University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8501
General Relativity I

Assignment Solution

Lecture Instructor: Professor Joseph Kapusta

Zong-En Chen
chen9613@umn.edu

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Question 1

The Lagrangian density for a scalar field ϕ is givesn by

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - U(\phi), \tag{1}$$

where $U(\phi)$ is a potential, typically $\lambda\phi^4$, and the metric is given. The action is

$$I_\phi = \int d^4x \sqrt{g} \mathcal{L}. \tag{2}$$

There are two ways to calculate the eneergy-momentum tensor.

(a) In field theory, it is usually calculated by varying the cation with respect to the field, yielding the formula

$$T^{\mu\nu} = g^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\kappa \phi g^{\kappa\nu}. \tag{3}$$

similar to classical mechanics. Calculate it this way.

(b) As discussed in lecture, it can also be calculated by varying the action with respect to the metric via the formula

$$\delta I_\phi = \frac{1}{2} \int d^4x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu}. \tag{4}$$

Calculate it this way. Does it agree with part (a)?

Answer

(a)

We have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\frac{1}{2} g^{\alpha\beta} \left(\delta^\mu_\alpha \partial_\beta \phi + \partial_\alpha \phi \delta^\mu_\beta \right) = -g^{\mu\beta} \partial_\beta \phi. \tag{5}$$

Hence,

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\kappa\phi g^{\kappa\nu} \tag{6}$$

$$= g^{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2 - U(\phi)\right) + g^{\mu\beta}\partial_\beta\phi\partial_\kappa\phi g^{\kappa\nu} \tag{7}$$

$$= g^{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2 - U(\phi)\right) + \partial^\mu\phi\partial^\nu\phi. \tag{8}$$

(b)

We have

$$\delta I_\phi = \int d^4x\,(\delta\sqrt{g}\mathcal{L} + \sqrt{g}\delta\mathcal{L}) \tag{9}$$

$$= \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\mathcal{L}\delta g_{\mu\nu} + \delta\mathcal{L}\right) \tag{10}$$

where we have used $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}$ (equation on page 364 in Weinberg's Gravitation and Cosmology). Next, we calculate $\delta\mathcal{L}$:

$$\delta\mathcal{L} = -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g^{\mu\nu}\delta(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}m^2\delta(\phi^2) - \delta U(\phi) \tag{11}$$

$$= -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{12}$$

$$= \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\partial_\mu\phi\partial_\nu\phi \tag{13}$$

$$= \frac{1}{2}\delta g_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi \tag{14}$$

$$= \frac{1}{2}\delta g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi, \tag{15}$$

since the other terms do not depend on the metric. Therefore, we have

$$\delta I_\phi = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\mathcal{L}\delta g_{\mu\nu} + \frac{1}{2}\delta g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\right) \tag{16}$$

$$= \frac{1}{2}\int d^4x\sqrt{g}\,(g^{\mu\nu}\mathcal{L} + \partial^\mu\phi\partial^\nu\phi)\,\delta g_{\mu\nu}. \tag{17}$$

Comparing this with

$$\delta I_\phi = \frac{1}{2}\int d^4x\sqrt{g}T^{\mu\nu}\delta g_{\mu\nu}, \tag{18}$$

we find

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} + \partial^\mu\phi\partial^\nu\phi, \tag{19}$$

which agrees with the result from part (a). $\square$

Question 2

In Newtonian mechanics a space probe in a circular orbit of radius r about the sun (for example) with mass M has a period given by

$$t_N = 2\pi\sqrt{\frac{r^3}{GM}}. \tag{20}$$

Consider a space probe in a circular orbit of radius $r > \frac{3}{2}GM$ in the plane $\theta = \frac{\pi}{2}$ in standard coordinates about Schwarzschild black hole of mass M . We need to solve the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{21}$$

for $\mu = t, r, \theta, \phi$. Yes, all four! The affine connection/Christoffel symbol can be found in various sources, but be sure their conventions align with ours. When solving these equations use the initial condition $t = 0$ and $\phi = 0$ at $\tau = 0$. Note that in this situation

$$d\tau^2 = \left(1 - \frac{R_s}{r}\right)dt^2 - r^2 d\phi^2. \tag{22}$$

(a) What is the period τ_p as measured by a clock in the space probe? How does it relate to t_N ? What happens as $r \to \frac{3}{2}R_s$?

(b) What is the period t_p in standard coordinates? How does it relate to t_N?

Answer

(a)

We first write down the non-zero Christoffel symbols in Schwarzschild coordinates:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{GM}{r(r-2GM)}, \tag{23}$$

$$\Gamma^r_{tt} = \frac{GM(r-2GM)}{r^3}, \tag{24}$$

$$\Gamma^r_{rr} = -\frac{GM}{r(r-2GM)}, \tag{25}$$

$$\Gamma^r_{\theta\theta} = -(r-2GM), \tag{26}$$

$$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\theta, \tag{27}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \tag{28}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \tag{29}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \tag{30}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{31}$$

For a circular orbit, we have $r = \text{constant}$ and $\theta = \frac{\pi}{2}$. Therefore, the geodesic equations for $\mu = t, r, \theta, \phi$ reduce to

$$\frac{d^2t}{d\tau^2} + 2\Gamma^t_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0, \qquad \text{t component}, \tag{32}$$

$$\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = 0, \qquad \text{r component}, \tag{33}$$

$$\Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 = 0, \qquad \theta \text{ component}, \tag{34}$$

$$\frac{d^2\phi}{d\tau^2} + 2\Gamma^\phi_{r\phi}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0, \qquad \phi \text{ component}. \tag{35}$$

The t and ϕ components are automatically satisfied since $dr/d\tau = 0$. The solution for t is straightforward:

$$\frac{d^2t}{d\tau^2} = 0 \implies \frac{dt}{d\tau} = \text{constant}. \tag{36}$$

The solution for ϕ is also straightforward:

$$\frac{d^2\phi}{d\tau^2} = 0 \implies \frac{d\phi}{d\tau} = \text{constant}. \tag{37}$$

The θ component is automatically satisfied since $\Gamma^\theta_{\phi\phi} = 0$ at $\theta = \frac{\pi}{2}$. The r component gives

$$\frac{GM(r-2GM)}{r^3}\left(\frac{dt}{d\tau}\right)^2 - (r-2GM)\left(\frac{d\phi}{d\tau}\right)^2 = 0. \tag{38}$$

Rearranging, we find

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^3}\left(\frac{dt}{d\tau}\right)^2. \tag{39}$$

Next, we use the relation

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - r^2 d\phi^2, \tag{40}$$

to express $d\tau$ in terms of dt and $d\phi$. Dividing both sides by $d\tau^2$, we get

$$1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2. \tag{41}$$

Substituting the expression for $\left(\frac{d\phi}{d\tau}\right)^2$, we have

$$1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - r^2 \cdot \frac{GM}{r^3}\left(\frac{dt}{d\tau}\right)^2. \tag{42}$$

Simplifying, we find

$$1 = \left(1 - \frac{3GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2. \tag{43}$$

Solving for $\frac{dt}{d\tau}$, we get

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{3GM}{r}}} = \frac{1}{\sqrt{1 - \frac{3R_s/2}{r}}}, \tag{44}$$

$$\implies \frac{d\phi}{d\tau} = \sqrt{\frac{GM}{r^3}} \cdot \frac{1}{\sqrt{1 - \frac{3R_s/2}{r}}}. \tag{45}$$

The period τ_p as measured by a clock in the space probe is given by

$$\tau_p = \frac{2\pi}{\frac{d\phi}{d\tau}}. \tag{46}$$

Using the relation between $\frac{d\phi}{d\tau}$ and $\frac{dt}{d\tau}$, we find

$$\tau_p = 2\pi\sqrt{\frac{r^3}{GM}}\sqrt{1 - \frac{3R_s/2}{r}} = t_N\sqrt{1 - \frac{3R_s/2}{r}}. \tag{47}$$

As $r \to \frac{3}{2}R_s$, we see that $\tau_p \to 0$. As $r \gg R_s$, we have $\tau_p \approx t_N$.

(b)

The period t_p in standard coordinates is given by

$$t_p = \frac{2\pi}{\frac{d\phi}{dt}}. \tag{48}$$

Using the chain rule, we have

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau} \cdot \frac{d\tau}{dt} = \frac{d\phi}{d\tau} \cdot \frac{1}{\frac{dt}{d\tau}}. \tag{49}$$

Therefore,

$$\frac{d\phi}{dt} = \sqrt{\frac{GM}{r^3}} \cdot \frac{1}{\sqrt{1 - \frac{3R_s/2}{r}}} \cdot \sqrt{1 - \frac{3R_s/2}{r}} \tag{50}$$

$$= \sqrt{\frac{GM}{r^3}}. \tag{51}$$

Thus, we find

$$t_p = 2\pi\sqrt{\frac{r^3}{GM}} = t_N. \tag{52}$$

□