ECE750T-28: Computer-aided Reasoning for Software Engineering

Lecture 17: SMT Solvers and the DPPL($\mathcal{T}$) Framework

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(Original notes from Isil Dillig)

SMT Solvers

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- This is the case because SMT solvers generalize SAT solvers; they can handle much richer theories than propositional logic

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- For instance, at Microsoft, there are at least two dozen projects that rely on the Z3 SMT solver

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- Also already know how to combine these theories using Nelson-Oppen technique
- Big missing piece: How to handle boolean structure of SMT formulas including disjunctions

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- In reality, this is completely impractical: DNF conversion can yield exponentially larger formula
- For many real problems, DNF conversion is prohibitively expensive
- Thus, we need a way to decide satisfiability of SMT formulas without expensive conversion to DNF

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- However, $\mathcal{T}$ can be a combination theory, such as $\mathcal{T}_= \cup \mathcal{T}_\mathbb{Z}$
- As before, solver for $\mathcal{T}_= \cup \mathcal{T}_\mathbb{Z}$ is obtained by using Nelson-Oppen technique

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- Now, use SAT solver to decide satisfiability of boolean abstraction

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- Main job of the theory solver is to check whether assignments made by SAT solver is Satisfiable Modulo Theory (SMT)
- If SAT solver finds assignment that is consistent with theory, then SMT formula is satisfiable

SMT Formulas and Boolean Abstraction

- SMT formula in theory $\mathcal{T}$ formed as usual (structural induction):

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- What is a sat assignment?

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- What is $\mathcal{B}^{-1}(A)$?

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- Query $\mathcal{T}$-solver for satisfiability of $\mathcal{B}^{-1}(A)$

SMT Solving: Simplest Version, cont

- If $\mathcal{T}$-solver decides $\mathcal{B}^{-1}(A)$ is sat, return SAT

SMT Solving: Simplest Version, cont

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- Why? Because we found an assignment that (i) both satisfies boolean structure, and (ii) consistent with theory axioms

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- In this case, construct new boolean abstraction $\mathcal{B}(F) \wedge \neg A$
- Repeat until we find assignment consistent with theory or until boolean abstraction is unsat

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- Does $B(F) \land \neg A$ still overapproximate satisfiability?
- Yes because since $\mathcal{B}^{-1}(A)$ is unsat $\mathcal{B}^{-1}(\neg A)$ is valid
- Thus, $F \land \mathcal{B}^{-1}(\neg A)$ is equivalent to F

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- Does $B(F) \land \neg A$ still overapproximate satisfiability?
- Yes because since $\mathcal{B}^{-1}(A)$ is unsat $\mathcal{B}^{-1}(\neg A)$ is valid
- Thus, $F \land \mathcal{B}^{-1}(\neg A)$ is equivalent to F
- Hence, $\mathcal{B}(F \land \mathcal{B}^{-1}(\neg A))$ (i.e., $\mathcal{B}(F) \land \neg A$) still overapproximates satisfiability

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- Formulas such as $\neg A$ that are $\mathcal{T}$-valid are called theory conflict clauses

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- Or all satisfying assignments contradict theory axioms $\Rightarrow$ UNSAT

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- In fact, there are 2^{98} unsat assignments containing $x = y \wedge x \neq y$ but $\neg A$ prevents only one of them!

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- Clearly, resulting ϕ is minimal unsat core of original formula

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- Repeat until we decide sat or unsat

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- Using simple strategy, we block just one assignment
- Using minimal unsat cores, we block many assignments using one theory conflict clause
- However, our strategy still not ideal because it waits for full assignment to boolean abstraction to generate conflict clause

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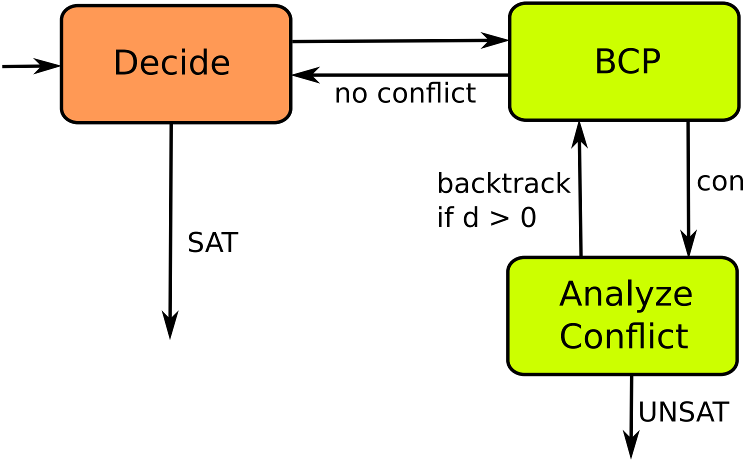
- Idea: Don't use SAT solver as "blackbox"
- Instead, integrate theory solver right into the DPLL algorithm

DPLL-Based SAT Solver Architecture



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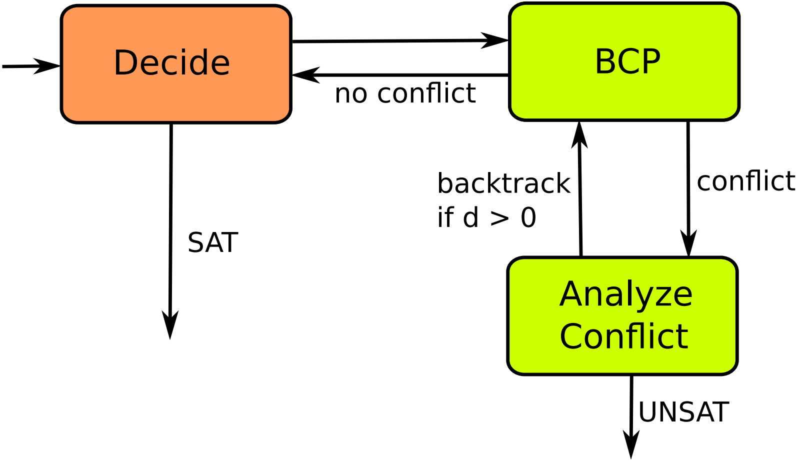
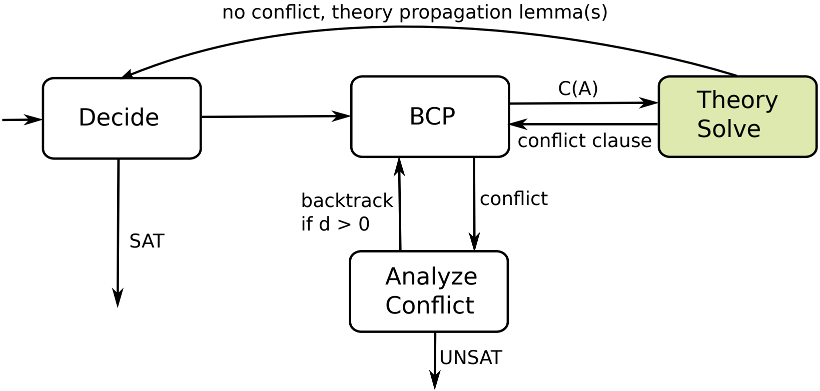


- Idea: Integrate theory solver right into this SAT solving loop!

DPLL($\mathcal{T}$) Framework



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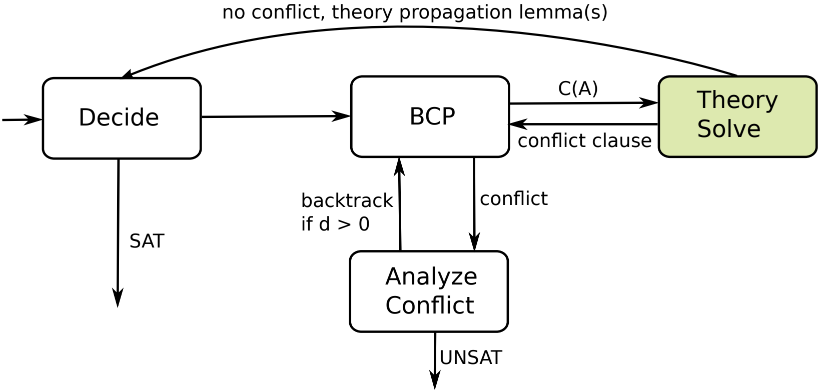

- Combination of DPLL-based SAT solver and decision procedure for conjunctive $\mathcal{T}$ formula called DPLL($\mathcal{T}$) framework

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- Or better, add negation of unsat core of A to clause database

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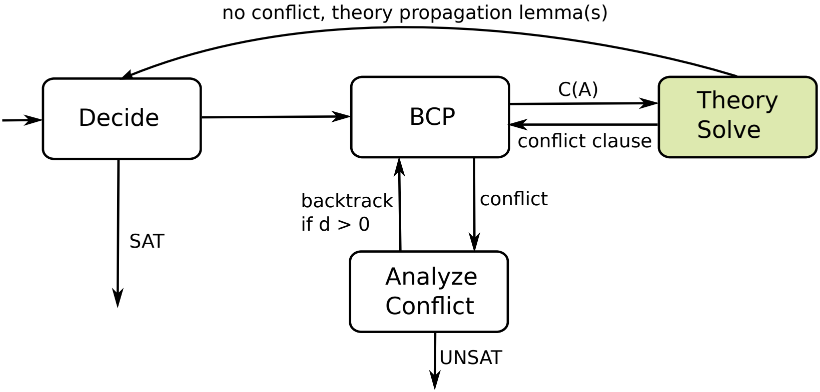

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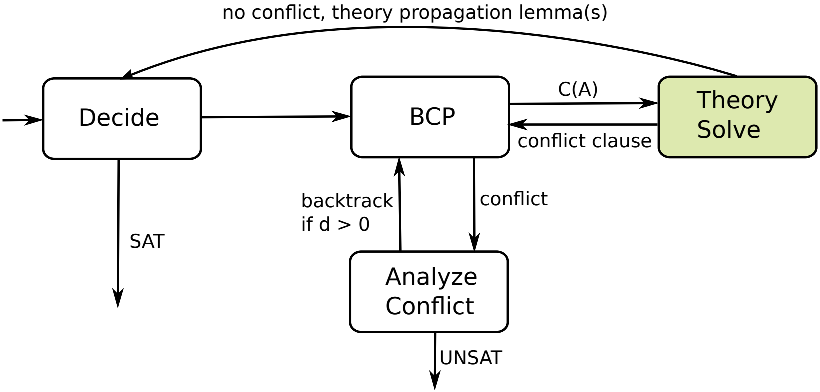

- Add theory conflict clause and continue doing BCP, which will detect conflict
- As before, AnalyzeConflict decides what level to backtrack to

Theory Propagation

- What we described so far is sufficient to solve SMT formulas, but we can much better!

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- But assignment $b_3 : \top$ is sub-optimal, b/c will lead to conflict in $\mathcal{T}$

Theory Propagation Lemma, cont

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- In our example, $\neg x \neq z$ implied by current partial assignment $x = y \wedge y = z$
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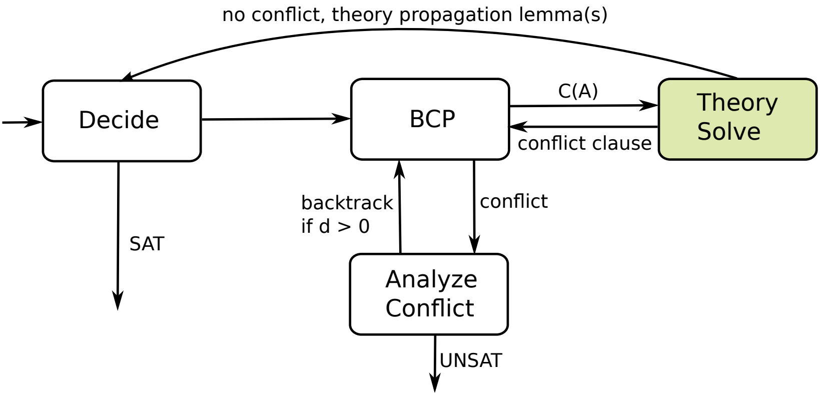
- These kinds of clauses implied by theory are called theory propagation lemmas

DPLL($\mathcal{T}$) Framework



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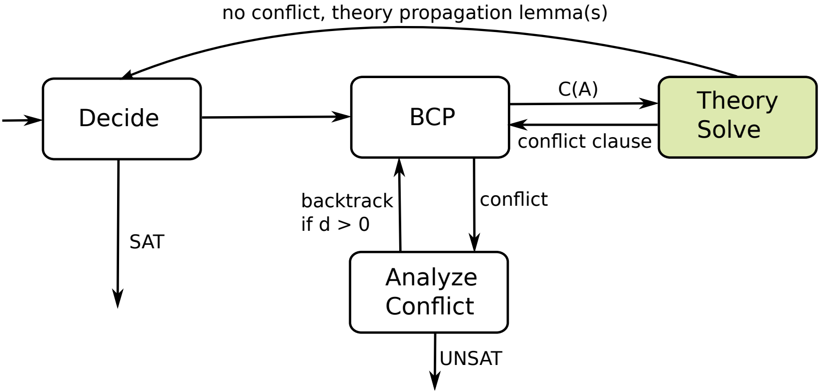

- After adding theory propagation lemma, continue doing BCP
- Adding theory propagation lemmas prevents bad assignments to boolean abstraction

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- Second option is considered more efficient, but have to figure out how to do this for each different theory

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- Solvers use different strategies to obtain simple-to-find implications

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- SMT solvers decide satisfiability in boolean combinations of different theories
- Instead of converting to DNF, they handle boolean structure using SAT solving technqiues
- Most common approach is to construct boolean abstraction and lazily infer theory conflict clauses
- To do this, can either consider SAT solver as blackbox or can integrate with it
- Latter strategy considered superior and known as DPLL($\mathcal{T}$) framework