ECE750T-28: Computer-aided Reasoning for Software Engineering

Lecture 13: Decision Procedure for the Theory of Rationals

Vijay Ganesh
(Original notes from Isil Dillig)

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- Today: Talk about how to decide satisfiability of the quantifier-free fragment of $T_{\mathbb{Q}}$

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- Example: $\exists x.\ (1+1)x = 1+1+1$ Is this formula valid in $T_{\mathbb{Q}}$? Yes
- Is it valid in $T_{\mathbb{Z}}$? No
- In general, every formula valid in $T_{\mathbb{Z}}$ is valid in $T_{\mathbb{Q}}$, but not vice versa

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- Full theory of rationals is decidable
- High-time complexity: $O(2^{2^{kn}})$ (k some positive integer)
- Conjunctive quantifier-free fragment efficiently decidable (polynomial time)

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- Most common technique for deciding satisfiability in $T_{\mathbb{Q}}$ is Simplex algorithm
- Simplex algorithm developed by Dantzig in 1949 for solving linear programming problems
- Since deciding satisfiability of qff conjunctive formulas is a special case of linear programming, we can use Simplex

The Plan

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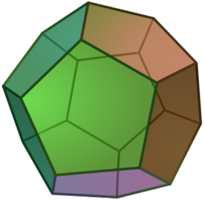
- Very important problem; applications in airline scheduling, transportation, telecommunications, finance, production management, marketing, networking, compilers ...

Geometric Formulation



- For $m \times n$ matrix A, the system $A\vec{x} \leq \vec{b}$ forms a convex polytope in n-dimensional space

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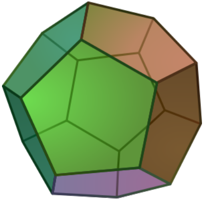

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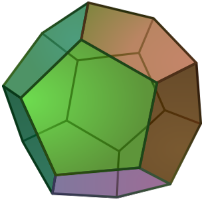

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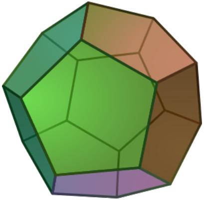

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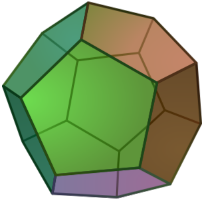

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- Goal of linear programming: Find a point that (i) lies inside the polytope, and (ii) maximizes the value of $\vec{c}^T\vec{x}$

Linear Programming Lingo

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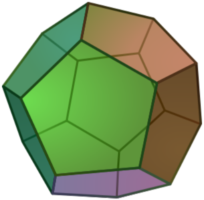
- If optimal solution is ∞, then problem is called unbounded

Geometric Interpretation



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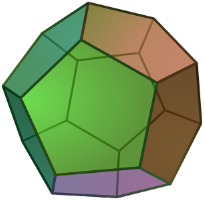

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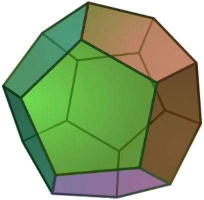

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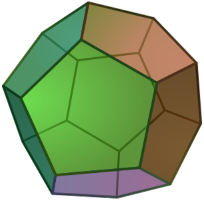

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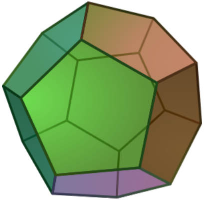

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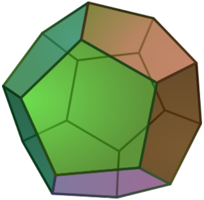

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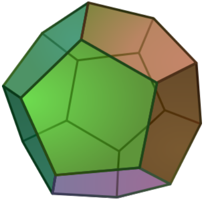

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- Question: If polytope is not closed, does this mean optimal solution is ∞? No!
- Since the polytope defined by $A\vec{x} \leq \vec{b}$ is convex, the optimal solution for bounded LP problem must lie on exterior boundary of polytope

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\vec{a}^T\vec{x} \neq c & \Rightarrow & \begin{array}{l}(\vec{a}^T\vec{x} + y \leq c \wedge y > 0) \vee \\ (-\vec{a}^T\vec{x} + y \leq -c \wedge y > 0)\end{array}
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- Why? If maximum value of y positive, we know $y > 0$ can be satisfied. If maximum value is ≤ 0, $y > 0$ cannot be satisfied.

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- Despite this, Simplex remains most popular and performs better for most problems of interest

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- Good news: We can convert every LP problem into an equisatisfiable standard form representation
- Equisat. means original problem has optimal objective value c iff problem in standard form has optimal objective value c

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- Thus, transformation yields equisatisfiable linear program and is in standard form

Standard Form Example

- Consider the following linear program:

Maximize $2x_1 - 3x_2$
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$$\begin{gathered} x_1 + x_2 \leq 7 \\ -x_1 - x_2 \leq -7 \\ x_1 - 2x_2 \leq 4 \\ x_1 \geq 0 \end{gathered}$$

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- Variable x_2 does not have non-negativity constraint; thus rewrite it as $x_2' - x_2''$
- Equisatisfiable system in standard form:

$$\begin{array}{ll} \text{Maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{Subject to:} & \end{array}$$

$$\begin{array}{c} x_1 + x_2' - x_2'' \leq 7 \\ -x_1 - x_2' + x_2'' \leq -7 \\ x_1 - 2x_2' + 2x_2'' \leq 4 \\ x_1, x_2', x_2'' \geq 0 \end{array}$$

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- New LP problem is equisatisfiable to the original one and in slack form

Slack Form Conversion Example

- Consider LP problem from previous example:

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\begin{array}{ll}
\text{Maximize} & 2x_1 - 3x_2 + 3x_3 \\
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- Basic solution called feasible basic solution if it doesn't violate non-negativity constraints

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- Thus, we'll talk about Phase II first

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- By convexity, local optimum is global optimum; thus algorithm can safely stop when local maximum is reached

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- If there is a term $c_j x_j$ with positive c_j, we can increase value of z by increasing x_j's value, i.e., by making x_j a basic variable

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- Recall: Objective function is $z = v + \sum_{x_j \in N} c_j x_j$
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- If there is a term $c_j x_j$ with positive c_j, we can increase value of z by increasing x_j's value, i.e., by making x_j a basic variable
- What if there are no positive c_j's?

Simplex Algorithm Optimization Phase

- When rewriting one slack form to another, goal is to increase value of objective function associated with basic solution
- Recall: Objective function is $z = v + \sum_{x_j \in N} c_j x_j$
- How can we increase value of z?
- If there is a term $c_j x_j$ with positive c_j, we can increase value of z by increasing x_j's value, i.e., by making x_j a basic variable
- What if there are no positive c_j's?
- Then, we know we can't increase value of z, thus we are done!

Simplex Algorithm Optimization Phase, cont

- Suppose we can increase objective value, i.e., there exists a term $c_j x_j$ with positive c_j

Simplex Algorithm Optimization Phase, cont

- Suppose we can increase objective value, i.e., there exists a term $c_j x_j$ with positive c_j
- We want to increase x_j's value, but is there a limit on how much we can increase x_j?

Simplex Algorithm Optimization Phase, cont

- Suppose we can increase objective value, i.e., there exists a term $c_j x_j$ with positive c_j
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- Thus, the amount by which we can increase x_j is limited by the smallest $\frac{b_i}{a_{ij}}$ among all i's
- If there is no positive coefficient a_{ij}, we can increase x_j (and thus z) without limit $\Rightarrow$ optimal solution $= \infty$

Simplex Algorithm Optimization Phase, cont

- Thus, given term $c_j x_j$ with positive c_j in objective function, we want to increase x_j as much as possible

Simplex Algorithm Optimization Phase, cont

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  1. x_j's coefficient a_{ij} is positive (otherwise doesn't limit x_j)

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- To increase x_j as much as possible, we find equality that most severely restricts how much we can increase x_j
- Equality that most severaly restricts x_j has following characteristics:
  1. x_j's coefficient a_{ij} is positive (otherwise doesn't limit x_j)
  2. has smallest value of $\frac{b_i}{a_{ij}}$ (most severely restricting)

Simplex Algorithm Optimization Phase, cont

- Suppose equality with basic var. x_i is most restrictive for x_j

Simplex Algorithm Optimization Phase, cont

- Suppose equality with basic var. x_i is most restrictive for x_j
- Swap roles of x_i and x_j by making x_j basic and x_i non-basic

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- Suppose equality with basic var. x_i is most restrictive for x_j
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- After performing this pivot operation, what is new value of x_j? $\frac{b_i}{a_{ij}}$
- Assuming b_i is non-zero, we have increased the value of x_j from 0 to $\frac{b_i}{a_{ij}}$

Simplex Algorithm Optimization Phase, cont

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- After performing this pivot operation, what is new value of x_j? $\frac{b_i}{a_{ij}}$
- Assuming b_i is non-zero, we have increased the value of x_j from 0 to $\frac{b_i}{a_{ij}}$
- Thus, after performing pivot we still have feasible solution but objective value is now greater

Simplex Optimization Phase Summary

- Pivot operation exchanges a basic variable with a non-basic variable to increase objective value of basic solution

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  1. All coefficients in objective function are negative $\Rightarrow$ optimal solution found

Simplex Optimization Phase Summary

- Pivot operation exchanges a basic variable with a non-basic variable to increase objective value of basic solution
- Simplex repeats this pivot operation until one of two conditions hold:
  1. All coefficients in objective function are negative $\Rightarrow$ optimal solution found
  2. There exists a non-basic variable x_j with positive coefficient c_j in objective function, but all coefficients a_{ij} are negative $\Rightarrow$ optimal solution $= \infty$

Example

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
$$

- How can we increase value of objective function?

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- How can we increase value of objective function?
- By increasing any of x_1, x_2, x_3; let's pick x_1

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- By increasing any of x_1, x_2, x_3; let's pick x_1
- Which equality restricts x_1 the most? x_6
- Rewrite x_1 in terms of x_6:

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

$$\begin{array}{ccccccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{4} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}$$

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

$$\begin{array}{lcllllllll}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
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x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}$$

- How can we increase value of z?

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

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\begin{array}{lllllllll}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
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x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
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- How can we increase value of z?
- Either by increasing x_2 or x_3, but not x_6; let's pick x_3

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

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\begin{array}{rclcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
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\end{array}
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- How can we increase value of z?
- Either by increasing x_2 or x_3, but not x_6; let's pick x_3
- Which equality restricts x_3 the most?

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

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\begin{array}{lclclclcl}
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x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

- How can we increase value of z?

- Either by increasing x_2 or x_3, but not x_6; let's pick x_3

- Which equality restricts x_3 the most? x_5

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

$$\begin{array}{lclclclcl}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{4} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}$$

- How can we increase value of z?
- Either by increasing x_2 or x_3, but not x_6; let's pick x_3
- Which equality restricts x_3 the most? x_5
- What is x_3 in terms of x_5, x_2, x_6?

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

$$
\begin{array}{lclclclcl}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{4} & - & \frac{x_6}{4} \\
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\end{array}
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- Either by increasing x_2 or x_3, but not x_6; let's pick x_3
- Which equality restricts x_3 the most? x_5
- What is x_3 in terms of x_5, x_2, x_6?

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$
\begin{array}{lcccccccc}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$\begin{array}{ccccccccc}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
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x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

- Can we increase z?

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$
\begin{array}{lllllllll}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
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x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

- Can we increase z? Yes, increase x_2

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$\begin{array}{lllllllll} z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\ x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\ x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\ x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16} \end{array}$$

- Can we increase z? Yes, increase x_2
- Which equality restricts x_2 the most?

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$
\begin{array}{lcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
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x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

- Can we increase z? Yes, increase x_2

- Which equality restricts x_2 the most?

- x_4 does not restrict; x_2 restricts by 132, x_3 restricts by $4 \Rightarrow x_3$

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$
\begin{array}{lllllllll}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

- Can we increase z? Yes, increase x_2
- Which equality restricts x_2 the most?
- x_4 does not restrict; x_2 restricts by 132, x_3 restricts by $4 \Rightarrow x_3$
- Solve x_2 in terms of x_3:

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$
\begin{array}{lclclclcl}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
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x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

- Can we increase z? Yes, increase x_2
- Which equality restricts x_2 the most?
- x_4 does not restrict; x_2 restricts by 132, x_3 restricts by $4 \Rightarrow x_3$
- Solve x_2 in terms of x_3:

$$x_2 = 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6$$

Example, cont.

- New slack form after making x_2 basic, x_3 non-basic:

$$
\begin{array}{lclclclcl}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Example, cont.

- New slack form after making x_2 basic, x_3 non-basic:

$$\begin{array}{lllllllll} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

- Can we increase objective value?

Example, cont.

- New slack form after making x_2 basic, x_3 non-basic:

$$\begin{array}{lcllllllll} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

- Can we increase objective value? No, Simplex terminates

Example, cont.

- New slack form after making x_2 basic, x_3 non-basic:

$$
\begin{array}{lclclclcl}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
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- What is optimal objective value?

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- Thus, Simplex never decreases value of the objective function!

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- Thus, the objective value does not decrease, but does not increase either!
- These kinds of problems where objective value can stay the same after pivoting are called degenerate problems

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- One such strategy is Bland's rule: If there are multiple variables with positive coefficients in objective function, always choose the variable with smallest index
- Example: If $z = 2x_1 + 5x_2 - 4x_3$, Bland's rule chooses x_1 as new basic variable since it has smallest index

Simplex Algorithm Phases

- Simplex algorithm has two phases:

  1. Phase I: Compute a feasible basic solution, if one exists
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  1. Phase I: Compute a feasible basic solution, if one exists
  2. Phase II: Optimize value of objective function ✓
- So far, we talked about the second phase, assuming we already have a feasible basic solution
- However, the initial basic solution might not feasible even if the linear program is feasible

Example of Infeasible Initial Basic Solution

- Consider the following linear program:

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\begin{aligned}
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- What is the initial basic solution? $(0, 0, 2, -4)$
- Is this solution feasible? No, violates non-negativity constraints
- Goal of Phase I of Simplex is to determine if a feasible basic solution exists, and if so, what it is

Overview of Phase I

- To find an initial basic solution, we construct an auxiliary linear program L_{aux}

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- Furthermore, original LP problem has a feasible solution if and only if the optimal objective value for L_{aux} is zero

Overview of Phase I

- To find an initial basic solution, we construct an auxiliary linear program L_{aux}
- This auxiliary linear program has the property that we can find a feasible basic solution for it after at most one pivot operation
- Furthermore, original LP problem has a feasible solution if and only if the optimal objective value for L_{aux} is zero
- If optimal value of L_{aux} is 0, we can extract basic feasible solution of original problem from optimal solution to L_{aux}

Constructing the Auxiliary Linear Program

- Consider the original LP problem:

Maximize $\sum_{j=1}^{n} c_j x_j$

Subject to:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \quad (i \in [1, m]) \\ x_j \geq 0 & \quad (j \in [1, n]) \end{aligned}$$

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- This problem is feasible iff the following LP problem L_{aux} has optimal value 0:

$$\begin{array}{ll} \text{Maximize} & -x_0 \\ \text{Subject to:} & \end{array}$$

$$\begin{array}{rll} \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i & & (i \in [1, m]) \\ x_j \geq 0 & & (j \in [0, n]) \end{array}$$

Justification for Auxiliary LP

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$\Leftarrow (a)$ Suppose original problem has feasible solution $\vec{x^*}$. Then $\vec{x^*}$ combined with $x_0 = 0$ is feasible solution for L_{aux}.

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$\Leftarrow (a)$ Suppose original problem has feasible solution $\vec{x^*}$. Then $\vec{x^*}$ combined with $x_0 = 0$ is feasible solution for L_{aux}.

$\Leftarrow (b)$ Due to the non-negativity constraint, $-x_0$ can be at most 0; thus, this solution is optimal for L_{aux}.

Finding Feasible Basic Solution for L_{aux}

- So far, we argued that original problem L has feasible solution iff L_{aux} has optimal value 0.

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- But we still need to figure out how to find feasible basic solution to L_{aux}.
- Next: We'll see how we can find feasible basic solution for L_{aux} after one pivot operation.

Auxiliary Problem in Slack Form

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\begin{aligned}
z &= -x_0 \\
x_i &= b_i + x_0 - \sum_{j=1}^{n} a_{ij} x_j
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- If all b_i's are positive, basic solution already feasible

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- If all b_i's are positive, basic solution already feasible
- If there is at least some negative b_i, find equality x_i with most negative b_i

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- Make x_0 new basic variable, and x_i non-basic

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- If there is at least some negative b_i, find equality x_i with most negative b_i
- Make x_0 new basic variable, and x_i non-basic
- Claim: After this one pivot operation, all b_i's are non-negative; thus basic solution is feasible

Why is This True?

- Suppose this equality has most negative b_i:

$$x_i = b_i + x_0 - \sum_{j=1}^{n} a_{ij} x_j$$

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$$x_0 = -b_i + x_i + \sum_{j=1}^{n} a_{ij} x_j$$

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- Thus, when we plug in equality for x_0 into other equations, their new constants will be positive
- Hence, we find a feasible basic solution after at most one pivot step

Example

- Consider the following linear program from earlier:

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- Which equation has most negative constant? x_4

- Swap x_4 and x_0:

$$x_0 = 4 + x_4 + x_1 - 5x_2$$

Example, cont

- After pivoting, we obtain the new slack form:

$$\begin{array}{rcl} z & = & -4 - x_4 - x_1 + 5x_2 \\ x_3 & = & 6 - x_1 - 4x_2 + x_4 \\ x_0 & = & 4 + x_4 + x_1 - 5x_2 \end{array}$$

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- What is current objective value? -4

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- What is current objective value? -4
- How can we increase it? increase x_2

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- Thus, Phase I returns the following slack form to Phase II:

$$\begin{array}{rcl} z & = & \frac{-4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\ x_2 & = & \frac{4}{5} - \frac{x_1}{5} + \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \end{array}$$

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- In first phase, we construct slack form such that it has a basic feasible solution
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- Although Simplex is a worst-case exponential, it is more popular than polynomial-time algorithms for LP