ECE750T-28:
Computer-aided Reasoning for Software Engineering

Lecture 9: Overview of First-Order Theories

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(Original notes from Isil Dillig)

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- First-order logic is very powerful and very general.
- But in many settings, we have a particular application in mind and do not need the full power of first order logic.
- For instance, instead of general predicates/functions, we might only need an equality predicate or arithmetic operations.
- Also, might want to disallow some interpretations that are allowed in first-order logic.

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- Today: Talk about what first-order theories are and look at some examples.
- Future lectures: Explore individual first-order theories in more detail and learn about specialized decision procedures

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- In our theory of heights, this interpretation is not legal b/c does not satisfy axioms

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- Theory T consists of all sentences that are valid in T.

Equivalence Modulo T

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- Falsifying interpretation: $U = \{\square, \triangle\}, I(=) : \{\langle \triangle, \square \rangle\}$

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- Unlike full first-order logic, many of the first-order theories we will study are decidable.
- For those that are not decidable, we are interested in fragments of that theory that are decidable.

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- A fragment of T is decidable if it is decidable whether $T \models F$ for every formula F in that fragment
- For some of the theories we will look at, the full theory is not decidable, but their quantifier-free fragment is (often efficiently) decidable and very useful in practice.

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- In the following lectures, we will further explore these theories and look at decision procedures.

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- Signature:

$$\Sigma_= : \{=, a, b, c, \cdots, f, g, h, \cdots, p, q, r, \cdots\}$$

  - $=$, a binary predicate, interpreted by axioms.
  - all constant, function, and predicate symbols.

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3. $\forall x, y, z.\ x = y \ \wedge \ y = z \ \rightarrow \ x = z$ (transitivity)

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Axioms of the Theory of Equality, cont.

- Function congruence:
  For any n-ary function f, two terms $f(\vec{x})$ and $f(\vec{y})$ are equal if $\vec{x}$ and $\vec{y}$ are equal:

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- If we derive contradiction in every branch, formula is valid modulo $T_=$.

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- Theory of integers: Equivalent in expressiveness to Presburger arithmetic, but more convenient notation

Peano Arithmetic Signature

- The theory of Peano arithmetic T_{PA} has signature:

$$\Sigma_{PA} : \{0, 1, +, \cdot, =\}$$

- $0, 1$ are constants
- $+, \cdot$ binary functions
- $=$ is a binary predicate

Peano Arithmetic Examples

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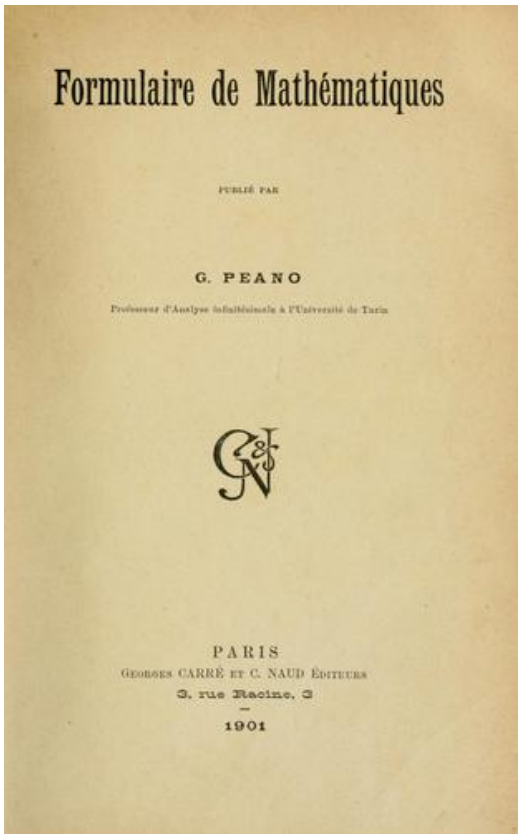
$$(1 + 1) \cdot x = y$$

Axioms of Peano Arithmetic



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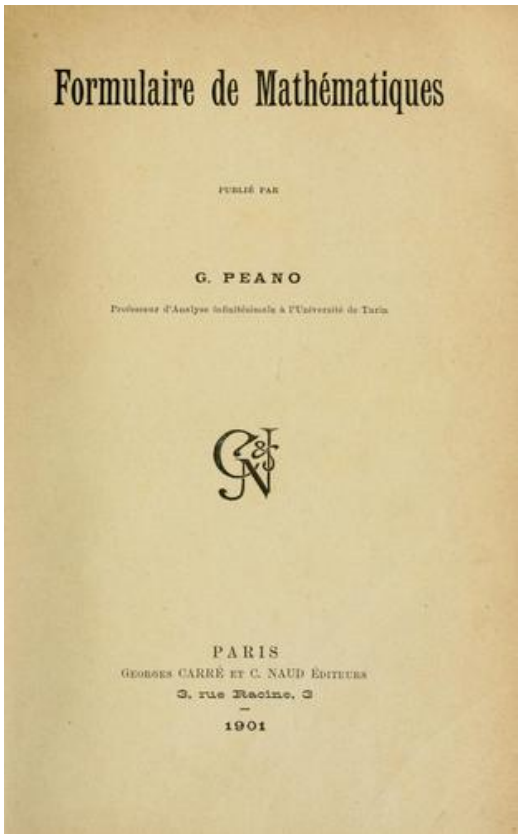

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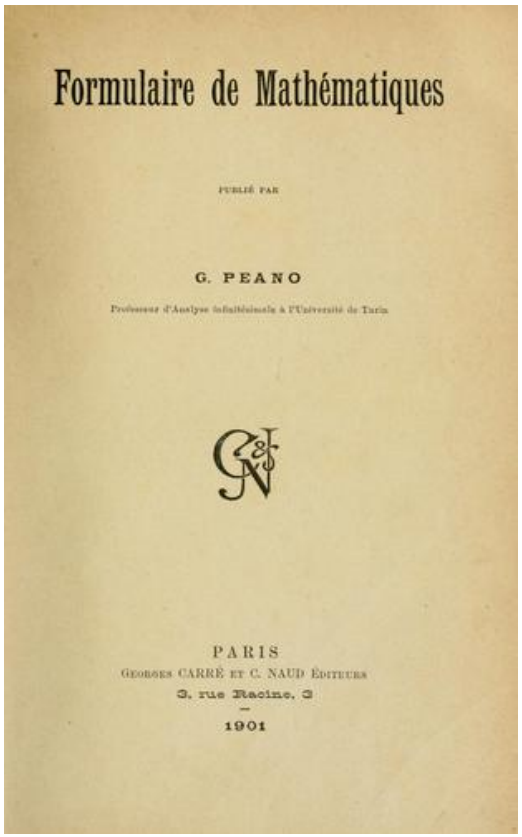

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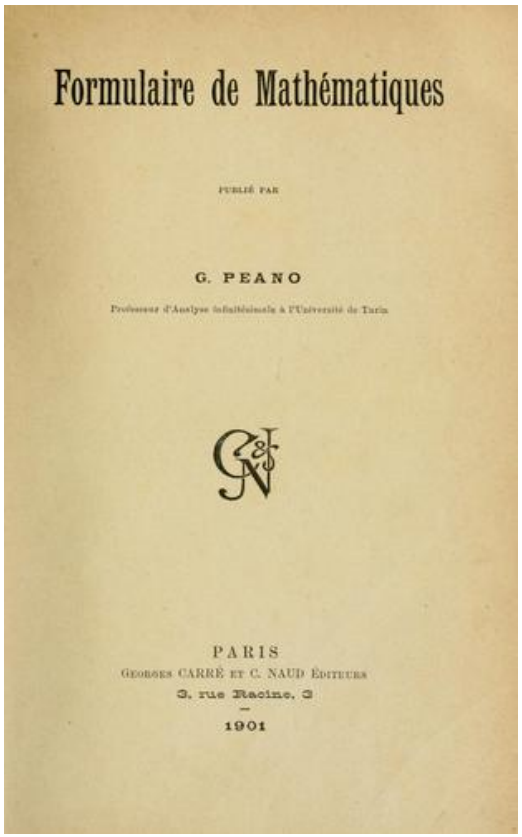

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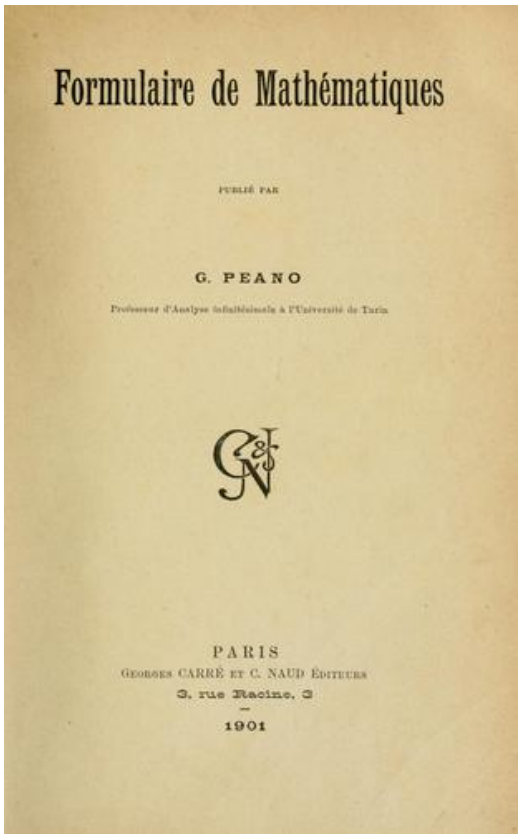

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- Unchanged since then, used to investigate consistency and completeness of number theory

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- To get decidability and completeness, we need to drop multiplication!

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1. $\forall x.\ \neg(x + 1 = 0)$ (zero)
2. $\forall x.\ x + 0 = x$ (plus zero)
3. $\forall x, y.\ x + 1 = y + 1 \ \rightarrow \ x = y$ (successor)
4. $\forall x, y.\ x + (y + 1) = (x + y) + 1$ (plus successor)
5. $F[0] \ \wedge \ (\forall x.\ F[x] \ \rightarrow \ F[x + 1]) \ \rightarrow \ \forall x.\ F[x]$ (induction)

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- Quantifier-free fragment NP-complete, full theory: $O(2^{2^{2^n}})$

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- Next week, will look at technique for deciding satisfiability of qff $T_{\mathbb{Q}}$ formula (Simplex)

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- We'll look at one example: theory of arrays
- Sometimes used in software verification

Theory of Arrays

Signature

$$\Sigma: \{\cdot[\cdot],\ \cdot\langle\cdot \lhd \cdot\rangle,\ =\}$$

where

- $a[i]$ binary function –
  read array a at index i ("read(a,i)")
- $a\langle i \lhd v\rangle$ ternary function –
  write value v to index i of array a ("write(a,i,e)")
- $a\langle i \lhd v\rangle$ represents the resulting array after writing value v at index i

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  3. $\forall a, v, i, j.\ i \neq j \ \rightarrow \ a\langle i \triangleleft v\rangle[j] = a[j]$ (read-over-write 2)

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- Assume there exists a model M and variable assignment σ that does not satisfy F and derive contradiction.

Example cont.

$$1. \qquad M, \sigma \quad \not\models \quad a[i] = e \rightarrow (\forall j.\ a\langle i \triangleleft e\rangle[j] = a[j]) \qquad \text{assumption}$$

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4.	$M, \sigma[j \mapsto k]$	$\not\models$	$a\langle i \triangleleft e\rangle[j] = a[j]$	3, $\forall$

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7.	$M, \sigma[j \mapsto k]$	$\models$	$a[i] = a[j]$	6, cong
8.	$M, \sigma[j \mapsto k]$	$\models$	$a\langle i \triangleleft e\rangle[j] = e$	6, r-o-w 1

Example cont.

1.	M, σ	$\not\models$	$a[i] = e \rightarrow (\forall j.\ a\langle i \triangleleft e\rangle[j] = a[j])$	assumption
2.	M, σ	$\models$	$a[i] = e$	1, $\rightarrow$
3.	M, σ	$\not\models$	$\forall j.\ a\langle i \triangleleft e\rangle[j] = a[j]$	1, $\rightarrow$
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- Example: array property fragment (disallows nested arrays, restrictions on where quantified variables can occur)

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- But in many applications, we need combined reasoning about several of these theories
- Example: The formula $f(x) + 3 = y$ isn't a well-formed formula in any individual theory, but belongs to combined theory $T_{\mathbb{Z}} \cup T_=$

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- Also, given decision procedures for T_1 and T_2, Nelson and Oppen's technique allows deciding satisfiability $T_1 \cup T_2$

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- Later in the course, we'll see about how to handle disjunctions much more efficiently