PS1

Exercise 1

- Couterexample:

$X = \{a, b, d\}$ $\qquad\qquad$ $\mathcal{B} = \{\underbrace{\{a,b\}}_{\substack{\rotatebox{90}{$\in$} \\ A_1}}, \underbrace{\{b,d\}}_{\substack{\rotatebox{90}{$\in$} \\ A_2}}, \underbrace{\{a,d\}}_{\substack{\rotatebox{90}{$\in$} \\ A_3}}\}$

$c(\{a,b\}) = a$

$c(\{b,d\}) = b$

$c(\{a,d\}) = d$

Why?

Suppose not. i.e. it is not an counterexample.

then there $\exists \succsim$, let's denote as $\succsim^1$ rationalize $c(\cdot)$

$\forall i \in \{1,2,3\}$ $\quad A_i \subseteq \mathcal{B}$

$c(A_i) = C(A_i, \succsim^1)$

$c(\overset{= A_1}{\{a,b\}}) = a \implies a \succsim^1 b$

$c(\overset{= A_2}{\{b,d\}}) = b \implies b \succsim^1 d$

$c(\overset{= A_3}{\{a,d\}}) = d \implies a \succsim^1 d$

$a \succsim^1 b,\ b \succsim^1 d$, but $a \succsim^1 d$

$\succsim^1$ fails transitivity. $\implies \succsim^1$ is not rational

contradiction! $\blacksquare$

- Question

Does WARP satisfy in my example?

✓ Yes. Just check the definition.

Exercise 2 (☆ core question)

- Preorder: reflexive and transitive

(a) $X = \{a, b, d, e\}$

The issue is that $\succsim$ not complete.

$$\begin{array}{ccccc} & & e & & \\ & & \curlyvee & & \\ a & \succsim & b & \succsim & d \end{array}$$

$a \succsim a$, $b \succsim b$, $d \succsim d$, $e \succsim e$

$a \succsim b$, $b \succsim d$, $a \succsim d$, $e \succsim b$, $e \succsim d$

since $u: X \to \mathbb{R}$, if exists,

then $u(a) \in \mathbb{R}$, $u(e) \in \mathbb{R}$

we know that $\mathbb{R}$ is complete

either $u(a) \geq u(e)$ or $u(e) \geq u(a)$

but we don't have $a \succsim e$ or $e \succsim a$. $\blacksquare$

(b) Start from a simple case (the easiest one I guess)

$X = \{a, b, d\}$

we have $\succsim$ define on s.t. $\succsim$ is reflexive and $a \succsim b$

Then construct the set of [complete] transitive extension of $\succsim$, $T^{j\text{th}}(\succsim)$

all the cases

$T^{1\text{st}}(\succsim)$: $a \succsim^1 b \succsim^1 c$

$T^{2\text{nd}}(\succsim)$: $c \succsim^2 a \succsim^2 b$

$T^{3\text{rd}}(\succsim)$: $a \succsim^3 c \succsim^3 b$

By URT, $\forall$ $j \in \{1, 2, 3\}$

there exists u^j s.t.

$$x \succsim^j y \iff u^j(a) \geq u^j(b)$$

It just means we have

$a \succsim a$ $b \succsim b$ $d \succsim d$

$a \succsim b$

Moreover, in terms of set

$a \succsim a \Rightarrow (a, a) \in \succsim$

$a \succsim b \Rightarrow (a, b) \in \succsim$

but $a \succsim b$ doesn't imply $(b, a) \in \succsim$

The general proof:

Denote $T(\succsim)$ where $\succsim \subseteq T(\succsim)$ is the complete and transitive extension of $\succsim$.

Theorem (Sziplrain) — check chapter 1.

Since I is finite, so there is finite such $T(\succsim)$, denote the cardinality as J.

$T^{j\text{th}}$ $j \in \{1, 2, \cdots, j, \cdots, J\}$

By the Utility Representation Theorem, for every j^{th} complete preorder on a finite set X, there is a utility function $u^j : X \to \mathbb{R}$.

such that $a \succsim^j b \Leftrightarrow u^j(a) \geq u^j(b)$ $\boxed{4}$

(c) Counterexample:

$X = \{a, b, d\}$

$a \succsim a$, $b \succsim b$, $d \succsim d$, $d \succ a$.

$C(\{a,b\}) = \{a,b\}$ $C(\{b,d\}) = \{b,d\}$ $C(\{a,d\}) = \{d\}$

$a R^D b R^D d$ but $d P^D a$ violate GARP.

(d) WTS:

$[C(A) \equiv \{a \in A \mid \text{there is no } b \in A, b \succ a\}]$

$\Rightarrow$

$\Bigg[$ WARNI $x, y \in A, B$.

$\left.\begin{array}{l} \forall\, x \in C(A) \\ \exists\, y \in C(B) \end{array}\right. \Rightarrow y \in C(A) \Bigg]$

$x \in C(A \sqcup \{x\} \cup \{y\}) \Leftrightarrow$ there is no $b \in A$, s.t. $b \succ x$.

i.e. $\neg\, b \succ x$

$y \in C(B) \Leftrightarrow \neg\, c \in B,\ c \succ y$

$\Rightarrow \neg\, x \succ y$

$$[\neg\, b \succ x \ \wedge\ \neg\, x \succ y]$$

$$\Leftrightarrow\ \neg [b \succ x \ \vee\ x \succ y]$$

$$\Rightarrow\ \neg\ b \succ y$$

thus $y \in C(A)$ $\blacksquare$