Section 2.1: Matrix Operations

<u>Note</u>: The ability to perform algebraic operations w/ matrices will greatly enhance our ability to analyze & solve Linear Systems ::

*Matrix Notation:

Consider an $m \times n$ matrix "A".
We already know that matrix A will have m-Rows & n-Columns:

$$A = \begin{array}{c c}
 & \text{Column } j,\ \vec{a_j} \\
\begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & & \vdots & & \vdots \\
a_{m1} & & a_{mj} & & a_{mn}
\end{bmatrix} & \begin{array}{l} \\ \\ *\text{Row } i \\ \\ \\ \end{array}
\end{array}$$

*<u>$(i,j)^{th}$ entry of A</u>: The scalar entry in the i^{th} row, j^{th} column.

→ <u>denoted</u>: a_{ij}

*<u>Columns of A</u>: Each column is a list of m-$\mathbb{R}$ #s, identifying a vector in $\mathbb{R}^m$.

→ <u>denoted</u>: $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_j, \ldots, \vec{a}_n$

*<u>Diagonal Entries</u>: The scalar entries that form the main digonal.

→ <u>consisting of</u>: $a_{11}, a_{22}, a_{33}, \ldots, a_{ij}, \ldots a_{mn}$

*<u>Diagonal Matrix</u>: A square, $n \times n$ matrix whose non-digonal entries are all zeros (Ex: Identity Matrix)

*<u>Zero Matrix</u>: An $m \times n$ matrix whose entries are all zero.

*Sums & Scalar Multiples:

Here we draw parallels btw vector arithmetic & matrix arithmetic :)

- Two (or more) matrices are said to be **Equal** if:
  - *They have the same size (i.e. $m \times n$)
  - *Their corresponding columns (& entries) are equal.

- If A & B are both $m \times n$ matrices, then their **Sum $A+B$** is:
  - *The $m \times n$ matrix whose columns are the sums of the corresponding columns in A & B
  - ***Note:** The Sum $A+B$ is only defined when A & B are the same size :)

- If "r" is a scalar & A is a matrix, then the **Scalar Multiple rA** is:
  - *The matrix whose columns are r-times the corresponding columns of A.
  - ***Note:** Subtract of matrices is thought of as: $A - B = A + (-1)B$

*Theorem // Properties:

Let $A, B,$ & C be matrices of the same size.
Let r & s be scalars ($\mathbb{R}$).

1. $A + B = B + A$
2. $(A+B) + C = A + (B+C)$
   *applies to 4 or more matrices too
3. $A + 0 = A$
4. $r(A+B) = rA + rB$
5. $(r+s)A = rA + sA$
6. $r(sA) = (rs)A$

Example (Sums & Scalar Multiples):

Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, & $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$

Find the following:

(a) $A + B$ (b) $A + C$ (c) $A - 2B$

Answer:

Note: Before we start, lets make note of the size of each matrix

* A & B are BOTH 2×3 matrices.
* C is a square 2×2 matrix.

Part (a):

Since A & B are the same size $\Rightarrow A + B \; \exists$:

$$A + B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} (4+1) & (0+1) & (5+1) \\ (-1+3) & (3+5) & (2+7) \end{bmatrix}$$

$$\boxed{\therefore A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}}$$

Part (b):

Since A & C are DIFFERENT sizes $\Rightarrow$ $\boxed{A + C \text{ DNE}}$

Part (c):

(i) $$2B = 2 \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(1) & 2(1) \\ 2(3) & 2(5) & 2(7) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

(ii) $$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} (4-2) & (0-2) & (5-2) \\ (-1-6) & (3-10) & (2-14) \end{bmatrix}$$

$$\boxed{\therefore A - 2B = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}}$$

*Matrix Multiplication:

Note: Multiplication of matrices corresponds to the composition of Linear Transformations :)

Definition:

If A is a $m \times n$ matrix & if B is an $n \times p$ matrix ST $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p]$, then the **Product AB** is the $m \times p$ matrix defined:

$$AB = A[\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p] = [A\vec{b}_1 \ \ A\vec{b}_2 \ \cdots \ A\vec{b}_p]$$

Important Observations:

- The # of Columns of A = the # of Rows in B.
- AB has the same # of rows as A (m-Rows) & the same # of columns as B (p-columns).
- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Ex: Consider the first column of $AB \Rightarrow$ "$A\vec{b}_1$"

* $A\vec{b}_1$ is a linear combination of the columns of A, using the entries of $\vec{b}_1$ as weights.

Deriving the Formula for Matrix Multiplication

Recall: When matrix B is multiplied by a vector $\vec{x}$, it transforms $\vec{x}$ into the vector $B\vec{x}$.

If we multiply '$B\vec{x}$' by a matrix A, then the result '$A(B\vec{x})$' is produced from $\vec{x}$ by a composition of mappings

→ IOW: $A(B\vec{x})$ is produced by a Linear Transformation!

Goal: Represent this linear transformation as multiplication by a single matrix (AB) ST: $A(B\vec{x}) = (AB)\vec{x}$

PF:

Let A be an $m \times n$ matrix.

Let B be an $n \times p$ matrix ST: $B = [\vec{b}_1 \; \vec{b}_2 \cdots \vec{b}_p]$.

Let $\vec{x}$ be a vector in $\mathbb{R}^p$ ST: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

*By Def., the vector-eq. $B\vec{x}$ is a Linear Combination of the columns of matrix B:

$$B\vec{x} = [\vec{b}_1 \; \vec{b}_2 \cdots \vec{b}_p] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_p\vec{b}_p$$

*By the Linearity of Multiplication:

$$\begin{aligned} A(B\vec{x}) &= A(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_p\vec{b}_p) \\ &= A(x_1\vec{b}_1) + A(x_2\vec{b}_2) + \cdots + A(x_p\vec{b}_p) \\ &= \underbrace{x_1(A\vec{b}_1) + x_2(A\vec{b}_2) + \cdots + x_p(A\vec{b}_p)}_{\text{*This is the def. of a Linear Combination!}} \end{aligned}$$

↘

Pf Continued...

* By Def., $A(B\vec{x})$ is a linear combination of the vectors $A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_p$ using the entries of $\vec{x}$ as weights.

IOW: $A(B\vec{x})$ is a Linear Combination of the columns of matrix $AB = [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p]$

$$A(B\vec{x}) = x_1(A\vec{b}_1) + x_2(A\vec{b}_2) + \cdots + x_p(A\vec{b}_p)$$

$$= [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$= (A[\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p])\vec{x}$$

$$= (AB)\vec{x} \ \checkmark$$

* When matrix AB is multiplied by vector $\vec{x}$, it transforms $\vec{x}$ into the vector $A(B\vec{x})$

$$\therefore \boxed{A(B\vec{x}) = (AB)\vec{x}, \text{ ST } AB = [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p]}$$

$\boxed{X}$

Example (Matrix Multiplication 1): Compute AB, where:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

Answer:

* **Recall:** AB $\exists$ IFF: $\left(\#\ \text{Columns of } A\right) = \left(\#\ \text{Rows of } B\right)$

* **Given:**

- A is a $2 \times \underline{2}$ matrix
- B is a $\underline{2} \times 3$ matrix

$\rightarrow$

- AB exists ✓
- AB will be a 2×3 matrix

* **By Definition:** $AB = A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix}$

- $A\vec{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

- $A\vec{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -6 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$

- $A\vec{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 9 \\ -15 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$

$$\therefore \boxed{AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}}$$

Answer.

*Row-Column Rule For Computing AB *

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from the row i of A & the column j of B.

IOW:

If A is an $m \times n$ matrix & $(AB)_{ij}$ denotes the $(i,j)^{th}$ entry of AB, then:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{jn}$$

Note: This is the same row-vector rule used for computing $A\vec{x}$ (i.e. The Dot Product Rule) :)

*Example (Matrix Multiplication 1) Revisited:

Compute $AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

*Use the Row-Column Rule :)

Answer:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2(4)+3(1) & 2(3)+3(-2) & 2(6)+3(3) \\ 1(4)-5(1) & 1(3)-5(-2) & 1(6)-5(3) \end{bmatrix}$$

$$= \begin{bmatrix} 8+3 & 6-6 & 12+9 \\ 4-5 & 3+10 & 6-15 \end{bmatrix}$$

$$\boxed{\therefore AB = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}}$$

Same answer as before ✓

Example (Matrix Multiplication 2):

If A is a 3×5 matrix & B is a 5×2 matrix, what are the sizes of AB and BA (if they $\exists$).

Answer:

Recall: For the product AB to be defined:

(# of Columns of A) = (# of Rows in B)

***The Product AB:**

- A is a $3\times\underline{5}$ matrix
- B is a $\underline{5}\times2$ matrix

$\Longrightarrow$ $\boxed{\therefore \text{ AB is defined \& will be a } 3\times2 \text{ matrix}}$

Recall: For the product BA to be defined:

(# of Columns of B) = (# of rows of A)

***The Product BA:**

- B is a $5\times\underline{2}$ matrix
- A is a $\underline{3}\times5$ matrix

$\Longrightarrow$ $\boxed{\therefore \text{ BA is NOT defined } (2\neq3)}$

Example: Compute each matrix sum or product, if it is defined. If it is not defined, explain why. Given

$$A=\begin{bmatrix}1 & 0 & -2\\ 3 & -4 & 2\end{bmatrix},\quad B=\begin{bmatrix}7 & -5 & 1\\ 1 & -3 & -4\end{bmatrix},\quad C=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix},\quad \& \quad D=\begin{bmatrix}3 & 4\\ -2 & 4\end{bmatrix},$$

Find the following: (a) $-2A$ (c) AC

(b) $B-2A$ (d) CD

Answer:

Part (a): Find the Scalar Multiple, $-2A$:

$$-2A = -2\begin{bmatrix}1 & 0 & -2\\ 3 & -4 & 2\end{bmatrix} = \begin{bmatrix}-2(1) & -2(0) & -2(-2)\\ -2(3) & -2(-4) & -2(2)\end{bmatrix} = \begin{bmatrix}-2 & 0 & 4\\ -6 & 8 & -4\end{bmatrix}$$

$$\boxed{\therefore -2A=\begin{bmatrix}-2 & 0 & 4\\ -6 & 8 & -4\end{bmatrix}}$$ Ans.

Part (b): Find the difference/sum of scalar multiples, $B-2A$:

$$B-2A = B+(-2A) = \begin{bmatrix}7 & -5 & 1\\ 1 & -3 & -4\end{bmatrix} + \begin{bmatrix}-2 & 0 & 4\\ -6 & 8 & -4\end{bmatrix}$$

$$= \begin{bmatrix}7-2 & -5+0 & 1+4\\ 1-6 & -3+8 & -4-4\end{bmatrix} = \begin{bmatrix}5 & -5 & 5\\ -5 & 5 & -8\end{bmatrix}$$

$$\boxed{\therefore B-2A = \begin{bmatrix}5 & -5 & 5\\ -5 & 5 & -8\end{bmatrix}}$$ Ans.

Example Continued...

Part (c): Find the Product, AC:

Note: * A is a 2×3 matrix
* C is a 2×2 matrix

$\Rightarrow$ Cannot multiply!

$\therefore$ AC is NOT defined b/c the (# of Col. of A) $\neq$ (# of rows of C)

Ans

Part (d): Find the Product, CD:

Notes: * C is a 2×2 matrix
* D is a 2×2 matrix

$\Rightarrow$ $\therefore$ CD is defined (d is a 2×2 matrix)

$$CD = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2(3)+3(-2) & 2(4)+3(4) \\ -1(3)+2(-2) & -1(4)+2(4) \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 8+12 \\ -3-4 & -4+8 \end{bmatrix} = \begin{bmatrix} 0 & 20 \\ -7 & 4 \end{bmatrix}$$

$$\boxed{\therefore CD = \begin{bmatrix} 0 & 20 \\ -7 & 4 \end{bmatrix}}$$

Ans

Example: Compute the product of AB by:

(a) The Definition of the Product of Matrices

(b) The Row-Column Rule for computing the Product

$$A = \begin{bmatrix} -2 & 4 \\ 3 & 3 \\ 5 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix}$$

Answer:

Note:
* Matrix A is $3\times\underline{\underline{2}}$ ✓ $\therefore$ AB is defined :)
* Matrix B is $\underline{\underline{2}}\times 2$

Part (a): Compute AB by the Definition:

$$\Rightarrow AB = A[\vec{b}_1 \ \ \vec{b}_2] = [A\vec{b}_1 \ \ A\vec{b}_2]$$

$$* A\vec{b}_1 = \begin{bmatrix} -2 & 4 \\ 3 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 4\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} - 3\begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \\ 20 \end{bmatrix} + \begin{bmatrix} -12 \\ -9 \\ 6 \end{bmatrix} = \begin{bmatrix} -20 \\ 3 \\ 26 \end{bmatrix}$$

$$* A\vec{b}_2 = \begin{bmatrix} -2 & 4 \\ 3 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -10 \end{bmatrix} + \begin{bmatrix} 12 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 16 \\ 3 \\ -16 \end{bmatrix}$$

$$\boxed{\therefore AB = [A\vec{b}_1 \ \ A\vec{b}_2] = \begin{bmatrix} -20 & 16 \\ 3 & 3 \\ 26 & -16 \end{bmatrix}}$$

Answer ✓

Example Continued...

Part (b): Compute AB by the Row-Column Rule:

$$AB = \begin{bmatrix} -2 & 4 \\ 3 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} (-2)(4) + 4(-3) & -2(-2) + 4(3) \\ 3(4) + 3(-3) & 3(-2) + 3(3) \\ 5(4) - 2(-3) & 5(-2) - 2(3) \end{bmatrix}$$

$$= \begin{bmatrix} -8 - 12 & 4 + 12 \\ 12 - 9 & -6 + 9 \\ 20 + 6 & -10 - 6 \end{bmatrix} = \begin{bmatrix} -20 & 16 \\ 3 & 3 \\ 26 & -16 \end{bmatrix}$$

woohoo! Same answer as part (a)

$$\boxed{\therefore AB = \begin{bmatrix} -20 & 16 \\ 3 & 3 \\ 26 & -16 \end{bmatrix}}$$

Answer

Example: Compute the product of AB by:

(a) The Definition of the Product of Matrices

(b) The Row-Column Rule for computing the product.

$$A = \begin{bmatrix} -2 & 2 \\ 1 & 4 \\ 6 & -3 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

Answer:

Note:
- Matrix A is $3\times\underline{\underline{2}}$ ✓ $\therefore AB \ \exists$ ⋮
- Matrix B is $\underline{\underline{2}}\times 2$

*** Part (a): Use the Definition to Find AB:**

Recall: By the Def: $AB = A[\vec{b}_1 \ \ \vec{b}_2] = [A\vec{b}_1 \ \ A\vec{b}_2]$

$$\bullet A\vec{b}_1 = \begin{bmatrix} -2 & 2 \\ 1 & 4 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 18 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \\ 21 \end{bmatrix}$$

$$\bullet A\vec{b}_2 = \begin{bmatrix} -2 & 2 \\ 1 & 4 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = -1\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 8 \\ 16 \\ -12 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ -18 \end{bmatrix}$$

$$\boxed{\therefore AB = [A\vec{b}_1 \ \ A\vec{b}_2] = \begin{bmatrix} -8 & 10 \\ -1 & 15 \\ 21 & -18 \end{bmatrix}}$$

Answer ✓ Woohoo!

*** Part (b): Use the Row-Column Picture.**

$$AB = \begin{bmatrix} -2 & 2 \\ 1 & 4 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -2(3)+2(-1) & -2(-1)+2(4) \\ 1(3)+4(-1) & 1(-1)+4(4) \\ 6(3)-3(-1) & 6(-1)-3(4) \end{bmatrix} = \boxed{\begin{bmatrix} -8 & 10 \\ -1 & 15 \\ 21 & -18 \end{bmatrix}}$$

Answer ✓

Example:

If a matrix "A" is 9×2 and the product "AB" is a 9×8 matrix, then what is the size of matrix B?

Answer:

Recall: $ that
- * Matrix A is $m \times \underline{n}$
- * Matrix B is $\underline{n} \times p$

$\Longrightarrow$ Then the product AB $\exists$ & is an $\underline{m \times p}$ matrix

* Same # of rows as A
* Same # of columns as B

* **Given:**

- Matrix A : 9×2
- Matrix AB : 9×8

$\Longrightarrow$ $\therefore$ Matrix B is $\underline{2 \times 8}$
- 2 rows
- 8 columns

Ans.

Example: How many rows does matrix B have if the Product BC is a 8×7 matrix?

Answer:

* **Recall:** A product of matrices "BC" has:
i) Same # of rows as B
ii) Same # of col. as C

* **Given:**

Matrix BC is 8×7 $\Longrightarrow$ $\boxed{\therefore \text{Matrix B has } \underline{8 \text{ rows}}}$

Ans.

*Theorem: Standard Properties of Matrix Multiplication:

Let A be an $m \times n$ matrix.

Let B & C be matrices of sizes ST the following are defined

① **The Associative Law of Multiplication:** $A(BC) = (AB)C$

② **The Left Distributive Law:** $A(B+C) = AB + AC$

③ **The Right Distributive Law:** $(B+C)A = BA + CA$

④ **Scalar Multiples:** $r(AB) = (rA)B = A(rB) \quad \forall\ r \in \mathbb{R}$

⑤ **Identity for Matrix Multiplication:** $I_m A = A = A I_n$

where: I_m is the $m \times m$ Identity Matrix ST $I_m \vec{x} = I_m \ \forall\ \vec{x} \in \mathbb{R}^m$

*CAUTION: Please be mindful of the following differences between regular algebra & matrix algebra.

① In General: $AB \neq BA$

Note: If $AB = BA$, then we say that A & B commute with each other :)

② Cancellation Laws do **NOT** hold for matrix multiplication!

⇒ IOW: If $AB = AC$, it is NOT necessarily true that $B = C$.

③ If a product AB is the zero matrix, we **CANNOT** conclude in general that $A = 0$ or $B = 0$

Right Distributive Law of Matrix Multiplication

Let A be an $m \times n$ matrix.

Let B & C have sizes for which the indicated sum/product exist. **Prove that:** $A(B+C) = AB + AC$

Proof:

Let B be an $n \times p$ matrix s.t. $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p]$.

Let C be an $n \times p$ matrix s.t. $C = [\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_p]$.

*By the Definition of the Sum of Matrices:

$$B + C = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p] + [\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_p]$$

$$= [(\vec{b}_1 + \vec{c}_1) \ (\vec{b}_2 + \vec{c}_2) \ \cdots \ (\vec{b}_p + \vec{c}_p)]$$

*Since A is an $m \times \underline{n}$ matrix ✓ & $(B+C)$ is an $\underline{n} \times p$ matrix $\Rightarrow$ The Product $A(B+C)$ $\exists$.

*By the Definition of Matrix Multiplication:

$$A(B+C) = [A(\vec{b}_1 + \vec{c}_1) \quad A(\vec{b}_2 + \vec{c}_2) \ \cdots \ A(\vec{b}_p + \vec{c}_p)]$$

$$= [A\vec{b}_1 + A\vec{c}_1 \quad A\vec{b}_2 + A\vec{c}_2 \ \cdots \ A\vec{b}_p + A\vec{c}_p]$$

$$= [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p] + [A\vec{c}_1 \ A\vec{c}_2 \ \cdots \ A\vec{c}_p]$$

$$= A[\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p] + A[\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_p]$$

$$= AB + AC \ \checkmark \qquad \therefore A(B+C) = AB + AC$$

Note: Showing "$(B+C)A = BA + CA$" works similarly. $\boxtimes$

*Alternative Proof For the Right Distributive Law:

Prove that: $A(B+C) = AB + AC$

Proof:

Let A be an $m \times n$ matrix.

Let B & C be matrices of valid size ST the sum/prod. $\exists$

*Consider the $(i,j)^{th}$ entry of $A(B+C)$: $\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$

*By algebra.

$$\sum_{k=1}^{n} a_{ik}\left(b_{kj} + c_{kj}\right) = \sum_{k=1}^{n}\left(a_{ik}b_{kj} + a_{ik}c_{kj}\right)$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

$\therefore$ $(i,j)^{th}$ entry of $A(B+C)$ = $(i,j)^{th}$ entry of $AB + AC$ ✓

$\Rightarrow \therefore A(B+C) = AB + AC$ $\boxtimes$

Left Distributive Law of Matrix Multiplication:

⇒ Prove that: $(B+C)A = BA + CA$

Proof:

Let A be an $m \times n$ matrix.

Let B & C be $p \times m$ matrices.

*Remember that "A" is being right-mult. by B & C here ⩶

*Consider the $(i,j)^{th}$ entry of $(B+C)A$: $\sum_{k=1}^{n} (b_{ik} + c_{ik}) a_{kj}$

*By Prop. of Sigma Algebra:

$$\underbrace{\sum_{k=1}^{n} (b_{ik} + c_{ik}) a_{kj}}_{*(i,j)^{th} \text{ entry of } (B+C)A} = \sum_{k=1}^{n} (b_{ik} a_{kj} + c_{ik} a_{kj})$$

$$= \underbrace{\sum_{k=1}^{n} b_{ik} a_{kj} + \sum_{k=1}^{n} c_{ik} a_{kj}}_{*(i,j)^{th} \text{ entry of } BA + CA}$$

∴ $(B+C)A = BA + BC$ ⊠

Example: Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

Show that these matrices do NOT commute.

Answer:

Note: If $AB \neq BA$, then A & B do NOT commute :)

*Find the Product AB:

reads: i) A is RIGHT multiplied by B -or- ii) B is LEFT multiplied by A

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 5(2)+1(4) & 5(0)+1(3) \\ 3(2)-2(4) & 3(0)-2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 10+4 & 0+3 \\ 6-8 & 0-6 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

*Find the Product BA:

*reads: i) B is Right multiplied by A -or- ii) A is Left multiplied by B

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2(5)+0(3) & 2(1)+0(-2) \\ 4(5)+3(3) & 4(1)+3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 10+0 & 2+0 \\ 20+9 & 4-6 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

*Conclusion:

$$AB \neq BC$$

$\therefore$ Since $\begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \neq \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$, A & B do NOT commute

Example: Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 8 \\ -4 & K \end{bmatrix}$. What value(s) of "K", if any, will make $AB = BA$?

Answer:

Recall: $AB \neq BA$ (in general)

*To Find "K" (if it $\exists$), Set $AB = BA$ & Solve for K:

- $AB = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 8 \\ -4 & K \end{bmatrix} = \begin{bmatrix} 3(2)+2(-4) & 3(8)+2K \\ -1(2)+2(-4) & -1(8)+2K \end{bmatrix} = \begin{bmatrix} -2 & 24+2K \\ -10 & -8+2K \end{bmatrix}$

- $BA = \begin{bmatrix} 2 & 8 \\ -4 & K \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3)+8(-1) & 2(2)+8(2) \\ -4(3)+K(-1) & -4(2)+K2 \end{bmatrix} = \begin{bmatrix} -2 & 20 \\ -12-K & -8+2K \end{bmatrix}$

- Set $AB = BC$ & then solve corresponding entries for K:

$$\begin{bmatrix} -2 & 24+2K \\ -10 & -8+2K \end{bmatrix} = \begin{bmatrix} -2 & 20 \\ -12-K & -8+2K \end{bmatrix}$$

- a_{11} ✓
- a_{12}: $24+2K=20$
  $2K=-4$
  $\boxed{K=-2}$
- a_{21}: $-10 = -12 - K$
  $2 = -K$
  $\boxed{-2 = K}$
- a_{22}: ✓

$\therefore$ $\boxed{AB = BA \text{ if } K = -2}$ Answer.

Example: Compute $A-3I_3$ & $(3I_3)A$, where:

$$A = \begin{bmatrix} 4 & -2 & 2 \\ -5 & 3 & -6 \\ -4 & 2 & 1 \end{bmatrix}$$

Answer:

Note: I_3 is the 3×3 Identity Matrix: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* **Compute $A-3I_3$:**

$$-3I_3 = -3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$A-3I_3 = \begin{bmatrix} 4 & -2 & 2 \\ -5 & 3 & -6 \\ -4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 4-3 & -2 & 2 \\ -5 & 3-3 & -6 \\ -4 & 2 & 1-3 \end{bmatrix}$$

$$\boxed{\therefore A-3I_3 = \begin{bmatrix} 1 & -2 & 2 \\ -5 & 0 & -6 \\ -4 & 2 & -2 \end{bmatrix}}$$

* **Compute $(3I_3)A$:**

- By the Scalar Multiples Rule: $(3I_3)A = 3(I_3A)$
- By the Identity Matrix Multiplication: $3(I_3A) = 3A$

$$\boxed{\therefore 3A = 3\begin{bmatrix} 4 & -2 & 2 \\ -5 & 3 & -6 \\ -4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 6 \\ -15 & 9 & -18 \\ -12 & 6 & 3 \end{bmatrix}}$$

Example: \$ the third column of B is the sum of the last 2 columns. Describe the 3rd column of the product of matrices AB.

Answer:

* Let A be an $n \times m$ matrix

* Let B be an $m \times p$ matrix

$\Rightarrow$ Since $m = m$, the product AB exist.

* Let $B = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \cdots \ b_p]$ ST $\vec{b}_3 = \vec{b}_{p-1} + \vec{b}_p$

* Using the Definition of Matrix Multiplication:

$$AB = [A\vec{b}_1 \ A\vec{b}_2 \ A\vec{b}_3 \ \cdots \ A\vec{b}_p]$$

$$\text{ST} \quad \boxed{A\vec{b}_3 = A(\vec{b}_{p-1} + \vec{b}_p) = A\vec{b}_{p-1} - A\vec{b}_p}$$

description of the 3rd column of matrix AB.

*Powers of a Matrix:

Let A be an $n \times n$ matrix.

Let K be a $\oplus$ integer: $K \in \mathbb{Z}$ ST $K \geq 0$.

The **Product of K-copies of A** is defined:

$$A^K = \underbrace{A \cdot A \cdot \cdots A}_{K\text{-factors}}$$

Note:

If A is nonzero & if $\vec{x} \in \mathbb{R}^n$, then "$A^K \vec{x}$" is the result of left multiplying $\vec{x}$ by matrix A repeatedly, K-times.

Fun/Quick Illustration: Find $A^K \vec{x}$ if $K=0$.

*IF $K=0$: $\boxed{A^0 \vec{x} = \vec{x}}$

↑

*The product of 0-copies of A

$\therefore$ A^0 is interpreted as the Identity Matrix

*The Transpose of a Matrix: A^T

Let A be an $m \times n$ matrix.

The **Transpose of A, A^T**, is an $n \times m$ matrix whose columns are formed from the corresponding rows of A.

*Theorem (Properties of the Transpose):

Let A & B be matrices whose sizes are appropriate for the following sums/products:

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(rA)^T = r(A^T)$, $\forall$ scalars $r \in \mathbb{R}$

④ $(AB)^T = B^T A^T$

*Note: (For products of more than 2) The transpose of a product of matrices equals the product of their transpose in the reverse order

*Caution(4): In general, $(AB)^T \neq A^T B^T$ (even when $A^T B^T$ $\exists$).

Prove that: For appropriate sized matrices A & B

$$(AB)^T = B^T A^T$$

Proof:

* Assume A & B are size-appropriate & the products $\exists$.

Then the $(i,j)^{th}$ entry of AB is: $\displaystyle\sum_{k=1}^{n} a_{ik} b_{kj}$

* By definition of the Transpose:

(i) The $(i,j)^{th}$ entry of $(AB)^T$ = $(j,i)^{th}$ entry of AB:

$$\sum_{k=1}^{n} a_{jk} b_{ki} = a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}$$

(ii) The i^{th}-row of B^T = The i^{th} column of B:

$$\sum_{k=1}^{n} b_{ki} = b_{1i} + b_{2i} + \cdots + b_{ni}$$

(iii) The j^{th}-column of A^T = The j^{th} row of A:

$$\sum_{k=1}^{n} a_{jk} = a_{j1} + a_{j2} + \cdots + a_{jn}$$

So, the $(i,j)^{th}$ entry in $B^T A^T$ is:

$$\Rightarrow a_{j1}b_{1i} + a_{j2}b_{2j} + \cdots + a_{jn}b_{ni} = \sum_{k=1}^{n} a_{jk}b_{ki} \quad \checkmark$$

$$\therefore (AB)^T = B^T A^T$$

Example (Transpose): Find the Transpose of each matrix:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} M & D \\ O & A \\ M & D \end{bmatrix}$, $C = \begin{bmatrix} y & a & n & k & o \\ w & s & k & a & s \end{bmatrix}$

Answer:

*Col 1 of A ⇒ Row 1 of A^T
*Col 2 of A ⇒ Row 2 of A^T

***Find A^T:**

IF $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then: $\boxed{A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}$

Find B^T:

IF $B = \begin{bmatrix} M & D \\ O & A \\ M & D \end{bmatrix}$, then: $\boxed{B^T = \begin{bmatrix} M & O & M \\ D & A & D \end{bmatrix}}$

***Find C^T:**

IF $C = \begin{bmatrix} y & a & n & k & o \\ w & s & k & a & s \end{bmatrix}$, then: $\boxed{C^T = \begin{bmatrix} y & w \\ a & s \\ n & k \\ k & a \\ o & s \end{bmatrix}}$

Example: Compute AD & DA. Explain how the columns or rows of A change when A is multiplied by D on the Right or on the Left. Find a 3×3 matrix B, not the identity matrix or zero matrix, such that $AB = BA$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 5 \\ 1 & 5 & 7 \end{bmatrix} \quad \& \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Answer:

Note: Both matrices A & D are 3×3 $\Rightarrow$ both AD & DA $\exists$ (3×3 also)

* Find the Product AD ("A is Right-Multiplied by D"):

$$AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 5 \\ 1 & 5 & 7 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+0 & 0+5+0 & 0+0+2 \\ 4+0+0 & 0+30+0 & 0+0+10 \\ 4+0+0 & 0+25+0 & 0+0+14 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 & 5 & 2 \\ 4 & 30 & 10 \\ 4 & 25 & 14 \end{bmatrix}}$$

* RH multiplication by "D": Multiplies each column of A by Diagonal entry of D

* Find the Product DA ("A is left-Multiplied by "D"):

$$DA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 5 \\ 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 4+0+0 & 4+0+0 & 4+0+0 \\ 0+5+0 & 0+30+0 & 0+25+0 \\ 0+0+2 & 0+0+10 & 0+0+14 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 & 4 & 4 \\ 5 & 30 & 25 \\ 2 & 10 & 14 \end{bmatrix}}$$

* LH multiplication by "D": Multiplies each row of A by Diagonal entry of D

* Have you noticed any other "fun" relationships btwn AD & DA?

Example Continued...

*Note: Did you notice that $(AD)^T = DA$ & $(DA)^T = AD$? :)

CAUTION: This ↑ is not alway true! See next page for further exploration.

*Find a new 3×3 matrix B ST $AB = BA$ ST $\begin{cases} B \neq I_3 \\ B \neq [0] \end{cases}$

Recall: $AI_3 = A = I_3 A \Rightarrow$ So, $AI_3 = I_3 A$

*While we cannot use the Identity Matrix I_3, any scalar multiple of I_3 will work!

- Let $B = rI_3$, where $r \in \mathbb{R}$ is any scalar
  *complex work too
- By Scalar Multiple Rule of Matrix Multiplication

i) $AB = A(rI_3) = r(AI_3) = rA$

ii) $BA = (rI_3)A = r(I_3A) = rA$

$$\boxed{\begin{array}{l} \therefore \forall \text{ scalars } r, \ AB = BA \ \text{ if } B = rI_3 \\ \Rightarrow \text{Infinitely Many Solutions of } B \ \exists. \end{array}}$$

Ans.

Example Continued...

Note: A naturally arising question after this example

$\rightarrow$ Is $(AD)^T = DA$?

where:
- Both A & D are square matrices
&
- D is a digonal matrix

*CAUTION: This is NOT true in general!!
The given matrix A here is unique & creates a special case :)

- By the Properties of Transposes, we know that:

(i) $(AB)^T = B^T A^T$

(ii) For some Diagonal Matrix D, $D^T = D$

So, $(AD)^T \stackrel{?}{=} DA$

$D^T A^T \stackrel{?}{=} DA$ * By Prop. (i) above *

$D A^T \stackrel{?}{=} DA$ * By Prop. (ii) above *

*This last step does not always hold true, but does here :)

$A^T = A$ $\Rightarrow$ *Note: When this occurs, matrix A is called "Symmetric" :)

$\boxed{\therefore (AD)^T = DA \text{ IFF } A \text{ is symmetric.}}$

2.) **Matrix Property:** Suppose that $CA = I_n$, where I_n is the $n\times n$ Identity Matrix. Show that $A\vec{x} = \vec{0}$ has only the trivial solution ($\vec{x} = \vec{0}$). Explain why matrix A cannot have more columns than rows.

Proof:

\$ that $CA = I_n$, where
- * C & A are size appropriate matrices (i.e. $n\times n$ Square Matrices)
- * $I_n \to$ $n\times n$ Identity matrix

Goal: Show that $A\vec{x} = \vec{0}$ has only the trivial sol, $\vec{x} = \vec{0}$ (& verify why A cannot have more col than rows)

Let $\vec{x} \in \mathbb{R}^n$ be some vector in $\mathbb{R}^n$.

*If $\vec{x} \in \mathbb{R}^n$ satisfies the Homogeneous Eq. $A\vec{x} = \vec{0}$, then:

$$C(A\vec{x}) = C(\vec{0}) \to CA\vec{x} = C\vec{0}$$
$$\to I_n\vec{x} = \vec{0}$$
$$\to \vec{x} = \vec{0} \text{ (the Trivial Solution) } \checkmark$$

$\boxed{\therefore \text{ If } CA = I_n, \text{ then } A\vec{x} = \vec{0} \text{ has only the trivial solution, } \vec{x} = \vec{0}.}$

*Since $A\vec{x} = \vec{0}$ has the trivial solution ONLY:

*IOW: Matrix A has n-pivot Columns.

$\Rightarrow$ $\boxed{\text{NO free variables } \exists}$ & *the Columns of A are Linearly Independent. $\therefore$ (# of Columns) $\leq$ (# of Rows)

Since each pivot must $\exists$ in a different row (#C) < (#R)

Recall: IF $\left(\begin{matrix}\#\text{ of}\\ \text{unknowns}\end{matrix}\right) > \left(\begin{matrix}\#\text{ of}\\ \text{eq.}\end{matrix}\right)$, then the Columns of A are Linearly Dependent (which would contradict the conclusion above :))

"Columns" "Rows"