Stochastic Revealed Preferences with Measurement Error

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Motivation

- A long-standing question about consumer behavior is whether it is consistent with the utility maximization theory (UMT):
  - Static utility maximization (Afriat (1967)).
  - The intertemporal consumption with exponential discounting (Browning (1989)).
- Measurement error is a well-known issue: Surveys, Experiments, Scanner.
- A key concern: deterministic revealed preferences (RP) tests may overreject the null hypothesis in the presence of measurement error.

Our contribution (methodological)

- New statistical framework to test for **any** RP model that can be characterized by first-order conditions.
  - Explicitly account for measurement error.
  - No parametric assumptions about preferences or heterogeneity.
  - No strong distributional assumptions about measurement error. Based on **centering** conditions.
  - *Measurement error in RP models* – Varian (1985), Tsur (1989), Hjerstrand (2013), Adams et.al (2014), Cherchye et. al (2017).
- General methodology to make **out-of-sample predictions** or **counterfactual** analysis with minimal assumptions.

Our contribution (empirical)

- We test for dynamic rationalizability with exponential discounting (ED) in a consumer panel data set from Spain.
  - Fail to reject the model for single-individual households.
  - Reject for couples' households.
- We test for static rationalizability (R) in a experimental data set (Ahn et al. (2014))
  - Fail to reject the model with measurement error in prices due to missperception.
  - Reject for the case of implementation error in consumption decisions (trembling hand).

Exponential-discounting model

- Dynamic rationalizability with ED. The paper is more general.
- $\mathcal{T} = \{0, \cdots, T\}$– finite horizon.
- $u(\cdot)$ – concave, locally nonsatiated, and continuous instantaneous utility function.
- $d \in (0, 1]$ – discount factor; $p_t \in \mathbb{R}^L_{++}$ – price vector.
- At time τ, the consumer is maximizing

$$u(c_\tau) + \sum_{j=1}^{T-\tau} d^j u(c_{\tau+j}),$$

s.t.

$$p_t^\mathsf{T} c_t - y_t + s_t - a_t = 0, \quad t = \tau, \ldots, T.$$

- y_t, s_t, a_t, r_t – income, savings, assets, and interest rate.
- Assets evolve according to $a_t = (1 + r_t)s_{t-1}$.
- $(p_t, r_t, c_t)_{t\in\mathcal{T}}$ is observed.

ED-rationalizability

Definition

(ED-rationalizability) A deterministic array $(p_t, r_t, c_t)_{t\in\mathcal{T}}$ *is* ED-rationalizable *if* $\exists$ *a concave, l.n.s., and cont. fnct* u*,* $(y_t)_{t\in\mathcal{T}} \in \mathbb{R}^{|\mathcal{T}|}_{++}$ *and* $a_0 \geq 0$ *such that the consumption stream* $(c_t)_{t\in\mathcal{T}}$ *solves:*

$$\max_{z\in\mathbb{R}^{L\times|\mathcal{T}|}_{+}} u(z_0) + \sum_{t=1}^{T} d^t u(z_t),$$

subject to

$$p_0^{\mathsf{T}} z_0 + \sum_{t=1}^{T} \frac{p_t^{\mathsf{T}} z_t}{\prod_{i=1}^{t}[1+r_i]} = \sum_{t=1}^{T} \frac{y_t}{\prod_{i=1}^{t}[1+r_i]} + a_0.$$

Fist-order-condition characterization

Browning (1989)

Let $\rho_t = p_t / \prod_{j=1}^{t}(1 + r_j)$. A deterministic array $(\rho_t, c_t)_{t \in \mathcal{T}}$ is ED*-rationalizable if and only if there exists a pair (u, d) such that*

1. $u : \mathbb{R}_+^L \to \mathbb{R}$ *is a concave, locally nonsatiated, and continuous function;*
2. $d \in (0, 1]$*;*
3. $\nabla u(c_t) \leq d^{-t} \rho_t$ *for every* $t \in \mathcal{T}$*. If* $c_{t,j} \neq 0$*, then* $\nabla u(c_t)_j = d^{-t} \rho_{t,j}$*, where* $c_{t,j}$*,* $\nabla u(c_t)_j$*, and* $\rho_{t,j}$ *are the* j*-th components of* c_t*,* $\nabla u(c_t)$*, and* ρ_t*, respectively.*

- Static utility maximization, quasilinear utility, cost minimization, hyperbolic discounting and many others can be written as FOC.
- $u(\cdot)$ is a latent infinite-dimensional parameter.
- Parametric utility vs. shape restrictions.

Statistical rationalizability

- Boldface font to denote random objects and regular font for deterministic ones.
- Each individual is an i.i.d. draw from some stochastic consumption rule.
- That is, every individual has her own $u(\cdot)$, d, income, prices etc.
- Similarly to the deterministic case

Definition

(s/ED-rationalizability) A random array $(\boldsymbol{\rho}_t, \mathbf{c}_t^*)_{t\in\mathcal{T}}$ is s/ED-rationalizable if there exists a pair $(\mathbf{u}, \mathbf{d})$ such that

1. $\mathbf{u}$ *is a random, concave, locally nonsatiated, and continuous function;*
2. $\mathbf{d}$ *is supported on or inside $(0, 1]$;*
3. $\nabla \boldsymbol{u}(\mathbf{c}_t^*) \leq \boldsymbol{d}^{-t}\boldsymbol{\rho}_t$ a.s. *for all $t \in \mathcal{T}$;*
4. *For every $j = 1, \ldots, L$ and $t \in \mathcal{T}$, it must be that* $\Pr(\mathbf{c}_{t,j}^* \neq 0, \nabla \boldsymbol{u}(\mathbf{c}_t^*)_j < \boldsymbol{d}^{-t}\boldsymbol{\rho}_{t,j}) = 0.$

Elimination of utility function

Lemma

For a given random array $(\boldsymbol{\rho}_t, \mathbf{c}_t^*)_{t\in\mathcal{T}}$, *the following are equivalent:*

1. *The random array* $(\boldsymbol{\rho}_t, \mathbf{c}_t^*)_{t\in\mathcal{T}}$ *is* s/ED-*rationalizable.*
2. *There exist a positive random* $(\mathbf{v}_t)_{t\in\mathcal{T}}$ *and* $\mathbf{d} \in (0, 1]$ *such that*

$$\mathbf{v}_t - \mathbf{v}_s \geq \boldsymbol{d}^{-t} \boldsymbol{\rho}_t^\top (\mathbf{c}_t^* - \mathbf{c}_s^*) \quad \text{a.s.}, \quad \forall s, t \in \mathcal{T}.$$

- Stochastic version of the deterministic case.
- BUT, the problem is still infinite-dimensional since the distribution of $\mathbf{v}$ and $\mathbf{d}$ is unknown.

Technical Point: Approximate rationalizability

- Consider the case of a consumer who maximizes

$$V_\tau(c) = u(c_\tau) + \beta \sum_{j=1}^{T-\tau} d^j u(c_{\tau+j}),$$

  where $\beta \in (0, 1]$ is the present-bias parameter.
- if $\beta \to 1$, then the consumption stream generated by this model is arbitrarily close to the ED-rationalizable behavior.

Definition

(Approximate s/ED*-rationalizability) We say that* $(\boldsymbol{\rho}_t, \mathbf{c}_t^*)_{t\in\mathcal{T}}$ *is* **approximately consistent** *with* s/ED*-rationalizability if there exists a sequence of random variables* $(\boldsymbol{d}_k, \mathbf{v}_k^\intercal)^\intercal \in (0, 1] \times \mathbb{R}_+^{|\mathcal{T}|}$, $k = 1, 2, \ldots,$ *such that*

$$\Pr\left(\mathbb{1}(\mathbf{v}_{k,t} - \mathbf{v}_{k,s} \geq \boldsymbol{d}_k^{-t} \boldsymbol{\rho}_t^\intercal [\mathbf{c}_t^* - \mathbf{c}_s^*]) = 1\right) \to_{k\to+\infty} 1,$$

for all $s, t \in \mathcal{T}$.

Measurement error

- Define the **measurement error** $\mathbf{w} = (\mathbf{w}_t)_{t \in \mathcal{T}} \in W$ as the difference between reported consumption and prices, $\mathbf{c} = (\mathbf{c}_t)_{t \in \mathcal{T}}$ and $\boldsymbol{\rho} = (\boldsymbol{\rho}_t)_{t \in \mathcal{T}}$; and true consumption and prices, $(\mathbf{c}_t^*)_{t \in \mathcal{T}}$ and $(\boldsymbol{\rho}_t^*)_{t \in \mathcal{T}}$. That is,

$$\mathbf{w}_t = \begin{pmatrix} \mathbf{w}_t^c \\ \mathbf{w}_t^p \end{pmatrix},$$

  where $\mathbf{w}_t^c = \mathbf{c}_t - \mathbf{c}_t^*$ and $\mathbf{w}_t^p = \boldsymbol{\rho}_t - \boldsymbol{\rho}_t^*$ for all $t \in \mathcal{T}$.

- As a result, a random array $\mathbf{x} = (\boldsymbol{\rho}_t, \mathbf{c}_t)_{t \in \mathcal{T}}$ can be s/ED-rationalized if and only if there exist $(\mathbf{d}, \mathbf{v}_t, \mathbf{w}_t)_{t \in \mathcal{T}}$, with $\mathbf{d}$ supported on or inside $(0, 1]$ such that

$$\mathbf{v}_t - \mathbf{v}_s \geq \mathbf{d}^{-t}(\boldsymbol{\rho}_t - \mathbf{w}_t^p)^\top(\mathbf{c}_t - \mathbf{c}_s + \mathbf{w}_s^c - \mathbf{w}_t^c) \quad \text{a.s.}, \forall s, t \in \mathcal{T}.$$

- Measurement error is an additional latent variable with unknown distribution.

Centering condition

Without additional restrictions on $\mathbf{w}$ the RP tests have no power: any data can be rationalized.

- $\mathbf{e} = (\mathbf{d}, \mathbf{w}^\intercal, \mathbf{v}^\intercal)^\intercal$ – latent random variables.
- $\mathbf{x}$ – vector of observed quantities.

Assumption

(Centered Measurement Error) (i) The random vector $\mathbf{e}$ *is supported on or inside* $E|X$. *(ii) There exists a known measurement error moment* $g_M : X \times E|X \to \mathbb{R}^{d_M}$ *such that*

$$\mathbb{E}[\, g_M(\mathbf{x}, \mathbf{e}) \,] = 0.$$

The choice of g_M depends on the application and the assumptions the researcher is willing to make given the knowledge about the nature of measurement error.

Centering Conditions: Survey Panel Datasets

For survey datasets, we consider measurement error in consumption:

- Measurement error does not alter the mean value of total expenditure:

$$\mathbb{E}\left[\,\boldsymbol{\rho}_t^{\intercal}\mathbf{c}_t\,\right] = \mathbb{E}\left[\,\boldsymbol{\rho}_t^{*\intercal}\mathbf{c}_t^{*}\,\right]$$

- Consumers' recall mistakes, in terms of expenditures, average out (Evidence from Denmark Abildgren et al. (2018) from validation data).
- This assumption is compatible with nonclassical measurement error in consumption.
- In our main application we assume that $\boldsymbol{\rho}_t^{*} = \boldsymbol{\rho}_t$ a.s. (error in consumption more severe than error in prices).

Assumption

(Mean Budget Neutrality) $\mathbb{E}\left[\,\boldsymbol{\rho}_t^{\intercal}\mathbf{w}_t^{c}\,\right] = 0$, *for all* $t \in \mathcal{T}$.

Econometric framework

- Define

$$
\begin{aligned}
g_{I,t,s}(\mathbf{x}, \mathbf{e}) &= \mathbb{1}(\mathbf{v}_t - \mathbf{v}_s - \mathbf{d}^{-t}\boldsymbol{\rho}_t^{\mathsf{T}}[\mathbf{c}_t - \mathbf{w}_t - \mathbf{c}_s + \mathbf{w}_s] \geq 0) - 1, \\
g_{M,t}(\mathbf{x}, \mathbf{e}) &= \boldsymbol{\rho}_t^{\mathsf{T}}\mathbf{w}_t, \\
g(\mathbf{x}, \mathbf{e}) &= (g_{I,t,s}(\mathbf{x}, \mathbf{e}), g_{M,t}(\mathbf{x}, \mathbf{e}))_{t,s}
\end{aligned}
$$

- $k = |\mathcal{T}|^2 - |\mathcal{T}|$ and $q = |\mathcal{T}|$ moment functions which correspond to inequality conditions (g_I) and the mean budget-neutrality conditions (g_M).

Econometric framework

Theorem

Suppose that the Mean Budget Neutrality condition is satisfied. Then the following are equivalent:

1. $\mathbf{x} = (\boldsymbol{\rho}_t, \mathbf{c}_t)_{t \in \mathcal{T}}$ *is approximately* s/ED*-rationalizable.*
2. $$\inf_{\mu \in \mathcal{P}_{E|X}} \left\| \mathbb{E}_{\mu \times \pi_0} \left[g(\mathbf{x}, \mathbf{e}) \right] \right\| = 0,$$

   where $\pi_0 \in \mathcal{P}_X$ *is the observed distribution of* $\mathbf{x}$*, and* $\mathcal{P}_{E|X}$ *is the set of all probability measures defined over the support of* $\mathbf{e}|\mathbf{x}$*.*

ELVIS

- Note that $\mathcal{P}_{E|X}$ is an infinite-dimensional space.
- We can use Entropic Variable Integration via Simulation (ELVIS) (Schennach (2014)).
- Define maximum-entropy moment condition (MEM) as

$$h(x;\gamma) = \frac{\int_{e\in E|X} g(x,e)\exp(\gamma^{\mathsf{T}} g(x,e))d\eta(e|x)}{\int_{e\in E|X}\exp(\gamma^{\mathsf{T}} g(x,e))d\eta(e|x)},$$

- $\gamma \in \mathbb{R}^{k+q}$ is a nuisance parameter.
- $\eta(\cdot|\cdot) \in \mathcal{P}_{E|X}$ is a user-input distribution function.
- The MEM depends only on the observable random variables.

ELVIS

Theorem

Suppose that the Mean Budget Neutrality condition is satisfied. Then the following are equivalent:

1. $\mathbf{x} = (\boldsymbol{\rho}_t, \mathbf{c}_t)_{t \in \mathcal{T}}$ *is approximately* s/ED*-rationalizable.*
2. $$\inf_{\gamma \in \mathbb{R}^{k+q}} \| \mathbb{E}_{\pi_0} [\, h(\mathbf{x}; \gamma) \,] \| = 0,$$

   where $\pi_0 \in \boldsymbol{P}_X$ *is the observed distribution of* $\mathbf{x}$.

- *Necessary and sufficient* conditions for the observed data to be s/ED-rationalizable.
- Distribution of unobservables is not identified.

Semi-analytic solution

- One can directly employ the MEM to test the model. However,
  1. The number of MEM is $k+q=|\mathcal{T}|^2$
  2. Nonstandard testing if $\|\gamma\|=\infty$

Theorem

Suppose that (i) $\mathbf{x}$ has a bounded support; (ii) the distribution of $\boldsymbol{\rho}_t^{\mathsf{T}}\mathbf{w}_t$ is nondegenerate for all $t\in\mathcal{T}$; (iii) the Mean Budget Neutrality condition is satisfied. Then the following are equivalent:

1. $\mathbf{x}=(\boldsymbol{\rho}_t,\mathbf{c}_t)_{t\in\mathcal{T}}$ *is approximately* s/ED-*rationalizable.*
2. $$\min_{\gamma_M\in\mathbb{R}^q}\left\|\mathbb{E}_{\pi_0}\left[\tilde{h}_M(\mathbf{x};\gamma_M)\right]\right\|=0,$$

   where

   $$\tilde{h}_M(x;\gamma)=\frac{\int_{e\in E|X} g_M(x,e)\exp(\gamma^{\mathsf{T}}g_M(x,e))\mathbb{1}(g_I(x,e)=0)d\eta(e|x)}{\int_{e\in E|X}\exp(\gamma^{\mathsf{T}}g_M(x,e))\mathbb{1}(g_I(x,e)=0)d\eta(e|x)}.$$

- We need to minimize the globally convex objective function over a much smaller parameter space.
- If the data is consistent with s/ED-rationalizability, then the minimizer has to be finite and unique.

Testing

- $\{\mathbf{x}_i\}_{i=1}^n$ – sample of size n
- Define sample analogues of the MEM and the MEM-variance matrix:

$$\hat{\tilde{h}}_M(\gamma) = \frac{1}{n}\sum_{i=1}^{n} \tilde{h}_M(\mathbf{x}_i, \gamma);$$

$$\hat{\tilde{\Omega}}(\gamma) = \frac{1}{n}\sum_{i=1}^{n} \tilde{h}_M(\mathbf{x}_i, \gamma)\tilde{h}_M(\mathbf{x}_i, \gamma)^{\mathsf{T}} - \hat{\tilde{h}}_M(\gamma)\hat{\tilde{h}}_M(\gamma)^{\mathsf{T}}.$$

- Test statistic:

$$\mathrm{TS}_n = n \inf_{\gamma \in \mathbb{R}^q} \hat{\tilde{h}}_M(\gamma)^{\mathsf{T}} \hat{\tilde{\Omega}}^{-}(\gamma) \hat{\tilde{h}}_M(\gamma).$$

Testing

Theorem

Suppose the above assumptions hold and $\{\mathbf{x}_i\}_{i=1}^n$ is i.i.d. Then under H_0 that the data is consistent with s/ED*-rationalizability, it follows that*

$$\lim_{n\to\infty} \Pr(\mathrm{TS}_n > \chi^2_{q,1-\alpha}) \leq \alpha,$$

for every $\alpha \in (0,1)$.
If, moreover, the eigenvalues of $\mathbb{V}[\tilde{h}_B(\mathbf{x},\gamma)]$ are bounded and bounded away from 0 uniformly in γ, then, under H_1, it follows that

$$\lim_{n\to\infty} \Pr(\mathrm{TS}_n > \chi^2_{q,1-\alpha}) = 1.$$

Inference on parameters/Counterfactuals

- Additional restrictions that come in the form of moments or support restrictions on the latent variables.
- The parameter of interest θ_0 is related to the model via the known function $g_R : X \times E \times \Theta \to \mathbb{R}^{d_R}$ such that

$$\mathbb{E}_{\mu \times \pi_0} \left[g_R(\mathbf{x}, \mathbf{e}; \theta_0) \right] = 0.$$

- Expected true consumption at time τ: $g_R(x, e; \theta_0) = c_\tau - w_\tau - \theta_0$.
- Support of the discount factor: $g_R(x, e; \theta_0) = \mathbb{1}(\theta_{01} \leq d \leq \theta_{02}) - 1$.
- Add additional moments to the original ones and compute the test statistic for a fixed θ_0.
- The $(1 - \alpha)$-confidence set for θ_0 is

$$\{\theta_0 \in \Theta : \mathrm{TS}_n(\theta_0) \leq \chi^2_{q+d_R,1-\alpha}\},$$

- We can construct confidence sets for counterfactuals and out-of-sample predictions.

Asymptotic Power

- Is the alternate hypothesis non-empty?
- Example: $\boldsymbol{\rho}_0 = (1,1)^\mathsf{T}$, $\boldsymbol{\rho}_1 = (2,2)^\mathsf{T}$, $\mathbf{c}_0 = (1,1)^\mathsf{T}$, $\mathbf{c}_1 = (2,2)^\mathsf{T}$.
- (Contradiction) $\exists\ \mathbf{d} \in (0,1]$, $\{\mathbf{w}_t^c\}_{t=0,1}$, $\{\mathbf{v}_t\}_{t=0,1}$ s.t.

$$\begin{aligned}
\mathbf{v}_1 - \mathbf{v}_0 &\geq \frac{\rho_1^\mathsf{T}}{\mathbf{d}}(c_1 - c_0) + \frac{\rho_1^\mathsf{T}}{\mathbf{d}}(\mathbf{w}_1 - \mathbf{w}_0) \text{ a.s.},\\
\mathbf{v}_0 - \mathbf{v}_1 &\geq \rho_0^\mathsf{T}(c_0 - c_1) + \rho_0^\mathsf{T}(\mathbf{w}_0 - \mathbf{w}_1) \text{ a.s.},\\
\mathbb{E}[\,\rho_0 \mathbf{w}_0\,] &= \mathbb{E}[\,\rho_1 \mathbf{w}_1\,] = 0.
\end{aligned}$$

- $\implies \frac{d}{2}(\mathbf{v}_1 - \mathbf{v}_0) \geq (\mathbf{v}_1 - \mathbf{v}_0)$ a.s. $\implies \mathbf{v}_0 \geq \mathbf{v}_1$ a.s..
- Taking expectations on the first inequality, applying Mean Budget Neutrality (contradiction):

$$0 \geq \mathbb{E}[\,\mathbf{d}(\mathbf{v}_1 - \mathbf{v}_0)\,] \geq 4 + \mathbb{E}\left[\,\rho_1^\mathsf{T}(\mathbf{w}_0 - \mathbf{w}_1)\,\right] = 4$$

Empirical application

- Data: The Spanish Continuous Family Expenditure Survey (1985-1997).
- 185 individuals, 2004 couples.
- Prices and expenditures for 17 categories of goods recorded over 4 consecutive quarters (e.g., all food and nonalcoholic drinks, all clothing, household services, public transport, petrol, food consumed outside the home).
- Nominal interest rate on consumer loans faced by the household in any particular quarter.

Empirical application

- We assume that $\mathbf{d}$ is supported on or inside $(0.1, 1]$.
- We also perform the deterministic test of Browning(1989).

Results

- Singles: Browning – 81.1% of rejections; our test fails to reject at 95% confidence level. (the same conclusion for $\mathbf{d} \in [0.99, 1)$)
- Couples: Browning – 88.5% of rejections; our test rejects at 95% confidence level (p-value< 0.001).

Conclusion

- Measurement error may lead to substantial overrejections in deterministic RP tests.
- We propose a methodology to test several RP models that can be characterized by FOC allowing for measurement error.
- We provide a general methodology to make out-of-sample predictions or counterfactual analysis with minimal assumptions.
- We do not make parametric assumptions about preferences or heterogeneity, nor impose strong distributional assumptions on measurement error.
- We find support for ED for single-individual households, and we reject the null hypothesis of ED for the case of couples.

Related literature

- *RP models* – Afrijat (1967), Rockafellar (1970), Browning (1989), Brown & Calsamiglia (2007), Forges & Minelli (2009), Beatty & Crawford (2011), Kitamura and Stoye (2016), Blow et. al (2017), Deb et al. (2018).

- *Measurement error in RP models* – Varian (1985), Tsur (1989), Hjerstrand (2013), Adams et.al (2014), Cherchye et. al (2017), Echenique, Lee, and Shum (2011)

- *Latent variable* – Galichon & Henry (2013), Ekeland at. al (2010), Schennach (2014).