ECE750T-28:
Computer-aided Reasoning for Software Engineering

Lecture 11: Theory of Equality with Uninterpreted Functions

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(Original notes from Isil Dillig)

Review

- Previous lecture: talked about signature and axioms of $T_=$

$$\Sigma_= : \; \{=, \; a, \; b, \; c, \; \ldots, \; f, \; g, \; h, \; \ldots, \; p, \; q, \; r, \; \ldots\}$$

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- Axioms:
  1. $\forall x.\ x = x$ (reflexivity)
  2. $\forall x, y.\ x = y \ \rightarrow\ y = x$ (symmetry)
  3. $\forall x, y, z.\ x = y \ \wedge\ y = z \ \rightarrow\ x = z$ (transitivity)
  4. $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ \bigwedge_i x_i = y_i \ \rightarrow\ f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ (congruence)
  5. for each positive integer n and n-ary predicate symbol p,
     $$\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ \bigwedge_i x_i = y_i \ \rightarrow\ (p(x_1, \ldots, x_n) \ \leftrightarrow\ p(y_1, \ldots, y_n))$$ (equivalence)

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- However, these "restrictions" are not real restrictions
- For formulas with disjunctions, can convert to DNF and check each clause separately (will consider efficient methods later)
- Furthermore, any formula containing predicates can be converted to equisatisfiable formula containing only functions!

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  4. $\forall x_1, \ldots, x_n, y_1, \ldots, y_n. \ \bigwedge_i x_i = y_i \ \rightarrow \ f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ (congruence)

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- What about $f(a) = a \wedge f(f(a)) \neq a$?

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- Reasoning: Substitute a for $f(f(f(a)))$ in second equality, this yields: $f(f(a)) = a$
- Since $f(f(a)) = a$, by congruence $f(f(f(a))) = f(a)$
- By first equality, we have $f(a) = a \Rightarrow$ contradiction!

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  3. transitive: $\forall s_1, s_2, s_3 \in S.\ s_1 R s_2 \ \wedge\ s_2 R s_3 \ \rightarrow\ s_1 R s_3$.

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- The relation $\equiv_2$ is equivalence relation over $\mathbb{Z}$
- A relation R is congruence relation over set S if it is an equivalence relation and for every n'ary function f:

$$\forall \vec{s}, \vec{t}.\ \bigwedge_{i=1}^{n} s_i R t_i \ \rightarrow\ f(\vec{s})\ R\ f(\vec{t})\ .$$

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Relation Refinements

- A binary relation R_1 is a refinement of another binary relation R_2, written $R_1 \prec R_2$, if

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- Example 2: Consider set $\mathbb{Z}$ and the relations:
$R_1 : \{xR_1y \;:\; x \text{ mod } 2 = y \text{ mod } 2\}$ $\quad R_2 : \{xR_2y \;:\; x \text{ mod } 4 = y \text{ mod } 4\}$

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- Thus, R^E is the smallest equivalence relation that includes R.

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- Thus, R^E needs to include all tuples in R and must obey reflexivity, symmetry, and transitivity.

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- What elements in R^E due to symmetry? $\langle b, a\rangle, \langle c, b\rangle$
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- What is R^E?

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  $R^E = \{\langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle, \langle b, a\rangle, \langle c, b\rangle, \langle a, c\rangle, \langle c, a\rangle\}$

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- Consider relation $R' = R^E \cup \{\langle c, d\rangle, \langle d, c\rangle, \langle b, d\rangle, \langle d, b\rangle, \langle a, d\rangle, \langle d, a\rangle\}$

Equivalence Closure Example, cont

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- $R^E = \{\langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle, \langle b, a\rangle, \langle c, b\rangle, \langle a, c\rangle, \langle c, a\rangle\}$
- Consider relation $R' = R^E \cup \{\langle c, d\rangle, \langle d, c\rangle, \langle b, d\rangle, \langle d, b\rangle, \langle a, d\rangle, \langle d, a\rangle\}$
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- Formally, the congruence closure R^C of a binary relation R over S is the congruence relation such that:

  1. R refines R^C, i.e. $R \prec R^C$;
  2. for all other congruence relations R' s.t. $R \prec R'$, either $R' = R^C$ or $R^C \prec R'$

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Congruence Closure Algorithm: Basic Idea

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- This represents the congruence closure over S_F.

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- Congruence closure: $\{\{a, f(a, b), f(f(a, b), b)\}, \{b\}\}$
- Is F satisfiable? No
- Since a and $f(f(a, b), b)$ are in same congruence class, we have $a \sim f(f(a, b), b)$
- This contradicts $f(f(a, b), b) \neq a$!

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- Next: Talk about Union-Find algorithm for computing congruence closures

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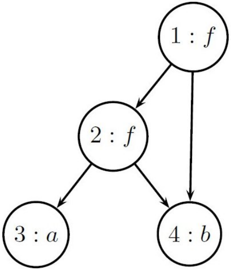

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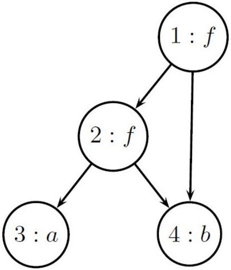

- Each node corresponds to a subterm and has unique id

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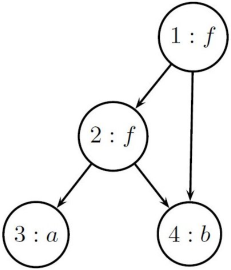

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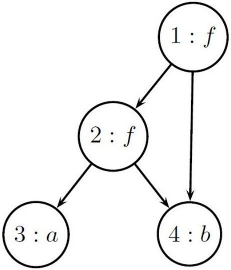

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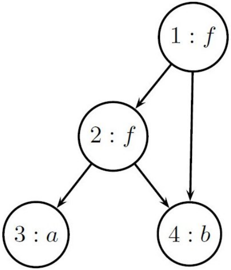

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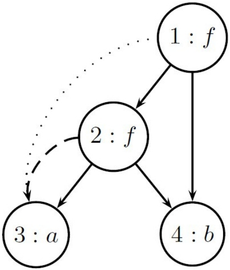

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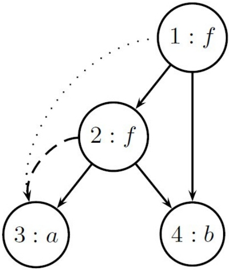

- Thus, for a given subterm, we need to be able to find the representative of its class
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- In this example, $a, f(a, b), f(f(a, b), b)$ are in same congruence class; a is the representative

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- Thus, keep pointer from representative of congruence class to parents of all subterms in the congruence class

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Processing Equalities, cont

To process equality $t_1 = t_2$:

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Observe: Processing one equality creates new equalities, which in turn might generate other new equalities!

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- That's why representative stores all parents for cong. class

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Algorithm to decide satisfiability of $T_=$ formula

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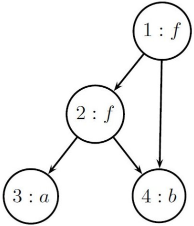

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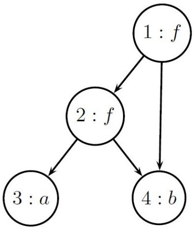

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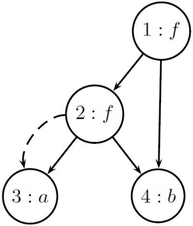

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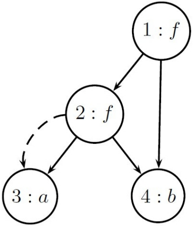

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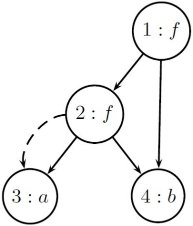

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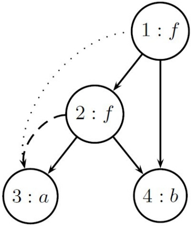

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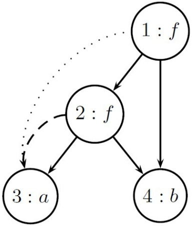

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- Formula unsatisfiable because $f(f(a, b), b)$ and a have same representative!

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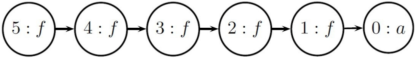

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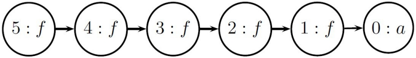

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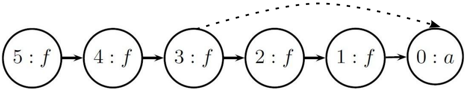

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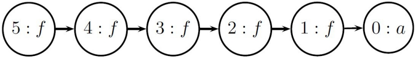

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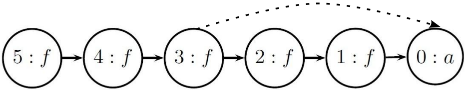

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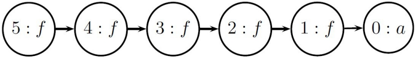

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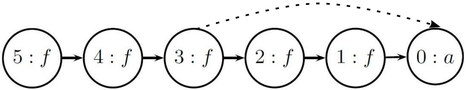

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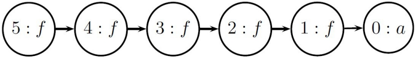

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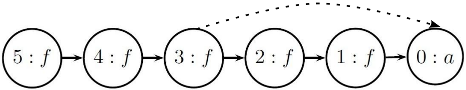

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Example II, cont

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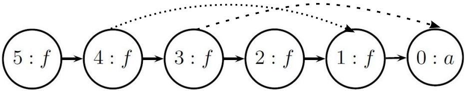

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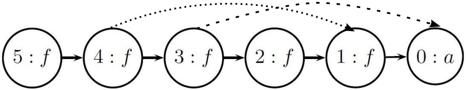

- Are $f^4(a)$'s and $f(a)$'s parents congruent?

Example II, cont

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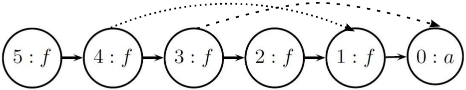

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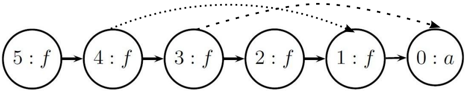

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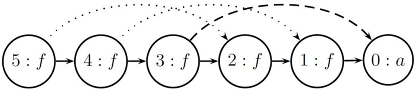
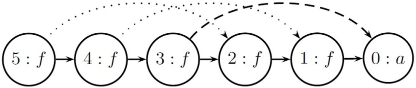


Example II, cont

- Formula: $F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$



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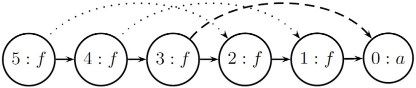

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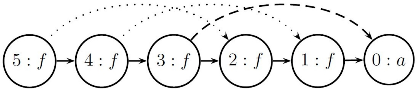

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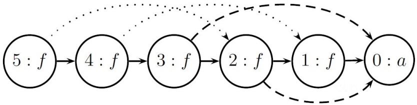

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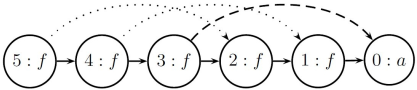

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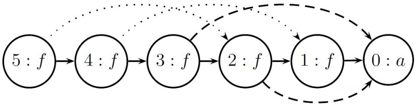

- Now, parents $f^3(a)$ and $f(a)$ congruent; process equality $f^3(a) = f(a)$

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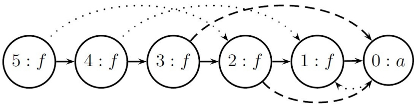

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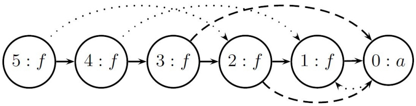

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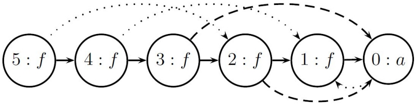

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- Reminder: Homework due next lecture!!