直交基底、直交行列、Gram–Schmidtの直行化法

正規直交

ベクトル$q_1 \cdots q_n$は

$$q_i^T q_j \begin{cases} 0 & i \neq j(\text{直交ベクトル}) \\ 1 & i = j(\text{単位ベクトル}:\|q_i\| = 1) \end{cases}$$

射影と最小**2**乗：正規直交の場合

もしAの列が正規直交ならば

$$A^T A = \begin{bmatrix} \cdots & a_1^T & \cdots \\ \cdots & a_2^T & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & a_n^T & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

これにより、射影行列は

$$P = \mathrm{AA}^T, \quad \bar{x} = A^T b$$

となる

3.3.1

```
A = [1,-2;1,-1;1,1;1,2]
```

```
A =
       1          -2
       1          -1
       1           1
       1           2
```

```
b = [-4;-3;-1;0]
```

```
b =
      -4
      -3
      -1
       0
```

```
dot(A(:,1),A(:,2))
```

```
ans =
       0
```

$$C = \frac{1}{2}c$$

$$c = 2C$$

$$D = \frac{1}{\sqrt{10}}d$$

$$d = \sqrt{10}\,D$$

```
A2 = [A(:,1)*(1/4),A(:,2)*(1/10)]
```

```
A2 =
       1/4           -1/5
       1/4           -1/10
       1/4            1/10
       1/4            1/5
```

```
x=A2'*b
```

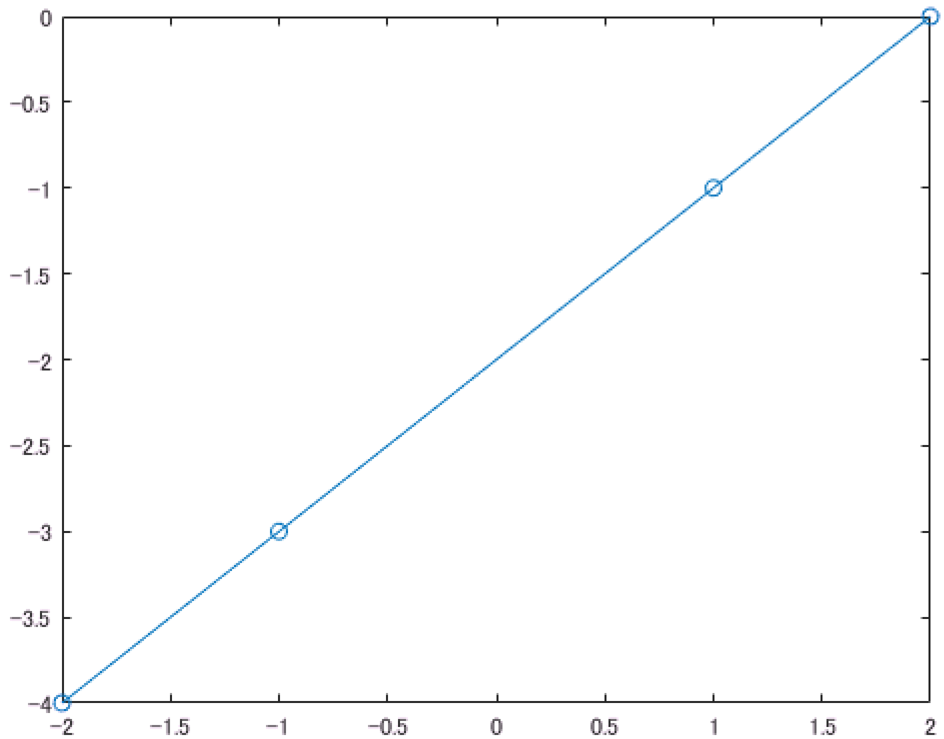
```
x =
      -2
       1
```

$$y = -2 + t$$

```
t = [-2,-1,1,2]
```

```
t =
      -2           -1            1            2
```

```
plot(t,-2+t,'-o')
```




```
A * x
```

```
ans =
      -4
      -3
      -1
       0
```

```
norm(A*x-b)
```

```
ans =
       0
```

3.3.2

```
b = [0;3;0]
```

```
b =
       0
       3
       0
```

```
a1 = [2/3;2/3;-1/3]
```

```
a1 =
       2/3
       2/3
      -1/3
```

```
a2 = [-1/3;2/3;2/3]
```

```
a2 =
      -1/3
       2/3
       2/3
```

```
a1 * a1'*b
```

```
ans =
       4/3
       4/3
      -2/3
```

```
a2 * a2' *b
```

```
ans =
      -2/3
       4/3
       4/3
```

```
A = [a1,a2]
```

```
A =
       2/3            -1/3
       2/3             2/3
      -1/3             2/3
```

```
A * A' * b
```

```
ans =
       2/3
       8/3
       2/3
```

3.3.3

```
a3 = [2/3;-1/3;2/3]
```

```
a3 =
       2/3
      -1/3
       2/3
```

```
a3 * a3' * b
```

```
ans =
      -2/3
       1/3
      -2/3
```

```
A = [a1,a2,a3]
```

```
A =
       2/3            -1/3             2/3
```

```
       2/3             2/3            -1/3
      -1/3             2/3             2/3
```

```
a1 * a1'*b+a2 * a2' *b + a3 * a3' * b
```

```
ans =
       0
       3
       0
```

直交行列

正規直交の列を持つ正方行列

$$Q^T Q = I$$
$$\mathrm{QQ}^T = I$$
$$Q^T = Q^{-1}$$

例1

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$Q^T = Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

例2

交換行列Pは直交行列

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$P^{-1} = P^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
$$P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$