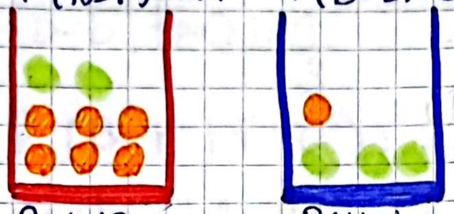
Ejercicios Probabilidades.

Priors:

$P(B=r)=0{,}4$ $\quad$ $P(B=b)=0{,}6$



Box: r $\quad$ Box: b

: Apple

: Orange

B : Box (Blue, Red) : b, r

$B \in \{r, b\}$

F: fruit (apple or Orange) : a, o

$F \in \{a, o\}$

Ejercicios

- $P(B=r)=0{,}4=\frac{4}{10}$
- $P(B=b)=0{,}6=\frac{6}{10}$
- $P(f=a \mid B=r)=\frac{2}{8}=\boxed{\frac{1}{4}}$ → (Me voy a la caja roja y miro sus elementos)
- $P(f=a \mid B=b)=\boxed{\frac{3}{4}}$
- $P(f=o \mid B=r)=\frac{6}{8}=\boxed{\frac{3}{4}}$
- $P(f=o \mid B=b)=\boxed{\frac{1}{4}}$
- $P(f=a)=\sum_{B_j \in \{r,b\}} P(F=a, B=B_j)$ → Regla de la suma.

Como no tenemos la conjunta vamos a expresarla en terminos de las condicionales esto lo podemos expresar con la Regla del producto entonces:

$$P(f=a)=P(f=a \mid B=r)\,P(B=r)+P(f=a \mid B=b)\,P(B=b)$$

$$P(f=a)=\frac{1}{4}\cdot\frac{4}{10}+\frac{3}{4}\cdot\frac{6}{10}=\frac{4}{40}+\frac{18}{40}$$

$$P(f=a)=\frac{1}{10}+\frac{9}{20}=\frac{20+90}{200}=\frac{110}{200}=\frac{11}{20}$$

$$P(f=a)=\frac{11}{20}$$

- $P(F=0) = \frac{9}{20}$

$$P(f=a) + P(f=0) = 1$$
$$P(f=0) = 1 - P(f=a)$$
$$P(f=0) = 1 - \frac{11}{20}$$
$$P(f=0) = \frac{20-11}{20} = \frac{9}{20}$$
$$P(f=0) = \frac{9}{20}$$

- $P(B=r \mid f=0) = \frac{2}{3}$.

Esto lo solucionamos por Bayes ya que podemos despejarlo a partir de la probabilidad de que ocurra $P(f=0 \mid B=r)$, entonces tenemos que:

$$P(B=r \mid f=0) = \frac{P(f=0 \mid B=r) \cdot P(B=r)}{P(f=0)}$$

$$P(B=r \mid f=0) = \frac{(3/4)(4/10)}{(9/20)} = \frac{\frac{12}{40}}{\frac{9}{20}}$$

$$P(B=r \mid f=0) = \frac{12 \cdot 20}{9 \cdot 40} = \frac{4}{3} \cdot \frac{2}{4} = \frac{8}{12} = \frac{2}{3}$$

$$P(B=r \mid f=0) = \frac{2}{3}$$

- $P(B=b \mid f=0) = \frac{1}{3}$.

$$P(B=b \mid f=0) = \frac{P(f=0 \mid B=b) \cdot P(B=b)}{P(f=0)}$$

$$P(B=b \mid f=0) = \frac{(1/4)(6/10)}{(9/20)} = \frac{(6/40)}{(9/20)} = \frac{\frac{6}{40}}{\frac{9}{20}} = \frac{6 \cdot 20}{9 \cdot 40}$$

$P(B=b \mid f=0) = \frac{12}{36} = \frac{1}{3}$

$P(B=b \mid f=0) = \frac{1}{3}$

- $P(B=r \mid F=a) = 2/11$

$$P(B=r \mid f=a) = \frac{P(f=a \mid B=r)\,P(B=r)}{P(f=a)}$$

$$P(B=r \mid f=a) = \frac{(1/4)(4/10)}{(11/20)}$$

$$P(B=r \mid f=a) = \frac{(4/40)}{(11/20)} = \frac{\frac{4}{40}}{\frac{11}{20}} = \frac{4 \cdot 20}{11 \cdot 40} = \frac{2}{11}$$

$P(B=r \mid f=a) = \frac{2}{11}$

- $P(B=b \mid F=a) = \frac{9}{11}$

$$P(B=b \mid f=a) = \frac{P(f=a \mid B=b)\ P(B=b)}{P(f=a)} = \frac{(3/4)(6/10)}{(11/20)}$$

$$P(B=b \mid f=a) = \frac{\frac{18}{40}}{\frac{11}{20}},\ \frac{18}{11} \cdot \frac{1}{2} = \frac{9}{11}$$

Tareas TAM. PRIMER CORTE.

Demostraciones

TAM

$Var\{x\} = \mathcal{E}\{x^2\} - \mathcal{E}^2\{x\}$

$$Var\{x\} = \mathcal{E}\{x^2\} - \mathcal{E}^2\{x\}$$

$$= \mathcal{E}\{(x-\mu_x)^2\}$$

$$= \mathcal{E}\{x^2 - 2x\mu_x + \mu_x^2\}$$

$$= \mathcal{E}\{x^2\} - 2\mu_x\mathcal{E}\{x\} + \mathcal{E}\{\mu_x^2\}$$

$$= \mathcal{E}\{x^2\} - 2\mathcal{E}\{x\}\mathcal{E}\{x\} + \mu_x^2$$

$$= \mathcal{E}\{x^2\} - 2\mathcal{E}^2\{x\} + \mathcal{E}^2\{x\}$$

$$= \boxed{\mathcal{E}\{x^2\} - \mathcal{E}^2\{x\}}$$

Nota: $\mu_x = \mathcal{E}\{x\}$

$COV\{x,y\} = \mathcal{E}_{xy}\{xy\} - \mathcal{E}\{x\}\mathcal{E}\{y\}$

$$COV\{x,y\} = \mathcal{E}_{xy}\{xy\} - \mathcal{E}\{x\}\mathcal{E}\{y\}$$

$$COV\{x,y\} = \mathcal{E}_{xy}\{$$

$$= \mathcal{E}_{x,y}\{xy - \mu_x y - x\mu_y + \mu_x\mu_y\}$$

$$= \mathcal{E}_{x,y}\{xy\} - \mu_x\mathcal{E}\{y\} - \mathcal{E}\{x\}\mu_y + \mu_x\mu_y\}$$

$$= \mathcal{E}_{x,y}\{xy\} - \mathcal{E}\{x\}\mathcal{E}\{y\} - \cancel{\mathcal{E}\{x\}\mathcal{E}\{y\}} + \cancel{\mathcal{E}\{x\}\mathcal{E}\{y\}}$$

$$= \boxed{\mathcal{E}_{x,y}\{xy\} - \mathcal{E}\{x\}\mathcal{E}\{y\}}$$

Nota: $\mu_x = \mathcal{E}\{x\}$

TAM.

Demostraciónes

$\mathrm{cov}\{x,y\} = \mathcal{E}_{x,y}\{xy^T\} - \mathcal{E}\{x\}\mathcal{E}\{y^T\}$

Matrices.

$$\mathrm{cov}\{x,y\} = \mathcal{E}_{xy}\{(x-\mu_x)(y-\mu_y)^T\}$$
$$= \mathcal{E}_{x,y^T}\{(x-\mu_x)(y^T-\mu_y^T)\}$$
$$= \mathcal{E}_{x,y}\{(xy^T - \mu_x y^T - x\mu_y^T + \mu_x\mu_y^T\}$$
$$= \mathcal{E}_{x,y}\{xy^T\} - \mu_x\mathcal{E}\{y^T\} - \mu_y^T\mathcal{E}\{x\} + \mu_x\mu_y^T$$
$$= \mathcal{E}_{x,y}\{xy^T\} - \mathcal{E}\{x\}\mathcal{E}\{y^T\} - \cancel{\mathcal{E}\{x\}\mathcal{E}\{y^T\}} + \cancel{\mathcal{E}\{x\}\mathcal{E}\{y^T\}}$$
$$= \boxed{\mathcal{E}_{x,y}\{xy^T\} - \mathcal{E}\{x\}\mathcal{E}\{y^T\}}$$

$\mathcal{E}\{x\} = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\,x\,dx = \mu$

$$\mathcal{E}\{x\} = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\,x\,dx = \mu$$
$$\mathcal{E}\{x\} = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\,x\,dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-\mu|^2}{2\sigma^2}\right) \cdot x\,dx.$$

sustituimos $z = x-\mu$;
$dz = dx$.

$$\mathcal{E}\{x\} = \int_{-\infty}^{\infty} \frac{(z+\mu)}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-|z|}{2\sigma^2}\right) dz.$$

$$\int_{-\infty}^{\infty} \underbrace{\frac{z}{\sqrt{2\pi\sigma^2}}}_{\text{impar}} \cdot \underbrace{\exp\left(\frac{-|z|}{2\sigma^2}\right)}_{\text{par}} dz + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-|z|^2}{2\sigma^2}\right) dz}_{\cancel{\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)dx} = 1}$$

$= 0$

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$$\boxed{\mathcal{E}\{x\} = \mu}$$

Demostraciónes

TAM.

$$\mathcal{E}\{x^2\} = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\, x^2\, dx = \mu^2 + \sigma^2$$

$$\mathcal{E}\{x^2\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-|x-\mu|^2}{2\sigma^2}\right) \cdot x^2\, dx.$$

Reemplazamos $z = x-\mu$; $dz = dx$.

$$\mathcal{E}\{x^2\} = \int_{-\infty}^{\infty} \frac{(z+\mu)^2}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-|z|^2}{2\sigma^2}\right) dz.$$

$$\mathcal{E}\{x^2\} = \underbrace{\int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-|z|^2}{2\sigma^2}\right) dz}_{\mathrm{var}\{x\} = \sigma^2} + \mu \underbrace{\int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-|z|^2}{2\sigma^2}\right) dz}_{0}$$

$$+ \mu^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-|z|^2}{2\sigma^2}\right) dz}$$

$$\boxed{\mathcal{E}\{x^2\} = \sigma^2 + \mu^2}$$

$\mathrm{var}\{x\} = \sigma^2$

$$\mathrm{var}\{x\} = \sigma^2$$

$$\mathrm{var}\{x\} = \mathcal{E}\{x^2\} - \mathcal{E}^2\{x\}$$

$$= \sigma^2 + \mu^2 - \mu^2.$$

$$\boxed{\mathrm{var}\{x\} = \sigma^2.}$$

Demostraciones

JAM

$$\mu_{ML} = \arg\max_{\mu} \log(P(x))$$

$$\frac{d \log(P(x))}{d\mu} = 0$$

$$\frac{d}{d\mu}\left(-\frac{N}{2}\log(2\pi)\right) - \frac{d}{d\mu}\left(\frac{N}{2}\log(\sigma^2)\right) - \frac{d}{d\mu}\left(\frac{1}{2\sigma^2}\sum_{n=1}^{N}|x_n - \mu|^2\right) = 0$$

$$\frac{d}{d\mu}\left(\frac{1}{2\sigma^2}\sum_{n=1}^{N}|x_n - \mu|^2\right) = 0.$$

$$\frac{1}{2\sigma^2}\cdot\sum_{n=1}^{N}\left(\frac{d}{d\mu}|x_n - \mu|^2\right) = 0$$

$$\frac{1}{2\sigma^2}\sum_{n=1}^{N} 2(x_n - \mu)(-1) = 0.$$

$$\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \mu = 0$$

$$N\cdot\mu = \sum_{n=1}^{N} x_n$$

$$\boxed{\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n.}$$ → Estimador de media muestral

Demostraciones

TAM.

$$\sigma_{ML}^2 = \arg\max_{\sigma^2} \log(P(x))$$

$$\frac{d}{d\sigma}(\log(P(x))) = 0$$

$$\frac{d}{d\sigma}\left(-\frac{N}{2}\log(2\pi)\right) - \frac{d}{d\sigma}\left(\frac{N}{2}\log(\sigma^2)\right) - \frac{d}{d\sigma}\left(\frac{1}{2\sigma^2}\sum_{n=1}^{N}|x_n - \mu|^2\right) = 0$$

$$\frac{N}{2}\cdot\frac{d}{d\sigma}(2\log(\sigma)) + \frac{1}{2}\frac{d}{d\sigma}\left(\frac{1}{\sigma^2}\right)\sum_{n=1}^{N}|x_n - \mu|^2 = 0.$$

$$\frac{N}{\sigma} + \frac{1}{2}\cdot\frac{-2}{\sigma^3}\cdot\sum_{n=1}^{N}|x_n - \mu|^2 = 0.$$

$$\sigma^3\left(\frac{N}{\sigma}\right) + \sigma^3\left(\frac{1}{2}\cdot\frac{-2}{\sigma^3}\right)\cdot\sum_{n=1}^{N}|x_n - \mu|^2 = 0$$

$$N\sigma^2 - \sum_{n=1}^{N}|x_n - \mu|^2 = 0$$

$$\boxed{\sigma_{ML}^2 = \frac{1}{N}\sum_{n=1}^{N}|x_n - \mu_{ML}|^2}$$

TAM

Demostraciones

Busqueda del mejor estimador

$A_1 = \frac{1}{N}\sum_{n=1}^{N} x_n$; $A_2 = x_1$

$ecm(\theta,\hat{\theta}) = var\{\hat{\theta}\} + b^2\{\theta\}$; A1 vs A2

Sesgos

$$\begin{aligned}
\bullet\, b(A_1) &= \mathcal{E}\{A_1\} - A \\
&= \frac{1}{N}\sum_{n=1}^{N} \mathcal{E}\{x_n\} - A \\
&= \frac{1}{N}\sum_{n=1}^{N} \mathcal{E}\{A + w_n\} - A \\
&= \frac{1}{N}\sum_{n=1}^{N} A - A \\
&= A - A = 0
\end{aligned}$$

$\bullet\, b(A_2) = \mathcal{E}\{A_2\} - A = \mathcal{E}\{x_1\} - A = A - A = 0$

Varianzas.

$$\begin{aligned}
\bullet\, Var\{A_1\} &= Var\left\{\frac{1}{N}\sum_{n=1}^{N}(A + w_n)\right\} \\
&= var\left\{\frac{1}{N}\sum_{n=1}^{N} A + \frac{1}{N}\sum_{n=1}^{N} w_n\right\} \\
&= var\left\{\frac{1}{N}\sum_{n=1}^{N} w_n\right\} \\
&= \sum_{n=1}^{N} var\left\{\frac{1}{N} w_n\right\} \\
&= \sum_{n=1}^{N} \frac{1}{N^2}\sigma^2
\end{aligned}$$

$\boxed{var\{A_1\} = \frac{\sigma^2}{N}}$ → Mejor estimador.

$A_1 = \frac{1}{N}\sum_{n=1}^{N} x_n$.

$$\begin{aligned}
Var\{A_2\} &= Var\{x_1\} \\
&= Var\{A + w_1\} \\
&= var\{w_1\}
\end{aligned}$$

$\boxed{Var\{A_2\} = \sigma^2}$

Demostraciones

TAM

$$\mathcal{E}\{\mu_{ML}\} = \mathcal{E}\left\{\frac{1}{N}\sum_{n=1}^{N} x_n\right\} = \mu$$

$$\mathcal{E}\{\mu_{ML}\} = \frac{1}{N}\sum_{n=1}^{N} \mathcal{E}\{x_n\}$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mu$$

$$= \frac{1}{\cancel{N}} \cdot \cancel{N} \cdot \mu$$

$$\boxed{\mathcal{E}\{\mu_{ML}\} = \mu}$$

$$\mathcal{E}\{\sigma^2_{ML}\} = \mathcal{E}\left\{\frac{1}{N}\sum_{n=1}^{N} |x_n - \mu_{ML}|^2\right\} = \left(\frac{N-1}{N}\right)\sigma^2$$

$$\mathcal{E}\{\sigma^2_{ML}\} = \mathcal{E}\left\{\frac{1}{N}\sum_{n=1}^{N} |x_n - \mu_{ML}|^2\right\}$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mathcal{E}(x_n^2 - 2\mathcal{E}\{x_n\}\,\mathcal{E}\{\mu_{ML}\} + \mathcal{E}\{\mu_{ML}^2\}).$$

$$= \frac{1}{N}\sum_{n=1}^{N} (\sigma^2 + \mu^2 - \mathcal{E}\{\mu_{ML}^2\})$$

$$\mathrm{var}\{\mu_{ML}\} = \mathcal{E}\{\mu_{ML}^2\} - \mathcal{E}^2\{\mu_{ML}\}$$

$$\mathcal{E}\{(\mu_{ML}^2)\} = \mathrm{var}\{\mu_{ML}\} + \mathcal{E}^2\{\mu_{ML}\}.$$

$$= \mathrm{var}\left\{\frac{1}{N}\sum_{n=1}^{N} x_n\right\} + \mu^2$$

$$= \frac{1}{N^2}\cdot N\sigma^2 + \mu^2.$$

$$= \frac{1}{N}\sigma^2 + \mu^2.$$

Así,

$$\mathcal{E}\{\sigma^2_{ML}\} = \frac{1}{N}\sum_{n=1}^{N}\left(\sigma^2 + \mu^2 - \frac{\sigma^2}{N} - \mu^2\right)$$

$$= \frac{1}{N}\cdot N\cdot\sigma^2\left(1 - \frac{1}{N}\right)$$

$$\boxed{\mathcal{E}\{\sigma^2_{ML}\} = \frac{N-1}{N}\sigma^2}$$

TAM.

Demostraciones

$S^2 = \Delta$

$$\phi^T\phi = V\Delta V^*$$

$$(USV^*)^T(USV^*) = V\Delta V^*$$

$$(V^*)^T S^T \underbrace{U^T U}_{I} S V^* = V\Delta V^*$$

$$V S^T S V^* = V\Delta V^*$$

$$V S^2 V^* = V\Delta V^*$$

Por lo tanto, $S^2 = \Delta$.

$\phi^T\phi = V\Delta V$
$\phi_{pca} = USV$
$U^TU = I$
$V^TV = I$

$\mu_{a|b} = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$
$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$.

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Identidad de la matriz inversa por partes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} M & -MBD^{-1} \\ -DCM & D^{-1} + D^{-1}CMBD^{-1} \end{bmatrix}$$

Siendo $M = (A - BD^{-1}C)^{-1}$

$$\begin{bmatrix} \Lambda_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{bmatrix} = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}^{-1} = \begin{bmatrix} M & -M\Sigma_{ab}\Sigma_{bb}^{-1} \\ -\Sigma_{bb}\Sigma_{ba} & \Sigma_{bb}^{-1} + \Sigma_{bb}\Sigma_{ba}M\Sigma_{ab}\Sigma_{bb}^{-1} \end{bmatrix}$$

donde $M = (\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}$

Así; $\Sigma_{a|b} = \Lambda_{aa}^{-1}$
$= M^{-1} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$

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$$\mu_{a|b} = \mu_a - \Sigma_{a|b}\,\Lambda_{ab}(\mu_b - x_b)$$
$$= \mu_a - M^{-1}(-M\,\Sigma_{ab}\,\Sigma_{bb}^{-1})(\mu_b - x_b)$$
$$= \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(-\mu_b - x_b)$$
$$= \mu_a - \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$$