Heapsort

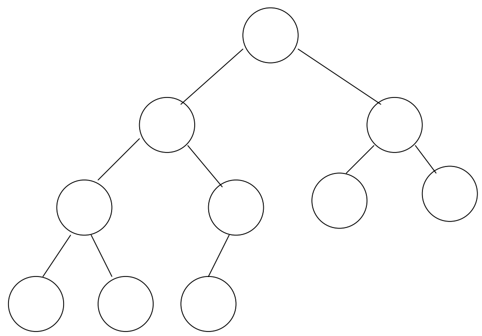
Reviews: Tree

- Tree
  - Rooted tree
    - parent, child, sibling, ancestor
- Binary tree
  - Left child, right child
- Some concepts
  - Height: # of edges on a longest simple path from node to a leaf

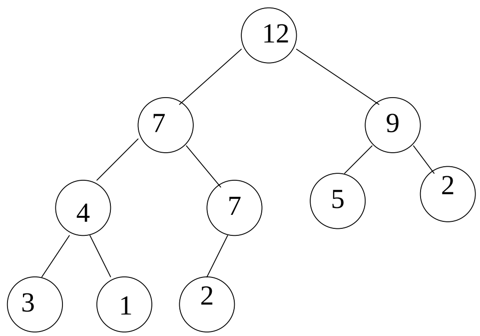
Heap Definition

- A heap is
  - An *essentially complete* *binary* tree which satisfies heap property.
- Binary tree
- Essentially complete binary tree
- Heap property
  - max-heap
    - The value (key) of each node in the heap is greater than or equal to the values (keys) of its children, if any.
  - min-heap
    - The value (key) of each node in the heap is less than or equal to the values (keys) of its children, if any.

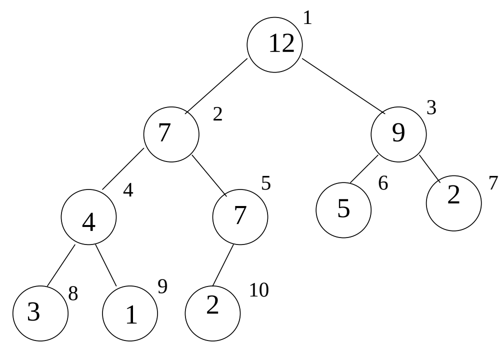
An essentially complete binary tree



A max-heap



A heap can be represented as an array



1	2	3	4	5	6	7	8	9	10
12	7	9	4	7	5	2	3	1	2

Some important properties of heaps

- Given a node *A[i]*
  - It's parent is *A[i/2]*, if *i>1*.
  - It's left child is *A[2*i]*, if *2*i<=n*.
  - It's right child is *A[2*i+1]*, if *2*i+1<=n*.

- The height of a heap containing n nodes is $\lfloor \lg n \rfloor$

- There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes with height h

Heapify

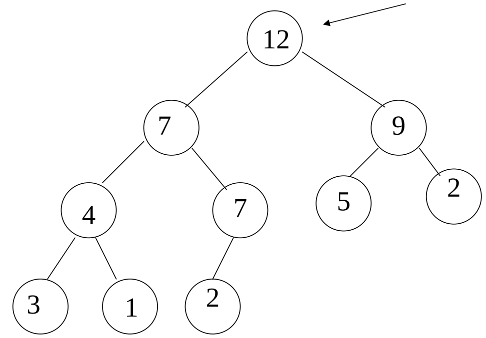
- Assume that the left and right subtrees of A[i] are already max-heaps
- A[i] may be less than its children a violation
- Goal: Make the subtree rooted at index i a max-heap
- Application
  - Call heapify(i) when the value of A[i] is decreased

Heapify

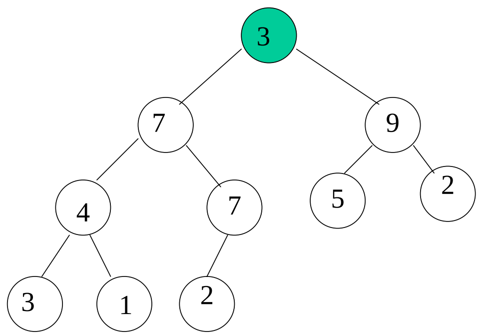
```
MAX-HEAPIFY(A, i, n)
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ n and A[l] > A[i]
   then largest ← l
   else  largest ← i
if r ≤ n and A[r] > A[largest]
   then largest ← r
if largest ≠ i
   then exchange A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest, n)
```

Example

A[1] is changed from 12 to 3

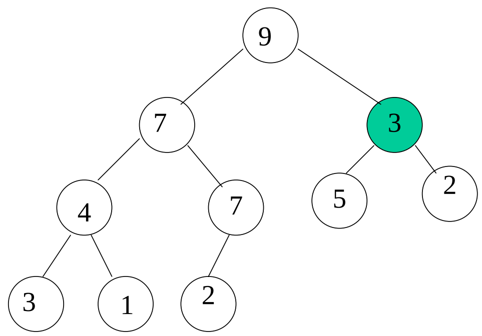


Example

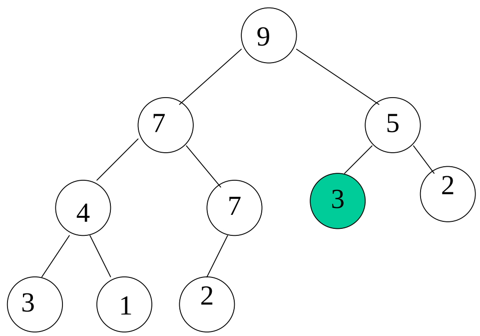


Call heapify(1)

Example



Example



Cost: O(log(n))

buildHeap

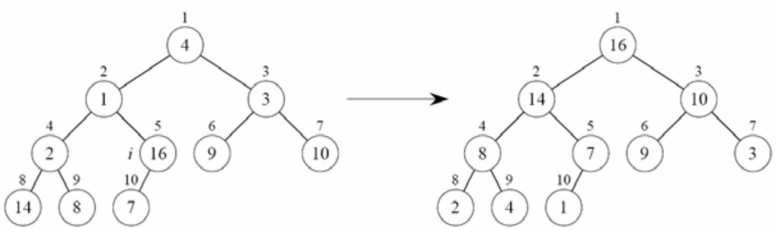
BUILD-MAX-HEAP(A, n)
for $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1
  do MAX-HEAPIFY(A, i, n)

- What's the idea here?
- Proof

Example

- i starts off as 5.
- MAX-HEAPIFY is applied to subtrees rooted at nodes (in order): 16, 2, 3, 1, 4.

	1	2	3	4	5	6	7	8	9	10
A	4	1	3	2	16	9	10	14	8	7



Correctness

Correctness

Loop invariant: At start of every iteration of **for** loop, each node $i + 1$, $i + 2, \ldots, n$ is root of a max-heap.

Initialization: By Exercise 6.1-7, we know that each node $\lfloor n/2 \rfloor + 1$, $\lfloor n/2 \rfloor + 2$, $\ldots, n$ is a leaf, which is the root of a trivial max-heap. Since $i = \lfloor n/2 \rfloor$ before the first iteration of the **for** loop, the invariant is initially true.

Maintenance: Children of node i are indexed higher than i, so by the loop invariant, they are both roots of max-heaps. Correctly assuming that $i+1, i+2, \ldots, n$ are all roots of max-heaps, MAX-HEAPIFY makes node i a max-heap root. Decrementing i reestablishes the loop invariant at each iteration.

Termination: When $i = 0$, the loop terminates. By the loop invariant, each node, notably node 1, is the root of a max-heap.

Analaysis

- O(n) calls to MAX-HEAPIFY
- Each takes O(log n)

Total time: O(n log n), not tight!

Analaysis

- The cost of heapify(i) for a node at height h is O(h)
- Number of nodes at height h is $\left\lceil \frac{n}{2^{h+1}} \right\rceil$
- The total cost is bounded by

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}) = O(n)$$

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \sum_{h=0}^{\infty} h(\frac{1}{2})^h = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

Heapsort Algorithm

- build a max-heap from the array
- starting from root, place maximum element into the correct place in array by swapping it with the element in the last position in the array
- “discard” the last node (knowing that it is in its correct place) by decreasing the heap size
- call MAX-HEAPIFY on new root
- repeat the “discarding” process until only one node remains

Heapsort Algorithm

HEAPSORT(A, n)

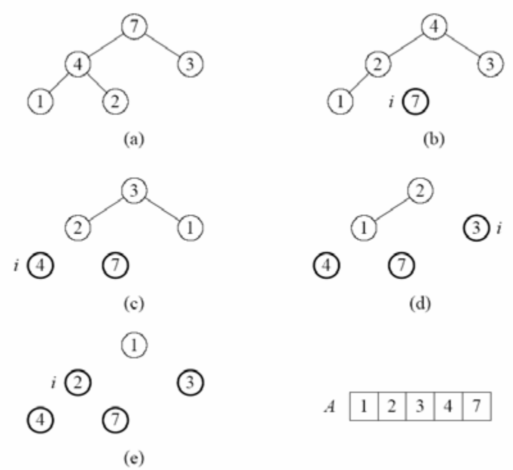
BUILD-MAX-HEAP(A, n)

for $i \leftarrow n$ **downto** 2

    do exchange $A[1] \leftrightarrow A[i]$

        MAX-HEAPIFY$(A, 1, i - 1)$

Example



Analysis

- BUILD-MAX-HEAP: O(n)
- For loop: n-1 times
- Exchange elements: O(1)
- MAX-HEAPFY” O(log n)

Total time: O(n log n)

Application: Priority Queues

Priority queue: a data structure for maintaining a set S of elements, each with an associated value called a key.

Applications: job scheduling, event-driven simulations

Operations of Priority Queues

- Insert(S, x)
  - Insert element x into S
- Maximum(S)
  - Return the element with the largest key
- Extract-Max(L)
  - Remove and return the largest element
- Increase-Key(S, x, k)
  - Increase the value of element x's key to the new value k

Finding the maximum element

- It's the root

HEAP-MAXIMUM(A)
    return A[1]

Time: $\Theta(1)$

Extracting maximum element

- Make sure heap is not empty
- Make a copy of the maximum element
- Make the last node in the tree the new root
- re-heapify the heap, with one fewer node
- Return the copy of the maximum element

```
HEAP-EXTRACT-MAX(A, n)
if n < 1
  then error "heap underflow"
max ← A[1]
A[1] ← A[n]
MAX-HEAPIFY(A, 1, n − 1)    ▷ remakes heap
return max
```

Time: constant time assignments plus time for MAX-HEAPIGY: O(log n)

Increasing key value

- make sure k >= x's current key
- Update x's key value to k
- Traverse tree upward comparing x to its parent and swapping keys if necessary, until x's key is smaller than its parent's key

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
  then error "new key is smaller than current key"
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i]
    do exchange A[i] ↔ A[PARENT(i)]
       i ← PARENT(i)
```

Time: O(log n) as upward path has length O(log n)

Inserting into heap

- Insert a new node in the last position of tree with key $-\infty$
- Increase the $-\infty$ key to k using the HEAP-INCREASE-KEY procedure

```
MAX-HEAP-INSERT(A, key, n)
A[n + 1] ← −∞
HEAP-INCREASE-KEY(A, n + 1, key)
```

Time: constant time assignments plus time for HEAP-INCREASE-KEY O(log n)