GRO-MECS PAPER

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1. Introduction

Rough outline for paper

1. Intro: Knots and mosaics, unknotting number, define hexagon unknotting number

2. Background: Mosaics, hex mosaics, unknotting number, hex unknotting, Results we're going to use

3. Relationship to regular unknotting number: upper and lower bounds, and extra lemmas related to bounds

4: Torus knots: 3n+1,3 and 2n+1,2 torus unknotting numbers

5: Subadditivity: hex unknotting subadditive, delta unknotting stuff

6: Future work: relate to other unknotting moves, subadditivity, unknotting game, other combos of unknotting moves

7: Appendix: list of examples and unknotting numbers we know

McLoud-Mann, et. al, discussed hexagonal mosaics in [4]. also there was stuff in [5]

2. Background

A *knot* K is an embedding of a closed continuous loop in three space. Knots are often studied through *diagrams*. A *knot diagram* D is a two dimensional projection of the knot with over and under crossings indicated by a break in the strand. Any two diagrams of the same knot can be related by a sequence of *Reidemeister moves*, shown in Figure 1.

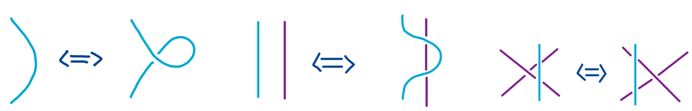


Figure 1. Reidemeister moves 1,2, and 3

Further basics of knot theory can be found in Adams [1] or Livingston [7].

A *mosaic* is a type of knot diagram constructed with tiles. A *square n-mosaic* is a diagram constructed using square tiles, shown in Figure 2 on an $n \times n$ grid. Knots mosaics have primarily been studied on square tiles. However, there are other tile shapes that can be used. In this research the focus has been on hexagon mosaics.

A *hexagon mosaic* is a knot diagram constructed using the hexagonal tiles shown in Figure 3, up to rotation, on a grid.

Date: June 2025.

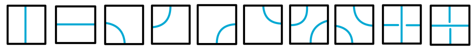


FIGURE 2. Tiles that can be used in a square mosaic diagram, excluding the empty tile

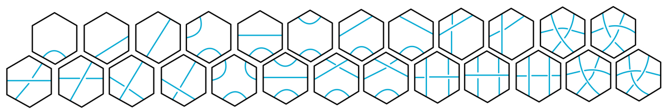


FIGURE 3. Tiles that can be used in a hexagon mosaic, up to rotation and excluding the empty tile. Unlike square mosaics, these include 2-crossing and 3-crossing tiles.

The *unknotting number* of a knot k is the minimum number of crossings that need to be changed on a diagram in order to produce the unknot. A crossing change allows us to swap the over- and under-strand of a specific crossing. Note that this is fundamentally changing our knot into something else, unlike Reidermeister moves, which only change the appearance of the knot, not its properties.

An *unknotting crossing* is a crossing that can be changed in order to unknot a knot. For knots with unknotting numbers larger than one, there are possibly many sets of unknotting crossings. These crossings may not exist on the minimal knot diagram, as shown by Nakanishi and Bleiler for $u(10_8) = 2$ [6].

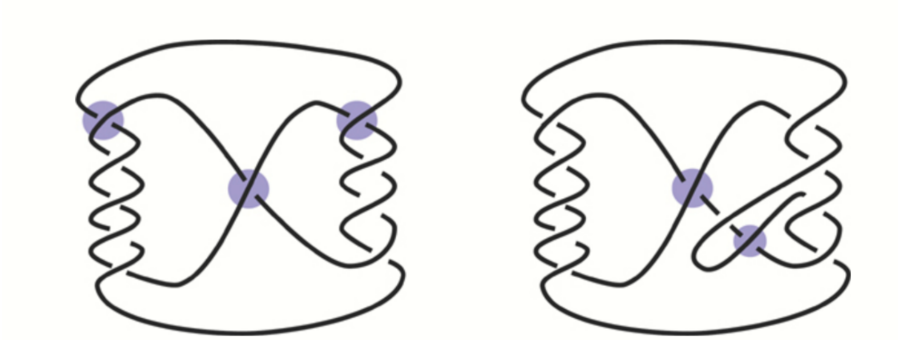


FIGURE 4. $u(10_8) = 2$ is only realized in a nonminimal crossing diagram. Finding the proper unknotting crossings to calculate $u(K)$ is often a challenging ordeal. Figure taken from [6].

The *square tile unknotting move*, shown in Figure 5, allows us to construct a different but analogous invariant to the classical unknotting number. The *square mosaic unknotting number*, denoted $u_s(K)$ is the minimum number of square tile unknotting moves that must be performed on some square mosaic diagram D of K to produce the unknot, where the minimum is taken over all square mosaic diagrams of K.

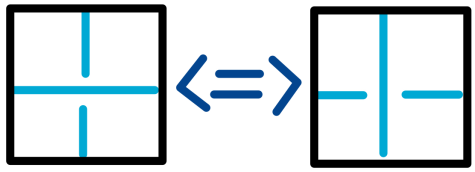


FIGURE 5. The square mosaic unknotting move is identical to the simple crossing change used to calculate $u(K)$. Thus $u_s(K) = u(K)$

Any knot can be drawn on square mosaics and crossings are isolated. Thus, $u_s(K) = u(K)$ for any knot K. This is true because, to unknot a knot on square mosaics, the tiles with the unknotting crossings are rotated one at a time and each tile has only one crossing. Similarly, over- and under- strands are swapped one crossing at a time for the classical unknotting number. Thus, both numbers will be the same.

This paper doesn't dedicate much time to the study of $u_s(K)$ because all of the work done in calculating $u(K)$ for knots still applies. Instead, we set our sights on a different type of mosaic.

Typically, when we think about unknotting a knot diagram, we are thinking about changing specific crossings that aren't necessarily laid out on a mosaic grid. In this paper, we approach these crossing changes on a square or hexagon grid by looking at the tiles containing crossings. Since hexagons can fit up to three crossings on a single tile, we define unknotting number of our diagram as the number of crossing tiles that need to be changed in order to produce the unknot, rather than the number of crossings that need to be changed. This allows us to cluster our crossing together to create the unknot through changes in a smaller area of the knot.

A *hexagon tile unknotting move* is **a tile swap created by a sequence of tile swaps across lines of symmetry** a rotation of a tile in sixty degree intervals or a reflection of the tile along a line of symmetry of the hexagon. Note that strands must connect the same connection points as the original tile. There are seven sets of tiles that follow these rules, shown in Figure 6. When rotating our hexagon crossing tiles, we sometimes end up with changes in the endpoints of each strand within the tile. This often results in the creation of links, rather than maintaining a knot diagram that can be transformed into the unknot. In order to avoid this, we require our tile changes to maintain *consistent connection points*. This means that if two sides of the original mosaic were connected, they should still be connected after a change in a tile. This allows us to change crossings without creating a link in the process.

The *hexagon unknotting number* of a knot K, denoted $u_h(K)$ is the minimum number of hexagon tile unknotting moves that must be performed on some hexagon diagram of K to produce the unknot, where the minimum is taken over all hexagon diagrams of K.

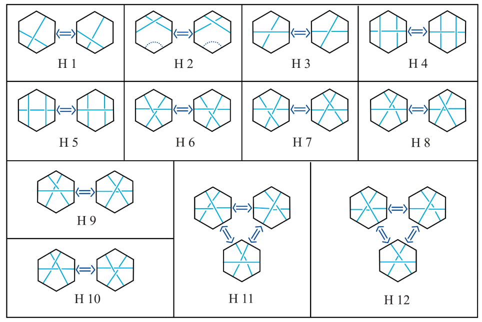


FIGURE 6. All legal tile change options when working with hexagon tiles. Unlike square mosiacs, these include 2-crossing and 3-crossing tile changes

Lemma 1. *Every unknotting hexagon tile change can be achieved through a sequence of planar symmetry changes.*

Since $u_h(K)$ should allow us to unknot more efficiently, our objective is to compare hex unknotting to square unknotting and therefore compare hex unknotting to classical unknotting.

First, we should determine how to construct 2-crossing tiles in which both crossings are unknotting crossings. These crossings must be consecutive. *Consecutive crossings* are crossings that occur directly next to each other. These crossings will share at least one strand. In a square mosaic diagram, this typically looks like two crossing tiles that are adjacent to each other.

Thus, our objective is to take specific unknotting crossings and make them consecutive. This will allow us to put two unknotting crossings on the same hexagon tile. Fortunately, we can use Redeimeister moves to achieve this.

Lemma 2. *We can always use Reidermeister moves to put any two crossings onto the same hexagon tile.*

Proof. In a classical knot diagram, choose two crossings. Between these two crossings, some orientation exists in which the crossings are connected by a sequence of strands that doesn't connect a crossing to itself first. Between any two unknotting crossings exist a finite number of under or overstrands.

For p strands between the two crossings, use $p+1$ Reidemester 3 moves to make the unknotting crossings consecutive. Utilize planar isotopy as needed. Observe two consecutive crossings.

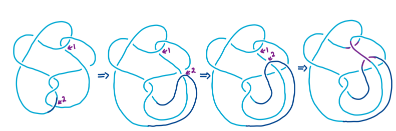


FIGURE 7. An example of moving two crossings (1 and 2) together by moving the dark blue strand, allowing the two crossings to be on the same tile when the knot K is drawn on a hexagonal grild.

□

Using these lemmas and ideas, we are equipped to analyze the effectiveness of 2-crossing tiles for unknotting.

3. Relationship to Classical Unknotting Number

With Lemma 4 and Lemma 5, we may establish an upper bound the the hex unknotting number.

Theorem 3. *For any knot, K, $u_h(K) \leq u(K)$.*

In the first place, square unknotting and hex unknotting produce the same result when $u(K) = 1$.

Lemma 4. *Given a knot K with unknotting number $u(K) = 1$, then $u_h(K) = u(K)$.*

Proof. When $u(K)$ is 1, it means that only one crossing needs to be changed to achieve the unknot. Therefore, when drawing the knot in a square mosaic, just one tile needs to be changed to unknot the knot, the one containing the unknotting crossing. It will always be possible to perform this change in one move because square tiles have up to one crossings per tile.

Similarly, when $u(K) = 1$, $u_h(K) = 1$. Firstly, the hexagon unknotting number cannot be smaller than 1, otherwise K would already be the unknot. With the hexagon representation, just one crossing will need to be changed. This crossing can either be on one tile, and thus changing that tile will be straightforward; or it can be in a multiple crossing tile, for which Reidermeister moves can be performed to isolate the unknotting crossing in one one-crossing tile and then be able to just change this one tile.

Thus, when $u(K) = 1$, then $u_h(K) = 1$. □

In a more visual explanation, since we are unknotting in clusters instead of specific crossing changes, unknotting 2 or 3 crossings at once won't necessarily improve the number of unknotting steps. Any fewer steps would just be the unknot.

Next, we examine cases in which hex unknotting and square unknotting yield different results.

Lemma 5. *Given a knot, K, with unknotting number $u(K) > 1$, then $u_h(K) < u(K)$.*

Proof. For the minimal mosaic diagram D, choose two unknotting crossings. Using Lemma 2, make these crossings consecutive and label this diagram D. From this design, construct a new mosaic diagram D' and from that construct a hexagon diagram D'_h, which converts the two consecutive unknotting crossings into a 2-crossing tile.

Since both unknotting crossings occur on the same tile, the number of hexagon tile changes derived from the diagram D'_h will be one less than $u_s(K)$.

Recall $u_h(K)$ is the minimum number of hexagon tile changes across all diagrams. While $u(K) - 1$ may not equal $u_h(K)$, it does function as a positive bound. Thus $u_h(K) \leq u(K) - 1$, which implies $u_h(K) < u(K)$. □

This argument cannot be used to calculate $u_h(K)$ for all knots with $u(K) > 1$, but it does produce a bound for what $u_h(K)$ is at most. 3-crossing tiles allow for even more efficient unknotting, but more on that later.

These theorems follow from the properties surrounding 2-crossing unknotting tiles. This type of unknotting is analogous to the Γ-unknotting number, denoted $u_\Gamma(K)$ and defined as the number of Γ-moves to unknot some knot [2].

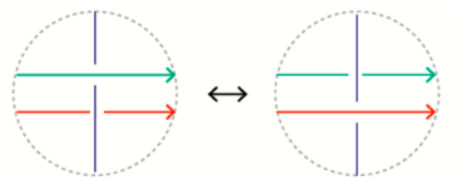


FIGURE 8. Γ-move example. This move is equivalent to H4 on a hexagon mosaic.

3.1. **3-crossing Tiles.**

3-crossing tiles have much more variety than 2-crossing tiles. They allow for some knots to achieve the lowest possible bound for the hexagon unknotting number. Since a 3-crossing tile changes at most 3 crossings at once, we have a lower bound for how much more efficient hex unknotting is.

Theorem 6. *Given a knot K with unknotting number $u_s(K)$, then $u_h(K) \geq \frac{1}{3}u(K)$*

Proof. When drawing a knot on a hexagonal mosaic diagram, there can be a maximum of 3 crossings on a single tile. This means that a hexagon unknotting move will only be able to change a maximum of 3 crossings at once. So, the greatest improvement that we can make to our unknotting number using hexagons is $\frac{1}{3}$ the classical or square unknotting number.

□

Corollary 7. $\frac{1}{3}u(K) \leq u_h(K) \leq u(K)$.

Proof. Theorem 3 defines an upper bound. Theorem 6 defines $\frac{1}{3}u(K) \leq u_h(K)$ as a lower bound. □

In some instances, knots are immediately able to be constructed with a 3-crossing unknotting tile. An example is taken from this square diagram of 8_{19}, in which three unknotting crossings form the necessary triangular configuration for a 3-crossing hexagon tile.

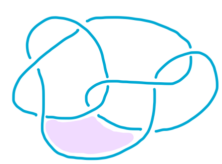
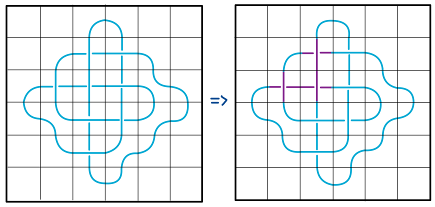


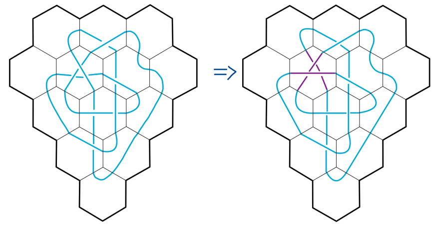
FIGURE 9. 8_{19} with highlighted triangular configuration

It is clear that $u_h(8_{19}) = 1$. In some instances, such as 10_{161}, a sequence of Reideimester moves and planar isotopies reveals the diagram in which this triangle configuration is clear. Conversion into a 3-crossing unknotting hexagon tile is easy, and we observe $u_h(10_{161}) = 1$.

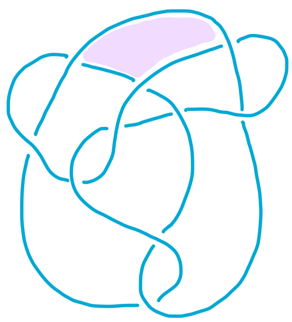
There are also some knots where there is a clear triangle of crossings, but these crossings don't allow us to unknot the knot in a single move. An example of this is



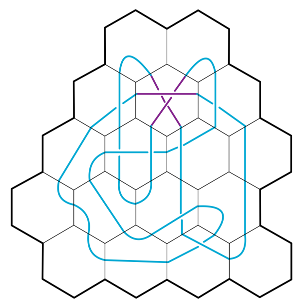
(A) 8_{19} has a set of three unknotting crossings forming a triangular configuration that produces the unknot when all crossings are changed.



(B) 8_{19} hexagonal mosaic derived from the square mosaic that converts a triangular configuration into a 3-crossing tile using H8. This calculates $u_h(8_{19}) = 1$



(A) 10_{161} diagram following a Redeimester 3 move.



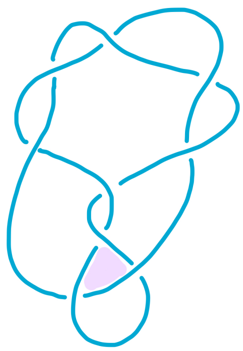
(B) 10_{161} hexagonal mosaic uses H8 to calculate $u_h(10_{161}) = 1$.

the 9_6 knot, which has an unknotting number of 3 and a clear triangle of crossings within the diagram. Unfortunately, the crossings that form the triangle aren't able to unknot the knot. In this case, the unknotting crossings need to be spread around the diagram more, making it difficult to put the unknotting crossings together in a small space.

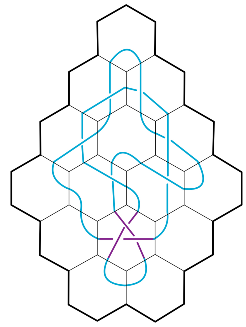
Some knots don't have a clear diagram nor an obvious sequence of Redeimester moves to solidify this 3-crossing unknotting tile. A good example of this is 9_3, which has no immediate 3-crossing configurations and a challenging or potentially impossible sequence of Redeimester moves to construct a hexagon unknotting diagram.

To work around, we compare the unknotting move that is analogous to the 3-crossing unknotting tile, which is the Δ-move and the Δ-unknotting number $u_\Delta(K)$, which is the minimum number of Δ-moves to convert some knot K into the unknot [2].

Nakanishi studied the Δ-move in depth and calculated $u_\Delta(K)$ for many knots [8]. With $L_x \to L_y$ being labeled as the Δ_{xy} move, Nakanishi used the basis of L_1, L_2, L_3, and L_4 [8]. We added L_a, L_b, L_c, and L_d to account for every possible 3-crossing that we are interested in, thus defining $x, y \in \{a, b, c, d, 1, 2, 3, 4\}$. Using this as a basis, we relate every possible 3-crossing hexagon unknotting move to every possible Δ-move. Δ_{14} and Δ_{23} are Δ-moves, while hex unknotting moves also include $\Delta_{ad}, \Delta_{bc}, \Delta_{13}$ and Δ_{24}. We know from Nakanishi that any two Δ_{xy}



(A) 9_6 minimal diagram with only triangular configuration highlighted



(B) 9_6 hexagonal mosaic. Unknotting the 3-crossing tile using H6 doesn't produce the unknot, so this may not be the unknotting diagram.

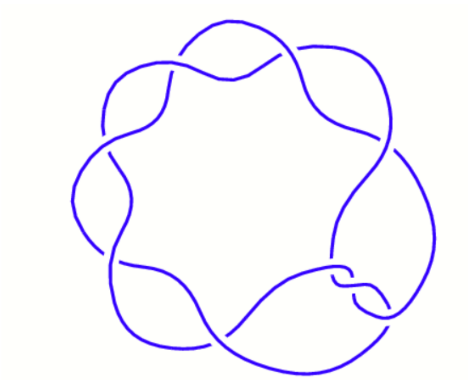


FIGURE 13. 9_3 diagram

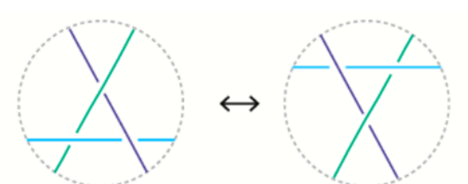


FIGURE 14. Δ-move example using H which can also be represented by H_{10}. For clarity in 17, we reference these by their Δ-move naming scheme as opposed to their H move naming scheme.

moves are related by a finite sequence of Δ-moves and Redeimester moves [8]. We constructed the additional explanation for $\Delta_{ad} \cong H_8$ and $\Delta_{bc} \cong H_9$.

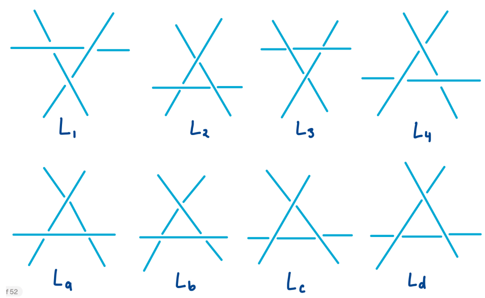


FIGURE 15. All possible 3-crossing links. We use these labels to define Δ labels. These are directly related to a valid hexagon move or indirectly related to any hexagon move using a sequence of those moves.

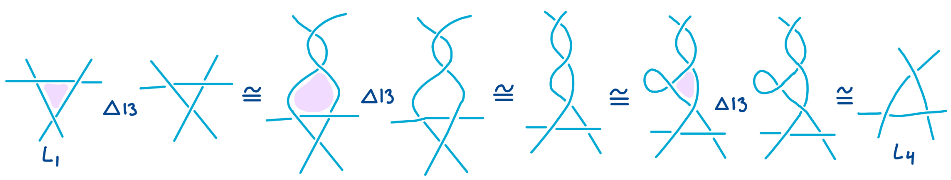


FIGURE 16. Δ_{14} as a sequence of Δ_{13}. We haven't found a method to show Δ_{13} as a sequence of Δ_{14}, but we confirm $\Delta_{14} \to 3\Delta_{12}$ and $\Delta_{12} \to 2H_6$, implying $\Delta_{14} \to 6H_6$, showing these moves are related sequentially by six moves of each other..

By counting the number of moves between each sequence, we relate the Δ-moves, which are Δ_{14} and Δ_{23}, to the hexagon unknotting moves, which are Δ_{ad}, Δ_{bc}, Δ_{13}, and Δ_{24}.

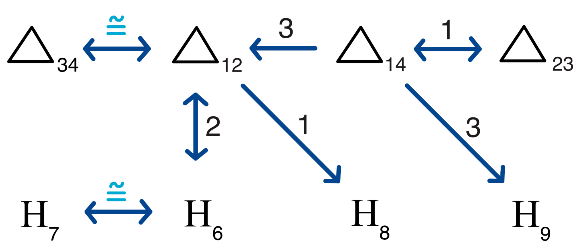


FIGURE 17. Δ_{14} and Δ_{23} are valid Δ-moves, while Δ_{12} and Δ_{34} are invalid Δ-moves. Through the sequencing method shown above, we generate these relations by computing them ourselves or using the work from [8]. Note that direction implies the sequencing that we proved, but every relation is symmetric since Δ and H moves are reversable.

The move count multiplies when taken across a Δ-move relation, since defining any move requires us to construct all other moves as a sequence of the chosen move. We conclude any H-move and any Δ-move is related by 6 or fewer moves.

Theorem 8. *Given some knot K with $u(K) = 3$, if $u_\Delta(K) > 6$, then $u_h(K) > 1$.*

Proof. Suppose the knot with $u_h(K) = 1$. This implies one hexagon move successfully unknots K. This move can be represented with some sequence of no more than 6 delta moves, so $u_\Delta(K) \leq 6$. However, $u_\Delta(K) > 6$, showing a contradiction. □

Since we have a theorem to assist us, we examine 9_3 and other knots with $u_s(K) = 3$. Since $u(9_3) = 3$ and $u_\Delta(9_3) = 9$, we may conclude $u_h(9_3) > 1$. By our earlier bounding formula, we know $u_h(9_3) < u(9_3) = 3$, implying $u_h(9_3) = 2$.

This result implies that no diagram exists in which a 3-crossing unknotting tile can be designed. Many knots behave this way, confirming our suspicions. This shows the vast difference in behavior for 2-crossing and 3-crossing unknotting. 2-crossing tiles will always be achievable, but 3-crossing tiles have significant limitations as defined by the Δ-unknotting number.

We apply this same rationale to a number of knots with under 10 crossings and categorize almost all of them. Two main outliers are 7_1 and 9_{16}. Both have square unknotting numbers of 3 and Δ-unknotting numbers of 6, so Theorem 8 doesn't apply. We still don't believe 7_1 has a 3-crossing unknotting tile, but we need a different method (or a sharper bound on the sequential Δ-move to H-move relation) to determine this unknotting number.

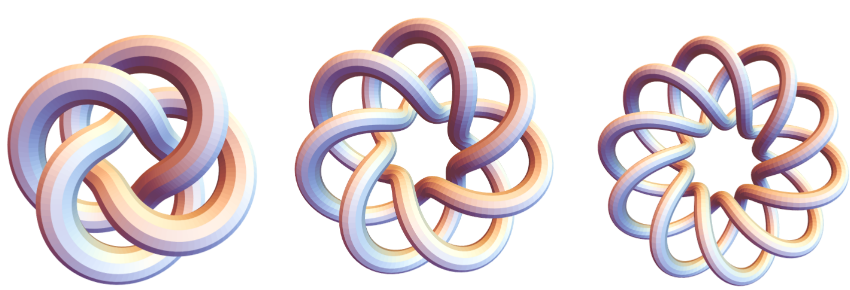
4. Torus Knots

Conjecture 9. *For $T(2, 2n+1)$ torus knots, $u_h(K) = \lceil \frac{u_s(K)}{2} \rceil$.*

Theorem 10. *For torus knots $T_{3n+1,3}$, $u_h(T_{3n+1,3}) = n$.*

Proof. It follows from the torus knot unknotting number formula that $u(T_{3n+1,3}) = 3n$, or $u_s(T_{3n+1,3}) = 3n$. Using the minimal crossing diagram of $T_{3n+1,3}$, find n triple intersections most close to the interior, and construct hex diagrams such that each triple intersection is contained in a singular tile. Unknot accordingly. □

When comparing torus knots, it seems that $T_{3n+1,3}$ torus knots maximize the efficiency of the hexagon diagram with the 3-crossing unknotting tiles.



(A) $T_{4,3}$ (B) $T_{7,3}$ (C) $T_{10,3}$

Figure 18. $T_{3n+1,3}$ knots generated using Wolfram Mathematica

5. Subadditivity

Theorem 11. *For a knot K with connected sum $K = K_1 \# K_2$, $u_h(K) \leq u_h(K_1) + u_h(K_2)$*

Proof. Suppose some set of unknotting crossings for K. For $K = K_1 \# K_2$, map the unknotting crossings of K to the respective crossings in K_1 or K_2. Since an additional crossing change in K_1 or K_2 may be necessary, we produce the following inequality. □

We can't discuss the unknotting number without analyzing unknotting across connected sums. For a connected sum $K = K_1 \# K_2$, it is well known among knot theorists that the unknotting number is subadditive across connected sum, or $u(K) \leq u(K_1) + u(K_2)$. It was conjectured that $u(K) = u(K_1) + u(K_2)$, but this was recently disproven [3]. We have a distinct disproof for this conjecture on the hexagon unknotting number. Take the connected sum of 9_3 and 3_1, yielding the following inequality:

$$u_h(9_3 \# 3_1) = 2 < 3 = 2 + 1 = u_h(9_3) + u_h(3_1)$$

Equality holds in some cases, such as 8_{19} and 3_1:

$$u_h(8_{19} \# 3_1) = 2 = 1 + 1 = u_h(8_{19}) + u_h(3_1)$$

Since some knots allow for 3-crossing unknotting tiles while others don't, we conclude the hexagon unknotting number must be strictly subadditive.

6. Future Work

Our research primarily examined low crossing number knots, meaning most of our knots had unknotting numbers of 3 or fewer. The only knot in our appendix with a larger square mosaic unknotting number than 3 is 9_1. So our unanswered question is on the properties of higher unknotting number knots.

Consider any knot with unknotting number 4. In this instance, we find a hexagon unknotting number of 2 using two 2-crossing unknotting tiles or one 3-crossing unknotting tile and one 1-crossing unknotting tile, if possible.

Now consider a knot with unknotting number 5 or more. We find a hexagon unknotting number of 2 with a 2-crossing and 3-crossing unknotting tile or two 3-crossing unknotting tiles. Both depend on the Δ-unknotting number. For a knot with $u_s(K) \in [4, 6]$, if $u_\Delta(K) > 12$, then $u_h(K) > 2$. We assert $u_s(K) \in [4, 6] \Rightarrow u_h(K) \in [2, 3]$ by generalizing our process from Lemma 5, or using the Γ-unknotting number. We also generalize the Δ-unknotting number bounding process for any $u_s(K)$ with a 3 divisor. This allows us to construct some bounding with these unknotting numbers.

Our goal is to describe a functional relationship from $u_h(K)$ to $u_\Gamma(K)$ and $u_\Delta(K)$. Assuming $u_\Gamma(K)$ halves the unknotting number and $u_\Delta(K)$ is bounded by five times the hexagon unknotting number - since six is the cusp - we suggest an upper bound.

Conjecture 12. $u_h(K) \leq \min\{u_\Gamma(K), \lceil \frac{u_\Delta(K)}{5} \rceil\}$

However, a more nuanced bound or even a formula could be constructed to identify the relationship between these unknotting numbers.

7. Appendix

K	$u(K)$	$u_\Delta(K)$	$u_h(K)$	K	$u(K)$	$u_\Delta(K)$	$u_h(K)$
3_1	1	1	1	9_8	2	2	1
4_1	1	1	1	9_9	3	8	2
5_1	2	3	1	9_{10}	3	8	2
5_2	1	2	1	9_{11}	2	4	1
6_1	1	2	1	9_{12}	1	1	1
6_2	1	1	1	9_{13}	3	7	2
6_3	1	1	1	9_{14}	1	1	1
7_1	3	6	[1,2]	9_{15}	2	2	1
7_2	1	3	1	9_{16}	3	6	[1,2]
7_3	2	5	1	9_{17}	2	2	1
7_4	2	4	1	9_{18}	2	6	1
7_5	2	4	1	9_{19}	1	2	1
7_6	1	1	1	9_{20}	2	2	1
7_7	1	1	1	9_{21}	1	3	1
8_1	1	3	1	9_{22}	1	1	1
8_2	2	2	1	9_{23}	2	5	1
8_3	2	4	1	9_{24}	1	1	1
8_4	2	3	1	9_{25}	2	2	1
8_5	2	3	1	9_{26}	1	2	1
8_6	2	2	1	9_{27}	1	2	1
8_7	1	2	1	9_{28}	1	1	1
8_8	2	2	1	9_{29}	2	1	1
8_9	1	2	1	9_{30}	1	1	1
8_{10}	2	3	1	9_{31}	2	2	1
8_{11}	1	1	1	9_{32}	2	1	1
8_{12}	2	3	1	9_{33}	1	1	1
8_{13}	1	1	1	9_{34}	1	1	1
8_{14}	1	2	1	9_{35}	3	7	2
8_{15}	2	4	1	9_{36}	2	3	1
8_{16}	2	1	1	9_{37}	2	3	1
8_{17}	1	1	1	9_{38}	3	6	1
8_{18}	2	1	1	9_{39}	1	2	1
8_{19}	3	5	1	9_{40}	2	1	1
8_{20}	1	2	1	9_{41}	2	2	1
8_{21}	1	2	1	9_{42}	1	2	1
9_1	4	10	2	9_{43}	2	3	1
9_2	1	4	1	9_{44}	1	2	1
9_3	3	9	2	9_{45}	1	2	1
9_4	2	7	1	9_{46}	2	2	1
9_5	2	6	1	9_{47}	2	1	1
9_6	3	7	2	9_{48}	2	3	1
9_7	2	5	1	9_{49}	3	6	1

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