On the strictly descending multi-unit auction*

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We analyze the bidding behavior in a strictly descending multi-unit auction where the price decreases continuously without going back to the initial start price once an object is sold. We prove that any equilibrium in the multi-unit descending auction is inefficient. We derive a symmetric equilibrium for general distribution functions as well as an arbitrary number of bidders and objects. Moreover, equilibrium bidding is characterized by a set of initial value problems. Our analysis thus generalizes previous results in the literature.

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1 Introduction

Bulow and Klemperer (1994) analyze a discriminatory descending multi-unit auction with single-unit demand. They assume that if demand exceeds supply, the price clock starts again at the initial price without selling any object (i.e., the auction is not strictly descending).[1] This assumption ensures an efficient outcome which simplifies the analysis through the use of the revenue-equivalence theorem: if multiple units are for sale to N risk neutral bidders with independent private values and unit demand, all mechanisms that allocate the objects efficiently yield the same bidder surplus and seller revenue. However, their efficiency result hinges on the crucial assumption that the price goes up again if demand exceeds supply. This is an important aspect as an upward adjustment of the price is not always possible. Here, we drop this assumption—just like Martínez-Pardina and Romeu (2011) who, independently of our work, developed a version similar to our setup and who speak of a "single-run descending-price auction".[2]

Our contribution to the literature is fourfold: by generalizing previous results in the literature for any distribution function and an arbitrary number of objects, we can (i) show that any equilibrium is inefficient. Moreover, we (ii) show the existence of symmetric equilibrium bidding strategies and (iii) characterize the equilibrium structure using the respective initial value problems. Last, we (iv) provide a closed-form solution for the uniform distribution in a situation with two objects for sale and three bidders. When comparing the strictly descending multi-unit auction to its (efficient) first-price sealed-bid counterpart, the inefficiency of the descending auction implies that the revenue in the sealed-bid auction is larger. This is a result worth stressing as it has been argued that bidding is the same in both formats which is based on the incorrect reasoning that bidders do not update their beliefs during the auction.[3]

In their closely related article, Martínez-Pardina and Romeu (2011) characterize

[1] See also Goeree et al. (2006) for an experimental investigation.

[2] Bulow and Klemperer (1994) mention this extension and give a short account of the case with a uniform distribution in section VI.

[3] See, e.g., Krishna (2009), sections 12.2.1 and 13.1.

the conditions for a monotone, symmetric equilibrium under the strictly descending auction for the case with N bidders and two units for sale. They propose a numerical solution method to obtain the bidding functions for the uniform distribution as well as for a generic distribution function. The authors also show that the strictly descending auction may result in faster sales and a smaller variance in prices which means that risk-averse and/or impatient sellers may have a preference for this type of auction. However, they derive a necessary equilibrium condition but do not show whether the solution to this condition indeed constitutes equilibrium bidding.

Our results explain observations in real-world markets that use this kind of auction (e.g., fish markets off the coast of Valencia, Spain, and fresh produce markets[4]). Other examples which have the above-mentioned characteristics are the sale of concert tickets which are often sold on a first-come-first-serve basis or the airline-overbooking problem where airlines look for volunteers among stranded passengers who accept a monetary compensation in exchange for their seat.[5]

The paper is organized as follows. In the next section, we set up the model and analyze equilibrium bidding behavior in the third section. In section 4, we illustrate the results for the special case of uniformly distributed valuations. The last section concludes.

2 Model

Consider a setting where K identical objects are for sale to N risk-neutral bidders (with $N > K$) who all wish to purchase a single unit. The reserve price for each of the units is denoted by r. Bidder i assigns a value of X_i to any of the K objects. This

[4]See Martínez-Pardina and Romeu (2011).

[5]The use of auctions as a solution to the airline-overbooking problem was suggested by Simon (1968). This solution is now at the heart of the so-called Volunteer Auction Scheme, a voluntary bumping plan for airlines mandated by the Civil Aeronautics Board (CAB) in 1978. Simon (1994) recalled the developments that had followed his first article and that had led to the airline auction scheme. There, he mentioned that “a cruder version is for the airline to cry a price and to ask for traders”. Effectively applying a reverse version of a strictly descending multi-unit auction, the airline personnel starts off with announcing a low price. If there are no or not sufficiently many volunteers who accept the current price in exchange for their seat, then the staff increases the price until a sufficiently high number of volunteers is found. This is actually what is being done by airlines for practical reasons (see, e.g., Rothstein, 1985).

value represents the maximum price bidder i is willing to pay for any of the units. The valuation is independently and identically distributed on the unit interval $[0, 1]$ according to an absolutely continuous distribution function F. It is assumed that the corresponding density function f is continuous on the real interval $[0, 1]$ (where $F' \equiv f$). Bidder i only knows his own realization x_i of X_i which is not affected by the valuation of the other bidders (independent private values). Each bidder wants to buy only one of the objects for sale (single-unit demand) and maximizes expected profits. Except for the realized values, everything else is common knowledge.

Following Bulow and Klemperer (1994), we use the efficient discriminatory first-price sealed-bid (or pay-as-bid) multi-unit auction as the benchmark of our analysis. In this format, a bidder wins one of the objects paying bid b if his bid is among the K highest bids. We compare its outcome with the discriminatory multi-unit open strictly descending (or Dutch) auction. In this multi-unit Dutch auction, a price clock starts at 1 and decreases continuously. Bidders decide when to stop the clock. A winning bidder has to pay the price b at which he stopped the price clock. The bidder who stopped the auction at price b to get one of the (remaining) objects leaves the auction. The other bidders observe that the clock has stopped and that a unit has been sold. They do not observe the identity of the winner of that unit. The clock then continues at b and the remaining bidders may stop the clock at any time. If more than one bidder stop the clock at the same time, either all bidders obtain a good (if sufficiently many objects are still available) or there is a lottery among those who stopped the clock where each bidder has the same probability of being chosen. Whenever we speak of the descending auction in the following, we refer to this version of the strictly descending Dutch auction just described (unless otherwise stated).

3 Equilibrium bidding and revenue

In this section, we first analyze equilibrium bidding strategies and then compare the revenue in the Dutch auction to its first-price sealed-bid counterpart.

Equilibrium bidding

In order to derive the equilibrium bidding strategy, consider the following history h_k when there are still k units available:

$$h_k = (b_{k+1}, \ldots, b_K)$$

where b_j (with $j \in \{k+1, \ldots, K\}$) denotes the price at which object j was sold. We write the pure measurable strategy σ^i for player i as the set $\{\sigma_k^i | k \in \{1, ..., K\}\}$ with the mappings σ_k^i defined as

$$\sigma_k^i(x_i, h_k) : [0,1]^{K-k+1} \longrightarrow [0,1].$$

This set of mappings specifies at what price bidder i with valuation x_i plans to stop the clock on object k given history h_k, i.e., $\sigma_k^i(x_i, h_k) = b \leq b_{k+1}$.

The following proposition highlights an important feature of any equilibrium:

Proposition 1. *Any equilibrium of the discriminatory strictly descending multi-unit auction with single-unit demand and independent private values is inefficient.*

Proof. Let σ be the equilibrium strategy. Suppose the first object is sold to bidder i at $b_K = \sigma_K^i(x_i, h_K)$ for some x_i. We will show that in any equilibrium of the descending auction, there is simultaneous bidding on all remaining objects with positive probability once the first object is sold. Such simultaneous bidding means that there is a pooling of types; as a consequence, the outcome is inefficient.

To show that simultaneous bidding occurs with positive probability, we show that if that was not the case, $\sigma_K^i(x_i, h_K)$ would not be a best reply when bidding for the first object. Suppose therefore that there is no simultaneous bidding on all remaining objects as soon as the first object is sold. Suppose furthermore that bidder i does not stop the clock at b_K but plans to stop the clock at $b_K - \epsilon$. It follows that either (i) none of the other bidders stops the clock and bidder i receives the object at $b_K - \epsilon$ or (ii) one of the other bidders stops the clock at a price b' with $b_K - \epsilon < b' < b_K$. As no simultaneous bidding occurs on all remaining objects, at least one object will remain unsold such that bidder i can buy this object at $b' < b_K$ with probability one. In either case, stopping the clock at b_k is not optimal. Thus, $\sigma_K^i(x_i, h_K) = b_K$ is not a best reply. It follows that simultaneous bidding is an equilibrium phenomenon. As a consequence, any equilibrium is inefficient. □

Efficiency means that those bidders with the K highest valuations receive the K units on offer. However, equilibrium bidding in the strictly descending multi-unit auction results in a set of bidders with different valuations who all have the same probability of winning the auction as they stop the price clock at the same time. Therefore, this format does not ensure that those bidders whose valuations for the objects are highest win the auction with certainty. The important observation here is that interestingly, pooling occurs for any object but the first. Note that simultaneous bidding means that bidders who submit a bid at the current price may not win any object for sale. Hence, if no simultaneous bidding took place, then it would never be optimal for a bidder to bid first because the bidder could gain by waiting and bidding second at a strictly lower price.

Before we derive the equilibrium of the descending auction, we state the following result:

Lemma 1. *In any equilibrium of the discriminatory strictly descending multi-unit auction with single-unit demand and independent private values, bidders use monotonic bidding strategies, i.e.,* $\sigma_k^i(x_i, h_k)$ *increases in* x_i*.*

Proof. Suppose that the strategy of bidder i—given history h_k—is such that he stops the clock at price b if his valuation is x_i and he stops the clock at price b' if his valuation is x_i', i.e., $\sigma_k^i(x_i, h_k) = b$ and $\sigma_k^i(x_i', h_k) = b'$. Suppose further that $b > b'$ and $x_i < x_i'$. Let the clock reach price b so that type x_i stops the clock. Now let q be the probability that type x_i wins the object (q might be smaller than one as there might be simultaneous bidding). Define by q' the probability that type x_i' (who does not stop the clock at the current price b) wins the object. Let p' denote the expected price type x_i' has to pay. It must hold that $q' < q$ and $p' < b$. As both types behave optimally, it must hold that $q(x_i - b) \geq q'(x_i - b')$ and at the same time $q(x_i' - b) \leq q'(x_i' - b')$. Thus $(q - q')(x_i - x_i') \geq 0$ and hence $x_i \geq x_i'$ which is a contradiction. $\square$

The explanation behind this result is that a bidder always wins an object with a higher probability if he stops the clock than if he does not. If a bidder with a low valuation receives the same expected surplus from two strategies with different probabilities of winning the auction, a higher-valuation bidder strictly prefers the strategy with the higher probability of winning (and vice versa). Hence, the higher-valuation bidder does not choose a strategy that results in a lower probability of receiving an object compared to the low-valuation types. As a consequence, this bidder's optimal strategy is to stop the clock if and only if his valuation exceeds some cut-off value.[6]

We now turn our attention to constructing a symmetric[7] equilibrium. In *Lemma 1*, we have shown that any equilibrium is monotonic meaning that types with lower valuations wait longer than types with higher valuations before they stop the clock. Then, it is reasonable to assume that the bidding behavior explicitly depends only on the current price and the price at which the last item was sold.[8] Thus, a possible symmetric equilibrium strategy σ can be written with a set of functions $\{\beta_1, \ldots, \beta_K\}$ and $\{c_1, \ldots, c_K\}$ such that

$$\sigma_k(x, h_k) = \begin{cases} \beta_k(x) & \text{if } x \leq c_k(b_{k+1}) \\ b_{k+1} & \text{if } x > c_k(b_{k+1}). \end{cases}$$

$\beta_k(x)$ denotes a differentiable, increasing function and $c_k(b_{k+1})$ is a cut-off value, i.e., all types $x > c_k(b_{k+1})$ bid as soon as the auction continues after a bidder purchased object $k+1$ at price b_{k+1}. Note that $\beta_1(x)$ is the well established equilibrium bidding strategy for the case with $N - K + 1$ bidders competing for a single object. The

[6]See Bulow and Klemperer (1994).

[7]We focus on symmetric strategies as well as symmetric equilibria and therefore we drop index i from mappings $\sigma_k(x, h_k)$. It cannot be ruled out per se that an asymmetric equilibrium exists in the auction game considered. However, this appears unlikely in the light of the results found by Maskin and Riley (2003).

[8]The equilibrium strategies implicitly also depend on the number of active bidders and remaining units.

following proposition establishes existence of such an equilibrium:

Proposition 2. *For any reserve price $r > 0$ and any number of units $K < N$ in the discriminatory strictly descending multi-unit auction with single-unit demand and independent private values, there exists a symmetric equilibrium. The equilibrium strategy σ can be written using a set of functions $\{\beta_1, \ldots, \beta_K\}$ and $\{c_1, \ldots, c_K\}$ where β_k is the well defined solution to an initial value problem and $c_k(b_{k+1})$ is the root of a continuous function.*

Proof. See the appendix. □

We use complete induction over K to prove this result. Starting at $K = 1$, the problem is just a single-unit descending auction and the equilibrium is well known. To perform the inductive step, we first look at a situation in which the first object was sold for a price b_K. The problem then reduces to a descending auction with $K - 1$ units—for which the existence of an equilibrium is ensured by the induction hypothesis—and a starting price of b_K. It is shown that an equilibrium in cut-off strategies exists for the subsequent subgame, i.e., all bidders with a valuation above the cut-off value bid at the starting price. To this end, we analyze the difference in utility from bidding at the given price and from waiting. The cut-off value is defined by setting this expression equal to zero. Making use of this result, we show that an increasing, differentiable bidding function for the first object exists by again considering the difference in expected utility underlying the decision problem of bidding at the current price b or waiting for the auction to continue for one more (infinitesimal) tick of the price clock. We can then derive a differential equation for the bidding function and show that a unique solution to this equation exists. Furthermore, it is shown that every solution of this differential equation constitutes an equilibrium bidding function for the first object.[9]

Note that the result in *Proposition 2* only holds if the reservation price is strictly larger than zero. However, we can provide a solution for arbitrarily small values of

[9]Note that the equilibrium we construct in *Proposition 2* is unique among the symmetric equilibria with differentiable bidding functions β where bidding only depends on the current price and the price at which the last object was sold. However, it is per se not possible to rule out equilibria in which one of these assumptions is violated.

the reservation value. Moreover, this (standard) assumption is a technical assumption that simplifies the proof to a great extent in that it helps to avoid singularities that would create problems for the existence and uniqueness of the solution to the differential equation. The underlying reasoning should hold for the case where $r = 0$: in the example below, an equilibrium indeed exists. Alternatively, one could assume that the distribution F of valuations has an atom at 0.

Revenue

The structure of the bidding behavior with a cut-off-value function $c_{K-1}(b_K)$ such that all types $x \in (c_{K-1}(b_K), \bar{x}_K]$ accept price b_K simultaneously implies that the discriminatory strictly descending multi-unit auction—unlike its first-price sealed-bid counterpart—is not efficient. One would thus expect different revenues for the seller under both formats. In order to compare the seller's expected revenues, we make the following assumption:

Assumption 1. *For the virtual valuation ϕ, defined as $\phi(x) := x - (1 - F(x))/f(x)$, it holds that $\partial\phi/\partial x > 0$.*

We can now establish the following result:[10]

Proposition 3. *Suppose* Assumption 1 *holds. Then, the expected revenue in the discriminatory multi-unit first-price sealed-bid auction with single-unit demand and independent private values exceeds the expected revenue in its descending counterpart.*

Proof. As we have shown in *Proposition 2*, an equilibrium of the discriminatory descending auction exists. Hence, we can apply the revelation principle and consider a direct mechanism (Q_d, M_d)[11] that leads to the same allocation as the descending auction (subscript d). As an efficient equilibrium exists in the first-price sealed-bid auction (subscript s), the revelation principle is also applicable. Therefore, a direct mechanism (Q_s, M_s) can be considered. Following Krishna (2009) (chapter 5), the

[10] See also Martínez-Pardina and Romeu (2011).

[11] Q_d denotes the allocation rule and M_d the payment rule in the direct mechanism.

revenue for the auctioneer in any direct mechanism can be written as

$$\sum_{i=1}^{N} m_{i,\{d,s\}}(0) + \int_{[0,1]^N} \left(\sum_{i=1}^{N} \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) Q_{\{d,s\}}(\mathbf{x}) \right) f(\mathbf{x}) d\mathbf{x},$$

where $m_{i,\{d,s\}}(0)$ denotes the expected payment of a bidder with valuation 0, which is 0 in both auctions. Given *Assumption 1*, it is now straightforward to see that the revenue in the first-price sealed-bid auction exceeds the revenue in the descending auction. In the first-price sealed-bid auction, Q_s puts weight only on the K highest valuations whereas in the descending auction, simultaneous bidding may occur for all but the first object. As by assumption the virtual valuation increases in x_i, the revenue of the first-price sealed-bid auction will be larger than in the descending auction. □

As the proposition suggests, the fact that information revelation during a descending auction results in an inefficient allocation makes the first-price sealed-bid auction more attractive from the auctioneer's point of view.

4 Example

Turning to the case where bidders' valuations are uniformly distributed on $[0, 1]$, Bulow and Klemperer (1994) show that if the distribution of values is uniform and an equilibrium exists, a bidder with valuation x stops the clock for the first object at the same price he submits in the discriminatory sealed-bid auction with x as the maximum valuation, i.e., a first-price sealed-bid auction where valuations y are distributed according to $\tilde{F}(y) = F(y)/F(x)$. The bidding strategy for the first object is thus $\beta_K(x) = E[Y_K^{(N-1)} | Y_1^{(N-1)} < x]$.[12] *Figure 1* illustrates the relationship of the discriminatory sealed-bid and descending auction for the uniform distribution and

[12] Note that $Y_K^{(N-1)}$ denotes the K-highest order statistic of $N-1$ independent draws and equilibrium bidding is characterized as $\beta_s(x) = E[Y_K^{(N-1)} | Y_1^{(N-1)} < x]$ in a first-price sealed-bid auction (see Krishna, 2009). Note further that the above-mentioned relation only holds for the uniform distribution. If the distribution of values is not uniform, this result no longer holds. It can be shown that bidding for the first object in the descending auction is different from $E[Y_K^{(N-1)} | Y_1^{(N-1)} < x]$.

$N = 3$ and $K = 2$. The equilibrium bidding strategies are given by

$$\beta_2(x) = \frac{x}{3}$$

in the present case and

$$\beta_s(x) = \frac{x\,(3 - 2x)}{3\,(2 - x)}$$

for the sealed-bid auction.

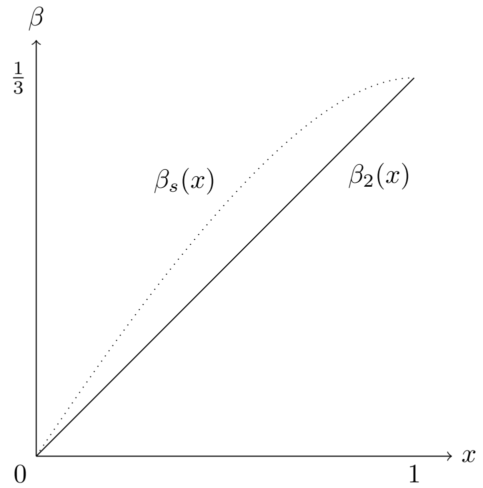

Figure 1: Equilibrium bidding in the descending auction $\beta_2(x)$ and the sealed-bid auction $\beta_s(x)$ for the uniform distribution, $N = 3$, and $K = k = 2$.

Now suppose that the first object is sold at a price $b_K = \beta_K(x)$. The remaining bidders may then decide whether to buy the remaining object at price b_K or wait for the auction to continue.[13] In this situation, all bidders with a valuation x below a cut-off value $c_{K-1}(b_K)$ (see condition (2) in the appendix) wait for the auction to continue and bid in the auction according to $\beta_{K-1}(x) = E[Y_{K-1}^{(N-2)}|Y_1^{N-2} < x]$. Moreover, all bidders with a valuation $x \geq c_{K-1}(b_K)$ bid instantly.[14] *Figure 2* illustrates this behavior for the case with $N = 3$ and $K = 2$ and where a bidder accepted price b_2 for the first unit. In this case, the equilibrium bidding function

[13]For more on the buy-now option, see Mathews and Katzman (2006).

[14]This observation is what Bulow and Klemperer (1994) call a frenzy. Analogously, the ensuing decrease of the price clock where none of the remaining bidders is willing to bid corresponds to a crash.

when there is only one item left is given by

$$\beta_1(x, b_2) = \begin{cases} \frac{x}{2} & \text{if } x \leq \frac{b_2}{2} \\ b_2 & \text{if } x > \frac{b_2}{2}. \end{cases}$$

Note that as $E[Y_K^{(N-1)}|Y_1^{(N-1)} < x] \leq E[Y_K^{(N-1)}|Y_K^{(N-1)} < x]$, bidders wait longer in the descending auction before they stop the clock for the first unit compared to the sealed-bid auction (except for the type with the maximum valuation). However, if someone buys this unit before them, then they rush for the second unit such that simultaneous bidding might occur. So the second unit might be sold at a larger price in the descending auction compared to a sealed bid auction. Still, as argued before, the expected revenue is always smaller in the descending auction.

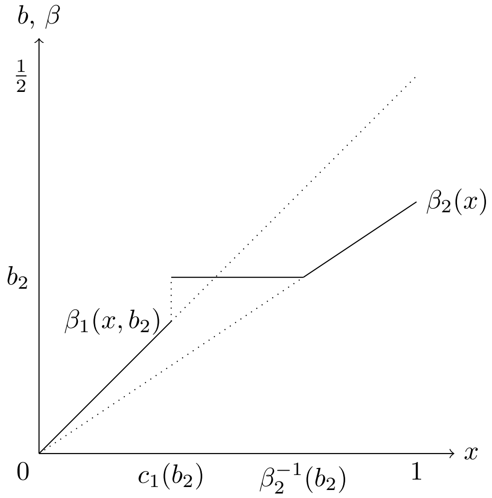

Figure 2: Equilibrium bidding strategies and simultaneous bidding for the uniform distribution, $N = 3$, and $K = 2$.

5 Summary

In this paper, we analyze bidders' optimal strategy in the discriminatory strictly descending multi-unit auction with independent private values and single-unit demand. Contrary to the arguments brought forward in the auction literature, the well established equilibrium in the first-price sealed-bid auction is not an equilibrium in the descending format. Analyzing the bidding behavior in the strictly descending multi-unit auction reveals that an equilibrium exists where once an object is sold, there is a set of bidder types who immediately accept the price in order to win the next object, i.e., simultaneous bidding occurs with positive probability. In this case, each of these bidders only receives the object with a certain probability which implies that the descending auction is not efficient. This inefficiency also has implications for the revenue comparison as revenue is always higher in the first-price sealed-bid auction.

Appendix

Proof of *Proposition 2*

Proof. To prove that for each K, there exists a symmetric equilibrium in the discriminatory multi-unit descending auction, we will use complete induction over K. We start the induction with $K = 1$. In this case, the problem reduces to a single-unit descending auction. From Krishna (2009) it is known that an equilibrium for this game exists and that this equilibrium is characterized by $\beta_1(x) = E[Y_1^{(N-1)} | Y_1^{(N-1)} < x]$.

The induction hypothesis can be stated as follows:

Induction hypothesis: *For each $k < K$, there exists a symmetric equilibrium in the discriminatory strictly descending multi-unit auction with single-unit demand and independent private values.*

To complete the induction, we have to show that if the induction hypothesis is true for all $k \leq K - 1$, then it is also true for all $k \leq K$. We divide the proof of the inductive step into four parts. More precisely, we show the following:

(i) If the first object is sold at a price b, the subsequent subgame has an equilibrium in cut-off strategies.

(ii) The necessary condition for a bidding function for the first object to be part of an equilibrium constitutes an initial value problem (IVP).

(iii) The IVP from part (ii) has a unique solution.

(iv) The solution to the IVP from part (iii) constitutes the equilibrium bidding function for the first object.

Ad (i): suppose K objects are for sale to N bidders and the first object was sold at a price b. In an increasing equilibrium the valuation of the bidder who has won the object becomes common knowledge. Denote this valuation by $\bar{x}$. Each of the remaining $N - 1$ bidders can observe that one of the objects was sold and is faced with the following decision problem: he can either bid for one of the $K-1$ remaining objects or wait and observe how many objects will be bought at price b and whether the auction will continue. We will consider an equilibrium in cut-off strategies in which each bidder with a valuation $x \geq c_{K-1}(b)$ bids immediately and bidders with valuation $x < c_{K-1}(b)$ wait for the auction to continue. If a bidder with valuation x decides to wait, either all the other objects are bought at b and the expected utility of waiting is 0 or $n < K-1$ objects are being bought and the expected utility is the same as in the descending auction with $K-1-n$ items conditional on the highest valuation bidder being $c^* := c_{K-1}(b)$. We will refer to this expected utility as $U^w_{K-1-n}(x, c^*)$.[15] If a bidder decides to bid immediately, he may not be the only one to do so. More precisely, each bidder with a valuation higher than c^* will also bid immediately. Therefore, the expected utility of bidding immediately is given by

$$\begin{aligned} U^b_{K-1}(x, c^*) = &\left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)(x-b) + \\ &\left(\sum_{n=K-1}^{N-2} \frac{K-1}{n+1} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)(x-b). \end{aligned} \quad (1)$$

The expected utility of waiting depends on how many of the objects were bought at price b and amounts to

$$U^w_{K-1}(x, c^*) = \left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} U^w_{K-1-n}(x, c^*, b)\right).$$

From the induction hypothesis we know that an equilibrium in each subsequent subgame exists. The necessary condition for c^* to be an equilibrium cut-off value is

$$\begin{aligned} & U^b_{K-1}(c^*, c^*) = U^w_{K-1}(c^*, c^*) \\ \Leftrightarrow\ & U^b_{K-1}(c^*, c^*) - U^w_{K-1}(c^*, c^*) = 0 \\ \Leftrightarrow\ & \left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)(c^* - b - U^w_{K-1-n}(c^*, c^*, b)) + \\ & \left(\sum_{n=K-1}^{N-2} \frac{K-1}{n+1} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)(c^* - b) = 0. \end{aligned} \quad (2)$$

Substituting $c^* = 0$ in the left-hand side of equation (2) yields $-b$. Substituting $c^* = \bar{x}$ yields $\bar{x} - b - U^w_{K-1}(\bar{x}, \bar{x}, b)$. If this expression is negative, the type with the highest valuation would never bid immediately, independent of the cut-off value. The problem reduces to a discriminatory descending auction with $K-1$ objects because every bidder would like to wait for the auction to continue.[16] If $\bar{x} - b - U^w_{K-1}(\bar{x}, \bar{x}, b)$ is non-negative, we can conclude from the intermediate value theorem that there exists

[15] Note that $U^w_{K-1-n}(x, c^*)$ depends on the reservation price r and therefore c^* will also depend on r.

[16] In this case, the induction hypothesis ensures existence of an equilibrium.

a c^* that solves equation (2). To show that c^* indeed constitutes an equilibrium, we prove that $U^b_{K-1}(x, c^*) - U^w_{K-1}(x, c^*)$ is increasing in x. Differentiating with respect to x yields

$$\left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} \right) \left(1 - \frac{\partial U^w_{K-1-n}(x, c^*, b)}{\partial x}\right) + \\ \left(\sum_{n=K-1}^{N-2} \frac{K-1}{n+1} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^n \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} \right).$$

Clearly, $\sum_{n=K-1}^{N-2}(K-1)/(n+1)\binom{N-2}{n}(1 - F(c^*)/F(\bar{x}))^n(F(c^*)/F(\bar{x}))^{N-n-2} \geq 0$ holds for all $c^* \in [0, \bar{x}]$. Therefore, in order to show that the derivative is positive, it suffices to show that $1 - \partial U^w_{K-1-n}(x, c^*, b)/\partial x \geq 0$ holds for each $n \leq K-1$ and each $x \in [0, \bar{x}]$. The induction hypothesis ensures the existence of an equilibrium in each of the subsequent subgames. Therefore, the revelation principle may be applied to each subgame. However, in a direct mechanism, the derivative of the expected utility of type x with respect to x is equal to his probability to win one of the objects.[17] Therefore, we can deduct $1 \geq \partial U^w_{K-1-n}(x, c^*, b)/\partial x$. This concludes part (i) of the inductive step.

Ad (ii): we have shown that if the first object is sold for a price b, an equilibrium in cut-off strategies for the subsequent subgame exists. To conclude the proof, we will show that an increasing, differentiable equilibrium bidding function $\beta_K(x)$ for the first object exists. Suppose such a function exists, all but bidder 1 follow the strategy $\beta_K(x_i)$, and bidder 1 intends to stop the auction at b. As $\beta_K(x)$ is increasing, it cannot be optimal for bidder 1 to stop the auction at a price b higher than $\beta_K(1)$. Hence, there exists a $z \in [0, 1]$ such that $b = \beta_2(z)$. In order to derive the properties of the optimal bidding strategy, we need to consider the difference in expected utility underlying the following decision problem. Suppose the price on the price clock has reached $b = \beta_K(z)$ and the bidder has to decide whether to stop the clock or let the auction continue. If he stops the clock, he gains $x_1 - \beta_K(z)$ for sure. Now suppose that he lets the auction continue for one more tick of the price clock. He then faces a trade-off of paying less for the unit and gaining $x_1 - \beta_K(z - \epsilon)$ compared to the risk of not receiving the object at all. Two cases are relevant: first, with probability $(F(z - \epsilon)/F(z))^{N-1}$, none of the other bidders stop the clock. In this case, he gains $x_1 - \beta_K(z - \epsilon)$ with probability 1. Second, with probability $1 - (F(z - \epsilon)/F(z))^{N-1}$, one of the other bidders stops the clock at a price $\beta_K(\tilde{z})$ with $\tilde{z} \in [z - \epsilon, z]$. As we have shown above, an equilibrium in cut-off strategies exists in the subsequent game. Therefore, bidder 1 would either bid immediately and expect a utility of $U^b_{K-1}(x_1, c(\beta_K(\tilde{z}))$ or he would wait and expect a utility of $U^w_{K-1}(x_1, c(\beta_K(\tilde{z}))$. Thus, the change in expected utility from waiting for another

[17] See Myerson (1981) for a proof.

tick of the price clock either amounts to

$$\Delta U^b = (x_1 - \beta_K(z-\epsilon)) \left(\frac{F(z-\epsilon)}{F(z)}\right)^{N-1} + \left(1 - \left(\frac{F(z-\epsilon)}{F(z)}\right)^{N-1}\right) E\left[U^b_{K-1}(x_1, c(\beta_K(\tilde{z}))\middle| z-\epsilon \leq \tilde{z} \leq z\right] - (x_1 - \beta_K(z))$$

if it is optimal to submit a bid immediately in the subsequent subgame or to

$$\Delta U^w = (x_1 - \beta_K(z-\epsilon)) \left(\frac{F(z-\epsilon)}{F(z)}\right)^{N-1} + \left(1 - \left(\frac{F(z-\epsilon)}{F(z)}\right)^{N-1}\right) E\left[U^w_{K-1}(x_1, c(\beta_K(\tilde{z}))\middle| z-\epsilon \leq \tilde{z} \leq z\right],$$

if it is optimal to wait in the subsequent subgame. Note that for any continuous function g and any random variable Y, which is distributed according to a distribution function F with continuous density f,

$$\lim_{\epsilon \to 0} E[g(Y)|x - \epsilon \leq Y \leq x] = g(x)$$

holds.

As $\partial U^{\{b,w\}}(z)/\partial z = \lim_{\epsilon \to 0^+} \Delta U^{\{b,w\}}/\epsilon$, the marginal change in expected utility can be written as

$$\frac{\partial U^b(z)}{\partial z} = -\beta_K'(z) + (x_1 - \beta_K(z)) \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} - \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} U^b_{K-1}(x_1, c(\beta_K(z))) \quad (3)$$

if it is optimal to submit a bid and as

$$\frac{\partial U^w(z)}{\partial z}(z) = -\beta_K'(z) + (x_1 - \beta_K(z)) \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} - \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} U^w_{K-1}(x_1, c(\beta_K(z))) \quad (4)$$

if it is optimal to wait. In a symmetric equilibrium, the expected profit is maximized at $z = x_1$. Moreover, if the price clock shows $\beta_K(x_1)$ in equilibrium, it is always optimal for bidder 1 to bid immediately in the subsequent subgame and expression (3) is relevant.[18] Thus, the first-order condition is $U'(x_1) = 0$. Rearranging equation

[18] Why will bidder 1 bid immediately? Suppose the opposite is true. As $\beta_K(x)$ is increasing, x_1 is the highest valuation in the auction. As we have shown above, the subsequent game has an equilibrium in cut-off strategies. Therefore, none of the other bidders would like to bid

(3) then gives

$$\beta_K'(x_1) = (x_1 - \beta_K(x_1))\,\frac{(N-1)f(x_1)F(x_1)^{N-2}}{F(x_1)^{N-1}} - \frac{(N-1)f(x_1)F(x_1)^{N-2}}{F(x_1)^{N-1}} U_{K-1}^{b}(x_1, c(\beta_K(x_1))). \quad (5)$$

Substituting expression (1) into the right-hand side of equation (5) yields

$$\beta_K'(x_1) = (x_1 - \beta_K(x_1)) \Bigg(\frac{(N-1)f(x_1)F(x_1)^{N-2}}{F(x_1)^{N-1}} - \frac{(N-1)f(x_1)F(x_1)^{N-2}}{F(x_1)^{N-1}} \left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^{n} \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} + \sum_{n=K-1}^{N-2} \frac{K-1}{n+1} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^{n} \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} \right) \Bigg). \quad (6)$$

Together with $\beta_K(r) = r$, expression (6) constitutes an initial value problem.

Ad(iii): if $r > 0$, it follows that $F(r) > 0$ and therefore

$$\Bigg(\frac{(N-1)f(x_1)F(x_1)^{N-2}}{F(x_1)^{N-1}} - \frac{(N-1)f(x_1)F(x_1)^{N-2}}{F(x_1)^{N-1}} \left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^{n} \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} + \sum_{n=K-1}^{N-2} \frac{K-1}{n+1} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^{n} \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} \right) \Bigg) \quad (7)$$

is bounded for all $x_1 \in [r, 1]$. Hence, we can deduct that the right-hand side of equation (6) is globally Lipschitz-continuous. From the Picard-Lindelöf theorem[19] it then follows that the initial value problem has a unique, differentiable solution $\beta_K(x_1)$. We have found that the first-order condition yields a unique, differentiable function. It remains to be shown that a solution of equation (6) is increasing and indeed optimal. Note that

$$\left(\sum_{n=0}^{K-2} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^{n} \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} + \sum_{n=K-1}^{N-2} \frac{K-1}{n+1} \binom{N-2}{n} \left(1 - \frac{F(c^*)}{F(\bar{x})}\right)^{n} \left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} \right) \quad (8)$$

immediately and bidder 1 would receive the object with probability 1 and pay a price less than $\beta_K(x_1)$. This implies that $\beta_K(x)$ is not optimal.

[19] A version of this theorem can be found in Coddington and Levinston (1955).

is a probability and therefore smaller than or equal to 1. Hence, expression (7) is positive and together with $x_1 - \beta_K(x_1) > 0$, it follows that $\beta'_K(x_1)$ is positive and thus $\beta_K(x_1)$ is increasing.

Ad (iv): suppose that z is such that equation (3) is relevant. Plugging expression (6) into equation (3) yields

$$\begin{aligned}\frac{\partial U^b(z)}{\partial z} = &-(z-\beta_K(z))\left(\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} - \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}}\right.\\ &\left(\sum_{n=0}^{K-2}\binom{N-2}{n}\left(1-\frac{F(c^*)}{F(\bar{x})}\right)^n\left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} + \right.\\ &\left.\left.\sum_{n=K-1}^{N-2}\frac{K-1}{n+1}\binom{N-2}{n}\left(1-\frac{F(c^*)}{F(\bar{x})}\right)^n\left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)\right)\\ &+(x_1-\beta_K(z))\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} - \\ &\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}}U^b_{K-1}(x_1, c(\beta_K(z))). \qquad (9)\end{aligned}$$

Using expression (1), equation (9) simplifies to

$$\begin{aligned}\frac{\partial U^b(z)}{\partial z} = &(x_1-z)\left(\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} - \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}}\right.\\ &\left(\sum_{n=0}^{K-2}\binom{N-2}{n}\left(1-\frac{F(c^*)}{F(\bar{x})}\right)^n\left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2} + \right.\\ &\left.\left.\sum_{n=K-1}^{N-2}\frac{K-1}{n+1}\binom{N-2}{n}\left(1-\frac{F(c^*)}{F(\bar{x})}\right)^n\left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)\right).\end{aligned}$$

As noted before, expression (7) is positive. Therefore, if $z < x_1$, then $\partial U^b(z)/\partial z > 0$ and if $z > x_1$, then $\partial U^b(z)/\partial z < 0$. It is thus clear that $z = x_1$ maximizes the expected utility. It remains to be checked whether $z = x_1$ is also optimal if equation (4) is relevant. It is clear that equation (4) can only be relevant if $z > x_1$.[20] Plugging

[20] If it is optimal to bid immediately in the subsequent auction when the price is $\beta_K(x_1)$, it is also optimal to bid immediately at any other lower price.

expression (6) into equation (4) yields

$$\begin{aligned}\frac{\partial U^w(z)}{\partial z} = &-(z-\beta_K(z))\left(\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}} - \frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}}\right.\\ &\left(\sum_{n=0}^{K-2}\binom{N-2}{n}\left(1-\frac{F(c^*)}{F(\bar{x})}\right)^n\left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}+\right.\\ &\left.\left.\sum_{n=K-1}^{N-2}\frac{K-1}{n+1}\binom{N-2}{n}\left(1-\frac{F(c^*)}{F(\bar{x})}\right)^n\left(\frac{F(c^*)}{F(\bar{x})}\right)^{N-n-2}\right)\right)\\ &+(x_1-\beta_K(z))\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}}-\\ &\frac{(N-1)f(z)F(z)^{N-2}}{F(z)^{N-1}}U^w_{K-1}(x_1,c(\beta_K(z))). \quad (10)\end{aligned}$$

If waiting in the subsequent subgame is optimal, then

$$U^b_{K-1}(x_1,c(\beta_K(z))) \leq U^w_{K-1}(x_1,c(\beta_K(z)))$$

holds. Therefore, the right-hand side of equation (9) is greater than the right-hand side of equation (10). It then follows that $\partial U^b(z)/\partial z < 0$ if $z > x_1$.

Summing up, we have shown that a unique solution to the initial value problem exists if $r > 0$ and that this solution indeed constitutes the equilibrium bidding function for the first object. □

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