SAT/SMT Solvers and Applications

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Winter 2013

The Goals of this Course

Introduction to SAT/SMT Solvers

- What are constraint solvers (e.g., Boolean SAT and SMT solvers)?
- Why should you care?
- Theoretical aspects (e.g., proof theoretic treatment of modern SAT algorithm)
- Practical aspects (e.g., Important SAT/SMT solver heuristics)

Applications

- Symbolic execution and solver-based automated bug finding
- Solver-based program analysis/synthesis
- Solver-based model-checking and theorem provers
- Debugging and equivalence-checking tools
- Applications in AI
- Solvers and computer security
- Solver-based modeling and requirements analysis

Goals of this Course
Lecture Schedule

- Modern CDCL SAT solvers: core ideas (VG)
- Modern SMT Solvers: core ideas (VG)
- MiniSAT: Anatomy of a modern CDCL SAT solver (VG and students)
- Symbolic execution and solver-based automated bug finding (VG and students)
- Solver-based program analysis/synthesis (Frank Tip)
- Solver-based model-checking and theorem provers (Shoham Ben-David)
- Debugging and equivalence-checking tools (VG and students)
- Applications in AI (Peter Van Beek)
- Solver-based modeling and requirements analysis (Czarnecki and Rayside)
- Solvers and computer security (Mahesh Tripunitara)
- Solvers and programming languages (VG and students)

What is a Constraint Solver?

Engineer/Mathematician's point of vew

- A "method" that takes as input a math formula, and produces a solution
- Examples: solving linear equations over the reals, polynomials, quadratic, Boolean logic,...
- Computing zeros of a polynomial

Theoretical computer scientist/logician's point of view

- A computer program called as a satisfiability procedure that solves a specific kind of decision problem, namely, the SAT problem
- The input formula is in a specified logic (e.g., Boolean, first-order, reals, integers,...)
- Output of a satisfiability procedure
  - UNSAT, if input has no satisfying assignments
  - SAT, otherwise

One Slide History of Constraint Solving Methods

Before modern conception of logic (Before Boole and Frege)

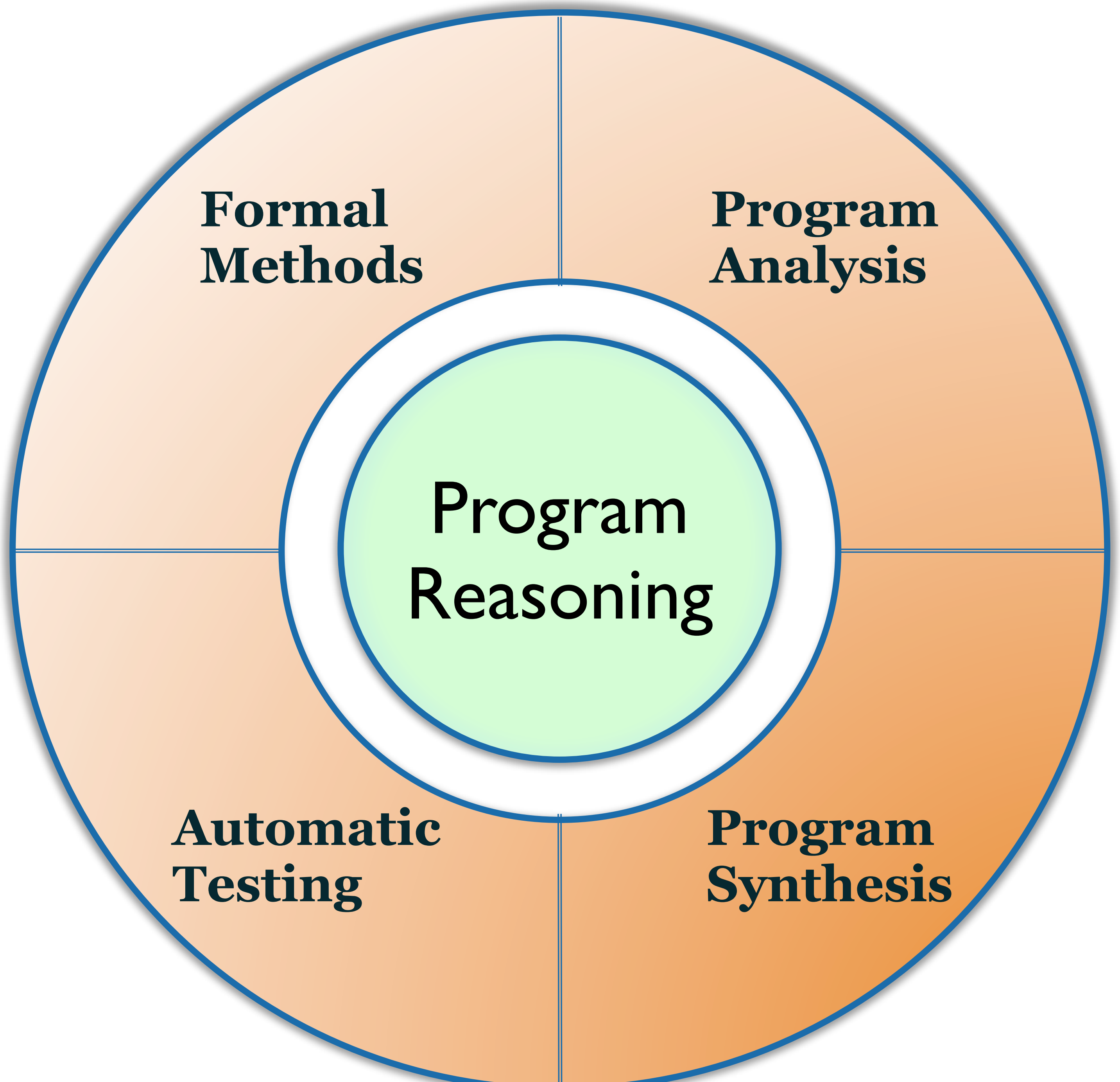
- From Babylon to late 1800's: Huge amount of work on methods to solve (find roots of) polynomials over reals, integers,...
- System of linear equations over the reals (Chinese methods, Cramer's method, Gauss elimination)
- These methods were typically not complete (e.g., worked for a special class of polynomials)

After modern conception of logic

- Systems of linear inequalities over the integers are solvable (Presburger, 1927)
- Peano arithmetic is undecidable (hence, not solvable) (Godel, 1931)
- First-order logic is undecidable (hence, not solvable) (Turing,1936. Church, 1937)
- A exponential-time algorithm for Boolean SAT problem (Davis, Putnam, Loveland, Loggeman in 1962)
- Systems of Diophantine equations are not solvable (Matiyasevich. 1970)
- Boolean SAT problem is NP-complete (Cook 1971)
- Many efficient, scalable SAT procedures since 1971 for a variety of mathematical theories

Foundation of Sofware Engineering

Logic Abstractions of Computation



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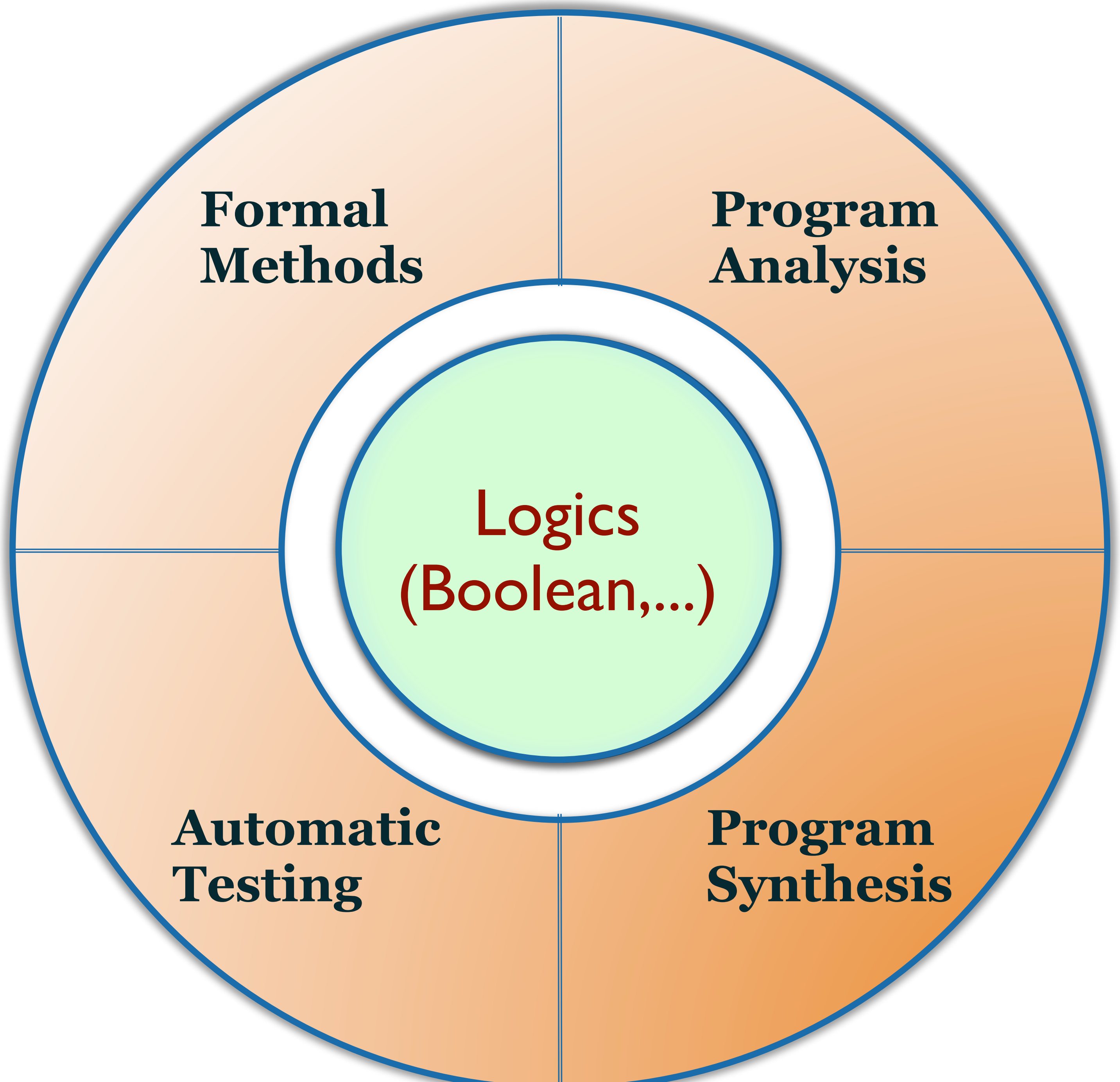
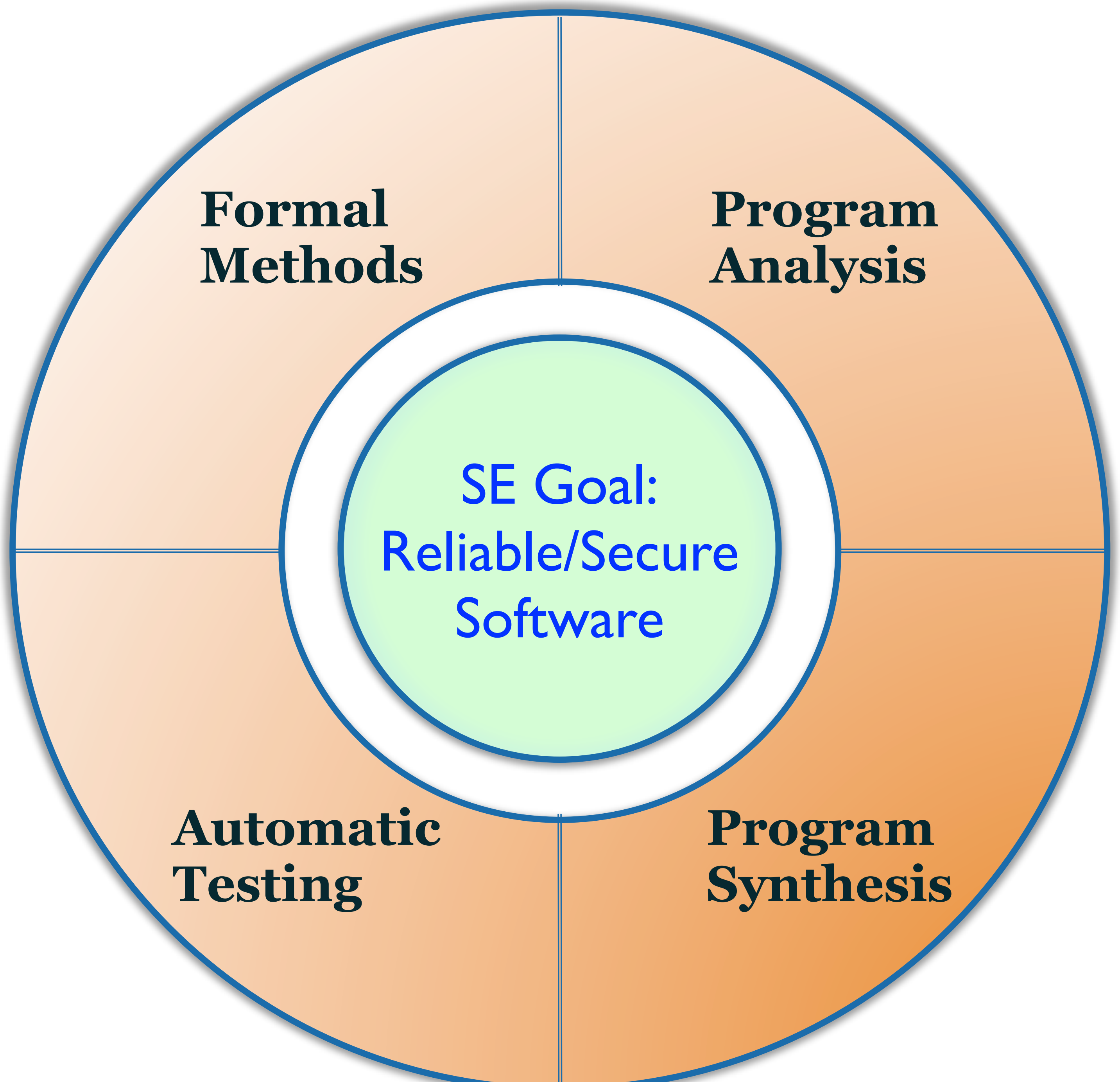


Software Engineering & SAT/SMT Solvers

An Indispensable Tactic for Any Strategy



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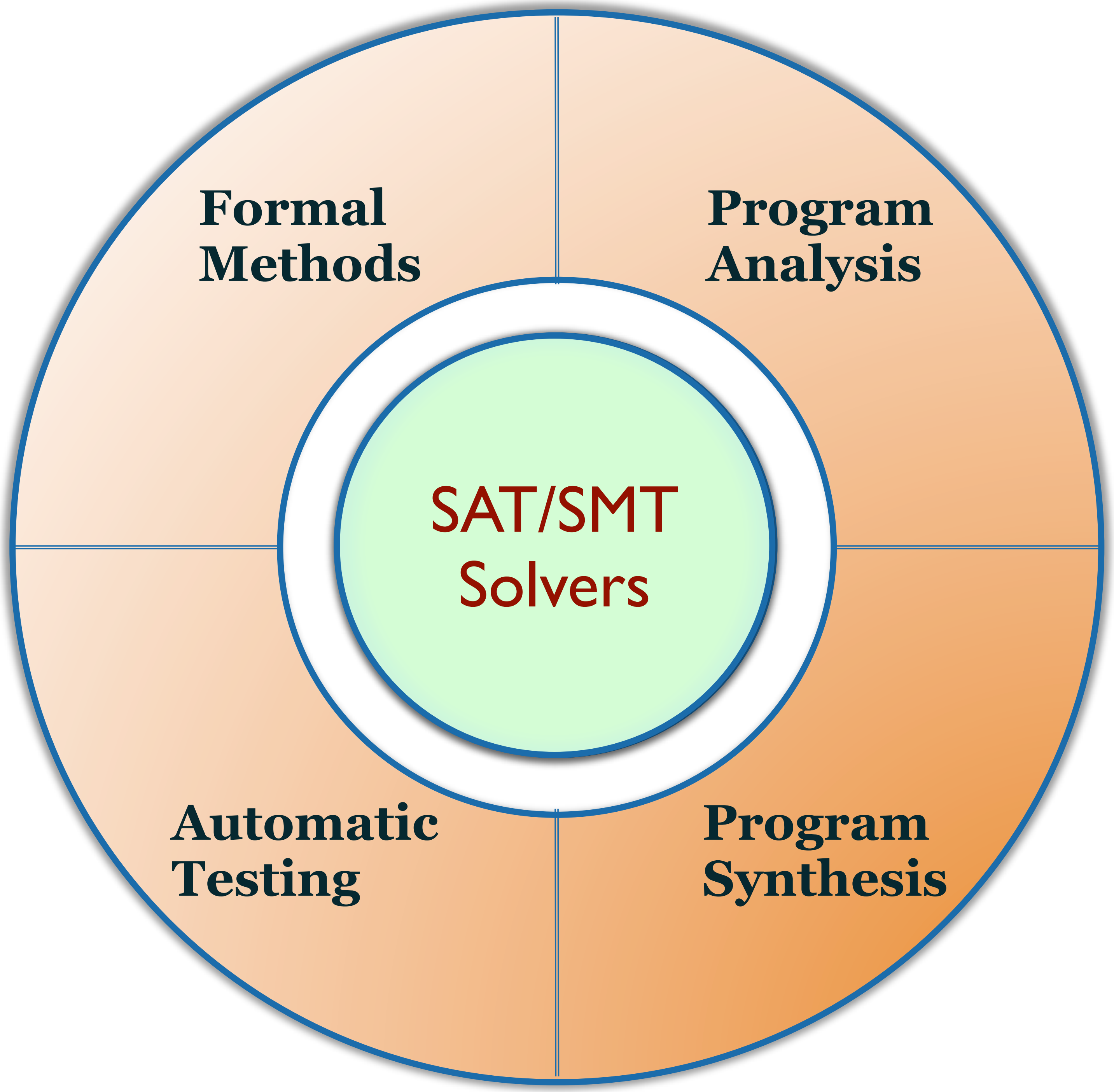
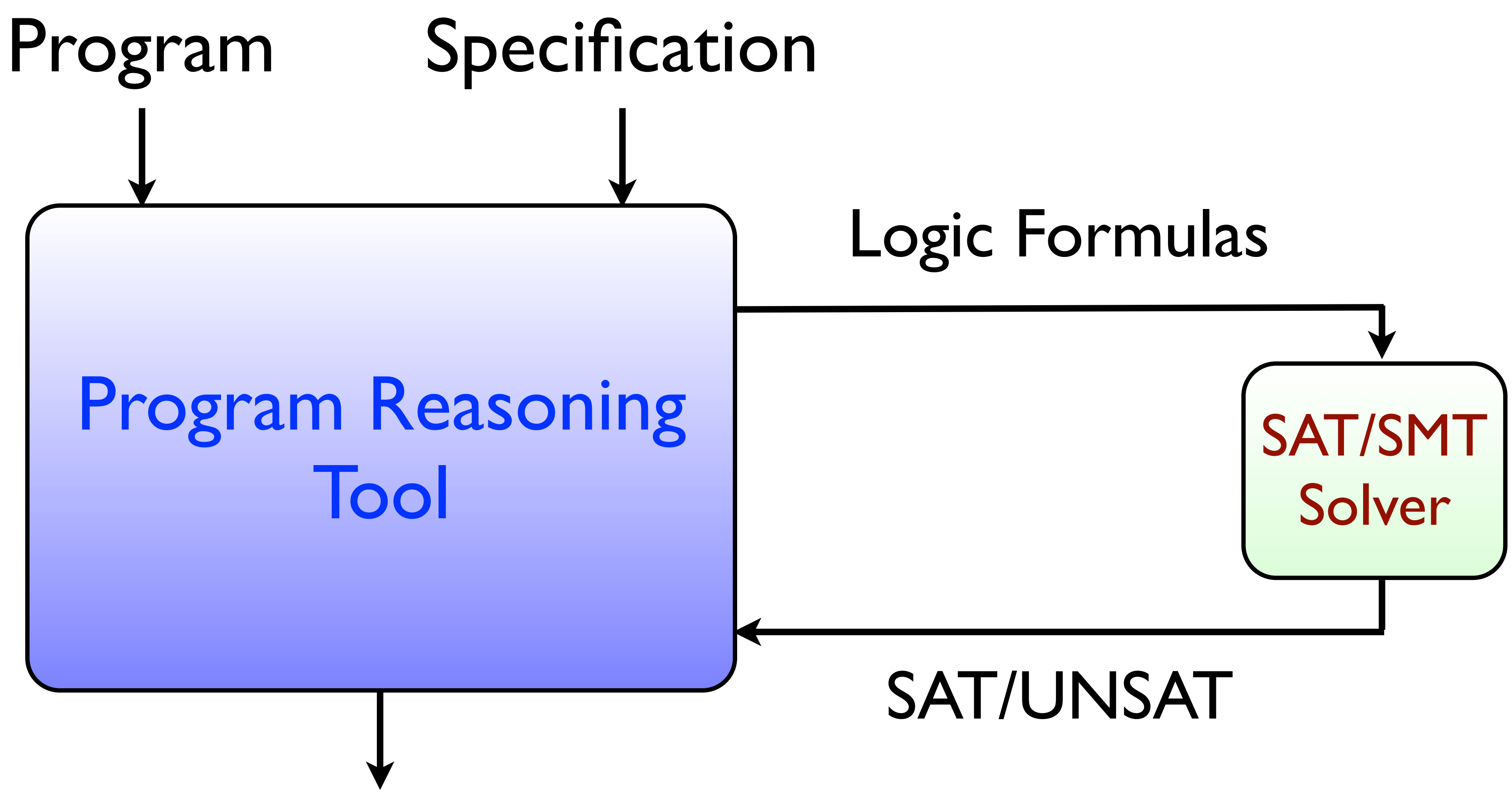


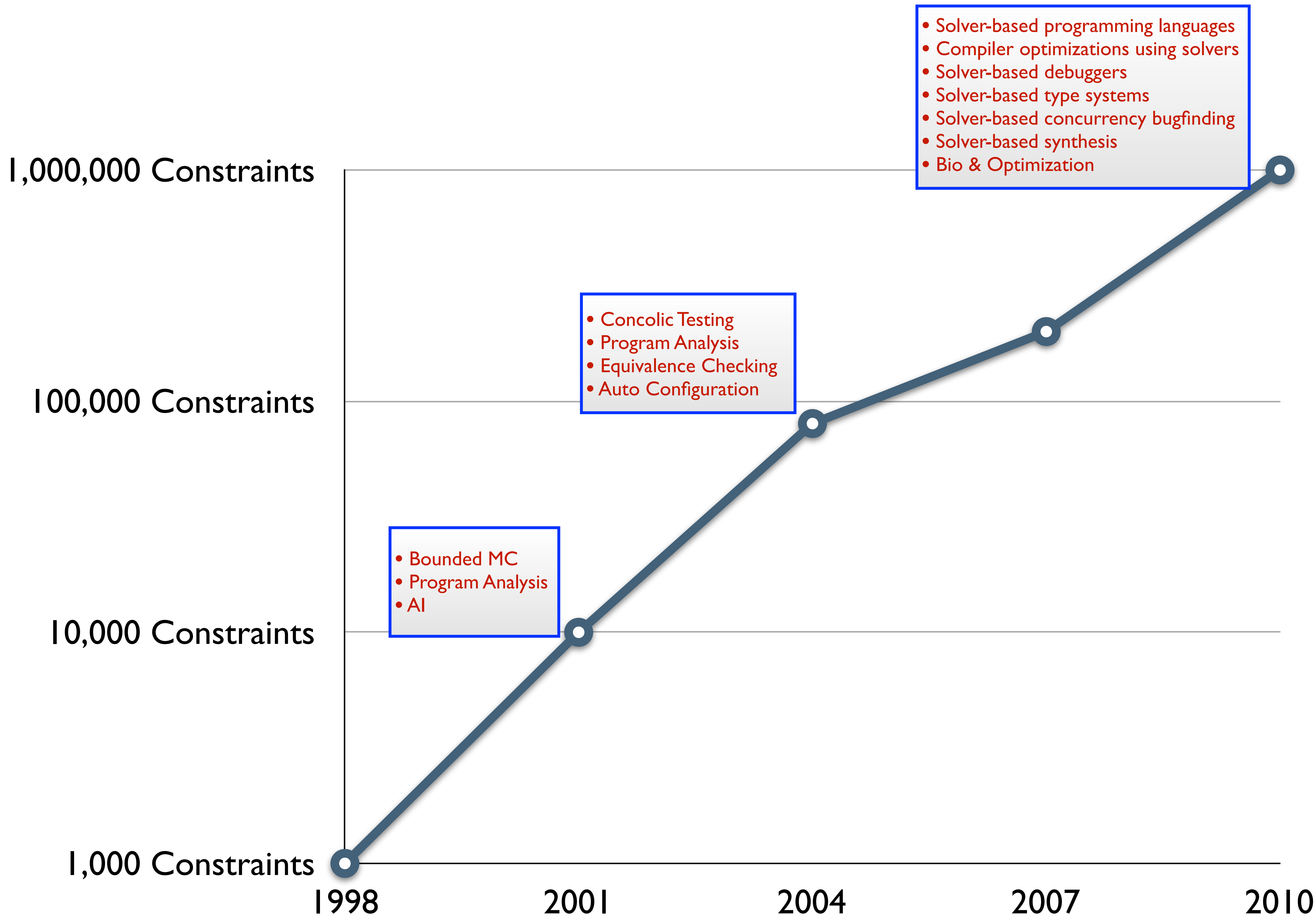
Software Engineering using Solvers
Engineering, Usability, Novelty



Program is Correct?
or Generate Counterexamples (Test cases)

SAT/SMT Solver Research Story

A 1000x Improvement



Why Should You Care?

SAT/SMT solver user

- Use solver as a black-box
- More importantly, solver algorithms are influencing algorithms in other areas
- Synthesis: Sketching by Armando Solar-Lezama uses SAT techniqes
- Analysis: Combining static and dynamic analysis has a flavor of the SAT algorithm
- Model-checking: The IC3 algorithm integrates SAT deeply into the model-checker

Some SAT/SMT solvers users at Waterloo

- Imeson, Tripunitara, Garg are using programmatic SAT for hardware security
- Krzysztof Czarnecki's group is using solvers for auto-configuration
- Derek Rayside's group is using solvers for software modelling through Alloy
- Frank Tip's group is using solvers for analysis
- Peter Van Beek's group works on CSPs, closely related to SAT
- Venkat Raman is working on complexity-theoretic aspects of the SAT problem
- Lin Tan's group is looking into symbolic-execution/solver-based bug finding

Brief overview of relevant Logic Concepts

What is a Logic?

What is a mathematical theory?

Notions of models, truth and proof

What is the connection between truth and proof?

What are the satisfiability and validity problems?

What is a decision procedure?

What is a satisfiability procedure?

What is the connection between satisfiability, validity and proof?

What is meant by soundness and completeness?

Logic, model, truth, assignments

- Study of valid modes of reasoning (inductive, deductive, ...)
- Formal language (e.g., Boolean logic, first-order logic,...)
- Rules for constructing well-formed formulas
- An associated proof system (axioms, inference rules,...)
- Model
  - Interpretation of connectives, functions, predicates over a domain
  - True, false
  - Assignment: Mapping of variables to elements of the domain

The SAT/SMT Problem

- Rich logics (Modular arithmetic, Arrays, Strings,...)
- NP-complete, PSPACE-complete,...
- Practical, scalable, usable, automatic
- Enable novel software reliability approaches

The SAT/SMT Problem

- Closely related to the Validity Problem
- Soundness, completeness, termination
- Connecting model theory and proof theory

Lecture Outline

Points already covered

- ☑ Motivation for SAT/SMT solvers in software engineering
- ☑ High-level description of the SAT/SMT problem & logics
- ☑ Defined logic, models, truth, proofs, SAT procedure, soundness, completeness

Rest of the lecture

- Modern CDCL SAT solver architecture & techniques
- SAT/SMT-based applications
- Future of SAT/SMT solvers
- Some history (who, when,...) and references sprinkled throughout the talk
- Non-CDCL SAT techniques

The Boolean SAT Problem

Basic Definitions and Format

A **literal** *p* is a Boolean variable *x* or its negation ¬*x*.

A **clause** *C* is a disjunction of literals: $x_2 \lor \neg x_{41} \lor x_{15}$

A **CNF** is a conjunction of clauses: $(x_2 \lor \neg x_1 \lor x_5) \land (x_6 \lor \neg x_2) \land (x_3 \lor \neg x_4 \lor \neg x_6)$

All Boolean formulas assumed to be in **CNF**

Assignment is a mapping (binding) from variables to Boolean values **(True, False).**

A **unit clause** *C* is a clause with a single unbound literal

The **SAT-problem** is:

Find an assignment s.t. each input clause has a true literal (aka input formula has a solution or is SAT)

OR establish input formula has no solution (aka input formula is UNSAT)

The Input formula is represented in **DIMACS Format:**

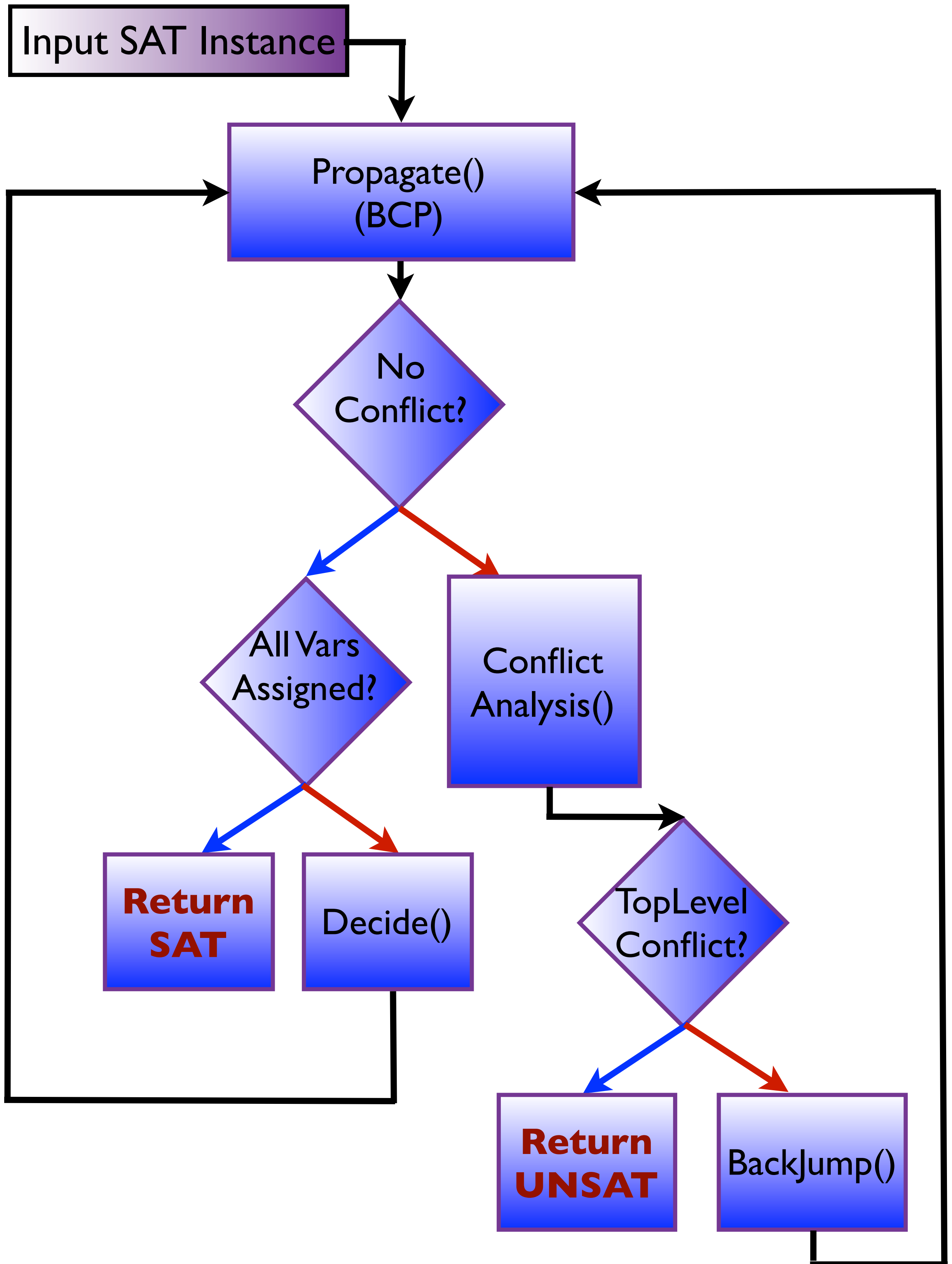
```
c DIMACS
p cnf 6 3
2 -1 5 0
6 -2 0
3 -4 -6 0
```

DPLL SAT Solver Architecture
The Basic Solver

```
DPLL(Θcnf, assign) {

    Propagate unit clauses;

    if "conflict": return FALSE;

    if "complete assign": return TRUE;

    "pick decision variable x";


    return
            DPLL(Θcnf | x=0, assign[x=0])
         || DPLL(Θcnf | x=1, assign[x=1]);

}
```

- Propagate (Boolean Constant Propagation):
  - Propagate inferences due to unit clauses
  - Most time in solving goes into this
- Detect Conflict:
  - Conflict: partial assignment is not satisfying
- Decide (Branch):
  - Choose a variable & assign some value
- Backtracking:
  - Implicitly done by the recursion

Modern CDCL SAT Solver Architecture
Key Steps and Data-structures



Key steps

- Decide()
- Propagate()
  (BCP: Boolean constraint propagation)
- Conflict analysis and learning()
- Backjump()
- Forget()
- Restart()

CDCL: Conflict-Driven Clause-Learning

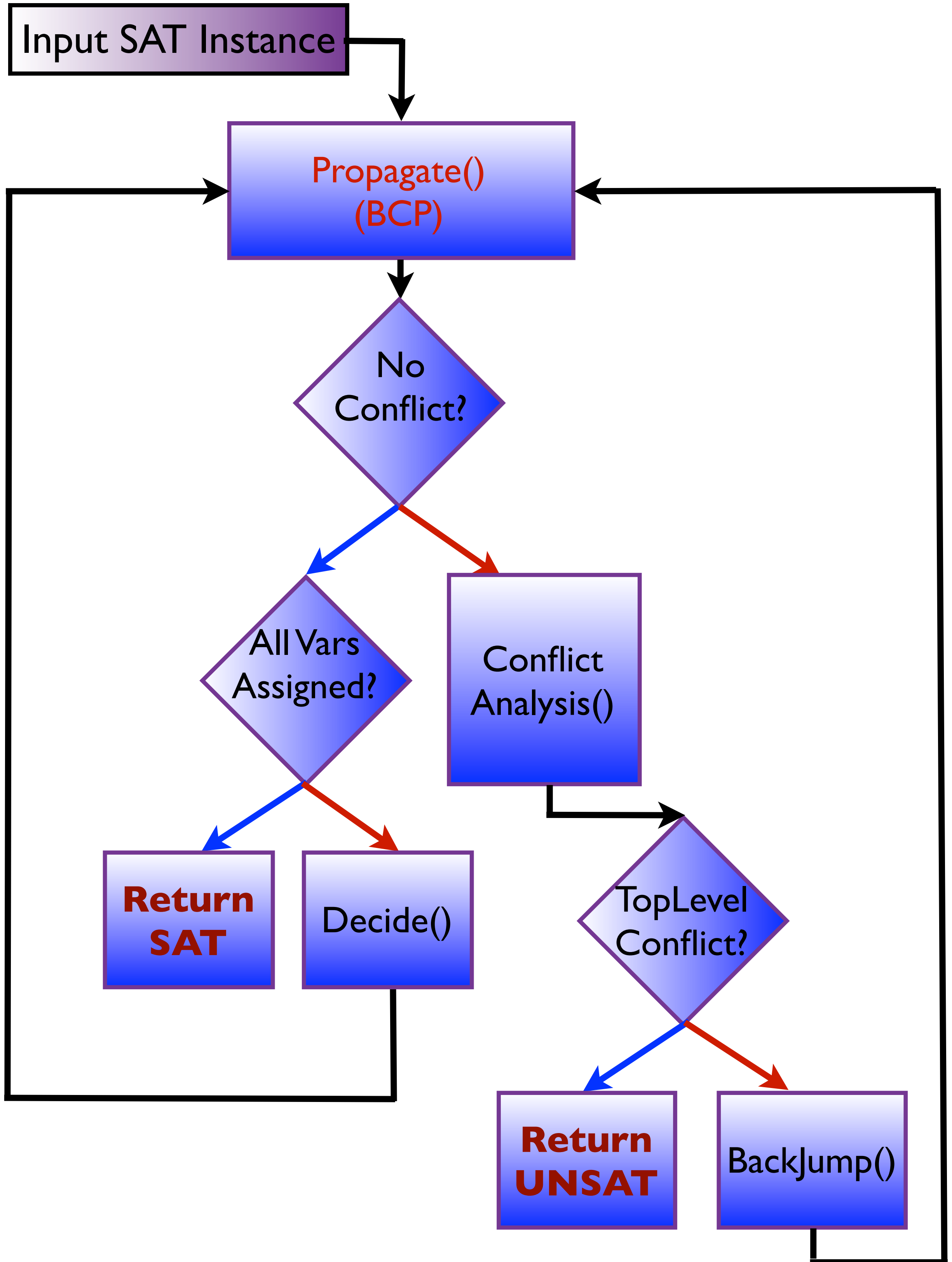
- Conflict analysis is a key step
- Results in learning a conflict clause
- Prunes the search space

Key data-structures (State):

- Stack or trail of partial assignments (AT)
- Input clause database
- Conflict clause database
- Conflict graph
- Decision level (DL) of a variable

Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()



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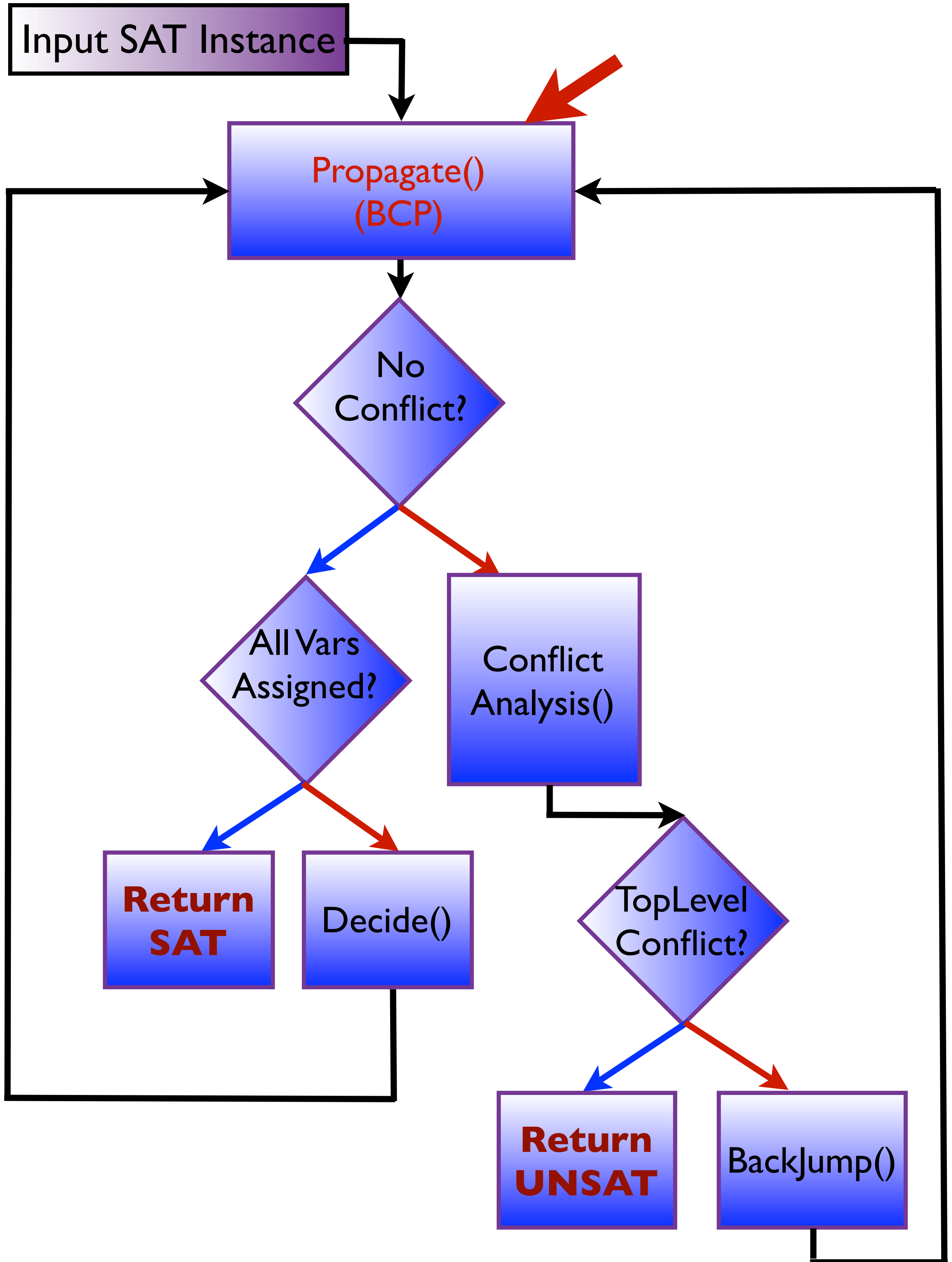

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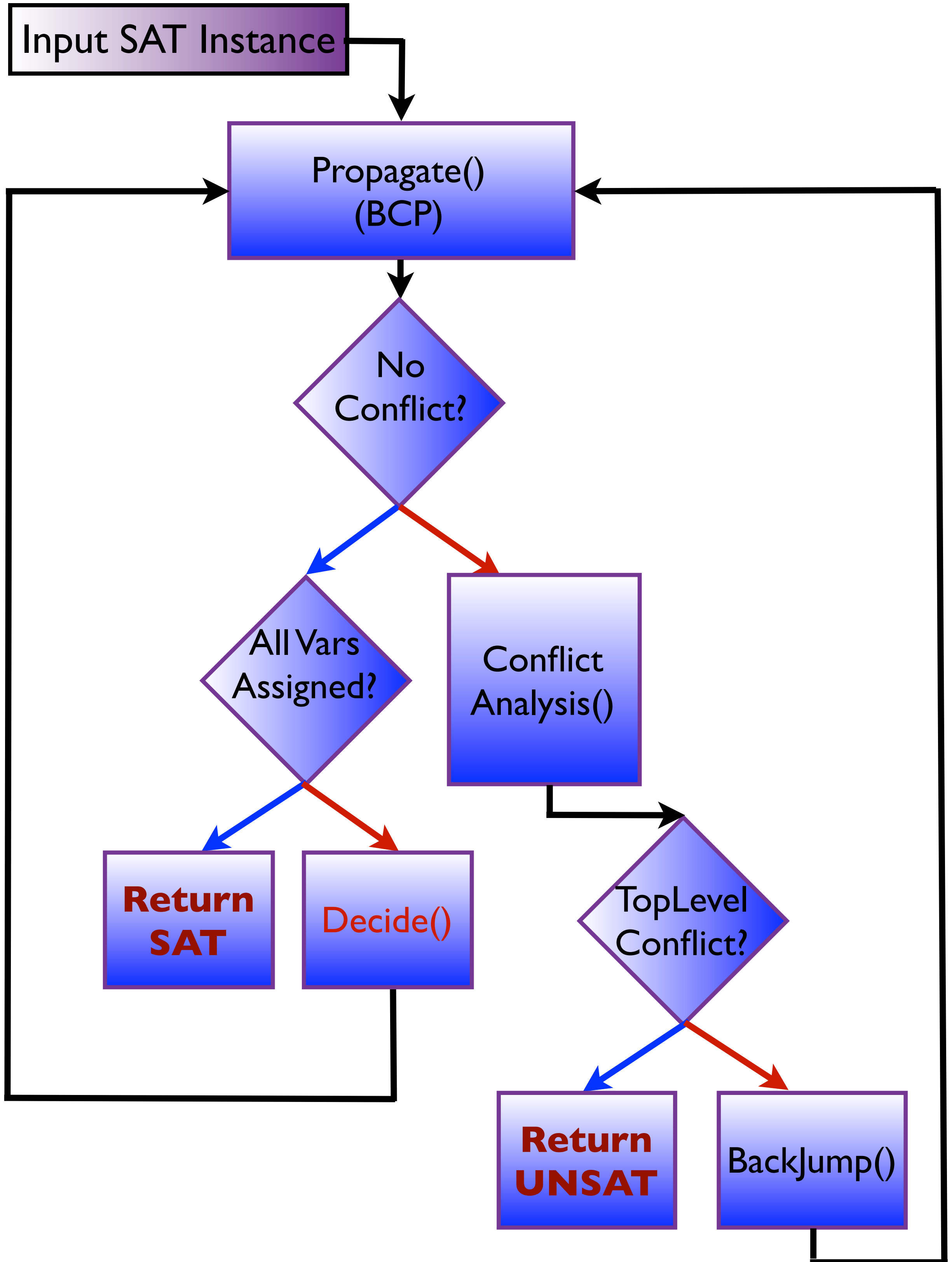

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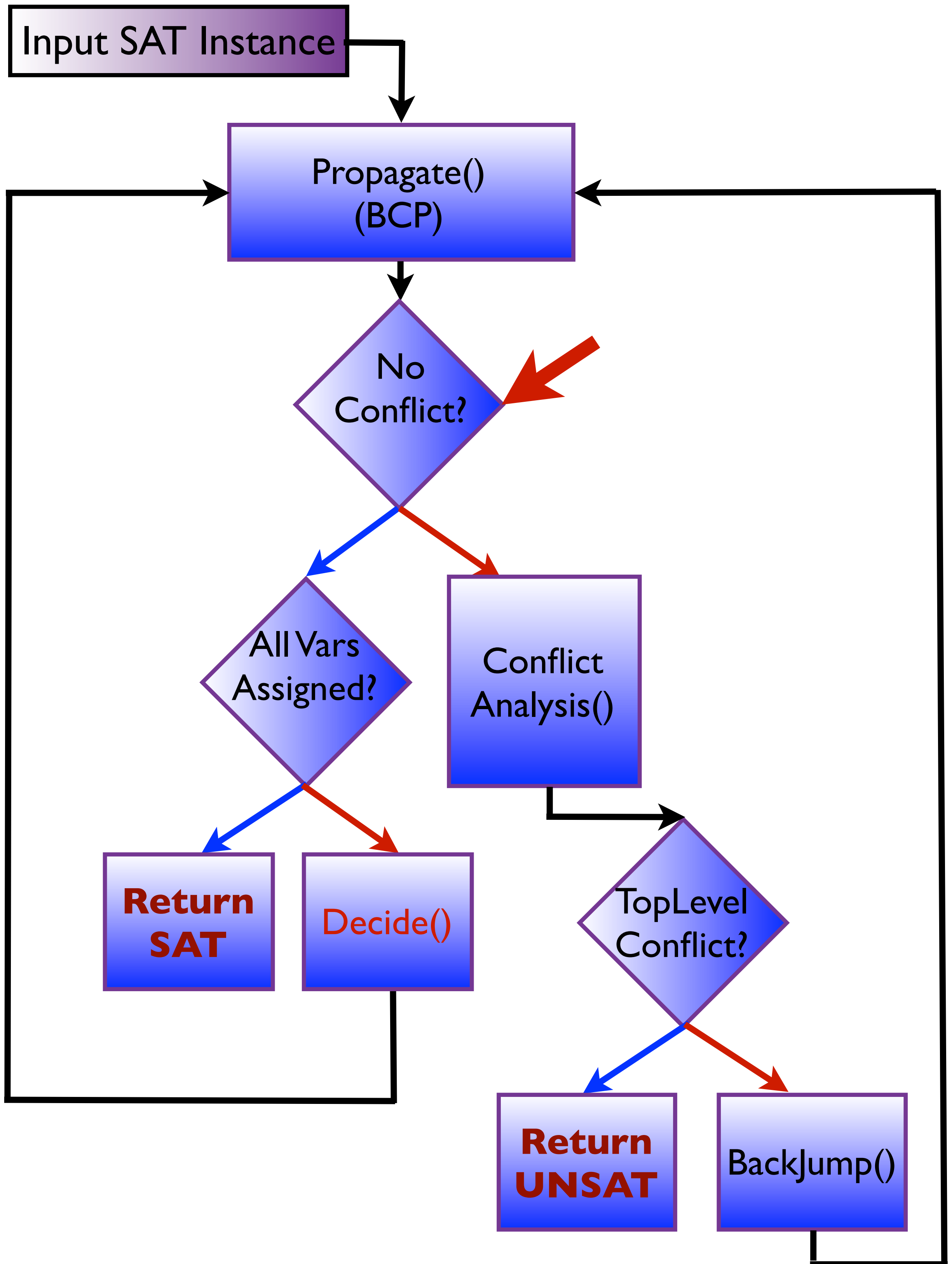

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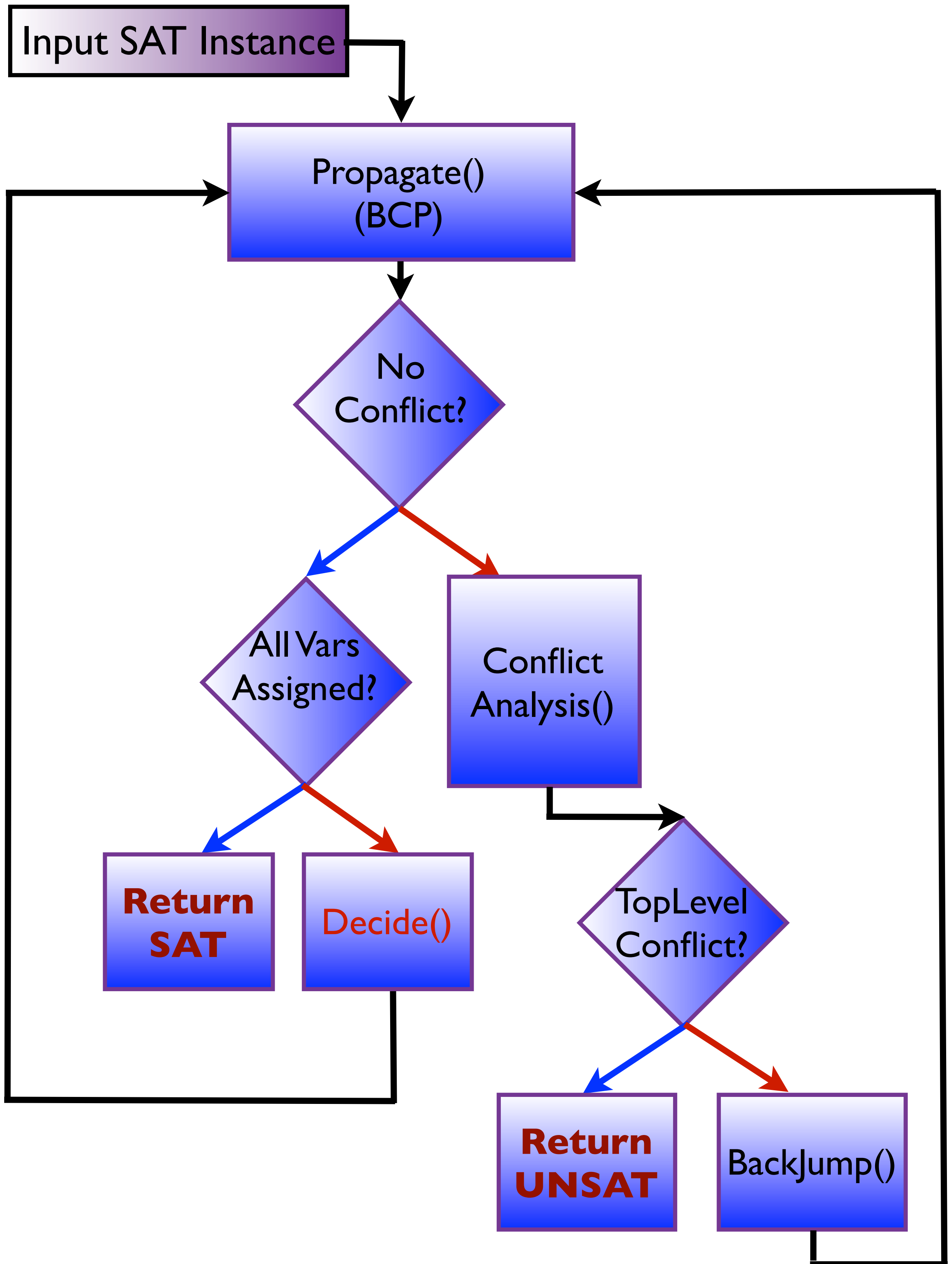

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  - Choose a variable & assign some value (decision)
  - Basic mechanism to do search
  - Imposes dynamic variable order
  - Decision Level (DL): variable ⇒ natural number

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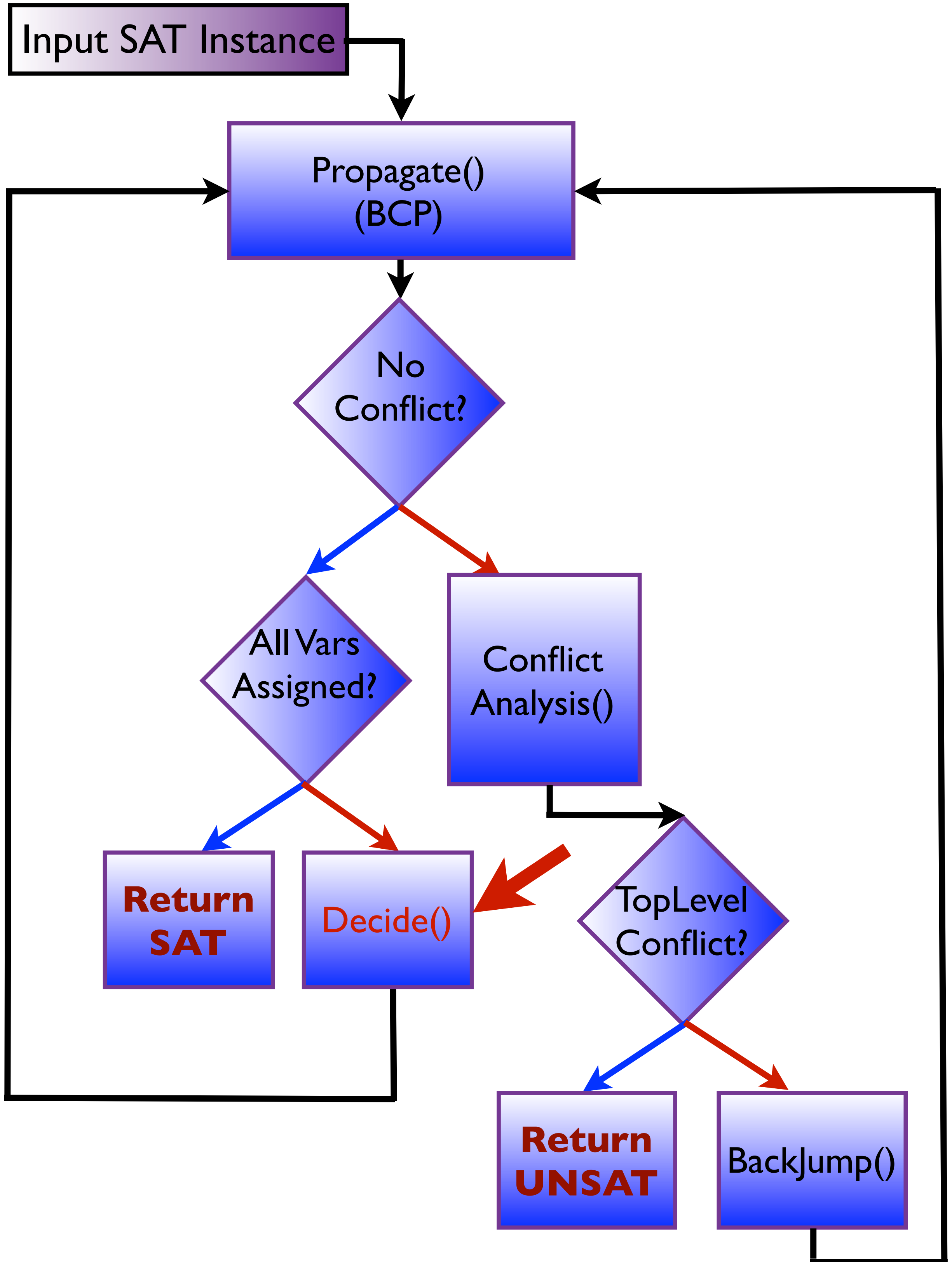

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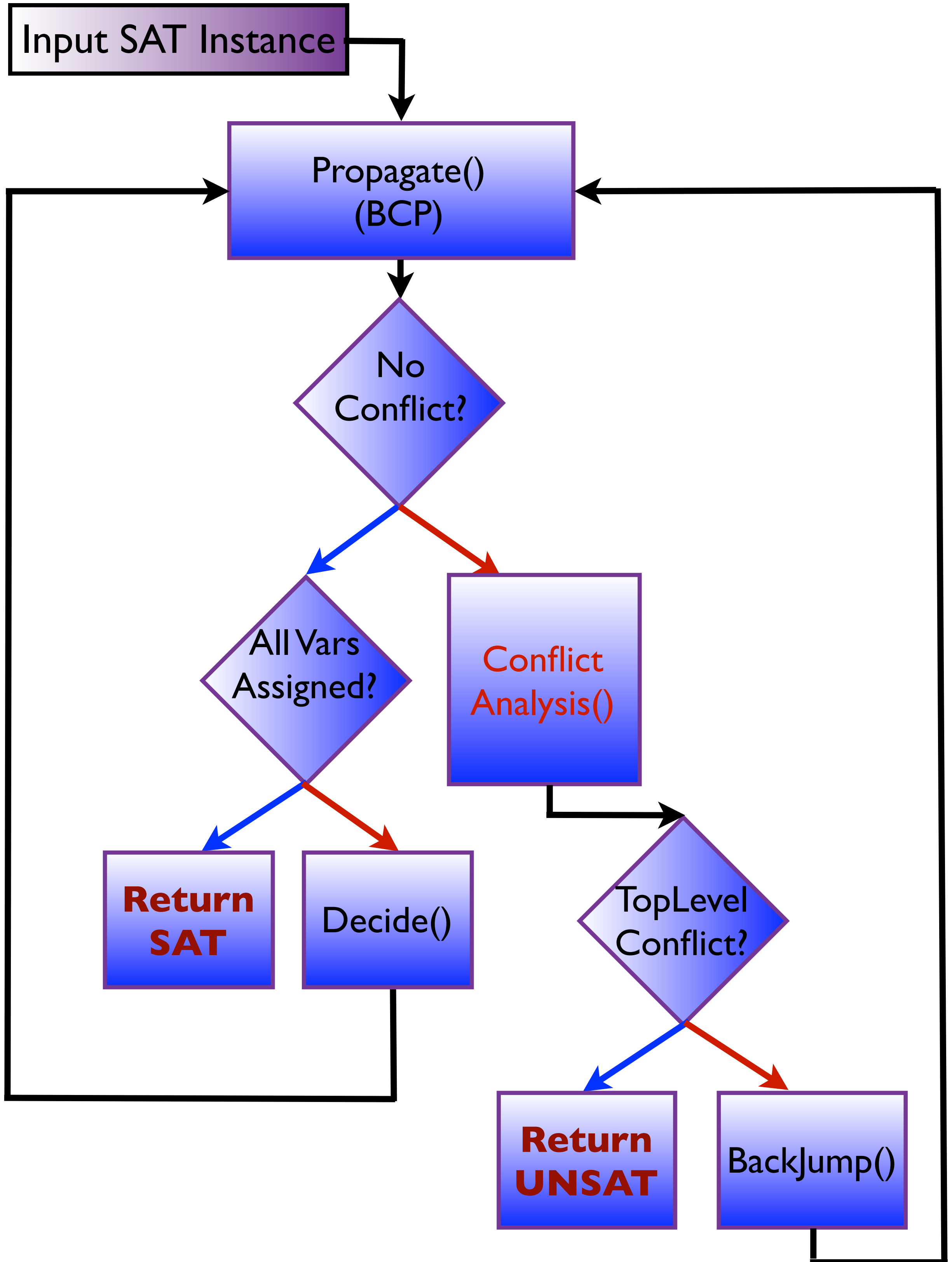

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  - Compute assignments that lead to conflict (analysis)
  - Construct conflict clause blocks the non-satisfying & a large set of other 'no-good' assignments (learning)
  - Marques-Silva & Sakallah (1996)

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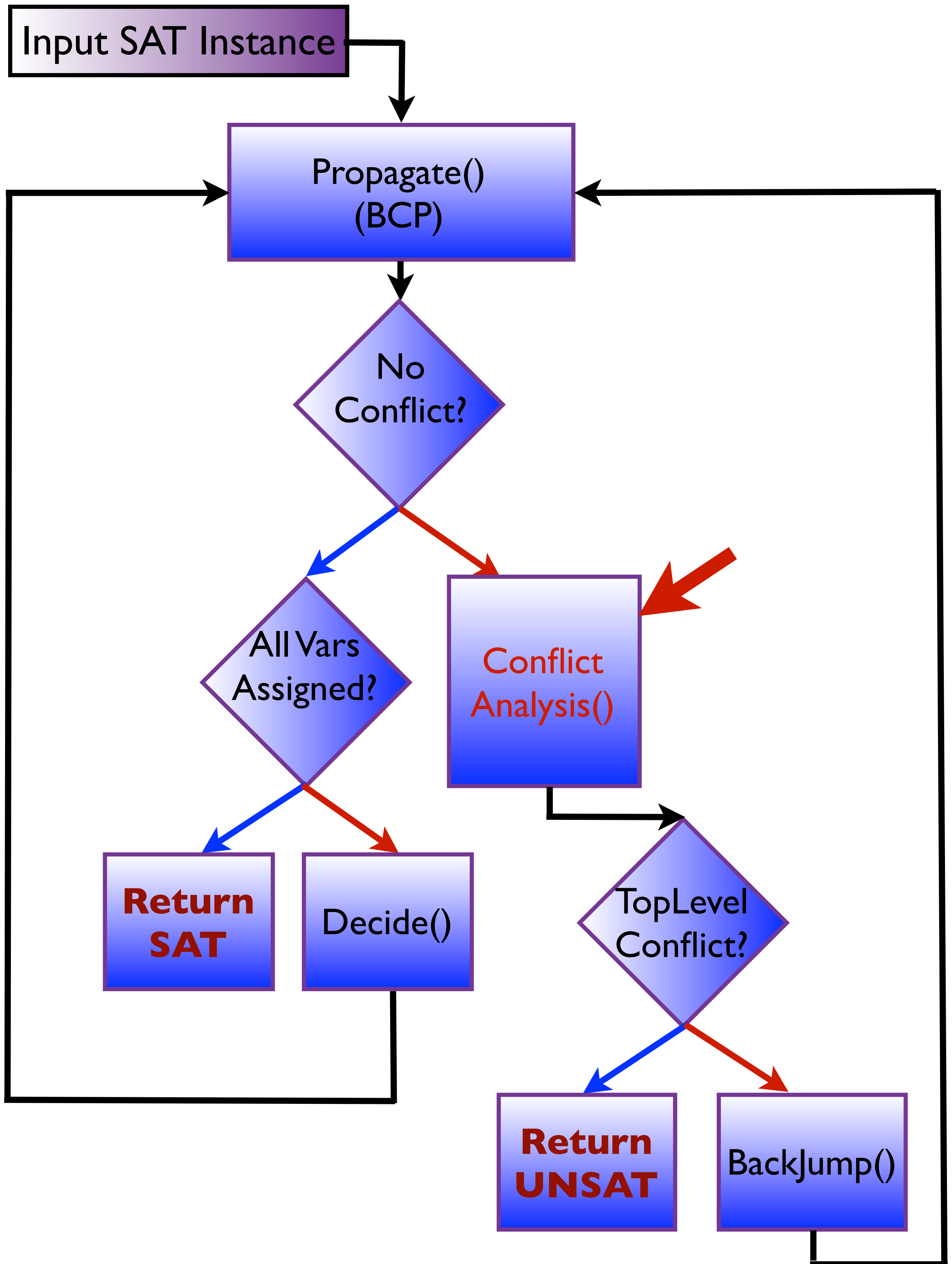

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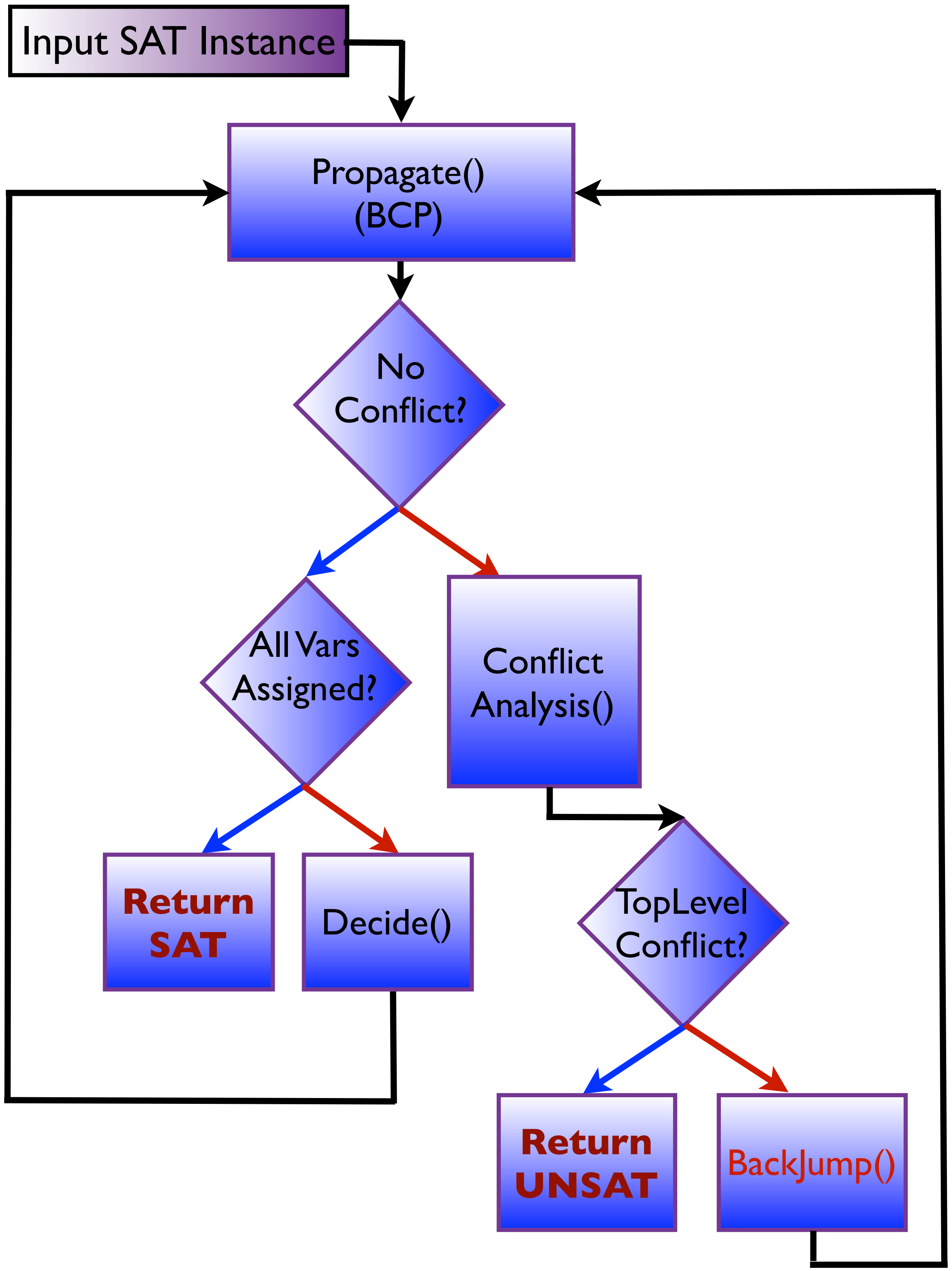

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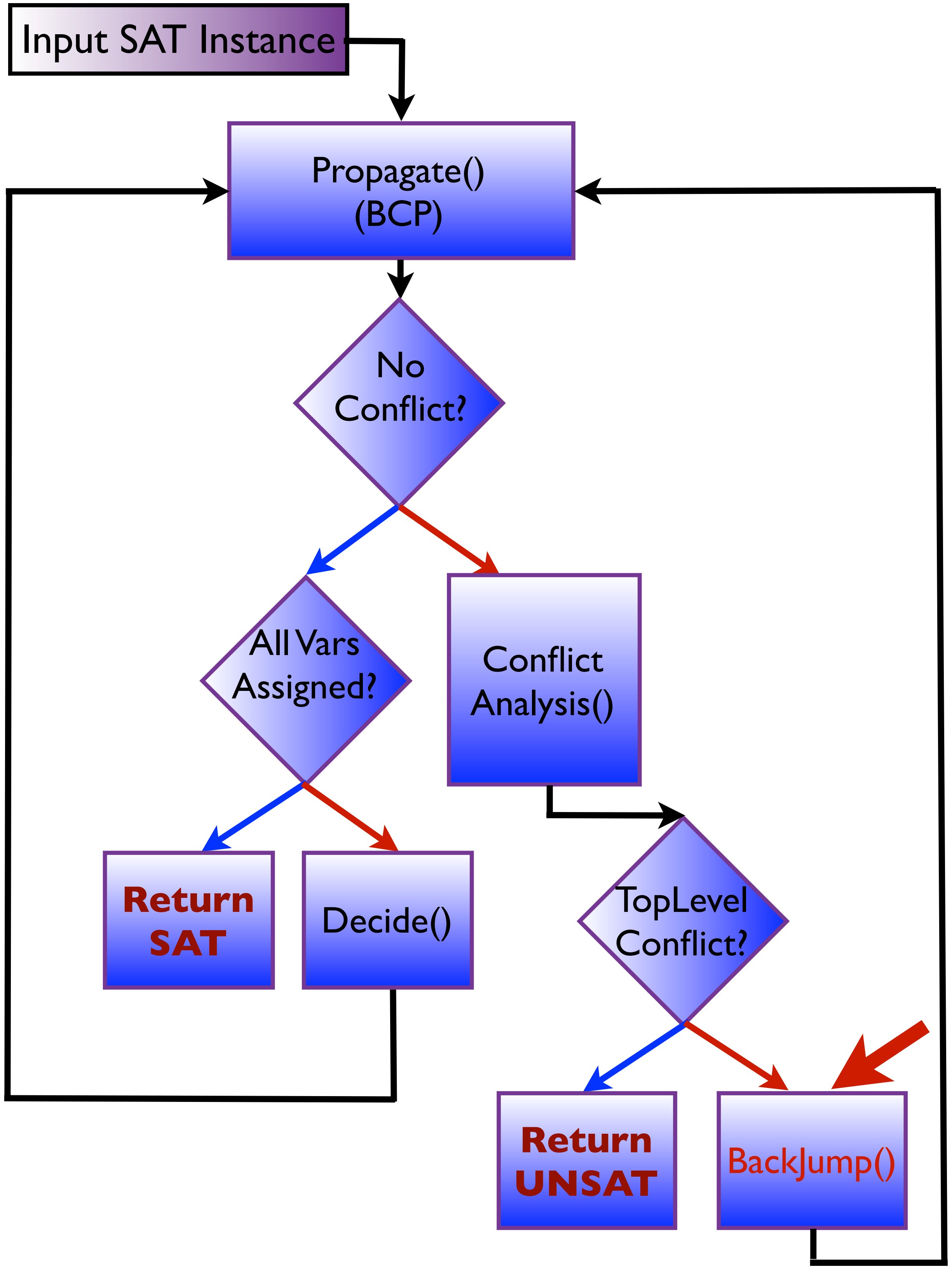
  - Marques-Silva & Sakallah (1996)
- Conflict-driven BackJump:
  - Undo the decision(s) that caused no-good assignment
  - Assign 'decision variables' different values
  - Go back several decision levels
  - Backjump: Marques-Silva, Sakallah (1999)
  - Backtrack: Davis, Putnam, Loveland, Logemann (1962)

Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()



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Modern CDCL SAT Solver Architecture

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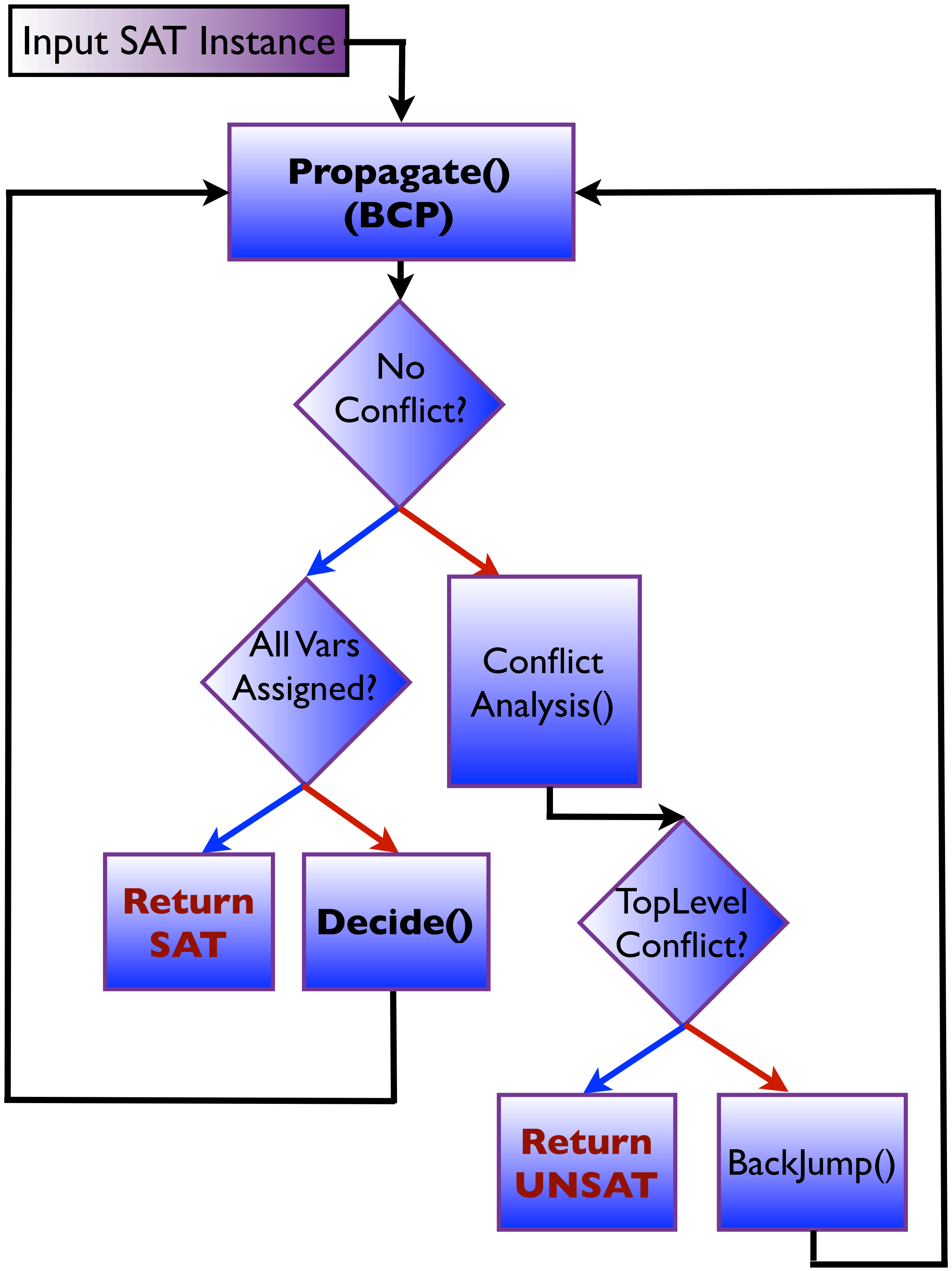
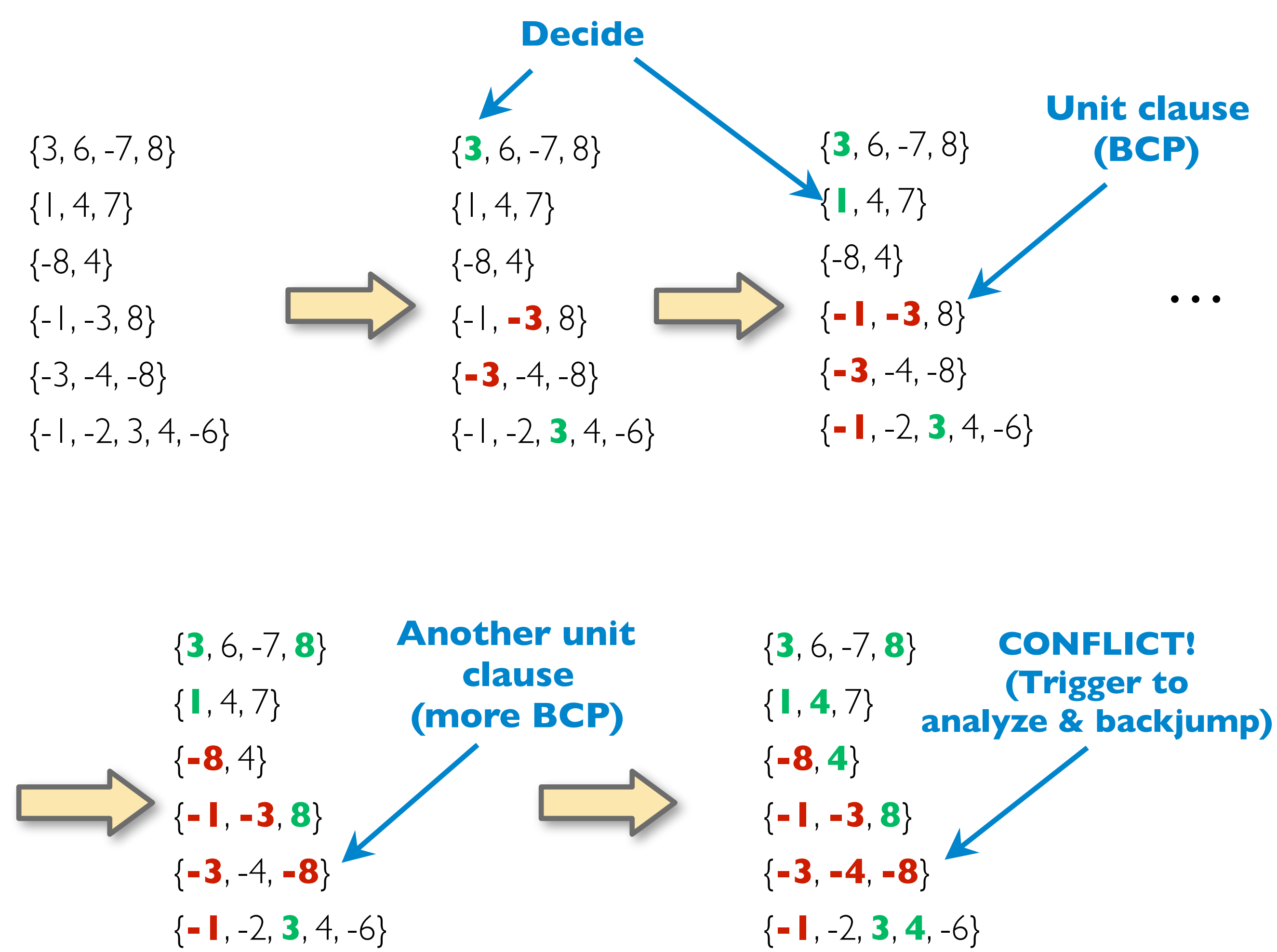
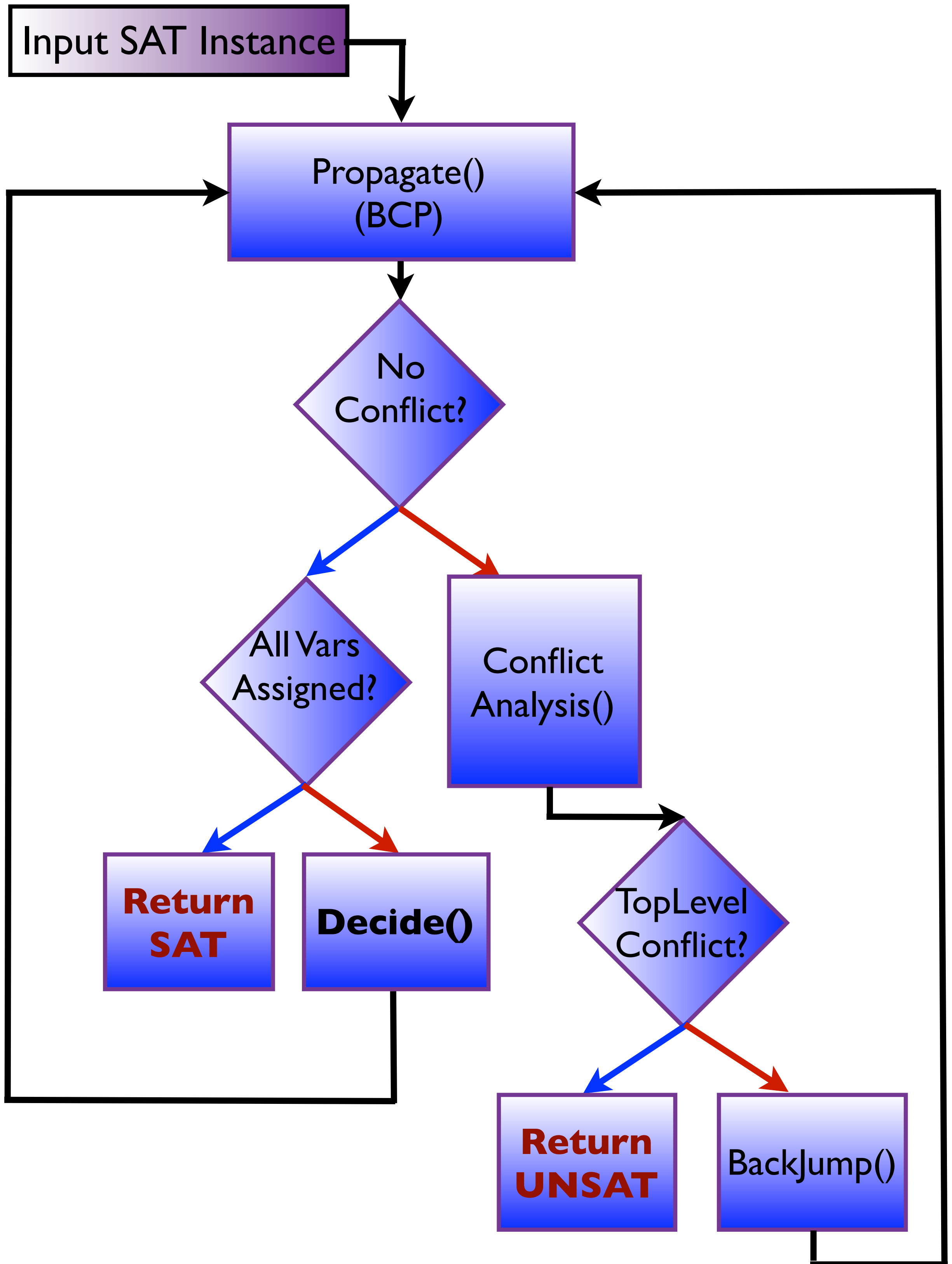




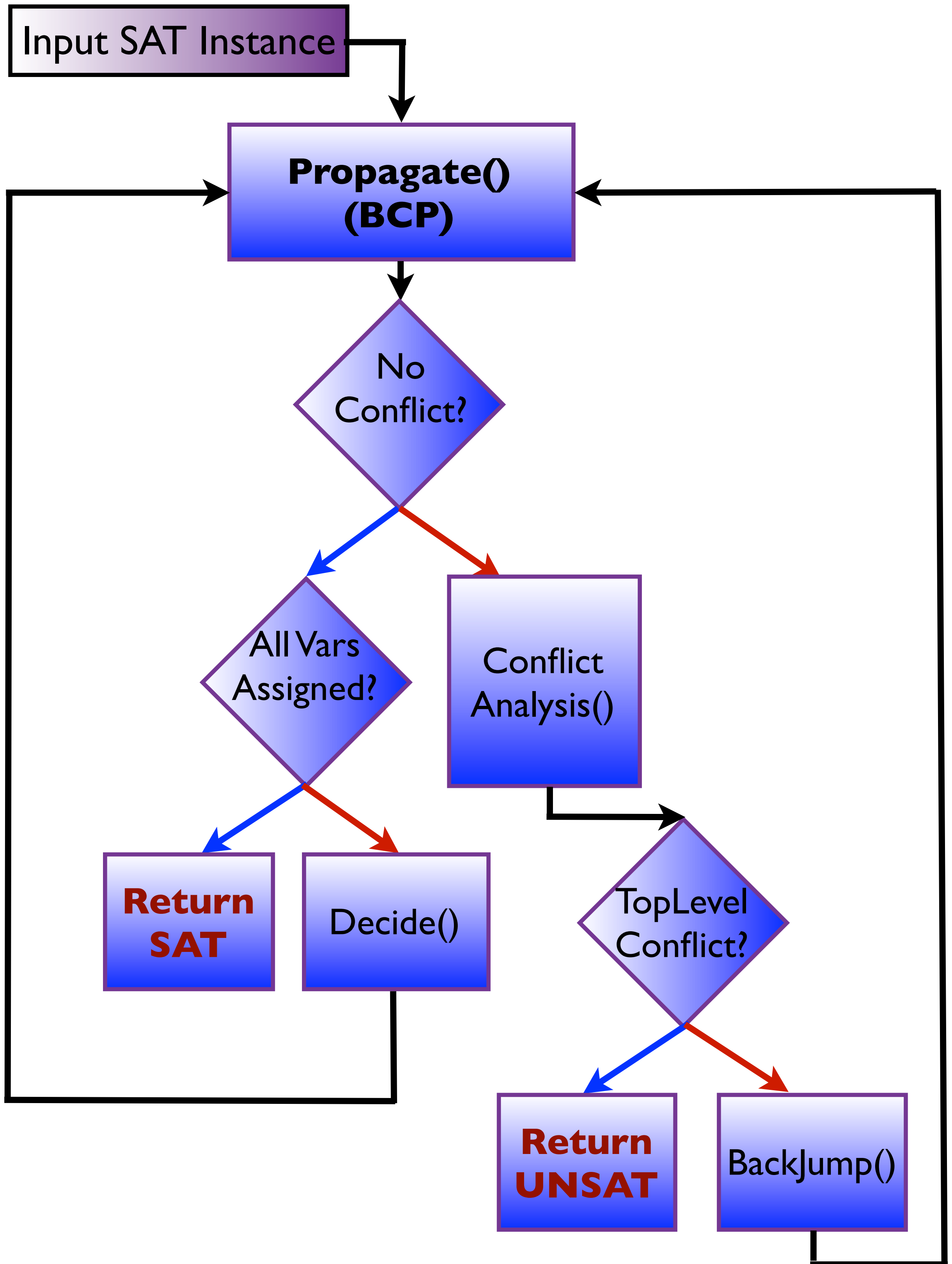
Modern CDCL SAT Solver Architecture
Decide() Details: VSIDS Heuristic



- Decide() or Branching():
  - Choose a variable & assign some value (decision)
  - Imposes dynamic variable order (Malik et al. 2001)
- How to choose a variable:
  - VSIDS heuristics
  - Each variable has an activity
  - Activity is bumped additively, if variable occurs in conflict clause
  - Activity of all variables is decayed by multiplying by const < 1
  - Next decision variable is the variable with highest activity
  - Over time, truly important variables get high activity
  - This is pure magic, and seems to work for many problems

Modern CDCL SAT Solver Architecture

Propagate() Details: Two-watched Literal Scheme



Decide

Unit clause (BCP)

{3, 6, -7, 8}	{**3**, 6, -7, 8}	{**3**, 6, -7, 8}
{1, 4, 7}	{1, 4, 7}	{**1**, 4, 7}
{-8, 4}	{-8, 4}	{-8, 4}
{-1, -3, 8}	{-1, **-3**, 8}	{**-1**, **-3**, 8}
{-3, -4, -8}	{**-3**, -4, -8}	{**-3**, -4, -8}
{-1, -2, 3, 4, -6}	{-1, -2, **3**, 4, -6}	{**-1**, -2, **3**, 4, -6}

…

Watched Literal	Watcher List
-1	{-1, -3, 8},...
-3	{-1, -3, 8},...
...	...

Watched Literal	Watcher List
-1	{-1, -3, 8},...
-3	...
8	{-1, -3, 8},...
...	...

Watched Literal	Watcher List
-1	...
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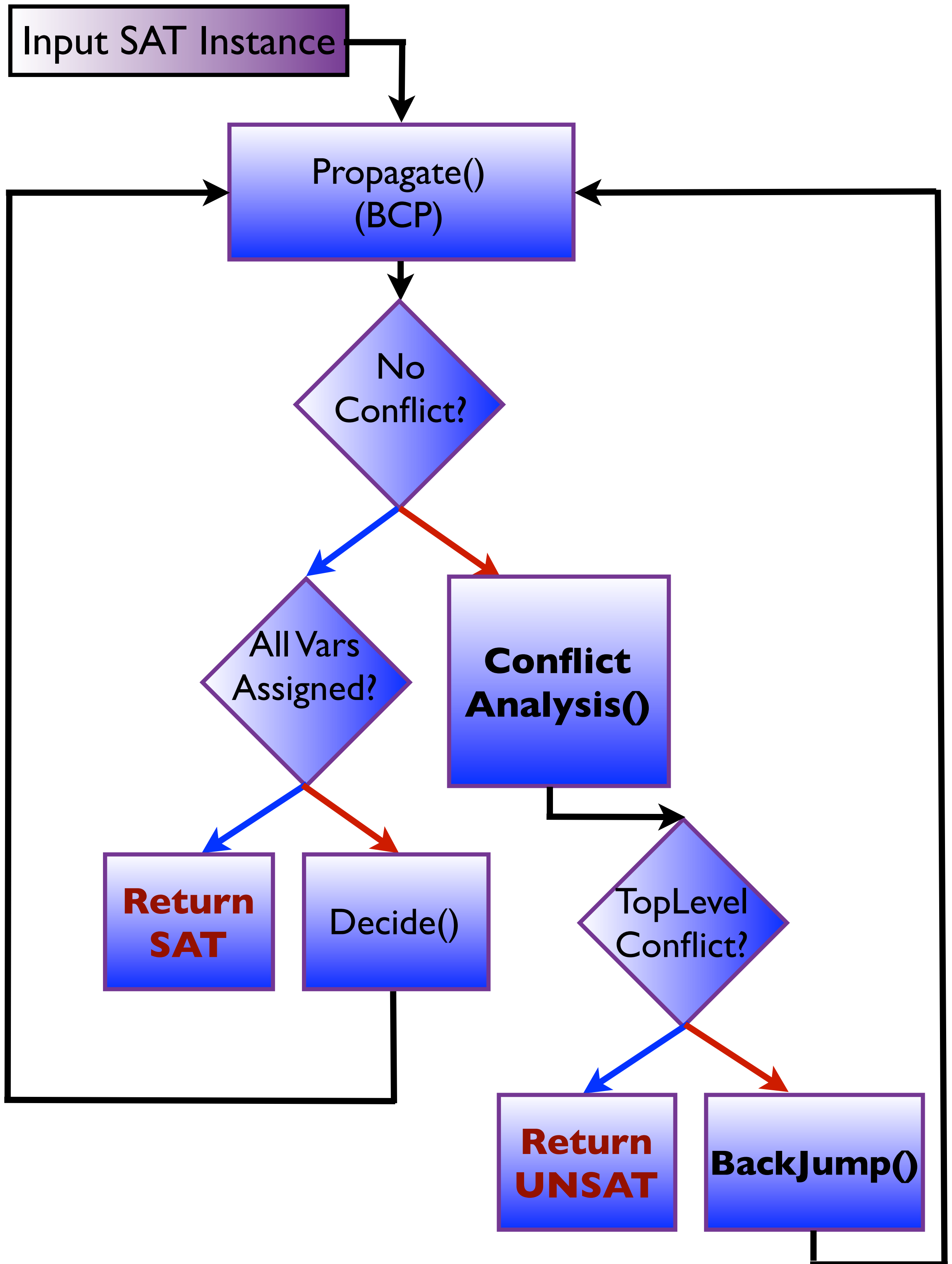
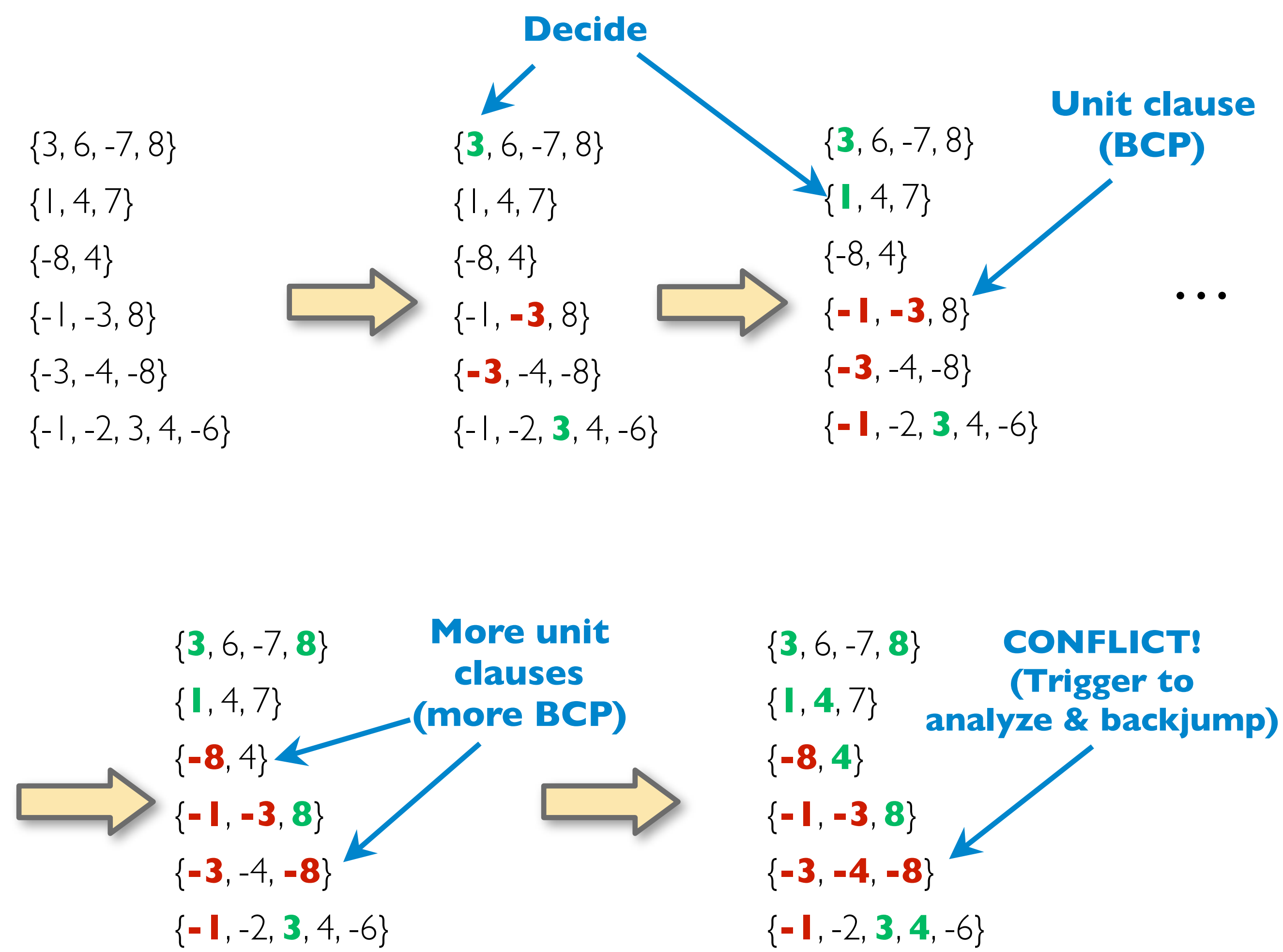
The constraint propagates 8

Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()

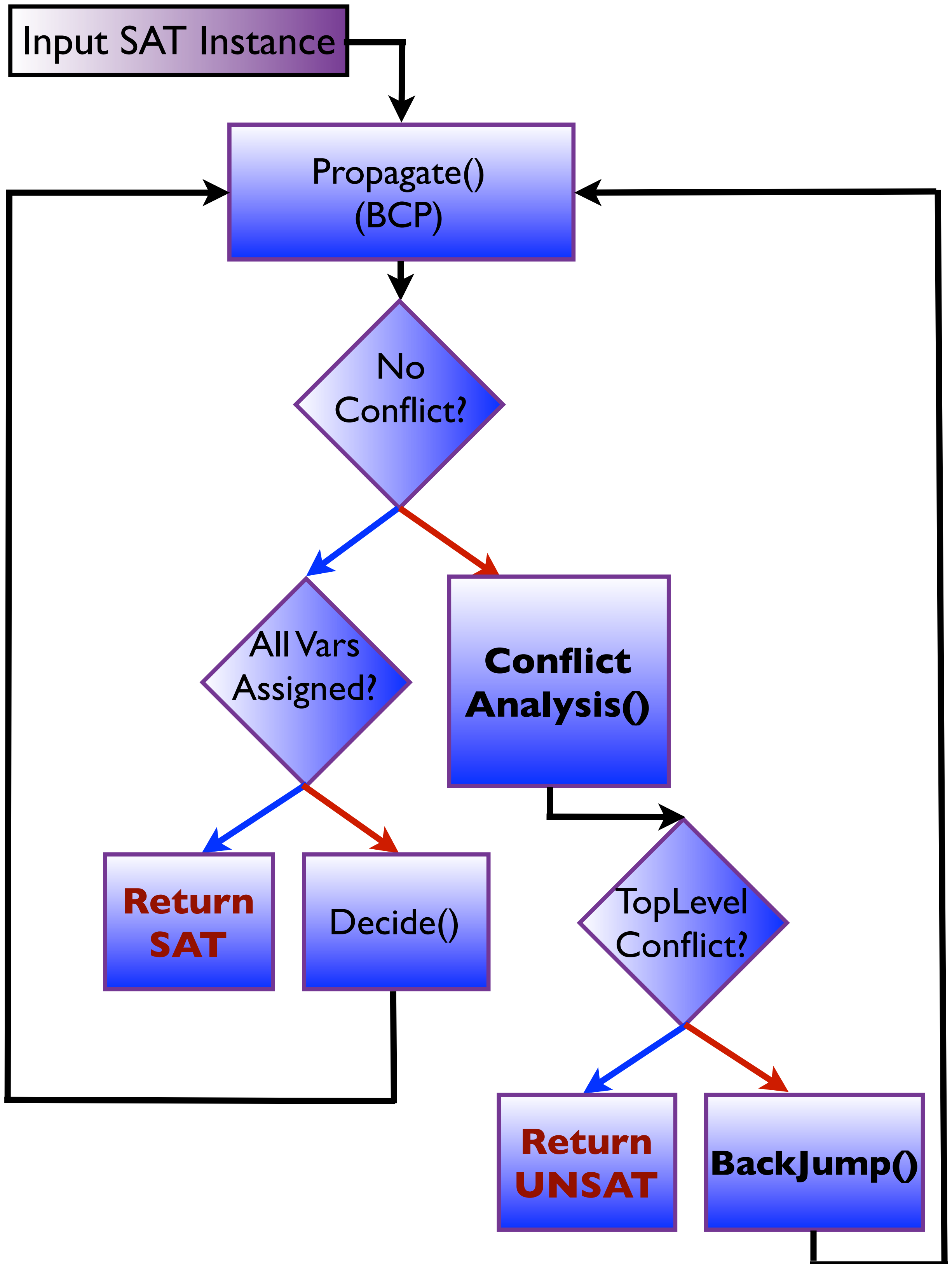




Basic Backtracking Search

- Flip the last decision 1
- Try setting 1 to False
- Highly inefficient
- No learning from mistakes

Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details



Some Definitions

- Decision Level (DL)
  - Map from Boolean variables in input to natural numbers
  - All unit clauses in input & resultant propagations get DL = 0
  - Every decision var gets a DL in increasing order >= 1
  - All propagations due to decision var at DL=x get the DL=x
- Conflict Graph (CG) or Implication Graph
  - Directed Graph that records decisions & propagations
  - Vertices: literals, Edge: unit clauses
- Conflict Clause (CC)
  - Clause returned by Conflict Analysis(), added to conflict DB
  - Implied by the input formula
  - A cut in the CG
  - Prunes the search
- Assignment Trail (AT)
  - A stack of partial assignment to literals, with DL info

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, \ldots\}$

Current decision: $\{X_1 = 1@6\}$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

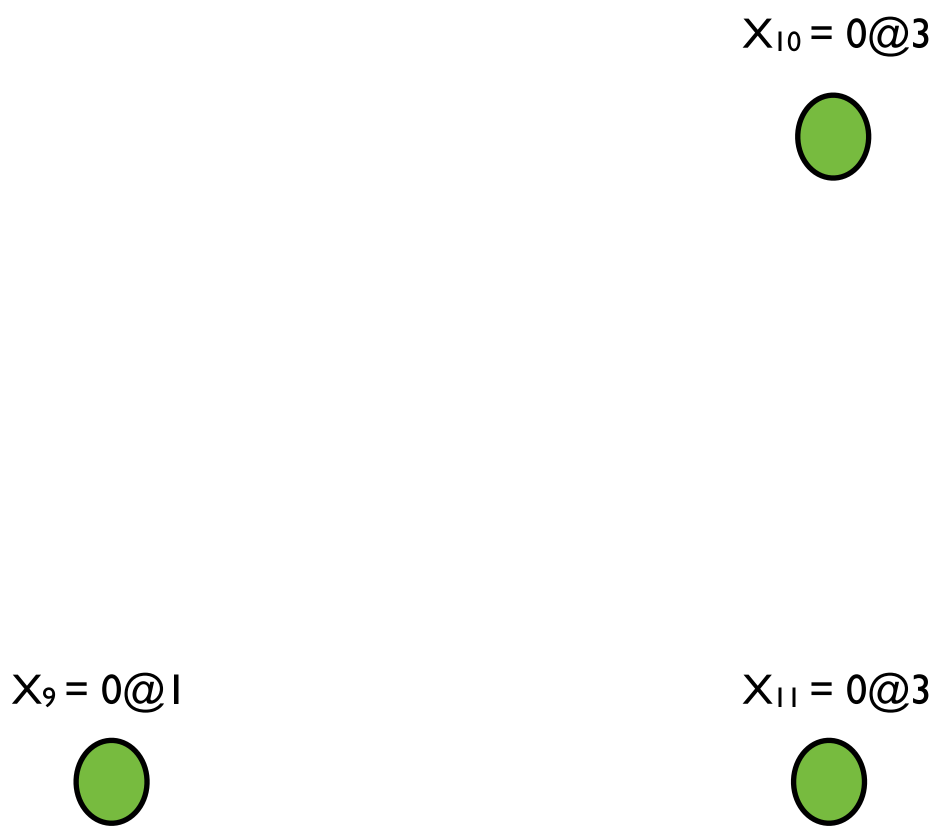
$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

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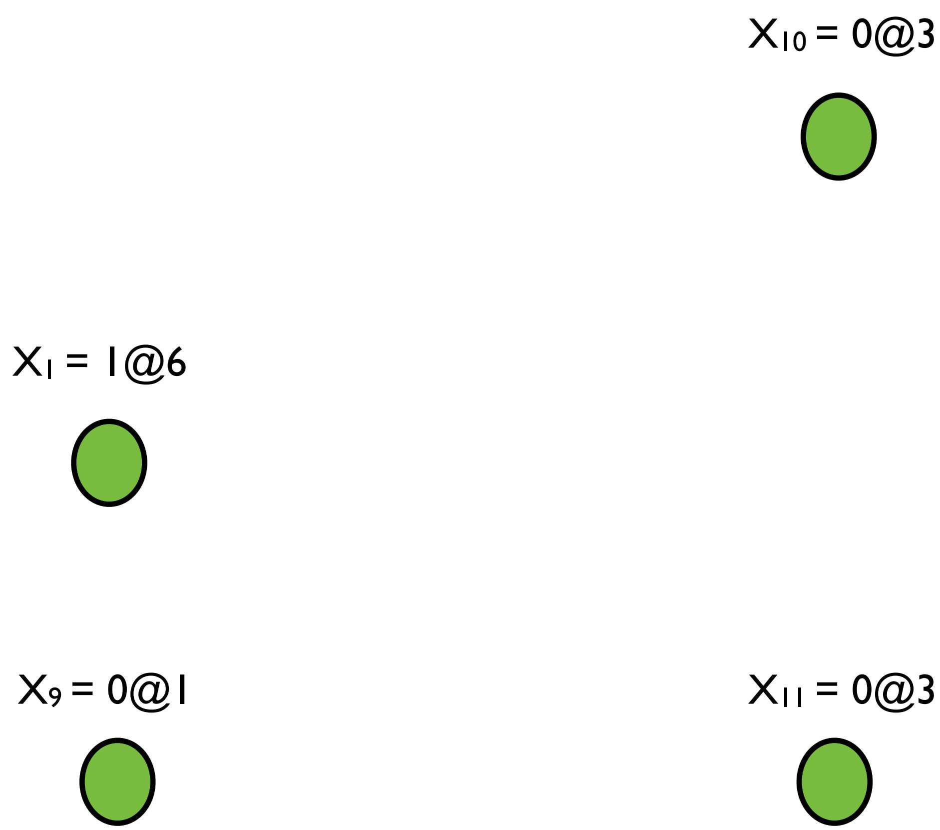
$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



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$W_4 = (\neg X_4 + X_5 + X_{10})$

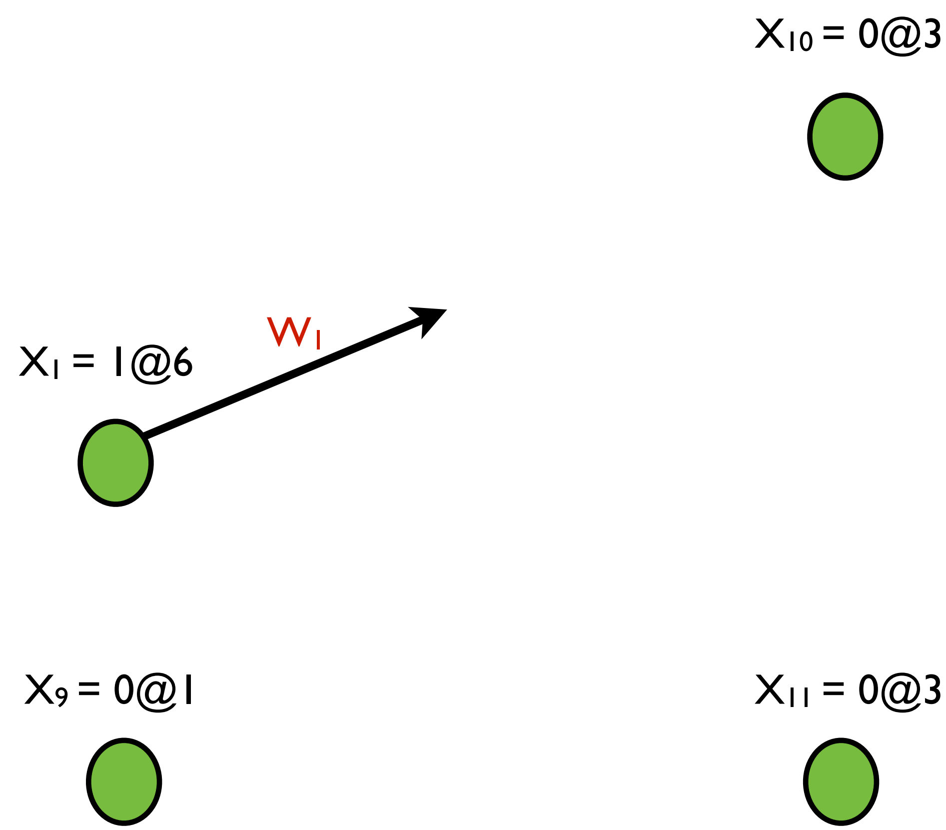
$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, ...\}$

Current decision: $\{X_1 = 1@6\}$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

$W_5 = (\neg X_4 + X_6 + X_{11})$

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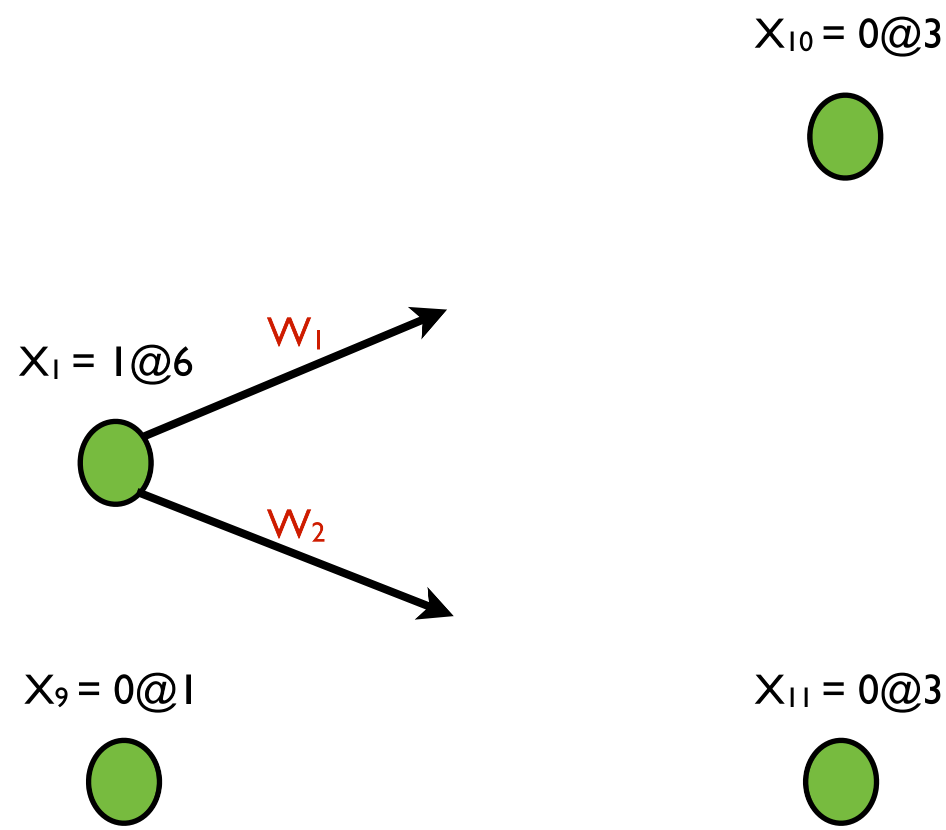

Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details: Implication Graph

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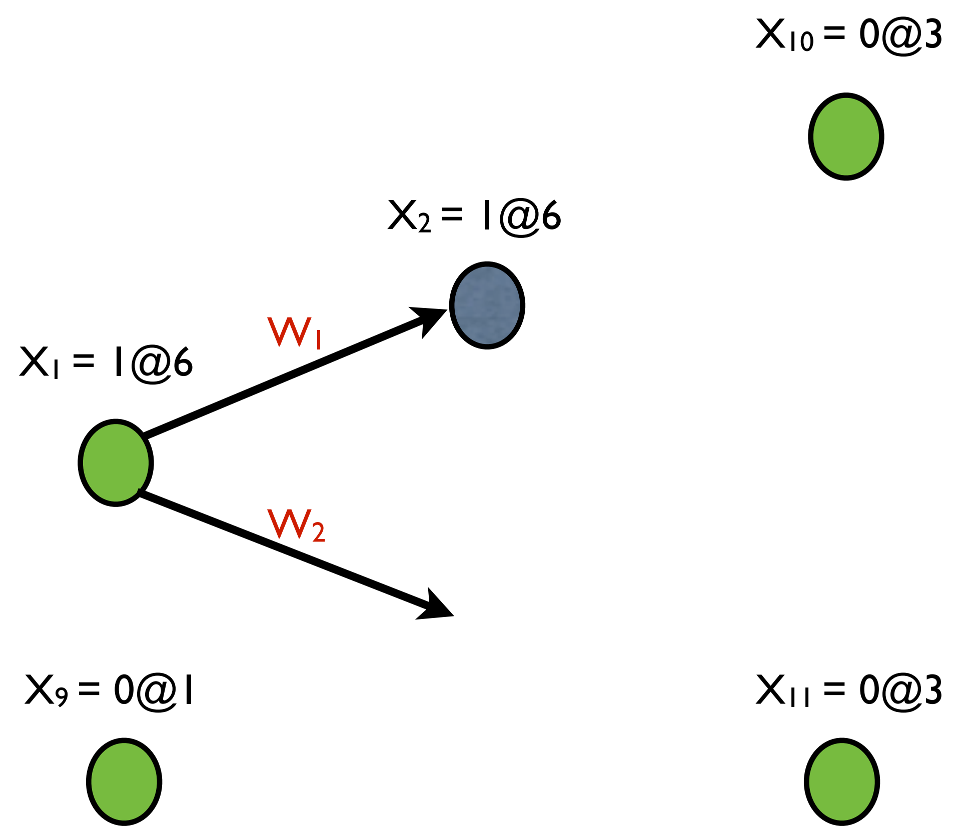

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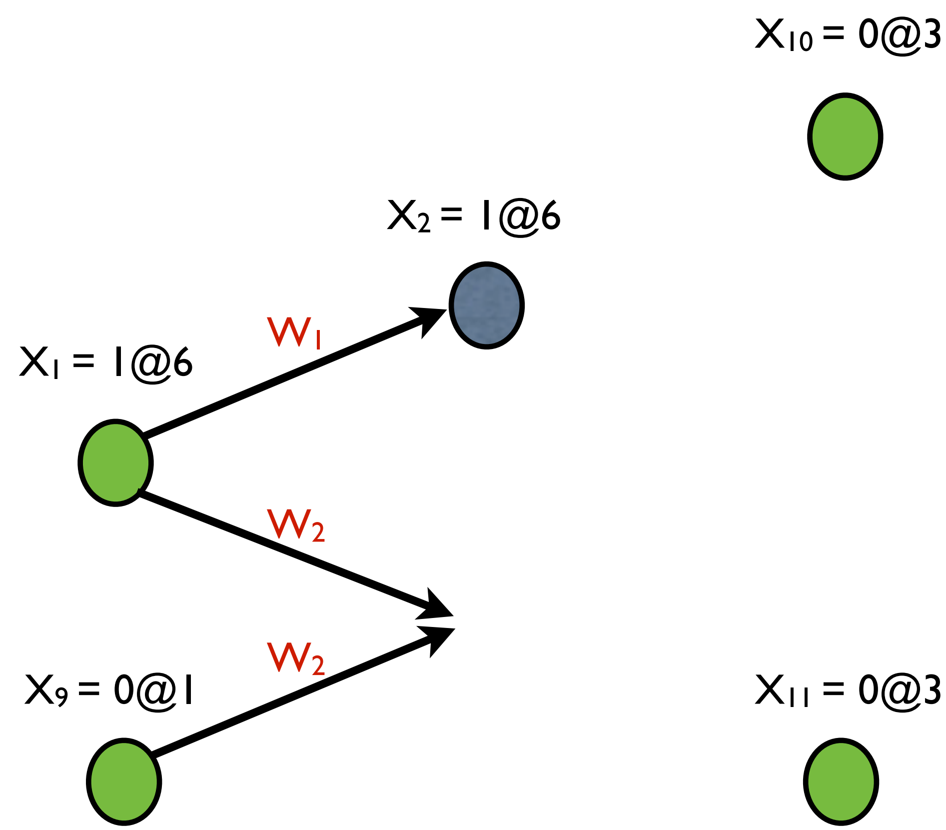

Modern CDCL SAT Solver Architecture
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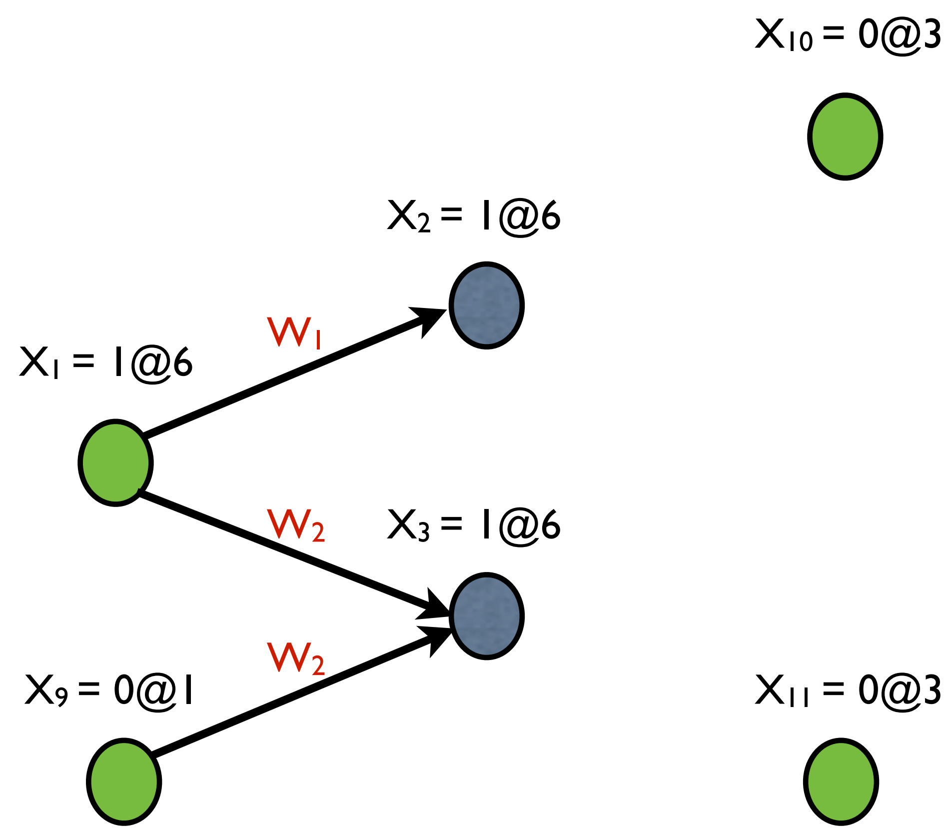

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

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Current decision: $\{X_1 = 1@6\}$

Clause DB

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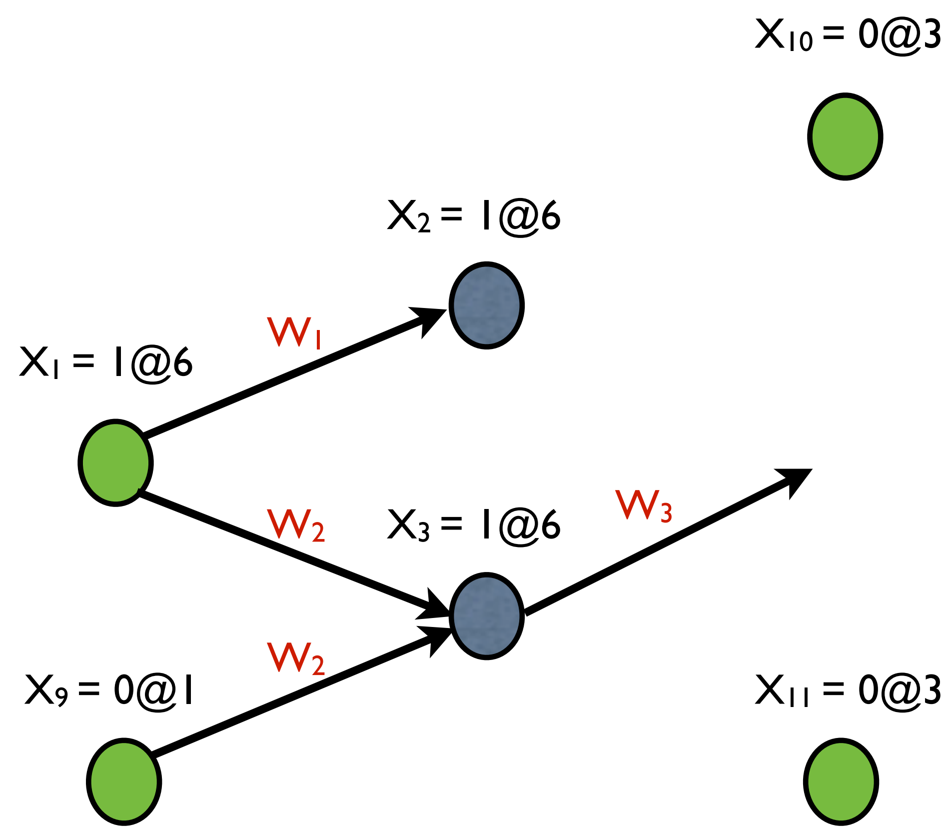

Wednesday, 9 January, 13

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, ...\}$

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Clause DB

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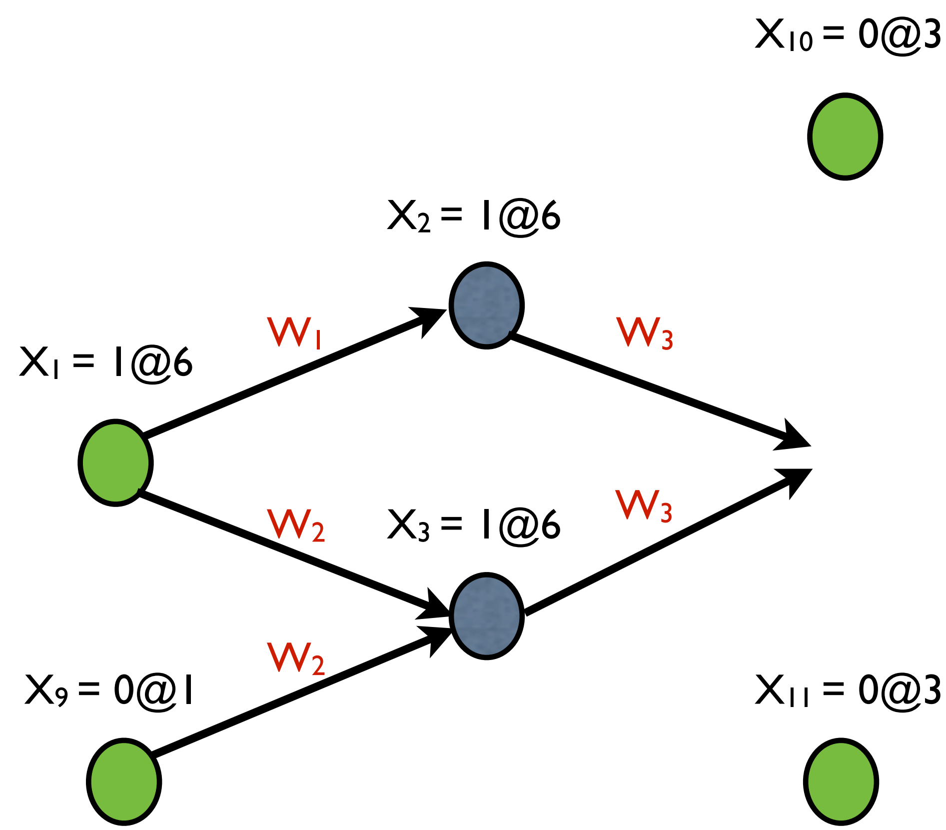
$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details: Implication Graph

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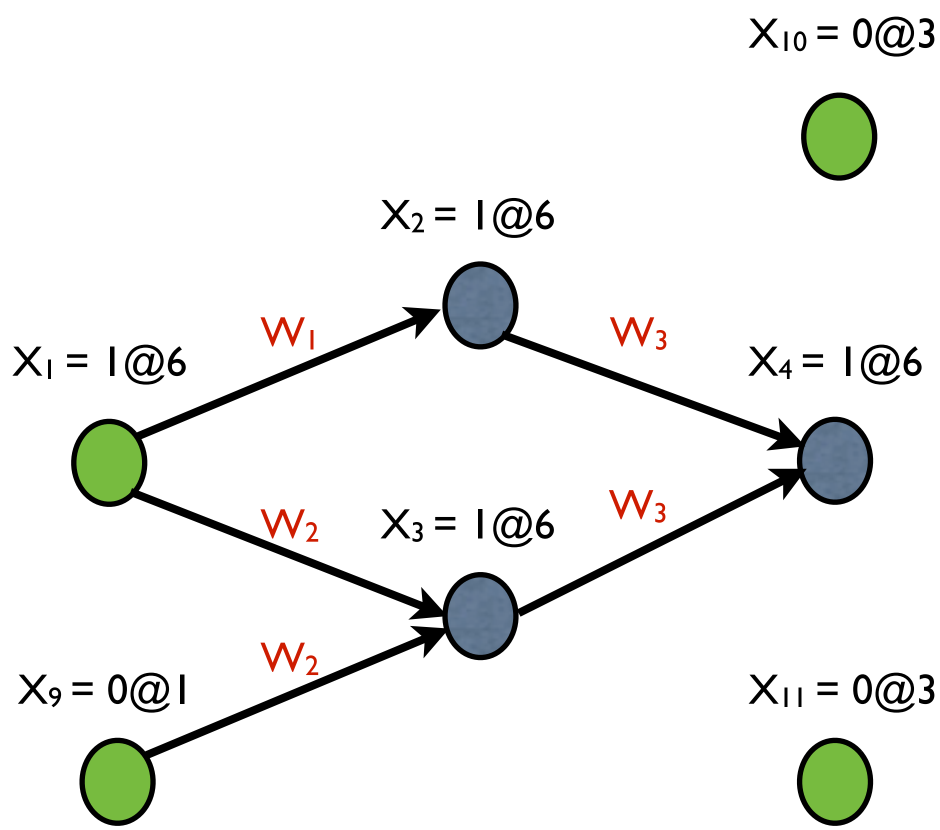

Modern CDCL SAT Solver Architecture

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Clause DB

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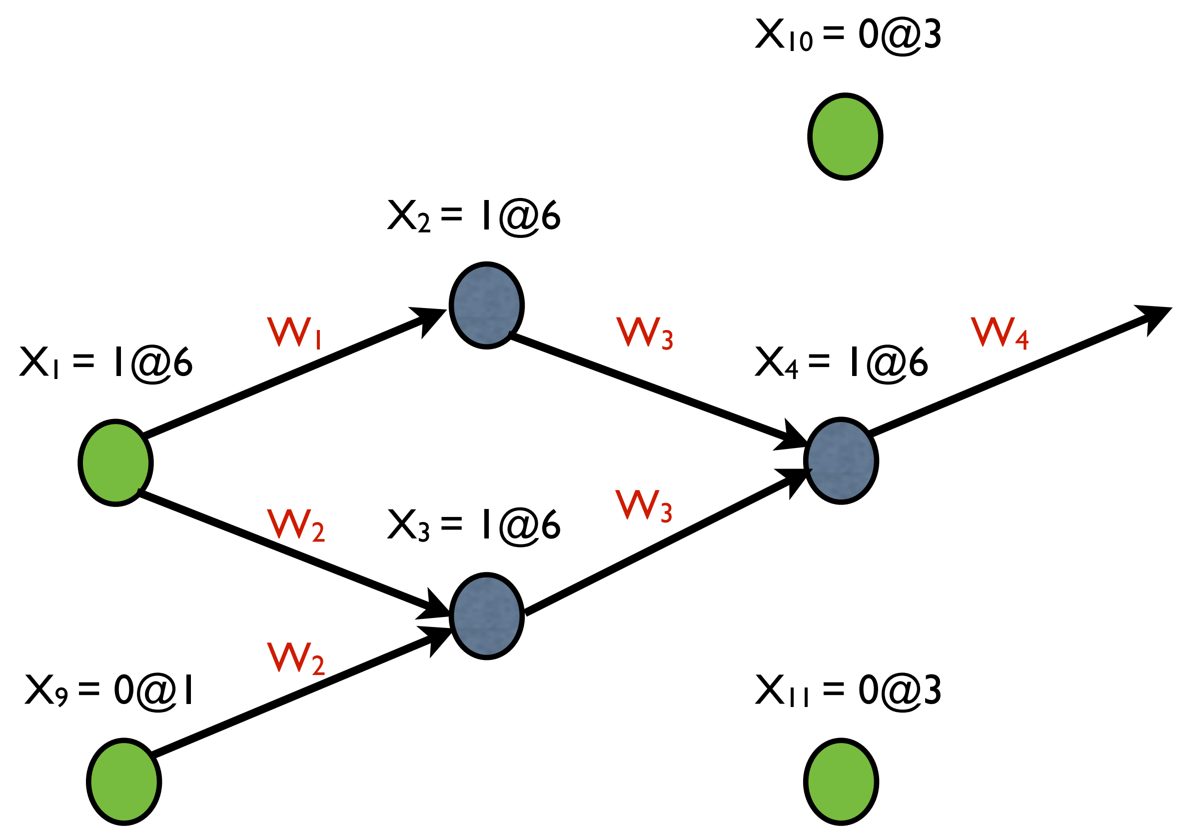

Modern CDCL SAT Solver Architecture
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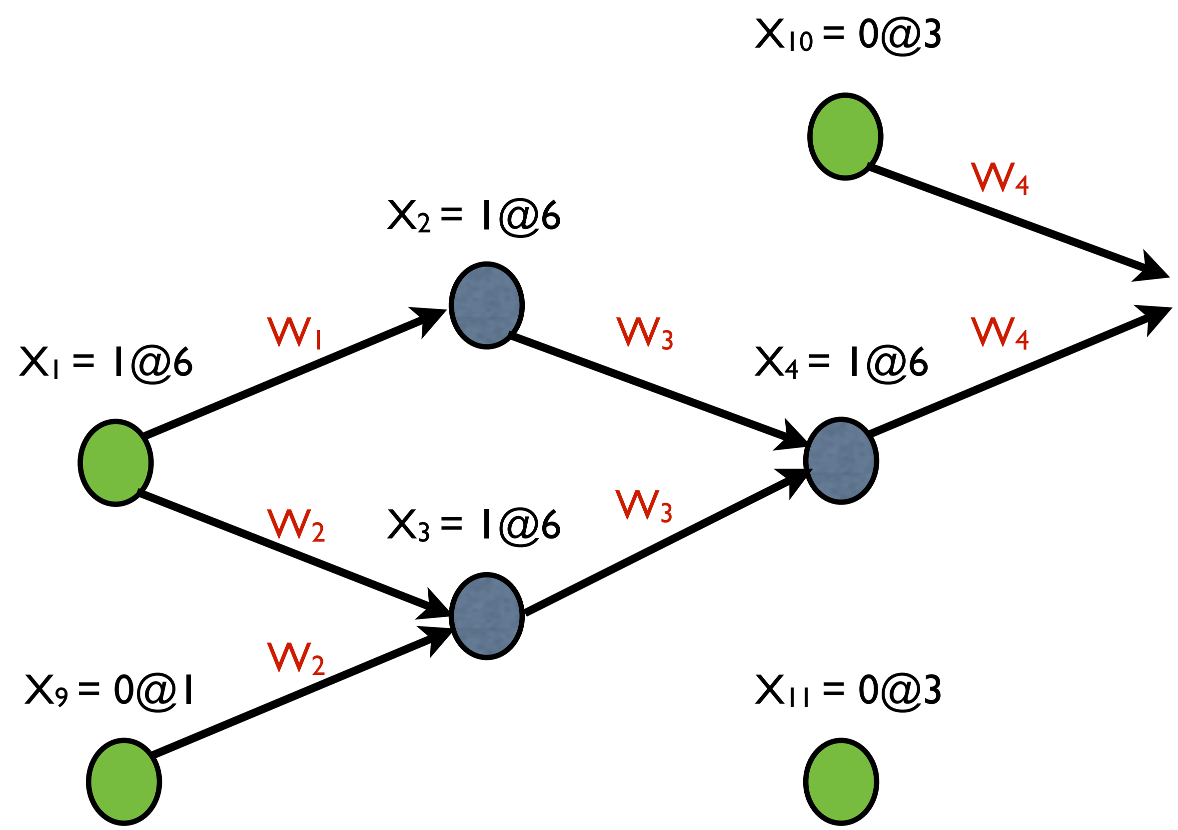

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

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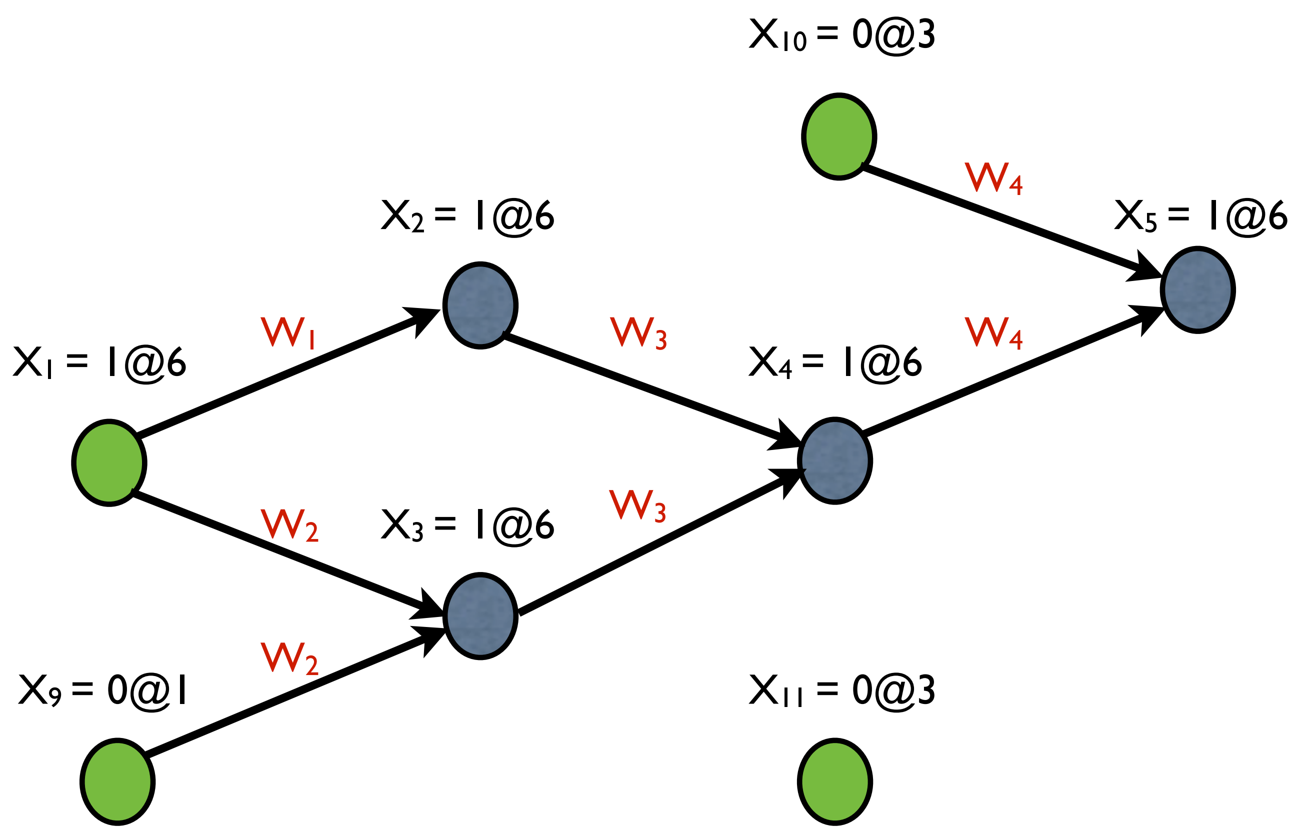

Modern CDCL SAT Solver Architecture

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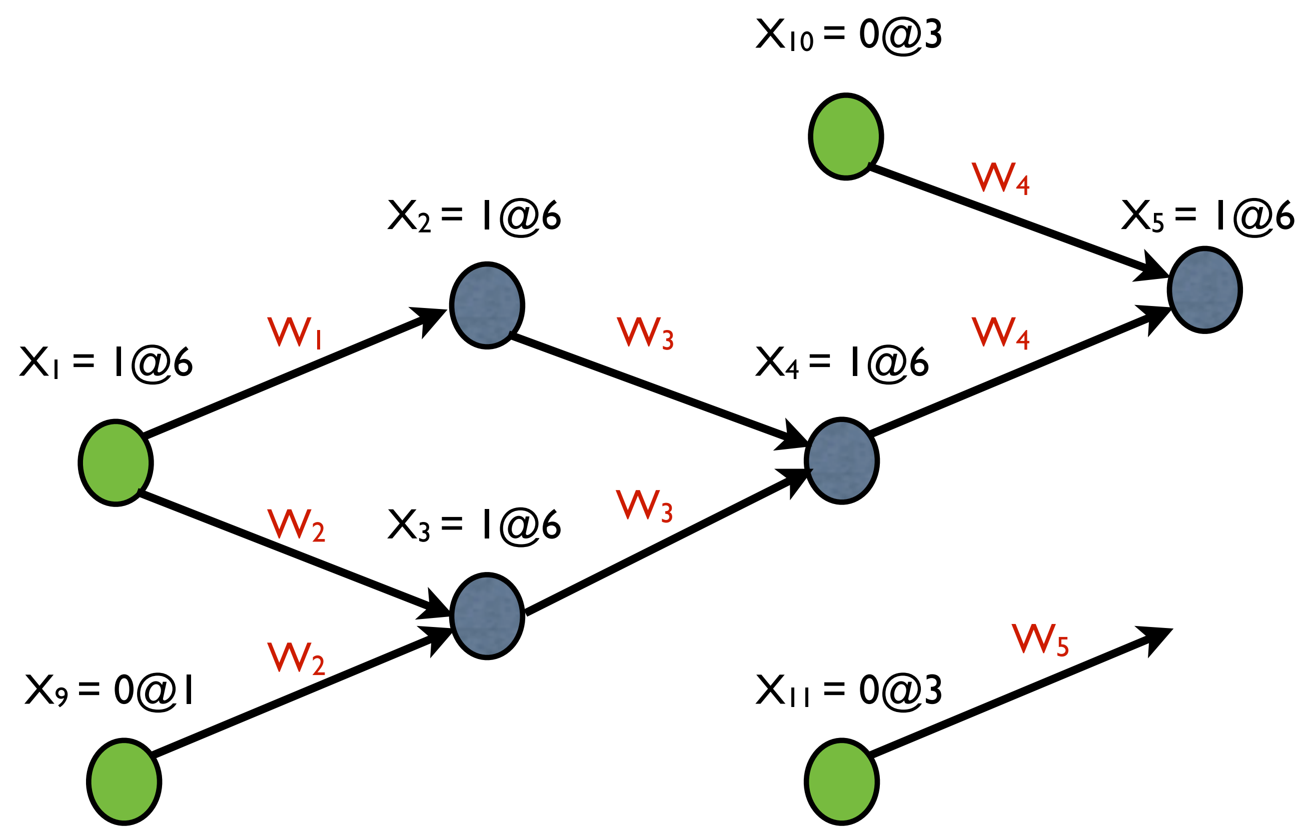

Modern CDCL SAT Solver Architecture

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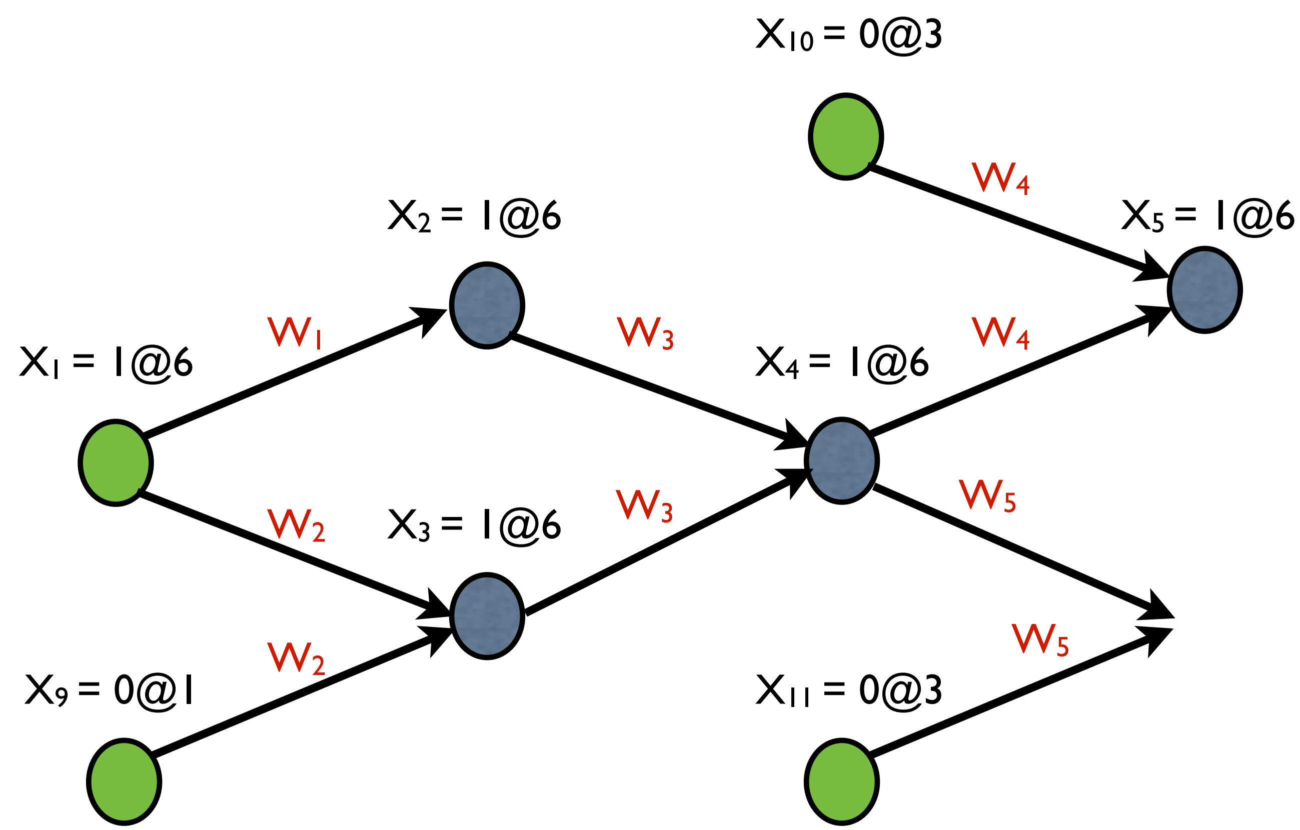

Modern CDCL SAT Solver Architecture

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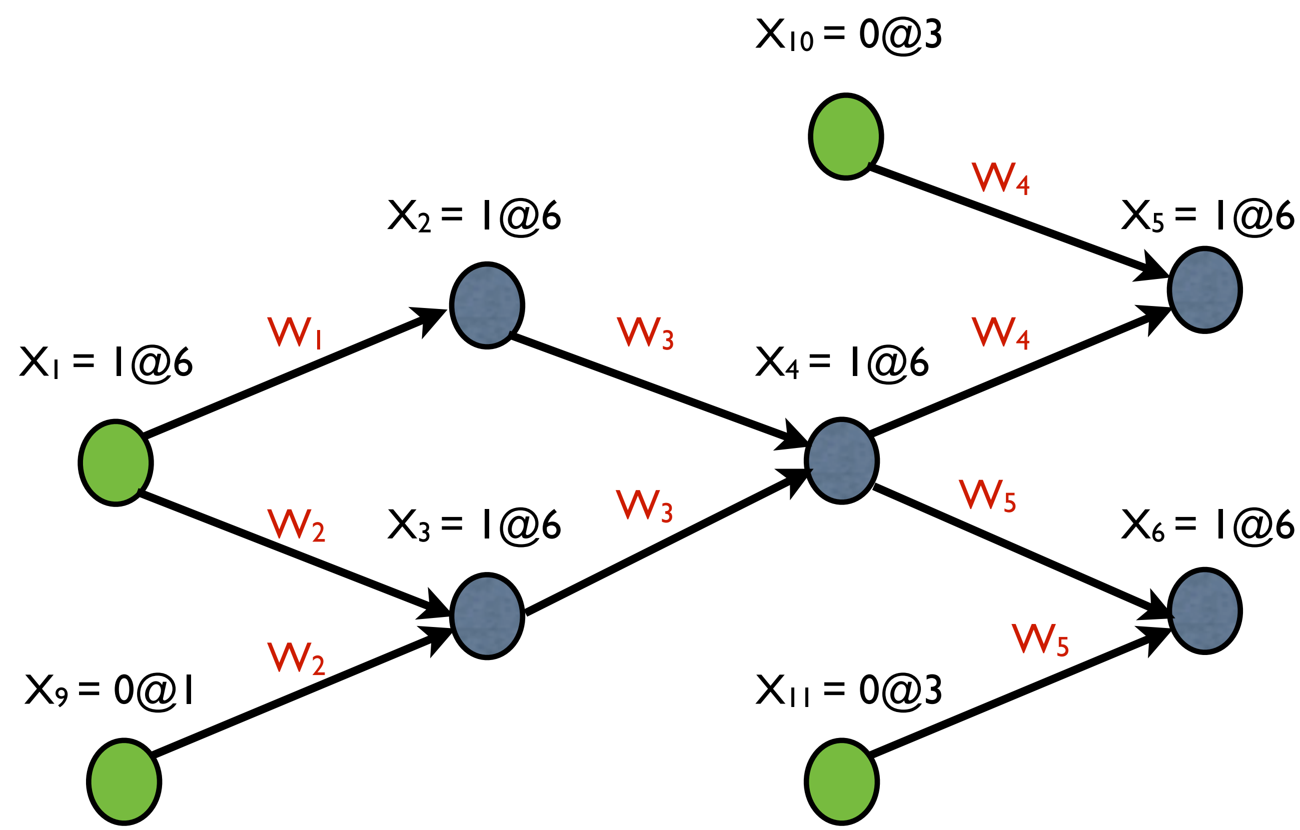

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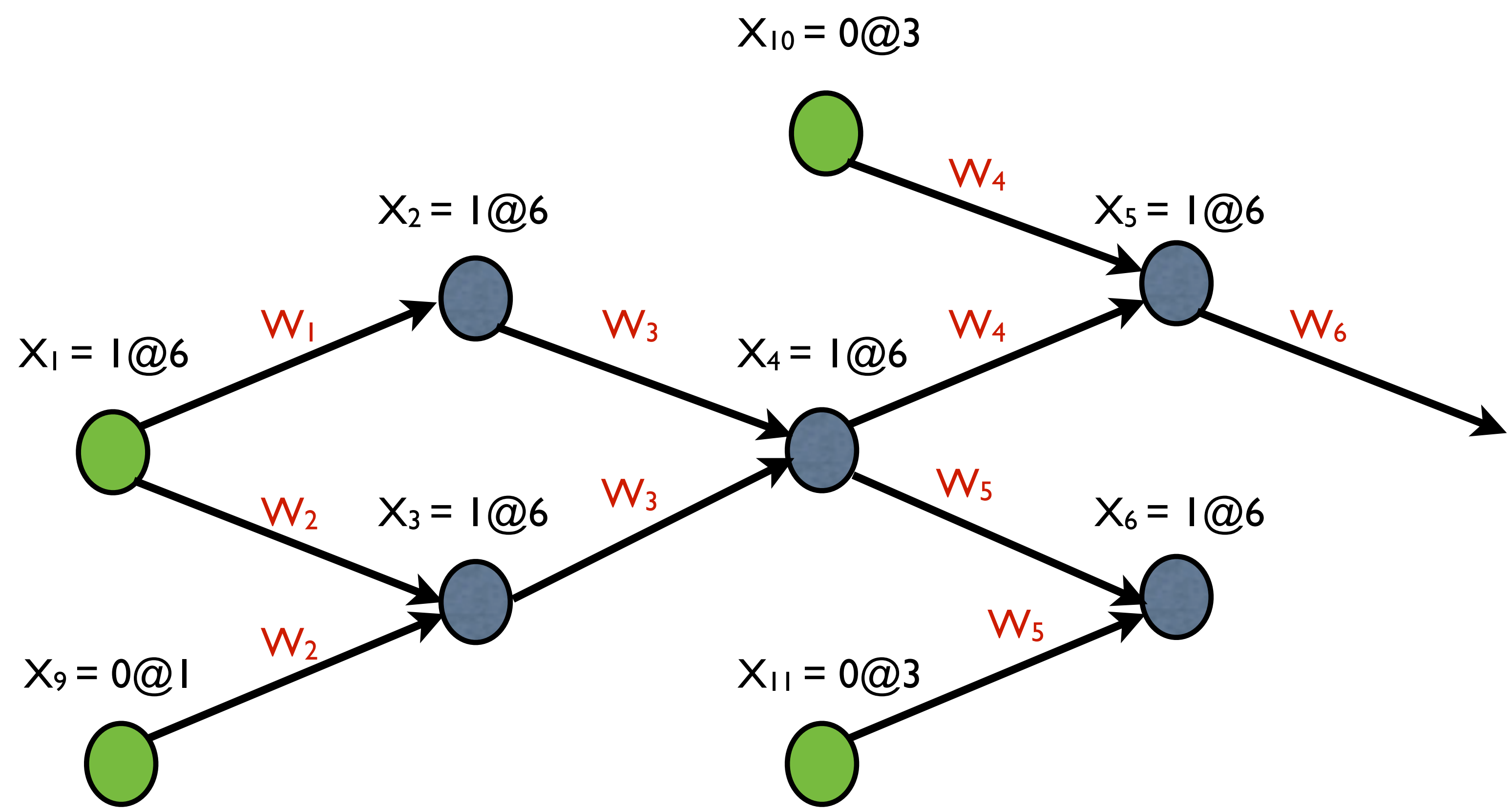

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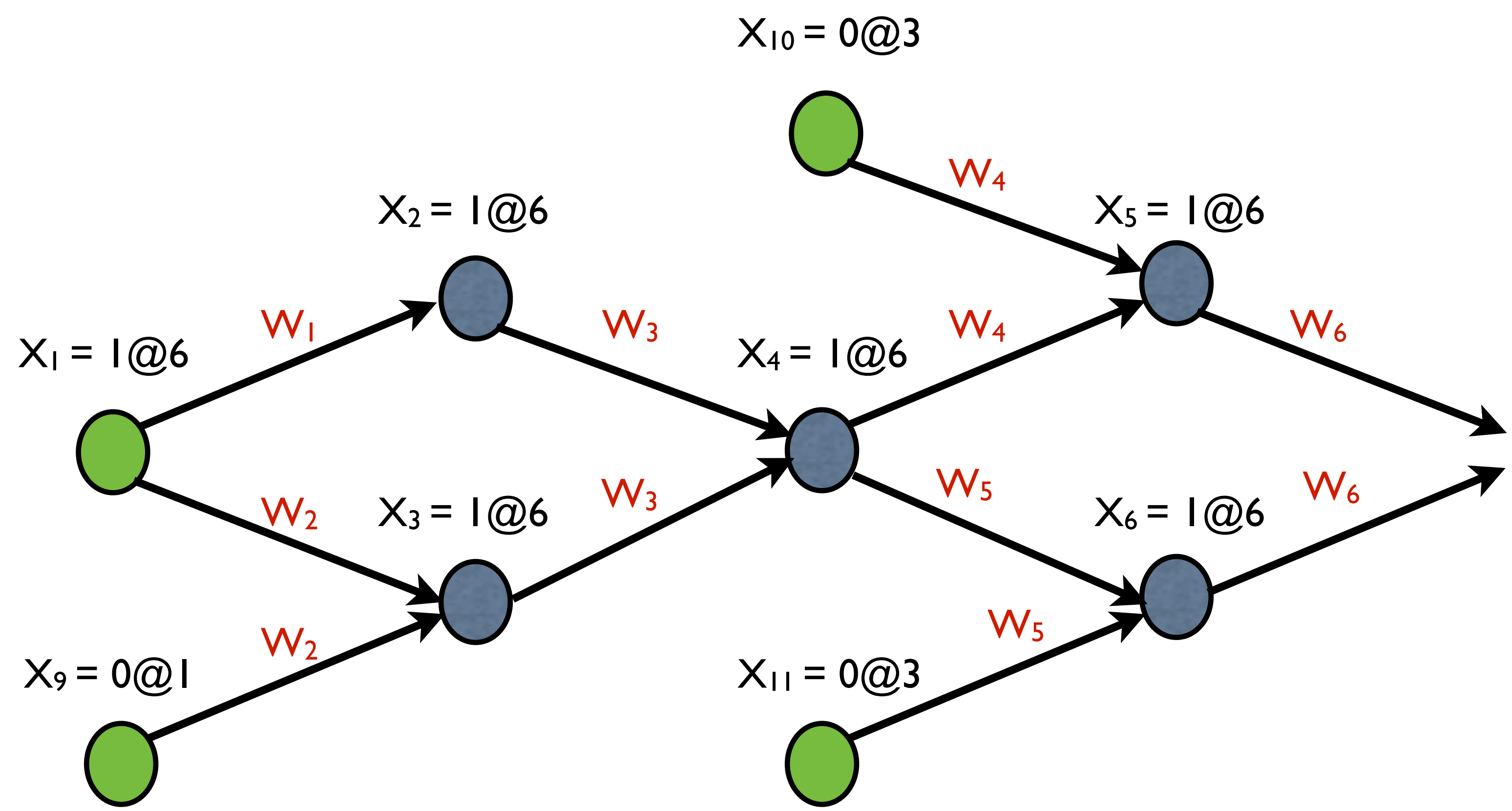

Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details: Implication Graph

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Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

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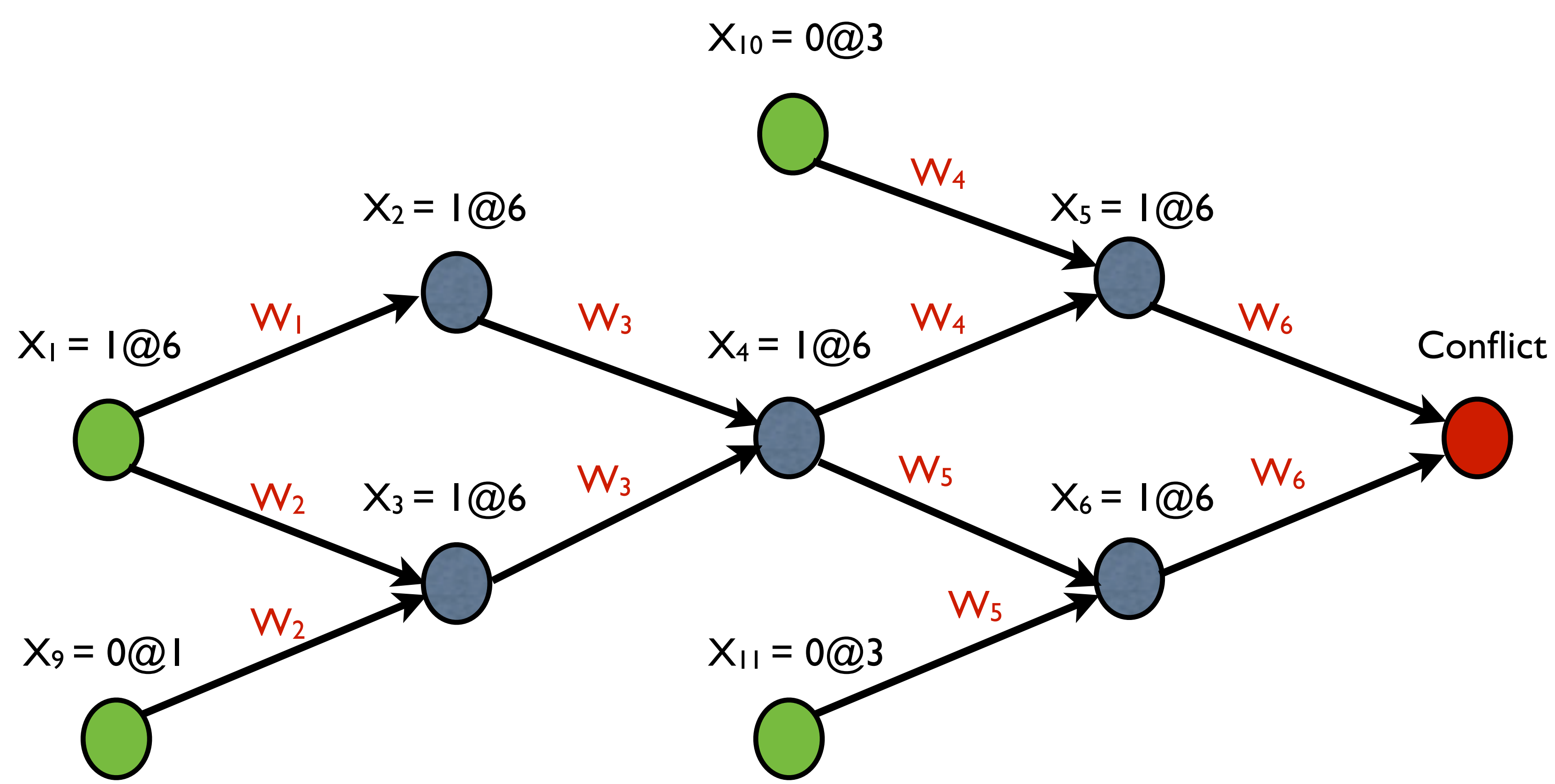

Modern CDCL SAT Solver Architecture

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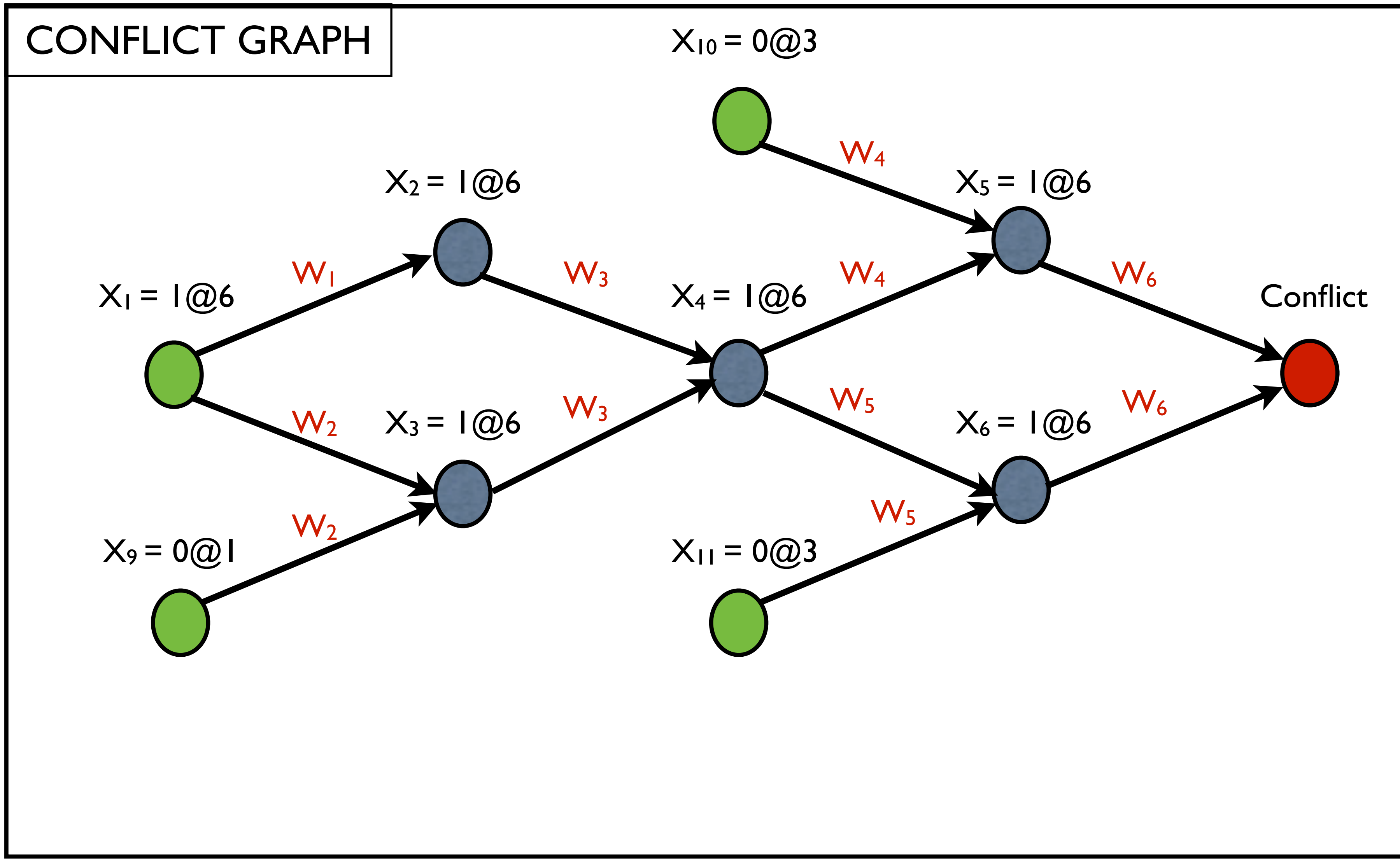

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Conflict Clause

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, \ldots\}$

Current Decision: $\{X_1 = 1@6\}$

Simplest strategy is to traverse the conflict graph backwards until decision variables: conflict clause includes only decision variables $(\neg X_1 + X_9 + X_{10} + X_{11})$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

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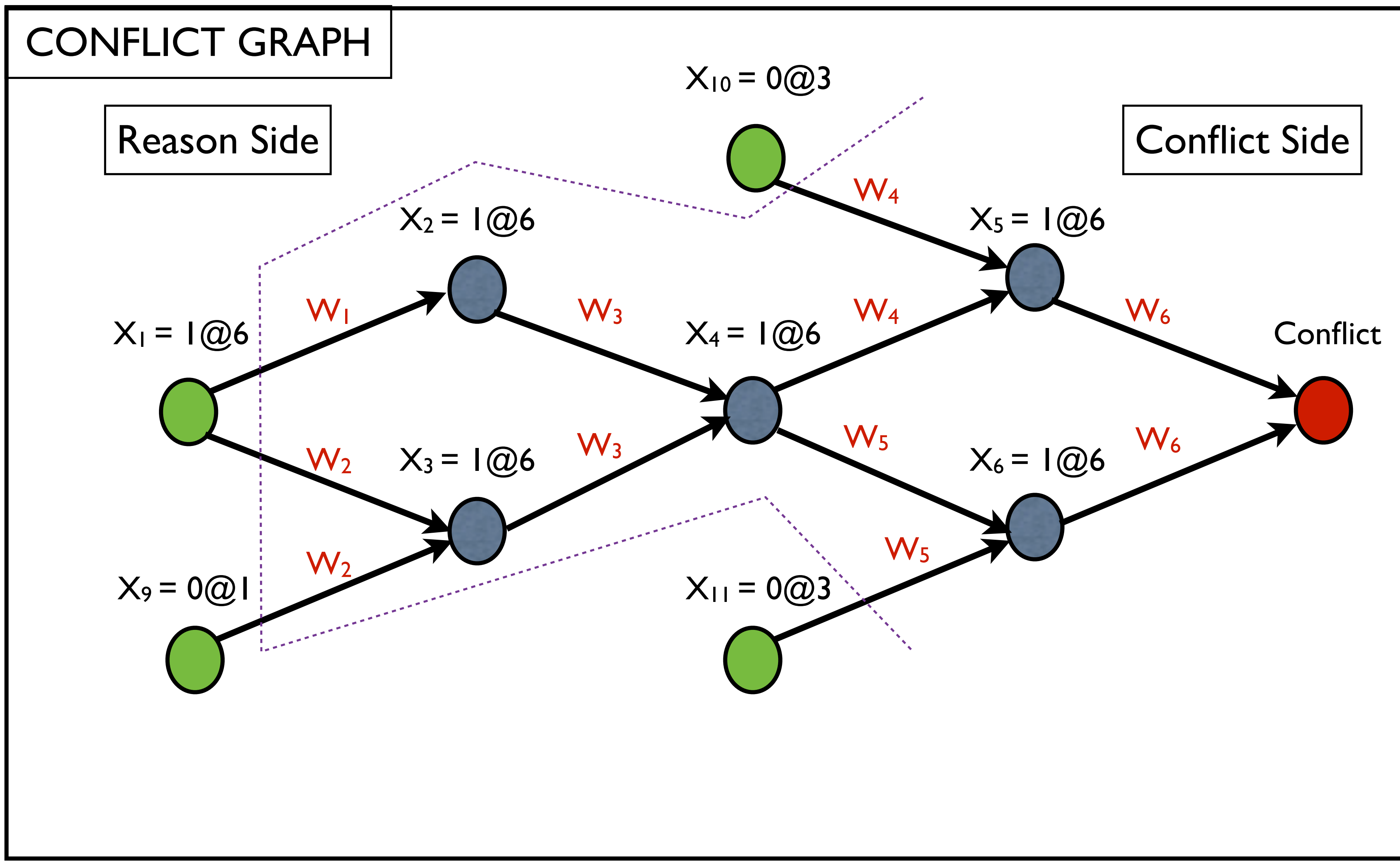

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Conflict Clause

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, \ldots\}$

Current Decision: $\{X_1 = 1@6\}$

Another strategy is to use First Unique Implicant Point (UIP):
Traverse graph backwards in breadth-first, expand literals of conflict, stop at first UIP

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

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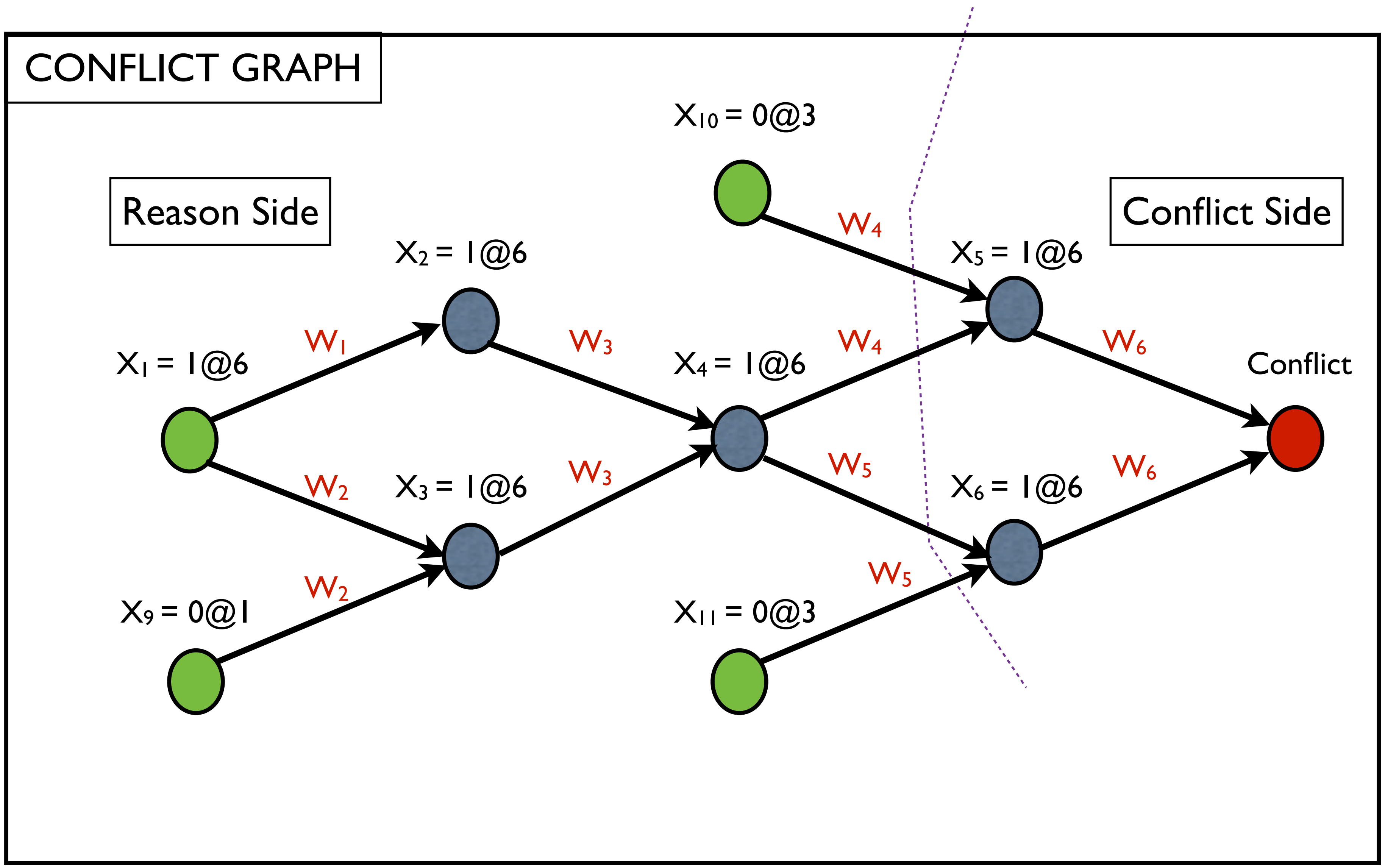

Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details: BackTrack

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, ...\}$

Current decision: $\{X_1 = 1@6\}$

Strategy: Closest decision level (DL) $\leq$ current DL for which conflict clause is unit. Undo $\{X_1 = 1@6\}$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

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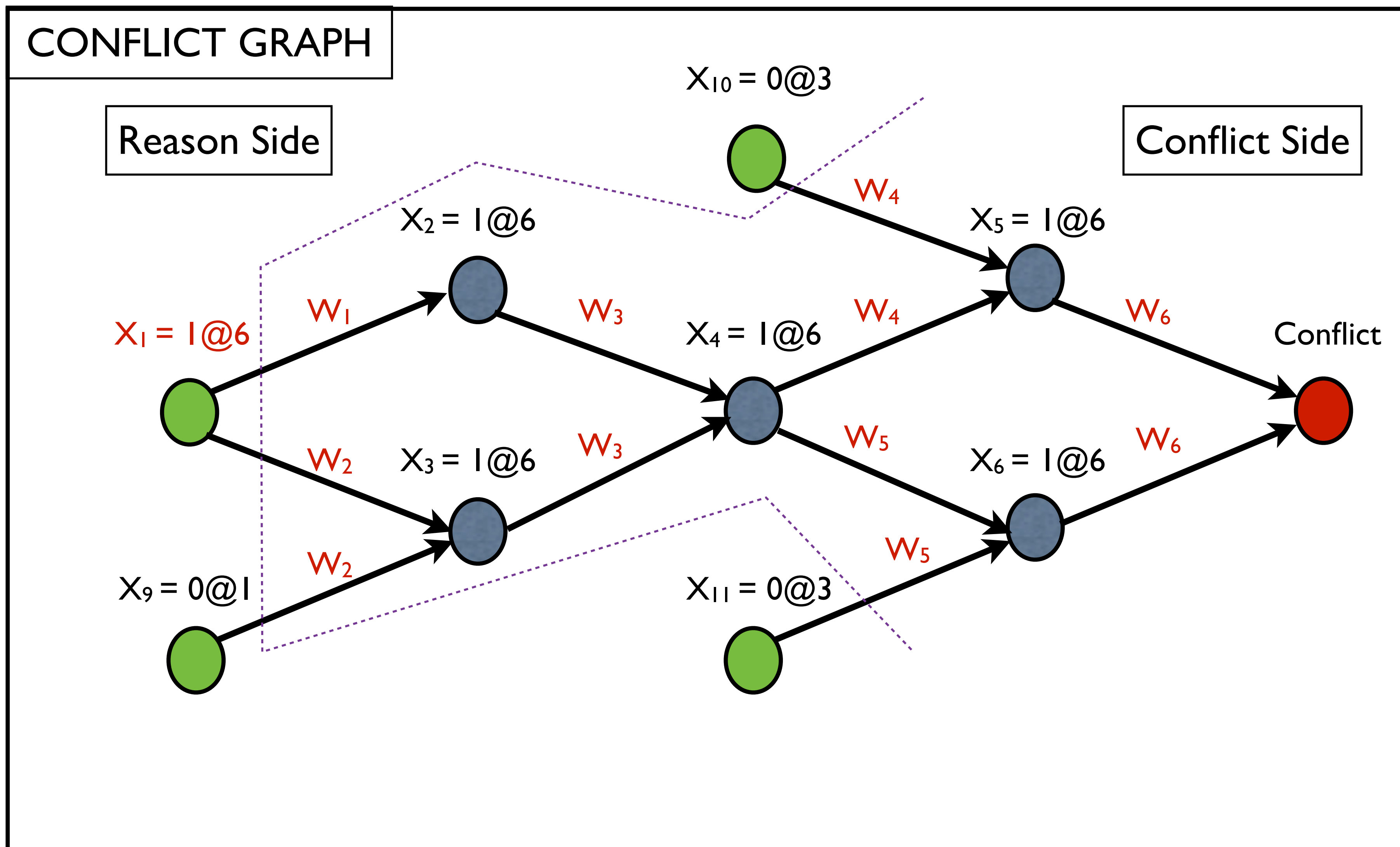


Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: BackJump

$\neg X_1$ was implied literal, leading to another conflict described below

Conflict clause: $(X_9 + X_{10} + X_{11} + \neg X_{12} + \neg X_{13})$

BackJump strategy: Closest decision level (DL) $\leq$ current DL for which conflict clause is unit. Undo $\{X_{10} = 0@3\}$

CLAUSE DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

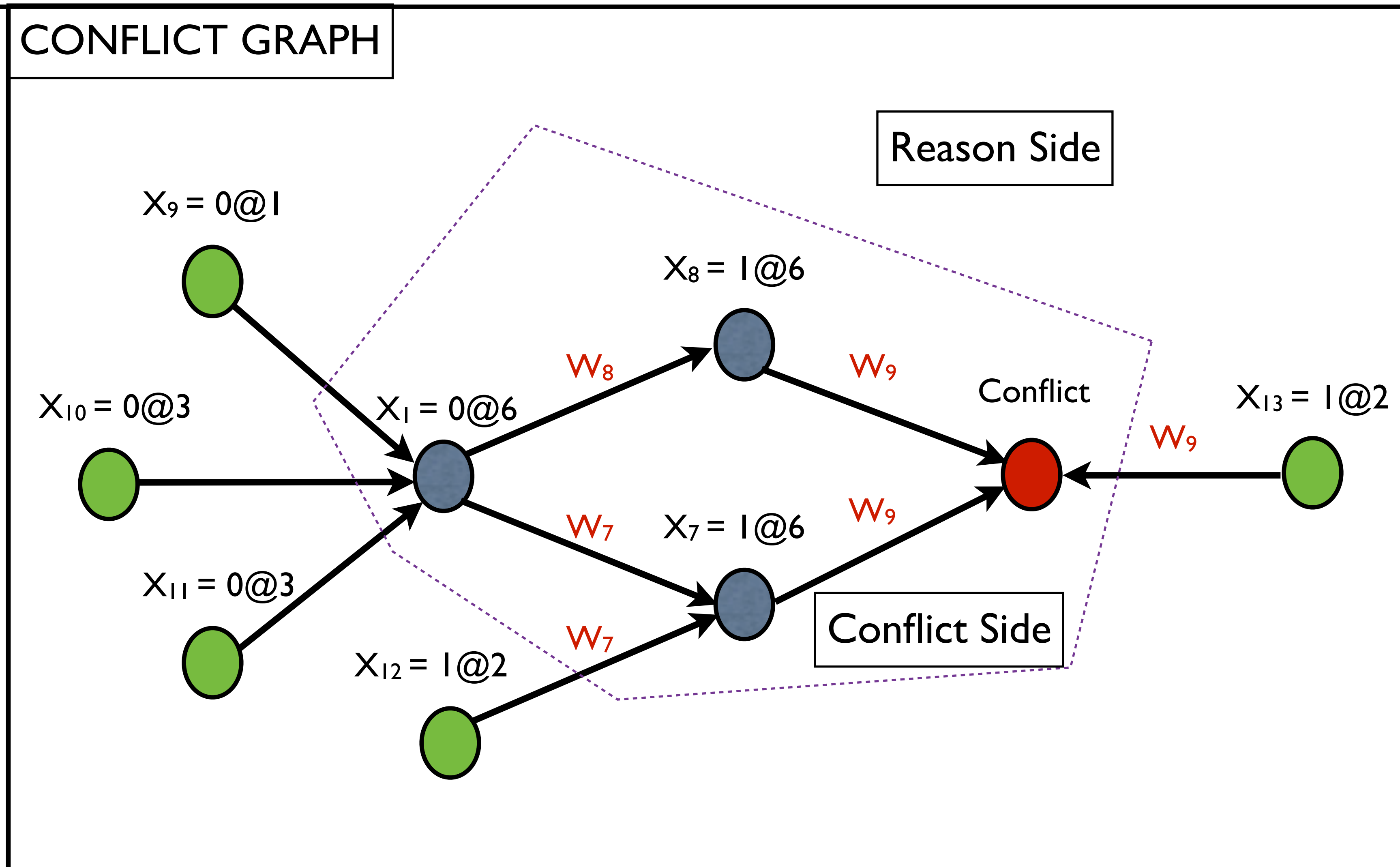
$W_5 = (\neg X_4 + X_6 + X_{11})$

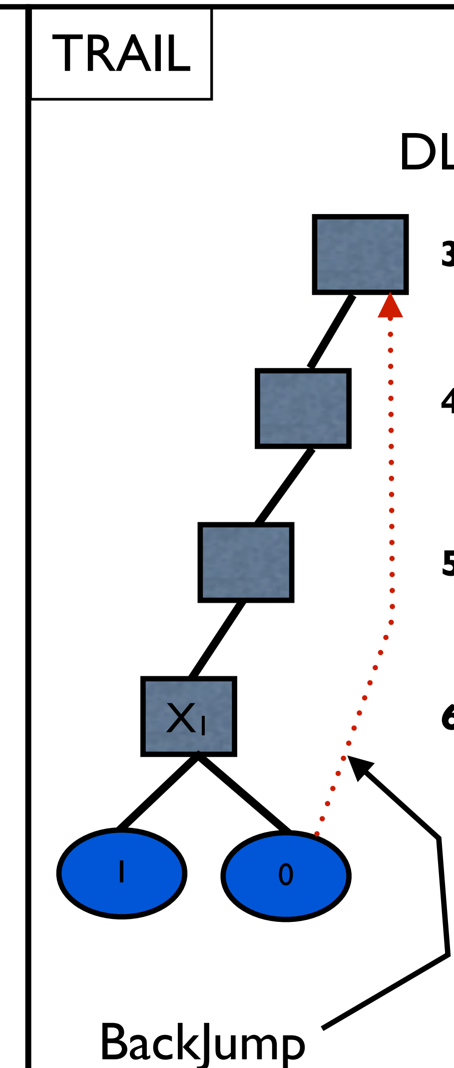
$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

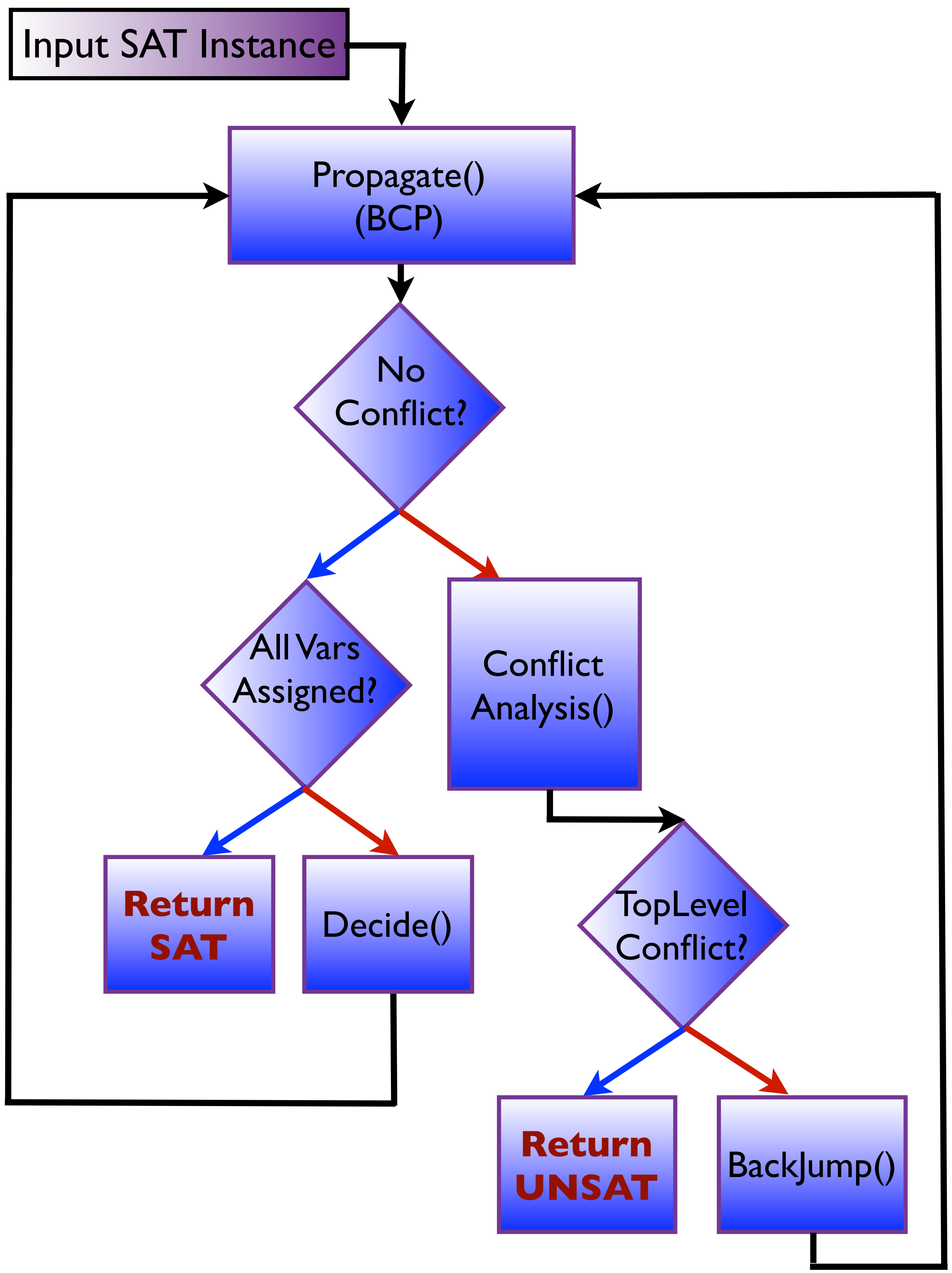
$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$





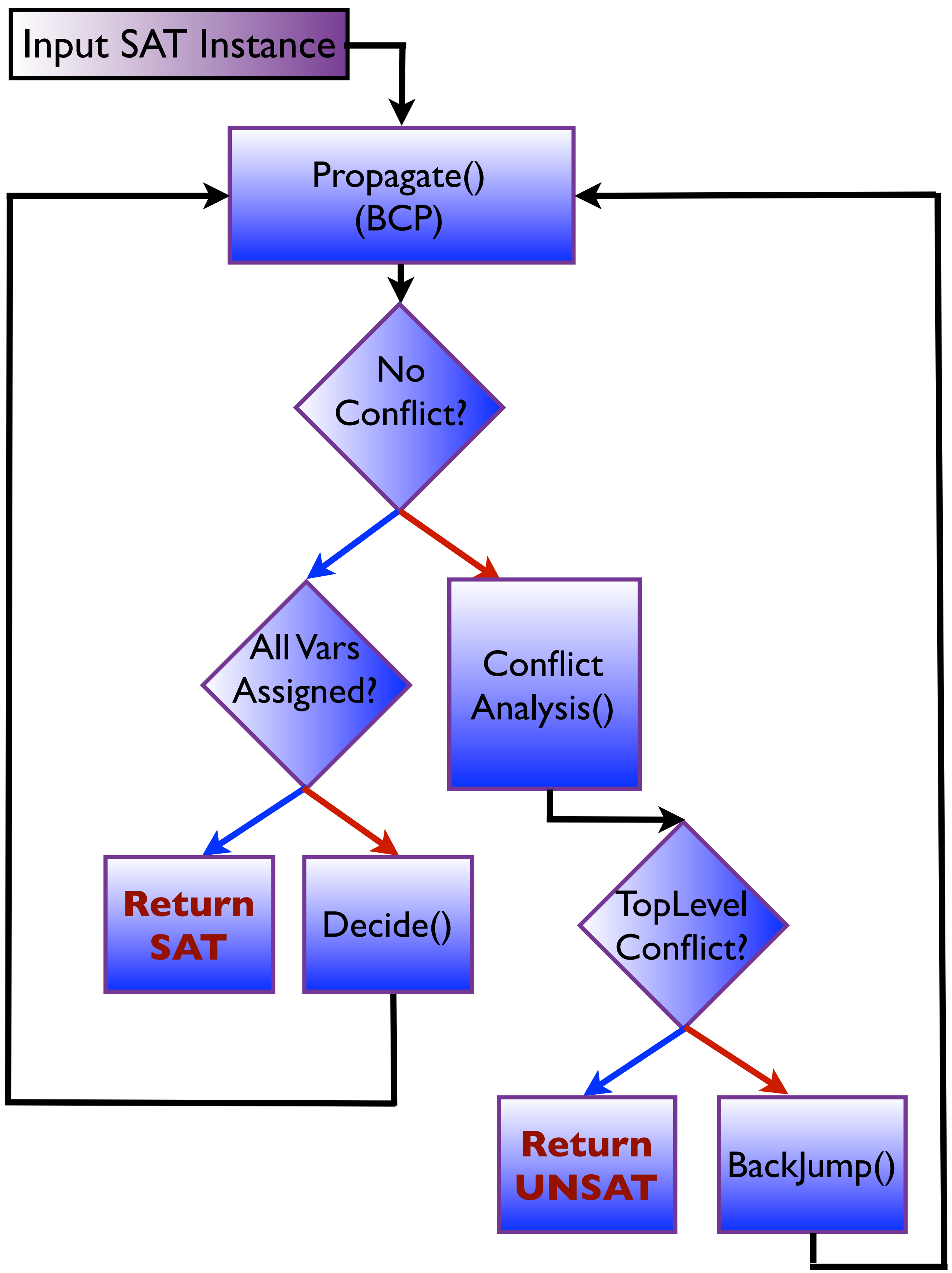
Modern CDCL SAT Solver Architecture

Restarts and Forget



- Restarts
  - Clear the Trail and start again
  - Start searching with a different variable order
  - Only Conflict Clause (CC) database is retained
- Forget: throw away less active learnt conflict clauses routinely
  - Routinely throw away very large CC
  - Logically CC are implied
  - Hence no loss in soundness/completeness
  - Time Savings: smaller DB means less work in propagation
  - Space savings

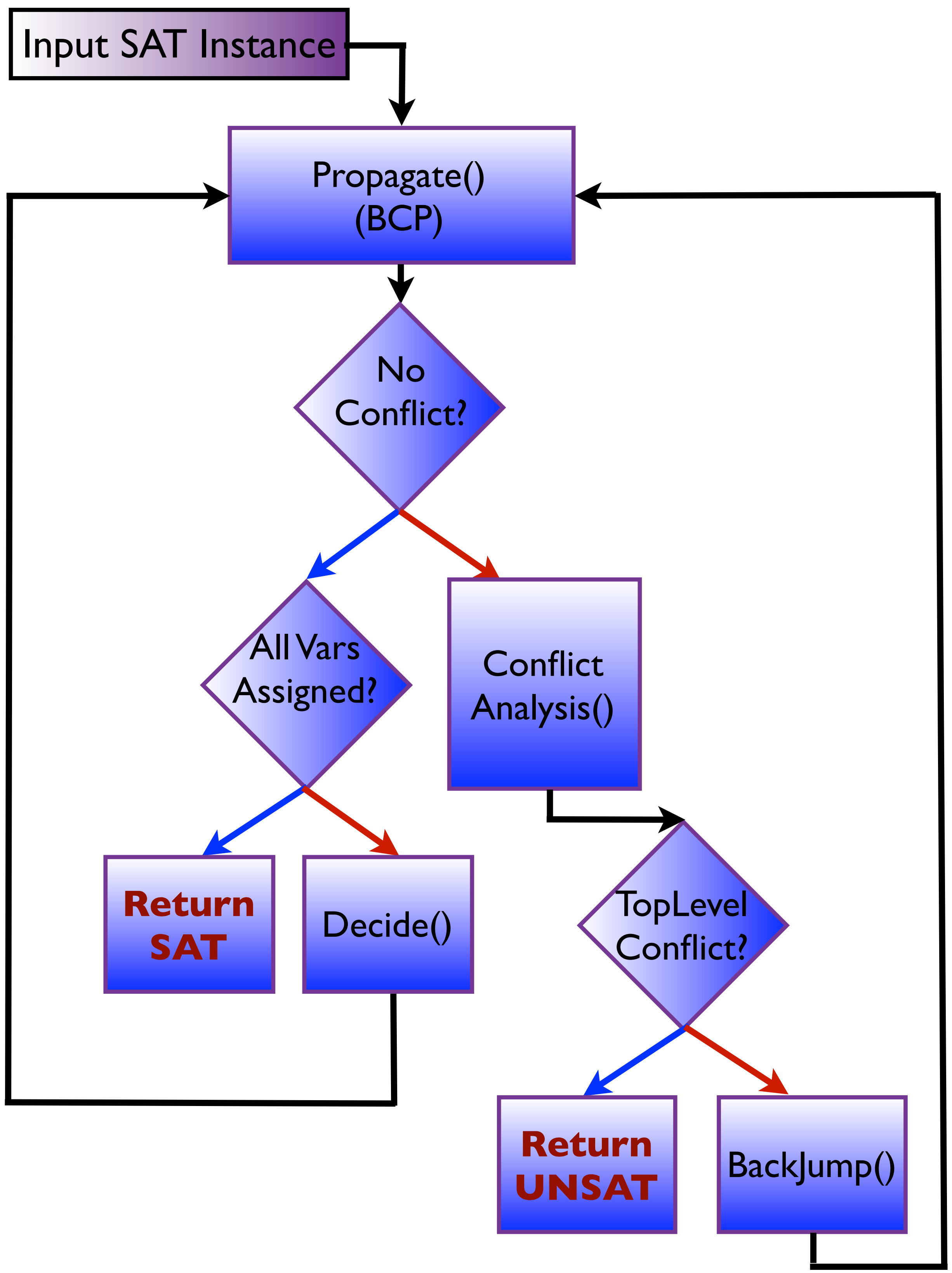
Modern CDCL SAT Solver Architecture
Why is SAT efficient?



- VSIDS branching heuristic and propagate (BCP)
- Conflict-Driven Clause-Learning (CDCL)
- Forget conflict clauses if DB goes too big
- BackJump
- Restarts
- All the above elements are needed for efficiency
- Deeper understanding lacking
- No predictive theory

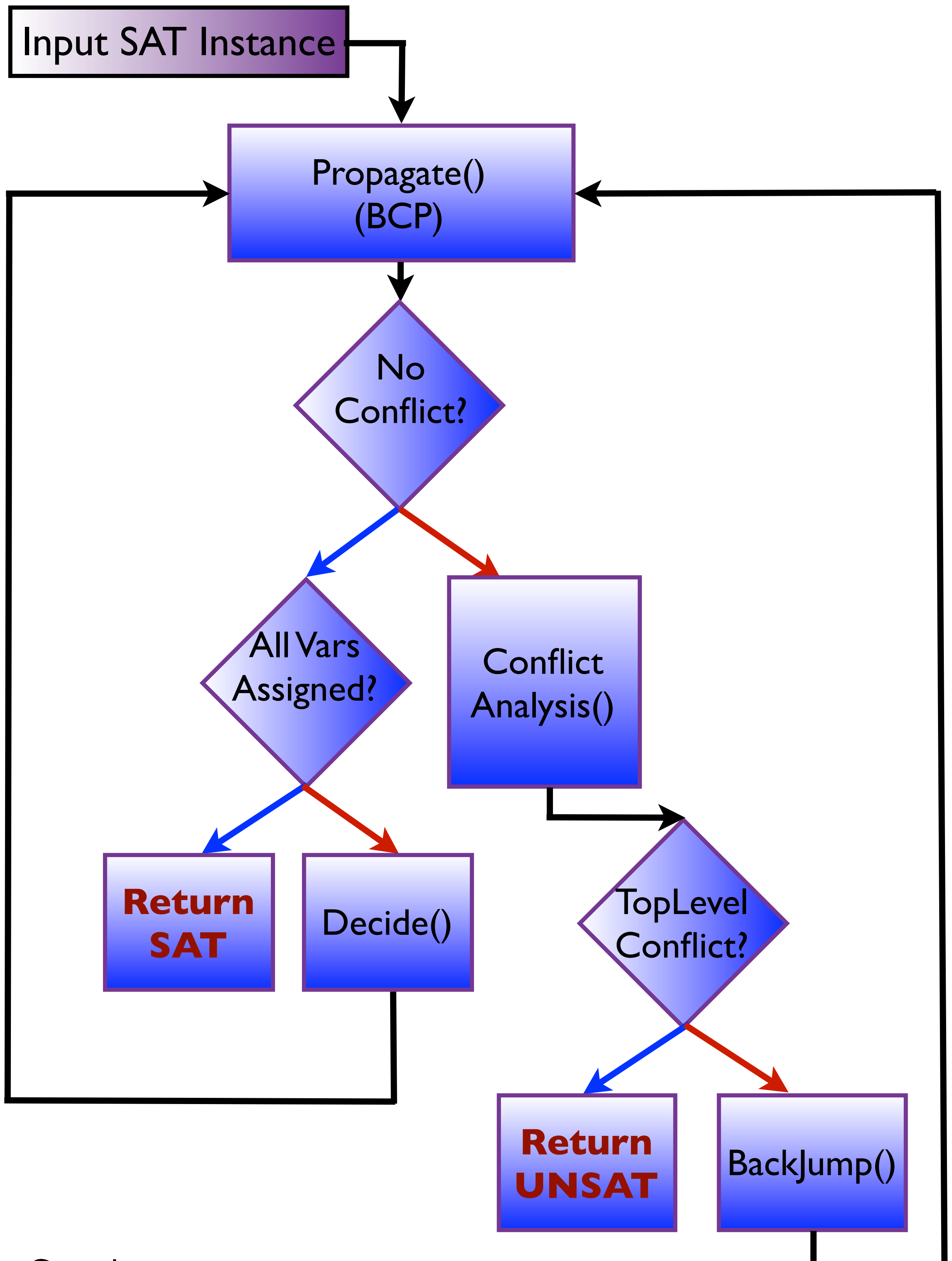
Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()



- Conflict-Driven Clause-Learning (CDCL)
  (Marques-Silva & Sakallah 1996)
- Decide/branch and propagate (BCP)
  (Malik et al. 2001, Zabih & McAllester 1988)
- BackJump
  (McAllester 1980, Marques-Silva & Sakallah 1999)
- Restarts
  (Selman & Gomes 2001)
- Follows MiniSAT
  (Een & Sorensson 2003)

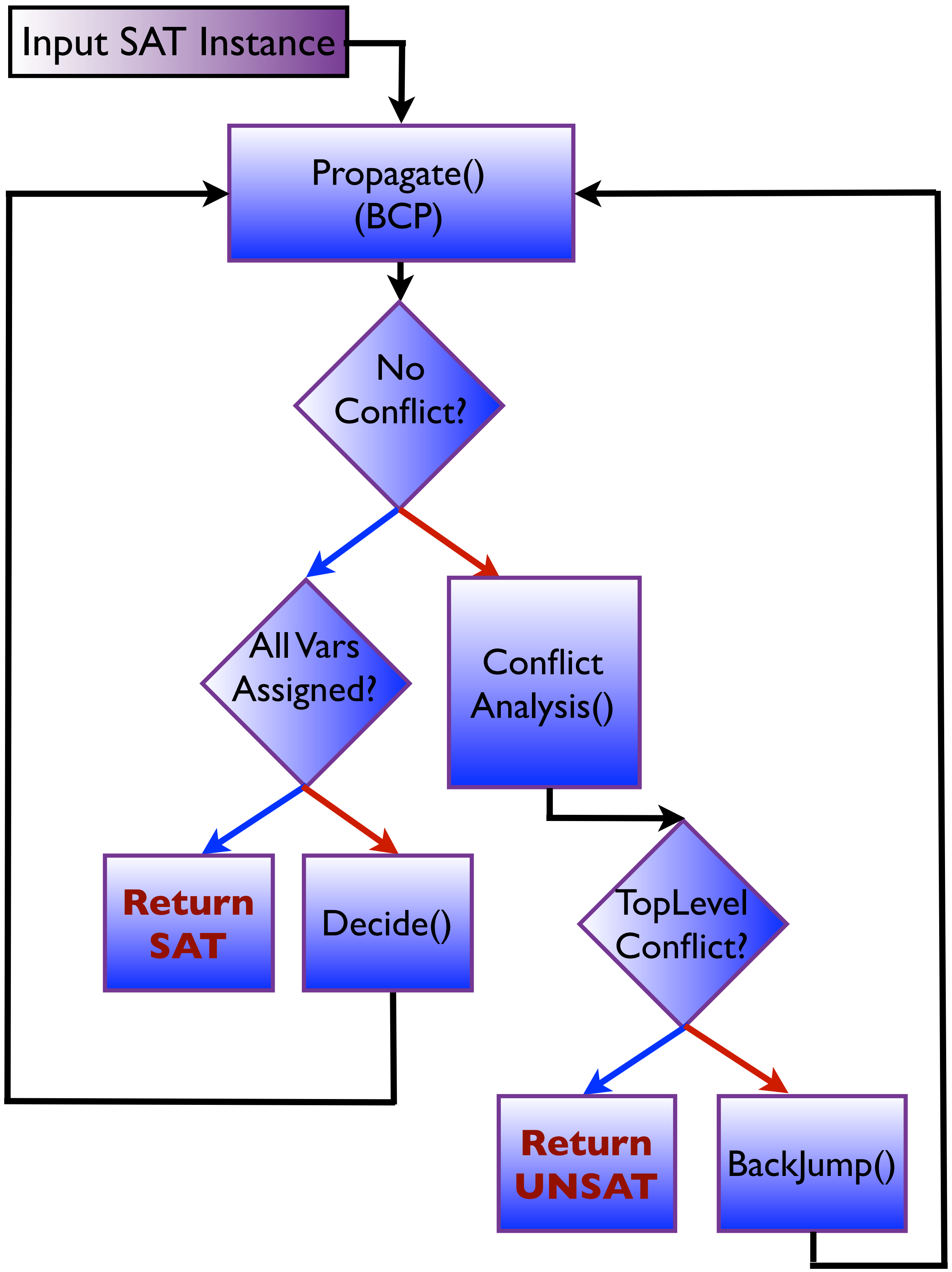
Modern CDCL SAT Solver Architecture
Soundness, Completeness & Termination



Soundness: A solver is said to be sound, if, for any input formula F, the solver terminates and produces a solution, then F is indeed SAT

Proof: (Easy) SAT is returned only when all vars have been assigned a value (True, False) by Decide or BCP, and solver checks the solution.

Modern CDCL SAT Solver Architecture
Soundness, Completeness & Termination



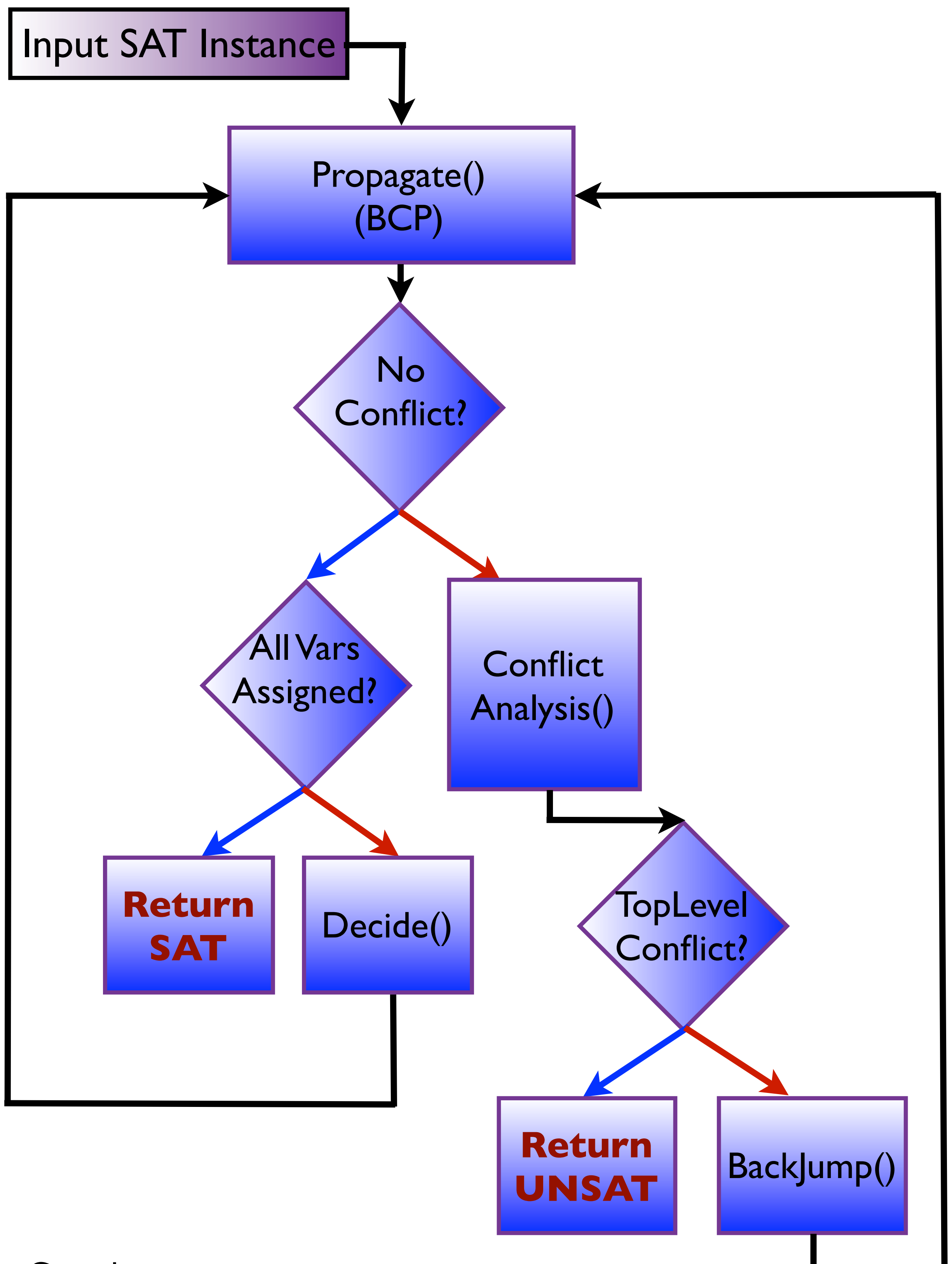
Completeness: A solver is said to be complete, if, for any input formula F that is SAT, the solver terminates and produces a solution (i.e., solver does not miss solutions)

Proof: (Harder)

- Backtracking + BCP + decide is complete (easy)
- Conflict clause is implied by input formula (easy)
- Only need to see backjumping does not skip assignments
  - Observe backjumping occurs only when conflict clause (CC) vars < decision level (DL) of conflicting var
  - Backjumping to max(DL of vars in CC)
  - Decision tree rooted at max(DL of vars in CC)+1 is guaranteed to not satisfy CC
  - Hence, backjumping will not skip assignments

Modern CDCL SAT Solver Architecture

Soundness, Completeness & Termination



Termination: Some measure decreases every iteration

Proof Sketch:

- Loop guarantees either conflict clause (CC) added OR assign extended
- CC added. What stops CC addition looping forever?
  - Recall that CC is remembered
  - No CC duplication possible
  - CC blocks UNSAT assign exploration in decision tree. No duplicate UNSAT assign exploration possible
  - Size of decision tree explored decreases for each CC add

Modern CDCL SAT Solver Architecture

References & Important SAT Solvers

1. Marques-Silva, J.P. and K.A. Sakallah. *GRASP: A Search Algorithm for Propositional Satisfiability.* IEEE Transactions on Computers 48(5), 1999, 506-521.

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4. L. Zhang, C. F. Madigan, M. H. Moskewicz and S. Malik. *Efficient Conflict Driven Learning in a Boolean Satisfiability Solver*. Proceedings of ICCAD, 2001, 279-285.

5. Armin Bierre, Marijn Heule, Hans van Maaren, and Toby Walsh (Editors). *Handbook of Satisfiability*. 2009. IOS Press. http://www.st.ewi.tudelft.nl/sat/handbook/

6. M. Davis, G. Logemann, and D. Loveland. *A machine program for theorem proving.* Communications of the ACM. 1962.

7. zChaff SAT Solver by Lintao Zhang 2002.

8. GRASP SAT Solver by Joao Marques-Silva and Karem Sakallah 1999.

9. MiniSAT Solver by Niklas Een and Niklas Sorenson 2005 - present

10. SAT Live: http://www.satlive.org/

11. SAT Competition: http://www.satcompetition.org/

12. SAT/SMT summer school: http://people.csail.mit.edu/vganesh/summerschool/

Modern CDCL SAT Solver Architecture
Important Ideas and Conclusions

1. SAT solvers are crucial for software engineering

2. Huge impact in formal methods, program analysis and testing

3. Key ideas that make SAT efficient

    1. Conflict-driven clause learning
    2. VSIDS (or similar) variable selection heuristics
    3. Backjumping
    4. Restarts

4. Techniques I didn't discuss

    1. Survey propagation (belief propagation) by Selman & Gomes
    2. Works well for randomized SAT, not yet for industrial instances
    3. Physics-inspired
    4. Combining CDCL with survey propagation (?)