ANALYSES OF TAPATAN AND PICARIA

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Abstract. Tapatan and Picaria are traditional abstract strategy games. In this paper, we present a standard combinatorial analysis and game-theoretic analysis of these games and their generalizations to polygonal boards as well as a less standard analysis of the expected outcomes when the players only look ahead a fixed number of moves rather than playing optimally. Some variations are also analyzed.

1. Introduction

Tapatan and Picaria are abstract strategy games with complete information played in many parts of the world. Tapatan is the name given to the game in the Philippines and is known by different names in different regions, such as Achi (Ghana), Marelle (France), Tant Fant (India), and Luk tsut k'i (China) [2]. Picaria comes from the Pueblo Nation in New Mexico and is believed to be an adaptation of a game brought by the Spanish colonizers [1]; it is also known, for instance, as luffarschack in Sweden [6], though this name seems to refer to multiple games. Both games call for three pieces to be placed on a square board containing nine vertices; they differ in the possible movements the players can make (Figures 1 and 2; please note that the four internal intersections in Figure 2 without dots are not playable vertices). The winning condition is the same for both games: one of the players must arrange their three pieces in a line. However, there is no way to achieve a stalemate. If neither player has won after all the pieces are placed on the board, then the game will continue until one of the players successfully makes a row of three pieces.

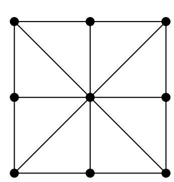


Figure 1. Standard Tapatan board.

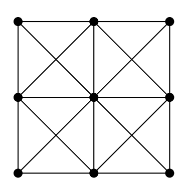


Figure 2. Standard Picaria board.

We must also note that there are different rules for the placement phase in Picaria: it is often the case that neither player is allowed to place a piece in the center during the placement phase [1]. In this paper, we will take the approach of Larsson and Rocha [6] and allow such placement to keep our primary analysis in line with that of Tapatan, which seems to have no such restrictions.

We begin by presenting some terminology that we will use to describe the game boards, their configurations, and the gameplay. We first discuss the types of *board* *configurations*, that is, the game board together with the placement of the pieces; we may sometimes refer to these simply as *boards* when there is no danger of confusion. We will only consider board configurations that represent legal and achievable placements of the pieces during an actual game of Tapatan or Picaria. Board configurations are considered legal and achievable if they can be produced when players alternate moves, move only to vertices (or, more informally, *nodes*) that are unoccupied, and occupy open vertices by moving along a single edge. In our illustrations, the red pieces will belong to Player 1 and the blue pieces to Player 2.

We will refer to the vertices as *corner vertices*, *edge vertices*, and the *center vertex*. The corner vertices are those at the corners of the board, the center vertex is that at the center, and all other vertices are edge vertices.

A *winning board configuration* is a board configuration in which exactly one player has three in a row and has thus won the game, and a *neutral board configuration* is any board configuration that is not a winning board configuration. When considering the point of view of a particular player, we may also speak of *losing board configurations*, which refer to boards for which their opponents have a winning strategy.

We note that it is not possible for both players to have three in a row simultaneously; we call board configurations in which this happens *double-win board configurations*, and we thus exclude these from our list of possible board configurations. Other board configurations that we will exclude are those that violate the rule of alternating turns among players. For instance, consider the two boards in Figure 3: given the rules of Tapatan, only the first one is a valid board configuration.

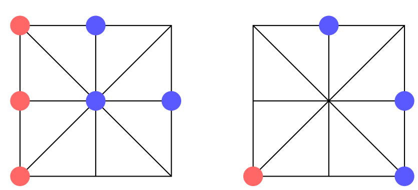
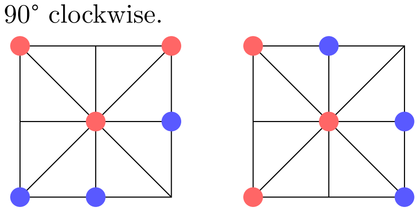
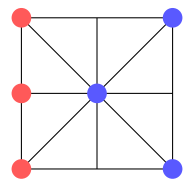


FIGURE 3. One valid board configuration (left) and one invalid board configuration (right).

Given two board configurations A and B, we say that A is *equivalent* to B if A can be transformed into B by a symmetry mapping T of the square. For example, the two boards below are equivalent because the board on the right is obtained by rotating the left board 90° clockwise.



A board is A is *self-reflective* if there is a symmetric mapping T such that $T(A) = A$. The board below provides an example of a self-reflective board for a horizontal reflection T about the center.



Since we may sometimes consider boards in the placement phase, we will call a board on which all six pieces have been played a *full board* in any circumstance where both types of boards are under consideration.

We will also consider the natural generalizations of these games to boards with other regular polygonal shapes; we will refer to such a game as n-Tapatan or n-Picaria. The boards for 6-Tapatan and 6-Picaria are illustrated in Figure 4.

FIGURE 4. Boards for 6-Tapatan (left) and 6-Picaria (right).

Larsson and Rocha presented examples of such Picaria boards in their concluding remarks at the end of [6] and mentioned that Player 1 had a winning strategy for n-Picaria whenever $n \neq 4$; however, no proof was provided.

Our paper is organized as follows. We present a combinatorial analysis of the number of board configurations for n-Tapatan and n-Picaria in Section 2. In Section 3, we discuss winning strategies for these games, and in Section 4, we discuss the consequences of playing with different strategies. Finally, in Section 5, we discuss some alternative versions of Tapatan and Picaria in which we seek to address some of the issues that arise in the standard games.

2. Combinatorial Analysis

The goal of this section is to determine formulas for the number of board configurations and winning configurations for n-Tapatan and n-Picaria. Since an n-Picaria board has precisely the same symmetries as an n-Tapatan board, the same analysis will apply for both games and we can simply call the board in question an *n-board.*

Remark 2.1. There are exactly $2n + 1$ vertices on an n-board. Furthermore, an n-Tapatan board has $4n$ edges, and an n-Picaria board has $5n$ edges.

We now turn our attention to the 4-board as an initial example. While Larsson and Rocha carried out these calculations in [6], we feel that it will be informative to include them here as well.

Proposition 2.2. *There are 1,680 total 4-board configurations and 225 total unique 4-board configurations.*

Proof. We first consider each and every configuration of six pieces separately. There are 9 nodes on the board, and each player takes up three of them in a full board configuration, so we have $\binom{9}{3} \cdot \binom{6}{3} = 1680$ possible board configurations.

Now we apply Burnside's Lemma to count the number of full board configurations. As the 4-board is a regular square, we let our group G be D_4. Since there are 1680 full board configurations, each reflection fixes 36 self-reflective boards, and there are four ways to reflect those 36 self-reflective boards, Burnside's Lemma gives us

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$
$$= \frac{1,680+4(36)}{8} = 228$$

full board configurations.

We note, however, that there are three distinguishable double-win board configurations included in this calculation: the one in which Player 1 and Player 2 each have a "win" on opposite sides of the board, the one in which Player 1 has a "win" on a side of the board and Player 2 has a "win" in the center, and the one in which Player 2 has a "win" on a side of the board and Player 1 has a "win" in the center. This leaves 225 distinguishable valid board configurations. □

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Proposition 2.3. *There are* 21 *total distinguishable full board winning configurations for each player in 4-Tapatan or 4-Picaria.*

Proof. A quick case analysis suffices. Without loss of generality, suppose Player 1 has won. If they have won with a horizontal or vertical line through the center of the board, Player 2 must have two pieces on one side of their winning line and one on the other; a quick count shows us that there are five such distinguishable configurations. The same reasoning shows that there are five distinguishable configurations if Player 1 has won with a diagonal line.

concerns about this count

If Player 1 has won with a line along an edge of the board, there are more possibilities: eight with two of Player 2's pieces along the center line parallel to the winning edge, and three with two of Player 2's pieces along the far edge. All of these possibilities combined give 21 distinguishable configurations. □

depends on previous result

Corollary 2.4. *There are 183 neutral board configurations in 4-Tapatan or 4-Picaria.*

We now turn our attention to 6-Tapatan and 6-Picaria as a further example.

Lemma 2.5. *There are 34,320 total 6-board configurations and 2,916 distinguishable 6-board configurations.*

Proof. As per Remark 2.1, there are 13 nodes on the board. Thus, one player must take three of them in a full 6-sided board configuration. This leaves 10 open nodes, three of which must then be taken up by the other player. Thus, we see there are $\binom{13}{3}\cdot\binom{10}{3} = 34320$ possible 6-board configurations.

Now we apply Burnside's Lemma to verify the number of full board configurations. As the six-sided board is a regular hexagon, we let our group G be D_6. There are exactly 12 unique board configurations whose 120° degree clockwise and counterclockwise rotations yield no change. Moreover, we know each reflection fixes 120 self-reflective boards and there are 6 ways to reflect those 120 self-reflective boards, so Burnside's Lemma gives us

$$\begin{aligned} |X/G| &= \frac{1}{|G|}\sum_{g\in G}|X^g| \\ &= \frac{34320+2(12)+6(120)}{12} = 2922 \end{aligned}$$

full board configurations.

We note, however, that there are six double-win board configurations included in this calculation. These configurations are shown in Figure 5 for simplicity.

After subtracting these six states, we find 2,916 legal and achievable unique full 6-board configurations. □

We now turn our attention to more general formulas for the number of board configurations for n-Tapatan and n-Picaria. We begin with some basic calculations on the number of configurations fixed by any rotation.

Lemma 2.6. *If $n \geq 4$ is not divisible by 2 nor 3, then each rotation fixes 0 n-sided board configurations.*

Proof. If n is prime, we assume for a contradiction that there exists a nontrivial rotation that fixes at least one n-sided board configuration. Then there exists an integer $k < n$ such that a board configuration is fixed after k turns. We can deduce that every kth corner and edge vertex of the board configuration must all have the same type of piece or all be empty. But since n is prime, no such value for k exists, as that would suggest n is divisible by k.

Now we suppose that n is composite. In this case, all of n's factors must be greater than 3. Thus, if an n-sided board configuration has rotational symmetry, then it must have an order of k where $k > 3$. This implies that for each piece on a vertex or edge, the same type of piece must occupy every kth corner or edge. Since each player can only place 3 pieces, then all of the edge and corner nodes must be empty in order to satisfy rotational symmetry. But this leaves only the center to be filled, and since 6 pieces must be placed in a full board configuration, a contradiction arises. □

Lemma 2.7. *For an n-sided board configuration, each reflection fixes $6(n-1)(n-2)$ self-reflective n-sided board configurations.*

Proof. Since each reflection line passes through exactly three of the $2n+1$ nodes on an n-sided board configurations, we see that there are $\frac{(2n+1)-3}{2} = n-1$ nodes on each side of the reflection line. Thus, we must count the number of combinations of $n-1$ reflective nodes and three stationary nodes. Since there must be exactly one pair of red pieces and one pair of blue pieces that reflect across the reflection line, this leaves one red piece, one blue piece, and a blank node occupying the vertices on the reflection line. We see that there are

$$\left[\binom{n-1}{1}\cdot\binom{n-2}{1}\right]\cdot 3! = 6(n-1)(n-2)$$

total such configurations. □

Lemma 2.8. *If $n \geq 3$ is odd, then there are $\frac{3(n-3)}{2}$ double-win configurations for an n-sided board.*

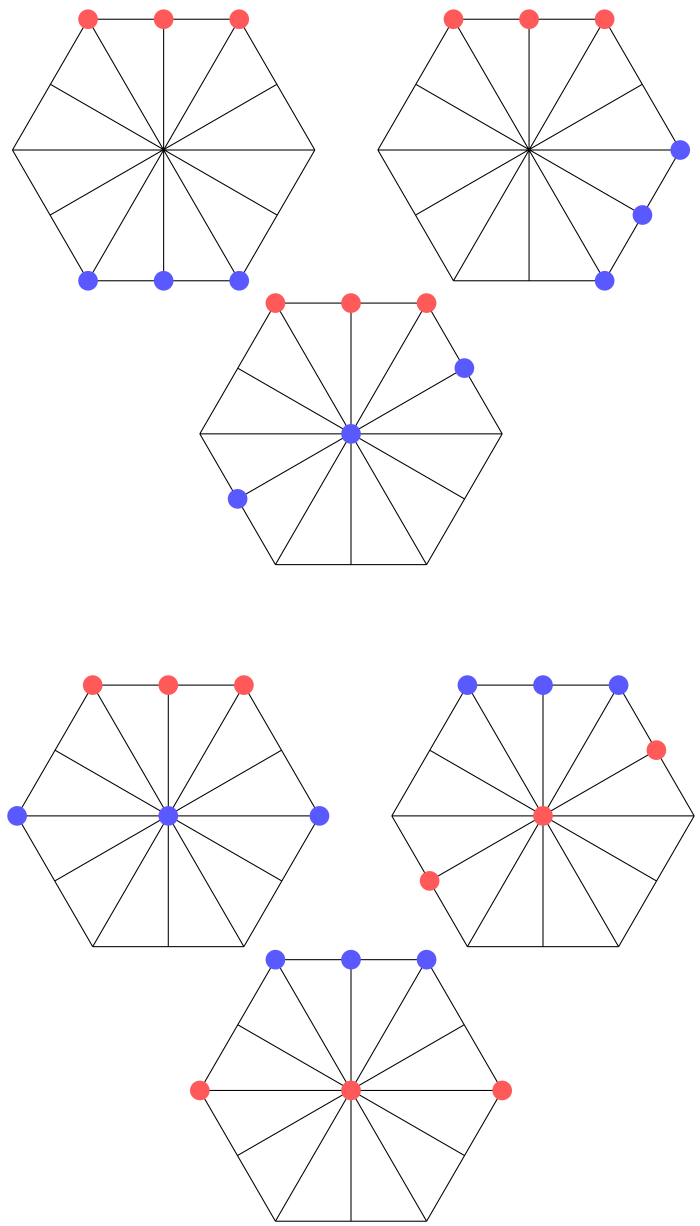


FIGURE 5. Double-win board configurations for 6-Tapatan.

Proof. First we will count the case when neither of the wins uses the central vertex. In this case, each win uses one side of the n-board. We can enumerate the sides of the n-gon with the numbers $1, 2, \ldots, n$ and, without loss of generality, we can assume Player 1's win uses the side numbered n. Observe that Player 2's win can't be on side 1 or $n-1$ since both of these share a vertex with the side 1. Then Player 2's win can be on any of the other $n-3$ sides of the board. However, if Player 2's win is on edge i, then by applying a reflection along the axis perpendicular to side n, we can obtain a configuration where Player 1's win is on side n and Player 2's win is on side $n-i$. Thus there are only $\frac{n-3}{2}$ double win configurations in this case.

Now we consider the case in which one of the wins uses the central vertex. If Player 1's win uses the central vertex, then it divides the board into "right" and "left" sides. Without loss of generality, we can assume Player 2's win is on the "right" side of the board where there are $\frac{n-1}{2}$ full sides of the n-gon. However, one of these has a vertex already occupied by Player 1, so there are only $\frac{n-3}{2}$ possible sides where Player 2's win can be. Therefore, there are $\frac{n-3}{2}$ double-win configurations in which Player 1 has the central vertex, and similarly there are $\frac{n-3}{2}$ double-win configurations in which Player 2 does.

Since these are the only possible cases for a double-win board, we get a total of $\frac{3(n-3)}{2}$ distinguishable double-win configurations for odd n. □

Lemma 2.9. *If $n \geq 4$ is even, then there are $3(\frac{n-2}{2})$ double win configurations.*

Proof. Once again, we consider the cases for the events of a double win configuration based on whether or not the central vertex is occupied.

Consider the cases which do not have a win in which the central vertex is used. We enumerate the sides of the n-gon with the numbers $1, 2, \ldots, n$. Without loss of generality, assume the win by Player 1 is on side n. Two win states cannot be on adjacent sides, so there are two sides which cannot have a Player 2 win given that side n has a Player 1 win. Thus there are $n-3$ double-win states without a center vertex. However, a double win configuration with Player 1 on side n and Player 2 on side i can be reflected along the axis perpendicular to side n so Player 2 wins on side $n-i$ instead (as long as $i \neq \frac{n}{2}$). Thus, the number of double wins without the center vertex is $\frac{n-4}{2} + 1 = \frac{n-2}{2}$. We note that the choice of Player 1 or 2 is irrelevant.

Now we consider the case in which a win uses the central vertex. There are two types of central vertex wins: ones which connect a corner vertex to a corner vertex, and ones which connect an edge vertex to an edge vertex.

First consider the edge-to-edge double wins. Without loss of generality, we can assume that Player 1's win is through the center vertex vertically, splitting the board into a left and right half, and that Player 2's win is on the right half. For an n-gon, there are $\frac{n-2}{2}$ sides on the right half, subtracting one to account for the side involving a Player 1 piece. However, some of the Player 2 win states are equivalent up to horizontal reflection. To account for this, we notice that when $n \equiv 0 \pmod 4$, an n-board has a side intersected by a horizontal line splitting the board in half, while when $n \equiv 2 \pmod 4$, it does not.

If $n \equiv 2 \pmod 4$, then we simply divide the number of lines by 2 to account for Player 2 wins that cannot be distinguished by reflection across a horizontal axis. Thus, the number of double wins is $\frac{n-2}{4}$ for this case.

If $n \equiv 0 \pmod 4$, then the intersected line does not have an corresponding Player 2 win to remove since a horizontal reflection returns the intersected line win to the same board position. Thus, we count a phantom side to be removed in the equation to account for this over-compensation in removing duplicates. This gives us

$$\frac{(n-1)/2 - 1}{2} + 1 = \frac{n}{4}$$

.

Now consider the case of Player 1 having a win from corner vertex to corner vertex through the center. Once again, without loss of generality, we assume the board is oriented so that the win is vertical and that Player 2's win is on the right half. There are $\frac{n-4}{2}$ full sides on the right, removing two sides as the central win intersects with two right sides. Now, as with the midpoint to midpoint case concerning horizontal reflections, we divide by two for the $n \equiv 2 \pmod 4$ case, resulting in

$$\frac{(n-4)/2-1}{2}+1=\frac{n-2}{4}$$

For the $n \equiv 0 \pmod 4$ case, we include the phantom full side to get[1] $\frac{n-4}{4}$.

In each situation, we find that if $n \geq 4$ is even, there are $\frac{3(n-2)}{2}$ double win states. ☐

Theorem 2.10. *If $n \geq 4$ and n is not divisible by 2 nor 3, then there are $(n-1)(n-2)[\frac{(2n+1)(2n-1)(2n-3)}{9}+3]-\frac{3(n-3)}{2}$ total full n-sided board configurations.*

Proof. We observe, as before, that there are $\binom{2n+1}{3} \cdot \binom{2n-2}{3}$ full n-sided board configurations. Lemma 2.6 tells us that there are no board configurations whose rotations yield no change. Moreover, we know from Lemma 2.7 that each reflection fixes $6(n-1)(n-2)$ self-reflective boards and there are n ways to reflect those $6(n-1)(n-2)$ self-reflective boards, so Burnside's Lemma then provides that the number of distinguishable configurations is

$$\frac{\binom{2n+1}{3}\binom{2n-2}{3}+6[n(n-1)(n-2)]}{2n}$$
$$=(n-1)(n-2)[\frac{(2n+1)(2n-1)(2n-3)}{9}+3]$$

However, Lemma 2.8 tells us that we must subtract $\frac{3(n-2)}{2}$ double-win configurations. So we see that there are $(n-1)(n-2)[\frac{(2n+1)(2n-1)(2n-3)}{9}+3]-\frac{3(n-3)}{2}$ total full n-sided board configurations if n is not divisible by 2 nor 3. ☐

3. Optimal play

We now turn our attention to the existence of winning strategies. As mentioned earlier, Larsson and Rocha proved that 4-Picaria is a draw and claimed that n-Picaria is not for $n \neq 4$ in [6]. Here we prove that Player 1 has a winning strategy in n-Tapatan for $n \geq 3$ and n-Picaria for $n \geq 4$. We begin with a discussion of the standard 4-Tapatan board and then extend this strategy to n-Tapatan for $n > 4$ and produce an alternative strategy for 3-Tapatan.

We begin by developing some notation for the vertices on an n-board. The bottom leftmost corner vertex is labeled $0,0$, and, going counter-clockwise, each subsequent corner vertex is labeled $1,0$, $2,0$, …, $n,0$. The edge vertex immediately counter-clockwise of $i,0$ is labeled $i,1$ for $0 \leq i \leq n$. The 4-Tapatan and 6-Tapatan boards appear in Figure 6:

We will also make use of the concept of an opposite vertex:

[1] fixed

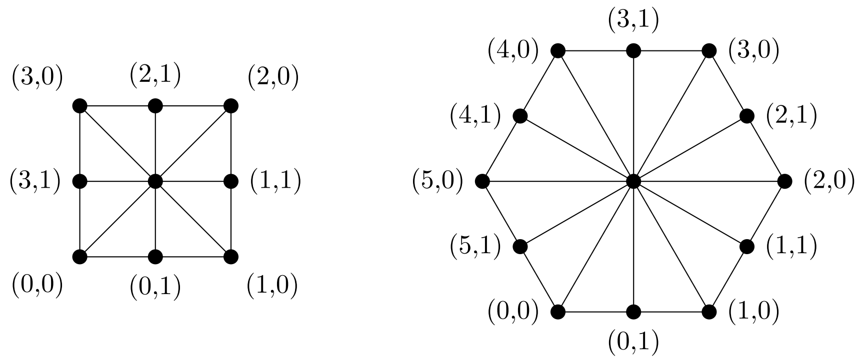


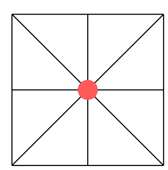
FIGURE 6. 4-Tapatan (left) and 6-Tapatan (right) boards with labeled vertices.

Definition 3.1. Let a vertex other than the center be given on an n-board. We define its *opposite vertex* to be the third vertex on the same straight line as our given vertex and the center.

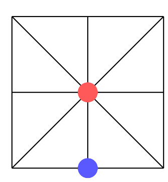
3.1. **Standard Tapatan Board.**

Theorem 3.2. *Player 1 can win 4-Tapatan in 9 moves.*

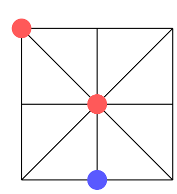
Proof. Player 1 begins by placing a piece in the center:



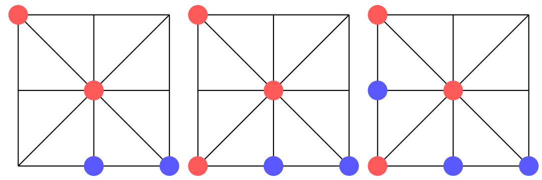
Player 2 has only two distinguishable moves due to symmetry: placement of a piece on a corner or an edge. We will first consider placement on an edge.



Player 1 now places their second piece on one of the two corners farthest from the edge where Player 2's piece lies:

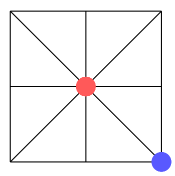


The next three total moves are all forced moves, since each player must place their piece so that they block their opponent from winning on the next moves. These three moves are shown as game states below in successive order: ADD ARROWS -JF

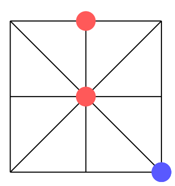


This last state marks the end of the placement phase and the beginning of the movement phase. Player 1 can now simply slide the piece on the top left corner to the top right corner in two moves, and they will have won; Player 2 will not be able to block this or win in any other way themselves in their one intermediate move.

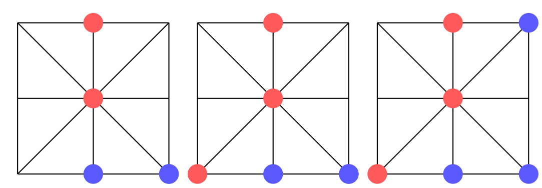
Now we turn our attention to the case in which Player 2 places their initial piece on a corner.



Similarly to the first case, Player 1 will choose to place their piece on one of the two edges farthest from Player 2's piece.



Again, the next three moves are forced as a result of each player needing to block the other from winning immediately: MORE ARROWS -JF



This marks the end of the placement phase and the start of the movement phase, and just as before, Player 2 is unable to prevent Player 1 from winning. Player 1 simply needs to slide over the two pieces in the middle column to the left column and they will win as before.

Therefore, Player 1 is able to force a win from the beginning game state in a total of 9 moves counting the placement phase no matter which moves Player 2 makes. □

3.2. **n-Tapatan, $n \geq 4$.** We now generalize this strategy for n-Tapatan for all $n \geq 4$.

Theorem 3.3. *Player 1 has a winning strategy in n-Tapatan for all $n \geq 4$.*

Proof. Just as in 4-Tapatan, Player 1 will occupy the center of the board with their first move, and then Player 2 has two possible distinguishable moves: a corner or an edge. We shall first address what happens if Player 2 places their first piece on an edge.

Without loss of generality, Player 2 places their piece at the point (0,1). Player 1 will now play their second piece exactly $n-1$ nodes in one direction along the outside from the node (0,1). We can say without loss of generality that Player 1 places their piece $n-1$ nodes counterclockwise from (0,1), which is $(\frac{n}{2}, 0)$ for even n and $(\frac{n-1}{2}, 1)$ for odd n (we will call this the *diagonal piece*).

Player 2 is now forced to play at the node (0,0) to prevent Player 1 from winning in the placement phase, and, similarly, Player 1 will place their third and final piece at (1,0), preventing Player 2 from winning across the "bottom" row. Now Player 2 has $2n-4$ different possible nodes to place their final piece. However, most of these can be dismissed immediately.

There are $n-3$ nodes counterclockwise between Player 1's piece on $(1,0)$ and their diagonal piece. If Player 2 places in any of these nodes, Player 1 may simply shift the diagonal piece counterclockwise by 2 nodes. Player 2 would be unable to do anything to prevent this since their closest piece lies on the node (0,0), which is exactly $n-2$ moves away from blocking. Because $n-2 \geq 2$ for all $n \geq 4$, Player 2 will be unable to block Player 1's win.

Next, consider the $n-4$ nodes immediately clockwise from (0,0). If Player 2 places their piece in any of these, they have at a minimum a win in 2 moves. This happens if they place at $(n-1, 0)$ and slide their two pieces one the "bottom" row 1 node counterclockwise each. However, Player 1 still has the same win in 2 moves as above, which Player 2 cannot block for the same reason.

This leaves only $(2n-4)-(n-3)-(n-4) = 3$ potential nodes for Player 2 to place their final piece, these being the three nodes directly counterclockwise from Player 1's diagonal piece. Suppose Player 2 plays at any of these nodes. But then Player 1 can simply slide this diagonal piece clockwise $n-4$ moves and finally move their center piece into the node (1,1), providing a win in $n-3$ moves. Player 2's quickest win is a win in $n-3$ moves at the minimum, and since Player 1 moves first, Player 1 wins once more.

There are a few things worth noting. First of all, Player 1 must not leave the center until their final move with which they end the game. This accomplishes two things: it prevents Player 2 from being able to block Player 1's pieces on their path to victory, and it prevents Player 2 from having access to a quicker path to their own victory via the center. Player 2, being unable to prevent Player 1 from winning, is forced to slide their pieces around the edge of the board in an attempt to win more quickly, which will always be impossible.

Additionally, the piece Player 1 placed at (1,0) will never move. This means that Player 2 cannot win along the bottom row, which forces them to spend 2 moves moving their two pieces one node clockwise each in an attempt to win along the edge immediately clockwise from the "bottom" row.

Therefore, Player 1's first sliding move must be the diagonal piece. This piece will either be slid counterclockwise twice or clockwise exactly $n-4$ times, depending on where Player 2 places their final piece as described above. Player 1's final move in the latter case will be to slide their center piece into the node (1,1) after sliding move $n-3$.

Now we consider the case in which Player 2 initially plays on a corner. Without loss of generality, Player 2 places their piece at the point (0,0).

Player 1 will now play their second piece exactly $n-1$ nodes in one direction along the outside from the node (0,0). Due to symmetry across the center line with one end at (0,0), we can say without loss of generality that Player 1 places their piece $n-1$ nodes counterclockwise from (0,0), which is the node $(\frac{n}{2}-1,1)$ for even n or $(\frac{n}{2}-1,0)$ for odd n. Once again, we call this the diagonal piece.

The next three moves will be forced as in 4-Tapatan. We first have Player 2's second move in which they are forced to play at the node $(n-1,1)$. Then Player 1 will place their third and final piece at the node $(n-1,0)$, preventing Player 2 from winning across the "bottom" row. Finally, Player 2 will place their final piece one node clockwise from Player 1's piece at either $(\frac{n}{2}-1,1)$ for even n or $(\frac{n}{2}-1,0)$ for odd n.

At this stage, both players have a win in exactly $n-2$ moves. This follows the exact same strategy described above when Player 2 places their first piece on an edge. Just as in that situation, because Player 1 moves first from this position, they are guaranteed to reach a win state first, and therefore they will win the game.

Therefore, Player 1 can force a win in n-Tapatan no matter what Player 2 does. □

3.3. **Tapatan, $n=3$.** Interestingly, 3-Tapatan behaves very differently from all other polygonal Tapatan boards. This is due to the fact that with only 7 nodes for the 6 pieces, it is common that a player may only have 1 legal move, which will lead to a loss for them immediately once the placement phase is complete. As a result, the winning strategy for Player 1 seems extremely counterintuitive compared to the winning strategy for $n \geq 4$.

It is worth stating and proving a few lemmas which will simplify the full proof later.

Lemma 3.4. *If Player 1's pieces lie on (0,1), (1,1), and (2,1), Player 2's pieces lie on (0,0), (1,0), and the center vertex, and it is Player 2's turn, Player 1 has a guaranteed win. We will refer to this configuration as Position A.*

Proof. There is only one legal move for Player 2: They must move their center piece to (2,0). Due to rotational symmetry, Player 1 also has only one legal move, which is to move any piece to the center. Without loss of generality, we may assume that they will move their piece on (0,1) into the center. Yet again, this time due to symmetry across the vertical axis, Player 2 has only one legal move, which is to move a piece to (0,1). Without loss of generality, we may assume they will move their piece from (0,0) to (0,1). Finally, Player 1 may simply move their piece on (2,1) to (0,0) and win the game. Therefore, Position A is a guaranteed win for Player 1. □

Lemma 3.5. *If Player 1's pieces Lie on (0,0), (1,0), and (2,0), Player 2's pieces Lie on (0,1), (1,1), and (2,1), and it is Player 1's turn, Player 1 has a guaranteed win. We will refer to this configuration as Position B.*

Proof. There is only one legal move for Player 1 due to rotational symmetry: They must move a piece to the center. Without loss of generality, we may assume they will move their piece on (2,0) into the center. Due to symmetry across the vertical axis, Player 2 also has only one legal move, which is to move a piece to (2,0). Without loss of generality we may assume that they will move their piece on (1,1) to the node (2,0). Finally, Player 1 may simply move their piece on (1,0) to (1,1) and win the game. Therefore, Position B is a guaranteed win for Player 1. □

Proposition 3.6. *Player 1 has a winning strategy in 3-Tapatan.*

Proof. Contrary to the winning strategy when $n > 3$, Player 1 must not play in the center, as this leads to a forced win for Player 2. We will show what happens if Player 1 first plays on the corner; for the sake of brevity, we leave it to the reader to explore how Player 1 may force a win if they play on the edge first.

Since all corner positions are identical on the empty board, we can say without loss of generality that Player 1 places their first piece on (0,0). Here, Player 2 has four unique possible moves: They can place either on one of the near edges, one of the two remaining corners, the center, or the edge opposite to Player 1's piece. We will see how Player 1 can respond in each scenario.

If Player 2 places on the far edge (1,1), then Player 1 will respond with placement of their second piece on one of the two remaining edge positions; without loss of generality, assume they place on (0,1). This leads to a series of forced moves: Player 2 blocks with a placement on (1,0), Player 1 blocks with a placement on (2,0), and finally Player 2 blocks with a placement in the center. Player 1 then may slide their piece on (0,0) into the empty node on (2,1), which is their only legal nonlosing move. But then Player 2 has only one move, which is to move their piece from the center to (0,0). Then Player 1 may simply slide their piece from (2,1) to the center and win the game. Thus, Player 1 will win if Player 2 places their first piece on the far edge.

If Player 2 places on a near edge (without loss of generality, the node (0,1)) Player 1 will respond with placement on the far edge (1,1). This, again, leads to a series of forced moves: Player 2 blocks with a placement in the center, and Player 1 blocks with a placement on (2,0). Here, Player 2 has the option of placing on either (1,0) or (2,1). Placement on (1,0) leads to Player 1 moving from (2,0) to (2,1); Player 2 must move from the center to (2,0), and Player 1 wins by moving from (2,1) to the center. So Player 2 must place their final piece on (2,1). But this forces Player 1 to move from (1,1) to (1,0), which in turn forces Player 2 to move from the center to (1,1). This is Position B, which is a known win for Player 1. Thus, Player 1 will win if Player 2 places their first piece on a near edge.

The next option is for Player 2 to place in the center, which, given how n-Tapatan for $n > 3$ proceeds, would appear to be the most logical choice. Player 1 will choose to respond with placement on the far edge, (1,1). Player 2 then has two options, these being placement on an edge ((0,1) without loss of generality) or a corner ((1,0) without loss of generality). Placement on (1,0) forces Player 1 to place on (2,1), which in turn forces Player 2 to place on (2,0). But this leads to Player 1 moving from (0,0) to (0,1), forcing Player 2 to enter Position B, a known win for Player 1. So Player 2 must place their second piece on (0,1). This forces Player 1 to place on (2,0), which leaves Player 2 with another two options: (1,0) and (2,1). Placement on (1,0) is met with movement from (2,0) to (2,1), forcing a move from the center to (2,0), and allowing Player 1 to win by moving from (2,1) to the center. However, placement on (2,1) forces Player 1 to move from (1,1) to (1,0), which in turn forces Player 2 to move from the center and enter Position B. Thus, Player 1 will win if Player 2 places their first piece in the center.

This leaves the final option: placement on a corner, which we will say is (1,0) without loss of generality. Player 1 responds with placement in the final corner, (2,0), which leads to a series of forced moves: Player 2 blocks with placement on (2,1), and Player 1 blocks with placement in the center. At this point, the game is lost; Player 2 has only one move due to reflection symmetry, that being (0,1). But Player 1 responds by moving from (2,0) to (1,1), winning the game once again.

Note that Position A was never reached, and Lemma 3.4 appears unnecessary. However, this position will be reached if Player 1 chooses to place first on an edge. □

3.4. **Standard Picaria.** As mentioned, Larsson and Rocha have previously shown that 4-Picaria is a draw [6]. We confirmed their result computationally by generating the state space of Picaria (the graph of all game states and moves between game states) and performing the minimax algorithm to a depth of 199 starting from the initial empty board; this is discussed in more detail in Section 4.

This depth is sufficient for the following reason. Note that any Picaria game is simply a path through the Picaria state space. During a game, players must traverse the empty board state, one state with 1 piece, one state with 2 pieces, and so on until 6 pieces. Picaria has 183 6-piece game states that are not win states. Thus, the longest path that does not result in a win has $1 + 5 + 183 = 189$ vertices. This counts the empty state, 5 states during the placement phase of the game, and traversing each of the 183 6-piece states. Since Picaria is an abstract strategy game, reaching the same game state at different points in the game makes no difference. Thus, minimax performed from the initial state to a depth of $199 \geq 189$ necessarily considers all possible strategies by both players. Since no winning strategy was found, it must be that none exist.

3.5. **Picaria, $n > 4$.** We need to consider the cases for even and odd n separately since their winning strategies are different. In both cases, the optimal first move for Player 1 is to place in the center.

The use of the opposite vertex (see Definition 3.1) is crucial in the winning strategy for n-Picaria. It should be noticed that when n is odd, the opposite vertex of a corner is an edge and vice versa; when n is even, the opposite vertex of a corner is a corner, and that of an edge is an edge.

Theorem 3.7. *Player 1 has a winning strategy in n-Picaria for $n > 4$.*

We first consider the case in which n is odd. Player 1 will place their first piece in the center. Player 2 has two possible moves up to rotational symmetry: either placing on a corner or placing on an edge.

Claim 3.8. *When Player 2 places their first piece on any corner vertex, Player 1 can win by placing their next piece immediately next to Player 2's first piece.*

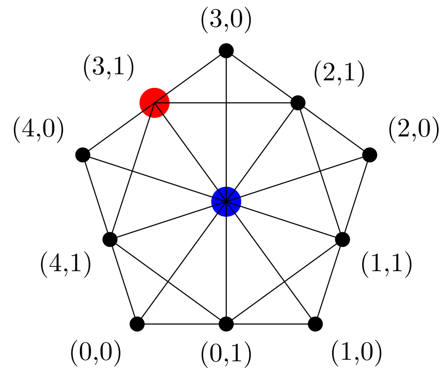
Proof. Suppose that the first piece of Player 2 is placed at $(a, 0)$ with opposite vertex $(b, 1)$. Without loss of generality, Player 1 places their second piece at $(a - 1, 1)$. Then Player 2 is forced to play at $(b, 0)$ to prevent Player 1 from winning. Player 1 then places their last piece on $(b + 1, 0)$, posing the threat of moving the piece at $(a - 1, 1)$ to $(a, 1)$ to win on their next turn. Again Player 2 has to block at $(a, 1)$.

In such a situation, the fastest way for Player 1 to win is to move the piece at $(a-1,1)$ along the lines that directly connect edge vertices until it reaches $(b+1,1)$; then the only move needed is to move the piece at the center to $(b+2,0)$. This process takes $\frac{n-1}{2}$ steps and cannot be interrupted by Player 2. The minimum number of steps for Player 2 to win is also $\frac{n-1}{2}$, and since Player 1 moves next, they will win.

□

Claim 3.9. *When Player 2 places their first piece on any edge vertex, Player 1 can win by placing their next piece at* $(b-2,1)$ *or* $(b+1,1)$*, where* $(b,0)$ *is the opposite vertex of the edge vertex that Player 2 placed on.*

Proof. Without loss of generality, let Player 1 place the next piece at $(b+1,1)$. Then Player 2 is forced to place the second piece at $(b-1,0)$. Player 1 then plays at $(b+1,0)$, posing the threat of moving the central piece to $(b+2,0)$ to win, so Player 2 has to block at $(b+2,0)$. Player 1 thus guarantees their victory by moving their piece at $(b+1,1)$ to $(b,1)$. Player 2 has no way to occupy $(b,0)$ to which Player 1 will move the central piece to win. Since Player 2 cannot win in one step, Player 1 wins. □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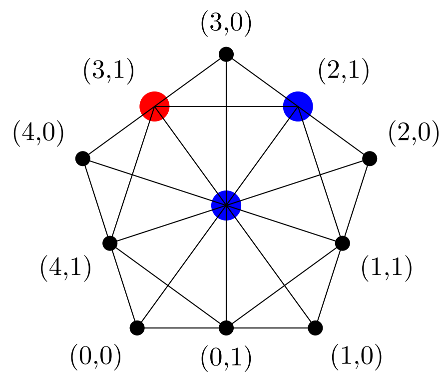
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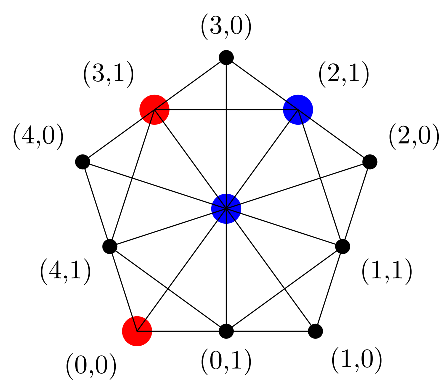
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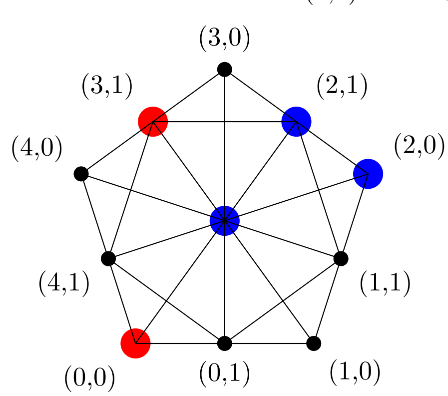
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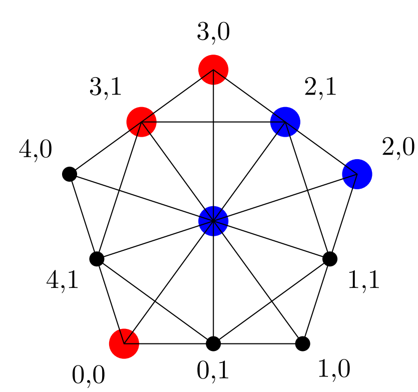
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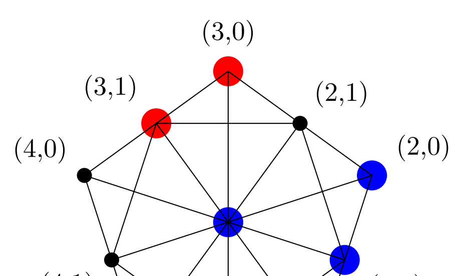
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(2,0)

Now we turn to the case in which n is even.

Claim 3.10. *When Player 2 places their first piece on a corner vertex* $(a, 0)$*, Player 1 can win by placing their second piece on* $(b+1, 1)$ *such that* $\mid a - b \mid = \frac{n}{2}$.

Proof. Without loss of generality, we can assume Player 2 places their first piece at $(0, 0)$. Player 1 will respond by placing their second piece at $(\frac{n+2}{2}, 1)$.Then Player 2 will have to make a forced move at $(1, 1)$ to prevent Player 1 from winning on the next turn. Then Player 1 can simply respond by placing their last piece on the corner next to their previous piece at $(\frac{n+2}{2}, 0)$. Player 2 will have to make another forced move onto the only corner left on that edge, which is $(\frac{n+4}{2}, 0)$. Player 1 is guaranteed a win since they are 2 steps away from a winning configuration and Player 2 is at least 2 steps away from a winning configuration (depending on the value of n). □

Are the skipped steps intentional? -JF

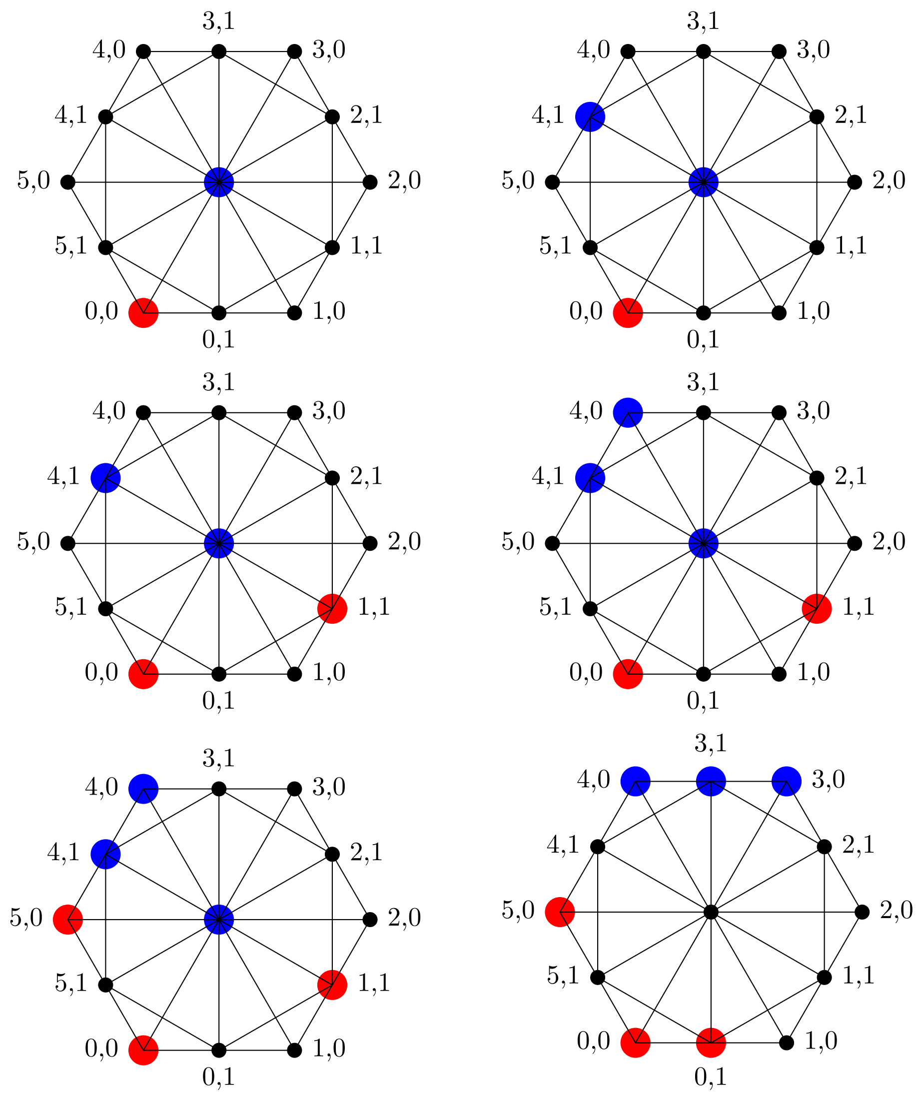

FIGURE 8. The Optimal Strategy for Player 1 When Player 2 Places on A Corner, Using 6-Picaria as an Example.

Claim 3.11. *When Player 2 places their first piece on any edge vertex* $(a, 1)$*, the optimal strategy for Player 1 is to respond by placing their second piece on* $(b+1, 1)$ *such that* $\mid a - b \mid = \frac{n}{2}$.

Proof. If Player 2 places their first piece on $(a, 1)$, Player 1 responds by placing their second piece on $(b+1, 1)$ for b such that $\mid a-b \mid = \frac{n}{2}$. Without loss of generality, we can assume Player 2 plays this piece at (0,1) and, as before, Player 1 places their next piece at $(\frac{n+2}{2}, 1)$.Then Player 2 will also have to make a forced move at the opposite vertex to $(\frac{n+2}{2}, 1)$ to prevent Player 1 from winning on their next turn: (1,1). Again, Player 1 places their last piece on $(\frac{n+2}{2}, 0)$, in which case Player 2 will again have to make a forced move at $(\frac{n+4}{2}, 0)$. At this point, Player 1 is, again, only 2 steps away from a winning configuration, while Player 2 is at least 3 steps away from a winning configuration (depending on the value of n). □

The notation we have chosen demonstrates why this optimal strategy does not work in 4-Picaria. For 4-Picaria, in the last step of placing the last piece on $(\frac{n+4}{2}, 0)$, we would go back to point $(0, 0)$, which is already occupied.

4. Strategies

We now turn our attention to the strategies that players who were unfamiliar with the winning strategies outlined above could employ and determine how successful they would be by implementing them in Python. We begin with the minimax algorithm and a variant, the multiminimax algorithm, before turning to random play modeled by Markov chains.

4.1. **Minimax.**

The *minimax* algorithm provides a method for finding the optimal move in a two-player strategy game given the immediate and delayed consequences of possible moves [7]. This algorithm is graph-based, implementing basic search trees and custom evaluation functions to tailor outcome values for accurate game representation. It considers both player's moves in the optimal manner, maximizing one player's outcome values given a board state while minimizing those of their opponent.

Mechanics. The minimax algorithm takes as input the current game state, a positive integer depth to determine optimal moves, and an evaluation function that transforms game states to real numbers representing winning, neutral, or losing moves. The output is a set of moves that the player could make from the current state, accompanied by a real number representing relative success should the player make a possible move.

The minimax algorithm first constructs a search tree by considering the states reachable from the current state, then the states reachable from those, and so on to the specified depth. Note that the same state may appear in the tree multiple times because it can be reached via different sequences of moves.

Once the tree has been constructed, the farthest states (the leaves of the search tree) are assigned a value using the evaluation function. In a method akin to backpropogation, each parent state except the current state is assigned a value equal to either the minimum or the maximum value of its children.

Suppose the minimax algorithm is constructed from the point of view of Player 1. The maximum is used if Player 1 is about to place a piece in the parent state, and the minimum is used if Player 2 is about to place a piece in the parent state. Note that if the current state is the root of the search tree and has depth 0, all states at odd depth are assigned values equal to the minimum of their children, and all states at even depth are assigned values equal to the maximum of their children.

Finally, the algorithm considers the values of the immediate children of the current state, returning all those that have the maximum value (in the event of ties).

Implementation. This implementation takes a Tapatan or Picaria board state and maps possible moves of both players for a given number of turns, displaying at each node the outcome of said move as 0 (neutral), 1 (winning), or -1 (losing). The search depth is defined as the number of turns taken from the original board state.

In the context of this project, the player uniformly randomly selects an optimal move found by the minimax algorithm.

We note that if the current player has a winning strategy, the minimax algorithm will always return the moves that allow the current player to force that win when run at a depth at least $2|S|-1$, where S is the sequence of moves that the current player could make that would force this win.

4.2. **Multiminimax.** The *multiminimax* algorithm takes as input the current game state, a positive integer depth to determine optimal moves, and an evaluation function that transforms game states to real numbers representing winning, neutral, or losing moves. multiminimax is an expansion of minimax, running the minimax algorithm multiple times at different depths, starting at depth 1 and terminating at target depth. After each minimax call, instead of selecting all moves with the maximum value, all moves that do not have the maximum value are removed. After all minimax calls have finished, the remaining moves are returned from multiminimax.

The following properties hold whenever multiminimax is allowed to run to at least a depth of the number of full board configurations plus 6 for both Tapatan and Picaria.

Property 4.1. *If a player is in a position where they are able to force a win and there are multiple paths which the player can take that have different lengths, then multiminimax will provide the shortest route to victory.*

This holds because multiminimax runs the minimax algorithm repeatedly at increasing depth. If the minimum depth at which a player can force a win is n, once multiminimax reaches depth n, it will remove the moves approaching the longer routes and only keep the shortest route to victory.

Property 4.2. *If a player is in a position where their opponent can force a win and there are multiple paths which the player can take that have different lengths, then multiminimax will provide the longest route to defeat.*

Multiminimax runs the minimax algorithm repeatedly at increasing depth. If there is a position in which a player's opponent can force a win and there are multiple paths with which to do so, then one win state occurs at depth n, while any others will have depth greater than n. Therefore, once multiminimax reaches depth n, it will remove the path approaching this win for the opponent in favor of a path n moves deep which is not yet a win for the opponent. This process will continue until the only path provided to the player is the one that stalls the opponent's win the most.

4.3. **Markov Chain Analysis.** Markov chains model scenarios with a fixed number of possible states, moving between in discrete steps such that for any two states σ and τ, the probability of going from σ to τ is constant no matter what happened previously. In this subsection, we mainly discuss the analyses using Markov chains on the Tapatan board under the random strategy.

We begin by recalling some basic terminology for Markov chains; for further information, please see [5]. A Markov chain has a finite set of states $S = \{s_1, s_2, ..., s_r\}$. The process begins with one of these states and moves in succession from one state to another. We write p_{ij} to indicate the probability of going from s_i to s_j in one step; this is called a *transition probability*.

We now turn our attention to absorbing Markov chains. A state is considered *absorbing* if once said state is entered, it is impossible to leave. A Markov chain is said to be *absorbing* if it has at least one absorbing state and it is possible to get to at least one absorbing state from any other state, and a nonabsorbing state in an absorbing Markov chain is called *transient*.

For an absorbing Markov chain, the entry n_{ij} of its fundamental matrix $\mathbf{N}$ gives the expected number of times that the process is in the transient state s_j if it is started in the transient state s_i.

Markov chains are often used to model gameplay in games of chance, see, for instance, [4, 8]. While neither Tapatan nor Picaria is such a game, we can still use Markov chains to model the use of memoryless strategies in abstract strategy games such as these. The states are the board configurations, and the winning board configurations are the absorbing states. Through the analyses with absorbing Markov chains, we can answer the following questions about random strategies for Tapatan and Picaria:

- How many times can we expect each board configuration to be obtained before a player wins?
- What is the expected number of moves before a player wins?
- What is the probability of a given player winning?

We use Markov chains to analyze random strategies: those in which both players arbitrarily move pieces without taking into account the position of any other pieces, including their own. Fill in with types of random strategies considered. -JF

Under the pure random strategy, players uniformly and randomly choose one of their possible moves each turn. On a 4-Tapatan board, there is a total of 744 unique board states: 63 absorbing states and 681 transient states. Explain how we get this number without code. -JF There can be up to 6 pieces on the board at once. Using Burnside's Lemma on a 3 by 3 board (with D4 symmetry), we determine that there is 1 unique board with 0 pieces, 3 boards with 1 piece, 12 boards with 2 pieces, 38 boards with 3 pieces, 108 boards with 4 pieces, 174 boards with 5 pieces, and 228 boards with 6 pieces. However, we manually count that there are 21 unique boards with 5 pieces are winning states and 42 unique boards with 6 pieces are winning states, hence there are 63 absorbing states. We also manually count out 3 double-win boards, since these states are impossible to reach. In total, there are 550 unique board states, 63 absorbing states and 487 transient states? -Alex

Here's some Scratch work. Not sure why the Burnside's lemma result is different from the results given by code?
There is 1 distinct board in the case of 1 piece.
For the case of 1 piece, the 3 distinct cases would be on the edge, corner and center.
In the case of 2 pieces, the only way to have a reflective board is when the two pieces are both on the reflection line. Using Burnside's lemma with D4 symmetry, there are a total of $\frac{9\times8+4\times3\times2}{8} = 12$ states.
In the case of 3 pieces, there are a total of 4 states that are fixed by 180 $^\circ$ rotation. As for reflection, the 3 pieces can either all be on the reflection axis, or there can be 1 distinct piece on the reflection axis and 2 identical pieces on the side. So there are a total of $4 \times (3 + 3 \times 3) = 48$ self-reflective states. Using Burnside's lemma, there are $\frac{\binom{9}{2}\times\binom{7}{1}+4+48}{8} = 38$ states.
In the case of 4 pieces, there are 12 states fixed by 180 $^\circ$ rotation, and 0 fixed by

90 ° or 270 ° rotations. In order for the board to be self-reflective, we can either have 2 of the same pieces on the reflection axis or one of each kind on each side of the reflection axis, so there are $4 \times (3 \times 3 \times 2 + 3 \times 2) = 96$ self-reflective states, leading to a total of $\frac{\binom{9}{2}\times\binom{7}{2}+12+96}{8} = 108$ distinct states.
Lastly, for the case of 5 pieces, there are again 12 states fixed by 180 ° rotation. For the self-reflective boards, we can either have 1 piece in the reflection axis or 3 pieces on the reflection axis, leading to a total of $4 \times (3 \times 3 \times 2 + 3 + 3 \times 3) = 120$ reflective boards. By Burnside's Lemma, there are a total of $\frac{\binom{9}{3}\times\binom{6}{2}+12+120}{8} = 174$ unique states. We calculated above that there are 228 unique states for 6 pieces. There are a total of 1 +3 +12 +38+108 +174 +228 = 564 states. Subtracting the 3 double-win states, there should be 561 distinct states.

5. Variations

We now turn to further variations on Tapatan and Picaria. These variations come in three basic forms: alterations of the rules, alterations of the pieces, and alterations of the boards. We address the first two types of variations here; the board alterations will be addressed in a future paper.

5.1. Rule alterations.

Our analyses of the standard versions of Tapatan and Picaria as well as their polygonal variations shows that Player 1 has a winning strategy except in one case and that the center node is a key component in any strategy. In this section, we propose two variants in an attempt to balance the game.

Dual-center Tapatan. To reduce the importance of the center node, we now allow it to hold two pieces from both players simultaneously and leave the rest of the rules unchanged. This is the standard 4-Tapatan game with the additional "house rule" that

> Both players can place one of their pieces at the central node simultaneously.

This removes the possibility of Player 1 blocking the center, a vital strategy in n-Tapatan for $n > 3$. We observe that in this variant, if a player has just positioned two of their pieces on opposite sides of the board, then they will win on their next move unless their opponent wins first.

We recall the following standard definition from Markov chain analysis to simplify our discussion of this variant.

Definition 5.1. A *closed communicating class* C is a set of states such that if $i \in C$ and $i \to j$ through an admissible transition, then $j \in C$.

Notice that in our context, the states represent board configurations and the admissible transitions are the legal moves for each player. In addition, since players can force moves on their opponents when playing optimally, we will refer to a communicating class as *closed by forced moves* if the players' legal moves remain within the elements of the class, either directly or by a sequence of forced moves. Furthermore, if all the possible outcomes of a set of board states eventually return to said set, then we will refer to it as a *semi-closed* communicating class. Notice that a game that enters into a board state that is part of a closed, closed by forced moves or semi-closed communicating class will end up in a tie. This is because the players will find themselves always coming back to the same board states. We can see that $Closed \Rightarrow\ Closed\ by\ forced\ moves \Rightarrow Semi-closed$.

Proposition 5.2. *If both players play optimally and begin with the center node, then the dual-center Tapatan game is a tie.*

We would like to show that this game will eventually fall into a communication class that is closed, closed by forced moves or semi-closed in which neither player can force a win. The proposed class is showed in Figure 9. Therefore, we first verify in Lemma 5.3 that it is, indeed, a communicating class closed by forced moves.

Lemma 5.3. *If the moves are restricted to optimal moves, then the five board configurations shown in Figure 9, up to symmetries and interchanged players, form a communicating class closed by forced moves.*

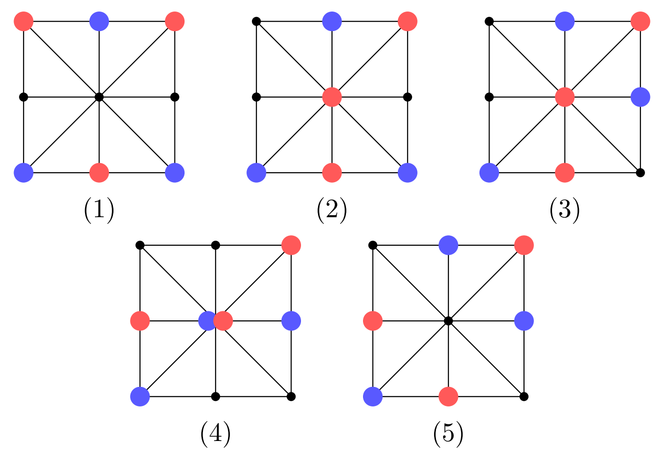

FIGURE 9. The communicating class closed by forced moves that both players will get into, given that both of their first moves were in the center node.

Proof. Using Figure 9 as a reference, we confirm that the admissible optimal moves for each board remain in the same class. Recall that Player 1 is red and Player 2 is blue.

Board (1): If it is Player 1's turn and they move any piece vertically, then Player 2 will be able to take two opposite nodes on the board. Hence, the only nonlosing move is to take the center node with either corner piece. The same applies if it is Player 2's turn and represents a transition (1) $\rightarrow$ (2) (mutatis mutandis).

Board (2): If it is Player 1's turn, the bottom piece cannot be moved at all, and moving their center piece horizontally or their top piece vertically would let Player 2 take over two opposite nodes. This leaves Player 1 with only one reasonable move, which is to move the center piece to the top right node; this represents the transition (2) $\rightarrow$ (1).

On the other hand, if it is Player 2's turn, then the middle top and the bottom left pieces cannot be moved without giving Player 1 a win. Therefore, they can only move the bottom right piece. If Player 2 moves it vertically we get the transition (2) $\rightarrow$ (3), while if they move it diagonally, it would be the transition (2) $\rightarrow$ (4).

Board (3): In this board, Player 2 cannot move their middle top and left bottom pieces without giving Player 1 a win. Moreover, moving their right middle piece horizontally or vertically will give us, respectively, the transitions (3) $\to$ (4) and (3) $\to$ (2).

Now, if it is Player 1's turn, then their top right piece does not have any moves at all. Moreover, moving their bottom piece horizontally or the center piece diagonally would allow Player 2 to take over two opposite nodes. Then, the remaining option is to move the center piece horizontally, which is the transition (3) $\to$ (5).

Board (4): As this board is symmetric, without loss of generality, we only consider Player 1. Notice that the upper right and middle left pieces cannot be moved without giving Player 2 a win. Hence, they can only move their center piece.

Moving the center piece to the bottom right node or top left node will end in a board configuration equivalent to (2), while moving it downwards leads to a board configuration equivalent to (3). Finally, if Player 1 moves to the top middle node, then we obtain the sequence of forced moves seen in Figure 10 that end in a board configuration equivalent to (2). In particular, notice that Player 1's move is forced, as it is the only way to pin Player 2's middle left piece.

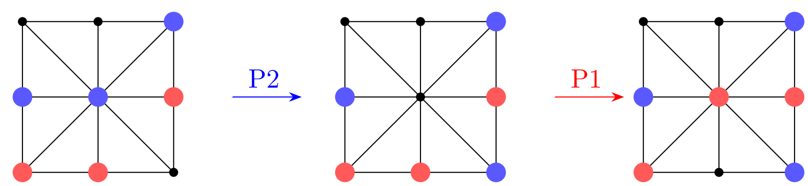

FIGURE 10. Forced moves for the indirect transition (4) $\to$ (2), given that Player 1 moved the center piece to the top middle node.

Board (5): Suppose it is Player 1's turn. Then moving any permitted piece to a corner will end in Player 2 taking over two opposite nodes in their next move. Thus, Player 1 can only move their pieces to the center, which is equivalent to (3). Since the board is symmetric, the same applies for Player 2's turn (mutatis mutandis). $\square$

Now we are ready to prove Proposition 5.2.

Proof. By hypothesis, Player 1 will place their first piece on the center. It is important to note that Player 2 must do the same; otherwise, Player 1 would force a win following the strategy for 4-Tapatan presented in Theorem 3.2. While Player 2 has an additional potential move into the center this case, the moves in the placement phase will still be forced, and Player 2 will still not be able to win before Player 1.

Up to symmetry, Player 1 has only two alternatives: placing on an edge or placing on a corner. In each case, Player 2's move will be forced. We exhaustively present the three different possibilities for gameplay in each of Figure 11 and Figure 12 by taking into account the legal moves available for Player 1 after Player 2's block. Notice that all of the outcomes fall into the communication class closed by forced moves seen in Figure 9 or are a win for Player 2.

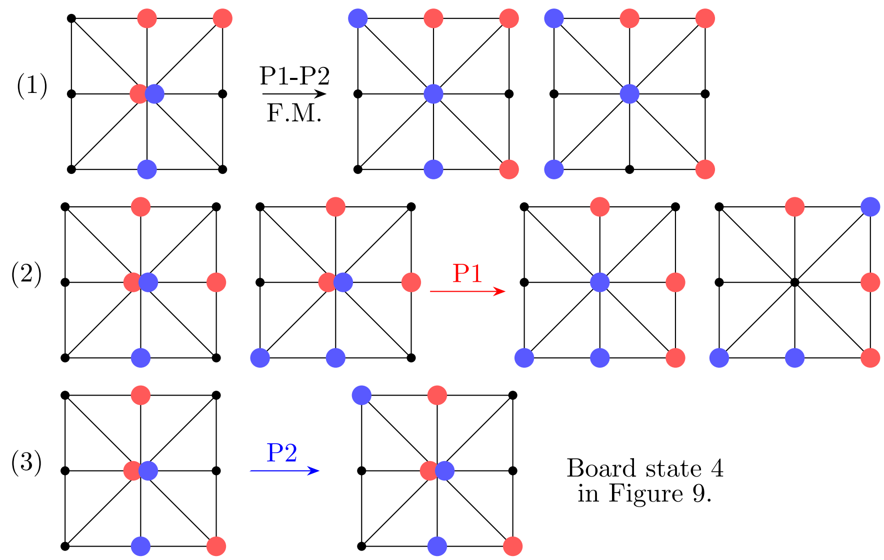

FIGURE 11. Dual-center Tapatan gameplay given that Player 1's second placement is on an edge.

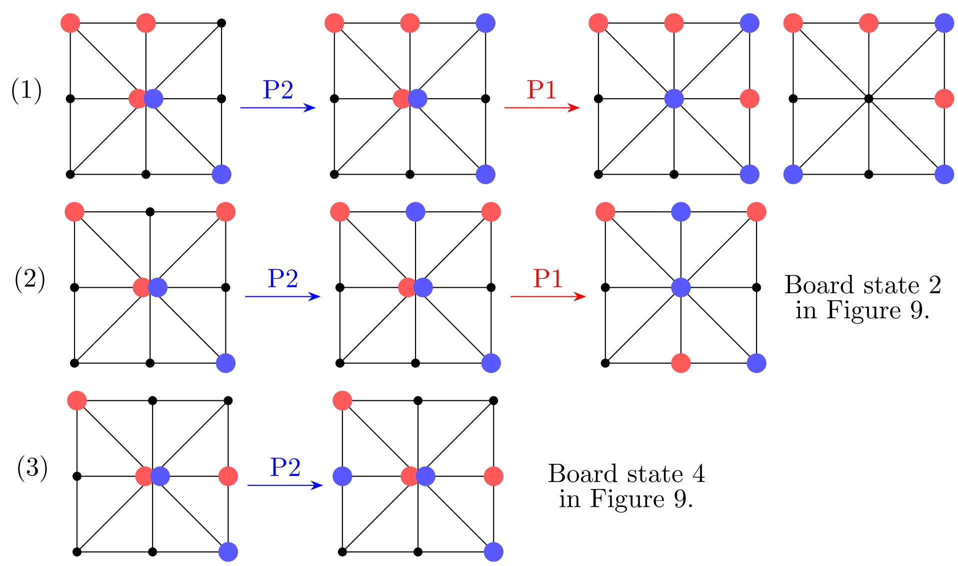

FIGURE 12. Dual-center Tapatan gameplay given that Player 1's third placement is on a corner.

Note that Player 1 can place their third piece so that the players will enter the semiclosed communicating class. □

Proposition 5.4. *If Player 1 places their first piece anywhere but the central node, then they have a winning strategy.*

Proof. If Player 1's first move is not in the center, then the occupied node has a single opposite node. Therefore, Player 2 must place their first piece on that opposite node to block Player 1's move since it is not possible for them to directly win. This will lead to the strategy presented in Figure 14 or that presented in Figure 13, depending on whether Player 1's first move is on an edge or a corner. In particular, if Player 1 starts on a corner, then all of Player 2's moves will be forced until their defeat.

Add arrow? -JF

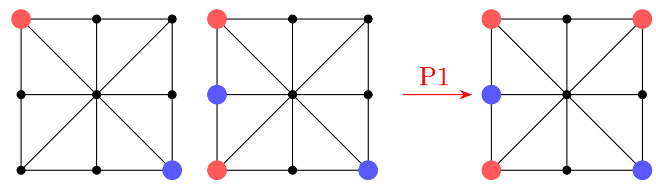

FIGURE 13. Player 1's winning strategy for the *dual-center Tapatan* variant by beginning on a corner.

Add P2's last forced move above: top center -JF

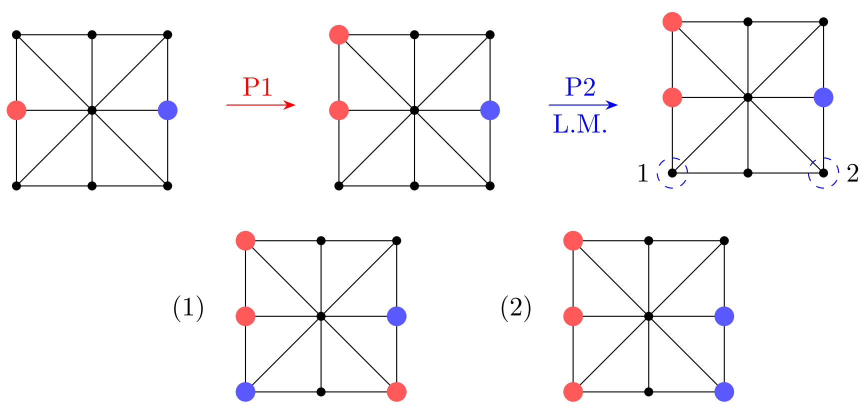

FIGURE 14. Player 1's winning strategy for the *dual-center Tapatan* variant by beginning on an edge.

□

It is interesting to observe that Player 1's optimal strategy is completely opposite to the strategy employed in the standard Tapatan game since, for dual-center Tapatan, the central node is the least relevant node. In particular, in this variant Player 1 can only guarantee their win by placing the first piece outside the center. Note that, even though this version deemphasizes the center, Player 1 still has a winning strategy. Moreover, Player 1 can ensure their win since the placement phase.

Avoid-the-center Tapatan: We now consider another variant on Tapatan that deemphasizes the center in a different way. In this case, we simply add the following rule:

Player 1 can't place their first piece on the center node.

This type of rule is found in other abstract strategy games. For instance, in mū tōrere, the first player is only allowed to choose from two of their four pieces for the first move [3].

Proposition 5.5. *If Player 1 places their first piece on an edge vertex, then Player 2 has a winning strategy for the avoid-the-center Tapatan variant.*

Proof. In this situation, Player 2 will place their first piece in the center. Then, there are only four possible options for Player 1's next move, up to symmetries. The winning strategy for each option is presented in Figure 15.

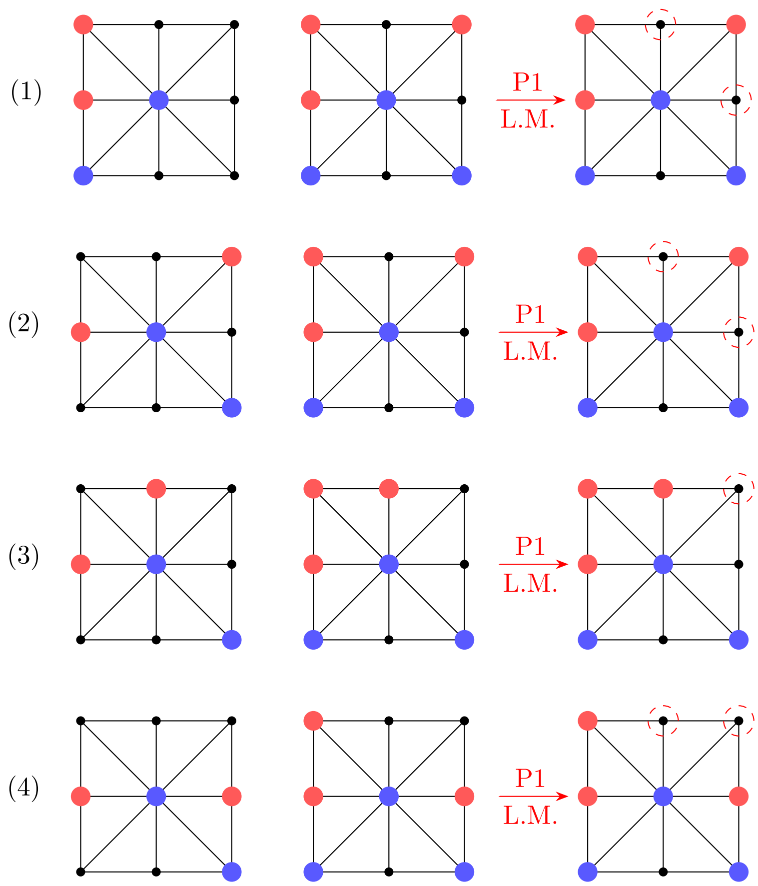

FIGURE 15. Player 2's winning strategy for avoid-the-center Tapatan, given that Player 1's first move is on an edge vertex.

□

If Player 1 starts in a corner instead, they will not have a winning strategy. Their best-case scenario is to force a tie, as seen in Proposition 5.7. Similarly to the *dual-center Tapatan* variant, to prove that the game ends up in a tie we show that Player 1 can force Player 2 into a board state that belongs to a semi-closed communicating class. Such a class is shown in Figure 16 and verified in Lemma 5.6.

Lemma 5.6. *Suppose that the players are only allowed to make optimal moves.*[2] *Then, the seven board configurations shown in Figure 16, up to symmetries and interchanged players, form a semi-closed communicating class.*

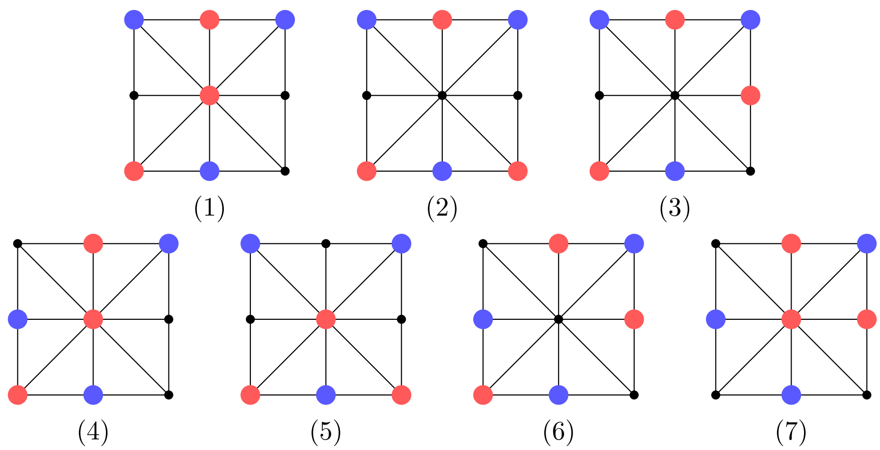

FIGURE 16. The semi-closed communicating class that both players will be forced into, given that Player 1's first move is in a corner.

Proof. Using Figure 16 as a reference, we proceed to confirm that the admissible optimal moves for each board remain in the same class. Recall that Player 1 is red and Player 2 is blue.

Board (1): If it is Player 1's turn, notice that the top piece does not have any legal moves available, while moving the center piece to the right diagonally or horizontally will result in a transition (1) → (2) or (1) → (3), respectively. Hence, the remaining options are to move the center piece to the left, or the bottom-left piece vertically. However, both scenarios will result in losing board configurations for Player 1. On the other hand, if it is Player 2's turn, we can see that that their top-right and bottom pieces are pinned. Therefore, Player 2's only option is to move the top-left piece vertically, which is the transition (1) → (4).

Board (2): Since the board is symmetric the outcomes are independent of whose turn it is. Then, without loss of generality, suppose that it is Player 1's turn. If Player 1 moves any of their bottom pieces either to the center or vertically, that will represent the transition (2) → (1) or (2) → (3), respectively. Moreover, moving their top piece to the center results in the transition (2) → (5).

Before verifying the remaining boards, we introduce in Figures 17-20 an auxiliary board configuration (AB1) and its corresponding outcomes. We do so since players will fall into it (or related boards) repeatedly. However, the board AB1 is not part of the class, as it is a losing board for Player 2 if it is Player 1's turn.

[2]We consider optimal gameplay in the sense that a player will not make a move that leads to board configurations where their opponent has a winning strategy in two combined moves ahead or less. One combined move consists of both players moving their pieces.

More specifically, Figure 17 shows the legal moves for Player 2 that arise from the proposed board state AB1. Notice that from board (1) Player 1 is forced into the board state (4) of the communicating class seen in Figure 16. On the other hand, from board (2), Player 1 has two legal moves that are equivalent under reflection.

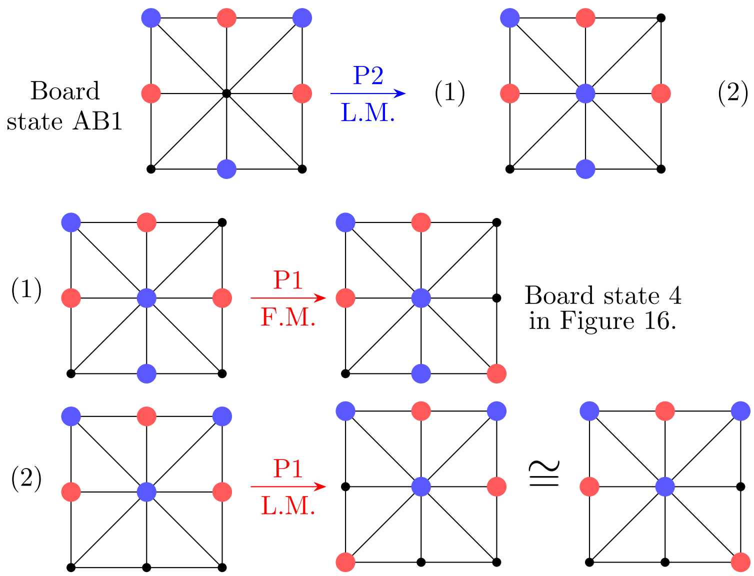

FIGURE 17. First set of board states: Generated by the proposed auxiliary board AB1.

Option (1) in Figure 17 already falls into the desired communicating class. Then, in Figure 18 we focus on the board configurations generated by the option (2). There we show that Player 2 has three possible legal moves. Nevertheless, options (2.2) and (2.3) fall into the desired communicating class.

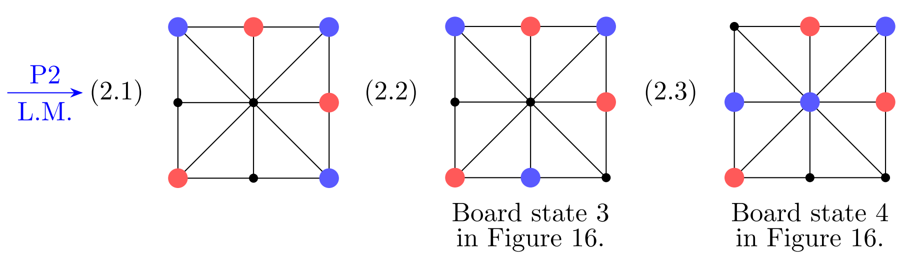

FIGURE 18. Second set of board states: Generated by Player 2's legal moves, starting from option (2) in Figure 17.

Consequently, Figure 19 continues with Player 1's legal moves starting off board (2.1) seen in Figure 18. Furthermore, in this figure we get that the board (2.1.1) forces Player 2 into a board state that belongs to the communicating class, while from board (2.1.2) they have two equivalent legal moves.

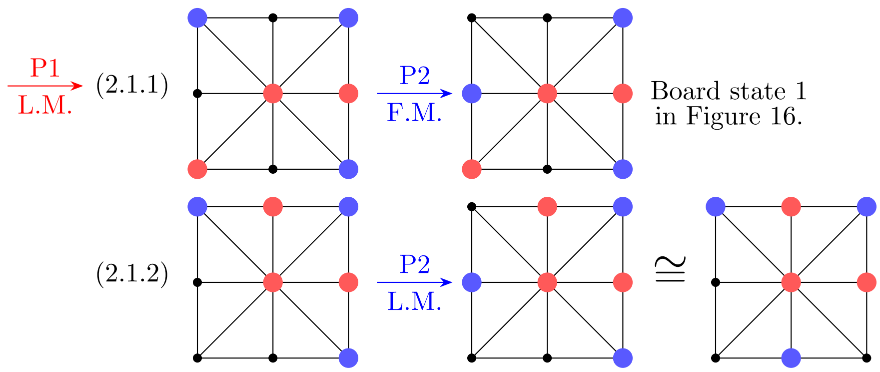

FIGURE 19. Third set of board states: Generated by Player 1's legal moves, starting from option (2.1) in Figure 18.

Finally, the only boards left are the ones generated by Player 1's legal moves following board (2.1.2) in Figure 19. However, Figure 20 shows that the board states of those legal moves either belong to the communicating class, or are equivalent to the auxiliary board AB1.

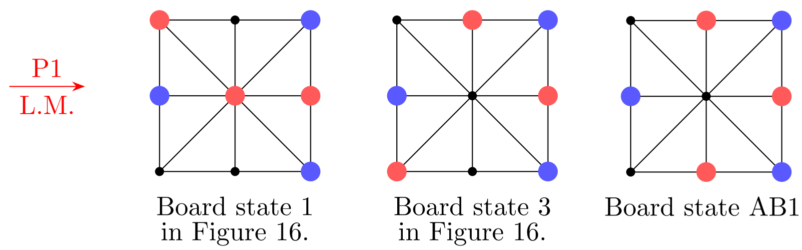

FIGURE 20. Fourth set of board states: Generated by Player 1's legal moves from option (2.1.2) in Figure 19.

Board (3): In this board Player 1 has four distinct possibilities. The first two include moving the right piece to the center or to the bottom-right corner. Nevertheless, these movements result in transitions $(3) \to (1)$ and $(3) \to (2)$, respectively. The next possibility includes moving their bottom-left piece vertically, but by doing this Player 1 falls into the board state AB1 seen in Figure 17. Finally, the last option is to move their top piece to the center. In this case, Player 2 has to block by moving their top-right piece vertically, obtaining an indirect transition $(3) \to (4)$.

Now, Player 2 has only three distinct possibilities, since moving their bottom piece to the center results in a losing board. If Player 2 moves their top-left piece to the center, it will be a transition of type $(3) \to (4)$. If they move their bottom piece to the right, they will fall into the board state (2.1) seen in Figure 18. Therefore, the only option left is to move the top-right piece to the center and force Player 2 to move their right piece downwards, which leads to the indirect transition $(3) \to (1)$.

Board (4): If it is Player 1's turn, the only moves that do not fall into a losing board are to move the center piece to the right or the bottom-right, resulting in the respective transitions $(4) \to (6)$ and $(4) \to (3)$. On the other hand, if it is Player

2's turn, they only have one possibility that avoids a losing board configuration, which is to move the left piece to the top. Thus we have the transition (4) $\rightarrow$ (1).

Board (5): For Player 1's turn, they can not move the center piece horizontally, as that results in a losing board configuration. Two legal moves are left, either moving the center piece or the bottom-left piece vertically. The former case leads to the transition (5) $\rightarrow$ (2), while the latter forces Player 2 to move their top-right piece down, obtaining an indirect transition (5) $\rightarrow$ (4). If it is Player 2's turn, they can only move their top pieces downwards, yet both cases are equivalent. They bring us to the board generated by the board state (2) in Figure 17.

Board (6): Since this board is symmetric, we can suppose that it is Player 1's turn without loss of generality. There are three distinct moves: the first is moving the top or right piece to the center, the second is to move the top piece to the left or the right piece downwards, and the third is moving the bottom-left corner piece to the center. Each distinct move will result, respectively, in the transition (6) $\rightarrow$ (1), (6) $\rightarrow$ (3) or (6) $\rightarrow$ (7).

Board (7): In Player 1's turn they have two possible moves, since moving their center piece to the top-left or bottom-right leads to losing board configurations. The first is to move the top piece to the left, which forces Player 2 to move their bottom piece to the right, resulting in the transition (7) $\rightarrow$ (1); the second is to move the center piece to the bottom-left corner and obtain the transition (7) $\rightarrow$ (6).

On the other hand, if it is Player 2's turn, they only have two different legal moves up to symmetry: to move their left piece up or down (bottom piece to the right or to the left). However, only in the former case Player 2 does not end up in a losing board configuration. In addition, this case is equivalent to the board generated by the board state (2.1.2) in Figure 19

□

Proposition 5.7. *If players are playing optimally and Player 1 starts with a corner vertex, then the avoid-the-center Tapatan variant will result in a tie.*

Proof. If Player 2 does not place their first piece at the center, then Player 1 can take it. In this case, Player 2 will not be able to force a victory. Hence, they rely on an unforced error by Player 1, which goes against the hypothesis of optimal gameplay. Furthermore, if Player 2 places their first piece at the center, then Player 1 can force a tie by the steps seen in Figure 21. Note that the outcome is a tie as Player 1 forces their way into the semi-closed communicating class showed in Lemma 5.6. In particular, for the last step, Player 2 must place their remaining piece on a corner, and the subsequent board is equivalent to the board state (1) seen in Figure 16.

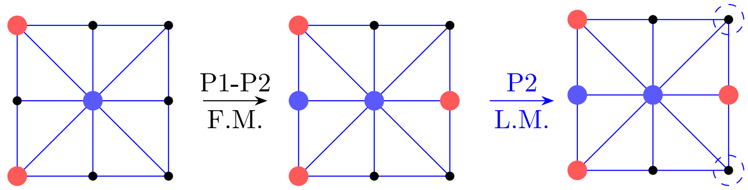

FIGURE 21. Player 1's tie strategy for the *avoid-the-center Tapatan* variant by beginning on a corner.

5.2. **Changes in the pieces.** We now turn our attention briefly to variants of Tapatan in which different pieces have different capabilities.

Weighted pieces. This variant of 4-Tapatan allows for larger pieces to be placed on top of smaller pieces. In this variant, each player is given 6 total pieces: 2 small, 2 medium, and 2 large. Permitted moves generally follow the same rules as in 4-Tapatan, but with the following additional restrictions.

- All six pieces need to be placed in the placement phase of the game.
- Pieces underneath other pieces cannot be moved, but if the top piece is moved, the piece that was immediately under that piece is back in play.
- A large piece cannot be placed on an empty node of the board.

One noticeable advantage to this variant is controlling the center of the board with a large piece. Since you cannot immediately place a large piece in the empty center node, it is logical for both players to avoid the center for as long as possible. Avoiding the center will delay giving your opponent the advantage of having the center of the board.

We observe that it is possible for both players to win the Weighted-Pieces variant at the same time.

Suppose all 6 pieces are placed on the Tapatan board such that Player 1 and Player 2 are each one move away from a winning position. It is possible for Player 1 to move one of their pieces into a winning position, but, in doing so, to uncover one of Player 2's pieces and thus leading to Player 2 winning as well.

6. Conclusion

References

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