Probability Assignment - I

Vaibhav Falgun Shah (AI23MTECH02007)

Question:

Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

Solution:

Let X denote the random variable representing numer of sixes.

To find the expectation of random variable, we can use the following formula:

$$E(X) = \sum_i x_i \cdot P(X = x_i)$$

Number of sixes when rolling two dice can be 0, 1 or 2.
Hence, Sample space of X = {0, 1, 2}
So for these values of X, Expectation can be written as:

$$E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2)$$

Now we have to find P(X=0), P(X=1) and P(X=2).

Let p be the probability of getting a six on a single die, and q be the probability of not getting a six on a single die.
Since each die has 6 possible outcomes and only one of them is a six, we have $p = \frac{1}{6}$ and $q = 1 - p = \frac{5}{6}$.

Parameter	Description	Value
p	Probability of getting a six on a single die	$\frac{1}{6}$
q	Probability of not getting a six on a single die	$\frac{5}{6}$
X	Number of sixes for two die rolls	{0 ,1, 2}

Input Variables

We have two independent dice rolls, so the probability of getting no sixes, $P(X = 0)$, is q^2

$$P(X = 0) = q \cdot q = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

We can get exactly one 6 in two ways: either we get six on first die and not on second, or we get six on second die and not on the first. So $P(X = 1)$ can be written as:

$$P(X = 1) = (p \cdot q) + (q \cdot p) = 2 \cdot (p \cdot q) = 2 \cdot \left(\frac{1}{6} \cdot \frac{5}{6}\right) = \frac{10}{36}$$

The probability of getting two sixes, $P(X = 2)$, is p^2 since we have two independent dice:

$$P(X = 2) = p \cdot p = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

Now we have all 3 probabilities required to calculate the expectation:

X	P(X)
0	$P(X = 0) = \frac{25}{36}$
1	$P(X = 1) = \frac{10}{36}$
2	$P(X = 2) = \frac{1}{36}$

Probabilities of dice outcomes

Substituting these values back into the expectation formula:

$$E(X) = 0 \cdot \left(\frac{25}{36}\right) + 1 \cdot \left(\frac{10}{36}\right) + 2 \cdot \left(\frac{1}{36}\right) = \frac{12}{36} = \frac{1}{3}$$

Therefore, the expectation of X is $\frac{1}{3}$

We can verify this expectation by simulating two dice rolls multiple times and taking the average of number of sixes obtained, as done in assignment1.py file:

```
import random
num_experiments = 1000000
both_dice_throws = [[random.randint(1, 6), random.randint(1, 6)] for i in range(num_experiments)]
count_6 = sum(one_throw.count(6) for one_throw in both_dice_throws)
mean_simulated = count_6 / num_experiments
print("The expectation of X using simulations is: " + str(mean_simulated))
```

Output:

```
The expectation of X using simulations is: 0.334089
```

We get mean very close to $\frac{1}{3}$ by simulating two dice rolls for a million times. Hence, expectation of $\frac{1}{3}$ is verified.