Homework 9

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Exercise 1

Prove that for every $\mathbf{A} \in \mathcal{M}$ *we have* $\|\mathbf{A}\| = \sqrt{\rho(\mathbf{A}'\mathbf{A})}$

Answer:

Before proving the result, let's introduce some notation. From Fact saw in class the spectral norm is equivalent to

$$\|A\| = \sup\left\{\|Ax\| : x \in B_{\mathbb{R}^n}\right\} \tag{1.1}$$

where $B_{\mathbb{R}^n} := \left\{x \in \mathbb{R}^n : \|x\| = 1\right\}$. Since,

- $x \mapsto \|Ax\|$ is continuous (composition of the norm, which is always continuous, with the linear map A that is always continuous in finite dimensional spaces)
- $B_{\mathbb{R}^n}$ is compact (closed and bounded on finite dimensional space)

we can rewrite (1.1) as alternative

$$\begin{aligned}
\|A\| &= \max\left\{\|Ax\| : x \in B_{\mathbb{R}^n}\right\} \\
&= \max_{\|x\|=1} \quad \left[(Ax)'(Ax)\right]^{0.5} \\
&= \max_{\|x\|=1} \quad \left[x'(A'A)x\right]^{0.5}
\end{aligned}$$

Consider the alternative problem, which has the same solution

$$\max_{\sum_i^n x_i^2 = 1} \quad x'(A'A)x$$

Since this is a constrained optimization problem, we can solve it using the Lagrangian method, which states that any solution must satisfy the following *foc*

$$(A'A)x = \lambda x$$

Notice that any *eigenvalue, eigenvector pair* (λ_i, x_i) of $A'A$ satisfy the *foc*. Pre-multiplying the last expression by x', we have

$$x'(A'A)x = \lambda x'x = \lambda \|x\|^2 = \lambda$$

So the actual solution is given by eigenvector x^* associated with the largest eigenvalue λ^* of $A'A$.[1] Moreover, the objective evaluated at x^* gives the desired result

$$\|A\|^2 = (\lambda^*) \Rightarrow \|A\| = \sqrt{\rho(A'A)}$$

Exercise 2

Making use of Gelfand's formula, show that if $\rho(\mathbf{A}) < 1$, then $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| < \infty$. In particular, show that under the stated condition there exists an $r < 1$ and $C \in \mathbb{N}$ such that $\|\mathbf{A}^k\| \leq r^k C \ \forall k \in \mathbb{N}$

Answer:

Gelfand's formula: $\|\mathbf{A}^k\|^{1/k} \to \rho(\mathbf{A})$

Let $0 < \epsilon < 0.5(1 - \rho(\mathbf{A}))$. From Gelfand's formula, we know that there exists $M \in \mathbb{N}$ such that

$$\begin{aligned} \|\mathbf{A}^k\|^{1/k} - \rho(\mathbf{A}) &\leq \epsilon \\ \|\mathbf{A}^k\|^{1/k} &\leq \underbrace{\epsilon + \rho(\mathbf{A})}_{\equiv r<1} \text{ for any } k \geq M \end{aligned}$$

From what we conclude $\|\mathbf{A}^k\| \leq r^k \ \forall k \in \mathbb{N}$ such that $k \geq M$. But that observation allow us to bound the entire sequence $\left(\|A^k\|\right)$ by

$$\|A^n\| \leq \frac{\max_{i \in \{1,\dots,M\}} \|A^i\|}{r^M} r^n = \frac{\|A^{i^*}\|}{r^M} r^n$$

To check it, note that

- if $n < M$ then

$$\|A^n\| \leq \|A^{i^*}\| \leq \|A^{i^*}\| \frac{r^n}{r^M}$$

[1] Here since all eigenvalues are real positive numbers - $A'A$ is symmetric and positive semi-definite - $\rho(A'A) = \lambda^* := \max_{i \in \{1,\dots,n\}} \lambda_i = \rho(A'A)$

- if $n \geq M$ then

$$\|A^n\| \leq r^n \leq \frac{\|A^{i^*}\|}{r^M} r^n$$

since $\frac{\|A^{i^*}\|}{r^M} \geq \frac{\|A^{i^*}\|}{r^{i^*}} > 1$, otherwise $M = 1$

Putting $C := \frac{\|A^{i^*}\|}{r^M}$, we have our desired result, namely

$$\|A^k\| \leq Cr^k \qquad \forall k \in \mathbb{N}$$

This implies that series resulting from the sequence $\left(\|A^k\|\right)$ converges, since $\sum_{i=1}^{m} \|A^k\|$ is a increasing sequence bounded above by

$$\sum_{k=0}^{\infty} \|\mathbf{A}^k\| \leq \sum_{k=0}^{\infty} Cr^k = \frac{C}{1-r} < \infty$$

Exercise 3

Show that if $\rho(\mathbf{A}) < 1$*, then* $\|\mathbf{A}^k\| \to 0$ without *using Gelfrand's formula. Assume that* $\mathbf{A}$ *is diagonalizable.*

Answer:

Suppose A is diagonalizable, i.e., it can be written on the form $P = P^{-1}\Lambda P$, where Λ is a diagonal $n \times n$ matrix with the eigenvalues of A as its entries and P is a nonsingular $n \times n$ matrix consisting of the eigenvectors corresponding to the eigenvalues in Λ.

From that decomposition, it is readily seen that $A^k = P^{-1}\Lambda^k P$. By submultiplicative property of the spectral norm we have $\|A^k\| \leq \|P^{-1}\|\|\Lambda^k\|\|P\|$. So we are done if we can show that $\|\Lambda^k\| \to 0$.

Applying the fact proved in Question 1 to the case of diagonal matrix, we have

$$\begin{aligned}
\|\Lambda^k\| &= \sqrt{\rho(\Lambda^{2k})} \\
&= \sqrt{\rho(\Lambda)^{2k}} \\
&= \rho(\Lambda)^k = \rho(A)^k \to 0
\end{aligned}$$

where second line uses the fact that eigenvalues of diagonal matrix are the entries itself, while the third comes from the diagonalization directly.

Exercise 4

Show that the set of nonnegative definite matrices is a closed subset of $\left(\mathcal{M}(n \times n), \|\cdot\|\right)$

Answer:

Let $\left(M_k\right)$ be a seq. of nonnegative definite matrices in $\left(\mathcal{M}(n \times n), \|\cdot\|\right)$ such that $M_k \to M$. We want to show that M is also a nonnegative definite matrix.

Let $x \in \mathbb{R}^n$ be an arbitrary vector and define $T_x : \mathcal{M}(n \times n) \to \mathbb{R}$ by $T_x(A) := x'Ax$. Therefore, we have $T_x(M_n) \geq 0 \quad \forall k \in \mathbb{N}$. If we can show that T_x is continuous we can conclude that $T_x(M) \geq 0$. Since x is arbitrary, that suffices to conclude that M is a nonnegative definite matrix, completing our proof.

Let x be as before and let $A_k \to A \in \mathcal{M}(n \times n)$.

$$\begin{aligned}
|T_x(A_k) - T_x(A)| &= |T_x(A_k - A)| \\
&= \langle x, (A_k - A)x \rangle \\
&\leq \|x\| \times \|(A_k - A)x\| \\
&\leq \|x\| \times \|(A_k - A)\| \|x\| \\
&= \|x\|^2 \|A_k - A\| \to 0
\end{aligned}$$

where first line follows for linearity, second from definition of T_x, third from Cauchy-Schwarz inequality, fourth from the definition of the spectral norm. Note that the norm $\|(A_k - A)x\|$ in third line is the usual euclidean norm, while $\|(A_k - A)\|$ in the fourth line is the spectral norm. The last line rearranges the terms and uses the fact that $A_k \to A$ with respect to the distance induced by spectral norm.

Exercise 5

Let $\mathbf{M}, \mathbf{A} \in \mathcal{M}(n \times n)$ *with* $\rho(\mathbf{A}) < 1$. *Let* $\mathbf{X}^*$ *be the unique solution to the Lyapunov equation*

$$X = AXA' + M$$

Show that

1. M *symmetric* $\Rightarrow$ X^* *symmetric*
2. M *nonnegative definite* $\Rightarrow$ X^* *nonnegative definite*
3. M *positive definite* $\Rightarrow$ X^* *positive definite*

Answer:

To prove (2) and (3) we are going to make use of the following theorem presented in lecture notes.

Theorem. *Let* $\mathbf{A}, \mathbf{M} \in \mathcal{M}(n \times n)$*. If* $\rho(\mathbf{A}) < 1$ *then the Lyapunov operator* $L : \mathcal{M}(n \times n) \to \mathcal{M}(n \times n)$ *defined by*

$$LX := AXA' + M \tag{5.1}$$

has a unique globally attracting fixed point X^**.*

1.

   Let X^* be solution of Lyapunov equation. Simple matrix algebra and the assumption that M is symmetric gives us that $(X^*)'$ is also a solution

   $$(X^*)' = (X^*A')'A' + M' = A(X^*)'A' + M$$

   However, we know from class that under the conditions stated on A, there exists a unique solution to the equation. Therefore $X^* = (X^*)'$

For (2), (3) we will use on top of theorem the following corollary

Corollary. *Let* L *satisfy the conditions of the theorem. If* S *is a closed subset of* $\mathcal{M}(n \times n)$ *and* $L(S) \subseteq S$*, then* $X^* \in S$*. Moreover, if* $L(S) \subseteq S' \subset S$*, then* $X^* \in S'$*.*

Proof. Let $X_0 \in S$ and consider the sequence $\left(L^n X_0\right)$. Since S is invariant under L, we have $\left(L^n X_0\right) \in S^\infty$. The theorem gives us that $L^n X_0 \to X^*$, but being S close we have $X^* \in S$. If is also true that L maps S to S', we have $X^* = LX^* \in S'$. $\square$

So all we need to check for (2), (3) is that (i) the set of nonnegatives definite matrices - which I will denote by $\mathcal{M}^+(n \times n) := \{A \in \mathcal{M}(n \times n) : A \succeq 0\}$ - is closed and (ii) that the Lyapunov operator maps the set of nonnegatives definite matrices into nonnegative definite/positive definite matrices.

2.

   First let show that $\mathcal{M}^+(n \times n)$ is closed. Let $x \in \mathbb{R}^n$ and consider the map $T_x : \mathcal{M}(n \times n) \to \mathbb{R}$ defined on question 4. Note that we can represent the set of nonneggative definite matrices as

   $$\mathcal{M}^+(n \times n) = \cap_{x \in \mathbb{R}^n} T_x^{-1}([0, \infty))$$

   which is closed. Why? $T_x^{-1}([0, \infty))$ is closed $\forall x \in \mathbb{R}^n$ by continuity of T_x and intersection of arbitrary number of closed sets is always closed.

So now we just need to show that $T_x(\mathcal{M}^+(n \times n)) \subseteq \mathcal{M}^+(n \times n)$ when $M \in \mathcal{M}^+(n \times n)$ and apply the corollary.

Let $X, M \in \mathcal{M}^+(n \times n)$ and consider and arbitrary $x \in \mathbb{R^n}$. Then

$$\begin{aligned} T_x(LX) &= x'(AXA' + M)x \\ &= (A'x)'X(A'x) + x'Mx \geq 0 \end{aligned}$$

Since both X, x are arbitrary, we have $LX \in \mathcal{M}^+(n \times n)$ which concludes our proof.

3.

For part (3), we only need to show that under the stronger assumption that $M \in \mathcal{M}^{++}(n \times n)$, our operator maps any nonneggative definite matrix to a positive definite matrix. Repeating the same step as in (2)

$$\begin{aligned} T_x(LX) &= x'(AXA' + M)x \\ &= \underbrace{(A'x)'X(A'x)}_{\geq 0} + \underbrace{x'Mx}_{>0} > 0 \end{aligned}$$

Again, since x is arbitrary, we have our desired result.