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SUBGROUP DECOMPOSITION OF THE GINI COEFFICIENT: A NEW SOLUTION TO AN OLD PROBLEM

VESA-MATTI HEIKKURI
Faculty of Management and Business, Tampere University

MATTHIAS SCHIEF
Economics Department, OECD

We derive a novel decomposition of the Gini coefficient into within- and between-group inequality terms that sum to the aggregate Gini coefficient. This decomposition is derived from a set of axioms that ensure desirable behavior for the within- and between-group inequality terms. The decomposition of the Gini coefficient is unique given our axioms, easy to compute, and can be interpreted geometrically.

KEYWORDS: Inequality, Gini coefficient, subgroup decomposition, axiomatic framework, Brunn–Minkowski theorem.

1. INTRODUCTION

EMPIRICAL ANALYSES OF INEQUALITY OFTEN aim to quantify the contributions of within-group inequality and between-group differences to aggregate inequality. This classic objective is known as inequality decomposition by population subgroups. Examples include analyses of inequality within and between countries (Sala-i-Martin (2006)), demographic subgroups (Cowell and Jenkins (1995)), or firms (SPGB+ (2019)).

The Gini coefficient is the most popular measure of inequality, yet it lacks a universally accepted decomposition formula.[1] Existing approaches to decomposing the Gini coefficient are criticized for producing within-group or between-group inequality terms that behave counter-intuitively. Further, these approaches often rely on introducing a third term that describes neither within-group nor between-group inequality. This lack of a satisfactory subgroup decomposition formula is arguably the most significant drawback of the Gini coefficient, which is otherwise valued for its intuitive arithmetic definition and geometric relation to the Lorenz curve.

In this paper, we derive a novel decomposition formula for the Gini coefficient from a set of axioms that ensure desirable behavior for the within-group and between-group inequality terms. We show that these axioms uniquely determine the decomposition formula

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Vesa-Matti Heikkuri: vesa-matti.heikkuri@tuni.fi
Matthias Schief: matthias.schief@oecd.org

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[1]The Gini coefficient is also one of the earliest measures of inequality, introduced by Corrado Gini in 1912. An overview of suggested decomposition formulas for the Gini coefficient can be found in Giorgi (2011).

for the Gini coefficient. The decomposition is easy to compute, and has both a geometric and an arithmetic interpretation. We demonstrate the decomposition by analyzing the evolution of household income inequality within and between demographic subgroups in the United States over the past six decades.[2]

Our paper contributes to the literature on decomposable inequality indices by proposing a standard for satisfactory decomposition that relies neither on the *aggregativity* requirement put forward by Bourguignon (1979) and Shorrocks (1980, 1984), nor on the *path independence* requirement introduced by Foster and Shneyerov (2000). While aggregativity characterizes generalized entropy indices (but is violated by Foster–Shneyerov indices) and path independence characterizes Foster–Shneyerov indices (but is violated by generalized entropy indices), neither standard allows decomposition of the Gini coefficient. Our definition of decomposability accommodates both the generalized entropy and the Foster–Shneyerov indices, and also enables decomposition of the Gini coefficient.

2. NOTATION AND DEFINITIONS

We use $\mathbb{R}_+$ to denote the interval $[0, \infty)$ and $\mathbb{R}_{++}$ to denote the interval $(0, \infty)$. An inequality index $I: \mathcal{D} \to \mathbb{R}_+$ is a function that maps a space of distributions $\mathcal{D}$ to the non-negative real numbers and satisfies the five standard axioms of anonymity, scale independence, population independence, normalization, and the Pigou–Dalton principle of transfers.[3]

Throughout the paper, we represent income distributions by their generalized Lorenz curves.[4] A generalized Lorenz curve is an increasing convex function $L: [0, 1] \to \mathbb{R}_+$ with $L(0) = 0$.[5] The value of the generalized Lorenz curve at point $p \in [0, 1]$ is equal to p times the mean income of individuals below the pth quantile, and $L(1)$ is equal to the overall mean income. Hence, the generalized Lorenz curve is obtained by multiplying the standard Lorenz curve by the mean income and thus preserves the information on the mean of the distribution. In addition, generalized Lorenz curves provide a more general representation of distributions than cumulative distribution functions.[6] In particular, distributions with perfect inequality can be represented by a generalized Lorenz curve that is zero everywhere except at 1, while a cumulative distribution function does not exist in this case.

The Gini coefficient is defined for any generalized Lorenz curve L as[7]

$$G(L) = 2\int_0^1 p - \frac{L(p)}{L(1)}\, dp.$$

[2]We provide a Stata command, `ginidecomp`, that implements the decomposition.

[3]Inclusion of the scale independence axiom means that we focus on indices of relative inequality rather than absolute inequality. Indices of relative inequality have the virtue of being independent of the unit of measurement.

[4]We frame our discussion in terms of income distributions, but income can be substituted with any non-negative real valued attribute.

[5]Note that by restricting generalized Lorenz curve to be increasing, we are ruling out negative incomes.

[6]For a distribution that has a cumulative distribution function, F, the generalized Lorenz curve can be expressed as $L(p) = \int_0^p F^{-1}(t)\, dt$, where F^{-1} is the generalized inverse of F (Gastwirth (1971)).

[7]If all incomes are zero, then the generalized Lorenz curve is a constant function at zero and the formula defining the Gini coefficient cannot be evaluated. We define the Gini coefficient to be zero in this case.

3. AXIOMATIC FRAMEWORK

We consider a population that is partitioned into K subgroups. Subgroup k's level of inequality, population size, and total income are denoted by I_k, n_k, and Y_k, respectively. An additive subgroup decomposition of an inequality index I is the sum of a within-group inequality term W and a between-group inequality term B:

$$I = W + B.$$

We introduce axioms that satisfactory within-group and between-group inequality terms must satisfy. We call an inequality measure decomposable if it admits an additive decomposition that satisfies these axioms.[8]

3.1. *Within-Group Inequality*

Within-group inequality summarizes how inequality within subgroups contributes to aggregate inequality. We require that within-group inequality depends only on subgroup inequality levels and aggregate characteristics, that is, each subgroup's total population and total income. Let $W_K: \mathbb{R}_+^{3K} \to \mathbb{R}_+$ denote within-group inequality for a population consisting of K subgroups. We posit the following axioms:

AXIOM 1—Regularity: *W_K is continuous, and strictly increasing in I_k if $n_k, Y_k > 0$.*

AXIOM 2—Symmetry: *For any permutation P and for all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$,*

$$W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) = W_K\big(P\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big)\big).$$

AXIOM 3—Scale and population independence:

$$W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) = W_K\big((I_1, an_1, bY_1), \dots, (I_K, an_K, bY_K)\big)$$

for all $a, b > 0$ and for all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$.

AXIOM 4—Normalization: *$W_K((0, n_1, Y_1), \dots, (0, n_K, Y_K)) = 0$ for all $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{2K}$.*

AXIOM 5—Weak reflexivity: *$W_K((I, a_1 n, a_1 Y), \dots, (I, a_K n, a_K Y)) = I$ for all $(I, n, Y) \in \mathbb{R}_{++}^3$ and for all $(a_k)_{k=1}^K \in \mathbb{R}_{++}^K$.*

AXIOM 6—Replacement: *For all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$ and $m \le K$,*

$$\begin{aligned} &W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) \\ &\quad = W_{K-m+1}\left(\left(\tilde{I}, \sum_{k=1}^m n_k, \sum_{k=1}^m Y_k\right), (I_{m+1}, n_{m+1}, Y_{m+1}), \dots, (I_K, n_K, Y_K)\right), \end{aligned}$$

where $\tilde{I} = W_m((I_1, n_1, Y_1), \dots, (I_m, n_m, Y_m))$.

[8] In Supplemental Appendix A (Heikkuri and Schief (2026)), we discuss how our definition of decomposability relates to other standards suggested in the literature.

Regularity ensures that within-group inequality increases continuously in subgroup inequality levels. Symmetry ensures that within-group inequality is independent of the labels given to each subgroup and is sometimes also called anonymity. Scale and population independence ensures that the decomposition is independent of the size of the population and the unit of measurement for income. Normalization ensures that within-group inequality is zero when there is no inequality within any subgroup.

Weak reflexivity states that when all subgroups have the same level of inequality and mean income, within-group inequality is equal to the common inequality level across the subgroups. Hence, weak reflexivity ensures that within-group inequality equals aggregate inequality when the population consists of only one subgroup. Moreover, it ensures that within-group inequality does not change if that subgroup is divided into smaller subgroups with identical income distributions. Without weak reflexivity, within-group inequality would not be comparable across populations with different numbers of subgroups.

Finally, the replacement axiom states that the level of within-group inequality remains unchanged if we replace any number of subgroups by a single subgroup with population size and total income equal to the combined population and income of the replaced subgroups, and with inequality equal to the level of within-group inequality among the replaced subgroups. Together with symmetry, replacement ensures a consistent aggregation property. For example, consider computing within-group inequality among the fifty U.S. states. Replacement guarantees that this can be done if we are given the level of within-group inequality in subaggregates, such as the eastern states and the western states, together with the population sizes and total incomes of these subaggregates. In particular, information on the inequality levels of individual states is not needed in this case. This is a natural property because the contribution to aggregate inequality stemming from inequality within individual states is already summarized in the levels of within-group inequality in the subaggregates.

Axioms 1 through 6 imply strong restrictions on the functional form that within-group inequality can have.

THEOREM 1: *Let* $(W_K)_{K=1}^{\infty}$ *be a sequence of functions* $W_K\colon \mathbb{R}_+^{3K} \to \mathbb{R}_+$. *Then,* (W_K) *satisfies Axioms* 1–6 *if and only if*

$$W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) = f^{-1}\left(\sum_{k=1}^{K} \pi_k^{1-\alpha} \theta_k^{\alpha} f(I_k)\right), \tag{1}$$

for all $K \geq 1$ *and for all* $(I_k, n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_+^{3K}$, *where* f *is some continuous and strictly increasing function with* $f(0) = 0$, $\alpha \in \mathbb{R}$ *is some real number,* $\pi_k = n_k / \sum_{k=1}^{K} n_k$, *and* $\theta_k = Y_k / \sum_{k=1}^{K} Y_k$.

Theorem 1 holds that within-group inequality must take the form of a quasi-arithmetic mean of subgroup inequalities with weights that do not necessarily sum to 1. Moreover, each subgroup's weight must itself be a weighted geometric average of the subgroup's population and income share. The proof proceeds by showing that, for given K and aggregate characteristics, the axioms imply that W_K satisfies bisymmetry. Bisymmetry has been used by Aczél (1948) and Münnich, Maksa, and Mokken (2000) to characterize quasi-arithmetic means. The rest of the proof shows that the generating function f and the form of the weight function do not depend on the number of subgroups or their aggregate characteristics. The complete proof is presented in the Appendix.

The functional form in Theorem 1 nests the expressions for within-group inequality in the standard decomposition formulas for the generalized entropy indices and the Foster–Shneyerov indices—two important classes of decomposable inequality indices (Bourguignon (1979), Shorrocks (1980), Cowell (1980), Shorrocks (1984), Foster and Shneyerov (2000)).[9] Interestingly, however, several decomposition formulas for the Gini coefficient suggested in the literature are ruled out by this theorem. For example, the most common decomposition formula for the Gini coefficient due to Bhattacharya and Mahalanobis (1967) is ruled out by Theorem 1 since the weight function is not a weighted geometric average, implying that the within-group inequality term violates weak reflexivity.[10] More recently, Shorrocks (2013) proposed an algorithm for inequality decomposition which results in a within-group inequality term that generally does not take the form of a quasi-arithmetic mean.

3.2. *Between-Group Inequality*

Between-group inequality summarizes how differences in income distributions across subgroups contribute to aggregate inequality. We define between-group inequality to be a function of subgroup income distributions, population sizes, and total incomes, that is, $B_K \colon (\mathcal{D} \times \mathbb{R}_+ \times \mathbb{R}_+)^K \to \mathbb{R}_+$. We posit the following axioms:

AXIOM 7—Normalization: *If all subgroups have the same income distribution, then B_K is equal to zero.*

AXIOM 8—Conditional distribution independence: *For given aggregate characteristics $(n_1, \dots, n_K, Y_1, \dots, Y_K) \in \mathbb{R}_+^{2K}$, if there exists a function $F \colon \mathbb{R}_+^K \to \mathbb{R}_+$ such that*

$$I(L) = F(I_1, \dots, I_K)$$

for all $(I_1, \dots, I_K) \in \mathbb{R}_+^K$, then B_K does not depend on the distribution of income within subgroups.

Normalization ensures that between-group inequality is zero when the subgroups' income distributions are identical so that aggregate inequality must be entirely due to income differences within subgroups.[11]

Conditional distribution independence generalizes a property imposed by Bourguignon (1979) and Shorrocks (1980), which requires that between-group inequality is unaffected by income transfers within subgroups. These authors impose distribution independence in the context of aggregative inequality indices, which are indices for which aggregate inequality is a function of subgroup population sizes, inequality levels, and average incomes. It is natural to require distribution independence for aggregative inequality indices

[9]The within-group inequality term in the standard decomposition formula for the generalized entropy index of order α is given by $\sum_{k=1}^K \pi_k^{1-\alpha} \theta_k^\alpha I_k$, and the within-group inequality term in the standard decomposition formula for the Foster–Shneyerov index of order q is given by $\sum_{k=1}^K \pi_k I_k$.

[10]The within-group inequality term in the Gini decomposition of Bhattacharya and Mahalanobis (1967) is given by $\sum_{k=1}^K \pi_k \theta_k I_k$.

[11]Note that Axiom 7 is implied by Axiom 5 (Weak reflexivity) and the requirement that aggregate inequality is the sum of within- and between-group inequality. In Supplemental Appendix A, we strengthen Axiom 7 to show that a similar axiomatic framework can be used to derive unique decompositions not only for the Gini coefficient but also other classes of inequality indices. The strengthened version of Axiom 7 is not implied by Axiom 5 and is satisfied by the Gini decomposition derived in this paper.

since the impact of within-group transfers on aggregate inequality is fully summarized by changes in subgroup inequality levels and should therefore be attributed to within-group inequality.

However, imposing unconditional distribution independence on nonaggregative inequality indices like the Gini coefficient is inappropriate, as within-group transfers only keep the means constant while aggregate inequality is generally also sensitive to whether these transfers make the subgroup distributions more or less similar in other moments. Clearly, changes in aggregate inequality due to subgroup distributions becoming more or less similar should be captured by between-group inequality, which therefore cannot be distribution independent.

Instead, *conditional* distribution independence only imposes distribution independence whenever, for given aggregate characteristics, the aggregate inequality index is a function of subgroup inequality indices alone. In these special cases, the inequality index is aggregative and therefore between-group inequality should indeed be independent of the income distribution within subgroups. As is well known, the Gini coefficient is aggregative when all but one subgroup have zero income or population share and the aggregate characteristics are therefore such that the subgroup income distributions cannot overlap.

4. DECOMPOSITION OF THE GINI COEFFICIENT

We next show that there exists a unique decomposition of the Gini coefficient into a within-group and a between-group term that satisfies all the axioms introduced in Section 3.

4.1. *Decomposition Formula*

THEOREM 2: *A decomposition for the Gini coefficient satisfies Axioms* 1–8 *if and only if the within-group inequality term is*

$$G^W = \left(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G_k} \right)^2, \tag{2}$$

where π_k, θ_k, and G_k are the population share, income share, and Gini coefficient of subgroup k, and the between-group inequality term is

$$G^B = \frac{1}{2\mu} \left(\Theta - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l} \right), \tag{3}$$

where μ is the aggregate mean, $\Theta = \mathbb{E}[\mathbf{1}_{\{g_i \neq g_j\}} |y_i - y_j|]$ is the cross mean absolute difference, $\Delta_k = \mathbb{E}[|y_i - y_j| | g_i = g_j = k]$ is the mean absolute difference within subgroup k, and y_i and g_i denote the income and subgroup affiliation of individual i.

Within-group inequality in the Gini decomposition is a weighted power mean of subgroup inequalities where each subgroup is weighted by the geometric mean of its income and population share.[12] Between-group inequality is the difference between the cross

[12]A power mean, also known as generalized mean or Hölder mean, with exponent p is a function $M_p(x_1, \dots, x_n) = (\frac{1}{n} \sum_{i=1}^{n} x_i^p)^{\frac{1}{p}}$ and includes as special cases the arithmetic, geometric, and harmonic means.

mean absolute difference, defined as the mean absolute difference of individuals drawn from different subgroups scaled by the share of cross-group pairs, and a weighted sum of geometric averages of subgroup mean absolute differences, divided by twice the aggregate mean. In practice, however, it is generally easier to compute the between-group inequality term as a residual, $G^B = G - G^W$.

The proof of Theorem 2 proceeds by first showing that the expressions for G^W and G^B sum to the aggregate Gini coefficient and that the decomposition satisfies Axioms 1–8. We then show uniqueness by leveraging the fact that the Gini coefficient is aggregative when subgroup income distributions cannot overlap. In these special cases, Axiom 8 together with Theorem 1 pin down the generating function f and the weight structure in equation (1), which uniquely determine the within-group and between-group inequality terms. The complete proof is presented in the Appendix.

In Supplemental Appendix A, we show that our axiomatic framework can also be used to uniquely determine the standard decomposition formulas for the generalized entropy and Foster–Shneyerov indices. In Supplemental Appendix B, we derive asymptotic confidence intervals for the within- and between-group inequality terms of the Gini decomposition. In Supplemental Appendix C, we show how the decomposition formula can be extended to the multivariate Gini coefficients introduced in Koshevoy and Mosler (1997).

4.2. *Discussion of the Gini Decomposition*

The within- and between-group inequality terms in the Gini decomposition of Theorem 2 satisfy several additional properties that we have not directly imposed as axioms. First, as a weighted power mean, the within-group inequality term is homogeneous.[13] That is, any redistribution of incomes within subgroups that reduces subgroup Gini coefficients by some given factor will also reduce within-group inequality by the same factor. Note, however, that the within-group inequality term is not linear in the subgroup Gini coefficients.

Second, the between-group inequality term summarizes the contribution to aggregate inequality stemming from differences in the means and the shapes of the subgroup income distributions. Unlike the between-group inequality term for the Theil index (and other generalized entropy indices) that depends only on differences in subgroup means, the between-group inequality term for the Gini coefficient depends on differences in all moments of the subgroup income distributions. This is natural since, unlike other inequality indices, the aggregate Gini coefficient itself depends on all the moments of the subgroup income distributions. Specifically, Proposition 1 shows that changes in the subgroup income distributions can affect the aggregate Gini coefficient even if all subgroup Gini coefficients as well as any number of subgroup moments are held fixed.[14]

PROPOSITION 1: *The aggregate Gini coefficient cannot be written as a function of subgroup population sizes, Gini coefficients, and any number of subgroup moments. That is, there does not exist a set of moments Ω and a function F, such that*

$$G(L) = F(G_1, \dots, G_K; \Omega_1, \dots, \Omega_K; n_1, \dots, n_K), \tag{4}$$

where Ω_k is the vector of moments for subgroup k.

[13]Homogeneity is in fact the only additional property that power means have over quasi-arithmetic means implied by Theorem 1.

[14]The claim that the between-group inequality term in the Gini decomposition is sensitive to any differences in subgroup income distributions is a corollary of Proposition 1.

How between-group inequality depends on differences in the shapes of the subgroup income distributions is also intuitive. First, as is shown in Proposition 8 of Section 4.3, the between-group inequality term is zero if and only if the distribution of income is identical across subgroups.

Second, the between-group inequality term becomes smaller as subgroup income distributions become more similar. Specifically, Proposition 2 states that between-group inequality is reduced when the subgroup affiliations for a random subset of individuals are permuted so that subgroup income distributions become unambiguously more similar to each other.[15] Moreover, the only case in which this operation does not make subgroup income distributions more similar is when the distributions are identical to begin with. Because the aggregate Gini coefficient is sensitive to all moments of the subgroup income distributions, this is the only case in which between-group inequality is not reduced.

PROPOSITION 2: *In an infinite population, randomly permuting the subgroup affiliations for a subset of randomly selected individuals with positive population share weakly reduces between-group inequality (while keeping the aggregate Gini coefficient constant). Between-group inequality remains constant if and only if all subgroups have the same distribution of income.*

Similarly, merging several subgroups into one should decrease between-group inequality as any differences between the merged subgroups can no longer contribute to overall between-group inequality. Proposition 3 states that this operation indeed reduces between-group inequality unless the income distributions of the merged subgroups are all identical.

PROPOSITION 3: *Merging any $m \leq K$ subgroups into one subgroup weakly reduces the between-group inequality term in the Gini decomposition. Between-group inequality is unaffected if and only if the merged subgroups have identical income distributions.*

There also exist operations that reduce both within-group inequality and between-group inequality. For example, redistributing incomes so that the difference between each individual's income and the average income is reduced by a given percentage clearly reduces aggregate inequality.[16] Moreover, as such redistribution at the same time compresses subgroup income distributions and brings them closer to each other, one may expect it to decrease within-group and between-group inequality by similar proportions. Proposition 4 shows that this is indeed the case.

PROPOSITION 4: *Let y_i denote the income of individual i, and let μ denote the average income in the population. For some $\alpha \in [0, 1]$, replacing every income y_i by $\tilde{y}_i = y_i - \alpha(y_i - \mu)$ reduces within-group inequality and between-group inequality in the Gini decomposition by a fraction α.*

It is often of interest to know by how much aggregate inequality would be reduced if all subgroup means were equalized while keeping the level of within-group inequality fixed.

[15] Equivalently, one may consider replacing the incomes of a fixed fraction of randomly sampled individuals in each subgroup with a random draw from the aggregate income distribution. Equivalency between these two operations is a direct consequence of the anonymity property of inequality indices.

[16] This can be achieved with a flat tax and a lump-sum transfer, which is the standard setup in the optimal linear tax literature (see, e.g., Piketty and Saez (2013)), in the absence of behavioral responses to the tax.

For many inequality indices, this question is not easily answered. For example, as was pointed out by Shorrocks (1980), there is no obvious operation for the Theil index that would eliminate differences in average incomes between subgroups and also keep the within-group inequality term fixed.[17] Proposition 5 shows that the decomposition of the Gini coefficient admits an operation that can be used to eliminate differences between subgroups in average incomes while keeping within-group inequality fixed. This property can be used to further decompose between-group inequality into a first part that reflects differences in means and a second part that reflects differences in the shape of the distribution. We implement such an exercise in Supplemental Appendix F.

PROPOSITION 5: *Lump-sum transfers between subgroups do not affect within-group inequality in the Gini decomposition*.

Finally, we note that scaling or translating all incomes does not affect the share of aggregate inequality that is attributed to within-group or between-group inequality in the Gini decomposition. While the scale independence of the decomposition follows directly from the scale independence of the Gini coefficient and therefore applies to the decomposition of all scale-independent inequality indices, translation independence is a special feature of the Gini decomposition. Translation independence is convenient if one wants to decompose the Gini coefficient of attributes that can be negative, such as wealth.

PROPOSITION 6: *The Gini decomposition is both scale- and translation-invariant*. *Specifically*, *changing incomes for each individual i from* y_i *to* $\tilde{y}_i = ay_i + b$ *does not affect the relative magnitude of within-group and between-group inequality for any* $a > 0$ *and* $b \in \mathbb{R}$ *such that each subgroup's mean income remains positive*.

4.3. *Geometric Interpretation*

The Gini coefficient is traditionally defined as twice the area between the Lorenz curve and the line of perfect equality. To derive a geometric interpretation for the Gini decomposition, it is useful to define the *Lorenz region* as the region within the unit square that is bounded by the Lorenz curve and its centrally reflected counterpart (see Figure 1).[18] Clearly, the Gini coefficient is equal to the area of the Lorenz region.

For a given partition of the population into subgroups, we can also define *subgroup Lorenz regions* by scaling each subgroup's Lorenz region with a vector (π_k, θ_k) where π_k and θ_k are subgroup k's population and income shares, respectively (see Figure 1). Importantly, Zagier (1983) shows that there is a geometric relationship between the aggregate and the subgroup Lorenz regions. Specifically, Proposition 7 (proven as part of Theorem 1 in Zagier (1983)) states that the Lorenz region of the aggregate population is the Minkowski sum of the subgroup Lorenz regions.[19]

[17]For example, scaling the incomes in each subgroup so that the subgroup means are equalized changes the subgroup income shares and thereby affects the within-group inequality term in the decomposition of the Theil index. The only generalized entropy index for which scaling incomes does not affect within-group inequality is the mean log deviation.

[18]This region is also known as the Lorenz zonoid; see Koshevoy and Mosler (1997).

[19]The Minkowski sum, also known as the vector sum, of two sets A and B is defined as $A \oplus B = \{a + b : a \in A, b \in B\}$.

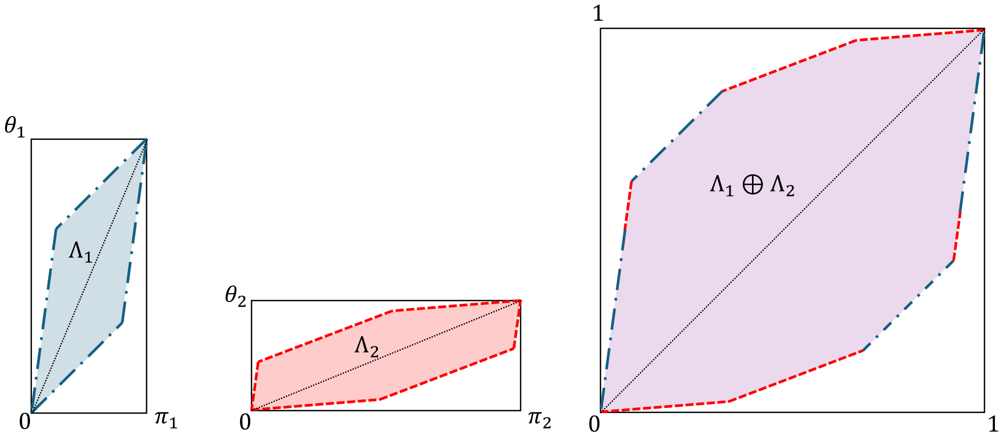

FIGURE 1.—Aggregation of subgroup Lorenz regions. In this example, there are two different levels of income in subgroup 1 and three different levels of income in subgroup 2. In the aggregate Lorenz curve, individuals from both subgroups are ordered by their income. Geometrically, this corresponds to arranging the linear segments of the scaled subgroup Lorenz curves (denoted by the dashed and dashed-dotted lines) in ascending order by their slopes. The resulting aggregate Lorenz region coincides with the Minkowski sum of the subgroup Lorenz regions.

PROPOSITION 7: *A population consisting of $K \geq 2$ subgroups with subgroup Lorenz regions $\Lambda_1, \Lambda_2, \dots, \Lambda_K$ has an aggregate Lorenz region*

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_K,$$

where $\oplus$ denotes the Minkowski sum of sets.

Figure 1 illustrates the Minkowski addition of two subgroup Lorenz regions in the case of discrete income distributions. In Supplemental Appendix E, we illustrate the Minkowski addition in the case of continuous income distributions.

A useful implication of representing the aggregate Gini coefficient as the area of the Minkowski sum of subgroup Lorenz regions is that we can make use of an important result relating the areas of compact sets: the Brunn–Minkowski theorem. Specifically, the Brunn–Minkowski theorem provides a lower bound for the aggregate Gini coefficient in terms of subgroup Gini coefficients and aggregate characteristics.

PROPOSITION 8—Brunn–Minkowski theorem: *For a population with Lorenz region Λ and Gini coefficient G consisting of K subgroups with subgroup Lorenz regions $\Lambda_1, \dots, \Lambda_K$, Gini coefficients $G_1, \dots, G_K$, population shares $\pi_1, \dots, \pi_K$, and income shares $\theta_1, \dots, \theta_K$, we have*

$$G = |\Lambda| \geq \left(\sum_{k=1}^{K} \sqrt{|\Lambda_k|} \right)^2 = \left(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G_k} \right)^2 = G^W,$$

where $|\cdot|$ is the Lebesgue measure. The inequality holds as equality if and only if $\Lambda_1, \dots, \Lambda_K$ are homothetic.

In light of Proposition 8, within-group inequality can be interpreted geometrically as the minimal area of the aggregate Lorenz region for given areas of the subgroup Lorenz regions. Similarly, between-group inequality is the excess area in the aggregate Lorenz region that is not explained by the areas of the subgroup Lorenz regions.

As the Minkowski sum of the subgroup Lorenz regions, the area of the aggregate Lorenz region depends both on the areas of the subgroup Lorenz regions as well as on how similar the shapes of the subgroup Lorenz regions are. The constrained minimum is achieved when the subgroup income distributions are all identical so that the subgroup Lorenz regions are homothetic. As a consequence, between-group inequality measures the excess area in the population Lorenz region resulting from non-homotheticity of the subgroup Lorenz regions. In other words, between-group inequality measures how differences in the shapes of the subgroup income distributions contribute to aggregate inequality.[20] In Supplemental Appendix D, we offer an arithmetic interpretation for the Gini decomposition.

The fact that the expression for the within-group inequality term is a lower bound for the aggregate Gini coefficient has previously been shown in Zagier (1983) using the Brunn–Minkowski inequality. Zagier (1983) studies the problem of bounding the aggregate Gini coefficient for given subgroup Gini coefficients, means, and population shares, and derives our expression for the within-group inequality term as one of several lower bounds for the aggregate Gini coefficient.[21] The paper also notes that aggregate inequality is smaller when subgroups are more similar in their income distributions. Because the paper considers the situation where subgroup Gini coefficients and aggregate characteristics are fixed, this is equivalent to noting that our between-group inequality term is smaller when the subgroup income distributions are more similar.

4.4. *Subgroup Consistency*

The Gini decomposition together with its geometric interpretation provides helpful insights into a notorious behavior of the Gini coefficient that is called *subgroup inconsistency*: an increase in inequality within subgroups, while keeping subgroup means and population sizes constant, can lead to a decrease in aggregate inequality.[22] Subgroup consistency, which rules out this behavior, has been proposed as a requirement for decomposability (Cowell (2000)). The results in this paper clarify that the Gini coefficient can violate subgroup consistency if an increase in inequality within subgroups also makes the subgroup income distributions more similar so that between-group inequality decreases more than within-group inequality increases.

Moreover, we note that the Gini coefficient does satisfy a weaker version of subgroup consistency. In particular, Proposition 9 states that any transfers within subgroups that increase subgroup inequalities according to the Lorenz criterion must also increase the

[20] In convex geometry, the excess area of the Minkowski sum relative to the Brunn–Minkowski lower bound is sometimes called the Brunn–Minkowski deficit, which has been shown to relate to the symmetry of the added sets. Figalli, Maggi, and Pratelli (2009), for example, show that the Brunn–Minkowski deficit is bounded from below by an increasing function of the relative asymmetry of the added sets.

[21] Note that the within-group inequality term for the Gini coefficient is not generally the best lower bound in Zagier's setup. Unlike in our setup where only the areas of the subgroup Lorenz regions (the product of π_k, θ_k, and G_k) are given, Zagier looks for the best lower bound for the aggregate Gini coefficient when the subgroup Gini coefficients, population shares, and means are given individually. The implied best lower bound therefore exceeds our within-group inequality term as it also incorporates the part of between-group inequality stemming from differences in subgroup means.

[22] Different examples of specific distributions and transfers that produce this behavior have been discussed in the literature (see, e.g., Cowell (1988)).

aggregate Gini coefficient.[23] It follows that if transfers that increase inequality within subgroups reduce the aggregate Gini coefficient, it must be the case that for at least one subgroup, the new Lorenz curve intersects with the old Lorenz curve. The proof of Proposition 9 makes use of the fact that outward shifts of the subgroup Lorenz curves must result in an outward shift of the aggregate Lorenz curve.

PROPOSITION 9—Weak subgroup consistency: *Transfers within one or more subgroups that increase the level of subgroup inequality according to the Lorenz criterion also increase aggregate inequality.*

5. EMPIRICAL APPLICATION

We demonstrate our decomposition of the Gini coefficient by analyzing the extent to which household income inequality in the United States reflects inequality within versus between demographic subgroups. In this decomposition, between-group inequality summarizes how much of overall inequality is "explained" by differences in income distributions between demographic subgroups, while within-group inequality is a measure of "residual inequality" (Cowell and Jenkins (1995), Juhn, Murphy, and Pierce (1993)).[24]

We use data on household income in the United States from the Current Population Surveys and define demographic subgroups by the age, education, sex, and race of the household head.[25] Figure 2 shows the evolution of within-group and between-group inequality for the years 1967–2021. Overall, and in line with the conclusion in Cowell and Jenkins (1995), a relatively minor share of aggregate inequality is explained by differences in income distributions between demographic subgroups. Moreover, over the past five decades, the aggregate Gini coefficient has increased from 0.39 to 0.50, and this increase was clearly driven by a rise in within-group inequality, which has increased from 0.29 to 0.41. At the same time, between-group inequality has decreased slightly from 0.10 to 0.09. As a consequence, the share of aggregate inequality that can be attributed to demographic characteristics has decreased from 25 to 17 percent. In Supplemental Appendix F, we show how the fact that lump-sum transfers between subgroups do not affect within-group inequality can be used to isolate the part of between-group inequality that is due to differences in average earnings between subgroups.[26]

[23]The Lorenz criterion states that inequality of distribution A exceeds that of distribution B if the Lorenz curve of A is always below that of B, and therefore any inequality index must judge inequality to be higher in distribution A than in distribution B.

[24]Cowell and Jenkins (1995) study how much of aggregate inequality in the United States can be "explained" by demographic characteristics using different Atkinson indices. Juhn, Murphy, and Pierce (1993) document rising wage inequality among men who are otherwise similar in terms of education and labor market experience. Since then, a large literature in labor economics devoted to explaining the rise in "residual inequality" has emerged.

[25]Specifically, we use the data produced by Heathcote, Perri, Violante, and Zhang (2023), who implement a procedure to deal with topcoding in the CPS data.

[26]In the same Supplemental Appendix, we also implement an additional empirical application focusing on gender earnings inequality in the United States. Both of our empirical applications apply the decomposition formula to study inequality in a single country over time. Alternatively, one could use the decomposition to compare inequality across countries. For example, if we observe that country A has a higher Gini coefficient than country B, and if the within-group inequality term is also higher in country A than in country B, then we can conclude that country A is more unequal, in part, because inequality within the subgroups contributes more to aggregate inequality in country A than in country B.

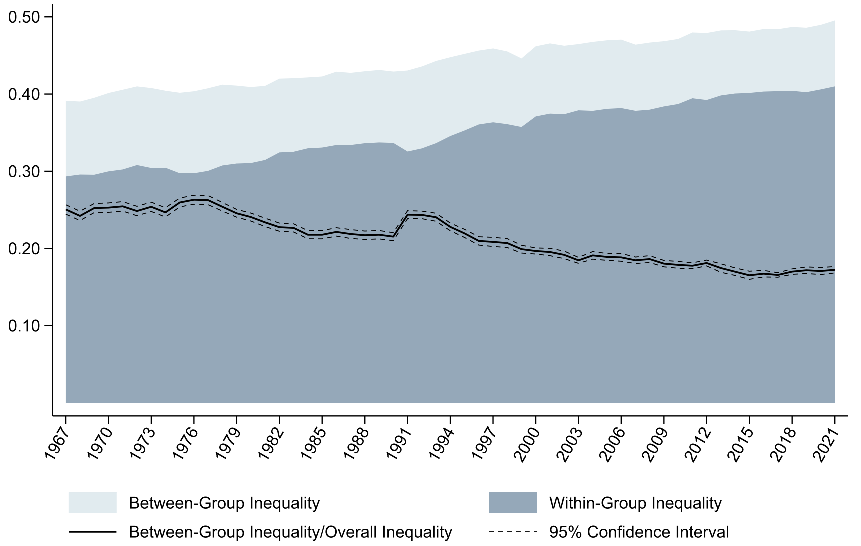

FIGURE 2.—Household-level income inequality within and between demographic subgroups. Households are grouped by the age ($\leq$35, 36–45, 46–55, 56–65, >65), education (no college, some college, Bachelor's degree, more than Bachelor's degree), sex (male, female), and race (Black, White, other) of the household head. Asymptotic confidence intervals are computed using the formulas derived in Supplemental Appendix B.

6. CONCLUDING REMARKS

The Gini coefficient is the most prominent measure of inequality. Yet, there has been much disagreement regarding its decomposition by population subgroups. As a consequence, researchers often rely on other inequality indices when assessing the contribution of within-group and between-group inequality to aggregate inequality, even if they would otherwise prefer to work with the Gini coefficient.

In this paper, we show that the Gini coefficient admits a satisfactory decomposition formula derived from a set of axioms that ensure desirable behavior for the within-group and between-group inequality terms. The decomposition is novel, unique given our axioms, and easy to compute. Moreover, it can be interpreted both geometrically and arithmetically. Given these advantages, the Gini decomposition derived in this paper can be a valuable tool for empirical research.

APPENDIX: PROOFS

A.1. *Proof of Theorem 1*

It is easy to verify that the functional form in (1) satisfies Axioms 1–6. The remainder of the proof shows that if (W_K) satisfies Axioms 1–6, then (1) holds. We first show that for given $K \geq 2$ and aggregate characteristics $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_{++}^{2K}$, W_K must have the form

$$W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) = f^{-1}\left(\sum_{k=1}^{K} a_k f(I_k)\right), \tag{5}$$

for all $(I_k)_{k=1}^K \in \mathbb{R}_+^K$, where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function such that $f(0) = 0$ and $(a_k)_{k=1}^K \in \mathbb{R}_{++}^K$.

Suppose $K \geq 2$ and $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_{++}^{2K}$ are fixed and let $B(x_1, \dots, x_K) = W_K((x_1, n_1, Y_1), \dots, (x_K, n_K, Y_K))$. Now, B satisfies the bisymmetry equation

$$B\big(B(x_{11}, \dots, x_{1K}), \dots, B(x_{K1}, \dots, x_{KK})\big) = B\big(B(x_{11}, \dots, x_{K1}), \dots, B(x_{1K}, \dots, x_{KK})\big)$$

for all $(x_{ij}) \in \mathbb{R}_+^{K \cdot K}$. To show this, we use Axioms 2, 3, and 6:

$$\begin{aligned}
&B\big(B(x_{11}, \dots, x_{1K}), \dots, B(x_{K1}, \dots, x_{KK})\big) \\
&\quad = W_K\big(\big(W_K\big((x_{11}, n_1, Y_1), \dots, (x_{1K}, n_K, Y_K)\big), n_1, Y_1\big), \dots \\
&\qquad \dots, \big(W_K\big((x_{K1}, n_1, Y_1), \dots, (x_{KK}, n_K, Y_K)\big), n_K, Y_K\big)\big) \\
&\quad = W_K\Big(\Big(W_K\Big(\Big(x_{11}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_1\Big), \dots, \Big(x_{1K}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_K\Big)\Big), n_1, Y_1\Big), \\
&\qquad \dots, \Big(W_K\Big(\Big(x_{K1}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_1\Big), \dots, \Big(x_{KK}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_K\Big)\Big), n_K, Y_K\Big)\Big) \\
&\quad = W_{K^2}\Big(\Big(x_{11}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_1\Big), \dots, \Big(x_{1K}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_K\Big), \\
&\qquad \dots, \Big(x_{K1}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_1\Big), \dots, \Big(x_{KK}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_K\Big)\Big) \\
&\quad = W_{K^2}\Big(\Big(x_{11}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_1\Big), \dots, \Big(x_{K1}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_1\Big), \\
&\qquad \dots, \Big(x_{1K}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_K\Big), \dots, \Big(x_{KK}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_K\Big)\Big) \\
&\quad = W_K\Big(\Big(W_K\Big(\Big(x_{11}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_1\Big), \dots, \Big(x_{K1}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_1, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_1\Big)\Big), n_1, Y_1\Big), \\
&\qquad \dots, \Big(W_K\Big(\Big(x_{1K}, \tfrac{n_1}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_1}{\sum_{k=1}^K Y_k} Y_K\Big), \dots, \Big(x_{KK}, \tfrac{n_K}{\sum_{k=1}^K n_k} n_K, \tfrac{Y_K}{\sum_{k=1}^K Y_k} Y_K\Big), n_K, Y_K\Big)\Big)\Big) \\
&\quad = W_K\big(\big(W_K\big((x_{11}, n_1, Y_1), \dots, (x_{K1}, n_K, Y_K)\big), n_1, Y_1\big), \\
&\qquad \dots, \big(W_K\big((x_{1K}, n_1, Y_1), \dots, (x_{KK}, n_K, Y_K)\big), n_K, Y_K\big)\big) \\
&\quad = B\big(B(x_{11}, \dots, x_{K1}), \dots, B(x_{1K}, \dots, x_{KK})\big).
\end{aligned}$$

Thus, B is a continuous function that is strictly increasing in each of its arguments, symmetric, and satisfies bisymmetry. As shown in Aczél (1948), B can be used to construct a function $M: \mathbb{R}_+^K \to \mathbb{R}_+$ that has the same properties but is also reflexive by defining $M(x_1, \dots, x_K) = F^{-1}(B(x_1, \dots, x_K))$, where $F(z) = B(z, \dots, z)$.[27] Hence, M satisfies the conditions of Theorem 2 in Münnich, Maksa, and Mokken (2000), which states that

$$M(x_1, \dots, x_K) = f^{-1}\left(\sum_{k=1}^K b_k f(x_k)\right), \tag{6}$$

[27] Aczél (1948) shows this in the case where B is a bivariate function, but the same proof generalizes to the case of K variables.

for some continuous increasing function $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$ and for some $b_k > 0$ such that $\sum_{k=1}^K b_k = 1$. Aczél (1948) shows that (6) implies

$$B(x_1, \dots, x_K) = f^{-1}\left(\sum_{k=1}^K a_k f(x_k) + b\right),$$

where $a_k = ab_k$ for some $a \neq 0$ and $b \in \mathbb{R}$. Since $f(0) = 0$, then by the normalization axiom, $B(0, \dots, 0) = f^{-1}(b) = 0$, which implies $b = 0$. Thus, we get equation (5).

In equation (5), the constants a_k and the generating function f can depend on K and $(n_k, Y_k)_{k=1}^K$. Note that f and cf generate the same W_K for any constant $c > 0$. Thus, if $g(x) = cf(x)$ for all x, we call f and g the same generating function. We next show that the generating function of W_K is independent of K and $(n_k, Y_k)_{k=1}^K$.

Let $K \geq 4$. By replacement and symmetry, we have

$$\begin{aligned} &W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) \\ &\quad = W_3\left((I_1, n_1, Y_1), (I_2, n_2, Y_2), \left(W_{K-2}\big((I_3, n_3, Y_3), \dots, (I_K, n_K, Y_K)\big), \sum_{k=3}^K n_k, \sum_{k=3}^K Y_k\right)\right) \end{aligned}$$

for all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$. Suppose $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_{++}^{2K}$ are given. By substituting equation (5) to both sides of the above equation, we get

$$f^{-1}\left(\sum_{k=1}^K a_k f(I_k)\right) = g^{-1}\left(b_1 g(I_1) + b_2 g(I_2) + b_3 g\left(h^{-1}\left(\sum_{k=3}^K c_k h(I_k)\right)\right)\right), \tag{7}$$

where f, g, h are continuous and strictly increasing functions with $f(0) = g(0) = h(0) = 0$. Now, set $I_k = 0$ for $k = 3, \dots, K$ and define $x = f(I_1)$, $y = f(I_2)$. Then, (7) implies

$$\phi(a_1 x + a_2 y) = b_1 \phi(x) + b_2 \phi(y)$$

for all $x, y \in \mathbb{R}_+$, where $\phi = g \circ f^{-1}$. By Theorem 2 on p. 67 in Aczél (1966), this equation has a solution only if $a_1 = b_1$ and $a_2 = b_2$ and the solution is $\phi(x) = cx$ for some constant $c \neq 0$. Thus, $g \circ f^{-1}(x) = cx$, which implies $g(x) = cf(x)$ for all $x \in \mathbb{R}_+$, that is, W_K and W_3 have the same generating function for any $K \geq 4$.

Inserting this result into equation (7) when $K = 4$, we get

$$f^{-1}\left(\sum_{k=1}^4 a_k f(I_k)\right) = f^{-1}\big(a_1 f(I_1) + a_2 f(I_2) + b_3 f\big(h^{-1}\big(c_1 h(I_3) + c_2 h(I_4)\big)\big)\big).$$

By defining $x = h(I_3)$ and $y = h(I_4)$, we can rewrite the above equation as

$$\frac{a_3}{b_3}\phi(x) + \frac{a_4}{b_3}\phi(y) = \phi(c_1 x + c_2 y)$$

for all $x, y \in \mathbb{R}_+$, where $\phi = f \circ h^{-1}$. This yields $c_1 = a_3/b_3$, $c_2 = a_4/b_3$, and $\phi(x) = f \circ h^{-1}(x) = cx$ for some constant $c \neq 0$, or $f(x) = ch(x)$ for all x. Thus, W_2 and W_4 have the same generating function. Hence, we get that the generating function of W_K is independent of K for $K \geq 2$. Moreover, since aggregate characteristics $(n_k, Y_k)_{k=1}^K$ were arbitrary, the generating function does not depend on them.

Finally, we show that the constants a_k in (5) have the form $a_k = \pi_k^{1-\alpha}\theta_k^{\alpha}$, where $\pi_k = n_k/\sum_{k=1}^{K} n_k$ and $\theta_k = Y_k/\sum_{k=1}^{K} Y_k$. Using replacement and equation (5), we get

$$f^{-1}\left(\sum_{k=1}^{K} a_k f(I_k)\right) = f^{-1}\left(b_1 \sum_{k=1}^{K-1} c_k f(I_k) + b_2 f(I_K)\right) \tag{8}$$

for all $K \geq 2$, $(I_k)_{k=1}^{K} \in \mathbb{R}_+^K$, and $(n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_{++}^{2K}$, where $a_k = a(n_k, Y_k; n_1, Y_1, \ldots, n_K, Y_K)$, $c_k = c(n_k, Y_k; n_1, Y_1, \ldots, n_{K-1}, Y_{K-1})$, $b_1 = b(\sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k; \sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k, n_K, Y_K)$, and $b_2 = b(n_K, Y_K; \sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k, n_K, Y_K)$ for some functions a, b, and c. By setting $I_k = 0$ for $k = 1, \ldots, K-1$, we get

$$a(n_K, Y_K; n_1, Y_1, \ldots, n_K, Y_K) = b\left(n_K, Y_K; \sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k, n_K, Y_K\right)$$

for all $(n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_+^{2K}$. By scale and population independence, we get

$$a(n_K, Y_K; n_1, Y_1, \ldots, n_K, Y_K) = b(\pi_K, \theta_K; 1-\pi_K, 1-\theta_K, \pi_K, \theta_K) = w(\pi_K, \theta_K) \tag{9}$$

for some function w. Due to symmetry, this result holds for any k. Moreover, since K is arbitrary, the same result applies to functions b and c. By substituting (9) into equation (8) and setting $I_j = 0$ for all $j \neq k$ for some $k = 1, \ldots, K-1$, we get

$$w\left(\frac{n_k}{\sum_{k=1}^{K} n_k}, \frac{Y_k}{\sum_{k=1}^{K} Y_k}\right) = w\left(\frac{\sum_{k=1}^{K-1} n_k}{\sum_{k=1}^{K} n_k}, \frac{\sum_{k=1}^{K-1} Y_k}{\sum_{k=1}^{K} Y_k}\right) w\left(\frac{n_k}{\sum_{k=1}^{K-1} n_k}, \frac{Y_k}{\sum_{k=1}^{K-1} Y_k}\right),$$

which generalizes to $w(ab, cd) = w(a, c)w(b, d)$ for all $a, b, d, c \in (0, 1)$. Let $a = e^{x_1}$, $b = e^{x_2}$, $c = e^{x_3}$, $d = e^{x_4}$. Then, we have

$$\varphi(x_1 + x_2, x_3 + x_4) = \varphi(x_1, x_3) + \varphi(x_2, x_4)$$

for all $x_1, x_2, x_3, x_4 \in (-\infty, 0)$ where $\varphi(x, y) = \ln w(e^x, e^y)$. By Theorem 1 on p. 215 in Aczél (1966), we get $\varphi(x, y) = d_1 x + d_2 y$ for some constants d_1, d_2, which implies $w(\pi, \theta) = \pi^{d_1}\theta^{d_2}$ for all $\pi, \theta \in (0, 1)$. Now, using weak reflexivity, we get

$$w(t\pi, t\theta) + w\big((1-t)\pi, (1-t)\theta\big) = w(\pi, \theta)$$

for any $t \in (0, 1)$, which simplifies to $t^{d_1+d_2} + (1-t)^{d_1+d_2} = 1$. Since the left-hand side is strictly increasing in $d_1 + d_2$, then $d_1 + d_2 = 1$ is the unique solution.

Thus, we have

$$W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) = f^{-1}\left(\sum_{k=1}^{K} \pi_k^{1-\alpha}\theta_k^{\alpha} f(I_k)\right)$$

for all $K \geq 2$, $(I_k)_{k=1}^{K} \in \mathbb{R}_+^K$, $(n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_{++}^{2K}$, and for some $\alpha \in \mathbb{R}$. Due to continuity of W_K, these results extend to cases where $n_k = 0$ or $Y_k = 0$ for some k.

A.2. *Proof of Proposition 1*

In the proof, we use Proposition 8 and the following lemma.

LEMMA 1: *There exist two distinct distributions that share the same moments and Gini coefficients, which are all finite.*

PROOF: Let X_c be a random variable with the density function

$$f_c(x) = \big(1 + c\sin\big(2\pi\ln(x)\big)\big)\frac{1}{\sqrt{2\pi}}\mathbf{1}_{[0,\infty)}(x)\frac{1}{x}e^{-\frac{(\ln(x))^2}{2}}$$

that depends on parameter $c \in \mathbb{R}$. Note that X_0 is distributed lognormally with parameters $\mu = 0$, $\sigma = 1$. It can be shown that for $c \in [-1, 1]$, all moments of X_c are finite and do not depend on c.[28] Moreover, we show that there are two distinct values of c such that the Gini coefficients of X_c are equal. Since the Gini coefficient can be written as $G = 1 - 2\int_0^1 L(x)/L(1)\,dx$, it suffices to show that $\int_0^1 L_c(x)\,dx$ is equal for two different values of c, where L_c is the generalized Lorenz curve of X_c.

Since $L_c(F_c(x)) = \int_0^x tf_c(t)\,dt$, where F_c is the CDF of X_c, we need to show that there exist two distinct values of c for which the integral of the generalized Lorenz curve, $A_c = \int_0^1\int_0^{F_c^{-1}(x)} tf_c(t)\,dt\,dx$, are equal. With some manipulation, we get

$$A_c = \frac{1}{\sqrt{2\pi}}\big(a + bc + dc^2\big),$$

where

$$\begin{aligned}
a &= \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}e^z\left(1 - \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds\right)dz,\\
b &= \int_{-\infty}^{\infty}\sin(2\pi z)e^{-\frac{z^2}{2}}e^z\,dz - \int_{-\infty}^{\infty}\sin(2\pi z)e^{-\frac{z^2}{2}}e^z\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds\,dz\\
&\quad - \int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}e^z\int_{-\infty}^{z}\sin(2\pi s)\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds\,dz,\\
d &= \int_{-\infty}^{\infty}\sin(2\pi z)e^{-\frac{z^2}{2}}e^z\int_{-\infty}^{z}\sin(2\pi s)\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds\,dz.
\end{aligned}$$

Now, if $b = 0$, then we have the result since $A_c = A_{-c}$ for any $c \in [-1, 1]$. First, note that

$$\int_{-\infty}^{\infty}\sin(2\pi z)e^{-\frac{z^2}{2}}e^z\,dz = \int_{-\infty}^{\infty}\sin(2\pi z)e^{-\frac{1}{2}(z-1)^2}e^{\frac{1}{2}}\,dz = e^{\frac{1}{2}}\int_{-\infty}^{\infty}\sin(2\pi x)e^{-\frac{1}{2}(x)^2}\,dx = 0$$

as an integral of an odd function. Thus,

$$\begin{aligned}
b &= -\int_{-\infty}^{\infty}\sin(2\pi z)e^{-\frac{z^2}{2}}e^z\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds\,dz - \int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}e^z\int_{-\infty}^{z}\sin(2\pi s)\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,ds\,dz\\
&= -e^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{z}e^{-\frac{1}{2}(z-1)^2}e^{-\frac{s^2}{2}}\big(\sin(2\pi z) + \sin(2\pi s)\big)\,ds\,dz
\end{aligned}$$

[28] See, for example, Schmüdgen (2017).

$$
\begin{aligned}
&= -2e^{\frac{1}{2}} \int_{-\infty}^{1} e^{-\frac{1}{4}t^2} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}t)^2} \left(\sin\left(2\pi\left(x + \frac{1}{2}t\right)\right) \cos(-\pi t) \right) dx\, dt \\
&= -2e^{\frac{1}{2}} \int_{-\infty}^{1} e^{-\frac{1}{4}t^2} \cos(-\pi t) \int_{-\infty}^{\infty} e^{-s^2} \sin(2\pi s)\, ds\, dt = 0,
\end{aligned}
$$

since $\int_{-\infty}^{\infty} e^{-s^2} \sin(2\pi s)\, ds = 0$ as an integral of an odd function. *Q.E.D.*

Suppose there exist a finite set of moments Ω and a function F such that (4) holds. Suppose L_1 and L_2 are generalized Lorenz curves of two distinct distributions that have the same values for the moments in Ω and the same Gini coefficients. Consider a population partitioned into K subgroups where each subgroup has generalized Lorenz curve L_1 and denote the vector of moments of L_1 by Ω_1. Then, the aggregate Gini coefficient of the population is equal to $G = F(G(L_1), \dots, G(L_1), \Omega_1, \dots, \Omega_1; \pi_1, \dots, \pi_K)$. By Proposition 8, we have

$$
F\big(G(L_1), \dots, G(L_1), \Omega_1, \dots, \Omega_1; \pi_1, \dots, \pi_K\big) = \left(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G(L_1)} \right)^2 = G(L_1),
$$

since subgroup Lorenz curves are homothetic, $\theta_k = \pi_k$ for all k, and $\sum_{k=1}^{K} \pi_k = 1$.

Next, consider another population partitioned into K subgroups, where one subgroup has generalized Lorenz curve L_1 and the other subgroups have generalized Lorenz curves L_2, and let $\tilde{G}$ denote the aggregate Gini coefficient of this population. Since the subgroup Gini coefficients and moments are the same as before, we have

$$
\tilde{G} = F\big(G(L_1), \dots, G(L_1), \Omega_1, \dots, \Omega_1; \pi_1, \dots, \pi_K\big) = G(L_1).
$$

But since the subgroup Lorenz curves are not all homothetic, then Proposition 8 implies $\tilde{G} > G(L_1)$, which is a contradiction. Therefore, (4) cannot hold.

Note that, as is shown in the proof of Lemma 1, there exist two distinct distributions that share the same sequence of moments and Gini coefficients. Therefore, equation (4) would not hold even if we allowed Ω to be a countably infinite set of moments.

A.3. *Proof of Theorem 2*

We first show that the within-group inequality term (2) and between-group inequality term (3) sum up to the aggregate Gini coefficient by showing that $G^B = G - G^W$. Let $\Delta = \mathbb{E}[|y_i - y_j|]$ denote the mean absolute difference in the population. Note that the aggregate Gini coefficient is equal to $\Delta/2\mu$ and the within-group inequality term $G^W = (\sum_{k=1}^{K} \pi_k \sqrt{\Delta_k})^2/2\mu$. Define $\Theta = \mathbb{E}[\mathbf{1}_{\{g_i \neq g_j\}} |y_i - y_j|]$. Now,

$$
\begin{aligned}
G - G^W &= \frac{1}{2\mu}\Delta - \frac{1}{2\mu}\left(\sum_{k=1}^{K} \pi_k \sqrt{\Delta_k}\right)^2 = \frac{1}{2\mu}\left(\Delta - \sum_{k=1}^{K} \pi_k^2 \Delta_k - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l}\right) \\
&= \frac{1}{2\mu}\left(\mathbb{E}\big[|y_i - y_j|\big] - \sum_{k=1}^{K} \mathbb{E}\big[\mathbf{1}_{\{g_i = k\}} \mathbf{1}_{\{g_j = k\}} |y_i - y_j|\big] - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l}\right) \\
&= \frac{1}{2\mu}\left(\Theta - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l}\right) = G^B.
\end{aligned}
$$

Next, we show that the decomposition satisfies all the axioms. Since the within-group inequality term has the form of (1) with $\alpha = \frac{1}{2}$ and $f(x) = x^{\frac{1}{2}}$, it satisfies Axioms 1–6 by Theorem 1. If all subgroups have identical income distributions, then the Gini coefficients are equal across subgroups and the income share of each subgroup equals its population share. In this case, both the aggregate Gini coefficient and the within-group inequality term in (2) are equal to the common subgroup Gini coefficient. Thus, between-group inequality is equal to zero and Axiom 7 holds.

Bhattacharya and Mahalanobis (1967) show that the aggregate Gini coefficient can be written as

$$G(L) = \sum_k \pi_k \theta_k G_k + G(\bar{L}) + R, \tag{10}$$

where $\bar{L}$ is the generalized Lorenz curve after replacing each individual's income with the respective subgroup mean and R is a residual term that depends on the amount of overlap between the subgroup income distributions. R is zero if and only if subgroup income distributions do not overlap. It is possible to create overlap with within-group transfers that keep subgroup Gini coefficients constant whenever at least two subgroups have positive population and income share. Hence, the aggregate Gini coefficient is aggregative if and only if at most one subgroup, j, has both strictly positive population and income share. In this special case, the between-group inequality term (3) reduces to $G^B = \pi_j + \theta_j - \pi_j \theta_j$, which does not depend on subgroup income distributions under fixed aggregate characteristics. Hence, Axiom 8 holds.

We next show that the decomposition of Gini coefficient into (2) and (3) is the unique decomposition that satisfies Axioms 1–8. Suppose W is a within-group inequality term for the Gini coefficient that satisfies all eight axioms. First, by Theorem 1,

$$W\big((G_1, n_1, Y_1), \ldots, (G_K, n_K, Y_K)\big) = f^{-1}\left(\sum_{k=1}^{K} \pi_k^{1-\alpha} \theta_k^{\alpha} f(G_k)\right)$$

for all $(G_k, n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_+^{3K}$, where f is a strictly monotonic function with $f(0) = 0$ and $\alpha \in \mathbb{R}$. Since cf generates the same W as f for any $c \neq 0$, we can assume $f(1) = 1$ without loss of generality.

Now, if there is only one subgroup, j, with strictly positive population and income share, then by Axiom 8 and equation (10), we get

$$W = f^{-1}\big(\pi_j^{1-\alpha} \theta_j^{\alpha} f(G_j)\big) = \pi_j \theta_j G_j$$

for all $\pi_j, \theta_j, G_j \in [0, 1]$. By taking f of both sides and setting $G_j = 1$, we get $\pi_j^{1-\alpha} \theta_j^{\alpha} = f(\pi_j \theta_j)$ for all $\pi_j, \theta_j \in [0, 1]$. By setting $\pi_j = 1$, we get $f(x) = x^{\alpha}$ for all $x \in [0, 1]$, and by setting $\theta_j = 1$, we get $f(x) = x^{1-\alpha}$ for all $x \in [0, 1]$. Thus, we have $\alpha = 1 - \alpha$ which implies $\alpha = \frac{1}{2}$ and $f(x) = x^{\frac{1}{2}}$ for all $x \in [0, 1]$. Therefore, the decomposition with the within-group inequality term (2) and between-group inequality term (3) is the unique decomposition for the Gini coefficient that satisfies Axioms 1–8.

A.4. *Proof of Proposition 2*

In the proof, we use the following lemma.

LEMMA 2: *Merging the first $m \leq K$ subgroups into one weakly increases the within-group inequality of the Gini coefficient. That is,*

$$G^W\big((G_1, \pi_1, \theta_1), \dots, (G_K, \pi_K, \theta_K)\big) \\ \leq G^W\big((\tilde{G}, \tilde{\pi}, \tilde{\theta}), (G_{m+1}, \pi_{m+1}, \theta_{m+1}), \dots, (G_K, \pi_K, \theta_K)\big),$$

where $\tilde{\pi} = \sum_{k=1}^m \pi_k$, $\tilde{\theta} = \sum_{k=1}^m \theta_k$, and $\tilde{G}$ is the Gini coefficient of the mixture of the first m subgroups' income distributions with weights $(\pi_k/\tilde{\pi})_{k=1}^m$. Merging does not affect within-group inequality if and only if the income distributions of the m subgroups are identical.

PROOF: Let G_m^W denote the within-group inequality among the first m subgroups and let $\tilde{G}^W$ denote the level of within-group inequality after the first m subgroups are merged into one. Because $\tilde{G} \geq G_m^W = \Big(\sum_{k=1}^m \sqrt{(\pi_k/\tilde{\pi})(\theta_k/\tilde{\theta})G_k}\Big)^2$, we have that

$$\tilde{G}^W = \left(\sqrt{\tilde{\pi}\tilde{\theta}\tilde{G}} + \sum_{k=m+1}^K \sqrt{\pi_k \theta_k G_k}\right)^2 \\ \geq \left(\sqrt{\tilde{\pi}\tilde{\theta}} \sum_{k=1}^m \sqrt{\frac{\pi_k}{\tilde{\pi}} \frac{\theta_k}{\tilde{\theta}} G_k} + \sum_{k=m+1}^K \sqrt{\pi_k \theta_k G_k}\right)^2 = \left(\sum_{k=1}^K \sqrt{\pi_k \theta_k G_k}\right)^2 = G^W.$$

Moreover, we have $\tilde{G} = G_m^W$ if and only if the income distributions of the first m subgroups are identical, in which case we have $\tilde{G}^W = G^W$. *Q.E.D.*

The proof proceeds by showing that within-group inequality increases under the operation while the aggregate Gini coefficient is unaffected. Consider first drawing a random sample of fraction $\alpha > 0$ from the population and assigning the sampled individuals to a new subgroup. This operation increases the within-group inequality term. To show this, we use equation (2) and the Brunn–Minkowski theorem (Proposition 8),

$$G^W\big((G_1, (1-\alpha)\pi_1, (1-\alpha)\theta_1), \dots, (G_K, (1-\alpha)\pi_K, (1-\alpha)\theta_K), (G, \alpha, \alpha)\big) \\ = \left((1-\alpha)\sum_{k=1}^K \sqrt{\pi_k \theta_k G_k} + \alpha\sqrt{G}\right)^2 \geq \left((1-\alpha)\sum_{k=1}^K \sqrt{\pi_k \theta_k G_k} + \alpha \sum_{k=1}^K \sqrt{\pi_k \theta_k G_k}\right)^2 \\ = \left(\sum_{k=1}^K \sqrt{\pi_k \theta_k G_k}\right)^2 = G^W\big((G_1, \pi_1, \theta_1), \dots, (G_K, \pi_K, \theta_K)\big), \tag{11}$$

where equality holds if and only if the subgroups have identical income distributions.

Now, by the weak reflexivity, symmetry, and replacement axioms, we have

$$G^W\big((G_1, (1-\alpha)\pi_1, (1-\alpha)\theta_1), \dots, (G_K, (1-\alpha)\pi_K, (1-\alpha)\theta_K), (G, \alpha, \alpha)\big) \\ = G^W\big((G_1, (1-\alpha)\pi_1, (1-\alpha)\theta_1), \dots, (G_K, (1-\alpha)\pi_K, (1-\alpha)\theta_K), \\ (G, \alpha\pi_1, \alpha\pi_1), \dots, (G, \alpha\pi_K, \alpha\pi_K)\big). \tag{12}$$

Let $\tilde{G}_k$ and $\tilde{\theta}_k$ denote the resulting Gini coefficient and income share after a $1-\alpha$ sample of subgroup k's distribution is merged with an $\alpha\pi_k$ sample of the aggregate distribution. By Lemma 2 and symmetry, the right-hand side of (12) is less than or equal to $G^W((\tilde{G}_1, \pi_1, \tilde{\theta}_1), \dots, (\tilde{G}_K, \pi_K, \tilde{\theta}_K))$. Equality holds only if all subgroups have the same distribution of income. Combining this with (11), we get

$$G^W\big((\tilde{G}_1, \pi_1, \tilde{\theta}_1), \dots, (\tilde{G}_K, \pi_K, \tilde{\theta}_K)\big) \geq G^W\big((G_1, \pi_1, \theta_1), \dots, (G_K, \pi_K, \theta_K)\big).$$

The left-hand side of the above equation corresponds to within-group inequality after replacing the incomes of a random α sample with a random draw from the aggregate income distribution. As the aggregate Gini coefficient is unchanged while the within-group inequality term increases, between-group inequality must decrease. Within-group inequality remains unchanged if and only if the subgroup income distributions are identical. In this case, between-group inequality is also unaffected.

A.5. *Proof of Proposition 3*

By Lemma 2 and symmetry, merging any $m \leq K$ subgroups weakly increases within-group inequality. Because the aggregate Gini coefficient is unaffected by merging subgroups, between-group inequality must weakly decrease. Moreover, by Lemma 2, within-group inequality does not change if and only if the m subgroups have identical income distributions. Then between-group inequality does not change either.

A.6. *Proof of Proposition 4*

Let $\tilde{G}$ denote the Gini coefficient after replacing y_i with $\tilde{y}_i$ for all i. First, note that the redistribution scheme reduces the aggregate Gini coefficient by α percent:

$$\begin{aligned}\tilde{G} &= \frac{\mathbb{E}\big[|\tilde{y}_i - \tilde{y}_j|\big]}{2\mu} = \frac{\mathbb{E}\big[|y_i - \alpha(y_i - \mu) - (y_j - \alpha(y_j - \mu))|\big]}{2\mu} = (1-\alpha)\frac{\mathbb{E}\big[|y_i - y_j|\big]}{2\mu} \\ &= (1-\alpha)G.\end{aligned}$$

It is easy to see that within-group inequality also decreases by α percent:

$$\begin{aligned}\tilde{G}^W &= \left(\sum_k \sqrt{\pi_k \tilde{\theta}_k \tilde{G}_k}\right)^2 = \left(\sum_k \sqrt{\pi_k \tilde{\theta}_k \frac{\mathbb{E}_k\big[|\tilde{y}_i - \tilde{y}_j|\big]}{2\tilde{\mu}_k}}\right)^2 \\ &= (1-\alpha)\left(\sum_k \sqrt{\pi_k \pi_k \frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu}}\right)^2 = (1-\alpha)\left(\sum_k \sqrt{\pi_k \theta_k \frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu_k}}\right)^2 \\ &= (1-\alpha)\left(\sum_k \sqrt{\pi_k \theta_k G_k}\right)^2 = (1-\alpha)G^W,\end{aligned}$$

where $\tilde{\theta}_k$ is subgroup k's income share after the replacement of incomes. Finally, since both the aggregate Gini coefficient as well as the within-group term decrease by fraction α, the between-group term must also decrease by fraction α.

A.7. *Proof of Proposition 5*

Note that the within-group inequality term can be written as

$$G^W = \left(\sum_k \sqrt{\pi_k \theta_k G_k}\right)^2 = \left(\sum_k \sqrt{\pi_k \theta_k \frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu_k}}\right)^2 = \left(\sum_k \pi_k \sqrt{\frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu}}\right)^2.$$

But since lump-sum transfers between subgroups affect neither the mean absolute difference within subgroups nor the mean income in the population, it follows that such transfers do not affect within-group inequality.

A.8. *Proof of Proposition 6*

Scale invariance follows directly from the scale invariance of within-group inequality and the inequality measure itself. To show translation invariance, note first that adding some fixed amount z to each income in a population with average income μ decreases the Gini coefficient by a factor of $\mu/(\mu+z)$. Let $\tilde{G}_k$ and $\tilde{\theta}_k$ denote the Gini coefficient and the income share of subgroup k after translation. Within-group inequality after translation can then be computed as

$$\begin{aligned}\tilde{G}^W &= \left(\sum_k \sqrt{\pi_k \tilde{\theta}_k \tilde{G}_k}\right)^2 = \left(\sum_k \sqrt{\pi_k \left(\pi_k \frac{\mu_k + z}{\mu + z}\right)\left(\frac{\mu_k}{\mu_k + z} G_k\right)}\right)^2 \\ &= \frac{1}{\mu + z}\left(\sum_k \sqrt{\pi_k^2 \mu_k G_k}\right)^2 = \frac{\mu}{\mu + z}\left(\sum_k \sqrt{\pi_k \theta_k G_k}\right)^2 = \frac{\mu}{\mu + z} G^W,\end{aligned}$$

that is, within-group inequality also decreases by a same factor of $\mu/(\mu + z)$. Hence, the ratio of within-group inequality to between-group inequality remains unchanged.

A.9. *Proof of Proposition 8*

The inequality together with the equality condition follows directly from the Brunn–Minkowski theorem. See, for example, Gardner (2002) for the proof of the Brunn–Minkowski theorem in the case of two sets. The theorem can be easily generalized to three sets. The general case of K sets follows by induction.

By associativity of Minkowski addition, we have

$$\begin{aligned}|\lambda_1 + \lambda_2 + \lambda_3| &= \big|(\lambda_1 + \lambda_2) + \lambda_3\big| \geq \left(|\lambda_1 + \lambda_2|^{\frac{1}{2}} + |\lambda_3|^{\frac{1}{2}}\right)^2 \\ &\geq \left(\left(\left(|\lambda_1|^{\frac{1}{2}} + |\lambda_2|^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} + |\lambda_3|^{\frac{1}{2}}\right)^2 = \left(|\lambda_1|^{\frac{1}{2}} + |\lambda_2|^{\frac{1}{2}} + |\lambda_3|^{\frac{1}{2}}\right)^2,\end{aligned}$$

where the inequalities hold as equalities when λ_1, λ_2, and λ_3 are homothetic. The last claim follows from the fact that homothety is closed under Minkowski addition.

A.10. *Proof of Proposition 9*

Transfers that increase subgroup inequality by the Lorenz criterion shift the subgroup Lorenz curves outwards so that the pre-transfers Lorenz regions are subsets of the posttransfers Lorenz regions. Since $A \oplus B \subseteq \tilde{A} \oplus \tilde{B}$ if $A \subseteq \tilde{A}$ and $B \subseteq \tilde{B}$, such transfers also shift the aggregate Lorenz curve outwards. Hence, aggregate inequality must increase.

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