i\omega(t_1+t_2+t_3+t_4)}+aaaa^\dagger e^{-i\omega(t_1+t_2+t_3)}e^{i\omega t_4}+aaa^\dagger ae^{-i\omega(t_1+t_2+t_4)}e^{i\omega t_3}\\
&+aaa^\dagger a^\dagger e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}+aa^\dagger aae^{-i\omega(t_1+t_3+t_4)}e^{i\omega t_2}+aa^\dagger aa^\dagger e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}\\
&+aa^\dagger a^\dagger ae^{-i\omega(t_1+t_4)}e^{i\omega(t_2+t_3)}+aa^\dagger a^\dagger a^\dagger e^{-i\omega t_1}e^{i\omega(t_2+t_3+t_4)}+a^\dagger aaae^{-i\omega(t_2+t_3+t_4)}e^{i\omega t_1}\\
&+a^\dagger aaa^\dagger e^{-i\omega(t_2+t_3)}e^{i\omega(t_1+t_4)}+a^\dagger aa^\dagger ae^{-i\omega(t_2+t_4)}e^{i\omega(t_1+t_3)}+a^\dagger aa^\dagger a^\dagger e^{-i\omega t_2}e^{i\omega(t_1+t_3+t_4)}\\
&+a^\dagger a^\dagger aae^{-i\omega(t_3+t_4)}e^{i\omega(t_1+t_2)}+a^\dagger a^\dagger aa^\dagger e^{-i\omega t_3}e^{i\omega(t_1+t_2+t_4)}+a^\dagger a^\dagger a^\dagger ae^{-i\omega t_4}e^{i\omega(t_1+t_2+t_3)}\\
&+a^\dagger a^\dagger a^\dagger a^\dagger e^{i\omega(t_1+t_2+t_3+t_4)}\Big]|0\rangle,\quad \text{red terms vanish but blue terms can survive,} && (379)\\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[aaa^\dagger a^\dagger e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}+aa^\dagger aa^\dagger e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}\Big]|0\rangle && (380)
\end{aligned}
$$

$$
\begin{aligned}
\langle 0|aaa^\dagger a^\dagger|0\rangle =& \langle 0|a(1+a^\dagger a)a^\dagger|0\rangle=\langle 0|aa^\dagger|0\rangle+\langle 0|aa^\dagger aa^\dagger|0\rangle && (381)\\
=&\langle 0|(1+a^\dagger a)|0\rangle+\langle 0|(1+a^\dagger a)(1+a^\dagger a)|0\rangle=2 && (382)\\
\langle 0|aa^\dagger aa^\dagger|0\rangle =& \langle 0|(1+a^\dagger a)(1+a^\dagger a)|0\rangle=1, && (383)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\langle 0|\mathrm{T}Q\left(t_1\right)Q\left(t_2\right)Q\left(t_3\right)Q\left(t_4\right)|0\rangle \\
=&\frac{\hbar^2}{4m^2\omega^2}\langle 0|\mathrm{T}\Big[2e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}+e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}\Big]|0\rangle &(384)\\
=&\frac{\hbar^2}{4m^2\omega^2}\Big[\langle 0|\mathrm{T}e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1+t_3)}e^{i\omega(t_2+t_4)}|0\rangle\Big] &(385)\\
=&\frac{\hbar^2}{4m^2\omega^2}\Big[\langle 0|\mathrm{T}e^{-i\omega(t_1-t_3)}e^{-i\omega(t_2-t_4)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1-t_4)}e^{-i\omega(t_2-t_3)}|0\rangle+\langle 0|\mathrm{T}e^{-i\omega(t_1-t_2)}e^{-i\omega(t_3-t_4)}|0\rangle\Big] &(386)\\
=&\frac{1}{4\omega^2}\Big[\Big(\theta(t_1-t_3)e^{-i\omega(t_1-t_3)}+\theta(t_3-t_1)e^{-i\omega(t_3-t_1)}\Big)\Big(\theta(t_2-t_4)e^{-i\omega(t_2-t_4)}+\theta(t_4-t_2)e^{-i\omega(t_4-t_2)}\Big)\\
&+\Big(\theta(t_1-t_4)e^{-i\omega(t_1-t_4)}+\theta(t_4-t_1)e^{-i\omega(t_4-t_1)}\Big)\Big(\theta(t_2-t_3)e^{-i\omega(t_2-t_3)}+\theta(t_3-t_2)e^{-i\omega(t_3-t_2)}\Big)\\
&+\Big(\theta(t_1-t_2)e^{-i\omega(t_1-t_2)}+\theta(t_2-t_1)e^{-i\omega(t_2-t_1)}\Big)\Big(\theta(t_3-t_4)e^{-i\omega(t_3-t_4)}+\theta(t_4-t_3)e^{-i\omega(t_4-t_3)}\Big)\Big], &(387)\\
&\quad\text{by setting } \hbar=m=1,\\
=&\frac{1}{i^2}\Big[G(t_1-t_3)G(t_2-t_4)+G(t_1-t_4)G(t_2-t_3)+G(t_1-t_2)G(t_3-t_4)\Big] &(388)
\end{aligned}
$$

where we have used the definition of $G(t)$ in eq. (7.14). This verifies eq. (7.17). $\square$

Question 4

Problem 7.4
Consider a harmonic oscillator in its ground state at $t = -\infty$. It is then subjected to an external force $f(t)$. Compute the probability $|\langle 0|0\rangle_f|^2$ that the oscillator is still in its ground state at $t = +\infty$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\tilde{f}(E) = \int_{-\infty}^{+\infty} e^{iEt} f(t)$. Your answer should not involve any other unevaluated integrals.

Answer

Recall eqs. (7.10), (7.11)and (7.14):

$$\begin{aligned}\langle 0|0\rangle_f =& \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right], & (7.10)\\ =& \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} dtdt' f(t)G(t-t')f(t')\right] & (7.11)\\ =& \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{i}{2\omega}e^{-i\omega|t-t'|}f(t')\right], \quad \text{by eq. (7.14)}\end{aligned}$$

$$|\langle 0|0\rangle_f|^2 = \langle 0|0\rangle_f \langle 0|0\rangle_f^* \tag{389}$$

$$= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{i}{2\omega}e^{-i\omega|t-t'|}f(t')\right]\exp\left[-\frac{i}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{-i}{2\omega}e^{i\omega|t-t'|}f(t')\right] \tag{390}$$

$$= \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}e^{-i\omega|t-t'|}f(t') - \frac{1}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}e^{i\omega|t-t'|}f(t')\right] \tag{391}$$

$$= \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}\left(e^{-i\omega|t-t'|}+e^{i\omega|t-t'|}\right)f(t')\right] \tag{392}$$

$$= \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{\omega}\cos\omega|t-t'|f(t')\right] \tag{393}$$

$$= \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}(e^{-i\omega(t-t')}+e^{i\omega(t-t')})f(t')\right] \tag{394}$$

By the definition of Fourier transform, we have

$$\tilde{f}(E) = \int_{-\infty}^{+\infty} e^{iEt} f(t)dt. \tag{395}$$

Thus, we have

$$\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}(e^{-i\omega(t-t')} + e^{i\omega(t-t')})f(t') \tag{396}$$

$$= \int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}e^{-i\omega(t-t')}f(t') + \int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}e^{i\omega(t-t')}f(t') \tag{397}$$

$$=\frac{1}{2\omega}\int_{-\infty}^{+\infty} dtdt' f(t)e^{-i\omega t}e^{i\omega t'}f(t') + \frac{1}{2\omega}\int_{-\infty}^{+\infty} dtdt' f(t)e^{i\omega t}e^{-i\omega t'}f(t') \tag{398}$$

$$=\frac{1}{2\omega}\Big(\int_{-\infty}^{+\infty} dt f(t)e^{-i\omega t}\Big)\Big(\int_{-\infty}^{+\infty} dt' f(t')e^{i\omega t'}\Big) + \frac{1}{2\omega}\Big(\int_{-\infty}^{+\infty} dt f(t)e^{i\omega t}\Big)\Big(\int_{-\infty}^{+\infty} dt' f(t')e^{-i\omega t'}\Big) \tag{399}$$

$$=\frac{1}{2\omega}\tilde{f}(-\omega)\tilde{f}(\omega) + \frac{1}{2\omega}\tilde{f}(\omega)\tilde{f}(-\omega) \tag{400}$$

$$=\frac{1}{\omega}\tilde{f}(\omega)\tilde{f}(-\omega) \tag{401}$$

Therefore, we have

$$|\langle 0|0\rangle_f|^2 = \exp\Big[-\frac{1}{2}\int_{-\infty}^{+\infty} dtdt' f(t)\frac{1}{2\omega}(e^{-i\omega(t-t')} + e^{i\omega(t-t')})f(t')\Big] \tag{402}$$

$$= \exp\Big[-\frac{1}{2}\cdot\frac{1}{\omega}\tilde{f}(\omega)\tilde{f}(-\omega)\Big] \tag{403}$$

$$= \exp\Big[-\frac{1}{2\omega}|\tilde{f}(\omega)|^2\Big], \tag{404}$$

where we have used the fact that $\tilde{f}(-\omega)$ is the complex conjugate of $\tilde{f}(\omega)$. This is a manifestly real expression and does not involve any other unevaluated integrals. $\square$

HW3 Due to October 21 11:59 PM

Question 1

Problem 8.7

Repeat the analysis of this section for the complex scalar field that was introduced in problem 3.5, and further studied in problem 5.1. Write your source term in the form $J^\dagger\varphi + J\varphi^\dagger$, and find an explicit formula, analogous to eq. (8.10), for $Z_0\left(J^\dagger, J\right)$. Write down the appropriate generalization of eq. (8.14), and use it to compute $\langle 0|\mathrm{T}\varphi\left(x_1\right)\varphi\left(x_2\right)|0\rangle$, $\langle 0|\mathrm{T}\varphi^\dagger\left(x_1\right)\varphi\left(x_2\right)|0\rangle$, and $\langle 0|\mathrm{T}\varphi^\dagger\left(x_1\right)\varphi^\dagger\left(x_2\right)|0\rangle$. Then verify your results by using the method of problem 8.4. Finally, give the appropriate generalization of eq. (8.17).

$$\mathcal{L} = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi.$$

$$\begin{aligned} Z_0(J) &= \exp\left[\frac{i}{2}\int d^4x d^4y J\Delta(x-y)J(y)\right] &(8.10)\\ &= \exp\left[\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\epsilon}\right], \\ \langle 0|T\varphi(x_1)\cdots|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots Z_0(J)|_{J=0} &(8.14)\\ \langle 0|T\varphi(x_1)\cdots\varphi(x_{2n})|0\rangle &= \frac{1}{i^n}\sum_{\text{all parings}}\Delta(x_{i_1}-x_{i_2})\cdots\Delta(x_{i_{2n-1}}-x_{i_{2n}}). &(8.17)\end{aligned}$$

Answer

We start from the lagrangian of complex scalar field:

$$\mathcal{L} = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi. \tag{405}$$

By fourier transformation, we have

$$\varphi(x) = \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\tilde{\varphi}(k), \quad \varphi^\dagger(x) = \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\tilde{\varphi}^\dagger(k). \tag{406}$$

$$J(x) = \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\tilde{J}(k), \quad J^\dagger(x) = \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\tilde{J}^\dagger(k). \tag{407}$$

Now we have

$$
\begin{aligned}
S_0 &= \int d^4x(\mathcal{L} + J\varphi^\dagger + J^\dagger\varphi) \\
&= \int d^4x \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} e^{i(k-k')\cdot x}\left[-\tilde{\varphi}^\dagger(k')(k\cdot k' + m^2)\tilde{\varphi}(k) + \tilde{J}^\dagger(k')\tilde{\varphi}(k) + \tilde{J}(k)\tilde{\varphi}^\dagger(k')\right] &(408)\\
&= \int \frac{d^4k}{(2\pi)^4}\left[-\tilde{\varphi}^\dagger(k)(k^2+m^2)\tilde{\varphi}(k) + \tilde{J}^\dagger(k)\tilde{\varphi}(k) + \tilde{J}(k)\tilde{\varphi}^\dagger(k)\right], &(409)
\end{aligned}
$$

where we have used the relation $\int d^4xe^{i(k-k')\cdot x} = (2\pi)^4\delta^4(k-k')$. Now we consider to change the variable of integration as

$$
\chi(k) = \tilde{\varphi}(k) - \frac{\tilde{J}(k)}{k^2+m^2}, \quad \chi^\dagger(k) = \tilde{\varphi}^\dagger(k) - \frac{\tilde{J}^\dagger(k)}{k^2+m^2}. \tag{410}
$$

Then we have

$$
S_0 = \int \frac{d^4k}{(2\pi)^4}\left[-\chi^\dagger(k)(k^2+m^2)\chi(k) + \frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2}\right]. \tag{411}
$$

Also, the measure is invariant under this shift, i.e. $\mathcal{D}\tilde{\varphi}\mathcal{D}\tilde{\varphi}^\dagger = \mathcal{D}\chi\mathcal{D}\chi^\dagger$. Therefore, the generating functional is given by

$$
\begin{aligned}
Z_0(J^\dagger, J) &= \int \mathcal{D}\tilde{\varphi}\mathcal{D}\tilde{\varphi}^\dagger \exp\left[iS_0\right] &(412)\\
&= \int \mathcal{D}\chi\mathcal{D}\chi^\dagger \exp\left[i\int\frac{d^4k}{(2\pi)^4}\left(-\chi^\dagger(k)(k^2+m^2)\chi(k) + \frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2}\right)\right] &(413)\\
&= \exp\left[i\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2-i\epsilon}\right]\int \mathcal{D}\chi\mathcal{D}\chi^\dagger \exp\left[-i\int\frac{d^4k}{(2\pi)^4}\chi^\dagger(k)(k^2+m^2-i\epsilon)\chi(k)\right] &(414)\\
&= \exp\left[i\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2-i\epsilon}\right] Z_0(0,0) &(415)\\
&= \exp\left[i\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2-i\epsilon}\right], &(416)
\end{aligned}
$$

where in the last line we have used the normalization condition $Z_0(0,0) = 1$, and we introduce a small real number ϵ to make the integral convergent. Thus we obtain the final result:

$$
\begin{aligned}
Z_0(J^\dagger, J) &= \exp\left[i\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2-i\epsilon}\right] &(417)\\
&= \exp\left[i\int d^4xd^4yJ^\dagger(x)\Delta(x-y)J(y)\right], &(418)
\end{aligned}
$$

where

$$\Delta(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-y)}}{k^2+m^2-i\epsilon}. \tag{419}$$

$\Delta(x-y)$ is the Feynman propagator for complex scalar field. Now we can compute the correlation functions by functional derivatives:

$$\begin{aligned}
\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle &= \frac{1}{i^2}\frac{\delta}{\delta J^\dagger(x_1)}\frac{\delta}{\delta J^\dagger(x_2)}Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0}=0 && (420)\\
&=\frac{1}{i^2}\frac{\delta}{\delta J^\dagger(x_1)}\Big[i\int d^4y\Delta(x_2-y)J(y)\Big]\exp\Big[i\int d^4xd^4yJ^\dagger(x)\Delta(x-y)J(y)\Big]\Big|_{J=J^\dagger=0} && (421)\\
&=\frac{1}{i^2}\Big[i\int d^4y\Delta(x_2-y)J(y)\Big]\Big[i\int d^4z\Delta(x_1-z)J(z)\Big]Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} && (422)\\
&=0, && (423)
\end{aligned}$$

$$\begin{aligned}
\langle 0|T\varphi^\dagger(x_1)\varphi(x_2)|0\rangle &= \frac{1}{i^2}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J^\dagger(x_1)}Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} && (424)\\
&=\frac{1}{i^2}\frac{\delta}{\delta J(x_2)}\Big[i\int d^4y\Delta(x_1-y)J(y)\Big]Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} && (425)\\
&=\frac{1}{i^2}i\Delta(x_1-x_2)Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} && (426)\\
&+\frac{1}{i^2}\Big[i\int d^4y\Delta(x_1-y)J(y)\Big]\Big[i\int d^4z\Delta(x_2-z)J^\dagger(z)\Big]Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} && (427)\\
&=\frac{1}{i}\Delta(x_1-x_2), && (428)
\end{aligned}$$

$$\begin{aligned}
\langle 0|T\varphi^\dagger(x_1)\varphi^\dagger(x_2)|0\rangle &= \frac{1}{i^2}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0}=0 && (429)\\
&=\frac{1}{i^2}\frac{\delta}{\delta J(x_1)}\Big[i\int d^4y\Delta(x_2-y)J^\dagger(y)\Big]\exp\Big[i\int d^4xd^4yJ^\dagger(x)\Delta(x-y)J(y)\Big]\Big|_{J=J^\dagger=0} && (430)\\
&=\frac{1}{i^2}\Big[i\int d^4y\Delta(x_2-y)J^\dagger(y)\Big]\Big[i\int d^4z\Delta(x_1-z)J^\dagger(z)\Big]Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0} && (431)\\
&=0. && (432)
\end{aligned}$$

Write down the appropriate generalization of eq. (8.14), we have

$$\langle 0|T\varphi(x_1)\cdots\varphi(x_n)\varphi^\dagger(y_1)\cdots\varphi^\dagger(y_m)|0\rangle=\frac{1}{i^{n+m}}\frac{\delta}{\delta J^\dagger(x_1)}\cdots\frac{\delta}{\delta J^\dagger(x_n)}\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_m)}Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0}. \tag{433}$$

For problem 8.4, we use extension equations from eqs. (3.19), (3.29) and (5.3) to verify eq.(8.15)

$$\varphi(x) = \int \widetilde{dk}\Big[a(\mathbf{k})e^{ik\cdot x} + b^\dagger(\mathbf{k})e^{-ik\cdot x}\Big], \quad \widetilde{dk} = \frac{d^3k}{(2\pi)^3 2\omega} \tag{3.19,3.18}$$

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}') \tag{3.29}$$

$$[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^3(\mathbf{k}-\mathbf{k}') \tag{3.29}$$

$$a(\mathbf{k})|0\rangle = 0, \quad b(\mathbf{k})|0\rangle = 0. \tag{5.3}$$

$$\langle 0|T\varphi(x_1)\varphi^\dagger(x_2)|0\rangle = \frac{1}{i}\Delta(x_1 - x_2) \tag{8.15}$$

Now we compute

$$\begin{aligned}
\langle 0|T\varphi(x_1)\varphi^\dagger(x_2)|0\rangle &= \langle 0|T\Big[\int \widetilde{dk_1}(a(\mathbf{k_1})e^{ik_1\cdot x_1} + b^\dagger(\mathbf{k_1})e^{-ik_1\cdot x_1}) \\
&\qquad \times \int \widetilde{dk_2}(a^\dagger(\mathbf{k_2})e^{-ik_2\cdot x_2} + b(\mathbf{k_2})e^{ik_2\cdot x_2})\Big]|0\rangle && (434)\\
&= \theta(x_1^0 - x_2^0)\langle 0|\varphi(x_1)\varphi^\dagger(x_2)|0\rangle + \theta(x_2^0 - x_1^0)\langle 0|\varphi^\dagger(x_2)\varphi(x_1)|0\rangle && (435)\\
&= \theta(x_1^0 - x_2^0)\int \widetilde{dk_1}\widetilde{dk_2}e^{ik_1\cdot x_1}e^{-ik_2\cdot x_2}\langle 0|a(\mathbf{k_1})a^\dagger(\mathbf{k_2})|0\rangle && (436)\\
&+ \theta(x_2^0 - x_1^0)\int \widetilde{dk_1}\widetilde{dk_2}e^{ik_2\cdot x_2}e^{-ik_1\cdot x_1}\langle 0|b(\mathbf{k_2})b^\dagger(\mathbf{k_1})|0\rangle && (437)\\
&= \theta(x_1^0 - x_2^0)\int \widetilde{dk_1}e^{ik_1\cdot(x_1 - x_2)} && (438)\\
&+ \theta(x_2^0 - x_1^0)\int \widetilde{dk_1}e^{-ik_1\cdot(x_1 - x_2)} && (439)\\
&= \frac{1}{i}\Delta(x_1 - x_2), \quad \text{by eq(8.13)}. && (440)
\end{aligned}$$

Thus we have verified eq.(8.15). Also, since anhilation operators always act on the vacuum state to give zero, we have

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = 0, \tag{441}$$

$$\langle 0|T\varphi^\dagger(x_1)\varphi^\dagger(x_2)|0\rangle = 0. \tag{442}$$

Last, we give the appropriate generalization of eq. (8.17):

$$\begin{aligned}
&\langle 0|T\varphi(x_1)\cdots\varphi(x_n)\varphi^\dagger(y_1)\cdots\varphi^\dagger(y_m)|0\rangle \\
&= \delta_{nm}\frac{1}{i^{2n}}\sum_{\text{all pairings}}\Delta(x_{i_1} - y_{j_1})\cdots\Delta(x_{i_n} - y_{j_n}). && (443)
\end{aligned}$$

Question 2

Prob1em 9.1

Compute the symmetry factor for each diagram in fig. (9.13). (You can then check your answers by consulting the earlier figures.)

Answer

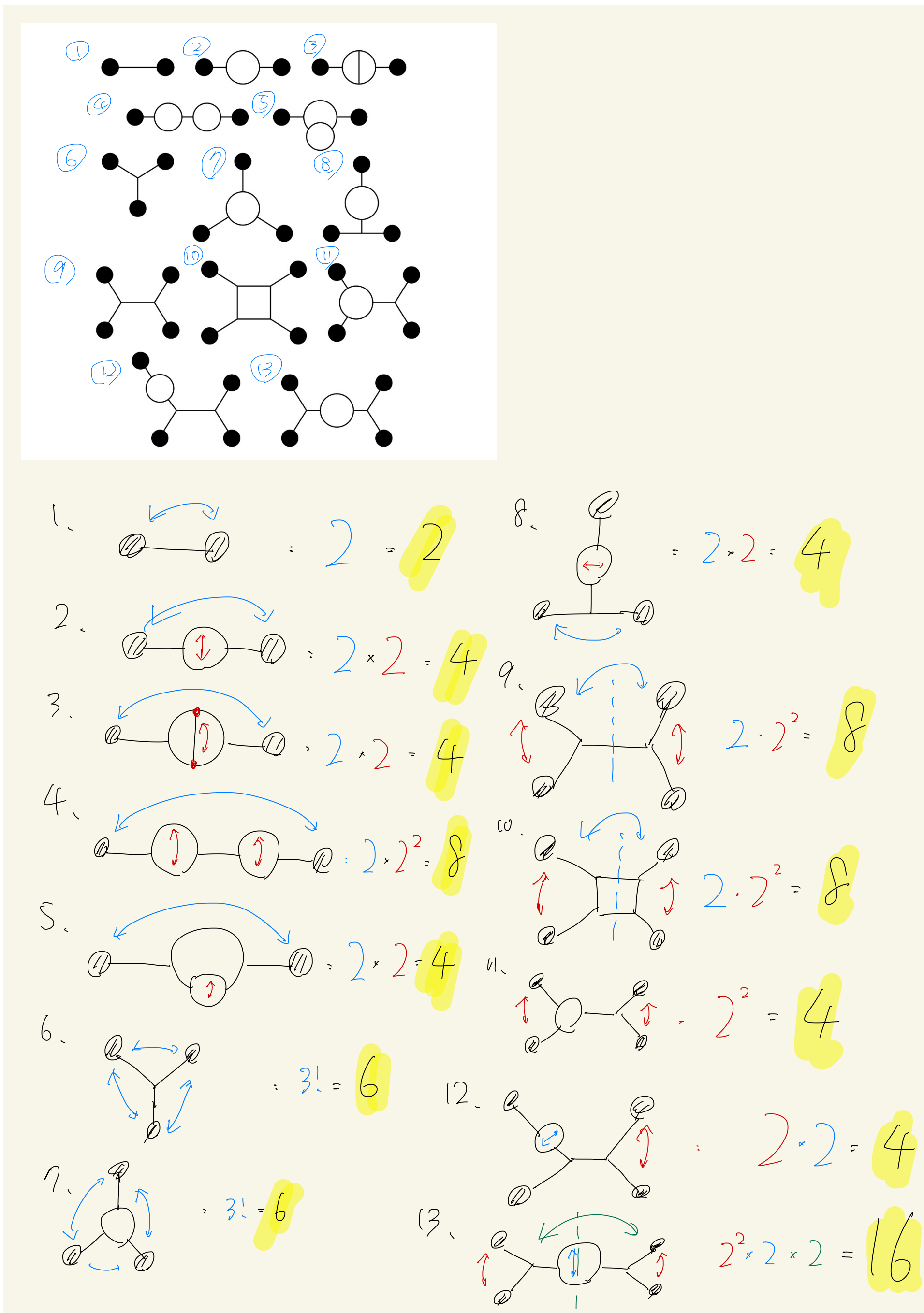


Figure 1: Feynman diagrams for problem 9.1

Question 3

Problem 9.5

The interaction picture. In this problem, we will derive a formula for $\langle 0|\mathrm{T}\varphi(x_n)\ldots\varphi(x_1)|0\rangle$ without using path integrals. Suppose we have a hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where $\mathcal{H}_0 = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2$, and $\mathcal{H}_1$ is a function of $\Pi(\mathbf{x}, 0)$ and $\varphi(\mathbf{x}, 0)$ and their spatial derivatives. (It should be chosen to preserve Lorentz invariance, but we will not be concerned with this issue.) We add a constant to H so that $H|0\rangle = 0$. Let $|\emptyset\rangle$ be the ground state of H_0, with a constant added to H_0 so that $H_0|\emptyset\rangle = 0$. (H_1 is then defined as $H - H_0$.) The Heisenberg-picture field is

$$\varphi(\mathbf{x}, t) \equiv e^{iHt}\varphi(\mathbf{x}, 0)e^{-iHt}. \tag{9.33}$$

We now define the interaction-picture field

$$\varphi_I(\mathbf{x}, t) \equiv e^{iH_0t}\varphi(\mathbf{x}, 0)e^{-iH_0t}. \tag{9.34}$$

(a) Show that $\varphi_I(x)$ obeys the Klein-Gordon equation, and hence is a free field.

(b) Show that $\varphi(x) = U^\dagger(t)\varphi_I(x)U(t)$, where $U(t) \equiv e^{iH_0t}e^{-iHt}$ is unitary.

(c) Show that $U(t)$ obeys the differential equation $i\frac{d}{dt}U(t) = H_I(t)U(t)$, where $H_I(t) = e^{iH_0t}H_1e^{-iH_0t}$ is the interaction hamiltonian in the interaction picture, and the boundary condition $U(0) = 1$.

(d) If $\mathcal{H}_1$ is specified by a particular function for the Schrodinger-picture field $\Pi(\mathbf{x}, 0)$ and $\varphi(\mathbf{x}, 0)$, show that $\mathcal{H}_I(t)$ is given by the same function of the interaction-picture fields $\Pi_I(\mathbf{x}, t)$ and $\varphi_I(\mathbf{x}, t)$.

(e) Show that, for $t > 0$,

$$U(t) = \mathcal{T}\exp\left[-i\int_0^t dt' H_I(t')\right] \tag{9.35}$$

obeys the differential equation and boundary condition of part (c). What is the comparable expression for $t < 0$? Hint: you may need fo define a new ordering symbol.

(f) Define $U(t_2, t_1) = U(t_2)U^\dagger(t_1)$. Show that, for $t_2 > t_1$,

$$U(t_2, t_1) = \mathcal{T}\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \tag{9.36}$$

(g) For any time ordering, show that $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$ and $U(t_1, t_2) = U^\dagger(t_2, t_1)$.

(h) Show that

$$\varphi(x_n)\cdots\varphi(x_1) = U^\dagger(t_n, 0)\varphi_I(x_n)U(t_n, t_{n-1})\cdots U^\dagger(t_2, t_1)\varphi_I(x_1)U(t_1, 0). \tag{9.37}$$

(i) Show that $U^\dagger(t_n,0) = U^\dagger(\infty,0)U(\infty,t_0)$ and also $U(t_1,0) = U^\dagger(t_1,-\infty)U(-\infty,0)$.

(j) Replace H_0 with $(1-i\epsilon)H_0$, and show that $\langle 0|U^\dagger(\infty,0) = \langle 0|\emptyset\rangle\langle\emptyset|$ and $U(-\infty,0)|0\rangle = |\emptyset\rangle\langle\emptyset|0\rangle$.

(k) Show that

$$\langle 0|\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle\emptyset|U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U(t_2,t_1)\varphi_I(x_1)U(t_1,-\infty)|\emptyset\rangle|\langle 0|\emptyset\rangle|^2 \tag{9.38}$$

(l) Show that

$$\langle 0|T\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle\emptyset|T\varphi(x_n)\cdots\varphi(x_1)\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle|\langle 0|\emptyset\rangle|^2. \tag{9.39}$$

(m) Show that

$$|\langle\emptyset|0\rangle|^2 = \frac{1}{\langle\emptyset|\mathcal{T}\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle}. \tag{9.40}$$

Thus we have

$$\langle 0|T\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \frac{\langle\emptyset|\mathcal{T}\varphi_I(x_n)\cdots\varphi_I(x_1)\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle}{\langle\emptyset|\mathcal{T}\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle}. \tag{9.41}$$

We can now Taylor expand the exponentials on the right-hand sides of eq. (9.41), and use free-field theory to compute the resulting correlation functions.

Note: We can skip parts in (f), (g).

Answer

(a)

By the definition of time derivatives in the interaction picture, we have

$$\partial_t\varphi_I(\mathbf{x},t) = ie^{iH_0t}[H_0,\varphi(\mathbf{x},0)]e^{-iH_0t} \tag{444}$$

$$\tag{445}$$

Then we can compute the $[H_0,\varphi(\mathbf{x},0)]$, with $H_0 = \int d^3y\mathcal{H}_0(y) = \int d^3y\left[\frac{1}{2}\Pi^2(y) + \frac{1}{2}(\nabla\varphi(y))^2 + \frac{1}{2}m^2\varphi^2(y)\right]$, and the canonical commutation relation $[\varphi(\mathbf{x},0),\Pi(\mathbf{y},0)] = i\delta^3(\mathbf{x}-\mathbf{y})$:

$$
\begin{aligned}
[H_0,\varphi(\mathbf{x},0)] &= \int d^3y\Big[\frac{1}{2}[\Pi^2(y),\varphi(\mathbf{x},0)] + \frac{1}{2}[(\nabla\varphi(y))^2,\varphi(\mathbf{x},0)] + \frac{1}{2}m^2[\varphi^2(y),\varphi(\mathbf{x},0)]\Big] && (446)\\
&= \int d^3y\Big[\frac{1}{2}(\Pi(y)[\Pi(y),\varphi(\mathbf{x},0)] + [\Pi(y),\varphi(\mathbf{x},0)]\Pi(y)) + 0 + 0\Big] && (447)\\
&= \int d^3y\Big[\frac{1}{2}(\Pi(y)(i\delta^3(\mathbf{y}-\mathbf{x})) + (i\delta^3(\mathbf{y}-\mathbf{x}))\Pi(y))\Big] && (448)\\
&= i\Pi(\mathbf{x},0). && (449)
\end{aligned}
$$

Thus we have

$$
\partial_t\varphi_I(\mathbf{x},t) = e^{iH_0t}\Pi(\mathbf{x},0)e^{-iH_0t} = \Pi_I(\mathbf{x},t). \tag{450}
$$

Taking another time derivative, we have

$$
\partial_t^2\varphi_I(\mathbf{x},t) = ie^{iH_0t}[H_0,\Pi(\mathbf{x},0)]e^{-iH_0t}. \tag{451}
$$

Now we compute $[H_0,\Pi(\mathbf{x},0)]$:

$$
\begin{aligned}
[H_0,\Pi(\mathbf{x},0)] &= \int d^3y\Big[\frac{1}{2}[\Pi^2(y),\Pi(\mathbf{x},0)] + \frac{1}{2}[(\nabla\varphi(y))^2,\Pi(\mathbf{x},0)] + \frac{1}{2}m^2[\varphi^2(y),\Pi(\mathbf{x},0)]\Big] && (452)\\
&= \int d^3y\Bigg[\frac{1}{2}(\nabla_y[\varphi(y),\Pi(\mathbf{x},0)]\cdot\nabla_y\varphi(y) + \nabla_y\varphi(y)\cdot\nabla_y[\varphi(y),\Pi(\mathbf{x},0)]) && (453)\\
&+ \frac{1}{2}m^2(\varphi(y)[\varphi(y),\Pi(\mathbf{x},0)] + [\varphi(y),\Pi(\mathbf{x},0)]\varphi(y))\Bigg] && (454)\\
&= \int d^3y\Bigg[\frac{1}{2}(\nabla_y(i\delta^3(\mathbf{y}-\mathbf{x}))\cdot\nabla_y\varphi(y) + \nabla_y\varphi(y)\cdot\nabla_y(i\delta^3(\mathbf{y}-\mathbf{x}))) && (455)\\
&+ \frac{1}{2}m^2(\varphi(y)(i\delta^3(\mathbf{y}-\mathbf{x})) + (i\delta^3(\mathbf{y}-\mathbf{x}))\varphi(y))\Bigg] && (456)\\
&= -i(\nabla_x^2 - m^2)\varphi(\mathbf{x},0). && (457)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\partial_t^2\varphi_I(\mathbf{x},t) &= -e^{iH_0t}(\nabla_x^2 - m^2)\varphi(\mathbf{x},0)e^{-iH_0t} && (458)\\
&= (\nabla_x^2 - m^2)\varphi_I(\mathbf{x},t). && (459)
\end{aligned}
$$

Therefore, we have shown that $\varphi_I(x)$ obeys the Klein-Gordon equation:

$$
(\partial_t^2 - \nabla_x^2 + m^2)\varphi_I(\mathbf{x},t) = (-\partial^2 + m^2)\varphi_I(x) = 0. \tag{460}
$$

(b)

By the definition of interaction picture field, we have

$$U^\dagger(t)\varphi_I(\mathbf{x},t)U(t) = e^{iHt}e^{-iH_0t}e^{iH_0t}\varphi(\mathbf{x},0)e^{-iH_0t}e^{iH_0t}e^{-iHt} \tag{461}$$

$$= e^{iHt}\varphi(\mathbf{x},0)e^{-iHt} \tag{462}$$

$$= \varphi(\mathbf{x},t). \tag{463}$$

(c)

Taking the time derivative of $U(t)$, we have

$$\frac{d}{dt}U(t) = \frac{d}{dt}\left(e^{iH_0t}e^{-iHt}\right) \tag{464}$$

$$= iH_0e^{iH_0t}e^{-iHt} - e^{iH_0t}iHe^{-iHt} \tag{465}$$

$$= -ie^{iH_0t}(H-H_0)e^{-iHt} \tag{466}$$

$$= -iH_1(t)U(t). \tag{467}$$

Also, at $t=0$, we have

$$U(0) = e^{iH_0\cdot 0}e^{-iH\cdot 0} = 1. \tag{468}$$

Therefore, we have shown that $U(t)$ obeys the differential equation $i\frac{d}{dt}U(t) = H_1(t)U(t)$ with the boundary condition $U(0)=1$.

(d)

By the definition of interaction picture Hamiltonian density, we have

$$\mathcal{H}_I(\mathbf{x},t) = e^{iH_0t}\mathcal{H}_1(\mathbf{x},0)e^{-iH_0t} \tag{469}$$

$$= \mathcal{H}_1\Big(e^{iH_0t}\Pi(\mathbf{x},0)e^{-iH_0t}, e^{iH_0t}\varphi(\mathbf{x},0)e^{-iH_0t}, \nabla(e^{iH_0t}\varphi(\mathbf{x},0)e^{-iH_0t})\Big) \tag{470}$$

$$= \mathcal{H}_1\Big(\Pi_I(\mathbf{x},t), \varphi_I(\mathbf{x},t), \nabla\varphi_I(\mathbf{x},t)\Big). \tag{471}$$

This is because we can insert the identity operator $e^{-iH_0t}e^{iH_0t}$ between any functions of $\Pi(\mathbf{x},0)$ and $\varphi(\mathbf{x},0)$ in $\mathcal{H}_1$.

(e)

We can verify that $U(t)$ defined in eq.(9.35) obeys the differential equation and boundary condition of part (c). Taking the time derivative of $U(t)$, we have

$$\frac{d}{dt}U(t) = \frac{d}{dt}\mathcal{T}\exp\left[-i\int_0^t dt' H_I(t')\right] \tag{472}$$

$$= -iH_I(t)\mathcal{T}\exp\left[-i\int_0^t dt' H_I(t')\right] \tag{473}$$

$$= -iH_I(t)U(t). \tag{474}$$

Also, at $t = 0$, we have

$$U(0) = \mathcal{T} \exp\left[-i \int_0^0 dt' H_I(t')\right] = 1. \tag{475}$$

Therefore, we have shown that $U(t)$ defined in eq.(9.35) obeys the differential equation $i\frac{d}{dt}U(t) = H_I(t)U(t)$ with the boundary condition $U(0) = 1$. For $t < 0$, we can define a new ordering symbol $\bar{\mathcal{T}}$ which orders operators with earlier times to the left. Then we have

$$U(t) = \bar{\mathcal{T}} \exp\left[-i \int_t^0 dt' H_I(t')\right]. \tag{476}$$

(f) **Skip**

(g) **Skip**

(h)

By repeatedly applying the result from part (b), we have

$$\begin{aligned} \varphi(x_n)\cdots\varphi(x_1) &= U^\dagger(t_n)\varphi_I(x_n)U(t_n)\cdots U^\dagger(t_1)\varphi_I(x_1)U(t_1) && (477)\\ &= U^\dagger(t_n,0)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U^\dagger(t_2,t_1)\varphi_I(x_1)U(t_1,0), && (478) \end{aligned}$$

where we have used the definition $U(t_2,t_1) = U(t_2)U^\dagger(t_1)$, and $U^\dagger(t_n) = 1 \cdot U^\dagger(t_n) = U(0)U^\dagger(t_n) = (U(t_0)U^\dagger(0))^\dagger = U^\dagger(t_n,0)$, and similarly for $U(t_1) = U(t_1,0)$.

(i)

By the definition of $U(t_2,t_1)$, we have

$$\begin{aligned} U^\dagger(t_n,0) &= U(0,t_n) = U(0,\infty)U(\infty,t_n) = U^\dagger(\infty,0)U(\infty,t_n), && (479)\\ U(t_1,0) &= U(t_1,-\infty)U(-\infty,0). && (480) \end{aligned}$$

(j)

By replacing H_0 with $(1-i\epsilon)H_0$, we have

$$\begin{aligned} U(-\infty,0)|0\rangle &= \lim_{t\to-\infty} e^{i(1-i\epsilon)H_0t}e^{-iHt}|0\rangle && (481)\\ &= \lim_{t\to-\infty} e^{i(1-i\epsilon)H_0t}|0\rangle, \quad \text{by } e^{iHt}|0\rangle = e^0|0\rangle = |0\rangle && (482)\\ &= \lim_{t\to-\infty} \sum_n e^{i(1-i\epsilon)H_0t}|n\rangle\langle n|0\rangle && (483)\\ &= \lim_{t\to-\infty} \sum_n e^{iE_nt}e^{\epsilon E_nt}|n\rangle\langle n|0\rangle && (484)\\ &= |\emptyset\rangle\langle\emptyset|0\rangle, \quad \text{since } E_n > 0 \text{ for excited states } |n\rangle. && (485) \end{aligned}$$

similarly, we have

$$\langle 0|U^\dagger(\infty,0) = \lim_{t\to\infty}\langle 0|e^{iHt}e^{-i(1-i\epsilon)H_0 t} \tag{486}$$
$$= \lim_{t\to\infty}\langle 0|e^{-i(1-i\epsilon)H_0 t} \tag{487}$$
$$= \lim_{t\to\infty}\sum_n \langle 0|n\rangle\langle n|e^{-i(1-i\epsilon)H_0 t} \tag{488}$$
$$= \lim_{t\to\infty}\sum_n \langle 0|n\rangle e^{-iE_n t}e^{-\epsilon E_n t}\langle n| \tag{489}$$
$$= \langle 0|\emptyset\rangle\langle\emptyset|, \quad \text{since } E_n > 0 \text{ for excited states } |n\rangle. \tag{490}$$

(k)

By substituting the results from parts (h) and (i) into the left-hand side, we have

$$\langle 0|\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle 0|U^\dagger(t_n,0)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U^\dagger(t_2,t_1)\varphi_I(x_1)U(t_1,0)|0\rangle \tag{491}$$
$$= \langle 0|U^\dagger(\infty,0)U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U^\dagger(t_2,t_1)\varphi_I(x_1)U(t_1,-\infty)U(-\infty,0)|0\rangle \tag{492}$$
$$= \langle 0|\emptyset\rangle\langle\emptyset|U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U^\dagger(t_2,t_1)\varphi_I(x_1)U(t_1,-\infty)|\emptyset\rangle\langle\emptyset|0\rangle \tag{493}$$
$$= \langle\emptyset|U(\infty,t_n)\varphi_I(x_n)U(t_n,t_{n-1})\cdots U^\dagger(t_2,t_1)\varphi_I(x_1)U(t_1,-\infty)|\emptyset\rangle|\langle 0|\emptyset\rangle|^2. \tag{494}$$

(l)

By substituting the result from part (f) into the right-hand side of part (k), we have

$$\langle 0|T\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle\emptyset|\mathcal{T}\varphi_I(x_n)\cdots\varphi_I(x_1)\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle|\langle 0|\emptyset\rangle|^2. \tag{495}$$

By expanding the time-ordered products in part (k), we can see that it is equivalent to the time-ordered product in part (l).

(m)

By setting $n = 0$ in part (l), we have

$$\langle 0|T1|0\rangle = \langle\emptyset|\mathcal{T}\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle|\langle 0|\emptyset\rangle|^2. \tag{496}$$

Since the left-hand side is equal to 1, we have

$$|\langle\emptyset|0\rangle|^2 = \frac{1}{\langle\emptyset|\mathcal{T}\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle}. \tag{497}$$

Therefore, we have derived the formula in eq.(9.41):

$$\langle 0|T\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \frac{\langle\emptyset|\mathcal{T}\varphi_I(x_n)\cdots\varphi_I(x_1)\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle}{\langle\emptyset|\mathcal{T}\exp\left[-i\int d^4x H_I(x)\right]|\emptyset\rangle}. \tag{498}$$

$\square$

Question 4

Problem 10.5

The scattering amplitudes should be unchanged if we make a *field redefinition.* Suppose, for example, we have

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2, \tag{10.15}$$

and we make the field redefinition

$$\varphi \to \varphi + \lambda\varphi^2, \tag{10.16}$$

Work out the lagrangian in terms of the redefined field, and the corresponding Feynman rules. Compute (at tree level) the $\varphi\varphi \to \varphi\varphi$ scattering amplitudes. You should get zero, because this is a free-field theory in disguise. (At the loop level, we also have to take into account the transformation of the functional measure $\mathcal{D}\varphi$; see section 85.)

Answer

By substituting the field redefinition in eq.(10.16) into the lagrangian in eq.(10.15), we have

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial^\mu(\varphi + \lambda\varphi^2)\partial_\mu(\varphi + \lambda\varphi^2) - \frac{1}{2}m^2(\varphi + \lambda\varphi^2)^2 && (499)\\
&= -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \lambda\partial^\mu\varphi\partial_\mu(\varphi^2) - \frac{\lambda^2}{2}\partial^\mu(\varphi^2)\partial_\mu(\varphi^2) - \frac{1}{2}m^2\varphi^2 - m^2\lambda\varphi^3 - \frac{m^2\lambda^2}{2}\varphi^4 && (500)\\
&= -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \lambda(2\varphi\partial^\mu\varphi\partial_\mu\varphi) - 2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - m^2\lambda\varphi^3 - \frac{m^2\lambda^2}{2}\varphi^4 && (501)\\
&= -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - 2\lambda\varphi\partial^\mu\varphi\partial_\mu\varphi - m^2\lambda\varphi^3 - 2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi - \frac{m^2\lambda^2}{2}\varphi^4 && (502)\\
&= \mathcal{L}_0 + \mathcal{L}_1, && (503)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2, && (504)\\
\mathcal{L}_1 &= -2\lambda\varphi\partial^\mu\varphi\partial_\mu\varphi - m^2\lambda\varphi^3 - 2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi - \frac{m^2\lambda^2}{2}\varphi^4 && (505)
\end{aligned}$$

We can treat the last four terms as interaction terms. The corresponding Feynman rules are:

- The propagator is the same as that of a free scalar field:

$$\frac{-i}{p^2 + m^2 - i\epsilon}. \tag{506}$$

- The three-point vertex from the $-m^2\lambda\varphi^3$ term contributes a factor of $-im^2\lambda \times 3! = -i6m^2\lambda$.

- The three-point vertex from the $-2\lambda\varphi\partial^\mu\varphi\partial_\mu\varphi$ term contributes a factor of $i2\lambda(p_1\cdot p_2+p_1\cdot p_3+p_2\cdot p_3)\times 2!$, where p_1, p_2, p_3 are the momenta of the three legs. However we can apply momenta conservation to rewrite it as $-i2\lambda(p_1^2+p_2^2+p_3^2)$ by $(p_1+p_2+p_3)^2=0$.

- The four-point vertex from the $-\frac{m^2\lambda^2}{2}\varphi^4$ term contributes a factor of $-im^2\lambda^2/2\times 4! = -i12m^2\lambda^2$.

- The four-point vertex from the $-2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi$ term contributes a factor of $i2\lambda^2(p_1\cdot p_2+p_1\cdot p_3+p_1\cdot p_4+p_2\cdot p_3+p_2\cdot p_4+p_3\cdot p_4)\times 2!2! = i8\lambda^2(p_1\cdot p_2+p_1\cdot p_3+p_1\cdot p_4+p_2\cdot p_3+p_2\cdot p_4+p_3\cdot p_4)$. Similarly, we can apply momenta conservation to rewrite it as $-i4\lambda^2(p_1^2+p_2^2+p_3^2+p_4^2)$ by $(p_1+p_2+p_3+p_4)^2=0$.

In summary, the Feynman rules are:

- Propagator: $\frac{-i}{p^2+m^2-i\epsilon}$.

- Three-point vertex: $-i6m^2\lambda - i2\lambda(p_1^2+p_2^2+p_3^2) = -i2\lambda(3m^2+p_1^2+p_2^2+p_3^2)$.

- Four-point vertex: $-i12m^2\lambda^2 - i4\lambda^2(p_1^2+p_2^2+p_3^2+p_4^2) = -i4\lambda^2(3m^2+p_1^2+p_2^2+p_3^2+p_4^2)$.

Now we can compute the tree-level $\varphi\varphi\to\varphi\varphi$ scattering amplitude. We assume the initial momenta are p_1 and p_2, and the final momenta are p_3 and p_4. The momentum of propagator will be labeled as q. Four momenta are all on-shell, so $p_1^2 = p_2^2 = p_3^2 = p_4^2 = -m^2$. We also ignore ϵ in the propagator since the propagator particle is off-shell. The Mandelstam variables are defined as:

$$s = -(p_1+p_2)^2 = -q^2, \tag{507}$$
$$t = -(p_1-p_3)^2, \tag{508}$$
$$u = -(p_1-p_4)^2. \tag{509}$$

The contributions to the scattering amplitude are:

- The s-channel diagram with two three-point vertices:

$$\mathcal{T}_s = \Big[-i2\lambda(3m^2+p_1^2+p_2^2+q^2)\Big]\frac{-i}{q^2+m^2}\Big[-i2\lambda(3m^2+p_3^2+p_4^2+q^2)\Big] \tag{510}$$
$$= \frac{4i\lambda^2(-s+m^2)^2}{-s+m^2}. \tag{511}$$

- The t-channel diagram with two three-point vertices:

$$\mathcal{T}_t = \Big[-i2\lambda(3m^2+p_1^2+p_3^2+q^2)\Big]\frac{-i}{q^2+m^2}\Big[-i2\lambda(3m^2+p_2^2+p_4^2+q^2)\Big] \tag{512}$$
$$= \frac{4i\lambda^2(-t+m^2)^2}{-t+m^2}. \tag{513}$$

- The u-channel diagram with two three-point vertices:

$$\mathcal{T}_u = \Big[-i2\lambda(3m^2 + p_1^2 + p_4^2 + q^2)\Big]\frac{-i}{q^2 + m^2}\Big[-i2\lambda(3m^2 + p_2^2 + p_3^2 + q^2)\Big] \tag{514}$$

$$= \frac{4i\lambda^2(-u + m^2)^2}{-u + m^2}. \tag{515}$$

- The four-point vertex diagram:

$$\mathcal{T}_4 = -i4\lambda^2(3m^2 + p_1^2 + p_2^2 + p_3^2 + p_4^2) = +4im^2\lambda^2 \tag{516}$$

Now we can sum up all contributions to get the total scattering amplitude:

$$\mathcal{T} = \mathcal{T}_s + \mathcal{T}_t + \mathcal{T}_u + \mathcal{T}_4 \tag{517}$$

$$= 4i\lambda^2\Big[\frac{(-s + m^2)^2}{-s + m^2} + \frac{(-t + m^2)^2}{-t + m^2} + \frac{(-u + m^2)^2}{-u + m^2} + m^2\Big] \tag{518}$$

$$= 4i\lambda^2\Big[(-s + m^2) + (-t + m^2) + (-u + m^2) + m^2\Big] \tag{519}$$

$$= 4i\lambda^2\Big[-(s + t + u) + 4m^2\Big] \tag{520}$$

$$= 0, \tag{521}$$

where we have used the relation $s + t + u = 4m^2$ for $2 \to 2$ scattering of identical particles. Therefore, we have shown that the tree-level $\varphi\varphi \to \varphi\varphi$ scattering amplitude is zero, as expected for a free-field theory. □

Question 5

Problem 11.2

Consider *Compton scattering*, in which a massless photon is scattered by an electron, initially at rest. (This is the FT frame.) In problem 59.1, we will compute $|\mathcal{T}|^2$ for this process (summed over the possible spin states of the scattered photon and electron, and averaged over the possible spin states of the initial photon and electron), with the result

$$|\mathcal{T}|^2 = 32\pi^2\alpha^2\Big[\frac{m^4 + m^2(3s+u) - su}{(m^2-s)^2} + \frac{m^4 + m^2(3u+s) - su}{(m^2-u)^2} + \frac{2m^2(s+u+2m^2)}{(m^2-s)(m^2-u)}\Big] + \mathcal{O}(\alpha^4) \tag{11.50}$$

where $\alpha = 1/137.036$ is the fine-structure constant.

(a) Express the Mandelstam variables s and u in terms of the initial and final photon energies ω and ω'

(b) Express the scattering angle $\theta_{\rm FT}$ between the initial and final photon three-momenta in terms of ω and ω'.

(c) Express the differential scattering cross section $d\sigma/d\Omega_{\rm FT}$ in terms of ω and ω'. Show that your result is equivalent to the *Klein-Nishina* formula

$$\frac{d\sigma}{d\Omega_{\rm FT}} = \frac{\alpha^2}{2m^2}\Big(\frac{\omega'}{\omega}\Big)^2\Big[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta_{\rm FT}\Big] \tag{11.51}$$

Answer

(a)

In the FT frame, the initial electron is at rest, so its four-momentum is $p = (m, \mathbf{0})$. The initial photon has four-momentum $k = (\omega, \mathbf{k})$, where $|\mathbf{k}| = \omega$. The final electron has four-momentum $p' = (E', \mathbf{p}')$, and the final photon has four-momentum $k' = (\omega', \mathbf{k}')$, where $|\mathbf{k}'| = \omega'$. The Mandelstam variables are defined as:

$$s = -(p+k)^2 = (m+\omega)^2 - |\mathbf{k}|^2 = m^2 + 2m\omega, \tag{522}$$

$$u = -(p-k')^2 = (m-\omega')^2 - |\mathbf{k}'|^2 = m^2 - 2m\omega'. \tag{523}$$

(b)

The scattering angle $\theta_{\rm FT}$ between the initial and final photon three-momenta can be expressed in terms of ω and ω' using the conservation of four-momentum:

$$p + k = p' + k'. \tag{524}$$

Taking the square of both sides, we have

$$p'^2 = (p+k-k')^2 = p^2 + k^2 + k'^2 + 2p\cdot(k-k') - 2k\cdot k'. \tag{525}$$

Since $p^2 = m^2$, $k^2 = 0$, and $k'^2 = 0$, we have

$$m^2 = m^2 + 2p \cdot (k - k') - 2k \cdot k'. \tag{526}$$

Rearranging, we have

$$2p \cdot (k - k') = 2k \cdot k'. \tag{527}$$

We know that

$$p \cdot k = -m\omega, \quad p \cdot k' = -m\omega', \quad k \cdot k' = \omega\omega'(\cos\theta_{\mathrm{FT}} - 1). \tag{528}$$

Substituting these into the previous equation, we have

$$2(-m\omega + m\omega') = 2\omega\omega'(\cos\theta_{\mathrm{FT}} - 1). \tag{529}$$

Rearranging, we have

$$\cos\theta_{\mathrm{FT}} = 1 + \frac{m(\omega' - \omega)}{\omega\omega'} = 1 + \frac{m}{\omega} - \frac{m}{\omega'}. \tag{530}$$

(c)

The differential scattering cross section in the CM frame is given by

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s|\mathbf{k}_{\mathrm{CM}}|^2}|\mathcal{T}|^2, \tag{11.34}$$

where t is the Mandelstam variable defined as

$$t = -(k - k')^2 = -2\omega\omega'(1 - \cos\theta_{\mathrm{FT}}).$$

, and $\mathcal{T}$ is given in eq.(11.50). By eq. (11.9), we have

$$m|\mathbf{k}_{\mathrm{FT}}| = \sqrt{s}|\mathbf{k}_{\mathrm{CM}}| \implies m^2|\mathbf{k}_{\mathrm{FT}}|^2 = s|\mathbf{k}_{\mathrm{CM}}|^2 \implies s|\mathbf{k}_{\mathrm{CM}}|^2 = m^2|\mathbf{k}_{\mathrm{FT}}|^2. \tag{531}$$

Next, we can express $d\sigma/d\Omega_{\mathrm{FT}}$ in terms of ω and ω' using the relation

$$\frac{d\sigma}{d\Omega_{\mathrm{FT}}} = \frac{d\sigma}{dt}\frac{dt}{d\Omega_{\mathrm{FT}}}. \tag{532}$$

Also, for $|\mathcal{T}|^2$, we can substitute the expressions for s and u from part (a) into eq.(11.50) to express it in terms of ω and ω':

$$|\mathcal{T}|^2 = 32\pi^2\alpha^2\Big[\frac{m^4 + m^2(3(m^2+2m\omega) + m^2 - 2m\omega') - (m^2+2m\omega)(m^2-2m\omega')}{(m^2-(m^2+2m\omega))^2} \tag{533}$$

$$+ \frac{m^4 + m^2(3(m^2-2m\omega') + m^2 + 2m\omega) - (m^2-2m\omega')(m^2+2m\omega)}{(m^2-(m^2-2m\omega'))^2} \tag{534}$$

$$+ \frac{2m^2((m^2+2m\omega) + (m^2-2m\omega') + 2m^2)}{(m^2-(m^2+2m\omega))(m^2-(m^2-2m\omega'))}\Big] \tag{535}$$

$$= 32\pi^2\alpha^2\Big[\frac{m^2}{w^2} - \frac{2m^2}{ww'} + \frac{m^2}{(w')^2} - \frac{2m}{w'} + \frac{2m}{w} + \frac{w'}{w} + \frac{w}{w'}\Big], \quad \text{by } \texttt{Mathematica}. \tag{536}$$

Note that

$$\cos\theta_{\rm FT} = 1 + \frac{m}{\omega} - \frac{m}{\omega'} \implies \sin^2\theta_{\rm FT} = 1 - \cos^2\theta_{\rm FT} = 1 - \Big(1 + \frac{m}{\omega} - \frac{m}{\omega'}\Big)^2 \tag{537}$$

$$\implies \sin^2\theta_{\rm FT} = \frac{2m}{\omega'} - \frac{2m}{\omega} - \frac{m^2}{\omega^2} - \frac{m^2}{\omega'^2} + \frac{2m^2}{\omega\omega'}. \tag{538}$$

Hence, we have

$$|\mathcal{T}|^2 = 32\pi^2\alpha^2\Big[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta_{\rm FT}\Big]. \tag{539}$$

Now, we can compute $dt/d\Omega_{\rm FT}$:

$$\frac{dt}{d\Omega_{\rm FT}} = \frac{dt}{d\cos\theta_{\rm FT}}\frac{d\cos\theta_{\rm FT}}{d\Omega_{\rm FT}} \tag{540}$$

$$= \frac{dt}{d\cos\theta_{\rm FT}}\frac{d\cos\theta_{\rm FT}}{2\pi d\cos\theta_{\rm FT}} \tag{541}$$

$$= \frac{1}{2\pi}\frac{dt}{d\cos\theta_{\rm FT}}. \tag{542}$$

Hence, we have

$$\frac{dt}{d\cos\theta_{\rm FT}} = \frac{d}{d\cos\theta_{\rm FT}}\Big(-2\omega\omega'(1-\cos\theta_{\rm FT})\Big) \tag{543}$$

$$=2\omega\omega' + (-2\omega)(1-\cos\theta_{\rm FT})\frac{d\omega'}{d\cos\theta_{\rm FT}} \tag{544}$$

$$=2\omega\omega' + (-2\omega)(1-\cos\theta_{\rm FT})\omega'^2/m, \quad \text{by differentiating } \cos\theta_{\rm FT} = 1 + \frac{m}{\omega} - \frac{m}{\omega'} \tag{545}$$

$$=2\omega\omega' - (2\omega)(\frac{m}{\omega'} - \frac{m}{\omega})\omega'^2/m \tag{546}$$

$$=2\omega'^2. \tag{547}$$

Therefore, we have

$$\frac{dt}{d\Omega_{\rm FT}} = \frac{1}{2\pi}\cdot 2\omega'^2 = \frac{\omega'^2}{\pi}. \tag{548}$$

Substituting the expressions for $|\mathcal{T}|^2$ and $dt/d\Omega_{\text{FT}}$ into the expression for $d\sigma/d\Omega_{\text{FT}}$, we have

$$\frac{d\sigma}{d\Omega_{\text{FT}}} = \frac{d\sigma}{dt}\frac{dt}{d\Omega_{\text{FT}}} \tag{549}$$

$$= \frac{1}{64\pi s|\mathbf{k}_{\text{CM}}|^2}|\mathcal{T}|^2 \cdot \frac{\omega'^2}{\pi} \tag{550}$$

$$= \frac{1}{64\pi^2 s|\mathbf{k}_{\text{CM}}|^2} \cdot 32\pi^2\alpha^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta_{\text{FT}}\right] \cdot \omega'^2 \tag{551}$$

$$= \frac{\alpha^2\omega'^2}{2s|\mathbf{k}_{\text{CM}}|^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta_{\text{FT}}\right] \tag{552}$$

$$= \frac{\alpha^2\omega'^2}{2m^2|\mathbf{k}_{\text{FT}}|^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta_{\text{FT}}\right], \quad \text{by } s|\mathbf{k}_{\text{CM}}|^2 = m^2|\mathbf{k}_{\text{FT}}|^2 \tag{553}$$

$$= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta_{\text{FT}}\right]. \tag{554}$$

This is exactly the Klein-Nishina formula in eq.(11.51). $\square$

HW4 Due to November 4 11:59 PM

Question 1

Problem 14.1

Derive a generalization of Feynman's formula,

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_i x_i^{\alpha_i - 1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}. \tag{555}$$

$$\int dF_n = (n-1)! \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \delta\left(\sum_{i=1}^n x_i - 1\right). \tag{556}$$

Hint: start with

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty dt\, t^{\alpha-1} e^{-tA}, \tag{557}$$

which defines the gamma function. Put an index on A, α and t, and take the product. Then multiply on the right-hand side by

$$1 = \int_0^\infty ds \delta(s - \sum_i t_i). \tag{558}$$

Make the change of variables $t_i = s x_i$ and carry out the integral over s.

Answer

By definesion of the gamma function, we have

$$\frac{1}{A_i^{\alpha_i}} = \frac{1}{\Gamma(\alpha_i)} \int_0^\infty dt_i\, t_i^{\alpha_i - 1} e^{-t_i A_i}. \tag{559}$$

Then we have

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} = \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} \int_0^\infty dt_i\, t_i^{\alpha_i - 1} e^{-t_i A_i} \tag{560}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_n \prod_{i=1}^n t_i^{\alpha_i - 1} e^{-t_i A_i} \tag{561}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_n \prod_{i=1}^n \left(t_i^{\alpha_i - 1} e^{-t_i A_i}\right) \int_0^\infty ds \delta(s - \sum_{i=1}^n t_i). \tag{562}$$

We make the change of variables $t_i = sx_i$, then we have

$$\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_n^{\alpha_n}} = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n \left((sx_i)^{\alpha_i - 1} e^{-sx_i A_i}\right) \delta(s - s\sum_{i=1}^n x_i) s \tag{563}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds\, s^{\sum_{i=1}^n \alpha_i - 1} e^{-s\sum_{i=1}^n x_i A_i} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \tag{564}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \int_0^\infty ds\, s^{\sum_{i=1}^n \alpha_i - 1} e^{-s\sum_{i=1}^n x_i A_i} \tag{565}$$

$$= \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \prod_{i=1}^n x_i^{\alpha_i - 1} \delta(1 - \sum_{i=1}^n x_i) \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{(\sum_{i=1}^n x_i A_i)^{\sum_{i=1}^n \alpha_i}} \tag{566}$$

$$= \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_{i=1}^n x_i^{\alpha_i - 1}}{(\sum_{i=1}^n x_i A_i)^{\sum_{i=1}^n \alpha_i}}. \tag{567}$$

Hence proved the formula. □

Question 2

Problem 14.2
Verify eq. (14.23).

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{568}$$

Answer

We start with the Gaussian integral in d dimensions,

$$I_d = \int d^d x \, e^{-\mathbf{x}^2}. \tag{569}$$

In catrtesian coordinates, we have

$$I_d = \left(\int_{-\infty}^{\infty} dx \, e^{-x^2} \right)^d = (\sqrt{\pi})^d = \pi^{d/2}. \tag{570}$$

In spherical coordinates, we have

$$I_d = \int_0^\infty dr \, r^{d-1} e^{-r^2} \int d\Omega_d = \Omega_d \int_0^\infty dr \, r^{d-1} e^{-r^2}. \tag{571}$$

Make the change of variable $t = r^2$, then we have

$$I_d = \frac{\Omega_d}{2} \int_0^\infty dt \, t^{d/2-1} e^{-t} = \frac{\Omega_d}{2} \Gamma(d/2), \tag{572}$$

where we have used the definition of the gamma function

$$\Gamma(\alpha) = \int_0^\infty dt \, t^{\alpha-1} e^{-t}. \tag{573}$$

Equating the two expressions for I_d, we have

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{574}$$

$\square$

Question 3

Problem 14.5

Compute the $O(\lambda)$ correction ot the propagator in φ^4 theory (see problem 9.2) in $d = 4 - \epsilon$ spacetime dimensions, and compute the $O(\lambda)$ terms in A and B.

Answer

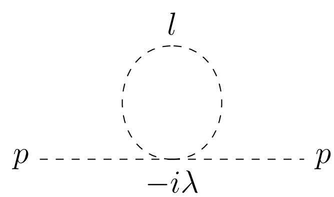


Figure 2: The Feynman diagram with the ϕ^4 propagator for 1-loop correction at $O(\lambda)$.

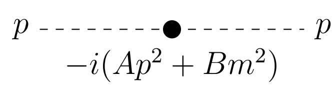


Figure 3: The Feynman diagram with the ϕ^4 propagator for 1-loop counter term at $O(\lambda)$.

First, we write down the Lagrangian for the φ^4 theory,

$$\mathcal{L} = \mathcal{L}_{\prime} + \mathcal{L}_I, \tag{575}$$

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2, \tag{576}$$

$$\mathcal{L}_I = -\frac{Z_\lambda}{4!}\lambda\varphi^4 + \mathcal{L}_{ct}, \tag{577}$$

$$\mathcal{L}_{ct} = -\frac{1}{2}(Z_\varphi - 1)\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2. \tag{578}$$

For the $O(\lambda)$ correction to the propagator, the Feynman diagram is shown in Figure 2. The corresponding amplitude is given by

$$i\Sigma(p) = \frac{1}{2}(-i\lambda)\frac{1}{i}\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2 + m^2 - i\epsilon} - i(Ap^2 + Bm^2) + O(\lambda^2), \tag{579}$$

where the factor $\frac{1}{2}$ is the symmetry factor for this diagram. Consider the wick rotation to Euclidean space ($d^4l \to id^4l_E$ and $l^2 \to l_E^2$), we have

$$\Sigma(p) = \frac{-\lambda}{2}\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{l_E^2 + m^2} - (Ap^2 + Bm^2) + O(\lambda^2), \tag{580}$$

where the $m = m - i\epsilon$ prescription is implied. Using the formula derived in Problem 14.1, we have

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 + m^2} = \int \frac{d^d l_E}{(2\pi)^d} \int_0^\infty dx\, e^{-x(l_E^2+m^2)} \tag{581}$$

$$= \int_0^\infty dx\, e^{-xm^2} \int \frac{d^d l_E}{(2\pi)^d} e^{-xl_E^2} \tag{582}$$

$$= \int_0^\infty dx\, e^{-xm^2} \frac{1}{(2\pi)^d} \left(\frac{\pi}{x}\right)^{d/2} \tag{583}$$

$$= \frac{1}{(4\pi)^{d/2}} \int_0^\infty dx\, x^{-d/2} e^{-xm^2} \tag{584}$$

$$= \frac{1}{(4\pi)^{d/2}} (m^2)^{d/2-1} \Gamma(1 - d/2). \tag{585}$$

Substituting $d = 4 - \epsilon$, we have

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 + m^2} = \frac{1}{(4\pi)^{2-\epsilon/2}} (m^2)^{1-\epsilon/2} \Gamma(-1 + \epsilon/2) \tag{586}$$

$$= \frac{m^2}{16\pi^2} \left(\frac{4\pi}{m^2}\right)^{\epsilon/2} \Gamma(-1 + \epsilon/2). \tag{587}$$

Also, when we consider dimension regularization, we need to introduce a mass scale μ to keep the coupling constant λ dimensionless. Thus, we have

$$\lambda \to \lambda \widetilde{\mu}^\epsilon. \tag{588}$$

Therefore, we have

$$\Sigma(p) = \frac{-\lambda \widetilde{\mu}^\epsilon}{2} \frac{m^2}{16\pi^2} \left(\frac{4\pi}{m^2}\right)^{\epsilon/2} \Gamma(-1 + \epsilon/2) - (Ap^2 + Bm^2) + O(\lambda^2) \tag{589}$$

$$= \frac{-\lambda m^2}{32\pi^2} \left(\frac{4\pi \widetilde{\mu}^2}{m^2}\right)^{\epsilon/2} \Gamma(-1 + \epsilon/2) - (Ap^2 + Bm^2) + O(\lambda^2). \tag{590}$$

Using the expansion of the gamma function around the pole at -1 and the $4\pi\widetilde{\mu}^2/m^2$ around $\epsilon = 0$,

$$\Gamma(-1 + \epsilon/2) = -\frac{2}{\epsilon} - 1 + \gamma + O(\epsilon), \tag{591}$$

$$\left(\frac{4\pi \widetilde{\mu}^2}{m^2}\right)^{\epsilon/2} = 1 + \frac{\epsilon}{2} \ln\left(\frac{4\pi \widetilde{\mu}^2}{m^2}\right) + O(\epsilon^2), \tag{592}$$

we have

$$\Sigma(p) = \frac{-\lambda m^2}{32\pi^2}\left[-\frac{2}{\epsilon} - 1 + \gamma\right]\left[1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)\right] - (Ap^2 + Bm^2) + O(\lambda^2) \tag{593}$$

$$= \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \ln\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)\right] - (Ap^2 + Bm^2) + O(\lambda^2) \tag{594}$$

$$= \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right] - (Ap^2 + Bm^2) + O(\lambda^2) \tag{595}$$

To satisfy the renormalization conditions,

$$\left.\frac{d}{dp^2}\Sigma(p)\right|_{p^2=-m^2} = 0, \tag{596}$$

$$\Sigma(p^2 = -m^2) = 0, \tag{597}$$

we have

$$A = 0 + O(\lambda^2), \tag{598}$$

$$B = \frac{\lambda}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right] + O(\lambda^2). \tag{599}$$

In summary, we have

$$\Sigma(p) = \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right] - (Ap^2 + Bm^2) + O(\lambda^2) = 0 + O(\lambda^2), \tag{600}$$

$$A = 0 + O(\lambda^2), \tag{601}$$

$$B = \frac{\lambda}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{4\pi\widetilde{\mu}^2}{e^\gamma m^2}\right)\right] + O(\lambda^2). \tag{602}$$

□

Question 4

Problem 16.1

Compute the $O(\lambda^2)$ correction in $\mathbf{V}_4$ in φ^4 theory in $d = 4 - \epsilon$ spacetime dimensions. Take $\mathbf{V}_4 = \lambda$ when all four external momenta are on shell, and $s = 4m^2$. What is the $O(\lambda)$ contribution to C?

Answer

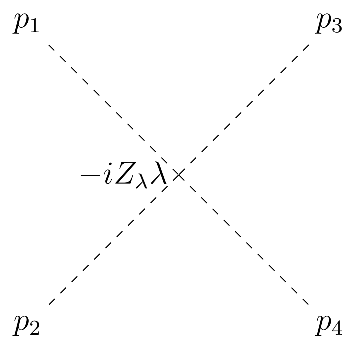


Figure 4: The Feynman diagram with the ϕ^4 vertex for tree level at $O(\lambda)$.

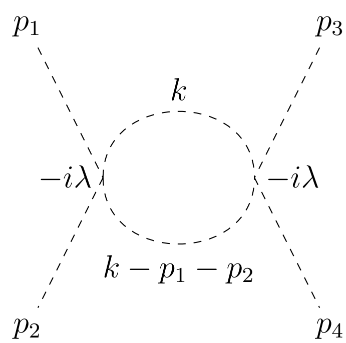


Figure 5: The Feynman diagram with the ϕ^4 vertex for 1-loop correction at $O(\lambda^2)$ in the s-channel.

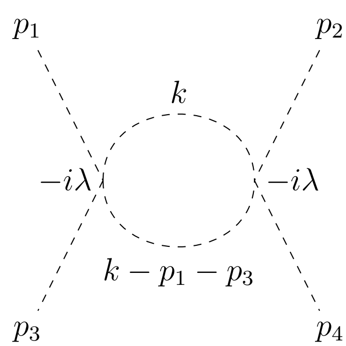


Figure 6: The Feynman diagram with the ϕ^4 vertex for 1-loop correction at $O(\lambda^2)$ in the t-channel.

At $O(\lambda^2)$, there are three Feynman diagrams contributing to the 1-loop correction to the ϕ^4 vertex, as shown in Figure 5, 6 and 7. The tree-level diagram is shown in Figure 4. The corresponding amplitude is

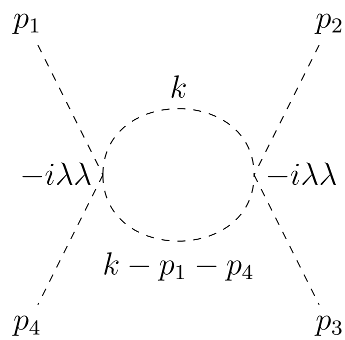


Figure 7: The Feynman diagram with the ϕ^4 vertex for 1-loop correction at $O(\lambda^2)$ in the u-channel.

given by

$$i\mathbf{V}_4 = iV_{tree} + iV_4^{(s)} + iV_4^{(t)} + iV_4^{(u)} \tag{603}$$

$$= -iZ_\lambda\lambda + \left(\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_2)^2+m^2} \tag{604}$$

$$+\left(\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_3)^2+m^2} \tag{605}$$

$$+\left(\frac{1}{2}\right)(-i\lambda)^2\left(\frac{1}{i}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_4)^2+m^2}. \tag{606}$$

Using Feynman parameterization, we have

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_i-p_j)^2+m^2} \tag{607}$$

$$=\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[k^2-2xk\cdot(p_i+p_j)+x(p_i+p_j)^2+m^2\right]^2} \tag{608}$$

$$=\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[(k-x(p_i+p_j))^2+x(1-x)(p_i+p_j)^2+m^2\right]^2} \tag{609}$$

$$=\int_0^1 dx\int\frac{d^4q}{(2\pi)^4}\frac{1}{\left[q^2+D_{ij}\right]^2} \tag{610}$$

$$=i\int_0^1 dx\int\frac{d^4q_E}{(2\pi)^4}\frac{1}{\left[q_E^2+D_{ij}\right]^2}, \tag{611}$$

where $D_{ij}=x(1-x)(p_i+p_j)^2+m^2$ and we have performed the wick rotation to Euclidean space ($d^4k\to id^4k_E$ and $k^2\to k_E^2$). Using the eq. (14.27) in textbook, we have

$$\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{(\bar{q}^2)^a}{\left[\bar{q}^2+D_{ij}\right]^b}=\frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)}D_{ij}^{-(b-a-d/2)}. \tag{612}$$

Thus, with $a = 0$, $b = 2$ and $d = 4 - \epsilon$, we have

$$\int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{[q_E^2 + D_{ij}]^2} = \frac{\Gamma(2-0-(4-\epsilon)/2)\Gamma(0+(4-\epsilon)/2)}{(4\pi)^{(4-\epsilon)/2}\Gamma(2)\Gamma((4-\epsilon)/2)} D_{ij}^{-(2-0-(4-\epsilon)/2)} \tag{613}$$

$$= \frac{\Gamma(\epsilon/2)\Gamma(2-\epsilon/2)}{(4\pi)^{2-\epsilon/2} \cdot 1 \cdot \Gamma(2-\epsilon/2)} D_{ij}^{-\epsilon/2} \tag{614}$$

$$= \frac{1}{(4\pi)^{2-\epsilon/2}} \Gamma(\epsilon/2) D_{ij}^{-\epsilon/2} \tag{615}$$

$$= \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{ij}} \right)^{\epsilon/2} \Gamma(\epsilon/2). \tag{616}$$

Besides, when we consider dimension regularization, we need to introduce a mass scale μ to keep the coupling constant λ dimensionless. Thus, we have

$$\lambda \to \lambda \widetilde{\mu}^{\epsilon}. \tag{617}$$

Therefore, we have

$$i\mathbf{V}_4 = -iZ_\lambda \lambda + \left(\frac{1}{2}\right) (-i\widetilde{\mu}^{\epsilon}\lambda)^2 \left(\frac{1}{i}\right)^2 i \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{12}}\right)^{\epsilon/2} \Gamma(\epsilon/2) \tag{618}$$

$$+ \left(\frac{1}{2}\right) (-i\widetilde{\mu}^{\epsilon}\lambda)^2 \left(\frac{1}{i}\right)^2 i \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{13}}\right)^{\epsilon/2} \Gamma(\epsilon/2) \tag{619}$$

$$+ \left(\frac{1}{2}\right) (-i\widetilde{\mu}^{\epsilon}\lambda)^2 \left(\frac{1}{i}\right)^2 i \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi}{D_{14}}\right)^{\epsilon/2} \Gamma(\epsilon/2) \tag{620}$$

$$= -iZ_\lambda \lambda - i(\widetilde{\mu}^{\epsilon}\lambda)^2 \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi}{D_{12}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{13}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{14}}\right)^{\epsilon/2} \right]. \tag{621}$$

In other words, we have

$$\mathbf{V}_4 = -Z_\lambda \lambda - (\widetilde{\mu}^{\epsilon}\lambda)^2 \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi}{D_{12}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{13}}\right)^{\epsilon/2} + \left(\frac{4\pi}{D_{14}}\right)^{\epsilon/2} \right]. \tag{622}$$

To satisfy the renormalization condition,

$$\mathbf{V}_4 = -\lambda \quad \text{at} \quad s = 4m^2, \tag{623}$$

we have

$$Z_\lambda = 1 + \lambda \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi\widetilde{\mu}^2}{D_{12}}\right)^{\epsilon/2} + \left(\frac{4\pi\widetilde{\mu}^2}{D_{13}}\right)^{\epsilon/2} + \left(\frac{4\pi\widetilde{\mu}^2}{D_{14}}\right)^{\epsilon/2} \right] + O(\lambda^2). \tag{624}$$

For $s = 4m^2$ and $t = u = 0$ (by assuming all $p_i = (m, \mathbf{0})$), we have

$$D_{12} = x(1-x)(-s) + m^2 = (1 - 4x(1-x))m^2 = (1-2x)^2 m^2, \tag{625}$$
$$D_{13} = x(1-x)(-t) + m^2 = m^2, \tag{626}$$
$$D_{14} = x(1-x)(-u) + m^2 = m^2. \tag{627}$$

Thus, we have

$$Z_\lambda = 1 + \lambda \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \int_0^1 dx \left[\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2} \right)^{\epsilon/2} + 2 \left(\frac{4\pi\widetilde{\mu}^2}{m^2} \right)^{\epsilon/2} \right] + O(\lambda^2) \tag{628}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \Gamma(\epsilon/2) \left[\int_0^1 dx \left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2} \right)^{\epsilon/2} + 2 \left(\frac{4\pi\widetilde{\mu}^2}{m^2} \right)^{\epsilon/2} \right] + O(\lambda^2). \tag{629}$$

For small ϵ, we have

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + O(\epsilon), \tag{630}$$
$$\left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2} \right)^{\epsilon/2} = 1 + \frac{\epsilon}{2} \ln \left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2} \right) + O(\epsilon^2), \tag{631}$$
$$\left(\frac{4\pi\widetilde{\mu}^2}{m^2} \right)^{\epsilon/2} = 1 + \frac{\epsilon}{2} \ln \left(\frac{4\pi\widetilde{\mu}^2}{m^2} \right) + O(\epsilon^2). \tag{632}$$

Substituting these expansions into the expression of Z_λ, we have

$$Z_\lambda = 1 + \lambda \frac{1}{2(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma \right) \left[\int_0^1 dx \left(1 + \frac{\epsilon}{2} \ln \left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2} \right) \right) + 2 \left(1 + \frac{\epsilon}{2} \ln \left(\frac{4\pi\widetilde{\mu}^2}{m^2} \right) \right) \right] + O(\lambda^2) \tag{633}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \left[\int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \ln \left(\frac{4\pi\widetilde{\mu}^2}{(1-2x)^2 m^2} \right) \right) + 2 \left(\frac{2}{\epsilon} - \gamma + \ln \left(\frac{4\pi\widetilde{\mu}^2}{m^2} \right) \right) \right] + O(\lambda^2) \tag{634}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \left[\int_0^1 dx \left(\frac{2}{\epsilon} + \ln \left(\frac{4\pi\widetilde{\mu}^2/e^\gamma}{(1-2x)^2 m^2} \right) \right) + 2 \left(\frac{2}{\epsilon} + \ln \left(\frac{4\pi\widetilde{\mu}^2/e^\gamma}{m^2} \right) \right) \right] + O(\lambda^2) \tag{635}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \left[\int_0^1 dx \left(\frac{2}{\epsilon} + \ln \left(\frac{\mu^2}{(1-2x)^2 m^2} \right) \right) + 2 \left(\frac{2}{\epsilon} + \ln \left(\frac{\mu^2}{m^2} \right) \right) \right] + O(\lambda^2) \tag{636}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \left[\int_0^1 dx \left(\frac{2}{\epsilon} + \ln \left(\frac{\mu^2/m^2}{(1-2x)^2} \right) \right) + 2 \left(\frac{2}{\epsilon} + 2 \ln \left(\frac{\mu}{m} \right) \right) \right] + O(\lambda^2) \tag{637}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \left[\left(\frac{2}{\epsilon} + 2 + 2 \ln(\mu/m) \right) + 2 \left(\frac{2}{\epsilon} + 2 \ln \left(\frac{\mu}{m} \right) \right) \right] + O(\lambda^2) \tag{638}$$
$$= 1 + \lambda \frac{1}{2(4\pi)^2} \left(\frac{6}{\epsilon} + 2 + 6 \ln(\mu/m) \right) + O(\lambda^2) \tag{639}$$
$$= 1 + \frac{3\lambda}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{1}{3} + \ln(\mu/m) \right) + O(\lambda^2). \tag{640}$$

In summary, we have

$$Z_\lambda - 1 = \frac{3\lambda}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{3} + \ln(\mu/m)\right) + O(\lambda^2), \tag{641}$$

$$i\mathbf{V}_4 = -i\lambda + i(Z_\lambda - 1)\lambda + O(\lambda^3) \tag{642}$$

$$= -i\lambda + \frac{3i\lambda^2}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{1}{3} + \ln(\mu/m)\right) + O(\lambda^3). \tag{643}$$

□

HW5 Due to November 18 11:59 PM

Question 1

Problem 18.1

In any number d of spacetime dimensions, a *Dirac field* Ψ_α carries a spin index α, and has a kinetic term of the form $i\overline{\Psi}\gamma^\mu\partial_\mu\Psi$, where we have suppressed the spin indices; the *gamma matrices* γ^μ are dimensionless, and $\overline{\Psi} = \Psi^\dagger\gamma^0$.

(a) What is the mass dimension $[\Psi]$ of the field Ψ.

(b) Consider interaction of the form $g_n(\overline{\Psi}\Psi)^n$, where $n \geq 2$ is an integer. What is the mass dimension $[g_n]$ of g_n?

(c) Consider interaction of the form $g_{m,n}\varphi^m(\overline{\Psi}\Psi)^n$, where φ is a scalar field, and $m, n > 0$ are integers. What is the mass dimension $[g_{m,n}]$ of $g_{m,n}$?

(d) In $d = 4$ spacetime dimensions, which of these interactions are allowed in a renormalizable theory?

Answer

(a)

We have in d spacetime dimensions, the action is dimensionless, so the Lagrangian density has mass dimension $[\mathcal{L}] = d$. The kinetic term for the Dirac field is given by:

$$\mathcal{L}_{\text{kin}} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi. \tag{644}$$

The derivative ∂_μ has mass dimension 1, and the gamma matrices γ^μ are dimensionless. Therefore, we can write:

$$[\mathcal{L}_{\text{kin}}] = [\overline{\Psi}] + [\partial_\mu] + [\Psi] = 2[\Psi] + 1. \tag{645}$$

Setting this equal to the mass dimension of the Lagrangian density, we have:

$$2[\Psi] + 1 = d \implies [\Psi] = \frac{d-1}{2}. \tag{646}$$

(b)

The interaction term is given by:

$$\mathcal{L}_{\text{int}} = g_n(\overline{\Psi}\Psi)^n. \tag{647}$$

The mass dimension of this term is:

$$[\mathcal{L}_{\text{int}}] = [g_n] + n([\overline{\Psi}] + [\Psi]) = [g_n] + 2n[\Psi]. \tag{648}$$

Setting this equal to the mass dimension of the Lagrangian density, we have:

$$[g_n] + 2n[\Psi] = d \implies [g_n] = d - 2n[\Psi] = d - 2n\left(\frac{d-1}{2}\right) = d - n(d-1) = d(1-n) + n. \tag{649}$$

(c)

The interaction term is given by:

$$\mathcal{L}_{\text{int}} = g_{m,n}\varphi^m(\overline{\Psi}\Psi)^n. \tag{650}$$

The mass dimension of this term is:

$$[\mathcal{L}_{\text{int}}] = [g_{m,n}] + m[\varphi] + n([\overline{\Psi}] + [\Psi]) = [g_{m,n}] + m[\varphi] + 2n[\Psi]. \tag{651}$$

Setting this equal to the mass dimension of the Lagrangian density, we have:

$$[g_{m,n}] + m[\varphi] + 2n[\Psi] = d \implies [g_{m,n}] = d - m[\varphi] - 2n[\Psi]. \tag{652}$$

The mass dimension of the scalar field φ in d dimensions is given by:

$$[\varphi] = \frac{d-2}{2}. \tag{653}$$

Substituting this and the expression for $[\Psi]$ into the equation for $[g_{m,n}]$, we get:

$$[g_{m,n}] = d - m\left(\frac{d-2}{2}\right) - 2n\left(\frac{d-1}{2}\right) = d - \frac{m(d-2)}{2} - n(d-1). \tag{654}$$

(d)

In $d = 4$ spacetime dimensions, we have:

$$[\Psi] = \frac{4-1}{2} = \frac{3}{2}. \tag{655}$$

For the interaction $g_n(\overline{\Psi}\Psi)^n$, we have:

$$[g_n] = 4(1-n) + n = 4 - 3n. \tag{656}$$

For the interaction $g_{m,n}\varphi^m(\overline{\Psi}\Psi)^n$, we have:

$$[g_{m,n}] = 4 - m\left(\frac{4-2}{2}\right) - 2n\left(\frac{4-1}{2}\right) = 4 - m - 3n. \tag{657}$$

For a theory to be renormalizable, the coupling constants must have non-negative mass dimensions. Therefore:

- For g_n:

$$4 - 3n \geq 0 \implies n \leq \frac{4}{3}. \tag{658}$$

  Since n is an integer and $n \geq 2$, there are no renormalizable interactions of this form.

- For $g_{m,n}$:

$$4 - m - 3n \geq 0 \implies m + 3n \leq 4. \tag{659}$$

  The possible integer pairs (m, n) that satisfy this inequality with $m, n > 0$ are:

  - $(m, n) = (1, 1)$
  - $(m, n) = (2, 1)$
  - $(m, n) = (1, 2)$

$\square$

Question 2

Problem 20.2
Compute the $\mathcal{O}(\alpha)$ correction to the two-particle scattering amplitude at *threshold*, that is, for $s = 4m^2$ and $t = u = 0$, corresponding to zero three-momentum for both the incoming and outgoing particles.
Hint: for 20.2, do not ues Eq. (20.12)-(20.19) they are in different unit.

Answer

Starting with the equation (20.2) in the Srednicki's textbook, we have

$$i\mathcal{T}_{1-loop} = \frac{1}{i}\Big(i[\mathbf{V}_3(s)]^2\widetilde{\mathbf{\Delta}}(-s) + i[\mathbf{V}_3(t)]^2\widetilde{\mathbf{\Delta}}(-t) + i[\mathbf{V}_3(u)]^2\widetilde{\mathbf{\Delta}}(-u)\Big) + i\mathbf{V}_4(s,t,u), \tag{660}$$

where

$$\tilde{\mathbf{\Delta}}(-s) = \frac{1}{-s + m^2 - \Pi(-s)} \tag{661}$$

$$\Pi(-s) = \frac{1}{2}\alpha\int_0^1 dx D_2(s)\ln\left(D_2(s)/D_0\right) - \frac{1}{12}\alpha\left(-s + m^2\right) \tag{662}$$

$$\mathbf{V}_3(s)/g = 1 - \frac{1}{2}\alpha\int dF_3 \ln\left(D_3(s)/m^2\right) \tag{663}$$

$$\mathbf{V}_4(s,t,u) = \frac{1}{6}g^2\alpha\int dF_4\left[\frac{1}{D_4(s,t)} + \frac{1}{D_4(t,u)} + \frac{1}{D_4(u,s)}\right]. \tag{664}$$

Also, we have

$$D_2(s) = -x(1-x)s + m^2 \tag{665}$$

$$D_0 = +[1 - x(1-x)]m^2 \tag{666}$$

$$D_3(s) = -x_1x_2s + \left[1 - (x_1+x_2)\,x_3\right]m^2 \tag{667}$$

$$D_4(s,t) = -x_1x_2s - x_3x_4t + \left[1 - (x_1+x_2)\,(x_3+x_4)\right]m^2 \tag{668}$$

and the integration measures are given by

$$\int dF_3 = 2\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\delta\left(x_1+x_2+x_3-1\right) \tag{669}$$

$$\int dF_4 = 6\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\int_0^1 dx_4\delta\left(x_1+x_2+x_3+x_4-1\right). \tag{670}$$

By *Mathematica* calculation, we have the following results at threshold $s = 4m^2$ and $t = u = 0$:

$$\tilde{\mathbf{\Delta}}(-4m^2) = -\frac{12}{\left(\left(2\sqrt{3}\pi - 9\right)\alpha + 36\right)m^2}, \tag{671}$$

$$\mathbf{V}_3(4m^2) = g - g\alpha\Big(-\frac{4}{3} + \frac{5i\pi}{12} + \frac{11\pi}{4\sqrt{3}}\Big), \tag{672}$$

$$\tilde{\mathbf{\Delta}}(0) = \frac{12}{\left(\left(2\sqrt{3}\pi - 11\right)\alpha + 12\right)m^2}, \tag{673}$$

$$\mathbf{V}_3(0) = g - g\alpha\Big(\frac{\pi}{2\sqrt{3}} - 1\Big), \tag{674}$$

$$\mathbf{V}_4(4m^2, 0, 0) = g^2\alpha\left(\frac{-6 + 6i\pi + 13\sqrt{3}\pi}{18m^2}\right) \tag{675}$$

Thus, substituting these results into the expression for $i\mathcal{T}_{1-loop}$, we obtain:

$$i\mathcal{T}_{1-loop} = \frac{5g^2}{3m^2} + \frac{\left((525 - 36i) + \pi\left((-36 + 30i) - (40 - 78i)\sqrt{3}\right)\right)\alpha g^2}{108m^2} \tag{676}$$

$$= \frac{1.66667g^2}{m^2} + \frac{(1.79858 + 4.46923i)\alpha g^2}{m^2} \tag{677}$$

Second line is numerical result.

Remark: My detail calculation in Mathematica. □

Question 3

Problem 27.1
Suppose that we have a theory with

$$\beta(\alpha) = b_1\alpha^2 + \mathcal{O}(\alpha^3), \tag{678}$$

$$\gamma_m(\alpha) = c_1\alpha + \mathcal{O}(\alpha^2). \tag{679}$$

Neglecting the higher-order terms, show that

$$m(\mu_2) = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{c_1/b_1} m(\mu_1). \tag{680}$$

Answer

The gamma and beta function are given by:

$$\gamma_m(\alpha) = \frac{d}{d\ln\mu}\ln m(\mu) = c_1\alpha, \tag{681}$$

$$\beta(\alpha) = \frac{d}{d\ln\mu}\alpha = b_1\alpha^2. \tag{682}$$

We can rearrange the beta function to express $d\ln\mu$ in terms of $d\alpha$:

$$d\ln\mu = \frac{d\alpha}{\beta(\alpha)} = \frac{d\alpha}{b_1\alpha^2}. \tag{683}$$

Substituting this into the expression for $\gamma_m(\alpha)$, we have:

$$\frac{d}{d\ln\mu}\ln m(\mu) = c_1\alpha \implies d\ln m(\mu) = c_1\alpha\, d\ln\mu = c_1\alpha\cdot\frac{d\alpha}{b_1\alpha^2} = \frac{c_1}{b_1}\frac{d\alpha}{\alpha}. \tag{684}$$

Integrating both sides from μ_1 to μ_2, we get:

$$\int_{m(\mu_1)}^{m(\mu_2)} d\ln m(\mu) = \frac{c_1}{b_1}\int_{\alpha(\mu_1)}^{\alpha(\mu_2)}\frac{d\alpha}{\alpha}. \tag{685}$$

This gives:

$$\ln\left(\frac{m(\mu_2)}{m(\mu_1)}\right) = \frac{c_1}{b_1}\ln\left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right). \tag{686}$$

Exponentiating both sides, we obtain:

$$\frac{m(\mu_2)}{m(\mu_1)} = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{c_1/b_1}. \tag{687}$$

Thus, we have shown that:

$$m(\mu_2) = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{c_1/b_1} m(\mu_1). \tag{688}$$

$\square$

Question 4

Problem 28.1
Consider φ^4 theory ,

$$\mathcal{L} = -Z_\varphi \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - Z_m \frac{1}{2} m^2 \varphi^2 - Z_\lambda \frac{\lambda \tilde{\mu}^\epsilon}{4!} \varphi^4, \tag{689}$$

in $d = 4 - \epsilon$ spacetime dimensions. Compute the beta function to $\mathcal{O}(\lambda^2)$, the anomalous dimension of m to $\mathcal{O}(\lambda)$, and the anomalous dimension of φ to $\mathcal{O}(\lambda)$.

Answer

We first write down the Lagrangian for ϕ^4 theory in $d = 4 - \epsilon$ dimensions:

$$\mathcal{L} = -Z_\varphi \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - Z_m \frac{1}{2} m^2 \varphi^2 - Z_\lambda \frac{\lambda \tilde{\mu}^\epsilon}{4!} \varphi^4, \tag{690}$$

where Z_φ, Z_m, and Z_λ are the renormalization constants for the field, mass, and coupling constant, respectively. We also write down the Lagrangian in terms of bare quantities:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \varphi_0)(\partial_\mu \varphi_0) - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{\lambda_0}{4!} \varphi_0^4, \tag{691}$$

where the bare quantities are related to the renormalized quantities by:

$$\varphi_0 = Z_\varphi^{1/2} \varphi, \tag{692}$$

$$m_0^2 = Z_m Z_\varphi^{-1} m^2, \tag{693}$$

$$\lambda_0 = Z_\lambda Z_\varphi^{-2} \lambda \tilde{\mu}^\epsilon. \tag{694}$$

From our previous calculations in ϕ^4 theory, we have the following results for the renormalization constants to the required orders:

$$Z_\varphi = 1 + \mathcal{O}(\lambda^2) = 1 + \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{\epsilon^n}, \tag{695}$$

$$Z_m = 1 + \frac{\lambda}{16\pi^2 \epsilon} + \mathcal{O}(\lambda^2) = 1 + \sum_{n=1}^{\infty} \frac{b_n(\lambda)}{\epsilon^n}, \tag{696}$$

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2 \epsilon} + \mathcal{O}(\lambda^2) = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n}. \tag{697}$$

We know $a_1(\lambda) = 0 + \mathcal{O}(\lambda^2)$, $b_1(\lambda) = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$, and $c_1(\lambda) = \frac{3\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)$. Using these results, we can compute the beta function and anomalous dimensions.

$$\ln \lambda_0 = \ln(Z_\lambda Z_\varphi^{-2}) + \ln \lambda + \epsilon \ln \widetilde{\mu}, \tag{698}$$

$$0 = \frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\ln\mu} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\ln\mu} + \frac{1}{\lambda} \frac{d\lambda}{d\ln\mu} + \epsilon, \tag{699}$$

$$\beta(\lambda) \equiv \frac{d\lambda}{d\ln\mu} = \lambda \left(-\epsilon - \frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\ln\mu} + \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\ln\mu} \right). \tag{700}$$

To compute $\frac{dZ_\lambda}{d\ln\mu}$ and $\frac{dZ_\varphi}{d\ln\mu}$, we use the chain rule:

$$\frac{dZ_\lambda}{d\ln\mu} = \frac{dZ_\lambda}{d\lambda} \frac{d\lambda}{d\ln\mu} = \frac{dZ_\lambda}{d\lambda} \beta(\lambda), \tag{701}$$

$$\frac{dZ_\varphi}{d\ln\mu} = \frac{dZ_\varphi}{d\lambda} \frac{d\lambda}{d\ln\mu} = \frac{dZ_\varphi}{d\lambda} \beta(\lambda). \tag{702}$$

Substituting these into the expression for $\beta(\lambda)$, we have:

$$\beta(\lambda) = \lambda \left(-\epsilon - \frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} \beta(\lambda) + \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \beta(\lambda) \right) \tag{703}$$

$$= \lambda \left(-\epsilon - \beta(\lambda) \left(\frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \right) \right). \tag{704}$$

We can solve for $\beta(\lambda)$:

$$\beta(\lambda) \left(1 + \lambda \left(\frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \right) \right) = -\lambda\epsilon. \tag{705}$$

In the limit $\epsilon \to 0$, we have:

$$\beta(\lambda) = -\lambda\epsilon \left(1 + \lambda \left(\frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \right) \right)^{-1} \tag{706}$$

$$= -\lambda\epsilon \left(1 - \lambda \left(\frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \right) + \mathcal{O}(\lambda^2) \right) \tag{707}$$

$$= -\lambda\epsilon + \lambda^2\epsilon \left(\frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \right) + \mathcal{O}(\lambda^3) \tag{708}$$

$$= -\lambda\epsilon + \lambda^2\epsilon \left(\frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\lambda} - \frac{2}{Z_\varphi} \frac{dZ_\varphi}{d\lambda} \right) + \mathcal{O}(\lambda^3) \tag{709}$$

$$= -\lambda\epsilon + \lambda^2\epsilon \left(\frac{d}{d\lambda} \left(\frac{3\lambda}{16\pi^2\epsilon} \right) - 0 \right) + \mathcal{O}(\lambda^3) \tag{710}$$

$$= -\lambda\epsilon + \lambda^2\epsilon \left(\frac{3}{16\pi^2\epsilon} \right) + \mathcal{O}(\lambda^3). \tag{711}$$

Expanding to $\mathcal{O}(\lambda^2)$, we find:

$$\beta(\lambda) = -\lambda\epsilon + \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3). \tag{712}$$

Next, we compute the anomalous dimension of the mass m:

$$\begin{aligned}
0 =& \frac{d}{d\ln\mu}\ln m_0 = \frac{d}{d\ln\mu}\ln(Z_m^{1/2} Z_\varphi^{-1/2} m) && (713)\\
=& \frac{1}{2Z_m}\frac{dZ_m}{d\ln\mu} - \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} && (714)\\
=& \frac{1}{2Z_m}\frac{dZ_m}{d\lambda}\beta(\lambda) - \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda) + \frac{1}{m}\frac{dm}{d\ln\mu}. && (715)
\end{aligned}$$

Solving for $\frac{dm}{d\ln\mu}$, we have:

$$\frac{dm}{d\ln\mu} = -m\beta(\lambda)\left(\frac{1}{2Z_m}\frac{dZ_m}{d\lambda} - \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\right). \tag{716}$$

Substituting the expressions for Z_m and Z_φ, we find:

$$\begin{aligned}
\frac{dm}{d\ln\mu} =& -m\left(-\lambda\epsilon + \frac{3\lambda^2}{16\pi^2}\right)\left(\frac{1}{2}\frac{d}{d\lambda}\left(\frac{\lambda}{16\pi^2\epsilon}\right) - 0\right) + \mathcal{O}(\lambda^2) && (717)\\
=& -m\left(-\lambda\epsilon + \frac{3\lambda^2}{16\pi^2}\right)\left(\frac{1}{2}\cdot\frac{1}{16\pi^2\epsilon}\right) + \mathcal{O}(\lambda^2) && (718)\\
=& -m\left(-\frac{\lambda}{32\pi^2} + \frac{3\lambda^2}{32\pi^4\epsilon}\right) + \mathcal{O}(\lambda^2) && (719)\\
=& \frac{\lambda m}{32\pi^2} + \mathcal{O}(\lambda^2). && (720)
\end{aligned}$$

Thus, the anomalous dimension of the mass m to $\mathcal{O}(\lambda)$ is:

$$\gamma_m(\lambda) = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2). \tag{721}$$

Finally, we compute the anomalous dimension of the field φ:

$$\begin{aligned}
0 =& \frac{d}{d\ln\mu}\ln\varphi_0 = \frac{d}{d\ln\mu}\ln(Z_\varphi^{1/2}\varphi) && (722)\\
=& \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\ln\mu} + \frac{1}{\varphi}\frac{d\varphi}{d\ln\mu} && (723)\\
=& \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda) + \frac{1}{\varphi}\frac{d\varphi}{d\ln\mu}. && (724)
\end{aligned}$$

Solving for $\frac{d\varphi}{d\ln\mu}$ ($\varphi = Z_\varphi^{-1/2}\varphi_0$), we have:

$$\frac{d\varphi}{d\ln\mu} = -\varphi \cdot \frac{1}{2Z_\varphi}\frac{dZ_\varphi}{d\lambda}\beta(\lambda). \tag{725}$$

Substituting the expressions for Z_φ and $\beta(\lambda)$, we find:

$$\frac{d\varphi}{d\ln\mu} = -\varphi \cdot \frac{1}{2} \cdot 0 \cdot \left(-\lambda\epsilon + \frac{3\lambda^2}{16\pi^2}\right) + \mathcal{O}(\lambda^2) \tag{726}$$

$$=0 + \mathcal{O}(\lambda^2). \tag{727}$$

Thus, the anomalous dimension of the field φ to $\mathcal{O}(\lambda)$ is:

$$\gamma_\varphi(\lambda) = \frac{1}{\varphi}\frac{d\varphi}{d\ln\mu} = 0 + \mathcal{O}(\lambda^2). \tag{728}$$

In summary, we have:

$$\beta(\lambda) = -\lambda\epsilon + \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3), \tag{729}$$

$$\gamma_m(\lambda) = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2), \tag{730}$$

$$\gamma_\varphi(\lambda) = 0 + \mathcal{O}(\lambda^2). \tag{731}$$

□

Question 5

Extra Problem

Dirac equation: Use find solutions of the matrices α_i and β of Dirac equation,

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \left(i\hbar c(\alpha^i)_{ab}\partial_i + (\beta)_{ab}mc^2\right)\Psi_b, \tag{732}$$

which makes the wavefunction Ψ satisfy the Klein-Gordon equation. (This is how Dirac discovered his equation.)

(a) What are the lowest dimensional matrices α^i and β for $m = 0$? Derive the solutions.

(b) Is the above solution unique? f not, can you write down another one of the same dimension?

(c) What are the lowest dimensional matrices α^i and β for $m \neq 0$? Derive the solutions.

(d) Is the above solution unique? If not, can you write down another one of the same dimension?

Answer

Starting with the Dirac equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \left(i\hbar c(\alpha^i)_{ab}\partial_i + (\beta)_{ab}mc^2\right)\Psi_b, \tag{733}$$

we want to find matrices α^i and β such that the wavefunction Ψ satisfies the Klein-Gordon equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\Psi = 0. \tag{734}$$

(a)

For $m = 0$, the Dirac equation simplifies to:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = i\hbar c(\alpha^i)_{ab}\partial_i\Psi_b. \tag{735}$$

To ensure that Ψ satisfies the Klein-Gordon equation, we require that the matrices α^i satisfy the anticommutation relations:

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}I, \tag{736}$$

where I is the identity matrix. The lowest dimensional matrices that satisfy these relations are the Pauli matrices, which are 2×2 matrices. Therefore, we can choose:

$$\alpha^1 = \sigma_x, \quad \alpha^2 = \sigma_y, \quad \alpha^3 = \sigma_z, \tag{737}$$

where σ_x, σ_y, and σ_z are the Pauli matrices.

(b)

The solution is not unique. Another set of 2×2 matrices that satisfy the same anticommutation relations can be obtained by multiplying the Pauli matrices by a unitary transformation. For example, we can choose:

$$\alpha^1 = U\sigma_x U^\dagger, \quad \alpha^2 = U\sigma_y U^\dagger, \quad \alpha^3 = U\sigma_z U^\dagger, \tag{738}$$

where U is any 2×2 unitary matrix.

(c)

For $m \neq 0$, the Dirac equation is:

$$i\hbar\frac{\partial}{\partial t}\Psi_a = \left(i\hbar c(\alpha^i)_{ab}\partial_i + (\beta)_{ab}mc^2\right)\Psi_b. \tag{739}$$

To ensure that Ψ satisfies the Klein-Gordon equation, we require that the matrices α^i and β satisfy the following relations:

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}I, \quad \{\alpha^i, \beta\} = 0, \quad \beta^2 = I. \tag{740}$$

The lowest dimensional matrices that satisfy these relations are the 4×4 Dirac matrices. This is because we need to accommodate both the spin and particle-antiparticle degrees of freedom. A common choice is:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{741}$$

where σ^i are the Pauli matrices and I is the 2×2 identity matrix.

(d)

The solution is not unique. Another set of 4×4 matrices that satisfy the same relations can be obtained by multiplying the Dirac matrices by a unitary transformation. For example, we can choose:

$$\alpha^i = U\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}U^\dagger, \quad \beta = U\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}U^\dagger, \tag{742}$$

where U is any 4×4 unitary matrix.
Remark: This representation is known as the Dirac representation. Other representations, such as the Weyl or Majorana representations, can also be used to express the Dirac matrices. □

HW6 Due to December 4 11:59 PM

Question 1

Problem 36.3

(a) Prove the *Fierz identities*

$$(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{36.58}$$

$$(\chi_1^\dagger\overline{\sigma^\mu}\chi_2)(\chi_3^\dagger\overline{\sigma_\mu}\chi_4) = (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) \tag{36.59}$$

(b) Define the Dirac fields

$$\Psi_1 \equiv \begin{pmatrix}\chi_i \\ \xi_i^\dagger\end{pmatrix}, \quad \Psi_i^C \equiv \begin{pmatrix}\xi_i \\ \chi_i^\dagger\end{pmatrix} \tag{36.60}$$

Use eqs. (36.58) and (36.59) to prove the Dirac form of the Fierz identities,

$$(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) = -2(\overline{\Psi}_1 P_R\Psi_3^C)(\overline{\Psi}_4^C P_L\Psi_2) \tag{36.61}$$

$$(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) = (\overline{\Psi}_1\gamma^\mu P_L\Psi_4)(\overline{\Psi}_3\gamma_\mu P_L\Psi_2) \tag{36.62}$$

(c) By writing both sides out in terms of Weyl fields, show that

$$\overline{\Psi}_1\gamma^\mu P_R\Psi_2 = -\overline{\Psi}_2^C\gamma^\mu P_L\Psi_1^C \tag{36.63}$$

$$\overline{\Psi}_1 P_L\Psi_2 = \overline{\Psi}_2^C P_L\Psi_1^C \tag{36.64}$$

$$\overline{\Psi}_1 P_R\Psi_2 = \overline{\Psi}_2^C P_R\Psi_1^C. \tag{36.65}$$

Combining equations (36.63–36.65) with equations (36.61–36.62) yields more useful forms of the Fierz identities.

Answer

(a)

We start from the left-hand side of equation (36.58):

$$(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) = (\chi_1^\dagger)_{\dot a}(\overline{\sigma}^\mu)^{\dot a b}(\chi_2)_b(\chi_3^\dagger)_{\dot c}(\overline{\sigma}_\mu)^{\dot c d}(\chi_4)_d \tag{743}$$

$$= (\chi_1^\dagger)_{\dot a}(\chi_2)_b(\chi_3^\dagger)_{\dot c}(\chi_4)_d(\overline{\sigma}^\mu)^{\dot a b}(\overline{\sigma}_\mu)^{\dot c d} \tag{744}$$

Using the identity in equations (35.4),(35.19)

$$(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} = -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \tag{35.4}$$
$$(\overline{\sigma}^\mu)^{\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}(\sigma_\mu)_{b\dot{b}} \tag{35.19}$$

we have

$$(\overline{\sigma}^\mu)^{\dot{a}b}(\overline{\sigma}_\mu)^{\dot{c}d} = \epsilon^{be}\epsilon^{\dot{a}\dot{f}}(\sigma^\mu)_{e\dot{f}}\epsilon^{dg}\epsilon^{\dot{c}\dot{h}}(\sigma_\mu)_{g\dot{h}} \tag{745}$$
$$= \epsilon^{be}\epsilon^{dg}\epsilon^{\dot{a}\dot{f}}\epsilon^{\dot{c}\dot{h}}(\sigma^\mu)_{e\dot{f}}(\sigma_\mu)_{g\dot{h}} \tag{746}$$
$$= -2\epsilon^{be}\epsilon^{dg}\epsilon^{\dot{a}\dot{f}}\epsilon^{\dot{c}\dot{h}}\epsilon_{eg}\epsilon_{\dot{f}\dot{h}} \tag{747}$$
$$= -2\epsilon^{be}\delta^d_e\epsilon^{\dot{a}\dot{f}}\delta^{\dot{c}}_{\dot{f}} \tag{748}$$
$$= -2\epsilon^{bd}\epsilon^{\dot{a}\dot{c}} \tag{749}$$

Substituting this back, we get

$$(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) = -2(\chi_1^\dagger)_{\dot{a}}(\chi_2)_b(\chi_3^\dagger)_{\dot{c}}(\chi_4)_d\epsilon^{bd}\epsilon^{\dot{a}\dot{c}} \tag{750}$$
$$= -2(\chi_1^\dagger)_{\dot{a}}(\chi_3^\dagger)_{\dot{c}}\epsilon^{\dot{a}\dot{c}}(\chi_2)_b(\chi_4)_d\epsilon^{bd} \tag{751}$$
$$= -2(\chi_1^\dagger)^{\dot{c}}(\chi_3^\dagger)_{\dot{c}}(\chi_2)_d(\chi_4)^d \tag{752}$$
$$= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4)\times(-1)(-1) \tag{753}$$
$$= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{754}$$

This proves equation (36.58). Similarly, we can prove equation (36.59):

$$(\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{755}$$
$$= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_4\chi_2) \tag{756}$$
$$= (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) \tag{757}$$

This proves equation (36.59).

(b)

First $\Psi = \begin{pmatrix} \chi_a \\ (\xi^\dagger)^{\dot{a}} \end{pmatrix}$, so $\overline{\Psi} = (\xi^a, (\chi^\dagger)_{\dot{a}})$. Also, $P_L\Psi = \begin{pmatrix} \chi_a \\ 0 \end{pmatrix}$ and $P_R\Psi = \begin{pmatrix} 0 \\ (\xi^\dagger)^{\dot{a}} \end{pmatrix}$. Thus,

$$\overline{\Psi}_1\gamma^\mu P_L\Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}})\begin{pmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \overline{\sigma}^{\mu\dot{a}b} & 0 \end{pmatrix}\begin{pmatrix} (\chi_2)_b \\ 0 \end{pmatrix} \tag{758}$$
$$= \xi_1^a\sigma^\mu_{a\dot{b}}(\chi_2)_b \tag{759}$$
$$= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2) \tag{760}$$

Similarly,

$$\overline{\Psi}_3\gamma_\mu P_L\Psi_4 = (\chi_3^\dagger\overline{\sigma}_\mu\chi_4) \tag{761}$$

Therefore,

$$\begin{aligned}(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) && (762)\\ &= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) && \text{(from (a))}\end{aligned}$$

Now, for the right-hand side of equation (36.61):

$$\begin{aligned}\overline{\Psi}_1 P_R\Psi_3^C &= (\xi_1^a, (\chi_1^\dagger)_{\dot a})\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}(\xi_3)_b\\ (\chi_3^\dagger)^{\dot b}\end{pmatrix} && (763)\\ &= (\chi_1^\dagger)_{\dot a}(\chi_3^\dagger)^{\dot a} && (764)\\ &= (\chi_1^\dagger\chi_3^\dagger) && (765)\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\Psi}_4^C P_L\Psi_2 &= ((\xi_4^a, (\chi_4^\dagger)_{\dot a})\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}(\chi_2)_b\\ (\xi_2^\dagger)^{\dot b}\end{pmatrix} && (766)\\ &= (\xi_4^a)(\chi_2)_a && (767)\\ &= (\chi_4\chi_2) = (\chi_2\chi_4) && (768)\end{aligned}$$

Thus,

$$-2(\overline{\Psi}_1 P_R\Psi_3^C)(\overline{\Psi}_4^C P_L\Psi_2) = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4) \tag{769}$$

This proves equation (36.61). Similarly, we can prove equation (36.62):

$$\begin{aligned}(\overline{\Psi}_1\gamma^\mu P_L\Psi_2)(\overline{\Psi}_3\gamma_\mu P_L\Psi_4) &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_2)(\chi_3^\dagger\overline{\sigma}_\mu\chi_4) && (770)\\ &= (\chi_1^\dagger\overline{\sigma}^\mu\chi_4)(\chi_3^\dagger\overline{\sigma}_\mu\chi_2) && \text{(from (a))}\\ &= (\overline{\Psi}_1\gamma^\mu P_L\Psi_4)(\overline{\Psi}_3\gamma_\mu P_L\Psi_2) && (771)\end{aligned}$$

This proves equation (36.62).

(c)

First, we compute the left-hand side of equation (36.63):

$$\begin{aligned}\overline{\Psi}_1\gamma^\mu P_R\Psi_2 &= (\xi_1^a, (\chi_1^\dagger)_{\dot a})\begin{pmatrix}0 & \sigma^\mu_{a\dot b}\\ \overline{\sigma}^{\mu\dot a b} & 0\end{pmatrix}\begin{pmatrix}0\\ (\xi_2^\dagger)^{\dot b}\end{pmatrix} && (772)\\ &= \xi_1^a\sigma^\mu_{a\dot b}(\xi_2^\dagger)^{\dot b} && (773)\end{aligned}$$

Next, we compute the right-hand side of equation (36.63):

$$\overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \overline{\sigma}^{\mu\dot{a}b} & 0 \end{pmatrix} \begin{pmatrix} (\xi_1)_b \\ 0 \end{pmatrix} \tag{774}$$

$$= (\xi_2^\dagger)_{\dot{a}} \overline{\sigma}^{\mu\dot{a}b} (\xi_1)_b \tag{775}$$

Using the identity

$$(\overline{\sigma}^\mu)^{\dot{a}a} \equiv \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} (\sigma_\mu)_{b\dot{b}} \tag{35.19}$$

we have

$$(\xi_2^\dagger)_{\dot{a}} \overline{\sigma}^{\mu\dot{a}b} (\xi_1)_b = (\xi_2^\dagger)_{\dot{a}} \epsilon^{bc} \epsilon^{\dot{a}\dot{b}} (\sigma_\mu)_{c\dot{b}} (\xi_1)_b \tag{776}$$

$$= -\xi_1^c \sigma^\mu_{c\dot{b}} (\xi_2^\dagger)^{\dot{b}} \tag{777}$$

$$= -\xi_1^a \sigma^\mu_{a\dot{b}} (\xi_2^\dagger)^{\dot{b}} \tag{778}$$

$$= \overline{\Psi}_2^C \gamma^\mu P_L \Psi_1^C = -\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 \tag{779}$$

This proves equation (36.63). Similarly, we can prove equations (36.64) and (36.65):

$$LHS = \overline{\Psi}_1 P_L \Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\chi_2)_a \\ (\xi_2^\dagger)^{\dot{a}} \end{pmatrix} \tag{780}$$

$$= \xi_1^a (\chi_2)_a = (\xi_1 \chi_2) \tag{781}$$

$$RHS = \overline{\Psi}_2^C P_L \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\xi_1)_a \\ (\chi_1^\dagger)^{\dot{a}} \end{pmatrix} \tag{782}$$

$$= (\chi_2)_a \xi_1^a = (\chi_2 \xi_1) = (\xi_1 \chi_2) \tag{783}$$

This proves equation (36.64).

$$LHS = \overline{\Psi}_1 P_R \Psi_2 = (\xi_1^a, (\chi_1^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\chi_2)_a \\ (\xi_2^\dagger)^{\dot{a}} \end{pmatrix} \tag{784}$$

$$= (\chi_1^\dagger)_{\dot{a}} (\xi_2^\dagger)^{\dot{a}} = (\chi_1^\dagger \xi_2^\dagger) \tag{785}$$

$$RHS = \overline{\Psi}_2^C P_R \Psi_1^C = (\chi_2^a, (\xi_2^\dagger)_{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\xi_1)_a \\ (\chi_1^\dagger)^{\dot{a}} \end{pmatrix} \tag{786}$$

$$= (\xi_2^\dagger)_{\dot{a}} (\chi_1^\dagger)^{\dot{a}} = (\xi_2^\dagger \chi_1^\dagger) = (\chi_1^\dagger \xi_2^\dagger) \tag{787}$$

This proves equation (36.65). $\square$

Question 2

38.1

Use equation (38.12) to compute $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ explicitly. Hint: Show that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1, and that, for any matrix A with eigenvalues ± 1, $\exp(cA) = \cosh(c) + A\sinh(c)$, where c is an arbitrary complex number.

Extra question: What is the expression in the large energy limit $E_{\mathbf{p}} \gg m$? Please write down the result.

$$u_s(\mathbf{p}) = \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)u_s(\mathbf{0}), \quad v_s(\mathbf{p}) = \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)v_s(\mathbf{0}) \tag{38.12}$$

Answer

We start by showing that the matrix $2i\hat{\mathbf{p}}\cdot\mathbf{K}$ has eigenvalues ± 1. The boost generators $\mathbf{K}$ in the Dirac representation are given by

$$K^j = \frac{i}{2}\gamma^j\gamma^0 = \frac{i}{2}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \tag{788}$$

Thus,

$$2i\hat{\mathbf{p}}\cdot\mathbf{K} = 2i\sum_{j=1}^{3}\hat{p}_j K^j = 2i\sum_{j=1}^{3}\hat{p}_j\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} = -\begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & -\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}, \tag{789}$$

where $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \hat{p}_3 & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 \end{pmatrix}$. Now we want to prove $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ has eigenvalues ± 1, since $\hat{\mathbf{p}}$ is a unit vector. The characteristic polynomial of $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ is given by

$$\det(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} - \lambda I) = \det\begin{pmatrix} \hat{p}_3 - \lambda & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 - \lambda \end{pmatrix} = (\hat{p}_3 - \lambda)(-\hat{p}_3 - \lambda) - (\hat{p}_1 - i\hat{p}_2)(\hat{p}_1 + i\hat{p}_2) \tag{790}$$

Simplifying this, we get

$$\det(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} - \lambda I) = \lambda^2 - (\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) = \lambda^2 - 1 \tag{791}$$

Setting the determinant to zero, we find the eigenvalues:

$$\lambda^2 - 1 = 0 \implies \lambda^2 = 1 \implies \lambda = \pm 1 \tag{792}$$

Thus, $\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}$ has eigenvalues ± 1. Consequently, the matrix $2i\hat{\mathbf{p}} \cdot \mathbf{K}$ has eigenvalues ± 1 as well. Now we can use the identity for any matrix A with eigenvalues ± 1:

$$\exp(cA) = \cosh(c) + A \sinh(c) \tag{793}$$

where c is an arbitrary complex number. Applying this to our case with $A = 2i\hat{\mathbf{p}} \cdot \mathbf{K}$ and $c = \eta/2$, we have

$$\begin{aligned}
\exp(i\eta\hat{\mathbf{p}} \cdot \mathbf{K}) &= \cosh\left(\frac{\eta}{2}\right) + (2i\hat{\mathbf{p}} \cdot \mathbf{K}) \sinh\left(\frac{\eta}{2}\right) && (794)\\
&= \cosh\left(\frac{\eta}{2}\right) - \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & 0 \\ 0 & -\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix} \sinh\left(\frac{\eta}{2}\right) && (795)\\
&= \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \sinh\left(\frac{\eta}{2}\right) & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \sinh\left(\frac{\eta}{2}\right) \end{pmatrix} && (796)
\end{aligned}$$

Now, we can compute $u_s(\mathbf{p})$,

$$\begin{aligned}
u_s ame(\mathbf{p}) &= \exp\left(i\eta\hat{\mathbf{p}} \cdot \mathbf{K}\right) u_s(\mathbf{0}) && (797)\\
&= \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \sinh\left(\frac{\eta}{2}\right) & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \sinh\left(\frac{\eta}{2}\right) \end{pmatrix} \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} && (798)\\
&= \sqrt{m} \begin{pmatrix} \left(\cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \sinh\left(\frac{\eta}{2}\right)\right) \xi_s \\ \left(\cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \sinh\left(\frac{\eta}{2}\right)\right) \xi_s \end{pmatrix}, && (799)
\end{aligned}$$

where $\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence, we have

$$\begin{aligned}
u_+(\mathbf{p}) &= \sqrt{m} \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{p}_3 \sinh\left(\frac{\eta}{2}\right) \\ -(\hat{p}_1 + i\hat{p}_2) \sinh\left(\frac{\eta}{2}\right) \\ \cosh\left(\frac{\eta}{2}\right) + \hat{p}_3 \sinh\left(\frac{\eta}{2}\right) \\ (\hat{p}_1 + i\hat{p}_2) \sinh\left(\frac{\eta}{2}\right) \end{pmatrix} && (800)\\
u_-(\mathbf{p}) &= \sqrt{m} \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2) \sinh\left(\frac{\eta}{2}\right) \\ \cosh\left(\frac{\eta}{2}\right) + \hat{p}_3 \sinh\left(\frac{\eta}{2}\right) \\ (\hat{p}_1 - i\hat{p}_2) \sinh\left(\frac{\eta}{2}\right) \\ \cosh\left(\frac{\eta}{2}\right) - \hat{p}_3 \sinh\left(\frac{\eta}{2}\right) \end{pmatrix} && (801)
\end{aligned}$$

Similarly, we can compute $v_s(\mathbf{p})$,

$$v_s(\mathbf{p}) = \exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right)v_s(\mathbf{0}) \tag{802}$$

$$= \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right) & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right) \end{pmatrix} \begin{pmatrix} \sqrt{m}\,\xi_s \\ -\sqrt{m}\,\xi_s \end{pmatrix} \tag{803}$$

$$= \sqrt{m}\begin{pmatrix} \left(\cosh\left(\frac{\eta}{2}\right) - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\right)\xi_s \\ -\left(\cosh\left(\frac{\eta}{2}\right) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\eta}{2}\right)\right)\xi_s \end{pmatrix}, \tag{804}$$

where $\xi_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi_- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Hence, we have

$$v_+(\mathbf{p}) = \sqrt{m}\begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \\ \cosh\left(\frac{\eta}{2}\right) + \hat{p}_3\sinh\left(\frac{\eta}{2}\right) \\ (\hat{p}_1 - i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \\ -\left(\cosh\left(\frac{\eta}{2}\right) - \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\right) \end{pmatrix} \tag{805}$$

$$v_-(\mathbf{p}) = \sqrt{m}\begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) - \hat{p}_3\sinh\left(\frac{\eta}{2}\right) \\ -(\hat{p}_1 + i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \\ -\left(\cosh\left(\frac{\eta}{2}\right) + \hat{p}_3\sinh\left(\frac{\eta}{2}\right)\right) \\ -(\hat{p}_1 + i\hat{p}_2)\sinh\left(\frac{\eta}{2}\right) \end{pmatrix} \tag{806}$$

Now, we can express $\cosh\left(\frac{\eta}{2}\right)$ and $\sinh\left(\frac{\eta}{2}\right)$ in terms of energy $E_{\mathbf{p}}$ and mass m. We know that

$$\cosh(\eta) = \frac{E_{\mathbf{p}}}{m}, \quad \sinh(\eta) = \frac{|\mathbf{p}|}{m} \tag{807}$$

Using the half-angle formulas for hyperbolic functions, we have

$$\cosh\left(\frac{\eta}{2}\right) = \sqrt{\frac{E_{\mathbf{p}} + m}{2m}}, \quad \sinh\left(\frac{\eta}{2}\right) = \sqrt{\frac{E_{\mathbf{p}} - m}{2m}} \tag{808}$$

Substituting these back into the expressions for $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$, we get

$$u_+(\mathbf{p}) = \begin{pmatrix} \sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -(\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ (\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \end{pmatrix} \tag{809}$$

$$u_-(\mathbf{p}) = \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ (\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \end{pmatrix} \tag{810}$$

$$v_+(\mathbf{p}) = \begin{pmatrix} -(\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ \sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ (\hat{p}_1 - i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -\left(\sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}}\right) \end{pmatrix} \tag{811}$$

$$v_-(\mathbf{p}) = \begin{pmatrix} \sqrt{\frac{E_\mathbf{p}+m}{2}} - \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -(\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \\ -\left(\sqrt{\frac{E_\mathbf{p}+m}{2}} + \hat{p}_3\sqrt{\frac{E_\mathbf{p}-m}{2}}\right) \\ -(\hat{p}_1 + i\hat{p}_2)\sqrt{\frac{E_\mathbf{p}-m}{2}} \end{pmatrix} \tag{812}$$

In the large energy limit $E_\mathbf{p} \gg m$, we have

$$\sqrt{\frac{E_\mathbf{p}+m}{2}} \approx \sqrt{\frac{E_\mathbf{p}}{2}}, \quad \sqrt{\frac{E_\mathbf{p}-m}{2}} \approx \sqrt{\frac{E_\mathbf{p}}{2}} \tag{813}$$

Thus, the expressions for $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ simplify to

$$u_+(\mathbf{p}) \approx \sqrt{\frac{E_{\mathbf{p}}}{2}} \begin{pmatrix} 1-\hat{p}_3 \\ -(\hat{p}_1+i\hat{p}_2) \\ 1+\hat{p}_3 \\ (\hat{p}_1+i\hat{p}_2) \end{pmatrix} \tag{814}$$

$$u_-(\mathbf{p}) \approx \sqrt{\frac{E_{\mathbf{p}}}{2}} \begin{pmatrix} -(\hat{p}_1-i\hat{p}_2) \\ 1+\hat{p}_3 \\ (\hat{p}_1-i\hat{p}_2) \\ 1-\hat{p}_3 \end{pmatrix} \tag{815}$$

$$v_+(\mathbf{p}) \approx \sqrt{\frac{E_{\mathbf{p}}}{2}} \begin{pmatrix} -(\hat{p}_1-i\hat{p}_2) \\ 1+\hat{p}_3 \\ (\hat{p}_1-i\hat{p}_2) \\ -(1-\hat{p}_3) \end{pmatrix} \tag{816}$$

$$v_-(\mathbf{p}) \approx \sqrt{\frac{E_{\mathbf{p}}}{2}} \begin{pmatrix} 1-\hat{p}_3 \\ -(\hat{p}_1+i\hat{p}_2) \\ -(1+\hat{p}_3) \\ -(\hat{p}_1+i\hat{p}_2) \end{pmatrix} \tag{817}$$

$\square$

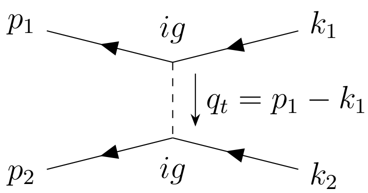
Question 3

45.2

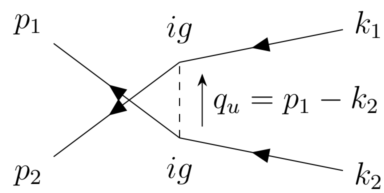
Use the Feynman rules to write down (at tree level) $i\mathcal{T}$ for the processes: $e^+e^+ \to e^+e^+$ and $\varphi\varphi \to e^+e^-\varphi$.
Remark: Do not write $\varphi\varphi \to e^+e^-$. Also, please draw Feynman diagrams when doing this problem. Remember the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + g\varphi\overline{\Psi}\Psi. \tag{818}$$

Answer

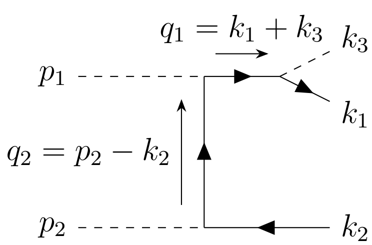


(a) t-channel diagram.

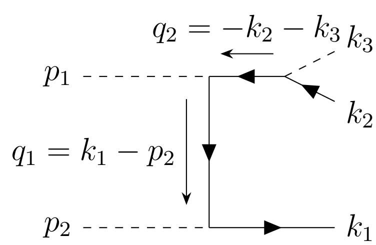


(b) u-channel diagram.

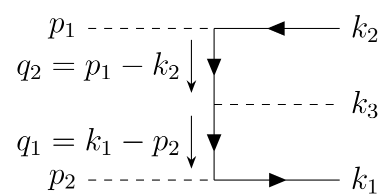
Figure 8: Feynman diagrams for $e^+e^+ \to e^+e^+$ at tree level (t and u channels).



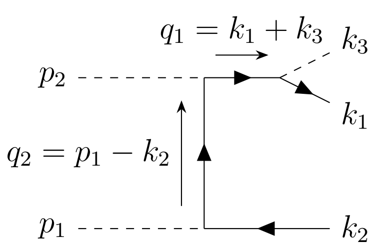
(a) Diagram 1 for $\varphi\varphi \to e^+e^-\varphi$.



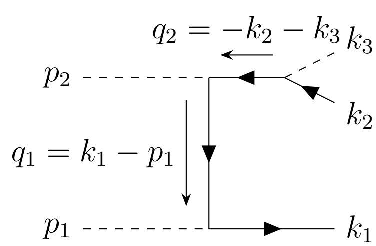
(b) Diagram 2 for $\varphi\varphi \to e^+e^-\varphi$.



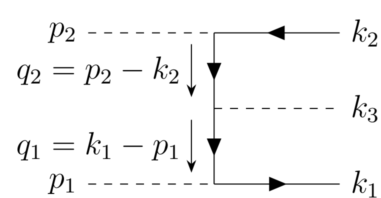
(c) Diagram 3 for $\varphi\varphi \to e^+e^-\varphi$.



(d) Diagram 1 for $\varphi\varphi \to e^+e^-\varphi$.



(e) Diagram 2 for $\varphi\varphi \to e^+e^-\varphi$.



(f) Diagram 3 for $\varphi\varphi \to e^+e^-\varphi$.

Figure 9: Two Feynman diagrams for $\varphi\varphi \to e^+e^-\varphi$ at tree level.

- **For the process** $e^+e^+ \to e^+e^+$**:**

The tree-level amplitude for the process $e^+e^+ \to e^+e^+$ consists of two diagrams: the t-channel and u-channel exchanges of a scalar particle. The total amplitude is given by the sum of the contributions from both channels. The amplitude for the t-channel diagram (Figure 8a) is

$$i\mathcal{T}_t = (ig)^2 \left[\overline{v}(p_2)v(k_2)\right] \frac{-i}{(p_1-k_1)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_1)v(k_1)\right]. \tag{819}$$

Similarly, the amplitude for the u-channel diagram (Figure 8b) is

$$i\mathcal{T}_u = (ig)^2 \left[\overline{v}(p_1)v(k_2)\right] \frac{-i}{(p_1-k_2)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_2)v(k_1)\right]. \tag{820}$$

Thus, the total amplitude for the process $e^+e^+ \to e^+e^+$ is

$$i\mathcal{T} = i\mathcal{T}_t - i\mathcal{T}_u \tag{821}$$

$$=(ig^2)\left[\overline{v}(p_2)v(k_2)\right] \frac{-i}{(p_1-k_1)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_1)v(k_1)\right] \tag{822}$$

$$-(ig^2)\left[\overline{v}(p_1)v(k_2)\right] \frac{-i}{(p_1-k_2)^2+m_\varphi^2-i\epsilon} \left[\overline{v}(p_2)v(k_1)\right]. \tag{823}$$

The minus sign arises due to the antisymmetry of the fermionic wavefunctions under exchange.

- **For the process $\varphi\varphi \to e^+e^-\varphi$:**

The tree-level amplitude for the process $\varphi\varphi \to e^+e^-\varphi$ consists of three diagrams, as shown in Figure 9. The total amplitude is given by the sum of the contributions from all three diagrams. The amplitude for Diagram 1 (Figure 9a) is

$$i\mathcal{T}_1 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{824}$$

where $q_1 = k_1 + k_3$ and $q_2 = p_2 - k_2$. The amplitude for Diagram 2 (Figure 9b) is

$$i\mathcal{T}_2 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{825}$$

where $q_1 = k_1 - p_2$ and $q_2 = -k_2 - k_3$. The amplitude for Diagram 3 (Figure 9c) is

$$i\mathcal{T}_3 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{826}$$

where $q_1 = k_1 - p_2$ and $q_2 = p_1 - k_2$. The amplitude for Diagram 4 (Figure 9d) is

$$i\mathcal{T}_4 = (ig)^3 \left[\overline{u}(k_1)\frac{-i(-\not{q}_1+m)}{q_1^2+m^2-i\epsilon}\frac{-i(-\not{q}_2+m)}{q_2^2+m^2-i\epsilon}v(k_2)\right], \tag{827}$$

where $q_1 = k_1 + k_3$ and $q_2 = p_1 - k_2$. The amplitude for Diagram 5 (Figure 9e) is

$$i\mathcal{T}_5 = (ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{q}_1 + m)}{q_1^2 + m^2 - i\epsilon} \frac{-i(-\not{q}_2 + m)}{q_2^2 + m^2 - i\epsilon} v(k_2) \right], \tag{828}$$

where $q_1 = k_1 - p_1$ and $q_2 = -k_2 - k_3$. The amplitude for Diagram 6 (Figure 9f) is

$$i\mathcal{T}_6 = (ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{q}_1 + m)}{q_1^2 + m^2 - i\epsilon} \frac{-i(-\not{q}_2 + m)}{q_2^2 + m^2 - i\epsilon} v(k_2) \right], \tag{829}$$

where $q_1 = k_1 - p_1$ and $q_2 = p_2 - k_2$. Thus, the total amplitude for the process $\varphi\varphi \to e^+e^-\varphi$ is

$$\begin{aligned}
i\mathcal{T} &= i\mathcal{T}_1 + i\mathcal{T}_2 + i\mathcal{T}_3 + i\mathcal{T}_4 + i\mathcal{T}_5 + i\mathcal{T}_6 && (830)\\
&=(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 - \not{k}_3 + m)}{(k_1 + k_3)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_2 + \not{k}_2 + m)}{(p_2 - k_2)^2 + m^2 - i\epsilon} v(k_2) \right] && (831)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_2 + m)}{(k_1 - p_2)^2 + m^2 - i\epsilon} \frac{-i(\not{k}_2 + \not{k}_3 + m)}{(-k_2 - k_3)^2 + m^2 - i\epsilon} v(k_2) \right] && (832)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_2 + m)}{(k_1 - p_2)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_1 + \not{k}_2 + m)}{(p_1 - k_2)^2 + m^2 - i\epsilon} v(k_2) \right] && (833)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 - \not{k}_3 + m)}{(k_1 + k_3)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_1 + \not{k}_2 + m)}{(p_1 - k_2)^2 + m^2 - i\epsilon} v(k_2) \right] && (834)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_1 + m)}{(k_1 - p_1)^2 + m^2 - i\epsilon} \frac{-i(\not{k}_2 + \not{k}_3 + m)}{(-k_2 - k_3)^2 + m^2 - i\epsilon} v(k_2) \right] && (835)\\
&+(ig)^3 \left[\overline{u}(k_1) \frac{-i(-\not{k}_1 + \not{p}_1 + m)}{(k_1 - p_1)^2 + m^2 - i\epsilon} \frac{-i(-\not{p}_2 + \not{k}_2 + m)}{(p_2 - k_2)^2 + m^2 - i\epsilon} v(k_2) \right]. && (836)
\end{aligned}$$

□

Question 4

48.2

Compute $\langle|\mathcal{T}|^2\rangle$ for $e^+e^- \to \varphi\varphi$. You should find that your result is the same as that for $e^-\varphi \to e^-\varphi$, but with $s \leftrightarrow t$, and an extra overall minus sign. This relationship is known as *crossing symmetry*. There is an overall minus sign for each fermion that is moved from the initial to the final state.

Remark: Please compute for $e^-\varphi \to e^-\varphi$, do not compute for $e^+e^- \to \varphi\varphi$. Please also draw Feynman diagrams. Remember the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + g\varphi\overline{\Psi}\Psi. \tag{837}$$

Answer

(a) s-channel diagram.

(b) u-channel diagram.

Figure 10: Feynman diagrams for $e^-\varphi \to e^-\varphi$ at tree level (s and u channels).

The tree-level amplitude for the process $e^-\varphi \to e^-\varphi$ consists of two diagrams: the s-channel and u-channel exchanges of a fermion. The total amplitude is given by the sum of the contributions from both channels. The amplitude for the s-channel diagram (Figure 10a) is

$$i\mathcal{T}_s = (ig)^2 \left[\overline{u}(p_2)\frac{-i(-\not{q}_s + m)}{q_s^2 + m^2 - i\epsilon}u(p_1)\right], \tag{838}$$

where $q_s = p_1 + k_1$. Similarly, the amplitude for the u-channel diagram (Figure 10b) is

$$i\mathcal{T}_u = (ig)^2 \left[\overline{u}(p_2)\frac{-i(-\not{q}_u + m)}{q_u^2 + m^2 - i\epsilon}u(p_1)\right], \tag{839}$$

where $q_u = p_1 - k_2$. Thus, the total amplitude for the process $e^-\varphi \to e^-\varphi$ is

$$\begin{aligned} i\mathcal{T} &= i\mathcal{T}_s + i\mathcal{T}_u & (840)\\ &=(ig^2)\left[\overline{u}(p_2)\frac{-i(-\not{q}_s + m)}{q_s^2 + m^2 - i\epsilon}u(p_1)\right] & (841)\\ &+(ig^2)\left[\overline{u}(p_2)\frac{-i(-\not{q}_u + m)}{q_u^2 + m^2 - i\epsilon}u(p_1)\right]. & (842)\end{aligned}$$

To compute $\langle|\mathcal{T}|^2\rangle$, we average over initial spins and sum over final spins:

$$\langle|\mathcal{T}|^2\rangle = \frac{1}{2}\sum_{\text{spins}}|\mathcal{T}|^2. \tag{843}$$

We define $\mathcal{A}$ as

$$\begin{aligned}
\mathcal{A} &= \left[\overline{u}_{s_2}(p_2)\frac{(-\not{q}_s + m)}{q_s^2 + m^2}u_{s_1}(p_1)\right] + \left[\overline{u}_{s_2}(p_2)\frac{(-\not{q}_u + m)}{q_u^2 + m^2}u_{s_1}(p_1)\right] && (844)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{q}_s + m}{q_s^2 + m^2} + \frac{-\not{q}_u + m}{q_u^2 + m^2}\right)u_{s_1}(p_1)\right] && (845)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-(\not{p}_1 + \not{k}_1) + m}{(p_1 + k_1)^2 + m^2} + \frac{-(\not{p}_1 - \not{k}_2) + m}{(p_1 - k_2)^2 + m^2}\right)u_{s_1}(p_1)\right] && (846)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{p}_1 - \not{k}_1 + m}{-s + m^2} + \frac{-\not{p}_1 + \not{k}_2 + m}{-u + m^2}\right)u_{s_1}(p_1)\right] && (847)\\
&= \left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{k}_1 + 2m}{-s + m^2} + \frac{+\not{k}_2 + 2m}{-u + m^2}\right)u_{s_1}(p_1)\right] && (848)
\end{aligned}$$

and its Hermitian conjugate $\mathcal{A}^\dagger$ as

$$\mathcal{A}^\dagger = \left[\overline{u}_{s_1}(p_1)\frac{-\not{k}_1 + 2m}{-s + m^2} + \frac{+\not{k}_2 + 2m}{-u + m^2}u_{s_2}(p_2)\right] \tag{849}$$

where s_1 and s_2 are the spin indices for the initial and final electrons, respectively, and we have used the Mandelstam variables:

$$s = -(p_1 + k_1)^2, \quad u = -(p_1 - k_2)^2. \tag{850}$$

Also, we apply the identity:

$$(\not{p} + m)u_s(p) = 0 \implies -\not{p}u_s(p) = +mu_s(p). \tag{851}$$

Then, we have

$$\langle|\mathcal{T}|^2\rangle = \frac{g^4}{2}\sum_{s_1,s_2}AA^\dagger. \tag{852}$$

Hence,

$$\sum_{s_1,s_2}\mathcal{A}\mathcal{A}^\dagger = \sum_{s_1,s_2}\left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_1}(p_1)\right] \tag{853}$$

$$\times\left[\overline{u}_{s_1}(p_1)\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_2}(p_2)\right] \tag{854}$$

$$=\sum_{s_1,s_2}\left[\overline{u}_{s_2}(p_2)\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_1}(p_1)\overline{u}_{s_1}(p_1)\right. \tag{855}$$

$$\left.\times\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_2}(p_2)\right] \tag{856}$$

$$=\mathrm{Tr}\left[\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_1}(p_1)\overline{u}_{s_1}(p_1)\right. \tag{857}$$

$$\left.\times\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)u_{s_2}(p_2)\overline{u}_{s_2}(p_2)\right] \tag{858}$$

$$=\mathrm{Tr}\left[\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)(-\not{p}_1+m)\right. \tag{859}$$

$$\left.\times\left(\frac{-\not{k}_1+2m}{-s+m^2}+\frac{+\not{k}_2+2m}{-u+m^2}\right)(-\not{p}_2+m)\right] \tag{860}$$

where we have used the completeness relation for spinors:

$$\sum_s u_s(p)\overline{u}_s(p) = -\not{p}+m. \tag{861}$$

Since only even numbers of gamma matrices contribute to the trace, we find

$$\sum_{s_1,s_2}\mathcal{A}\mathcal{A}^\dagger = \frac{1}{(-s+m^2)^2}\mathrm{Tr}\Big[(-\not{k}_1+2m)(-\not{p}_1+m)(-\not{k}_1+2m)(-\not{p}_2+m)\Big] \tag{862}$$

$$+\frac{1}{(-u+m^2)^2}\mathrm{Tr}\Big[(+\not{k}_2+2m)(-\not{p}_1+m)(+\not{k}_2+2m)(-\not{p}_2+m)\Big] \tag{863}$$

$$+\frac{1}{(-s+m^2)(-u+m^2)}\mathrm{Tr}\Big[(-\not{k}_1+2m)(-\not{p}_1+m)(+\not{k}_2+2m)(-\not{p}_2+m)\Big] \tag{864}$$

$$+\frac{1}{(-u+m^2)(-s+m^2)}\mathrm{Tr}\Big[(+\not{k}_2+2m)(-\not{p}_1+m)(-\not{k}_1+2m)(-\not{p}_2+m)\Big] \tag{865}$$

$$=\frac{1}{(s-m^2)^2}\mathrm{Tr}\Big[(\not{k}_1-2m)(\not{p}_1-m)(\not{k}_1-2m)(\not{p}_2-m)\Big] \tag{866}$$

$$+\frac{1}{(u-m^2)^2}\mathrm{Tr}\Big[(\not{k}_2+2m)(\not{p}_1-m)(\not{k}_2+2m)(\not{p}_2-m)\Big] \tag{867}$$

$$+\frac{-1}{(s-m^2)(u-m^2)}\mathrm{Tr}\Big[(\not{k}_1-2m)(\not{p}_1-m)(+\not{k}_2+2m)(\not{p}_2-m)\Big] \tag{868}$$

$$+\frac{-1}{(u-m^2)(s-m^2)}\mathrm{Tr}\Big[(\not{k}_2+2m)(\not{p}_1-m)(\not{k}_1-2m)(\not{p}_2-m)\Big]. \tag{869}$$

Now, we compute each trace term separately:

$$\text{First term: } \mathrm{Tr}\Big[(\not{k}_1 - 2m)(\not{p}_1 - m)(\not{k}_1 - 2m)(\not{p}_2 - m)\Big] \tag{870}$$

$$= \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_1\not{k}_1 + 4m^2\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{k}_1 + 4m^4\Big] \tag{871}$$

$$\text{Second term: } \mathrm{Tr}\Big[(\not{k}_2 + 2m)(\not{p}_1 - m)(\not{k}_2 + 2m)(\not{p}_2 - m)\Big] \tag{872}$$

$$= \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_2\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_2\not{k}_2 - 4m^2\not{k}_2\not{p}_2 - 4m^2\not{p}_1\not{k}_2 + 4m^4\Big] \tag{873}$$

$$\text{Third term: } \mathrm{Tr}\Big[(\not{k}_1 - 2m)(\not{p}_1 - m)(\not{k}_2 + 2m)(\not{p}_2 - m)\Big] \tag{874}$$

$$= \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_2\not{p}_2 + m^2\not{k}_1\not{k}_2 - 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) - 4m^2\not{p}_1\not{p}_2 - 4m^4\Big] \tag{875}$$

$$\text{Fourth term: } \mathrm{Tr}\Big[(\not{k}_2 + 2m)(\not{p}_1 - m)(\not{k}_1 - 2m)(\not{p}_2 - m)\Big] \tag{876}$$

$$= \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_1\not{p}_2 + m^2\not{k}_2\not{k}_1 - 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) - 4m^2\not{p}_1\not{p}_2 - 4m^4\Big]. \tag{877}$$

Using the trace identities:

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(ad)(bc) - (ac)(bd) + (ab)(cd)], \quad \mathrm{Tr}[\not{a}\not{b}] = -4(ab), \quad \mathrm{Tr}[\mathcal{I}] = 4, \tag{878}$$

and the on-shell conditions $p_1^2 = p_2^2 = -m^2$ and $k_1^2 = k_2^2 = -m_\varphi^2$, we find

$$\text{First term: } \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_1\not{k}_1 + 4m^2\not{k}_1\not{p}_2 + 4m^2\not{p}_1\not{k}_1 + 4m^4\Big] \tag{879}$$

$$=4\Big[(k_1p_2)(p_1k_1) - (k_1k_1)(p_1p_2) + (k_1p_1)(k_1p_2) - 4m^2(p_1p_2) \tag{880}$$

$$-m^2(k_1k_1) - 4m^2(k_1p_2) - 4m^2(p_1k_1) + 4m^2\Big] \tag{881}$$

$$=4\Big[2(k_1p_2)(p_1k_1) + (m_\varphi^2 - 4m^2)(p_1p_2) + m^2m_\varphi^2 - 4m^2(k_1p_2) - 4m^2(p_1k_1) + 4m^4\Big] \tag{882}$$

$$\text{Second term: } \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_2\not{p}_2 + 4m^2\not{p}_1\not{p}_2 + m^2\not{k}_2\not{k}_2 - 4m^2\not{k}_2\not{p}_2 - 4m^2\not{p}_1\not{k}_2 + 4m^4\Big] \tag{883}$$

$$=4\Big[(k_2p_2)(p_1k_2) - (k_2k_2)(p_1p_2) + (k_2p_1)(k_2p_2) - 4m^2(p_1p_2) \tag{884}$$

$$-m^2(k_2k_2) + 4m^2(k_2p_2) + 4m^2(p_1k_2) + 4m^2\Big] \tag{885}$$

$$=4\Big[2(k_2p_2)(p_1k_2) + (m_\varphi^2 - 4m^2)(p_1p_2) + m^2m_\varphi^2 + 4m^2(k_2p_2) + 4m^2(p_1k_2) + 4m^4\Big] \tag{886}$$

$$\text{Third term: } \mathrm{Tr}\Big[\not{k}_1\not{p}_1\not{k}_2\not{p}_2 + m^2\not{k}_1\not{k}_2 - 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) - 4m^2\not{p}_1\not{p}_2 - 4m^4\Big] \tag{887}$$

$$=4\Big[(k_1p_2)(p_1k_2) - (k_1k_2)(p_1p_2) + (k_1p_1)(k_2p_2) - m^2(k_1k_2) + 2m^2(k_1(p_1 + p_2)) \tag{888}$$

$$- 2m^2(k_2(p_1 + p_2)) + 4m^2(p_1p_2) - 4m^4\Big] \tag{889}$$

$$=4\Big[(k_1p_2)(p_1k_2) - (k_1k_2)(p_1p_2) + (k_1p_1)(k_2p_2) - m^2(k_1k_2) \tag{890}$$

$$+2m^2(k_1p_1 + k_1p_2 - k_2p_1 - k_2p_2) - 4m^2(p_1p_2) - 4m^4\Big] \tag{891}$$

$$\text{Fourth term: } \mathrm{Tr}\Big[\not{k}_2\not{p}_1\not{k}_1\not{p}_2 + m^2\not{k}_2\not{k}_1 - 2m^2\not{k}_2(\not{p}_1 + \not{p}_2) + 2m^2\not{k}_1(\not{p}_1 + \not{p}_2) + 4m^2\not{p}_1\not{p}_2 - 4m^4\Big] \tag{892}$$

$$=4\Big[(k_2p_2)(p_1k_1) - (k_2k_1)(p_1p_2) + (k_2p_1)(k_1p_2) - m^2(k_2k_1) + 2m^2(k_2(p_1 + p_2)) \tag{893}$$

$$- 2m^2(k_1(p_1 + p_2)) + 4m^2(p_1p_2) - 4m^4\Big] \tag{894}$$

$$=4\Big[(k_2p_2)(p_1k_1) - (k_2k_1)(p_1p_2) + (k_2p_1)(k_1p_2) - m^2(k_2k_1) \tag{895}$$

$$+2m^2(k_2p_1 + k_2p_2 - k_1p_1 - k_1p_2) + 4m^2(p_1p_2) - 4m^4\Big]. \tag{896}$$

We can express the dot products in terms of the Mandelstam variables:

$$s = -(p_1 + k_1)^2 = -(p_2 + k_2)^2 = m^2 + m_\varphi^2 - 2(p_1k_1) = m^2 + m_\varphi^2 - 2(p_2k_2), \tag{897}$$

$$u = -(p_1 - k_2)^2 = -(p_2 - k_1)^2 = m^2 + m_\varphi^2 + 2(k_2p_1) = m^2 + m_\varphi^2 + 2(k_1p_2), \tag{898}$$

$$t = -(k_1 - k_2)^2 = -(p_1 - p_2)^2 = 2m^2 + 2(p_1p_2) = 2m_\varphi^2 + 2(k_1k_2). \tag{899}$$

Thus, by $s+t+u=2m^2+2m_\varphi^2$, we have

$$(p_1k_1)=\frac{m^2+m_\varphi^2-s}{2}, \tag{900}$$
$$(p_2k_2)=\frac{m^2+m_\varphi^2-s}{2}, \tag{901}$$
$$(k_2p_1)=\frac{u-m^2-m_\varphi^2}{2}, \tag{902}$$
$$(k_1p_2)=\frac{u-m^2-m_\varphi^2}{2}, \tag{903}$$
$$(p_1p_2)=\frac{t-2m^2}{2}=\frac{-(s+u)+2m_\varphi^2}{2}, \tag{904}$$
$$(k_1k_2)=\frac{t-2m_\varphi^2}{2}=\frac{-(s+u)+2m^2}{2}. \tag{905}$$

Substituting these expressions back into the sum, we find (by Mathematica):

$$\sum_{s_1,s_2}\mathcal{A}\mathcal{A}^\dagger=2\times\left[\frac{7m^4+m^2\left(-8m_\varphi^2+9s+u\right)+m_\varphi^4-su}{\left(m^2-s\right)^2}\right. \tag{906}$$
$$+\frac{7m^4+m^2\left(-8m_\varphi^2+s+9u\right)+m_\varphi^4-su}{\left(m^2-u\right)^2} \tag{907}$$
$$\left.+\frac{2\left(9m^4+m^2\left(3(s+u)-8m_\varphi^2\right)-m_\varphi^4+su\right)}{\left(m^2-s\right)\left(m^2-u\right)}\right]. \tag{908}$$

Therefore, the final result for $\langle|\mathcal{T}|^2\rangle$ is

$$\langle|\mathcal{T}|^2\rangle=\frac{g^4}{2}\sum_{s_1,s_2}\mathcal{A}\mathcal{A}^\dagger \tag{909}$$
$$=g^4\left[\frac{7m^4+m^2\left(-8m_\varphi^2+9s+u\right)+m_\varphi^4-su}{\left(m^2-s\right)^2}\right. \tag{910}$$
$$+\frac{7m^4+m^2\left(-8m_\varphi^2+s+9u\right)+m_\varphi^4-su}{\left(m^2-u\right)^2} \tag{911}$$
$$\left.+\frac{2\left(9m^4+m^2\left(3(s+u)-8m_\varphi^2\right)-m_\varphi^4+su\right)}{\left(m^2-s\right)\left(m^2-u\right)}\right]. \tag{912}$$

$\square$

Final Due to December 16 11:59 PM

Question 1

Problem 48.5

The charged pion π^- is represented by a complex scalar field φ, the *muon* μ^- by a Dirac field $\mathcal{M}$, and the *muon neutrino* ν_μ by a spin-projected Dirac field $P_L\mathcal{N}$, where $P_L = \frac{1}{2}(1-\gamma_5)$. The charged pion can decay to a muon and a muon antineutrino via the interaction

$$\mathcal{L}_1 = 2c_1 G_F f_\pi \partial_\mu \varphi \overline{\mathcal{M}} \gamma^\mu P_L \mathcal{N} + h.c, \tag{913}$$

where c_1 is the consine of the *Cabibbo angle*, G_F is the *Fermi constant*, and f_π is the *pion decay constant*.

(a) Compute the charged pion decay rate Γ.

(b) The charged pion mass is $m_\pi = 139.6$ MeV, the muon mass is $m_\mu = 105.7$ MeV, and the muon neutrino mass is massless. The Fermi constant is $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, and the cosine of the Cabibbo angle is measured in nuclear beta decays to be $c_1 = 0.974$. The measured value of the charged pion life time is $\tau = 2.6033 \times 10^{-8}$ s. Determine the value of f_π in MeV. Your result is too large by 0.8%, due to neglect of electromagnetic loop corrections.

(c) The previous parts assume π^- always decay into $\mu^-\overline{\nu}_\mu$, but actually π^- can also decay into $e^-\overline{\nu}_e$. The charged pion, electron by a Dirac field $\mathcal{M}_e$, and the electron neutrino by a spin-projected Dirac field $P_L\mathcal{N}_e$ have the form of interaction

$$\mathcal{L}_2 = 2c_2 G_F f_\pi \partial_\mu \varphi \overline{\mathcal{M}}_e \gamma^\mu P_L \mathcal{N}_e + h.c. \tag{914}$$

Given the decay branching ratio of $\pi^- \to e^-\overline{\nu}_e$ is 1.230×10^{-4}, the decay branching ratio of $\pi^- \to \mu^-\overline{\nu}_\mu$ is 99.9877%. Find the value of c_2. For example, the electronic decay branching ratio is

$$\mathrm{Br}(\pi^- \to e^-\overline{\nu}_e) = \frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu) + \Gamma(\pi^- \to e^-\overline{\nu}_e)}. \tag{915}$$

The coupling of pion-electron is similar with the coupling of pion-muon, why pion favoring decay into muon instead of electron? ($m_e = 0.511$ MeV.)

Answer

(a)

First, we analyze the decay $\pi^- \to \mu^-\overline{\nu}_\mu$. The Feynman diagram is shown in Fig. 11a. Now, we can write

(a) π^- decay diagram.

(b) π^- decay diagram.

Figure 11: Feynman diagram for π^- decay into (a) muon and muon antineutrino; (b) electron and electron antineutrino.

down the amplitude:

$$i\mathcal{T} = 2ic_1 G_F f_\pi k^\mu \overline{u}_{s_1}(p_1)\gamma_\mu P_L v_{s_2}(p_2) \tag{916}$$

$$= 2ic_1 G_F f_\pi \overline{u}_{s_1}(p_1)\not{k} P_L v_{s_2}(p_2) \tag{917}$$

$$= ic_1 G_F f_\pi \overline{u}_{s_1}(p_1)\not{k}(1-\gamma_5) v_{s_2}(p_2), \tag{918}$$

where k^μ, p_1^μ, and p_2^μ are the four-momenta of π^-, μ^-, and $\overline{\nu}_\mu$, respectively. s_1 and s_2 are the spin indices of μ^- and $\overline{\nu}_\mu$. We can write $k^\mu = p_1^\mu + p_2^\mu$ due to momentum conservation. Thus, the amplitude can be further simplified as:

$$i\mathcal{T} = ic_1 G_F f_\pi \overline{u}_{s_1}(p_1)(\not{p}_1 + \not{p}_2)(1-\gamma_5) v_{s_2}(p_2) \tag{919}$$

$$= ic_1 G_F f_\pi \Big[\overline{u}_{s_1}(p_1)\not{p}_1(1-\gamma_5) v_{s_2}(p_2) + \overline{u}_{s_1}(p_1)\not{p}_2(1-\gamma_5) v_{s_2}(p_2)\Big] \tag{920}$$

$$= ic_1 G_F f_\pi \Big[(-m)\overline{u}_{s_1}(p_1)(1-\gamma_5) v_{s_2}(p_2) + 0\Big] \tag{921}$$

where we have used the Dirac equation $\overline{u}_{s_1}(p_1)(\not{p}_1 + m) = 0$ and the massless neutrino condition $\not{p}_2 v_{s_2}(p_2) = 0$. Therefore, the amplitude becomes:

$$i\mathcal{T} = -ic_1 G_F f_\pi m \overline{u}_{s_1}(p_1)(1-\gamma_5) v_{s_2}(p_2). \tag{922}$$

Next, we can write down the Hermitian conjugate of the amplitude:

$$-i\mathcal{T}^* = ic_1 G_F f_\pi m \Big[\overline{u}_{s_1}(p_1)(1-\gamma_5) v_{s_2}(p_2)\Big]^\dagger \tag{923}$$

$$= ic_1 G_F f_\pi m \Big[v_{s_2}^\dagger(p_2)(1-\gamma_5)^\dagger \overline{u}_{s_1}^\dagger(p_1)\Big] \tag{924}$$

$$= ic_1 G_F f_\pi m \Big[v_{s_2}^\dagger(p_2)\gamma^0(1-\gamma_5)^\dagger \gamma^0 u_{s_1}(p_1)\Big] \tag{925}$$

$$= ic_1 G_F f_\pi m \Big[\overline{v}_{s_2}(p_2)(1+\gamma_5) u_{s_1}(p_1)\Big], \tag{926}$$

where we have used the relation $\overline{u} = u^\dagger\gamma^0$ and the Hermitian property of γ_5 (that is, $\gamma_5^\dagger = \gamma_5$) to get the last line. Now, we can compute the squared amplitude averaged over initial spins and summed over final spins:

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle &= \frac{1}{1}\sum_{s_1,s_2}\mathcal{T}\mathcal{T}^* && (927)\\
&= c_1^2G_F^2f_\pi^2m^2\sum_{s_1,s_2}\Big[\overline{u}_{s_1}(p_1)(1-\gamma_5)v_{s_2}(p_2)\Big]\Big[\overline{v}_{s_2}(p_2)(1+\gamma_5)u_{s_1}(p_1)\Big] && (928)\\
&= c_1^2G_F^2f_\pi^2m^2\sum_{s_1,s_2}\mathrm{Tr}\Big[(1+\gamma_5)u_{s_1}(p_1)\overline{u}_{s_1}(p_1)(1-\gamma_5)v_{s_2}(p_2)\overline{v}_{s_2}(p_2)\Big] && (929)\\
&= c_1^2G_F^2f_\pi^2m^2\mathrm{Tr}\Big[(1+\gamma_5)(-\not{p}_1+m)(1-\gamma_5)(-\not{p}_2)\Big] && (930)
\end{aligned}$$

where we have used the completeness relations for spinors and the trace properties of gamma matrices:

$$\begin{aligned}
\sum_s u_s(p)\overline{u}_s(p) &= -\not{p}+m, && (931)\\
\sum_s v_s(p)\overline{v}_s(p) &= -\not{p}-m, && (932)\\
\mathrm{Tr}(\not{a}\not{b}) &= -4(a\cdot b), && (933)\\
\mathrm{Tr}(\text{odd number of }\gamma\text{-matrices}) &= 0, && (934)\\
\gamma_5^2 &= 1. && (935)
\end{aligned}$$

We can expand the trace:

$$\begin{aligned}
(1+\gamma_5)(-\not{p}_1+m)(1-\gamma_5)(-\not{p}_2) &= (1+\gamma_5)(-\not{p}_1)(1-\gamma_5)(-\not{p}_2)+(1+\gamma_5)m(1-\gamma_5)(-\not{p}_2) && (936)\\
&= (1+\gamma_5)(-\not{p}_1)(1-\gamma_5)(-\not{p}_2)+0 && (937)\\
&= (1+\gamma_5)(\not{p}_1)(1-\gamma_5)(\not{p}_2) && (938)\\
&= (\not{p}_1)(\not{p}_2)+(\not{p}_1)(-\gamma_5)(\not{p}_2)+\gamma_5(\not{p}_1)(\not{p}_2)+\gamma_5(\not{p}_1)(-\gamma_5)(\not{p}_2) && (939)\\
&= 2(\not{p}_1)(\not{p}_2)+2\gamma_5(\not{p}_1)(\not{p}_2) && (940)
\end{aligned}$$

where we have used the anticommutation relation $\{\gamma_5,\gamma^\mu\}=0$ to get the last line. Therefore, we have:

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle &= c_1^2G_F^2f_\pi^2m^2\mathrm{Tr}\Big[2(\not{p}_1)(\not{p}_2)+2\gamma_5(\not{p}_1)(\not{p}_2)\Big] && (941)\\
&= 2c_1^2G_F^2f_\pi^2m^2\Big[\mathrm{Tr}(\not{p}_1\not{p}_2)+\mathrm{Tr}(\gamma_5\not{p}_1\not{p}_2)\Big] && (942)\\
&= 2c_1^2G_F^2f_\pi^2m^2\Big[-4(p_1\cdot p_2)+0\Big] && (943)\\
&= -8c_1^2G_F^2f_\pi^2m^2(p_1\cdot p_2) && (944)
\end{aligned}$$

where we have used the trace properties of gamma matrices again. In the rest frame of π^-, we have:

$$k^2 = -m_\pi^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2(p_1 \cdot p_2) = -m^2 + 0 + 2(p_1 \cdot p_2) \tag{945}$$

$$\Rightarrow -(p_1 \cdot p_2) = \frac{m_\pi^2 - m^2}{2} \tag{946}$$

Thus, the squared amplitude becomes:

$$\langle |\mathcal{T}|^2 \rangle = 4c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2) \tag{947}$$

Note that the m is the muon mass. Finally, we can compute the decay rate:

$$\Gamma = \frac{1}{2m_\pi} \int d\Phi_2 \langle |\mathcal{T}|^2 \rangle, \tag{948}$$

where the two-body phase space integral is:

$$\int d\Phi_2 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(k - p_1 - p_2) \tag{949}$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \delta^3(\mathbf{0} - \mathbf{p}_1 - \mathbf{p}_2) \tag{950}$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \tag{951}$$

$$= \int \frac{4\pi p_1^2 dp_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(m_\pi - E_1 - E_2) \tag{952}$$

$$= \int \frac{4\pi p_1^2 dp_1}{(2\pi)^2 4E_1 E_2} \delta(m_\pi - E_1 - E_2) \tag{953}$$

where we have used the delta function to perform the $\mathbf{p}_2$ integral. In the rest frame of π^-, we have $\mathbf{p}_2 = -\mathbf{p}_1$ and $E_2 = |\mathbf{p}_2| = |\mathbf{p}_1|$. Thus, we can write $E_1 + E_2 - m_\pi = \sqrt{p_1^2 + m^2} + p_1 - m_\pi$. The root of the equation $E_1 + E_2 - m_\pi = 0$ is:

$$p_1 = \frac{m_\pi^2 - m^2}{2m_\pi} \tag{954}$$

Also, we can compute the derivative:

$$\frac{d}{dp_1}(E_1 + E_2 - m_\pi) = \frac{p_1}{\sqrt{p_1^2 + m^2}} + 1 = \frac{E_1 + E_2}{E_1} = \frac{m_\pi}{E_1} \tag{955}$$

Therefore, the phase space integral becomes:

$$\int d\Phi_2 = \frac{4\pi p_1^2}{(2\pi)^2 4E_1E_2}\frac{E_1}{m_\pi} \tag{956}$$

$$= \frac{p_1^2}{4\pi m_\pi E_2} = \frac{p_1}{4\pi m_\pi} \tag{957}$$

$$= \frac{m_\pi^2 - m^2}{8\pi m_\pi^2} \tag{958}$$

Finally, the decay rate is:

$$\Gamma_{\pi^- \to \mu\overline{\nu}_\mu} = \frac{1}{2m_\pi}\langle|\mathcal{T}|^2\rangle \int d\Phi_2 \tag{959}$$

$$= \frac{1}{2m_\pi}\left[4c_1^2 G_F^2 f_\pi^2 m^2(m_\pi^2 - m^2)\right]\left[\frac{m_\pi^2 - m^2}{8\pi m_\pi^2}\right] \tag{960}$$

$$= \frac{c_1^2 G_F^2 f_\pi^2 m^2 (m_\pi^2 - m^2)^2}{4\pi m_\pi^3} \tag{961}$$

$$= \frac{c_1^2 G_F^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{4\pi m_\pi^3} \tag{962}$$

(b)

The charged pion life time is related to the decay rate by $\tau = 1/\Gamma$. Note that 2.6033×10^{-8} s $\approx 3.955 \times 10^{16}$ MeV^{-1}. Thus, we can solve for f_π:

$$f_\pi = \sqrt{\frac{4\pi m_\pi^3}{c_1^2 G_F^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2 \tau}} \tag{963}$$

$$= \sqrt{\frac{4\pi(139.6 \text{ MeV})^3}{(0.974)^2(1.166\times 10^{-5}\text{ GeV}^{-2})^2(105.7\text{ MeV})^2((139.6\text{ MeV})^2 - (105.7\text{ MeV})^2)^2(3.955\times 10^{16}\text{ MeV}^{-1})}} \tag{964}$$

$$\approx 0.09314 \text{ GeV} = 93.14 \text{ MeV}. \tag{965}$$

(c)

Now we analyze the decay $\pi^- \to e^-\overline{\nu}_e$. The Feynman diagram is shown in Fig. 11b. By following the same procedure as in part (a), we can write down the decay rate:

$$\Gamma_{\pi^- \to e\overline{\nu}_e} = \frac{c_2^2 G_F^2 f_\pi^2 m_e^2 (m_\pi^2 - m_e^2)^2}{4\pi m_\pi^3}. \tag{966}$$

Given the branching ratio $\mathrm{Br}(\pi^- \to e^-\overline{\nu}_e) = 1.230 \times 10^{-4}$, we have:

$$\begin{aligned}
\mathrm{Br}(\pi^- \to e^-\overline{\nu}_e) &= \frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu) + \Gamma(\pi^- \to e^-\overline{\nu}_e)} && (967)\\
&\approx \frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} && (968)\\
&= \frac{c_2^2 m_e^2 (m_\pi^2 - m_e^2)^2}{c_1^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2} && (969)
\end{aligned}$$

where we have used the fact that $\Gamma(\pi^- \to e^-\overline{\nu}_e) \ll \Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)$ to get the second line. Therefore, we can solve for c_2:

$$\begin{aligned}
c_2 &= c_1 \sqrt{\mathrm{Br}(\pi^- \to e^-\overline{\nu}_e) \frac{m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{m_e^2 (m_\pi^2 - m_e^2)^2}} && (970)\\
&= 0.974 \sqrt{(1.230 \times 10^{-4}) \frac{(105.7 \text{ MeV})^2((139.6 \text{ MeV})^2 - (105.7 \text{ MeV})^2)^2}{(0.511 \text{ MeV})^2((139.6 \text{ MeV})^2 - (0.511 \text{ MeV})^2)^2}} && (971)\\
&\approx 0.95345. && (972)
\end{aligned}$$

The most obvious reason that pion favoring decay into muon instead of electron is that the muon mass is much larger than the electron mass. Since the $\Gamma \propto m_l^2$ $(l = e, \mu)$, the decay rate into muon is greatly enhanced compared to that into electron.

Remark: This phenomenon is known as *helicity suppression*. But I think the explanation above is sufficient for this problem. □

Question 2

Consider QED with both electron and muon:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{l=e,\mu}(i\overline{\Psi}_l\partial\!\!\!/\Psi_l - m_l\overline{\Psi}_l\Psi_l + \frac{g}{2}\overline{\Psi}_l\gamma^\mu\Psi_l A_\mu), \tag{973}$$

where both Ψ_e and Ψ_μ are Dirac fields. Compute the $\langle|\mathcal{T}^2|\rangle$ for $e^+e^- \to \mu^+\mu^-$. Then, compute its cross section σ. Eq. (11.22) and Eq. (11.30) should be useful.

Answer

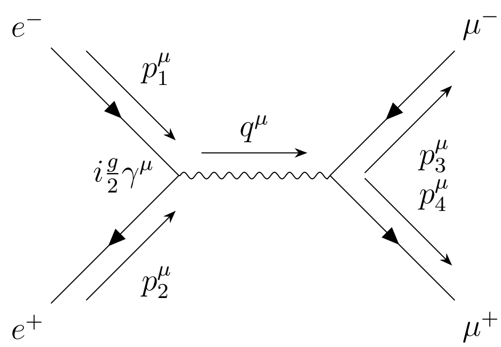


Figure 12: Feynman diagram for $e^+e^- \to \mu^+\mu^-$.

The Feynman diagram for $e^+e^- \to \mu^+\mu^-$ is shown in Fig. 12. We can write down the amplitude:

$$\begin{aligned} i\mathcal{T} &= \overline{v}_{s_2}(p_2)(i\frac{g}{2}\gamma^\mu)u_{s_1}(p_1)\frac{-ig_{\mu\nu}}{q^2}\overline{u}_{s_3}(p_3)(i\frac{g}{2}\gamma^\nu)v_{s_4}(p_4) & (974)\\ &= i\frac{g^2}{4q^2}\Big[\overline{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\Big]\Big[\overline{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4)\Big], & (975)\end{aligned}$$

where p_1^μ, p_2^μ, p_3^μ, and p_4^μ are the four-momenta of e^-, e^+, μ^-, and μ^+, respectively. s_1, s_2, s_3, and s_4 are the spin indices of e^-, e^+, μ^-, and μ^+, respectively. Also, we have defined $q^\mu = p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$. Next, we can write down the Hermitian conjugate of the amplitude:

$$\begin{aligned} -i\mathcal{T}^* &= -i\frac{g^2}{4q^2}\Big[\overline{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\Big]^\dagger\Big[\overline{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4)\Big]^\dagger & (976)\\ &= -i\frac{g^2}{4q^2}\Big[u_{s_1}^\dagger(p_1)\gamma^{\mu\dagger}(\gamma^0)^\dagger v_{s_2}(p_2)\Big]\Big[v_{s_4}^\dagger(p_4)\gamma_\mu^\dagger(\gamma^0)^\dagger u_{s_3}(p_3)\Big] & (977)\\ &= -i\frac{g^2}{4q^2}\Big[\overline{u}_{s_1}(p_1)\gamma^\mu v_{s_2}(p_2)\Big]\Big[\overline{v}_{s_4}(p_4)\gamma_\mu u_{s_3}(p_3)\Big]. & (978)\end{aligned}$$

Therefore, we can compute the squared amplitude averaged over initial spins and summed over final spins:

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle &= \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}\mathcal{T}\mathcal{T}^* && (979)\\
&= \frac{g^4}{64q^4}\sum_{s_1,s_2,s_3,s_4}\Big[\overline{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\Big]\Big[\overline{u}_{s_3}(p_3)\gamma_\mu v_{s_4}(p_4)\Big]\Big[\overline{u}_{s_1}(p_1)\gamma^\nu v_{s_2}(p_2)\Big]\Big[\overline{v}_{s_4}(p_4)\gamma_\nu u_{s_3}(p_3)\Big] && (980)\\
&= \frac{g^4}{64q^4}\sum_{s_1,s_2,s_3,s_4}\mathrm{Tr}\Big[\gamma^\mu u_{s_1}(p_1)\overline{u}_{s_1}(p_1)\gamma^\nu v_{s_2}(p_2)\overline{v}_{s_2}(p_2)\Big]\mathrm{Tr}\Big[\gamma_\mu u_{s_3}(p_3)\overline{u}_{s_3}(p_3)\gamma_\nu v_{s_4}(p_4)\overline{v}_{s_4}(p_4)\Big] && (981)\\
&= \frac{g^4}{64q^4}\mathrm{Tr}\Big[\gamma^\mu(-\not{p}_1+m_e)\gamma^\nu(-\not{p}_2-m_e)\Big]\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_3+m_\mu)\gamma_\nu(-\not{p}_4-m_\mu)\Big], && (982)
\end{aligned}$$

where we have used the completeness relations for spinors and the trace properties of gamma matrices:

$$\begin{aligned}
\sum_s u_s(p)\overline{u}_s(p) &= -\not{p}+m, && (983)\\
\sum_s v_s(p)\overline{v}_s(p) &= -\not{p}-m, && (984)\\
\mathrm{Tr}(\not{a}\not{b}) &= -4(a\cdot b), && (985)\\
\mathrm{Tr}(\text{odd number of }\gamma\text{-matrices}) &= 0 && (986)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= -4g^{\mu\nu} && (987)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}) && (988)\\
g_{\mu\nu}g^{\mu\nu} &= 4 && (989)
\end{aligned}$$

We can expand the traces:

$$\begin{aligned}
\mathrm{Tr}\Big[\gamma^\mu(-\not{p}_1+m_e)\gamma^\nu(-\not{p}_2-m_e)\Big] &= \mathrm{Tr}\Big[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\Big]-m_e^2\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\Big] && (990)\\
=&(p_1)_\alpha(p_2)_\beta\mathrm{Tr}\Big[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\Big]+4m_e^2g^{\mu\nu} && (991)\\
=&(p_1)_\alpha(p_2)_\beta 4(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\nu}g^{\alpha\beta}+g^{\mu\beta}g^{\nu\alpha})+4m_e^2g^{\mu\nu} && (992)\\
=&4\Big[p_1^\mu p_2^\nu-g^{\mu\nu}(p_1\cdot p_2)+p_1^\nu p_2^\mu+m_e^2g^{\mu\nu}\Big] && (993)\\
=&4\Big[p_1^\mu p_2^\nu+p_1^\nu p_2^\mu-g^{\mu\nu}(p_1\cdot p_2-m_e^2)\Big], && (994)
\end{aligned}$$

and

$$\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_3+m_\mu)\gamma_\nu(-\not{p}_4-m_\mu)\Big] = \mathrm{Tr}\Big[\gamma_\mu\not{p}_3\gamma_\nu\not{p}_4\Big] - m_\mu^2\mathrm{Tr}\Big[\gamma_\mu\gamma_\nu\Big] \tag{995}$$

$$=(p_3)^\rho(p_4)^\sigma\mathrm{Tr}\Big[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\Big] + 4m_\mu^2 g_{\mu\nu} \tag{996}$$

$$=(p_3)^\rho(p_4)^\sigma 4(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho}) + 4m_\mu^2 g_{\mu\nu} \tag{997}$$

$$=4\Big[p_{3\mu}p_{4\nu} - g_{\mu\nu}(p_3\cdot p_4) + p_{3\nu}p_{4\mu} + m_\mu^2 g_{\mu\nu}\Big] \tag{998}$$

$$=4\Big[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - g_{\mu\nu}(p_3\cdot p_4 - m_\mu^2)\Big]. \tag{999}$$

Therefore, the squared amplitude becomes:

$$\langle|\mathcal{T}|^2\rangle = \frac{g^4}{64q^4}16\Big[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1\cdot p_2 - m_e^2)\Big]\Big[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - g_{\mu\nu}(p_3\cdot p_4 - m_\mu^2)\Big] \tag{1000}$$

$$= \frac{g^4}{4q^4}\Big[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4 - m_\mu^2) \tag{1001}$$

$$+ (p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4 - m_\mu^2) \tag{1002}$$

$$- (p_1\cdot p_2 - m_e^2)(p_3\cdot p_4) - (p_1\cdot p_2 - m_e^2)(p_3\cdot p_4) + 4(p_1\cdot p_2 - m_e^2)(p_3\cdot p_4 - m_\mu^2)\Big] \tag{1003}$$

$$= \frac{g^4}{4q^4}\Big[(p_1p_3)(p_2p_4) + (p_1p_4)(p_2p_3) - (p_1p_2)(p_3p_4) + (p_1p_2)m_\mu^2 \tag{1004}$$

$$+ (p_1p_4)(p_2p_3) + (p_1p_3)(p_2p_4) - (p_1p_2)(p_3p_4) + (p_1p_2)m_\mu^2 \tag{1005}$$

$$- (p_1p_2)(p_3p_4) + m_e^2(p_3p_4) - (p_1p_2)(p_3p_4) + m_e^2(p_3p_4) \tag{1006}$$

$$+ 4(p_1p_2)(p_3p_4) - 4m_\mu^2(p_1p_2) - 4m_e^2(p_3p_4) + 4m_e^2m_\mu^2\Big] \tag{1007}$$

$$= \frac{g^4}{4q^4}\Big[2(p_1p_3)(p_2p_4) + 2(p_1p_4)(p_2p_3) - 2(p_1p_2)m_\mu^2 - 2m_e^2(p_3p_4) + 4m_e^2m_\mu^2\Big] \tag{1008}$$

Next, we can compute the cross section in the center-of-mass frame. In this frame, we have:

$$p_1^\mu = (E, 0, 0, p), \tag{1009}$$

$$p_2^\mu = (E, 0, 0, -p), \tag{1010}$$

$$p_3^\mu = (E,\ p'\sin\theta,\ 0,\ p'\cos\theta), \tag{1011}$$

$$p_4^\mu = (E,\ -p'\sin\theta,\ 0,\ -p'\cos\theta), \tag{1012}$$

where $E = \sqrt{p^2 + m_e^2} = \sqrt{p'^2 + m_\mu^2} = \frac{\sqrt{s}}{2}$. We can compute the dot products:

$$(p_1 \cdot p_2) = -E^2 + (-p^2) = m_e^2 - 2E^2 \tag{1013}$$
$$(p_3 \cdot p_4) = -E^2 + (-p'^2) = m_\mu^2 - 2E^2 \tag{1014}$$
$$(p_1 \cdot p_3) = -E^2 + pp' \cos\theta \tag{1015}$$
$$(p_2 \cdot p_4) = -E^2 + pp' \cos\theta \tag{1016}$$
$$(p_1 \cdot p_4) = -E^2 - pp' \cos\theta \tag{1017}$$
$$(p_2 \cdot p_3) = -E^2 - pp' \cos\theta \tag{1018}$$

Therefore, the squared amplitude becomes:

$$\langle|\mathcal{T}|^2\rangle = \frac{g^4}{4q^4}\Big[2(-E^2 + pp'\cos\theta)^2 + 2(-E^2 - pp'\cos\theta)^2 - 2(m_e^2 - 2E^2)m_\mu^2 - 2m_e^2(m_\mu^2 - 2E^2) + 4m_e^2 m_\mu^2\Big] \tag{1019}$$
$$= \frac{g^4}{4q^4}\Big[4(E^4 + p^2p'^2\cos^2\theta) + 4E^2(m_e^2 + m_\mu^2)\Big] \tag{1020}$$
$$= \frac{g^4}{q^4}\Big[E^4 + p^2p'^2\cos^2\theta + E^2(m_e^2 + m_\mu^2)\Big]. \tag{1021}$$

Next, we can compute the cross section (by eq. (11.31) in the textbook):

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}\langle|\mathcal{T}|^2\rangle \tag{1022}$$
$$= \frac{1}{64\pi^2 s}\frac{p'}{p}\frac{g^4}{q^4}\Big[E^4 + p^2p'^2\cos^2\theta + E^2(m_e^2 + m_\mu^2)\Big] \tag{1023}$$
$$= \frac{g^4}{64\pi^2 s^3}\frac{p'}{p}\Big[E^4 + p^2p'^2\cos^2\theta + E^2(m_e^2 + m_\mu^2)\Big], \tag{1024}$$

where we have used the fact that $q^2 = (p_1 + p_2)^2 = -s = -4E^2$. Finally, we can integrate over the solid angle to get the total cross section:

$$\int d\Omega = 4\pi, \tag{1025}$$
$$\int d\Omega \cos^2\theta = 2\pi \int_{-1}^{1} d\cos\theta \cos^2\theta = \frac{4\pi}{3}, \tag{1026}$$

thus,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega \tag{1027}$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[E^4 \int d\Omega + p^2 p'^2 \int d\Omega \cos^2\theta + E^2(m_e^2 + m_\mu^2) \int d\Omega \right] \tag{1028}$$

$$= \frac{g^4}{64\pi^2 s^3} \frac{p'}{p} \left[4\pi E^4 + \frac{4\pi}{3} p^2 p'^2 + 4\pi E^2 (m_e^2 + m_\mu^2) \right] \tag{1029}$$

$$= \frac{g^4}{16\pi s^3} \frac{p'}{p} \left[E^4 + \frac{1}{3} p^2 p'^2 + E^2 (m_e^2 + m_\mu^2) \right]. \tag{1030}$$

Now, we can express p and p' in terms of s:

$$p = \sqrt{E^2 - m_e^2} = \sqrt{\frac{s}{4} - m_e^2}, \tag{1031}$$

$$p' = \sqrt{E^2 - m_\mu^2} = \sqrt{\frac{s}{4} - m_\mu^2}, \tag{1032}$$

$$E = \frac{\sqrt{s}}{2}. \tag{1033}$$

Therefore, the final expression for the cross section is:

$$\sigma = \frac{g^4}{16\pi s^3} \frac{\sqrt{\frac{s}{4} - m_\mu^2}}{\sqrt{\frac{s}{4} - m_e^2}} \left[\left(\frac{s}{4}\right)^2 + \frac{1}{3}\left(\frac{s}{4} - m_e^2\right)\left(\frac{s}{4} - m_\mu^2\right) + \frac{s}{4}(m_e^2 + m_\mu^2) \right] \tag{1034}$$

$$= \frac{g^4}{192\pi s^3} \frac{\sqrt{s - 4m_\mu^2}}{\sqrt{s - 4m_e^2}} \left[(s + 2m_e^2)(s + 2m_\mu^2) \right], \quad \text{in terms of } s, \tag{1035}$$

$$= \frac{g^4}{3072\pi E^6} \frac{\sqrt{E^2 - m_\mu^2}}{\sqrt{E^2 - m_e^2}} \left[(2E^2 + m_e^2)(2E^2 + m_\mu^2) \right], \quad \text{in terms of } E. \tag{1036}$$

$\square$

Question 3

Consider classical field theory with two real scalar fields in (3+1)-dimension spacetime:

$$\mathcal{L}(x) = \sum_{a=1}^{2}\Big(-\frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a\Big) - V(x), \tag{1037}$$

$$V(x) = -\sum_{a=1}^{2}\Big(\frac{1}{2}\mu^2\phi_a\phi_a\Big) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2, \tag{1038}$$

where μ and λ are positive real constants.

(a) Show that the Lagrangian has an $SO(2)$ transformation symmetry:

$$\phi_1(x) \to \phi_1'(x) = \phi_1(x)\cos\alpha_0 - \phi_2(x)\sin\alpha_0, \tag{1039}$$

$$\phi_2(x) \to \phi_2'(x) = \phi_1(x)\sin\alpha_0 + \phi_2(x)\cos\alpha_0, \tag{1040}$$

(b) Find the conjugate momentum $\Pi_1(x)$, $\Pi_2(x)$ of $\phi_1(x)$, $\phi_2(x)$. Find the Hamiltonian density $\mathcal{H}(x)$ in the terms of $\phi_a(x)$, $\Pi_a(x)$, and $\partial_i\phi_a(x)$.

(c) Find the ground state in the basis of $\{\phi_r(x), \phi_\theta(x)\}$ where

$$\phi_1(x) = \phi_r(x)\cos(\phi_\theta(x)), \tag{1041}$$

$$\phi_2(x) = \phi_r(x)\sin(\phi_\theta(x)), \tag{1042}$$

with $\phi_r(x) \geq 0$ and $\phi_\theta(x) \in [0, 2\pi)$. Is the Lagrangian $\mathcal{L}$ invariant under a continuous shift symmetry of $\phi_\theta(x) \to \phi_\theta(x) + \alpha_0$?

Hint: In general, finding the ground state is to find $\phi(x)$ s.t. minimize $H = \int \mathcal{H}(x)d^3x$; but for this problem, finding $\phi(x)$ to minimize $\mathcal{H}(x)$ is the same. If you have trouble with the above procedure, given the Lagrangian of this problem, one can simply find $\phi(x)$ s.t. minimize $V(x)$, which is the same as minimizing $\mathcal{H}$ for this problem.

(d) Now let's study the system's dynamics around the ground state.
$\phi_r(x)$ should fluctuate around $\sqrt{\frac{\mu^2}{\lambda}}$: $\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x)$. $\phi_\theta(x)$ should fluctuate within $[0, 2\pi)$.
Show that $f_r(x)$ is a massive field and find its mass. Taking $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}}\phi_\theta(x)$ as the other scalar field, does $f_\theta(x)$ have a mass? Does $\mathcal{L}$ have a continuous shift symmetry of $f_\theta(x) \to f_\theta(x) + \Lambda_0$?
Remark: This problem paves the road for your understanding of spontaneous symmetry breaking. We also see again that the symmetry groups of $SO(2)$ and $U(1)$ are isomorphic.
Remark: More to think about after solving the problems above: Note that we reparametrized the field into a non-linear realization, where you see the $U(1)$ symmetry explicitly. How do you interpret the kinetic term? How do you interpret the $f_r(x)$ field-dependent kinetic terms for $f_\theta(x)$? Is it canonically normalized? How does the field $f_\theta(x)$ relate to the original $SO(2)$ field $\phi_a(x)$? And again, is the ratio of

the field a linear redefinition of the field configuration? It is a non-linear realization because all powers of $f_\theta(x)/\sqrt{\frac{\mu^2}{\lambda}}$ need to enter. There is only a region of validity, that is $f_r(x) \ll \sqrt{\frac{\mu^2}{\lambda}}$

Answer

(a)

The Lagrangian can be separated into the kinetic term and the potential term:

$$\mathcal{L}(x) = \sum_{a=1}^{2}\Big(-\frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a\Big) - V(x) \tag{1043}$$

$$= -\frac{1}{2}\begin{pmatrix}\partial^\mu\phi_1 & \partial^\mu\phi_2\end{pmatrix}\begin{pmatrix}\partial_\mu\phi_1\\ \partial_\mu\phi_2\end{pmatrix} - \left[-\frac{\mu^2}{2}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \frac{\lambda}{4}\left(\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)^2\right] \tag{1044}$$

Under the $SO(2)$ transformation, the fields transform as:

$$\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = \begin{pmatrix}\cos\alpha_0 & -\sin\alpha_0\\ \sin\alpha_0 & \cos\alpha_0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \equiv R(\alpha_0)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{1045}$$

where $R(\alpha_0)$ is the rotation matrix. The kinetic term transforms as:

$$-\frac{1}{2}\begin{pmatrix}\partial^\mu\phi_1' & \partial^\mu\phi_2'\end{pmatrix}\begin{pmatrix}\partial_\mu\phi_1'\\ \partial_\mu\phi_2'\end{pmatrix} \tag{1046}$$

$$= -\frac{1}{2}\begin{pmatrix}\partial^\mu(R_{11}\phi_1 + R_{12}\phi_2) & \partial^\mu(R_{21}\phi_1 + R_{22}\phi_2)\end{pmatrix}\begin{pmatrix}\partial_\mu(R_{11}\phi_1 + R_{12}\phi_2)\\ \partial_\mu(R_{21}\phi_1 + R_{22}\phi_2)\end{pmatrix} \tag{1047}$$

$$= -\frac{1}{2}\begin{pmatrix}R_{11}\partial^\mu\phi_1 + R_{12}\partial^\mu\phi_2 & R_{21}\partial^\mu\phi_1 + R_{22}\partial^\mu\phi_2\end{pmatrix}\begin{pmatrix}R_{11}\partial_\mu\phi_1 + R_{12}\partial_\mu\phi_2\\ R_{21}\partial_\mu\phi_1 + R_{22}\partial_\mu\phi_2\end{pmatrix} \tag{1048}$$

$$= -\frac{1}{2}\begin{pmatrix}\partial^\mu\phi_1 & \partial^\mu\phi_2\end{pmatrix}R^T(\alpha_0)R(\alpha_0)\begin{pmatrix}\partial_\mu\phi_1\\ \partial_\mu\phi_2\end{pmatrix} \tag{1049}$$

$$= -\frac{1}{2}\begin{pmatrix}\partial^\mu\phi_1 & \partial^\mu\phi_2\end{pmatrix}\begin{pmatrix}\partial_\mu\phi_1\\ \partial_\mu\phi_2\end{pmatrix}, \tag{1050}$$

where we have used the orthogonality of the rotation matrix: $R^T(\alpha_0)R(\alpha_0) = I$. Similarly, the potential term transforms as:

$$-\frac{\mu^2}{2}\begin{pmatrix}\phi_1' & \phi_2'\end{pmatrix}\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}+\frac{\lambda}{4}\left(\begin{pmatrix}\phi_1' & \phi_2'\end{pmatrix}\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}\right)^2 \tag{1051}$$

$$=-\frac{\mu^2}{2}\begin{pmatrix}R_{11}\phi_1+R_{12}\phi_2 & R_{21}\phi_1+R_{22}\phi_2\end{pmatrix}\begin{pmatrix}R_{11}\phi_1+R_{12}\phi_2\\ R_{21}\phi_1+R_{22}\phi_2\end{pmatrix} \tag{1052}$$

$$+\frac{\lambda}{4}\left(\begin{pmatrix}R_{11}\phi_1+R_{12}\phi_2 & R_{21}\phi_1+R_{22}\phi_2\end{pmatrix}\begin{pmatrix}R_{11}\phi_1+R_{12}\phi_2\\ R_{21}\phi_1+R_{22}\phi_2\end{pmatrix}\right)^2 \tag{1053}$$

$$=-\frac{\mu^2}{2}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}R^T(\alpha_0)R(\alpha_0)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}+\frac{\lambda}{4}\left(\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}R^T(\alpha_0)R(\alpha_0)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)^2 \tag{1054}$$

$$=-\frac{\mu^2}{2}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}+\frac{\lambda}{4}\left(\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)^2 \tag{1055}$$

Thus, the Lagrangian is invariant under the $SO(2)$ transformation.

(b)

The conjugate momenta are given by:

$$\Pi_1(x)=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_1)}=-\frac{1}{2}\cdot(-2)\cdot(\partial^0\phi_1)=\partial^0\phi_1, \tag{1056}$$

$$\Pi_2(x)=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_2)}=-\frac{1}{2}\cdot(-2)\cdot(\partial^0\phi_2)=\partial^0\phi_2. \tag{1057}$$

The Hamiltonian density is given by:

$$\mathcal{H}(x)=\Pi_1(x)\partial_0\phi_1+\Pi_2(x)\partial_0\phi_2-\mathcal{L}(x) \tag{1058}$$

$$=(\partial^0\phi_1)(\partial_0\phi_1)+(\partial^0\phi_2)(\partial_0\phi_2)-\left[-\frac{1}{2}(\partial^\mu\phi_1\partial_\mu\phi_1+\partial^\mu\phi_2\partial_\mu\phi_2)-V(x)\right] \tag{1059}$$

$$=(\partial^0\phi_1)^2+(\partial^0\phi_2)^2+\frac{1}{2}(\partial^\mu\phi_1\partial_\mu\phi_1+\partial^\mu\phi_2\partial_\mu\phi_2)+V(x) \tag{1060}$$

$$=(\partial^0\phi_1)^2+(\partial^0\phi_2)^2+\frac{1}{2}\Big(-(\partial^0\phi_1)^2+(\partial^i\phi_1)^2-(\partial^0\phi_2)^2+(\partial^i\phi_2)^2\Big)+V(x) \tag{1061}$$

$$=\frac{1}{2}(\partial^0\phi_1)^2+\frac{1}{2}(\partial^i\phi_1)^2+\frac{1}{2}(\partial^0\phi_2)^2+\frac{1}{2}(\partial^i\phi_2)^2+V(x) \tag{1062}$$

$$=\frac{1}{2}\Pi_1^2+\frac{1}{2}(\nabla\phi_1)^2+\frac{1}{2}\Pi_2^2+\frac{1}{2}(\nabla\phi_2)^2+V(x). \tag{1063}$$

(c)

To find the ground state, we need to minimize the potential $V(x)$:

$$V=-\frac{\mu^2}{2}(\phi_1^2+\phi_2^2)+\frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2 \tag{1064}$$

$$=-\frac{\mu^2}{2}\phi_r^2+\frac{\lambda}{4}\phi_r^4, \tag{1065}$$

where we have used the transformation:

$$\phi_1(x) = \phi_r(x)\cos(\phi_\theta(x)), \tag{1066}$$

$$\phi_2(x) = \phi_r(x)\sin(\phi_\theta(x)). \tag{1067}$$

To minimize V, we take the derivative with respect to ϕ_r and set it to zero:

$$\frac{dV}{d\phi_r} = -\mu^2\phi_r + \lambda\phi_r^3 = 0 \tag{1068}$$

$$\Rightarrow \phi_r(\lambda\phi_r^2 - \mu^2) = 0. \tag{1069}$$

The solutions are:

$$\phi_r = 0, \quad \text{or} \quad \phi_r = \sqrt{\frac{\mu^2}{\lambda}}. \tag{1070}$$

To determine which solution corresponds to the ground state, we evaluate the second derivative of V:

$$\frac{d^2V}{d\phi_r^2} = -\mu^2 + 3\lambda\phi_r^2. \tag{1071}$$

At $\phi_r = 0$:

$$\left.\frac{d^2V}{d\phi_r^2}\right|_{\phi_r=0} = -\mu^2 < 0, \tag{1072}$$

indicating a local maximum. At $\phi_r = \sqrt{\frac{\mu^2}{\lambda}}$:

$$\left.\frac{d^2V}{d\phi_r^2}\right|_{\phi_r=\sqrt{\frac{\mu^2}{\lambda}}} = -\mu^2 + 3\lambda\left(\frac{\mu^2}{\lambda}\right) = 2\mu^2 > 0, \tag{1073}$$

indicating a local minimum. Therefore, the ground state is at:

$$\phi_r = \sqrt{\frac{\mu^2}{\lambda}}, \quad \phi_\theta \text{ is arbitrary.} \tag{1074}$$

Next, we check if the Lagrangian is invariant under the continuous shift symmetry $\phi_\theta(x) \to \phi_\theta(x) + \alpha_0$. We can rewrite the kinetic term in terms of ϕ_r and ϕ_θ:

$$\partial^\mu\phi_1\partial_\mu\phi_1 + \partial^\mu\phi_2\partial_\mu\phi_2 = (\partial^\mu(\phi_r\cos\phi_\theta))(\partial_\mu(\phi_r\cos\phi_\theta)) + (\partial^\mu(\phi_r\sin\phi_\theta))(\partial_\mu(\phi_r\sin\phi_\theta)) \tag{1075}$$

$$= (\partial^\mu\phi_r\cos\phi_\theta - \phi_r\sin\phi_\theta\partial^\mu\phi_\theta)(\partial_\mu\phi_r\cos\phi_\theta - \phi_r\sin\phi_\theta\partial_\mu\phi_\theta) \tag{1076}$$

$$+ (\partial^\mu\phi_r\sin\phi_\theta + \phi_r\cos\phi_\theta\partial^\mu\phi_\theta)(\partial_\mu\phi_r\sin\phi_\theta + \phi_r\cos\phi_\theta\partial_\mu\phi_\theta) \tag{1077}$$

$$= (\partial^\mu\phi_r)(\partial_\mu\phi_r) + \phi_r^2(\partial^\mu\phi_\theta)(\partial_\mu\phi_\theta). \tag{1078}$$

The potential term depends only on ϕ_r:

$$V = -\frac{\mu^2}{2}\phi_r^2 + \frac{\lambda}{4}\phi_r^4. \tag{1079}$$

Thus, the Lagrangian in terms of ϕ_r and ϕ_θ is:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \phi_r)(\partial_\mu \phi_r) - \frac{1}{2}\phi_r^2(\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \frac{\mu^2}{2}\phi_r^2 - \frac{\lambda}{4}\phi_r^4. \tag{1080}$$

This Lagrangian is invariant under the shift $\phi_\theta(x) \to \phi_\theta(x) + \alpha_0$ since ϕ_θ appears only through its derivatives. **Therefore, the Lagrangian has a continuous shift symmetry in ϕ_θ.**

(d)

We expand $\phi_r(x)$ around its vacuum expectation value:

$$\phi_r(x) = \sqrt{\frac{\mu^2}{\lambda}} + f_r(x). \tag{1081}$$

Substituting this into the Lagrangian, we have:

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) - \frac{1}{2}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^2 (\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \frac{\mu^2}{2}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^2 - \frac{\lambda}{4}\left(\sqrt{\frac{\mu^2}{\lambda}} + f_r\right)^4 \tag{1082}$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) - \frac{1}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right)(\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right) \tag{1083}$$

$$-\frac{\lambda}{4}\left(\frac{\mu^4}{\lambda^2} + 4\frac{\mu^2}{\lambda}\sqrt{\frac{\mu^2}{\lambda}}f_r + 6\frac{\mu^2}{\lambda}f_r^2 + 4f_r^3\sqrt{\frac{\mu^2}{\lambda}} + f_r^4\right) \tag{1084}$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) + \left(-\frac{\mu^2}{2\lambda} - \sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{1}{2}f_r^2\right)(\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \left(\frac{\mu^4}{2\lambda} + \mu^2\sqrt{\frac{\mu^2}{\lambda}}f_r + \frac{\mu^2}{2}f_r^2\right) \tag{1085}$$

$$+\left(-\frac{\mu^4}{4\lambda} - \mu^2\sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{3\mu^2}{2}f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right) \tag{1086}$$

$$= -\frac{1}{2}(\partial^\mu f_r)(\partial_\mu f_r) + \left(-\frac{\mu^2}{2\lambda} - \sqrt{\frac{\mu^2}{\lambda}}f_r - \frac{1}{2}f_r^2\right)(\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) + \left(\frac{\mu^4}{4\lambda} - \mu^2 f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right). \tag{1087}$$

The mass term for f_r can be identified from the potential part of the Lagrangian:

$$V(f_r) = -\left(\frac{\mu^4}{4\lambda} - \mu^2 f_r^2 - \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} - \frac{\lambda}{4}f_r^4\right) \tag{1088}$$

$$= -\frac{\mu^4}{4\lambda} + \mu^2 f_r^2 + \lambda f_r^3\sqrt{\frac{\mu^2}{\lambda}} + \frac{\lambda}{4}f_r^4. \tag{1089}$$

The mass term for f_r is given by the coefficient of the f_r^2 term:

$$m_{f_r}^2 = 2\mu^2. \tag{1090}$$

Thus, $f_r(x)$ is a massive field with mass $m_{f_r} = \sqrt{2}\mu$. For the field $f_\theta(x) \equiv \sqrt{\frac{\mu^2}{\lambda}}\phi_\theta(x)$, we can rewrite the kinetic term involving ϕ_θ as:

$$-\frac{1}{2}\left(\frac{\mu^2}{\lambda} + 2\sqrt{\frac{\mu^2}{\lambda}}f_r + f_r^2\right)(\partial^\mu \phi_\theta)(\partial_\mu \phi_\theta) = -\frac{1}{2}\left(1 + \frac{2f_r}{\sqrt{\frac{\mu^2}{\lambda}}} + \frac{f_r^2}{\frac{\mu^2}{\lambda}}\right)(\partial^\mu f_\theta)(\partial_\mu f_\theta). \tag{1091}$$

The field $f_\theta(x)$ does not have a mass term, as there is no term proportional to f_θ^2 in the potential. Therefore, $f_\theta(x)$ is a massless field. The Lagrangian remains invariant under the continuous shift symmetry $f_\theta(x) \to f_\theta(x) + \Lambda_0$, since f_θ appears only through its derivatives. **Thus, the shift symmetry is preserved**. $\square$

Question 4

Problem 66.3
Use the result of problem 66.2 to compute the anomalous dimension of m and the beta function for e in spionor electrodynamics in R_ξ gauge. You should find that the results are independent of ξ.
Remark:

$$\widetilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} + (\xi - 1)k^\mu k^\nu / k^2}{k^2 - i\epsilon} \tag{1092}$$

The book only choose the Feynman gauge ($\xi = 1$) to show the loop calculation and get $Z_{1,2,3,m}$. For arbitrary gauge choice ξ, we can repeat the calculation and get:

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\Big(\frac{1}{\epsilon} + \text{finite}\Big) + \mathcal{O}(e^4), \quad \text{derived from photon propagator loop correction} \tag{1093}$$

$$Z_2 = 1 - \xi\frac{e^2}{8\pi^2}\Big(\frac{1}{\epsilon} + \text{finite}\Big) + \mathcal{O}(e^4), \quad \text{derived from fermion propagator loop correction} \tag{1094}$$

$$Z_m = 1 - (3+\xi)\frac{e^2}{8\pi^2}\Big(\frac{1}{\epsilon} + \text{finite}\Big) + \mathcal{O}(e^4), \quad \text{derived from fermion mass loop correction} \tag{1095}$$

$$Z_1 = 1 - \xi\frac{e^2}{8\pi^2}\Big(\frac{1}{\epsilon} + \text{finite}\Big) + \mathcal{O}(e^4), \quad \text{derived from vertex loop correction} \tag{1096}$$

Use the above to finish this problem.

Answer

Now, let's write down the bare Lagrangian and the renormalized Lagrangian:

$$\mathcal{L}_{bare} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \overline{\Psi}_0(i\not{D}_0 - m_0)\Psi_0 \tag{1097}$$

$$= -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \overline{\Psi}_0(i\not{\partial} - e_0\not{A}_0 - m_0)\Psi_0, \tag{1098}$$

$$= -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + i\overline{\Psi}_0\not{\partial}\Psi_0 - e_0\overline{\Psi}_0\not{A}_0\Psi_0 - m_0\overline{\Psi}_0\Psi_0, \tag{1099}$$

and

$$\mathcal{L}_{re} = \mathcal{L}_0 + \mathcal{L}_1, \tag{1100}$$

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\Psi}\not{\partial}\Psi - m\overline{\Psi}\Psi \tag{1101}$$

$$\mathcal{L}_1 = Z_1 e\overline{\Psi}\not{A}\Psi + \mathcal{L}_{ct}, \tag{1102}$$

$$\mathcal{L}_{ct} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} + i(Z_2 - 1)\overline{\Psi}\not{\partial}\Psi - (Z_m - 1)m\overline{\Psi}\Psi, \tag{1103}$$

Hence, we have

$$\mathcal{L}_{bare} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + i\overline{\Psi}_0\partial\!\!\!/\Psi_0 - e_0\overline{\Psi}_0 A\!\!\!/_0\Psi_0 - m_0\overline{\Psi}_0\Psi_0, \tag{1104}$$

$$\mathcal{L}_{re} = -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\overline{\Psi}\partial\!\!\!/\Psi + Z_1e\overline{\Psi}A\!\!\!/\Psi - Z_m m\overline{\Psi}\Psi. \tag{1105}$$

From the above two equations, we can identify the relations between bare and renormalized quantities:

$$A_{0\mu} = \sqrt{Z_3}A_\mu, \tag{1106}$$

$$\Psi_0 = \sqrt{Z_2}\Psi, \tag{1107}$$

$$e_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}e\widetilde{\mu}^{\epsilon/2}, \tag{1108}$$

$$m_0 = \frac{Z_m}{Z_2}m. \tag{1109}$$

Note that $\widetilde{\mu}$ is the renormalization scale introduced in dimensional regularization to keep the coupling constant dimensionless in $d = 4 - \epsilon$ dimensions. We first compute the beta function for e:

$$0 = \frac{d\log e_0}{d\log\mu} = \frac{d}{d\log\mu}\left(\log Z_1 - \log Z_2 - \frac{1}{2}\log Z_3 + \log e + \frac{\epsilon}{2}\log\widetilde{\mu}\right), \tag{1110}$$

which gives

$$\beta(e) = \frac{de}{d\log\mu} = e\left(-\frac{d\log Z_1}{d\log\mu} + \frac{d\log Z_2}{d\log\mu} + \frac{1}{2}\frac{d\log Z_3}{d\log\mu} - \frac{\epsilon}{2}\right). \tag{1111}$$

To compute the derivatives of the Z factors, we use the expressions given in the problem statement:

$$\frac{d\log Z_2}{d\log\mu} = \frac{1}{Z_2}\frac{dZ_2}{d\log\mu} = \frac{1}{Z_2}\frac{dZ_2}{de}\frac{de}{d\log\mu} = \frac{1}{Z_2}\frac{dZ_2}{de}\beta(e), \tag{1112}$$

$$\frac{d\log Z_1}{d\log\mu} = \frac{1}{Z_1}\frac{dZ_1}{d\log\mu} = \frac{1}{Z_1}\frac{dZ_1}{de}\frac{de}{d\log\mu} = \frac{1}{Z_1}\frac{dZ_1}{de}\beta(e), \tag{1113}$$

$$\frac{d\log Z_3}{d\log\mu} = \frac{1}{Z_3}\frac{dZ_3}{d\log\mu} = \frac{1}{Z_3}\frac{dZ_3}{de}\frac{de}{d\log\mu} = \frac{1}{Z_3}\frac{dZ_3}{de}\beta(e). \tag{1114}$$

Substituting these into the expression for $\beta(e)$, we have:

$$\beta(e) = e\left(-\frac{1}{Z_1}\frac{dZ_1}{de}\beta(e) + \frac{1}{Z_2}\frac{dZ_2}{de}\beta(e) + \frac{1}{2}\frac{1}{Z_3}\frac{dZ_3}{de}\beta(e) - \frac{\epsilon}{2}\right) \tag{1115}$$

$$\implies \beta(e)\left(1 + e\left(\frac{1}{Z_1}\frac{dZ_1}{de} - \frac{1}{Z_2}\frac{dZ_2}{de} - \frac{1}{2}\frac{1}{Z_3}\frac{dZ_3}{de}\right)\right) = -\frac{\epsilon}{2}e. \tag{1116}$$

$$\implies \beta(e) = -\frac{\epsilon}{2}e\left(1 + e\left(\frac{1}{Z_1}\frac{dZ_1}{de} - \frac{1}{Z_2}\frac{dZ_2}{de} - \frac{1}{2}\frac{1}{Z_3}\frac{dZ_3}{de}\right)\right)^{-1} \tag{1117}$$

$$= -\frac{\epsilon}{2}e\left(1 - e\left(\frac{1}{Z_1}\frac{dZ_1}{de} - \frac{1}{Z_2}\frac{dZ_2}{de} - \frac{1}{2}\frac{1}{Z_3}\frac{dZ_3}{de}\right)\right) + \mathcal{O}(e^4) \tag{1118}$$

$$= -\frac{\epsilon}{2}e + \frac{\epsilon}{2}e^2\left(\frac{1}{Z_1}\frac{dZ_1}{de} - \frac{1}{Z_2}\frac{dZ_2}{de} - \frac{1}{2}\frac{1}{Z_3}\frac{dZ_3}{de}\right) + \mathcal{O}(e^4). \tag{1119}$$

Now we can apply the expressions for Z_1, Z_2, and Z_3 (also Z_m) given in the problem statement to compute the derivatives:

$$\frac{1}{Z_1}\frac{dZ_1}{de} = -\xi\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3), \tag{1120}$$

$$\frac{1}{Z_2}\frac{dZ_2}{de} = -\xi\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3), \tag{1121}$$

$$\frac{1}{Z_3}\frac{dZ_3}{de} = -\frac{e}{3\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3 \tag{1122}$$

$$\frac{1}{Z_m}\frac{dZ_m}{de} = -(3+\xi)\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3). \tag{1123}$$

Substituting these into the expression for $\beta(e)$, we have:

$$\beta(e) = -\frac{\epsilon}{2}e + \frac{\epsilon}{2}e^2\left(-\xi\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \xi\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \frac{1}{2}\cdot\frac{e}{3\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right)\right) + \mathcal{O}(e^4) \tag{1124}$$

$$= -\frac{\epsilon}{2}e + \frac{\epsilon}{2}e^2\cdot\frac{e}{6\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4) \tag{1125}$$

$$= -\frac{\epsilon}{2}e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4). \tag{1126}$$

Now we can compute the anomalous dimension of m:

$$0 = \frac{d\log m_0}{d\log\mu} = \frac{d}{d\log\mu}\left(\log Z_m - \log Z_2 + \log m\right), \tag{1127}$$

which gives

$$\gamma_m = \frac{d\log m}{d\log\mu} = -\frac{d\log Z_m}{d\log\mu} + \frac{d\log Z_2}{d\log\mu}. \tag{1128}$$

Using the expressions for Z_m and Z_2, we have:

$$\frac{d\log Z_m}{d\log\mu} = \frac{1}{Z_m}\frac{dZ_m}{d\log\mu} = \frac{1}{Z_m}\frac{dZ_m}{de}\frac{de}{d\log\mu} = \frac{1}{Z_m}\frac{dZ_m}{de}\beta(e), \tag{1129}$$

$$\frac{d\log Z_2}{d\log\mu} = \frac{1}{Z_2}\frac{dZ_2}{d\log\mu} = \frac{1}{Z_2}\frac{dZ_2}{de}\frac{de}{d\log\mu} = \frac{1}{Z_2}\frac{dZ_2}{de}\beta(e). \tag{1130}$$

Substituting these into the expression for γ_m, we have:

$$\gamma_m = -\frac{1}{Z_m}\frac{dZ_m}{de}\beta(e) + \frac{1}{Z_2}\frac{dZ_2}{de}\beta(e) \tag{1131}$$

$$=\beta(e)\left(-\frac{1}{Z_m}\frac{dZ_m}{de} + \frac{1}{Z_2}\frac{dZ_2}{de}\right) \tag{1132}$$

$$= \left(-\frac{\epsilon}{2}e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4)\right)\left(-\left(-(3+\xi)\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3)\right) + \left(-\xi\frac{e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3)\right)\right) \tag{1133}$$

$$= \left(-\frac{\epsilon}{2}e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4)\right)\left(\frac{3e}{4\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^3)\right) \tag{1134}$$

$$= -\frac{3e^2}{8\pi^2} + \mathcal{O}(e^4). \tag{1135}$$

Thus, we have found that the beta function for e is:

$$\beta(e) = -\frac{\epsilon}{2}e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^4), \tag{1136}$$

and the anomalous dimension of m is:

$$\gamma_m = -\frac{3e^2}{8\pi^2} + \mathcal{O}(e^4). \tag{1137}$$

Remark: Notice that both results are independent of the gauge parameter ξ. □

Question 5

Consider the following theory:

$$\mathcal{L} = \mathcal{L}_\phi^0 + \mathcal{L}_\Psi^0 + \mathcal{L}_A^0 + \mathcal{L}_I \tag{1138}$$

$$= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \overline{\Psi}(i\not{D} - m_\Psi)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + y\phi\overline{\Psi}\Psi. \tag{1139}$$

The Dirac field Ψ is charged under a $U(1)$ gauge symmetry with a charge Q, and the gauge interaction strength is e. The $U(1)$ gauge field is A_μ, whose kinetic term is $\mathcal{L}_A^0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. (This is part of the real-world calculation for the discovery mode for the Higgs boson, which gone through heroic phenomenological studies on predicting the Higgs properties.)

(a) Draw the leading diagrams that enable $\phi \to \gamma\gamma$ decay. (The gauge field A_μ is identified as the photon field γ.)

(b) In the ϕ rest frame, write down the amplitude in the general d dimension. No need to carry out the loop intergral at this point, but need to simplify the trace. (Notice that $k_\mu\epsilon^\mu(k) = 0$ in Lorenz gauge.)

(c) Does the integral have a UV divergence in $d = 4$ dimension (loop momentum goes to ∞)? Answer Yes or No with a few lines of argument.

(d) Does the integral have a singularity in $d = 4$ dimension when the Euclidean loop momentum squared $\overline{q}^2$ go to $-D$? Answer Yes or No with a few lines of argument. (For simplicity, assume that D is real and can be zero for some configuration of x_1, x_2, x_3.)

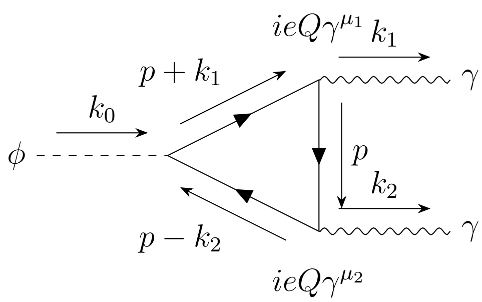
(e) For $m_\Psi = 0$, calculate using dimensional regularization in $d = 4 - \epsilon$. Write down your final answer in the simplest form. (The final answer would be short.)

(f) Carry out the full calculation of the amplitude in Part b using dimensional regularization in $d = 4 - \epsilon$. Write down your final answer in the simplest form. (The full answer would be a long calculation.)
Hint: The following few equations, identities, and tricks, and the discussion around them might be helpful for you: Eq. (62.18), Eq. (47.18), Eq. (67.2).
Remark: No need to answer this, but one can think about it for fun. Recall that taking $\epsilon \to 0$ (from plus or minus direction?) get you back to d = 4. In such a limit, contrast your result in Part f and Part c and think about why.

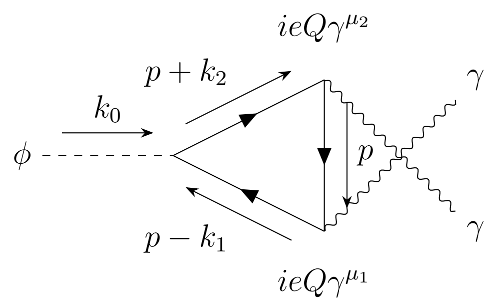
Answer

(a)

We have two leading diagrams that contribute to the decay $\phi \to \gamma\gamma$, as shown in Fig. 13a and Fig. 13b. Both diagrams involve a fermion loop with two photon vertices and one scalar vertex.



(a) Leading diagram for $\phi \to \gamma\gamma$ decay.



(b) Leading diagram for $\phi \to \gamma\gamma$ decay.

Figure 13: Leading diagrams for $\phi \to \gamma\gamma$ decay.

(b)

The amplitude for the decay $\phi \to \gamma\gamma$ can be written as:

$$i\mathcal{M} = i\mathcal{M}_a + i\mathcal{M}_b, \tag{1140}$$

where $i\mathcal{M}_a$ and $i\mathcal{M}_b$ are the contributions from the two diagrams. The contribution from the first diagram (Fig. 13a) is:

$$i\mathcal{M}_a = (-1)(iy)\int \frac{d^dp}{(2\pi)^d}\left[\frac{-i(-(\not{p}+\not{k}_1)+m_\Psi)}{(p+k_1)^2+m_\Psi^2-i\epsilon}(ieQ\gamma^{\mu_1})\epsilon_{\mu_1}(k_1)\frac{-i(-\not{p}+m_\Psi)}{p^2+m_\Psi^2-i\epsilon}\right. \tag{1141}$$

$$\left.\times\,(ieQ\gamma^{\mu_2})\epsilon_{\mu_2}(k_2)\frac{-i(-(\not{p}-\not{k}_2)+m_\Psi)}{(p-k_2)^2+m_\Psi^2-i\epsilon}\right]_{\mathrm{Tr}} \tag{1142}$$

$$= -y(eQ)^2\int \frac{d^dp}{(2\pi)^d}\frac{\mathrm{Tr}\Big[(-(\not{p}+\not{k}_1)+m_\Psi)\gamma^{\mu_1}\epsilon_{\mu_1}(k_1)(-\not{p}+m_\Psi)\gamma^{\mu_2}\epsilon_{\mu_2}(k_2)(-(\not{p}-\not{k}_2)+m_\Psi)\Big]}{\big((p+k_1)^2+m_\Psi^2-i\epsilon\big)\big(p^2+m_\Psi^2-i\epsilon\big)\big((p-k_2)^2+m_\Psi^2-i\epsilon\big)} \tag{1143}$$

$$= ye^2Q^2\int \frac{d^dp}{(2\pi)^d}\frac{\mathrm{Tr}\Big[((\not{p}+\not{k}_1)-m_\Psi)\gamma^{\mu_1}(\not{p}-m_\Psi)\gamma^{\mu_2}((\not{p}-\not{k}_2)-m_\Psi)\Big]}{\big((p+k_1)^2+m_\Psi^2-i\epsilon\big)\big(p^2+m_\Psi^2-i\epsilon\big)\big((p-k_2)^2+m_\Psi^2-i\epsilon\big)}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2). \tag{1144}$$

Note that the factor of (-1) comes from the fermion loop. The contribution from the second diagram (Fig. 13b) is similar, with the photon vertices interchanged(i.e., $k_1 \leftrightarrow k_2$ and $\mu_1 \leftrightarrow \mu_2$). Thus, the total amplitude is:

$$i\mathcal{M}_b = ye^2Q^2\int \frac{d^dp}{(2\pi)^d}\frac{\mathrm{Tr}\Big[((\not{p}+\not{k}_2)-m_\Psi)\gamma^{\mu_2}(\not{p}-m_\Psi)\gamma^{\mu_1}((\not{p}-\not{k}_1)-m_\Psi)\Big]}{\big((p+k_2)^2+m_\Psi^2-i\epsilon\big)\big(p^2+m_\Psi^2-i\epsilon\big)\big((p-k_1)^2+m_\Psi^2-i\epsilon\big)}\epsilon_{\mu_2}(k_2)\epsilon_{\mu_1}(k_1). \tag{1145}$$

Then we can try to simplify the trace in $i\mathcal{M}_a$. Expanding the trace, we only need to keep terms with an even number of gamma matrices, as the trace of an odd number of gamma matrices vanishes. After simplification, we find:

$$\mathrm{Tr}\Big[((\not{p}+\not{k}_1)-m_\Psi)\gamma^{\mu_1}(\not{p}-m_\Psi)\gamma^{\mu_2}((\not{p}-\not{k}_2)-m_\Psi)\Big] \tag{1146}$$

$$=\mathrm{Tr}\Big[(-m_\Psi)\gamma^{\mu_1}(\not{p})\gamma^{\mu_2}(\not{p}-\not{k}_2)\Big]+\mathrm{Tr}\Big((\not{p}+\not{k}_1)\gamma^{\mu_1}(-m_\Psi)\gamma^{\mu_2}(\not{p}-\not{k}_2)\Big) \tag{1147}$$

$$+\mathrm{Tr}\Big((\not{p}+\not{k}_1)\gamma^{\mu_1}(\not{p})\gamma^{\mu_2}(-m_\Psi)\Big)+\mathrm{Tr}\Big((-m_\Psi)\gamma^{\mu_1}(-m_\Psi)\gamma^{\mu_2}(-m_\Psi)\Big) \tag{1148}$$

$$=-m_\Psi p_\alpha(p_\beta-k_{2\beta})\mathrm{Tr}\Big[\gamma^{\mu_1}\gamma^\alpha\gamma^{\mu_2}\gamma^\beta\Big]-m_\Psi(p_\alpha+k_{1\alpha})(p_\beta-k_{2\beta})\mathrm{Tr}\Big[\gamma^\alpha\gamma^{\mu_1}\gamma^{\mu_2}\gamma^\beta\Big] \tag{1149}$$

$$-m_\Psi(p_\alpha+k_{1\alpha})p_\beta\mathrm{Tr}\Big[\gamma^\alpha\gamma^{\mu_1}\gamma^\beta\gamma^{\mu_2}\Big]-m_\Psi^3\mathrm{Tr}\Big[\gamma^{\mu_1}\gamma^{\mu_2}\Big]. \tag{1150}$$

We have the following trace identities in d dimensions:

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu]=-dg^{\mu\nu}, \tag{1151}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]=d(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}). \tag{1152}$$

Using these identities, we can simplify the trace further:

$$-m_\Psi p_\alpha(p_\beta-k_{2\beta})\mathrm{Tr}\Big[\gamma^{\mu_1}\gamma^\alpha\gamma^{\mu_2}\gamma^\beta\Big]=-4m_\Psi p_\alpha(p_\beta-k_{2\beta})(g^{\mu_1\alpha}g^{\mu_2\beta}-g^{\mu_1\mu_2}g^{\alpha\beta}+g^{\mu_1\beta}g^{\alpha\mu_2}) \tag{1153}$$

$$=-dm_\Psi\Big(p^{\mu_1}(p^{\mu_2}-k_2^{\mu_2})-g^{\mu_1\mu_2}p\cdot(p-k_2)+p^{\mu_2}(p^{\mu_1}-k_2^{\mu_1})\Big) \tag{1154}$$

$$=-dm_\Psi\Big(2p^{\mu_1}p^{\mu_2}-p^{\mu_1}k_2^{\mu_2}-p^{\mu_2}k_2^{\mu_1}-g^{\mu_1\mu_2}(p^2-p\cdot k_2)\Big), \tag{1155}$$

$$-m_\Psi(p_\alpha+k_{1\alpha})(p_\beta-k_{2\beta})\mathrm{Tr}\Big[\gamma^\alpha\gamma^{\mu_1}\gamma^{\mu_2}\gamma^\beta\Big]=-dm_\Psi(p_\alpha+k_{1\alpha})(p_\beta-k_{2\beta})(g^{\alpha\mu_1}g^{\mu_2\beta}-g^{\alpha\mu_2}g^{\mu_1\beta}+g^{\alpha\beta}g^{\mu_1\mu_2}) \tag{1156}$$

$$=-dm_\Psi\Big((p^{\mu_1}+k_1^{\mu_1})(p^{\mu_2}-k_2^{\mu_2})-(p^{\mu_2}+k_1^{\mu_2})(p^{\mu_1}-k_2^{\mu_1})+(p+k_1)\cdot(p-k_2)g^{\mu_1\mu_2}\Big) \tag{1157}$$

$$=-dm_\Psi\Big(p^{\mu_1}p^{\mu_2}-p^{\mu_1}k_2^{\mu_2}+k_1^{\mu_1}p^{\mu_2}-k_1^{\mu_1}k_2^{\mu_2}-p^{\mu_2}p^{\mu_1}+p^{\mu_2}k_2^{\mu_1}-k_1^{\mu_2}p^{\mu_1}+k_1^{\mu_2}k_2^{\mu_1} \tag{1158}$$

$$+(p^2+p\cdot(k_1-k_2)-k_1\cdot k_2)g^{\mu_1\mu_2}\Big) \tag{1159}$$

$$=-dm_\Psi\Big(-p^{\mu_1}k_2^{\mu_2}+k_1^{\mu_1}p^{\mu_2}+p^{\mu_2}k_2^{\mu_1}-k_1^{\mu_2}p^{\mu_1}-k_1^{\mu_1}k_2^{\mu_2}+k_1^{\mu_2}k_2^{\mu_1} \tag{1160}$$

$$+(p^2+p\cdot k_1-p\cdot k_2-k_1\cdot k_2)g^{\mu_1\mu_2}\Big), \tag{1161}$$

$$-m_\Psi(p_\alpha+k_{1\alpha})p_\beta\mathrm{Tr}\Big[\gamma^\alpha\gamma^{\mu_1}\gamma^\beta\gamma^{\mu_2}\Big]=-dm_\Psi(p_\alpha+k_{1\alpha})p_\beta(g^{\alpha\mu_1}g^{\beta\mu_2}-g^{\alpha\beta}g^{\mu_1\mu_2}+g^{\alpha\mu_2}g^{\beta\mu_1}) \tag{1162}$$

$$=-dm_\Psi\Big((p^{\mu_1}+k_1^{\mu_1})p^{\mu_2}-g^{\mu_1\mu_2}(p+k_1)\cdot p+(p^{\mu_2}+k_1^{\mu_2})p^{\mu_1}\Big) \tag{1163}$$

$$=-dm_\Psi\Big(2p^{\mu_1}p^{\mu_2}+p^{\mu_1}k_1^{\mu_2}+p^{\mu_2}k_1^{\mu_1}-g^{\mu_1\mu_2}(p^2+p\cdot k_1)\Big), \tag{1164}$$

$$-m_\Psi^3 \mathrm{Tr}\Big[\gamma^{\mu_1}\gamma^{\mu_2}\Big] = dm_\Psi^3 g^{\mu_1\mu_2}. \tag{1165}$$

For the first three terms, we have:

$$-dm_\Psi\Big(2p^{\mu_1}p^{\mu_2} - p^{\mu_1}k_2^{\mu_2} - p^{\mu_2}k_2^{\mu_1} - g^{\mu_1\mu_2}(p^2 - p\cdot k_2)\Big) \tag{1166}$$

$$-dm_\Psi\Big(- p^{\mu_1}k_2^{\mu_2} + k_1^{\mu_1}p^{\mu_2} + p^{\mu_2}k_2^{\mu_1} - k_1^{\mu_2}p^{\mu_1} - k_1^{\mu_1}k_2^{\mu_2} + k_1^{\mu_2}k_2^{\mu_1} \tag{1167}$$

$$+ (p^2 + p\cdot k_1 - p\cdot k_2 - k_1\cdot k_2)g^{\mu_1\mu_2}\Big) \tag{1168}$$

$$-dm_\Psi\Big(2p^{\mu_1}p^{\mu_2} + p^{\mu_1}k_1^{\mu_2} + p^{\mu_2}k_1^{\mu_1} - g^{\mu_1\mu_2}(p^2 + p\cdot k_1)\Big) \tag{1169}$$

$$= - dm_\Psi\Big(4p^{\mu_1}p^{\mu_2} - 2p^{\mu_1}k_2^{\mu_2} + 2k_1^{\mu_1}p^{\mu_2} - k_1^{\mu_1}k_2^{\mu_2} + k_1^{\mu_2}k_2^{\mu_1} + g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2)\Big), \tag{1170}$$

With the fourth term, we have:

$$-dm_\Psi\Big(4p^{\mu_1}p^{\mu_2} - 2p^{\mu_1}k_2^{\mu_2} + 2k_1^{\mu_1}p^{\mu_2} - k_1^{\mu_1}k_2^{\mu_2} + k_1^{\mu_2}k_2^{\mu_1} + g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2)\Big) + dm_\Psi^3 g^{\mu_1\mu_2} \tag{1171}$$

$$= - dm_\Psi\Big(4p^{\mu_1}p^{\mu_2} - 2p^{\mu_1}k_2^{\mu_2} + 2k_1^{\mu_1}p^{\mu_2} - k_1^{\mu_1}k_2^{\mu_2} + k_1^{\mu_2}k_2^{\mu_1} + g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2 - m_\Psi^2)\Big). \tag{1172}$$

Contracting with the photon polarization vectors, by the Lorenz gauge condition $k_\mu\epsilon^\mu(k) = 0$, we get:

$$\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2)\Big(4p^{\mu_1}p^{\mu_2} - 2p^{\mu_1}k_2^{\mu_2} + 2k_1^{\mu_1}p^{\mu_2} - k_1^{\mu_1}k_2^{\mu_2} + k_1^{\mu_2}k_2^{\mu_1} + g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2 - m_\Psi^2)\Big) \tag{1173}$$

$$=4p^{\mu_1}p^{\mu_2}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) - 2p^{\mu_1}k_2^{\mu_2}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) + 2k_1^{\mu_1}p^{\mu_2}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) \tag{1174}$$

$$-k_1^{\mu_1}k_2^{\mu_2}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) + k_1^{\mu_2}k_2^{\mu_1}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) \tag{1175}$$

$$+g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2 - m_\Psi^2)\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) \tag{1176}$$

$$=4p^{\mu_1}p^{\mu_2}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) + k_1^{\mu_2}k_2^{\mu_1}\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) + g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2 - m_\Psi^2)\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) \tag{1177}$$

$$=(4p^{\mu_1}p^{\mu_2} + k_1^{\mu_2}k_2^{\mu_1} + g^{\mu_1\mu_2}(-p^2 - k_1\cdot k_2 - m_\Psi^2))\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2) \tag{1178}$$

$$=(4p^\mu p^\nu + k_1^\nu k_2^\mu + g^{\mu\nu}(-p^2 - k_1\cdot k_2 - m_\Psi^2))\epsilon_\mu(k_1)\epsilon_\nu(k_2). \tag{1179}$$

where we have used the fact that $k_\mu\epsilon^\mu(k) = 0$ and the properties of the photon polarization vectors. Therefore, the amplitude in the general d dimension is:

$$i\mathcal{M}_a = - y(eQ)^2(-dm_\Psi)\int\frac{d^dp}{(2\pi)^d}\frac{(4p^\mu p^\nu + k_1^\nu k_2^\mu + g^{\mu\nu}(-p^2 - k_1\cdot k_2 - m_\Psi^2))}{\big((p+k_1)^2 + m_\Psi^2 - i\epsilon\big)\big(p^2 + m_\Psi^2 - i\epsilon\big)\big((p-k_2)^2 + m_\Psi^2 - i\epsilon\big)}\epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1180}$$

$$=ye^2Q^2dm_\Psi\int\frac{d^dp}{(2\pi)^d}\frac{(4p^\mu p^\nu + k_1^\nu k_2^\mu + g^{\mu\nu}(-p^2 - k_1\cdot k_2 - m_\Psi^2))}{\big((p+k_1)^2 + m_\Psi^2 - i\epsilon\big)\big(p^2 + m_\Psi^2 - i\epsilon\big)\big((p-k_2)^2 + m_\Psi^2 - i\epsilon\big)}\epsilon_\mu(k_1)\epsilon_\nu(k_2). \tag{1181}$$

Similarly, for the second diagram, we have:

$$i\mathcal{M}_b =ye^2Q^2dm_\Psi\int\frac{d^dp}{(2\pi)^d}\frac{(4p^\mu p^\nu + k_2^\nu k_1^\mu + g^{\mu\nu}(-p^2 - k_2\cdot k_1 - m_\Psi^2))}{\big((p+k_2)^2 + m_\Psi^2 - i\epsilon\big)\big(p^2 + m_\Psi^2 - i\epsilon\big)\big((p-k_1)^2 + m_\Psi^2 - i\epsilon\big)}\epsilon_\mu(k_2)\epsilon_\nu(k_1). \tag{1182}$$

Now, the total amplitude is:

$$i\mathcal{M} = ye^2Q^2dm_\Psi \int \frac{d^dp}{(2\pi)^d} \frac{4p^\mu p^\nu + k_1^\nu k_2^\mu + g^{\mu\nu}(-p^2 - k_1 \cdot k_2 - m_\Psi^2)}{\left((p+k_1)^2 + m_\Psi^2\right)\left((p-k_2)^2 + m_\Psi^2\right)\left(p^2 + m_\Psi^2\right)} \epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1183}$$

$$+ye^2Q^2dm_\Psi \int \frac{d^dp}{(2\pi)^d} \frac{4p^\mu p^\nu + k_2^\nu k_1^\mu + g^{\mu\nu}(-p^2 - k_2 \cdot k_1 - m_\Psi^2)}{\left((p+k_2)^2 + m_\Psi^2\right)\left((p-k_1)^2 + m_\Psi^2\right)\left(p^2 + m_\Psi^2\right)} \epsilon_\mu(k_2)\epsilon_\nu(k_1). \tag{1184}$$

where we omit the $-i\epsilon$ terms for simplicity. Note that both integrals have the same form in the numerator but different denominators. Moreover, we can apply eq. (62.18)

$$\int d^dq q^\mu q^\nu f(q^2) = \frac{1}{d} g^{\mu\nu} \int d^dq q^2 f(q^2) \tag{1185}$$

$$\implies dq^\mu q^\nu = g^{\mu\nu} q^2, \tag{1186}$$

to simplify the integral. In our case (if $d = 4$):

$$4p^\mu p^\nu = g^{\mu\nu} p^2. \tag{1187}$$

Hence, in $d = 4$ dimensions, the amplitude simplifies to:

$$i\mathcal{M} = ye^2Q^2 4m_\Psi \int \frac{d^4p}{(2\pi)^4} \frac{k_1^\nu k_2^\mu + g^{\mu\nu}(-k_1 \cdot k_2 - m_\Psi^2)}{\left((p+k_1)^2 + m_\Psi^2\right)\left((p-k_2)^2 + m_\Psi^2\right)\left(p^2 + m_\Psi^2\right)} \epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1188}$$

$$+ye^2Q^2 4m_\Psi \int \frac{d^4p}{(2\pi)^4} \frac{k_2^\nu k_1^\mu + g^{\mu\nu}(-k_2 \cdot k_1 - m_\Psi^2)}{\left((p+k_2)^2 + m_\Psi^2\right)\left((p-k_1)^2 + m_\Psi^2\right)\left(p^2 + m_\Psi^2\right)} \epsilon_\mu(k_2)\epsilon_\nu(k_1) \tag{1189}$$

$$= ye^2Q^2 4m_\Psi \int \frac{d^4p}{(2\pi)^4} \frac{k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{\left((p+k_1)^2 + m_\Psi^2\right)\left((p-k_2)^2 + m_\Psi^2\right)\left(p^2 + m_\Psi^2\right)} \epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1190}$$

$$+ye^2Q^2 4m_\Psi \int \frac{d^4p}{(2\pi)^4} \frac{k_2^\nu k_1^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{\left((p+k_2)^2 + m_\Psi^2\right)\left((p-k_1)^2 + m_\Psi^2\right)\left(p^2 + m_\Psi^2\right)} \epsilon_\mu(k_2)\epsilon_\nu(k_1), \tag{1191}$$

where we apply the on-shell condition: $k_1^2 = 0$, $k_2^2 = 0$, $k_0^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2 = -m_\phi^2$.

(c)

To determine if the integral has a UV divergence in $d = 4$ dimensions, we need to analyze the behavior of the integrand as the loop momentum p goes to infinity. The powers of p in the numerator and denominator are:

$$\text{Numerator} \sim p^0, \tag{1192}$$

$$\text{Denominator} \sim p^6. \tag{1193}$$

The integral is:

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^6}. \tag{1194}$$

This integral converges in $d = 4$ dimensions, as the integrand falls off faster than $1/p^4$ for large p. Therefore,

there is no UV divergence in $d = 4$ dimensions.

(d)

We can apply Feynman's trick to evaluate the integral:

$$\frac{1}{A_1 A_2 A_3} = (2!) \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(x_1 + x_2 + x_3 - 1)(x_1 A_1 + x_2 A_2 + x_3 A_3)^{-3} \tag{1195}$$

$$=(2!) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(x_1 A_1 + x_2 A_2 + (1 - x_1 - x_2) A_3)^3}. \tag{1196}$$

Hence,

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{((p + k_1)^2 + m_\Psi^2)(p^2 + m_\Psi^2)((p - k_2)^2 + m_\Psi^2)} \tag{1197}$$

$$=2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^4p}{(2\pi)^4} \Big[x_1((p + k_1)^2 + m_\Psi^2) + x_2((p - k_2)^2 + m_\Psi^2) + (1 - x_1 - x_2)(p^2 + m_\Psi^2) \Big]^{-3}. \tag{1198}$$

We can complete the square for the quadratic term in p:

$$x_1((p + k_1)^2 + m_\Psi^2) + x_2((p - k_2)^2 + m_\Psi^2) + (1 - x_1 - x_2)(p^2 + m_\Psi^2) \tag{1199}$$

$$=p^2 + p\,(2k_1 x_1 - 2k_2 x_2) + \left(k_1^2 x_1 + k_2^2 x_2 + m_\Psi^2\right) \tag{1200}$$

$$=p^2 + p\,(2k_1 x_1 - 2k_2 x_2) + m_\Psi^2 \tag{1201}$$

$$= (p + (k_1 x_1 - k_2 x_2))^2 - (k_1 x_1 - k_2 x_2)^2 + m_\Psi^2 \tag{1202}$$

$$= (p + (k_1 x_1 - k_2 x_2))^2 - k_1^2 x_1^2 + 2k_1 k_2 x_2 x_1 - k_2^2 x_2^2 + m_\Psi^2 \tag{1203}$$

$$= (p + (k_1 x_1 - k_2 x_2))^2 + m_\Psi^2 - m_\phi^2 x_1 x_2, \tag{1204}$$

where we apply the on-shell condition: $k_1^2 = 0$, $k_2^2 = 0$, $k_0^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2 = -m_\phi^2$. We can define a new variable q:

$$q_{k_1 k_2} = q_{12} = q = p + (k_1 x_1 - k_2 x_2)\,, \tag{1205}$$

$$D = m_\Psi^2 - m_\phi^2 x_1 x_2. \tag{1206}$$

Thus, the integral becomes:

$$2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^4p}{(2\pi)^4} \frac{k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{(q^2 + D)^3} \tag{1207}$$

We can change variables to q, and the integral over p becomes an integral over q:

$$2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^4q}{(2\pi)^4} \frac{k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{(q^2 + D)^3}. \tag{1208}$$

Now we can change q to $\overline{q}$, where $q^0 = i\overline{q}^0$, $\mathbf{q} = \overline{\mathbf{q}}$, $d^4q = id^4\overline{q}$, and $q^2 = -(q^0)^2 + \mathbf{q}^2 = (\overline{q}^0)^2 + \overline{\mathbf{q}}^2 = \overline{q}^2$. The integral over q becomes an integral over $\overline{q}$:

$$2i\int_0^1 dx_1\int_0^{1-x_1} dx_2\int\frac{d^4\overline{q}}{(2\pi)^4}\frac{k_1^\nu k_2^\mu+g^{\mu\nu}(m_\phi^2/2-m_\Psi^2)}{(\overline{q}^2+D)^3}. \tag{1209}$$

Now we can apply eq. (14.27)

$$\int\frac{d^d\overline{q}}{(2\pi)^d}\frac{(\overline{q}^2)^a}{(\overline{q}^2+D)^b}=\frac{\Gamma(b-a-\frac{d}{2})\Gamma(a+\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})(4\pi)^{d/2}}\frac{1}{D^{b-a-\frac{d}{2}}}. \tag{1210}$$

with $a=0$, $b=3$, and $d=4$:

$$\int\frac{d^4\overline{q}}{(2\pi)^4}\frac{1}{(\overline{q}^2+D)^3}=\frac{\Gamma(3-\frac{4}{2})\Gamma(\frac{4}{2})}{\Gamma(3)\Gamma(\frac{4}{2})(4\pi)^{4/2}}\frac{1}{D^{3-\frac{4}{2}}}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(3)\Gamma(2)(4\pi)^2}\frac{1}{D^{1/2}}=\frac{1}{2(4\pi)^2D}. \tag{1211}$$

Thus, the integral over $\overline{q}$ is:

$$2i\int_0^1 dx_1\int_0^{1-x_1} dx_2\frac{1}{2(4\pi)^2D}\left(k_1^\nu k_2^\mu+g^{\mu\nu}(m_\phi^2/2-m_\Psi^2)\right) \tag{1212}$$

$$=\frac{i}{(4\pi)^2}\int_0^1 dx_1\int_0^{1-x_1} dx_2\frac{1}{D}\left(k_1^\nu k_2^\mu+g^{\mu\nu}(m_\phi^2/2-m_\Psi^2)\right). \tag{1213}$$

Now we can discuss D:

$$D=m_\Psi^2-m_\phi^2x_1x_2,\quad \text{with}\quad x_1x_2\leq\frac{1}{4}.\implies D\geq\frac{1}{4}(4m_\Psi^2-m_\phi^2). \tag{1214}$$

Hence if $4m_\Psi^2-m_\phi^2>0$, that is, $m_\phi<2m_\Psi$, then D is always positive, and there is no singularities in the integral over D. If $4m_\Psi^2-m_\phi^2<0$, that is, $m_\phi>2m_\Psi$, then D can be negative, and the integral over D has singularities. **Therefore, the integral over D has singularities if $m_\phi>2m_\Psi$.**

Remark: This result is physical reasonable because if $m_\phi>2m_\Psi$, the intermediate state with mass m_ϕ can decay into two particles with mass m_Ψ each, which is cosistent with the Lehmann-Källén theorem. This means the amplitude can have both real and imaginary parts.

(e)

Now, if $m_\Psi=0$, the amplitude $i\mathcal{M}=0$ since

$$i\mathcal{M}=4ye^2Q^2m_\Psi\int\frac{d^4p}{(2\pi)^4}\frac{k_1^\nu k_2^\mu+g^{\mu\nu}(m_\phi^2/2-m_\Psi^2)}{\left((p+k_1)^2+m_\Psi^2\right)\left((p-k_2)^2+m_\Psi^2\right)\left(p^2+m_\Psi^2\right)}\epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1215}$$

$$+4ye^2Q^2m_\Psi\int\frac{d^4p}{(2\pi)^4}\frac{k_2^\nu k_1^\mu+g^{\mu\nu}(m_\phi^2/2-m_\Psi^2)}{\left((p+k_2)^2+m_\Psi^2\right)\left((p-k_1)^2+m_\Psi^2\right)\left(p^2+m_\Psi^2\right)}\epsilon_\mu(k_2)\epsilon_\nu(k_1) \tag{1216}$$

$$\implies i\mathcal{M}\propto m_\Psi. \tag{1217}$$

This result is independent of $d=4-\epsilon$, $d=4$ or any other dimension. **Therefore, the amplitude $i\mathcal{M}=0$ if $m_\Psi=0$.**

(f)

Now we can evaluate the integral:

$$iM =4ye^2Q^2m_\Psi \int \frac{d^4p}{(2\pi)^4} \frac{k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{\left((p+k_1)^2+m_\Psi^2\right)\left((p-k_2)^2+m_\Psi^2\right)\left(p^2+m_\Psi^2\right)}\epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1218}$$

$$+4ye^2Q^2m_\Psi \int \frac{d^4p}{(2\pi)^4} \frac{k_2^\nu k_1^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{\left((p+k_2)^2+m_\Psi^2\right)\left((p-k_1)^2+m_\Psi^2\right)\left(p^2+m_\Psi^2\right)}\epsilon_\mu(k_2)\epsilon_\nu(k_1) \tag{1219}$$

$$=8ye^2Q^2m_\Psi 2i\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^4\overline{q}}{(2\pi)^4}\frac{k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)}{(\overline{q}^2+D)^3}\epsilon_\mu(k_1)\epsilon_\nu(k_2). \tag{1220}$$

Now if we consider $d = 4-\epsilon$ dimensions, the integral over $\overline{q}$ is:

$$\int \frac{d^{4-\epsilon}\overline{q}}{(2\pi)^{4-\epsilon}}\frac{1}{(\overline{q}^2+D)^3} = \frac{\Gamma(3-\frac{4-\epsilon}{2})\Gamma(\frac{4-\epsilon}{2})}{\Gamma(3)\Gamma(\frac{4-\epsilon}{2})(4\pi)^{(4-\epsilon)/2}}\frac{1}{D^{3-\frac{4-\epsilon}{2}}} = \frac{\Gamma(1+\frac{\epsilon}{2})\Gamma(2-\frac{\epsilon}{2})}{\Gamma(3)\Gamma(2-\frac{\epsilon}{2})(4\pi)^{(4-\epsilon)/2}}\frac{1}{D^{1+\frac{\epsilon}{2}}}. \tag{1221}$$

The gamma function has the following property:

$$\Gamma(n+1) = n!, \tag{1222}$$

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\frac{(2n)!}{2^n n!} \tag{1223}$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n}\frac{1}{k} + \mathcal{O}(x)\right], \quad \text{for small} \quad x. \tag{1224}$$

However, all gamma function in the numerator and denominator are positive for $n \geq 0$ and $x > 0$. Therefore, the integral over $\overline{q}$ is:

$$\int \frac{d^{4-\epsilon}\overline{q}}{(2\pi)^{4-\epsilon}}\frac{1}{(\overline{q}^2+D)^3} \to \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)\Gamma(2)(4\pi)^2}\frac{1}{D^1} = \frac{1}{2(4\pi)^2 D} \quad \text{as} \quad \epsilon \to 0. \tag{1225}$$

Thus, the integral over $\overline{q}$ is:

$$iM =8ye^2Q^2m_\Psi 2i\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{2(4\pi)^2 D}\left(k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)\right)\epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1226}$$

$$=\frac{8iye^2Q^2m_\Psi}{(4\pi)^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{D}\left(k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)\right)\epsilon_\mu(k_1)\epsilon_\nu(k_2) \tag{1227}$$

$$=\frac{iye^2Q^2m_\Psi}{2\pi^2}\left[\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{D}\right]\left(k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)\right)\epsilon_\mu(k_1)\epsilon_\nu(k_2). \tag{1228}$$

Now we can discuss $D = m_\Psi^2 - m_\phi^2 x_1 x_2$:

$$
\begin{aligned}
&\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{D} && (1229)\\
=&\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{m_\Psi^2 - m_\phi^2 x_1 x_2} && (1230)\\
=&\frac{1}{m_\Psi^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{1-\frac{m_\phi^2}{m_\Psi^2}x_1x_2} = \frac{1}{m_\Psi^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{1-\lambda x_1 x_2}, \quad \text{with} \quad \lambda = \frac{m_\phi^2}{m_\Psi^2} && (1231)\\
=&\frac{1}{m_\Psi^2}\frac{\mathrm{Li}_2\left(\frac{1}{2}\left(\lambda - \sqrt{\lambda-4}\sqrt{\lambda}\right)\right) + \mathrm{Li}_2\left(\frac{1}{2}\left(\lambda + \sqrt{\lambda-4}\sqrt{\lambda}\right)\right)}{\lambda}, \quad \text{with} \quad \lambda = \frac{m_\phi^2}{m_\Psi^2}, && (1232)
\end{aligned}
$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function. **Note that if $\lambda = \frac{m_\phi^2}{m_\Psi^2} > 4$, that is $m_\phi > 2m_\Psi$, then result has both real and imaginary parts, consistent with the interpretation of the previous part (d).** Therefore, the amplitude $i\mathcal{M}$ becomes:

$$
i\mathcal{M} = \frac{iye^2Q^2}{2\pi^2 m_\Psi}\left(\frac{\mathrm{Li}_2\left(\frac{1}{2}\left(\lambda - \sqrt{\lambda-4}\sqrt{\lambda}\right)\right) + \mathrm{Li}_2\left(\frac{1}{2}\left(\lambda + \sqrt{\lambda-4}\sqrt{\lambda}\right)\right)}{\lambda}\right)\left(k_1^\nu k_2^\mu + g^{\mu\nu}(m_\phi^2/2 - m_\Psi^2)\right)\epsilon_\mu(k_1)\epsilon_\nu(k_2). \tag{1233}
$$

where $\lambda = \frac{m_\phi^2}{m_\Psi^2}$. This is the final result for the amplitude in $d = 4-\epsilon$ or $d = 4$ dimensions. □