Topics in Computational Economics - Problem Set 9

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Exercise 1

Instead of the hint, I am going to use the Spectral Theorem for self-adjoint (in our specific case, symmetric) matrices:

Theorem 1. *Spectral Theorem: Let Q be a self-adjoint linear operator in a finite dimensional vector space. Then there exist $\lambda_1 \ldots \lambda_k$ real numbers (nonnegative if Q is real) and $P_1 \ldots P_k$ orthogonal projections with $k \leq dimV$ such that:*

- $\sum_i P_i = I$
- $\sum_i \lambda_i P_i = Q$

Proof of the exercise:

Show that: $||A||^2 \leq \rho(A'A)$

Take any $x \in \mathbb{R}^n$ st.: $< Ax, Ax >= \|Ax\|^2 = \|A\|^2 \cdot \|x\|^2$ Then using the properties of symmetric linear operators, the Spectral Theorem and that orthogonal projections are idempotent and symmetric:

$$\begin{aligned} < Ax, Ax >= &\quad < A'Ax, x > \\ = &\quad < \sum_i \lambda_i P_i x, x > \\ = &\quad \sum_i \lambda_i < P_i x, x > \\ = &\quad \sum_i \lambda_i < P_i P_i x, x > \\ = &\quad \sum_i \lambda_i < P_i' P_i x, x > \\ = &\quad \sum_i \lambda_i < P_i x, P_i x > \\ = &\quad \sum_i \lambda_i \|P_i x\| \\ \leq &\quad \lambda_{max} \sum_i \|P_i x\| \end{aligned}$$

where the last inequality is true as all the eigenvalues are nonnegative and the norm is non-negative. Continuing and using the second part of the Spectral Theorem:

$$\begin{aligned}
\lambda_{max} \sum_i \|P_i x\| = & \quad \lambda_{max} < \textstyle\sum_i P_i x, x > \\
= & \quad \lambda_{max} < \textstyle\sum_i P_i x, x > \\
= & \quad \lambda_{max} < x, x > \\
= & \quad \lambda_{max} \|x\|^2
\end{aligned}$$

Proving that $\|A\|^2 \leq \rho(A'A)$

Show that: $||A||^2 \geq \rho(A'A)$**:**

Take any $x \in \mathbb{R}^n$ st.: $A'Ax = \rho(A'A)x$. Then:

$$\begin{aligned}
\rho(A'A)\|x\|^2 = & \quad < \rho(A'A)x, x > \\
= & \quad < Ax, Ax > \\
= & \quad \|Ax\|^2 \\
\leq & \quad \|A\|^2 \cdot \|x\|^2
\end{aligned}$$

Proving that $\|A\|^2 \geq \rho(A'A)$
Overall we have that $\|A\|^2 = \rho(A'A)$

Exercise 2

Gelfand's formula states that $\forall A \in \mathcal{M}(n \times n)$ we have that $\|A^k\|^{1/k} \to \rho(A)$ as $k \to \infty$. This means that for any $\varepsilon > 0$ there exists a $K \in \mathbb{N}$ such that $\forall k \geq K$ we have $|\|A^k\|^{1/k} - \rho(A)| < \varepsilon$. This means that $\forall k \geq K$

$$\begin{aligned}
\|A^k\|^{1/k} - \rho(A) < \varepsilon \\
\|A^k\|^{1/k} < \rho(A) + \varepsilon \\
\|A^k\| < (\rho(A) + \varepsilon)^k
\end{aligned}$$

Since $\rho(A) < 1$ and we can set ε arbitrarily small, let $r = \rho(A) + \varepsilon < 1$. Then we have that $\|A^k\| < r^k C$, where in this case $C = 1$.

Now, we have

$$\sum_{k=0}^{\infty}\|A^k\| = \sum_{k=0}^{K}\|A^k\| + \sum_{k=K+1}^{\infty}\|A^k\| < \sum_{k=0}^{K}\|A^k\| + \sum_{k=K+1}^{\infty} r^k = \sum_{k=0}^{K}\|A^k\| + \frac{r^{K+1}}{1-r} < \infty$$

where the result of the second summation in the final equality follows from the properties of partial geometric sums. The final inequality follows from the facts that: $\|A^k\|$ is well defined for any matrix, and so is finite for any k and a finite sum of finite values is itself finite; $\frac{r^{K+1}}{1-r} < \infty$ as long as $r \neq 1$.

Exercise 3

If A is diagonalizable, we can express it as $A = PVP^{-1}$, where P is matrix whose columns are the eigenvectors of A and form an orthogonal basis of $\mathbb{R}^n$, and where V is a diagonal matrix with diagonal elements equal to the eigenvalues of A. We can thus write

$$\|A^k\| = \|(PVP^{-1})^k\| = \|PV^kP^{-1}\| \leq \|P\|\|V^k\|\|P^{-1}\| \leq \|P\|\|P^{-1}\|\|V\|^k$$

Now using exercise 1 (and the fact that V is symmetric)

$$\|V\| = \rho(V) = \rho(A)$$

Now observe that

$$\|A^k\| \leq \|P\|\|P^{-1}\|\|V\|^k = \|P\|\|P^{-1}\|\rho(A)^k$$

and since $\rho(A) < 1$, we have that $\|A^k\| \to 0$ as $k \to \infty$.

Exercise 4

We want to show that the set of positive semidefinite matrices ($\mathcal{PS}(n \times n)$ from here on) is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$. To do this, we will show that for any sequence of positive semidefinite matrices $\{M_k\}$, its limit point, M, is also a positive semidefinite matrix.

First, note that for any matrix, A, positive semidefiniteness means that $z'Az \geq 0$ for all $z \in \mathbb{R}^n/\{0\}$. Notice, also, that quadratic form is a continuous function in A, i.e. $f(A) := z'Az$ for some z. To see this suppose $A_k \to A$ in matrix norm but then $|f(A) - f(A_k)| = |z'Az - z'A_kz| = |z'(A - A_k)z| \leq |\|z'\|\|(A - A_k)\|\|z\|| < \epsilon(z, \delta)$ where $\|(A - A_k)\| < \delta$.

Now, let M_k be a sequence of positive semidefinite matrices such that $M_k \to M$ as $k \to \infty$ under the matrix norm, $\|\cdot\|$. We will work towards a contradiction. Suppose the limit point M is such that $z'Mz$ for some $z \in \mathbb{R}^n/\{0\}$ (i.e. M is not positive semidefinite).

$M_k \to M$ means that $\|M_k - M\| < \varepsilon \; \forall k \geq K \in \mathbb{N}$. Because $f(\cdot)$ is a continuous function we know that for all $\varepsilon > 0$ and all $k \geq K$, there exists a $\delta > 0$ such that $\|M_k - M\| < \delta$ implies that $|f(M_k) - f(M)| < \varepsilon$.

Now, since ε is arbitrary, let $\varepsilon = \frac{z'Mz}{2}$. Then $|f(M_k) - f(M)| = |z'M_kz - z'Mz| < \frac{z'Mz}{2}$ implies that $-\frac{z'Mz}{2} < z'M_kz - z'Mz < \frac{z'Mz}{2}$. Rearranging, we find that $\frac{z'Mz}{2} < z'M_kz < \frac{3z'Mz}{2}$. Since we assumed that $z'Mz < 0$, this implies that $z'M_kz < 0$, which is a contradiction.

Therefore, it must be that the limit point, M, is also a positive semidefinite matrix. Thus, all sequences in the set of positive semidefinite matrices have limit points that also fall inside the set. Therefore, the set of positive semidefinite matrices is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$.

Exercise 5

For this question, we will make use of the following theorem:

Theorem 2 (Corollary to the Contraction Mapping Theorem)**.** *If T is a uniform contraction on a Banach space $(S, d(\cdot))$, $T : S' \to S'$, and $S' \underset{closed}{\subset} S$, then the fixed point $Tv = v$ is such that $v \in S'$. Further, if $TS' \subseteq S'' \subset S$, then $v \in S''$.*

Now, let L be the Lyapunov operator on $(\mathcal{M}(n \times n), \|\cdot\|)$, where

$$LX = AXA' + M \tag{1}$$

From the lecture notes we know that L is a uniform contraction on $(\mathcal{M}(n \times n), \|\cdot\|)$, and so has a unique fixed point X^*.

Part 1: M symmetric $\Rightarrow$ X^* symmetric

Consider the solution to the Lyapunov equation, X^*. We have $X^* = AX^*A' + M$. We know that M is symmetric. We know that for any $X^* \in \mathcal{M}(n \times n)$, it will be the case that

$$(X^*)' = (AX^*A' + M)' = (AX^*A')' + M' = A(X^*)'A' + M$$

This means that there is another solution to the Lyapunov equation, $(X^*)'$, but we know that there is only a unique solution, therefore $X^* = (X^*)'$. And so we have showed that if M is symmetric, then so is X^*.

Part 2: M positive semidefinite $\Rightarrow$ X^* positive semidefinite

Denote the set of positive semidefinite matrices as $\mathcal{PS}(n \times n)$. As we saw in the previous problem, a positive semidefinite matrix B has the property that $z'Bz \geq 0$ for all $z \in \mathbb{R}^n / \{0\}$. Let M be a positive semidefinite matrix. Consider our Lyapunov operator in (1). Note that for any $X \in \mathcal{M}(n \times n)$, AXA' is positive semidefinite because AXA' is a quadratic form, and so is automatically positive semidefinite. Because the set of positive semidefinite matrices is also a linear subspace, it is closed under addition, and so $AXA' + M$ is also positive semidefinite. Thus, we can see that if X is positive semidefinite, the Lyapunov operator is such that: $L : \mathcal{PS}(n \times n) \to \mathcal{PS}(n \times n)$.

As I proved in exercise 4, $\mathcal{PS}(n\times n) \underset{closed}{\subset} \mathcal{M}(n\times n)$. So we can apply the Corollary to the Contraction Mapping Theorem again, and we immediately see that the fixed point of the Lyapunov operator, X^*, is contained in $\mathcal{PS}(n\times n)(n\times n)$. Thus, M positive semidefinite $\Rightarrow X^*$ positive semidefinite, as we wanted to show.

Part 3: M positive definite $\Rightarrow X^*$ positive definite

Denote the subset of positive definite matrices as $\mathcal{P}(n\times n)$. A positive definite matrix B has the property that $z'Bz > 0$ for all $z \in \mathbb{R}^n/\{0\}$. Let M be a positive definite matrix. Consider our Lyapunov operator in (1). Note that for any $X \in \mathcal{M}(n\times n)$, AXA' is positive semidefinite because AXA' is a quadratic form, and so is automatically positive semidefinite.

Now, note that $z'(AXA'+M)z = z'AXA'z + z'Mz$. We argued that AXA' is positive semidefinite, and since $z'AXA'z$ is another quadratic form and must be greater than or equal to zero. So we have $z'AXA'z \geq 0$ and $z'Mz > 0$ and so $z'AXA'z + z'Mz > 0$. Thus, we know that the Lyapunov operator takes the set of positive semidefinite matrices to the set of positive definite matrices $L : \mathcal{PS}(n\times n) \to \mathcal{P}(n\times n)$.

Again from above, we know that $P.S.D(n\times n) \underset{closed}{\subset} \mathcal{M}(n\times n)$. Note, too, that that $\mathcal{P}(n\times n) \subset \mathcal{PS}(n\times n)$.

This time we can apply the second part of the Corollary to the Contraction Mapping Theorem: $L(\mathcal{PS}(n\times n)) \subseteq \mathcal{P}(n\times n) \subset \mathcal{PS}(n\times n)$ implies that the fixed point, X^*, is contained in $\mathcal{P}(n\times n)$. Thus, M positive definite $\Rightarrow X^*$ positive definite, as we wanted to show.

Exercise 6

```
In [1]: import numpy as np
        import matplotlib
        import matplotlib.cm as cm
        import matplotlib.mlab as mlab
        import matplotlib.pyplot as plt
        import scipy.linalg as sp_linalg
        % matplotlib inline
```

```
In [2]: A = np.array([[0.8, -0.1],[-0.1,0.8]])
        C = np.eye(2,2)
        D = np.eye(2,2)
        beta = 0.9
```

```
In [3]: def contraction(P,beta,A,C,D):
            return beta * A.T @ (D + P) @ A
```

In [4]:
```
def Ljapunov(beta,A,C,D,P = None,tol = 1e-10):
    w,v = sp_linalg.eig(A)
    if np.max(np.abs(w)) >= 1.0 / beta ** (1/2):
        print("The largest eigenvalue is too large")
        Res = np.max(np.abs(w))
    else:
        if P is None:
            P = np.eye(A.shape[0],A.shape[0])
        conv = 1.0
        while conv > tol:
            P_next = contraction(P,beta,A,C,D)
            conv = np.sum(np.abs(P_next - P))
            np.copyto(P,P_next)
        delta = beta / (1.0 - beta) * np.trace(C.T @ (D + P) @ C)
        Res = P ,  delta
    return Res
```

In [5]:
```
P , delta = Ljapunov(beta,A,C,D)
```

In [6]:
```
def price(x,P,delta):
    return x.T @ P @ x + delta
```

In [7]:
```
price(np.array([1.1,1.0]),P,delta)
```

Out[7]: 51.063505005906883

In [8]:
```
matplotlib.rcParams['xtick.direction'] = 'out'
matplotlib.rcParams['ytick.direction'] = 'out'
plt.figure() # Create a new figure window
n = 100
xlist = np.linspace(-10.0, 10.0, n) # Create 1-D arrays for x,y dimensions
ylist = np.linspace(-10.0, 10.0, n)
X,Y = np.meshgrid(xlist, ylist) # Create 2-D grid xlist,ylist values
Z = np.empty(X.shape)
for i in range(len(xlist)):
    for j in range(len(ylist)):
        Z[i ,j] = price(np.array([X[i,j],Y[i,j]]),P,delta)
CS = plt.contourf(X, Y, Z,5)
```

```
plt.clabel(CS, inline=0, colors = 'b', fontsize=10)
plt.title('Equilibrium price function')
plt.show()
```