1. Nash Equilibrium and Maximin

Definition 1.1 (Nash Equilibrium): A strategy profile (π_i, π_{-i}) forms a *Nash equilibrium* if none of the players benefit by deviating from their policy.

$$\forall i \in \mathcal{N}, \forall \pi'_i : R_i^{\pi_i, \pi_{-i}} \geq R_i^{\pi'_i, \pi_{-i}}$$

Lemma 1.1: Strategy profile (π_i, π_{-i}) forms a Nash equilibrium if and only if

$$\forall i \in \mathcal{N} : \pi_i \text{ is a best response to } \pi_{-i}$$

Where *best response* to π_{-i} is $\pi_i^\star = \arg\max_{\pi_i} R_i^{\pi_i, \pi_{-i}}$.

Definition 1.2 (Maximin Policy): Maximin policy of a player i is:

$$\arg\max_{\pi_i} \min_{\pi_{-i}} R_i^{\pi_i, \pi_{-i}}$$

Theorem 1.2 (Minimax):

$$\max_{\pi_i} \min_{\pi_{-i}} R_i^{\pi_i, \pi_{-i}} = \min_{\pi_{-i}} \max_{\pi_i} R_i^{\pi_i, \pi_{-i}}$$

We will denote $\underline{v_i} = \max_{\pi_i} \min_{\pi_{-i}} R_i^{\pi_i, \pi_{-i}} = -\max_{\pi_i} \min_{\pi_{-i}} R_{-i}^{\pi_i, \pi_{-i}} = \underline{v_{-i}}$.

Theorem 1.3 (Nash is Maximin): For a two player zero-sum game:

$$(\pi_i, \pi_{-i}) \text{ is a Nash equilibrium} \Rightarrow \pi_i \text{ is a maximin policy} \land \pi_{-i} \text{ is a maximin policy}$$

Proof: WLOG we will talk only about player i. First we can see that since π_{-i} is the best response from Lemma 1.1. Suppose that there is an another policy π'_i that does better than π_i in the worst case. That $\min_{\pi^\star_{-i}} R_i^{\pi'_i, \pi^\star_{-i}} > \min_{\pi^\star_{-i}} R_i^{\pi_i, \pi^\star_{-i}}$. So even:

$$R_i^{\pi'_i, \pi_{-i}} \geq \min_{\pi^\star_{-i}} R_i^{\pi'_i, \pi^\star_{-i}} > \min_{\pi^\star_{-i}} R_i^{\pi_i, \pi^\star_{-i}} \overset{\text{Lemma 1.1}}{=} R_i^{\pi_i, \pi_{-i}} \overset{\text{Definition 1.1}}{\geq} R_i^{\pi'_i, \pi_{-i}}$$

This is a contradiction. Which concludes the proof. □

Theorem 1.4 (Maximin is Nash): For a two player zero-sum game:

$$\pi_i^\star \text{ is a maximin policy} \land \pi_{-i}^\star \text{ is a maximin policy} \Rightarrow (\pi_i^\star, \pi_{-i}^\star) \text{ is a Nash equilibrium}$$

Proof: Suppose that (π_i, π_{-i}) is not Nash equilibrium. Then there is a policy π_i' such that $R_i^{\pi'_i, \pi^\star_{-i}} > R_i^{\pi^\star_i, \pi^\star_{-i}}$. And with that immediately follows:

$$\underline{v_i} \overset{\text{Theorem 1.2}}{=} -\underline{v_{-i}} \overset{\pi^\star_{-i} \text{ is maximin}}{\geq} -R_{-i}^{\pi'_i, \pi^\star_{-i}} \overset{\text{zero-sum game}}{=} R_i^{\pi'_i, \pi^\star_{-i}} > R_i^{\pi^\star_i, \pi^\star_{-i}} \geq \underline{v_i}$$

This is a contradiction. Which concludes the proof. □