Predictive Inference Tools for Researchers

by

Voyze G. Harris III

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THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Master's Committee, we certify that we have read the thesis prepared by Voyze Gabriel Harris III, titled *[Enter Thesis Title]* and recommend that it be accepted as fulfilling the dissertation requirement for the Master's Degree.

____ Date: _____________
Dr. Dean Billheimer

____ Date: _____________
Dr. Edward Bedrick

____ Date: _____________
Dr. Walter Piegorsch

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to the Graduate College.

I hereby certify that I have read this thesis prepared under my direction and recommend that it be accepted as fulfilling the Master's requirement.

____ Date: _____________
Dr. Dean Billheimer
Master's Thesis Committee Chair
Biostatistics

ARIZONA

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2.2.2 When is parametric inference more useful?

[examples, comparisons]

2.3 The Bayesian Parametric Prediction Format

[Geisser p. 49]

Let

$$f\left(x^{(N)}, x_{(M)}|\theta\right) = f\left(x_{(M)}|x^{(N)}, \theta\right) f\left(x^{(N)}|\theta\right).$$

Here $x^{(N)}$ represents observed events and $x_{(M)}$ are future events. We calculate

$$f\left(x_{(M)}, x^{(N)}\right) = \int f\left(x^{(N)}, x_{(M)}|\theta\right) p(\theta) d\theta$$

where $p(\theta)$ is the prior density and

$$f\left(x_{(M)}|x^{(N)}\right) = \frac{f\left(x_{(M)}, x^{(N)}\right)}{f\left(x^{(N)}\right)} = \int f\left(x_{(M)}|\theta\right) p\left(\theta|x^{(N)}\right) d\theta$$

where

$$p\left(\theta|x^{(N)}\right) \propto f\left(x^{(N)}|\theta\right) p(\theta).$$

2.4 [Maybe] Example of Difference between results from Plug-in estimator and results using Predictive Inference

3 Chapter 1: Predictive Problems with Conjugate Priors

[Problems with closed-form solutions. These problems will be what the R package is designed for. Use problems from Geisser, Casella & Berger (Bayesian chapter), other sources. Regression problem–predictive distributions of models that include and exclude some predictor]

3.1 Prediction of Future Successes: Beta-Binomial (Geisser p. 73)

3.1.1 Derivation

Let X_i be independent binary variables with $\Pr(X_i = 1) = \theta$, and let $T = \sum X_i$. Then T has probability

$$\binom{N}{t}\theta^t(1-\theta)^{N-t}.$$

Assume $\theta \sim \text{Beta}(\alpha, \beta)$, so

$$p(\theta) = \frac{\Gamma(\alpha+\beta)\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}.$$

Then

$$p\left(\theta|X^{(N)}\right) = \frac{\Gamma(N+\alpha+\beta)\theta^{t+\alpha-1}(1-\theta)^{N-t+\beta-1}}{\Gamma(t+\alpha)\Gamma(N-t+\beta)}$$

So for $R = \sum_{i=1}^{M} X_{N+i}$ we have Beta-Binomial predictive distribution

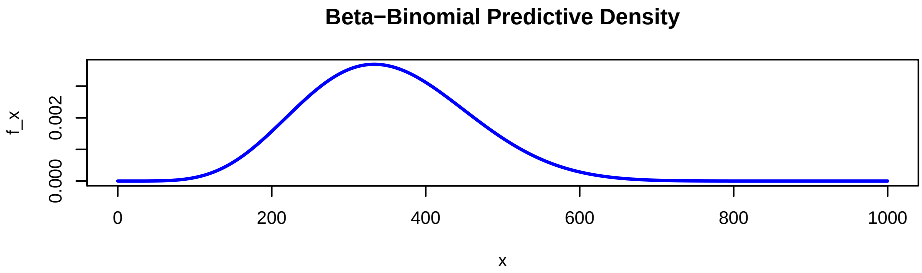
$$\begin{aligned}
\Pr[R=r|t] &= \int \binom{M}{r}\theta^r(1-\theta)^{M-r}p\left(\theta|X^{(N)}\right)d\theta \\
&= \binom{M}{r}\int \theta^r(1-\theta)^{M-r}\frac{\Gamma(N+\alpha+\beta)}{\Gamma(t+\alpha)\Gamma(N-t+\beta)}\theta^{t+\alpha-1}(1-\theta)^{N-t+\beta-1}d\theta \\
&= \frac{M!}{r!(M-r)!}\frac{\Gamma(N+\alpha+\beta)}{\Gamma(t+\alpha)\Gamma(N-t+\beta)}\int \theta^{r+t+\alpha-1}(1-\theta)^{M-r+N-t+\beta-1}d\theta \\
&= \frac{\Gamma(M+1)\Gamma(N+\alpha+\beta)\Gamma(r+t+\alpha)\Gamma(M-r+N-t+\beta)}{\Gamma(r+1)\Gamma(M-r+1)\Gamma(t+\alpha)\Gamma(N-t+\beta)\Gamma(M+N+\alpha+\beta)}
\end{aligned}$$

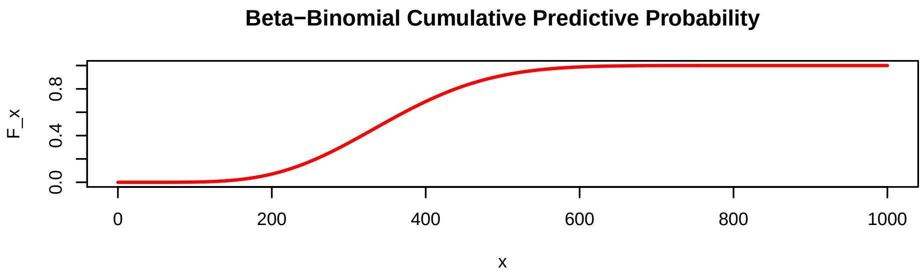
3.1.2 R Implementation

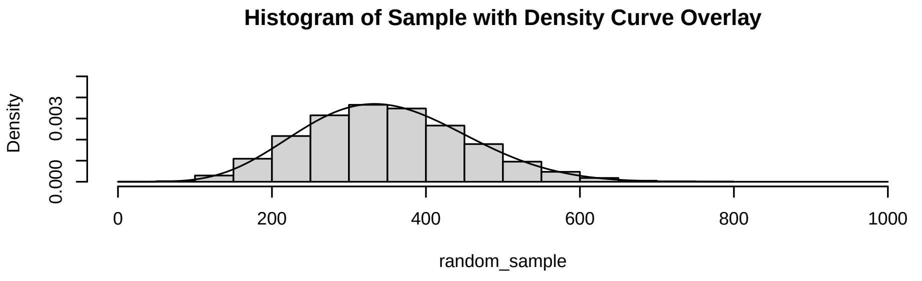
This result has been used to create "standard" R functions dpredBB(), ppredBB(), and rpredBB() for the Beta-Binomial distribtuion for density, cumulative probability, and random sampling, respectively (see appendix). These functions are exercised in the following example.

3.1.3 Example

Suppose $t = 5$ successes have been observed out of $N = 10$ binary events, $\alpha = 2$ and $\beta = 8$. For $M = 1000$ future observations, the figures below show the predictive distribution from dpredBB(), the cumulative distribution from ppredBB(), and a histogram of random draws from rpredBB().







3.2 Survival Time: Exponential-Gamma (Geisser p. 74)

3.2.1 Derivation

Suppose $X^{(N)} = \left(X^{(d)}, X^{(N-d)}\right)$ where $X^{(d)}$ represents copies fully observed from an exponential survival time density

$$f(x|\theta) = \theta e^{-\theta x}$$

and $X^{(N-d)}$ represents copies censored at $x_{d+1}, ..., x_N$, respectively. Hence

$$L(\theta) \propto \theta^d e^{-\theta N\bar{x}}$$

when $N\bar{x} = \sum_1^N x_i$, as shown below.

The usual exponential likelihood is used for the fully observed copies, whereas for the censored copies we need $\Pr(x > \theta) = 1 - \Pr(x \le \theta) = 1 - F(x|\theta) = 1 - (1 - e^{-\theta x}) = e^{-\theta x}$. Thus the overall likelihood is

$$L(\theta|x) = \prod_{i=1}^{d} \theta e^{-\theta x_i} \prod_{i=d+1}^{N} e^{-\theta x_i} = \theta^d e^{-\theta N\bar{x}}$$

Assuming a Gamma(δ, γ) prior for θ,

$$p(\theta) = \frac{\gamma^\delta \theta^{\delta-1} e^{-\gamma\theta}}{\Gamma(\delta)}$$

we obtain the posterior

$$\begin{aligned}
p\left(\theta|X^{(N)}\right) &= \frac{p\left(x^{(N)}|\theta\right)p(\theta)}{\int p\left(X^{(N)}|\theta\right)p(\theta)d\theta} \\
&= \frac{\theta^d e^{-\theta N\bar{x}} \cdot \frac{\gamma^\delta \theta^{\delta-1} e^{-\gamma\theta}}{\Gamma(\delta)}}{\int \left(\theta^d e^{-\theta N\bar{x}} \cdot \frac{\gamma^\delta \theta^{\delta-1} e^{-\gamma\theta}}{\Gamma(\delta)}\right) d\theta} \\
&= \frac{\cancel{\frac{\gamma^\delta}{\Gamma(\delta)}} \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right)}{\cancel{\frac{\gamma^\delta}{\Gamma(\delta)}} \int \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right) d\theta} \\
&= \frac{\frac{(\gamma+N\bar{x})^{d+\delta}}{\Gamma(d+\delta)} \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right)}{\cancel{\frac{(\gamma+N\bar{x})^{d+\delta}}{\Gamma(d+\delta)} \int \left(\theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}\right) d\theta}} \\
&= \frac{(\gamma + N\bar{x})^{d+\delta} \theta^{d+\delta-1} e^{-\theta(\gamma+N\bar{x})}}{\Gamma(d+\delta)}
\end{aligned}$$

with the Gamma$(d + \delta, \gamma + N\bar{x})$ density in the next to last step integrating to 1.

Thus the survival time predictive probability is

$$
\begin{aligned}
P\left(X=x|\theta, X^{(N)}\right) &= \int p\left(\theta|X^{(N)}\right) p(x|\theta) d\theta \\
&= \int \frac{(\gamma+N\bar{x})^{d+\delta}\theta^{d+\delta-1}e^{-\theta(\gamma+N\bar{x})}}{\Gamma(d+\delta)} \cdot \theta e^{-\theta x} d\theta \\
&= (d+\delta)(\gamma+N\bar{x})^{d+\delta} \int \frac{\theta^{(d+\delta+1)-1}e^{-\theta(\gamma+N\bar{x}+x)}}{(d+\delta)\Gamma(d+\delta)} d\theta \\
&= \frac{(d+\delta)(\gamma+N\bar{x})^{d+\delta}}{(\gamma+N\bar{x}+x)^{d+\delta+1}} \int \frac{(\gamma+N\bar{x}+x)^{d+\delta+1}\,\theta^{(d+\delta+1)-1}e^{-\theta(\gamma+N\bar{x}+x)}}{\Gamma(d+\delta+1)} d\theta \\
&= \frac{(d+\delta)(\gamma+N\bar{x})^{d+\delta}}{(\gamma+N\bar{x}+x)^{d+\delta+1}}
\end{aligned}
$$

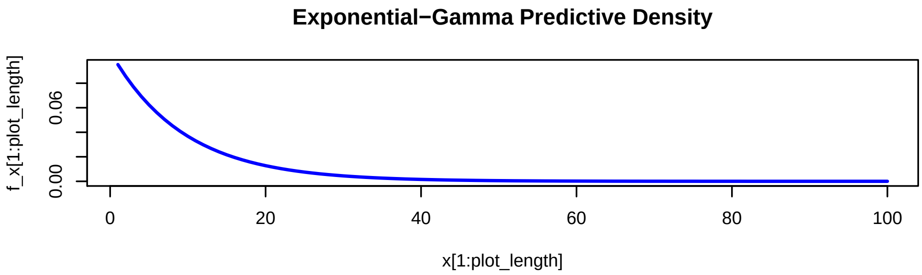
(simplifying by constructing a Gamma$(d+\delta+1, \gamma+N\bar{x}+x)$ density in the final integrand.)

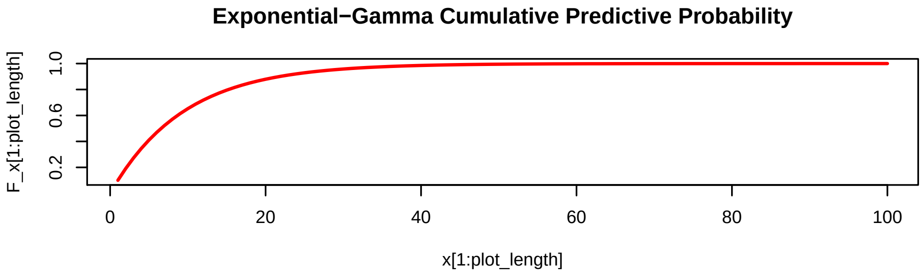
3.2.2 R Implementation

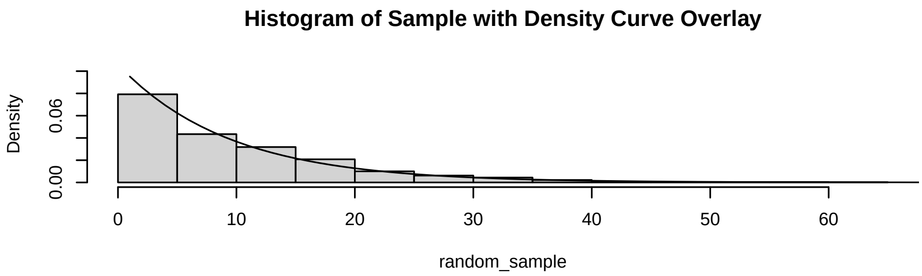
This result has been used to create standard format R functions dpredEG(), ppredEG(), and rpredEG() for the Gamma-Exponential distribtuion for density, cumulative probability, and random sampling, respectively (see appendix). These functions are exercised in the following example.

3.2.3 Example

Suppose $d = 800$ out of $N = 1000$ copies have been observed, and the remaining 200 censored. Say $\delta = 20$, $\gamma = 5$, and we are interested in the number of survivors out of $M = 1000$ future observations. The figures below illustrate the predictive probability using dpredEG() and rpredEG(), along with a histogram of a random sample taken using rpredEG().


Exponential−Gamma Predictive Density
f_x[1:plot_length]
0.00
0.06
0
20
40
60
80
100
x[1:plot_length]


Exponential−Gamma Cumulative Predictive Probability
F_x[1:plot_length]
0.2
0.6
1.0
0
20
40
60
80
100
x[1:plot_length]


Histogram of Sample with Density Curve Overlay
Density
0.00
0.06
0
10
20
30
40
50
60
random_sample

3.3 Poisson-Gamma Model (Hoff p. 43ff)

3.3.1 Derivation

[using Hoff's notation and variable names below. Should I convert this to Geisser's $x^{(N)}, x_{(M)}$ convention for uniformity throughout my thesis?]

Suppose $Y_1, ..., Y_n|\theta \overset{i.i.d.}{\sim}$ Poisson(θ) with Gamma prior $\theta \sim$ Gamma(α, β). That is,

$$P\left(Y_1 = y_1, ..., Y_n = y_n|\theta\right) = \prod_{i=1}^{n} p\left(y_i|\theta\right)$$

$$= \prod_{i=1}^{n} \frac{1}{y!}\theta^{y_i} e^{-\theta}$$

$$= \left(\prod_{i=1}^{n} \frac{1}{y!}\right) \theta^{\sum y_i} e^{-n\theta}$$

$$= c\left(y_1, ..., y_n\right) \theta^{\sum y_i} e^{-n\theta}$$

and

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \theta, \alpha, \beta > 0.$$

Then we have posterior distribution

$$p\left(\theta|y_1, ..., y_n\right) = \frac{p\left(y_1, ..., y_n|\theta\right) p(\theta)}{\int_\theta p\left(y_1, ..., y_n|\theta\right) p(\theta)}$$

$$= \frac{p\left(y_1, ..., y_n|\theta\right) p(\theta)}{p\left(y_1, ..., y_n\right)}$$

$$= \frac{1}{p\left(y_1, ..., y_n\right)} \theta^{\sum y_i} e^{-n\theta} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$= C\left(y_1, ..., y_n, \alpha, \beta\right) \theta^{\alpha + \sum y_i - 1} e^{-(\beta+n)\theta}$$

$$\sim \text{Gamma}\left(\alpha + \sum y_i, \beta + n\right).$$

Here

$$C\left(y_1,...,y_n,\alpha,\beta\right)=\frac{1}{p\left(y_1,...,y_n\right)}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}$$

$$=\frac{1}{\int_\theta p\left(y_1,...,y_n|\theta\right)p(\theta)}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}$$

$$=\frac{1}{\int_\theta\left(\prod\frac{1}{y_i!}\right)\theta^{\sum y_i}e^{-n\theta}\cancel{\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)}\theta^{\alpha-1}e^{-\beta\theta}}\cdot\cancel{\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)}$$

$$=\frac{1}{\left(\prod\frac{1}{y_i!}\right)\frac{\Gamma(\alpha+\sum y_i)}{(\beta+n)^{\alpha+\sum y_i}}\int_\theta\frac{(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}\theta^{\sum y_i+\alpha-1}e^{-(\beta+n)\theta}}$$

$$=\frac{\prod_{i=1}^n y_i!(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}$$

Call this constant C_n (for n observations).

Note that an additional observation $y_{n+1}=\tilde{y}$ the constant becomes

$$C_{n+1}=\frac{\prod_{i=1}^{n+1}y_i!(\beta+n+1)^{\alpha+\sum_{i=1}^{n+1}y_i}}{\Gamma(\alpha+\sum_{i=1}^{n+1}y_i)}.$$

Also note that the marginal joint distribution of k observations is

$$p\left(\tilde{y}|y_1,...,y_k\right)=\frac{1}{C_k}\frac{\beta^\alpha}{\Gamma(\alpha)}.$$

For future observation $\tilde{y}$, then, we compute predictive distribution

$$p\left(\tilde{y}|y_1,...,y_n\right)=\frac{p\left(y_1,...,y_n,\tilde{y}\right)}{p\left(y_1,...,y_n\right)}=\frac{p\left(y_1,...,y_{n+1}\right)}{p\left(y_1,...,y_n\right)}=\frac{\frac{1}{C_{n+1}}\cancel{\frac{\beta^\alpha}{\Gamma(\alpha)}}}{\frac{1}{C_n}\cancel{\frac{\beta^\alpha}{\Gamma(\alpha)}}}=\frac{C_n}{C_{n+1}}$$

$$=\frac{\dfrac{\prod_{i=1}^n y_i!(\beta+n)^{\alpha+\sum_{i=1}^n y_i}}{\Gamma(\alpha+\sum_{i=1}^n y_i)}}{\dfrac{\prod_{i=1}^{n+1} y_i!(\beta+n+1)^{\alpha+\sum_{i=1}^{n+1} y_i}}{\Gamma(\alpha+\sum_{i=1}^{n+1} y_i)}}$$

$$=\frac{\Gamma\left(\alpha+\sum_{i=1}^{n+1} y_i\right)(\beta+n)^{\alpha+\sum_{i=1}^n y_i}}{(y_{n+1}!)\,\Gamma\left(\alpha+\sum_{i=1}^n y_i\right)(\beta+n+1)^{\alpha+\sum_{i=1}^{n+1} y_i}}$$

$$=\frac{\Gamma\left(\alpha+\sum_{i=1}^n y_i+\tilde{y}\right)(\beta+n)^{\alpha+\sum_{i=1}^n y_i}}{(\tilde{y}!)\,\Gamma\left(\alpha+\sum_{i=1}^n y_i\right)(\beta+n+1)^{\alpha+\sum_{i=1}^n y_i+\tilde{y}}}$$

$$=\frac{\Gamma\left(\alpha+\sum y_i+\tilde{y}\right)}{\Gamma(\tilde{y}+1)\Gamma(\alpha+\sum y_i)}\cdot\left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum y_i}\cdot\left(\frac{1}{\beta+n+1}\right)^{\tilde{y}}$$

This is a negative binomial distribution: $\tilde{y}\sim NB\left(\alpha+\sum y_i,\beta+n\right)$, for which

$$E\left[\tilde{Y}|y_1,...,y_n\right]=\frac{a+\sum y_i}{b+n}=E\left[\theta|y_1,...,y_n\right];$$

$$\begin{aligned}\mathrm{Var}\left[\tilde{Y}|y_1,...,y_n\right]&=\frac{a+\sum y_i}{b+n}\frac{b+n+1}{b+n}\\&=\mathrm{Var}\left[\theta|y_1,...,y_n\right]\times(b+n+1)\\&=E\left[\theta|y_1,...,y_n\right]\times\frac{b+n+1}{b+n}\end{aligned}$$

[Showing here that it is indeed a NB distribution]

$$\theta\sim NB(\alpha,\beta)\Rightarrow p(\theta)=\binom{\theta+\alpha-1}{\alpha-1}\left(\frac{\beta}{\beta+1}\right)^\alpha\left(\frac{1}{\beta+1}\right)^\theta$$

so

$$\tilde{y} \sim NB\left(\alpha + \sum y_i), \beta + n\right) \Rightarrow p(\tilde{y}) = \binom{\tilde{y} + \alpha + \sum y_i - 1}{\alpha + \sum y_i - 1} \left(\frac{\beta + n}{\beta + n + 1}\right)^{\alpha + \sum y_i} \left(\frac{1}{\beta + n + 1}\right)^{\tilde{y}}$$

$$= \frac{(\alpha + \sum y_i + \tilde{y} - 1)!}{(\alpha + \sum y_i - 1)!\,(\tilde{y})!} \left(\frac{\beta + n}{\beta + n + 1}\right)^{\alpha + \sum y_i} \left(\frac{1}{\beta + n + 1}\right)^{\tilde{y}}$$

$$= \frac{\Gamma\left(\alpha + \sum y_i + \tilde{y}\right)}{\Gamma\left(\alpha + \sum y_i\right)\Gamma\left(\tilde{y} + 1\right)} \left(\frac{\beta + n}{\beta + n + 1}\right)^{\alpha + \sum y_i} \left(\frac{1}{\beta + n + 1}\right)^{\tilde{y}}$$

[This is the result in Hoff. The straightforward derivation below is off by a constant multiple. Need to figure out what went awry.]

$$p\left(\tilde{y}|y_1, ..., y_n\right) = \int_0^\infty p\left(\tilde{y}|\theta, y_1, ..., y_n\right) p\left(\theta|y_1, ..., y_n\right) d\theta$$

$$= \int p\left(\tilde{y}|\theta\right) p\left(\theta|y_1, ..., y_n\right) d\theta$$

$$= C \int \left(\frac{1}{\tilde{y}!}\theta^{\tilde{y}} e^{-\theta}\right) \theta^{\alpha + \sum y_i - 1} e^{-(\beta + n)\theta} d\theta$$

$$= \frac{C}{\tilde{y}!} \int \theta^{\tilde{y} + \alpha + \sum y_i - 1} e^{-(\beta + n + 1)\theta} d\theta$$

$$= \frac{C\Gamma\left(\tilde{y} + \alpha + \sum y_i\right)}{\Gamma\left(\tilde{y} + 1\right)\left(\beta + n + 1\right)^{\tilde{y} + \alpha + \sum y_i}} \int \frac{\left(\beta + n + 1\right)^{\tilde{y} + \alpha + \sum y_i}}{\Gamma\left(\tilde{y} + \alpha + \sum y_i\right)} \theta^{\tilde{y} + \alpha + \sum y_i - 1} e^{-(\beta + n + 1)\theta)} d\theta$$

$$= C \cdot \frac{\Gamma\left(\tilde{y} + \alpha + \sum y_i\right)}{\Gamma\left(\tilde{y} + 1\right)\left(\beta + n + 1\right)^{\tilde{y} + \alpha + \sum y_i}}$$

$$= \frac{\prod_{i=1}^n y_i! (\beta + n)^{\alpha + \sum y_i}}{\Gamma(\alpha + \sum y_i)} \cdot \frac{\Gamma\left(\tilde{y} + \alpha + \sum y_i\right)}{\Gamma\left(\tilde{y} + 1\right)\left(\beta + n + 1\right)^{\tilde{y} + \alpha + \sum y_i}}$$

$$= \prod_{i=1}^n y_i! \cdot \frac{\Gamma\left(\tilde{y} + \alpha + \sum y_i\right)}{\Gamma(\tilde{y} + 1)\Gamma(\alpha + \sum y_i)} \cdot \left(\frac{\beta + n}{\beta + n + 1}\right)^{\alpha + \sum y_i} \cdot \left(\frac{1}{\beta + n + 1}\right)^{\tilde{y}}$$

Hoff p.47:

- b is interpreted as the number of prior observations
- a is interpreted as the sum of counts from b prior observations

Hoff p. 49 (Birth rate example): $a = 2, b = 1$.

3.3.2 R Implementation

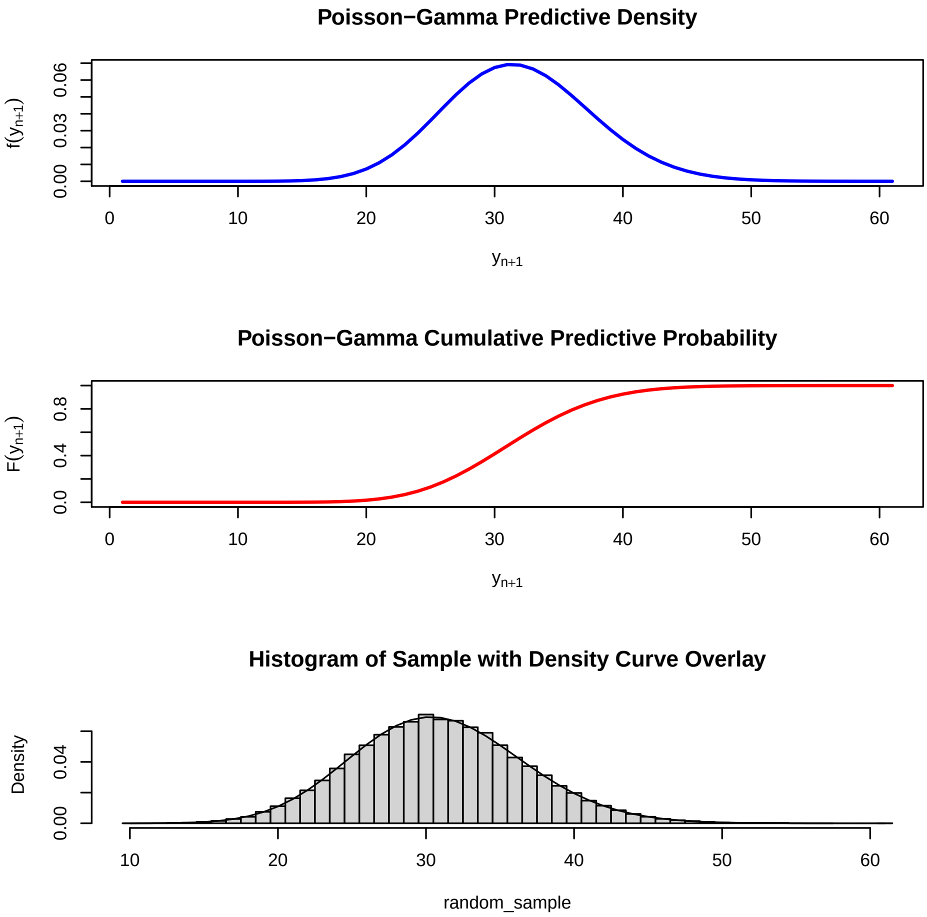
This result has been used to create standard format R functions dpredPG(), ppredPG(), and rpredPG() for the Poisson-Gamma distribution for density, cumulative probability, and random sampling, respectively (see appendix). These functions are exercised in the following example.

Developing the random sample function rpredPG(): I need to establish the support of the predictive distribution f_x from which to sample. the uniroot() function is not working because it keeps feeding non-integer values to dnbinom(). Strategy: a modified bisection method as follows:

1. set a desired tolerance ϵ.
2. Find the expected value E_x (closed formula, see above).
3. Step to the right of E_x by whole integers, in the sequence $E_x + \{1, 2, 4, ...2^n\}$, stopping at $U = f_x(E_x + 2^n) < 0$. This is the upper bound for the bisection method.
4. Bisect the interval, rounding to the nearest integer. Call the resulting mid-interval number B.
5. If B is positive, test whether $0 \leq f_x(B) \leq \epsilon$. If so, DONE. If not:
6. Establish new interval, choosing endpoints from E_x, B, and U so that the interval straddles 0, and repeat the steps until the condition in step 5 is reached.

3.3.3 Example

Suppose we have 10 prior observations with counts 27, 79, 21, 100, 8, 4, 37, 15, 3, 97. Let $\alpha = 11$ and $\beta = 3$. For $\tilde{y} = 1 : 100$ possible future occurrences, the figures below show the predictive distribution from dpredPG(), the cumulative distribution from ppredPG(), and a histogram of random draws from rpredPG().


Poisson−Gamma Predictive Density
f(yn+1)
0.00
0.03
0.06
0
10
20
30
40
50
60
yn+1
Poisson−Gamma Cumulative Predictive Probability
F(yn+1)
0.0
0.4
0.8
0
10
20
30
40
50
60
yn+1
Histogram of Sample with Density Curve Overlay
Density
0.00
0.04
10
20
30
40
50
60
random_sample

3.4 Normal Observation with Normal-Inverse Gamma Prior

3.4.1 One sample

3.4.1.1 Derivation [Hoff p. 69ff]

Let $\{Y_1, ..., Y_n|\theta, \sigma^2\} \overset{i.i.d.}{\sim} N(\theta, \sigma^2)$. Then the joint sampling density is

$$
\begin{aligned}
p\left(y_1, ..., y_n|\theta, \sigma^2\right) &= \prod_{i=1}^{n} p\left(y_i|\theta, \sigma^2\right) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y_i-\theta}{\sigma}\right)^2} \\
&= \left(2\pi\sigma^2\right)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i-\theta}{\sigma}\right)^2}.
\end{aligned}
$$

It can be shown that $\{\sum y_i^2, \sum y_i\}$ and hence $\{\bar{y}, s^2\}$ are sufficient statistics, where $\bar{y} = \sum y_i/n$ and $s^2 = \sum \left(y_i - \bar{y}\right)^2/(n-1)$.

$\vdots$

Following Hoff (p. 74ff), for joint inference on both θ and σ, assume priors

$$
\begin{aligned}
\frac{1}{\sigma^2} &\sim \text{gamma}\left({}^{\nu_0}/_2, {}^{\nu_0\sigma_0^2}/_2\right) \\
\theta|\sigma^2 &\sim \text{normal}\left(\mu_0, {}^{\sigma^2}/_{\kappa_0}\right)
\end{aligned}
$$

where (σ_0^2, ν_0) are the sample variance and sample size of prior observations, and (μ_o, κ_0) are the sample mean and sample size of prior observations.

Note: μ_0, κ_0, ν_0, and σ_0^2 come from prior knowledge. [in the Hoff example (Midge Wing Length), κ_0 and ν_0 are both set to 1 so that "our prior distributions are only weakly centered around these estimates from other populations."]

From this we derive joint posterior

$$
\begin{aligned}
\left\{\theta|y_1, ..., y_n, \sigma^2\right\} &\sim \text{normal}\left(\mu_n, {}^{\sigma^2}/_{\kappa_n}\right) \\
\left\{\sigma^2|y_1, ..., y_n\right\} &\sim \text{inverse-gamma}\left({}^{\nu_n}/_2, {}^{\sigma_n^2\nu_n}/_2\right).
\end{aligned}
$$

where

$$\kappa_n = \kappa_0 + n$$

$$\mu_n = \frac{\kappa_0 \mu_0 + n\bar{y}}{\kappa_n}$$

$$\nu_n = \nu_0 + n$$

$$\sigma_n^2 = \frac{1}{\nu_n}\left[\nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_n}\left(\bar{y} - \mu_0\right)^2\right].$$

Here $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is the sample mean and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$ is the sample variance.

From the joint posterior distribution we generate marginal samples by means of the Monte Carlo method (Hoff, p. 77):

$$\begin{array}{ll} \sigma^{2(1)} \sim \text{inverse-gamma}\left(\nu_n/2, \sigma_n^2 \nu_n/2\right), & \theta^{(1)} \sim \text{normal}\left(\mu_n, \sigma^{2(1)}/\kappa_n\right) \\ \vdots & \vdots \\ \sigma^{2(S)} \sim \text{inverse-gamma}\left(\nu_n/2, \sigma_n^2 \nu_n/2\right), & \theta^{(S)} \sim \text{normal}\left(\mu_n, \sigma^{2(S)}/\kappa_n\right) \end{array}$$

For prediction of future $\tilde{y}|y_1, ..., y_n, \theta, \sigma^2$, generate $\tilde{y}_i \sim \text{normal}\left(\theta^{(i)}, \sigma^{2(i)}\right)$.

3.4.1.2 R Implementation Standard format R functions dpredNormIG(), ppredNormIG(), and rpredNormIG() have been created for the Normal-Inverse Gamma distribution for density, cumulative probability, and random sampling, respectively (see appendix). For the random sampler rpredNormIG(), the Monte-Carlo method described above was directly employed. The predictive density and cumulative density functions depend on the random sample, and utilize Kernel Density Estimation (KDE) and R's built-in density() function. The KDE is computed by definition, using a normal kernel:

$$\hat{f}_K(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h} K\left(\frac{x - X_i}{h}\right),$$

where

X_i is the random sample generated using rpredNormIG()

K is Normal(0,1)

h is the bandwidth from R's density() function (that is, $h =$ density(X_i)$bw)

These functions are exercised in the following example.

Example (Hoff p. 72ff, using data from Grogan and Wirth (1981)): Midge wing length

Grogan and Wirth (1981) provide 9 measurements of midge wing length, in millimeters: $y = \{1.64, 1.7, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08\}$. Prior studies suggest values $\mu_0 = 1.9$ and $\sigma_0^2 = 0.01$. We choose $\kappa_0 = \nu_0 = 1$ "...so that our prior distributions are only weakly centered around these estimates from other populations" (Hoff p. 76). We compute

$$\bar{y} = 1.804$$

$$\mathrm{var}(y) = 0.0169$$

$$\kappa_n = 1 + 9 = 10$$

$$\mu_n = \frac{1 \cdot 1.9 + 9 \cdot 1.804}{10} = 1.814$$

$$\nu_n = 1 + 9 = 10$$

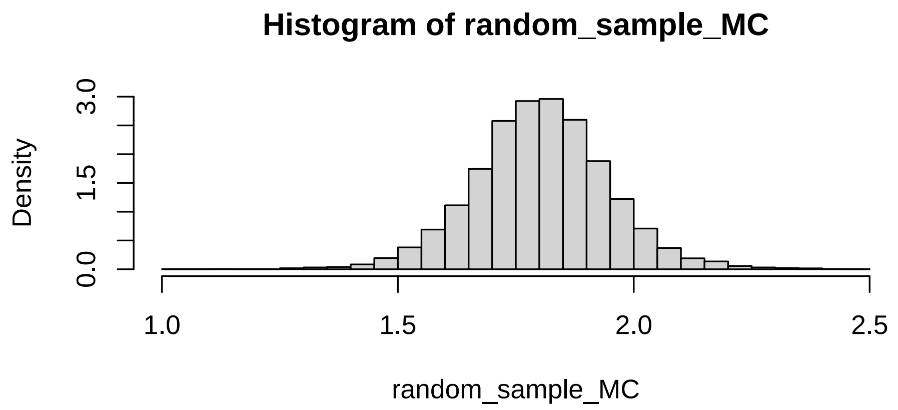
$$\sigma_n^2 = \frac{1}{10}\left[1 \cdot 0.01 + (9-1) \cdot 0.0169 + \frac{1 \cdot 9}{10}(1.804 - 1)^2\right] = 0.0153$$

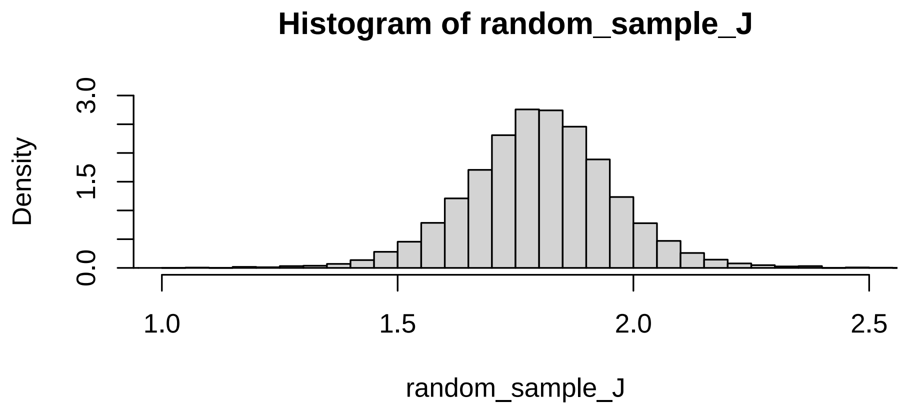
Thus $\nu_n/2 = 5$ and $\nu_n\sigma_n^2/2 = 0.7662$ and we have posteriors

$$\left\{\theta | y_1, ..., y_n, \sigma^2\right\} \sim \text{normal}\left(1.814, \sigma^2/10\right)$$

$$\left\{\sigma^2 | y_1, ..., y_n\right\} \sim \text{inverse-gamma}(5, 0.7662)$$

```
          [,1]
kn  10.0000000
mun  1.8040000
     0.1204326
nun 10.0000000
```


Histogram of random_sample_MC
Density
0.0
1.5
3.0
1.0
1.5
2.0
2.5
random_sample_MC


Histogram of random_sample_J
Density
0.0
1.5
3.0
1.0
1.5
2.0
2.5
random_sample_J

Jeffrey's Prior: $p(\theta, \sigma^2) = 1/\sigma^2$.

⋮

posteriors given Jeffrey's Prior:

$$\left\{1/\sigma^2 | y_1, ..., y_n\right\} \sim \text{gamma}\left(\frac{n-1}{2}, \frac{1}{2}\sum (y_i - \bar{y})^2\right)$$

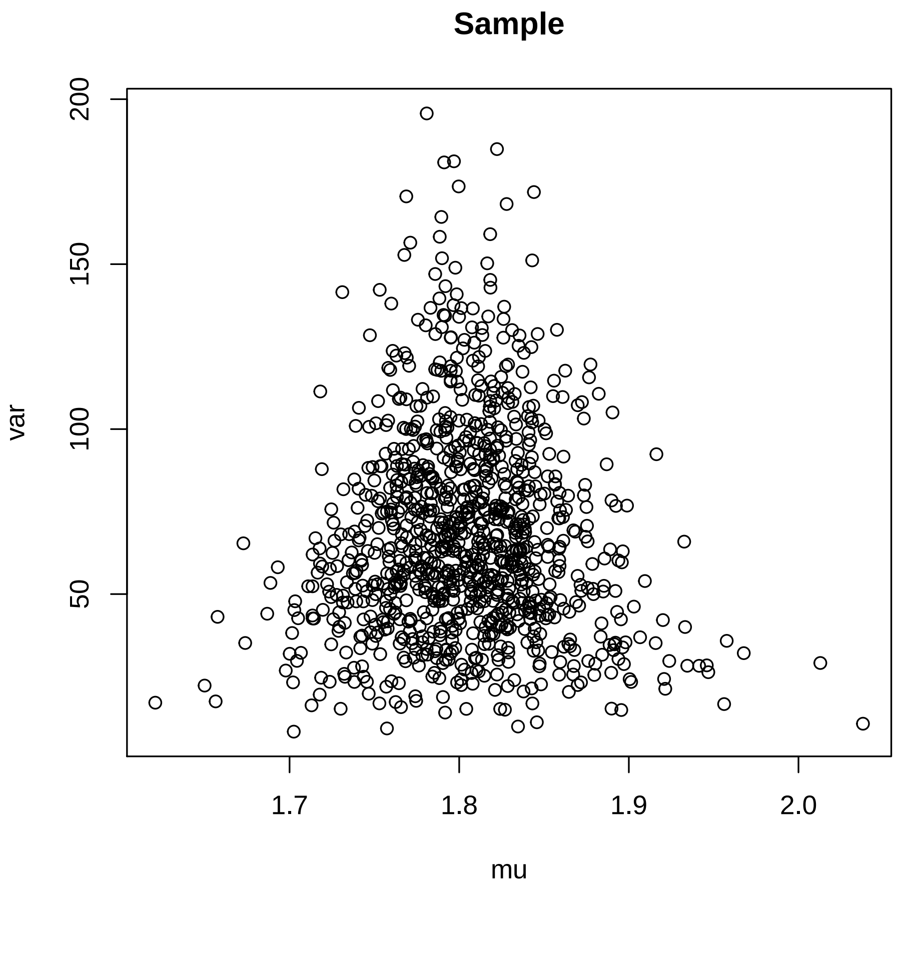
$$\left\{\theta | \sigma^2, y_1, ..., y_n\right\} \sim \text{normal}\left(\bar{y}, \frac{\sigma^2}{n}\right)$$

This leads to

$$\frac{\theta - \bar{y}}{s/\sqrt{n}} | y_1, ..., y_n \sim t_{n-1} \text{ (Hoff p. 79)}$$

So we have predictive distribution

$$\tilde{y} \sim t_{n-1} \text{ with location } \bar{y} \text{ and scale } s\sqrt{\left(1 + \frac{1}{n}\right)} \text{(Gelman et. al. p. 66)}$$



3.4.1.3 Example

3.4.2 Two samples

3.4.2.1 Derivation

3.4.2.2 R Implementation

3.4.2.3 Example

3.4.3 k samples

3.4.3.1 Derivation

3.4.3.2 R Implementation

3.4.3.3 Example

3.4.3.4 Ranking Treatments

4 Chapter 2: Normal Regression with Zellner's g-prior

4.0.0.1 Derivation

4.0.0.2 R Implementation

4.0.0.3 Example

5 Conclusion