SAT/SMT Solvers and Applications

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Today's Lecture

Lessons learnt so far

- Implementation-related attacks (control-hazard, malware,...)
- Program analysis techniques can detect them, prevent them
- How do you build such program analysis-based detectors and bug-finders
- Often, using constraint solvers (aka SAT/SMT solvers)

Constraint solvers

- What are constraint solvers?
- Why you should care?
- Security and software engineering applications

What is a Constraint Solver?

Engineer/Mathematician's point of vew

- A "method" that takes as input a math formula, and produces a solution
- Examples: solving linear equations over the reals, polynomials, quadratic, Boolean logic,...
- Computing zeros of a polynomial

Theoretical computer scientist/logician's point of view

- A computer program called as a satisfiability procedure that solves a specific kind of decision problem, namely, the SAT problem
- The input formula is in a specified logic (e.g., Boolean, first-order, reals, integers,...)
- Output of a satisfiability procedure
  - UNSAT, if input has no satisfying assignments
  - SAT, otherwise

One Slide History of Constraint Solving Methods

Before modern conception of logic (Before Boole and Frege)

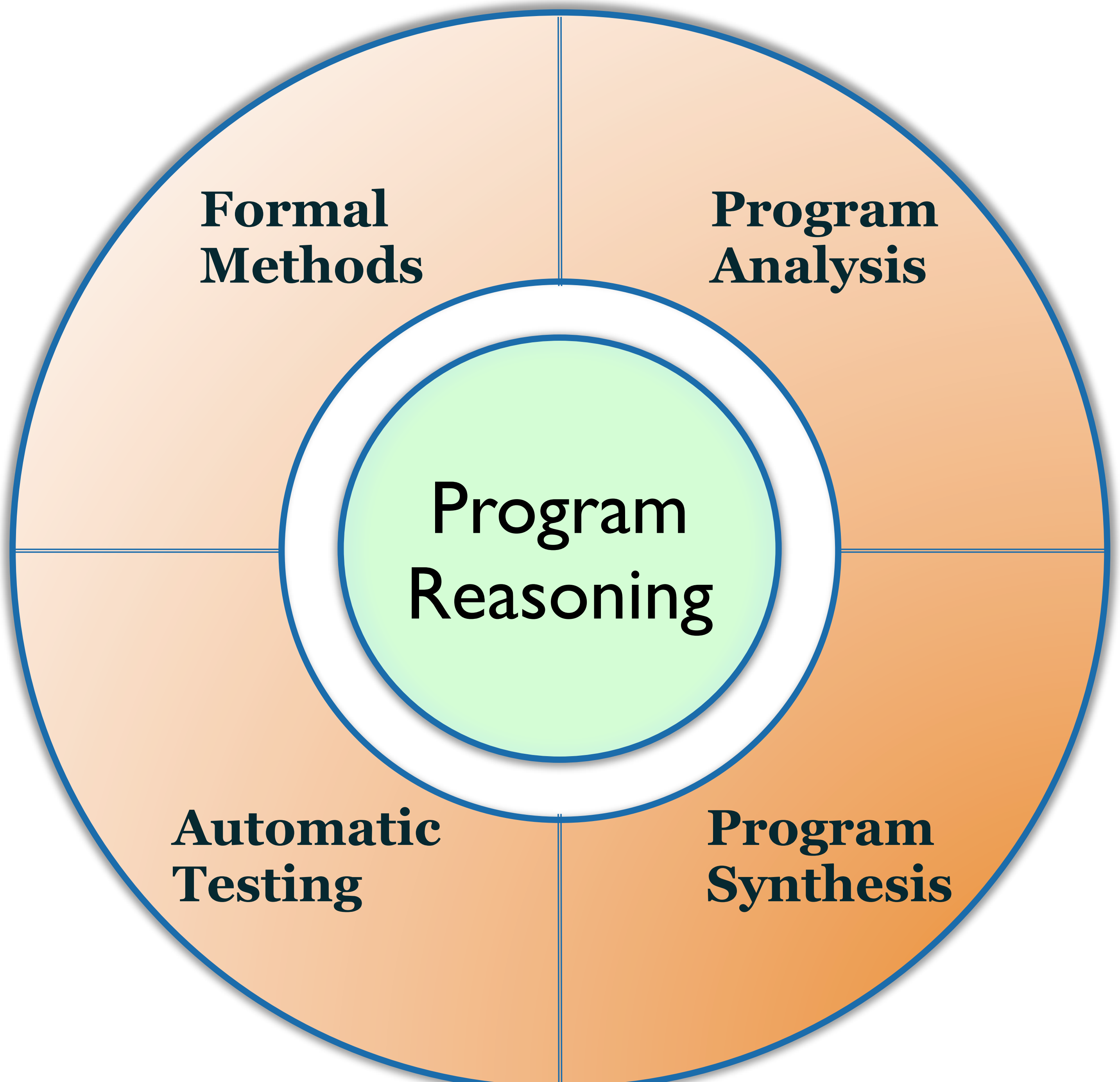
- From Babylon to late 1800's: Huge amount of work on methods to solve (find roots of) polynomials over reals, integers,...
- System of linear equations over the reals (Chinese methods, Cramer's method, Gauss elimination)
- These methods were typically not complete (e.g., worked for a special class of polynomials)

After modern conception of logic

- Systems of linear inequalities over the integers are solvable (Presburger, 1927)
- Peano arithmetic is undecidable (hence, not solvable) (Godel, 1931)
- First-order logic is undecidable (hence, not solvable) (Turing,1936. Church, 1937)
- A exponential-time algorithm for Boolean SAT problem (Davis, Putnam, Loveland, Loggeman in 1962)
- Systems of Diophantine equations are not solvable (Matiyasevich. 1970)
- Boolean SAT problem is NP-complete (Cook 1971)
- Many efficient, scalable SAT procedures since 1971 for a variety of mathematical theories

Foundation of Sofware Engineering

Logic Abstractions of Computation



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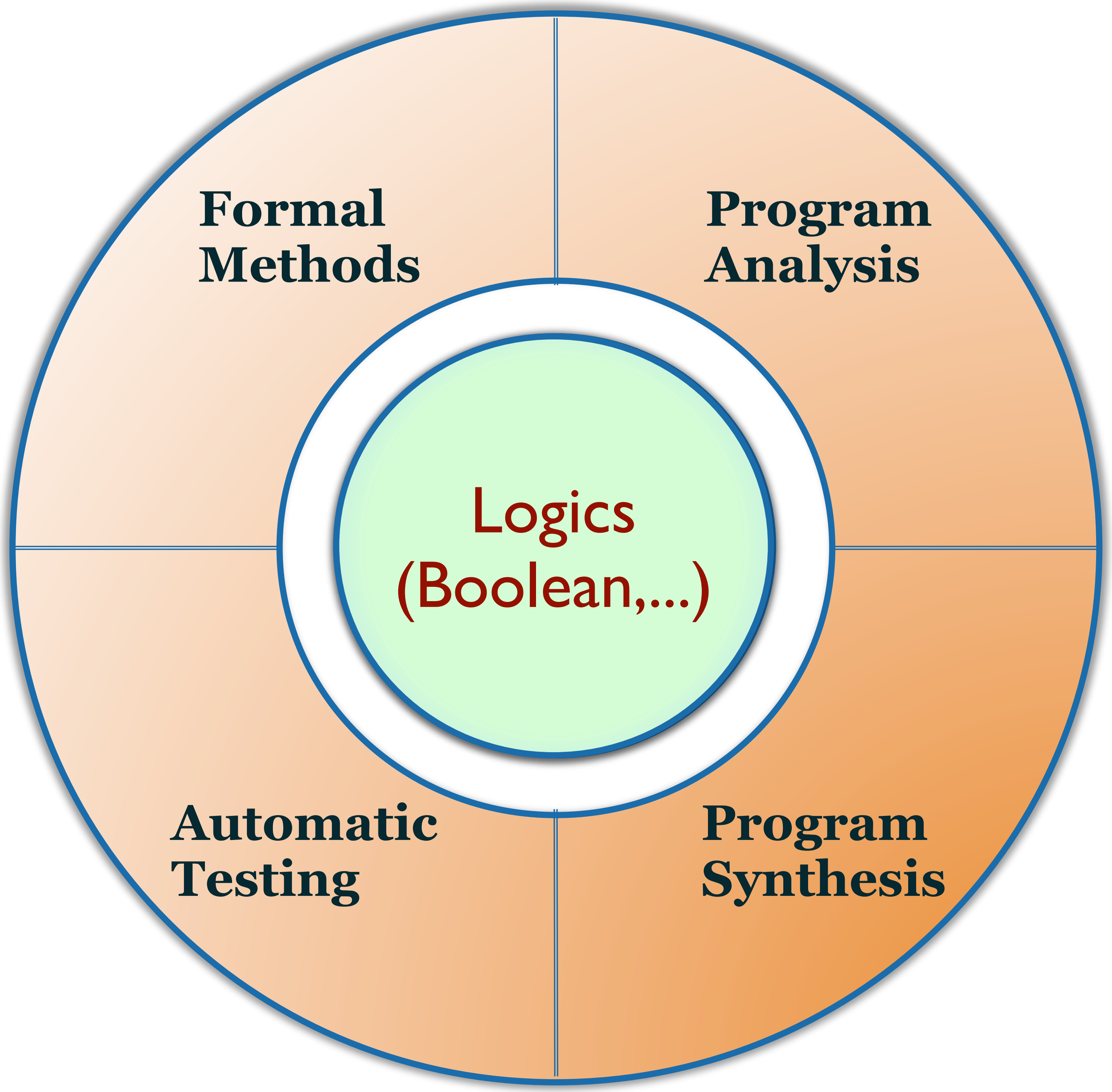
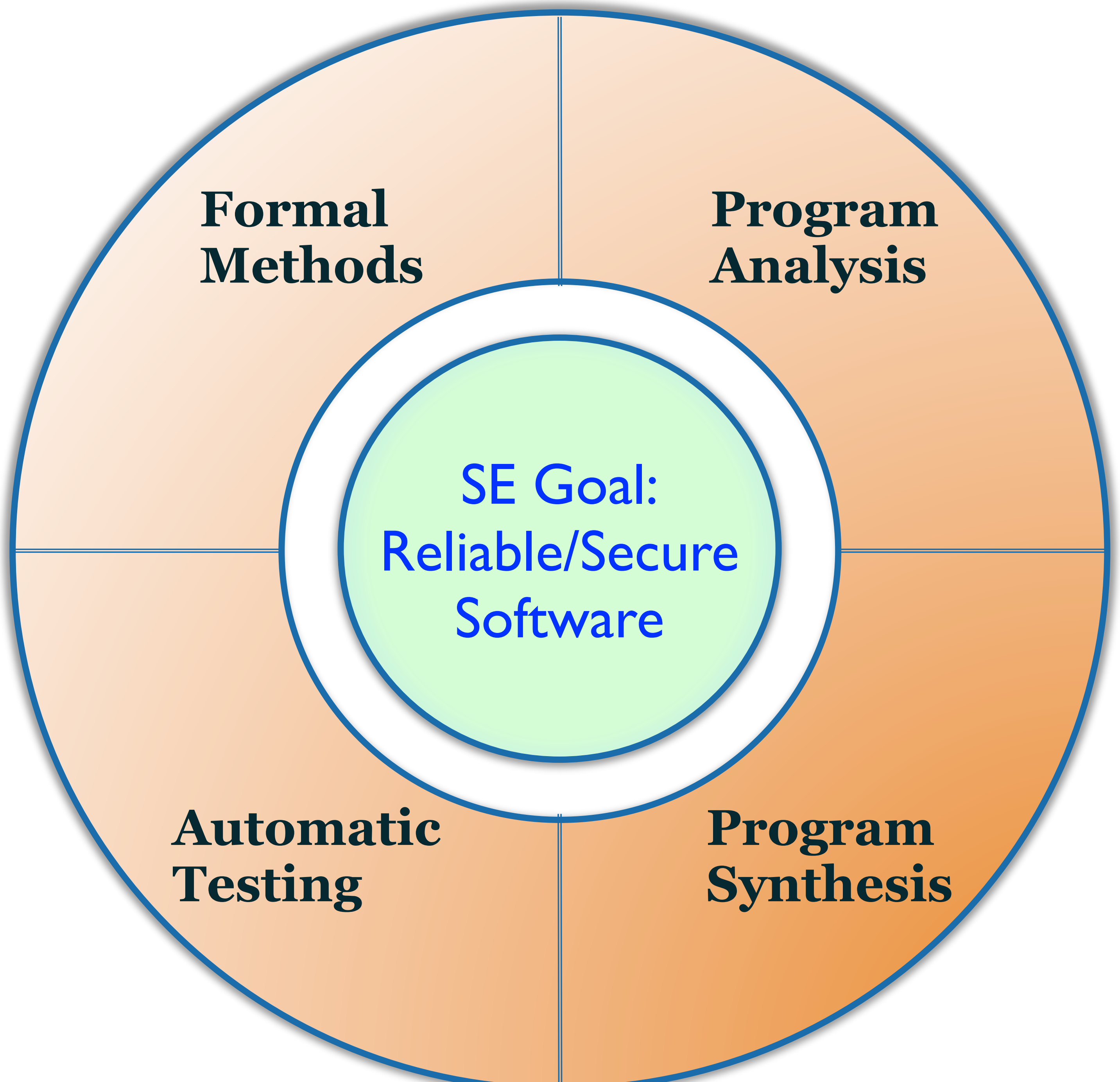


Software Engineering & SAT/SMT Solvers

An Indispensable Tactic for Any Strategy



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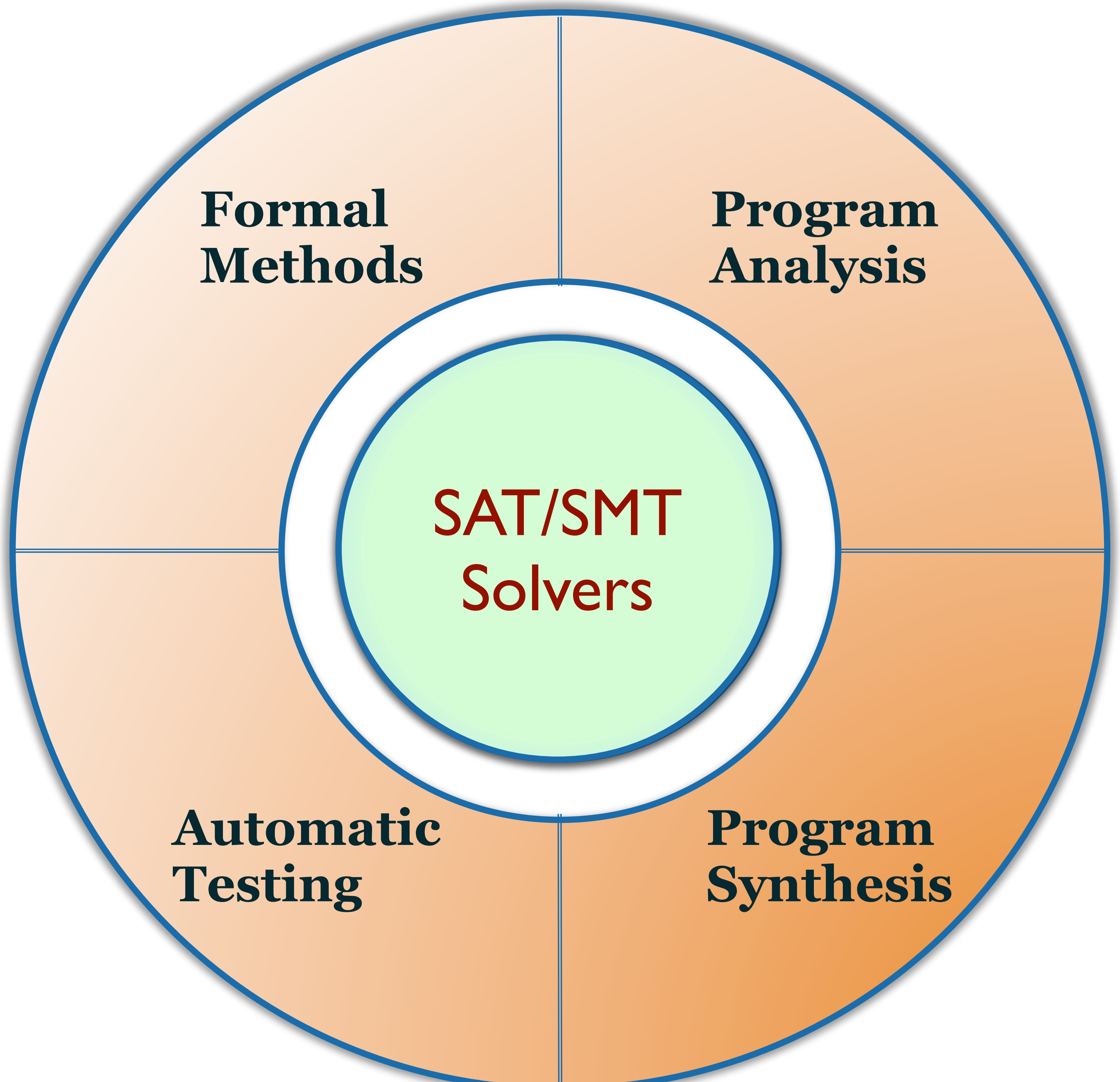
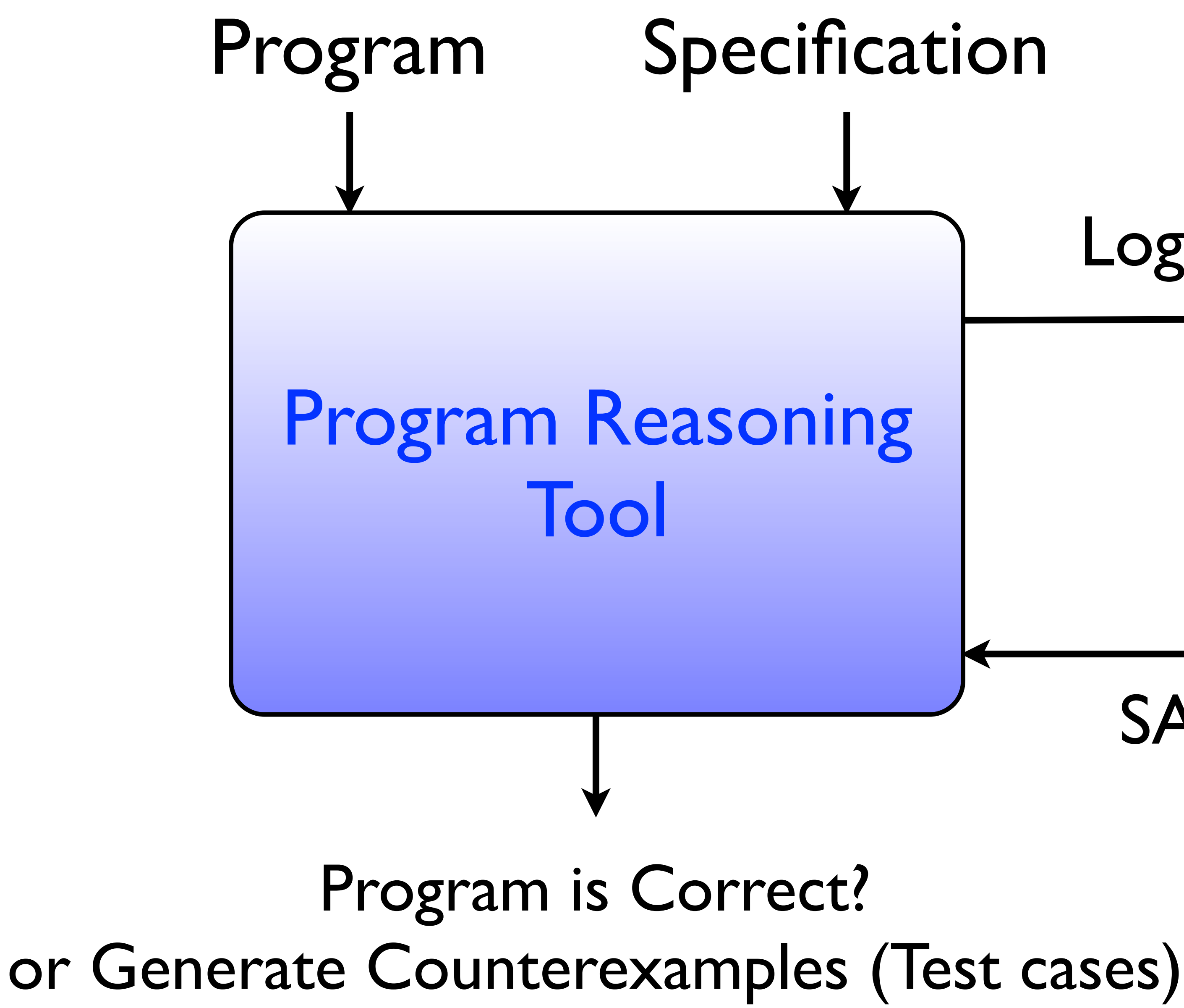
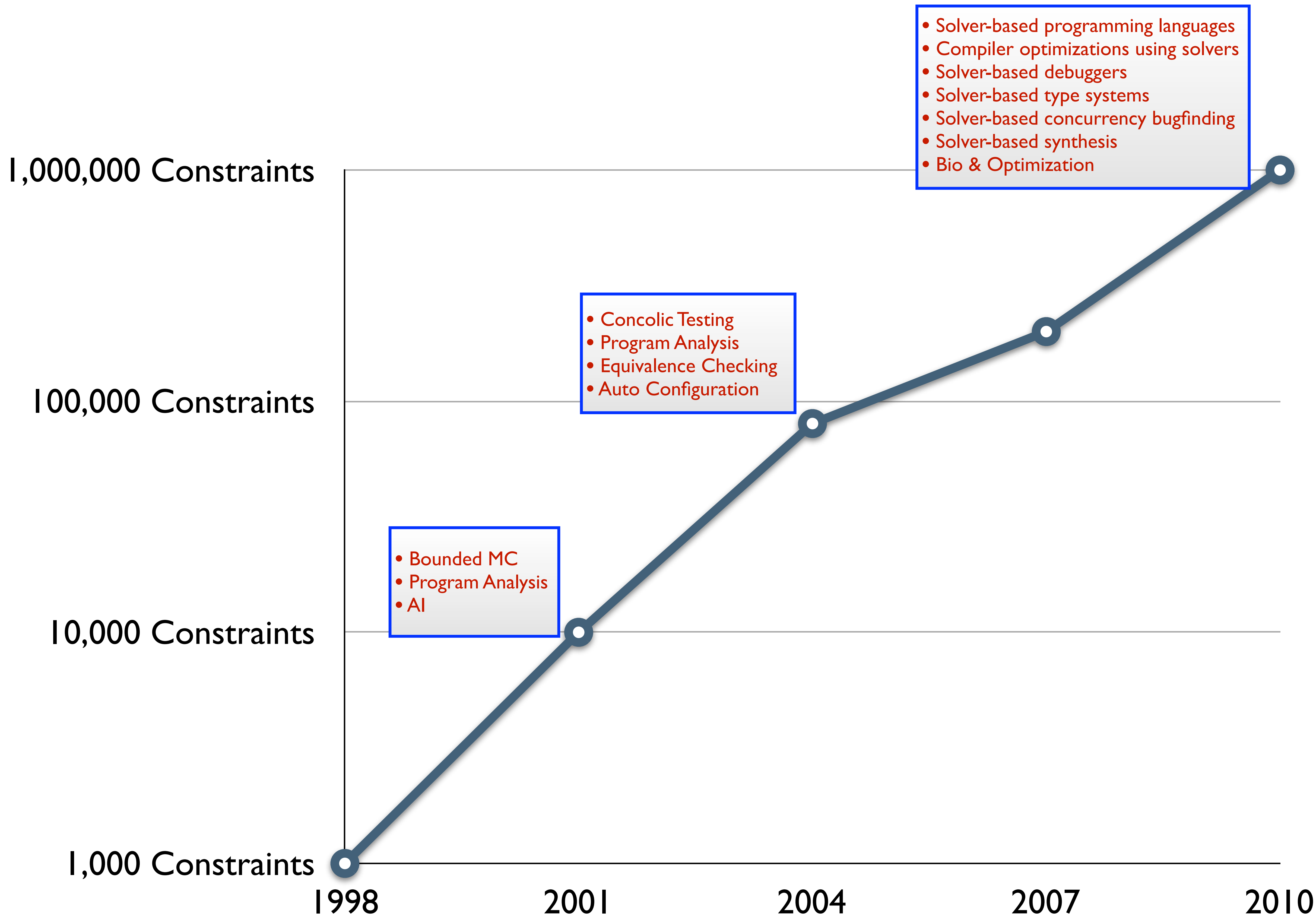


Software Engineering using Solvers

Engineering, Usability, Novelty



SAT/SMT Solver Research Story
A 1000x Improvement



The SAT/SMT Problem

- Rich logics (Modular arithmetic, Arrays, Strings,...)
- NP-complete, PSPACE-complete,...
- Practical, scalable, usable, automatic
- Enable novel software reliability approaches

The SAT/SMT Problem

- Closely related to the Validity Problem
- Soundness, completeness, termination
- Connecting model theory and proof theory

Lecture Outline

Points already covered

- ☑ Motivation for SAT/SMT solvers in software engineering
- ☑ High-level description of the SAT/SMT problem & logics
- ☑ Defined logic, models, truth, proofs, SAT procedure, soundness, completeness

Rest of the lecture

- Modern CDCL SAT solver architecture & techniques
- SAT/SMT-based applications
- Future of SAT/SMT solvers
- Some history (who, when,...) and references sprinkled throughout the talk
- Non-CDCL SAT techniques

Concolic Testing: Example

```
int double (int v) {
    return 2*v;
}

void testme (int x, int y) {
    z = double (y);
    if (z == x) {
        if (x > y+10) {
            ERROR;
        }
    }
}
```

The concolic testing slides are courtesy Koushik Sen

Concolic Testing: Example

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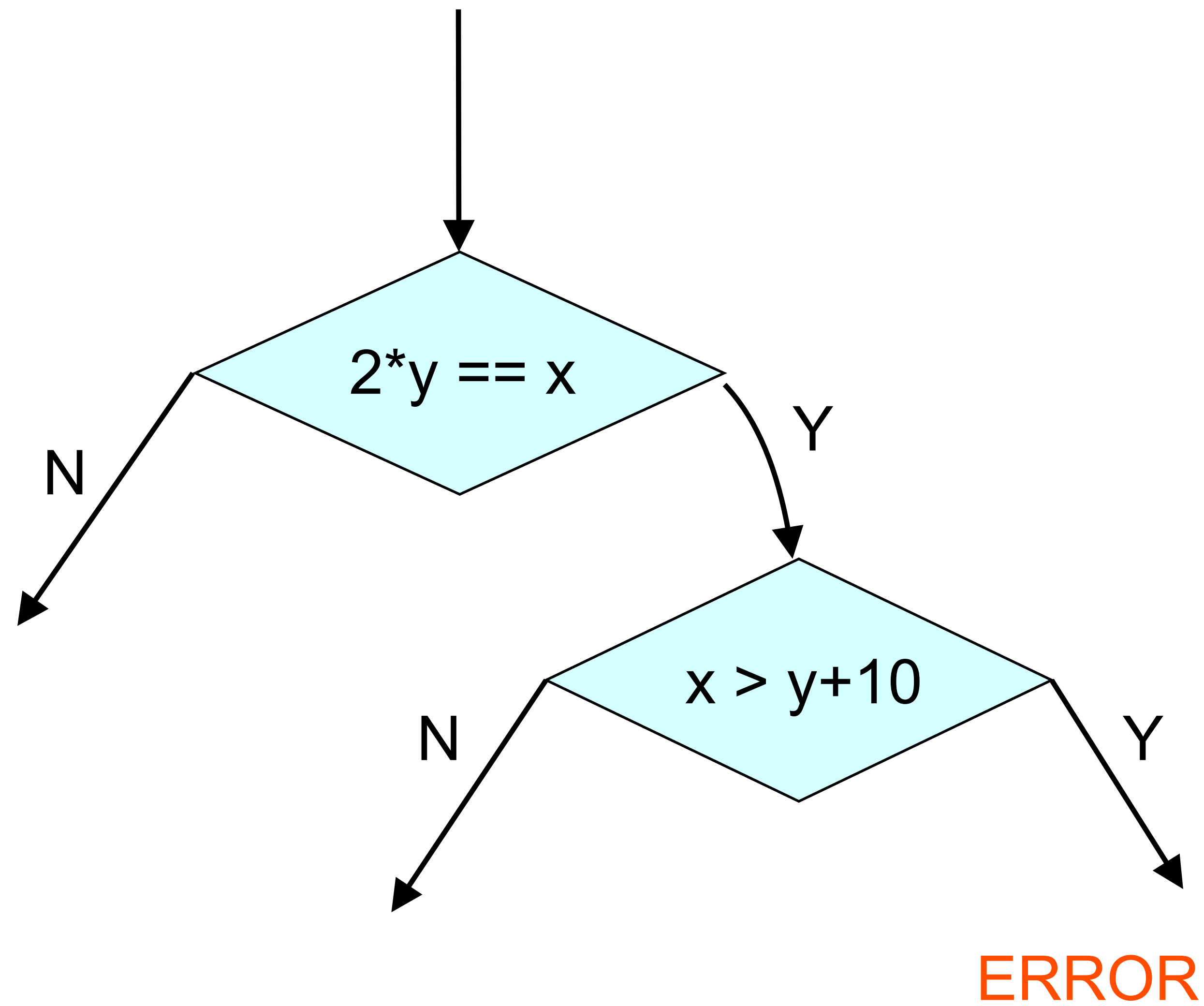

Concolic Testing Approach

```
int double (int v) {
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```

	Concrete Execution	Symbolic Execution	
	concrete state	symbolic state	path condition
	$x = 22, y = 7$	$x = x_0, y = y_0$	

Concolic Testing Approach

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int double (int v) {
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```

Concrete Execution	Symbolic Execution	
concrete state	symbolic state	path condition
x = 22, y = 7,	$x = x_0$, $y = y_0$,	
z = 14	$z = 2^*y_0$	

Concolic Testing Approach

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int double (int v) {
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```

	Concrete Execution	Symbolic Execution	
	concrete state	**symbolic state**	**path condition**
			$2*y_0 \mathrel{!=} x_0$
	x = 22, y = 7, z = 14	$x = x_0$, $y = y_0$, $z = 2*y_0$	

Concolic Testing Approach

```
int double (int v) {
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void testme (int x, int y) {
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```

Concrete Execution		Symbolic Execution	
concrete state		symbolic state	path condition
			$2*y_0 != x_0$
$x = 22$, $y = 7$, $z = 14$		$x = x_0$, $y = y_0$, $z = 2*y_0$	

Solve: $2*y_0 == x_0$

Solution: $x_0 = 2$, $y_0 = 1$

Concolic Testing Approach: Use-case for SAT

```
int double (int v) {
    return 2*v;
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void testme (int x, int y) {
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Concrete Execution		Symbolic Execution	
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			$2*y_0 \mathrel{!=} x_0$
$x = 22$, $y = 7$, $z = 14$		$x = x_0$, $y = y_0$, $z = 2*y_0$	

Solve: $2*y_0 == x_0$

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Concolic Testing Approach: Use-case for SAT

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Concrete Execution		Symbolic Execution	
concrete state		symbolic state	path condition
$x = 22$, $y = 7$, $z = 14$		$x = x_0$, $y = y_0$, $z = 2*y_0$	$2*y_0 \neq x_0$

Solve: $2*y_0 == x_0$

Solution: $x_0 = 2$, $y_0 = 1$

Observe that digital programs with finite loops are Boolean logic circuits

The Boolean SAT Problem

Basic Definitions and Format

A **literal** p is a Boolean variable x or its negation $\neg x$.

A **clause** C is a disjunction of literals: $x_2 \vee \neg x_{41} \vee x_{15}$

A **CNF** is a conjunction of clauses: $(x_2 \vee \neg x_1 \vee x_5) \wedge (x_6 \vee \neg x_2) \wedge (x_3 \vee \neg x_4 \vee \neg x_6)$

All Boolean formulas assumed to be in **CNF**

Assignment is a mapping (binding) from variables to Boolean values **(True, False).**

A **unit clause** C is a clause with a single unbound literal

The **SAT-problem** is:

Find an assignment s.t. each input clause has a true literal (aka input formula has a solution or is SAT)

OR establish input formula has no solution (aka input formula is UNSAT)

The Input formula is represented in **DIMACS Format:**

```
c DIMACS
p cnf 6 3
2 -1 5 0
6 -2 0
3 -4 -6 0
```

DPLL SAT Solver Architecture
The Basic Solver

DPLL(Θ_{cnf}, assign) {

Propagate unit clauses;

if *"conflict"*: **return** FALSE;

if *"complete assign"*: **return** TRUE;

"pick decision variable x";

return

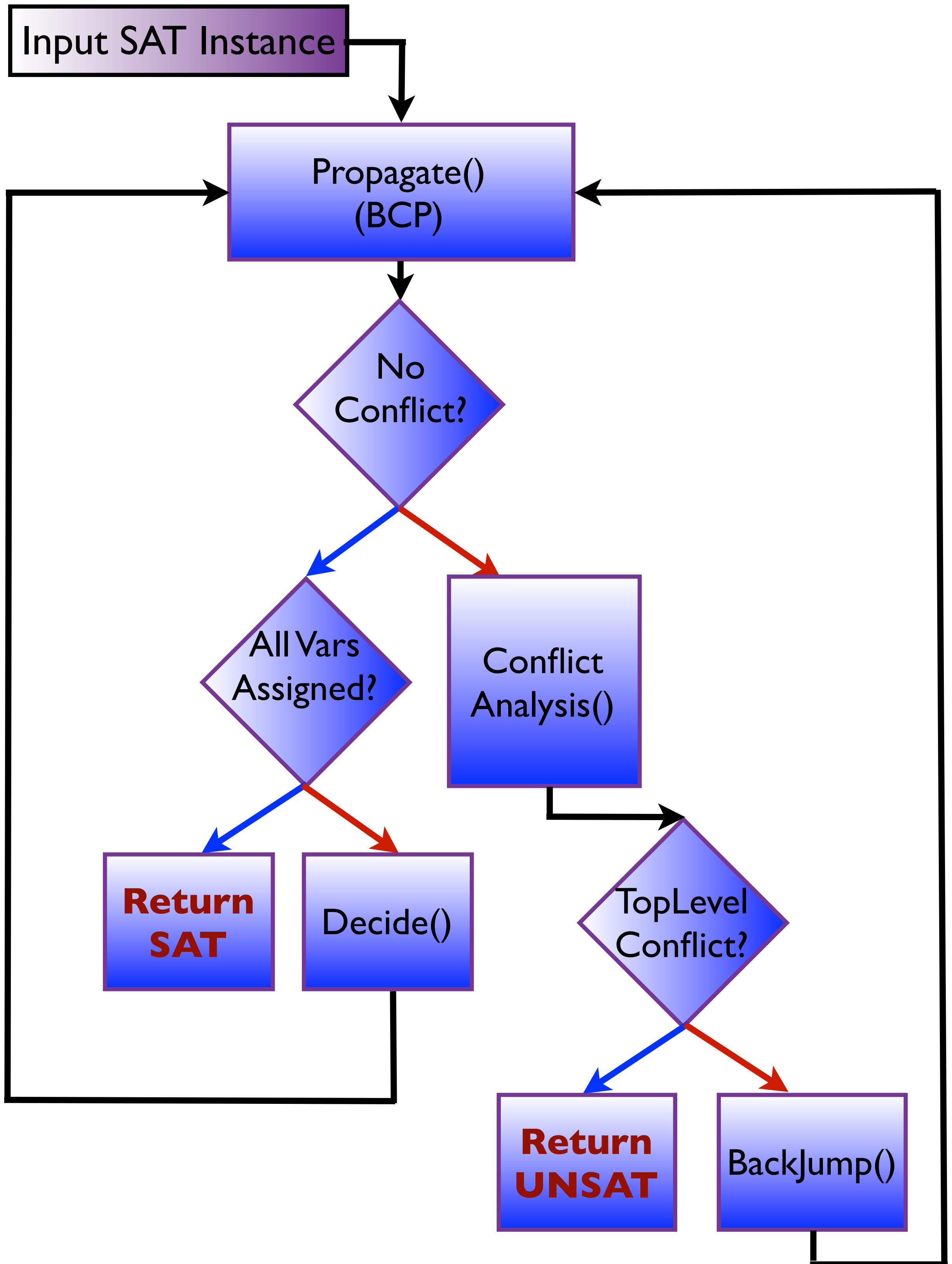
DPLL(Θ_{cnf}□ $_{x=0}$, assign[x=0])

|| DPLL(Θ_{cnf}□ $_{x=1}$, assign[x=1]);

}

- Propagate (Boolean Constant Propagation):
  - Propagate inferences due to unit clauses
  - Most time in solving goes into this
- Detect Conflict:
  - Conflict: partial assignment is not satisfying
- Decide (Branch):
  - Choose a variable & assign some value
- Backtracking:
  - Implicitly done by the recursion

Modern CDCL SAT Solver Architecture
Key Steps and Data-structures



Key steps

- Decide()
- Propagate()
  (BCP: Boolean constraint propagation)
- Conflict analysis and learning()
- Backjump()
- Forget()
- Restart()

CDCL: Conflict-Driven Clause-Learning

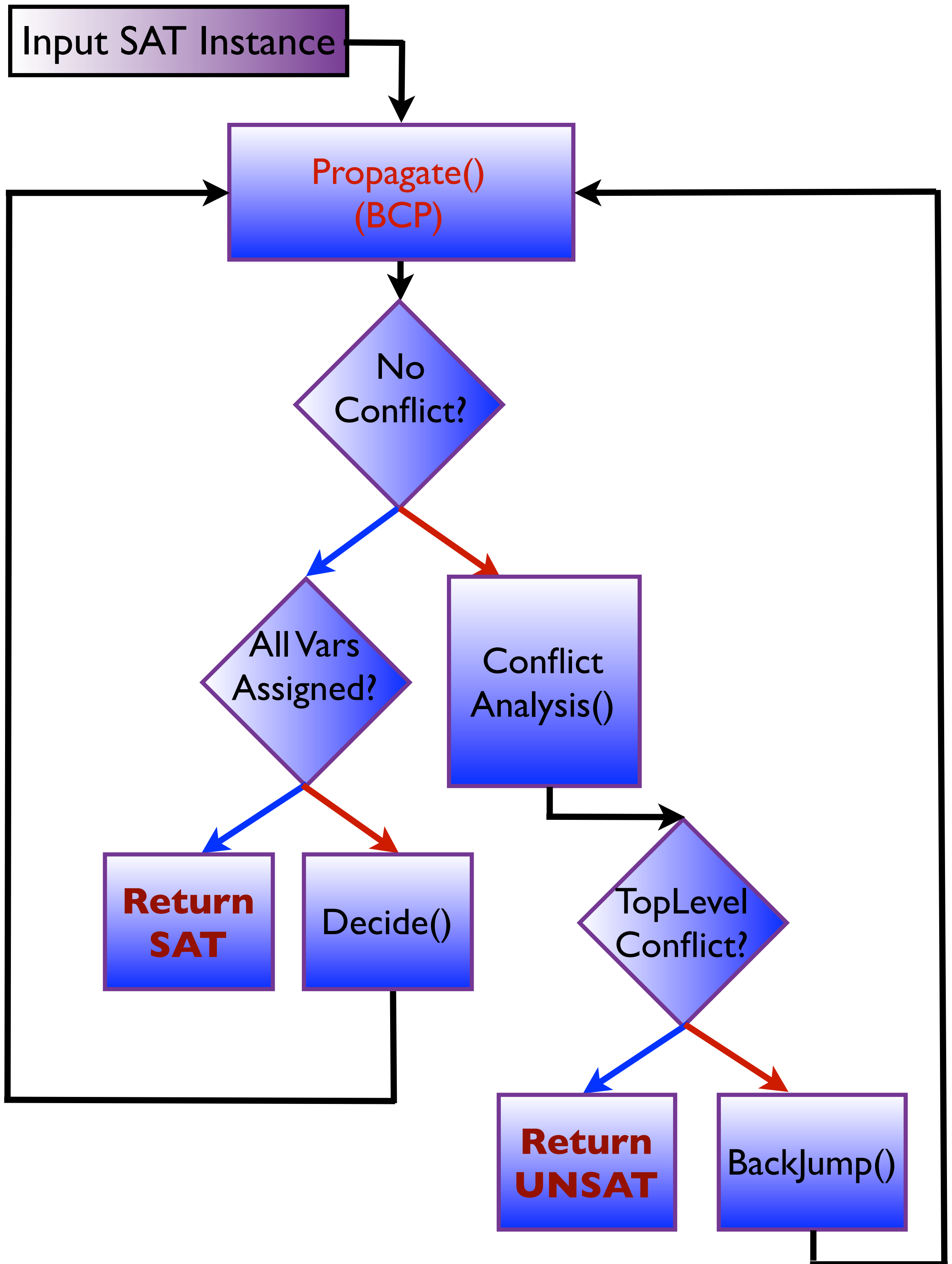
- Conflict analysis is a key step
- Results in learning a conflict clause
- Prunes the search space

Key data-structures (State):

- Stack or trail of partial assignments (AT)
- Input clause database
- Conflict clause database
- Conflict graph
- Decision level (DL) of a variable

Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()



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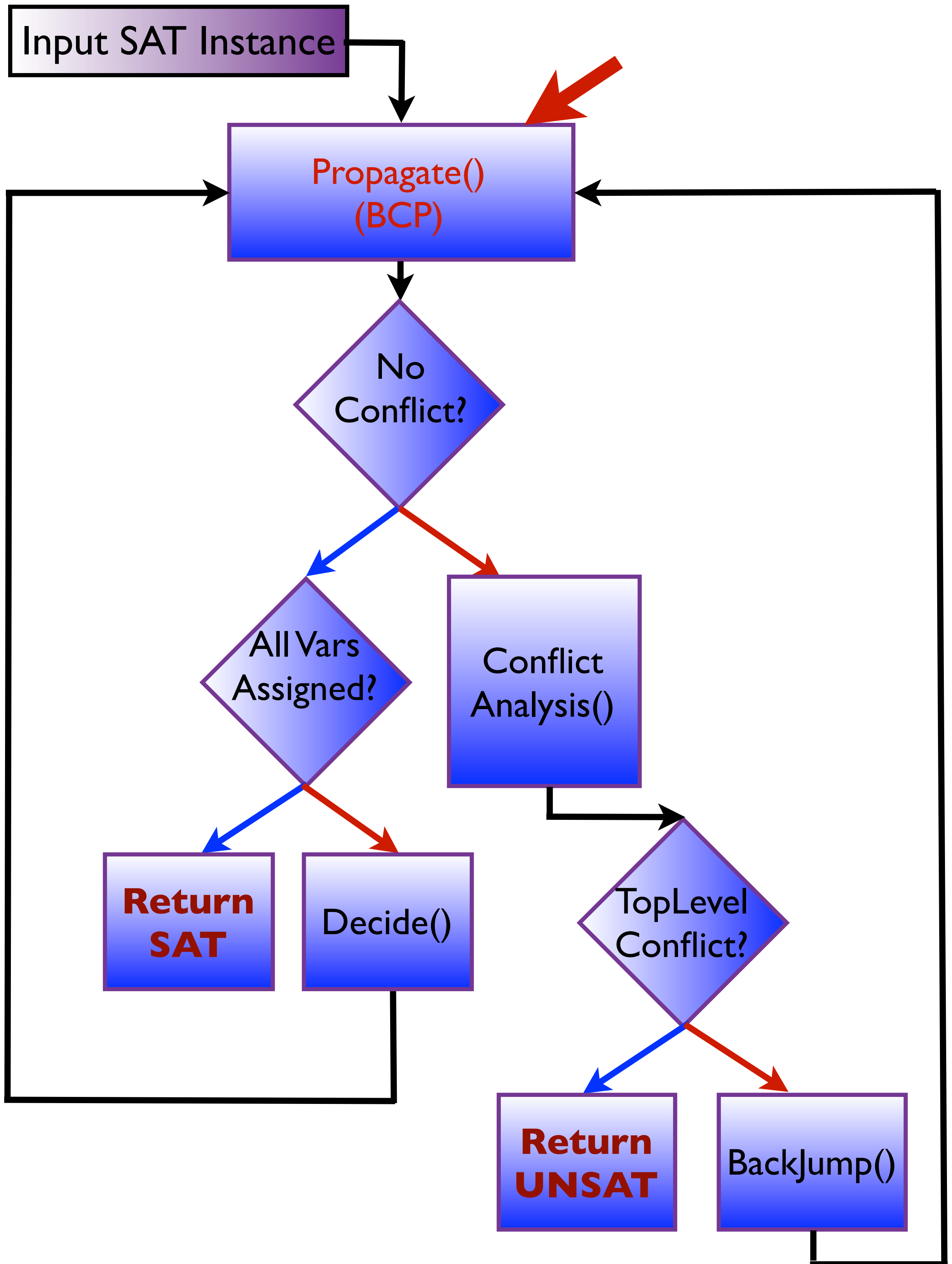

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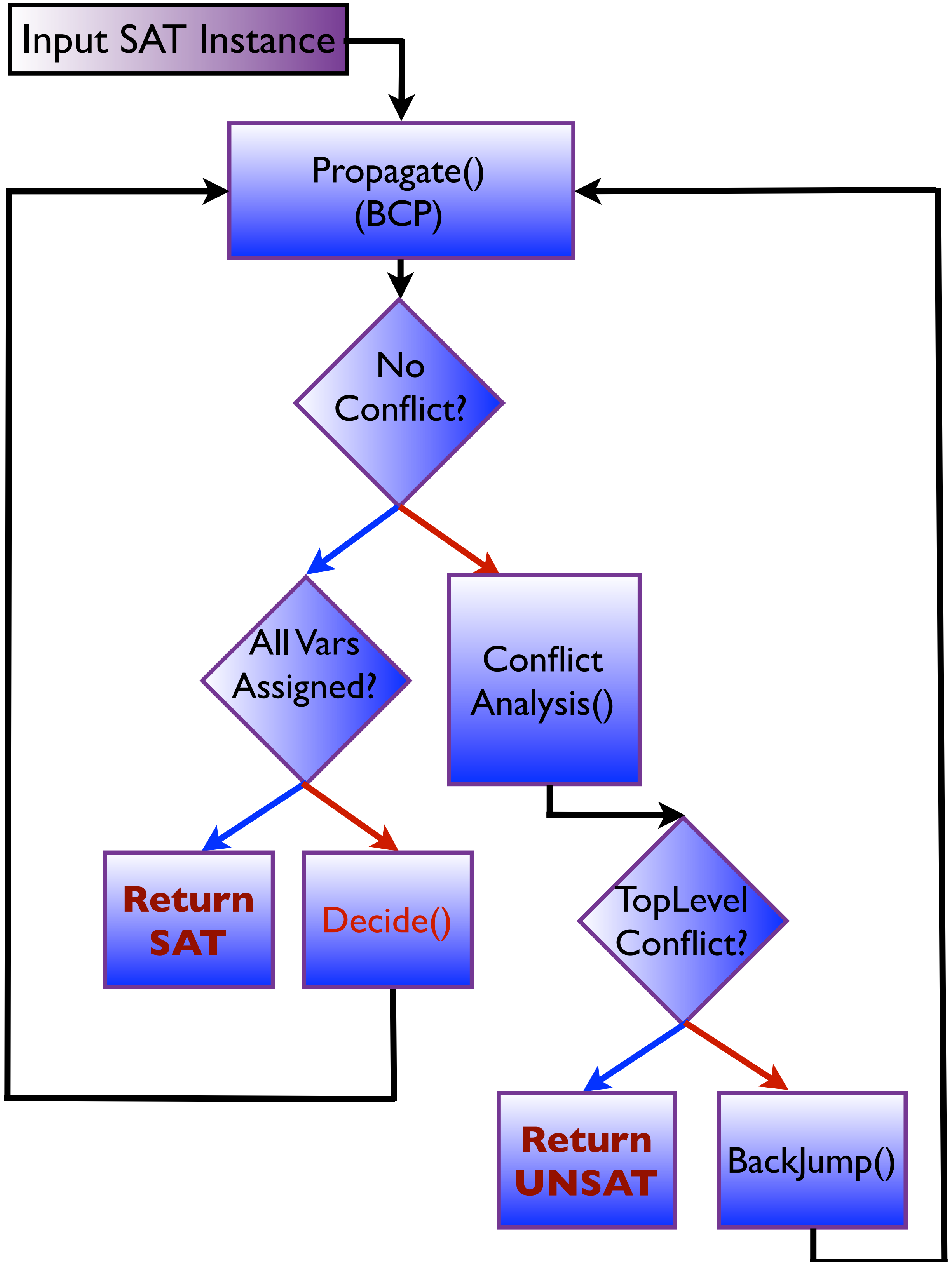

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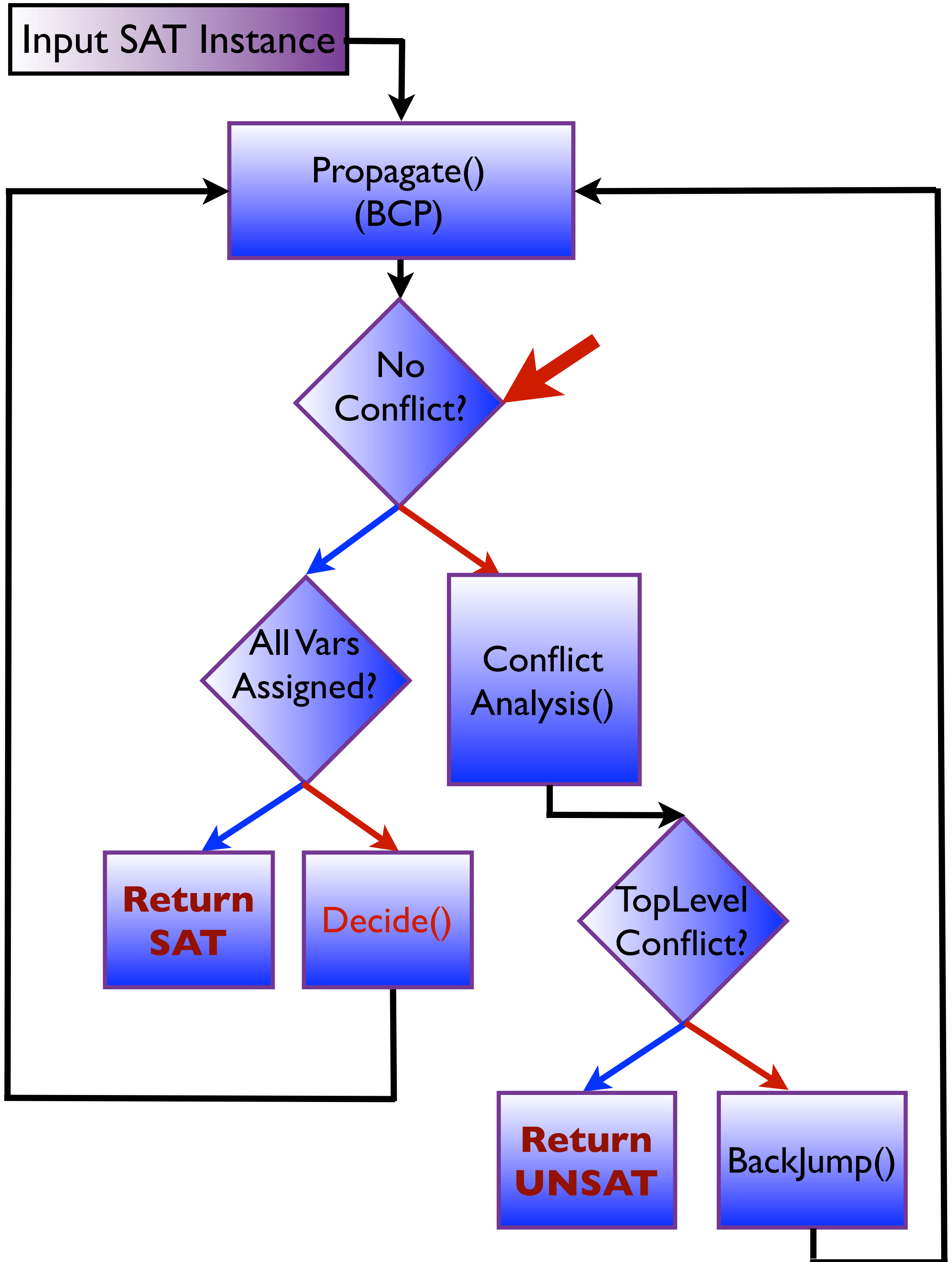

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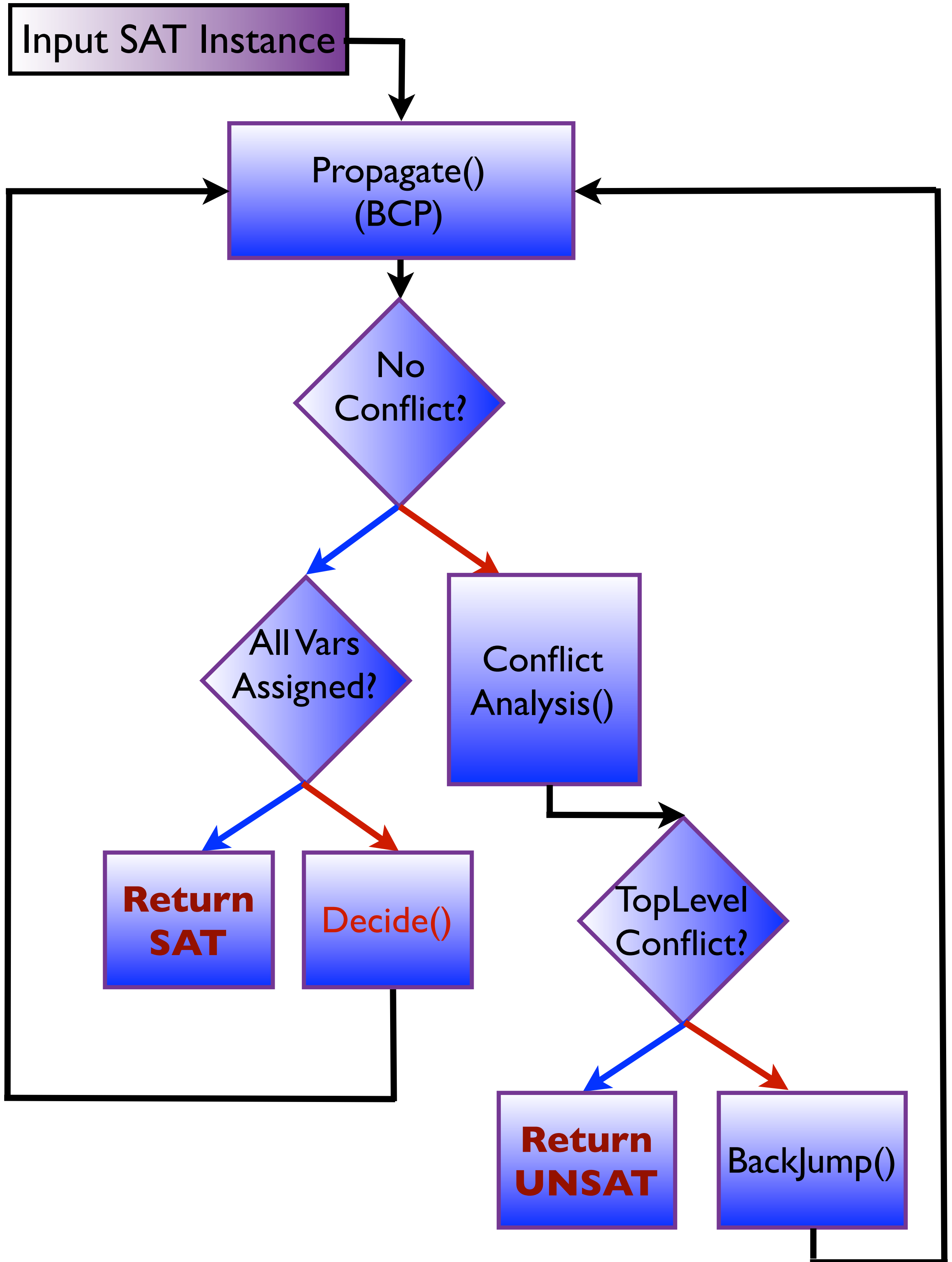

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  - Imposes dynamic variable order
  - Decision Level (DL): variable ⇒ natural number

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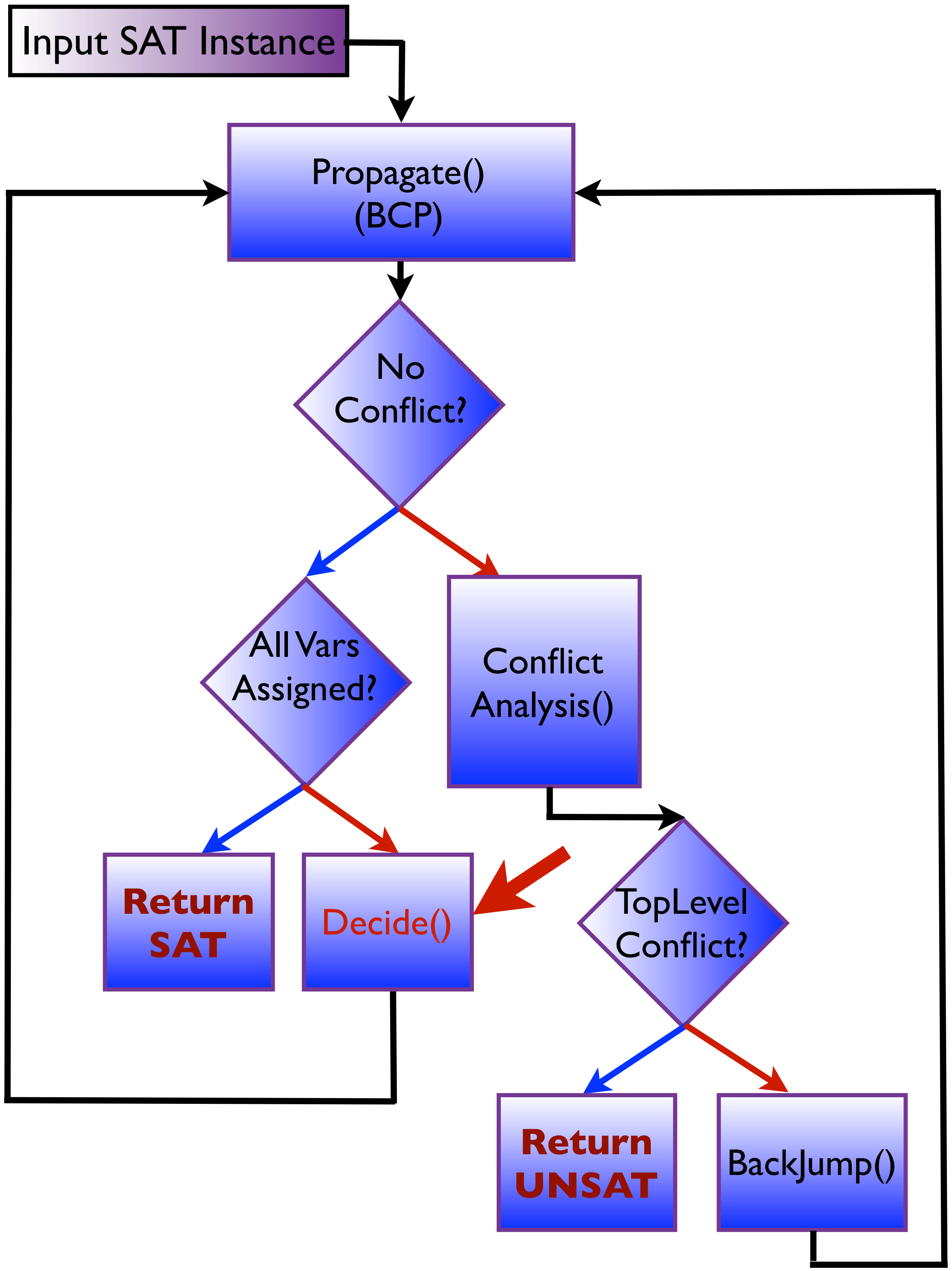

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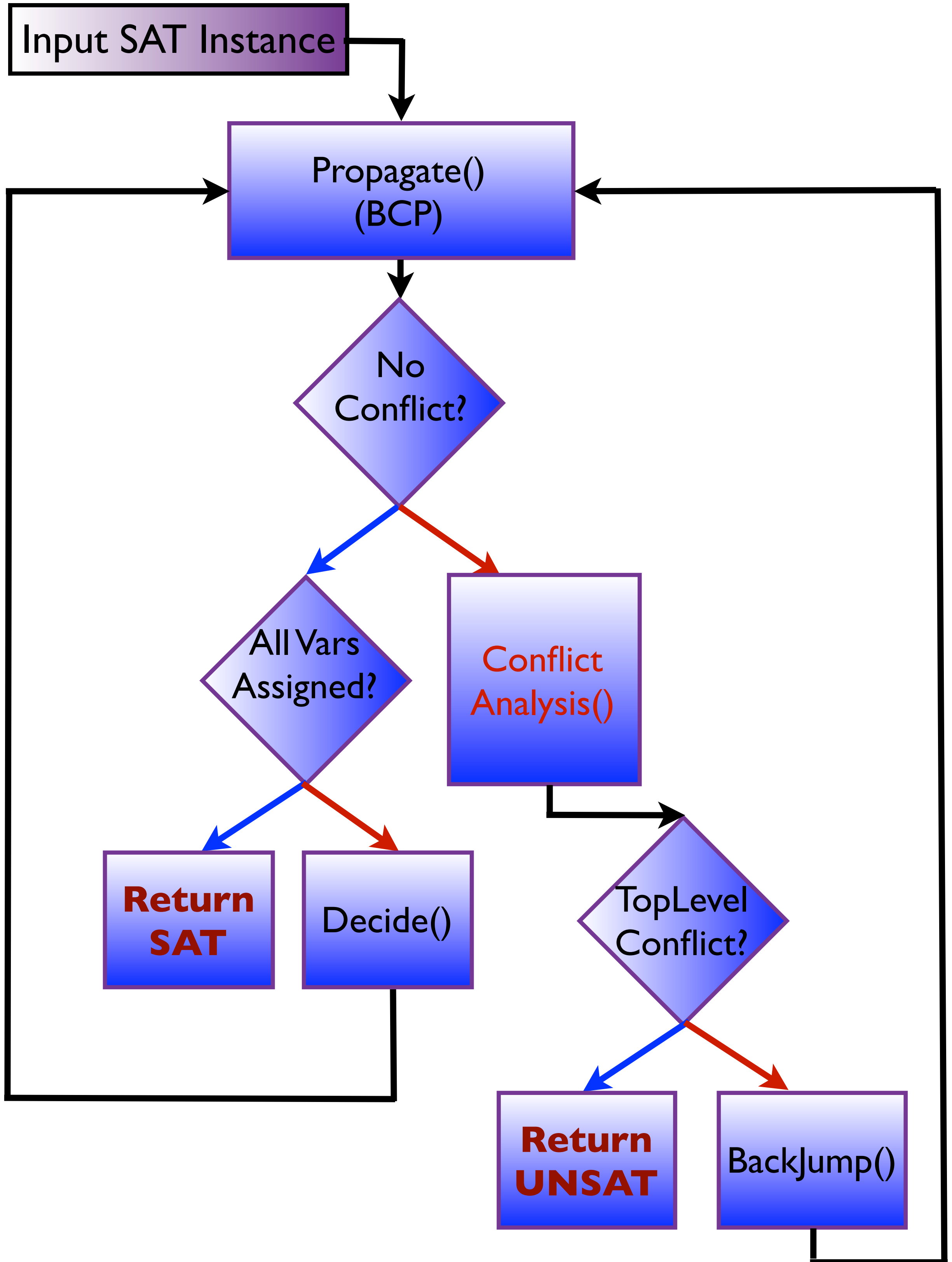

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Modern CDCL SAT Solver Architecture

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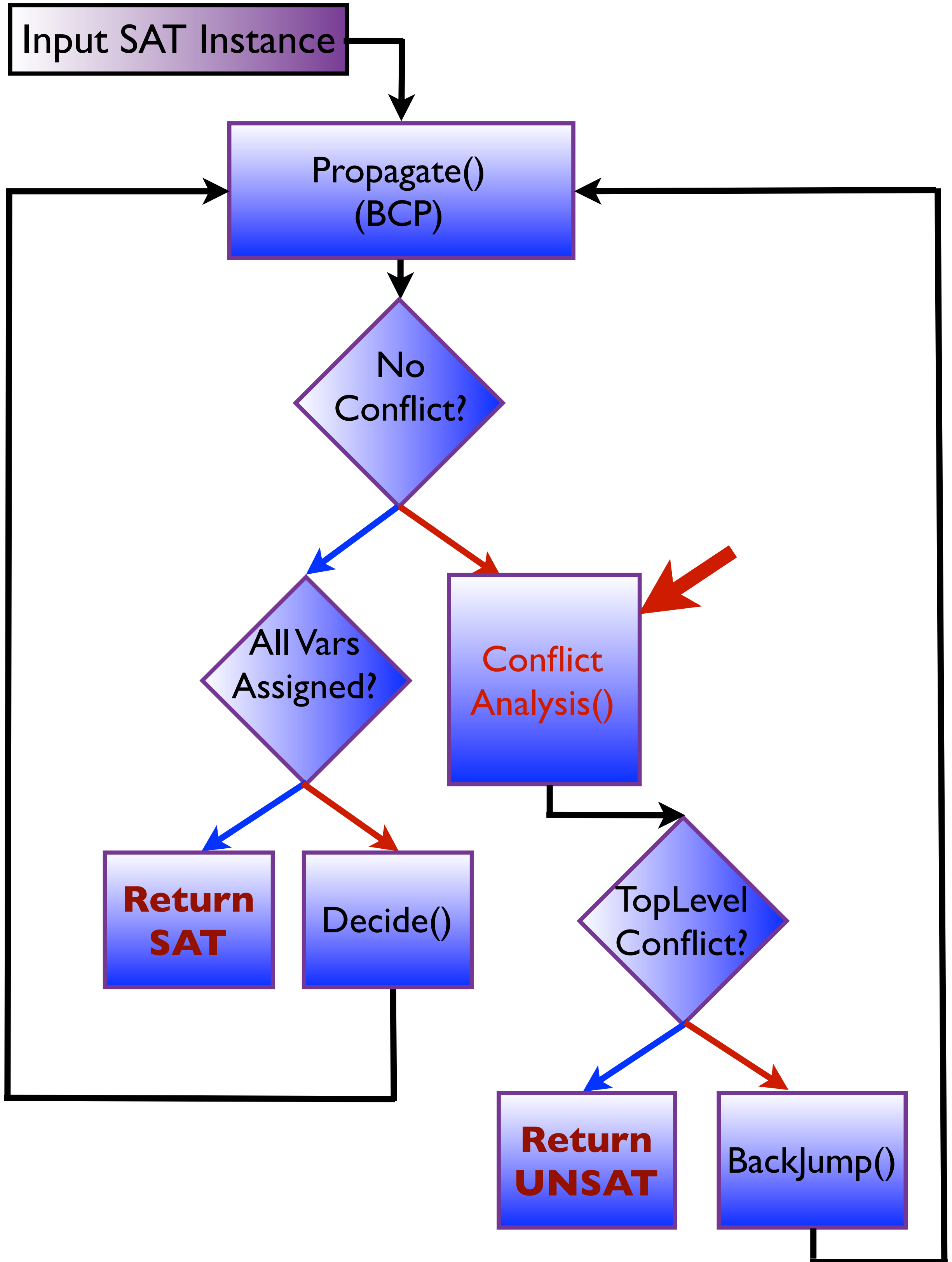

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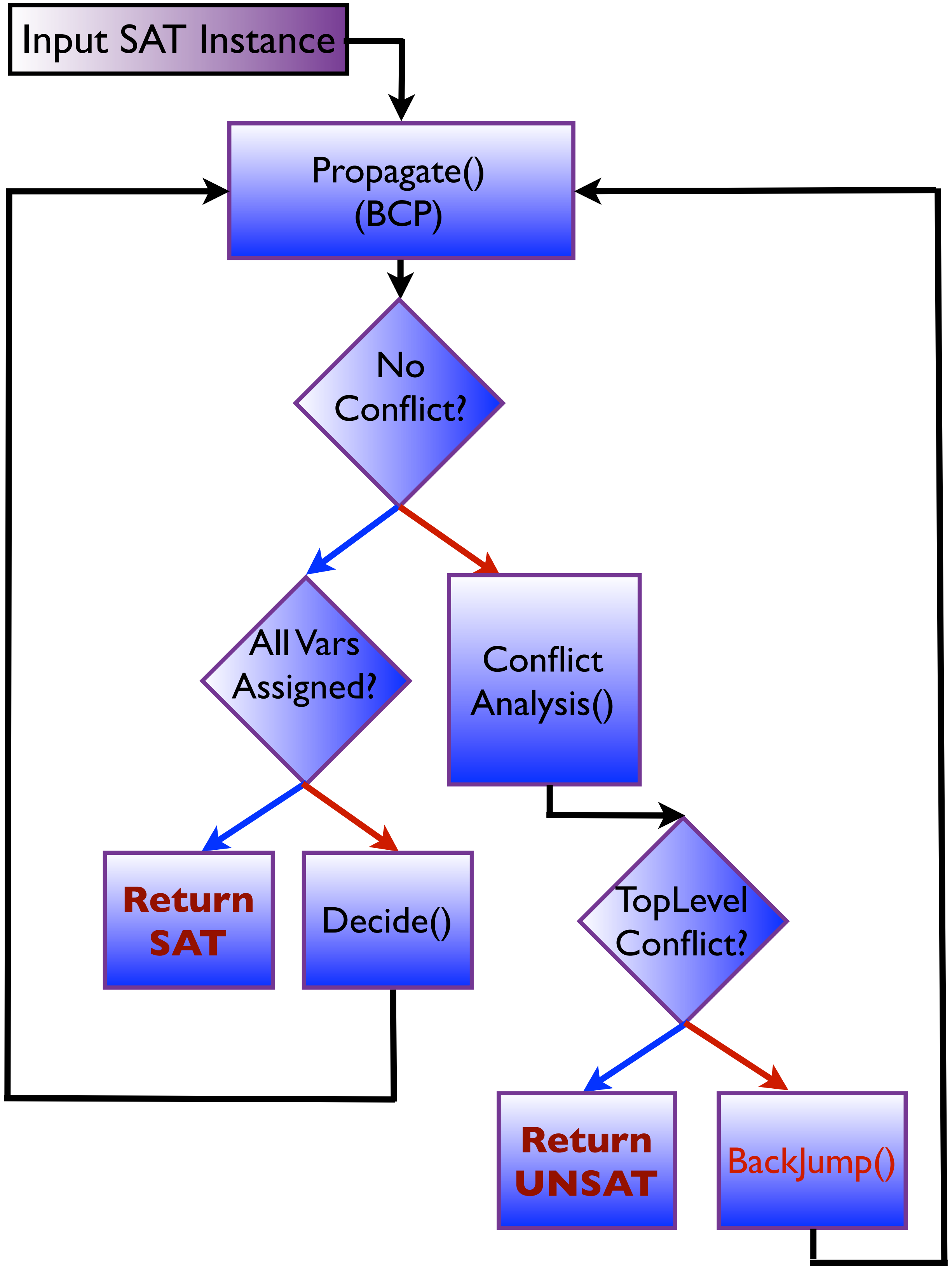

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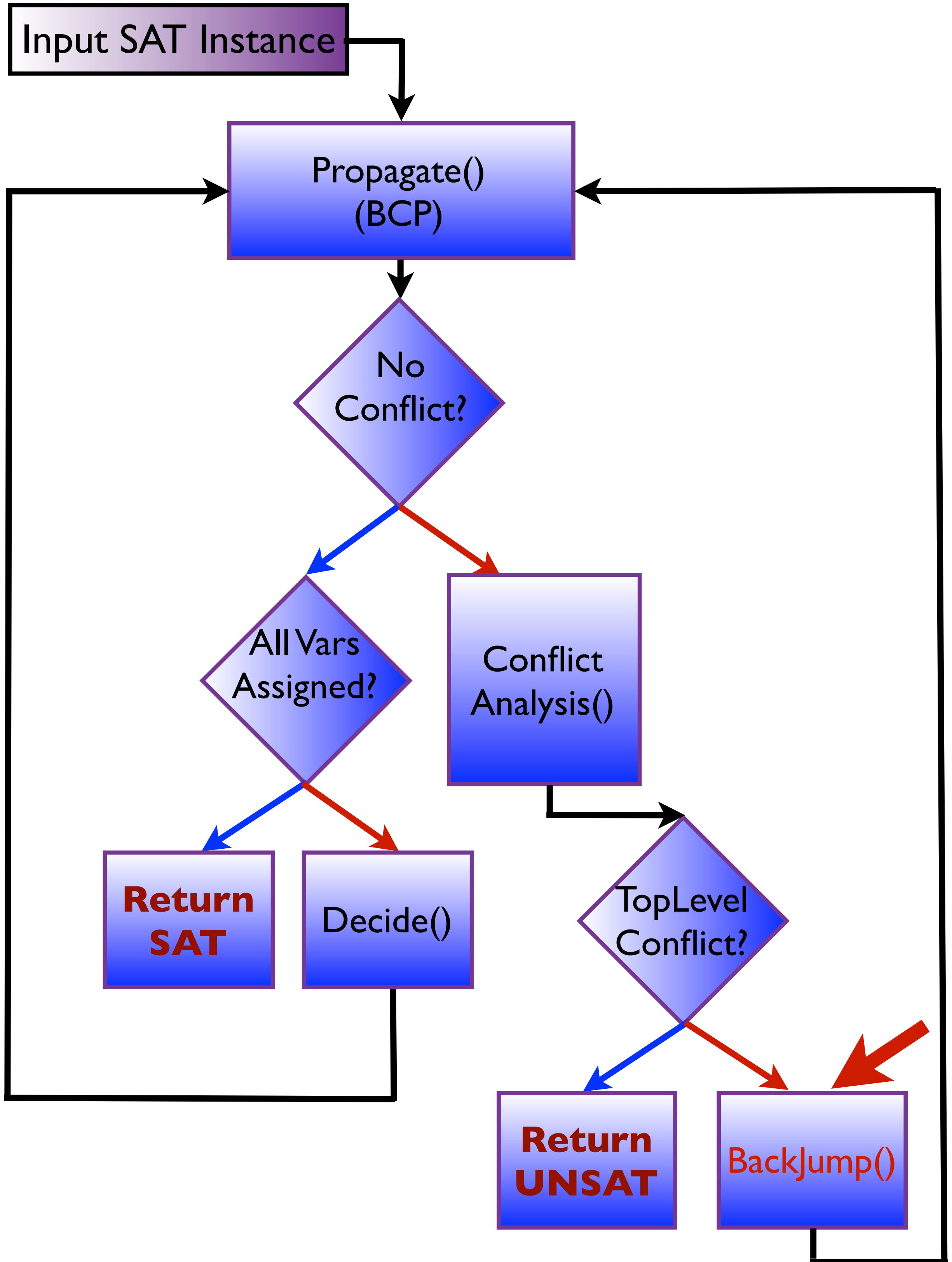
- Conflict-driven BackJump:
  - Undo the decision(s) that caused no-good assignment
  - Assign 'decision variables' different values
  - Go back several decision levels
  - Backjump: Marques-Silva, Sakallah (1999)
  - Backtrack: Davis, Putnam, Loveland, Logemann (1962)

Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()



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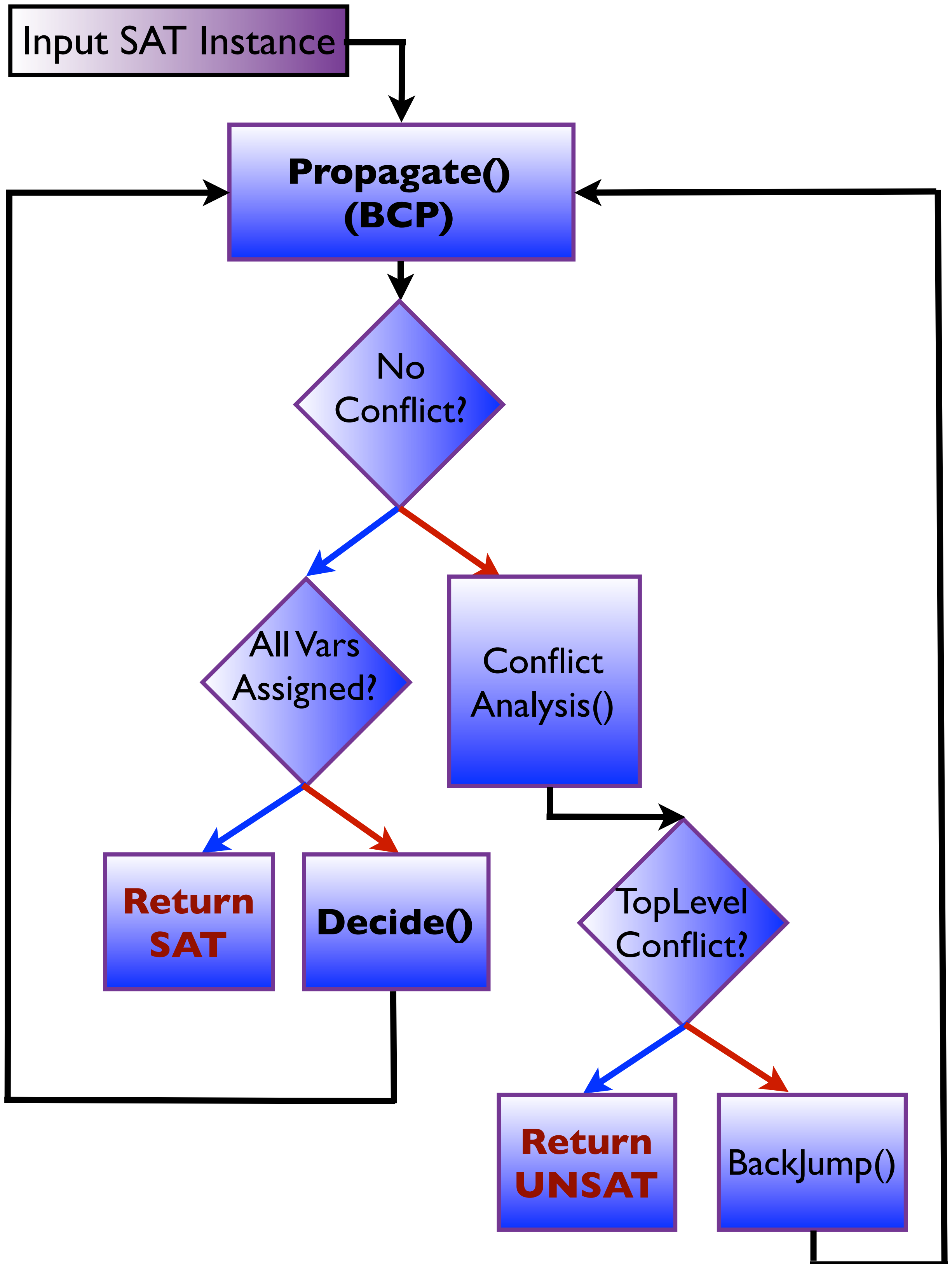
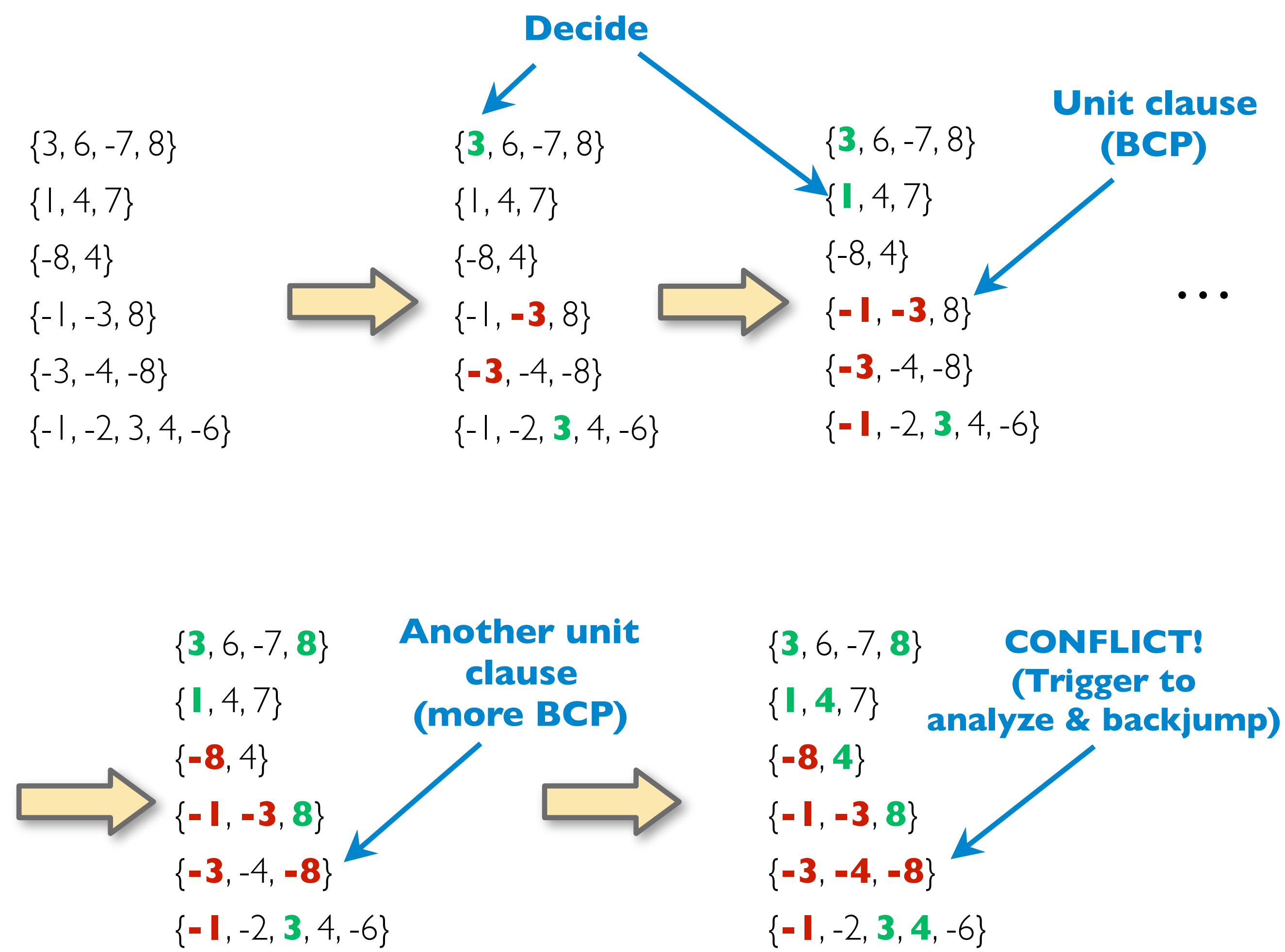
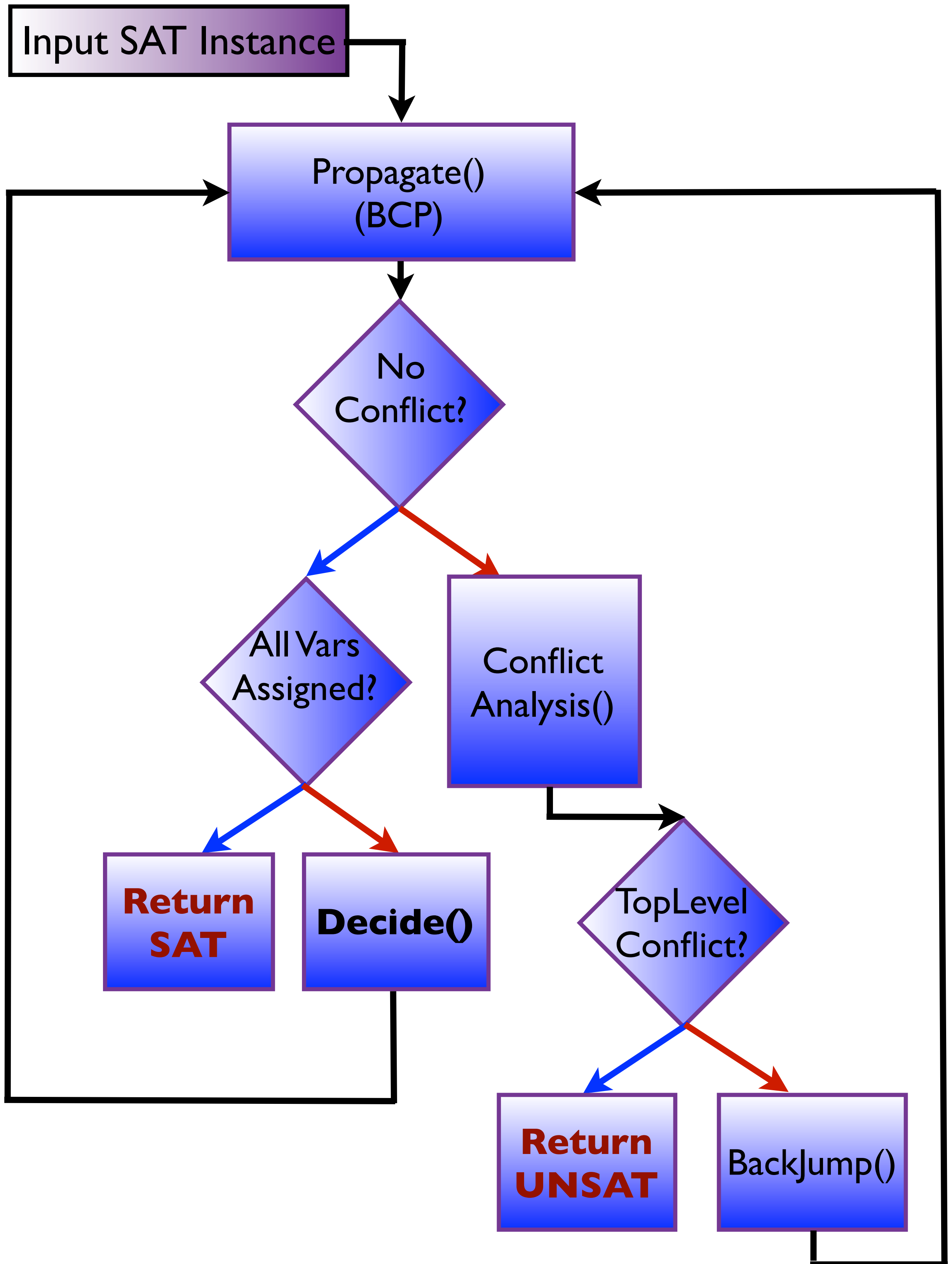




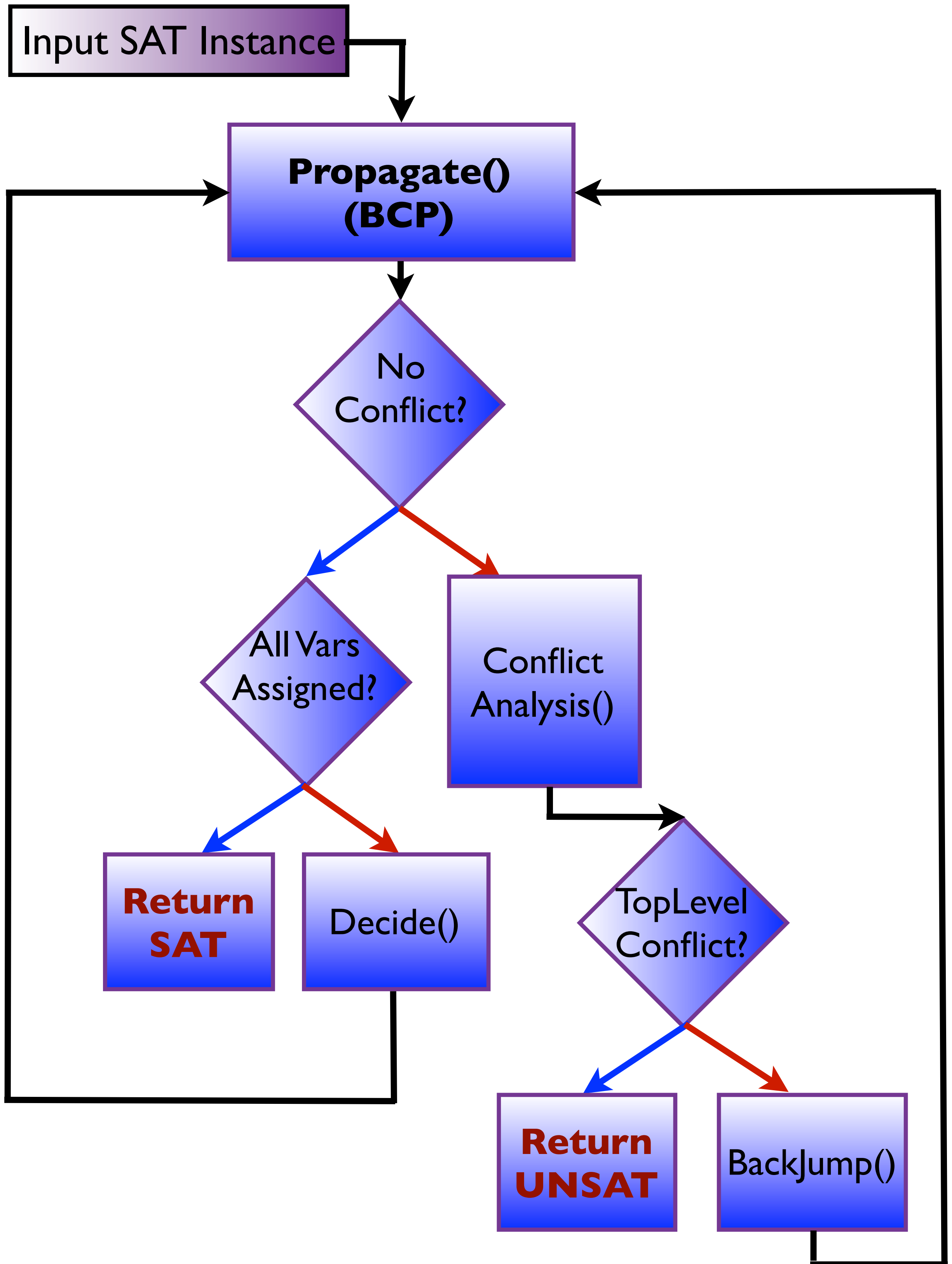
Modern CDCL SAT Solver Architecture
Decide() Details: VSIDS Heuristic



- Decide() or Branching():
  - Choose a variable & assign some value (decision)
  - Imposes dynamic variable order (Malik et al. 2001)
- How to choose a variable:
  - VSIDS heuristics
  - Each variable has an activity
  - Activity is bumped additively, if variable occurs in conflict clause
  - Activity of all variables is decayed by multiplying by const < 1
  - Next decision variable is the variable with highest activity
  - Over time, truly important variables get high activity
  - This is pure magic, and seems to work for many problems

Modern CDCL SAT Solver Architecture

Propagate() Details: Two-watched Literal Scheme



	Decide	Unit clause (BCP)
{3, 6, -7, 8}	{**3**, 6, -7, 8}	{**3**, 6, -7, 8}
{1, 4, 7}	{1, 4, 7}	{**1**, 4, 7}
{-8, 4}	{-8, 4}	{-8, 4}
{-1, -3, 8}	{-1, **-3**, 8}	{**-1**, **-3**, 8}
{-3, -4, -8}	{**-3**, -4, -8}	{**-3**, -4, -8}
{-1, -2, 3, 4, -6}	{-1, -2, **3**, 4, -6}	{**-1**, -2, **3**, 4, -6}

…

Watched Literal	Watcher List
-1	{-1, -3, 8},...
-3	{-1, -3, 8},...
...	...

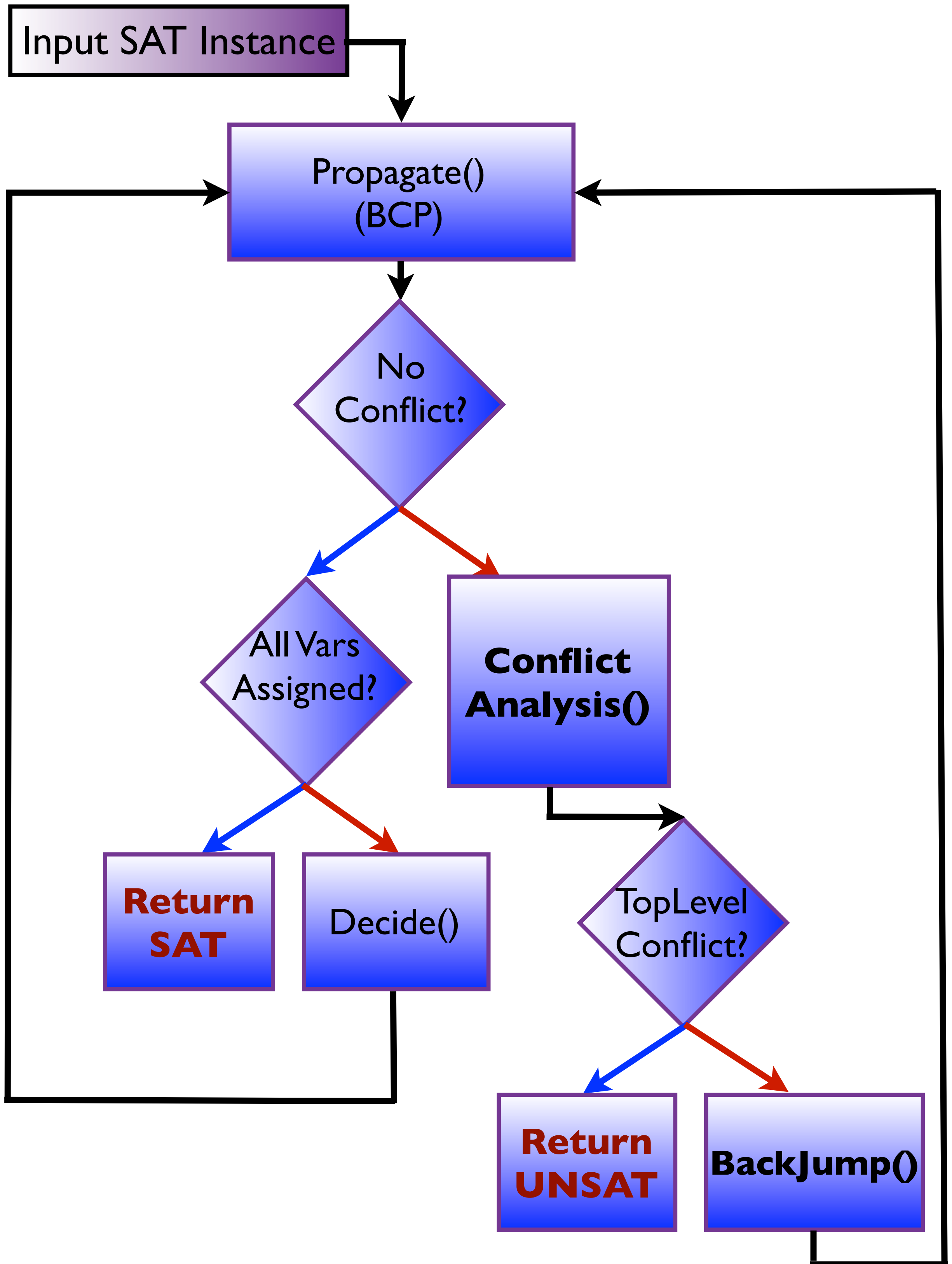
Watched Literal	Watcher List
-1	{-1, -3, 8},...
-3	...
8	{-1, -3, 8},...
...	...

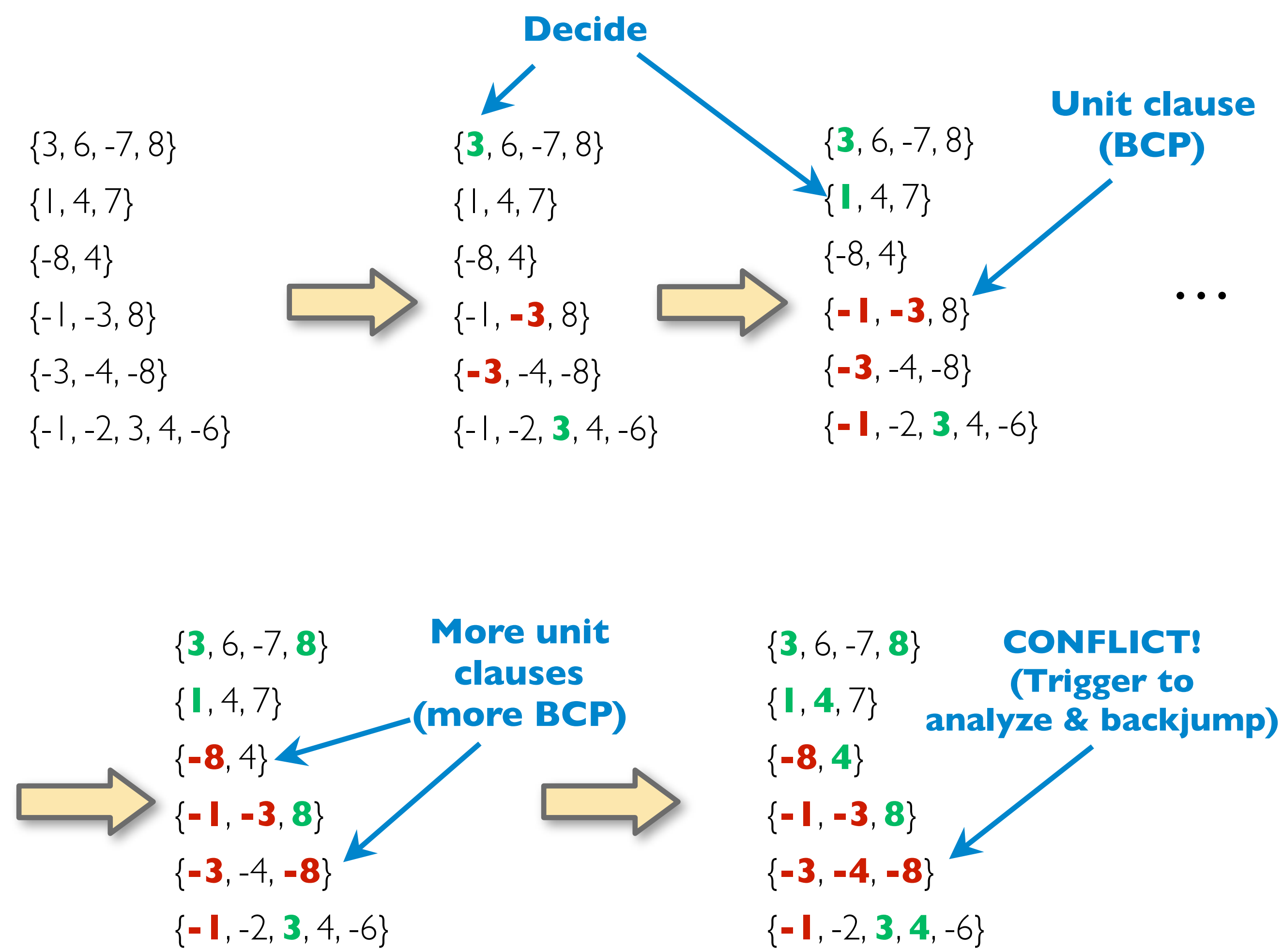
Watched Literal	Watcher List
-1	...
-3	...
8	{-1, -3, 8},...
...	...

The constraint propagates 8

Modern CDCL SAT Solver Architecture

Propagate(), Decide(), Analyze/Learn(), BackJump()

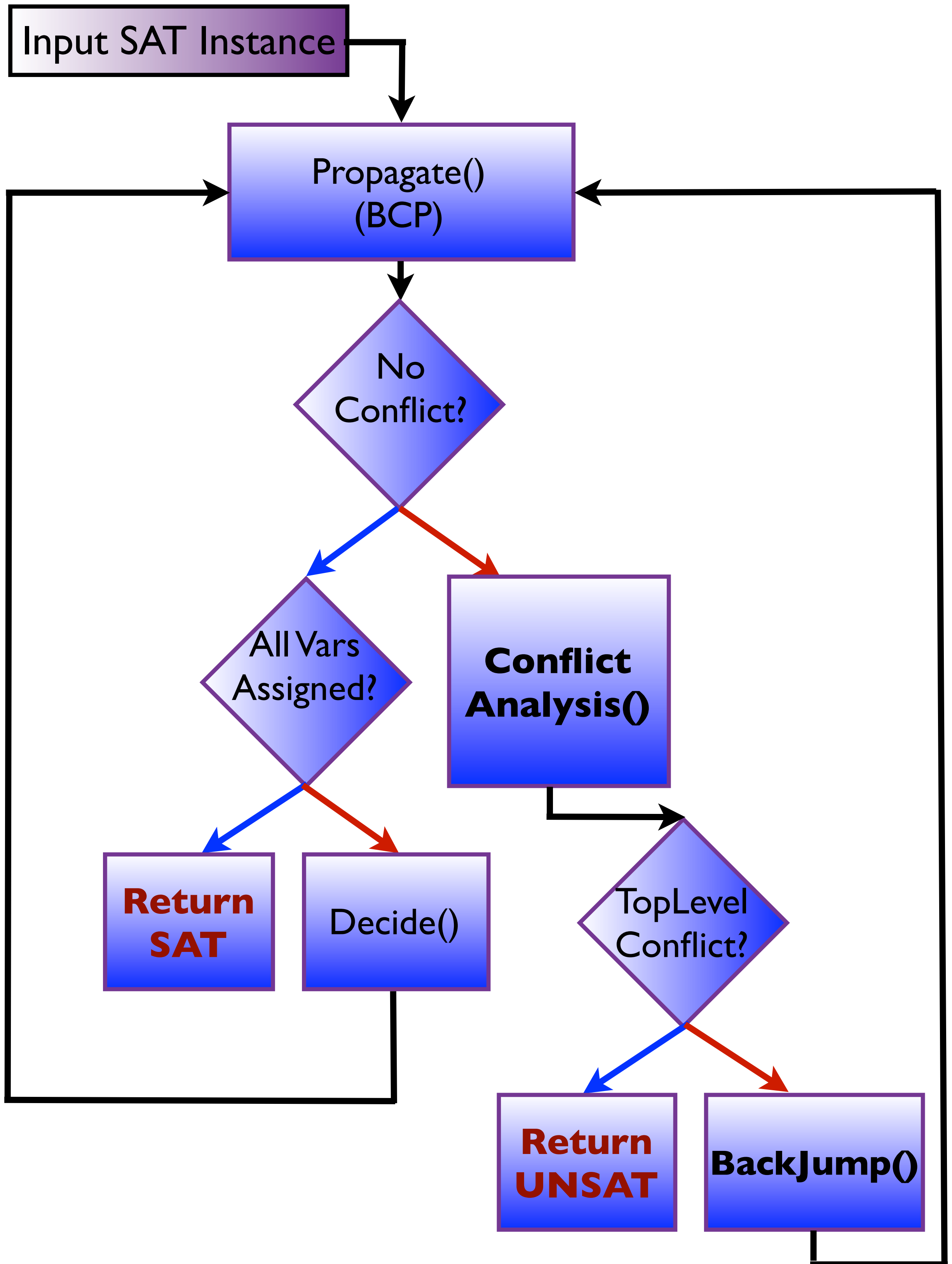




Basic Backtracking Search

- Flip the last decision 1
- Try setting 1 to False
- Highly inefficient
- No learning from mistakes

Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details



Some Definitions

- Decision Level (DL)
  - Map from Boolean variables in input to natural numbers
  - All unit clauses in input & resultant propagations get DL = 0
  - Every decision var gets a DL in increasing order >= 1
  - All propagations due to decision var at DL=x get the DL=x
- Conflict Graph (CG) or Implication Graph
  - Directed Graph that records decisions & propagations
  - Vertices: literals, Edge: unit clauses
- Conflict Clause (CC)
  - Clause returned by Conflict Analysis(), added to conflict DB
  - Implied by the input formula
  - A cut in the CG
  - Prunes the search
- Assignment Trail (AT)
  - A stack of partial assignment to literals, with DL info

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Implication Graph

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, ...\}$

Current decision: $\{X_1 = 1@6\}$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

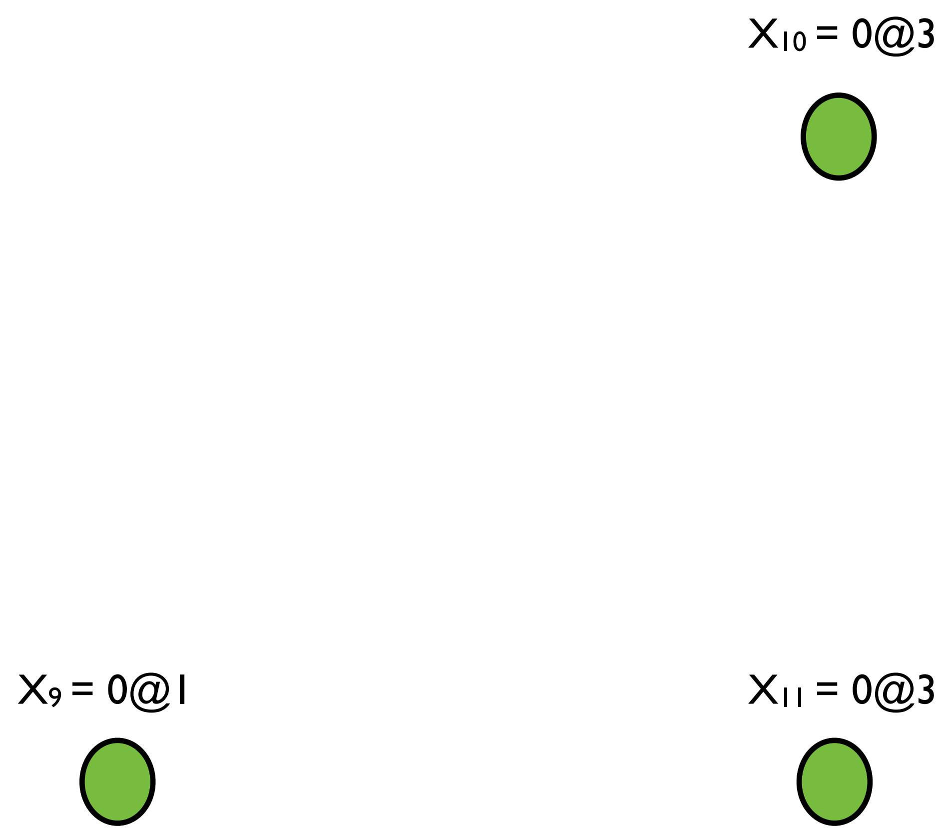
$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details: Implication Graph

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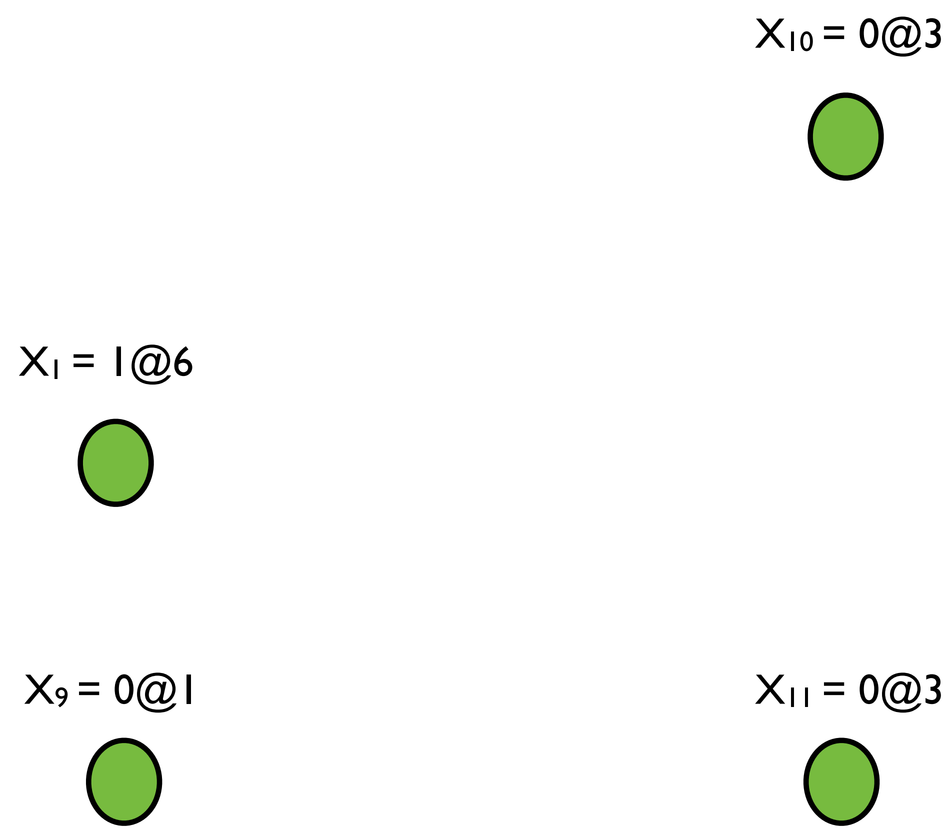

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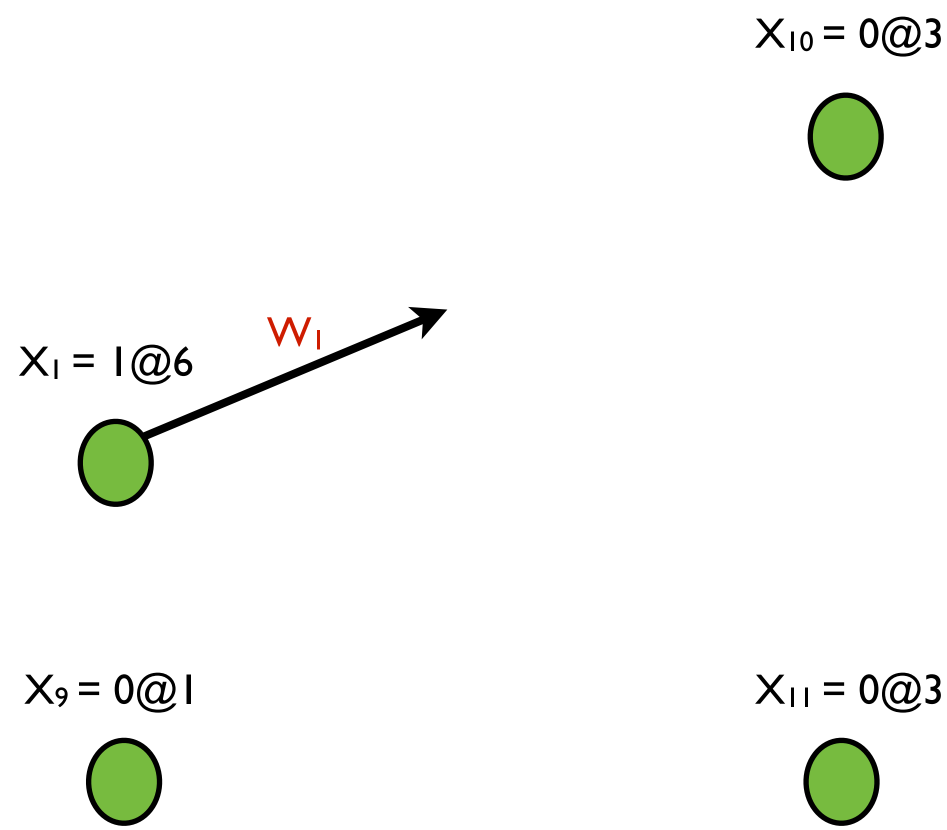

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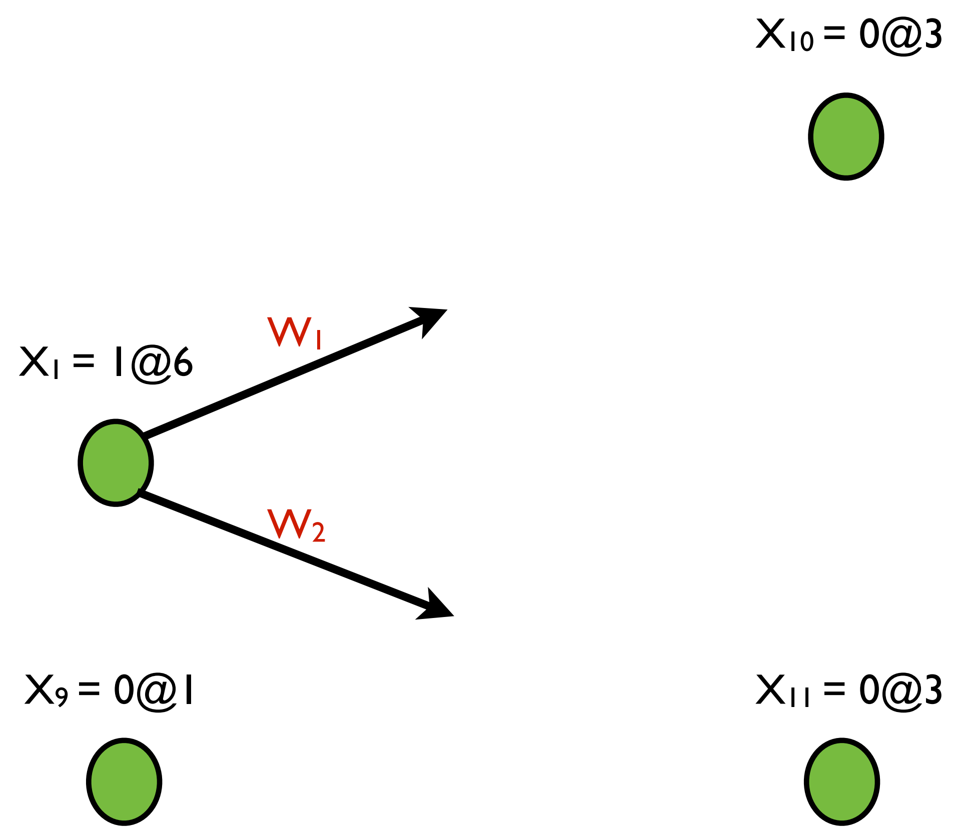

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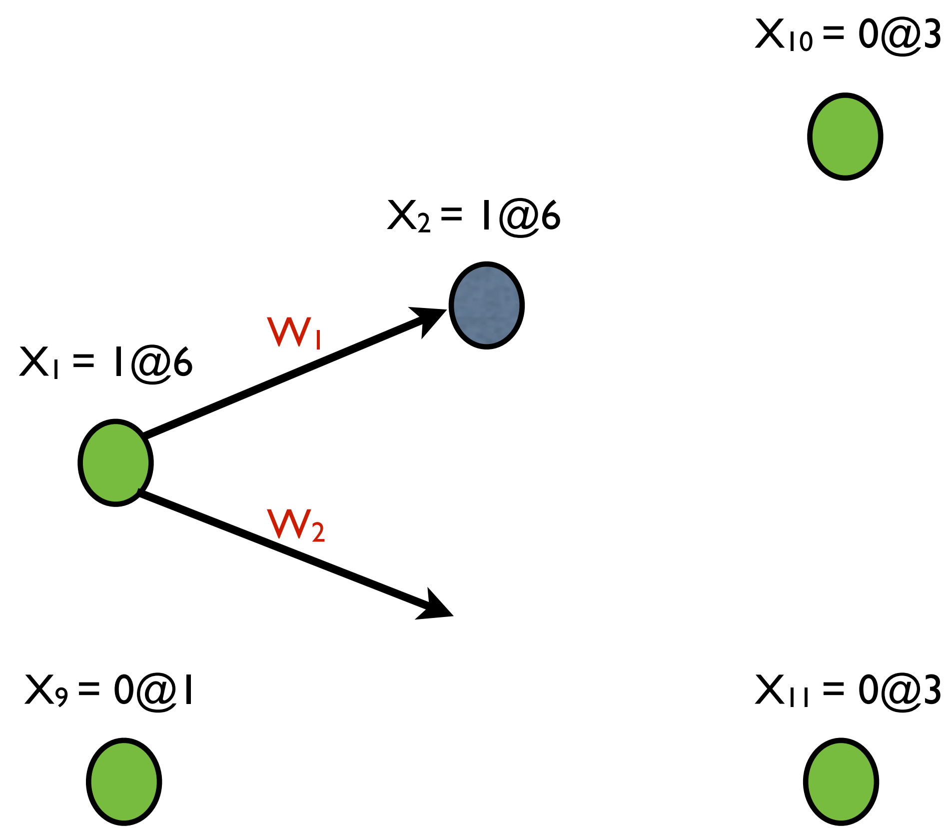

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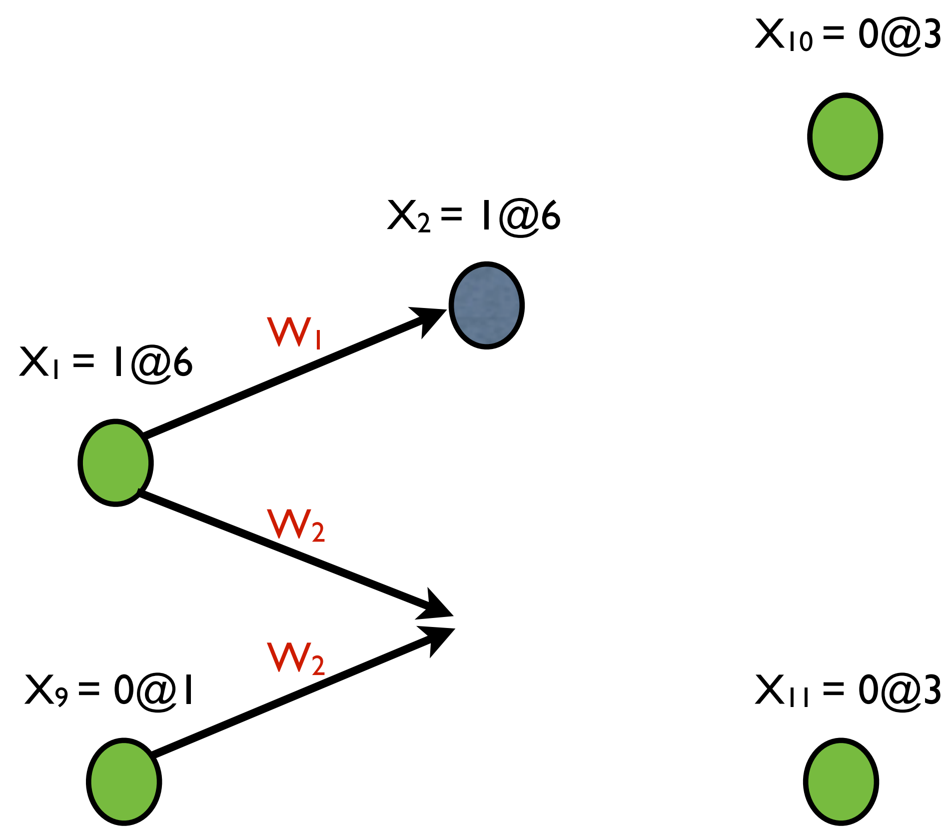

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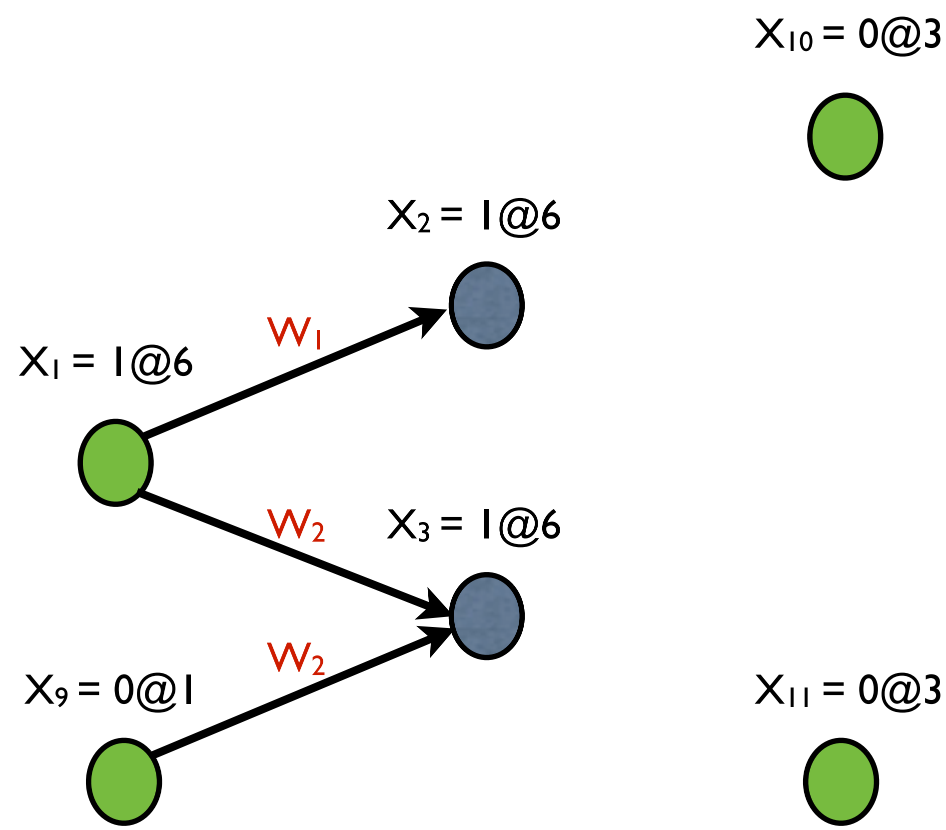

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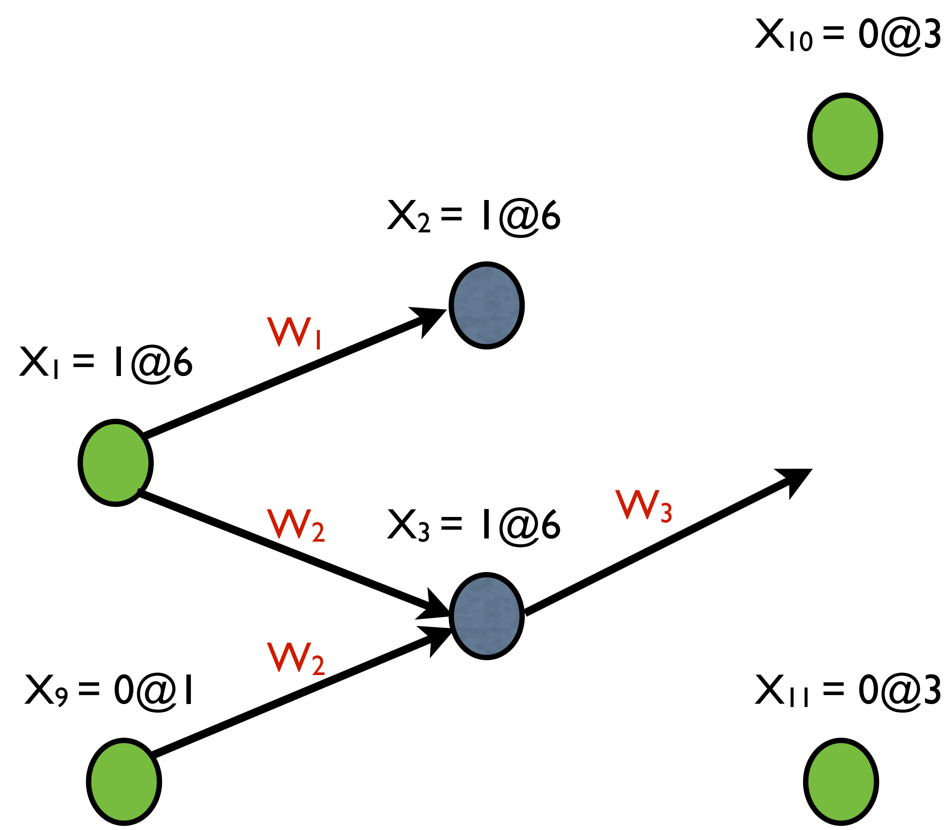

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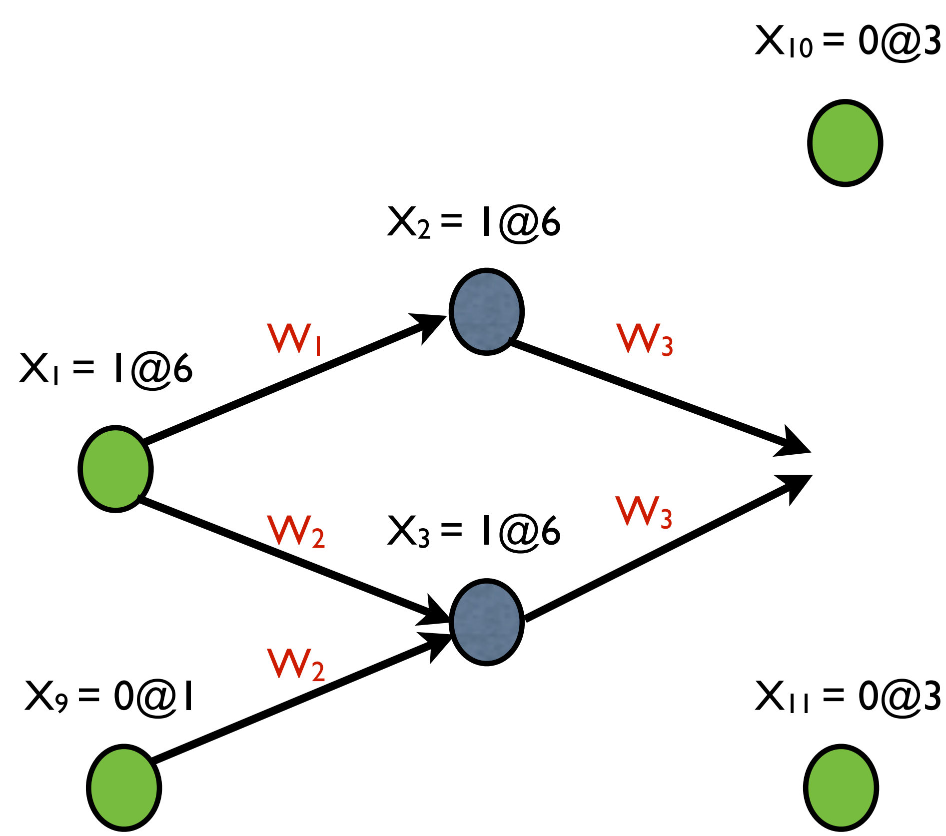

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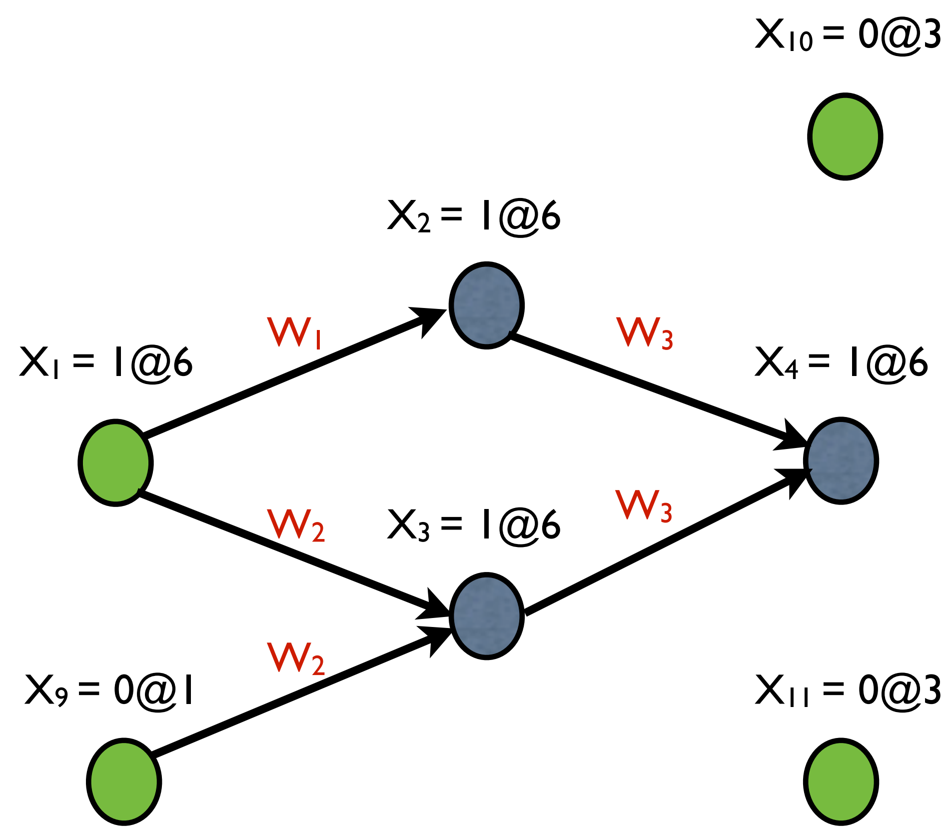

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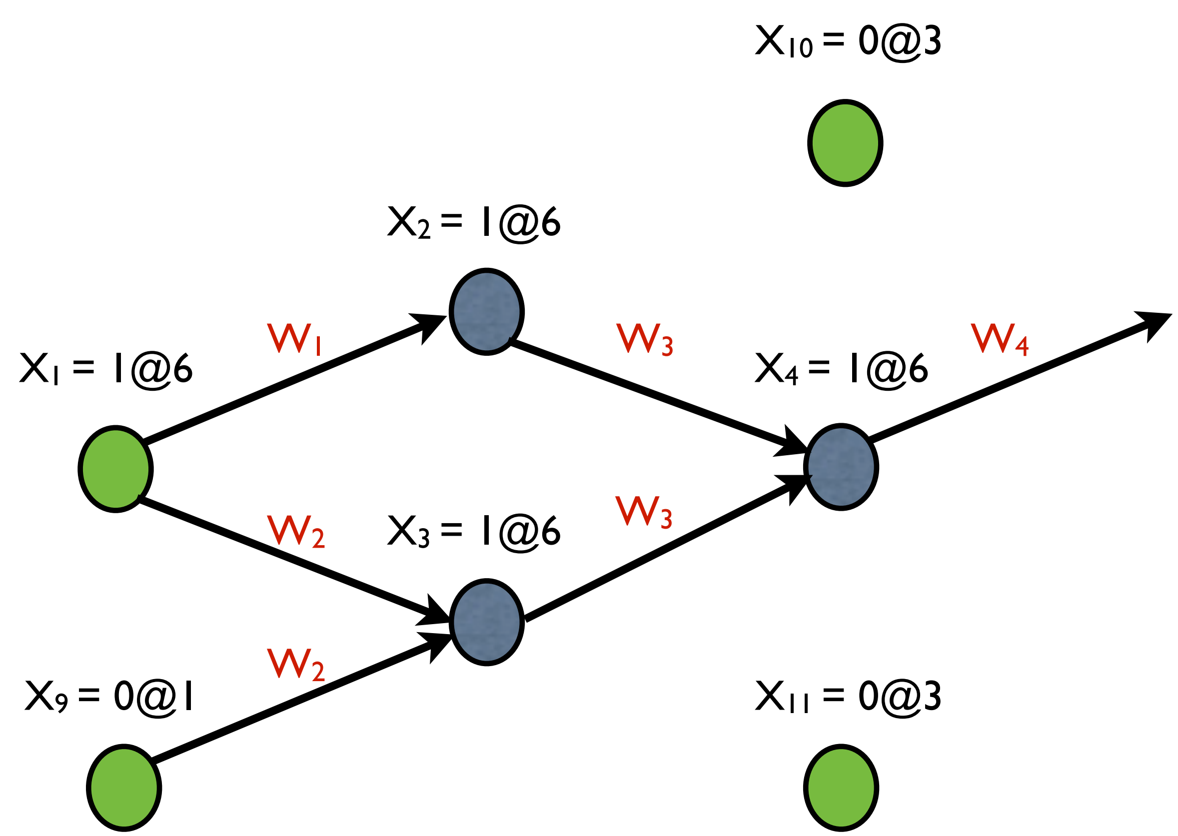

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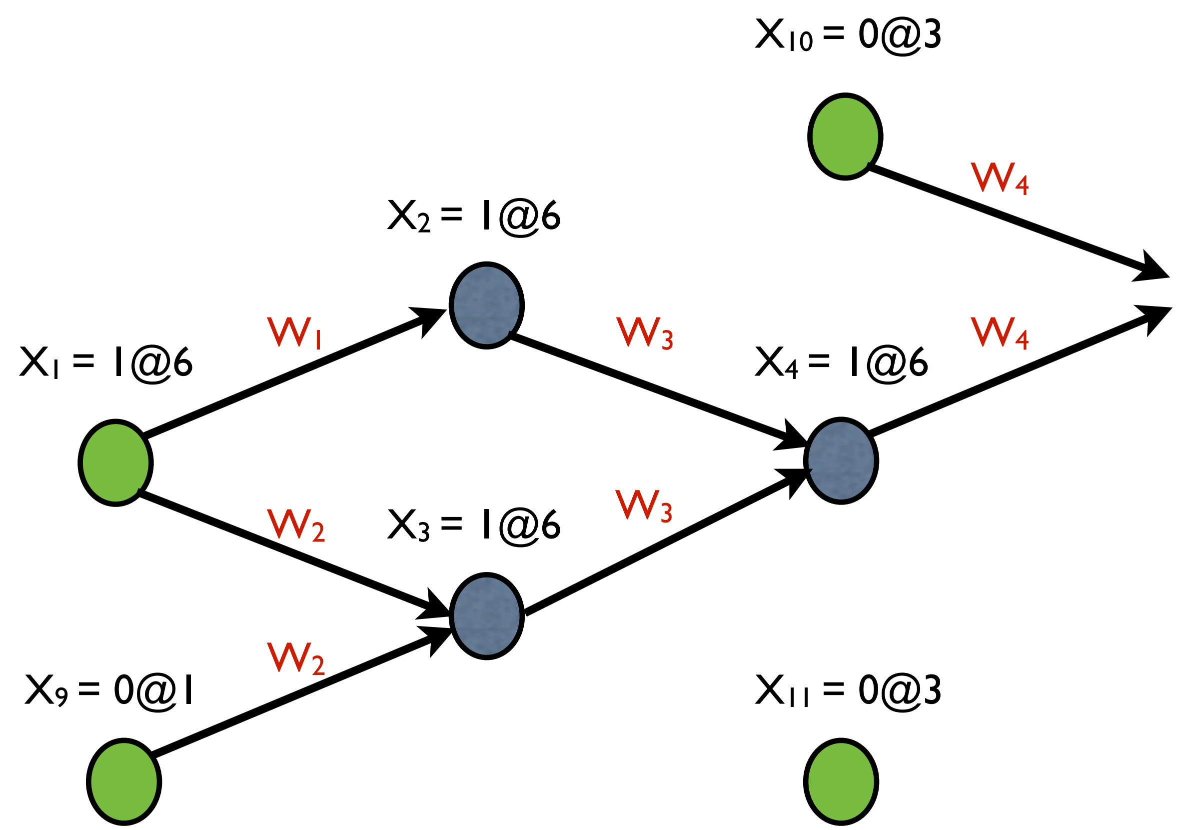

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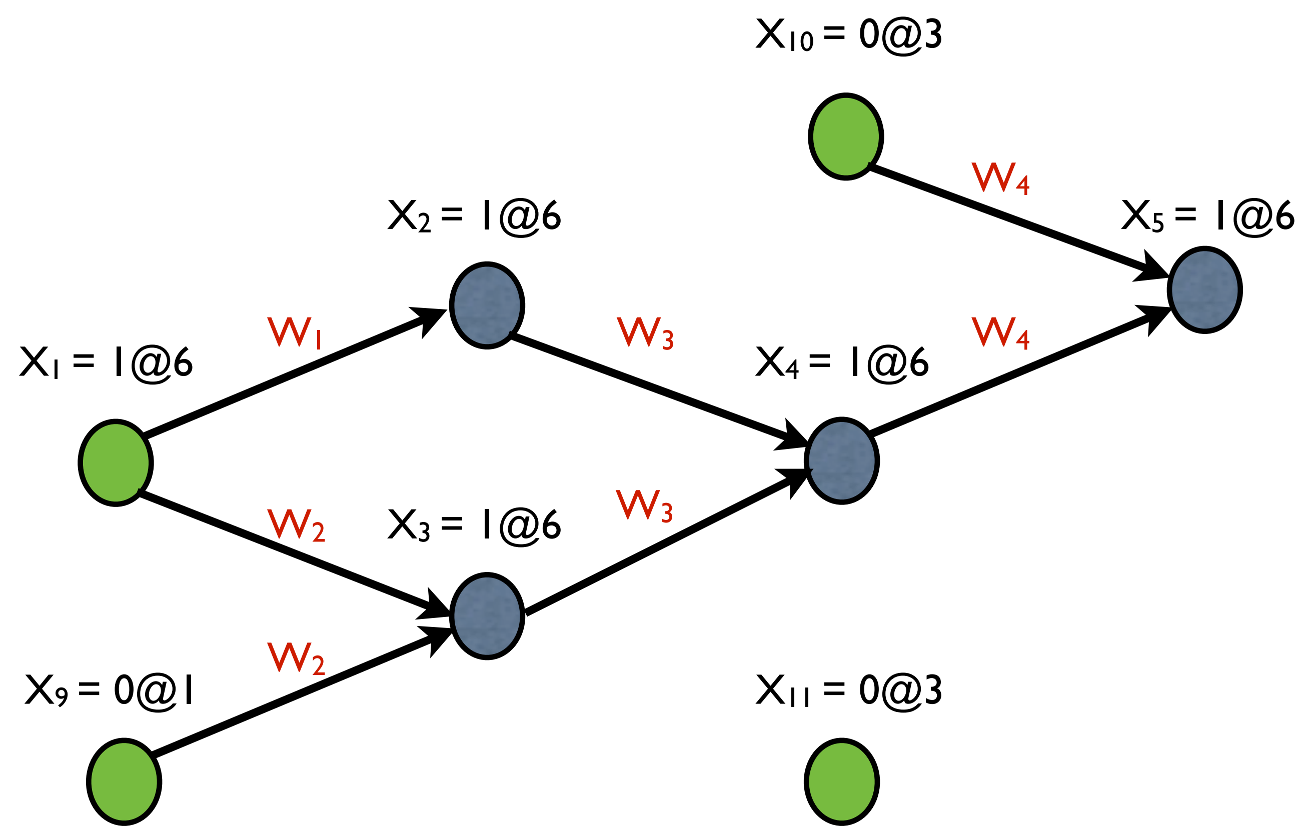

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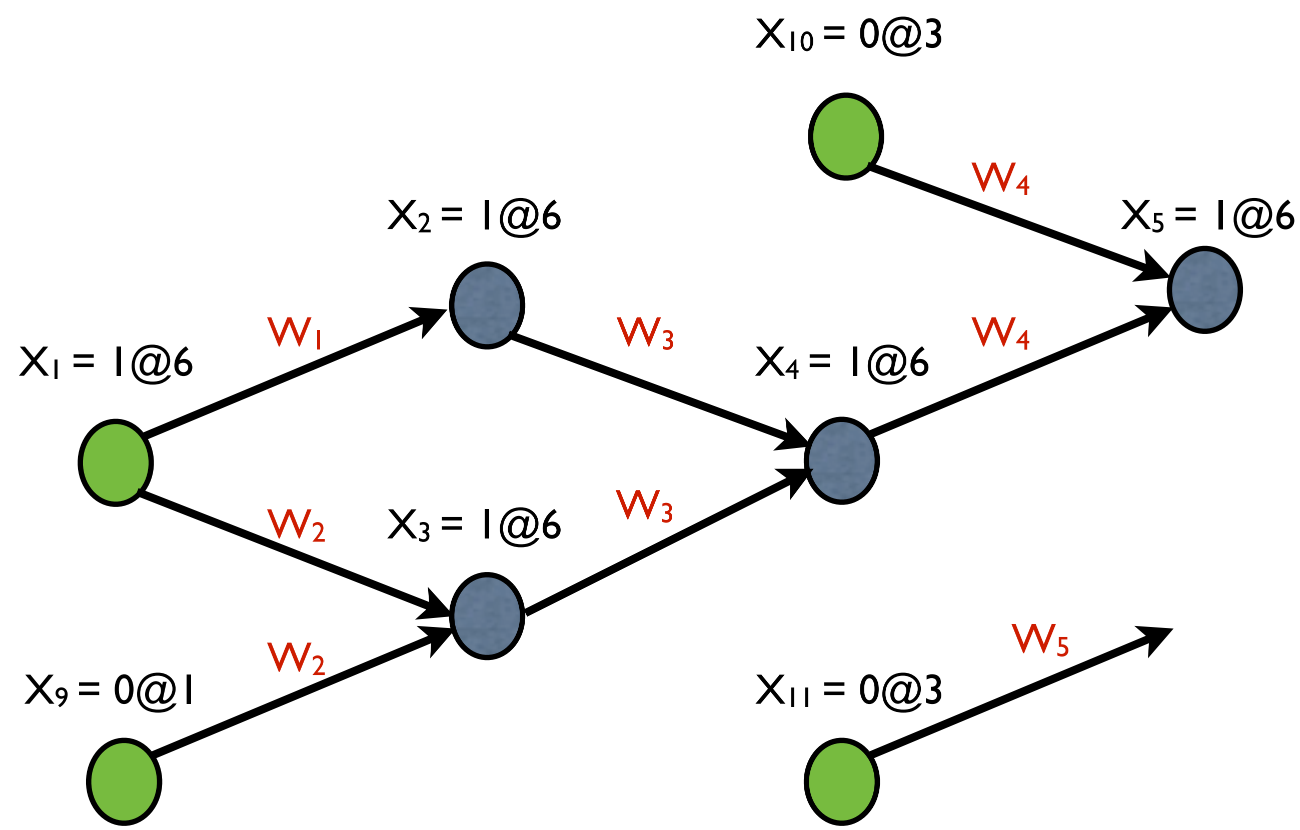

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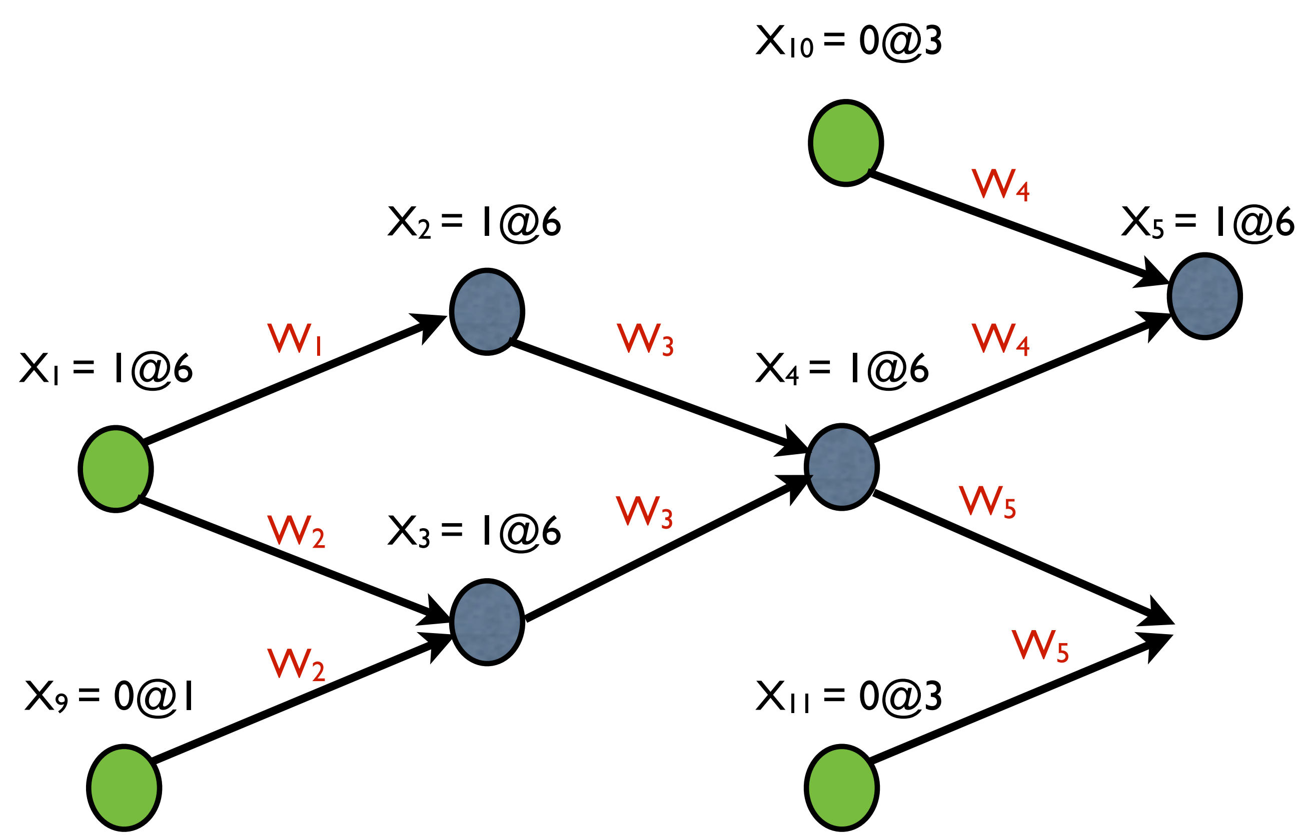

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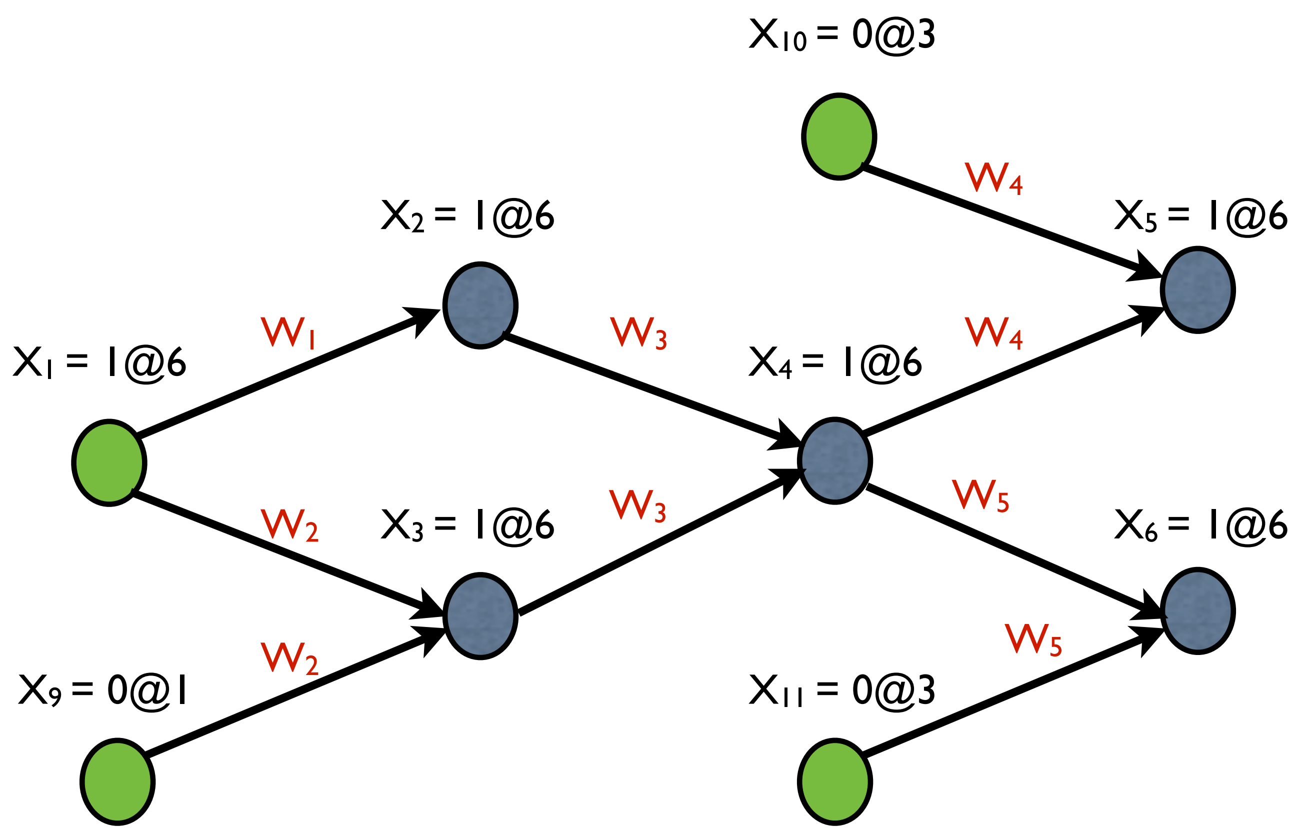

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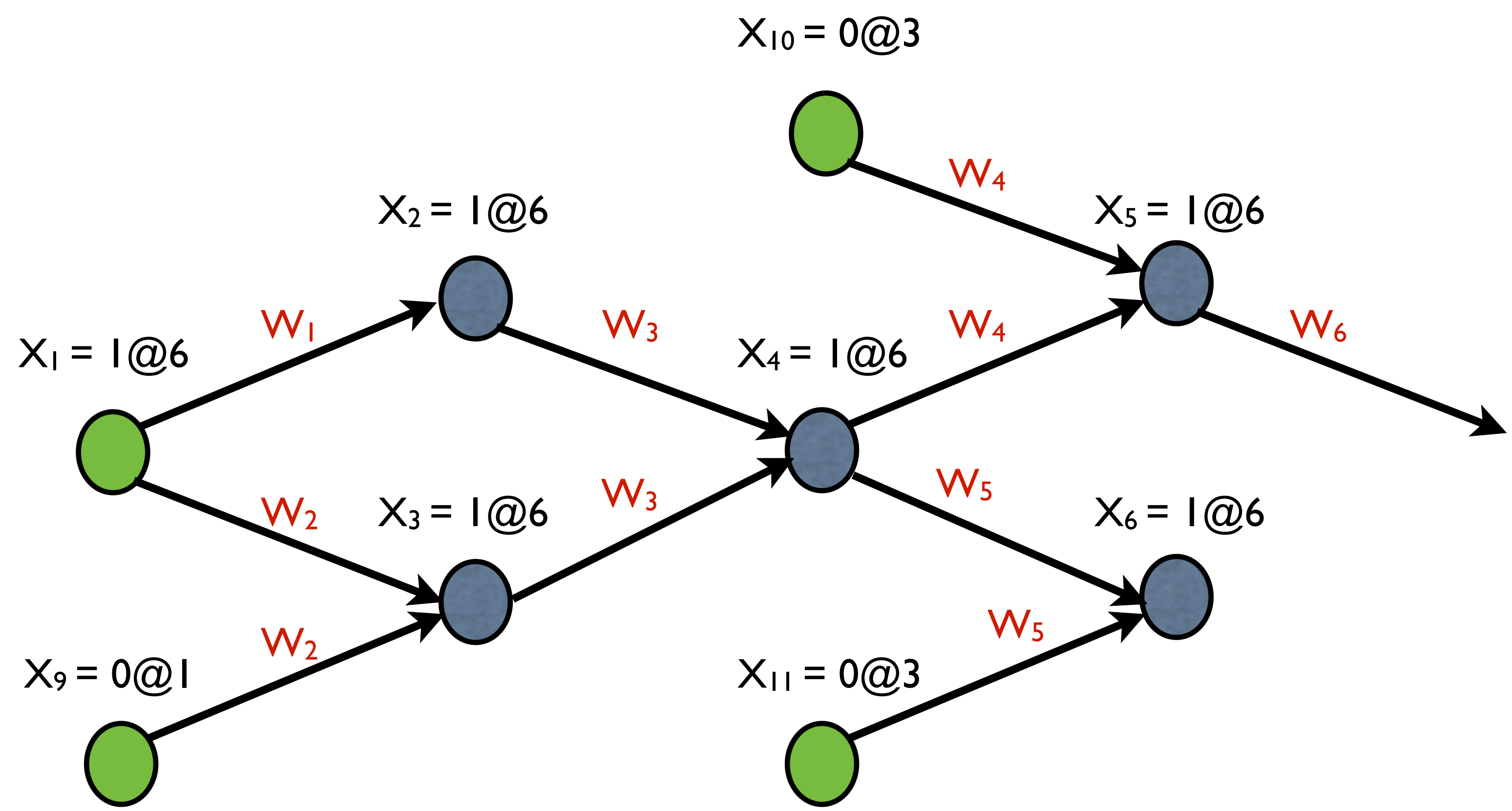

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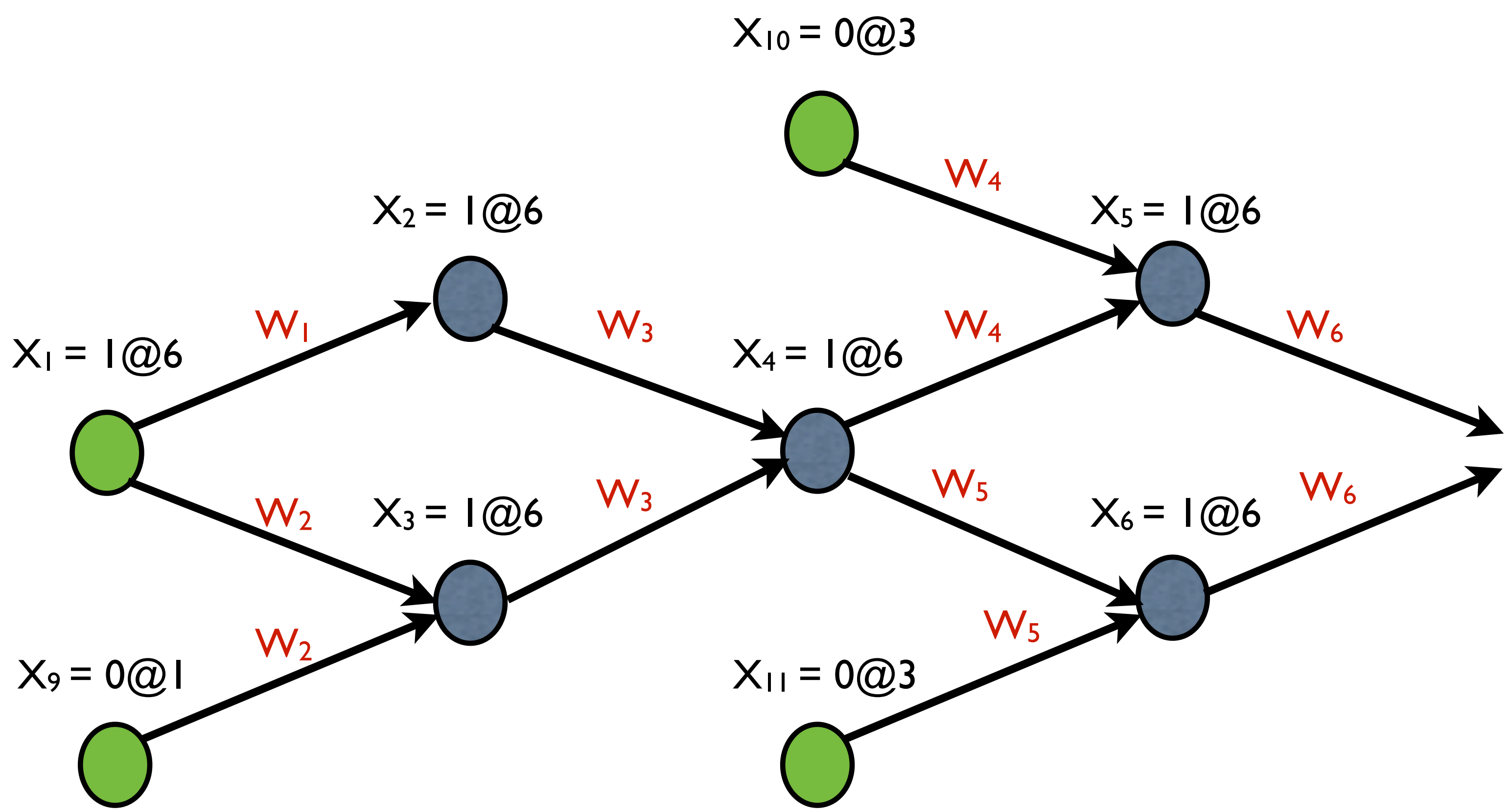

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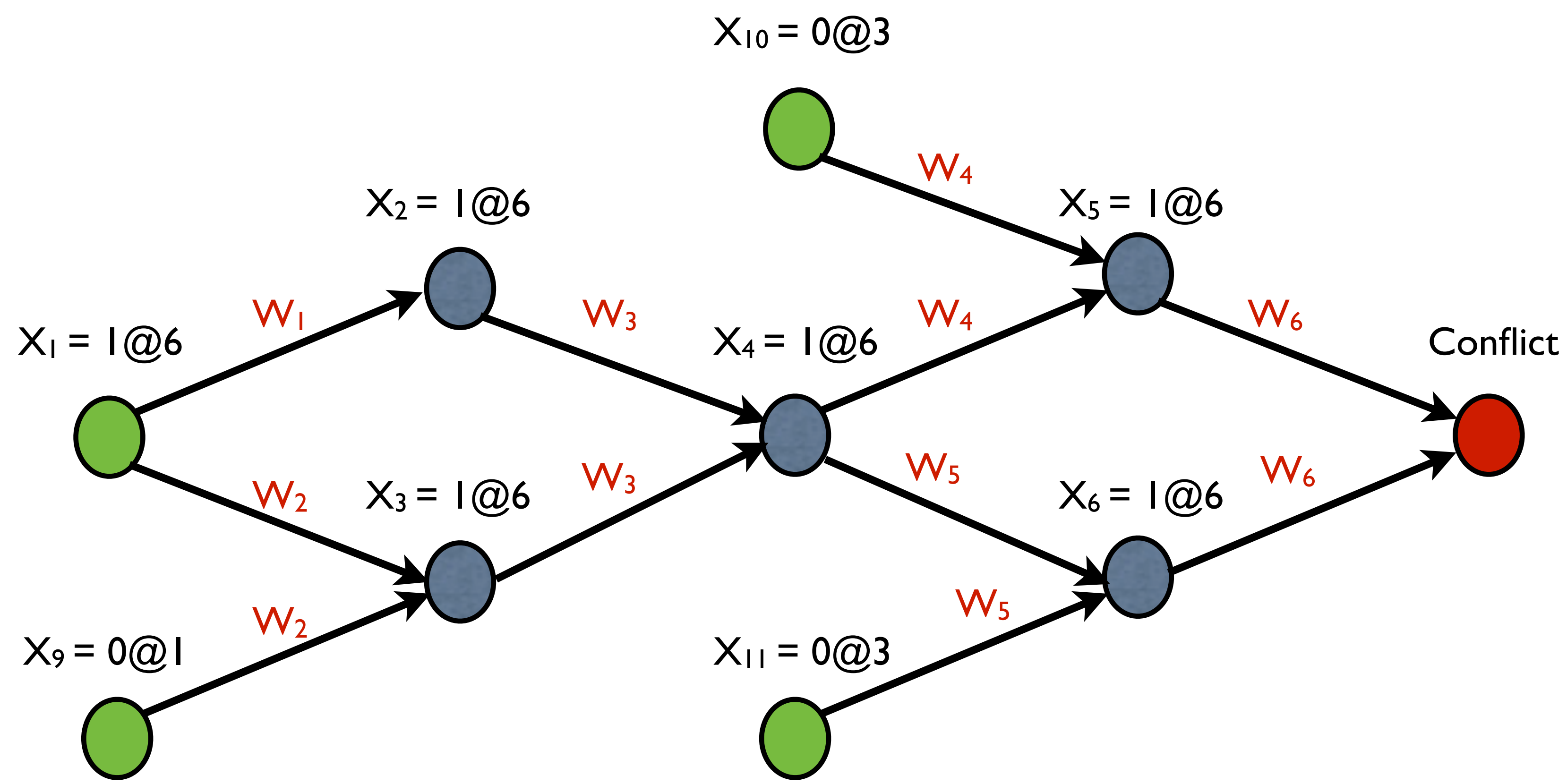

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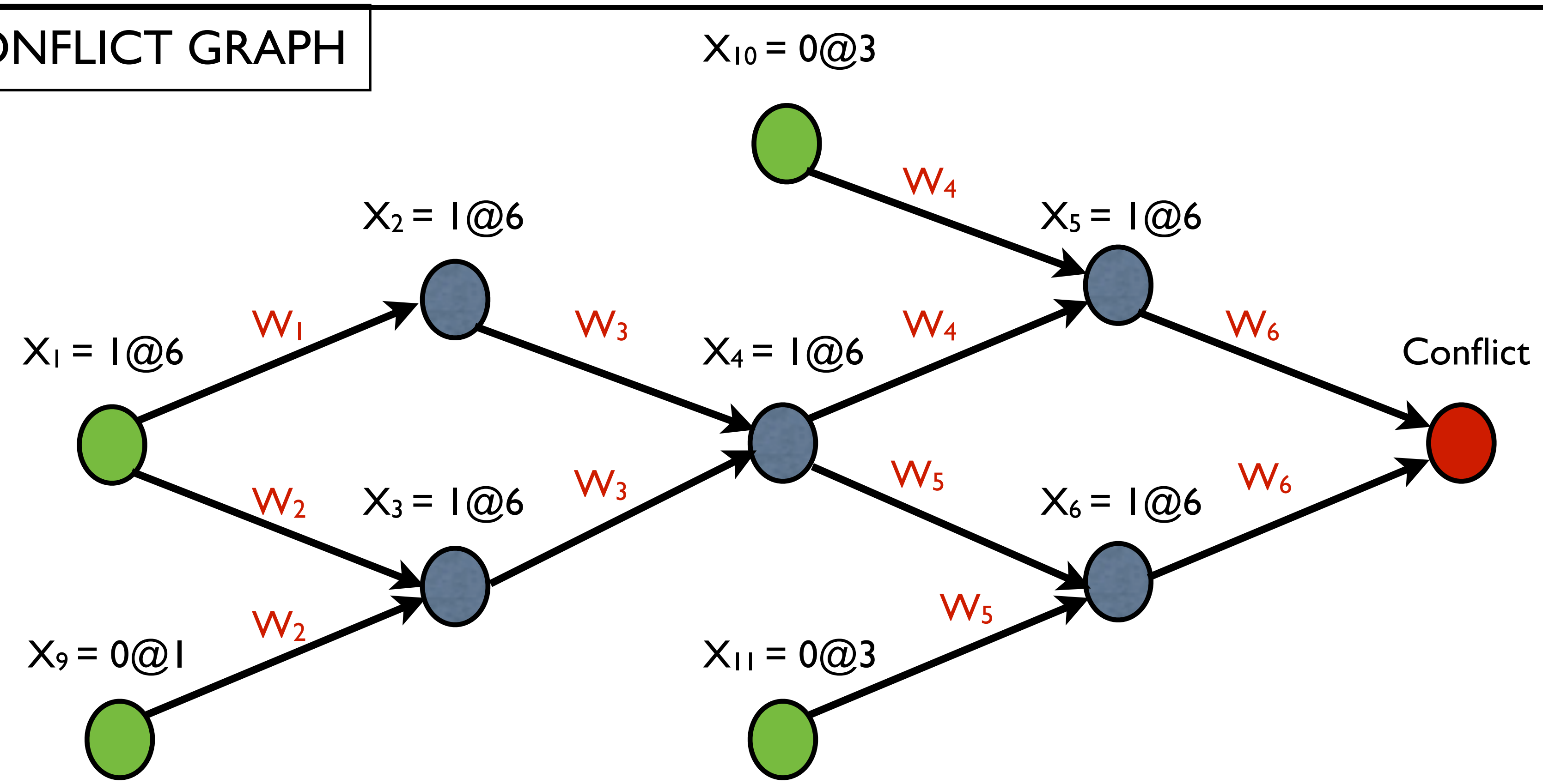
CONFLICT GRAPH



Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Conflict Clause

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, \ldots\}$

Current Decision: $\{X_1 = 1@6\}$

Simplest strategy is to traverse the conflict graph backwards until decision variables: conflict clause includes only decision variables $(\neg X_1 + X_9 + X_{10} + X_{11})$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

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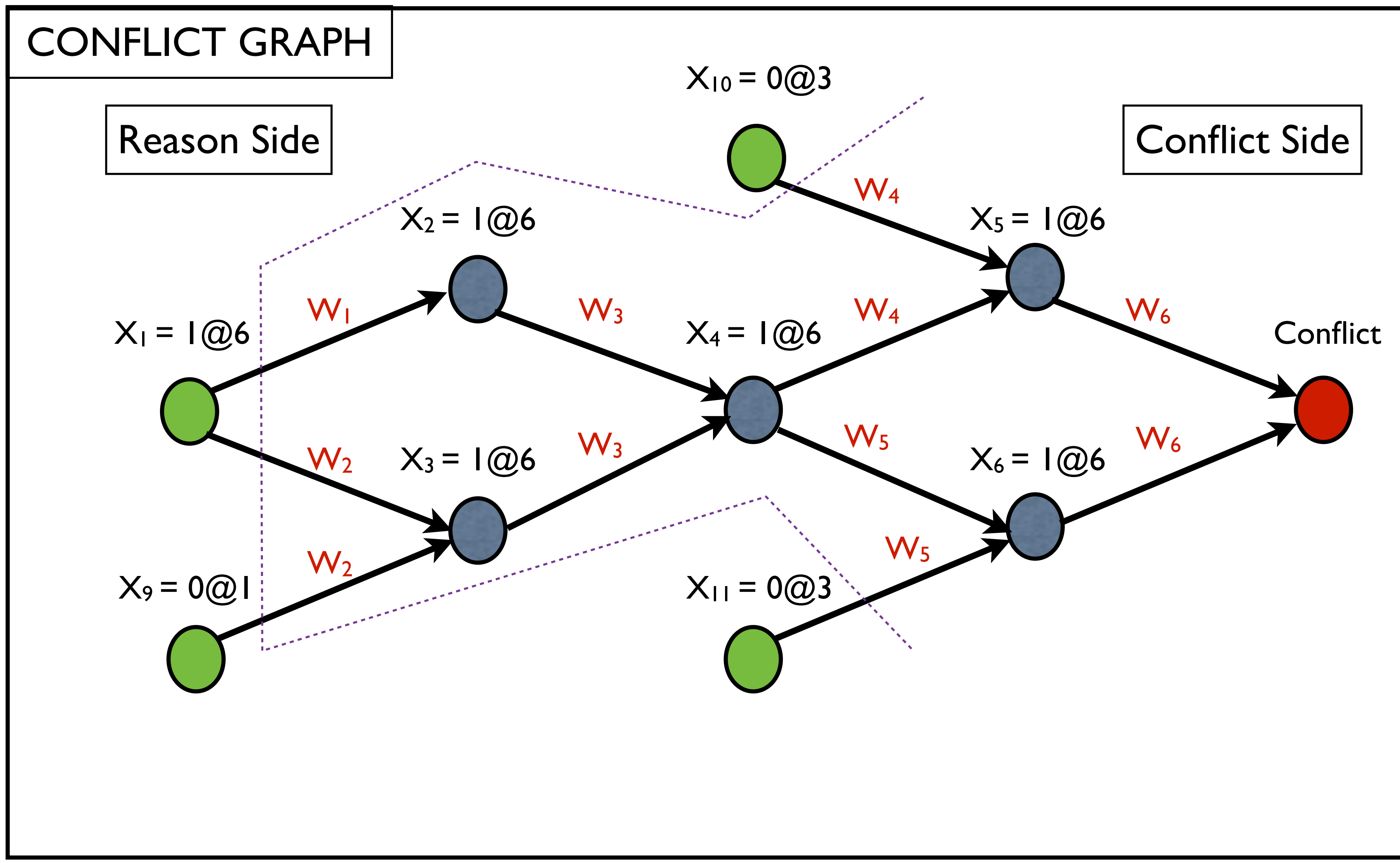

Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: Conflict Clause

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Current Decision: $\{X_1 = 1@6\}$

Another strategy is to use First Unique Implicant Point (UIP):
Traverse graph backwards in breadth-first, expand literals of conflict, stop at first UIP

Clause DB

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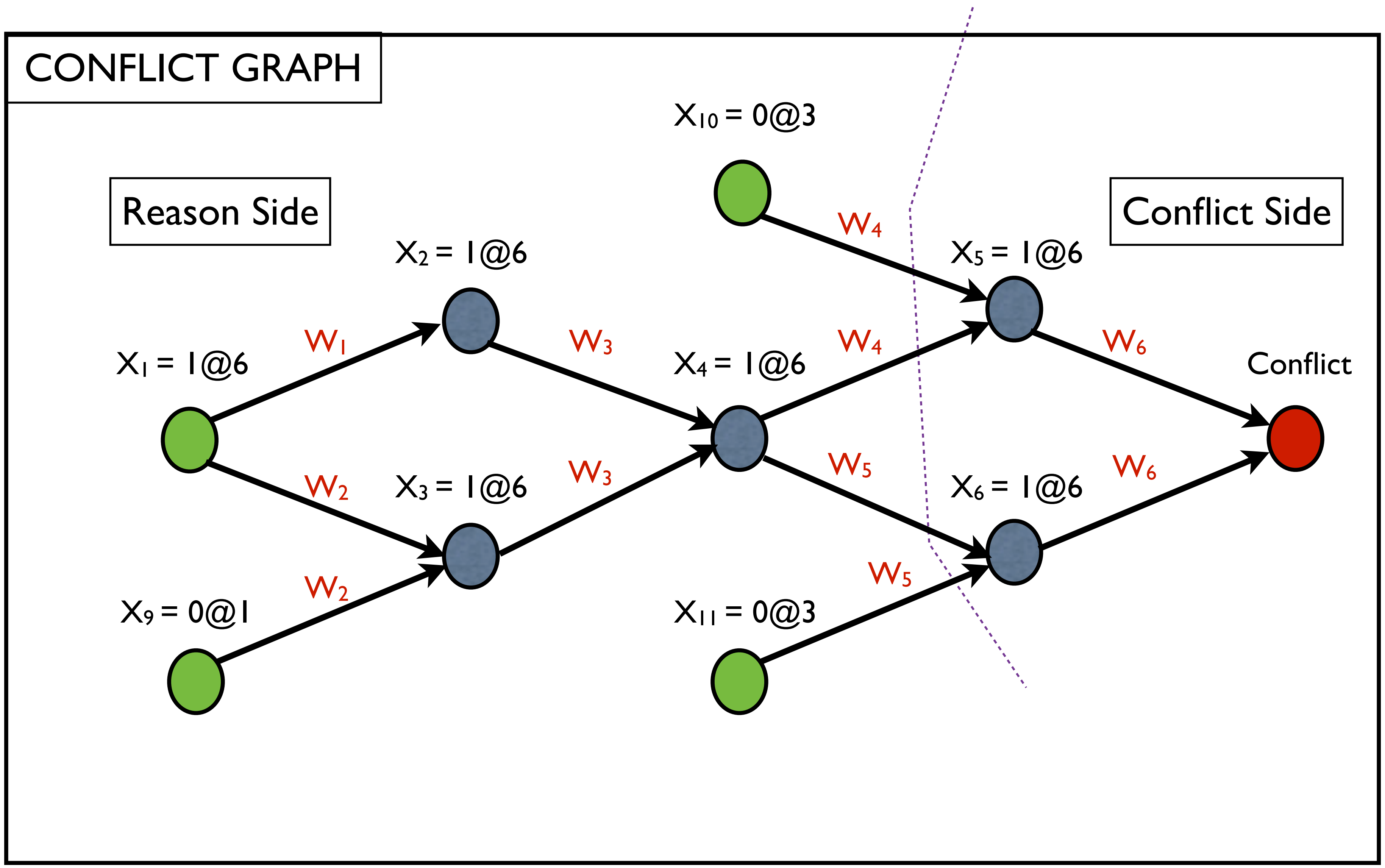
$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



Modern CDCL SAT Solver Architecture
Conflict Analysis/Learn() Details: BackTrack

Current Assignment Trail: $\{X_9 = 0@1, X_{10} = 0@3, X_{11} = 0@3, X_{12} = 1@2, X_{13} = 1@2, \ldots\}$

Current decision: $\{X_1 = 1@6\}$

Strategy: Closest decision level (DL) $\leq$ current DL for which conflict clause is unit. Undo $\{X_1 = 1@6\}$

Clause DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

$W_5 = (\neg X_4 + X_6 + X_{11})$

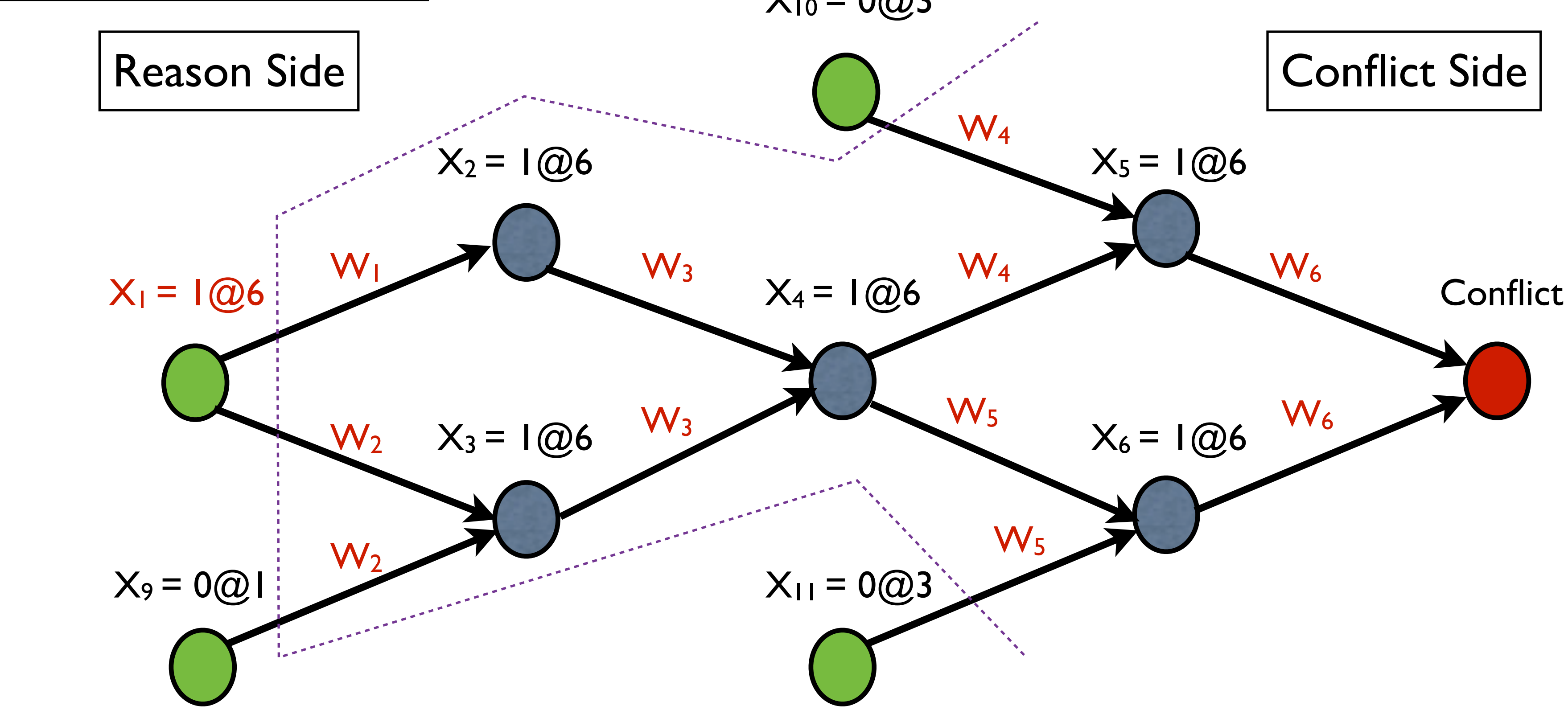
$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$

CONFLICT GRAPH



Modern CDCL SAT Solver Architecture

Conflict Analysis/Learn() Details: BackJump

$\neg X_1$ was implied literal, leading to another conflict described below

Conflict clause: $(X_9 + X_{10} + X_{11} + \neg X_{12} + \neg X_{13})$

BackJump strategy: Closest decision level (DL) $\leq$ current DL for which conflict clause is unit. Undo $\{X_{10} = 0@3\}$

CLAUSE DB

$W_1 = (\neg X_1 + X_2)$

$W_2 = (\neg X_1 + X_3 + X_9)$

$W_3 = (\neg X_2 + \neg X_3 + X_4)$

$W_4 = (\neg X_4 + X_5 + X_{10})$

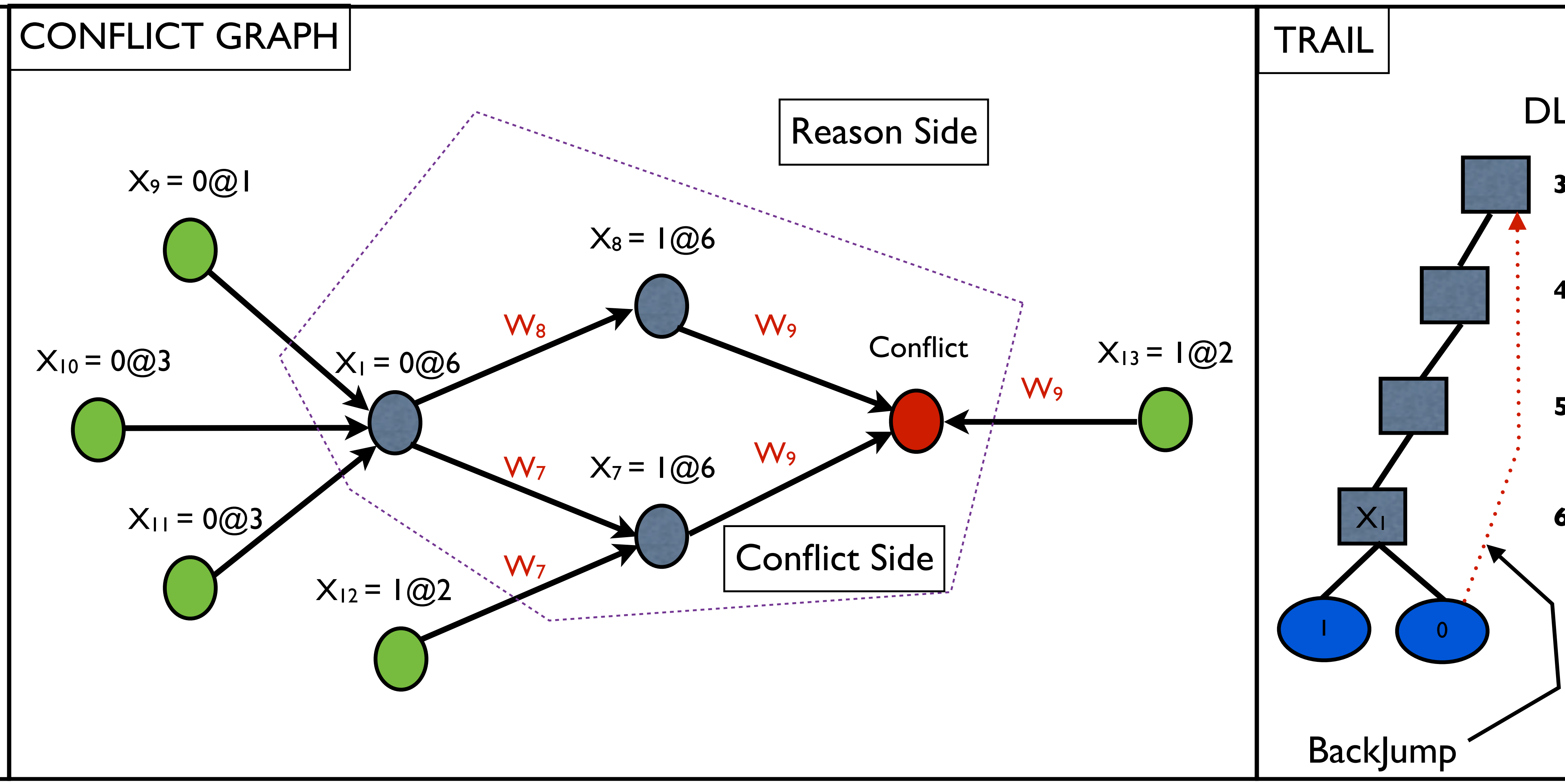
$W_5 = (\neg X_4 + X_6 + X_{11})$

$W_6 = (\neg X_5 + \neg X_6)$

$W_7 = (X_1 + X_7 + \neg X_{12})$

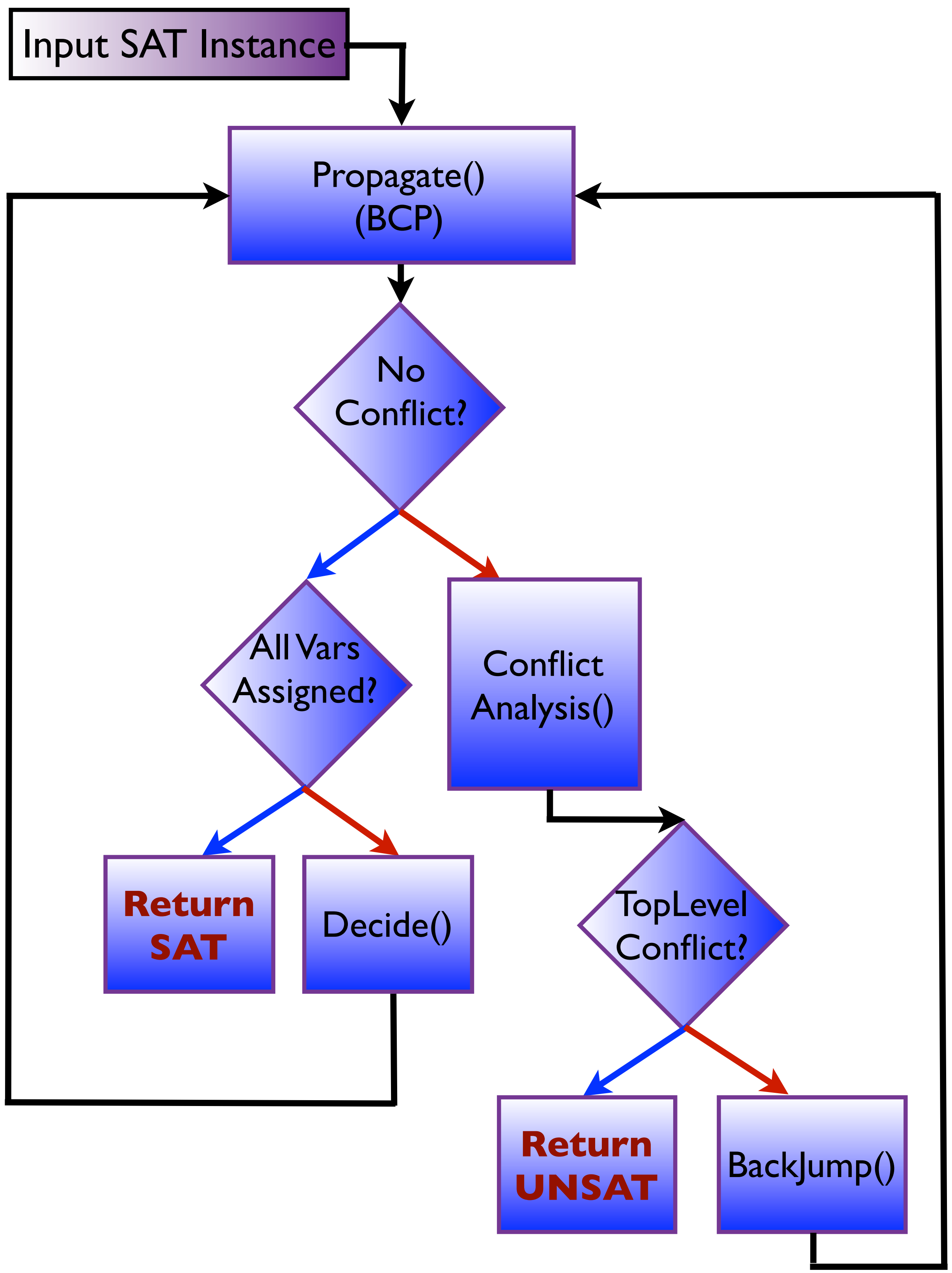
$W_8 = (X_1 + X_8)$

$W_9 = (\neg X_7 + \neg X_8 + \neg X_{13})$



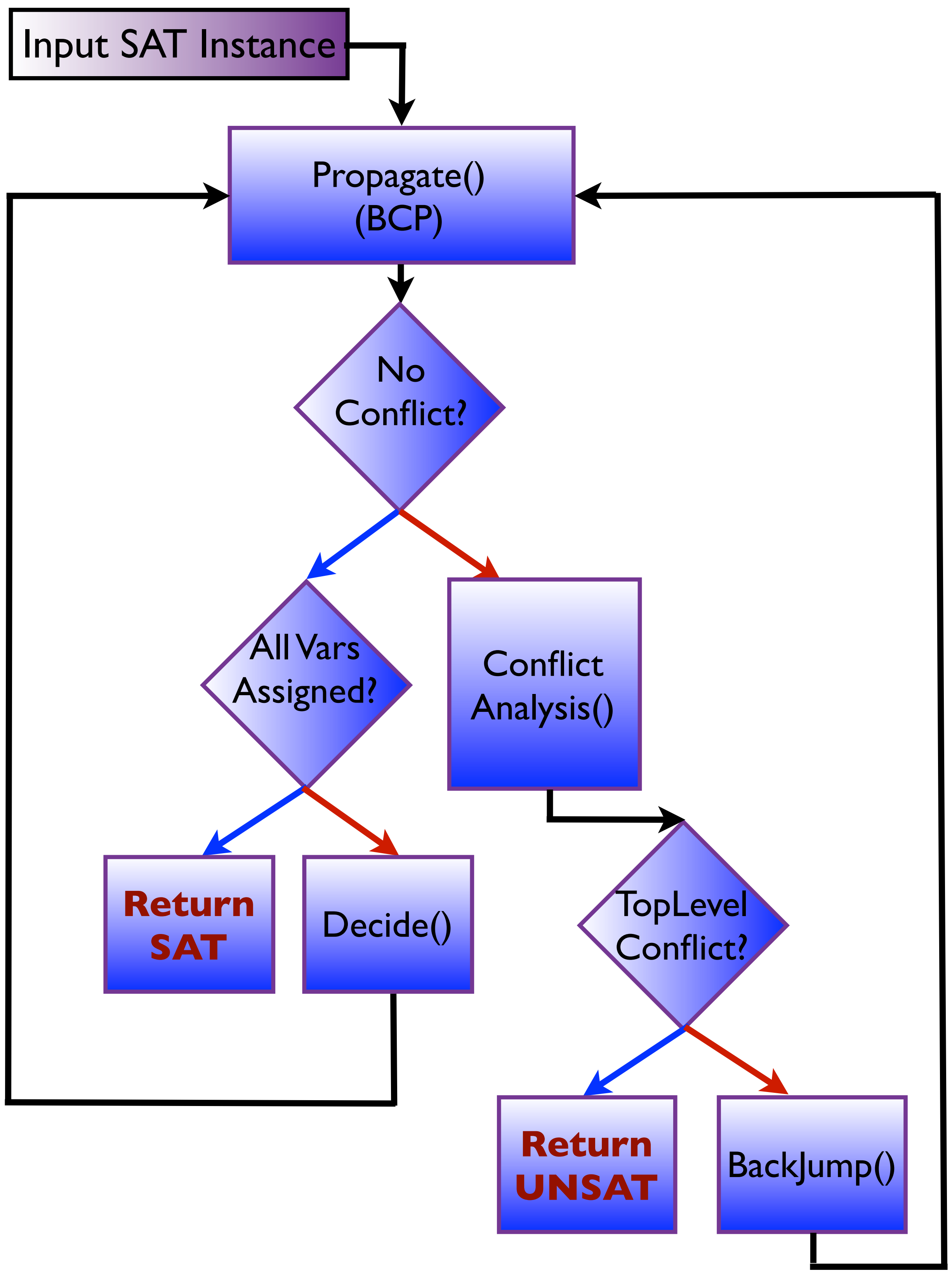
Modern CDCL SAT Solver Architecture

Restarts and Forget



- Restarts
  - Clear the Trail and start again
  - Start searching with a different variable order
  - Only Conflict Clause (CC) database is retained
- Forget: throw away less active learnt conflict clauses routinely
  - Routinely throw away very large CC
  - Logically CC are implied
  - Hence no loss in soundness/completeness
  - Time Savings: smaller DB means less work in propagation
  - Space savings

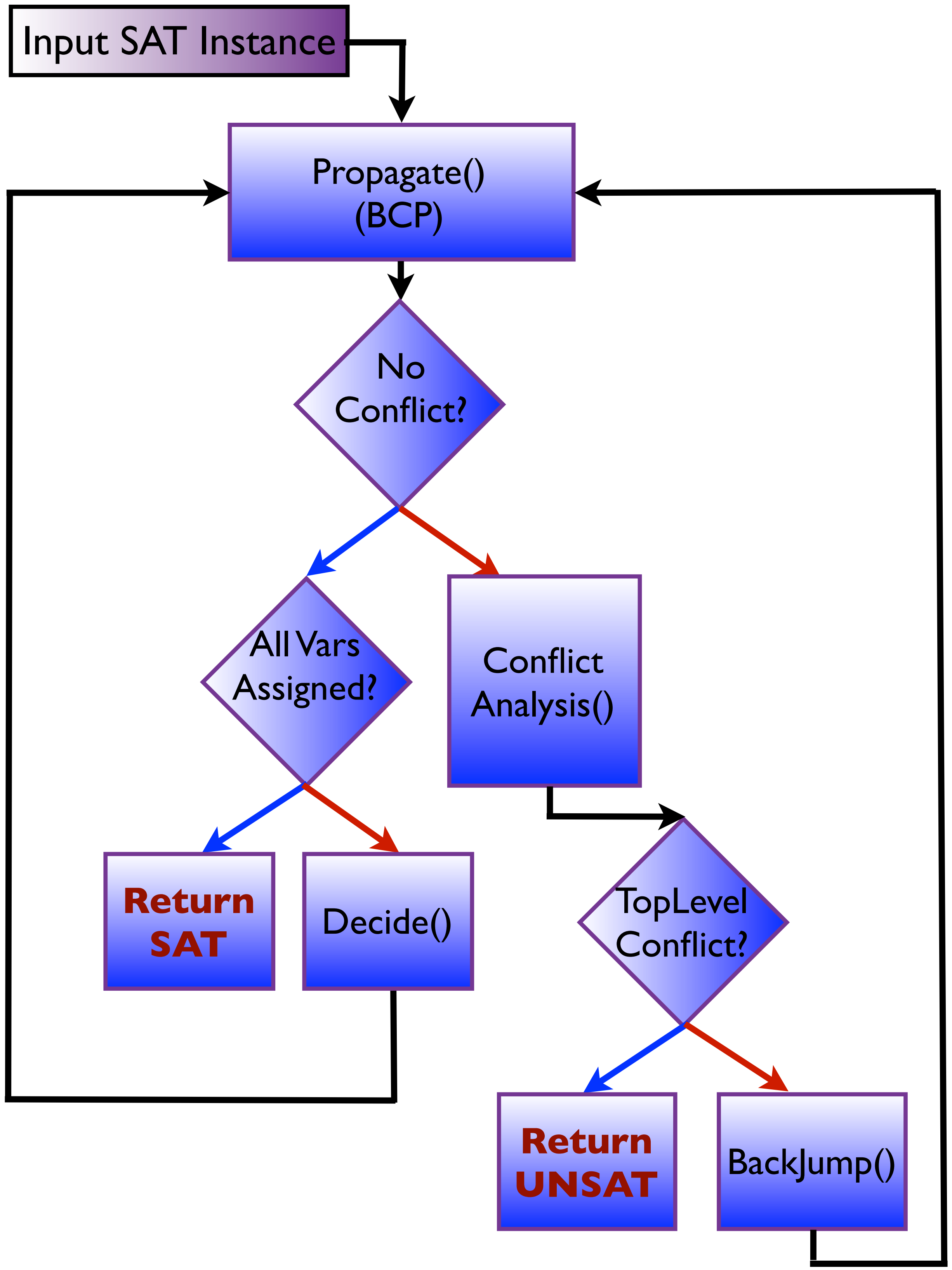
Modern CDCL SAT Solver Architecture
Why is SAT efficient?



- VSIDS branching heuristic and propagate (BCP)
- Conflict-Driven Clause-Learning (CDCL)
- Forget conflict clauses if DB goes too big
- BackJump
- Restarts
- All the above elements are needed for efficiency
- Deeper understanding lacking
- No predictive theory

Modern CDCL SAT Solver Architecture

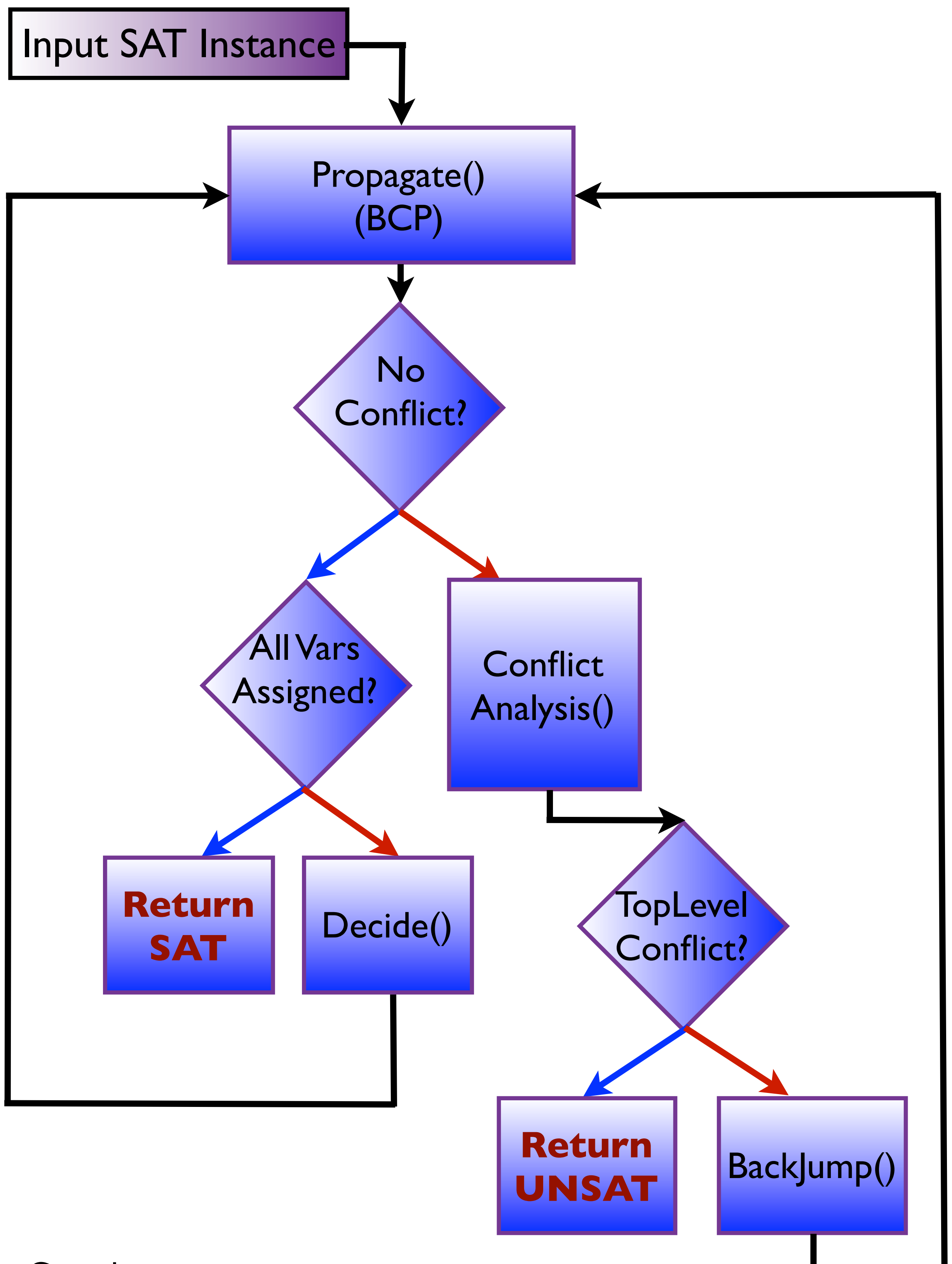
Propagate(), Decide(), Analyze/Learn(), BackJump()



- Conflict-Driven Clause-Learning (CDCL)
  (Marques-Silva & Sakallah 1996)
- Decide/branch and propagate (BCP)
  (Malik et al. 2001, Zabih & McAllester 1988)
- BackJump
  (McAllester 1980, Marques-Silva & Sakallah 1999)
- Restarts
  (Selman & Gomes 2001)
- Follows MiniSAT
  (Een & Sorensson 2003)

Modern CDCL SAT Solver Architecture

Soundness, Completeness & Termination

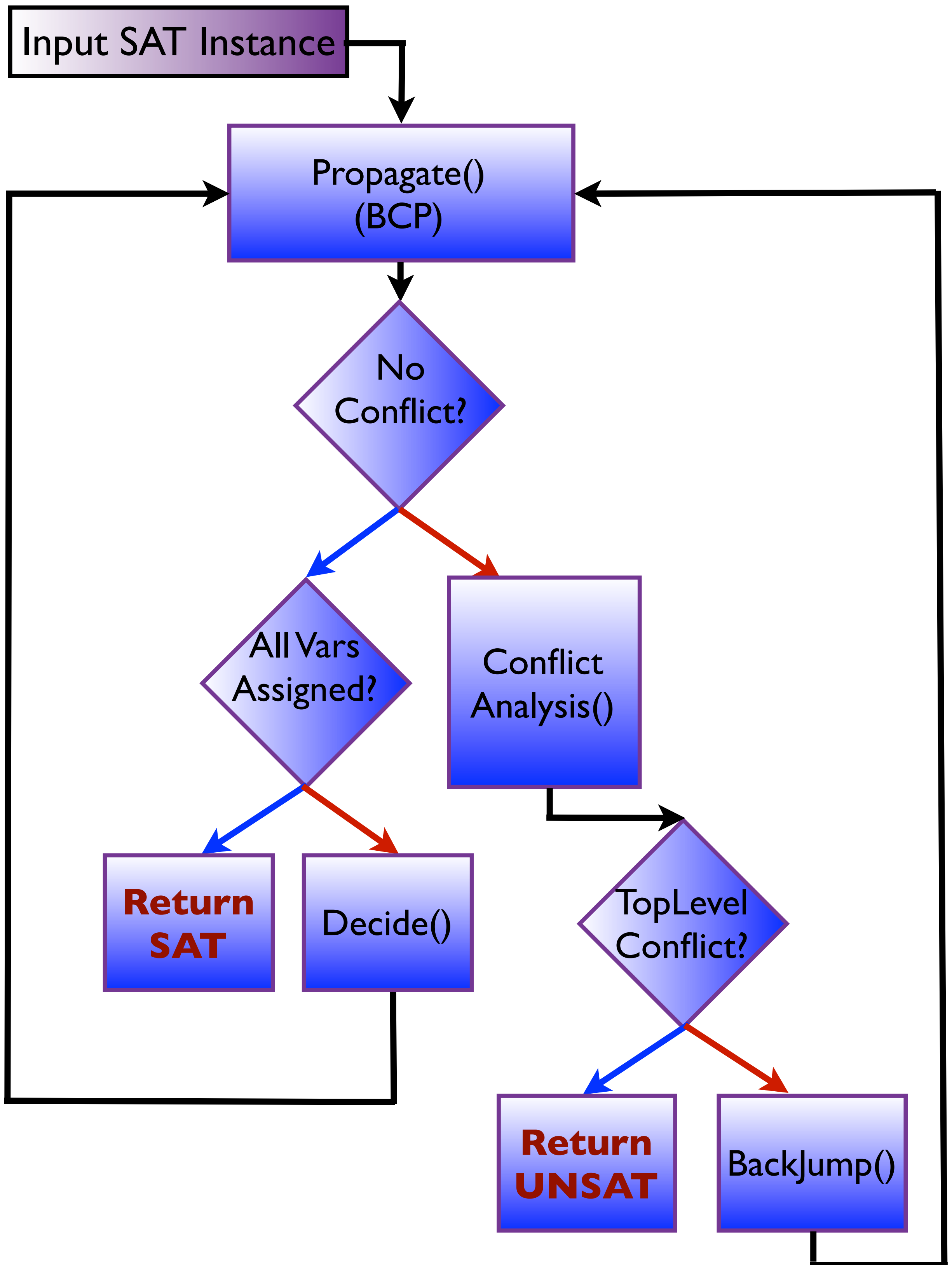


Soundness: A solver is said to be sound, if, for any input formula F, the solver terminates and produces a solution, then F is indeed SAT

Proof: (Easy) SAT is returned only when all vars have been assigned a value (True, False) by Decide or BCP, and solver checks the solution.

Modern CDCL SAT Solver Architecture

Soundness, Completeness & Termination



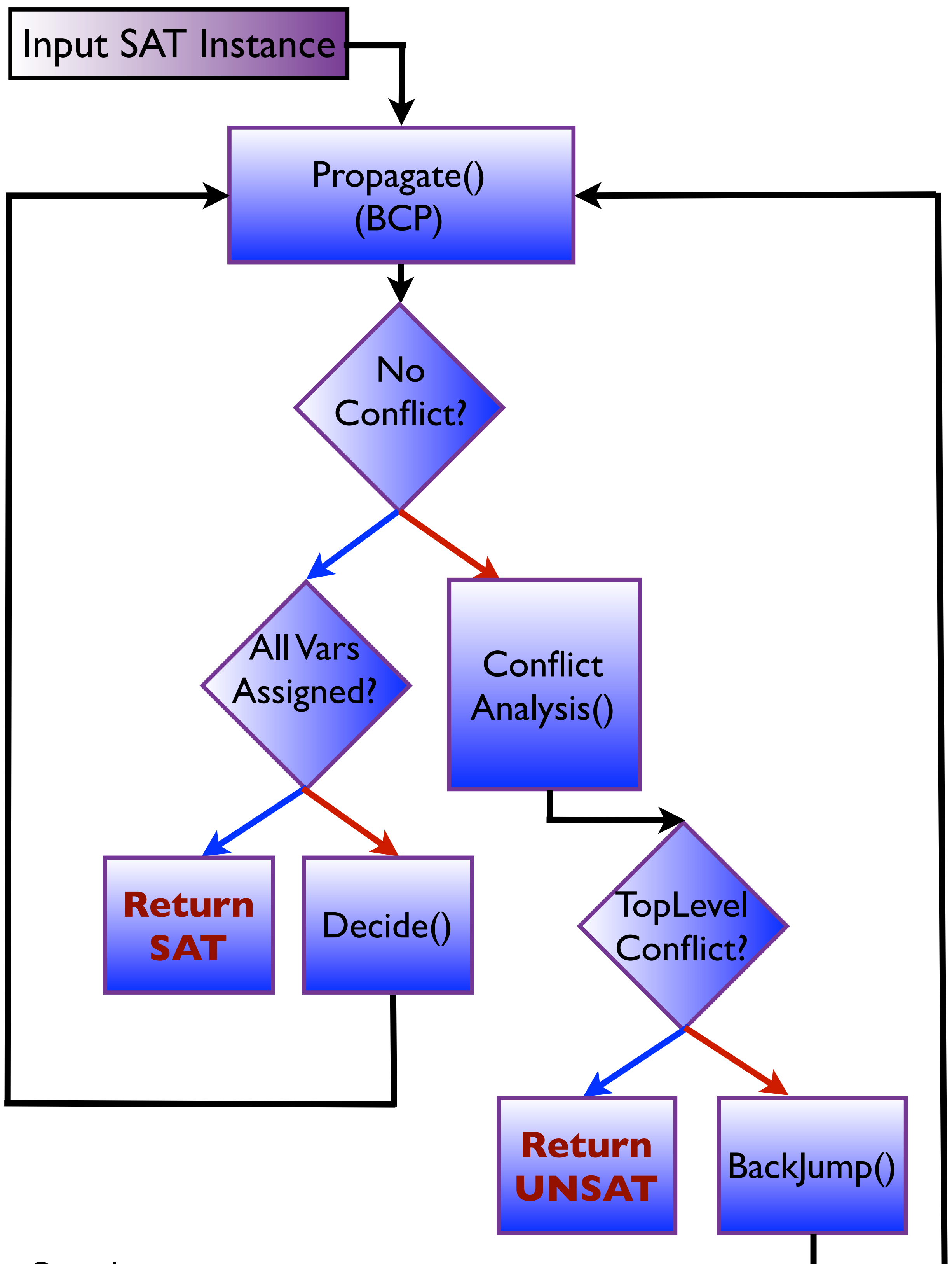
Completeness: A solver is said to be complete, if, for any input formula F that is SAT, the solver terminates and produces a solution (i.e., solver does not miss solutions)

Proof: (Harder)

- Backtracking + BCP + decide is complete (easy)
- Conflict clause is implied by input formula (easy)
- Only need to see backjumping does not skip assignments
  - Observe backjumping occurs only when conflict clause (CC) vars < decision level (DL) of conflicting var
  - Backjumping to max(DL of vars in CC)
  - Decision tree rooted at max(DL of vars in CC)+1 is guaranteed to not satisfy CC
  - Hence, backjumping will not skip assignments

Modern CDCL SAT Solver Architecture

Soundness, Completeness & Termination



Termination: Some measure decreases every iteration

Proof Sketch:

- Loop guarantees either conflict clause (CC) added OR assign extended
- CC added. What stops CC addition looping forever?
  - Recall that CC is remembered
  - No CC duplication possible
  - CC blocks UNSAT assign exploration in decision tree. No duplicate UNSAT assign exploration possible
  - Size of decision tree explored decreases for each CC add

Modern CDCL SAT Solver Architecture
References & Important SAT Solvers

1. Marques-Silva, J.P. and K.A. Sakallah. *GRASP: A Search Algorithm for Propositional Satisfiability.* IEEE Transactions on Computers 48(5), 1999, 506-521.

2. Marques-Silva, J.P. and K.A. Sakallah. *GRASP: A Search Algorithm for Propositional Satisfiability*. Proceedings of ICCAD, 1996.

3. M. Moskewicz, C. Madigan, Y. Zhao, L. Zhang, and S. Malik. *CHAFF: Engineering an efficient SAT solver*. Proceedings of the Design Automation Conference (DAC), 2001, 530-535.

4. L. Zhang, C. F. Madigan, M. H. Moskewicz and S. Malik. *Efficient Conflict Driven Learning in a Boolean Satisfiability Solver*. Proceedings of ICCAD, 2001, 279-285.

5. Armin Bierre, Marijn Heule, Hans van Maaren, and Toby Walsh (Editors). *Handbook of Satisfiability*. 2009. IOS Press. http://www.st.ewi.tudelft.nl/sat/handbook/

6. M. Davis, G. Logemann, and D. Loveland. *A machine program for theorem proving.* Communications of the ACM. 1962.

7. zChaff SAT Solver by Lintao Zhang 2002.

8. GRASP SAT Solver by Joao Marques-Silva and Karem Sakallah 1999.

9. MiniSAT Solver by Niklas Een and Niklas Sorenson 2005 - present

10. SAT Live: http://www.satlive.org/

11. SAT Competition: http://www.satcompetition.org/

12. SAT/SMT summer school: http://people.csail.mit.edu/vganesh/summerschool/

Modern CDCL SAT Solver Architecture
Important Ideas and Conclusions

1. SAT solvers are crucial for software engineering

2. Huge impact in formal methods, program analysis and testing

3. Key ideas that make SAT efficient

    1. Conflict-driven clause learning
    2. VSIDS (or similar) variable selection heuristics
    3. Backjumping
    4. Restarts

4. Techniques I didn't discuss

    1. Survey propagation (belief propagation) by Selman & Gomes
    2. Works well for randomized SAT, not yet for industrial instances
    3. Physics-inspired
    4. Combining CDCL with survey propagation (?)