Game On: Tiling, Playing, and Counting with N-ominoes

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Abstract

This paper explores a student-driven investigation into generalizing classic tile-based games like dominoes and triominoes to a broader type of game using ***N***-ominoes: regular polyhedral tiles with ***N*** sides. Three students take on the challenge of determining the best value of ***N*** as well as the number of pips allowed on each tile. Along the way, they grapple with questions about fairness, strategy, complexity and enumeration: How many tiles will be in the game? How many pips should be allowed on a tile? How many points are in a game? What makes a game "good", and how can mathematics help us decide?

Keywords: Combinatorics, Game Theory, N-ominoes, Table Top Games, Tessellations

MSC Classification: 05A19 , 05B45 , 91A99

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1 Introduction

Dee, Vee, and Zee look forward to game night every semester. And every semester they prepare in similar ways. Seven days out: time to schedule out homework and get ahead - no distractions allowed on Thursday night. Five days out: a trip to the library to check out some games. Three days out: start thinking about strategies. One day out: oops, better get that homework done! Finally, game night has arrived, which game shall we play? Inevitably every semester it is the same tradition, the same preparation, and sadly, the same outcome: no one can agree on which game to play.

Dee wants to play *Settlers of Catan* but Vee hates how the dice always seem to roll 6's instead of 8's. Vee wants to play *Scrabble*, but Zee is still mad that "hotdish" doesn't count as a valid word but "undersea" does. Zee wants to play *dominoes* and magically, everyone agrees! Oh no, they forgot to bring the game. Everyone is bummed out.

Enter Drs. S and W. "Why so bummed out? I thought you all loved game night." After some discussion, Dr. W suggests that they build their own domino set out of cardboard. Not to be outdone, Dr. S remarks that they are not even bound by the restrictions of dominoes. They can make dominoes with seven dots on one side! They could even make three-sided ominoes: tri-ominoes! Or four-sided ominoes: quad-ominoes?[1] Even N-ominoes!

Needless to say, Dee, Vee, and Zee never got around to playing a game that night. Instead, they spent the evening discussing all the possible variants for the game, which variants would be best, and most importantly, why.

1.1 N-ominoes for small N

Most readers will know the game of 2-ominoes by its more common name: dominoes. Typically, dominoes are represented as flat, rectangular tiles with markings (pips) on each side. Dominoes are divided down the middle by a line, creating two squares. Each square has a number of pips, typically ranging from zero to six, and points are earned for all pips on each tile a player plays.

Perhaps less familiar to the reader is that the game of 3-ominoes also exists. A popular version of this game is marketed as Tri-Ominos by the Pressman Toy Corp. Pressman also has released a game known as Quad-Ominos (i.e. 4-ominoes) but it is now out of print. In each of these games the N-ominoes are represented by flat tiles in regular polygonal shapes (triangles and squares). In lieu of pips, Tri-Ominos tiles have a number inscribed near each corner, 0 through 5. Similarly, Quad-Ominos also have one of six numbers on each corner, but this time 1 through 6. In this article, we resolve this discrepancy by utilizing the indexing from dominoes and Tri-Ominos. We will still refer to the numbers on the -ominoes as pips. That is, the lowest pip-value will be 0 and the max will be n (in the case of the Pressman Toy Corp game, $n = 5$ and typically in dominoes $n = 6$). We will later see that we can create table-top N-omino games for $N = 3, 4, 6$. For other values of N we would have to resort to playing on a hyperbolic or spherical-shaped board (or table).

[1] Three- and four-sided ominoes do in fact already exist. The games are trademarked and the fictional characters in this article did not play any role in developing the game.

2 Tile Directionality

In the game of dominoes, there is no "tile directionality" to consider. That is, the five-two tile and the two-five tile are in fact the same tile by rotating 180°. For $N \geq 3$, we are presented with more options. The one-two-three tile can be rotated 120° and 240° to present itself as a two-three-one or three-one-two tile, always reading the numbers in a clockwise fashion. However, since the tiles are single-sided and non-transparent, there is no orientation of this tile that would allow the numbers to be read clockwise in the order one-three-two. As such, these two are different tiles and the students must consider whether to include them both.

Zee feels very strongly that only a clockwise non-decreasing order should be allowed, arguing that if both directions are allowed, there would be way too many tiles. Zee continues, saying that without the restriction, the number of tiles with three distinct pip values in the 3-ominoes game would double. That is, the number of tiles that have three distinct pip values would increase from $\binom{n}{3}$ to $2\binom{n}{3}$. In general, as the number of sides, N, increases, there is a corresponding disparity that would increase exponentially from $\binom{n}{N}$ to $(N-1)!\binom{n}{N}$. Yikes! The others are easily convinced: the tiles should be restricted to those with a non-decreasing pip sequence read in the clockwise direction. Otherwise, all the extra tiles would extend the game while making each tile easier to place. No one likes a long, boring game!

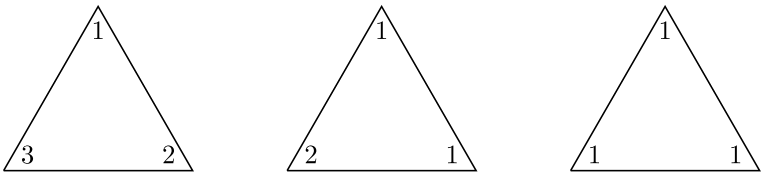

Fig. 1 Three allowed triominoes.

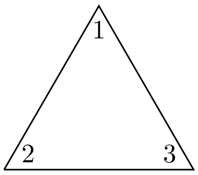

Fig. 2 A forbidden triomino.

3 Tile Shape

Having agreed to only include tiles that can be presented in a clockwise non-decreasing manner, the group now focuses on determining the best value of N. That is, how many sides should the optimal N-omino have? The students started constructing regular polygons and they soon conjectured that only $N = 3, 4, 6$ can tile the plane. In fact, this is true: the interior angles of an N-sided regular polygon sum up to $(N-2)\times 180°$. Thus, each interior angle must be $\frac{(N-2)\times 180°}{N}$. To find out how many tiles will meet at a time to tesselate the plane we divide $360°$ by $\frac{(N-2)\times 180°}{N}$. Simplified, this yields

$$d(N) = \frac{2N}{N-2},$$

where $d = d(N)$ is determined by N and is defined to be the number of tiles meeting at a time to tesselate a surface.

The students noticed that $d(N)$ is always greater than 2 and $d(7) = 2.8$. Since $d(x) = \frac{2x}{x-2}$ has a negative derivative, $d(N)$ is a decreasing function. That is, if $N \geq 7$ then $2.8 \geq d(N) > 2$. Therefore, $d(N)$ is not an integer when $N \geq 7$. They quickly checked $N = 3, 4, 5, 6$ and discovered that with $N = 3, 4, 6$, the polygons tessellate the plane, but with $N = 5$ they do not. In fact, $d(5) = \frac{10}{3}$ and it is impossible for a non-integer number of tiles to meet at a point (on a surface with zero curvature). Vee points out that having exactly three pentagonal tiles meet at a corner is possible if they play on a surface with positive curvature, such as a dodecahedron (it turns out all the Platonic solids can be constructed this way!). Dee says they could also try to force four (or more) regular pentagons together but this would create the need for a surface with negative curvature, something akin to the hyperbolic plane.

Zee noted that the relationship between d and N could also be expressed as

$$(d-2)(N-2) = 4$$

and pondered the significance of the value 4 for a while. However, Dee argued that the simplified equation lost information and set out to correct it. After all, $360°$ was used in the derivation of the formula, but $360°$ is only valid for a plane, but not for the Platonic solids or a surface with negative curvature. Dee argued that the formula should be $d = \frac{D°N}{(N-2)180°}$ which can be expressed as

$$(d-2)(N-2) = 4 + \frac{(D° - 360°)N}{180°}.$$

In the case of the plane, $D = 360$. But if they were to play on a hyperbolic plane then $D > 360$ and on a closed surface $D < 360$. That is, the curvature (positive, negative, zero) of the playing surface can be determined by the following formula

$$C(d, N) = (d-2)(N-2) - 4$$

After much discussion they decided that creating a hyperbolic playing board would be too difficult. So, $C(d, N) \geq 0$.

They first considered playing on a tetrahedron and wondered if it would be possible to place tiles on all four sides while respecting the clockwise non-decreasing rule. To determine this, they labeled the vertices a, b, c, d as in Figure 3. Since each of the faces must obey the clockwise non-decreasing rule they note that several inequalities must

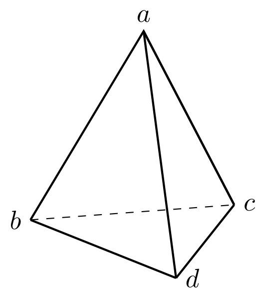

Fig. 3 A labeled tetrahedon.

be simultaneously true. For example, the bottom face must satisfy

$$b \leq d \leq c \quad \text{or} \quad d \leq c \leq b \quad \text{or} \quad c \leq b \leq d.$$

Vee suggested they switch to a more concise notation by letting xyz denote $x \leq y \leq z$, by replacing the word "or" with $\vee$ and the word "and" with $\wedge$. Furthermore, Zee proposed they refer to the posterior, left, right, and bottom tiles with P, L, R, B (respectively). Thus, for a proper tiling of the tetrahedron to exist they note they must have

$$\mathrm{P} \wedge \mathrm{L} \wedge \mathrm{R} \wedge \mathrm{B} = (abc \vee bca \vee cab) \wedge (adb \vee dba \vee bad) \wedge (acd \vee cda \vee dac) \wedge (bdc \vee dcb \vee cbd)$$

Dee noted that by rotating the tetrahedron and relabeling the vertices they could assume that $a = \min\{a, b, c, d\}$ and $b = \min\{b, c, d\}$. These assumed conditions can be written as :

$$A = abcd \vee abdc$$

Thus, a proper tiling of the tetrahedron exists exactly when P, L, R, B, and A are simultaneously true. Fortunately, they note they don't have to look at all of those conditions together to see that the tetrahedron is not a great playing surface:

$$L \wedge A = (adb \vee dba \vee bad) \wedge (abcd \vee abdc)$$

$$\begin{array}{lccccc} = & (adb \wedge (abcd \vee abdc)) & \vee & (dba \wedge (abcd \vee abdc)) & \vee & (bad \wedge (abcd \vee abdc)) \\ \Rightarrow & (b = d) & \vee & (a = b = d) & \vee & (b = a) \\ \Rightarrow & & (b = d) & & \vee & (b = a). \end{array}$$

Hence, the tiled tetrahedron has to be one of those shown in Figure 4.

On the first tetrahedron, the right tile shows acb which implies $b = c$. Hence, they would need three tiles of the form abb (the right, left, and posterior tiles). On the second tetrahedron, the right tile and bottom tile reveal $acd \wedge adc$ which implies $c = d$. Hence they would need two tiles of the form acc and two tiles of the form aac. In either case the game could not be played without repeating tiles. The students did not want to create repeated tiles so they exclude the tetrahedron from contention.

The students then set out to investigate the other Platonic solids. After much effort they weren't able to find a proper tiling of any of the solids. In Figure 5 you will see

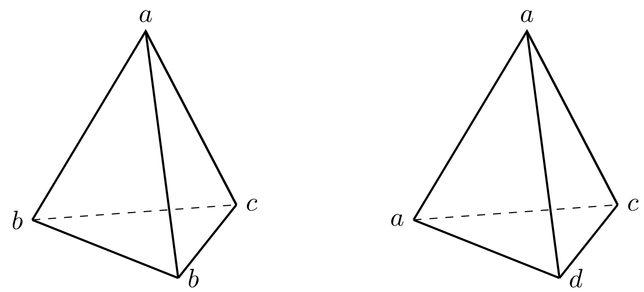

Fig. 4 The two possible tetrahedral games.

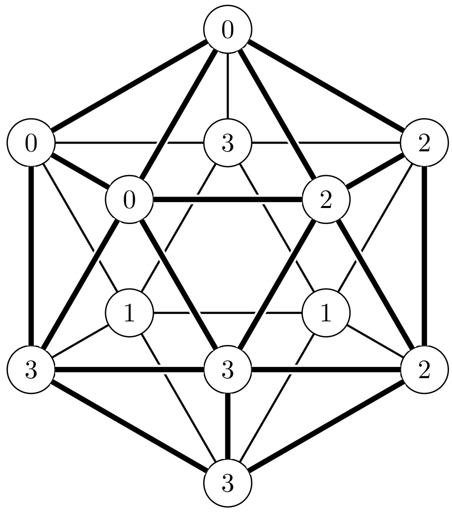

Fig. 5 An icosahedral playing surface.

a failed attempt at tiling the icosahedron – can you spot the errors? [2] Ultimately, the group decided that $C(d, N) > 0$ was not practical since the Platonic solids had very few sides to play on and they hadn't been able to find a proper tiling. As such, they agreed to play with regular triangles, squares, or hexagons on a flat surface.

The students constructed sets and played a few games with each shape. They made some observations. The first observation was that the easiest, and therefore most frequent, plays would attach the new tile to only one existing tile. They then noticed that playing in such a way would increase the number of available edges for the next player. In particular, with triangular tiles, they would often create one more available edge, with squares they would often create two more available edges, and with hexagons they would often create four more available edges. They called these placements "typical placements." There are also some atypical placements. Atypical placements are those that place a tile so that it is adjacent to not just one, but two or

[2] We have conjectured that it is impossible to completely tile any of the Platonic solids using distinct non-decreasing tiles. We'd love for one of our readers to solve this!

more tiles, including the possibilities of either "creating a bridge" (bridging together two distant tiles) or "completing the circuit" (placing the last tile so that all 360° are included). Table 1 summarizes the most common change in the number of edges when making typical, bridge, and circuit completion moves. Other possibilities exist, including plugging a hole, or when completing a circuit simultaneously while either creating a bridge or completing a second (or third) circuit.

Table 1 Change in Available Playing Locations

	Triangle	Square	Hexagon
Typical Placement	+1	+2	+4
Creating a Bridge	+1	0	+2
Completing a singular Circuit	−1	0	+2

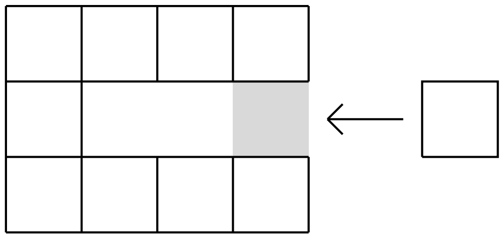

Fig. 6 Creating a bridge using square tiles.

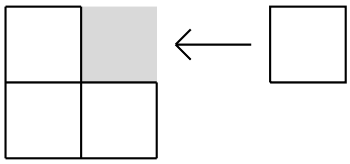

Fig. 7 Completing a circuit using square tiles.

In playing with squares or hexagons, they felt the game was easier since the number of placement options rarely decreased, whereas with triangles the number of placement options often increased, but could decrease more easily. The team decided that $N = 3$ would probably be best, but stopped short of decreeing triangles to be the champion. They shifted their interest to a different question: How many pips should be allowed on a tile? Should it be $n = 6$, like in dominoes? Perhaps that would be too many tiles.

4 Maximum Pip Value

Dee, Vee, and Zee have a new question of interest: how many tiles will be in the game? Certainly, this depends on the choices of N and n. They considered N to be a fixed number – probably 3, but still leaving the door open – and proceeded to see how the

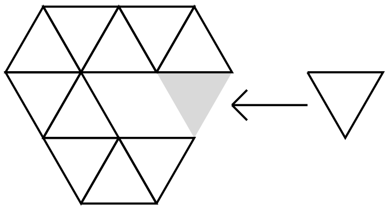

Fig. 8 Creating a bridge using triangular tiles.

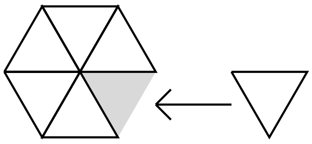

Fig. 9 Completing a circuit using triangular tiles.

choice of the maximum pip value, n, impacts the number of tiles. Zee suggests they categorize each tile by the number of distinct pip values it uses, and then count the number of tiles in each category. For $N = 3$, Zee says there are only three categories to consider: those that have the same pip value on each corner: xxx, those that have two distinct pip values (one of them repeated): xxy, and those with all three pip values distinct: xyz. In this categorization Zee assumes $x < y < z$. As such, they figured out that the number of tiles would be $\binom{n+1}{1}$ for xxx, $\binom{n+1}{2}$ for xxy, and $\binom{n+1}{3}$ for xyz. As diligent students, they set out to check their work for $n = 5$. According to their calculation there should be $\binom{5+1}{1} + \binom{5+1}{2} + \binom{5+1}{3} = 41$ tiles. But they quickly note that this is incorrect as they have already created 56 distinct tiles.

After a bit of inspection they found the error. Certain combinations of integers need to be double counted since the 223 and the 233 tiles are distinct! That is, there are four categories: xxx, xxy, xyy, and xyz. The number of categories doubles to eight for $N = 4$: $wwww$, $wwwx$, $wwxx$, $wxxx$, $wwxy$, $wxxy$, $wxyy$, and $wxyz$. Dee observes that the fact that the number of categories doubled is not a coincidence! One could create four of the eight categories by appending a "w" to the left of the four categories for $N = 3$ to get $xxx \to wxxx$, $xxy \to wxxy$, $xyy \to wxyy$, and $xyz \to wxyz$; and then get the other four by first shifting each letter down one unit and then appending the succeeding letter on the right: $xxx \to wwwx$, $xxy \to wwxy$, $xyy \to wxxy$, and $xyz \to wxyz$. For $N = 6$ the number of categories is 32. In general, assuming that n is large enough, the number of categories for N is represented exponentially by 2^{N-1} (yet another reason why $N = 3$ may be the best number of sides).

Dee objected to having so many categories; after all, 223 and the 233 tiles are distinct but they could still be grouped together. Dee observed that the number of non-decreasing strings of length N using k different characters can be counted combinatorially. Consider any such string: it will have $k - 1$ transitions from one character to the next. For example, $xxyzz$ transitions after the second character and after the third character. Thus, a string of string of length N has $N - 1$ possible character transition locations, of which $k - 1$ are being selected. Zee notes this means that there are $\binom{N-1}{k-1}$ different representations of a string with k distinct characters, and they can assign the characters in $\binom{n+1}{k}$ ways (here $n+1$ is used because there are $n+1$ characters when counting from zero to n). Hence, the number of tiles is

$$\sum_{k=1}^{N} \binom{N-1}{k-1}\binom{n+1}{k}.$$

Still not satisfied, the students set out to see if they could count the number of tiles in a different way to get a simpler formula. After some effort the three of them had a great idea! The problem with counting tiles, and the issue that caused them to accidentally under count, is that tiles could repeat numbers. That is, the numbers on the tiles had to obey a non-decreasing rule. It would be a lot easier if the tiles had to follow a strictly increasing rule. Fortunately, that can be solved: take the non-decreasing numbers that appear on the tile and write them down, for instance "w w x z." Now, to the first entry, add 1; to the second entry, add 2; to the third entry, add 3; and so on:

$$\text{“}w+1 \qquad w+2 \qquad x+3 \qquad z+4.\text{”}$$

Now these numbers are guaranteed to be unique positive integers: $w+1 < w+2 < x+3 < z+4$. In general, assuming $n \geq N$ this creates every possible string of length N from the numbers in $\{1, 2, 3, \ldots, n+N\}$ that is strictly increasing. Hence, the number of tiles must be $\binom{n+N}{N}$. That is

$$\sum_{k=1}^{N} \binom{N-1}{k-1}\binom{n+1}{k} = \binom{n+N}{N}.$$

Table 2 illustrates the number of tiles in a game, based on $N \in \{3, 4, 6\}$ and $n \geq N$.

Table 2 Number of Tiles in Game

$N\backslash n$	3	4	5	6	7	8	9
3	20	35	56	84	120	165	220
4		70	126	210	330	495	715
6				924	1716	3003	5005

Once again $N = 3$ appeared to be a favorite, perhaps $N = 4$ but certainly not $N = 6$ (those numbers are huge). After some discussion the students concluded that playing a game with over 100 pieces would be exhausting, thus concluding that $(N, n) \in \{(3, 3), (3, 4), (3, 5), (3, 6), (4, 4)\}$. As such, they set out to make the necessary 154 pieces and played games with each set.

They weren't yet satisfied with any game as it appeared that the winner was usually determined on luck of drawing good tiles. Something was missing to make the game more strategic. Dee proposed that there should be some special placements that earn bonuses, specifically when a player creates a bridge they should get B bonus points and when a player completes a circuit they should get C bonus points. They just needed to determine the values of B and C, and perhaps, those values should vary with N and n.

5 Choosing B and C, how many points will there be in a game?

Before choosing values for the bonus points B and C, Dee says they want to know how many total points are available if all the tiles are played, assuming that, as in dominoes, the total pip value of the tile is the number of points awarded for playing the tile. Vee thought this was a good idea and collectively they devised a strategy to calculate total points, $T := T(n, N)$, for the entire tile set. For an N-omino tile set S and maximum pip-value n, they constructed a bijection $\phi : S \to S$. Given a tile considered as an ordered N-tuple, $(x_1, x_2, ..., x_N)$, with $0 \leq x_1 \leq x_2 \leq ... \leq x_N \leq n$, the function $\phi(x_1, x_2, ..., x_N) = (n-x_N, n-x_{N-1}, ..., n-x_1)$ pairs each tile to another tile in the set. The students note that since $0 \leq x_1 \leq x_2 \leq ... \leq x_N \leq n$ implies $0 \leq n - x_N \leq n - x_{N-1} \leq ... \leq n - x_1 \leq n$, the function ϕ is well defined. They also observe that if:

$$\phi(y_1, y_2, ..., y_N) = \phi(z_1, z_2, ..., z_N),$$

then

$$(n - y_N, n - y_{N-1}, ..., n - y_1) = (n - z_N, n - z_{N-1}, ..., n - z_1)$$

and

$$(y_1, y_2, ..., y_N) = (z_1, z_2, ..., z_N).$$

They thus conclude ϕ is an injective function from S to itself. Since $|S|$ is finite, ϕ must also be surjective. Hence, ϕ is a permutation of the tiles in S.

The students then direct their attention to solving for T, assuming there are no bonus points and that every tile is played. Zee observes that, using Dee's function, they can more easily tally up the total points in the game. Indeed, pretending to have two sets of tiles now makes the calculation easy using the bijection from ϕ. They note that

$$2T = \sum_{(x_1,x_2,...,x_N)\in S} (x_1+x_2+...+x_N) + \sum_{\phi(x_1,x_2,...,x_N)\in S} (n-x_N+n-x_{N-1}+\cdots+n-x_1).$$

Simplifying the right hand side of this gives them

$$2T = \sum_{(x_1,x_2,...,x_N)\in S} Nn = |S|Nn,$$

which yields

$$T = \binom{n+N}{N} N\frac{n}{2}.$$

This formula allows the students to calculate the maximum possible points in a game for various values of N and n, as can be seen in Table 3.

After playing several games they found that setting both B and C to approximately $\frac{T}{10}$ incorporated just a enough strategy into the game while not allowing one move to completely dictate the winner. As such, they proposed the bonus structure in Table 4.

Table 3 Max Points in a Game

$N \backslash n$	3	4	5	6
3	90	210	420	756
4		420		

Table 4 Bonus Structure

$N \backslash n$	3	4	5	6
3	10	20	50	75
4		50		

6 Conclusion

Ultimately, the team never determined the best choices for the values of N and n. Instead, they concluded that having options was good. Often, in between classes they would play a quick game using $N = n = 3$ and play to 100 points. On game nights, they would play several games and let the parameters vary. Once, over a long weekend, they played a marathon game with $N = 3$ and $n = 6$ and played to $10,000$ (it took 27 games!).

What started as a disappointing game night turned into a captivating mathematical adventure. By carefully studying the shapes of the tiles, the possible values, the board geometry, and scoring strategies, Dee, Vee, and Zee transformed a simple game night into an exploration of combinatorics and geometry. While they never insisted on a single "best" version of the game, their discoveries highlighted how mathematics can contribute to game design, from ensuring fairness to balancing complexity to creating engaging game mechanics. The trio learned that creating a good game isn't just about finding the perfect formula; it's about asking the right questions, exploring the possibilities, and having fun in the process.

There are still many directions to explore. How can probability help determine the best game strategies? Combining triangles and squares can create some interesting planar tessellations. Would combining different N values in a single game be worthwhile? Can a semi-regular polyhedral board be fully tiled while avoiding duplicate pieces?