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1.1 Series de Fourier

Primero, queremos mostrar que la sumatoria

$$\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

converge uniformemente. Para esto necesitamos que

$$|g_n(t)| \leq M_n \quad \forall n \geq 1$$

donde

$$\sum_{n=1}^{\infty} M_n \text{ converge.}$$

Si tomamos

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \text{ converge}$$

Se tiene que

$$f(t) = \sum_{n=-\infty}^{0} C_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{in\omega_0 t}$$

$$= \sum_{n=0}^{\infty} C(-n) e^{-in\omega_0 t} + \sum_{n=y}^{\infty} C_n e^{in\omega_0 t}$$

$$= C_0 + \sum_{n=y}^{\infty} C(-n) e^{-in\omega_0 t} + \sum_{n=y}^{\infty} C_n e^{in\omega_0 t}$$

Entonces $g_n(t) = C_n e^{in\omega_0 t} \rightarrow |g_n(t)| = |C_n|$

Pero sabemos que $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$, que se trata de una integral convergente y acotada.

Por el teorema de Parseval

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt \leq \left| \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt \right|$$

$$\leq \frac{1}{T}\int_{-T/2}^{T/2} |f(t)| \, dt \leq \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt$$

concluyendo que $\sum_{n=-\infty}^{\infty} |C_n|^2$ es convergente

$M_n = |C_n|^2 \to |C_n e^{i n w_0 t}| \leq |C_n|^2 \Rightarrow$ Converge

$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + b_n \sin(n w_0 t)$

$$\frac{d}{dt} f(t) = \sum_{n=1}^{\infty} \frac{d}{dt} (a_n \cos(n w_0 t) + b_n \sin(n w_0 t))$$

$$= \sum_{n=1}^{\infty} n w_0 (-a_n \sin(n w_0 t) + b_n \cos(n w_0 t))$$ $\blacksquare$

$$\Rightarrow \int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} \left(\sum_{n=1}^{\infty} (a_n \cos(n w_0 t) + b_n \sin(n w_0 t) + \frac{a_0}{2} \right) dt$$

$$\int_{t_1}^{t_2} f(t) = \frac{a_0}{2}(t_2 - t_1) + \sum_{n=1}^{\infty} \frac{1}{n w_0} \left(-b_n (\cos(n w_0 t_2) - \cos(n w_0 t_1)) + a_n (\sin(n w_0 t_2) - \sin(n w_0 t_1))\right)$$ $\blacksquare$

1.2 Presentación de funciones

$f(t) = t \; ; \; (-\pi, \pi) ; \; f(t + 2\pi) = f(t)$

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{\pi}\frac{t^2}{2}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos\left(\frac{n\pi t}{\pi}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt)\,dt$

$= \frac{1}{\pi}\left(\frac{1}{n}t\,\text{sen}(nt)\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\text{sen}(nt)\,dt\right)$

$= \frac{1}{\pi n^2}\left(\cos(n\pi) - \cos(n\pi)\right) = 0$

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,\text{sen}\left(\frac{n\pi t}{\pi}\right)dt$

$= \frac{1}{\pi}\left(-\frac{1}{n}t\cos(nt)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nt)\,dt\right.$

$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos(n\pi) + \cancel{\frac{1}{n^2}\text{sen}(n\pi)} - \cancel{\frac{1}{n^2}\text{sen}(-n\pi)}\right)$

$\cos(n\pi) = (-1)^n$

$= -\frac{2}{n}(-1)^n$

$\Rightarrow f(t) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\text{sen}(nt)$

1.3 Función $\zeta(s)$ de Riemann

1. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2$

$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} t^2\cos(n w_0 t)\,dt$

$= \frac{1}{\pi}\left[t^2\frac{1}{n}\text{sen}(nt) - \frac{2}{n}\int_{-\pi}^{\pi}\text{sen}(nt)t\,dt\right]$

$= \frac{1}{n\pi}\left[t^2\text{sen}(nt) + \frac{2}{n}t\cos(nt) - \frac{2}{n^2}\text{sen}(nt)\right]\Big|_{-\pi}^{\pi}$

$$= \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \operatorname{sen}(nt)\,dt = 0$$

función par → Función impar

$$\Rightarrow f(t) = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt)$$

$$\int_{-\pi}^{\pi} f(t) = \int_{-\pi}^{\pi} \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{-1^n}{n^2}\int_{-\pi}^{\pi}\cos(nt) \to 0$$

$$\Rightarrow \frac{1}{12}t(t^2-\pi^2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\sin(nt)$$

∴ Usando la identidad de Parseval

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{12}t(t^2-\pi^2)\right)^2 dt = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^6}$$