⇒ Probablity Distributions

Random Variable (R.V)

Discrete R.V (mass function)	Continuous R.V. (density function)
↓	↓
Discrete Probablity Distribution	Continuous Probablity Distribution
↓	↓
Principles of DPS (mean/variance/mode)	Principles of C.P.S. (mean/variance/mode)
↓	↓
Special DPD: Binomial, Poission	Special C.P.D: Normal, Exponential

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⇒ Mathematical Expectation

Let 'X' be a discrete random variable which can assume the values $x_1, x_2, x_3 \cdots x_n$ with corresponding probablities $P(x_1), P(x_2), P(x_3) \cdots P(x_n)$. Then the mathematical expectation of ~~capital~~ 'X' is denoted by E(X) and is defined as

$$E(X) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) \cdots x_n P(x_n)$$

$$= \sum_{i=1}^{n} x_i P(x_i) \quad \text{where} \quad \sum_{i=1}^{n} P(x_i) = 1$$

Similarly, if X is a continuous random variable with probablity density function $f(x)$ [ie $f(X=x)$], then the mathematical expectation of X is given by

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \quad \text{where} \quad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

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$$E(X) = \begin{cases} \sum_{i=1}^{n} x_i p(x_i) & \text{if } X \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} x f(x)\, dx & \text{if } X \text{ is c.r.v.} \end{cases}$$

Physical Interpretation of mathematical expectation.

Let us consider the frequency distribution of the random variable 'X' as below

C.I.	Mid value $X = x_i$	frequencies $f(x_i)$
0 – 10	x_1	$f(x_1)$
11 – 20	x_2	$f(x_2)$
21 – 30	x_3	$f(x_3)$
⋮	⋮	⋮
	x_n	$f(x_n)$
		$\sum_{i=1}^{n} f(x_i) = N$

we know that, Mean of 'X' for the frequency table (distribution) is given by

$$\text{mean of } X = \frac{\sum_{i=1}^{n} x_i f(x_i)}{\sum_{i=1}^{n} f(x_i)}$$

$$= \frac{x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n)}{N}$$

$$= x_1 \frac{f(x_1)}{N} + x_2 \frac{f(x_2)}{N} + \cdots + x_n \frac{f(x_n)}{N}$$

$$= x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n)$$

$$= \sum_{i=1}^{n} x_i p(x_i)$$

$$= E(X)$$

$\therefore$ Mean of $X = E(X)$

i.e. the mathematical expectation of random variable 'x' is nothing but it's mean.

Note 1: we know that, in any random experiment, the values of random variable 'X' may be changed according to certain expression, but the corresponding probablities will not change. i.e.

$$E[\psi(X)] = \begin{cases} \sum_{i=1}^{n} \psi(x_i) \cdot P(x_i) & \text{where } \sum_{i=1}^{n} P(x_i) = 1 \\ \int_{-\infty}^{\infty} \psi(x) f(x)\, dx & \int_{-\infty}^{\infty} f(x)\, dx = 1 \end{cases}$$

example:

$$E(X^2) = \begin{cases} \sum_{i=1}^{n} x_i^2\, p(x_i) \\ \int_{-\infty}^{\infty} x^2 f(x)\, dx \end{cases}$$

$$E(2X^2+3X+4) = \begin{cases} \sum_{i=1}^{n} (2x_i^2+3x_i+4)\, P(x_i) \\ \int_{-\infty}^{\infty} (2x^2+3x+4) f(x)\, dx \end{cases}$$

Note 2:

- $E(C) = C$ where C is a constant
- $E(cX) = c \cdot E(X)$
- $E(aX \pm bY) = aE(X) \pm bE(Y)$
  where a, b are any two constants and Y is another expression of X, like $Y = X^2$; $Y = 2X^2+3X+4$; etc.
- $E(X \cdot Y) = E(X) \cdot E(Y)$ provided X, Y must be independent variables
- $E(X_1 \cdot X_2 \cdot X_3 \cdots X_n) = E(X_1) \cdot E(X_2) \cdot E(X_3) \cdots E(X_n)$
  where $X_1, X_2, X_3 \cdots X_n$ are independent variables.

→ Variance and Standard Deviation

The variance of the random variable 'X' is denoted by Var(X) or σ_x^2 ; and is defined as

$$Var(X) = \sigma_x^2 = E[(X-\mu)^2] = \begin{cases} \sum_{i=1}^{n} (x_i-\mu)^2 \, p(x_i) \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \end{cases}$$

where $\mu = E(X) =$ mean of X

The positive square root of variance is known as 'Standard Deviation'. i.e.

$$S.D. \text{ of } X = \sqrt{Var. \text{ of } X} = \sigma_x$$

Result 1:

$$Var(X) = E[(X-\mu)^2] = E(X^2) - [E(X)]^2$$

Proof: $Var(X) = E[(X-\mu)^2]$

$= E[X^2 - 2X\mu + \mu^2]$

$= E(X^2) - E(2X\mu) + E(\mu^2)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) + \mu^2 - 2\mu E(X)$

$= E(X^2) + \mu^2 - 2\mu^2$ —($E(X) = \mu$)

$= E(X^2) - \mu^2$

$Var(X) = E(X^2) - [E(X)]^2$

Result 2:

$Var(aX+b) = a^2 \cdot Var(X)$ *

where a, b are constants.

Proof:

Let $Y = aX + b$ —(i)

$\therefore E(Y) = E(aX+b)$

$= E(aX) + E(b)$

$= aE(X) + b$ —(ii)

taking (i) − (ii)

$Y - E(Y) = aX + \cancel{b} - aE(X) - \cancel{b}$

$= a(X - E(X))$

squaring and take 'E' on both side, we get

$$E[(Y-E(Y))^2] = E[a^2[X-E(X)]^2]$$
$$= a^2 E[(X-\mu)^2]$$
$$Var(Y) = a^2 Var(X)$$

$$\therefore \quad Var(aX+b) = a^2 Var(X)$$

<u>Observations</u>:

- $Var(aX) = a^2 Var(X)$
- $Var(b) = 0$

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⇒ Mode

Mode is a value of X at which the probablity function $[P(X=x_i)$ of $f(X=x)]$ is maximum.

There may exist a unique mode or more than one mode also.

⇒ Co-variance

The co-variance of any two random variables X, Y is denoted by Cov(X, Y) and is defined as

$$cov(X,Y) = E\{[X-E(X)]\cdot[Y-E(Y)]\}$$

<u>Observations</u>:

$$cov(X,Y) = E\{[X-E(X)]\cdot[Y-E(Y)]\}$$
$$= E\{XY - XE(Y) - YE(X) + E(X)E(Y)\}$$
$$= E(XY) - E(Y)\cdot E(X) - E(X)\cdot E(Y) + E(X)E(Y)$$
$$= E(XY) - E(X)\cdot E(Y)$$

#Note:

(1) If X, Y are independent, then, covariance

$cov(X,Y) = 0$,

since $cov(x,y) = E(X \cdot Y) - E(X)E(Y)$

$= E(X) \cdot E(Y) - E(X)E(Y)$

$= 0$

-(since independent)

(2) $cov(aX, bY) = ab\, cov(X,Y)$

(3) $cov(X+a, Y+b) = cov(X,Y)$

(4) $Var(X \pm Y) = Var(X) + Var(Y) \pm 2\, cov(X,Y)$

Example 1:

Find $E(X)$, $E(X^2)$, mean(X), var(X), mode(X) $E(2X^2+3X+4)$ for following discrete probablity distribution

$X = x_i$	0	1	2
$P(X=x_i)$	1/4	2/4	1/4

Also find the probablity distribution, mean, and variance of random ~~ex~~ variable 'Y', where $Y = X^2 + 3X - 5$

Sol^n. The given probablity distribution is

$X = x_i$	0	1	2
$P(X=x_i)$	1/4	2/4	1/4

we know that, X is a discrete random variable, then

• $E(X) = \sum_{i=1}^{n} x_i P(x_i)$

$= (0 \times 1/4) + (1 \times 2/4) + (2/4)$

$= 1$

• $E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$

$= \cancel{\Sigma}$

$= (0^2 \times 1/4) + (1^2 \times 2/4) + (2^2 \times 1/4)$

$= 0 + 2/4 + 4/4$

$= 3/2$

- mean of X $= E(X) = 1$

- $var(X) = \sigma^2 = E[(X-\mu)^2]$ where $E(x) = \mu = 1$

$= E[(X-1)^2]$

$= \sum_{i=1}^{n} (x_i - 1)^2 P(x_i)$

$= (0-1)^2(1/4) + (1-1)^2(2/4) + (2-1)^2(1/4)$

$= 1/4 + 0 + 1/4$

$= 1/2$

< Observation:- $var(X) = E(X^2) - [E(X)]^2$

$= 3/2 - (1)^2$

$= 3/2 - 1$

$= 1/2$ >

- Mode

By defination, it is clear that there exists a unique mode at $x = 1$

- $E(2X^2+3X+4) = \sum_{i=1}^{n} (2x_i^2+3x_i+4) P(x_i)$

$= (2(0)^2+3(0)+4)(1/4)$

$+ (2(1)^2+3(1)+4)(2/4) + (2(2)^2+3(2)+4)(1/4)$

$= (4 \times 1/4) + (9 \times 2/4) + (18 \times 1/4)$

$= 40/4 = 10$

< observation: $E(2X^2+3X-4) = 2E(X^2)+3E(X)+4$

$= 2(3/2) + 3(1) + 4$

$= 10$ >

- The probablity distribution of $Y = X^2 - 3X - 5$ is given by

$X = x_i$	0	1	2
$Y = X^2+3X-5$	-5	-1	5
$P(Y) = P(X)$	$1/4$	$2/4$	$1/4$

- $E(Y) = \text{mean of } Y = \sum_{i=1}^{n} y_i P(y_i)$

$$= (-5)(1/4) + (-1)(2/4) + (5)(1/4)$$
$$= -1/2$$

- $\text{var}(Y) = E(Y^2) - [E(Y)]^2$

$$E(Y^2) = \sum_{i=1}^{n} y_i^2 P(y_i)$$
$$= (-5)^2(1/4) + (-1)^2(2/4) + (5)^2(1/4)$$
$$= 25/4 + 2/4 + 25/4$$
$$= 52/4 = 13$$

$$\therefore \text{Var}(Y) = E(Y^2) - [E(Y)]^2$$
$$= 13 - [-1/2]^2$$
$$= 13 - 1/4$$
$$= 51/4$$

Example 2.

The probablity density function of a random variable 'X' is given by $f(X) = x/2 \quad 0 \leq x \leq 2$; then find
- expected value of X,
- expected value of $3X^2 - 2X$ and
- standard deviation.

Solⁿ : The probablity density function of continuous random variable 'x' is given by

$$f(x) = \begin{cases} x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Also, we know that, for continuous random variable 'X', $E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$$= \int_{-\infty}^{0} x \cdot f(x)\, dx + \int_{0}^{2} x \cdot f(x)\, dx + \int_{2}^{\infty} x \cdot f(x)\, dx$$

$$= 0 + \int_{0}^{2} x \cdot (x/2)\, dx + 0$$

$$= 1/2 \left[x^3/3 \right]_0^2 = 1/2 \left[8/3 - 0/3 \right]$$

$$= 8/6 = 4/3$$

Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \int_0^2 x^2 (x/2)\, dx = 1/2 \int_0^2 x^3\, dx$$

$$= 1/2 \cdot 1/4 \left[x^4 \right]_0^2 = 1/8 \left[16 - 0 \right]$$

$$= 16/8 = 2$$

Also, $E(3X^2 - 2X) = 3E(X^2) - 2E(X)$

$$= 3(2) - 2(4/3)$$

$$= 6 - 8/3 = 10/3$$

Variance of $X = \sigma_x^2 = E(X^2) - [E(x)]^2$

$$= 2 - (4/3)^2$$

$$= 2 - 16/9 = 2/9$$

Standard deviation of $X = \sigma = \sqrt{Var(X)}$

$$= \sqrt{2/9} = \sqrt{2}/3$$

⇒ Probablity Distributions.

Random Variable

In general, a random variable means a real number associated with outcomes of a random experiment.

Example: Consider a random experiment, "tossing of two coins." we consider the random variable 'X' which is the no. of heads when tossing two coins.

Clearly, outcomes = {HH, HT, TH, TT}

Values of X = {2, 1, 1, 0}

$x \to 0, 1, 2$

- **Discrete Random Variable**

If a Random Variable 'X' assumes discrete values, then, it is known as discrete random variable.

- **Probablity mass function**

Suppose, 'X' is a discrete random variable which assumes the discrete values, $x_1, x_2, x_3, \dots, x_n$; with corresponding probablities $P(X=x_1), P(X=x_2), \dots P(X=x_n)$

OR $P(x_i)$ $i = 1, 2, 3, \dots, n$

Then, the probablity function $P(x_i)$ is said to be probablity mass function OR discrete probablity function, if it satisfies the following two conditions.

i) $0 \leq P(x_i) \leq 1 \quad \forall i$

ii) $\sum_{i=1}^{n} P(x_i) = 1$

then, the probablity distribution of discrete random variable 'X' is given by

$\{x_i, P(x_i)\}$ $i = 1,2,3,\cdots,n$

OR.

$X = x_i$	x_1	x_2	x_3	$\cdots$	x_n
$P(X=x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$		$P(x_n)$

⇒ Cummulative Probablity Function

OR

Distributive Probablity function of discrete random variable.

If 'X' is a discrete random variable which can take the discrete values, $x_1, x_2, x_3 \cdots x_n$ with correspond probablities $P(x_1), P(x_2), P(x_3) \cdots P(x_n)$, then the cummulative probablity function OR distributive probablity function 'X' is denoted by $F(X=x_i)$ and is defined as $F(X=x_i) \overset{=}{\leqslant} P(X \leqslant x_i)$. i.e.

$$F(X=x_i) = P(X \leqslant x_i) = P(X=x_i) + P(X=x_2) \cdots \cancel{P(X=x_i)} \cdots P(X=x_i)$$

$$F(X=x_{i-1}) = P(X \leqslant x_{i-1}) = P(X=x_1) + P(X=x_2) + \cdots + P(X=x_{i-1})$$

$$\therefore \; P(X=x_i) = F(X=x_i) - F(X=x_{i-1})$$

Example:

$X = x_i$	0	1	2	3
$P(X=x_i)$	1/8	3/8	3/8	1/8

Clearly, the above example is a discrete probablity distribution; since, it satisfies both conditions $0 \leqslant P(X=x_i) \leqslant 1$ and $\sum_{i=1}^{n} P(X=x_i) = 1$

Also, the corresponding cummulative probablity function or probablity distribution function is given by $F(X=0) = P(X \leqslant 0) = 1/8 = P(X=0)$

$F(X=1) = P(X \leqslant 1) = P(X=1) + P(X=0) = 3/8 + 1/8 = 4/8$

$F(X=2) = P(X \leqslant 2) = P(X=2) + P(X=1) + P(X=0) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$

and $F(X=3) = P(X \leq 3) = P(X=3) + P(X=2) + P(X=1) + P(X=0)$

$F(X=3) = 1/8 + 3/8 + 3/8 + 1/8 = 8/8 = 1$

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$X = x_i$	0	1	2	3
$P(X=x_i)$	1/8	3/8	3/8	1/8
$F(X=x_i)$	1/8	4/8	7/8	1

⇒ Continuous Random Variable.

A random variable 'X' is said to be continuous, if it can take all possible values between certain limits. In other words, a random variable 'X' is said to be continuous when it's different values cannot be put in one-one correspondance with a set of positive integers.

⇒ Probablity Density function

The probablity density function, of a continuous random variable 'X' is denoted by $F(X=x)$ or $f(x)$ in simple way; and it satisfies following two properties.

- $0 \leq f(x) \leq 1$ for $\forall x \in R$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

⇒ Continuous Cummulative Probablity function OR Continuous Probablity Distribution Function

If 'X' is a continuous random variable with probablity density function $f(x)$, then cummulative probablity function, or probablity distribution function is denoted by $F(X=x)$,

$$F(X=x) = \int_{-\infty}^{x} f(x)\,dx$$

Example 1

A student is to match three historical events, ie. Mahatma Gandhi's Birthday, Indian Independance ~~Day~~, and first world war with 3 years, ie 1947, 1914, 1869. If he guesses with no knowledge of correct answers, what is the probablity distribution of the number of answers he gets correclty. Also find the corresponding probablity distribution function.

Sol? Let 'X' denote the no. of correct matches done by the student. Clearly, X will take values {0, 1, 2, 3}

The possible matching's are as follows.

1869	1914	1947	X
G	I	W	1
G	W	I	3
I	G	W	0
I	W	G	1
W	G	I	1
W	I	G	0

G ≡ Gandhi's Birthday
I ≡ Independence
W ≡ World war I

Clearly, 'X' is a discrete random variable, and corresponding probablity mass function is given by

$P(X=0) = 2/6$ $\quad$ $P(X=2) = 0$

$P(X=1) = 3/6$ $\quad$ $P(X=3) = 1/6$

Therefore, the probablity distribution, and distributive function of 'X' are given by

$X = x_i$	0	1	2	3
$P(X=x_i)$	2/6	3/6	0	1/6
$F(X=x_i)$	2/6	5/6	5/6	6/6 = 1

Example 2

Let 'X' be a random variable such that
$P(X=-2) = P(X=-1)$; $P(X=2) = P(X=1)$
and $P(X>0) = P(X<0) = P(X=0)$
Then, obtain probablity mass function of 'X' and it's distributive function.

Sol^n.

Clearly, X will take the values

$-2, -1, 0, 1, 2$,

ie X is a discrete random variable.

Given that,

$P(X=-2) = P(X=-1) = a$ (say)

$P(X=2) = P(X=1) = b$ (say)

$P(X>0) = P(X<0) = P(X=0)$

$P(X=1)+P(X=2) = P(X=-1)+P(X=-2) = P(X=0)$

$a + a = b + b = P(X=0)$

$2a = 2b = P(X=0)$

$\therefore a=b$ and $P(X=0) = 2b$
$= 2a$

Also, we know that, total probablity $=1$

$P(X=-2)+P(X=-1)+P(X=0)+P(X=1)+P(X=2) = 1$

$a+a+a+a+2a = 1$

$\therefore 6a = 1$

$a = 1/6 = 1/6$

$\therefore$ The probablity mass functions are given by

$P(X=-2) = 1/6$ $\quad$ $P(X=2) = 1/6$

$P(X=-1) = 1/6$ $\quad$ $P(X=1) = 1/6$

$P(X=0) = 2/6$

Therefore, the probablity distribution of 'X' is given by

$X = x_i$	-2	-1	0	1	2
$P(X = x_i)$	$1/6$	$1/6$	$2/6$	$1/6$	$1/6$
$F(X = x_i)$	$1/6$	$2/6$	$4/6$	$5/6$	$6/6 = 1$

Example 3

Let $P(x)$ be the probablity function of a ~~de~~ discrete random variable 'X' which assumes the values x_1, x_2, x_3, x_4 ; such that $2P(x_1) = 3P(x_2) = P(x_3) = 5P(x_4)$ find the probablity distribution and cummulative probablity function.

Sol^n

Let $2P(x_1) = 3P(x_2) = P(x_3) = 5P(x_4) = A$.

$\therefore$ $P(x_1) = A/2$ $\quad$ $P(x_2) = A/3$ $\quad$ $P(x_3) = A$

$P(x_4) = A/5$

Also, we know that, total probablity is 1

$\therefore$ $P(x_1) + P(x_2) + P(x_3) + P(x_4) = 1$

$A/2 + A/3 + A + A/5 = 1$

$A(1/2 + 1/3 + 1 + 1/5) = 1$

$\therefore$ $A \cdot (15/30 + 10/30 + 30/30 + 6/30) = 1$

$A(61/30) = 1$

$\therefore$ $A = 30/61$

Therefore, the probablity distribution of 'X' and distribution function of 'X' is given by

$X = x_i$	x_1	x_2	x_3	x_4
$P(X = x_i)$	$15/61$	$10/61$	$30/61$	$6/61$
$F(X = x_i)$	$15/61$	$25/61$	$55/61$	$61/61 = 1$

Example 4

Let 'X' be a continuous random variable, with probablity density function $f(x)$; where

$$f(x) = \begin{cases} ax & 0 \leq x \leq 1 \\ a & 1 \leq x \leq 2 \\ -ax+3a & 2 \leq x \leq 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Then, (i) find 'a' (ii) Compute $F(x = 1.5)$

Sol?

We know that, for a continuous random variable X, total probablity = 1

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (-ax+3a)dx + 0 = 1$$

$$\frac{a[1-0]^2}{2} + a[2-1] + \left[\frac{-ax^2}{2} + 3ax\right]_2^3 + 0 = 1$$

$$\frac{a}{2} + a \left[-a\left(\frac{9}{2}\right) + 3a(3) + a\left(\frac{4}{2}\right) - 3a(2)\right] = 1$$

$$\frac{a}{2} + a - \frac{9a}{2} + 9a + 2a - 6a = 1$$

$$\frac{a}{2} + \frac{2a}{2} - \frac{9a}{2} + \frac{18a}{2} + \frac{4a}{2} - \frac{12a}{2} = 1$$

$$\frac{4a}{2} = 1 \qquad a = 1/2$$

Also,

$$F(X = x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{at } x = 1.5$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$$

$$= 0 + \int_0^1 ax \cdot dx + \int_1^{1.5} a\, dx$$ (17)

$$= 0 + a\left[\frac{1}{2}\right] + a\left[\frac{1.5}{-1}\right]$$

$$= a/2 + (3a/2 - a/2) = a/2 + a/2$$

$F(X=1.5) = a = 1/2$.

Example 5

The amount of bread in hundreds of pounds, 'X', that a certain bakery has available to sale in a day, is found to be a numerical value random phenomenon, with a function $f(x)$, where

$$f(x) = \begin{cases} Ax & 0 \leq x \leq 5 \\ A(10-x) & 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

(i) find 'A' such that $f(x)$ is a probablity density function

(ii) what is the Probablity that the number of pounds of bread that will be sold tomorrow, is a) more than 500 pounds
b) less than 500 pounds
c) between 250 → 750 pounds

Soln.

Let 'X' denote the amount of bread in hundreds of pounds.

(i)

we know that, for continuous random variable 'X' total probablity = 1

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{5} f(x)\,dx + \int_{5}^{10} f(x)\,dx + \int_{10}^{\infty} f(x)\,dx = 1$$

$$0 + \int_0^5 Ax\,dx + \int_5^{10} A(10-x)\,dx + 0 = 1$$

$$A\left[x^2/2\right]_0^5 + A\left[10x - x^2/2\right]_5^{10} = 1$$

$$A\left[25/2\right] + A\left[100 - 100/2 - 50 + 25/2\right] = 1$$

$$\frac{25A}{2} + \frac{25A}{2} = 1 \qquad 25A = 1$$

$$\therefore A = 1/25$$

(ii)

a) $$P(x>5) = \int_5^{\infty} f(x)dx$$

$$= \int_5^{10} A(10-x)dx + \int_{10}^{\infty} f(x)dx$$

$$= A\left[10x - x^2/2\right]_5^{10}$$

$$= A\left[100 - 100/2 - 50 + 25/2\right]$$

$$= A\left[25/2\right]$$

$$= \frac{1}{25} \times \frac{25}{2}$$

$$\therefore P(x>5) = 1/2$$

b) $$P(x<5) = 1 - P(x>5)$$

$$= 1 - 1/2$$

$$= 1/2$$

< can also be obtained by $\int_{-\infty}^{5} f(x)dx$ >

c) $$P(2.5 \leq x \leq 7.5) = \int_{2.5}^{7.5} f(x)dx$$

$$= \int_{2.5}^{5} f(x)dx + \int_{5}^{7.5} f(x)dx$$

$$= \int_{2.5}^{5} A(x)dx + \int_{5}^{7.5} A(10-x)dx$$

$$= A\left[\frac{x^2}{2}\right]_{2.5}^{5} + A\left\{10\left[x\right]_5^{7.5} - \left[\frac{x^2}{2}\right]_5^{7.5}\right\}$$

$$= \frac{A}{2}(5^2 - 2\cdot 5^2) + A\left\{10(7.5-5) - \frac{1}{2}(7.5^2 - 5^2)\right\}$$

$$= \frac{A}{2}(18.75) + A(25 - 31.25/2)$$ 19

$$= \frac{37.5A}{2} = \frac{37.5}{2} \times \frac{1}{25} = 0.75$$

$$\therefore P(2.5 \leq x \leq 7.5) = 0.75$$

20th August, 2015

Example 6

The probablity density function of a continuous Random variable 'X' is given by $f(x) = \sin x \quad x \in [0, \pi/2]$, then find mean, variance, mode.

Solⁿ

Consider $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$ — (x is continuous)

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\pi/2} x f(x)\,dx + \int_{\pi/2}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\pi/2} x \cdot \sin x\,dx$$

$$= \left[x(-\cos x) - \int (1)(-\cos x)\,dx\right]_0^{\pi/2}$$

$$= \left[-x\cos x + \sin x\right]_0^{\pi/2}$$

$$= [0+1] - [-0 \cancel{+ \pi/2}]$$

$$= 1$$

Similarly $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_0^{\pi/2} x^2 \sin x = \int_0^{\pi/2}\left[(x^2(-\cos x) \mp \int (2x)(-\cos x)\,dx\right]$$

$$= \left[-x^2\cos x + 2\{(x)(\sin x) \mp \int (1)\sin x\,dx\}\right]_0^{\pi/2}$$

$$= \left[-x^2\cos x + 2\{x\sin x + \cos x\}\right]_0^{\pi/2}$$

$$= \left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi/2} = \pi - 2$$

Therefore,

Mean $= E(X) = 1$

Variance $= E(X^2) - [E(X)]^2$

$= (\pi - 2) - (1)^2$

$= \pi - 3$

Mode $\Rightarrow$ $f(x) = \sin x$, $x \in [0, \pi/2]$

$f'(x) = \cos x$

$f''(x) = -\sin x$

Then, $f'(x) = 0$

$\therefore \cos x = 0 \quad \therefore x = \pi/2$

Now, $f''(x)\Big|_{x=\pi/2} -\sin \pi/2 = -1 < 0$

$\therefore f(x)$ has maxima at $x = \pi/2$

$\therefore$ There exists a unique mode at $x = \pi/2$

Note:

Median : Median is the value of x at which total probablity is splitted into exactly two equal parts.

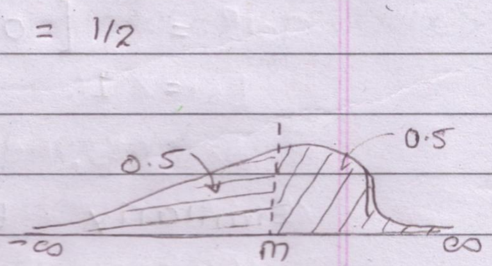
further, we know that,

Total area = Total probablity $= 1 = \int_{-\infty}^{\infty} f(x)\,dx$

There exists $x = m$ such that

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = 1/2$$

$$\boxed{\int_{-\infty}^{m} f(x)\,dx = 1/2.}$$



Then, $x = m$ is known as median.

example:: from above example,

$f(x) = \sin x \quad x \in [0, \pi/2]$

$$\int_{-\infty}^{m} f(x)\,dx = 1/2$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{m} \sin x\,dx = 1/2.$$

$m \in [0, \pi/2]$

~~Exa~~ $\left[-\cos x\right]_0^m = 1/2$

$\therefore \quad -\left[\cos m - 1\right] = 1/2$

$\therefore \cos m = -1/2 + 1$

$\cos m = 1/2$

$\therefore m = \pi/3 \; (= 60^\circ)$

$\therefore$ median $m = \pi/3$

$\Rightarrow$ Special Probablity Distributions.

Z - Transforms

Let u_n be a function defined for positive discrete values ie; $n = 0, 1, 2, 3, \dots$ then the Z-Transform of u_n is ~~defined~~ denoted by $Z\{u_n\}$ and defined as

$$\boxed{Z\{u_n\} = \sum_{n=0}^{\infty} u_n \cdot z^{-n} = \bar{u}(z)}$$

Provided the R.H.S. infinite series Converges.

The inverse Z transform is defined as

$$\boxed{Z^{-1}\{\bar{u}(z)\} = u_n}$$

Ex ①: $Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$

$$\cancel{= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}}$$

$$= \frac{1}{0!} z^{-0} + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \frac{1}{4!} z^{-4} + \dots$$

$$= 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{4!}\left(\frac{1}{z}\right)^4 + \dots$$

$$= 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \frac{(1/z)^4}{4!} + \dots$$

$$= e^{1/z} \qquad \left[\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right]$$

$$\therefore Z\left\{\frac{1}{n!}\right\} = e^{1/z}$$

$$\text{Illy}, \quad Z^{-1}\{e^{1/z}\} = \frac{1}{n!}$$

Note:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

Some Properties of Z-Transforms:

①. If c is any constant then $Z\{c \cdot u_n\} = c \cdot Z\{u_n\}$

②. If c_1, c_2 are any two constants and u_n, v_n are any two functions defined for $n = 0, 1, 2, 3, \ldots$ then

$$Z\{c_1 u_n \pm c_2 v_n\} = c_1 Z\{u_n\} \pm c_2 Z\{v_n\}$$

③. $$Z\{a^n\} = \frac{z}{z-a}$$

Proof: $$Z\{a^n\} = \sum_{n=0}^{\infty} a^n \cdot z^{-n} \quad [\because \text{By def.}]$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \left[1 - \frac{a}{z}\right]^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1} = \frac{z}{z-a}$$

$$\therefore \boxed{Z\{a^n\} = \frac{z}{z-a}}$$

Ex ①: $Z\{1\} = Z\{1^n\} = \dfrac{z}{z-1}$ $[\because a = 1 \text{ in the above}]$

②. $Z\{2^n\} = \dfrac{z}{z-2}$ $[\because a = 2]$

③. $Z\{e^{-an}\} = Z\{(e^{-a})^n\} = \dfrac{z}{z-e^{-a}}$ $[\because (x^m)^n = x^{mn}$

Obs: $$Z[e^{-an}] = \sum_{n=0}^{\infty} e^{-an} \cdot z^{-n} \quad [\because \text{By def.}]$$

$$= \sum_{n=0}^{\infty} (e^{-a})^n \cdot (z^{-1})^n$$

$$= \sum_{n=0}^{\infty} (e^{-a} \cdot z^{-1})^n = 1 + (e^{-a} \cdot z^{-1}) + (e^{-a} \cdot z^{-1})^2 + (e^{-a} \cdot z^{-1})^3 + \cdots$$

$$= [1 - e^{-a} z^{-1}]^{-1} = \left[1 - \frac{e^{-a}}{z}\right]^{-1} = \left[\frac{z - e^{-a}}{z}\right]^{-1} = \frac{z}{z - e^{-a}}$$

Result ⓞ: (a). $Z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$ ③

(b). $Z\{\sin n\theta\} = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

Proof: we know that

$Z\{a^n\} = \dfrac{z}{z-a}$, Put $a = e^{i\theta}$ on both sides

$\Rightarrow Z\{e^{i\theta n}\} = \dfrac{z}{z-e^{i\theta}}$

$\Rightarrow Z\{\cos n\theta + i\sin n\theta\} = \dfrac{z}{z-(\cos\theta+i\sin\theta)}$

$\Rightarrow Z\{\cos n\theta\} + i\,Z\{\sin n\theta\} = \dfrac{z}{(z-\cos\theta)-i\sin\theta} \times \dfrac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$

$= \dfrac{z(z-\cos\theta)+iz\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$ $[\because i^2=-1]$

$= \dfrac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}$

$= \dfrac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$

$= \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\,\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

Campare real and imaginary parts on both sides, we get

$Z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$Z\{\sin n\theta\} = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

Result ②: $Z\{n^p\} = -z \dfrac{d}{dz} Z\{n^{p-1}\}$, where p being a positive number.

Proof:

$$Z\{n^p\} = \sum_{n=0}^{\infty} n^p \cdot z^{-n} \qquad [\because \text{By def. of Z Transform}]$$

$$= \sum_{n=0}^{\infty} n \cdot n^{p-1} \cdot z \cdot z^{-n-1}$$

$$= z \sum_{n=0}^{\infty} n \cdot n^{p-1} \cdot z^{-n-1} \longrightarrow (i)$$

lly,

$$Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} \cdot z^{-n} \qquad [\because \text{Again, by def.}]$$

diff. w.r.to z on both sides,

$$\Rightarrow \frac{d}{dz} Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} \cdot (-n) \cdot z^{-n-1}$$

$$= -\sum_{n=0}^{\infty} n \cdot n^{p-1} \cdot z^{-n-1}$$

$$= -\frac{1}{z} \cdot Z\{n^p\} \qquad [\because \text{use equ. } (i)]$$

$$\Rightarrow \therefore \boxed{Z\{n^p\} = -z \cdot \frac{d}{dz} Z\{n^{p-1}\}}$$

Ex ①: Show that $Z\{n\} = \dfrac{z}{(z-1)^2}$

$$Z\{n^2\} = \frac{z^2+z}{(z-1)^3}$$

$$Z\{n^3\} = \frac{z^3+4z^2+z}{(z-1)^4}$$

Sol: we know that,

$$Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\} \longrightarrow (*)$$

Put $p=1$ in equ. $(*)$, we get

$$Z\{n\} = -z \frac{d}{dz} Z\{1\} = -z \frac{d}{dz}\left(\frac{z}{z-1}\right) \qquad \Big|\because Z\{1\} = \frac{z}{z-1}$$

5

$$= -z \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z \cdot \frac{(z-1)\cdot 1 - z(1)}{(z-1)^2}$$

$$= -z \cdot \frac{\cancel{z}-1-\cancel{z}}{(z-1)^2} = \frac{z}{(z-1)^2}$$

$$\therefore Z\{n\} = \frac{z}{(z-1)^2}$$

Again, put $p=2$ in eqn. (*), we get

$$Z\{n^2\} = -z \cdot \frac{d}{dz} Z\{n\}$$

$$= -z \cdot \frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\} \quad [\because \text{previous answer}]$$

$$= -z \cdot \frac{(z-1)^2 \cdot 1 - z \cdot 2(z-1)}{(z-1)^4}$$

$$= -z \; \frac{(z-1)\,[z-1-2z]}{(z-1)^4}$$

$$= -z \cdot \frac{(z-1)(-1-z)}{(z-1)^4} = -z \cdot \frac{\cancel{(z-1)}(z+1)(-1)}{(z-1)^{\cancel{4}\,3}}$$

$$\cancel{= -z \cdot \frac{(z^2-1)(-1)}{(z-1)^4}} = \cancel{\frac{z(z^2-1)}{(z-1)^4}}$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\therefore Z\{n^2\} = \frac{z^2+z}{(z-1)^3}$$

Ily, we can prove $Z\{n^3\} = \dfrac{z^3+4z^2+z}{(z-1)^4}$

and $Z\{n^4\} = \dfrac{z^4+11z^3+11z^2+z}{(z-1)^5}$

Result ③: If $Z\{u_n\} = \bar{u}(z)$ then i) $Z\{a^{-n}.u_n\} = \bar{u}(az)$

ii) $Z\{a^{n}.u_n\} = \bar{u}(z/a)$

Proof: we know that, by def. of Z-Transformation,

$$Z\{u_n\} = \sum_{n=0}^{\infty} u_n . z^{-n} = \bar{u}(z)$$

$$\Rightarrow Z\{a^{-n}.u_n\} = \sum_{n=0}^{\infty} (a^{-n} u_n) . z^{-n}$$

$$= \sum_{n=0}^{\infty} u_n . (a^{-n} z^{-n})$$

$$= \sum_{n=0}^{\infty} u_n . (az)^{-n}$$

$$\therefore \boxed{Z\{a^{-n}. u_n\} = \bar{u}(az)}$$

$$\text{lly} \quad \boxed{Z\{a^{n}. u_n\} = \bar{u}\left(\frac{z}{a}\right)}$$

Ex: i) $Z\{n\} = \dfrac{z}{(z-1)^2} = \bar{u}(z)$ say

then $Z\{a^n.n\} = \bar{u}\left(\dfrac{z}{a}\right) = \dfrac{z/a}{(z/a - 1)^2} = \dfrac{z/a}{\left(\frac{z-a}{a}\right)^2} = \dfrac{z}{\cancel{a}} . \dfrac{a^{\cancel{2}}}{(z-a)^2}$

$$= \frac{az}{(z-a)^2}$$

lly Try for the following:

ii) $Z\{a^n . n^2\} = \dfrac{az^2 + a^2 z}{(z-a)^3}$

iii) $Z\{a^n . \cos n\theta\} = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$

$Z\{a^n \sin n\theta\} = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$