ECE750T-28: Computer-aided Reasoning for Software Engineering

Lecture 6: First Order Logic Syntax and Semantics

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(Original notes from Isil Dillig)

Overview

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- Propositional logic is simple and easy to automate, but not very expressive
- Today: First order logic, also known as relational logic, predicate logic, or first-order predicate calculus
- Much richer and more expressive, but does not admit completely automated reasoning (more on this later)

The Plan

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- Resolution and first-order theorem proving (fourth lecture)

Constants in First-Order Logic

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- In first order logic, constants are more involved.
- Three kinds of constants:
  1. object constants
  2. function constants
  3. relation constants

Object Constants

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- Example: a, *art*, *beth*, 1 etc. refer to object constants.

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- An object constant is really a special case of a function constant with arity 0

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- Example: *loves* has arity 2, *ishappy* has arity 1 etc.

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- Examples: $mary$, x, $sister(mary)$, $price(x, macys)$, $age(mother(y))$, ...

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- If F is a formula, then so is $\neg F$
- If F is a formula and x a variable, so are $\forall x.F$ (asserts facts about *all* objects) and $\exists x.F$ (asserts facts about *some* object)

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- $f(p(x))$, $p(p(x))$ etc. not valid in FOL!

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- An occurrence of a variable is bound if it is in the scope of some quantifier.
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Free vs. Bound Variable Example

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- Is $\forall x.p(x)$ ground? no

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- $\exists x.\forall y.friend(x, y)$
- If you flip the quantifiers, completely different meaning!

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- *"No student has a friend unless he/she is at William & Mary."*
- $\forall x.((student(x) \wedge \neg atWM(x)) \rightarrow \neg\exists y.friend(x, y))$

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- Another way of saying this:

  $$\forall x.(student(x) \land atWM(x)) \to (\forall y.(friends(x, y) \to atWM(y))$$

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- Another way of saying this:

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- These two formulas are actually semantically equivalent

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$$\forall x. \quad ((atWM(x) \land student(x)) \rightarrow \\ \exists y.(friends(x, y) \land \neg atWM(y)))$$

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$$\forall n. n > 2 \ \rightarrow \ \neg \exists a, b, c.\ a > 0 \land b > 0 \land c > 0 \land a^n + b^n = c^n$$

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- This cannot be expressed in FOL because it requires quantification over relation constants!
- But it can, however, be expressed in second-order logic:

$$\forall x, y. friend(x, y) \rightarrow \exists p. p(x) \land p(y)$$

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- In FOL, these concepts are a bit more involved ...
- To give semantics to FOL, we need to talk about a universe of discourse (also sometimes called just "universe" or "domain")

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  - Students in this class: finite

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- I maps every $c \in C$ to some member of U: $I(c) \in U$
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- Observe: A first-order interpretation does not talk about variables (only constants)

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- Observe: Different object constants do not have to map to distinct objects in U!

Another Example

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- $I(f) = \{\square \mapsto \square, \triangle \mapsto \square\}$, or $I(f) = \{\square \mapsto \square, \triangle \mapsto \triangle\}$, or $I(f) = \{\square \mapsto \triangle, \triangle \mapsto \square\}$ or $I(f) = \{\square \mapsto \triangle, \triangle \mapsto \triangle\}$

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- For a unary relation constant r, what are all the possible interpretations?
- $I(r) = \{\}$, $I(r) = \{\langle\square\rangle\}$, or $I(r) = \{\langle\triangle\rangle\}$, or $I(r) = \{\langle\square\rangle, \langle\triangle\rangle\}$

First-Order Structures and Variable Assignments

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- Example: Given $U = \{\Box, \triangle\}$, a possible variable assignment for x: $\sigma(x) = \triangle$
- Observe: σ does not map variables to object constants but to objects in U!

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$$\langle I, \sigma \rangle(f(t_1, \ldots, t_k)) = I(f)(\langle I, \sigma \rangle(t_1), \ldots, \langle I, \sigma \rangle(t_k))$$

Example: Evaluation of Terms

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- We define the semantics of $\models$ inductively.

Evaluation of Formulas, Bases Cases

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$$U, I, \sigma \models p(t_1, \ldots, t_k) \ \text{ iff } \ \langle\ \langle I, \sigma \rangle(t_1), \ldots, \langle I, \sigma \rangle(t_k)\ \rangle \in I(p)$$

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- Consider universe $\{\star, \bullet\}$, variable assignment $\sigma : \{x \mapsto \star\}$, and interpretation I:

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Satisfiability and Validity of First-Order Formulas

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- Intuition: Consider any object o. If $p(o, o)$ is false, then implication satisfied. If $p(o, o)$ is true, there there exists a y (namely o) s.t $p(x, y)$ is also true.

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Summary

- Today: Syntax and formal semantics of FOL
- Next lecture:
  - Semantic argument method for FOL
  - Properties of first-order logic: decidability results, compactness