<div align="center">

CMPSC 623 Problem Set 5, Solution.
by Prof. Honggang Zhang

Out: November 2, 2006
Due: November 9, 2006, before class.

</div>

Problem 1. Suppose we use quadratic probing

$$h(k, i) = (h'(k) + ci^2) \mod m.$$

Show that if the table size m is a prime number, the first $\lfloor m/2 \rfloor$ probes are distinct.

Solution:

Proof by contradiction. Assume the claim is false, then there exists $i \neq j$ such that $0 \leq i < j < \lfloor m/2 \rfloor$ and i-th and j-th probe will probe the same slot.

Then, we have:

$$
\begin{align}
h(x, i) = h(x, j) \;\Rightarrow\; & h'(x) + i^2 = h'(x) + j^2 (mod \quad m) \tag{1} \\
\Rightarrow\; & i^2 = j^2 (mod \quad m) \tag{2} \\
\Rightarrow\; & i^2 - j^2 = 0 (mod \quad m) \tag{3} \\
\Rightarrow\; & (i - j)(i + j) = 0 (mod \quad m) \tag{4}
\end{align}
$$

As m is a prime number, we must have either $(i - j) = 0(mod \quad m)$ or $(i + j) = 0(mod \quad m)$. However, either case cannot be true, thus, our assumption is false. Therefore, the claim is correct.

Problem 2. Page 244, Exercise 11.4-1. Only do double hashing (Page 240).

Solution:

((10 mod 11) + 0*(1+ (10 mod 10))) mode 11 = 10

((22 mod 11) + 0*(1+ (22 mod 10))) mode 11 = 0

((31 mod 11) + 0*(1+ (31 mod 10))) mode 11 = 9

((4 mod 11) + 0*(1+ (4 mod 10))) mode 11 = 4

((15 mod 11) + 0*(1+ (15 mod 10))) mode 11 = 4, collision

((15 mod 11) + 1*(1+ (15 mod 10))) mode 11 = 10, collision

((15 mod 11) + 2*(1+ (15 mod 10))) mode 11 = 5

((28 mod 11) + 0*(1+ (28 mod 10))) mode 11 = 6

((17 mod 11) + 0*(1+ (17 mod 10))) mode 11 = 6, collision

((17 mod 11) + 1*(1+ (17 mod 10))) mode 11 = 3

((88 mod 11) + 0*(1+ (88 mod 10))) mode 11 = 0, collision

((88 mod 11) + 1*(1+ (88 mod 10))) mode 11 = 9, collision

((88 mod 11) + 2*(1+ (88 mod 10))) mode 11 = 7

((59 mod 11) + 0*(1+ (59 mod 10))) mode 11 = 4, collision

((59 mod 11) + 1*(1+ (59 mod 10))) mode 11 = 3, collision

((59 mod 11) + 2*(1+ (59 mod 10))) mode 11 = 2

Problem 3. Page 250, Problem 11-2 (a) (b). Only do (b) if you did (a) in the previous homework.

Solution:

(a). Note that here the table size is n. A given key is hashed to a chosen slot with probability $1/n$. For a given slot, if we choose k keys, then the probability that all of them are hashed into that slot and all other other keys are hashed to other slots is: $(1/n)^k(1 - 1/n)^{(n-k)}$. As there are C_k^n possibilities, thus, we get:

$$C_k^n(1/n)^k(1 - 1/n)^{(n-k)}.$$

(b). Let x_i be the number of keys in slot i. Then, we have:

$$
\begin{aligned}
P_k &= P\{M = k\} & (5)\\
&= P\{(\max_i \ x_i) = k\} & (6)\\
&= P\{(\exists i \ \text{ such that } \ x_i = k) \ \text{ and } \ (\forall i, x_i \le k)\} & (7)\\
&\le P\{(\exists i \ \text{ such that } \ x_i = k)\} & (8)\\
&= P\{(x_1 = k) \ \text{ or } \ (x_2 = k) \ \text{ or } \ (x_3 = k) \ \text{ or } \ ... \ \text{ or } \ (x_n = k)\} & (9)\\
&\le \sum_i P\{x_i = k\} & (10)\\
&= nQ_k & (11)
\end{aligned}
$$

Problem 4. Suppose that a hash table has m slots, and we resolve collisions by chaining. There are n keys. Assume that our hashing is simple uniform hashing. If you randomly pick up a slot from the table, what is the probability that you find that slot is not empty? What is the probability that you find at least two keys in that slot? If you pick up a key and identify the slot where this key is in, then what is the probability that some other key(s) will be in this slot?

Solution:

Probability that a key is in a randomly picked slot is $q = \frac{1}{m}$, and the probability that a key is in other remaining slots is $1 - q$.

Probability that a randomly picked slot is not empty is: $1 - (1 - q)^n = 1 - (\frac{m-1}{m})^n$.

Probability that a randomly picked slot has at least two keys:

$$
1 - (1 - q)^n - n * q * (1 - q)^{(n-1)}
$$
$$
= 1 - (\tfrac{m-1}{m})^n - n * (1/m) * (\tfrac{m-1}{m})^{(n-1)}
$$

If you identify the slot where a given key is in, then the probability that some other key(s) will be in this slot is $1 - ((m - 1)/m)^{n-1}$.