Asset Pricing - Empirical Application 2 -
Factorial analysis of the Yield Curve in the extended version of Nelson and Siegel

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Abstract

In this project, we estimate the yield curve (YC) for the US economy using the Nelson-Siegel framework, specifically focusing on a yield-only model. This approach centers solely on the dynamics of the yield curve factors without incorporating additional macroeconomic inputs, effectively capturing the fundamental shape and shifts of the curve. Robustness checks confirm that the yield-only model provides a reliable representation of yield curve dynamics. Although exploring extensions such as incorporating macroeconomic factors, as suggested in methodologies like Diebold et al. (2006), was considered, the analysis in this project is confined to the yield-only approach to maintain precision and focus. This streamlined model highlights the Nelson-Siegel framework's strength in capturing yield curve behavior independently of broader macroeconomic influences..

Introduction

The Nelson-Siegel (NS) model, first introduced in 1987 by Charles Nelson and Andrew Siegel, established a foundational framework for modeling the yield curve, which represents the relationship between bond yields and their maturities. The model's three factors—Level, Slope, and Curvature—are designed to capture the primary shapes of the yield curve, effectively approximating trends, steepness, and curvatures across different maturities. Since its inception, the NS model has become a cornerstone in finance, widely used for understanding and forecasting interest rates.

This project builds on the NS model and its evolution, particularly as expanded by Francis X. Diebold and Glenn Rudebusch in 2006 in their work, *The

Macroeconomy and the Yield Curve: A Dynamic Latent Factor Approach*. They developed a dynamic version of the NS model that incorporates macroeconomic factors, linking changes in the yield curve to broader economic indicators. Their model, which applies a state-space framework and Kalman filter, captures time-varying dynamics in Level, Slope, and Curvature and integrates economic conditions, highlighting how macroeconomic forces shape interest rates over time.

The structure of this project is organized into two main sections. The first section, presents the **yields-only model**, where we analyze the core factors—Level, Slope, and Curvature—without external economic influences, focusing on the yield curve itself. In the second and section, we introduce **macro factors into the yield model**, developing a yield-macro model that incorporates broader economic indicators.

The data used in this study include yield curve data and macroeconomic indicators for the United States, covering the period from **2000 to 2020**. The yield data, obtained from the **US Department of the Treasury,** consist of daily Treasury par yields with maturities of 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, and 20 years. To align with the frequency of macroeconomic data, we selected the yield on the first day of each month. The macroeconomic data, sourced on a monthly basis, include the same indicators used by Diebold et al. (2006): manufacturing capacity utilization, the federal funds rate, and the annual Consumer Price Index (CPI) from the **FRED**. These indicators were chosen to capture key economic conditions that may influence the shape and dynamics of the yield curve. This approach allows us to explore both a yield-only model and a yield-macro model within the Nelson-Siegel framework, facilitating an analysis of the potential interactions between economic conditions and yield curve behavior.

1 Yield only model

1.1 Modele calibration

The foundation for the dynamic latent factor model used in this study is based on the Nelson-Siegel (1987) functional form, a compact and effective three-component exponential model for yield curves. Diebold and al.(2006) reformulated the original Nelson-Siegel model to express the yield curve as:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

where $y_t(\tau)$ represents the continuously compounded nominal yield at maturity τ. In the Nelson-Siegel model, the yield curve corresponds to a discount curve that starts at zero maturity and approaches zero as maturity becomes infinite. The parameters β_{1t}, β_{2t}, and β_{3t} are time-varying and capture different components of the yield curve, while λ controls the rate of exponential decay. Smaller values of λ lead to a slower decay rate that better fits longer-term maturities, while larger values yield a faster decay that is more suitable for short-term maturities. This decay parameter λ also determines where the loading on β_{3t} reaches its maximum.

The three time-varying parameters can be interpreted as dynamic latent factors. The loading on β_{1t} is constant (equal to 1), meaning it affects all maturities uniformly and is typically interpreted as the "level" factor (L_t), representing long-term yield trends. The loading on β_{2t} is

$$\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right),$$

a function that starts at 1 and decays gradually to zero. An increase in β_{2t} primarily impacts short-term yields, influencing the slope of the yield curve, which is defined as the difference between short- and long-term yields. This factor is commonly interpreted as the "slope" factor (S_t).

The loading on β_{3t} is

$$\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}-e^{-\lambda\tau}\right),$$

which starts at zero, increases to a maximum, and then decays back to zero. An increase in β_{3t} has minimal impact on yields at very short and long maturities but significantly affects medium-term yields. This factor is interpreted as the "curvature" factor (C_t), influencing the shape of the yield curve, particularly in terms of upward or downward sloping and humped curves.

Together, the three parameters—β_{1t}, β_{2t}, and β_{3t}, or equivalently L_t, S_t, and C_t, along with the decay parameter λ—can capture a wide variety of yield curve shapes over time, including upward sloping, downward sloping, and inverted curves. This factor representation aggregates information from multiple yields, ensuring that the representation is robust to the choice of yields.

To estimate the yield curve parameters β_1, β_2, β_3, and λ for each date, we fit the Nelson-Siegel model to observed yields across various maturities (e.g., 3 months, 6 months, 1 year, etc.). The goal is to find parameter values that closely approximate the observed yield curve for each date.

Optimization Process

For each date, the model parameters are estimated by minimizing the difference between the observed yields and the model-predicted yields. The parameters β_1, β_2, β_3, and λ are adjusted to minimize the sum of squared errors between the observed yields (actual market rates for each maturity) and the model's predictions. This process ensures that the fitted parameters align the model as closely as possible with the real yield curve data.

Regularization for Stability

To ensure that the parameters evolve smoothly over time, a regularization term, or "smoothness penalty," is introduced. This penalty discourages large changes in parameter values from one date to the next, enhancing stability in the estimates and preventing overfitting to short-term fluctuations. A penalty weight α controls the strength of this regularization, balancing accuracy with stability.

Sequential Training

The model is trained sequentially over dates. The initial parameters for the first date are set with reasonable starting values. For each subsequent date, the previously estimated parameters are used as the starting point, incorporating the smoothness penalty to promote gradual changes in the estimates. This approach allows the model to track shifts in the yield curve over time realistically.

1.2 Results

After estimating the Nelson-Siegel parameters β_1, β_2, β_3, and λ for each date, we can visualize their time-series evolution to better understand the dynamic behavior of the yield curve. The following graphs display the estimated β parameters over the study period, allowing us to observe how the level, slope, and curvature components of the yield curve respond to changing economic conditions.

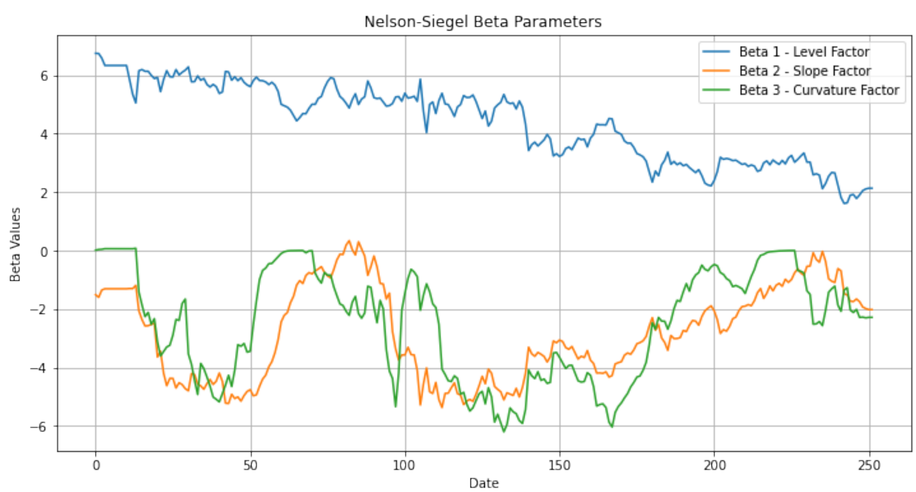

Figure 1: Beta estimations from the Yield Only model

Figure Descriptions

In **Figure 1**, we plot the three estimated Nelson-Siegel factors—level (β_1), slope (β_2), and curvature (β_3)—together to facilitate a comparative assessment of their dynamics over time. The level factor is relatively stable and primarily positive, reflecting the long-term trend of interest rates, while the slope and curvature factors fluctuate around zero, taking on both positive and negative values. This behavior is consistent with the properties of yield curves, where the level factor is typically persistent, while the slope and curvature reflect more short-term dynamics.

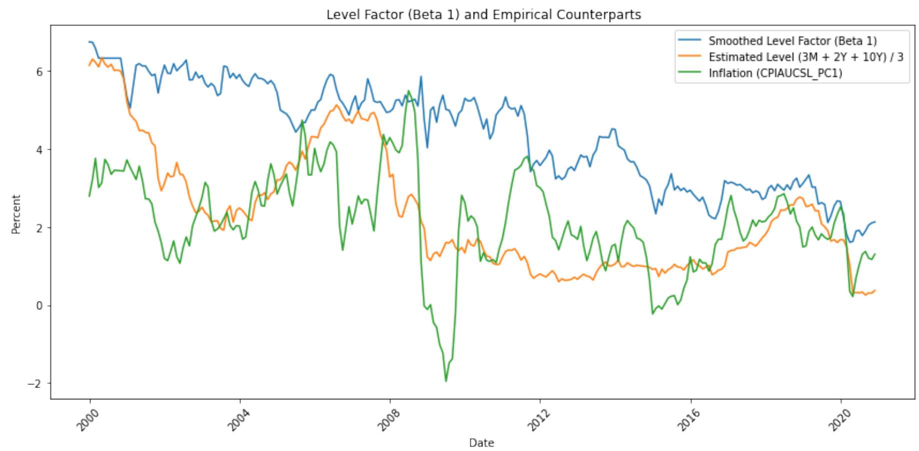

Figure 2: Yields-Only Model Level Factor and Empirical Counterparts

In **Figure 2**, we isolate the level factor (β_1) and compare it to two empirical proxies: an estimated average of short-, medium-, and long-term yields ((3M+2Y+10Y)/3) and inflation (as measured by the 12-month percent change in CPI). The high correlation between β_1 and the yield average supports its interpretation as a long-term level factor. The correlation with inflation suggests that the level of the yield curve may be influenced by inflationary expectations, aligning with theories such as the Fisher equation, which links interest rates to inflation.

In **Figure 3**, we present the slope factor (β_2) alongside two comparison series: a typical empirical slope proxy (3M − 10Y) and capacity utilization, a measure of macroeconomic activity. The close correlation between β_2 and the yield curve slope proxy validates its role as a measure of the yield curve's tilt, reflecting monetary policy effects and economic expectations. The slope factor's correlation with capacity utilization suggests that changes in the yield curve slope are closely linked to the cyclical behavior of the economy.

Finally, in **Figure 4**, we plot the curvature factor (β_3) and compare it to a standard empirical curvature proxy ($2 \times 2\text{Y} - 10\text{Y} - 3\text{M}$). The strong correlation between β_3 and this curvature proxy lends support to its interpretation as a measure of the yield curve's medium-term shape. However, unlike the level and slope factors, we find no significant macroeconomic variable closely linked to curvature, suggesting that the curvature factor may primarily capture market expectations and short-term volatility rather than a direct economic indicator.

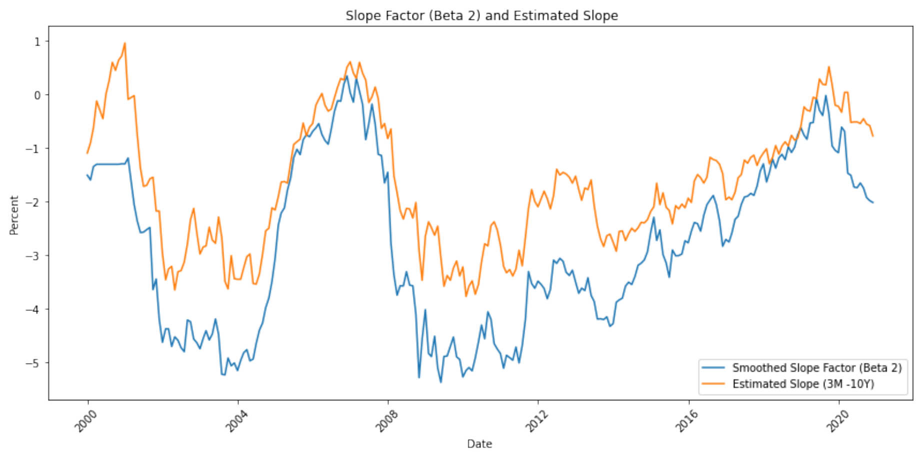

Figure 3: Yields-Only Model Slope Factor and Empirical Counterparts

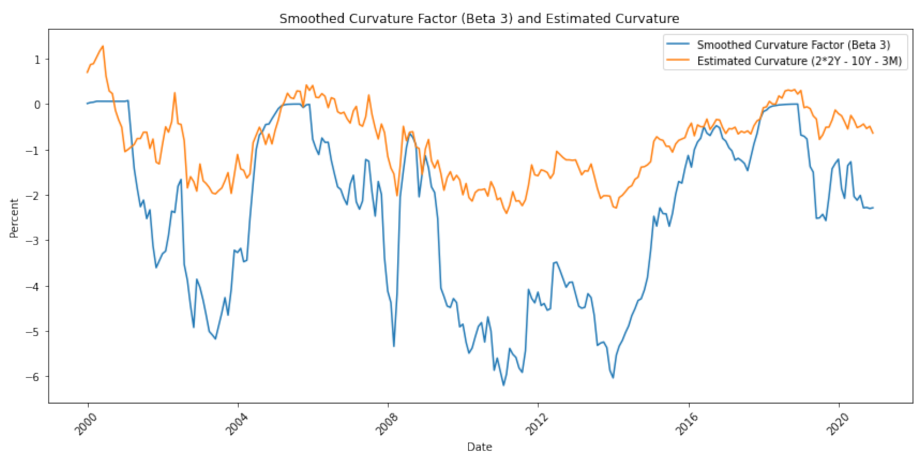

Figure 4: Yields-Only Model Curvature Factor and Empirical Counterpart

1.3 Model Validation

Interpolation Validation Method

In this validation section, we conduct a simple linear interpolation check to evaluate how well the Nelson-Siegel model captures the yield curve on randomly selected dates. By selecting different dates, we aim to assess whether the model effectively fits the observed yield curve under various interest rate environments. For practical purposes, only two dates are displayed in this paper: January 1, 2000, and December 1, 2020. These dates represent distinct economic phases, allowing us to examine the model's performance under different conditions.

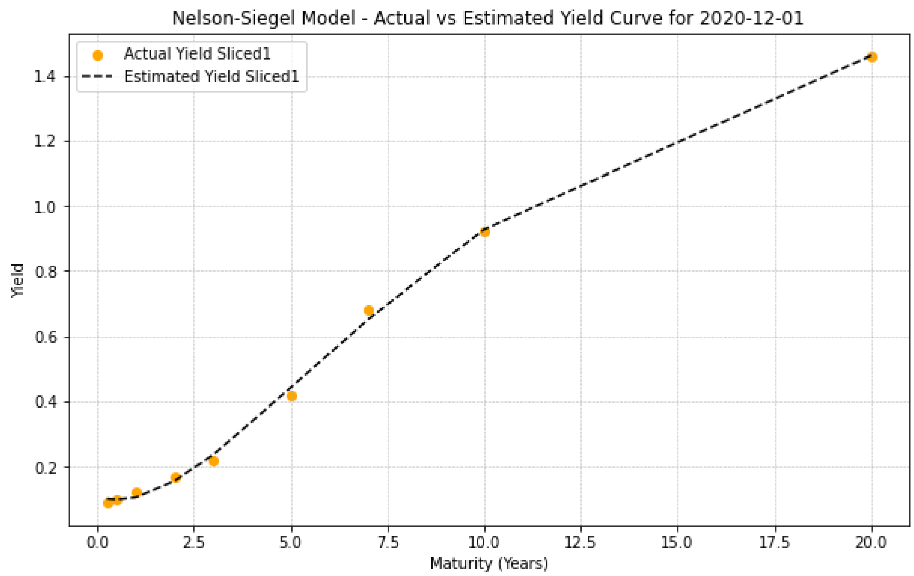

Figure 5: Actual vs Estimated Yield Curve for 2020-12-01

In **Figure 5**, we display the actual and estimated yield curves for December 1, 2020. The Nelson-Siegel model appears to fit the yield curve well for this date, accurately capturing the upward slope and shape of the curve across maturities. The actual and estimated yields are closely aligned, indicating that the model effectively represents the yield curve structure in a low-interest-rate environment, characteristic of this period.

In contrast, **Figure 6** shows the yield curve fit for January 1, 2000. While the model still captures the general shape of the yield curve, noticeable discrepancies appear, particularly at intermediate maturities. Specifically, the model struggles to estimate the curve accurately when the 5-year yield is higher than the 7-year yield, a pattern that is inconsistent with a typical, "normal" yield curve (non-inverted). This misalignment is expected, as linear interpolation is not designed to account for such irregular deviations. This result highlights the limitations of the Nelson-Siegel model in capturing certain specific details of the yield curve, particularly in market configurations where maturities do not follow a monotonic trend.

Rolling Window Validation Results

To further validate the Nelson-Siegel model, we implemented a rolling window approach, which provides a more rigorous assessment of the model's predictive performance over time. In this approach:

1. **Window Selection**: A training window of 120 months (10 years) was used. For each iteration, the model was trained on this 10-year period.

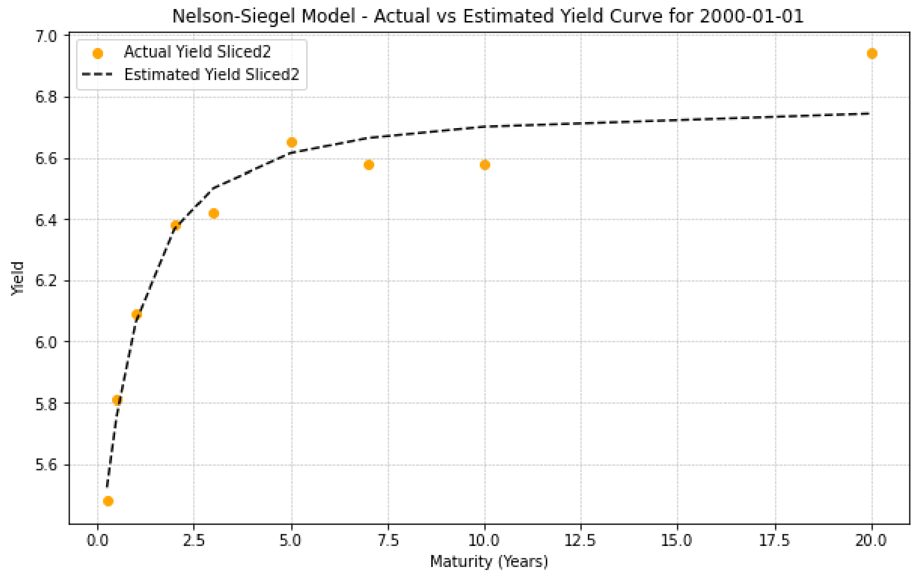

Figure 6: Actual vs Estimated Yield Curve for 2000-01-01

Table 1: Summary Statistics for Rolling Window Errors

Metric	MSE	MAE	RMSE
Count	132	132	132
Mean	0.001153	0.026365	0.031366
Std	0.000928	0.011437	0.013660
Min	0.000127	0.008889	0.011288
25%	0.000421	0.016783	0.020516
50%	0.000789	0.023678	0.028844
75%	0.001738	0.035656	0.041694
Max	0.004943	0.058198	0.070304

2. **Out-of-Sample Prediction**: After training, we evaluated the model by predicting the yield curve for the next month (out-of-sample). This procedure was repeated by rolling the 10-year training window forward by one month each time, generating a sequence of out-of-sample forecasts.

3. **Error Metrics**: For each test month, we calculated three error metrics:

- **MSE (Mean Squared Error)**: Reflects the average squared difference between the predicted and actual yields.
- **MAE (Mean Absolute Error)**: Measures the average absolute difference, providing insight into typical prediction deviations.
- **RMSE (Root Mean Squared Error)**: A commonly used error metric

that penalizes larger errors, giving a measure of typical prediction error magnitude.

This rolling window method enables us to see how well the model generalizes over time and helps to capture potential changes in predictive accuracy across different market conditions.

Interpretation of Results

From the summary table:

- **Mean MSE, MAE, and RMSE**: The mean MSE (0.001153), mean MAE (0.026365), and mean RMSE (0.031366) suggest that, on average, the model produces relatively small errors across the entire validation period.
- **Minimum and Maximum Errors**: The minimum and maximum values for each error metric indicate some variability in model accuracy over time. For instance, the maximum RMSE of 0.070304 suggests that there were periods where the model's predictive accuracy was notably lower.
- **Quartiles**: The 25%, 50% (median), and 75% values provide additional insights into the distribution of errors. For example, 50% of the RMSE values fall below 0.028844, indicating a generally good fit in most cases.

Conclusion

The rolling window results indicate that the Nelson-Siegel yield-only model demonstrates stable and accurate performance across the study period, with relatively low average errors. However, the range between minimum and maximum error values suggests that model accuracy can fluctuate, likely due to shifts in market conditions or changes in the yield curve structure that are not fully captured by the model.

These findings confirm the robustness of the Nelson-Siegel model for yield curve estimation while highlighting areas where accuracy may be sensitive to certain economic conditions. The model effectively fits the yield curve over time, maintaining a reasonably low and consistent error rate. Nonetheless, further refinements, such as accounting for more complex dynamics or external factors, could enhance its performance during periods of heightened economic volatility.

By focusing exclusively on the yield-only framework, this analysis underscores the model's strength in capturing the fundamental dynamics of the yield curve without reliance on macroeconomic inputs. While incorporating such factors may offer additional insights, this project demonstrates that the yield-only approach provides a solid foundation for yield curve modeling and remains highly effective in capturing its behavior over time.