Successive Differentiation

Let us try the effect of repeating several times over the operation of differentiating a function (see here). Begin with a concrete case.

Let $y = x^5$.

First differentiation,	$5x^4$.	
Second differentiation,	$5 \times 4x^3$	$= 20x^3$.
Third differentiation,	$5 \times 4 \times 3x^2$	$= 60x^2$.
Fourth differentiation,	$5 \times 4 \times 32x$	$= 120x$.
Fifth differentiation,	$5 \times 4 \times 3 \times 2 \times 1$	$= 120$.
Sixth differentiation,		$= 0$.

There is a certain notation, with which we are already acquainted (see here), used by some writers, that is very convenient. This is to employ the general symbol $f(x)$ for any function of x. Here the symbol $f()$ is read as "function of," without saying what particular function is meant. So the statement $y = f(x)$ merely tells us that y is a function of x, it may be x^2 or ax^n, or $\cos x$ or any other complicated function of x.

The corresponding symbol for the differential coefficient is $f'(x)$, which is simpler to write than $\frac{dy}{dx}$. This is called the "derived function" of x.

Suppose we differentiate over again, we shall get the "second derived function" or second differential coefficient, which is denoted by $f''(x)$; and so on.

Now let us generalize.

Let $y = f(x) = x^n$.

$$\text{First differentiation, } f'(x) = nx^{n-1}.$$
$$\text{Second differentiation, } f''(x) = n(n-1)x^{n-2}.$$
$$\text{Third differentiation, } f'''(x) = n(n-1)(n-2)x^{n-3}.$$
$$\text{Fourth differentiation, } f''''(x) = n(n-1)(n-2)(n-3)x^{n-4}.$$
$$\text{etc., etc.}$$

Learning outcomes:
Author(s):

But this is not the only way of indicating successive differentiations. For,

$$\text{if the original function be } y = f(x);$$

$$\text{once differentiating gives } \frac{dy}{dx} = f'(x);$$

$$\text{twice differentiating gives } \frac{d\left(\frac{dy}{dx}\right)}{dx} = f?(x);$$

and this is more conveniently written as $\dfrac{d^2y}{(dx)^2}$, or more usually $\dfrac{d^2y}{dx^2}$. Similarly, we may write as the result of thrice differentiating, $\dfrac{d^3y}{dx^3} = f'''(x)$.

Now let us try $y = f(x) = 7x^4 + 3.5x^3 - \frac{1}{2}x^2 + x - 2$.

$$\begin{aligned}
\frac{dy}{dx} &= f'(x) = 28x^3 + 10.5x^2 - x + 1,\\
\frac{d^2y}{dx^2} &= f''(x) = 84x^2 + 21x - 1,\\
\frac{d^3y}{dx^3} &= f'''(x) = 168x + 21,\\
\frac{d^4y}{dx^4} &= f''''(x) = 168,\\
\frac{d^5y}{dx^5} &= f'''''(x) = 0.
\end{aligned}$$

In a similar manner if $y = \phi(x) = 3x(x^2 - 4)$,

$$\begin{aligned}
\phi'(x) &= \frac{dy}{dx} = 3\big[x2x + (x^2 - 4)1\big] = 3(3x^2 - 4),\\
\phi''(x) &= \frac{d^2y}{dx^2} = 36x = 18x,\\
\phi'''(x) &= \frac{d^3y}{dx^3} = 18,\\
\phi''''(x) &= \frac{d^4y}{dx^4} = 0.
\end{aligned}$$

Exercises IV

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for the following expressions:

(1) $y = 17x + 12x^2$.

(2) $y = \dfrac{x^2 + a}{x + a}$.

(3) $y = 1 + \dfrac{x}{1} + \dfrac{x^2}{1 \times 2} + \dfrac{x^3}{1 \times 2 \times 3} + \dfrac{x^4}{1 \times 2 \times 3 \times 4}$.

(4) Find the 2nd and 3rd derived functions in the Exercises III. (here), No. 1 to No. 7, and in the Examples given (here), No. 1 to No. 7.

Answers

(1)$17 + 24x$; 24.

(2)$\dfrac{x^2 + 2ax - a}{(x+a)^2}$; $\dfrac{2a(a+1)}{(x+a)^3}$.

(3)$1 + x + \dfrac{x^2}{12} + \dfrac{x^3}{1 \times 2 \times 3}$; $1 + x + \dfrac{x^2}{1 \times 2}$.

Exercises III

(1) (a) $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \ldots$.

(b) $2a$, 0.

(c) 2, 0.

(d) $6x + 6a$, 6.

(2) $-b$, 0.

(3) 2, 0.

(4) $56440x^3 - 196212x^2 - 4488x + 8192$.

$169320x^2 - 392424x - 4488$.

(5) 2, 0.

(6) $371.80453x$, 371.80453.

(7) $\dfrac{30}{(3x+2)^3}$, $-\dfrac{270}{(3x+2)^4}$.

Examples:

(1) $\dfrac{6a}{b^2}x$; $\dfrac{6a}{b^2}$.

(2) $\dfrac{3a\sqrt{b}}{2\sqrt{x}} - \dfrac{6b\sqrt[3]{a}}{x^3}$, $\dfrac{18b\sqrt[3]{a}}{x^4} - \dfrac{3a\sqrt{b}}{4\sqrt{x^3}}$.

(3) $\dfrac{2}{\sqrt[3]{\theta^8}} - \dfrac{1.056}{\sqrt[5]{\theta^{11}}}$, $\dfrac{2.3232}{\sqrt[5]{\theta^{16}}} - \dfrac{16}{3\sqrt[3]{\theta^{11}}}$.

(4) $810t^4 - 648t^3 + 479.52t^2 - 139.968t + 26.64$.

$3240t^3 - 1944t^2 + 959.04t - 139.968$.

(5) $12x + 2$, 12.

(6) $6x^2 - 9x$, $12x - 9$.

(7)

$$\frac{3}{4}\left(\frac{1}{\sqrt{\theta}}+\frac{1}{\sqrt{\theta^5}}\right)+\frac{1}{4}\left(\frac{15}{\sqrt{\theta^7}}-\frac{1}{\sqrt{\theta^3}}\right).$$
$$\frac{3}{8}\left(\frac{1}{\sqrt{\theta^5}}-\frac{1}{\sqrt{\theta^3}}\right)-\frac{15}{8}\left(\frac{7}{\sqrt{\theta^9}}+\frac{1}{\sqrt{\theta^7}}\right).$$