Notes on Finite Difference Method for Cosserat Rod Dynamics

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Abstract

These are my notes on the finite difference method for Cosserat rod dynamics.

Contents

1 Introduction

Cosserat rod are an important model for soft slender robotic systems such as continuum manipulators. These soft robotic systems have drawn recent research interest due to the natural compliance with the environment as well as the opportunities for dynamic modeling/numerical simulation [2, 15, 22, 23] novel state estimation [12, 13, 26], and control [1, 8]. Soft slender robots have found applications in biomedicine [7] with is use in surgical robotics, mobile robotics [25] and human robot interaction [16].

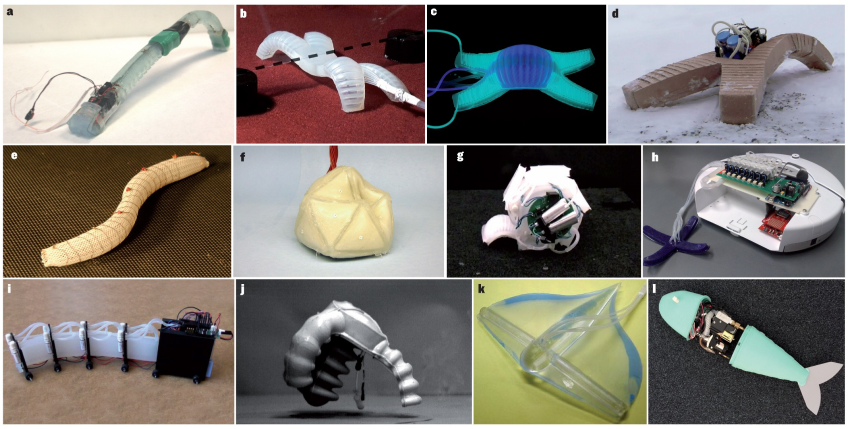

Figure 1: Examples of soft robotic systems [20].

These notes are meant the be a quick refresher, or introduction to Cosserat rod theory from the modern geometric mechanics perspective [10, 14]. In the pursuit to be self-contained, we very briefly review mathematical preliminaries used to derive the equations of motion, this includes the Special Euclidean group and some identities from the calculus from Lie theory [10]. We state the equations of motion first in the Lagrangian formalism. We perform the finite difference method on the Cosserat PDEs by introducing a second-order backward difference for the time dependent variables and then solving the resulting boundary value problem with the shooting method[9, 21, 22]. These notes are self contained, but some ideas are drawn/inspired from the following sources: geometric mechanics[10, 11, 14], Cosserat theory [3, 4, 6, 17, 18, 19].

2 Preliminaries

To make these notes somewhat self contained, we review some basic definitions of rigid motion groups and we review the strong form of the dynamic Cosserat equations of motion based on Lie group theory.

2.1 Special Euclidean Group

Rigid Rotations The Special Orthogonal group $SO(3)$ is the matrix group of rigid rotations defined by

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, R^T R = I, \quad \det R = 1\}. \tag{1}$$

with inverse $R^{-1} = R^T$. It's Lie algebra $\mathfrak{so}(3) \subset \mathbb{R}^{3\times 3}$ is the set of all skew-symmetric matrices $\widehat{\boldsymbol{\Omega}} + \widehat{\boldsymbol{\Omega}} = \mathbf{0}$ which can be shown to be tangent space at the identity $\mathfrak{so}(3) = T_I SO(3)$ [11, 14]. There exists an isomorphism $.^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ defined implicitly $\widehat{\boldsymbol{\Omega}}\boldsymbol{r} = \boldsymbol{\Omega} \times \boldsymbol{r}$ for any $\boldsymbol{r} \in \mathbb{R}^3$. The hat map $\widehat{.} : \mathbb{R}^3 \to \mathfrak{so}(3)$ preserves the Lie algebra structure of $(\mathbb{R}^3, \times)$,

$$\widehat{\boldsymbol{\Omega} \times \boldsymbol{r}} = \widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{r}} - \widehat{\boldsymbol{r}}\widehat{\boldsymbol{\Omega}}. \tag{2}$$

Rigid Motion The Special Euclidean group $SE(3)$ is the matrix group of all rigid motions (rotation and translations) of $\mathbb{R}^3$. It is defined as the semi-direct product $SE(3) = SO(3) \times \mathbb{R}^3$ represented by 4×4 matrices

$$g = \begin{bmatrix} R & \boldsymbol{r} \\ \mathbf{0} & 1 \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} R^T & -R^T\boldsymbol{r} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{3}$$

The Lie algebra $\mathfrak{se}(3) = \mathfrak{so}(3) \times \mathbb{R}^3$ is the set of all twists represented by 4×4 matrices

$$\widehat{\boldsymbol{X}} = \begin{bmatrix} \widehat{\boldsymbol{\Omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \widehat{\boldsymbol{X}}^\vee = \begin{bmatrix} \boldsymbol{\Omega} \\ \boldsymbol{v} \end{bmatrix}, \tag{4}$$

where $.^\vee : \mathfrak{se}(3) \to \mathbb{R}^6$ is the inverse hat map for $\mathfrak{se}(3)$ defined in terms of the inverse hat map for $\mathfrak{so}(3)$. For $g = (R, \boldsymbol{r}) \in SE(3)$, the Adjoint map $\mathrm{Ad}_g : \mathfrak{se}(3)^\vee \to \mathfrak{se}(3)^\vee$ and for $\widehat{\boldsymbol{X}} \in \mathfrak{se}(3)$, the adjoint map $\mathrm{ad}_{\hat{\boldsymbol{X}}} : \mathfrak{se}(3) \to \mathfrak{se}(3)$ are represented by 6×6 matrices

$$[\mathrm{Ad}_g] = \begin{bmatrix} R & \widehat{\boldsymbol{r}}R \\ \mathbb{O} & R \end{bmatrix}, \quad [\mathrm{ad}_{\widehat{\boldsymbol{X}}}] = \begin{bmatrix} \widehat{\boldsymbol{\Omega}} & \widehat{\boldsymbol{v}} \\ \mathbb{O} & \widehat{\boldsymbol{\Omega}} \end{bmatrix} \tag{5}$$

where $\widehat{\boldsymbol{X}}^\vee = (\boldsymbol{\Omega}, \boldsymbol{v}) \in \mathbb{R}^6$. The corresponding coAdjoint and coadjoint maps acting on the dual Lie algebra $\mathfrak{se}(3)^*$ are given by

$$[\mathrm{Ad}_g^*] = \begin{bmatrix} R^T & \mathbb{O} \\ -R^T\widehat{\boldsymbol{r}} & R^T \end{bmatrix}, \quad [\mathrm{ad}_{\widehat{\boldsymbol{X}}}^*] = \begin{bmatrix} -\widehat{\boldsymbol{\Omega}} & \mathbb{O} \\ -\widehat{\boldsymbol{v}} & -\widehat{\boldsymbol{\Omega}} \end{bmatrix} \tag{6}$$

2.2 Cosserat Rod Dynamics

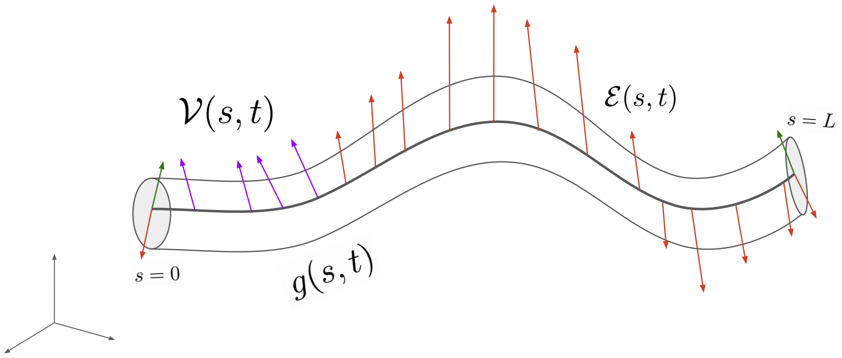

Figure 2: Illustration of a Cosserat rod

The configuration space for a Cosserat rod of length L is $\mathcal{Q} = \{g(s) \in SE(3), | \, s \in [0, L]\}$, the space of all embedded smooth curves in the Special Euclidean group. A motion $g(s,t)$ of the rod is a smooth two-parameter hypersurface in $SE(3)$. We define the body (material) frame strain field on the curve to be left translation of the spatial derivative

$$\mathcal{E}(s,t) = \left(g^{-1}(s,t)\frac{\partial g}{\partial s}(s,t)\right)^{\vee} = \begin{bmatrix} \boldsymbol{K}(s,t) \\ \boldsymbol{\Gamma}(s,t) \end{bmatrix} \tag{7}$$

where $\boldsymbol{K}(s,t)$ is the angular strain field and $\boldsymbol{\Gamma}(s,t)$ is the linear strain field. The body velocity field of the motion is given by

$$\mathcal{V}(s,t) = \left(g^{-1}(s,t)\frac{\partial g}{\partial t}(s,t)\right)^{\vee} = \begin{bmatrix} \boldsymbol{\Omega}(s,t) \\ \boldsymbol{V}(s,t) \end{bmatrix}, \tag{8}$$

where $\boldsymbol{\Omega}(s,t)$ is the body angular velocity field and $\boldsymbol{V}(s,t)$ is the body linear velocity field.

Lemma 1. *The compatibility conditions for a motion $g(s,t)$ to be $C^2(SE(3))$ is stated as a differential relationship between the velocity and strain fields*

$$\frac{\partial \mathcal{V}}{\partial s} = ad_{\mathcal{V}}\mathcal{E} + \frac{\partial \mathcal{E}}{\partial t} \tag{9}$$

Proof. Follows from a simple computation. □

An expression for the (reduced) Lagrangian density under the inverse hat map is given by

$$L(\mathcal{V}, \mathcal{E}) = \frac{1}{2}\mathcal{V}^T \mathbb{M}(s)\, \mathcal{V} - (\mathcal{E} - \mathcal{E}^*)^T\, \mathbb{K}(s)\; (\mathcal{E} - \mathcal{E}^*) \tag{10}$$

where $\mathbb{M}(s)$ is the mass matrix, and $\mathbb{K}(s)$ is the stiffness matrix both defined per unit length, and $\mathcal{E}^*$ is the reference strain. The former defined the total kinetic energy and the later describes the elastic internal energy.

Lemma 2. *Let $\widehat{\boldsymbol{\psi}} = g^{-1}\delta g \in \mathfrak{se}(3)$ be the variation of the motion $g(s,t)$ induced by the left action of $SE(3)$ i.e., $\delta g = T_e L_g(\widehat{\boldsymbol{\psi}}) = g\widehat{\boldsymbol{\psi}}$. Then, the variations of the velocity and strain fields satisfy*

$$\delta\mathcal{V} = \frac{\partial \boldsymbol{\psi}}{\partial t} + ad_{\mathcal{V}}\boldsymbol{\psi}, \quad \delta\mathcal{E} = \frac{\partial \boldsymbol{\psi}}{\partial s} + ad_{\mathcal{E}}\boldsymbol{\psi}. \tag{11}$$

Proof. A straightforward computation. □

We also assume the work (density) W_{ext} exerted by external force density $\boldsymbol{F}_{ext}$ and viscoelastic material $\mathbb{D}\dot{\mathcal{E}}$ proportional to the time variation of the strain and an active constitutive law $\boldsymbol{F}_a(s,t)$. By the principle of virtual work, the variation of the work (density) δW_{ext} satisfies

$$\delta W_{ext} = \int_0^L \langle \boldsymbol{F}_{ext}, \psi_g\rangle + \langle \boldsymbol{\Lambda}_{ext}, \delta\mathcal{E}_h\rangle\; ds \tag{12}$$

$$+ \big\langle \boldsymbol{F}_{ext,BC}(0), \boldsymbol{\psi}_g(0)\big\rangle - \big\langle \boldsymbol{F}_{ext,BC}(L), \boldsymbol{\psi}_g(L)\big\rangle \tag{13}$$

where the *external stains* given by $\boldsymbol{\Lambda}_{ext} = -\mathbb{D}\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\Lambda}_a$ are visoelastic strains and actuation inputs.

Theorem 1 (Strong Cosserat Equations of Motion)**.** *Let $g(s,t) \in Q$ be a motion of a Cosserat rod and let $\mathcal{V}(s,t)$ and $\mathcal{E}(s,t)$ be its velocity and strain fields respectively. Assume that the system is subjected to external work W_{ext} defined by the forces $\boldsymbol{F}_{ext}$, a (linear) visoelatic material $\mathbb{B}\dot{\mathcal{E}}$ and active constitutive law $\boldsymbol{\Lambda}_a$. Then, the strong form of equations of motion for a Cosserat rod are given by the system*

$$\frac{\partial}{\partial s} g = g\widehat{\mathcal{E}}, \quad \frac{\partial}{\partial t} g = g\widehat{\mathcal{V}} \tag{14}$$

$$\frac{\partial}{\partial s}\mathcal{V} = \frac{\partial}{\partial t}\mathcal{E} + ad_{\mathcal{V}}\mathcal{E} \tag{15}$$

$$\frac{\partial}{\partial s}\boldsymbol{\Lambda} = \frac{\partial}{\partial t}\mathbb{M}\mathcal{V} - ad^*_{\mathcal{V}}\mathbb{M}\mathcal{V} + ad^*_{\mathcal{E}}\boldsymbol{\Lambda} - \boldsymbol{F}_{ext} \tag{16}$$

where $\boldsymbol{\Lambda} = -\mathbb{K}(\mathcal{E} - \mathcal{E}^*) - \mathbb{D}\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\Lambda}_a$ is the stress field associated to the strain field $\mathcal{E}$ and actuation force $\boldsymbol{\Lambda}_a$. The system of Cosserat PDEs are either subjected to one of the following boundary conditions:

1. *Cantilevered Boundary conditions:*

$$g(0,t) = g_0(t), \quad \mathcal{V}(0,t) = \mathcal{V}_0(t), \quad \mathbf{\Lambda}(L,t) = \boldsymbol{F}_{BC,L}(t) \tag{17}$$

2. *Free-Free Boundary conditions:*

$$\mathbf{\Lambda}(0,t) = -\boldsymbol{F}_{BC,0}(t), \quad \mathbf{\Lambda}(L,t) = \boldsymbol{F}_{BC,L}(t). \tag{18}$$

Proof. Follows from Hamilton's variational principle. □

3 Finite Difference Method

The finite difference method for two parameter PDEs such as the Cosserat Rod is to discretize the time derivatives in the system (15)-(16) by a second order backward difference

$$\frac{\partial}{\partial t}\mathcal{E}(s,t) \approx \frac{1}{\Delta t}\left(\frac{3}{2}\mathcal{E}(s,t) - 2\mathcal{E}(s,t-\Delta t) + \frac{1}{2}\mathcal{E}(s,t-2\Delta t)\right), \tag{19}$$

$$\frac{\partial}{\partial t}\mathcal{V}(s,t) \approx \frac{1}{\Delta t}\left(\frac{3}{2}\mathcal{V}(s,t) - 2\mathcal{V}(s,t-\Delta t) + \frac{1}{2}\mathcal{V}(s,t-2\Delta t)\right). \tag{20}$$

To simplify notation, we collect the terms that depend on previous time steps Δt,

$$\frac{\partial}{\partial t}\mathcal{E}(s,t) \approx c_0\mathcal{E}(s,t) + \mathcal{E}^h(s,t), \tag{21}$$

$$\frac{\partial}{\partial t}\mathcal{V}(s,t) \approx c_0\mathcal{V}(s,t) + \mathcal{V}^h(s,t) \tag{22}$$

$$c_0 = \frac{3}{2\Delta t}, \quad c_1 = -\frac{2}{\Delta t}, \quad c_2 = \frac{1}{2\Delta t} \tag{23}$$

Then, having discretized the time dependence, we arrive at a system of ordinary differential equations

$$\frac{\partial}{\partial s}g = g\widehat{\mathcal{E}} \tag{24}$$

$$\frac{\partial}{\partial s}\mathcal{V} = c_0\,\mathcal{E}(s,t) + \mathcal{E}^h(s,t) + \mathrm{ad}_{\mathcal{V}}\mathcal{E} \tag{25}$$

$$\frac{\partial}{\partial s}\mathbf{\Lambda} = \mathbb{M}\left(c_0\,\mathcal{V}(s,t) + \mathcal{V}^h(s,t)\right) - \mathrm{ad}^*_{\mathcal{V}}\mathbb{M}\mathcal{V} + \mathrm{ad}^*_{\mathcal{E}}\mathbf{\Lambda} - \boldsymbol{F}_{ext} \tag{26}$$

$$\mathcal{E}(s,t) = (\mathbb{K} + c_0\mathbb{D})^{-1}\left(\mathbf{\Lambda} - \mathbf{\Lambda}_a + \mathbb{K}\mathcal{E}^*(s) - \mathbb{D}\mathcal{E}^h(s,t)\right) \tag{27}$$

subject to one of the following boundary conditions

1. Cantilevered Boundary conditions:

$$g(0,t) = g_0(t), \quad \mathcal{V}(0,t) = \mathcal{V}_0(t), \quad \mathbf{\Lambda}(L,t) = \boldsymbol{F}_{BC,L}(t) \tag{28}$$

2. Free-Free Boundary conditions:

$$\mathbf{\Lambda}(0,t) = -\boldsymbol{F}_{BC,0}(t), \quad \mathbf{\Lambda}(L,t) = \boldsymbol{F}_{BC,L}(t). \tag{29}$$

3.1 Boundary Value Problem

3.1.1 Cantilevered B.C.

We consider the boundary value problem for the cantilevered boundary conditions. To solve the boundary value problem, we use the shooting method. At each time step t, we are given the boundary forces at the tip $F_{BC,L}(t)$, and initial conditions $g_0(t)$, and $\mathcal{V}_0(t)$ for the cantilevered configuration, we perform an optimization on the initial conditions $R(s=0,t)$ such that the cost

$$\mathcal{L} = ||\mathbf{\Lambda}_{integrated}(g_0, \mathcal{V}_0, \mathbf{\Lambda}_{0,guess}) - \boldsymbol{F}_{BC,L}||^2. \tag{30}$$

The term 'shooting method' is used to mean that we optimize a guess of initial conditions $(g_0(t), \mathcal{V}_0(t), \mathbf{\Lambda}(0,t))$ then integrate the equations of motion in such a way that the boundary conditions $\mathbf{\Lambda}(L,t) = \boldsymbol{F}_{BC,L}(t)$ are satisfied. The optimization may be computed using Levenberg–Marquardt algorithm.

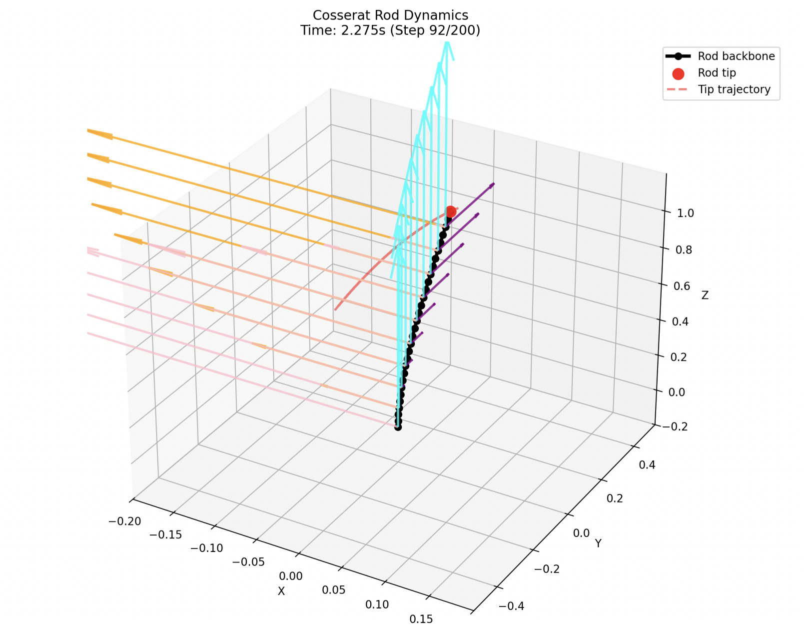

Figure 3: Finite Difference simulation of a Cosserat Rod

4 Conclusion

In these notes we reviewed the Cosserat rod equations and their numerical simulation using the finite difference method applied to the matrix Lie group representation. We stated two boundary value problems: the cantilevered case for soft robotic manipulators, and free-free boundary conditions for soft mobile robotic systems. We formulated the shooting method solution to the cantilevered rod. In future work, we formulate the shooting method for the free-free boundary conditions study numerical simulations of the Cosserat equations to include contact forces [24]. The study of numerical stability criterion for finite difference methods is an important direction in order for reliably applications of this method to soft robotic systems [5].

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