直交基底、直交行列、Gram–Schmidtの直行化法

正規直交

ベクトル$q_1 \cdots q_n$は

$$q_i^T q_j \begin{cases} 0 & i \neq j(\textbf{直交ベクトル}) \\ 1 & i = j(\textbf{単位ベクトル}: \|q_i\| = 1) \end{cases}$$

射影と最小**2**乗：正規直交の場合

もしAの列が正規直交ならば

$$A^T A = \begin{bmatrix} \cdots & a_1^T & \cdots \\ \cdots & a_2^T & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & a_n^T & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

これにより、射影行列は

$$P = \mathrm{AA}^T, \ \ \overline{x} = A^T b$$

となる

3.3.1

```
A = [1,-2;1,-1;1,1;1,2]
```

```
A =

     1    -2
     1    -1
     1     1
     1     2
```

```
b = [-4;-3;-1;0]
```

```
b =

    -4
    -3
    -1
     0
```

```
dot(A(:,1),A(:,2))
```

```
ans =

     0
```

$$C = \frac{1}{2}c$$

$$c = 2C$$

$$D = \frac{1}{\sqrt{10}}d$$

$$d = \sqrt{10}\,D$$

```
A2 = [A(:,1)*(1/4),A(:,2)*(1/10)]
```

```
A2 =
       1/4           -1/5
       1/4           -1/10
       1/4            1/10
       1/4            1/5
```

```
x=A2'*b
```

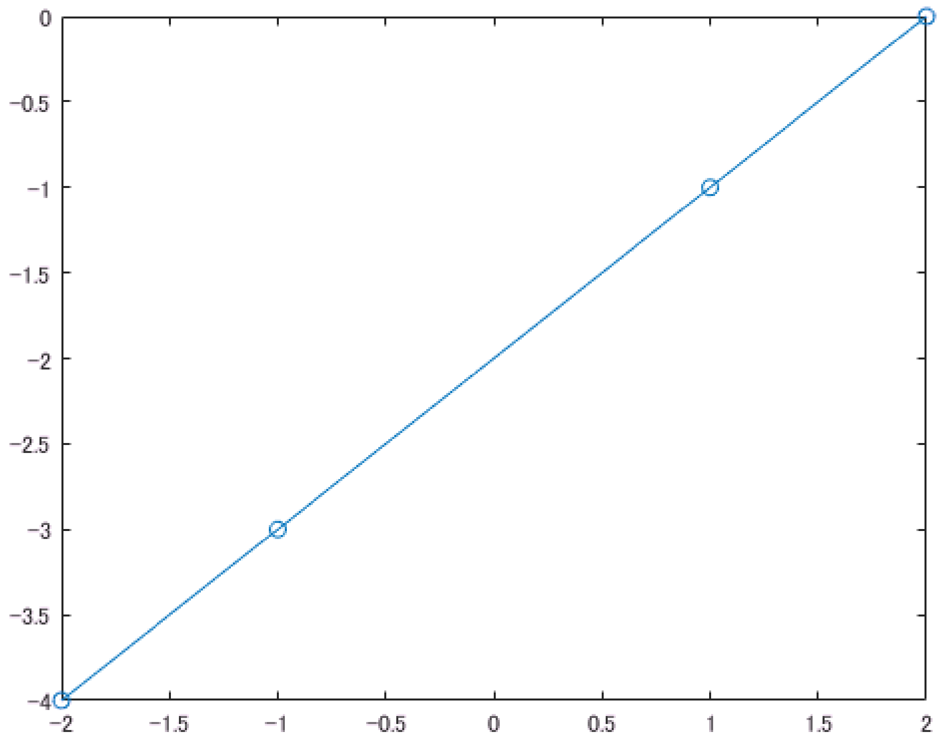
```
x =
      -2
       1
```

$$y = -2 + t$$

```
t = [-2,-1,1,2]
```

```
t =
      -2          -1          1          2
```

```
plot(t,-2+t,'-o')
```




```
A * x
```

```
ans =
      -4
      -3
      -1
       0
```

```
norm(A*x-b)
```

```
ans =
       0
```

3.3.2

```
b = [0;3;0]
```

```
b =
       0
       3
       0
```

```
a1 = [2/3;2/3;-1/3]
```

```
a1 =
       2/3
       2/3
      -1/3
```

```
a2 = [-1/3;2/3;2/3]
```

```
a2 =
     -1/3
      2/3
      2/3
```

```
a1 * a1'*b
```

```
ans =
      4/3
      4/3
     -2/3
```

```
a2 * a2' *b
```

```
ans =
     -2/3
      4/3
      4/3
```

```
A = [a1,a2]
```

```
A =
      2/3           -1/3
      2/3            2/3
     -1/3            2/3
```

```
A * A' * b
```

```
ans =
      2/3
      8/3
      2/3
```

3.3.3

```
a3 = [2/3;-1/3;2/3]
```

```
a3 =
      2/3
     -1/3
      2/3
```

```
a3 * a3' * b
```

```
ans =
     -2/3
      1/3
     -2/3
```

```
A = [a1,a2,a3]
```

```
A =
      2/3           -1/3            2/3
```

```
     2/3         2/3         -1/3
    -1/3         2/3          2/3
```

```
a1 * a1'*b+a2 * a2' *b + a3 * a3' * b
```

```
ans =
     0
     3
     0
```

直交行列

正規直交の列を持つ正方行列

$$Q^T Q = I$$
$$\mathrm{QQ}^T = I$$
$$Q^T = Q^{-1}$$

例1

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$Q^T = Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

例2

交換行列Pは直交行列

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$P^{-1} = P^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
$$P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

直行行列の最も重要な性質

直行行列による積は長さを保存する

$$||Qx|| = ||x||$$

また、内積も保存される

$$(Qx)^T(Qy) = x^T y$$

3.3.4

$$(Q_1Q_2)^T Q_1Q_2 = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T I Q_2 = Q_2^T Q_2 = I$$

$$Q_1Q_2(Q_1Q_2)^T = Q_1Q_2Q_2^TQ_1^T = Q_1Q^T = I$$

$$(Q_1Q_2)^T = Q_2^TQ_1^T = Q_2^{-1}Q_1^{-1} = (Q_1Q_2)^{-1}$$

3.3.5

$$Q^T = \left(I - 2uu^T\right)^T = I - 2\left(uu^T\right)^T = I - 2uu^T$$

$$QQ^T = \left(I - 2uu^T\right)\left(I - 2uu^T\right) = I - 4uu^T + 4uu^Tuu^T = I$$

```
clear
u = [1/sqrt(3);1/sqrt(3);1/sqrt(3)]
```

```
u =

    780/1351
    780/1351
    780/1351
```

```
Q = eye(3)-2*u*u'
```

```
Q =

      1/3            -2/3            -2/3
     -2/3             1/3            -2/3
     -2/3            -2/3             1/3
```

3.3.6

```
clear
A = [1/sqrt(3),1/sqrt(3),1/sqrt(3);1/sqrt(2),0,-1/sqrt(2)]
```

```
A =

    780/1351       780/1351       780/1351
    985/1393         0           -985/1393
```

```
E21 = eye(2);
E21(2,1) = -sqrt(3)/sqrt(2)
```

```
E21 =

      1              0
  -1079/881          1
```

```
U = E21 * A
```

```
U =
     780/1351        780/1351        780/1351
       *            -985/1393      -1393/985
```

```
E12 = eye(2)
```

```
E12 =
       1               0
       0               1
```

```
E12(1,2) = sqrt(2)/sqrt(3)
```

```
E12 =
       1             881/1079
       0               1
```

```
U = E12 * U
```

```
U =
     780/1351          0            -780/1351
       *            -985/1393      -1393/985
```

```

```

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\frac{1}{\sqrt{3}} a = \frac{1}{\sqrt{3}} c$$

$$a = c$$

$$-\frac{1}{\sqrt{2}} b = \sqrt{2}\, c$$

$$b = -2c$$

$$x = \begin{bmatrix} a \\ -2a \\ a \end{bmatrix} = \pm \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

3.3.7

ベクトルが正規直行の場合

$$q_i^T q_j \begin{cases} 0 & i \neq j (\text{直交ベクトル}) \\ 1 & i = j (\text{単位ベクトル}:\|q_i\| = 1) \end{cases}$$

より、

$$
\begin{aligned}
vv^T &= \left(q_1x_1+\cdots+q_nx_n\right)\left(q_1x_1+\cdots+q_nx_n\right)^T \\
&= x_1^2+\cdots+x_n^2
\end{aligned}
$$

3.3.8

```
clear
Q = [1/2,1/2,1/2,1/2;
    1/2,1/2,-1/2,-1/2;
    1/2,-1/2,-1/2,1/2;
    1/2,-1/2,1/2,-1/2;]
```

```
Q =
       1/2            1/2            1/2            1/2
       1/2            1/2           -1/2           -1/2
       1/2           -1/2           -1/2            1/2
       1/2           -1/2            1/2           -1/2
```

```
Q*Q'
```

```
ans =
       1              0              0              0
       0              1              0              0
       0              0              1              0
       0              0              0              1
```

```
Q'*Q
```

```
ans =
       1              0              0              0
       0              1              0              0
       0              0              1              0
       0              0              0              1
```

```
Q'
```

```
ans =
       1/2            1/2            1/2            1/2
       1/2            1/2           -1/2           -1/2
       1/2           -1/2           -1/2            1/2
       1/2           -1/2            1/2           -1/2
```

```
Q^-1
```

```
ans =
       1/2            1/2            1/2            1/2
       1/2            1/2           -1/2           -1/2
       1/2           -1/2           -1/2            1/2
       1/2           -1/2            1/2           -1/2
```

直交行列

Gram-Schmidtの直交化法

列を正規直交にする方法

任意の線形独立なベクトルの集合$a_{1,} \cdots , a_n$をGram-Schmidtの過程によって直交ベクトルの集合へ変換

$$v_1 = a_1$$
$$v_i = a_i - \frac{v_1^T a_i}{v_1^T v_1} v_1 - \cdots - \frac{v_{i-1}^T a_i}{v_{i-1}^T v_{i-1}} v_{i-1}$$
$$q_i = \frac{v_i}{\|v_i\|} \text{ は正規直交}$$

例

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_1 = a_1$$
$$v_2 = a_2 - \frac{1}{2} v_1$$
$$v_3 = a_3 - \frac{1}{2} v_1 - \frac{1}{3} v_2$$

$$q_1 = \frac{v_1}{\|v_1\|} = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$q_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$
$$q_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{3}{4}} \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$a_1 = v_1$$
$$a_2 = \frac{1}{2} v_1 + v_2$$
$$a_3 = \frac{1}{3} v_1 + \frac{1}{2} v_2 + v_3$$

若しくは

$$a_1 = \sqrt{2}\, q_1$$
$$a_2 = \sqrt{\frac{1}{2}}\, q_1 + \sqrt{\frac{3}{2}}\, q_2$$
$$a_3 = \sqrt{\frac{1}{2}}\, q_1 + \sqrt{\frac{1}{6}}\, q_2 + \sqrt{\frac{4}{3}}\, q_3$$

以上より

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{6}} \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{bmatrix}$$

$$A = QR$$

3.3.9

```
a1 = [0;0;1]
```

```
a1 =
       0
       0
       1
```

```
a2 = [0;1;1]
```

```
a2 =
       0
       1
       1
```

```
a3 = [1;1;1]
```

```
a3 =
       1
       1
       1
```

```
v1 = a1
```

```
v1 =
       0
       0
       1
```

```
v2 = a2 - (a2' * v1)/(v1' * v1)*v1
```

```
v2 =

     0
     1
     0
```

```
v3 = a3 - (a3' * v1) / (v1' * v1)*v1 - (a3' * v2)/(v2' * v2)*v2
```

```
v3 =

     1
     0
     0
```

```
q1 = v1 / norm(v1)
```

```
q1 =

     0
     0
     1
```

```
q2 = v2/norm(v2)
```

```
q2 =

     0
     1
     0
```

```
q3 = v3/norm(v3)
```

```
q3 =

     1
     0
     0
```

```
A = [a1,a2,a3]
```

```
A =

     0     0     1
     0     1     1
     1     1     1
```

```
Q = [q1,q2,q3]
```

```
Q =

     0     0     1
     0     1     0
     1     0     0
```

```
R = [1,1,1;0,1,1;0,0,1]
```

```
R =

     1     1     1
     0     1     1
     0     0     1
```

```
Q * R
```

```
ans =
       0              0              1
       0              1              1
       1              1              1
```

3.3.10

```
clear
A = [3,0;4,5]
```

```
A =
       3              0
       4              5
```

```
a1 = [3;4]
```

```
a1 =
       3
       4
```

```
a2 = [0;5]
```

```
a2 =
       0
       5
```

```
v1 = a1
```

```
v1 =
       3
       4
```

```
v2 = a2 - (a2' * v1)/(v1' * v1)*v1
```

```
v2 =
     -12/5
       9/5
```

```
q1 = v1 / norm(v1)
```

```
q1 =
       3/5
       4/5
```

```
q2 = v2 / norm(v2)
```

```
q2 =
      -4/5
       3/5
```

```
Q = [q1,q2]
```

```
Q =
       3/5           -4/5
```

```
        4/5              3/5
```

```
R = [5,4;0,3]
```

```
R =
        5                4
        0                3
```

```
Q * R
```

```
ans =
        3                *
        4                5
```

3.3.11

```
clear
a1 = [1;2;2]
```

```
a1 =
        1
        2
        2
```

```
a2 = [1;3;1]
```

```
a2 =
        1
        3
        1
```

```
v1 = a1
```

```
v1 =
        1
        2
        2
```

```
v2 = a2 - (a2' * v1)/(v1' * v1)*v1
```

```
v2 =
        0
        1
       -1
```

```
q1 = v1 / norm(v1)
```

```
q1 =
        1/3
        2/3
        2/3
```

```
q2 = v2 / norm(v2)
```

```
q2 =
```

```
        0
  985/1393
 -985/1393
```

```
Q = [q1,q2]
```

```
Q =
     1/3           0
     2/3       985/1393
     2/3      -985/1393
```

```
R = [3,3;0,sqrt(2)]
```

```
R =
     3             3
     0         1393/985
```

```
Q * R
```

```
ans =
     1             1
     2             3
     2             1
```

$$A = m \times n$$

ならば

$$Q = m \times n$$
$$R = n \times n$$

3.3.12

```
clear
a1 = [1;2;2]
```

```
a1 =
     1
     2
     2
```

```
a2 = [1;3;1]
```

```
a2 =
     1
     3
     1
```

```
b = [1;1;1]
```

```
b =
     1
     1
     1
```

```
A = [a1,a2]
```

```
A =
       1              1
       2              3
       2              1
```

```
v1 = a1
```

```
v1 =
       1
       2
       2
```

```
v2 = a2 - (a2' * v1)/(v1' * v1)*v1
```

```
v2 =
       0
       1
      -1
```

```
q1 = v1 / norm(v1)
```

```
q1 =
       1/3
       2/3
       2/3
```

```
q2 = v2 / norm(v2)
```

```
q2 =
       0
     985/1393
    -985/1393
```

```
Q = [q1,q2]
```

```
Q =
       1/3            0
       2/3          985/1393
       2/3         -985/1393
```

```
R = [3,3;0,sqrt(2)]
```

```
R =
       3              3
       0           1393/985
```

```
R^-1 * Q' * b
```

```
ans =
       5/9
       0
```

3.3.13

$$
\begin{aligned}
P &= QR\left(R^TQ^TQR\right)^{-1}R^TQ^T \\
&= QRR^{-1}\left(R^T\right)^{-1}R^TQ^T \\
&= QQ^T
\end{aligned}
$$

3.3.14

$$v_2^T\left(c - \frac{v_1^Tc}{v_1^Tv_1}v_1\right) = v_2^Tc - \frac{v_1^Tc}{v_1^Tv_1}v_2^Tv_1 = v_2^T$$

関数空間と**Furier**級数

Hilbert空間

ベクトル$(v_1, v_2 \cdots)$はその長さ$\|v\|$が有限であるとき、かつその時に限り、無限次元の「ヒルベルト空間」にある

関数の区間の長さ

例えば、区間$0 \le x \le 2\pi$の関数$f(x)$の長さを求める場合

$$\|f\|^2 = \int_0^{2\pi} \left(f(x)\right)^2 dx$$

Furier級数

$$y(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + \cdots$$

このとき、係数b_1を求めるには、両辺に$\sin(x)$をかけて、0から2πまで積分する

$$\int_0^{2\pi} y(x)\sin(x)dx = a_0\int_0^{2\pi} \sin(x)dx + a_1\int_0^{2\pi} \cos(x)\sin(x)dx + b_1\int_0^{2\pi} \left(\sin(x)\right)^2dx + \cdots$$

sinとcosは互いに直交なので、sincosは0そのほかもb_1の値以外はすべて積分すると0になるので

$$f(x) = \sin(x)$$

$$b_1 = \frac{\int_0^{2\pi} y(x)\sin(x)dx}{\int_0^{2\pi} \left(\sin(x)\right)^2dx} = \frac{y^Tf}{f^Tf}$$

もしくは

$$b_1 \sin(x) = \frac{y^T f}{f^T f} f$$

3.3.15

$$\|v\|^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

$$\int_0^1 e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{1}{2}\left(e^2 - 1\right)$$

$$\int_0^1 e^x e^{-x} \mathrm{dx} = 1$$

3.3.16

$$b_1 = \frac{\int_0^{2\pi} y(x) \sin(x) dx}{\int_0^{2\pi} (\sin(x))^2 dx}$$

3.3.17

$$a_0 = \int_0^{2\pi} y = \frac{1}{2}$$

$$a_1 = \frac{\int_0^{\pi} y \cos(x) dx + \int_{\pi}^{2\pi} y \cos(x) dx}{\int_0^{\pi} (\cos(x))^2 dx + \int_{\pi}^{2\pi} (\cos(x))^2 dx} = 0$$

$$b_1 = \frac{\int_0^{\pi} \mathrm{y} \sin(x) dx + \int_{\pi}^{2\pi} y \sin(x) dx}{\int_0^{\pi} (\sin(x))^2 dx + \int_{\pi}^{2\pi} (\sin(x))^2 dx} = \frac{2}{\pi}$$

3.3.18

$$v_4 = x^3 - \frac{\int_{-1}^{1} x^3 x dx = \frac{2}{5}}{\int_{-1}^{1} x x dx = \frac{2}{3}} x = x^3 - \frac{3}{5} x$$

3.3.19

$$y = \frac{x^T 1}{1^T 1} + \frac{x^{2T} x}{x^T x} x = \frac{\left(\frac{2}{3}\right)}{2} + 0 = \frac{1}{3}$$