Entry and Exit in Heterogeneous Firm Models

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October 29, 2020

Abstract

I describe a general model framework for entry and exit in heterogeneous agent models; typically firms in most applications. I describe the timing assumptions involved, and how (endogenous and exogenous) entry and exit affect the value function, law-of-motion for agent distribution, and general equilibrium conditions. I then provide pseudocode describing the VFI Toolkit implementation based on entry/exit framework. The result is a codebase capable of solving a wide range (but not all) of heterogeneous agent general equilibrium problems with entry and exit.

Keywords: Entry and Exit, Heterogeneous agent models, Numerical methods, General Equilibrium.

JEL Classification: E00; C68; C63; C62

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Contents

1 Introduction

Entry and exit are important concepts for many economic issues. From the role of entrepreneurship and the creation of new firms in economic growth, to the implications of policies that sustain zombie-firms (avoid exit). From the implications for China's future of abolishing the one-child policy (allowing greater 'entry' of newborns) for the costs of aging, to the impacts of health-policy (reducing 'exit' due to death) on the sustainability of pensions.

This paper provides a standardized description of such entry and exit problems —both endogenous and exogenous— and provides definitions of stationary equilibrium and transtitions paths in such models. We frame these models as a value function problem, an equation for iterating on the distribution of agents, and a general equilibrium condition. Defining the models in this way allows us to solve them using the VFI Toolkit (vfitoolkit.com), and we provide links to codes implementing a variety of examples. Relevant theory on existence of equilibrium, convergence, and similar issues is summarized.

Important examples of firm models which are encompased by the model framework given here are **?** and **?** (both have been replicated using codes implementing this framework in VFI Toolkit). **?** does not fit the model due to timing of the exit decision, although since it does not contain any endogenous state variables it can actually be solved using essentially the same algorithms as the timing is rendered irrelevant for most aspects of the model when there are only exogenous states. Some models such as **?** and **?** are often interpreted as having entry and exit, but from the mathematical perspective of the models described here they have neither. The framework developed in this paper considers entry and exit to involve changes in the total mass of agents, while the 'entry/exit' in these later models would be viewed by this framework as simply a change in the value of an endogenous state variable[1] and there is no change in the total mass of agents.

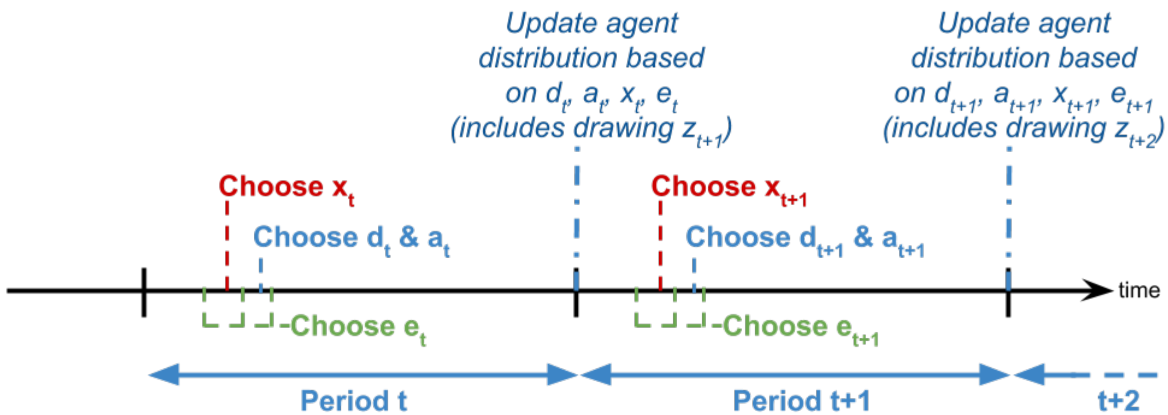
The most important limitation to the generality of the framework implemented by the VFI Toolkit relates to the timing of entry. The timing of exit by default is start-of-period, but end-of-period exit with a mixture of endogenous and exogenous exit is also possible, see Appendix **??** for details. Figure 1 illustrates the assumed timing, and a full description is now given. Agents that choose to enter in the current period do not appear until next period. Agents that choose to exit are unable to take further decisions this period, but don't get removed from the current distribution of agents until the end of this period; that is, the exit decision is made at the beginning of the period (prior to any other decisions).[2] This timing is explicit in the formulations of the value function and the law-of-motion of the agent distribution and so will not be described further as interested

[1] Worker/entrepreneur are the names typically given to the values of the endogenous state in these Lucas-span-of-control models.

[2] The choice of these timings is both common in the literature on heterogeneous firms, and has advantages in terms of codes. Different timing of the entry decision would make transition paths difficult, and require treating new entrants as genuinely distinct from otherwise identical existing agents. Different timing of the exit decision would require a whole new set of functions to evaluate the characteristics of exiting agents.

readers will be able to 'see' it there. Note that with (endogenous) exit decisions being made at the beginning of the period but the actual exit not occouring until the end of the period, models of exogenous exit are best understood as having exit occour between periods (often informally referred to as 'overnight').

Figure 1: Timing of Entry and Exit



d_t and a_t are decision variables d and a' in the recursive notation for the value function problem associated with time t, the current state (a, z) is a_{t-1} and z_t. x is the exit decision and e is the entry decision. Note that while exit decision x_t is taken before d_t and a_t the actual exit from the distribution does not occour until after. The 'three' points for the entry decision indicate that it is irrelevant at which point in time relative to the exit and usual decisions it is being made, note further that the actual entry into the distribution does not occour until after them.

When both entry and exit are exogenous only exit appears in the value function problem of existing agents (exit appears as a change in the discount factor on the future[3]). Both entry and exit appear in the simulation of agents distribution. No general equilibrium condition is required.

Making entry endogenous adds a general equilibrium condition, but does not affect the value function or agent simulation problems. Making exit endogenous requires adding an exit decision to the value function problem, and adds a further general equilibrium condition, but does not affect the agent simulation problem.

This perspective on setting up and solving an entry-exit model has the advantage of making it clear exactly how to modify standard algorithms for Bewley-Huggett-Aiyagari heterogeneous agent models to incorporate entry-exit. Figure 2 summarizes the changes necessary to the algorithms, as described by the previous few paragraphs. Extending these algorithms to general equilibrium transition paths can then be done analagous to standard Bewley-Huggett-Aiyagari models, and we

[3]To be clear, while entry does not enter directly, it may have indirect effects in a general equilibrium model via various prices. Similarly to how in the model of **?** the distribution of agents is not irrelevant, simply that prices (specifically the interest rate) contain all relevant information about the distribution of agents.

explain how to do so in Section **AAA**

Figure 2: Modifications to standard BHA-solution due to Entry/Exit.

		Stage of BHA-solution algorithm		
	Stage:	Bellman Equation	Iteration on Agent Dist.	General Eqm
	Output:	(Value fn, Policy fn)	(Stationary Dist.)	(Evaluate GE cond^n)
Entry	Endo.	-	$\mu' = P\mu + \eta$	entry condition
	Exog.	-	$\mu' = P\mu + \eta$	-
Exit	Endog.	decision variable	exit probability, P^{exit} from g^{exit}	-
	Exog.	discount rate	exit probability, P^{exit}	-

'-' indicates no change from standard problem without entry/exit. η is the distribution of entrants. In standard heterogeneous firm models 'entry condition' is the free-entry condition, but may be different in other models, such as endogenous fertility in OLG models; in all cases it is a general equilibrium condition that depends on the distribution of entrants. In all cases the 'mass' of the agent distribution also has to be kept track of during the iteration on agent distribution stage.

A brief observation on what 'endogenous' exit means is worthwhile. A model in which agents have a 'level of health' which they can increase or decrease by investing in medicine, and where the (conditional) probability of dying depends on their current level of health is considered exogenous exit. This is because exit is not a deliberate decision of agents. Endogenous exit involves a deliberate decision to exit, and must be modelled as a binary choice to stay (0) or exit (1). The use of this in a model of firms is clear —an unprofitable firm may choose to close— in a household model it is less clear and only seems likely in models wishing to consider either suicide or euthanasia.[4]

What kinds of entry and exit are not allowed for? We do not allow for the probability of exit to depend on current decisions directly (only indirectly via their influence on next periods endogenous state).

It is worth noting that the landmark model of **?**, which covers endogenous entry and exit, is often referred to as partial equilibrium because it does not consider equilibrium in, e.g., capital and labour markets. From our perspective this is incorrect. The model has general equilibrium in the endogenous entry of firms. As with any general equilibrium model one can always add 'more' general equilibrium; e.g., in the model of **?** we can add endogenous labour, which adds another general equilibrium condition (albeit one which is trivial in stationary competitive equilibria), or add endogenous human capital or a second asset.[5] In the context of **?** the addition of capital and labour markets, with general equilibrium conditions on these, is then referred to as making

[4]Whether or not utility maximizing rational expectations models are suitable/useful for modelling such decisions is an empirical/scientific question and beyond the current paper which is purely technical. Your author suspects it is not very useful for understanding causes, but may still be useful for assessing, e.g., the likely impact of legalising euthanasia on health care spending.

[5]It is also worth noting that heterogeneous firm models are almost always incomplete markets models, but because firms are typically modelled as risk-neutral this is irrelevant to the actual decisions of firms and so goes unmentioned. More accurately, it remains unspecificed whether markets are complete or incomplete because with risk-neutral agents the model solution is anyway identical.

the model general equilibrium, but from our perspective this is really just adding 'more' general equilibrium.

2 Model Environment and Definitions

We now describe a general Entry-Exit model. We first describe the model environment and provide the definition of a stationary competitive equilibrium. While this definition is standard we use it to introduce notation to index the stationary competitive equilibrium by the parameters and general equilibrium price vector, the later as there are possibly multiple competitive equilibria and it is important for the theory and algorithms that we are explicit about which equilibria.

2.1 The Model Environment

The models are those which can be expressed as follows: Let $A \subseteq \mathbb{R}^{n_a}$ be the endogenous state variable, $D = D_1 \times D_2 \subseteq A \times \mathbb{R}^{n_d}$ be the choice variable, and $Z \subseteq \mathbb{R}^{n_z}$ be the exogenous state variable. Let $\Theta \subseteq \mathbb{R}^q$ be a parameter space, and $\theta \in \Theta$ a parameter vector.[6] The state of an agent is then a pair (a, z). A value function maps $V_{p,\theta} : A \times Z \to \mathbb{R}$. A policy function maps $g_{p,\theta} : A \times Z \to D$. Let $S = A \times Z$, and let $\mathcal{S}$ be it's Borel σ-field. The measure of agents $\mu_{p,\theta}$ is a probability distribution over $(S, \mathcal{S})$. The return function maps $F_{p,\theta} : D \times A \times Z \to \mathbb{R}$, and the discount factor is $0 < \beta < 1$ (β is considered to be an element of θ).

Aggregate variables are $M \in \mathbb{M} \subseteq \mathbb{R}^m$. A price vector is $p \in \mathbb{P} \subseteq \mathbb{R}^p$. The exogenous shock follows a Markov-chain with transition function Q mapping from Z to Z. The aggregation function maps $\mathcal{M} : \mathcal{M}(S, \mathcal{S}) \to \mathbb{R}^m$, where $\mathcal{M}(S, \mathcal{S})$ is the space of probability measures on $(S, \mathcal{S})$. The general equilibrium function maps $\lambda : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$.

The timing of decisions/events is important for entry/exit models. If exit is endogenous we assume that at the exit decision occours at the beginning of the period, but that the actual exit does not occour until the end of the period. Thus exiting firms exist this period but their actions are solely those of 'exit', which in many models carries some kind of cost. If exit is exogenous then the exit shock does not occour until the end of the period and firms continue to make standard decisions during the present period. Entry decisions are made at the beginning of the period, immediately after the exit shock, and firms essentially 'sit-out' the present period and are able to make standard decisions from the period after entering. This eliminates the possibility of a firm entering and exiting within the same single period; they must enter in one period, and exit the next, but they need not do anything else in between entry and exit (i.e., no standard production decision need take place).

The literal translations of exit and entry to Greek are exodus and 'eisodos', but unfortunately

[6] The parameter vector consists of all the parameters of the model, not just those relating to the agents problem.

this did not help come up with a clean notation, so will use ζ for exit and η for entry; ζ will be a parameter/policy function depending on whether exit is exogenous/endogenous, η a distribution.

2.2 The Value Function

The value function problem is that facing an existing agent, a typical example might be the expected present discounted profits of a firm. We begin by defining the value function with exogenous exit, and then the (different) value function problem with endogenous exit. Let ζ be the exogenous conditional probability of exit, so $1-\zeta$ is the conditional probability of survival. As a minimum this survival probability is conditional on being 'alive' in the present time period, but we further allow for ζ to depend on both the current endogenous state, a, and exogenous state z (so ζ may depend on $s=(a,z)$; we will sometimes denote this explicitly as $\zeta(a,z)$).

Given prices p, the agents value function $V_{p,\theta}$ and policy function $g_{p,\theta}$ solve the agents problem

$$V_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta} \left\{ F_{p,\theta}(d,a',a,z) + \beta(1-\zeta(a,z)) \int V_{p,\theta}(a',z') Q_\theta(z,dz') \right\} \tag{1}$$

Notice that exogenous exit, $\zeta(a,z)$, appears in the discount factor, while exogenous entry does not appear in the value function problem. Since the conditional survival probability, $1-\zeta(a,z)$, will always be a real valued scalar between zero and one all the standard results on value functions continue to apply.

Endogenous exit Let $\zeta \in \{0,1\}$ be the exit decision, with $\zeta=0$ being to remain and $\zeta=1$ being to exit. The value function problem to be solved becomes,

$$V_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta \zeta\in\{0,1\}} \left\{ (1-\zeta) F_{p,\theta}(d,a',a,z) + \zeta F^{\zeta}_{p,\theta}(,a,z) + \beta(1-\zeta) \int V_{p,\theta}(a',z') Q_\theta(z,dz') \right\} \tag{2}$$

Notice that the value of exit, $F^{\zeta}_{p,\theta}(,a,z)$ is assumed to be independent of the other current decisions ($y=(d,a')$), but may depend on current state $s=(a,z)$.[7] Note also how the discount rate with endogenous exit looks a lot like that with exogenous exit (namely, it equals $1-\zeta$); this similarity between endogenous and exogenous exit be useful when implementing the agents distribution.

2.2.1 Pseudo-code implementing value function iteration

As mentioned previously, entry (endogenous or exogenous) plays no role in the value function problem. When exit is exogenous the exit probability is simply incorporated into the discount factor as a 'conditional probability of survival'. The algorithm implementing this is therefore the standard pure discretized value function iteration used by the VFI Toolkit, the outline of which is given by,

[7] This is equivalent to a timing assumption that the endogenous exit decision is made at the start of the period and only after that are the rest of the agents decisions made.

Algorithm 1 VFI with exogenous exit

Declare initial value V_0.
Declare iteration count $n = 0$.
while $||V_{n+1} - V_n|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for $z = 1, ..., n_z$ **do**
    for $a = 1, ..., n_a$ **do**
      Calculate $E[V_{old}|z]$ $\triangleright$ Using quadrature method based on finite transition matrix for z.
      Calculate $V_n(a, z) = \max_{d=1,...,n_d,\, a'=1,...,n_a} F(d, a', a, z) + \beta(1 - \zeta(a, z))E[V_{old}(a', z')|z]$
      Calculate $g_n(a, z) = \arg\max_{d=1,...,n_d,\, a'=1,...,n_a} F(d, a', a, z) + \beta(1 - \zeta(a, z))E[V_{old}(a', z')|z]$
    end for
  end for
end while

Notice the use of discount factor $\beta(1 - \zeta(a, z))$ to discount the future.

When exit is endogenous the value function problem now requires evaluating $V_n(a, z)$ based on remaining (no exit), and the return to exit $F^{\zeta}(, a, z)$, and then comparing the two. This gives the following algorithm, The VFI Toolkit commands for value function iteration will output 'Policy'

Algorithm 2 VFI with endogenous exit

Declare initial value V_0.
Declare iteration count $n = 0$.
while $||V_{n+1} - V_n|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for $z = 1, ..., n_z$ **do**
    for $a = 1, ..., n_a$ **do**
      Calculate $E[V_{old}|z]$ $\triangleright$ Using quadrature method based on finite transition matrix for z.
      Calculate $V_n(a, z) = \max_{d=1,...,n_d,\, a'=1,...,n_a} F(d, a', a, z) + \beta E[V_{old}(a', z')|z]$
      Calculate $g_n(a, z) = \arg\max_{d=1,...,n_d,\, a'=1,...,n_a} F(d, a', a, z) + \beta E[V_{old}(a', z')|z]$
      Calculate $F^{\zeta}(a, z)$.
      if $V_n(a, z) < F^{\zeta}(, a, z)$ **then** $\zeta(a, z) = 1$, $V_n(a, z) = F^{\zeta}(a, z)$, $g_n(a, z) = 0$.
      else $\zeta(a, z) = 0$.
      end if
    end for
  end for
end while

which contains $g_n(a, z)$ (to be precise, it outputs the policy in the form of an index, not as a value; setting $vfoptions.polindorval = 2$ will change this) and will also output 'ExitPolicy' which contains $\zeta(a, z)$. Notice that by setting 'Policy' (i.e., $g_n(a, z)$) to a zero index means that it will generate errors if the user tries to use it inappropriately.[8]

In both cases the actual implementation includes use of Howards improvement algorithm, and

[8]Matlab indexes from 1; there is no zero index. Julia would do the same, but Python is different (worse ;) and indexes from zero, so some other value would be better there.

the use of loops vs vectorization, as well as single cpu, parallel cpus, or gpu varies depending on various internal options.

2.3 Agents Distribution

Let $\mu(a,z)$ be the agents distribution. We are interested in the evolution of the agents distribution over time, that is, in how $\mu'(a,z)$ is determined. We will consider iteration on the agents distribution. This will depend on the

$$\mu'_{p,\theta}(a,z) = \frac{N_E}{N}\eta(a,z) + \int\int\left[\int 1_{a'=g^{a'}_{p,\theta}(\hat{a},z)}(1-\zeta(\hat{a},z))\mu_{p,\theta}(\hat{a},z)Q_\theta(z,dz')\right]d\hat{a}dz \tag{3}$$

where N_E is the mass of new-entrants, and N is the mass of agents. We also need to keep track of the mass of agents,

$$N' = N_E + N\int\int(1-\zeta(a,z))\mu_{p,\theta}(a,z)dadz \tag{4}$$

The intuition is clearer in the case where ζ is constant (is independent of (a,z)) and this equation simplies to $N' = N_E + (1-\zeta)N$.

When computing the stationary distribution we perform the iteration described by equations (3) and (4). We start from an initial distribution simply given by the new-entrants distribution, $\eta(a,s)$, and then iterate until convergence. In principle however any initial guess for the agent stationary distribution could be used.[9]

Note that neither making entry endogenous, nor making exit endogenous will have any effect on this stage of the problem. However when actually writing code to implement the agent simulation it turns out to be useful to know whether exit is endogenous or exogenous (due to how the policy function $g^{a'}_{p,\theta}(a,z)$ is stored in the case of endogenous exit). And so while mathematically it is unimportant the codes do require this information to be input.

Note also that equation (3) is dependent on our assumption about the timing of entry and exit decisions/shocks. Importantly it means that functions which depend on policies be evaluated directly on $\mu(s,z)$, as new entrants in $\mu(s,z)$ are those that decided to enter last period, and thus get to make all the standard decisions of existing agents in the present period (potentially including exit). (It does not contain those who make a decision to enter in the current period.)

[9]When doing the computation of this the VFI Toolkit actually works internally with the 'mass*pdf', rather than the pdf and mass seperately. This is because by combining equations (3) and (4) we avoid having to compute the integral in the later. Note also that when implementing this using 'iteration on the whole distribution' methods, the policy function, the exogenous shock process, and the conditional survival probability, can all be combined into a single 'transition probability matrix' (it is not really a transition probability matrix as the rows do not sum to one due to the probability of death/exit).

2.3.1 Pseudo-code implementing Stationary Agent Distribution

I describe a single algorithm for implementing the Stationary Agent Distribution with both entry and exit; whether they are endogenous or exogenous is irrelevant.

Algorithm 3 Stationary Agent Distribution with entry and exit

Declare initial distribution μ_0

Create $P = [P_{ij}]$, a matrix with rows indexing (a, z) and columns indexing (a', z').

$\triangleright\ P_{ij} = (1 - \zeta(a, z))\mathbb{1}_{a'=g(a,z)}\pi(z'|z)$

Declare iteration count $n = 0$.

while $||\mu_{n+1} - \mu_n|| >$'tolerance' **do**

  Increment n.

  Calculate $\mu_n = P'\mu + N_e\eta$

end while

Return the mass of μ_n and a normalized version of μ_n (a pdf)

Implementation uses $\mu_0 = \eta$ (initial distribution is the entry distribution). The VFI Toolkit command output is 'StationaryDist', which contains both *StationaryDist.mass* and *StationaryDist.pdf*. I choose this as the pdf is both more directly useful for certain purposes, and because it then has known properties (summing to one) which can be exploited. Notice that P incorporates exit because of $(1-\zeta(a, z))$, and so the rows will not sum to one as they would for a standard transition matrix; $(1 - \zeta(a, z))$ is the conditional probability of survival, when exit is endogenous these probabilities will be 0 or 1.[10] Depending on internal options P may be a sparse matrix; so might μ.

2.4 Stationary Competitive Equilibrium with Entry and Exit

We now provide the relevant defintions for Stationary Competitive Equilibrium with Entry and Exit for each of the possible combinations of endogenous/exogenous entry and exit. Any of these can be extended by the addition of further general equilibrium conditions of the kind that are standard in representative agent and heterogeneous household models. We deliberately ignore these kinds of additional standard general equilibrium conditions here so as to focus on directly on the roles of entry and exit in defining stationary competitive equilibrium.

***Exogenous entry and exogenous exit*:** If both the entry and exit are exogenous then there is no general equilibrium condition. We begin with a description of stationary competitive equilibrium in this case.

Definition 1. *A Stationary Competitive Equilibrium is an agents value function $V_{p,\theta}$; agents policy function g; vector of prices p; measure of agents $N\mu_{p,\theta}$; measure of entrants $N_e\eta_{p,\theta}$; such that*

1. *Given prices p and exit ζ, the agents value function $V_{p,\theta}$ and policy function $g_{p,\theta}$ solve the*

[10] P' is the transpose of P.

agents problem

$$V_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta} \left\{ F_{p,\theta}(d,a',a,z) + \beta(1-\zeta(a,z)) \int V_{p,\theta}(a',z') Q_\theta(z,dz') \right\} \tag{5}$$

2. *The measure of agents is invariant:*

$$\mu_{p,\theta}(a,z) = \frac{N_E}{N}\eta(a,z) + \int\int \left[\int 1_{a'=g^{a'}_{p,\theta}(\hat{a},z)} (1-\zeta(\hat{a},z)) \mu_{p,\theta}(\hat{a},z) Q_\theta(z,dz') \right] d\hat{a}dz \tag{6}$$

$$N = N_E + N \int\int (1-\zeta(a,z)) \mu_{p,\theta}(a,z) dadz \tag{7}$$

This —the case of exogenous entry and exogenous exit— is just a partial equilibrium, hence we need only solve the value function and stationary distribution. It is possible to add general equilibrium conditions based on embedding this model in more complicated structures, but since these have nothing to do with entry and exit per se this possibility is ignored here as setting up and solving them is just the same as in any representative agent of heterogeneous household model.

***Endogenous exit*:** The only relevant change to the Stationary Competitive Equilibrium definition is in the value function problem, as exit is now a policy function, rather than a parameter. From the perspective of the measure of agents however it can still be treated as a parameter, and it does not impose any general equilibrium conditions.

Definition 2. *A Stationary Competitive Equilibrium is an agents value function $V_{p,\theta}$; agents policy function g and exit policy function ζ; vector of prices p; measure of agents $N\mu_{p,\theta}$; measure of entrants $N_e\eta_{p,\theta}$; such that*

1. *Given prices p , the agents value function $V_{p,\theta}$, policy function $g_{p,\theta}$, and exit policy function $\zeta_{p,\theta}$ solve the agents problem*

$$V_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta,\, \zeta\in\{0,1\}} \left\{ F_{p,\theta}(d,a',a,z) + \beta(1-\zeta) \int V_{p,\theta}(a',z') Q_\theta(z,dz') \right\} \tag{8}$$

2. *The measure of agents is invariant:*

$$\mu_{p,\theta}(a,z) = \frac{N_E}{N}\eta(a,z) + \int\int \left[\int 1_{a'=g^{a'}_{p,\theta}(\hat{a},z)} (1-\zeta(\hat{a},z)) \mu_{p,\theta}(\hat{a},z) Q_\theta(z,dz') \right] d\hat{a}dz \tag{9}$$

$$N = N_E + N \int\int (1-\zeta(a,z)) \mu_{p,\theta}(a,z) dadz \tag{10}$$

This —the case of exogenous entry and endogenous exit— is just a partial equilibrium, hence we need only solve the value function and stationary distribution. It is possible to add general equilibrium conditions based on embedding this model in more complicated structures, but since these have nothing to do with entry and exit per se this possibility is ignored here as setting up and solving them is just the same as in any representative agent of heterogeneous household model.

***Endogenous entry*:** Endogenous entry changes the Stationary Competitive Equilibrium definition by adding a general equilibrium, namely the free-entry condition. This general equilibrium condition will typically require special treatment as it depends not on the distribution of agents, but on the distribution of potential entrants.

Definition 3. *A Stationary Competitive Equilibrium is an agents value function $V_{p,\theta}$; agents policy function g; vector of prices p; measure of agents $N\mu_{p,\theta}$; measure of entrants $N_e\eta_{p,\theta}$; such that*

1. *Given prices p and exit ζ, the agents value function $V_{p,\theta}$ and policy function $g_{p,\theta}$ solve the agents problem*
$$V_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta} \left\{ F_{p,\theta}(d,a',a,z) + \beta(1-\zeta(a,z)) \int V_{p,\theta}(a',z')Q_\theta(z,dz') \right\} \tag{11}$$

2. *Free-entry condition: $\int(\textit{net-value-of-entry}(a,z))N_e d\eta$.*

3. *The measure of agents is invariant:*
$$\mu_{p,\theta}(a,z) = \frac{N_E}{N}\eta(a,z) + \int\int\left[\int 1_{a'=g^{a'}_{p,\theta}(\hat{a},z)}(1-\zeta(\hat{a},z))\mu_{p,\theta}(\hat{a},z)Q_\theta(z,dz')\right] d\hat{a}dz \tag{12}$$
$$N = N_E + N\int\int(1-\zeta(a,z))\mu_{p,\theta}(a,z)dadz \tag{13}$$

This —the case of endogenous entry and exogenous exit— is a general equilibrium, hence we need to solve the value function and stationary distribution, and then further solve the fixed point problem over these imposed by the general equilibrium condition. It is possible to add further general equilibrium conditions based on embedding this model in more complicated structures, but since these have nothing to do with entry and exit per se this possibility is ignored here as setting up and solving them is just the same as in any representative agent of heterogeneous household model. Any further general equilibrium conditions are likely to take the form of standard general equilibrium conditions and involve evaluation of functions on the agents distribution; they will typically not require special treatment. The free-entry condition requires special treatment on account of being evaluated on the potential entrants distribution. The free-entry condition above is written to be as general as possible, a standard implementation would be something like $\beta E_\eta[V] - c_e$, where c_e is a fixed-cost of entry. *net-value-of-entry*(a,z) should be understood as the net value of entry for a firm that ends up entering with initial value (a,z).

***Endogenous entry and endogenous exit*:** Endogenous entry changes the Stationary Competitive Equilibrium definition by adding a general equilibrium, namely the free-entry condition. This general equilibrium condition will typically require special treatment as it depends not on the distribution of agents, but on the distribution of potential entrants. Endogenous exit changes the value function problem to include the exit policy function as a decision, rather than a parameter.

Definition 4. *A Stationary Competitive Equilibrium is an agents value function $V_{p,\theta}$; agents policy function g and exit policy function ζ; vector of prices p; measure of agents $N\mu_{p,\theta}$; measure of entrants $N_e\eta_{p,\theta}$; such that*

1. *Given prices p, the agents value function $V_{p,\theta}$ policy function $g_{p,\theta}$, and exit policy function $\zeta_{p,\theta}$ solve the agents problem*

$$V_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta,\, \zeta\in\{0,1\}} \left\{ F_{p,\theta}(d,a',a,z) + \beta(1-\zeta)\int V_{p,\theta}(a',z')Q_\theta(z,dz') \right\} \quad (14)$$

2. *Free-entry condition: $\int(\text{net-value-of-entry}(a,z))N_e d\eta$.*

3. *The measure of agents is invariant:*

$$\mu_{p,\theta}(a,z) = \frac{N_E}{N}\eta(a,z) + \int\int\left[\int 1_{a'=g^{a'}_{p,\theta}(\hat{a},z)}(1-\zeta(\hat{a},z))\mu_{p,\theta}(\hat{a},z)Q_\theta(z,dz')\right] d\hat{a}dz \quad (15)$$

$$N = N_E + N\int\int(1-\zeta(a,z))\mu_{p,\theta}(a,z)dadz \quad (16)$$

This —the case of endogenous entry and endogenous exit— is a general equilibrium, hence we need to solve the value function and stationary distribution, and then further solve the fixed point problem over these imposed by the general equilibrium condition. It is possible to add further general equilibrium conditions based on embedding this model in more complicated structures, but since these have nothing to do with entry and exit per se this possibility is ignored here as setting up and solving them is just the same as in any representative agent or heterogeneous household model. Any further general equilibrium conditions are likely to take the form of standard general equilibrium conditions and involve evaluation of functions on the agents distribution; they will typically not require special treatment. The free-entry condition requires special treatment on account of being evaluated on the potential entrants distribution. The free-entry condition above is written to be as general as possible, a standard implementation would be something like $\beta E_\eta[V]-c_e$, where c_e is a fixed-cost of entry. *net-value-of-entry*(a,z) should be understood as the net value of entry for a firm that ends up entering with initial value (a,z).

2.4.1 Pseudo-code implementing stationary competitive equilibrium

Since the case of exogenous entry and exogenous exit simply involves solving the value function iteration and finding the stationary agent distribution (pseudo-code for both having been given in earlier sections) they will be skipped here; similarly for exogenous entry with endogenous exit.

For endogenous entry with endogenous entry the pseudocode for finding the Stationary Competitive equilibrium (Definition 4) is, A few comments: Steps 4 & 5 are about solving a fixed-point

Algorithm 4 Stationary Competitive Equilibrium

Declare initial 'equilibrium price' p_0.
Step 1: Solve value function iteration problem (to get value fn, policy fn, exit policy fn) given p_n.
Step 2: Solve Agents stationary distribution problem (to get μ)
Step 3: Evaluate the 'free-entry' general equilibrium condition.
[Evaluate any other general equilibrium condition.]
Step 4: Update equilibrium price p_{n+1} based on the general equilibrium conditions.
Step 5: Repeat steps 1-4 until $|p_{n+1} - p_n| <$'tolerance'.

problem, and standard optimization routines can be used (think of it as minimizing the absolute value of the general equilibrium conditions). A standard alternative algorithm is to simply evaluate the general eqm conditions on a grid and then take the minimum (absolute) value; this won't work for models with conditional entry conditions (Section 2.4.2) as it does not ensure that the general equilibrium conditions relating to entry and conditional entry hold. Grid based methods do still work for most entry-exit models namely any with standard endogenous/exogenous entry/exit.[11] Step 3, the 'free-entry' condition is a special kind of general equilibrium condition as it involves valuing a function on the distribution of potential entrants, while all other general equilibrium conditions are valued on the stationary distribution of agents.[12]

Notice that this algorithm is simply about finding the equilibrium prices (the final value of p_n). You will likely then want to calculate the value function (and policy fn and exit policy fn), and the stationary agent distribution.

The final case, endogenous entry with exogenous exit also uses this same pseudocode, except that the implementation of solving the value function is different, as covered earlier.

2.4.2 Conditional Entry in Endogenous Entry Problems

When using endogenous entry, the standard form of a free-entry condition assumes that agents pay a fixed entry cost before knowing their initial state (they have an expectation over a distribution of possible initial states). Some models, such as **?**, add the possibility of 'conditional entry decisions', that after a free-entry decision is made to enter and the agent draws the state in which they will enter they can decide to abort/cancel their entry. Since this later decision is made conditional on the actual state in which they will enter we refer to it as a conditional entry decision. The conditional entry decision involves choosing $\bar{e}(a,z) \in \{0,1\}$ to maximize $\bar{e}(a,z)$*conditional-net-value-of-entry*(a,z), for each (a,z). A few observations: (i) the decision is conditional on (a,z)

[11] VFI Toolkit still implements the grids which still work in the absence of a conditional entry decision. In VFI Toolkit these grid-based methods are used by setting a non-zero value for n_p. While the grids approach cannot be used to solve the model it can still be useful to explore the model properties.

[12] VFI Toolkit needs to be told which general equilibrium conditions are 'entry' conditions using *heteroagentoptions.specialgeneqmcondn*. It also needs to be told what the entrants distribution is which is done as part of the problem setup. See the example codes.

(hence no integrals), (ii) *conditional-net-value-of-entry*(a, z) will typically be similar to but not the same as the *net-value-of-entry*(a, z) used in the free-entry condition.

When conditional entry is used, there is also an important change in the free-entry condition and in the law-of-motion of the agent distribution. Both of these depend on $N_e\eta$, the distribution of entrants. This distribution is no longer exogenous, it is now a combination of the conditional entry decision $\bar{e}(a, z)$ and an exogenous distribution of potential agents, η^{pot}. The codes implementing this framework in the VFI Toolkit make these changes automatically based on the assumption that the exogenously given distribution of agents that relates to entry, which is normally the entrants distribution, should be reinterpreted as the distribution of potential entrants, and that the actual entry distribution is now $N_e\bar{e}\eta^{pot}$; note that the actual mass of this entry distribution is then the integral (or sum) of this actual entry distribution.

Obviously $\bar{e}(a, z) = 1$ is interpreted as that an agent with state (a, z) chooses to continue with entry. While $\bar{e}(a, z) = 0$ is interpreted as that upon knowing their state the agent chooses not to enter, aborting/cancelling the entry decision they made at the stage of the free-entry decision. In terms of timing these conditional entry decisions are assumed to be made immediately after the standard entry decision and prior to anything else occouring.

When solving a model with a conditional entry condition there are two changes to the Algorithm 0. The first is simply a matter of adding a 'Step 1.5' in Algorithm 0. Step 1.5 uses the value function computed in Step 1 to evaluate the conditional entry condition and calculate the conditional entry decision based on this. By implementing this step at this stage of the algorithm the conditional entry decision is always holding in any equilibrium calculation and hence we do not need to take explicit account of it when doing Steps 4 & 5.[13] In some sense there is also a minor modification to both Steps 2 and 3, in that both continue to use the distribution of new agents as before, but that this should now be distinguished from the distribution of potential new agents. But from a practical perspective this involves little to no change to the computation being performed; we can compute the distribution of new agents directly from the distribution of potential agents together with the conditional entry decisions, as described two paragraphs above, and then perform the exact same computations as in the case without conditional entry.

The implementation of this in VFI Toolkit requires being told which general equilibrium condition is a conditional entry problem using $heteroagentoptions.specialgeneqmcondn$ and then specifying which 'parameter' is keeping track of the conditional entry decision using $EntryExitParamNames.CondlEntry$ For an illustration of this in practice see the example based on **?**. I am not aware of any theoretical results that ensure this approach works. But it appears to in practice (it was used to replicate **?**).

[13] The VFI Toolkit always reports a 'zero' as the distance between the new and old conditional entry decision, which is really just stating that it always holds, rather than that it has converged.

3 Conclusion

References

A Existence of Stationary Equilibrium

This section provide the relevant theoretical results that establish the existence of stationary equilibria in this model; the results are not novel, it is simply intended as a convenient summary of existing results.

B End-of-period Exit with both Endogenous and Exogenous Exit

The exit modeled here occours at the end of period, but agents know/decide that they will exit at the beginning of the period and take actions during the current period prior to the actual exit occurring. To the best of my knowledge this originates with **?**. The VFI Toolkit allows for a mixture of both endogenous and exogenous exit, and using just one or the other is an easy subcase.

The timing is summed up in Figure AAA.

The key to this approach is that at the beginning of period t an agent draws an 'exit shock'. With some probability the agent faces no exit decision, with some probability the agent faces an endogenous exit decision, and with some probability the agent faces exogenous exit; let π_{nx}, π_{gx}, and π_{xx} be these three probabilities, respectively.

From the perspective of the VFI Toolkit there are just two changes that need to be made: you need a 'return to exit function' that depends on (d,a',a,z) and you need to specify the exit probabilities (both $vfoptions.exitprobabilities$ and $simoptions.exitprobabilities$, which you would typically set to the same value). As with standard endogenous exit a continuation cost has to be given in $vfoptions.continuationcost$. The contents of $CondlSurvivalProb$ are also now understood as being the exit decision of those firms facing the endogenous exit decision. The conditional survival probability prior to drawing an 'exit shock' can be calculated as $\pi_{gx} * CondlSurvivalProb + \pi_{xx}$.

Examples of codes using this setup based on **?** are given by

Value function problem:
From the perspective of solving the value function problem this looks like having to solve three problems within a given time period (no exit, endogenous exit, known exogenous exit) and then putting these together. This problem is now given explicitly following closely along the lines of those earlier in this paper so as to make it easier to see the similarities and differences. The value function problem is that facing an existing agent, a typical example might be the expected present discounted profits of a firm. Let ζ be the endogenous conditional decision on exit; taking a value of one when choosing exit, zero when not exiting. As a minimum this exit decision is conditional on being 'alive' in the present time period, but we further allow for ζ to depend on both the current endogenous state, a, and exogenous state z (so ζ may depend on $s = (a, z)$; we will sometimes denote this explicitly as $\zeta(a, z)$).

In what follows the value function problem is described as three seperate intraperiod problems —no exit, endogenous exit, and known exogenous exit— which are then combined to form the value function itself. When using the value function iteration command of the VFI Toolkit the outputs are the combined value function, called V in codes, the policy function for 'no exit' $g^{nx}_{p,\theta}$ called $Policy$ in codes, the exit decisions of those agents facing endogenous exit decision called $ExitPolicy$ in codes, and the policy function for firms that are exiting (whether exogenously or by endogenous decision) called *PolicyWhenExiting* in codes.[14]

Given prices p, let the agents value function be $V_{p,\theta}$. $V_{p,\theta}$ is a simple combination of the value functions for 'no exit' (nx), 'endogenous exit' (gx), and 'exogenous exit' (xx),

$$V_{p,\theta}(a,z) = \pi_{nx} V^{nx}_{p,\theta}(a,z) + \pi_{gx} V^{gx}_{p,\theta}(a,z) + \pi_{xx} V^{xx}_{p,\theta}(a,z) \tag{17}$$

The problem with no exit is standard. Given prices p, the agents 'no exit' value function $V^{nx}_{p,\theta}$ and policy function $g^{nx}_{p,\theta}$ solve the agents problem

$$V^{nx}_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta} \left\{ F_{p,\theta}(d,a',a,z) + \beta \int V_{p,\theta}(a',z') Q_\theta(z,dz') \right\} \tag{18}$$

the associated *argmax* of this gives $g^{nx}_{p,\theta}$.

The problem with exogenous exit is trivial. Given prices p, the agents value function $V^{xx}_{p,\theta}$ and policy function $g^{xx}_{p,\theta}$ solve the agents problem

$$V^{xx}_{p,\theta}(a,z) = \max_{y=(d,a')\in Y_\theta} F^{\zeta}_{p,\theta}(d,a',a,z) \tag{19}$$

the associated *argmax* of this gives $g^{xx}_{p,\theta}$. Notice that the return function for agents facing exit is allowed to differ from those that do not exit ($F^\zeta(\cdot)$ rather than $F(\cdot)$).

The problem with endogenous exit is more involved. Let $\zeta \in \{0,1\}$ be the exit decision, with $\zeta = 0$ being to remain and $\zeta = 1$ being to exit. The value function problem to be solved becomes,

$$V_{p,\theta}(a,z) = \max_{\zeta(a,z)\in\{0,1\} y=(d,a')\in Y_\theta} \Big\{ (1 - \mathbb{1}_{\{\zeta==1\}}) F_{p,\theta}(d,a',a,z) + \mathbb{1}_{\{\zeta==1\}} (F^{\zeta}_{p,\theta}(d,a',a,z) - \phi_{gx}) + \beta(1 - \mathbb{1}_{\{\zeta==1\}}) \int \tag{20}$$

where ϕ_{gx} is a constant continuation cost, payable only if the firm chooses to remain. From a practical perspective notice that we only need to solve this problem to get the exit decision $\zeta(a,z)$ itself. Many of the the objects on the right-hand side are already being calculated as part of the 'no exit' and 'exogenous exit' problems, and the policy function for the 'other' variables is simply $g^{gx}_{p,\theta} = (1 - \zeta(a,z)) g^{nx}_{p,\theta} + \zeta(a,z) g^{xx}_{p,\theta}$ so the VFI Toolkit does not calculate nor store this seperately as it can be trivally recovered later from $\zeta(a,z)$, $g^{nx}_{p,\theta}$, and $g^{xx}_{p,\theta}$.

With a mixture of endogenous and exogenous exit the value function problem continues to have the same general structure with a complication of the innermost steps around computing $V_n(a,z)$ based on remaining (no exit), the return to exit $F^\zeta(d,a',a,z)$, and comparison of the two. There is a need to keep both the policy of those who exit and of those who choose to remain, as well as the exit decisions themselves. This gives the following algorithm, The VFI Toolkit commands for value function iteration will output 'Policy' which contains $g^{nx}_n(a,z)$ (to be precise, it outputs the

[14] The framework used here assumes that the problems faced by agents who exit are identical whether that exit is endogenous or exogenous.

Algorithm 5 VFI with both exogenous exit and endogenous exit

Declare initial value V_0.
Declare iteration count $n = 0$.
while $||V_{n+1} - V_n|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for $z = 1, ..., n_z$ **do**
    for $a = 1, ..., n_a$ **do**
      Calculate $E[V_{old}|z]$ ▷ Using quadrature method based on finite transition matrix for z.
      Calculate $V_n^{nx}(a, z) = \max_{d=1,...,n_d,\, a'=1,...,n_a} F(d, a', a, z) + \beta E[V_{old}(a', z')|z]$ and associated argmax $g_n^{nx}(a, z)$.
      Calculate $V_n^{xx}(a, z) = \max_{d=1,...,n_d,\, a'=1,...,n_a} F^{\zeta}(d, a', a, z)$.
      if $V_n(a, z) - \phi < F^{\zeta}(d, a', a, z)$ **then** $\zeta(a, z) = 1$
      else $\zeta(a, z) = 0$.
      end if
      Calculate $V_n^{gx}(a, z) = (1 - \zeta(a, z))(V_n^{nx}(a, z) - \phi) + \zeta(a, z)V_n^{xx}(a, z)$
      Implicitly $g_n^{gx}(a, z) = (1 - \zeta(a, z))(g_n^{nx}(a, z) - \phi) + \zeta(a, z)g_n^{xx}(a, z)$ ▷ Not actually calculated or stored.
      Calculate $V_n(a, z) = \pi_{nx}V_n^{nx}(a, z) + \pi_{gx}V_n^{gx}(a, z) + \pi_{xx}V_n^{xx}(a, z)$.
    end for
  end for
end while

policy in the form of an index, not as a value; setting $vfoptions.polindorval = 2$ will change this) and 'PolicyWhenExiting' which contains $g_n^{xx}(a, z)$ and will also output 'ExitPolicy' which contains $\zeta(a, z)$. Note that while the policy under endogenous exit is not explicitly stored it can easily be recovered.

In both cases the actual implementation includes use of Howards improvement algorithm, and the use of loops vs vectorization, as well as single cpu, parallel cpus, or gpu varies depending on various internal options.

Agent Distribution:
Calculating the agent distribution with both endogenous and exogenous exit is a simple combination of how it is done when there is just one of the two. A very brief explanation will therefore suffice. The only change is in the calculation of P, the transition matrix for the agent distribution. First calculate $P0$ the transition matrix for non-exiting firms.[15] Then $P = \pi_{nx}P0 + \pi_{gx}(1 - \zeta)P0$. Two observations: (i) π_{xx} exogenous exiting firms are being removed simply by ommision, and (ii) the multiplication $(1 - \zeta)P0$ is non-trival but follows the obvious logic (each element of $(1 - \zeta)$ is being applied to the corresponding row in $P0$).

Model statistics (e.g., aggregate variables):
Because of the timing changes the agents distribution now contains (π_{nx}) firms that won't be

[15] This is standard so I omit detailed description, the documentation of VFI Toolkit for models without exit nor entry explain this step in detail for those interested.

exiting, (π_{gx}) firms that face an endogenous exit decision and choose to remain or exit, and (π_{xx}) that will exogenously exit. To ease calculation of certain model statistics there is an (optional) $FnsToEvaluateParamNames(1).ExitStatus = [1, 1, 1, 1]$ which specifices that the function being evaluated should be calculated on each of these four kinds of firms, respectively. By default (when not explicitly specified) it takes the value $[1, 1, 1, 1]$ which means it will be evaluated on all four kinds of firms. So, for example $[0, 0, 1, 0]$ would specify that this statistic was calculated only for firms facing an endogenous exit decision and which choose to exit.

Stationary General Equilibrium:
Beyond the changes to the value function problem and the stationary agent distribution just described, the use of mixed exit makes no change to the computation of stationary general equilibrium beyond those already detailed in the main body of this document for 'simpler' cases of entry-exit models.

C Hopenhayn and Rogerson (1993) - Job Turnover and Policy Evaluation: A General Equilibrium Analysis

? study the impact of factor misallocation, specifically the misallocation of labor due to firing costs, in a model of heterogeneous firms. The model has both endogenous entry and endogenous exit of firms.

This appendix describes the model of **?** both as they write it and then rewritten in the framework described by this paper as used by the VFI Toolkit. It contains links to example and replication codes implementing the model, and discusses which aspects of the model that are most awkward to deal with from the perspective of this framework.

This model fits naturally into the framework described in this paper, and so very little change from the original formulation will be needed. The only thing in the model that is awkward to deal with is the decision by **?** that "we are assuming that a new entrant bears only the fixed cost of entry and does not pay the cost cf" (Footnote 5 on page 922). This requires us to make a slight modification to the existing firm's problem for new entrants and an additional 'new entrants' grid point in the assets grid is used to do this. It is worth observing that this is in direct conflict with the statement that "Once this [entry] cost has been paid the entrant is in the same position as an incumbent that has chosen to remain in the productive sector and had zero employees in the previous period." on pg 919 of **?** which would not require any change from the standard framework described in this paper.

I here use V for what **?** call W, and z for s.

The problem of an existing firm is to choose employment to maximize (present discounted value of expected) profits, subject to firing costs which must be payed if employment is lower this period than last period. Hiring is costless. Firms have a decreasing-returns-to-scale production function, with labor being the only factor of production, and face a fixed cost of production. Alternatively, a firm can decide to exit in which case it incurs the firing costs of all remaining employees and ceases to exist. The technology level in the production function, z, follows an AR(1) process in logs.

In the original setup of **?** the (existing) firm's problem is thus,

$$\begin{aligned} V(n,z) = \max_{n'} pz(n')^{\alpha} - wn' - pc_f - \tau(\mathbb{1}_{n'<n}(n-n'))) \\ +\beta \max(E[V(n',z')|z], -\tau n) \end{aligned} \tag{21}$$

where n is 'lag' of employment (last period employment; so n' is this employment this period); z is (idiosyncratic) technology level; $x \in \{0,1\}$ is the exit decision; p is the price level; w is the wage; c_f is the fixed-cost of production. Note how in this formulation the exit decision is set up as a nesting of two maximization problems, while when we convert into our framework we will instead express it as a single/joint maximization problem.

Rewritten for our framework the (existing) firm's problem is thus,

$$V(n,z) = \max_{n',x} \mathbb{1}_{\{x==0\}}(pz(n')^{\alpha} - wn' - pc_f - \tau(\mathbb{1}_{n'<n}(n-n'))) - \mathbb{1}_{\{x==1\}}\tau n \\ +\beta\mathbb{1}_{\{x==0\}}E[V(n',z')|z] \quad (22)$$

where n is 'lag' of employment (last period employment; so n' is this employment this period); z is (idiosyncratic) technology level; $x \in \{0,1\}$ is the exit decision; p is the price level; w is the wage; c_f is the fixed-cost of production.

Since nothing else is written any differently between the setup described in **?** and the form in which it should be written for our framework we will now proceed to write it out just once.[16]

The problem facing potential entrant firms is to decide whether or not to enter. A firm that decides to enter must pay a fixed cost of entry, c_e, and will enter as with technology level z drawn from a distribution of entrants η calibrated as a uniform distribution over the lower two-thirds of the range of possible z values, and a 'lagged employment' value of zero.[17] "Once this [entry] cost has been paid, the entrant is in the same position as an incumbent that has chosen to remain in the productive sector and had zero employees in the previous period", according to **?** on pg 919; however, in direct conflict with this their Footnote 5 on page 922 states that "Note that we are assuming that a new entrant bears only the fixed cost of entry and does not pay the cost cf"; meaning need to make a slight modification to the existing firm's problem for new entrants (I will ignore this dependence notationally elsewhere). This replication reports the results with Footnote 5 imposed; the codes allow for it to be turned on and off. The mass of new entrants is given by parameter N_e. The problem faced by potential entrants is thus to choose to enter or not based on whether or not $E_\eta[V] > c_e$.

The model also includes a representative household with utility function $\sum_{t=1}^{\infty} \beta^t [log(c) - aN]$; note that there is no aggregate uncertainty in this model. This model gives us the 'demand function' (or equally, the condition for goods market clearance), which enters the model as a general equilibrium condition. It provides household-side assumptions that deliver the infinitely elastic labor-supply (guaranteeing that the labour market clears, regardless of labour demand), and allows for welfare interpretations. It does necessitate additional calibrations, namely A and β to deliver the model values of r and fraction-of-time-worked.[18]

[16] In their formal definition of equilibrium **?** emphasize the labour market equilibrium while we emphasize the goods market equilibrium; but because of Walras' law both of these definitions are equivalent.

[17] "We found that a uniform distribution on the lower part of the interval in which realizations of z lie produced a reasonable fit.", pg 930 of Hopenhayn & Rogerson (1993), but no mention of what constitutes 'lower part'. Martin Flodén concludes that roughly the bottom 2/3 (0.65 to be precise) is a good definiting of 'lower' (pg 5): http://martinfloden.net/files/macrolab.pdf Also, "Once this cost has been paid the entrant is in the same position as an incumbent that has chosen to remain in the productive sector and had zero employees in the previous period.", pg 919 of Hopenhayn & Rogerson (1993) tells us that all entrants have zero 'lagged employees'.

[18] To derive the goods-market clearance condition you can first derive $A/C = p$ from the household problem and then combine this with C = real output, and definition of real output as $\int zn^{\alpha}d\mu$ (note: Y is nominal output in notation of **?**, so real output can also be found from Y/p. For household side you can get $A/C = p$ from solving $\max_{c,N}[log(c) - AN]$ $s.t.$ $pC = wN + T$, where T is just all lump-sum transfers and other wealth (is not part of

A stationary competitive equilibrium of this model is given by,

Definition 5. *A Stationary Competitive Equilibrium is an agents value function V; agents policy function g; agents exit decision g^x; price of goods p; mass of new entrants N_e; and measure of agents μ; such that*

1. *Given interest rate r, the agents value function V, policy function g, and exit decision g^x, solve the agents problem, as given by equation (22).*
2. *Equilibrium in goods market: $p = \frac{A}{\int z(g^n)^\alpha d\mu}$.*
3. *Free-entry condition: $\int V\eta(n,z)dndz - c_e = 0$*
4. *The measure of agents is invariant:*

$$\mu_(n,z) = \int\int\left[\int 1_{n=g(\hat{n},z)} 1_{g^x(\hat{n},z)=0}\mu(\hat{n},z)Q(z,dz')\right]d\hat{n}dz + N_e\eta(n,z) \tag{23}$$

Note that the measure of agents μ is not a probability density function, as it includes the mass, N, of agents. Also note that this stationary competitive equilibrium condition can also be defined in terms of finding c_e and treating p as exogenous; **?** follow this later in baseline case, and the alternative (p) in other two calibrations with positive values of τ that they solve.

standard model notation); note that it is just the standard intratemporal labour/leisure tradeoff condition $-\frac{u_c}{u_N} = \frac{p}{w}$, together with normalization of $w = 1$.

D Restuccia and Rogerson (2008) - Policy Distortions and Aggregate Productivity with Heterogeneous Plants

The model of **?** studies the role of idiosyncratic distortions to firms on aggregate outcomes. The focus is on comparing outcomes between different stationary competitive equilibria. Because all of the shocks are permanent, while the model is in principle dynamic in practice it is effectively static. **?** take an approach to solving the model that exploits this static nature of the solution, and additionally exploits that it is known to be linear in the mass of entrants. Since the framework described in this paper is inherently dynamic and stochastic it will not attempt to exploit either of these (static, largely deterministic). The codes that implement the example will exploit the linearity in the mass of entrants, but most of the replication code does not exploit this linearity.

This appendix describes the model of **?** rewritten in the framework described by this paper as used by the VFI Toolkit. It contains links to example and replication codes implementing the model, and discusses which aspects of the model that are most awkward to deal with from the perspective of this framework.

Other than being 'too simple', in the sense that the model is largely static and deterministic, and so a dynamic stochastic notation is overkill, the model of **?** fits perfectly in the framework described by this paper. Because the model in their paper takes so much advantage of the simplifications I do not repeat it here, I simply provide the version of the model rewritten for the present framework and readers should consult **?** for the numerous simplifications that can be made and the resulting model.

The model has both a standard endogenous entry condition (which introduces a free-entry general equilibrium condition) and a further 'conditional entry' condition. The Conditional entry condition is that after deciding to enter firms draw their initial state, (s, τ), and then *conditional* on this firms can decide whether or not they will actually enter; this might also be thought of as a 'conditional decision to abort entry'. This conditional entry decision is additional to the standard endogenous entry condition and imposes a further general equilibrium condition. In the model of RR2008 we thus have both the free-entry condition and a condition relating to the conditional entry condition on $\bar{e}(s, \tau)$ (which they denoted $\bar{x}(s, \tau)$). There is a third general equilibrium condition, namely choosing the mass of entrants N_e (which they denoted E) to ensure labour market clearance; because the model is linear in N_e this can simply be ignored at first when solving the other two general equilibrium conditions, and then imposed by renormalizing the solutions. This renormalization is the approach taken by RR2008, and which the replication follows for the baseline case. When solving the other cases there is typically an additional requirement that the subsidy rate is determined in general equilibrium to keep aggregate capital equal to it's value in the baseline (see RR2008 for explanation of why they wish to do so), and in this case the renormalization relating N_e to the labour market clearance cannot be imposed afterwards (would change aggregate capital stock to no longer equal baseline value) and so this 'renormalization' general eqm condition is treated by replication codes as a standard general equilibrium condition for all the non-baseline cases.

Exit is exogenous, and the rest of the model is standard. The optimal choices of capital and labour (and hence profits) can all be solved in closed form as functions of the exogenous state and the equations are in RR2008, I provide the full derivation for these below (*kbar* and *nbar*).

The problem of an existing firm is to choose capital and labour inputs to maximize expected present value of profits conditional on their idiosyncratic values of productivity, s, and tax/subsidy τ,

$$
\begin{aligned}
V(s,\tau) &= \max_{n,k} \pi(n,k) + \frac{1}{1+r}\lambda E V(s',\tau') \\
\text{s.t} \quad & \pi(n,z) = (1-\tau)y - wn - rk - c_f \\
& y = s k^{\alpha} n^{\gamma} \\
& s' = s, \tau' = \tau
\end{aligned}
$$

where π is (period) profit; r is the interest rate (in this model equivalent to the rental rate of capital and to the return on capital); n is labour input, k is capital input; w is wage; c_f is fixed-cost of production.[19] since this problem is essentially static (as every period is independent) it can be written as such and solved directly (see RR2008). However to determine entry we still need to calculate the expected present discounted value (as this is what matters for the entry decision) and so the replication codes simply solve the full dynamic programing problem directly, rather than following RR2008 who used based results from summation of sequences to massively simplify this. This simplified decision uses the static case to solve analytically for k and n and give results in the following simplified problem,

$$
\begin{aligned}
V(s,\tau) &= \pi(n,k) + \frac{1}{1+r}\lambda E V(s',\tau') \\
\text{s.t} \quad & \pi(n,z) = (1-\tau)y - wn - rk - c_f \\
& k = \left(\frac{\alpha}{r}\right)^{\frac{1-\gamma}{1-\gamma-\alpha}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma-\alpha}} ((1-\tau)s)^{\frac{1}{1-\gamma-\alpha}} \quad n = (\frac{(1-\tau)s\gamma}{w})^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \quad s' = s, \tau' = \tau
\end{aligned}
$$

(The replication code uses slight further simplifications of expressions for n, y, and π which are derived in full below.)

The calculation of the stationary agents distribution is standard for cases of endogenous entry and exogenous exit, with conditional entry adding the minor change that the distribution of entrants used when iterating the agents distribution is the distribution of potential entrants times the conditional entry decision; again RR2008 avoid this iterating on the agents distribution by taking

[19] Since c_f is a lump-sum and exit is exogenous it does not change any decisions of existing firms. From the perspective of potential entrants it is no different to the fixed-cost of entry c_e, and so plays no (seperate) role there. But it does effect the 'conditional entry' decisions (which c_e does not).

advantage of the fact that there are no time-varying idiosyncratic shocks, and the exogenous and constant value of the probability of exit to avoid iterating on the agent distribution and instead again just using results from summing infinite sequences (this time on distributions).

The conditional entry decision is $\max_{\bar{e}(s,\tau)}\{\bar{e}(s,\tau)\beta V(s,\tau), 0\}$, for each (s,τ). It is a general equilibrium condition on $\bar{e}$.

The free-entry condition is $\beta\int V(s,\tau)\bar{e}(s,\tau)dg(s,\tau)-c_e=0$. Notice that this is the requirement that the (discounted) expected value of being a new entrant is equal to the (fixed) cost of entry; as otherwise there would be more (or less) entry if this did not hold with equality. Notice that the expectation is taken across the distribution of actual entrants (the conditional entry decision times the distribution of potential entrants). This general equilibrium condition is used to determine the wage w (as this in turn determines the value function and so can be chosen to ensure that the free-entry condition holds; other models instead commonly use this condition to instead determine the fixed cost of entry).

There is a (representative) household side of the economy, which essentially provides two general equilibrium conditions. The first can be reduced to just a calibration issue, that $r = 1/\beta-(1-\delta)$, the second is that labour supply equals one (households have endowment of one unit of time which they supply perfectly inelastically), and so leads to the labour market clearance condition that labour demand of the firms must equal labour supply (which equals one). RR2008 solve this by simply renormalizing the mass of potential entrants after solving all the other general equilibrium conditions (which works for this specific model as the total mass is linear in potential entrants, and the decisions are unaffected); the replication codes use this for some but not all cases.

Many of the experiments performed by RR2008 involve setting the subsidy rate so that the aggregate capital remains equal to it's baseline level, which is computationally equivalent to condsidering this (aggregate capital minus baseline aggregate capital equals zero) as an additional 'general eqm' condition. This is the approach taken by the replication codes.

We thus have three general equilibrium conditions: conditional entry, free entry, and labour market clearance. (In the baseline model we can just solve for the first two, the third can be done as a renormalization, and the fourth is not relevant.)

The definition of stationary competitive eqm in this model is standard. It involves finding the parameters w, $\bar{e}$, and N_e ($\bar{e}$ can be thought of as a decision, or just as an equilibrium parameter, denoted $\bar{x}$ by RR2008; N_e is the mass of potential entrants, denoted E by RR2008). These are chosen to satisfy the conditional entry condition, the free-entry condition, labour market clearance.

Replication involves computing many different stationary competitive equilibria and comparing various model outputs from these. However it typically also involves adding a condition that is to choose the subsidy rate τ_s to satisfy the requirement that capital remains equal to it's baseline value. This is done in the codes by simply considering this requirement as if it were a fourth general equilibrium condition.

Derive nbar and kbar: With the output tax, take the FOCs of $(1-\tau)sk^{\alpha}n^{\gamma} - wn - rk - c_f$ w.r.t. k and n to get:

$$\alpha(1-\tau)sk^{\alpha-1}n^{\gamma} - r = 0$$
$$\gamma(1-\tau)sk^{\alpha}n^{\gamma-1} - w = 0$$

rearranging the second of these

$$n^{1-\gamma} = \frac{\gamma(1-\tau)sk^{\alpha}}{w}$$

so

$$n = \left(\frac{(1-\tau)s\gamma}{w}\right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \tag{24}$$

That is nbar done. Now, rearrange first FOC to get

$$k^{1-\alpha} = \frac{\alpha(1-\tau)sn^{\gamma}}{r}$$

substitute our expression for nbar in here for n to get

$$k^{1-\alpha} = ((1-\tau)s)\frac{\alpha}{r}((1-\tau)s)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} k^{\frac{\alpha\gamma}{1-\gamma}}$$

so

$$k = ((1-\tau)s)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\alpha-\gamma}} \left(\frac{\alpha}{r}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \tag{25}$$

by returning to sub this into the eqn for *nbar* we can get the alternative formula,

$$n = ((1-\tau)s)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{1-\alpha}{1-\alpha-\gamma}} \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \tag{26}$$

that completes our derivation of formulae for *kbar* and *nbar* with an output tax. We can also substitute these into the production function, $sk^{\alpha}n^{\gamma}$ to get

$$y = (1-\tau)^{\frac{\alpha+\gamma}{1-\alpha-\gamma}} s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\alpha-\gamma}} \tag{27}$$