Predictive Inference Tools for Researchers

by

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THE UNIVERSITY OF ARIZONA
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As members of the Master's Committee, we certify that we have read the thesis prepared by Voyze Gabriel Harris III, titled *[Enter Thesis Title]* and recommend that it be accepted as fulfilling the dissertation requirement for the Master's Degree.

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Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to the Graduate College.

I hereby certify that I have read this thesis prepared under my direction and recommend that it be accepted as fulfilling the Master's requirement.

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- (paragraph) Explanation of R package

2 Introduction: Predictive Inference

2.1 Why is predictive inference important?

2.2 Difference between parametric inference and predictive inference

2.2.1 When is predictive inference more useful?

2.2.2 When is parametric inference more useful?

[examples, comparisons]

2.3 The Bayesian Parametric Prediction Format

[Geisser p. 49]

Let
$$f\left(x^{(N)}, x_{(M)}|\theta\right) = f\left(x_{(M)}|x^{(N)}, \theta\right) f\left(x^{(N)}|\theta\right).$$
Here $x^{(N)}$ represents observed events and $x_{(M)}$ are future events. We calculate
$$f\left(x_{(M)}, x^{(N)}\right) = \int f\left(x^{(N)}, x_{(M)}|\theta\right) p(\theta) d\theta$$
where $p(\theta)$ is the prior density and
$$f\left(x_{(M)}|x^{(N)}\right) = \frac{f\left(x_{(M)}, x^{(N)}\right)}{f\left(x^{(N)}\right)} = \int f\left(x_{(M)}|\theta\right) p\left(\theta|x^{(N)}\right) d\theta$$
where
$$p\left(\theta|x^{(N)}\right) \propto f\left(x^{(N)}|\theta\right) p(\theta).$$

2.4 [Maybe] Example of Difference between results from Plug-in estimator and results using Predictive Inference

3 Chapter 1: Predictive Problems with Conjugate Priors

[Problems with closed-form solutions. These problems will be what the R package is designed for. Use problems from Geisser, Casella & Berger (Bayesian chapter), other sources. Regression problem–predictive distributions of models that include and exclude some predictor]

3.1 Prediction of Future Successes: Beta-Binomial (Geisser p. 73)

3.1.1 Derivation

Let X_i be independent binary variables with $\Pr(X_i = 1) = \theta$, and let $T = \sum X_i$. Then T has probability

$$\binom{N}{t}\theta^t(1-\theta)^{N-t}.$$

Assume $\theta \sim \text{Beta}(\alpha, \beta)$, so

$$p(\theta) = \frac{\Gamma(\alpha+\beta)\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}.$$

Then

$$p\left(\theta|X^{(N)}\right) = \frac{\Gamma(N+\alpha+\beta)\theta^{t+\alpha-1}(1-\theta)^{N-t+\beta-1}}{\Gamma(t+\alpha)\Gamma(N-t+\beta)}$$

So for $R = \sum_{i=1}^{M} X_{N+i}$ we have Beta-Binomial predictive distribution

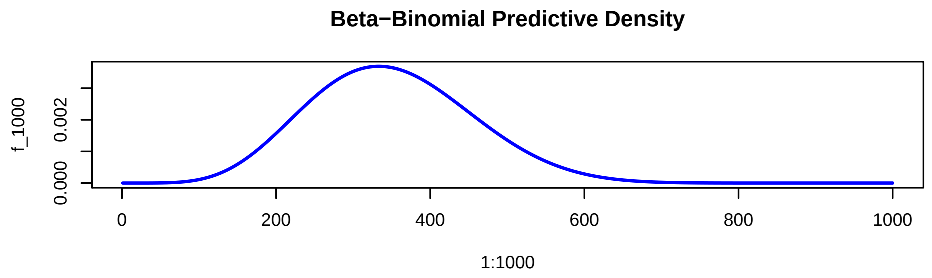
$$\begin{aligned}
\Pr[R = r|t] &= \int \binom{M}{r}\theta^r(1-\theta)^{M-r} p\left(\theta|X^{(N)}\right) d\theta \\
&= \binom{M}{r}\int \theta^r(1-\theta)^{M-r}\frac{\Gamma(N+\alpha+\beta)}{\Gamma(t+\alpha)\Gamma(N-t+\beta)}\theta^{t+\alpha-1}(1-\theta)^{N-t+\beta-1}d\theta \\
&= \frac{M!}{r!(M-r)!}\frac{\Gamma(N+\alpha+\beta)}{\Gamma(t+\alpha)\Gamma(N-t+\beta)}\int \theta^{r+t+\alpha-1}(1-\theta)^{M-r+N-t+\beta-1}d\theta \\
&= \frac{\Gamma(M+1)\Gamma(N+\alpha+\beta)\Gamma(r+t+\alpha)\Gamma(M-r+N-t+\beta)}{\Gamma(r+1)\Gamma(M-r+1)\Gamma(t+\alpha)\Gamma(N-t+\beta)\Gamma(M+N+\alpha+\beta)}
\end{aligned}$$

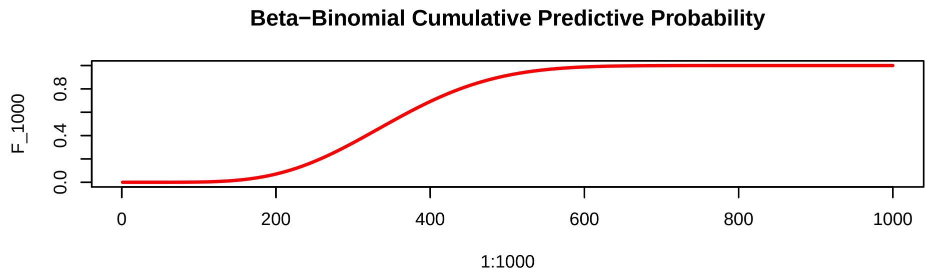
3.1.2 R Implementation

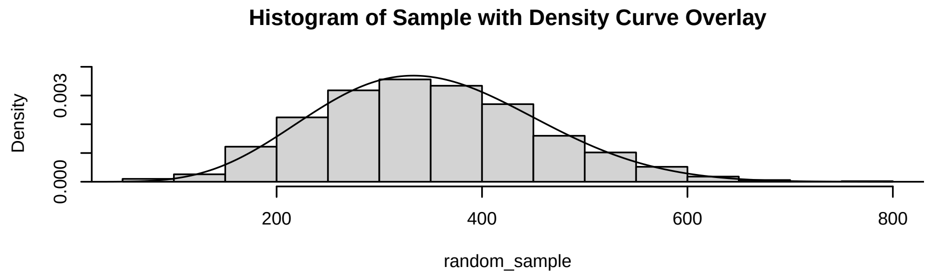
This result has been used to create "standard" R functions dpredBB(), ppredBB(), and rpredBB() for the Beta-Binomial distribtuion for density, cumulative probability, and random sampling, respectively (see appendix). These functions are exercised in the following example.

3.1.3 Example

Suppose $t = 5$ successes have been observed out of $N = 10$ binary events, $\alpha = 2$ and $\beta = 8$. For $M = 1000$ future observations, the figures below show the predictive distribution from dpredBB(), the cumulative distribution from ppredBB(), and a histogram of random draws from rpredBB().







3.2 Survival Time: Exponential-Gamma (Geisser p. 74)

3.2.1 Derivation

Suppose $X^{(N)} = \left(X^{(d)}, X^{(N-d)}\right)$ where $X^{(d)}$ represents copies fully observed from an exponential survival time density

$$f(x|\theta) = \theta e^{-\theta x}$$

and $X^{(N-d)}$ represents copies censored at $x_{d+1}, ..., x_N$, respectively. Hence

$$L(\theta) \propto \theta^d e^{-\theta N\bar{x}}$$

when $N\bar{x} = \sum_1^N x_i$, as shown below.

The usual exponential likelihood is used for the fully observed copies, whereas for the censored copies we need $\Pr(x > \theta) = 1 - \Pr(x \leq \theta) = 1 - F(x|\theta) = 1 - (1 - e^{-\theta x}) = e^{-\theta x}$. Thus the overall likelihood is

$$L(\theta|x) = \prod_{i=1}^{d} \theta e^{-\theta x_i} \prod_{i=d+1}^{N} e^{-\theta x_i} = \theta^d e^{-\theta N\bar{x}}$$

Assuming a Gamma(δ, γ) prior for θ,

$$p(\theta) = \frac{\gamma^\delta \theta^{\delta-1} e^{-\gamma\theta}}{\Gamma(\delta)}$$

we obtain the posterior

$$p\left(\theta|X^{(N)}\right) = \frac{p\left(x^{(N)}|\theta\right) p(\theta)}{\int p\left(X^{(N)}|\theta\right) p(\theta) d\theta}$$

$$= \frac{\theta^d e^{-\theta N\bar{x}} \cdot \frac{\gamma^\delta \theta^{\delta-1} e^{-\gamma\theta}}{\Gamma(\delta)}}{\int \left(\theta^d e^{-\theta N\bar{x}} \cdot \frac{\gamma^\delta \theta^{\delta-1} e^{-\gamma\theta}}{\Gamma(\delta)}\right) d\theta}$$

$$= \frac{\cancel{\frac{\gamma^\delta}{\Gamma(\delta)}} \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right)}{\cancel{\frac{\gamma^\delta}{\Gamma(\delta)}} \int \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right) d\theta}$$

$$= \frac{\frac{(\gamma+N\bar{x})^{d+\delta}}{\Gamma(d+\delta)} \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right)}{\cancel{\frac{(\gamma+N\bar{x})^{d+\delta}}{\Gamma(d+\delta)} \int \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right) d\theta}}$$

$$= \frac{(\gamma + N\bar{x})^{d+\delta} \theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}}{\Gamma(d+\delta)}$$

with the Gamma$(d + \delta, \gamma + N\bar{x})$ density in the next to last step integrating to 1.

Thus the survival time predictive probability is

$$
\begin{aligned}
P\left(X=x|\theta, X^{(N)}\right) &= \int p\left(\theta|X^{(N)}\right) p(x|\theta) d\theta \\
&= \int \frac{(\gamma+N\bar{x})^{d+\delta}\theta^{d+\delta-1}e^{-\theta(\gamma+N\bar{x})}}{\Gamma(d+\delta)} \cdot \theta e^{-\theta x} d\theta \\
&= (d+\delta)(\gamma+N\bar{x})^{d+\delta} \int \frac{\theta^{(d+\delta+1)-1}e^{-\theta(\gamma+N\bar{x}+x)}}{(d+\delta)\Gamma(d+\delta)} d\theta \\
&= \frac{(d+\delta)(\gamma+N\bar{x})^{d+\delta}}{(\gamma+N\bar{x}+x)^{d+\delta+1}} \int \frac{(\gamma+N\bar{x}+x)^{d+\delta+1}\,\theta^{(d+\delta+1)-1}e^{-\theta(\gamma+N\bar{x}+x)}}{\Gamma(d+\delta+1)} d\theta \\
&= \frac{(d+\delta)(\gamma+N\bar{x})^{d+\delta}}{(\gamma+N\bar{x}+x)^{d+\delta+1}}
\end{aligned}
$$

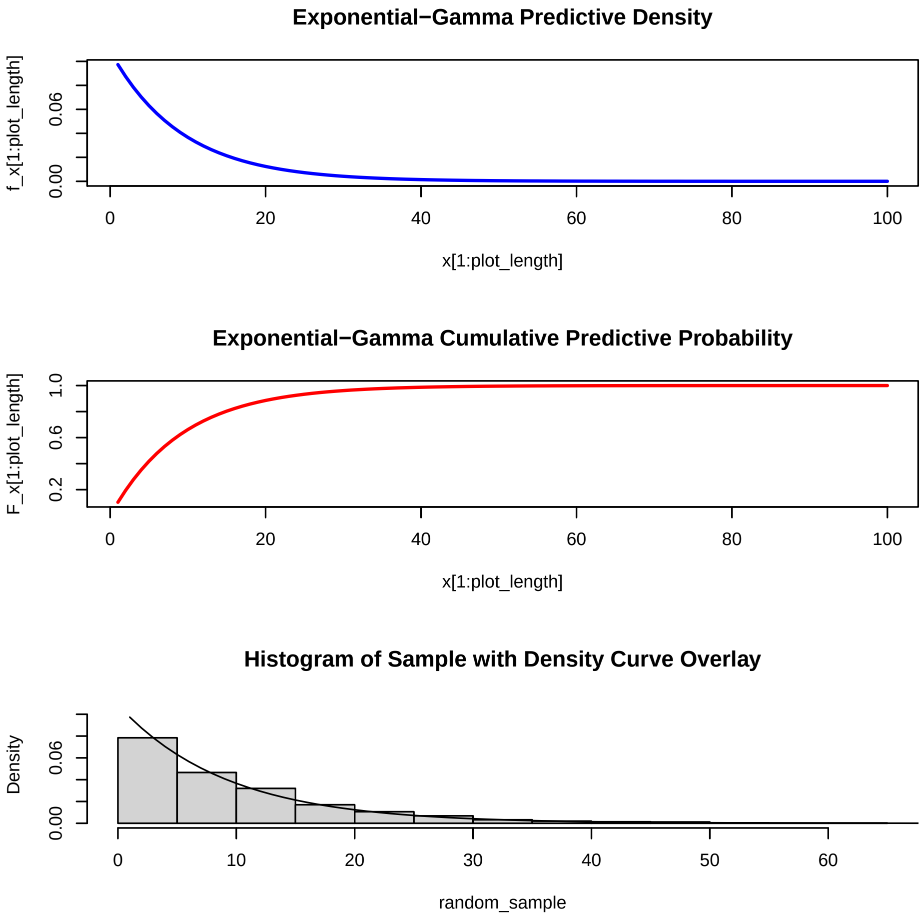
(simplifying by constructing a Gamma$(d+\delta+1, \gamma+N\bar{x}+x)$ density in the final integrand.)

3.2.2 R Implementation

This result has been used to create standard format R functions dpredEG(), ppredEG(), and rpredEG() for the Gamma-Exponential distribtuion for density, cumulative probability, and random sampling, respectively (see appendix). These functions are exercised in the following example.

3.2.3 Example

Suppose $d = 800$ out of $N = 1000$ copies have been observed, and the remaining 200 censored. Say $\delta = 20$, $\gamma = 5$, and we are interested in the number of survivors out of $M = 1000$ future observations. The figures below illustrate the predictive probability using dpredEG() and rpredEG(), along with a histogram of a random sample taken using rpredEG().


Exponential−Gamma Predictive Density
f_x[1:plot_length]
0.00
0.06
0
20
40
60
80
100
x[1:plot_length]
Exponential−Gamma Cumulative Predictive Probability
F_x[1:plot_length]
0.2
0.6
1.0
0
20
40
60
80
100
x[1:plot_length]
Histogram of Sample with Density Curve Overlay
Density
0.00
0.06
0
10
20
30
40
50
60
random_sample

3.3 Poisson-Gamma Model (Hoff p. 43ff)

3.3.1 Derivation

[using Hoff's notation and variable names below. Should I convert this to Geisser's $x^{(N)}, x_{(M)}$ convention for uniformity throughout my thesis?]

Suppose $Y_1, ..., Y_n|\theta \overset{i.i.d.}{\sim}$ Poisson(θ) with Gamma prior $\theta \sim$ Gamma(α, β). That is,

$$
\begin{aligned}
P\left(Y_1 = y_1, ..., Y_n = y_n|\theta\right) &= \prod_{i=1}^{n} p\left(y_i|\theta\right) \\
&= \prod_{i=1}^{n} \frac{1}{y!}\theta^{y_i}e^{-\theta} \\
&= \left(\prod_{i=1}^{n} \frac{1}{y!}\right)\theta^{\sum y_i}e^{-n\theta} \\
&= c\left(y_1, ..., y_n\right)\theta^{\sum y_i}e^{-n\theta}
\end{aligned}
$$

and

$$
p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \theta, \alpha, \beta > 0.
$$

Then we have posterior distribution

$$
\begin{aligned}
p\left(\theta|y_1, ..., y_n\right) &= \frac{p\left(y_1, ..., y_n|\theta\right)p(\theta)}{\sum_\theta p\left(y_1, ..., y_n|\theta\right)p(\theta)} \\
&= \frac{p\left(y_1, ..., y_n|\theta\right)p(\theta)}{p\left(y_1, ..., y_n\right)} \\
&= \frac{1}{p\left(y_1, ..., y_n\right)}\theta^{\sum y_i}e^{-n\theta}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \\
&= C\left(y_1, ..., y_n, \alpha, \beta\right)\theta^{\alpha+\sum y_i-1}e^{-(\beta+n)\theta} \\
&\sim \text{Gamma}\left(\alpha + \sum y_i, \beta + n\right).
\end{aligned}
$$

Here

$$C\left(y_1,...,y_n,\alpha,\beta\right)=\frac{1}{p\left(y_1,...,y_n\right)}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}$$

$$=\frac{1}{\sum_\theta p\left(y_1,...,y_n|\theta\right)p(\theta)}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}$$

$$=\frac{1}{\sum_\theta\left(\prod\frac{1}{y_i!}\right)\theta^{\sum y_i}e^{-n\theta}\cancel{\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)}\theta^{\alpha-1}e^{-\beta\theta}}\cdot\cancel{\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)}$$

$$=\frac{1}{\left(\prod\frac{1}{y_i!}\right)\frac{\Gamma(\alpha+\sum y_i)}{(\beta+n)^{\alpha+\sum y_i}}\sum_\theta\frac{(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}\theta^{\sum y_i+\alpha-1}e^{-(\beta+n)\theta}}$$

$$=\frac{\prod_{i=1}^n y_i!(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}$$

Making a tiny change to test my ability to commit.
Adding just a little bit more.
Still able to commit?
Yes. Will try pushing again later.

For future observation $\tilde{y}$, then, we compute predictive distribution

$$
\begin{aligned}
p\left(\tilde{y}|y_1,...,y_n\right) &= \int_0^\infty p\left(\tilde{y}|\theta,y_1,...,y_n\right) p\left(\theta|y_1,...,y_n\right) d\theta \\
&= \int p\left(\tilde{y}|\theta\right) p\left(\theta|y_1,...,y_n\right) d\theta \\
&= C\int \left(\frac{1}{\tilde{y}!}\theta^{\tilde{y}}e^{-\theta}\right)\theta^{\alpha+\sum y_i-1}e^{-(\beta+n)\theta}d\theta \\
&= \frac{C}{\tilde{y}!}\int \theta^{\tilde{y}+\alpha+\sum y_i-1}e^{-(\beta+n+1)\theta}d\theta \\
&= \frac{C\Gamma\left(\tilde{y}+\alpha+\sum y_i\right)}{\Gamma\left(\tilde{y}+1\right)\left(\beta+n+1\right)^{\tilde{y}+\alpha+\sum y_i}}\int \frac{\left(\beta+n+1\right)^{\tilde{y}+\alpha+\sum y_i}}{\Gamma\left(\tilde{y}+\alpha+\sum y_i\right)}\theta^{\tilde{y}+\alpha+\sum y_i-1}e^{-(\beta+n+1)\theta)}d\theta \\
&= C\cdot\frac{\Gamma\left(\tilde{y}+\alpha+\sum y_i\right)}{\Gamma\left(\tilde{y}+1\right)\left(\beta+n+1\right)^{\tilde{y}+\alpha+\sum y_i}} \\
&= \frac{\prod_{i=1}^n y_i!(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}\cdot\frac{\Gamma\left(\tilde{y}+\alpha+\sum y_i\right)}{\Gamma\left(\tilde{y}+1\right)\left(\beta+n+1\right)^{\tilde{y}+\alpha+\sum y_i}} \\
&= \prod_{i=1}^n y_i!\cdot\frac{\Gamma(\tilde{y}+\alpha+\sum y_i)}{\Gamma(\tilde{y}+1)\Gamma(\alpha+\sum y_i)}\cdot\left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum y_i}\cdot\left(\frac{1}{\beta+n+1}\right)^{\tilde{y}}
\end{aligned}
$$

3.3.2 R Implementation

3.3.3 Example

3.4 Normal Observation with Normal-Inverse Gamma Prior

3.4.1 One sample

3.4.1.1 Derivation

3.4.1.2 R Implementation

3.4.1.3 Example

3.4.2 Two samples

3.4.2.1 Derivation

3.4.2.2 R Implementation

3.4.2.3 Example

3.4.3 k samples

3.4.3.1 Derivation

3.4.3.2 R Implementation

3.4.3.3 Example

3.4.3.4 Ranking Treatments

4 Chapter 2: Normal Regression with Zellner's g-prior

4.0.0.1 Derivation

4.0.0.2 R Implementation

4.0.0.3 Example

5 Conclusion