(Un)-Decidability Results for Word Equations with Length and Regular Expression Constraints

Vijay Ganesh

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Web Security:
A Motivation for Theories over Strings, Length and RE

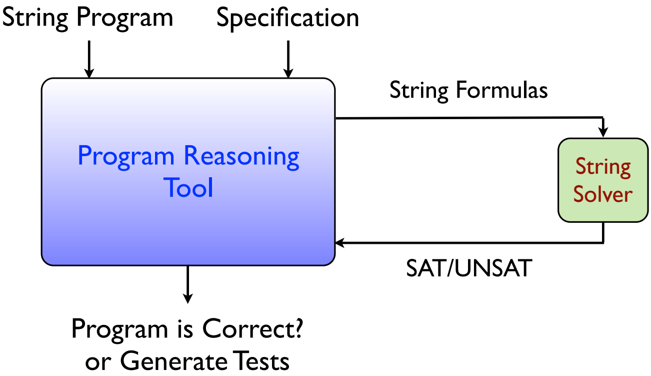
Motivating problem:

- Web apps are plagued by security errors
- Examples include SQL injection, XSS, and CSRF attacks
- These security errors often involve some computation over strings, numbers and regular expressions
- Question:
  - How can we analyze code automatically to detect/find these class of errors?

Solution:

- String analysis (static, dynamic, symbolic) that uses a backend string solver

String Solver Usage Scenario: For Security Analysis, Verification and Bug-finding



Web Security:
A Motivation for Theories over Strings and Numbers

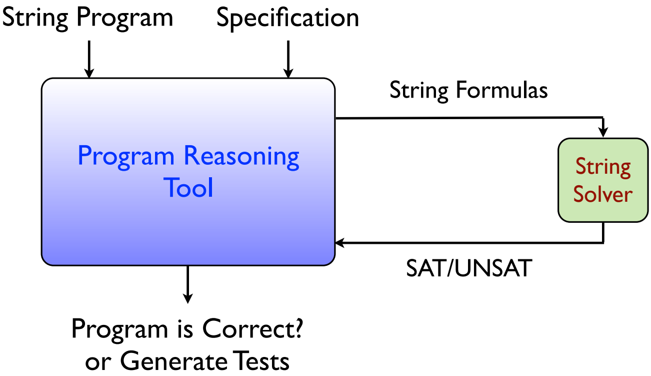
How are strings, length and RE used in Web applications?

- Construct SQL commands
  - string concatenations to create strings from input forms
- Sanity check
  - membership in regexp for checking input against known vectors
- Length check
  - string length comparisons for protecting against overflow errors

Powerful language over strings & numbers

- String operations such as concatenation, extraction
- Length function: strings -> int
- Regular expressions and context-free grammars

String Solver Usage Scenario: For Security Analysis, Verification and Bug-finding



Web Security:
A Motivation for Theories over Strings and Numbers

SAT procedures for theories of strings, length and regular expressions

- HAMPI [G. et al. (ISSTA 2009, CAV 2011, TOSEM 2012)]
  - Supports bounded strings, bounded regular expressions and context-free grammars (finite languages)
- Rex [Veanes et al.]
  - Unbounded strings and length
- Kaluza [Saxena et al.]
  - Bounded strings and length
- Z3-str [Zheng, Zhang and G. (FSE 2013)]
  - Unbounded strings and length

A Rich Language for Strings (Words), Length and RE: For Security, Verification and Bug-finding Applications

	string sort	**number sort**
Constants	finite strings/words over finite alphabet Σ $\epsilon, a, b, ab, \ldots \in \Sigma^*$	numbers disjoint from Σ 0,1,2,...
Variables	denoted by X,Y,...	N,M,...
Functions	string concatenation $\cdot$	add, length $+$, len(str-term)
Predicates	word equations str-term = str-term	less-than-or-equal-to num-term $\leq$ num-term
Predicates	regexp membership str-term $\in$ RegExp	

Formulas constructed inductively in the usual way

The Meaning of Formulas in the Language of Strings, Length and RE

Sample formulas

- $abX = Xba$, where X is a string variable
- $(abX = Xba) \wedge (X \in (ab \mid ba)(ab)^* a) \wedge (len(X) \leq 5))$

Informal semantics

- A solution to a word equation is a mapping from variables to string constants
- A word equation may have infinitely many solutions
- For example: the solutions to the equation Xa = aX can be represented by the set a^*
- Example of a formula with RE: $aX \in (a + b)^*$
- Observe that word equations can represent regular sets or even CFGs

The Meaning of Formulas in the Language of Strings, Length and RE

More sample formulas

- $abX = Xba$, where X is a string variable
  - Solution: X := a
- $(abX = Xba) \wedge (X \in (ab \mid ba)(ab)^{*}a) \wedge (len(X) \leq 5))$
  - Solution: X := aba
- $Xa = bX$
  - UNSAT
- $XabX = XbaX$
  - UNSAT
- Difficulty: Overlapping variables

Outline of the Rest of the Talk

- Undecidability result #1
  - $\forall\exists$-fragment of positive word equations is undecidable
- Decidability result #2
  - The SAT problem for word equations in solved form + length constraints is decidable
- Decidability result #3
  - The SAT problem for word equations in solved form + length + RE constraints is decidable
- Practical utility
  - Most word equations in applications are already in solved form
- Solvers: HAMPI and Z3-str
- Future directions

Decision problem #1: Decidability Question for Theory of Word Equations

Boundary between decidability and undecidability for word equations

- Fully quantified theory with only word equations is undecidable (Quine 1946)
- Quantifier-free (QF) theory of word equations is decidable (Makanin 1977)
- QF fragment with only word equations is PSPACE (Plandowski 2006)
- How many quantifier-alternation for theory to be undecidable? (**Our Answer: 1**)

Decision problem #2: SAT Problem for QF Word Equations and Length

Decidability of SAT problem of QF fragment with word equations and integer constraints over length function

- Open problem for 70 years
  (Studied by Post, Matiyasevich and Plandowski)
- If shown undecidable, lead to new proof of Matiyasevich's theorem
  (Matiyasevich 2006)
- Matiyasevich showed that Hilbert's Tenth problem is unsolvable
  (Matiyasevich 1971)
- **We provide a conditional solution important in practice**

Decision problem #3: SAT Problem for QF Word Equations, Length, Regexp

Decidability of SAT problem of QF fragment with word equations, length function and regular expression membership predicate

- Still open
- Word equations, membership predicate over regexp is decidable (Schulz 1992. Extending Makanin's 1977 result)
- Restriction: $var \in regexp$
- Strictly more general than Makanin's result
- **We provide a conditional solution important in practice**

Result #1:
Undecidability of $\forall\exists$-fragment over Word Equations

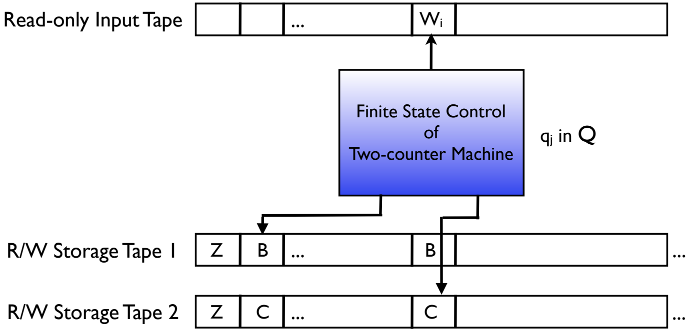
Theorem:

Validity problem for $\forall\exists$-sentences over positive word equations, with at most two occurrences of any variable is undecidable

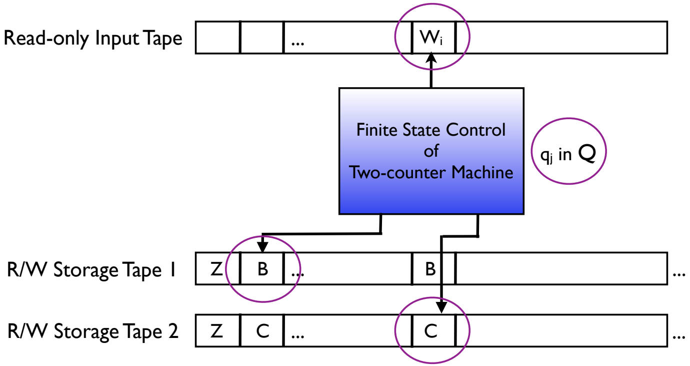
Proof Idea:

- Reduction from the halting problem for two-counter machines to the problem-under-consideration
- Two-counter machines can simulate arbitrary Turing machines
- Hence, halting problem for two-counter machines is undecidable
- Choice of two-counter machines is crucial for our proof

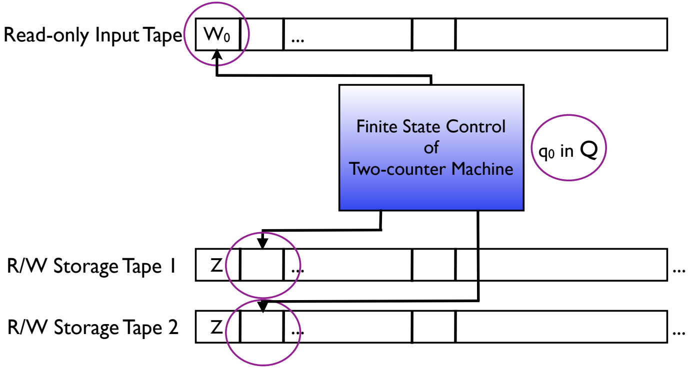
Undecidability Result: Recalling Two-counter Machines



Undecidability Result: Instantaneous Description (ID) and Strings

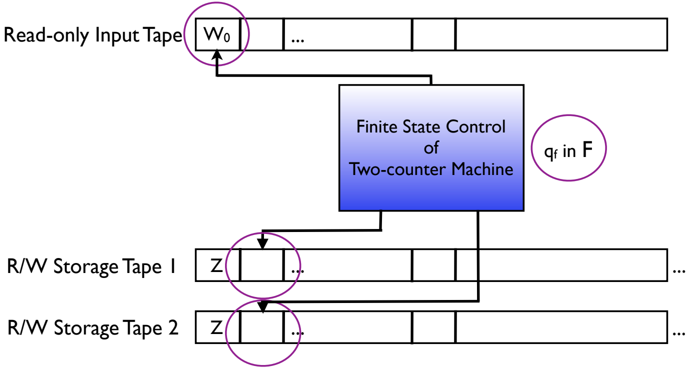


Undecidability Result: Initial Instantaneous Description (ID)



Initial ID = <q_0, W_0, ϵ, ϵ>

Undecidability Result:
Final Instantaneous Descriptions (ID)



Final ID = $<q_f, W_0, \epsilon, \epsilon>$

Undecidability Result: Computational History of a Two-counter Machine

Given a machine M and an input string W, define well-formed computational history as a string:

$\#q_0W0\epsilon\epsilon\#...ID_i\#ID_{i+1}\#...$

Accepting computational history:

$\#q_0W0\epsilon\epsilon\#...\#ID_i\#ID_{i+1}\#...\#q_fW0\epsilon\epsilon$

Computational history can be accepting, rejecting, non well-formed or non-terminating

Undecidability Result: Revisiting the Proof Strategy

By reduction from halting problem of two-counter machines to the validity problem:

given a machine M and an input string w, construct a string formula θ such that

- Every assignment to the θ is a non-accepting computational history
- M does not halt on w $\iff$ θ is valid
- Intuition: M does not halt on w iff no finite computational history is an accepting history

String alphabet over which θ is defined

- Σ_0 := $\#q_i w N_j : q_i \in Q, 0 \leq j <| w |$
- Σ_1 := b
- Σ_2 := c

Undecidability Result: How does the Reduction Work? The structure of θ

given a machine M and an input string w, the string formula θ satisfied by any non-accepting computational history:

For any string S, there exists partitions such that:
$(\forall S \exists parts.\theta(S, parts)$ is valid $\iff$ (M, w) does not accept and halt)

- Does not begin with an Initial ID
  OR
- Does not end in a Final ID
  OR
- ID_{i+1} does not follow ID_i according to transition function of M
  OR
- Is not a well-formed sequence of IDs

Undecidability Result: How does the Reduction Work? The structure of θ

θ accepts assignments that are not well-formed sequences of IDs

This sub-formula illustrates the value of two-counter machines over Turing

- Well-formed ID $\in$ regexp $\Sigma_0 b^* c^*$
- Well-formed sequence of IDs $\in$ regexp $(\Sigma_0 b^* c^*)^*$
- Not a well-formed sequence of IDs $\in \Sigma^* - (\Sigma_0 b^* c^*)^*$
- Regexp can be eliminated by word equations: $(S = \epsilon) \vee (S = S_1 \cdot c \cdot b \cdot S_4)$

Lesson confirmation: good to have multiple representations and proofs!

Recap of Result #1: Undecidability of $\forall\exists$-fragment over Word Equations

Theorem:

Validity problem for $\forall\exists$-sentences over positive word equations, with at most two occurrences of any variable is undecidable

Proof Idea:

- Reduction from the halting problem for two-counter machines to the problem-in-question
- The halting problem for two-counter machines is known to be undecidable
- Encode computation histories as solutions to word equations
- Choice of two-counter machines is crucial
- Given M,w, construct θ such that M does not halt and accept w $\iff$ θ valid

Result #2: The SAT Problem for Word Equations and Length

The Problem:

Is the SAT problem for the QF theory of word equations and length constraints decidable?

Example:

$abX = Xba \wedge X = abY \wedge len(X) < 2$

Importance:

- Problem studied for 70 years, still open
- Directly relevant to JavaScript bug-finding
- If proven undecidable, provides a new simpler proof for Matiyasevich's theorem (Matiyasevich 2006)

Result #2: Difficulty of Resolving SAT Problem for Word Equations and Length

The Problem:

Is the SAT problem for the QF theory of word equations and length constraints decidable?

Difficulty:

- Word equations, by themselves, are decidable (Makanin)
- Length constraints are essentially Presburger arithmetic
- However, no known finite way to characterize length constraints implied by word equations
- No known theorem that states that implied length constraints cannot be finitized

Result #2: Conditional Decidability of SAT Problem for Word Equations and Length

Theorem: This SAT problem is decidable, if word equations can be converted into solved form

Solved form for conjunction of word equations

- Every word equation can be written as $X = t$
- t is a concatenation of string constants, int parameters and new variables
- Variable can occur exactly once, and only on the left handside

Example: $Xa = aY \wedge Ya = Xa$

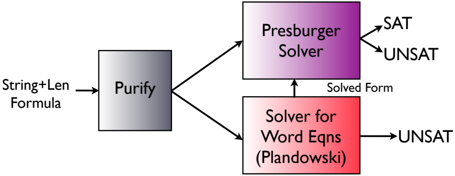
- **Solved form:** $X = a^i, Y = a^i$ where $i \geq 0$

Results #2 and #3: Conditional SAT Procedure for Word Equations and Length

Solved form: finite representation of implied length constraints

Result #3: Easy extension when regular expressions are added

Satisfiability procedure: Suffices to consider conjunction of literals



Results #2 and #3:
Relevance of Solved Form to Practice

Kaluza: A solver for word equations and length constraints (Saxena, Akhawe, Song)

- 50,000+ constraints from JavaScript bug-finding applications
- Categorized into SAT and UNSAT constraints
- Over 75% and 87% of the word equations in solved form
- Uses the HAMPI string solver (G. et al)

Related Work

Undecidability of free semi-groups (Durnev 1995, Marchenkov 1982)

Differences:

- Our proof uses standard well-understood two-counter machines
- Durnev uses fewer variables, more occurrences
- Free semi-groups don't have identity operator

Undecidability of bit-vectors with unbounded concatenation, extraction and equality
(Moller 1998)

Differences:

- Our result is stronger, because the theory we consider is weaker

HAMPI String Solver

TOSEM 2012
CAV 2011
ISSTA 2009

- X = concat("SELECT...",v) AND (X $\in$ SQL_grammar)
- JavaScript, PHP, ... string expressions
- NP-complete
- ACM Distinguished Paper Award 2009
- Google Faculty Research Award 2011

Theory of Strings
The Hampi Language

PHP/JavaScript/C++...	HAMPI: Theory of Strings	Notes
Var a; $a = 'name'	Var a : 1...20; a = 'name'	Bounded String Variables String Constants
string_expr." is "	concat(string_expr," is ");	Concat Function
substr(string_expr,1,3)	string_expr[1:3]	Extract Function

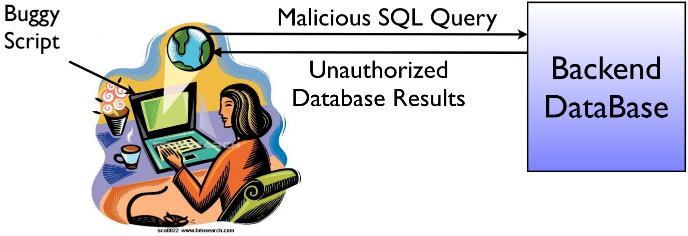
assignments/strcmp
a = string_expr;
a /= string_expr;
	equality
	a = string_expr;
	a /= string_expr;
		Equality Predicate

Sanity check in regular expression RE Sanity check in context-free grammar CFG	string_expr in RE string_expr in SQL string_expr NOT in SQL	Membership Predicate
string_expr contains a sub_str string_expr does not contain a sub_str	string_expr contains sub_str string_expr NOT?contains sub_str	Contains Predicate (Substring Predicate)

HAMPI and Z3-str String Solvers

HAMPI Solver Motivating Example

SQL Injection Vulnerabilities



SELECT m FROM messages WHERE id='1' OR 1 = 1

HAMPI Solver Motivating Example

SQL Injection Vulnerabilities

Buggy Script

```
if (input in regexp("[0-9]+"))
   query := "SELECT m FROM messages WHERE id=' " + input + " ' ")
```

- input passes validation (regular expression check)
- query is syntactically-valid SQL
- query can potentially contain an attack substring (e.g., 1' OR '1' = '1)

HAMPI Solver Motivating Example

SQL Injection Vulnerabilities

Should be: “^[0-9]+$”

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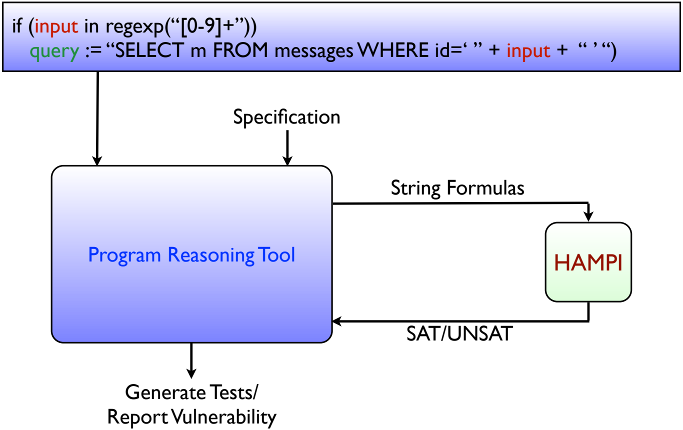

Expressing the Problem in HAMPI

SQL Injection Vulnerabilities

Input String

```
Var v : 12;
```

SQL Grammar

cfg *SqlSmall* := "SELECT " [a-z]+ " FROM " [a-z]+ " WHERE " *Cond*;

cfg *Cond* := *Val* "=" *Val* | *Cond* " OR " *Cond*;

cfg *Val* := [a-z]+ | "'" [a-z0-9]* "'" | [0-9]+;

SQL Query

val q := concat("SELECT msg FROM messages WHERE topicid='", v, "'");

assert v in [0-9]+;

SQLI attack conditions

assert q in *SqlSmall*; **"q is a valid SQL query"**

assert q contains "OR '1'='1'"; **"q contains an attack vector"**

Expressing the Problem in HAMPI

SQL Injection Vulnerabilities

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Hampi Key Conceptual Idea

Bounding, expressiveness and efficiency

L_i	Complexity of $\varnothing = L_1 \cap \ldots \cap L_n$	Current Solvers
Context-free	Undecidable	n/a
Regular	PSPACE-complete	Quantified Boolean Logic
Bounded	NP-complete	SAT Efficient in practice

HAMPI and Z3-str String Solvers

Hampi Key Idea: Bounded Logics

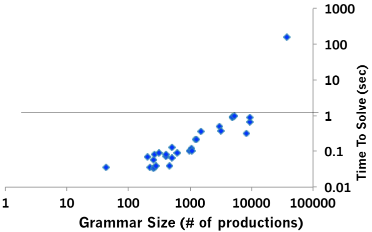
Testing, Analysis, Vulnerability Detection,...

- Finding SAT assignment is key
- Short assignments are sufficient

- Bounding is sufficient
- Bounded logics easier to decide

HAMPI: Result 1

Static SQL Injection Analysis



- 1367 string constraints from Wasserman & Su [PLDI'07]
- Hampi scales to large grammars
- Hampi solved 99.7% of constraints in < 1sec
- All solvable constraints had short solutions

HAMPI: Result 2
Security Testing and XSS

- Attackers inject client-side script into web pages
- Somehow circumvent same-origin policy in websites
- echo "Thank you $my_poster for using the message board";
- Unsanitized $my_poster
- Can be JavaScript
- Execution can be bad

HAMPI: Result 2
Security Testing

- Hampi used to build Ardilla security tester [Kiezun et al., ICSE'09]

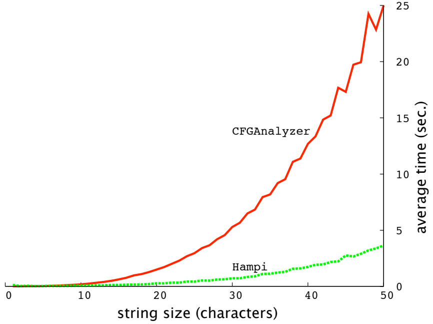
- 60 new vulnerabilities on 5 PHP applications (300+ kLOC)
  - 23 SQL injection
  - 37 cross-site scripting (XSS)

5 added to
US National Vulnerability DB

- 46% of constraints solved in < 1 second per constraint

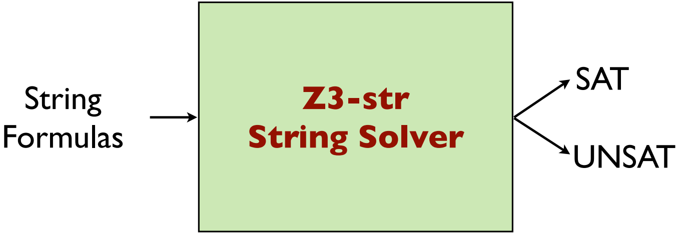
- 100% of constraints solved in <10 seconds per constraint

HAMPI: Result 3
Comparison with Competing Tools



- HAMPI vs. CFGAnalyzer (U. Munich): HAMPI ~7x faster for strings of size 50+
- HAMPI vs. Rex (Microsoft Research): HAMPI ~100x faster for strings of size 100+
- HAMPI vs. DPRLE (U.Virginia): HAMPI ~1000x faster for strings of size 100+

Z3-str String Solver*



- Quantifier-free theory of word equations and length function
- Status: unknown
- Our partial decidability technique
  - Given a word equation partition its solutions space into finite buckets
  - Leverage Z3 for identifying equivalent expressions and length consistency checks
  - Approximate by heuristically solving "overlapping" equations

*** Joint work with Xiangyu Zhang and Yunhui Zheng (Purdue University)**

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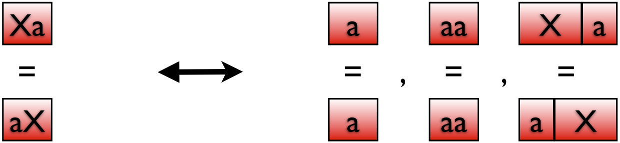

HAMPI and Z3-str String Solvers

Key Contributions

https://ece.uwaterloo.ca/~vganesh

Name	Key Concept	Impact	Pubs
STP Bit-vector & Array Solver[1,2]	Abstraction-refinement for Solving	Concolic Testing	CAV 2007 CCS 2006 TISSEC 2008
HAMPI String Solver[1]	App-driven Bounding for Solving	Analysis of Web Apps	ISSTA 2009[3] TOSEM 2012 CAV 2011
(Un)Decidability results for Strings	Reduction from two-counter machine halting problem		HVC 2012
Taint-based Fuzzing	Information flow is cheaper than concolic	Scales better than concolic	ICSE 2009
Automatic Input Rectification	Acceptability Envelope: Fix the input, not the program	New way of approaching SE	ICSE 2012

1. **STP won the SMTCOMP 2006 and 2010 competitions for bit-vector solvers**
2. **HAMPI: ACM Best Paper Award 2009**
3. **Google Award 2011**
4. Retargetable Compiler (DATE 1999)
5. Proof-producing decision procedures (TACAS 2003)
6. Error-finding in ARBAC policies (CCS 2011)
7. Programmatic SAT Solvers (SAT 2012)

Summary of Results

0) Motivated by Web security

- Considered powerful theory over word eqns, length, and regexp

1) Undecidability of $\forall\exists$ fragment over word equations

- Interesting use of two-counter machines

2) Conditional decidability of SAT for QF word eqns and length

- Relies on solved form
- Empirically observed the value of solved form in practice

3) Extended result #2 to QF word eqns, length, regexp

4) HAMPI and Z3-str

5) Formal methods for counterexample construction