Exercise 1. (Integrability)

(i) Walras' law and $P_2 = 1$ (HDO)

$$P_1 x_1(P_1, W) + x_2(P_1, W) = W$$

$$\Rightarrow x_2(P_1, W) = W - \beta P_1 + \alpha P_1^2$$

(ii) Method 1 (Recommend)

With HDO and Walras' law

$S(P,W)P = 0$

Recall (from Tutorial 1), with $S(P,W)P$, we can reduce the Slutsky Matrix to a 1×1 matrix.

The reduced Slutsky Matrix $S_{1,1} = -\alpha$

it is symmetric (because it is scalar)

and is NSD if $\alpha \geq 0$

With Walras' law the conditions for HW Theorem hold.

Method 2:

Say P_1 stands for the relative price, we denote $\bar{P}_1$ and $\bar{P}_2$ as the original prices. i.e. $P_1 = \frac{\bar{P}_1}{\bar{P}_2}$

Then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \beta - \alpha \frac{\bar{P}_1}{\bar{P}_2} \\ \frac{W}{\bar{P}_2} - \beta \frac{\bar{P}_1}{\bar{P}_2} + \alpha \frac{\bar{P}_1^2}{\bar{P}_2^2} \end{pmatrix}$$

It is easy to get

$$D_p x(\bar{P},W) = \begin{pmatrix} -\frac{\alpha}{\bar{P}_2} & \alpha\frac{\bar{P}_1}{\bar{P}_2^2} \\ -\beta\frac{1}{\bar{P}_2} + 2\alpha\frac{\bar{P}_1}{\bar{P}_2^2} & -\frac{W}{\bar{P}_2^2} + \beta\frac{\bar{P}_1}{\bar{P}_2^2} - 2\alpha\frac{\bar{P}_1^2}{\bar{P}_2^3} \end{pmatrix}$$

+

$$D_w x(\bar{P},W)\, x(\bar{P},W')' = \begin{pmatrix} 0 \\ \frac{1}{\bar{P}_2} \end{pmatrix} \left(\beta - \alpha\frac{\bar{P}_1}{\bar{P}_2},\ \frac{W}{\bar{P}_2} - \beta\frac{\bar{P}_1}{\bar{P}_2} + \alpha\frac{\bar{P}_1^2}{\bar{P}_2^2}\right)$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{\beta}{\bar{P}_2} - \alpha\frac{\bar{P}_1}{\bar{P}_2^2} & \frac{W}{\bar{P}_2^2} - \beta\frac{\bar{P}_1}{\bar{P}_2^2} + \alpha\frac{\bar{P}_1^2}{\bar{P}_2^3} \end{pmatrix}$$

$$S(\bar{P},W) = \begin{pmatrix} -\frac{\alpha}{\bar{P}_2} & \alpha\frac{\bar{P}_1}{\bar{P}_2^2} \\ \alpha\frac{\bar{P}_1}{\bar{P}_2^2} & -\alpha\frac{\bar{P}_1^2}{\bar{P}_2^3} \end{pmatrix}$$

$$\underset{\bar{P}_1 = P_1}{\xrightarrow{\text{Plug in } \bar{P}_2 = 1}} \begin{pmatrix} -\alpha & \alpha P_1 \\ \alpha P_1 & -\alpha P_1^2 \end{pmatrix}$$

Easy to check

$$S(\bar{P},W) = S(\bar{P},W)' \qquad S(\bar{P},W) \leq 0$$

Mathematical technique : ODE (Ordinary differential equation)

(iii) Usually we are solving the UMP to get x.

For example: $U = x_1^{\alpha} x_2^{1-\alpha}$

$$\text{s.t. } P_1 x_1 + P_2 x_2 \leq W$$

If I give you x_1 and x_2 solved from an UMP,

how to get U ?

HU Theorem opens the possibility of integral.

$$W(P_1) = P_1 x_1 + x_2$$

I changed the notation a little bit, to help me explain.

$$W'(P_1) = x_1(P, W) = \beta - \alpha P_1$$

$$W(P_1) = \beta P_1 - \frac{1}{2}\alpha P_1^2 + C$$

For a initial condition (P_1^0, W_1^0) — Note that actually the initial condtion is actually the data.

$$W^0 = \beta P_1^0 - \frac{1}{2}\alpha (P_1^0)^2 + C$$

$$\Rightarrow C = W^0 - \beta P_1^0 + \frac{1}{2}\alpha (P_1^0)^2$$

So $W(P_1^*; P_1^0, W^0) = \beta P_1^* - \frac{\alpha}{2} {P_1^*}^2 + W^0 - \beta P_1^0 + \frac{1}{2}\alpha (P_1^0)^2$

To define $V(,)$, we choose $P_1^* = 0$, we have

$$V(P_1^0, W^0) = W(0, P_1^0, W^0) = W^0 - \beta P_1^0 + \frac{1}{2}\alpha (P_1^0)^2$$

So $V(P_1, W) = W - \beta P_1 + \frac{1}{2}\alpha (P_1)^2$

Given (x_1, x_2)

Since $x_1 = \beta - \alpha P_1$ $\quad x_2 = W - \beta P_1 + \alpha P_1^2$

$$\Rightarrow p_1 = \frac{\beta - x_1}{\alpha} \qquad w = x_2 + \beta p_1 + \alpha p_1^2 = x_2 + \beta \frac{\beta - x_1}{\alpha} - \alpha \left(\frac{\beta - x_1}{\alpha}\right)^2$$

Thus, $$u(x_1, x_2) = V(p, w) = V\left(\frac{\beta - x_1}{\alpha},\ x_2 + \beta \frac{\beta - x_1}{\alpha} - \alpha \left(\frac{\beta - x_1}{\alpha}\right)^2 \right)$$

$$= x_2 - \frac{(\beta - x_1)^2}{2\alpha}$$

Basics of Consumer Theory (Revisit)

- Three Axioms:
  - Walras' law
  - No money illusion / Homogeneity of Degree 0 (HD0)
  - Weak Axiom of Revealed Preference (WARP)

- Slutsky Matrix

$$S(P,W) = D_p x(P,W) + D_w x(P,W) \cdot x(P,W)'$$

Slutsky compensated variation of prices: leaves the previous consumption boundle affordable

Some implications:

<1> x satisfy Walras' law

$\Longrightarrow$

(i) $P' D_w x(P,W) = 1$

(ii) $P' D_p x(P,W) = -x(P,W)'$

(iii) $P' S(P,W) = 0$

<2> Walras' law + HD0

$\Longrightarrow$

$S(P,W) \cdot P = 0$

<3> Walras' law + WARP

$\Longrightarrow$

CLD (Compensated Law of Demand)

$\Longrightarrow$ $S(P,W)$ is NSD, i.e. $S(P,W) \leq 0$

<1> x satisfy Walras' law

$\Longrightarrow$

(i) $p' D_w x(p,w) = 1$

(ii) $p' D_p x(p,w) = -x(p,w)'$

(iii) $p' S(p,w) = 0$

Proof. x satisfies Walras' law $\Longleftrightarrow$ $p' x(p,w) = w$

(i) Take derivative w.r.t. w in both sides

$$p' D_w x(p,w) = 1$$

Why? LHS $=$ $p' x(p,w) = \sum_\ell p_\ell x_\ell(p,w)$

$$D_w p' x(p,w) = \Sigma_\ell p_\ell \partial_w x_\ell(p,w)$$
$$= p' D_w x(p,w)$$

(ii) Take derivative w.r.t. p in both sides

$$p' x(p,w) = \sum_\ell p_\ell x_\ell(p,w)$$

$$D_p p' x(p,w) = \begin{bmatrix} x_1(p,w) + & p_1 \partial p_1 x_1(p,w) + p_2 \partial p_1 x_2(p,w) + \cdots \\ \vdots & \\ x_\ell(p,w) + & p_1 \partial p_\ell x_1(p,w) + p_2 \partial p_\ell x_2(p,w) + \cdots \\ \vdots & \end{bmatrix}$$

$$= x(p,w) + [\, p' D_p x(p,w)]'$$

$$= [D_p w = 0]$$

$$= 0$$

Thus we got (ii)

(iii) Recall $S(P,W) = D_p x(P,W) + D_w x(P,W) \cdot x(P,W)'$

$$P' S(P,W) = \underbrace{P' D_p x(P,W)}_{\substack{\| \text{ by (ii)} \\ -x(P,W)'}} + \underbrace{P' D_w x(P,W)}_{\substack{\| \text{ by (i)} \\ 1}} x(P,W)'$$

$$= -x(P,W)' + x(P,W)'$$

$$= 0.$$

$\boxed{///}$

<2> Walras' law + HDO

$\Longrightarrow$

$S(P,W) \cdot P = 0$

Proof. By HDO we have that

$$x(\lambda P, \lambda W) = x(P,W) \quad \forall \lambda \geq 0$$

Take derivative w.r.t. λ in both side, we obtain

$$D_p x(\lambda P, \lambda W)\, p + D_w x(\lambda P, \lambda W)\, W = 0$$

By Walras' law, we have $P' x(P,W) = W = x'(P,W) \cdot P$

Then we got

$$\underbrace{\left(D_p x(\lambda P, \lambda W) + D_w x(\lambda P, \lambda W)\, x'(P,W)\right)}_{\substack{\| \text{ By definition} \\ S(P,W)}} P = 0$$

$\boxed{///}$

<3> Walras' law + WARP

① $\Longrightarrow$

CLD (Compensated Law of Demand)

② $\Longrightarrow$ $S(p,w)$ is NSD, i.e. $S(p,w) \leq 0$

Proof.

(CLD) We say that x satisfies the CLD

if and only if

$(p^* - p)'(x(p^*,w^*) - x(p,w)) < 0$ when

$x(p^*,w^*) \neq x(p,w)$ and $w^* = (p^*)' x(p,w)$

①

Let $(p^*)' x(p,w) = w^*$ (1)

also by Walras' law $(p^*)' x(p^*,w^*) = w^*$ (2)

$p' x(p,w) = w$ (3)

From WARP, we get $p' x(p^*,w^*) > w$ (4) (Why?)

From (1) and (2) [(2) − (1)]

$(p^*)'(x(p^*,w^*) - x(p,w)) = 0$ (5)

From (3) and (4) [(4) − (3)]

$p'(x(p^*,w^*) - x(p,w)) > 0$ (6)

(5) − (6), we got

$(p^* - p)'(x(p^*,w^*) - x(p,w)) < 0$

② Very easy, just need write down the differential version.

▨