Student: Phong Vo **Date:** 02/19/20	**Instructor:** Erica Yankowskas **Course:** Linear Algebra I (Spring 2020)	**Assignment:** Section 1.8 Homework

1. Let $A = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 5 \\ 20 \\ -25 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} c \\ b \\ k \end{bmatrix}$. Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

Select the correct answer below and fill in any answer boxes within your choice.
(Simplify your answers.)

○ **A.** $T(\mathbf{u}) = [\quad\quad\quad]$

◉ **B.** $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$

Select the correct answer below and fill in any answer boxes within your choice.
(Simplify your answers.)

◉ **A.** $T(\mathbf{v}) = \begin{bmatrix} \frac{c}{5} \\ \frac{b}{5} \\ \frac{k}{5} \end{bmatrix}$

○ **B.** $T(\mathbf{v}) = [\quad\quad]$

2. If T is defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique. Let $A = \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -2 \\ 2 & -9 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ -6 \\ -2 \end{bmatrix}$.

Find a single vector $\mathbf{x}$ whose image under T is $\mathbf{b}$.

$\mathbf{x} = \begin{bmatrix} -32 \\ -8 \\ -1 \end{bmatrix}$

Is the vector $\mathbf{x}$ found in the previous step unique?

○ **A.** Yes, because there is a free variable in the system of equations.
○ **B.** No, because there is a free variable in the system of equations.
○ **C.** No, because there are no free variables in the system of equations.
◉ **D.** Yes, because there are no free variables in the system of equations.

3. If T is defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique. Let $A = \begin{bmatrix} 1 & -6 & -16 \\ -3 & 10 & 24 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$.

Find a single vector $\mathbf{x}$ whose image under T is $\mathbf{b}$.

$\mathbf{x} = \begin{bmatrix} 2 - 2x_3 \\ 1 - 3x_3 \\ x_3 \end{bmatrix}$

Is the vector $\mathbf{x}$ found in the previous step unique?

○ **A.** Yes, because there are no free variables in the system of equations.
◉ **B.** No, because there is a free variable in the system of equations.
○ **C.** Yes, because there is a free variable in the system of equations.
○ **D.** No, because there are no free variables in the system of equations.

4. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^4$ into $\mathbb{R}^8$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

Choose the correct answer below.

○ **A.** The matrix A must have 4 rows and 4 columns.
◉ **B.** The matrix A must have 8 rows and 4 columns.
○ **C.** The matrix A must have 8 rows and 8 columns.
○ **D.** The matrix A must have 4 rows and 8 columns.

5. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix $\begin{bmatrix} 1 & -2 & 6 & -6 \\ 0 & 1 & -3 & 6 \\ 2 & -2 & 6 & -5 \end{bmatrix}$. Is **b** in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

Is **b** in the range of the linear transformation? Why or why not?

- **A.** No, **b** is not in the range of the linear transformation because the system represented by the augmented matrix [A **b**] is inconsistent.
- **B.** No, **b** is not in the range of the linear transformation because the system represented by the augmented matrix [A **b**] is consistent.
- **C.** Yes, **b** is in the range of the linear transformation because the system represented by the augmented matrix [A **b**] is inconsistent.
- **D.** Yes, **b** is in the range of the linear transformation because the system represented by the augmented matrix [A **b**] is consistent.

6. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Use the fact that T is linear to find the images under T of 4**u**, 2**v**, and 4**u** + 2**v**.

What is the image of 4**u**?

- **A.** $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$
- **B.** $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$
- **C.** $\begin{bmatrix} 16 \\ 4 \end{bmatrix}$
- **D.** $\begin{bmatrix} 4 \\ 16 \end{bmatrix}$

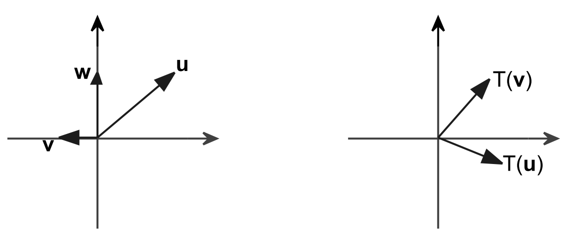
What is the image of 2**v**?

- **A.** $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$
- **B.** $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$
- **C.** $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$
- **D.** $\begin{bmatrix} 16 \\ 4 \end{bmatrix}$

What is the image of 4**u** + 2**v**?

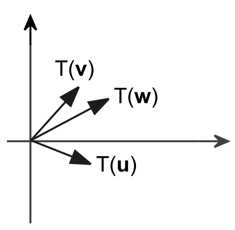
- **A.** $\begin{bmatrix} -12 \\ 14 \end{bmatrix}$
- **B.** $\begin{bmatrix} -14 \\ -12 \end{bmatrix}$
- **C.** $\begin{bmatrix} 14 \\ 12 \end{bmatrix}$
- **D.** $\begin{bmatrix} 12 \\ 14 \end{bmatrix}$

7. The figure shows vectors **u**, **v**, and **w**, along with the images T(**u**) and T(**v**) under the action of a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$. Copy this figure carefully, and draw the image T(**w**) as accurately as possible.

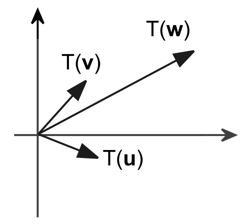


Which figure displays the correct image of T(**w**)?

A.



B.



8. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, and let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Which is the correct image of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$?

A. $\begin{bmatrix} 11 \\ -21 \end{bmatrix}$

B. $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$

C. $\begin{bmatrix} 11 \\ 21 \end{bmatrix}$

D. $\begin{bmatrix} 21 \\ 11 \end{bmatrix}$

Which is the correct image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$?

A. $\begin{bmatrix} 3x_1 - 2x_2 \\ 7x_1 + 8x_2 \end{bmatrix}$

B. $\begin{bmatrix} 3x_1 + 2x_2 \\ 7x_1 - 8x_2 \end{bmatrix}$

C. $\begin{bmatrix} 7x_1 - 8x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$

D. $\begin{bmatrix} 7x_1 - 2x_2 \\ 3x_1 + 8x_2 \end{bmatrix}$

9. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, and let T : $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that T($\mathbf{x}$) is A$\mathbf{x}$ for each $\mathbf{x}$.

A = $\begin{bmatrix} 7 & 5 \\ -3 & 8 \end{bmatrix}$

10. Determine whether each statement below is true or false. Justify each answer.

a. A linear transformation is a special type of function.

A. False. A linear transformation is not a function because it maps more than one vector $\mathbf{x}$ to the same vector T($\mathbf{x}$).
B. False. A linear transformation is not a function because it maps one vector $\mathbf{x}$ to more than one vector T($\mathbf{x}$).
C. True. A linear transformation is a function from $\mathbb{R}$ to $\mathbb{R}$ that assigns to each vector $\mathbf{x}$ in $\mathbb{R}$ a vector T($\mathbf{x}$) in $\mathbb{R}$.
D. True. A linear transformation is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector T($\mathbf{x}$) in $\mathbb{R}^m$.

b. If A is a 3×5 matrix and T is a transformation defined by T($\mathbf{x}$) = A$\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

A. False. The domain is actually $\mathbb{R}^5$, because in the product A$\mathbf{x}$, if A is an $m \times n$ matrix then $\mathbf{x}$ must be a vector in $\mathbb{R}^n$.
B. True. The domain is $\mathbb{R}^3$ because A has 3 columns, because in the product A$\mathbf{x}$, if A is an $m \times n$ matrix then $\mathbf{x}$ must be a vector in $\mathbb{R}^m$.
C. False. The domain is actually $\mathbb{R}$, because in the product A$\mathbf{x}$, if A is an $m \times n$ matrix then $\mathbf{x}$ must be a vector in $\mathbb{R}$.
D. True. The domain is $\mathbb{R}^3$ because A has 3 rows, because in the product A$\mathbf{x}$, if A is an $m \times n$ matrix then $\mathbf{x}$ must be a vector in $\mathbb{R}^m$.

c. If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

A. True. The range of the transformation is $\mathbb{R}^m$, because each vector in $\mathbb{R}^m$ is a linear combination of the rows of A.
B. True. The range of the transformation is $\mathbb{R}^m$, because each vector in $\mathbb{R}^m$ is a linear combination of the columns of A.
C. False. The range of the transformation is the set of all linear combinations of the columns of A, because each image of the transformation is of the form A$\mathbf{x}$.
D. False. The range of the transformation is $\mathbb{R}^n$ because the domain of the transformation is $\mathbb{R}^m$.

d. Every linear transformation is a matrix transformation.

A. False. A matrix transformation not a linear transformation because multiplication of a matrix A by a vector $\mathbf{x}$ is not linear.
B. False. A matrix transformation is a special linear transformation of the form $\mathbf{x} \mapsto A\mathbf{x}$ where A is a matrix.
C. True. Every linear transformation T($\mathbf{x}$) can be expressed as a multiplication of a matrix A by a vector $\mathbf{x}$ such as A$\mathbf{x}$.
D. True. Every linear transformation T($\mathbf{x}$) can be expressed as a multiplication of a vector A by a matrix $\mathbf{x}$ such as A$\mathbf{x}$.

e. A transformation T is linear if and only if $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain of T and for all scalars c_1 and c_2.

A. False. A transformation T is linear if and only if T($\mathbf{0}$) = $\mathbf{0}$.
B. False. A transformation T is linear if and only if T(c$\mathbf{u}$) = cT($\mathbf{u}$) for all scalars c and all $\mathbf{u}$ in the domain of T.
C. False. A transformation T is linear if and only if T($\mathbf{u}$ + $\mathbf{v}$) = T($\mathbf{u}$) + T($\mathbf{v}$) for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T.
D. True. This equation correctly summarizes the properties necessary for a transformation to be linear.

11. Mark each statement True or False. Justify each answer. Complete parts a through e.

a. The range of the transformation **x** Ax is the set of all linear combinations of the columns of A.

- **A.** False; each image T(**x**) is of the form A**x**. Thus, the range is not the set of all linear combinations of the columns of A.
- **B.** True; each image T(**x**) is not of the form A**x**. Thus, the range is not the set of all linear combinations of the columns of A.
- **C.** False; each image T(**x**) is not of the form A**x**. Thus, the range is not the set of all linear combinations of the columns of A.
- **D.** True; each image T(**x**) is of the form A**x**. Thus, the range is the set of all linear combinations of the columns of A.

b. Every matrix transformation is a linear transformation.

- **A.** True; every matrix transformation has the property T(**u** + **v**) = T(**u**) + T(**v**), but not all matrix transformations have the property T(c**u**) = cT(**u**) for all **u** and **v**, in the domain of
- **B.** True; every matrix transformation has the properties T(**u** + **v**) = T(**u**) + T(**v**) and T(c**u**) = cT(**u**) for all **u** and **v**, in the domain of T and all scalars c.
- **C.** False; not every matrix transformation has the properties T(**u** + **v**) = T(**u**) + T(**v**) and T(c**u**) = cT(**u**) for all **u** and **v**, in the domain of T and all scalars c.
- **D.** False; every matrix transformation has the properties T(**u** + **v**) = T(**u**) + T(**v**) and T(c**u**) = cT(**u**) for all **u** and **v**, in the domain of T and all scalars c.

c. If T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if **c** is in $\mathbb{R}^m$, then a uniqueness question is "Is **c** in the range of T?"

- **A.** True; the question "is **c** in the range of T?" is the same as"does there exist an **x** whose image is **c**?" This is a uniqueness question.
- **B.** False; the question "is **c** in the range of T?" is the same as "is **c** the image of a unique **x** in $\mathbb{R}^n$?" This is an existence question.
- **C.** True; the question "is **c** in the range of T?" is the same as "is **c** the image of a unique **x** in $\mathbb{R}^n$?" This is a uniqueness question.
- **D.** False; the question "is **c** in the range of T?" is the same as "does there exist an **x** whose image is **c**?" This is an existence question.

d. A linear transformation preserves the operations of vector addition and scalar multiplication.

- **A.** False; The linear transformation T(c**u** + d**v**) is not the same as cT(**u**) + dT(**v**) in $\mathbb{R}^m$. Therefore, vector addition and scalar multiplication are not preserved.
- **B.** False; The linear transformation T(c**u** + d**v**) is the same as cT(**u**) + dT(**v**) in $\mathbb{R}^m$. Therefore, vector addition and scalar multiplication are not preserved.
- **C.** True; The linear transformation T(c**u** + d**v**) is the same as cT(**u**) + dT(**v**) in $\mathbb{R}^m$. Therefore, vector addition and scalar multiplication are preserved.
- **D.** True; The linear transformation T(c**u** + d**v**) is not the same as cT(**u**) + dT(**v**) in $\mathbb{R}^m$. Therefore, vector addition and scalar multiplication are preserved.

e. A linear transformation T: $\mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$.

- **A.** True; for a linear transformation, T(**0**) is equal to **0**.
- **B.** False; for a linear transformation, T(**0**) is equal to **0**.
- **C.** False; for a linear transformation, T(**0**) does not equal **0**.
- **D.** True; for a linear transformation, T(**0**) does not equal **0**.

12. Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} be a linearly dependent set in $\mathbb{R}^n$. Explain why the set {T($\mathbf{v}_1$), T($\mathbf{v}_2$), T($\mathbf{v}_3$)} is linearly dependent.

Choose the correct answer below.

- **A.** Given that the set {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is linearly dependent, there exist c_1, c_2, c_3, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. It follows that $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) \neq \mathbf{0}$. Therefore
- **B.** Given that the set {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is linearly dependent, there exist c_1, c_2, c_3, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \neq \mathbf{0}$. It follows that $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) \neq \mathbf{0}$. Therefore
- **C.** Given that the set {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is linearly dependent, there exist c_1, c_2, c_3, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. It follows that $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$. Therefore
- **D.** Given that the set {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is linearly dependent, there exist c_1, c_2, c_3, all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. It follows that $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$. Therefore, th