Solutions tutorial 4, week 9

BUST10134

19/03/2020

Problem 1

Initialization. After setting $Z^* = -\infty$, we form the LP relaxation of this problem by *deleting* the set of constraints that x_j is an integer for $j = 1, 2, 3$. Applying the simplex method to this LP relaxation yields its optimal solution below.

$$\text{LP relaxation of whole problem: } (x_1, x_2, x_3, x_4) = \left(\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 0\right), \qquad \text{with } Z = 14\frac{1}{4}.$$

Because it has *feasible* solutions and this optimal solution has *noninteger* values for its integer-restricted variables, the whole problem is not fathomed, so the algorithm continues with the first full iteration below.

Iteration 1. In this optimal solution for the LP relaxation, the *first* integer-restricted variable that has a noninteger value is $x_1 = \frac{5}{4}$, so x_1 becomes the branching variable. Branching from the *All* node (*all* feasible solutions) with this branching variable then creates the following two subproblems:

Subproblem 1:
Original problem plus additional constraint

$$x_1 \leq 1.$$

Subproblem 2:
Original problem plus additional constraint

$$x_1 \geq 2.$$

Deleting the set of integer constraints again and solving the resulting LP relaxations of these two subproblems yield the following results.

Subproblem 1:

Optimal solution for LP relaxation: $(x_1, x_2, x_3, x_4) = \left(1, \frac{6}{5}, \frac{9}{5}, 0\right)$, with $Z = 14\frac{1}{5}$.

Bound: $Z \leq 14\frac{1}{5}$.

Subproblem 2:

LP relaxation: No feasible solutions.

This outcome for subproblem 2 means that it is fathomed by test 2. However, just as for the whole problem, subproblem 1 fails all fathoming tests.

Iteration 2. With only one remaining subproblem, corresponding to the $x_1 \leq 1$ node in Fig. 12.11, the next branching is from this node. Examining its LP relaxation's optimal solution given above, we see that this node reveals that the *branching variable* is x_2, because $x_2 = \frac{6}{5}$ is the first integer-restricted variable that has a noninteger value. Adding one of the constraints $x_2 \leq 1$ or $x_2 \geq 2$ then creates the following two new subproblems.

Subproblem 3:
Original problem plus additional constraints

$$x_1 \leq 1, \qquad x_2 \leq 1.$$

Subproblem 4:
Original problem plus additional constraints

$$x_1 \leq 1, \qquad x_2 \geq 2.$$

Solving their LP relaxations gives the following results.

Subproblem 3:

Optimal solution for LP relaxation: $(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 1, \frac{11}{6}, 0\right)$, with $Z = 14\frac{1}{6}$.

Bound: $Z \leq 14\frac{1}{6}$.

Subproblem 4:

Optimal solution for LP relaxation: $(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 2, \frac{11}{6}, 0\right)$, with $Z = 12\frac{1}{6}$.

Bound: $Z \leq 12\frac{1}{6}$.

Because both solutions exist (feasible solutions) and have noninteger values for integer-restricted variables, neither subproblem is fathomed. (Test 1 still is not operational, since $Z^* = -\infty$ until the first incumbent is found.)

Iteration 3. With two remaining subproblems (3 and 4) that were created simultaneously, the one with the larger bound (subproblem 3, with $14\frac{1}{6} > 12\frac{1}{6}$) is selected for the next branching. Because $x_1 = \frac{5}{6}$ has a noninteger value in the optimal solution for this subproblem's LP relaxation, x_1 becomes the branching variable. (Note that x_1 now is a *recurring* branching variable, since it also was chosen at iteration 1.) This leads to the following new subproblems.

Subproblem 5:
Original problem plus additional constraints

$$
\begin{aligned}
&x_1 \leq 1 \\
&x_2 \leq 1 \\
&x_1 \leq 0 \qquad (\text{so } x_1 = 0).
\end{aligned}
$$

Subproblem 6:
Original problem plus additional constraints

$$
\begin{aligned}
&x_1 \leq 1 \\
&x_2 \leq 1 \\
&x_1 \geq 1 \qquad (\text{so } x_1 = 1).
\end{aligned}
$$

The results from solving their LP relaxations are given below.

Subproblem 5:

Optimal solution for LP relaxation: $(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right)$, with $Z = 13\frac{1}{2}$.

Bound: $Z \leq 13\frac{1}{2}$.

Subproblem 6:

LP relaxation: No feasible solutions.

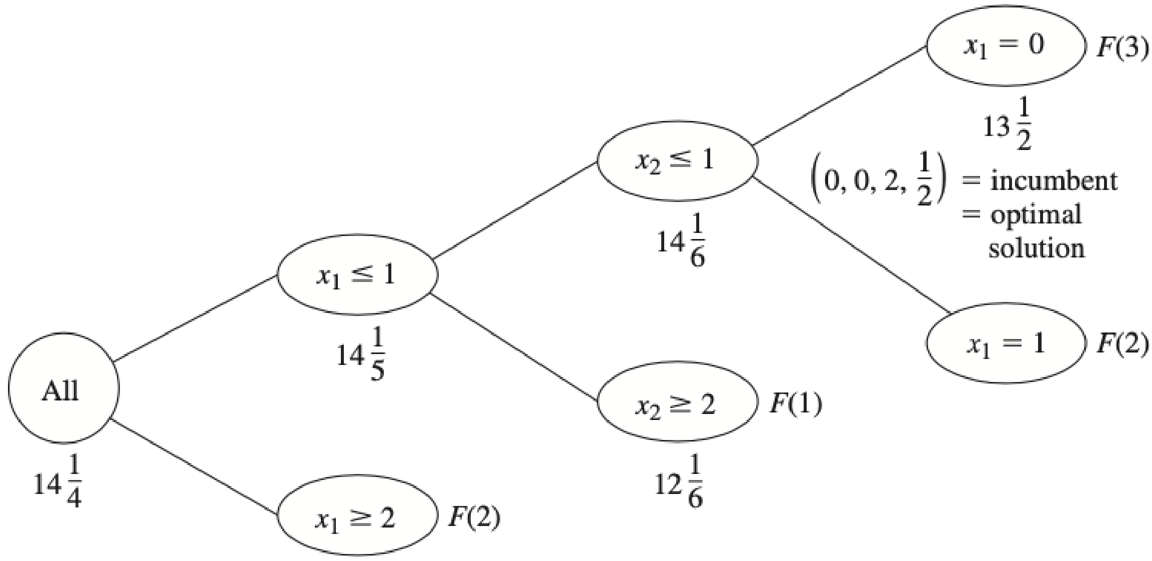
Subproblem 6 is immediately fathomed by test 2. However, note that subproblem 5 also can be fathomed. Test 3 passes because the optimal solution for its LP relaxation has integer values ($x_1 = 0$, $x_2 = 0$, $x_3 = 2$) for all three integer-restricted variables. (It does not matter that $x_4 = \frac{1}{2}$, since x_4 is not integer-restricted.) This *feasible* solution for the original problem becomes our first incumbent:

$$\text{Incumbent} = \left(0, 0, 2, \frac{1}{2}\right) \qquad \text{with } Z^* = 13\frac{1}{2}.$$

Using this Z^* to reapply fathoming test 1 to the only other subproblem (subproblem 4) is successful, because its bound $12\frac{1}{6} \leq Z^*$.

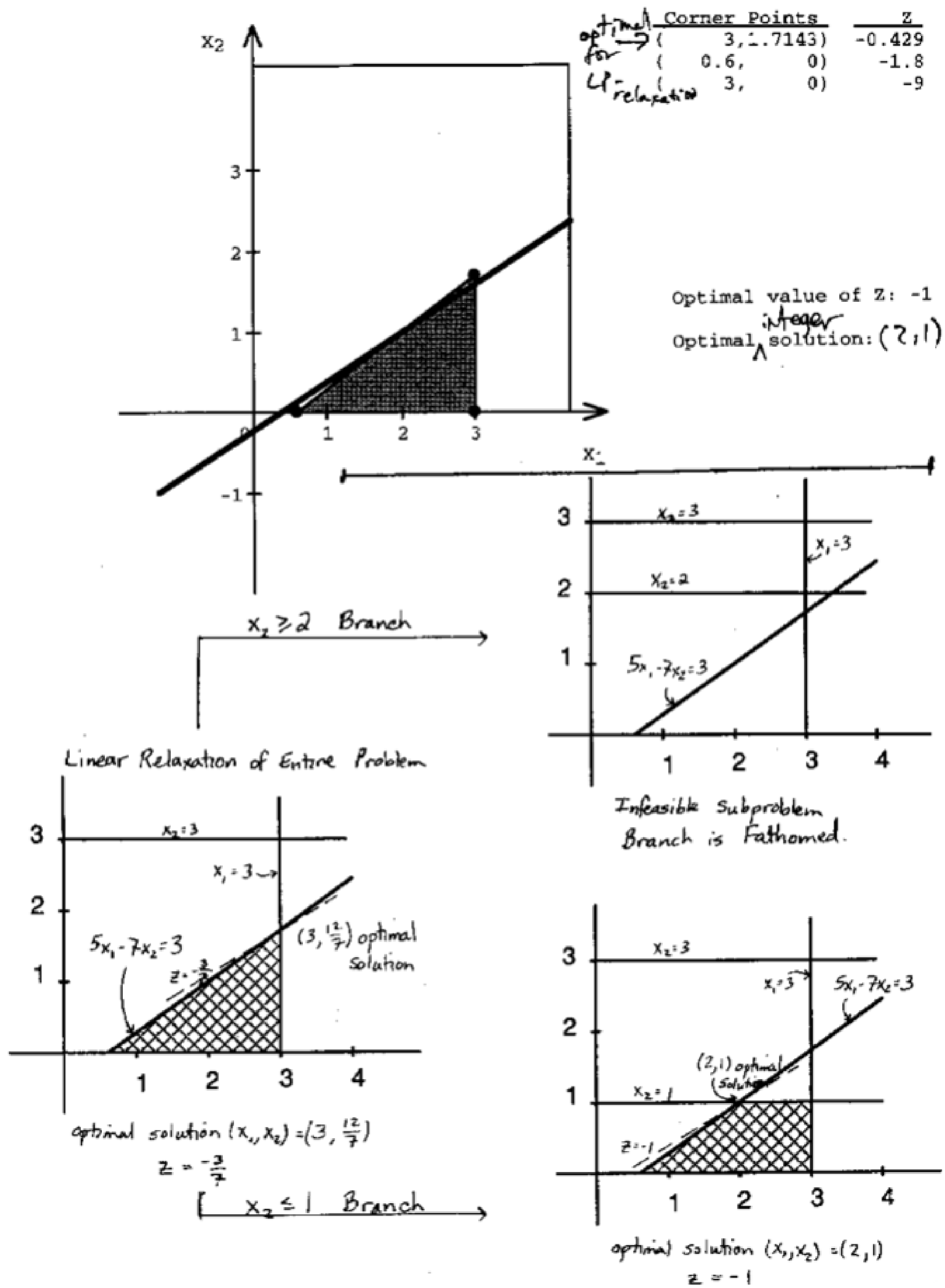
This iteration has succeeded in fathoming subproblems in all three possible ways. Furthermore, there now are no remaining subproblems, so the current incumbent is optimal.

$$\text{Optimal solution} = \left(0, 0, 2, \frac{1}{2}\right) \qquad \text{with } Z = 13\frac{1}{2}.$$



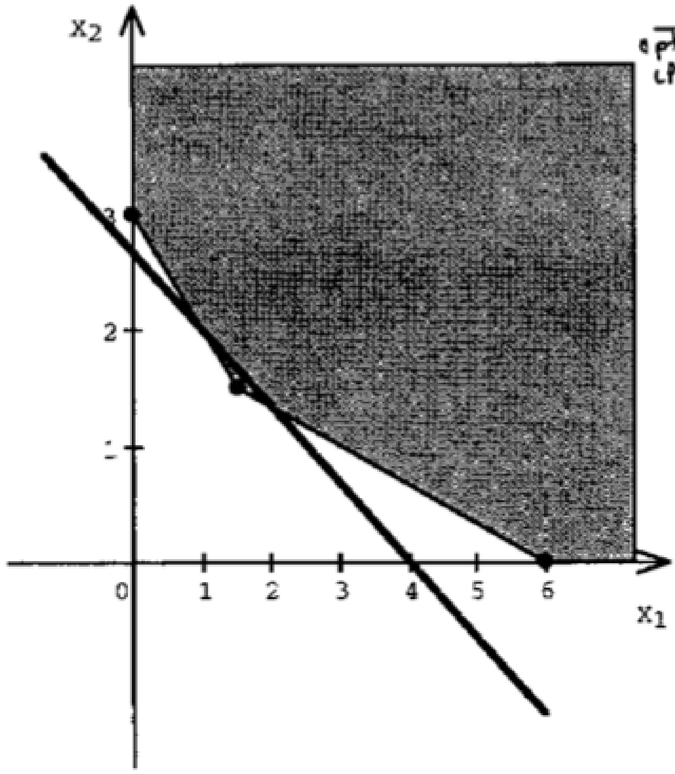
All
$14\frac{1}{4}$
$x_1 \leq 1$
$14\frac{1}{5}$
$x_1 \geq 2$
$F(2)$
$x_2 \leq 1$
$14\frac{1}{6}$
$x_2 \geq 2$
$F(1)$
$12\frac{1}{6}$
$x_1 = 0$
$F(3)$
$13\frac{1}{2}$
$\left(0, 0, 2, \frac{1}{2}\right)$ = incumbent
= optimal
solution
$x_1 = 1$
$F(2)$

Problem 2



optimal for LP-relaxation
Corner Points Z
(3, 1.7143) -0.429
(0.6, 0) -1.8
(3, 0) -9
Optimal value of Z: -1
Optimal integer solution: (2,1)
x_2
x_1
$x_2 \geq 2$ Branch
$x_2=3$
$x_1=3$
$x_2=2$
$5x_1-7x_2=3$
Infeasible Subproblem
Branch is Fathomed.
Linear Relaxation of Entire Problem
$(3, \frac{12}{7})$ optimal solution
$z=-\frac{3}{7}$
optimal solution $(x_1,x_2)=(3,\frac{12}{7})$
$z=-\frac{3}{7}$
$x_2 \leq 1$ Branch
$x_2=1$
(2,1) optimal Solution
$z=-1$
optimal solution $(x_1,x_2)=(2,1)$
$z=-1$

Problem 3

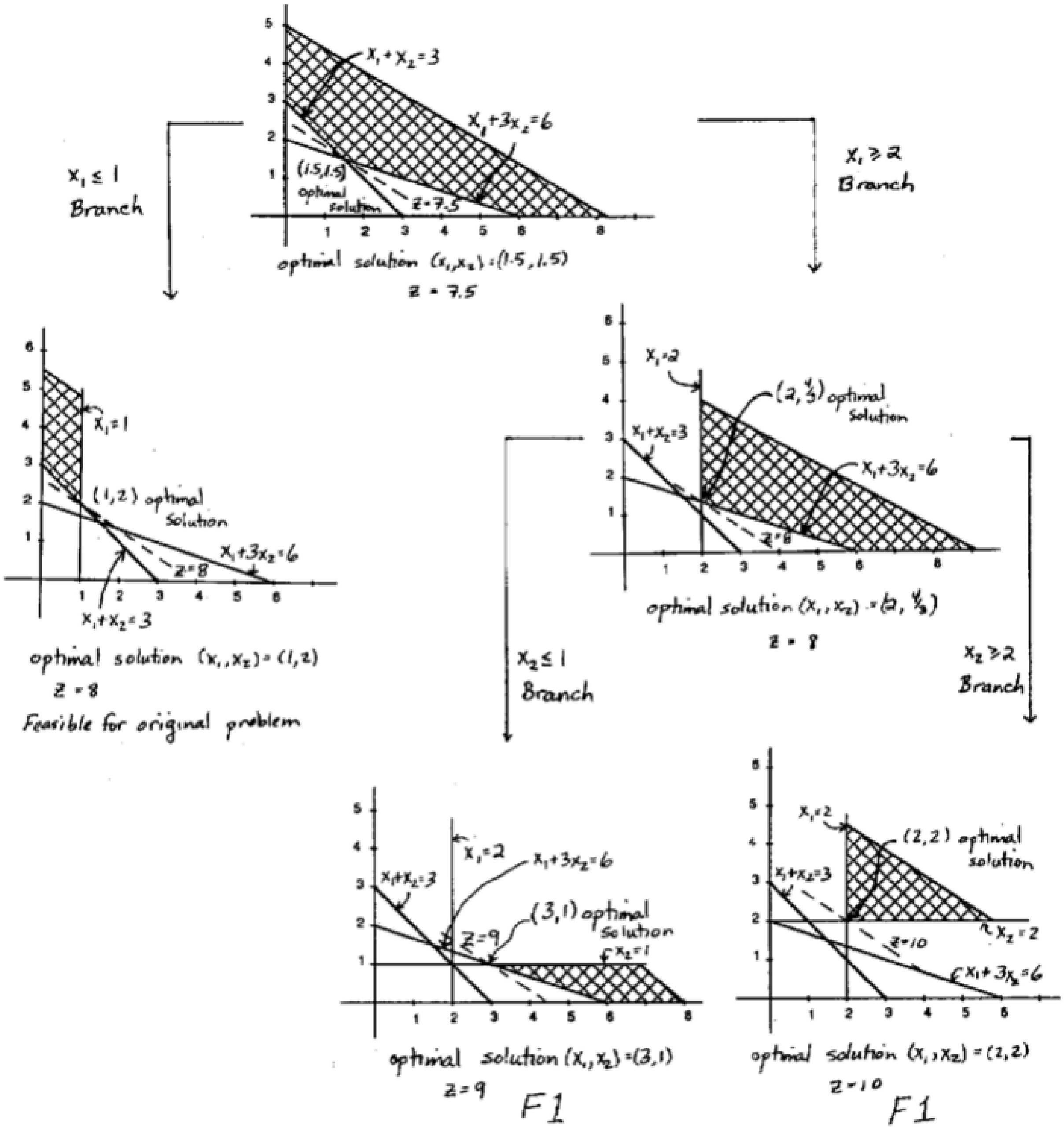


optimal LP-relax.

Corner Points		Z
(1.5,	1.5)	7.5
(0,	3)	9
(6,	0)	12

Optimal value of Z: 8

Optimal integer solution: (1,2)



The optimal solution to the original problem is:

$$(x_1, x_2) = (1, 2) \quad \text{with} \quad z = 8$$