ECE750T-28: Computer-aided Reasoning for Software Engineering

Lecture 1: Introduction to Logic in SE

Vijay Ganesh
(Original notes from Isil Dillig)

What is this Course About?

- This course is about computational logic and its application to software engineering.

What is this Course About?

- This course is about computational logic and its application to software engineering.
- Explore various logical theories widely used in computer science.

What is this Course About?

- This course is about computational logic and its application to software engineering.
- Explore various logical theories widely used in computer science.
- Learn about decision procedures, provers, solvers.

What is this Course About?

- This course is about computational logic and its application to software engineering.
- Explore various logical theories widely used in computer science.
- Learn about decision procedures, provers, solvers.
- Learn about applications such as concolic testing, model checking, analysis, fault localization, synthesis and programming languages.

Why Should You Care?

Logic is a fundamental part of computer science:

Why Should You Care?

Logic is a fundamental part of computer science:

- Computation, irrespective of representation, can be very complex to understand/process in all its gory detail.

Why Should You Care?

Logic is a fundamental part of computer science:

- Computation, irrespective of representation, can be very complex to understand/process in all its gory detail.
- Hence, we need abstractions.

Why Should You Care?

Logic is a fundamental part of computer science:

- Computation, irrespective of representation, can be very complex to understand/process in all its gory detail.
- Hence, we need abstractions.
- Logics are precise languages that allow us to represent/manipulate/process/morph abstractions of computations.

Why Should You Care?

Logic is a fundamental part of computer science:

- Computation, irrespective of representation, can be very complex to understand/process in all its gory detail.
- Hence, we need abstractions.
- Logics are precise languages that allow us to represent/manipulate/process/morph abstractions of computations.
- Examples include Boolean logic (aka propostional or sentential calculus), predicate logic, first-order theories,...

Why Should You Care?

Logic is a fundamental part of computer science:

Why Should You Care?

Logic is a fundamental part of computer science:

- Artificial intelligence: constraint satisfaction, automated game playing, planning, ...

Why Should You Care?

Logic is a fundamental part of computer science:

- Artificial intelligence: constraint satisfaction, automated game playing, planning, ...
- Programming Languages: logic programming, type systems, programming language theory ...

Why Should You Care?

Logic is a fundamental part of computer science:

- Artificial intelligence: constraint satisfaction, automated game playing, planning, ...
- Programming Languages: logic programming, type systems, programming language theory ...
- Hardware verification and synthesis: correctness of circuits, ATPG, ...

Why Should You Care?

Logic is a fundamental part of computer science:

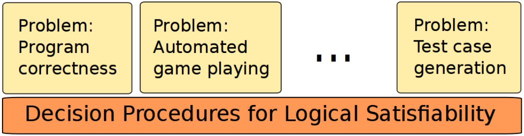
- Artificial intelligence: constraint satisfaction, automated game playing, planning, ...
- Programming Languages: logic programming, type systems, programming language theory ...
- Hardware verification and synthesis: correctness of circuits, ATPG, ...
- Program analysis, verification and synthesis: Static analysis, software verification, test case generation, program understanding, ...

Why Should You Care?

- No matter what your research area or interest is, the techniques we cover in this course are likely to be relevant.

Why Should You Care?

- No matter what your research area or interest is, the techniques we cover in this course are likely to be relevant.
- Very good tool kit because many difficult problems can be reduced deciding satisfiabilty of formulas in logic.



Topics Covered in the Course

- Review of propositional logic

Topics Covered in the Course

- Review of propositional logic
- Modern SAT solvers

Topics Covered in the Course

- Review of propositional logic
- Modern SAT solvers
- Complexity theory basics, reductions, classes,...

Topics Covered in the Course

- Review of propositional logic
- Modern SAT solvers
- Complexity theory basics, reductions, classes,...
- First-order theorem provers

Topics Covered in the Course

- Review of propositional logic
- Modern SAT solvers
- Complexity theory basics, reductions, classes,...
- First-order theorem provers
- Theory of uninterpreted functions

Topics Covered in the Course

- Review of propositional logic
- Modern SAT solvers
- Complexity theory basics, reductions, classes,...
- First-order theorem provers
- Theory of uninterpreted functions
- Linear inequalities over reals (Simplex) and integers

Topics Covered in the Course

- Review of propositional logic
- Modern SAT solvers
- Complexity theory basics, reductions, classes,...
- First-order theorem provers
- Theory of uninterpreted functions
- Linear inequalities over reals (Simplex) and integers
- Theories of bit-vectors, arrays and strings

Topics Covered in the Course

- Combining decision procedures (Nelson-Oppen)

Topics Covered in the Course

- Combining decision procedures (Nelson-Oppen)
- SMT Solvers and the DPLL(T) framwork

Topics Covered in the Course

- Combining decision procedures (Nelson-Oppen)
- SMT Solvers and the DPLL(T) framwork
- Constraint Simplification

Topics Covered in the Course

- Combining decision procedures (Nelson-Oppen)
- SMT Solvers and the DPLL(T) framwork
- Constraint Simplification
- Quantifier elimination

Topics Covered in the Course

- Combining decision procedures (Nelson-Oppen)
- SMT Solvers and the DPLL(T) framwork
- Constraint Simplification
- Quantifier elimination
- Applications: concolic testing, analysis, formal methods

Logistics

- Class meets every Friday from 11:30 AM to 2:20 PM

Logistics

- Class meets every Friday from 11:30 AM to 2:20 PM
- All lectures will be held in EIT 3141

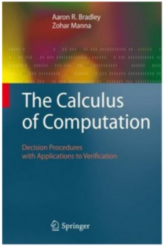
Logistics

- Class meets every Friday from 11:30 AM to 2:20 PM
- All lectures will be held in EIT 3141
- All the material for the class (lecture slides, homework, reading, announcements) will be posted on the course website:

  `https://ece.uwaterloo.ca/~vganesh/teaching.html`

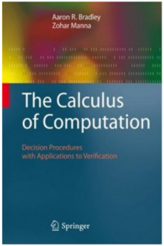
Recommended Books

- The Calculus of Computation by Aaron Bradley and Zohar Manna



Recommended Books

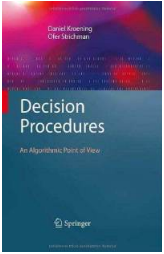
- The Calculus of Computation by Aaron Bradley and Zohar Manna



- Warning: Will cover many topics not in the Bradley & Manna textbook and will skip some chapters of this textbook

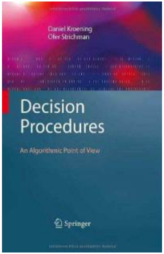
Another Recommended Book

- Decision Procedures: An Algorithmic Point of View by Daniel Kroening and Ofer Strichman



Another Recommended Book

- Decision Procedures: An Algorithmic Point of View by Daniel Kroening and Ofer Strichman



- Mostly I will follow papers, and these papers will be cited on the website.

Requirements

- Two homework assignments (15% of the final grade)

Requirements

- Two homework assignments (15% of the final grade)
- You get 1-2 weeks to complete the assignment from the day it is handed out

Requirements

- Two homework assignments (15% of the final grade)
- You get 1-2 weeks to complete the assignment from the day it is handed out
- No late submissions

Requirements

- Two homework assignments (15% of the final grade)
- You get 1-2 weeks to complete the assignment from the day it is handed out
- No late submissions
- All assignments should be done individually

Final Exam

- One final exam (50% of the final grade)

Final Exam

- One final exam (50% of the final grade)
- No mid-term exam

Final Exam

- One final exam (50% of the final grade)
- No mid-term exam
- All exams closed-book, closed-notes, closed-laptop, closed-phone etc.

Final Exam

- One final exam (50% of the final grade)
- No mid-term exam
- All exams closed-book, closed-notes, closed-laptop, closed-phone etc.
- Date fixed by registrar. Non-negotiable.

Research Projects

- Research Project (35% of the final grade)

Research Projects

- Research Project (35% of the final grade)
- Maximum 2 people per research project group

Research Projects

- Research Project (35% of the final grade)
- Maximum 2 people per research project group
- Ideally, new research that is publishable. Both theoretical or practical projects are acceptable

Research Projects

- Research Project (35% of the final grade)
- Maximum 2 people per research project group
- Ideally, new research that is publishable. Both theoretical or practical projects are acceptable
- Examples include: Novel solving technique, decidability/complexity result, feature in solver/prover, application of logics

Research Projects

- Research Project (35% of the final grade)
- Maximum 2 people per research project group
- Ideally, new research that is publishable. Both theoretical or practical projects are acceptable
- Examples include: Novel solving technique, decidability/complexity result, feature in solver/prover, application of logics
- Must get approval of the project idea from instructor by October 4th, 2013

Research Projects

- Research Project (35% of the final grade)
- Maximum 2 people per research project group
- Ideally, new research that is publishable. Both theoretical or practical projects are acceptable
- Examples include: Novel solving technique, decidability/complexity result, feature in solver/prover, application of logics
- Must get approval of the project idea from instructor by October 4th, 2013
- Must submit 2-page project proposal with title, names, abstract, problem statement, solution description, impact

Grading

- Final exam: 50%

Grading

- Final exam: 50%
- Homeworks and class participation: 15%

Grading

- Final exam: 50%
- Homeworks and class participation: 15%
- Respect honor code on exams and homework

Grading

- Final exam: 50%
- Homeworks and class participation: 15%
- Respect honor code on exams and homework
- You can consult other students on the homework, but write-up must be your own

Grading

- Final exam: 50%
- Homeworks and class participation: 15%
- Respect honor code on exams and homework
- You can consult other students on the homework, but write-up must be your own
- Also, write-up must mention names of consultants/collaborators

Grading

- Final exam: 50%
- Homeworks and class participation: 15%
- Respect honor code on exams and homework
- You can consult other students on the homework, but write-up must be your own
- Also, write-up must mention names of consultants/collaborators
- Research project: 35%

Let's get started!

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas
- Semantics: Interpretation/models

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas
- Semantics: Interpretation/models
- Forms of valid reasoning: Deductive, inductive, abductive,...

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas
- Semantics: Interpretation/models
- Forms of valid reasoning: Deductive, inductive, abductive,...
- Proof systems: Intuinistic vs. classical

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas
- Semantics: Interpretation/models
- Forms of valid reasoning: Deductive, inductive, abductive,...
- Proof systems: Intuinistic vs. classical
- Fields of study: proof, model, set, recursion, and type theory

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas
- Semantics: Interpretation/models
- Forms of valid reasoning: Deductive, inductive, abductive,...
- Proof systems: Intuinistic vs. classical
- Fields of study: proof, model, set, recursion, and type theory
- Questions studied: Proof and truth, Provability, Decidability, Complexity, Foundations,...

What are Logics?

- Precise mathematical languages with well-defined syntax and semantics
- Syntax: Meta-rules for defining well-formed formulas
- Semantics: Interpretation/models
- Forms of valid reasoning: Deductive, inductive, abductive,...
- Proof systems: Intuinistic vs. classical
- Fields of study: proof, model, set, recursion, and type theory
- Questions studied: Proof and truth, Provability, Decidability, Complexity, Foundations,...
- Properties of logics: Soundness, completeness, compactness, expressive power, decidability,...

Review of Propositional Logic: Syntax

Atom truth symbols $\top$ ("true") and $\bot$ ("false")
propositional variables $p, q, r, p_1, q_1, r_1, \cdots$

Review of Propositional Logic: Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false")
	propositional variables $p, q, r, p_1, q_1, r_1, \cdots$
Literal	atom α or its negation $\neg\alpha$

Review of Propositional Logic: Syntax

Atom	truth symbols $\top$ ("true") and $\perp$ ("false") propositional variables $p, q, r, p_1, q_1, r_1, \cdots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a logical connective to formulae F, F_1, F_2

Review of Propositional Logic: Syntax

Atom truth symbols $\top$ ("true") and $\bot$ ("false")
propositional variables $p, q, r, p_1, q_1, r_1, \cdots$

Literal atom α or its negation $\neg\alpha$

Formula literal or application of a
logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)

Review of Propositional Logic: Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false") propositional variables $p, q, r, p_1, q_1, r_1, \cdots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)

Review of Propositional Logic: Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false") propositional variables $p, q, r, p_1, q_1, r_1, \cdots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)

Review of Propositional Logic: Syntax

Atom truth symbols $\top$ ("true") and $\bot$ ("false")
propositional variables $p, q, r, p_1, q_1, r_1, \cdots$

Literal atom α or its negation $\neg\alpha$

Formula literal or application of a
logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \rightarrow F_2$	"implies"	(implication)

Review of Propositional Logic: Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false") propositional variables $p, q, r, p_1, q_1, r_1, \cdots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \rightarrow F_2$	"implies"	(implication)
$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Interpretations in Propositional Logic

- An interpretation I for a formula F in propositional logic is a mapping from each propositional variables in F to exactly one truth value

$$I : \{p \mapsto \top, q \mapsto \bot, \cdots\}$$

Interpretations in Propositional Logic

- An interpretation I for a formula F in propositional logic is a mapping from each propositional variables in F to exactly one truth value

$$I : \{p \mapsto \top, q \mapsto \bot, \cdots\}$$

- For a formula F with 2 propositional variables, how many interpretations are there?

Interpretations in Propositional Logic

- An interpretation I for a formula F in propositional logic is a mapping from each propositional variables in F to exactly one truth value

$$I : \{p \mapsto \top, q \mapsto \bot, \cdots\}$$

- For a formula F with 2 propositional variables, how many interpretations are there?
- In general, for formula with n propositional variables, how many interpretations?

Entailment

- Under an interpretation, every propositional formula evaluates to T or F

Formula F + Interpretation I = Truth value

Entailment

- Under an interpretation, every propositional formula evaluates to T or F

$$\text{Formula } F + \text{Interpretation } I = \text{Truth value}$$

- We write $I \models F$ if F evaluates to $\top$ under I (satisfying interpretation)

Entailment

- Under an interpretation, every propositional formula evaluates to T or F

$$\text{Formula } F + \text{Interpretation } I = \text{Truth value}$$

- We write $I \models F$ if F evaluates to $\top$ under I (satisfying interpretation)
- Similarly, $I \not\models F$ if F evaluates to $\bot$ under I (falsifying interpretation).

Inductive Definition of Propositional Semantics

Base Cases:

$I \models \top$

Inductive Definition of Propositional Semantics

<u>Base Cases</u>:

$I \models \top \qquad I \not\models \bot$

Inductive Definition of Propositional Semantics

<u>Base Cases</u>:

$$I \models \top \qquad I \not\models \bot$$

$$I \models p \quad \text{iff} \quad I[p] = \top$$

Inductive Definition of Propositional Semantics

Base Cases:

$$I \models \top \qquad I \not\models \bot$$

$$I \models p \quad \text{iff} \quad I[p] = \top$$
$$I \not\models p \quad \text{iff} \quad I[p] = \bot$$

Inductive Definition of Propositional Semantics

<u>Base Cases</u>:

$$I \models \top \qquad I \not\models \bot$$

$$I \models p \quad \text{iff} \quad I[p] = \top$$
$$I \not\models p \quad \text{iff} \quad I[p] = \bot$$

<u>Inductive Cases</u>:

Inductive Definition of Propositional Semantics

Base Cases:

$$I \models \top \qquad I \not\models \bot$$

$$I \models p \quad \text{iff} \quad I[p] = \top$$
$$I \not\models p \quad \text{iff} \quad I[p] = \bot$$

Inductive Cases:

$$I \models \neg F \qquad \text{iff } I \not\models F$$

Inductive Definition of Propositional Semantics

<u>Base Cases</u>:

$$I \models \top \qquad I \not\models \bot$$

$$I \models p \quad \text{iff} \quad I[p] = \top$$
$$I \not\models p \quad \text{iff} \quad I[p] = \bot$$

<u>Inductive Cases</u>:

$$I \models \neg F \qquad \text{iff } I \not\models F$$
$$I \models F_1 \wedge F_2 \qquad \text{iff } I \models F_1 \text{ and } I \models F_2$$

Inductive Definition of Propositional Semantics

Base Cases:

$I \models \top \qquad I \not\models \bot$

$I \models p \quad \text{iff} \quad I[p] = \top$

$I \not\models p \quad \text{iff} \quad I[p] = \bot$

Inductive Cases:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \land F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \lor F_2$ iff $I \models F_1$ or $I \models F_2$

Inductive Definition of Propositional Semantics

Base Cases:

$I \models \top \qquad I \not\models \bot$

$I \models p$ iff $I[p] = \top$

$I \not\models p$ iff $I[p] = \bot$

Inductive Cases:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$

$I \models F_1 \rightarrow F_2$

Inductive Definition of Propositional Semantics

Base Cases:

$I \models \top \qquad I \not\models \bot$

$I \models p$ iff $I[p] = \top$

$I \not\models p$ iff $I[p] = \bot$

Inductive Cases:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$

$I \models F_1 \rightarrow F_2$ iff, $I \not\models F_1$ or $I \models F_2$

Inductive Definition of Propositional Semantics

<u>Base Cases</u>:

$$I \models \top \qquad I \not\models \bot$$

$$I \models p \quad \text{iff} \quad I[p] = \top$$
$$I \not\models p \quad \text{iff} \quad I[p] = \bot$$

<u>Inductive Cases</u>:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$

$I \models F_1 \rightarrow F_2$ iff, $I \not\models F_1$ or $I \models F_2$

$I \models F_1 \leftrightarrow F_2$

Inductive Definition of Propositional Semantics

Base Cases:

$I \models \top \qquad I \not\models \bot$

$I \models p \quad \text{iff} \quad I[p] = \top$

$I \not\models p \quad \text{iff} \quad I[p] = \bot$

Inductive Cases:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$

$I \models F_1 \rightarrow F_2$ iff, $I \not\models F_1$ or $I \models F_2$

$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$

Inductive Definition of Propositional Semantics

<u>Base Cases</u>:

$I \models \top \qquad I \not\models \bot$

$I \models p \quad \text{iff} \quad I[p] = \top$
$I \not\models p \quad \text{iff} \quad I[p] = \bot$

<u>Inductive Cases</u>:

$I \models \neg F$ iff $I \not\models F$
$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$ iff, $I \not\models F_1$ or $I \models F_2$
$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$
or $I \not\models F_1$ and $I \not\models F_2$

Simple Example

$$F: \ (p \land q) \rightarrow (p \lor \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

Simple Example

$$F : \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I : \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1. $I \quad \models ? \quad p$

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$
$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1. $I \quad \models \quad p$ since $I[p] = \top$

Simple Example

$$F : \ (p \land q) \rightarrow (p \lor \neg q)$$

$$I : \ \{p \mapsto \top, \ q \mapsto \bot\}$$

$$\begin{array}{llll} 1. & I & \models & p & \quad \text{since } I[p] = \top \\ 2. & I & \models ? & q & \end{array}$$

Simple Example

$$F: (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \{p \mapsto \top, \ q \mapsto \bot\}$$

1. $I \models p$ since $I[p] = \top$
2. $I \not\models q$ since $I[q] = \bot$

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

$$\begin{array}{llll} 1. & I & \models & p \\ 2. & I & \not\models & q \\ 3. & I & \models ? & \neg q \end{array} \qquad \begin{array}{l} \text{since } I[p] = \top \\ \text{since } I[q] = \bot \\ \end{array}$$

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1.	I	$\models$	p	since $I[p] = \top$
2.	I	$\not\models$	q	since $I[q] = \bot$
3.	I	$\models$	$\neg q$	by 2 and $\neg$

Simple Example

$$F: \ (p \land q) \rightarrow (p \lor \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

$$\begin{array}{llll} 1. & I & \models & p & \text{since } I[p] = \top \\ 2. & I & \not\models & q & \text{since } I[q] = \bot \\ 3. & I & \models & \neg q & \text{by 2 and } \neg \\ 4. & I & \color{red}{\models ?} & p \land q & \end{array}$$

Simple Example

$$F: \ (p \land q) \to (p \lor \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1.	I	$\models$	p	since $I[p] = \top$
2.	I	$\not\models$	q	since $I[q] = \bot$
3.	I	$\models$	$\neg q$	by 2 and $\neg$
4.	I	$\not\models$	$p \land q$	by 2 and $\land$

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1.	I	$\models$	p	since $I[p] = \top$
2.	I	$\not\models$	q	since $I[q] = \bot$
3.	I	$\models$	$\neg q$	by 2 and $\neg$
4.	I	$\not\models$	$p \wedge q$	by 2 and $\wedge$
5.	I	$\models ?$	$p \vee \neg q$	

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

$$\begin{array}{llll} 1. & I & \models & p & \text{since } I[p] = \top \\ 2. & I & \not\models & q & \text{since } I[q] = \bot \\ 3. & I & \models & \neg q & \text{by 2 and } \neg \\ 4. & I & \not\models & p \wedge q & \text{by 2 and } \wedge \\ 5. & I & \models & p \vee \neg q & \text{by 1 and } \vee \end{array}$$

Simple Example

$$F : \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I : \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1.	I	$\models$	p	since $I[p] = \top$
2.	I	$\not\models$	q	since $I[q] = \bot$
3.	I	$\models$	$\neg q$	by 2 and $\neg$
4.	I	$\not\models$	$p \wedge q$	by 2 and $\wedge$
5.	I	$\models$	$p \vee \neg q$	by 1 and $\vee$
6.	I	$\models ?$	F	

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1.	I	$\models$	p	since $I[p] = \top$
2.	I	$\not\models$	q	since $I[q] = \bot$
3.	I	$\models$	$\neg q$	by 2 and $\neg$
4.	I	$\not\models$	$p \wedge q$	by 2 and $\wedge$
5.	I	$\models$	$p \vee \neg q$	by 1 and $\vee$
6.	I	$\models$	F	by 4 and $\rightarrow$

Simple Example

$$F: \ (p \wedge q) \rightarrow (p \vee \neg q)$$

$$I: \ \{p \mapsto \top, \ q \mapsto \bot\}$$

1.	I	$\models$	p	since $I[p] = \top$
2.	I	$\not\models$	q	since $I[q] = \bot$
3.	I	$\models$	$\neg q$	by 2 and $\neg$
4.	I	$\not\models$	$p \wedge q$	by 2 and $\wedge$
5.	I	$\models$	$p \vee \neg q$	by 1 and $\vee$
6.	I	$\models$	F	by 4 and $\rightarrow$

Thus, F is true under I.

Another Example

- What does the formula

$$F:\ (p \leftrightarrow \neg q) \rightarrow (q \rightarrow \neg r)$$

evaluate to under this interpretation?

$$I = \{p \mapsto \bot,\ q \mapsto \top, r \mapsto \top\}$$

Another Example

- What does the formula

$$F: \ (p \leftrightarrow \neg q) \rightarrow (q \rightarrow \neg r)$$

  evaluate to under this interpretation?

$$I = \{p \mapsto \bot, \ q \mapsto \top, r \mapsto \top\}$$

- $I \not\models F$

Satisfiability and Validity

- F is satisfiable iff there exists an interpretation I such that $I \models F$.

Satisfiability and Validity

- F is satisfiable iff there exists an interpretation I such that $I \models F$.
- F valid iff for all interpretations I, $I \models F$.

Satisfiability and Validity

- F is satisfiable iff there exists an interpretation I such that $I \models F$.
- F valid iff for all interpretations I, $I \models F$.
- F is contingent if it is satisfiable but not valid.

Satisfiability and Validity

- F is satisfiable iff there exists an interpretation I such that $I \models F$.
- F valid iff for all interpretations I, $I \models F$.
- F is contingent if it is satisfiable but not valid.
- Duality between satisfiability and validity:

$$\boxed{F \text{ is valid iff } \neg F \text{ is unsatisfiable}}$$

Satisfiability and Validity

- F is satisfiable iff there exists an interpretation I such that $I \models F$.
- F valid iff for all interpretations I, $I \models F$.
- F is contingent if it is satisfiable but not valid.
- Duality between satisfiability and validity:

$$\boxed{F \text{ is valid iff } \neg F \text{ is unsatisfiable}}$$

- Thus, if we have a procedure for checking satisfiability, this also allows us to decide validity

Deciding Satisfiability and Validity

- Before we talk about practical algorithms for deciding satisfiability, let's review some simple techniques

Deciding Satisfiability and Validity

- Before we talk about practical algorithms for deciding satisfiability, let's review some simple techniques
- Two very simple techniques:

Deciding Satisfiability and Validity

- Before we talk about practical algorithms for deciding satisfiability, let's review some simple techniques
- Two very simple techniques:
  - Truth table method: essentially a search-based technique

Deciding Satisfiability and Validity

- Before we talk about practical algorithms for deciding satisfiability, let's review some simple techniques
- Two very simple techniques:
  - Truth table method: essentially a search-based technique
  - Semantic argument method: deductive way of deciding satisfiability

Deciding Satisfiability and Validity

- Before we talk about practical algorithms for deciding satisfiability, let's review some simple techniques
- Two very simple techniques:
  - Truth table method: essentially a search-based technique
  - Semantic argument method: deductive way of deciding satisfiability
- Completely different, but complementary techniques

Deciding Satisfiability and Validity

- Before we talk about practical algorithms for deciding satisfiability, let's review some simple techniques
- Two very simple techniques:
  - Truth table method: essentially a search-based technique
  - Semantic argument method: deductive way of deciding satisfiability
- Completely different, but complementary techniques
- In fact, as we'll see later, modern SAT solvers combine both search-based and deductive techniques!

Method 1: Truth Tables

Example $F : (p \ \land \ q) \ \rightarrow \ (p \ \lor \ \neg q)$

Method 1: Truth Tables

Example $F : (p \land q) \rightarrow (p \lor \neg q)$

p q	$p \land q$	$\neg q$	$p \lor \neg q$	F
0 0	0	1	1	1
0 1	0	0	0	1
1 0	0	1	1	1
1 1	1	0	1	1

Method 1: Truth Tables

<u>Example</u> $F : (p \ \land \ q) \ \rightarrow \ (p \ \lor \ \neg q)$

p q	$p \land q$	$\neg q$	$p \lor \neg q$	F
0 0	0	1	1	1
0 1	0	0	0	1
1 0	0	1	1	1
1 1	1	0	1	1

Thus F is valid.

Another Example

$$F : (p \lor q) \rightarrow (p \land q)$$

p q	$p \lor q$	$p \land q$	F	
0 0	0	0	1	$\leftarrow$ satisfying I
0 1	1	0	0	$\leftarrow$ falsifying I
1 0	1	0	0	
1 1	1	1	1	

Another Example

$$F : (p \ \vee \ q) \ \rightarrow \ (p \ \wedge \ q)$$

p q	$p \vee q$	$p \wedge q$	F	
0 0	0	0	1	$\leftarrow$ satisfying I
0 1	1	0	0	$\leftarrow$ falsifying I
1 0	1	0	0	
1 1	1	1	1	

Thus F is satisfiable, but invalid.

Summary: Truth Tables

- List all interpretations $\Rightarrow$ If all interpretations satisfy formula, then valid. If no interpretation satisfies it, unsatisfiable.

Summary: Truth Tables

- List all interpretations $\Rightarrow$ If all interpretations satisfy formula, then valid. If no interpretation satisfies it, unsatisfiable.
- Completely brute-force, impractical: requires explicitly listing all 2^n interpretations in the worst-case!

Summary: Truth Tables

- List all interpretations $\Rightarrow$ If all interpretations satisfy formula, then valid. If no interpretation satisfies it, unsatisfiable.
- Completely brute-force, impractical: requires explicitly listing all 2^n interpretations in the worst-case!
- Method does not work for any logic where domain is not finite (e.g., first-order logic)

Method 2: Semantic Argument

- Semantic argument method is essentially a proof by contradiction, and is also applicable for theories with non-finite domain.

Method 2: Semantic Argument

- Semantic argument method is essentially a proof by contradiction, and is also applicable for theories with non-finite domain.

- Main idea: Assume F is not valid $\Rightarrow$ there exists some falsifying interpretation I such that $I \not\models F$

Method 2: Semantic Argument

- Semantic argument method is essentially a proof by contradiction, and is also applicable for theories with non-finite domain.
- Main idea: Assume F is not valid $\Rightarrow$ there exists some falsifying interpretation I such that $I \not\models F$
- Apply proof rules.

Method 2: Semantic Argument

- Semantic argument method is essentially a proof by contradiction, and is also applicable for theories with non-finite domain.
- Main idea: Assume F is not valid $\Rightarrow$ there exists some falsifying interpretation I such that $I \not\models F$
- Apply proof rules.
- If we derive a contradiction in every branch of the proof, then F is valid.

Method 2: Semantic Argument

- Semantic argument method is essentially a proof by contradiction, and is also applicable for theories with non-finite domain.
- Main idea: Assume F is not valid $\Rightarrow$ there exists some falsifying interpretation I such that $I \not\models F$
- Apply proof rules.
- If we derive a contradiction in every branch of the proof, then F is valid.
- If there exists some branch where we cannot derive a contradiction (after exhaustively applying all proof rules), then F is not valid.

The Proof Rules (I)

- According to semantics of negation, from $I \models \neg F$, we can deduce $I \not\models F$:

$$\frac{I \models \neg F}{I \not\models F}$$

The Proof Rules (I)

- According to semantics of negation, from $I \models \neg F$, we can deduce $I \not\models F$:

$$\frac{I \models \neg F}{I \not\models F}$$

- Similarly, from $I \not\models \neg F$, we can deduce:

The Proof Rules (I)

- According to semantics of negation, from $I \models \neg F$, we can deduce $I \not\models F$:

$$\frac{I \models \neg F}{I \not\models F}$$

- Similarly, from $I \not\models \neg F$, we can deduce:

$$\frac{I \not\models \neg F}{I \models F}$$

The Proof Rules (II)

- According to semantics of conjunction, from $I \models F \wedge G$, we can deduce:

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \textsf{and}$$

The Proof Rules (II)

- According to semantics of conjunction, from $I \models F \wedge G$, we can deduce:

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array} \leftarrow \text{and}}$$

- Similarly, from $I \not\models F \wedge G$, we can deduce:

The Proof Rules (II)

- According to semantics of conjunction, from $I \models F \wedge G$, we can deduce:

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \text{and}$$

- Similarly, from $I \not\models F \wedge G$, we can deduce:

$$\frac{I \not\models F \wedge G}{I \not\models F \quad | \quad I \not\models G}$$

The Proof Rules (II)

- According to semantics of conjunction, from $I \models F \land G$, we can deduce:

$$\frac{I \models F \land G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \text{and}$$

- Similarly, from $I \not\models F \land G$, we can deduce:

$$\frac{I \not\models F \land G}{I \not\models F \quad | \quad I \not\models G}$$

- The second deduction results in a branch in the proof, so each case has to be examined separately!

The Proof Rules (III)

- According to semantics of disjunction, from $I \models F \vee G$, we can deduce:

$$\frac{I \models F \vee G}{I \models F \quad | \quad I \models G}$$

The Proof Rules (III)

- According to semantics of disjunction, from $I \models F \vee G$, we can deduce:

$$\frac{I \models F \vee G}{I \models F \quad | \quad I \models G}$$

- Similarly, from $I \not\models F \vee G$, we can deduce:

The Proof Rules (III)

- According to semantics of disjunction, from $I \models F \vee G$, we can deduce:

$$\frac{I \models F \vee G}{I \models F \quad | \quad I \models G}$$

- Similarly, from $I \not\models F \vee G$, we can deduce:

$$\frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

The Proof Rules (IV)

- According to semantics of implication:

$$\frac{I \models F \to G}{}$$

The Proof Rules (IV)

- According to semantics of implication:

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G}$$

The Proof Rules (IV)

- According to semantics of implication:

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G}$$

- And:

$$\frac{I \not\models F \rightarrow G}{}$$

The Proof Rules (IV)

- According to semantics of implication:

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G}$$

- And:

$$\frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

The Proof Rules (V)

- According to semantics of iff:

$$\frac{I \models F \leftrightarrow G}{\quad}$$

The Proof Rules (V)

- According to semantics of iff:

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G}$$

The Proof Rules (V)

- According to semantics of iff:

$$\frac{I \ \models \ F \leftrightarrow G}{I \ \models \ F \wedge G \quad | \quad I \ \models \ \neg F \wedge \neg G}$$

The Proof Rules (V)

- According to semantics of iff:

$$\frac{I \;\models\; F \leftrightarrow G}{I \;\models\; F \wedge G \quad | \quad I \;\models\; \neg F \wedge \neg G}$$

- And:

$$\frac{I \;\not\models\; F \leftrightarrow G}{}$$

The Proof Rules (V)

- According to semantics of iff:

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \quad | \quad I \models \neg F \wedge \neg G}$$

- And:

$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \quad | \quad I \models \neg F \wedge G}$$

The Proof Rules (Contradiction)

- Finally, we derive a contradiction, when I both entails F and does not entail F:

$$\frac{\begin{array}{ccc} I & \models & F \\ I & \not\models & F \end{array}}{\begin{array}{ccc} I & \models & \perp \end{array}}$$

An Example

Prove $F: \ (p \ \wedge \ q) \ \rightarrow \ (p \ \vee \ \neg q)$ is valid.

An Example

Prove $F: (p \land q) \rightarrow (p \lor \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

An Example

Prove $F : (p \wedge q) \rightarrow (p \vee \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1. $I \not\models (p \wedge q) \rightarrow (p \vee \neg q)$ assumption

An Example

Prove $F: (p \wedge q) \rightarrow (p \vee \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$(p \wedge q) \rightarrow (p \vee \neg q)$	assumption
2.	I	$\models$	$p \wedge q$	1 and $\rightarrow$
3.	I	$\not\models$	$p \vee \neg q$	1 and $\rightarrow$

An Example

Prove $F:\ (p \land q) \rightarrow (p \lor \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$(p \land q) \rightarrow (p \lor \neg q)$	assumption
2.	I	$\models$	$p \land q$	1 and $\rightarrow$
3.	I	$\not\models$	$p \lor \neg q$	1 and $\rightarrow$
4.	I	$\models$	p	2 and $\land$
5.	I	$\models$	q	2 and $\land$

An Example

Prove $F: (p \wedge q) \rightarrow (p \vee \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$(p \wedge q) \rightarrow (p \vee \neg q)$	assumption
2.	I	$\models$	$p \wedge q$	1 and $\rightarrow$
3.	I	$\not\models$	$p \vee \neg q$	1 and $\rightarrow$
4.	I	$\models$	p	2 and $\wedge$
5.	I	$\models$	q	2 and $\wedge$
6.	I	$\not\models$	p	3 and $\vee$
7.	I	$\not\models$	$\neg q$	3 and $\vee$

An Example

Prove $F : (p \wedge q) \rightarrow (p \vee \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$(p \wedge q) \rightarrow (p \vee \neg q)$	assumption
2.	I	$\models$	$p \wedge q$	1 and $\rightarrow$
3.	I	$\not\models$	$p \vee \neg q$	1 and $\rightarrow$
4.	I	$\models$	p	2 and $\wedge$
5.	I	$\models$	q	2 and $\wedge$
6.	I	$\not\models$	p	3 and $\vee$
7.	I	$\not\models$	$\neg q$	3 and $\vee$
8.	I	$\models$	$\perp$	4 and 6 are contradictory

An Example

Prove $F:\ (p \land q) \rightarrow (p \lor \neg q)$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$(p \land q) \rightarrow (p \lor \neg q)$	assumption
2.	I	$\models$	$p \land q$	1 and $\rightarrow$
3.	I	$\not\models$	$p \lor \neg q$	1 and $\rightarrow$
4.	I	$\models$	p	2 and $\land$
5.	I	$\models$	q	2 and $\land$
6.	I	$\not\models$	p	3 and $\lor$
7.	I	$\not\models$	$\neg q$	3 and $\lor$
8.	I	$\models$	$\perp$	4 and 6 are contradictory

$\Rightarrow$ Thus F is valid.

Another Example

- Prove that the following formula is valid using semantic argument method:

$$F: \quad ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Equivalence

- Formulas F_1 and F_2 are equivalent (written $F_1 \Leftrightarrow F_2$) iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

$$\boxed{F_1 \Leftrightarrow F_2 \text{ iff } F_1 \leftrightarrow F_2 \text{ is valid}}$$

Equivalence

- Formulas F_1 and F_2 are equivalent (written $F_1 \Leftrightarrow F_2$) iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

$$\boxed{F_1 \Leftrightarrow F_2 \text{ iff } F_1 \leftrightarrow F_2 \text{ is valid}}$$

- Thus, if we have a procedure for checking satisfiability, we can also check equivalence.

Implication

- Formula F_1 implies F_2 (written $F_1 \Rightarrow F_2$) iff for all interpretations I, $I \models F_1 \to F_2$

$$\boxed{F_1 \Rightarrow F_2 \text{ iff } F_1 \to F_2 \text{ is valid}}$$

Implication

- Formula F_1 implies F_2 (written $F_1 \Rightarrow F_2$) iff for all interpretations I, $I \models F_1 \rightarrow F_2$

$$\boxed{F_1 \Rightarrow F_2 \text{ iff } F_1 \rightarrow F_2 \text{ is valid}}$$

- Thus, if we have a procedure for checking satisfiability, we can also check implication

Implication

- Formula F_1 implies F_2 (written $F_1 \Rightarrow F_2$) iff for all interpretations I, $I \models F_1 \rightarrow F_2$

$$\boxed{F_1 \Rightarrow F_2 \text{ iff } F_1 \rightarrow F_2 \text{ is valid}}$$

- Thus, if we have a procedure for checking satisfiability, we can also check implication
- Caveat: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulas (they are not part of PL syntax); they are semantic judgments!

Example

- Prove that $F_1 \wedge (\neg F_1 \vee F_2)$ implies F_2 using semantic argument method.

Summary

- Today:

  Review of basic concepts underlying propositional logic

Summary

- Today:

  Review of basic concepts underlying propositional logic

- Next lecture:

  Normal forms and algorithms for deciding satisfiability

Summary

- Today:

  Review of basic concepts underlying propositional logic

- Next lecture:

  Normal forms and algorithms for deciding satisfiability

- Reading:

  Bradley & Manna texbook until Section 1.6