Computational Economics
HW9 – Solutions

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Exercise 1

CLAIM: For every $\mathbf{A} \in \mathcal{M}(n \times n)$ we have $\|\mathbf{A}\| = \sqrt{\rho(\mathbf{A}'\mathbf{A})}$.

PROOF:
Notice that in the definition of the spectral norm we can restrict our attention to vectors with unit norm without loss in generality[1]

$$\max\left\{\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} : \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}\right\} = \max\left\{\|\mathbf{A}\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\right\}$$

Moreover, since the quadratic function is a strictly monotone transformation on the positive orthant, it preserves the extrama and we can write the Lagrangian for the constraint maximization as follows

$$\max_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x}\|^2 - \lambda(\|\mathbf{x}\|^2 - 1)$$

where $\lambda \geq 0$ is the multiplier and the necessary first-order condition is

$$\begin{aligned} \mathbf{A}'\mathbf{A}\mathbf{x} - \lambda\mathbf{x} &= 0 \\ \mathbf{A}'\mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \end{aligned}$$

that is, the maximizer $(\mathbf{x}^*, \lambda^*)$ must be an eigenvector-eigenvalue pair of $\mathbf{A}'\mathbf{A}$. Moreover, premultiplying both sides by $\mathbf{x}'$ gives us

$$\begin{aligned} \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} &= \lambda\mathbf{x}'\mathbf{x} \\ \|\mathbf{A}\mathbf{x}\| &= \sqrt{\lambda} \end{aligned}$$

[1] From the norm properties and from $\mathbf{x} \neq \mathbf{0}$ it follows that $\|\mathbf{x}\| > 0$ and $\alpha(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} > 0$, hence $\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} = \alpha(\mathbf{x})\|\mathbf{A}\mathbf{x}\| = \|\mathbf{A}\alpha(\mathbf{x})\mathbf{x}\|$, where $\|\alpha(\mathbf{x})\mathbf{x}\| = 1$.

Notice that the LHS is the objective function of the definition and so λ^* must be the maximal eigenvalue of $\mathbf{A}'\mathbf{A}$ with the eigenvector $\mathbf{x}^*$ of unit length. Therefore

$$\|\mathbf{A}\| = \|\mathbf{A}\mathbf{x}^*\| = \sqrt{\lambda^*} = \sqrt{\rho(\mathbf{A}'\mathbf{A})}$$

Exercise 2

CLAIM: If $\rho(\mathbf{A}) < 1$, then there exists an $r < 1$ and a $C \in \mathbb{N}$ such that $\|\mathbf{A}^k\| \leq r^k C$ for all $k \in \mathbb{N}$. Consequently, $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| < \infty$.

PROOF:
Let $r \in (\rho(\mathbf{A}), 1) \cap \mathbb{R}$. According to Gelfand's formula, $\exists N_r \in \mathbb{N}$, s.t. $\forall n \geq N_r$, we have

$$\|\mathbf{A}^n\|^{\frac{1}{n}} \leq r \quad \Rightarrow \quad \|\mathbf{A}^n\| \leq r^n \leq r^{N_r}$$

Moreover, let C be defined as

$$C \equiv \min\left\{ z \in \mathbb{N} : z \geq \frac{1}{r^{N_r}} \max\left\{\|\mathbf{A}^j\| : j \in \{0, 1, 2, \ldots, N_r\}\right\}\right\}$$

then we have that

if $k \leq N_r$, then $\|\mathbf{A}^k\| \leq r^k C$, because $r^{k-N_r} \geq 1$

if $k \geq N_r$, then $\|\mathbf{A}^k\| \leq r^k C$, because $C \geq 1$

Finally, $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| < \infty$ follows from the fact that $\sum_{k=0}^{\infty} r^k C = C\frac{1}{1-r} < \infty$.

Exercise 3

CLAIM: If $\rho(\mathbf{A}) < 1$ and $\mathbf{A}$ is diagonalizable, then $\|\mathbf{A}^k\| \to 0$ as $k \to \infty$.

PROOF:
Since $\mathbf{A}$ is diagonalizable, there is an invertible matrix $\mathbf{P}$, such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}$, where the diagonal matrix $\mathbf{\Lambda}$ contains the eigenvalues of $\mathbf{A}$. Because $\rho(\mathbf{A}) < 1$, we know that $|\mathbf{\Lambda}_{ii}| < 1$, for all $i = 1, 2, \ldots, n$. This implies that

$$\mathbf{A}^k = \mathbf{P}^{-1}\mathbf{\Lambda}^k\mathbf{P} \to \mathbf{P}^{-1}\mathbf{0}_{n\times n}\mathbf{P} = \mathbf{0}_{n\times n} \quad \text{as} \quad k \to \infty$$

Since the norm is a continuous functional, we have

$$\|\mathbf{A}^k\| \to \|\mathbf{0}_{n\times n}\| = 0 \quad \text{as} \quad k \to \infty$$

Exercise 4

CLAIM: The set of nonnegative definite matrices is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$

PROOF:
Take a convergent sequence of nonnegative definite matrices $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(n \times n)$ with the limit $\mathbf{B} \in \mathcal{M}(n\times n)$, that is $\|\mathbf{A}_n - \mathbf{B}\| \to 0$. For a fixed $\mathbf{x} \in \mathbb{R}^n$ define the functional $f_{\mathbf{x}} : \mathcal{M}(n\times n) \to \mathbb{R}$ as $f_{\mathbf{x}}(\mathbf{C}) = \mathbf{x}'\mathbf{C}\mathbf{x}$. Evidently, this functional is continuous given that

$$|f_{\mathbf{x}}(\mathbf{A}_n) - f_{\mathbf{x}}(\mathbf{B})| = |\mathbf{x}'(\mathbf{A}_n - \mathbf{B})\mathbf{x}| \leq \|\mathbf{x}\|\|(\mathbf{A}_n - \mathbf{B})\mathbf{x}\| \leq \|\mathbf{x}\|^2\|\mathbf{A}_n - \mathbf{B}\| \to 0$$

where the first inequality is the Cauchy-Schwartz inequality. In other words, $f_{\mathbf{x}}(\mathbf{A}_n) \to f_{\mathbf{x}}(\mathbf{B})$. Notice that the positive semidefiniteness of the $\mathbf{A}_n$'s implies that $f_{\mathbf{x}}(\mathbf{A}_n) \geq 0$, $\forall n \in \mathbb{N}$. This leads to $f_{\mathbf{x}}(\mathbf{B}) \geq 0$, $\forall \mathbf{x} \in \mathbb{R}^n$, consequently, $\mathbf{B}$ is nonnegative definite.

Exercise 5

CLAIM: Let $\mathbf{M}, \mathbf{A} \in \mathcal{M}(n \times n)$ with $\rho(\mathbf{A}) < 1$. Let $\mathbf{X}^*$ be the unique solution of the Lyapunov equation $\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}' + \mathbf{M}$. Then

1. $\mathbf{M}$ is symmetric $\Longrightarrow$ $\mathbf{X}^*$ is symmetric
   PROOF:

$$\mathbf{X}^* = \mathbf{A}\mathbf{X}^*\mathbf{A}' + \mathbf{M} = \big(\mathbf{A}(\mathbf{X}^*)'\mathbf{A}' + \mathbf{M}'\big)' = \big(\mathbf{A}(\mathbf{X}^*)'\mathbf{A}' + \mathbf{M}\big)' = (\mathbf{X}^*)'$$

2. $\mathbf{M}$ is nonnegative definite $\Longrightarrow$ $\mathbf{X}^*$ is nonnegative definite
   PROOF:
   The set of nonnegative definite matrices is a closed subset of the Banach space $(\mathcal{M}(n \times n), \|\cdot\|)$ and due to $\rho(\mathbf{A}) < 1$, the operator L^k is a uniform contraction for some $k \in \mathbb{N}$, where $L\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}' + \mathbf{M}$. We can show that L^k is actually a self-mapping on the positive semidefinite cone. That is, if $\mathbf{Z}$ is positive semidefinite, then $\forall \mathbf{x} \in \mathbb{R}^n$

$$\mathbf{x}'L^k\mathbf{Z}\mathbf{x} = \mathbf{x}'\mathbf{A}^k\mathbf{Z}\left(\mathbf{A}^k\right)'\mathbf{x} + \mathbf{x}'\mathbf{A}^{k-1}\mathbf{M}\left(\mathbf{A}^{k-1}\right)'\mathbf{x} + \cdots + \mathbf{x}'\mathbf{M}\mathbf{x} \geq 0$$

   because all terms on the RHS are nonnegative real numbers. This implies that $L^k\mathbf{Z}$ is positive semidefinite, so L^k is indeed a self-map and $\mathbf{X}^*$ is positive semidefinite.[2]

[2] This is an easy and well-known corollary of the contraction mapping theorem: L^k has a globally stable fixed point, so if we start with an element of a closed set and the iteration does not lead us out of the set, then the fixed point must also belong to the set.

3. $\mathbf{M}$ is positive definite $\Longrightarrow \mathbf{X}^*$ is positive definite

   Proof:

   I will show that if $\mathbf{M}$ is positive definite, then L^k maps nonnegative definite matrices into positive definite matrices. Let $\mathbf{Z}$ is nonnegative definite. Then $\forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$

$$\mathbf{x}' L^k \mathbf{Z} \mathbf{x} = \mathbf{x}' \mathbf{A}^k \mathbf{Z} \left(\mathbf{A}^k\right)' \mathbf{x} + \mathbf{x}' \mathbf{A}^{k-1} \mathbf{M} \left(\mathbf{A}^{k-1}\right)' \mathbf{x} + \cdots + \mathbf{x}' \mathbf{M} \mathbf{x} > 0$$

   because all terms on the RHS are nonnegative and the last term is positive. In other words, L^k maps the closed set of nonnegative definite matrices into positive definite matrices. By definition, the unique fixed point must satisfy $\mathbf{X}^* = L^k \mathbf{X}^*$, hence $\mathbf{X}^*$ is positive definite.