WIENER INTEGRAL MONTE CARLO APPROACH TO ANALYZE THE FUNDAMENTAL MODE IN COMPLEX TRANSMISSION LINES

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A Wiener Integral approach to compute the scalar potential of transverse electromagnetic modes in complex (multiconductor) transmission lines and its application to characteristic impedance computation via stationary (variational) formulas are presented. The computation of the involved Wiener functional integrals is accomplished by means of Monte Carlo methods.

1 - INTRODUCTION.

The analysis of transverse electromagnetic (henceforth TEM) modes plays an important role in uniform transmission lines theory [1]. It is well known (Collin, 1960) that the fields can be found from a (static) scalar potential, which is a solution of Laplace equation in the transverse plane. Although distributed-circuit theory (Collin, 1960) allows the study of excitation and propagation of current and voltage waves on a transmission line without the need to know field distribution in detail, this latter is essential for evaluating the fundamental distributed-circuit parameters (e.g. characteristic impedance). Moreover, knowledge of the detailed field distribution is needed in a number of applications, such as power microwave systems (power handling, dielectric breakdown, heating, choice of materials).

However, the scalar potential problem can be analytically solved only in a few simple (separable) geometries. Whenever the transverse geometry is complex, one must resort to numerical methods, such as Method of Moments (henceforth MoM) (Harrington, 1968) or Finite Elements Method (henceforth FEM) (Silvester and Ferrari, 1990). These techniques are relatively expensive in terms of memory and CPU requirements since they ultimately reduce to the inversion of a large matrix. If one is interested only in characteristic impedance computation, a possible approach can be based on *conformal transformations* (Collin, 1960), yielding in some cases simple analytical approximations. But this technique is not completely applicable in arbitrary geometries, and it often requires tedious calculations and/or numerical approaches for the inversion.

[1] Moreover a *quasi-TEM* approximation is also useful to study the fundamental mode of inhomogeneous lines in the low-frequency limit (Collin, 1960).

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In particularly complex geometries, a good tradeoff between computational burden and accuracy can be provided by combining an efficient numerical technique to compute the scalar potential and a variational method (Collin, 1960) for impedance evaluation. This ensures a second order error on the impedance for the field distribution correct to only the first order.

Recently we introduced in electromagnetic problems a new class of numerical methods based on Functional Integration and Monte Carlo method (Galdi et al., 1997). These methods represent an attractive alternative to the usual ones in terms of power and computational budget.

In this paper we present an algorithm for computing the scalar potential and characteristic impedance of the dominant (TEM) mode in multiconductor transmission lines of general cross-section geometry, based on Wiener Integration (henceforth WI) and numerically implemented by means of Monte Carlo methods. The paper is organized as follows. In *Section 2* we review the principal features of TEM modes and present a variational formula for the characteristic impedance. In *Section 3* we introduce the WI solution for the scalar potential and in *Section 4* we discuss its numerical (Monte Carlo) implementation. In *Section 5* we outline a comparison between the proposed method and the standard ones and in *Section 6* we present some representative results. Conclusions follow under *Section 7*. The Wiener Integral concept is heuristically presented in the Appendix.

2 - TEM MODES IN TRANSMISSION LINES.

A z-uniform transmission line is considered with arbitrary (multiply connected) cross-section delimited by perfect conductors (and otherwise partially open) filled by an uniform dielectric medium (*vacuum*). A time harmonic $\exp(j\omega t)$ depencence will be assumed and dropped. The fundamental mode (zero cut-off freq.) in such a structure is TEM ($E_z = H_z = 0$). The fields can be related to a scalar potential (Collin, 1960):

$$\left\{ \begin{array}{l} \vec{E}(x,y,z) = -\nabla_t \Phi(x,y)\; e^{\mp j k_0 z}, \\ \vec{H}(x,y,z) = \mp Z_0^{-1} \hat{u}_z \times \nabla_t \Phi(x,y)\; e^{\mp j k_0 z}, \end{array} \right. \tag{1}$$

where $k_0 = \omega(\epsilon_0\mu_0)^{1/2}$, $Z_0 = (\mu_0/\epsilon_0)^{1/2}$, the $-$ ($+$) sign refers to forward (backward) propagation waves, and the scalar potential Φ satisfies the Laplace equation with suitable boundary conditions:

$$\left\{ \begin{array}{ll} \nabla_t^2 \Phi(x,y) = 0 & in\;\; \mathcal{D}, \\ \Phi = costant & on\;\; \partial\mathcal{D}. \end{array} \right. \tag{2}$$

It can be easily shown (Collin, 1960) that a non-trivial solution is always possible when two or more conductors are present and the potential does not take equal (constant) values on all of them [2]:

$$\begin{array}{l} \Phi(x,y) = V_i \quad (x,y) \in \partial\mathcal{D}_i \;\;, \quad i = 1, \ldots, m, \\ \partial\mathcal{D} = \bigcup_{i=1}^{m} \partial\mathcal{D}_i \;\;, \quad \partial\mathcal{D}_i \bigcap \partial\mathcal{D}_k = \emptyset \quad i \neq k. \end{array} \tag{3}$$

For two conductors there is an unique voltage-current wave associated with the electromagnetic field (Collin, 1960):

[2] For open structures one has to consider a *regularity-at-infinity* boundary condition as well.

$$\begin{cases} V = V_0 \; e^{-jk_0 z}, \\ I = I_0 \; e^{-jk_0 z}, \end{cases} \tag{4}$$

V_0 being the (static) potential difference between the conductors and I_0 the current at $z = 0$, related by the *characteristic impedance* (Collin, 1960):

$$Z_c = \frac{V_0}{I_0} = \frac{\epsilon_0 Z_0}{C}. \tag{5}$$

Thus knowledge of the (static) capacitance per unit lenght C alone suffices to determine the characteristic impedance. A variational expression for the capacitance and, hence, for the characteristic impedance, can be obtained for a two-conductor line in the form (Collin, 1960):

$$\frac{C}{\epsilon_0 Z_0} = \frac{1}{Z_c} = \frac{\displaystyle\int\!\!\int \nabla_t \Phi \cdot \nabla_t \Phi \;\; dx \; dy}{(\mu_0/\epsilon_0)^{1/2} \;\; V_0^2}. \tag{6}$$

Generalization to multiconductor lines is straightforward (Pozar, 1993). It is stressed that the above formula is particularly advantageous whenever approximate (e.g. numerical) solution for the scalar potential are available.

3 - WIENER INTEGRALS AND LAPLACE EQUATION.

In this Section the connection is explored among Wiener processes (see Appendix), functional integrals and Dirichlet problems for Laplace equation. This deep connection, well-established in probability theory (Ventsel, 1983), has not been yet properly exploited in electromagnetics problems. Following (Ventsel, 1983), the solution of the Dirichlet problem (2)-(3) can be expressed as:

$$\Phi(x, y) = E_{(x,y)} \left[f \left[w_x(\tau), w_y(\tau) \right] \right], \tag{7}$$

$$\begin{aligned} & f(x, y) = V_i \quad (x, y) \in \partial\mathcal{D}_i \;\; , \quad i = 1, ..., m, \\ & \partial\mathcal{D} = \textstyle\bigcup_{i=1}^{m} \partial\mathcal{D}_i \quad , \quad \partial\mathcal{D}_i \textstyle\bigcap \partial\mathcal{D}_k = \emptyset \quad i \neq k. \end{aligned} \tag{8}$$

where $E_{(x,y)}$ represents a Wiener Integral, viz., an *expectation value* with respect to the probability measure associated to the Wiener processes (w_x, w_y) starting at (x, y) at time $t = 0$ (see Appendix), and τ is the *first exit time* from $\mathcal{D}$ (Ventsel, 1983):

$$\tau = inf \left\{ t \; : \; (w_x(t), w_y(t)) \notin \mathcal{D} \right\}. \tag{9}$$

For unbounded domains $\mathcal{D}$, typical of *open* transmission lines, and regularity-at-infinity conditions, a WI solution is possible upon introducing a suitable absorbing ($\Phi = 0$) boundary surrounding the structure at sufficient distance from it.

4 - MONTE CARLO IMPLEMENTATION.

The Wiener Integral (7) can be computed without any restriction on the transverse geometry complexity using Monte Carlo methods (Sobol, 1975),(Kloeden and Platen, 1991). The underlying idea can be explained as follows. First introduced is a time discretization with (e.g. constant) step size Δ. Next the classical time discrete *Euler approximation* is resorted to (Kloeden and Platen, 1991) for the Wiener processes involved :

$$\begin{cases} w_x(t_{k+1}) = w_x(t_k) + \sqrt{2}\,\Delta_{W_{xk}} \quad , \quad w_x(0) = x, \\ w_y(t_{k+1}) = w_y(t_k) + \sqrt{2}\,\Delta_{W_{yk}} \quad , \quad w_y(0) = y, \end{cases} \tag{10}$$

where $t_{k+1} - t_k = \Delta$, and $\Delta_{W_{xk}}, \Delta_{W_{yk}}$ are independent gaussian distributed random variables with means and variances:

$$E(\Delta_{W_{xk}}) = E(\Delta_{W_{yk}}) = 0 \quad , \quad E[(\Delta_{W_{xk}})^2] = E[(\Delta_{W_{yk}})^2] = \Delta. \tag{11}$$

The process is evolved, using equations (10) and (11), until the time τ_+ , immediately after reaching the boundary $\partial\mathcal{D}$, viz.:

$$(w_x(\tau_+ - \Delta), w_y(\tau_+ - \Delta)) \in \mathcal{D} \quad and \quad (w_x(\tau_+), w_y(\tau_+)) \notin \mathcal{D}. \tag{12}$$

The *exit point* (x_e, y_e) can be computed solving the following system:

$$\begin{cases} y = h_{int}(x), \\ y = h_{\partial\mathcal{D}}(x), \end{cases} \quad , \quad x \in [min\{w_x(\tau_+ - \Delta), w_x(\tau_+)\}, max\{w_x(\tau_+ - \Delta), w_x(\tau_+)\}], \tag{13}$$

where $h_{int}(x)$ is the linear interpolation between two process samples :

$$h_{int}(x) = \frac{[w_y(\tau_+) - w_y(\tau_+ - \Delta)][x - w_x(\tau_+ - \Delta)]}{w_x(\tau_+) - w_x(\tau_+ - \Delta)} + w_y(\tau_+ - \Delta), \tag{14}$$

and $h_{\partial\mathcal{D}}(x)$ defines (locally) the boundary $\partial\mathcal{D}$.

Following the classical Monte Carlo method (Sobol, 1975), the Wiener Integral (expectation value) (7) can be computed iterating the above described procedure and evaluating the arithmetical means:

$$\mu^{(1)}_{M,\Delta} = M^{-1} \sum_{j=1}^{M} f(x_e^j, y_e^j), \tag{15}$$

where (x_e^j, y_e^j) denotes the j-th realization of the random variables (x_e, y_e). Hence :

$$\Phi(x, y) = E_{(x,y)}\,[f[w_x(\tau), w_y(\tau)]] = \lim_{\Delta \to 0} \lim_{M \to \infty} \mu^{(1)}_{M,\Delta}. \tag{16}$$

Obviously, for any finite Δ and M the result will be affected by i) a systematic error due to the effect of discretization (average over piecewise linear paths instead of general paths), for which (Kloeden and Platen, 1991):

$$\epsilon_{sys}(\Delta) \sim \mathcal{O}(\Delta), \tag{17}$$

and *ii)* a statistical error, which in view of the Central Limit theorem is asymptotically gaussian, with zero average and r.m.s deviation (Sobol, 1975),(Kloeden and Platen, 1991):

$$\epsilon_{stat}(M,\Delta) \sim \left\{M^{-1} var\left[f(x_e,y_e)\right]\right\}^{1/2}, \tag{18}$$

which depends (weakly) on Δ, as well as (strongly) on M.

By computing the second moment :

$$\mu^{(2)}_{M,\Delta} = M^{-1}\sum_{j=1}^{M}\left[f(x^j_e,y^j_e)\right]^2, \tag{19}$$

one can estimate the confidence interval of the result (Sobol, 1975):

$$\delta_{M,\Delta} = \mu^{(1)}_{M,\Delta} \pm \alpha M^{-1/2}(\mu^{(2)}_{M,\Delta} - {\mu^{(1)}_{M,\Delta}}^2)^{1/2}, \tag{20}$$

where α depends on the sought confidence level.

5 - COMPARISON WITH OTHER METHODS.

In this Section a comparison is outlined between the proposed Wiener Integral Monte Carlo (henceforth WIMC) method and other alternative standard techniques like MoM and FEM to compute the scalar potential, in terms of computational budget. The main differences between WIMC and MoM/FEM can be summarized as follows:

- WIMC has very mild memory requirements, irrespective of the complexity and size of the problem. In contrast, MoM and FEM require the storage of a large (possibly block-Toeplitz) matrix.
- WIMC does *not* require any meshing algorithm.
- WIMC is intrinsecally parallelizable.
- Tight accuracy bounds are easily obtained.

On the other hand, the main drawback of WIMC is related to the relatively slow convergency rate ($\propto M^{-1/2}$). Thus, WIMC should be seriously considered for complicated geometries i.e. , whenever fast (parallel) computing engines and relatively little memory are available.

Moreover, as remarked in previous Sections, WIMC can be used as an auxilary tool for characteristic impedance computation. In this connection one could evaluate the scalar potential on a suitable spatial lattice and then resort to suitable interpolators (e.g. polinomial, splines (Press et al., 1992)) to obtain the fields all over the domain $\mathcal{D}$. A variational formula like eq. (6) can provide a very good approximation with a restricted number of potential samples. We remark that WIMC accuracy on each potential sample does *not* depend on the spatial discretization and the computation of the needed potential samples can be *easily* distributed among *parallel* processors.

6 - COMPUTATIONAL RESULTS.

As a test of the accuracy of the proposed method, first to be considered is a circular coaxial cable (*Fig. 1a*), for which the exact analytical solution is known (Collin, 1960). In order to obtain an indicative test, we ignored the circular symmetry of the structure, and computed the scalar potential (in one quadrant of the structure only) on 100 points arranged on a 10×10 grid. We applied the variational formula (6) to compute the characteristic impedance, using a bicubic interpolation (Press et al., 1992) of the potential samples. In *Fig. 1b* the exact and WIMC normalized characteristic impedance versus radius-ratio are compared. The computed errors are displayed in *Fig. 1c*, and were always below 0.3%.

As a further step in complication, in *Fig. 2a* a structure is considered with a *stadium-shaped* outer conductor and an off-centered regular hexagonal inner one. In this case we computed the scalar potential at 200 points arranged on a 20×10 grid covering one half of the structure, because of its symmetry. *Fig. 2b* shows the WIMC-computed normalized characteristic impedance (obtained via formula (6) with bicubic interpolation) as a function of the off-center parameter:

$$\gamma = \frac{|x_{off}|}{a+b-r} \quad , 0 \leq \gamma \leq 1. \tag{21}$$

A typical potential contour-plot is shown in *Fig. 2c.*

Finally we considered a shielded coupled stripline with coplanar strips (*Fig. 3a*). The potential was computed at 100 points arranged in 10×10 grid covering one quadrant of the geometry, for symmetry, and a bicubic interpolation has been used to compute characteristic impedance, generalizing eq. (6) to three-conductor lines (Pozar, 1993). The WIMC-computed normalized characteristic impedance for even ($V_1 = V_2$) and odd ($V_1 = -V_2$) modes as a function of normalized strips distance is shown in *Fig. 3b*. Note that both characteristic impedances above are measured (defined) between one strip and the outer (ground) shield. Typical potential countour-plots are displayed in *Fig.s 3c-3d.*

In order to obtain all graphs above, a typical number $M \sim 10^5$ paths with an adaptive time-step[3] have been used. The typical average confidence interval halfwidth on potential (eq. (20), $\alpha = 3$) was always less than 1%.

7 - CONCLUSIONS.

We introduced a new method for computing the scalar potential and characteristic impedance of a TEM mode in complex (multiconductors) transmission lines based on Wiener Integral and Monte Carlo method. Application to a number of structures has been presented. The method seems attractive by comparison with standard techniques in terms of computational budget and ease.

[3] $\Delta \sim 10^{-5}$ close to the conductors (where one has to solve eq. (13)), $\Delta \sim 10^{-3}$ elsewhere.

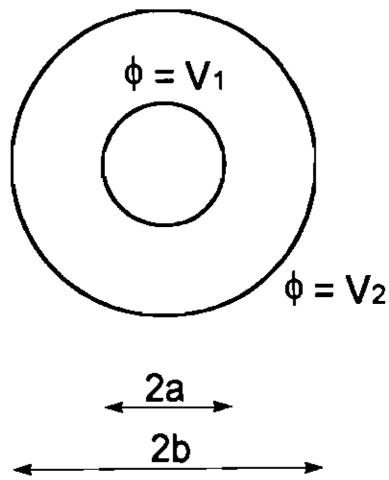

FIGURE 1a - Coaxial circular transmission line.

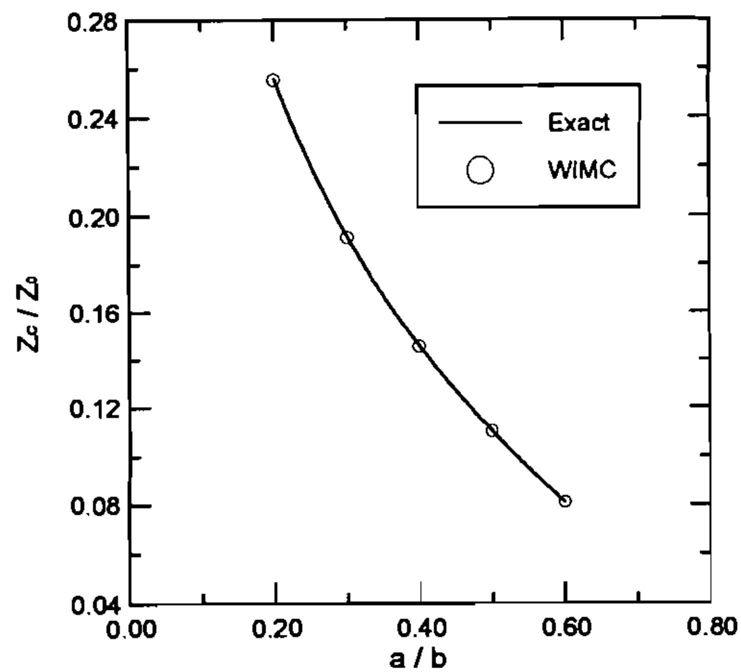

FIGURE 1b - Exact and WIMC normalized characteristic impedance vs. a/b.

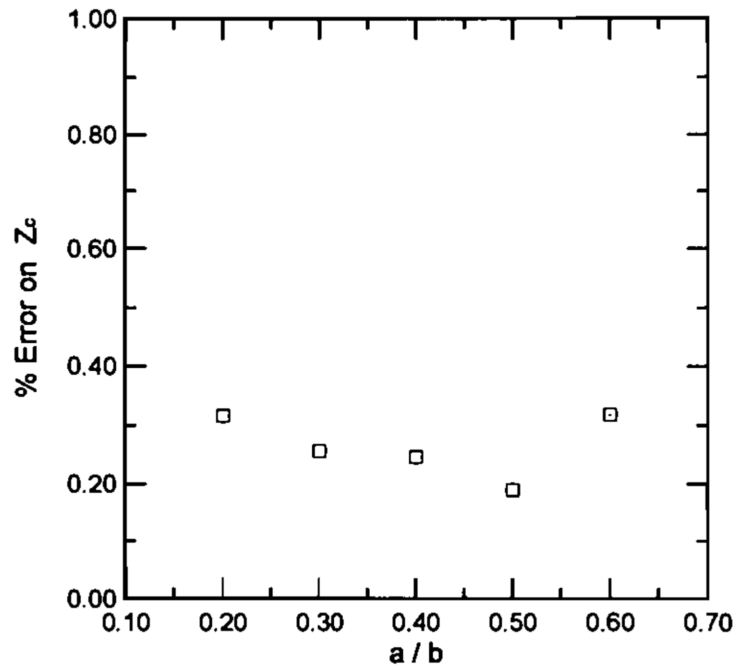

FIGURE 1c - % error on characteristic impedance vs. a/b.

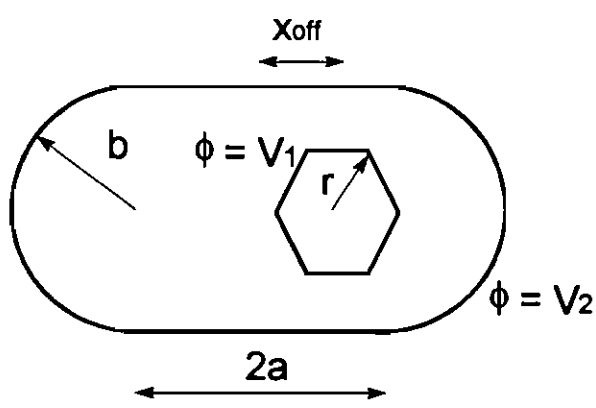

FIGURE 2a - Stadium shaped transmission line with off-centered hexagonal inner conductor.

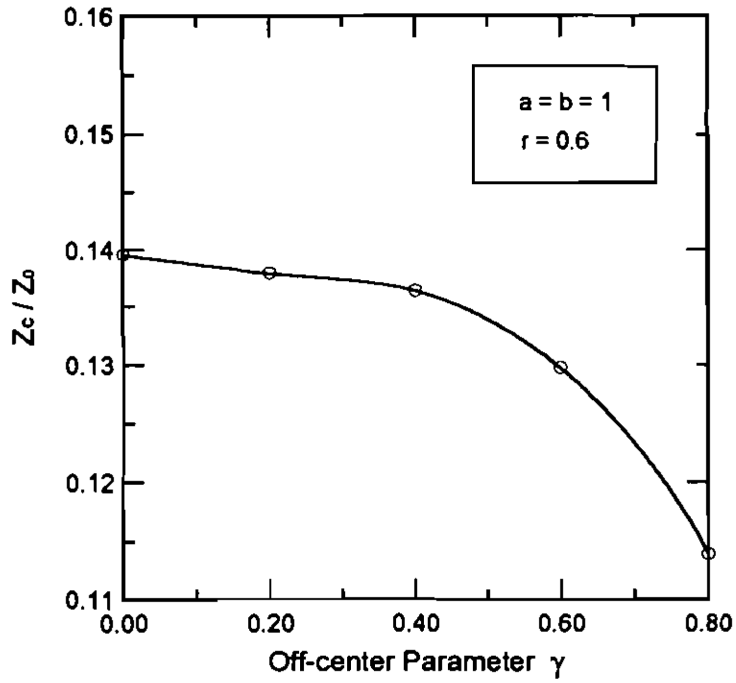

FIGURE 2b - WIMC normalized characteristic impedance vs. off-center parameter $\gamma = |x_{off}|/(a+b-r)$.

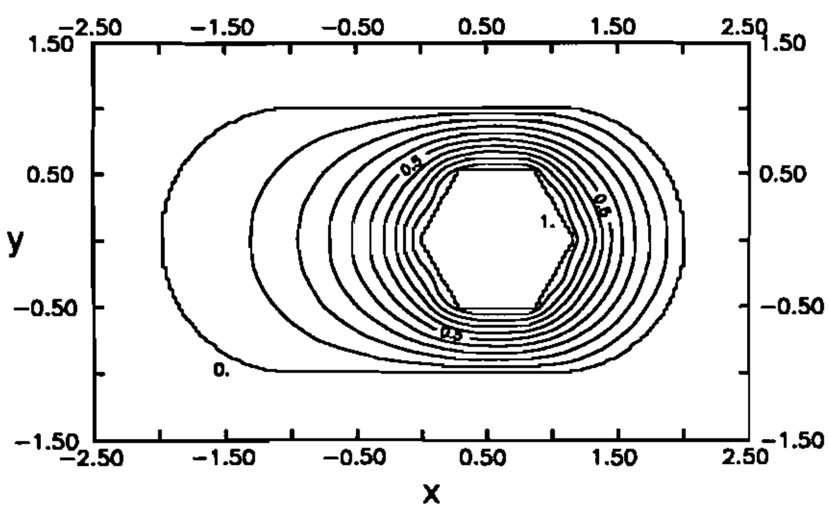

FIGURE 2c - Potential contour-plot:
$\gamma = 0.4$, $a = b = 1$, $r = 0.6$, $V_1 = 1$, $V_2 = 0$.

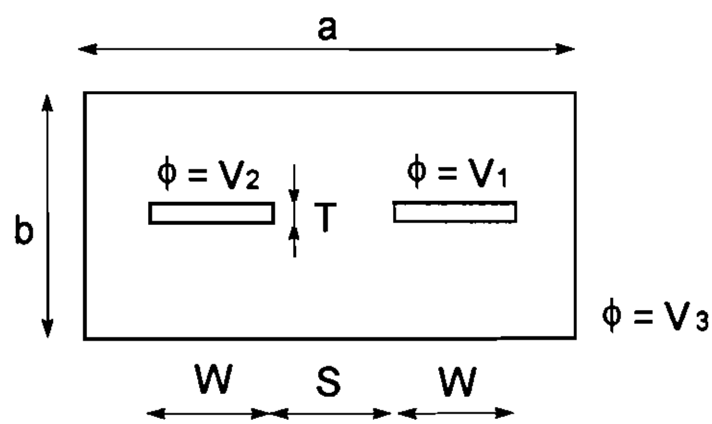

FIGURE 3a - Shielded coupled striplines with coplanar strips.

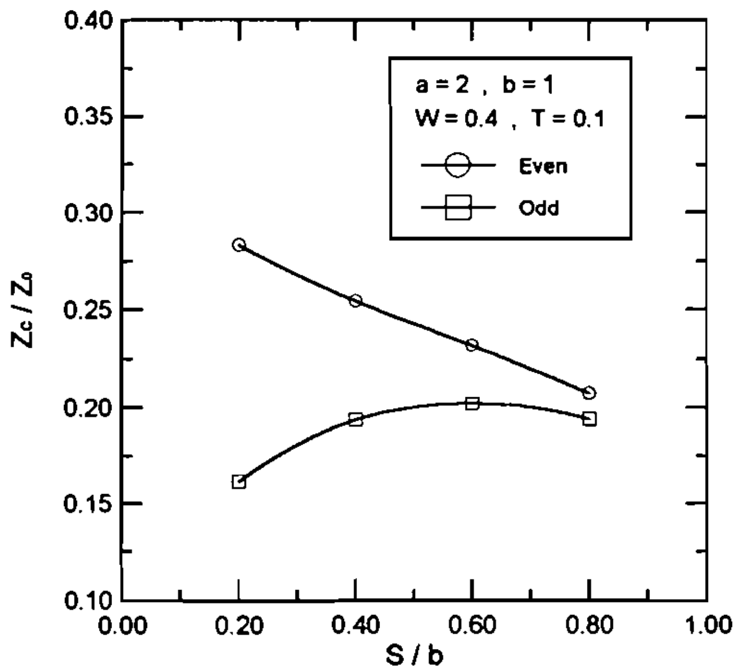

FIGURE 3b - WIMC normalized characteristic impedance for even ($V_1 = V_2$) and odd ($V_1 = -V_2$) modes vs. normalized strips distance S/b.

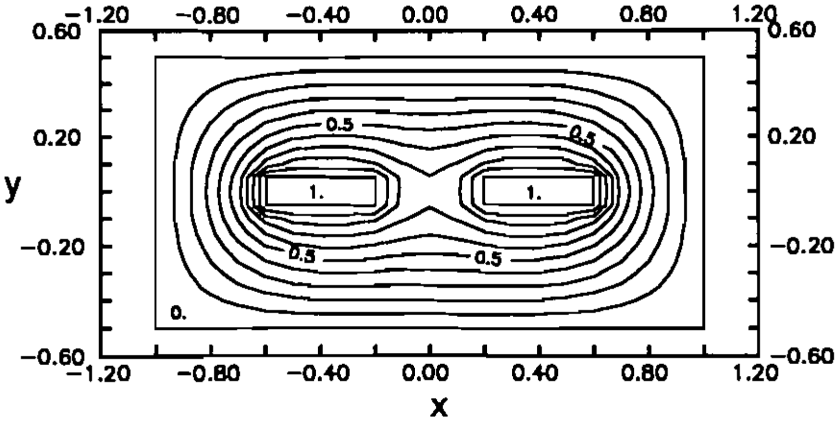

FIGURE 3c - Potential contour-plot for even mode:
$V_1 = V_2 = 1$, $V_3 = 0$, $a = 2$, $b = 1$, $W = 0.4$, $T = 0.1$, $S/b = 0.4$.

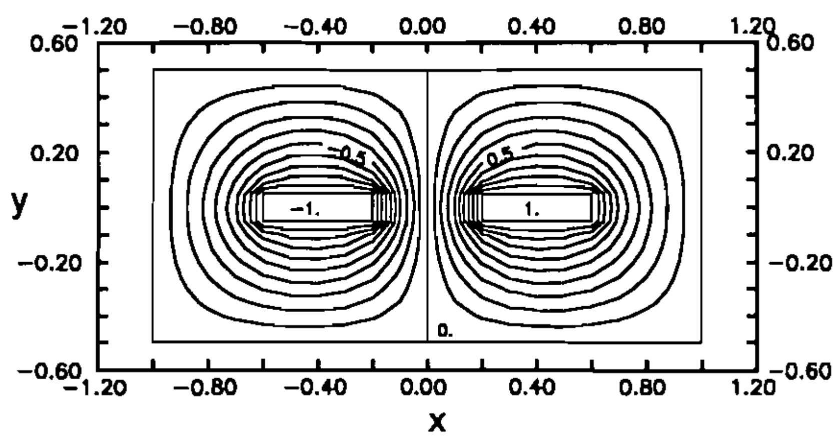

FIGURE 3d - Potential contour-plot for odd mode:
$V_1 = -V_2 = 1$, $V_3 = 0$, $a = 2$, $b = 1$, $W = 0.4$, $T = 0.1$, $S/b = 0.4$.

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APPENDIX - ABOUT WIENER INTEGRALS.

We define a standard (scalar) Wiener process $w = \{w(t), t \geq 0\}$ originating from x_0 at $t = 0$ as a gaussian process with independent increments such that (Kloeden and Platen, 1991),(Gardiner, 1983):

$$w(0) = x_0, \quad E[w(t) - w(s)] = 0, \quad var[w(t) - w(s)] = 2D(t-s), \quad t \geq s. \quad (A1)$$

where D is the so-called *diffusion coefficient* [4].

We can also consider n-dimensional Wiener processes, whose components $\{w_1, \ w_2, \ ..., w_n\}$ are independent scalar Wiener processes with respect to a common family of $\sigma-$algebras (Gardiner, 1983). It can be shown that the sample paths of a Wiener process are continuous but nowhere differentiable functions of time (Gardiner, 1983). The transition probability density:

$$p(x_0, s; x, t) = [4\pi D(t-s)]^{-1/2} \exp\left[-\frac{(x-x_0)^2}{4D(t-s)}\right] \quad (A2)$$

satisfies the Fokker-Planck equation (Gardiner, 1983):

$$\frac{\partial p}{\partial t} = D\nabla^2 p. \quad (A3)$$

Wiener was able to define a *probability measure* associated to the process defined above and demonstrated that for a wide class of regular functionals there exists an *integral* over it (Gelfand and Yaglom, 1960), (Schulman, 1981). Such an integral admits an immediate interpretation as an *average* of the functional over the Wiener paths (Schulman, 1981).

[4] Note that in *Sections 3-4* we reference to Wiener processes with diffusion coefficient $D = 1$.