Section 2.2: The Inverse of a Matrix

Note: Here we explore the matrix equivalent of the "reciprocal" (or multiplicative inverse :)) of a nonzero #.

⇒ Recall: $2(2^{-1}) = 1$ &/or $(2^{-1})2 = 1$

Invertible Matrices

An $n \times n$ matrix 'A' is said to be **invertible** if $\exists$ an $n \times n$ matrix 'C' ST: $CA = I$ & $AC = I$

where: $I = I_n \rightarrow$ The $n \times n$ Identity Matrix.

* Here, C is called an "Inverse of A"

- Denoted: A^{-1} ST $\begin{cases} * A(A^{-1}) = I \\ * (A^{-1})A = I \end{cases}$

Notes:

- An invertible matrix is called a: Nonsingular Matrix
- A matrix that is NOT invertible is called a Singular Matrix.

Example: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ & $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

Compute the Following:

(a) AC & (b) CA

Answer:

Part (a): Compute AC

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2(-7)+5(3) & 2(-5)+5(2) \\ -3(-7)-7(3) & -3(-5)-7(2) \end{bmatrix}$$

$$= \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Part (b) Compute CA:

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -7(2)-5(-3) & -7(5)-5(-7) \\ 3(2)+2(-3) & 3(5)+2(-7) \end{bmatrix}$$

$$= \begin{bmatrix} -14+15 & -35+35 \\ 6-6 & 15-14 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Note: Since $AC = I_2 = CA$, C is the inverse of A

$\Rightarrow C = A^{-1}$:)

Note: The following theorem provides us with an easy way to find the inverse of a 2×2 matrix, as well as, a test to determine if an inverse exists :)

***Theorem:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(i) If $(ad - bc) \neq 0$, then A is invertible &

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* **Note:** $(ad-bc)$ is called the '<u>Determinant of A</u>'

$\Rightarrow$ **Denoted:** $\det(A) = ad - bc$

(ii) If $(ad-bc) = 0$, then A is NOT invertible.

Proof for Part (i): Let $(ad-bc) = \det(A)$ & $\det(A) \neq 0$

* **Goal:** Show that A is invertible & formula for A^{-1} holds true :)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* Compute $\frac{1}{\det(A)}(AC)$:

$$AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & \overset{0}{\cancel{-ab + ba}} \\ \underset{0}{\cancel{cd - cd}} & -bc + ad \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

Pf. For Part (i) Continued...

$$\frac{1}{\det(A)}(AC) = \frac{1}{(ad-bc)}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} \frac{(ad-bc)}{(ad-bc)} & 0 \\ 0 & \frac{(ad-bc)}{(ad-bc)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

* Compute $\frac{1}{\det(A)}(CA)$:

- $CA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & \cancel{db-bd}^{0} \\ \cancel{-ac+ac}_{0} & -cb+ad \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

- $\frac{1}{\det(A)}(CA) = \frac{1}{(ad-bc)}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$

∴ Therefore:

Since

$$\frac{1}{(ad-bc)}(AC) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{(ad-bc)}(CA),$$

A is invertible & $A^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}(C)$

holds true $\boxtimes$

Example: Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

Answer:

* **Given:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

* **Want:** $A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = ?$

- **Find the det(A):**

$$\det(A) = (ad-bc) = 3(6) - 4(5) = 18 - 20 = -2$$

$$\boxed{\therefore \det(A) = -2}$$

- **By scalar-multiplication:**

$$\frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -6/2 & +4/2 \\ +5/2 & -3/2 \end{bmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}}$$

Answer:

To Check:

- $AA^{-1} = I_2$?
- $A^{-1}A = I$?

Example: Find the inverse of the matrix : $A = \begin{bmatrix} 9 & 6 \\ 8 & 2 \end{bmatrix}$

Answer:

*Recall: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then:

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

• Given: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 8 & 2 \end{bmatrix}$

*Check the determinant:

$$\det(A) = ad - bc = (9)(2) - (6)(8) = 18 - 48 = -30 \checkmark$$

*Find the Inverse of A:

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{30} \begin{bmatrix} 2 & -6 \\ -8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{30} & +\frac{6}{30} \\ +\frac{8}{30} & -\frac{9}{30} \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{15} & \frac{1}{5} \\ \frac{4}{15} & -\frac{3}{10} \end{bmatrix} = A^{-1}}$$

Answer.

Example: Find the inverse of the matrix: $A = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}$

Answer:

*Recall: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*Given:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}$$

*Check the determinant:

$$ad - bc = (7)(-7) - (12)(-4) = -49 + 48 = -1 \neq 0 \checkmark$$

*Find the inverse of A:

$$A^{-1} = \frac{1}{(-1)} \begin{bmatrix} -7 & -12 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}}$$

Answer ✓

Example: Find the inverse of the matrix, if it exists:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}$$

Answer:

***Recall:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $(ad-bc) \neq 0$, then

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

***Given:**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}$$

***Check the determinant.**

$$\det(A) = ad - bc = 28 - 6 = 22 \neq 0 \checkmark$$

***Find the inverse:**

$$A^{-1} = \frac{1}{22} \begin{bmatrix} 7 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 7/22 & -1/11 \\ -3/22 & 2/11 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 7/22 & -1/11 \\ -3/22 & 2/11 \end{bmatrix}}$$

Answer.

Example: Find the inverse of the matrix, if it $\exists$:

$$A = \begin{bmatrix} 1 & -3 \\ 6 & -9 \end{bmatrix}$$

Answer:

* **Recall:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $(ad-bc) \neq 0$, then

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* **Given:**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 6 & -9 \end{bmatrix}$$

* **Check the Determinant:**

$$\det(A) = (ad-bc) = -9+18 = 9 \neq 0 \checkmark$$

* **Find the Inverse:**

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -9 & +3 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1/3 \\ -2/3 & 1/9 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} -1 & 1/3 \\ -2/3 & 1/9 \end{bmatrix}}$$ Answer ✓

Note: To prove the 2nd part of theorem 1, we must first observe the following statement ∴

* Theorem 2:

If A is an invertible $n \times n$ matrix, then $\forall$ $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Proof:

Let 'A' be an invertible, $n \times n$ matrix.

Then by definition:

$\exists$ an $n \times n$ matrix A^{-1} ST: $AA^{-1} = I_n = A^{-1}A$

where I_n = Identity Matrix in $\mathbb{R}^n$.

Let $\vec{b}$ be any arbitrary vector in $\mathbb{R}^n$, $\vec{b} \in \mathbb{R}^n$ (Show $\vec{x} = A^{-1}\vec{b}$ $\exists$)

Substitute $\vec{x} = A^{-1}\vec{b}$ into the matrix eq (to verify $A\vec{x} = \vec{b}$):

$$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n\vec{b} = \vec{b}$$

∴ $\boxed{\text{The nonhomogeneous eq. } A\vec{x} = \vec{b} \text{ has the solution } \vec{x} = A^{-1}\vec{b} \ \checkmark}$

Now show $\vec{x}$ is unique

\$ $\vec{x} = \vec{u}$ for some $\vec{u} \in \mathbb{R}^n$ ST $A\vec{u} = \vec{b}$. (Show $\vec{u} = \vec{x} = A^{-1}\vec{b}$)

Multiply both sides of the eq. by "A^{-1}":

$$A\vec{u} = \vec{b} \longrightarrow A^{-1}(A\vec{u}) = A^{-1}\vec{b} \longrightarrow (A^{-1}A)\vec{u} = A^{-1}\vec{b}$$

$I_n\vec{u} = A^{-1}\vec{b} \longrightarrow$ ∴ $\boxed{\vec{u} = A^{-1}\vec{b} = \vec{x}. \text{ So } \vec{x} \text{ is unique}}$ (X)

Note: While solving the nonhomogeneous equation $A\vec{x}=\vec{b}$ by row-reduction is often the fastest method, solving by the property seen in theorem 2 is easier when we know that matrix A is invertible ∵

Example: Use the inverse of matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ (as seen in a previous ex.) to solve the following system:
$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases}$$

Answer:

*Convert the given system to $A\vec{x}=\vec{b}$:

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases} \iff \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Recall: In a previous example, we found the inverse of

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

*Since A is a 2×2 invertible matrix, then $A\vec{x}=\vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$:

Ans

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = 3\begin{bmatrix} -3 \\ 5/2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} -9+14 \\ \frac{15}{2} - \frac{21}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -3 \end{bmatrix}}$$

Example: Use the given inverse of the coefficient matrix to solve the following system:

$$\begin{cases} 7x_1 + 3x_2 = 6 \\ -6x_1 - 3x_2 = 2 \end{cases} ; \quad A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}$$

Answer:

* **Recall:** IF A is an $n \times n$ invertible matrix, then $\forall \vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$

* Rewrite the Given System as the equivalent nonhomogeneous equation, $A\vec{x} = \vec{b}$:

$$\begin{cases} 7x_1 + 3x_2 = 6 \\ -6x_1 - 3x_2 = 2 \end{cases} \Longleftrightarrow \underbrace{\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ 2 \end{bmatrix}}_{\vec{b}}$$

* Since matrix A is invertible, then:

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -7/3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -12 \end{bmatrix} + \begin{bmatrix} 2 \\ -14/3 \end{bmatrix} = \begin{bmatrix} 6 + 2 \\ \frac{-36 - 14}{3} \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ -50/3 \end{bmatrix} \Longleftrightarrow \begin{cases} x_1 = 8 \\ x_2 = -50/3 \end{cases}}$$

Answer ✓

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$,

$\vec{b}_3 = \begin{bmatrix} 4 \\ 18 \end{bmatrix}$ & $\vec{b}_4 = \begin{bmatrix} 6 \\ 22 \end{bmatrix}$. Find the Following:

(a) <u>Find A^{-1} & use it to solve the four equations:</u>
$A\vec{x} = \vec{b}_1$, $A\vec{x} = \vec{b}_2$, $A\vec{x} = \vec{b}_3$, & $A\vec{x} = \vec{b}_4$.

(b) The 4 equations in part (a) can be solved by the same set of operations, since the coefficient matrix is the same in each case. <u>Solve the 4 equations in part (a) by row-reducing the augmented matrix:</u> $[A \,\vdots\, \vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \vec{b}_4]$

Answer:

*<u>Recall</u>: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $(ad-bc) \neq 0$, then: $A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(a)

*<u>Given</u>:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$$

- <u>Check the determinant:</u>

$$(ad-bc) = 12 - 10 = 2 \neq 0 \checkmark$$

- <u>Find A^{-1}:</u>

$$\boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5/2 & 1/2 \end{bmatrix}}$$ Ans.

Example Continued... *(a) Continued.*

*Solve the 4 Nonhomogeneous Equations:

Recall: If A is an $n\times n$ invertible matrix, then $\forall\ \vec{b}\in\mathbb{R}^n$, $A\vec{x}=\vec{b}$ has a unique solution: $\vec{x}=A^{-1}\vec{b}$

(i) Solve $A\vec{x}=\vec{b}_1$:

$$\vec{x}=\begin{bmatrix}6 & -1\\ -\frac{5}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\8\end{bmatrix}=0\begin{bmatrix}6\\-\frac{5}{2}\end{bmatrix}+8\begin{bmatrix}-1\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}0-8\\0+4\end{bmatrix}=\boxed{\begin{bmatrix}-8\\4\end{bmatrix}}$$

(ii) Solve $A\vec{x}=\vec{b}_2$:

$$\vec{x}=\begin{bmatrix}6 & -1\\ -5/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}=1\begin{bmatrix}6\\-\frac{5}{2}\end{bmatrix}+(-3)\begin{bmatrix}-1\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}6+3\\-\frac{5}{2}-\frac{3}{2}\end{bmatrix}=\boxed{\begin{bmatrix}9\\-4\end{bmatrix}}$$

(iii) Solve $A\vec{x}=\vec{b}_3$:

$$\vec{x}=\begin{bmatrix}6 & -1\\ -5/2 & 1/2\end{bmatrix}\begin{bmatrix}4\\18\end{bmatrix}=4\begin{bmatrix}6\\-\frac{5}{2}\end{bmatrix}+18\begin{bmatrix}-1\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}24-18\\-10+9\end{bmatrix}=\boxed{\begin{bmatrix}6\\-1\end{bmatrix}}$$

(iv) Solve $A\vec{x}=\vec{b}_4$:

$$\vec{x}=\begin{bmatrix}6 & -1\\ -5/2 & 1/2\end{bmatrix}\begin{bmatrix}6\\22\end{bmatrix}=6\begin{bmatrix}6\\-5/2\end{bmatrix}+22\begin{bmatrix}-1\\ 1/2\end{bmatrix}=\begin{bmatrix}36-22\\-15+11\end{bmatrix}=\boxed{\begin{bmatrix}14\\-4\end{bmatrix}}$$

Example Continued...

(b) Solve the 4 eq. by Row-Reducing $[A \,\vdots\, \vec{b}_1 \; \vec{b}_2 \; \vec{b}_3 \; \vec{b}_4]$:

$$\Rightarrow \left[\begin{array}{cc|cccc} 1 & 2 & 0 & 1 & 4 & 6 \\ 5 & 12 & 8 & -3 & 18 & 22 \end{array}\right]$$

*Use 1st pivot to eliminate the other entries in Col. 1.

* $-5R_1 + R_2$ / new R_2

$$\sim \left[\begin{array}{cc|cccc} 1 & 2 & 0 & 1 & 4 & 6 \\ 0 & 2 & 8 & -8 & -2 & -8 \end{array}\right]$$

* $\frac{1}{2}R_2$

$$\sim \left[\begin{array}{cc|cccc} 1 & 2 & 0 & 1 & 4 & 6 \\ 0 & 1 & 4 & -4 & -1 & -4 \end{array}\right]$$

*Use 2nd pivot to eliminate other entries in Col. 2

* $-2R_2 + R_1$ / new R_1

$$\sim \boxed{\left[\begin{array}{cc|cccc} 1 & 0 & -8 & 9 & 6 & 14 \\ 0 & 1 & 4 & -4 & -1 & -4 \end{array}\right]}$$

Answer ✓

Note: The Solutions are equivalent to those found in (a).

*Thm: Properties of Invertible Matrices *

① If A is an invertible matrix, then A^{-1} is invertible, and:

$$\rightarrow \quad (A^{-1})^{-1} = A$$

② If A & B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses in reverse order:

$$\rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$$

③ If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}

$$\rightarrow \quad (A^T)^{-1} = (A^{-1})^T$$

<u>Note</u>: Proofs for these properties can be found on the following pages :)

Example: Use matrix algebra to show that if:

"A is an invertible matrix & D is a matrix that satisfies $AD = I$, then $D = A^{-1}$."

Answer: Note: This helps to prove prop. # 1 :)

* Let A be some $n \times n$ invertible matrix.

* Let D be some $n \times n$ matrix ST: $AD = I$

  where: I = Identity Matrix

Goal: Show that $D = A^{-1}$

(i) Left-Multiply "$AD = I$" by "A^{-1}":

$$A^{-1}(AD) = A^{-1}(I)$$

$$(A^{-1}A)D = A^{-1}$$

* By Properties of Matrix Mult.
  - Associative Law (LHS)
  - Identity for Matrix Mult. (RHS)

$$ID = A^{-1}$$

$$D = A^{-1} \checkmark$$

Proof (Property #2):

If A & B are $n \times n$ invertible matrices, then so is AB, & the inverse of AB is the product:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

$\$$ that AB & $(AB)^{-1}$ are inverses.

* Goal: Show that
  (i) $(AB)(AB)^{-1} = I_n$ & (ii) $(AB)^{-1}(AB) = I_n$

Case 1: Show that $(AB)(AB)^{-1} = I_n$:

$$\begin{aligned}(AB)(AB)^{-1} &= (AB)B^{-1}A^{-1} && \text{*by theorem}\\ &= A(BB^{-1})A^{-1} && \text{*grouping middle terms}\\ &= A I_n A^{-1} && \text{*By Def. of an Inverse}\\ &= AA^{-1}\\ &= I_n \checkmark\end{aligned}$$

Case 2: Show that $(AB)^{-1}(AB) = I_n$:

$$\begin{aligned}(AB)^{-1}(AB) &= B^{-1}A^{-1}(AB) && \text{*By Theorem}\\ &= B^{-1}(A^{-1}A)B && \text{*Grouping Middle Terms}\\ &= B^{-1} I_n B && \text{*By Def. of an Inverse}\\ &= B^{-1}B\\ &= I \checkmark\end{aligned}$$

Proof (Property # 3):

If A is an $n \times n$ invertible matrix, then so is the transpose of A, A^T, and so:

$$(A^T)^{-1} = (A^{-1})^T$$

Proof:

\$ that A^T and $(A^T)^{-1}$ are inverses.

Goal: Show that

(i) $A^T(A^T)^{-1} = I_n$ & (ii) $(A^T)^{-1} A^T = I_n$

Recall: By the properties of the transpose, we know

$$\Rightarrow \quad (AB)^T = B^T A^T$$

Case 1: Show that $A^T(A^T)^{-1} = I_n$:

$$\begin{aligned} A^T(A^T)^{-1} &= A^T(A^{-1})^T && \text{* By Theorem} \\ &= (A^{-1}A)^T && \text{* By Prop. of the Transposes} \\ &= (I_n)^T && \text{* By Def. of Inverses} \\ &= I_n \checkmark && \text{* By Prop. of Transposes} \end{aligned}$$

Case 2: Show that $(A^T)^{-1}A^T = I_n$:

$$\begin{aligned} (A^T)^{-1} A^T &= (A^{-1})^T A^T \\ &= (AA^{-1})^T = (I_n)^T = I_n \checkmark \end{aligned}$$

* Elementary Matrices *

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix :

> If an elementary row-operation is performed on a $m \times m$ matrix A, the resulting matrix can be written a EA, where the $m \times m$ matrix E is created by performing the same row operation of the Identity Matrix I_m.

Example: Compute E_1A, E_2A, E_3A and describe how these products can be obtained by elementary row-operations on A :

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & F \\ g & h & i \end{bmatrix}$$

Answer:

* Compute $E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & F \\ g & h & i \end{bmatrix}$

$$= \boxed{\begin{bmatrix} a & b & c \\ d & e & F \\ -4a+g & -4b+h & -4c+i \end{bmatrix}}$$

Note:
Keeping the Row-Column Rule of Matrix Multiplication in mind, we see that the only change comes from row 3 of E_1!

$\Rightarrow$ We need: $\left\{ \begin{matrix} -4R_1 \text{ (of } A) \\ + \quad R_3 \text{ (of } A) \end{matrix} \right. = \text{NEW } R_3 \text{ OF } E_1A$

Example Continued...

* **Compute $E_2 A$:** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Row-Operations on A:

* Interchanging R_1 & R_2 of A produce R_1 & R_2 of $E_2 A$ $\Rightarrow$ $= \boxed{\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}}$

* **Compute $E_3 A$:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Note: Keeping the 'Row Column' Rule of Matrix Multiplication, we see that the only change comes from the 3rd row of E_3!

* $5R_3$ (of A) $\Rightarrow$ new R_3 of $E_3 A$

$$\therefore = \boxed{\begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}}$$

*Elementary Matrices Continued... *

Note: Since elementary row-operations are reversible, elementary matrix operations are inverted :)

IOW:

* If a row-operation on the Identity Matrix, I, can produce some matrix A...

$\Updownarrow$

* Then, a row operation can be performed on a matrix A to reproduce the Identity Matrix, I.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Theorem: A $n \times n$ matrix A is invertible IFF A is row equivalent to I_n, & in any case, any sequence of row operations that reduces A to I_n also transforms I_n to A^{-1}.

Note: This thm ↑ provides us w/ an algorithm to find the inverse of a matrix :)

Example:

Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Answer:

Note: To find the inverse of E_1, E_1^{-1}, we want to transform matrix E_1 into the Identity Matrix I.

* To transform E_1 into I:

$$\begin{array}{r} 4R_1 \\ + \; R_3 \\ \hline \text{new } R_3 \end{array} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

$\therefore$ The Inverse of E_1, E_1^{-1}, is the elementary matrix:

$$\boxed{E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}}$$

Ans.

Example: Use the Algorithm for finding A^{-1} to find the inverses of the given matrices. Let A be the corresponding $n \times n$ matrix, & let B be its inverse. Guess the form of B, & then show that $AB = I$.

a) $\begin{bmatrix} 8 & 0 & 0 \\ 8 & 8 & 0 \\ 8 & 8 & 8 \end{bmatrix}$ & b) $\begin{bmatrix} 8 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 \\ 8 & 8 & 8 & 0 \\ 8 & 8 & 8 & 8 \end{bmatrix}$

Answer:

Part (a): Row-Reduce $[A \,\vdots\, I]$ to rref

$$\left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 1 & 0 & 0 \\ 8 & 8 & 0 & 0 & 1 & 0 \\ 8 & 8 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \frac{1}{8}R_1 \\ \frac{1}{8}R_2 \\ \frac{1}{8}R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 1 & 1 & 0 & 0 & \frac{1}{8} & 0 \\ 1 & 1 & 1 & 0 & 0 & \frac{1}{8} \end{array}\right]$$

$*$ $\begin{array}{r} -R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array}$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{8} & 0 \\ 1 & 1 & 1 & 0 & 0 & \frac{1}{8} \end{array}\right]$

$*$ $\begin{array}{r} -R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array}$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{8} & 0 \\ 0 & 1 & 1 & -\frac{1}{8} & 0 & \frac{1}{8} \end{array}\right]$

Example Continued...

$*\ \begin{array}{l} -R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{8} \end{array}\right]$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{8} & 0 \\ 0 & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}}$$

Note: Using this ↑ solution, find A^{-1} for the next matrix w/o computation (if possible)

(b)

$$A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 \\ 8 & 8 & 8 & 0 \\ 8 & 8 & 8 & 8 \end{bmatrix} \Rightarrow \boxed{A^{-1} = \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{8} & \frac{1}{8} & 0 & 0 \\ 0 & -\frac{1}{8} & \frac{1}{8} & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}}$$

*An Algorithm For Finding A^{-1} *

<u>Note:</u> If we place matrix A & the Identity Matrix I side-by-side to create the augmented matrix $[A \mid I]$, then the row-operations on this matrix produce identical operations on both A & I.

Algorithm to Find A^{-1}:

- Row-reduce the augmented matrix $[A \mid I]$
- <u>If A is row-equivalented to I</u>, then $[A \mid I]$ is row-equivalent to $[I \mid A^{-1}]$
- <u>If A is NOT row-equivalent to I</u>, then A does <u>not</u> have an inverse.

Example: Find the inverse of the matrix A, if it exists.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

Answer:

* Row-reduce the Augmented Matrix $[A \,\vdots\, I]$ to rref:

$$[A \,\vdots\, I] = \left[\begin{array}{ccc:ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

*Interchange R_1 & R_2

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

*Now use 1st pivot to eliminate all entries in Column 1:

$$\begin{array}{r} -4R_1 \\ +\ R_3 \\ \hline \text{new } R_1 \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

*Use the 2nd pivot to eliminate all other entries in Column 2.

$$\begin{array}{r} 3R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

*Use the 3rd pivot to eliminate all other entries in Column 3.

Example Continued...

$$* \quad \frac{\begin{array}{r}-R_3\\ +R_2\end{array}}{\text{new } R_2} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\frac{1}{2}R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$* \quad \frac{\begin{array}{r}-3R_3\\ +R_1\end{array}}{\text{new } R_1} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$[I \,\vdots\, A^{-1}]$:)

$$\therefore \boxed{A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}}$$

Answer

Note: Since A is invertible, we do not need to find $AA^{-1} = I$ ... BUT it is a great way to check your work :)

Example: Find the inverse of the given matrix, if it $\exists$:

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 1 & 2 \\ -4 & 4 & 2 \end{bmatrix}$$

Answer:

* Recall (Algorithm for Finding A^{-1}):
Row-reduce $[A \,\vdots\, I]$. If A is row-equivalent to I, then $[A \,\vdots\, I]$ is row-equivalent to $[I \,\vdots\, A^{-1}]$
$\Rightarrow$ Otherwise A does not have an inverse.

* Row-reduce $[A \,\vdots\, I]$ to rref:

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ -4 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

* Use 1^{st} pivot to eliminate other entries in Col. 1

* $-3R_1 + R_2$ / new R_2

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 11 & -3 & 1 & 0 \\ -4 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

* $4R_1 + R_3$ / new R_3

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 11 & -3 & 1 & 0 \\ 0 & 4 & -10 & 4 & 0 & 1 \end{array}\right]$$

*Use the 2^{nd} pivot to eliminate all other entries in Col. 2

Example Continued...

$$* \;\; \frac{\begin{array}{r} -4R_2 \\ + \;\; R_3 \end{array}}{\text{new } R_3} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 11 & -3 & 1 & 0 \\ 0 & 0 & -54 & 16 & -4 & 1 \end{array}\right]$$

* Use the 3rd pivot to eliminate all other entries in Col. 3

$$* \;\; -\frac{1}{18}R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 11 & -3 & 1 & 0 \\ 0 & 0 & 3 & \frac{-8}{9} & \frac{2}{9} & -\frac{1}{18} \end{array}\right]$$

$$* \;\; \frac{\begin{array}{r} R_3 \\ + \; R_1 \end{array}}{\text{new } R_1} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & -\frac{1}{18} \\ 0 & 1 & 11 & -3 & 1 & 0 \\ 0 & 0 & 3 & \frac{-8}{9} & \frac{2}{9} & \frac{-1}{18} \end{array}\right]$$

$$* \;\; \frac{1}{3}R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & 2/9 & -1/18 \\ 0 & 1 & 11 & -3 & 1 & 0 \\ 0 & 0 & 1 & -8/27 & 2/27 & -1/54 \end{array}\right]$$

$$* \;\; \frac{\begin{array}{r} -11R_3 \\ + \;\; R_2 \end{array}}{\text{new } R_2} \quad \boxed{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & 2/9 & -1/18 \\ 0 & 1 & 0 & 7/27 & 5/27 & 11/54 \\ 0 & 0 & 1 & -8/27 & 2/27 & -1/54 \end{array}\right]}$$

Example: Let $A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}$. Construct a 4×2 matrix D (using only 1s & 0s) ST $AD = I_2$

Answer:

* Note: Since matrix A is 2×4, we need to find a 4×2 matrix D ST: $AD = I_2$

Find "D" ST: $AD = I_2$:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* By the Row-Column Rule:

$$\begin{bmatrix} -d_{11} + d_{21} + d_{31} + 0d_{41} & -d_{12} + d_{22} + d_{32} + 0d_{42} \\ d_{11} + 0d_{21} - d_{31} + d_{41} & d_{12} + 0d_{22} - d_{32} + d_{42} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* d_{11} & d_{31} cancel in R_1 & R_2

↳ Let entries be "1"

* $d_{41} = 0$ & $d_{21} = 1$

* d_{12} & d_{32} cancel in R_1 & R_2

↳ Let entries be '1'

* $d_{22} = 0$ & $d_{42} = 1$

$$\boxed{\therefore D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}$$

Answer

*Property 1: \$ A is an $n \times n$ matrix & the equation $A\vec{x} = \vec{0}$ has on the trivial solution. Explain why A has n-pivots & A is row-equivalent to I_n.

*Note: This property appears again in the all encompassing, "Invertible Matrix Thm"

Proof/Verification:

* Given an $n \times n$ matrix A ST $A\vec{x} = \vec{0}$ has the Trivial Sol. ONLY

$\Rightarrow$ The Columns of matrix A are Linearly Independent

* NO Free Variables $\exists$ $\Rightarrow$ n-pivot Columns

⋮

* Since the Eq. $A\vec{x} = \vec{0}$ has at least one solution $(\vec{x} = \vec{0})$

$\Rightarrow$ A pivot position $\exists$ in each row (No Free var.)

$\therefore$ Since matrix A has n-rows, matrix A has n-pivots

$\Rightarrow$ Since matrix A is square $(n \times n)$ & each pivot must $\exists$ in a different row, the pivot positions of the rref-matrix must be along the main diagonal $\Leftrightarrow$ Identity Matrix, I_n

***Prop #2:** $ A is an $n \times n$ matrix & $A\vec{x} = \vec{b}$ has a solution $\forall\, \vec{b} \in \mathbb{R}^n$. Explain why matrix A must be invertible.

(*Note: Again, this property plays an important role in helping us to prove the "Invertible Matrix Thm" (2.3)

Proof/Verify:

* Let A be some $n \times n$ matrix ST the nonhomogeneous eq. $A\vec{x} = \vec{b}$ has a solution $\forall\, \vec{b} \in \mathbb{R}^n$.

1 $\Rightarrow$ Since $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ is consistent, then a pivot position exists in every row (Logical Equivalence Thm)

2 $\Rightarrow$ Since matrix A is square $(n \times n)$ & $\exists$ 'n' pivots (one per. row), then the pivot-positions of the rref of A must exist along the main diagonal.

3 $\Rightarrow$ rref(A) is row-equivalent to the Identity Matrix I_n

$\therefore$ A must be invertible $\boxtimes$

Matrix Prop. #2: $\exists$ $AD = I_n$ (the $n\times n$ Identity matrix).

Show that $\forall \vec{b} \in \mathbb{R}^n$, the nonhomogeneous equation, $A\vec{x} = \vec{b}$, has a solution.

Proof:

$\exists$ $AD = I_n$, where:
$\begin{cases} * A = \text{some } n\times m \text{ matrix} \\ * D = \text{some } m\times n \text{ matrix} \\ * I_n = n\times n \text{ Identity Matrix} \end{cases}$

Goal: Show that $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ is consistent ~~AND~~ verify why (# rows of A) $\leq$ (# of Col of A)

$\nexists$ not. (show reverse).

$\exists$ that $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ is consistent.

Recall: These 4 statements are logically equivalent.

Then, $\Leftrightarrow$ (i) $\vec{b}$ is a Linear Combination of the Col. of A.

$\Leftrightarrow$ (ii) Columns of A span $\mathbb{R}^n$.

$\Leftrightarrow$ (iii) $\exists$ a pivot position in each row

* Since A is a square, $n\times n$ matrix: $\exists$ n-pivot positions.

* Since each pivot must $\exists$ in a different row: the n-pivots must $\exists$ along the main diagonal of A.

$\Rightarrow$ In which case, rref(A) will be row-equivalent to the Identity Matrix I_n

So, A is invertible $\Longrightarrow$ $\therefore AD = I_n$ ✓ $\boxtimes$