Case Study: Maternal Labor Supply and Child Academic Outcomes

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Background

This case study focuses on how a single mother, Lisa, allocates her time between work, child care, and leisure. The objective is to understand how different welfare reform policies might affect both Lisa's labor supply and her child's academic outcomes.

Scenario

Meet Lisa, a single mother in her early 30s, who is contemplating how to allocate her time between work, child care, and leisure as she navigates various welfare reform policies. The time frame of the decision is one week (168 hours).

Setting Up the Problem

Given:

- c = Consumption goods.
- L = Leisure time measured in hours.
- m = Time spent on maternal care measured in hours.
- l = Time spent on work measured in hours.

- T = Total available time in a week, such that $m + l + L = T$ (typically $T = 168$ hours).
- p = Price of consumption goods, assumed constant.
- w = Wage rate per hour.
- B = Welfare benefits (converted to weekly income).
- θ = Academic outcome of the child, given by $\theta = m^b$ for $b \in (0, 1)$.

Utility Function:

Lisa's utility function $U(c, L, \theta)$ captures her preferences for consumption, leisure, and her child's academic outcomes. The utility function is:

$$U(c, L, \theta) = c^a + L^a + \theta^a$$

Here, $a \neq 1$, is a preference parameter.

Budget Constraint:

Before Reform: Lisa consumes c at prices p, given labor income $W \times l$ before reform.

After Reform: Lisa consumes c at prices p, given labor income $W \times l + B$ after reform.

Time Constraint:

Before and After Reform:

$$m + l + L = T$$

Questions

1 Without Welfare Reform

1. **Budget and Time Constraints**: Write down Lisa's budget and time constraints before welfare reform. Explain the meaning of the budget.

Explain how θ affects the utility of Lisa $U(c, L, \theta)$. Are the preference concave, monotone? (If yes, for what values of a).

2. **Optimization**: Use the Lagrangian method to solve for Lisa's optimal choices of c, L, m, θ and l before and after reform. You can reduce the problem to 3 variables but not less than that. Given the utility function, you are asked to use, all the optimal quantities will be positive. Call the optimal values of this situation 1: $c^1, L^1, m^1, \theta^1, l^1$.

Answer:

1.1. $pc + w(m + L) \leq wT$, the utility $U(c, L, \theta) = c^a + L^a + \theta^a$ it s a CES utility function it is concave for $a \leq 1$ and and convex for $a > 1$, it is monotone as it increases its value with positive quantities of consumption of c, L, θ.

1.2. I write down the problem in terms of m replacing the equation $\theta = bm$

$\mathcal{L}(c, L, m) = c^a + L^a + (bm)^a + (wT - pc - w(m + L))$

We proceed to obtain the derivatives of $\mathcal{L}$ with respect to c, L, m:

$\frac{\partial \mathcal{L}}{\partial c} = ac^{-1+a} - p\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial L} = aL^{-1+a} - w\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial m} = ab(bm)^{-1+a} - w\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = wT - pc - w(m + L) = 0$

Now, we proceed to find optimal values of c, L, m in terms of the parameters of the problem, t his is done by solving the system of equatios implied by the FOC of the Lagrangian.

In this particular case, echoing what we have done in class, we write down two MRS conditions the first MRS(L, c) and MRS(m, c), then use that to obtain L in terms of c and parameters, and m in terms of c and parameters. Then we replace back in the budget constraint to obtain the optimal value of c in terms of parameters alone and then solve the whole problem.

The two MRS conditions are:

$$c^{-1+a}L^{1-a} = \frac{p}{w}$$

$$c^{-1+a}m(bm)^{-a} = \frac{p}{w}$$

From this I obtain:

$$L = c\left(\frac{p}{w}\right)^{\frac{1}{1-a}}$$

$$m = c\left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}$$

Then replace in the budget constraint:

$$Tw - c\left(p + \left(\left(\frac{p}{w}\right)^{\frac{1}{1-a}} + \left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}\right) w\right) = 0$$

Obtain the optimal quantities, as you can see LHS is full of parameters and no variable remains, also I have verified that this quantities satisfy the budget contraint:

$$c^{*1} = \frac{Tw}{p + \left(\left(\frac{p}{w}\right)^{\frac{1}{1-a}} + \left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}\right) w}$$

$$L^{*1} = \frac{Tw\left(\frac{p}{w}\right)^{\frac{1}{1-a}}}{p + \left(\left(\frac{p}{w}\right)^{\frac{1}{1-a}} + \left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}\right) w}$$

$$m^{*1} = \frac{Tw}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}}\left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}.$$

$$\theta^{*1} = b\frac{Tw}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}}\left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}$$

2 With the Welfare Reform

1. Write down Lisa's budget and time constraints after the welfare reform takes place. Explain the meaning of the budget.

2. **Optimization**: Use the Lagrangian method to solve for Lisa's optimal choices of c, L, m, and l before and after reform. You can reduce the problem to 3 variables but not less than that. Given the utility function, you are asked to use, all the optimal quantities will be positive. Call the optimal values $c^{*2}, L^{*2}, m^{*2}, \theta^{*2}$ and l^{*2}.

3. **Policy Analysis**: Based on the results before the reform, how generous should the welfare benefit B be in order to double the academic output ($\theta^{*2} = 2\theta^{*1}$)?

Answer:

2.1. When Lisa has $T - L - m > 0$ then the budget is $pc + w(m + L) \leq wT + B$, when $T - L - m = 0$ then the budget is $pc \leq B$. The explanation of this budget can echo the analogous discussion in Chapter 5.

2.2. $\mathcal{L}(c, L, m) = c^a + L^a + (bm)^a + (wT + B - pc - w(m + L))$

The solution for this case is completely analogous to the first part so I will omit and only write down the solutions in your case put all your steps.

$$c^{*2} = \frac{Tw + B}{p + \left(\left(\frac{p}{w}\right)^{\frac{1}{1-a}} + \left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}\right) w}$$

$$L^{*2} = \frac{(Tw + B)\left(\frac{p}{w}\right)^{\frac{1}{1-a}}}{p + \left(\left(\frac{p}{w}\right)^{\frac{1}{1-a}} + \left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}\right) w}$$

$$m^{*2} = \frac{Tw + B}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}} \left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}.$$

We use the equations of $l^{*2} = T - L^{*2} - m^{*2}$, and essentially assume that $T - L - m > 0$, in general the CES produces interior solutions so (for most parameters) this is fine. Then obtain $\theta^{*2} = bm^{*2}$.

$$l^{*2} = \frac{-B + \frac{p(B+Tw)}{p + \left(\left(\frac{p}{w}\right)^{\frac{1}{1-a}} + \left(\frac{b^a p}{w}\right)^{\frac{1}{1-a}}\right) w}}{w}$$

$$\theta^{*2} = \frac{b(B + Tw)}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}} \left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}$$

2.3. The quantity of transfer needed to obtain a doubling of the academic outcome is

$$\theta^{*1} = b \frac{Tw}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}} \left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}$$

$$\theta^{*2} = 2b\frac{Tw}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}}\left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}$$

$$\frac{b(B+Tw)}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}}\left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)} = 2b\frac{Tw}{w + \left(\frac{b^a p}{w}\right)^{\frac{1}{-1+a}}\left(p + \left(\frac{p}{w}\right)^{\frac{1}{1-a}} w\right)}$$

$$B + Tw = 2Tw$$

$$B = Tw.$$

That is the wage time the total time T.

3 Policy Implications and Real-world connections.

1. **Optimal Welfare Reform:** Assume the policy-maker aims to optimize their own utility function Z that reflects a weighted sum of the log of household income (through labor) and the log of the academic outcomes of children. For simplicity set $w = 1$, $p = 1$, $a = 1/2$, $b = 1/2$ and $T = 168$.

Formally, this can be represented as:

$$Z(B) = (l^{*2}(B))^{\delta}(\theta^{*2}(B))^{(1-\delta)}$$

where δ is a weight parameter ($0 \leq \delta \leq 1$) that captures the policy-maker's preference between the two objectives.

Here, $l^{*2}(B)$ and $\theta^{*}(B)$ are the optimal labor supply and academic outcome, respectively, that depend on B, as solved in the earlier questions (situation 2).

How should the policy-maker optimally choose B to maximize Z? Provide the mathematical formulation and solve it. Hint: This is equivalent to maximizing a univariate concave function without constraints that you learned in Calculus. Notice in this case there is no constraints (we are assuming away government financing here). Optimize for $\delta = \frac{1}{3}$, explain the value, and obtain the optimal transfer under this parameter (it should be a positive quantity $B > 0$), explain the economical logic on this result.

Answer:

Replacing all the parameters we obtain:

$$l^{*2}(B) = \frac{2(168 + B)}{5} - B$$

$$\theta^{*2}(B) = \frac{168 + B}{10}$$

Then we obtain for $\delta = \frac{1}{3}$

$$Z(B) = \frac{1}{5}(336 - 3B)^{\frac{1}{3}}(84 + \frac{B}{2})^{2/3}$$

To maximize it we obtain its first derivative with respect to B, and verify it is a concave function then:

$$\frac{\partial Z}{\partial B} = \frac{56 - 3B}{5(672 - 6B)^{\frac{2}{3}}(168 + B)^{\frac{1}{3}}} = 0$$

$$B = \frac{56}{3}.$$

2. **Real-world Relevance:** Link the findings of this model to the referenced paper on "Designing Cash Transfers in the Presence of Children's Human Capital Formation." from Mullins. Explain how this simplified model helps us understand implications about the optimal welfare reform reported in this paper.

This is the abstract of that paper: "This paper finds that accounting for the human capital development of children has a quantitatively large effect on the true costs and benefits of providing cash assistance to single mothers in the United States. A dynamic model of work, welfare participation, and parental investment in children introduces a formal apparatus for calculating costs and benefits when individuals respond to incentives. The model provides a tractable outcome equation in which a policy's effect on child skills can be understood through its impact on two economic resources in the household – time and money – and the share of each resource as factors in the production of skills. These key causal parameters are cleanly identified by policy variation through the 1990s. The model also admits simple and interpretable formulae for optimal nonlinear transfers in the style of Mirrlees (1971), with novel features arising when child skill formation is accounted for. Using a broadly conservative empirical strategy, estimates imply that optimal transfers are about 20% more generous than the US benchmark, and shaped very differently. In contrast to current policies, the optimal policy discourages labor supply at the bottom of the income distribution due to the costly estimated impacts of work on child development. The finding underscores the importance of reconciling results in the literature on the developmental effects of maternal employment. Finally, a counterfactual model exercise suggests that changes to the welfare and tax environment after 1996 had negative average effects both on maternal welfare and child skill outcomes, with a significant degree of redistribution across latent dimensions."

This part I will leave it for you to think but a good answer will be concise yet able to convey your knowledge of the class, and the understanding of the case.