

University of Minnesota
School of Physics and Astronomy

2025 Fall Physics 8901
Elementary Particle Physics I
Assignment Solution

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Question 1

Three-photon decay of a scalar particle

Consider the decay $X \rightarrow 3\gamma$, where X is a scalar particle of mass M . Assume the decay amplitude \mathcal{M}_{fi} is approximately constant (i.e. independent of photon energies and angles) and can be written as $\mathcal{M}_{fi} = A$. This is a good approximation for the decay of orthopositronium in its ground state.

- (a) Derive the differential decay rate $\frac{d\Gamma}{d\omega}$ corresponding to the measured energy ω of a single photon in the rest frame of X .
- (b) Express the total decay rate Γ in terms of the constant amplitude A .

Answer

We can start from the general expression for the decay rate of a particle decaying into three massless particles:

$$d\Gamma = \frac{1}{2M} \frac{1}{3!} |\mathcal{M}_{fi}|^2 d\tau_3, \quad (1)$$

where $d\tau_3$ is the three-body phase space element, M is the mass of the decaying particle, $1/3!$ accounts for the identical photons in the final state and $\mathcal{M}_{fi} = A$ is the invariant matrix element. The three-body phase space element for massless particles can be expressed as:

$$d\tau_3 = (2\pi)^4 \delta^4(P - p_1 - p_2 - p_3) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3}, \quad (2)$$

where P is the four-momentum of the decaying particle, and p_i and E_i are the momenta and energies of the final state photons, respectively. If we consider the rest frame of the decaying particle, we have $P = (M, 0, 0, 0)$. Besides, we can apply the splitting formula for three-body phase space:

$$d\tau_3 = d\tau_2(M \rightarrow p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \rightarrow p_2 + p_3), \quad (3)$$

where $q = p_2 + p_3$ is the combined four-momentum of photons 2 and 3. The two-body phase space elements can be expressed as:

$$d\tau_2(M \rightarrow p_1 + q) = (2\pi)^4 \delta^4(P - p_1 - q) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 q}{(2\pi)^3 2E_q}, \quad (4)$$

$$d\tau_2(q \rightarrow p_2 + p_3) = (2\pi)^4 \delta^4(q - p_2 - p_3) \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3}. \quad (5)$$

To find the differential decay rate with respect to the energy of one photon, say $\omega = E_1$, we can integrate over the other variables. The total energy conservation gives us:

$$M = E_1 + E_2 + E_3. \quad (6)$$

Since the photons are massless, we have $E_i = |\vec{p}_i|$.

$$d\Gamma = \frac{1}{2M} \frac{1}{3!} |A|^2 d\tau_3 \quad (7)$$

$$= \frac{1}{2M} \frac{1}{3!} |A|^2 d\tau_2(M \rightarrow p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \rightarrow p_2 + p_3). \quad (8)$$

The two-body phase space elements can be evaluated in their respective rest frames. In this frame, $q = (\sqrt{q^2}, 0, 0, 0)$, and the energies of the photons are $E_2 = |\vec{p}_2|$ and $E_3 = |\vec{p}_3|$.

$$\int d\tau_2(q \rightarrow p_2 + p_3) = \int (2\pi)^4 \delta(\sqrt{q^2} - E_2 - E_3) \delta^3(\vec{0} - \vec{p}_2 - \vec{p}_3) \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3} \quad (9)$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_2 - E_3) \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{1}{(2\pi)^3 2E_2} \quad (10)$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_2) \frac{4\pi E_2^2 dE_2}{(2\pi)^3 2E_2} \frac{1}{(2\pi)^3 2E_2} \quad (11)$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_2) \frac{4\pi}{4(2\pi)^6} dE_2 \quad (12)$$

$$= (2\pi)^4 \frac{\pi}{(2\pi)^6} \frac{1}{2} \quad (13)$$

$$= \frac{1}{8\pi}, \quad (14)$$

where we have used the delta function to perform the integral over E_2 to get extra factor of $1/2$. Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of M . In this frame, $P = (M, 0, 0, 0)$, and the energies are $p_1^\mu = (\omega, \vec{p}_1)$ and $q^\mu = (M - \omega, \vec{q}) = (E_q, \vec{q})$.

$$\int d\tau_2(M \rightarrow p_1 + q) = \int (2\pi)^4 \delta(M - E_1 - E_q) \delta^3(\vec{0} - \vec{p}_1 - \vec{q}) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 q}{(2\pi)^3 2E_q} \quad (15)$$

$$= \int (2\pi)^4 \delta(M - \omega - E_q) \frac{d^3 p_1}{(2\pi)^3 2\omega} \frac{1}{(2\pi)^3 2E_q} \quad (16)$$

$$= \int (2\pi)^4 \delta(M - \omega - E_q) \frac{4\pi \omega^2 d\omega}{(2\pi)^3 2\omega} \frac{1}{(2\pi)^3 2E_q} \quad (17)$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega \quad (18)$$

$$= \frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega. \quad (19)$$

Thus, we have

$$\int d\tau_2(M \rightarrow p_1 + q) = \frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega. \quad (20)$$

Now we can combine the results to get the differential decay rate:

$$d\Gamma = \frac{1}{2M} \frac{1}{3!} |A|^2 \left(\frac{1}{4\pi} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \right) \quad (21)$$

$$= \frac{|A|^2}{768\pi^3 M} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega dq^2. \quad (22)$$

We have to be careful when we use the delta function to perform the integral over ω since $E_q = \sqrt{(\vec{q})^2 + q^2} = \sqrt{\omega^2 + q^2}$

$$\frac{d}{dq^2}(\omega + E_q) = \frac{d}{dq^2}(\omega + \sqrt{\omega^2 + q^2}) = \frac{1}{2\sqrt{\omega^2 + q^2}} = \frac{1}{2E_q}. \quad (23)$$

Thus, we have

$$d\Gamma = \frac{|A|^2}{768\pi^3 M} \int \delta(M - \omega - E_q) \frac{\omega}{E_q} d\omega dq^2 \quad (24)$$

$$= \frac{|A|^2}{768\pi^3 M} \frac{\omega}{E_q} 2E_q d\omega \quad (25)$$

$$= \frac{|A|^2}{384\pi^3 M} \omega d\omega. \quad (26)$$

However, we cannot distinguish which photon we are measuring, so we have to multiply by a factor of 3. Therefore, the final expression for the differential decay rate is:

$$\frac{d\Gamma}{d\omega} = \frac{|A|^2}{128\pi^3 M} \omega, \quad 0 \leq \omega \leq \frac{M}{2}. \quad (27)$$

To find the total decay rate, we can integrate over the allowed range of ω :

$$\Gamma = \int_0^{M/2} \frac{d\Gamma}{d\omega} d\omega \quad (28)$$

$$= \int_0^{M/2} \frac{|A|^2}{128\pi^3 M} \omega d\omega \quad (29)$$

$$= \frac{|A|^2}{128\pi^3 M} \left[\frac{\omega^2}{2} \right]_0^{M/2} \quad (30)$$

$$= \frac{|A|^2}{128\pi^3 M} \frac{M^2}{8} \quad (31)$$

$$= \frac{|A|^2 M}{1024\pi^3}. \quad (32)$$

□

Question 2

Find the total Lorentz-invariant three-body phase space τ_3 for a final state containing one particle of mass m and two massless particles, produced from an initial particle of mass M . Express your final result in terms of the Mandelstam variable s and m .

Answer

From the definition of the three-body phase space, we have

$$\tau_3 = \int (2\pi)^4 \delta^4(P - p_1 - p_2 - p_3) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3}, \quad (33)$$

where P is the four-momentum of the initial particle, p_1 is the four-momentum of the massive particle with mass m , and p_2 and p_3 are the four-momenta of the two massless particles. We can use the splitting formula for three-body phase space:

$$\tau_3 = \int d\tau_2(M \rightarrow p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \rightarrow p_2 + p_3), \quad (34)$$

where $q = p_2 + p_3$ is the combined four-momentum of the two massless particles. The two-body phase space elements can be expressed as:

$$d\tau_2(M \rightarrow p_1 + q) = (2\pi)^4 \delta^4(P - p_1 - q) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 q}{(2\pi)^3 2E_q}, \quad (35)$$

$$d\tau_2(q \rightarrow p_2 + p_3) = (2\pi)^4 \delta^4(q - p_2 - p_3) \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3}. \quad (36)$$

For the second two-body phase space element, we can quote the result from the previous problem, since both particles are massless:

$$\int d\tau_2(q \rightarrow p_2 + p_3) = \frac{1}{8\pi}. \quad (37)$$

Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame of M . In this frame, $P = (M, 0, 0, 0)$, and the energies are $E_1 = \sqrt{|\vec{p}_1|^2 + m^2}$ and $E_q = M - E_1 = \sqrt{|\vec{q}|^2 + q^2}$.

$$\int d\tau_2(M \rightarrow p_1 + q) = \int (2\pi)^4 \delta(M - E_1 - E_q) \delta^3(\vec{0} - \vec{p}_1 - \vec{q}) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 q}{(2\pi)^3 2E_q} \quad (38)$$

$$= \int (2\pi)^4 \delta(M - E_1 - E_q) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_q} \quad (39)$$

$$= \int (2\pi)^4 \delta(M - E_1 - E_q) \frac{4\pi |\vec{p}_1|^2 d|\vec{p}_1|}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_q} \quad (40)$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| \quad (41)$$

$$= \frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1|. \quad (42)$$

Thus, we have

$$\int d\tau_2(M \rightarrow p_1 + q) = \frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1|. \quad (43)$$

Now we can combine the results to get the total three-body phase space:

$$\tau_3 = \int d\tau_2(M \rightarrow p_1 + q) \frac{dq^2}{2\pi} d\tau_2(q \rightarrow p_2 + p_3) \quad (44)$$

$$= \int \left(\frac{1}{4\pi} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \right) \quad (45)$$

$$= \frac{1}{64\pi^3} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| dq^2. \quad (46)$$

To perform the integral over $|\vec{p}_1|$, we need to express E_q in terms of $|\vec{p}_1|$ and q^2 :

$$E_q = \sqrt{|\vec{q}|^2 + q^2} = \sqrt{|\vec{p}_1|^2 + q^2} \quad (47)$$

$$E_1 = \sqrt{|\vec{p}_1|^2 + m^2}. \quad (48)$$

We also need to compute the derivative of $(E_1 + E_q)$ with respect to $|\vec{p}_1|$:

$$\frac{d}{d|\vec{p}_1|}(E_1 + E_q) = \frac{d}{d|\vec{p}_1|} \left(\sqrt{|\vec{p}_1|^2 + m^2} + \sqrt{|\vec{p}_1|^2 + q^2} \right) = \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + m^2}} + \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + q^2}} = \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_1|}{E_q} \quad (49)$$

$$= |\vec{p}_1| \left(\frac{E_1 + E_q}{E_1 E_q} \right) = |\vec{p}_1| \left(\frac{M}{E_1 E_q} \right). \quad (50)$$

Thus, we have

$$\tau_3 = \frac{1}{64\pi^3} \int \delta(M - E_1 - E_q) \frac{|\vec{p}_1|^2}{E_1 E_q} d|\vec{p}_1| dq^2 \quad (51)$$

$$= \frac{1}{64\pi^3} \int \frac{|\vec{p}_1|^2}{E_1 E_q} \frac{E_1 E_q}{M |\vec{p}_1|} dq^2 \quad (52)$$

$$= \frac{1}{64\pi^3 M} \int |\vec{p}_1| dq^2. \quad (53)$$

We can apply the relation between q^2 and $|\vec{p}_1|$ to change the integration variable in the rest frame of M , where $P^\mu = (M, 0, 0, 0)$ and $p_1^\mu = (E_1, \vec{p}_1)$, $q^\mu = P^\mu - p_1^\mu$:

$$q^2 = (P - p_1)^2 = M^2 + m^2 - 2ME_1 = M^2 + m^2 - 2M\sqrt{|\vec{p}_1|^2 + m^2}, \quad (54)$$

$$\frac{dq^2}{d|\vec{p}_1|} = -2M \frac{|\vec{p}_1|}{\sqrt{|\vec{p}_1|^2 + m^2}} = -2M \frac{|\vec{p}_1|}{E_1}. \quad (55)$$

By $E_1^2 = |\vec{p}_1|^2 + m^2$, we have $dE_1 = \frac{|\vec{p}_1|}{E_1} d|\vec{p}_1|$. Thus, we have

$$\tau_3 = \frac{1}{64\pi^3 M} \int_{q_{min}}^{q_{max}} |\vec{p}_1| dq^2 \quad (56)$$

$$= \frac{1}{64\pi^3 M} \int_{|\vec{p}_1|_{min}}^{|\vec{p}_1|_{max}} |\vec{p}_1| \left(-2M \frac{|\vec{p}_1|}{E_1} \right) d|\vec{p}_1| \quad (57)$$

$$= \frac{1}{32\pi^3} \int_{|\vec{p}_1|_{min}}^{|\vec{p}_1|_{max}} \frac{|\vec{p}_1|^2}{E_1} d|\vec{p}_1| \quad (58)$$

$$= \frac{1}{32\pi^3} \int_{E_{1,min}}^{E_{1,max}} \sqrt{E_1^2 - m^2} dE_1. \quad (59)$$

The limits of integration for E_1 can be found from the kinematic constraints. The minimum energy occurs when the two massless particles are emitted back-to-back with maximum energy, and the maximum energy occurs when the massive particle is at rest:

$$E_{1,min} = m, \quad (60)$$

$$E_{1,max} = \frac{M^2 + m^2}{2M} = \frac{s + m^2}{2\sqrt{s}}, \text{ from } q_{min}^2 = 0 \Rightarrow M^2 + m^2 - 2ME_{1,max} = 0. \quad (61)$$

Thus, we have (by *Mathematica*)

$$\tau_3 = \frac{1}{32\pi^3} \int_m^{\frac{s+m^2}{2\sqrt{s}}} \sqrt{E_1^2 - m^2} dE_1 \quad (62)$$

$$= \frac{1}{32\pi^3} \frac{-m^4 + 2m^2s \log\left(\frac{m^2}{s}\right) + s^2}{8s} \quad (63)$$

$$= \frac{-m^4 + 2m^2s \log\left(\frac{m^2}{s}\right) + s^2}{256\pi^3 s}. \quad (64)$$

□

Question 3

Hadronic Transitions in Quarkonium

- (a) The decay amplitude for the transition $\psi(2S) \rightarrow J/\psi(1S)\pi^+\pi^-$ can be approximated by

$$M_{fi} = a_\psi \sqrt{4m_{\psi(2S)}m_{J/\psi}}(q^2 - 4.5m_\pi^2), \quad (65)$$

where q is the total four-momentum of the emitted pion pair, and a_ψ is a dimensionful coupling constant. Using the experimental decay rate (performing a numerical phase-space integration if necessary), determine the absolute value $|a_\psi|$ in appropriate units of GeV.

- (b) Perform the same analysis for the decay $\Upsilon(2S) \rightarrow \Upsilon(1S)\pi^+\pi^-$, for which the phenomenological amplitude is

$$M_{fi} = a_\Upsilon \sqrt{4m_{\Upsilon(2S)}m_{\Upsilon(1S)}}(q^2 - 3.2m_\pi^2). \quad (66)$$

Compare the extracted magnitudes of $|a_\psi|$ and $|a_\Upsilon|$. What can you infer about the relative spatial extent of the charmonium and bottomonium bound states?

Note: The constants a_ψ and a_Υ reflect overlap integrals between the $2S$ and $1S$ quarkonium wavefunctions and scale with the mean-square radius $\langle r^2 \rangle$ of the bound state.

Answer

- (a)

The decay rate for the process $\psi(2S) \rightarrow J/\psi(1S)\pi^+\pi^-$ can be expressed as:

$$d\Gamma = \frac{1}{2m_{\psi(2S)}} |M_{fi}|^2 d\tau_3, \quad (67)$$

where $d\tau_3$ is the three-body phase space element for the final state particles. The three-body phase space element can be expressed as:

$$d\tau_3 = (2\pi)^4 \delta^4(P - p_{J/\psi} - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}} \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}}, \quad (68)$$

where P is the four-momentum of the initial $\psi(2S)$ particle, and $p_{J/\psi}$, p_{π^+} , and p_{π^-} are the four-momenta of the final state particles. In the rest frame of $\psi(2S)$, we have $P = (m_{\psi(2S)}, 0, 0, 0)$. The invariant matrix element is given by:

$$M_{fi} = a_\psi \sqrt{4m_{\psi(2S)}m_{J/\psi}}(q^2 - 4.5m_\pi^2), \quad (69)$$

where $q = p_{\pi^+} + p_{\pi^-}$ is the combined four-momentum of the pion pair. To find the total decay rate, we need to integrate over the three-body phase space:

$$\Gamma = \int d\Gamma = \frac{1}{2m_{\psi(2S)}} |a_\psi|^2 4m_{\psi(2S)} m_{J/\psi} \int (q^2 - 4.5m_\pi^2)^2 d\tau_3. \quad (70)$$

For $d\tau_3$, we can use the splitting formula for three-body phase space:

$$d\tau_3 = d\tau_2(m_{\psi(2S)} \rightarrow p_{J/\psi} + q) \frac{dq^2}{2\pi} d\tau_2(q \rightarrow p_{\pi^+} + p_{\pi^-}), \quad (71)$$

where $q = p_{\pi^+} + p_{\pi^-}$ is the combined four-momentum of the pion pair. The two-body phase space elements can be expressed as:

$$d\tau_2(m_{\psi(2S)} \rightarrow p_{J/\psi} + q) = (2\pi)^4 \delta^4(P - p_{J/\psi} - q) \frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}} \frac{d^3 q}{(2\pi)^3 2E_q}, \quad (72)$$

$$d\tau_2(q \rightarrow p_{\pi^+} + p_{\pi^-}) = (2\pi)^4 \delta^4(q - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}}. \quad (73)$$

Now we have to evaluate the two-body phase space elements. For the second two-body phase space element, and we can set $q = (\sqrt{q^2}, 0, 0, 0)$,

$$\int d\tau_2(q \rightarrow p_{\pi^+} + p_{\pi^-}) = \int (2\pi)^4 \delta^4(q - p_{\pi^+} - p_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}} \quad (74)$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_{\pi^+} - E_{\pi^-}) \delta^3(\vec{0} - \vec{p}_{\pi^+} - \vec{p}_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{d^3 p_{\pi^-}}{(2\pi)^3 2E_{\pi^-}} \quad (75)$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - E_{\pi^+} - E_{\pi^-}) \frac{d^3 p_{\pi^+}}{(2\pi)^3 2E_{\pi^+}} \frac{1}{(2\pi)^3 2E_{\pi^-}}, \quad \text{where } E_{\pi^+} = E_{\pi^-} \quad (76)$$

$$= \int (2\pi)^4 \delta(\sqrt{q^2} - 2E_{\pi^+}) \frac{4\pi |p|^2 d|p|}{(2\pi)^3 2E_{\pi^+}} \frac{1}{(2\pi)^3 2E_{\pi^+}}, \quad \text{where } |p| = |\vec{p}_{\pi^+}| \quad (77)$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} dE, \quad E = E_{\pi^+} \quad (78)$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} dE \quad (79)$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{|p|^2}{E^2} \frac{E}{|p|} dE \quad \text{using } dE = \frac{|p|}{E} dE \quad (80)$$

$$= \frac{1}{4\pi} \int \delta(\sqrt{q^2} - 2E_\pi) \frac{\sqrt{E^2 - m_\pi^2}}{E} dE \quad (81)$$

$$= \frac{1}{4\pi} \frac{\sqrt{\frac{q^2}{4} - m_\pi^2}}{\frac{q^2}{2}} \frac{1}{2} \quad \text{using } E = \frac{\sqrt{q^2}}{2} \quad (82)$$

$$= \frac{1}{8\pi} \sqrt{1 - \frac{4m_\pi^2}{q^2}}. \quad (83)$$

Next, we evaluate the other two-body phase space element, and we can evaluate in the rest frame

of $\psi(2S)$. In this frame, $P = (m_{\psi(2S)}, 0, 0, 0)$, and the energies are $E_{J/\psi} = \sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}$ and $E_q = m_{\psi(2S)} - E_{J/\psi} = \sqrt{|\vec{q}|^2 + q^2}$.

$$\int d\tau_2(m_{\psi(2S)} \rightarrow p_{J/\psi} + q) = \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q) \delta^3(\vec{0} - \vec{p}_{J/\psi} - \vec{q}) \frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}} \frac{d^3 q}{(2\pi)^3 2E_q} \quad (84)$$

$$= \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q) \frac{d^3 p_{J/\psi}}{(2\pi)^3 2E_{J/\psi}} \frac{1}{(2\pi)^3 2E_q} \quad (85)$$

$$= \int (2\pi)^4 \delta(m_{\psi(2S)} - E_{J/\psi} - E_q) \frac{4\pi |\vec{p}_{J/\psi}|^2 d|\vec{p}_{J/\psi}|}{(2\pi)^3 2E_{J/\psi}} \frac{1}{(2\pi)^3 2E_q} \quad (86)$$

$$= \frac{4\pi}{4(2\pi)^2} \int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q) \frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi} E_q} d|\vec{p}_{J/\psi}| \quad (87)$$

$$= \frac{1}{4\pi} \int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q) \frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi} E_q} d|\vec{p}_{J/\psi}|. \quad (88)$$

$E_{J/\psi}$ and E_q can be expressed in terms of $|\vec{p}_{J/\psi}|$ and q^2 :

$$E_q = \sqrt{|\vec{q}|^2 + q^2} = \sqrt{|\vec{p}_{J/\psi}|^2 + q^2}, \quad (89)$$

$$E_{J/\psi} = \sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}. \quad (90)$$

We also need to compute the derivative of $(E_{J/\psi} + E_q)$ with respect to $|\vec{p}_{J/\psi}|$:

$$\frac{d}{d|\vec{p}_{J/\psi}|} (E_{J/\psi} + E_q) = \frac{d}{d|\vec{p}_{J/\psi}|} \left(\sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2} + \sqrt{|\vec{p}_{J/\psi}|^2 + q^2} \right) \quad (91)$$

$$= \frac{|\vec{p}_{J/\psi}|}{\sqrt{|\vec{p}_{J/\psi}|^2 + m_{J/\psi}^2}} + \frac{|\vec{p}_{J/\psi}|}{\sqrt{|\vec{p}_{J/\psi}|^2 + q^2}} = \frac{|\vec{p}_{J/\psi}|}{E_{J/\psi}} + \frac{|\vec{p}_{J/\psi}|}{E_q} \quad (92)$$

$$= |\vec{p}_{J/\psi}| \left(\frac{E_{J/\psi} + E_q}{E_{J/\psi} E_q} \right) = |\vec{p}_{J/\psi}| \left(\frac{m_{\psi(2S)}}{E_{J/\psi} E_q} \right). \quad (93)$$

Thus, we have

$$\frac{1}{4\pi} \int \delta(m_{\psi(2S)} - E_{J/\psi} - E_q) \frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi} E_q} d|\vec{p}_{J/\psi}| \quad (94)$$

$$= \frac{1}{4\pi} \frac{|\vec{p}_{J/\psi}|^2}{E_{J/\psi} E_q} \frac{E_{J/\psi} E_q}{m_{\psi(2S)} |\vec{p}_{J/\psi}|} \quad (95)$$

$$= \frac{1}{4\pi} \frac{|\vec{p}_{J/\psi}|}{m_{\psi(2S)}}. \quad (96)$$

Now we can combine the results to get the total three-body phase space:

$$d\Gamma = \frac{1}{2m_{\psi(2S)}} |a_\psi|^2 4m_{\psi(2S)} m_{J/\psi} \int \left(\frac{1}{4\pi} \frac{|\vec{p}_{J/\psi}|}{m_{\psi(2S)}} \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \sqrt{1 - \frac{4m_\pi^2}{q^2}} \right) (q^2 - 4.5m_\pi^2)^2 \quad (97)$$

$$= \frac{|a_\psi|^2 m_{J/\psi}}{32\pi^3 m_{\psi(2S)}} \int |\vec{p}_{J/\psi}| \sqrt{1 - \frac{4m_\pi^2}{q^2}} (q^2 - 4.5m_\pi^2)^2 dq^2. \quad (98)$$

We can express $|\vec{p}_{J/\psi}|$ in terms of q^2 :

$$m_{J/\psi}^2 = E_{J/\psi}^2 - |\vec{p}_{J/\psi}|^2 = (m_{\psi(2S)} - E_q)^2 - |\vec{p}_{J/\psi}|^2 = (m_{\psi(2S)} - \sqrt{|\vec{p}_{J/\psi}|^2 + q^2})^2 - |\vec{p}_{J/\psi}|^2, \quad (99)$$

which gives (by *Mathematica*):

$$|\vec{p}_{J/\psi}| = \frac{\sqrt{(M - (m - q))(M + (m - q))(M - (m + q))(M + (m + q))}}{2M} \quad (100)$$

$$= \frac{\sqrt{(M^2 - (m + q)^2)(M^2 - (m - q)^2)}}{2M}, \quad \text{where } M = m_{\psi(2S)}, m = m_{J/\psi}, q = \sqrt{q^2} \quad (101)$$

$$= \frac{\sqrt{(m_{\psi(2S)}^2 - (m_{J/\psi} + \sqrt{q^2})^2)(m_{\psi(2S)}^2 - (m_{J/\psi} - \sqrt{q^2})^2)}}{2m_{\psi(2S)}}. \quad (102)$$

We can discuss the limits of integration for q^2 . $q^2 = (p_{\pi^+} + p_{\pi^-})^2 = (p_{\psi(2S)} - p_{J/\psi})^2$, which is the invariant mass squared of the pion pair. The minimum value of q^2 occurs when the two pions are produced at rest in their center-of-mass frame, which gives:

$$q_{min}^2 = (2m_\pi)^2 = 4m_\pi^2. \quad (103)$$

The maximum value of q^2 occurs when the J/ψ is produced at rest in the $\psi(2S)$ rest frame, which gives:

$$q_{max}^2 = (m_{\psi(2S)} - m_{J/\psi})^2. \quad (104)$$

Thus, we have with ($m_{J/\psi} = 3.096$ GeV, $m_{\psi(2S)} = 3.686$ GeV, $m_\pi = 0.13957$ GeV, and the experimental decay rate $\Gamma_{exp} = 101.64$ keV = 1.01×10^{-4} GeV):

$$\Gamma = \frac{|a_\psi|^2 m_{J/\psi}}{32\pi^3 m_{\psi(2S)}} \int_{4m_\pi^2}^{(m_{\psi(2S)} - m_{J/\psi})^2} |\vec{p}_{J/\psi}| \sqrt{1 - \frac{4m_\pi^2}{q^2}} (q^2 - 4.5m_\pi^2)^2 dq^2 \quad (105)$$

$$= |a_\psi|^2 \times 8.82391 \times 10^{-7} \text{GeV}^5 = 1.01 \times 10^{-4} \text{ GeV} \quad (106)$$

$$\Rightarrow |a_\psi| = 10.6987 \text{ GeV}^{-3}. \quad (107)$$

(b)

We can perform a similar analysis for the decay $\Upsilon(2S) \rightarrow \Upsilon(1S)\pi^+\pi^-$. The decay rate can be expressed

as:

$$\Gamma = \frac{1}{2m_{\Upsilon(2S)}} |a_\Upsilon|^2 4m_{\Upsilon(2S)} m_{\Upsilon(1S)} \int \left(\frac{1}{4\pi} \frac{|\vec{p}_{\Upsilon(1S)}|}{m_{\Upsilon(2S)}} \right) \frac{dq^2}{2\pi} \left(\frac{1}{8\pi} \sqrt{1 - \frac{4m_\pi^2}{q^2}} \right) (q^2 - 3.2m_\pi^2)^2 \quad (108)$$

$$= \frac{|a_\Upsilon|^2 m_{\Upsilon(1S)}}{32\pi^3 m_{\Upsilon(2S)}} \int |\vec{p}_{\Upsilon(1S)}| \sqrt{1 - \frac{4m_\pi^2}{q^2}} (q^2 - 3.2m_\pi^2)^2 dq^2. \quad (109)$$

We can express $|\vec{p}_{\Upsilon(1S)}|$ in terms of q^2 :

$$|\vec{p}_{\Upsilon(1S)}| = \frac{\sqrt{(m_{\Upsilon(2S)}^2 - (m_{\Upsilon(1S)} + \sqrt{q^2})^2)(m_{\Upsilon(2S)}^2 - (m_{\Upsilon(1S)} - \sqrt{q^2})^2)}}{2m_{\Upsilon(2S)}}. \quad (110)$$

The limits of integration for q^2 are:

$$q_{min}^2 = 4m_\pi^2, \quad (111)$$

$$q_{max}^2 = (m_{\Upsilon(2S)} - m_{\Upsilon(1S)})^2. \quad (112)$$

With ($m_{\Upsilon(1S)} = 9.460$ GeV, $m_{\Upsilon(2S)} = 10.023$ GeV, $m_\pi = 0.13957$ GeV, and the experimental decay rate $\Gamma_{exp} = 5.71$ keV = 5.71×10^{-6} GeV):

$$\Gamma = \frac{|a_\Upsilon|^2 m_{\Upsilon(1S)}}{32\pi^3 m_{\Upsilon(2S)}} \int_{4m_\pi^2}^{(m_{\Upsilon(2S)} - m_{\Upsilon(1S)})^2} |\vec{p}_{\Upsilon(1S)}| \sqrt{1 - \frac{4m_\pi^2}{q^2}} (q^2 - 3.2m_\pi^2)^2 dq^2 \quad (113)$$

$$= |a_\Upsilon|^2 \times 9.36631 \times 10^{-7} \text{GeV}^5 = 5.71 \times 10^{-6} \text{ GeV} \quad (114)$$

$$\Rightarrow |a_\Upsilon| = 2.46907 \text{ GeV}^{-3}. \quad (115)$$

Comparing the extracted magnitudes of $|a_\psi|$ and $|a_\Upsilon|$, we find that $|a_\psi|$ is significantly larger than $|a_\Upsilon|$. Since the constants a_ψ and a_Υ reflect overlap integrals between the $2S$ and $1S$ quarkonium wavefunctions and scale with the mean-square radius $\langle r^2 \rangle$ of the bound state, we can infer that the charmonium bound state (associated with a_ψ) has a larger spatial extent compared to the bottomonium bound state (associated with a_Υ). This suggests that charmonium states are more loosely bound and have a larger size than bottomonium states, which are more tightly bound and compact. \square