University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

December 8, 2025

Assignment 13 due on Monday December 8 at 10PM

The metric for an electrically charged and rotating black hole is called the Kerr-Newman metric. It is a simple extension of the Kerr metric but with the addition of the electromagnetic field. Read pp. 261-267 of Carroll before starting this problem.

Question 1

In class we derived the thermodynamic-like relationship

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J, \tag{1}$$

for the Kerr solution. For the Kerr-Newman solution, there should be another term on the right hand sides of the form $\mu\delta Q$. In this case, what are κ, Ω_H, and μ?

Answer

We start from the Kerr-Newman metric in Boyer-Lindquist coordinates:

$$ds^2 = -\left(1 - \frac{2GMr - GQ^2}{\rho^2}\right)dt^2 - \frac{2a(2GMr - GQ^2)\sin^2\theta}{\rho^2}dtd\phi \tag{2}$$

$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{(2GMr - GQ^2)a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \tag{3}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2GMr + a^2 + GQ^2$, and $a = \frac{J}{M}$ is the specific angular momentum. The event horizon is located at $r_+ = GM + \sqrt{(GM)^2 - a^2 - GQ^2}$. The surface gravity κ is given by

$$\kappa = \frac{r_+ - GM}{r_+^2 + a^2} = \frac{\sqrt{(GM)^2 - a^2 - GQ^2}}{r_+^2 + a^2}. \tag{4}$$

The angular velocity of the horizon Ω_H is

$$\Omega_H = \frac{a}{r_+^2 + a^2}. \tag{5}$$

The electrostatic potential μ at the horizon is given by

$$\mu = \frac{Qr_+}{r_+^2 + a^2}. \tag{6}$$

Thus, the first law of black hole mechanics for the Kerr-Newman black hole is

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J + \mu\delta Q. \tag{7}$$