"The Jealous Husbands" and "The Missionaries and Cannibals"

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The jealous husbands and The missionaries and cannibals

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The classical river crossing problem of the jealous husbands involves three couples who have to cross a river using a boat that holds just two people. The jealousy of the husbands requires that no wife can be in the presence of another man without her husband being present. This can be accomplished in 11 crossings (i.e. one-way trips). Tartaglia gave a sketchy solution for four couples but Bachet pointed out that this was erroneous and that four couples could not get across the river. In 1879, De Fontenay pointed out that four or more couples could cross the river if there was an island in the river and gave a solution for n couples in $8n - 8$ crossings. Dudeney improved the solution for $n = 4$ and Ball noted that this gives $6n - 7$ crossings for n couples.

From the results of a computer search, we have discovered solutions in 16 crossings for $n = 4$ and in $4n + 1$ crossings for $n > 4$ and we have proven that these are the minimal number of crossings. We have also found that De Fontenay's solution should be in $8n - 6$ crossings and that this is the minimal number of crossings when trips from bank to bank are prohibited.

The more recent missionaries and cannibals problem has n of each type of person and the conditions are that the cannibals must never outnumber the missionaries at any location. This is a proper weakening of the jealous husbands problem. When bank-to-bank crossings are prohibited, De Fontenay's method already uses the least possible number of crossings, even disregarding any conditions, hence is also optimal for this version of the problem. When bank-to-bank crossings are permitted, the 16 crossing solution for the jealous husbands can be reduced to 15 and this generates a solution in $4n - 1$ crossings, which is the minimal number of crossings for $n \geqslant 3$.

1. Historical introduction

River crossing problems first appear in the *Propositiones ad Acuendos Juvenes*, traditionally attributed to Alcuin of York (c. 732–804). The earliest extant manuscript is from the 9th century. Alcuin essentially gives three problems. The first is the problem of the three jealous husbands, the second is the problem of the wolf, the goat and the cabbage, and the third is the problem of the two adults and the two children, where the children weigh half as much as the adults.

Alcuin's phrasing of the problem is much more direct than later versions, so we present a translation, kindly done by John Hadley.

> *Three men, each with a sister, needed to cross a river. Each one of them coveted the sister of another. At the river, they found only a small boat, in which only two of them could cross at a time. How did they cross the river, without any of the women being defiled by the men?*

The Latin is a bit ambiguous about who covets whom, but the usual interpretation is that we cannot allow any sister to be with another man without the protection of her brother. Alcuin gives a correct solution in 11 crossings "without anything untoward happening."

These problems apparently were widely disseminated as they appear in many of the problem collections of the next centuries, with the notable exception of Fibonacci. Ahrens gives a verse solution which may be as much as 1000 years old:

> Binae, sola, duae, mulier, duo, vir mulierque,
> Bini, sola, duae, solus, vir cum mulier.

An approximate English version is:

> Women, woman, women, wife, men, man and wife,
> Men, woman, women, man, man and wife.

The first person to consider more couples appears to be Luca Pacioli, in his unpublished manuscript *De Viribus Quantitatis*, where he asserts that 4 or 5 couples requires a three person boat. However, Tartaglia, in 1556 asserts that 4 couples can cross in a two person boat and gives a sketchy solution. Bachet pointed out Tartaglia's error and showed that 4 couples cannot cross in a two person boat. Tartaglia's error is worth examining since it clarifies what is considered acceptable. Tartaglia transfers the four women to the other side, then sends one back. She remains and two men other than her husband get in

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Gleanings

A new dimension in motoring

"We are constantly improving our product and space is one area we are continuing to look at." From a Citroën brochure, sent in by A. R. G. Burrows.

the boat. When they reach the far side, the unaccompanied women gets in the boat and returns with it. This is the point where Bachet and other authors object—when the men reach the far side, there are two men and three women there, which is not permitted by the usual interpretation of the jealousy conditions.

2. *De Fontenay's generalisation*

In the late 19th century, Lucas stimulated a revival of interest in recreational problems among French mathematicians and students. In 1879, a young student, M. Cadet De Fontenay, observed that 4 or more couples could cross if there were an island in the river. He gave a solution for 4 couples in 24 crossings, which extends to n couples in $8n - 8$ crossings. Below, we show De Fontenay's solution in the usual and convenient diagrammatic form, though we do not show each of the dispositions of people. The couples are denoted $A\ a$, with A the husband, etc. The solution has three stages.

	Left bank		*Island*		*Right bank*
STAGE I					
	$A\ a\ B\ b\ C\ c\ D\ d$				
1.		— $a\ b$ →			
2.		← b —			
3.		— $b\ c$ →			
4.		← c —			
	$A\ \ B\ \ C\ c\ D\ d$		$a\ b$ ———		
5.		— $A\ B$ →			
6.				— $a\ b$ →	
7.				← b —	
8.				— $A\ B$ →	
9.				← B —	
	$C\ c\ D\ d$		$B\ b$		$A\ a$
STAGE II					
10.		← B —			
11.		— $c\ d$ →			
12.		← d —			
13.		— $B\ C$ →			
	$D\ d$		$B\ b\ C\ c$		$A\ a$
14.				— $B\ C$ →	
15.				← a —	
16.				— $a\ b$ →	
17.				← C —	
	$D\ d$		$C\ c$		$A\ a\ B\ b$

STAGE III

18.	← *C* —		
19.	— *C D* →		
20.			— *C D* →
21.			← *b* —
22.			— *b c* →
23.	←————	*b*	————
24.	————	*b d*	————→
			A a B b C c D d

The stages take 9, 8, 7 moves. Noting that the difference between the beginning and the end of Stage II is simply equivalent to moving one couple across the river, we see that n couples can be moved across the river by repeating Stage II $n-3$ times, giving a total of $8(n-3)+16=8n-8$ crossings. (Lucas says $4n-4$ voyages, which must mean two-way trips.)

The position at the beginning of Stage III is symmetric to the position at the end of Stage I, so it seemed to us, at first, that these stages should take the same number of crossings. Indeed, a little work shows that Stages I and II can each be reduced to 7 crossings by using bank-to-bank crossings as in moves 23 and 24 as above. If bank-to-bank moves are prohibited, then moves 23 and 24 must each be done in two steps, making a total of $8n-6$ moves. We now see that this is the best possible.

THEOREM 1. The above method is optimal; i.e. the minimal number of crossings to ferry n couples across a river with an island, using a two-person boat and with no bank-to-bank crossings, is $8n-6$.

Proof. Consider any method of ferrying $2n$ people across the river without bank-to-bank crossings. Each departure and return from the left bank results in at most one person being taken from the left bank to the island, except that two persons can go on the last departure. Hence there must be at least $2(2n-2)+1=4n-3$ crossings involved. The same holds for the transits from the island to the right bank, so overall there must be at least $8n-6$ crossings in any ferrying of $2n$ people. As we have seen, the (amended) method of De Fontenay manages to ferry jealous couples in this number of moves, so the method must be optimal.

3. *Dudeney's solution with bank-to-bank crossings*

The only later author who worked on De Fontenay's problem appears to be Dudeney, who used bank-to-bank crossings and obtained a solution in 17

Acutely cold

"An avalanche zone is a slope of between 27° and 45° (– 3° and 7°C)." From a tourist guide to Alaska, sent in by Dennis Archer.

crossings for 4 couples. We present his solution below. Again, it has three stages:

	Left bank		*Island*		*Right bank*
STAGE I					
	A a B b C c D d				
1.		———	*a b*	———→	
2.		←———	*b*	———	
3.		— *b c* →			
4.		← *c* —			
5.		———	*A B*	———→	
6.		←———	*B*	———	
	B C c D d		*b*		*A a*
STAGE II					
7.		— *c d* →			
8.		← *d* —			
9.		———	*B C*	———→	
10.				← *a* —	
11.				— *a c* →	
12.		←———	*B*	———	
	B D d		*b*		*A a C c*
STAGE III					
13.		———	*B D*	———→	
14.				← *c* —	
15.				— *b c* →	
16.		←———	*c*	———	
17.		———	*c d*	———→	
					A a B b C c D d

Ball discusses these "Ferry boat problems" in his book *Mathematical recreations and essays*. In the 1920 edition he says: "It would, however, seem that if n is greater than 3, we need not require more than $6n-7$ passages from land to land." Examining Dudeney's solution above, we see that Stage II has the effect of moving one couple across the river in 6 crossings. Repeating Stage II $n-3$ times, we therefore have a solution for n couples in $6n-7$ crossings as asserted by Ball.

4. *Better solutions with bank-to-bank crossings*

An initial observation about the problem is that there are $2n+1$ objects (counting the boat) which can be in any one of three places, so there are 3^{2n+1} distinct positions that might be considered. For $n=4$, this gives $3^9 = 19\,683$ positions. Obviously many of these violate the jealousy restraints and many are clearly equivalent. Also, the boat cannot be somewhere where there are no people! We recognised that an equivalence class of permissible positions is completely determined by the number of men and women at each site. A simple computer count of such distributions revealed rather small numbers and then we were able to evaluate the number exactly by somewhat tedious enumeration of cases. The result is that there are $3n(n^2 + 10n - 1)/2$ inequivalent permissible positions. For $n = 4$, this gives 330 such positions, which is much less than the 19 683 initially considered. The fact that the number was only a cubic in n made us feel that almost any searching method would find optimal solutions.

We were delighted to find that a standard graph-theoretic algorithm on the computer found a solution for $n = 4$ in 16 crossings. This involved an ingenious step which Dudeney apparently failed to consider. In trying to understand this solution, we found a solution for n couples that used $4n + 1$ crossings, and then found that, for $n > 4$, this is again optimal:

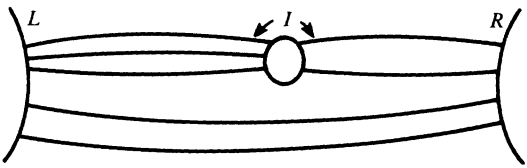
	Left bank		*Island*		*Right bank*
STAGE I					
	A a B b C c D d				
1.		— *a b* →			
2.		← *b* —			
3.		— *b c* →			
4.		← *c* —			
	A B C c D d		*a b*		
STAGE II					
5.		———	*A B*	———→	
6.		←———	*B*	———	
7.		———	*C c*	———→	
8.		←———	*A*	———	
	A B D d		*a b*		*C c*
STAGE II′					
9.		———	*A B*	———→	
10.		←———	*B*	———	
11.		———	*D d*	———→	
12.		←———	*A*	———	
	A B		*a b*		*C c D d*
STAGE III					
13.		———	*A B*	———→	
14.				← *c* —	
15.				— *b c* →	
16.				← *b* —	
17.				— *a b* →	
					A a B b C c D d

Observing that Stage II moves a couple across the river, we see that it can be repeated $n-2$ times to move n couples across the river in $4n+1$ moves. The simplicity of this method startled us and we are quite surprised that it was not found sooner. One simply parks two wives on the island and uses the two unaccompanied men to ferry couples across.

Note that for four couples 16 crossings will do, and it makes a nice exercise for the reader. However, for $n > 4$ $4n+1$ is the best possible, as we now outline.

THEOREM 2. The above method is optimal for $n > 4$; i.e. the minimal number of crossings to ferry n (>4) couples across a river with an island, using a two-person boat and bank-to-bank crossings, is $4n+1$.

Proof. First we need a little notation:



L, I, R = no. of lines ending at left, island and right

Now

$$T = \text{total number of crossings} = \tfrac{1}{2}(L + I + R).$$

By the argument given in the proof of Theorem 1, $L \geqslant 4n-3$, the optimal value of L is attained if and only if each departure from L (except the last) reduces the number of people at the left by one and the last departure reduces it by two: there are similar arguments for R.

Hence

$$T = \tfrac{1}{2}(L + R + I) \geqslant 4n - 3 + \tfrac{1}{2}I$$

where $\frac{1}{2}I$ is the number of visits to the island. It is fairly easy to see (as Bachet proved for $n = 4$ in the 17th century) that for $n \geqslant 4$ no solution exists without the use of an island. Hence $\frac{1}{2}I \geqslant 1$. Clearly $\frac{1}{2}I = 1$ would be silly because a single visit to the island either abandons someone or is a wasted visit. Hence in any optimal solution $\frac{1}{2}I \geqslant 2$.

So to beat our value of $I = 4n+1$ we must have one of the following sets of

Hypermarket?

"The company has applied for outline permission for an extension of 55 486 feet at the rear of the present shop." From the *Devon Advertiser* of 4 November 1988, sent in by Frank Tapson.

values of L, R and I:

$$\begin{array}{llll} 4n+1 > T = & \tfrac{1}{2}L & + \quad \tfrac{1}{2}R & + \tfrac{1}{2}I \\ & \geqslant \tfrac{1}{2}(4n-3) & \geqslant \tfrac{1}{2}(4n-3) & \geqslant 2 \end{array}$$

$\frac{1}{2}L$	$\frac{1}{2}R$	$\frac{1}{2}I$	T
$\frac{1}{2}(4n-3)$	$\frac{1}{2}(4n-3)$	2	$4n-1$
$\frac{1}{2}(4n-1)$	$\frac{1}{2}(4n-3)$	2	$4n$
$\frac{1}{2}(4n-3)$	$\frac{1}{2}(4n-1)$	2	$4n$
$\frac{1}{2}(4n-3)$	$\frac{1}{2}(4n-3)$	3	$4n$

Note that $\frac{1}{2}I = 2$ means the most that can happen at the island is that a person is dropped there and collected later: $\frac{1}{2}I = 3$ means that the most that can happen is that one person is deposited there, later he or she is collected and another person deposited, and later still the remaining person is collected.

Consider now the situation after four people have left from the left bank and the boat has returned to the left. In order for the position there to be legal the situation must be one of the following types:

Left bank	*Island*	*Right bank*
B C + couples	A	a b c
	or	
A B C D + couples		a b c d
	or	
A B C D + couples	a	b c d
	or	
couples		A a B b
	or	
couples	a	A B b

We can now see that none of these situations is possible whilst maintaining L, R and I within the constraints imposed in the above table. For example, in the first three of these five situations there are at least three women on the right bank. Moving any away again would eventually make R too high. But then there is no way of getting a man to the right bank: whether a man arrives by himself, with another man, or with his wife there will be an unescorted woman at the right bank. Ruling out the last two situations is a little harder and we leave those to the interested reader.

5. *Missionaries and cannibals*

There are several variants of our problem which have appeared relatively recently. In the 19th century G. Tarry suggested the problem with n men travelling with their harems of m wives each. Obviously this would be in some culture where women were not permitted to row! Contemplation indicates that this may not be a serious problem proposal. However, it seems to be the first example where the ability to row is introduced as a further condition. For the traditional problem with $n = 3$ couples and no island, one needs 3 people who can row in order to get a solution in 11 crossings. If there are only 2 rowers, a solution takes 13 crossings. We have not yet pursued this topic, beyond noting that the $4n + 1$ solution can be done with n rowers A, B, c and one from each other couple.

Pacioli seems to be the first to suggest use of a larger boat! Bachet discussed this in the 17th century (although it has been suggested that the material is due to the 19th century editor). Of more interest is a change of character of the problem. The first example that we know of is in a 1624 book under the name of van Etten, but popularly attributed to Leurechon. Here there are three masters and three valets, but the men hate the other valets and will beat them if left with them. This is exactly the problem of the jealous husbands except that it is now based on class discrimination rather than sex discrimination.

In 1881, *Cassell's book of in-door amusements* gives a reversal of the situation where the servants are dishonest and will rob the masters if the servants ever outnumber the masters at any location. This makes a subtle change in the problem, in that the connection of a master to a particular servant is no longer relevant. By the end of the 19th century the problem was stated in the form of missionaries and cannibals, where the cannibals must never outnumber the missionaries. The earliest version of this form that we know of is due to Pocock who describes the problem as well known in 1891. This formulation has remained popular and has also been varied to explorers and natives. (Comment added at proof stage: we have since found a verson of the problem in Mittenzwey's *Mathematische Kurzweil*, which first appeared in 1879—but our edition is 1918.)

We now state, without proof, missionary/cannibal versions of our theorems.

THEOREM 3. The minimal number of crossing to ferry n missionaries and n cannibals across a river with an island, using a two-person boat and with no bank-to-bank crossings, is $8n - 6$.

THEOREM 4. The minimal number of crossings to ferry $n \geqslant 3$ missionaries and n cannibals across a river with an island, using a two-person boat and bank-to-bank crossings, is $4n - 1$.

A more detailed version of this article, together with extensive bibliography is available from David Singmaster.

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