ECE750T-28:
Computer-aided Reasoning for Software Engineering

Lecture 2:
Normal Forms and DPLL

Vijay Ganesh
(Original notes from Isil Dillig)

Overview

- Last lecture:
  - Two simple techniques for proving satisfiability and validity in propositional logic: truth tables and semantic argument

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  - An algorithm called DPLL for determining satisfiability
  - Many SAT solvers used today based on DPLL (more precisely, conflict-driven clause-learning)
  - However, requires converting formulas to a respresentation called normal forms
- The plan: First talk about normal forms, then discuss DPLL

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  - Disjunctive Normal Form (DNF)
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- How do we express $F_1 \leftrightarrow F_2$ using only $\neg, \wedge . \vee$?

$$\boxed{F_1 \leftrightarrow F_2 \;\Leftrightarrow\; (\neg F_1 \vee F_2) \wedge (\neg F_2 \vee F_1)}$$

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$$F_1: \quad \neg(\neg p \vee (p \wedge q))$$

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$$\begin{array}{ll} F_1 : & \neg(\neg p \vee (p \wedge q)) \\ F_2 : & \neg\neg p \wedge \neg(p \wedge q) \end{array}$$

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Convert $F: \neg(p \rightarrow (p \wedge q))$ to NNF

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F_4 is equivalent to F and is in NNF

Disjunctive Normal Form (DNF)

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- Question: If a formula is in DNF, is it also in NNF?

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$$(F_1 \ \lor \ F_2) \ \land \ F_3 \quad \Leftrightarrow \quad (F_1 \ \land \ F_3) \ \lor \ (F_2 \ \land \ F_3)$$
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$$\begin{array}{rcl}(F_1 \vee F_2) \wedge F_3 & \Leftrightarrow & (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) & \Leftrightarrow & (F_1 \wedge F_2) \vee (F_1 \wedge F_3)\end{array}$$

Example

Convert $F:\ (q_1 \ \vee \ \neg\neg q_2) \ \wedge \ (\neg r_1 \ \rightarrow \ r_2)$ into DNF

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F_2: \; (q_1 \;\vee\; q_2) \;\wedge\; (r_1 \vee r_2) & \text{in NNF}
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\begin{array}{lll}
F_1: & (q_1 \ \lor \ \neg\neg q_2) \ \land \ (\neg\neg r_1 \lor r_2) & \text{remove} \rightarrow \\
F_2: & (q_1 \ \lor \ q_2) \ \land \ (r_1 \lor r_2) & \text{in NNF} \\
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- Idea: To determine satisfiability, convert formula to DNF and just do a syntactic check.

DNF and Blow-up in formula size

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- Moral: DNF conversion causes exponential blow-up in size!
- Checking satisfiability by converting to DNF is almost as bad as truth tables!

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- Then, distribute $\lor$ over $\land$:

$$(F_1 \land F_2) \lor F_3 \quad \Leftrightarrow \quad (F_1 \lor F_3) \land (F_2 \lor F_3)$$
$$F_1 \lor (F_2 \land F_3)$$

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F_1: & (p \rightarrow (q \rightarrow r)) \wedge ((q \rightarrow r) \rightarrow p) & \text{remove } \leftrightarrow \\
F_2: & (\neg p \vee (q \rightarrow r)) \wedge (\neg (q \rightarrow r) \vee p) & \text{remove } \rightarrow
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\end{array}
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CNF Conversion Example

Convert $F: \ (p \leftrightarrow (q \rightarrow r))$ into CNF

$F_1:$	$(p \rightarrow (q \rightarrow r)) \wedge ((q \rightarrow r) \rightarrow p)$	remove $\leftrightarrow$
$F_2:$	$(\neg p \vee (q \rightarrow r)) \wedge (\neg(q \rightarrow r) \vee p)$	remove $\rightarrow$
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$F_5:$	$(\neg p \vee \neg q \vee r) \wedge (q \vee p) \wedge (\neg r \vee p)$	Distribute $\vee$ over $\wedge$

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Convert $F : \ (p \leftrightarrow (q \rightarrow r))$ into CNF

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F_5 : & (\neg p \lor \neg q \lor r) \land (q \lor p) \land (\neg r \lor p) & \text{Distribute } \lor \text{ over } \land
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F_5 is equivalent to F and is in CNF

DNF vs. CNF

- Fact: Unlike DNF, it is not trivial to determine satisfiability of formula in CNF.

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- Does CNF conversion cause exponential blow-up in size? Yes
- News: But almost all SAT solvers first convert formula to CNF before solving!

Why CNF?

- Interesting Question: If it is just as expensive to convert formula to CNF as to DNF, why do solvers convert to CNF although it is much easier to determine satisfiability in DNF?

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  1. Possible to convert to equisatisfiable (not equivalent) CNF formula with only linear increase in size!

  2. CNF makes it possible to perform interesting deductions (resolution)

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- But useful if all we want to do is determine satisfiability.

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- Use an algorithm (DPLL) to decide satisfiability of F'
- Since F' is equisatisfiable to F, F is satifiable iff algorithm decides F' is satisfiable
- Big question: How do we convert formula to equisatisfiable formula without causing exponential blow-up in size?

Tseitin's Transformation

Tseitin's transformation converts formula F
to equisatisfiable formula F' in CNF
with only a linear increase in size.

Tseitin's Transformation I

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- p_{G_1} is said to be representative of G_1 and p_{G_2} is representative of G_2.

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- Observe: Since $p_G \leftrightarrow p_{G_1} \circ p_{G_2}$ contains at most three propositional variables and exactly two connectives, size of this formula in CNF is bound by a constant.

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- Formula is also in CNF because conjunction of CNF formulas is in CNF.

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- Thus, trasformation causes only linear increase in formula size.
- More precisely, the size of resulting formula is bound by $30n + 2$ where n is size of original formula

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Convert $F : (p \vee q) \to (p \wedge \neg r)$ to equisatisfiable CNF formula.

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Convert $F : (p \lor q) \to (p \land \neg r)$ to equisatisfiable CNF formula.

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p_1 \leftrightarrow (p_2 \to p_3) & \Rightarrow F_1 : (\neg p_1 \lor \neg p_2 \lor p_3) \land (p_2 \lor p_1) \land (\neg p_3 \lor p_1) \\
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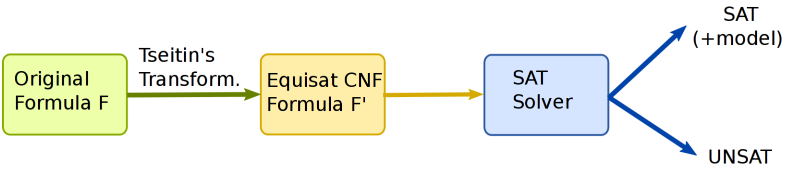
3. The formula

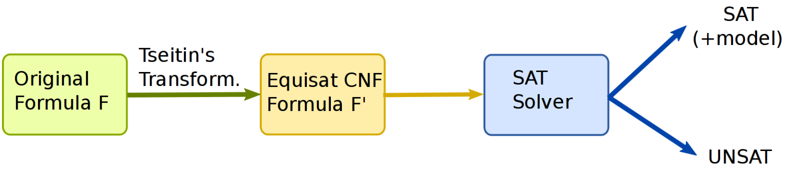
$$p_1 \land F_1 \land F_2 \land F_3 \land F_4$$

is equisatisfiable to F and is in CNF.

SAT Solvers


Original
Formula F
Tseitin's
Transform.
Equisat CNF
Formula F'
SAT
Solver
SAT
(+model)
UNSAT

SAT Solvers



- Almost all SAT solvers today are based on an algorithm called DPLL (Davis-Putnam-Logemann-Loveland)

DPLL: Historical Perspective

- 1962: the original algorithm known as DP (Davis-Putnam)
  $\Rightarrow$ "simple" procedure for automated theorem proving

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- Not all of their ideas worked out as planned ⇒ refined algorithm to what is known today as DPLL

DPLL insight

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- DPLL combines search and deduction in a very effective way!

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- Resolution can only be applied to formulas in CNF
- SAT solvers convert formulas to CNF to be able to perform resolution

Propositional Resolution

- Consider two clauses in CNF:

$$C_1 : \quad (l_1 \lor \ldots p \ldots \lor l_k) \qquad C_2 : \quad (l_1' \lor \ldots \neg p \ldots \lor l_n')$$

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- From these, we can deduce a new clause C_3, called resolvent:

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  - Suppose p is assigned $\top$: Since C_2 must be satisfied and since $\neg p$ is $\bot$, $(l_1' \vee \ldots\ldots \vee l_n')$ must be true.

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- Consider two clauses in CNF:

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  - Thus, C_3 must be true.

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- In DPLL, all possible applications of unit resolution called Boolean Constraint Propagation (BCP).

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- Resolvent of first and second clause: q
- New formula: $q \land (r \lor \neg q \lor s)$
- Apply unit resolution again: $(r \lor s)$
- No more unit resolution possible, so this is the result of BCP.

Basic DPLL

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bool DPLL(φ)
{



}
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- Recursive procedure; input is formula in CNF

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  5. if(DPLL(φ'[p ↦ ⊤])) then return SAT;

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  6. else return (DPLL(φ'[p ↦ ⊥]));
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An Optimization: Pure Literal Propagation

- If variable p occurs only positively in the formula (i.e., no $\neg p$), p must be set to $\top$
- Similarly, if p occurs only negatively (i.e., only appears as $\neg p$), p must be set to $\perp$
- This is known as Pure Literal Propagation (PLP).

DPLL with Pure Literal Propagation

```
bool DPLL(φ)
{
   1. φ' = BCP(φ)
   2. φ'' = PLP(φ')
   3. if(φ'' = ⊤) then return SAT;
   4. else if(φ'' = ⊥) then return UNSAT;
   5. p = choose_var(φ'');
   6. if(DPLL(φ''[p ↦ ⊤])) then return SAT;
   7. else return (DPLL(φ''[p ↦ ⊥]));
}
```

Example

$$F: \ (\neg p \ \vee \ q \ \vee \ r) \ \wedge \ (\neg q \ \vee \ r) \ \wedge \ (\neg q \ \vee \ \neg r) \ \wedge \ (p \ \vee \ \neg q \ \vee \ \neg r)$$

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- No BCP possible because no unit clause
- No PLP possible because there are no pure literals
- Choose variable q to branch on:

$$F[q \mapsto \top] : \ (r) \ \wedge \ (\neg r) \ \wedge \ (p \ \vee \ \neg r)$$

Example

$F: \; (\neg p \;\vee\; q \;\vee\; r) \;\wedge\; (\neg q \;\vee\; r) \;\wedge\; (\neg q \;\vee\; \neg r) \;\wedge\; (p \;\vee\; \neg q \;\vee\; \neg r)$

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- Choose variable q to branch on:

$$F[q \mapsto \top]: \; (r) \;\wedge\; (\neg r) \;\wedge\; (p \;\vee\; \neg r)$$

- Unit resolution using (r) and $(\neg r)$ deduces $\perp \Rightarrow$ backtrack

Example Cont.

$$F: \ (\neg p \ \lor \ q \ \lor \ r) \ \land \ (\neg q \ \lor \ r) \ \land \ (\neg q \ \lor \ \neg r) \ \land \ (p \ \lor \ \neg q \ \lor \ \neg r)$$

- Now, try $q = \bot$

$$F[q \mapsto \bot]: \ (\neg p \lor r)$$

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- By PLP, set p to $\bot$ and r to $\top$

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- Now, try $q = \bot$

$$F[q \mapsto \bot] : \ (\neg p \lor r)$$

- By PLP, set p to $\bot$ and r to $\top$
- $F[q \mapsto \bot, p \mapsto \bot, r \mapsto \top] : \top$

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$$F: (\neg p \ \vee \ q \ \vee \ r) \ \wedge \ (\neg q \ \vee \ r) \ \wedge \ (\neg q \ \vee \ \neg r) \ \wedge \ (p \ \vee \ \neg q \ \vee \ \neg r)$$

- Now, try $q = \bot$

$$F[q \mapsto \bot] : \ (\neg p \vee r)$$

- By PLP, set p to $\bot$ and r to $\top$
- $F[q \mapsto \bot, p \mapsto \bot, r \mapsto \top] : \top$
- Thus, F is satisfiable and the assignment $[q \mapsto \bot, p \mapsto \bot, r \mapsto \top]$ is a model (i.e., a satisfying interpretation) of F.

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- Almost all SAT solvers work on CNF formulas to perform BCP
- DPLL basis of most state-of-the-art SAT solvers

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