INTEGRAL DE RIEMANN

Seja f uma função limitada no intervalo $[a,b]$.

Uma partição de $[a,b]$ é um conjunto $P = \{t_0, t_1, \ldots, t_n\}$

tal que $\quad a = t_0 < t_1 < \cdots < t_n = b$.

$\|P\| = \text{máx}\{ t_i - t_{i-1} : i = 1, \ldots, n \}$. CALIBRE DA PARTIÇÃO

$m_i = \text{mín}\{ f(x) : t_{i-1} \leq x \leq t_i \}$

$M_i = \text{máx}\{ f(x) : t_{i-1} \leq x \leq t_i \}$.

$L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$ SOMA DE RIEMANN INFERIOR

$U(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$ " " " SUPERIOR

f se diz integrável em $[a,b]$ se $\lim_{\|P\| \to 0} L(f,P) = \lim_{\|P\| \to 0} U(f,P)$

Em tal caso o valor comum do limite denota-se $\int_a^b f$.

Teorema: Se f é contínua em $[a,b]$, então é integrável.

Demonstração: f contínua no compacto $[a,b] \Rightarrow$ uniformemente contínua. Seja $\varepsilon > 0$ arbitrário.

$$|U(f,P) - L(f,P)| = U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

f contínua no compacto $[t_{i-1}, t_i] \Rightarrow M_i = f(\beta_i)$ e

$m_i = f(\alpha_i)$ com $\alpha_i, \beta_i \in [t_{i-1}, t_i]$.

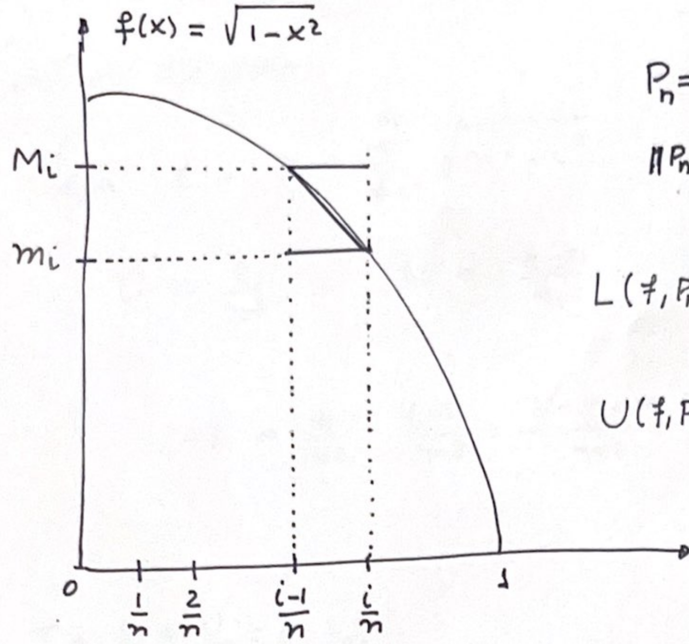
$\therefore$

$$|U(f,P) - L(f,P)| = \sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| \; (t_i - t_{i-1}).$$

f unif cont. $\Rightarrow \exists\, \delta > 0: |x-y| < \delta \Rightarrow |f(x) - f(y)| < \dfrac{\varepsilon}{b-a}$

$\therefore$ se $\|P\| < \delta$ $\quad |\alpha_i - \beta_i| \leq |t_i - t_{i-1}| \leq \|P\| < \delta$

$$\therefore \quad |U(f,P) - L(f,P)| < \frac{\varepsilon}{b-a} \underbrace{\sum_{i=1}^{n} (t_i - t_{i-1})}_{= t_n - t_0 = b-a} = \varepsilon.$$



$$P_n = \left\{ \frac{i}{n} : i = 0, 1, \dots, n \right\}.$$

$$\|P_n\| = \frac{1}{n}.$$

$$L(f, P_n) = \sum_{i=1}^{n} f(i/n) \cdot \frac{1}{n}$$

$$U(f, P_n) = \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right) \frac{1}{n}.$$

$$\therefore \quad L(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} \sqrt{1 - (i/n)^2} = \frac{1}{n^2} \sum_{i=1}^{n} \underbrace{\sqrt{n^2 - i^2}}_{= 0 \text{ se } i = n}. \implies$$

$$U(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} \sqrt{1 - \left(\frac{i-1}{n}\right)^2} = \frac{1}{n^2} \sum_{i=1}^{n} \sqrt{n^2 - (i-1)^2}.$$

$$= \frac{1}{n^2} \left[n + \sum_{i=1}^{n-1} \sqrt{n^2 - i^2} \right].$$

$$\implies L(f, P_n) = \frac{1}{n^2} \sum_{i=1}^{n-1} \sqrt{n^2 - i^2}$$

$f = \sqrt{1-x^2}$ é contínua em $[0,1] \implies$ integrável.

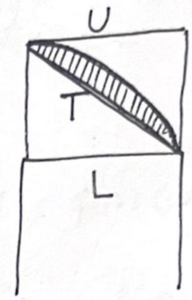
$$\therefore \quad \frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\, dx = \lim_{n \to \infty} U(f, P_n).$$

Controle do erro:

Seja $t = n + \sum_{k=1}^{n-1} \sqrt{n^2 - i^2}$.

$\therefore \quad U = \frac{t}{n^2} \; ; \quad L = \frac{t-n}{n^2}$.

$\therefore \quad U - L = \frac{t}{n^2} - \frac{t-n}{n^2} = \frac{\cancel{t} - \cancel{t} + n}{n^2} = \frac{1}{n} \; (*) \Rightarrow \varepsilon_n < \frac{1}{n}$.



$$T = L + \frac{U-L}{2} = \frac{L+U}{2}$$

$$= \frac{1}{2}\left(\frac{t}{n^2} + \frac{t-n}{n^2}\right) = \frac{t+t-n}{2n^2} = \frac{2t-n}{2n^2}.$$

$$\therefore \quad \pi \approx \overset{2}{\cancel{4}} \cdot \frac{2t-n}{\cancel{2}n^2} = \frac{4t-2n}{n^2} = \frac{\overset{2}{\cancel{4}}n + 4\Sigma - \cancel{2n}}{n^2}$$

$$= \frac{2}{n} + \frac{4}{n^2} \sum_{k=1}^{n-1} \sqrt{\underset{(n+k)(n-k).}{n^2 - k^2}} \qquad \text{com } 4\varepsilon_n < \frac{4}{n}.$$

Estimativa do erro com T:

$$\varepsilon_n < U - T = U - \frac{U+L}{2} = \frac{\cancel{2}U - \cancel{U} - L}{2} = \frac{U-L}{2} \overset{(*)}{=} \frac{1}{2n}.$$

$\therefore$ erro para $\pi = 4\varepsilon_n < \frac{4}{2n} = \frac{2}{n}$.

Em particular se $n = 2^p$, então err $< \frac{2}{2^p} = \frac{1}{2^{p-1}}$

TEOREMA FUNDAMENTAL DO CÁLCULO

Versão para funções contínuas.

Lema: Se f é contínua em $[a,b]$, então existe $\xi \in [a,b]$ tal que $f(\xi) = \frac{1}{b-a}\int_a^b f$.

Demonstração: f contínua em $[a,b] \Rightarrow f$ integrável.

f contínua no compacto $[a,b] \Rightarrow \exists\, \alpha, \beta \in [a,b]$ tais que

$f(\alpha) \leq f(x) \leq f(\beta)$.

$\therefore$ com $P = \{a, b\}$ tem-se $f(\alpha)(b-a) \leq \int_a^b f \leq f(\beta)(b-a)$

$$\therefore \quad f(\alpha) \leq \frac{1}{b-a}\int_a^b f \leq f(\beta).$$

Se vale algum $=$, basta tomar $\xi = \alpha$ ou $\xi = \beta$.

Caso contrário: $f(\alpha) < \frac{1}{b-a}\int_a^b f < f(\beta)$

$\therefore$ Pela propriedade dos valores intermediários,

existe $\xi \in (a,b)$: $f(\xi) = \frac{1}{b-a}\int_a^b f$. QED.

Teorema: Se f é contínua em $[a,b]$, então $F(x) = \int_a^x f$ é diferenciável em $[a,b]$ e tem-se $F'(x) = f(x)\ \forall x \in [a,b]$.

Demonstração: $\dfrac{F(x+h) - F(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f = f(\xi)$, $\xi \in [x, x+h]$

$\therefore$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(\xi) = f(\lim_{h \to 0} \xi)$$

$$= f(x), \text{ pois } x \leq \xi \leq x+h. \qquad \text{QED}$$

Corolário: Seja f é contínua em $[a,b]$. Se existe g tal que $g' = f$, então $\int_a^b f = g(b) - g(a)$.

Demonstração: Pelo teorema anterior, $F'(x) = f = g'(x)$

$\therefore \quad F(x) = g(x) + c.$

$$g(a) + c = F(a) = \int_a^a f = 0 \Rightarrow c = -g(a).$$

$\therefore \quad F(x) = g(x) - g(a) \quad \forall x \in [a,b].$

Em particular, se $x = b$, tem-se: $\int_a^b f = F(b) = g(b) - g(a)$

Versão para funções integráveis

Teorema: Seja f integrável em $[a,b]$. Então, a função $F(x) = \int_a^x f$ é diferenciável nos pontos $c \in [a,b]$ que são pontos de continuidade de f e tem-se $F'(c) = f(c)$

(Se $c=a$ ou $c=b$, trata-se da derivada lateral).

Demonstração: Seja $c \in [a,b]$ ponto de continuidade de f. Observe que: $\dfrac{F(c+h) - F(c)}{h} = \dfrac{1}{h}\int_c^{c+h} f$. Considere:

$$m(h) = \min\{f(x) : c \leq x \leq c+h\}$$
$$M(h) = \max\{f(x) : c \leq x \leq c+h\}.$$

Com a partição $P = \{c, c+h\}$ tem-se:

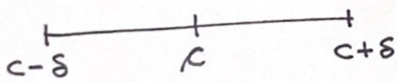
$$m(h)\cdot h \leq \int_c^{c+h} f \leq M(h)\cdot h \Rightarrow m(h) \leq \frac{1}{h}\int_c^{c+h} f \leq M(h).$$

Afirmação: $\lim\limits_{h\to 0} m(h) = \lim\limits_{h\to 0} M(h) = f(c)$.

Dado $\varepsilon > 0$ arbitrário, existe $\delta > 0$ tal que

$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \Leftrightarrow$

$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$.



Se $|h| < \delta \Rightarrow -\delta < h < \delta \Rightarrow c-\delta < c+h < c+\delta$.

$\therefore \quad M(h) \leq f(c) + \varepsilon \Rightarrow 0 \leq M - f(c) \leq \varepsilon$, como também

$f(c) - \varepsilon \leq m(h) \Rightarrow 0 \leq f(c) - m(h) \leq \varepsilon$.

Logo, pela afirmação anterior tem-se:

$$\underset{\substack{\downarrow \\ f(c)}}{m(h)} \le \frac{1}{h}\int_c^{c+h} f \le \underset{\substack{\downarrow \\ f(c)}}{M(h)} \quad \text{se } h \to 0.$$

$\therefore$ Pelo teorema confronto, tem-se:

$$F'(c) = \lim_{h \to 0} \frac{F(c+h)-F(c)}{h} = \lim_{h \to 0} \frac{1}{h}\int_c^{c+h} f = f(c).$$

INTEGRAÇÃO POR PARTES

Derivada de um produto: $(f \cdot g)' = f'g + fg' \Rightarrow$

$$\Rightarrow \quad fg = \int (fg)' = \int f'g + \int fg' \Rightarrow \boxed{\int f'g = fg - \int fg'}$$

FÓRMULA DE INTEGRAÇÃO POR PARTES

Exemplo: $\int x e^x dx = xe^x - \int e^x = xe^x - e^x = e^x(x-1)$

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x = x^2 e^x - 2\left(e^x(x-1)\right)$$

$$= e^x(x^2 - 2x + 2).$$

Exemplo: $\int sen^3 x\, dx = \int sen^2 x . sen\, x\, dx$

$= (1 - sen^2 x)$

$$= -\cos x\, sen^2 x + 2\int \cos^2 x\, sen\, x\, dx$$

$$= -\cos x\, sen^2 x + 2\int sen\, x\, dx - 2\int sen^3 x\, dx$$

$\therefore$

$$3\int sen^3 x\, dx = -\cos x . sen^2 x - 2\cos x$$

$$= -\cos x\,(sen^2 x + 2) = -\cos x\,(1 - \cos^2 x + 2)$$

$$= \frac{\cos^3 x - 3\cos x}{3} = \frac{1}{3}\cos^3 x - \cos x.$$

Exercício: $\underbrace{\int x^n e^{ax}\,dx}_{=:\, I_n} = x^n \dfrac{e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}\cdot e^{ax}\,dx$

$\therefore$

$$I_n = x^n\frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

$$= x^n\frac{e^{ax}}{a} - \frac{n}{a}\left(x^{n-1}\frac{e^{ax}}{a} - \frac{n-1}{a} I_{n-2}\right)$$

$$= \frac{e^{ax}}{a}\left(x^n - \frac{n}{a}x^{n-1}\right) + \frac{n(n-1)}{a^2} I_{n-2}$$

$$= \frac{e^{ax}}{a}\left(x^n - \frac{n}{a}x^{n-1}\right) + \frac{n(n-1)}{a^2}\left(x^{n-2}\frac{e^{ax}}{a} - \frac{n-2}{a} I_{n-3}\right)$$

$$= \frac{e^{ax}}{a}\left(x^n - \frac{n}{a}x^{n-1} + \frac{n(n-1)}{a^2}x^{n-2}\right) - \frac{n(n-1)(n-2)}{a^3} I_{n-3}$$

$$= \frac{e^{ax}}{a}\left(x^n - \frac{n}{a}x^{n-1} + \frac{n(n-1)}{a^2}x^{n-2} - \frac{n(n-1)(n-2)}{a^3}x^{n-3} + \cdots + (-1)^k\frac{n\cdots(n-k+1)}{a^k}x^{n-k} + \cdots +\right) + \frac{n!}{a^n}\underbrace{I_0}_{=\frac{e^{ax}}{a}}$$

$$= \frac{e^{ax}}{a}\sum_{k=0}^{n}\frac{(-1)^k}{a^k}\cdot\frac{n!}{(n-k)!}\cdot x^{n-k}.$$

Exercício: Determine explicitamente o valor de

$$I_n := \int_0^{\pi/2} \operatorname{sen}^{2n+1} x \, dx .$$

$$\int \operatorname{sen}^{2n+1} x \, dx = \int \operatorname{sen}^{2n} x \,.\, \operatorname{sen} x \, dx$$

$$= -\cos x \,.\, \operatorname{sen}^{2n} x + 2n \int \cos^2 x \operatorname{sen}^{2n-1} x \, dx$$

$$= -\cos x \operatorname{sen}^{2n} x + 2n \int \operatorname{sen}^{2n-1} x \, dx - 2n \int \operatorname{sen}^{2n+1} x \, dx$$

$\therefore$

$$2n+1 \int \operatorname{sen}^{2n+1} x \, dx = -\cos x \,.\, \operatorname{sen}^{2n} x + 2n \int \operatorname{sen}^{2n-1} x \, dx$$

$$\therefore \; (2n+1)\, I_n = -\cos x \,.\, \operatorname{sen}^{2n} x \Big|_0^{\pi/2} + 2n \, I_{n-1}$$

$$\therefore \; I_n = \frac{2n}{2n+1} I_{n-1}$$

$$= \frac{2n}{2n+1} \frac{2(n-1)}{2n-1} I_{n-2}$$

$$= \frac{2n}{2n+1} \frac{2n-2}{2n-1} \frac{2n-4}{2n-3} \cdots \frac{2}{3} I_0$$

$$I_0 = \int_0^{\pi/2} \operatorname{sen} x \, dx = -\cos x \Big|_0^{\pi/2} = \cos 0 = 1.$$

Alternativamente:

$$I_n = \frac{2^n \, n!}{(2n+1)!!} = \frac{4^n (n!)^2}{(2n)!} \frac{1}{2n+1} = \frac{A_n}{2n+1}$$

$$= \frac{(2n+1)!}{2^n . n!} \qquad \text{onde } \frac{1}{A_n} = \frac{1}{4^n} \binom{2n}{n}.$$

Exercício: Deduza uma fórmula análoga para

$$I_n = \int_0^{\pi/2} \operatorname{sen}^{2n} x \, dx$$

e prove a fórmula do produto de Wallis:

$$\frac{\pi}{2} = \lim_{n \to \infty} \left(\frac{2 . 4 . 6 . \cdots . 2n}{1 . 3 . 5 \cdots (2n-1)} \cdot \frac{1}{2n+1} \right)^2 . (2n+1)$$

INTEGRAÇÃO POR SUBSTITUIÇÃO

Derivada de uma função composta: $(f \circ g)' = f'(g) \cdot g'$

$$\therefore \quad \int f'(g(x)) \cdot g'(x)\, dx = \int (f \circ g)' = f \circ g$$

ou denotando $h = f'$: $\int h(g(x)) \cdot g'(x)\, dx = \int h \circ g$.

$$\int h(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\,dx}_{du} = \int h(u)\, du$$

e depois compor com g, ou seja, voltar de u para x.

$u = g(x)$

$du = g'(x)\, dx$

Exemplo: $\int \cos^2 x \,\text{sen}\, x \, dx \underbrace{=}_{\boxed{\begin{array}{l} u = \cos x \\ du = -\text{sen}\, x\, dx \end{array}}} -\int u^2 du = -\frac{u^3}{3} = -\frac{\cos^3 x}{3}$

Exemplo: $\int \sqrt{1-x^2}\, dx \underbrace{=}_{\boxed{\begin{array}{l} x = \text{sen}\, u \\ dx = \cos u\, du \end{array}}} \int \cos^2 u \, du = \int \frac{1+\cos 2u}{2} du$

$$= \frac{1}{2}\left[\int du + \int \cos 2u\, du\right] = \frac{1}{2}\left[u + \frac{1}{2}\,\text{sen}\, 2u\right]$$

$\boxed{\begin{array}{l} 2u = y \\ 2\,du = dy \end{array}}$ $\int \cos 2u\, du = \int \cos y \frac{dy}{2}$

$$= \frac{1}{2}\left[u + \frac{\cancel{2}\,\text{sen}\, u \cos u}{\cancel{2}}\right]$$

$$= \frac{1}{2}\left[\text{arcsen}\, x + x\sqrt{1-x^2}\right].$$

Exemplo: $\int \sqrt{1-x^2}\,dx = -\int sen^2 u\,du = -\int \frac{1-\cos 2u}{2}\,du$

$x = \cos u$

$dx = -sen u\,du$

$= \frac{1}{2}\left[\underbrace{\int \cos 2u\,du}_{= \frac{sen\,2u}{2}} - \int du \right] = \frac{1}{2}\left[sen u . \cos u - u \right].$

$= \frac{1}{2}\left[x\sqrt{1-x^2} - \arccos x \right]$

$\therefore$ $\quad arcsen\,x + \arccos x = c$

$c = arcsen\,0 + \arccos 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$

$\therefore$ $\boxed{arcsen\,x + \arccos x = \pi/2.}$

FÓRMULA EQUIVALENTE AO POSTULADO DAS PARALELAS DE EUCLIDES.

Exemplo: Decomposição em frações simples

$$\int \frac{dx}{1-x^2} \underset{\substack{x=iu \\ dx=i\,du}}{=} \int \frac{i\,du}{1+u^2} = i \operatorname{arctg} u = i \operatorname{arctg}(-ix).$$

$$\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} \overset{?}{=} \frac{A}{1+x} + \frac{B}{1-x}$$

$$= \frac{A(1-x) + B(1+x)}{(1+x)(1-x)} = \frac{A+B-Ax+Bx}{}$$

$-A+B = 0 \Rightarrow A = B$

$A+B = 1 \Rightarrow 2A = 1 \Rightarrow A = 1/2 = B.$

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

$$\therefore \int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1+x} + \frac{1}{2}\int \frac{dx}{1-x} = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x$$

$$= \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

$$\therefore \frac{1}{2}\ln\frac{1+x}{1-x} = i\operatorname{arctg}(-ix) + C$$

se $x=0 \Rightarrow 0 = 0 + C \Rightarrow C = 0.$

$$\therefore \boxed{\frac{1}{2}\ln\frac{1+x}{1-x} = i\operatorname{arctg}(-ix)}$$

$$u = \operatorname{arctg}(-ix) \;\Rightarrow\; iu = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\left(\frac{1+x}{1-x}\right)^{1/2}.$$

$$\therefore \quad e^{iu} = \left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\operatorname{tg}u = -ix \Rightarrow x = i\operatorname{tg}u \Rightarrow 1+x = 1+i\operatorname{tg}u$$
$$= 1 + i\frac{\operatorname{sen}u}{\cos u}$$
$$= \frac{\cos u + i\operatorname{sen}u}{\cos u}.$$

Analogamente: $1-x = 1-i\operatorname{tg}u = \dfrac{\cos u - i\operatorname{sen}u}{\cos u}$

$$\therefore \quad \frac{1+x}{1-x} = \frac{\overbrace{\cos u + i\operatorname{sen}u}^{=:\,z}}{\cos u - i\operatorname{sen}u} = \frac{z}{\bar{z}}\cdot\frac{z}{z} = \frac{z^2}{|z|^2} = z^2.$$

$$|z|^2 = \cos^2 u + \operatorname{sen}^2 u = 1.$$

$$\therefore \quad e^{iu} = \left(\frac{1+x}{1-x}\right)^{1/2} = (z^2)^{1/2} = z = \cos u + i\operatorname{sen}u.$$

$$\boxed{e^{iu} = \cos u + i\operatorname{sen}u}$$ FÓRMULA DE EULER.

$u = \pi$ $\quad e^{i\pi} = \underbrace{\cos\pi}_{=-1} + i\underbrace{\operatorname{sen}\pi}_{=0} = -1 \Rightarrow \boxed{e^{i\pi} + 1 = 0}$

IDENTIDADE DE EULER.

INTEGRAIS IMPRÓPRIAS

$$\int_a^{\infty} f = \lim_{R\to\infty} \int_a^R f \quad \text{se o limite existir.}$$

Exemplo: $\displaystyle\int_1^{\infty} x^{-s}\,dx = \lim_{R\to\infty} \int_1^R x^{-s}\,dx = \frac{x^{-s+1}}{-s+1}\Bigg|_1^R$

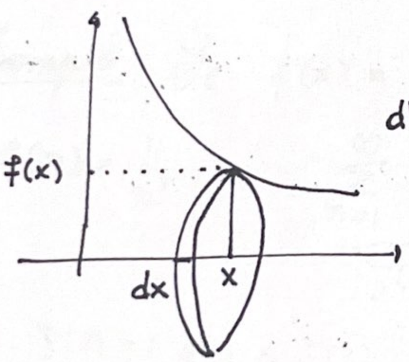
$$= \frac{1}{s-1}\left(1 - R^{-s+1}\right) = \frac{1}{s-1}\left(1 - e^{(-s+1)\ln R}\right)$$

$\to 0$ quando $R \to +\infty$

desde que $-s+1<0$

$\therefore \displaystyle\int_1^{\infty} x^{-s}\,dx = \frac{1}{s-1}$; se $s>1$, De fato $\Leftrightarrow$ $s>1$.

Em particular, $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = 1$.



$$dV = \pi f(x)^2 . dx \Rightarrow \boxed{V = \pi \int f(x)^2 dx}$$

VOLUME DE REVOLUÇÃO

Exemplo: Pelo exemplo anterior, $f(x) = x^{-3/4}$ possui área infinita no intervalo $[1,+\infty)$ pois $s = 3/4 < 1$.
Por outro lado o volume é dado por $V = \pi \displaystyle\int_1^{\infty} x^{-3/2}\,dx$

$$= \frac{\pi}{\frac{3}{2}-1} = 2\pi \text{, pois } s = 3/2 > 1.$$

Teorema: Critério da integral: Seja f positiva e decresente no intervalo $[1,+\infty)$ tal que $f(n) = a_n \ \forall n \in \mathbb{N}$.

Então $\{a_n\}$ é somável se e somente se $\int_1^\infty f$ existe.

<u>Dem</u>: Observe que $\int_1^n f = \sum_{k=1}^{n-1} \int_k^{k+1} f$.

$$a_{k+1} = f(k+1) \leqslant f(x) \leqslant f(k) = a_k \quad \forall x \in [k, k+1]$$

$$\therefore \quad a_{k+1} \leqslant \int_k^{k+1} f \leqslant a_k .$$

Observação: $\sum_{k=1}^{n-1} a_{k+1} \leqslant \sum_{k=1}^{n-1} \int_k^{k+1} f \leqslant \sum_{k=1}^{n-1} a_k$

$$\therefore \quad \sum_{k=1}^{\infty} a_k - a_1 \leqslant \int_1^\infty f \leqslant \sum_{k=1}^{\infty} a_k .$$

<u>Exemplo</u>: Se $f(x) = \frac{1}{x^2}$ pelo exemplo anterior $\int_1^\infty f = 1$.

$f(n) = \frac{1}{n^2} \quad \therefore \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\zeta(2)}$ é somável e tem-se:

$$\zeta(2) - 1 \leqslant 1 \leqslant \zeta(2) \rightarrow 1 \leqslant \zeta(2) \leqslant 2.$$

Exemplo: Integral imprópria de segunda espécie.

$$\int_{a^+}^{b} f = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} f \quad \text{se o limite existir.}$$

<u>Exemplo</u>: $$\int_{\epsilon}^{b} x^{-s}\,dx = \begin{cases} \dfrac{b^{-s+1} - \epsilon^{-s+1}}{-s+1} & \text{se } s \neq 1 \\ \ln b - \ln \epsilon & \text{se } s = 1. \end{cases}$$

$$\epsilon^{-s+1} = e^{(1-s)\ln\epsilon} \quad \text{converge} \Leftrightarrow \quad 1-s > 0 \Leftrightarrow s < 1.$$

com valor $\displaystyle\int_{0^+}^{b} x^{-s}dx = \frac{b^{1-s}}{1-s}$.

Alternativamente, observe que:

$$\int_{\epsilon}^{b} x^{-s}\,dx = -\int_{1/\epsilon}^{1/b} \left(\frac{1}{u}\right)^{-s} \frac{du}{u^2} = \int_{1/b}^{1/\epsilon} u^{s-2}\,du$$

$u = \dfrac{1}{x}$

$du = -\dfrac{dx}{x^2}$

$$\therefore \lim_{\epsilon\to 0^+} \int_{\epsilon}^{b} x^{-s}dx = \lim_{\epsilon\to 0^+} \int_{1/b}^{1/\epsilon} u^{s-2}du = \lim_{R\to+\infty} \int_{1/b}^{R} u^{s-2}du$$

$$\text{converge} \Leftrightarrow \quad 2-s > 1 \Leftrightarrow 1 > s.$$

Exemplo: No intervalo $[0,1]$ $f(x) = x^{-3/4}$ possui área finita $\displaystyle\int_{0^+}^{1} x^{-3/4} = \frac{1}{1-\frac{3}{4}} = 4$, mas volume infinito

$$V = \pi \int_{0^+}^{1} x^{-3/2} \quad \text{pois}$$

$s = 3/2 > 1$.

PROBLEMA DE BASSEL

Por um exemplo anterior sabe-se que $\arcsin x = \sum_{k=0}^{\infty} \underbrace{\frac{1}{4^k}\binom{2k}{k}}_{=: \frac{1}{A_n}} \frac{x^{2k+1}}{2k+1}$.

$\therefore$

$$t = \arcsin(\operatorname{sen} t) \Rightarrow$$

$$\int_0^{\pi/2} t\,dt = \int_0^{\pi/2} \arcsin(\operatorname{sen} t)\,dt = \sum_{n=0}^{\infty} \frac{1}{A_n}\frac{1}{2n+1}\int_0^{\pi/2} \operatorname{sen}^{2n+1} t\,dt$$

$$\underbrace{\frac{t^2}{2}\Big|_0^{\pi/2}}_{\frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}} = \sum_{n=0}^{\infty} \frac{1}{\cancel{A_n}}\frac{1}{2n+1}\frac{\cancel{A_n}}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

$$\therefore \boxed{\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}}$$

Por outro lado, $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{=\zeta(2)}$$

$$\therefore \underbrace{\left(1-\frac{1}{4}\right)}_{=\frac{3}{4}}\zeta(2) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\zeta(2) = \frac{4}{3}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\cancel{4}}{3}\cdot\frac{\pi^2}{\cancel{8}_2} = \frac{\pi^2}{6}.$$

$$f(x) = ax^2 \ ; \ a \neq 0.$$

$$\therefore \ f'(x) = 2ax.$$

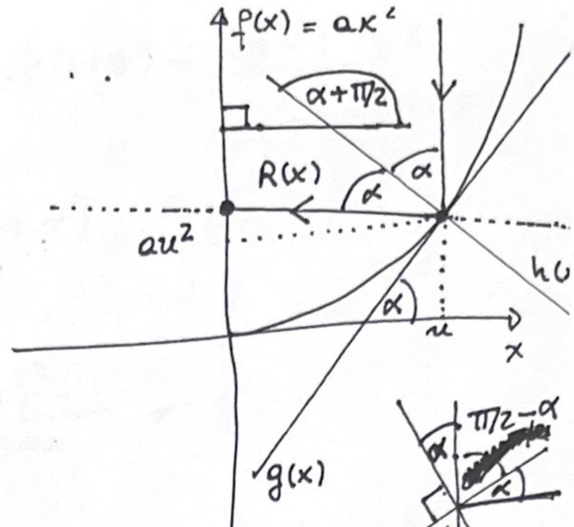
$$\therefore \ g(x) = 2au\,x + b$$

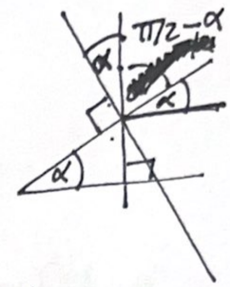
$$x = u \Rightarrow \ au^2 = 2au^2 + b$$

$$\therefore \ g(x) = 2au(x-u) + au^2$$

$$= 2aux - \cancel{2au^2} + \cancel{au^2}$$

$$= 2aux - au^2.$$





<u>Observação</u>: $$tg(\alpha+\pi/2) = \frac{sen(\alpha+\pi/2)}{cos(\alpha+\pi/2)} = \frac{sen\,\alpha.\underbrace{cos\,\pi/2}_{=0} + \underbrace{sen\,\pi/2}_{=1}\,cos\,\alpha}{cos\,\alpha.\underbrace{cos\,\pi/2}_{=0} - sen\,\alpha.\underbrace{sen\,\pi/2}_{=1}}$$

$$= \frac{cos\,\alpha}{-sen\,\alpha} = -\frac{1}{tg\,\alpha}.$$

$$\therefore \ h(x) = -\frac{1}{2au}\,x + B$$

$$au^2 = -\frac{1}{2a\cancel{u}}.\cancel{u} + B \Rightarrow B = au^2 + \frac{1}{2a}$$

$$\therefore \ h(x) = -\frac{1}{2au}\,x + au^2 + \frac{1}{2a}\frac{u}{u}$$

$$= -\frac{1}{2au}(x-u) + au^2.$$

Observação: $tg(x+y) = \dfrac{tgx + tgy}{1 - tgx\, tgy}$. *

$$\therefore\ tg\left(\alpha + \pi/2 + \alpha\right) = \frac{-\frac{1}{2au} + 2au}{1 + \frac{1}{\cancel{2au}} \cdot \cancel{2au}}$$

$$= \frac{1}{2}\,\frac{(2au)^2 - 1}{2au} =: \rho$$

$$\therefore\ R(x) = \rho x + \beta$$

$$au^2 = \rho u + \beta \Rightarrow \beta = au^2 - \rho u$$

$$= au^2 - \frac{1}{2}\,\frac{(2au)^2 - 1}{2a\cancel{u}} \cdot \cancel{u}$$

$$= \frac{\cancel{4a^2u^2} - \cancel{(2au)^2} + 1}{4a} = \frac{1}{4a}$$

$$\therefore\ R(0) = \beta = \frac{1}{4a}.$$

$\therefore$ $R(0)$ é o <u>foco</u> da parábola $f(x) = ax^2$.

$$* \quad tg(x+y) = \frac{sen(x+y)}{cos(x+y)} = \frac{sen x . cos y + sen y . cos x}{cos x . cos y - sen x . sen y}$$

$$= \frac{\frac{sen x . cos y}{cos x . cos y} + \frac{sen y . cos x}{cos x . cos y}}{\frac{cos x . cos y}{cos x . cos y} - \frac{sen x . sen y}{cos x . cos y}} = \frac{tgx + tgy}{1 - tgx . tgy}$$

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int x^2 \frac{x}{\sqrt{1+x^2}}\,dx = x^2\sqrt{1+x^2} - 2\int x\sqrt{1+x^2}\,dx$$

$x = iu$

$dx = i\,du$

$$= x^2\sqrt{1+x^2} - 2\int \overset{=-1}{i^2}\, u\sqrt{1-u^2}\,du$$

$y = 1-u^2$

$dy = -2u\,du$

$$= x^2\sqrt{1+x^2} - \int y^{1/2}\,dy$$

$$= x^2\sqrt{1+x^2} - \frac{y^{3/2}}{3/2} = x^2\sqrt{1+x^2} - \frac{2}{3}\left(\sqrt{1+x^2}\right)^3$$

$$= \sqrt{1+x^2}\left(x^2 - \frac{2}{3}(1+x^2)\right) = \frac{1}{3}\sqrt{1+x^2}\,(x^2-2).$$

Derivando para checar: $\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}(x^2-2) + \sqrt{1+x^2}\,.\,2x\right) =$

$$= \frac{1}{3}\frac{1}{\sqrt{1+x^2}}\left(x^3 - \cancel{2x} + \cancel{2x} + 2x^3\right) = \frac{1}{\cancel{3}}\frac{\cancel{3}x^3}{\sqrt{1+x^2}}.$$

O/F:

$$\int \frac{x^3\,dx}{\sqrt{1+x^2}} = \int \cancel{\cos u} \cdot \frac{\operatorname{sen}^3 u}{\cos^3 u} \cdot \frac{du}{\cos^2 u} = \int \frac{\operatorname{sen}^3 u}{\cos^4 u}\,du$$

$x = tg\,u$ $\qquad$ $c^2+s^2=1$

$dx = \dfrac{du}{\cos^2 u}$ $\qquad$ $1+T^2 = \dfrac{1}{c^2}$

$$= \int \frac{\operatorname{sen} u}{\cos^4 u} \cdot \operatorname{sen}^2 u\,du = \frac{\cos^{-3} u}{3} \cdot \operatorname{sen}^2 u - \frac{2}{3}\int \frac{\operatorname{sen} u \cdot \cancel{\cos u}}{\cos^{\cancel{4}3} u}\,du$$

$$= \frac{\operatorname{sen}^2 u}{3\cos^3 u} - \frac{2}{3}\cdot\frac{1}{\cos u} = \frac{1}{3}\frac{1}{\cos u}\left(tg^2 u - 2\right)$$

$$= \frac{1}{3}\sqrt{1+x^2}\left(x^2-2\right)$$

O/F:

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{u^2-1}{\cancel{u}}\,\cancel{u}\,du = \frac{u^3}{3} - u$$

$u^2 = 1+x^2$

$\cancel{2}u\,du = \cancel{2}x\,dx$

$$= \frac{(1+x^2)^{3/2}}{3} - \sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\left(\frac{\cancel{1}+x^2}{3} - \cancel{1}\right) = \frac{1}{3}\sqrt{1+x^2}\left(1+x^2-3\right)$$

$$= \frac{1}{3}\sqrt{1+x^2}\left(x^2-2\right).$$

O/F:

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{u-1}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{2}\int (u-1)\,u^{-1/2}\,du =$$

$u = 1+x^2$

$du = 2x\,dx$

$$= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du = \frac{1}{2}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right) = \frac{\cancel{2}}{\cancel{2}}u^{1/2}\left(\frac{u}{3}-1\right).$$

$$= \frac{1}{3}\sqrt{1+x^2}\left(1+x^2-3\right) = \frac{1}{3}\sqrt{1+x^2}\left(x^2-2\right).$$

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \frac{x^{-3}}{-3}\cdot\sqrt{1-x^2} - \int \frac{1}{\cancel{-}3x^{\cancel{3}\,2}}\cdot\frac{\cancel{-2x}}{\cancel{2}\sqrt{1-x^2}}\,dx$$

$$= -\frac{\sqrt{1-x^2}}{3x^3} - \frac{1}{3}\int\frac{dx}{x^2\sqrt{1-x^2}} \qquad (*)$$

$$\int\frac{dx}{x^2\sqrt{1-x^2}} = \int\frac{\cancel{\cos u}\,du}{\operatorname{sen}^2 u\cdot\cancel{\cos u}} = \int\frac{du}{\operatorname{sen}^2 u} = -\frac{\cos u}{\operatorname{sen} u}$$

$x = \operatorname{sen} u$
$dx = \cos u\,du$

$$(*) = -\frac{\sqrt{1-x^2}}{3x^3} + \frac{1}{3}\,\frac{\sqrt{1-x^2}}{x} = \frac{\sqrt{1-x^2}}{3x^3}\,(x^2-1)$$

$$= -\frac{(1-x^2)^{3/2}}{3x^3}.$$

Derivando para checar: $+\frac{1}{\cancel{3}}\,\dfrac{\frac{\cancel{3}}{\cancel{2}}(1-x^2)^{1/2}(-\cancel{2}x)x^3 + (1-x^2)^{3/2}\cdot\cancel{3}x^2}{x^6}$

$$= \frac{\sqrt{1-x^2}}{x^{\cancel{6}\,4}}\,(\cancel{x^4} + x^{\cancel{2}} - \cancel{x^4}) = \frac{\sqrt{1-x^2}}{x^4}.$$

$$c^2+s^2 = 1 \Rightarrow \left(\frac{c}{s}\right)^2 + 1 = \frac{1}{s^2}$$

$$\left(\frac{c}{s}\right)' = \frac{-s^2-c^2}{s^2} = -\frac{1}{s^2}.$$

O/F:

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx \underset{\substack{x=\text{sen}\,u \\ dx=\cos u\,du}}{=} \int \frac{\cos^2 u}{\text{sen}^4 u}\,du = \int \frac{\cos u}{\text{sen}^4 u}\cdot \cos u\,du$$

$$= \frac{\text{sen}^{-3} u}{-3}\cdot \cos u - \int \frac{\overset{-2}{\cancel{\text{sen}^{-3}}} u}{\cancel{-}3}(\cancel{-}\,\text{sen}\,u)\,du$$

$$= -\frac{1}{3}\frac{\cos u}{\text{sen}^3 u} - \frac{1}{3}\int \frac{1}{\text{sen}^2 u}\,du = -\frac{1}{3}\frac{\cos u}{\text{sen}^3 u} + \frac{1}{3}\frac{\cos u}{\text{sen}\,u}$$

$$= \frac{1}{3}\frac{\cos u}{\text{sen}\,u}\left(1-\frac{1}{\text{sen}^2 u}\right) = -\frac{1}{3}\frac{\cos u}{\text{sen}^3 u}\overbrace{(1-\text{sen}^2 u)}^{\cos^2 u}$$

$$= -\frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3}.$$

$$u^2 = \frac{1-x^2}{x^2} = x^{-2}-1$$

O/F:

$$\cancel{2}u\,du = -\cancel{2}x^{-3}\,dx$$

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \sqrt{\frac{1-x^2}{x^2}}\,\frac{dx}{x^3} = \int u\,(-u\,du) = -\frac{u^3}{3}$$

$$= -\frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3}.$$

O/F:

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx \underset{\substack{x=\text{sen}\,u \\ dx=\cos u\,du}}{=} \int \frac{\cos^2 u}{\text{sen}^4 u}\,du = \int \frac{\cos^2 u}{\text{sen}^2 u}\,\frac{du}{\text{sen}^2 u} =$$

$y = \text{cotg}\,u$

$dy = \frac{-du}{\text{sen}^2 u}$

$$= -\int y^2\,dy = -\frac{y^3}{3} = -\frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3}.$$

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\sqrt{u}}{(1-u)^2}\cdot\frac{-du}{2\sqrt{1-u}} = -\frac{1}{2}\int \frac{\sqrt{u}\,du}{(1-u)^{5/2}}.$$

$$u = 1-x^2 \Rightarrow x^2 = 1-u \Rightarrow x = \pm\sqrt{1-u}.$$

$$du = -2x\,dx \Rightarrow \frac{-du}{2x} = dx$$

$$= -\frac{1}{2}\left[\frac{(1-u)^{-3/2}}{-3/2}\sqrt{u} + \int \frac{(1-u)^{-3/2}}{\cancel{+}\frac{3}{\cancel{2}}}\cdot \underbrace{(\sqrt{u})'}_{=\frac{1}{\cancel{2}\sqrt{u}}}\,du\right]$$

$$= \cancel{+}\frac{1}{\cancel{2}}\cdot\cancel{+}\frac{\cancel{2}}{3}(1-u)^{-3/2}\sqrt{u} - \frac{1}{2}\cdot\frac{1}{3}\int (1-u)^{-3/2}\frac{du}{\sqrt{u}}$$

$$= \frac{1}{3}(1-u)^{-3/2}\sqrt{u} - \frac{1}{6}\int (1-u)^{-3/2}u^{-1/2}\,du.$$

$x = \sqrt{u}$, $dx = \frac{du}{2\sqrt{u}}$ $\qquad$ $x = \text{sen}\,y$, $dx = \cos y\,dy$

$$\int (1-u)^{-3/2}u^{-1/2}\,du = \int \frac{1}{(1-u)^{3/2}}\cdot\frac{du}{\sqrt{u}} = 2\int \frac{dx}{(1-x^2)^{3/2}} =$$

$$= 2\int \frac{\cancel{\cos y}\,dy}{\cos^{\cancel{3}\,2} y} = 2\int \frac{dy}{\cos^2 y} = 2.\,\text{tg}\,y = 2.\frac{\text{sen}\,y}{\cos y}$$

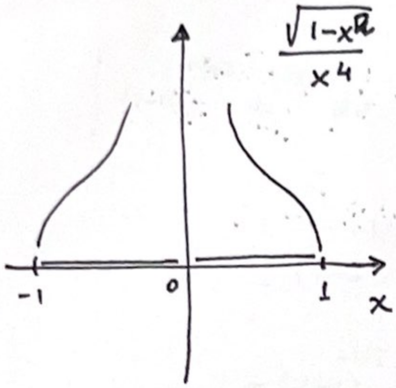
$$= 2\frac{x}{\sqrt{1-x^2}} = 2\frac{\sqrt{u}}{\sqrt{1-u}}.$$

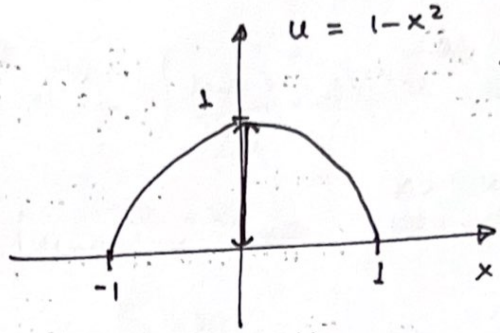
$$\therefore \int \frac{\sqrt{1-x^2}}{x^4}\,dx = \frac{1}{3}(1-u)^{-3/2}\sqrt{u} - \frac{1}{\cancel{6}_{3}}\cancel{2}\frac{\sqrt{u}}{(1-u)^{1/2}}$$

$$= \frac{1}{3}(1-u)^{-3/2}\sqrt{u}\,(\cancel{1} - \cancel{1} + u) = \frac{1}{3}(1-u)^{-3/2}\cdot u^{3/2}$$

$$= \frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3} \quad \text{se} \quad x = \sqrt{1-u}.$$

$$= -\frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3} \quad \text{se} \quad x = -\sqrt{1-u}.$$





$$\left.\begin{array}{l}\sqrt{u} \Rightarrow u \geq 0 \\ x=\sqrt{1-u} \Rightarrow 1-u \geq 0 \Rightarrow u \leq 1\end{array}\right\} \Rightarrow 0 \leq u \leq 1$$

$$(1-u)^{-3/2}\, u^{-1/2} \Rightarrow 0 < u < 1.$$

$$\left(2\sqrt{\frac{u}{1-u}}\right)' = \cancel{2}\,\frac{1}{\cancel{2}\sqrt{\frac{u}{1-u}}}\cdot\frac{1-\cancel{u}+\cancel{u}}{(1-u)^2} = \frac{1}{\sqrt{u}\cdot(1-u)^{3/2}} = (1-u)^{-3/2}\,u^{-1/2}.$$

$$\int (1-u)^{-3/2} u^{-1/2}\, du = \frac{(1-u)^{-1/2}}{-1/2} u^{-1/2} - \int \frac{(1-u)^{-1/2}}{\cancel{+1/2}} \cdot \cancel{+1/2}\, u^{-3/2}\, du$$

$y = 1-u$
$dy = -du$

$$= -2(1-u)^{-1/2} u^{-1/2} - \int (1-u)^{-1/2} u^{-3/2}\, du$$

$$= -2(1-u)^{-1/2} u^{-1/2} + \int y^{-1/2} (1-y)^{-3/2}\, dy\,.$$

$$\int (1-u)^{-3/2} u^{-1/2} + (1-u)^{-1/2} u^{-3/2}\, du = -2(1-u)^{-1/2} u^{-1/2}.$$

$$\frac{1}{(1-u)\sqrt{1-u}\,\sqrt{u}} + \frac{1}{\sqrt{1-u}\,\sqrt{u}\; u} = \frac{\cancel{u} + 1 - \cancel{u}}{(1-u)\sqrt{1-u}\,\sqrt{u}\,.\,u} = (1-u)^{-3/2} u^{-3/2}.$$

$$\therefore \int (1-u)^{-3/2} u^{-3/2}\, du = -2(1-u)^{-1/2} u^{-1/2}.$$

No entanto, tem-se: $\left[-2(1-u)^{-1/2} u^{-1/2}\right]' = +\cancel{2} \cdot \frac{1}{\cancel{2}} (1-u)^{-3/2} u^{-3/2} \cdot (1-2u)$

$$= (1-u)^{-3/2} u^{-3/2} (1-2u).$$

$$\frac{1}{6}\int (1-u)^{-3/2}\, u^{-1/2}\, du = \frac{1}{3}\,(1-u)^{-3/2}\sqrt{u} - \int \frac{\sqrt{1-x^2}}{x^4}\,dx$$

$$= \frac{1}{3}\,(1-u)^{-3/2}\sqrt{u} \pm \frac{1}{3}\,\frac{(1-x^2)^{3/2}}{x^3}$$

$$= \frac{1}{3}\,(1-u)^{-3/2}\sqrt{u} \pm \frac{1}{3}\, u^{3/2}\,(1-u)^{-3/2}$$

$$= \frac{1}{3}\,(1-u)^{-3/2}\sqrt{u}\,(1 \pm u)$$

$$= \frac{1}{3}\,\frac{\sqrt{u}}{\sqrt{1-u}\,(1-u)}\,(1 \pm u)$$

$$\therefore \int (1-u)^{-3/2}\, u^{-1/2}\, du = 2\,\frac{\sqrt{u}}{\sqrt{1-u}}$$ *com o sinal menos, ou seja,* $x = +\sqrt{1-u}$.

Com sinal mais, tem-se: $\left[\sqrt{u}\,(1-u)^{-3/2}\,(1+u)\right]' =$

$$= \frac{1}{2\sqrt{u}}\,(1-u)^{-3/2}(1+u) + \sqrt{u}\cdot\frac{3}{2}\,(1-u)^{-5/2}\,(1+u) + \sqrt{u}\,(1-u)^{-3/2}$$

$$= \frac{(1-u)^{-3/2}}{2\sqrt{u}}\left[\,1+u + 3u\,\frac{(1+u)}{1-u} + 2u\,\right]$$

$$= \frac{(1-u)^{-3/2}}{2\sqrt{u}}\;\frac{\overset{1-\cancel{u^2}}{(1+u)(1-u)} + \overset{3u+3\cancel{u^2}}{3u(1+u)} + \overset{2u-2\cancel{u^2}}{2u(1-u)}}{1-u}$$

$$= \frac{(1-u)^{-3/2}}{2\sqrt{u}}\cdot\frac{5u+1}{1-u}.$$