Section 2.5: Matrix Factorization

<u>Note</u>: A Factorization of a matrix A is an equation that expresses A as a product of 2 or more matrices.

* *Matrix Multiplication involves the synthesis of data (i.e. combining the effects of 2 or more Linear Transformations into a single matrix)
* *Matrix Factorization involves the analysis of data.

The LU Factorization: Introduction

The LU Factorization is motivated by the problem of solving a sequence of equations, all with the same coefficient matrix:

$$A\vec{x}=\vec{b}_1, \quad A\vec{x}=\vec{b}_2, \quad \dots, \quad A\vec{x}=\vec{b}_p$$

<u>Note</u>: While if A is invertible, we could find A^{-1} & then compute $A^{-1}\vec{b}_1$, $A^{-1}\vec{b}_2$, .. etc. ⇒ It is MORE EFFICIENT to solve the 1st eq. by row-reduction & obtain an LU Factorization of A simultaneously ∴ We can then solve the remaining eq. w/ this!

① Assume A is an $m \times n$ matrix that can be row-reduced to echelon form (<u>without interchanging rows</u>)

② Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix w/ 1s along the main diagonal & U is an $m \times n$ upper triangular matrix in echelon form.

*An LU Factorization of A:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L \text{ *Lower}} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U \text{ *Upper}}$$

* L is invertible & is called a "<u>Unit Lower Triangular Matrix</u>"

$*$ → "ANY scalar" ; $\blacksquare$ → Nonzero Leading Entries

*Why are LU Factorizations Useful??

Because they are triangular :)

- When $A = LU$, the equation $A\vec{x} = \vec{b}$ can be written:

$$L(U\vec{x}) = \vec{b}$$

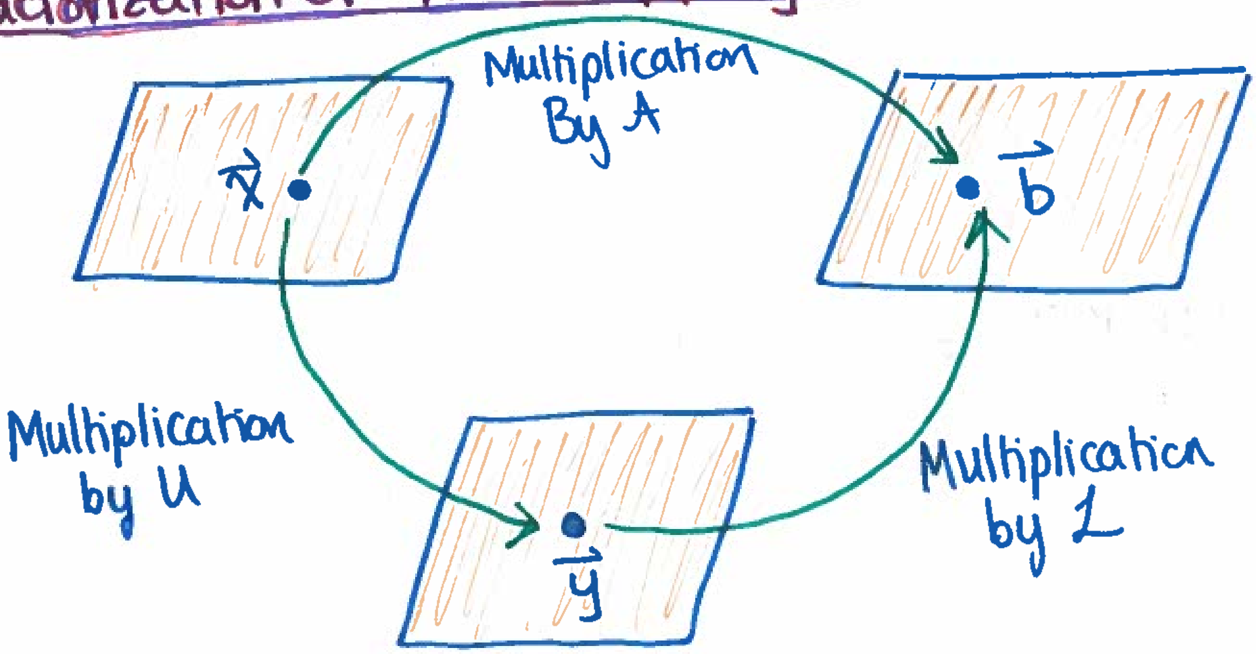
- Writing $\vec{y}$ for $U\vec{x}$, we can solve for $\vec{x}$ by solving the pair of equations:

$$\begin{cases} L\vec{y} = \vec{b} \\ U\vec{x} = \vec{y} \end{cases}$$

- First solve $L\vec{y} = \vec{b}$ for $\vec{y}$, & then solve $U\vec{x} = \vec{y}$ for $\vec{x}$

→ Each equation is easy to solve b/c L & U are triangular :)

*Factorization of the Mapping: $\vec{x} \mapsto A\vec{x}$



Example (Why LU Factorization is Useful):

It can be verified that

$$A=\begin{bmatrix}3&-7&-2&2\\-3&5&1&0\\6&-4&0&-5\\-9&5&-5&12\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\2&-5&1&0\\-3&8&3&1\end{bmatrix}\begin{bmatrix}3&-7&-2&2\\0&-2&-1&2\\0&0&-1&1\\0&0&0&-1\end{bmatrix}=LU$$

Use this factorization of A to solve $A\vec{x}=\vec{b}$, where:

$$\vec{b}=\begin{bmatrix}-9\\5\\7\\11\end{bmatrix}$$

* While @ a quick glance, this may not seem useful/quick, notice that we found each solution in less than 10 steps. ...Whereas solving $[A \,\vdots\, \vec{b}]$ takes close to 70 steps! EW.

Answer:

Note: First we solve $L\vec{y}=\vec{b}$ & then we solve $U\vec{x}=\vec{y}$:)

*Solve $L\vec{y}=\vec{b}$: Row reduced $[L \,\vdots\, \vec{b}]$ to $[I \,\vdots\, \vec{y}]$:

$$[L \,\vdots\, \vec{b}]=\left[\begin{array}{cccc|c}1&0&0&0&-9\\-1&1&0&0&5\\2&-5&1&0&7\\-3&8&3&1&11\end{array}\right]\underset{\text{N.}R_2}{\overset{+R_1+R_2}{\sim}}\left[\begin{array}{cccc|c}1&0&0&0&-9\\0&1&0&0&-4\\2&-5&1&0&7\\-3&8&3&1&11\end{array}\right]\underset{\text{N.}R_3}{\overset{-2R_1+R_3}{\sim}}\left[\begin{array}{cccc|c}1&0&0&0&-9\\0&1&0&0&-4\\0&-5&1&0&25\\-3&8&3&1&11\end{array}\right]$$

$$\underset{\text{N.}R_4}{\overset{3R_1+R_4}{\sim}}\left[\begin{array}{cccc|c}1&0&0&0&-9\\0&1&0&0&-4\\0&-5&1&0&25\\0&8&3&1&-16\end{array}\right]\underset{\text{N.}R_3}{\overset{5R_2+R_3}{\sim}}\left[\begin{array}{cccc|c}1&0&0&0&-9\\0&1&0&0&-4\\0&0&1&0&5\\0&8&3&1&-16\end{array}\right]\underset{\text{N.}R_4}{\overset{-8R_2+R_4}{\sim}}\left[\begin{array}{cccc|c}1&0&0&0&-9\\0&1&0&0&-4\\0&0&1&0&5\\0&0&3&1&16\end{array}\right]$$

Ex. Continued... (Why LU is Useful)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 1 & 16 \end{array}\right] \overset{\substack{-3R_3 \\ +R_4 \\ \overline{N.R_4}}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \boxed{\vec{y} = \begin{bmatrix} -9 \\ -4 \\ 5 \\ 1 \end{bmatrix}}$$

Answer ♥

*Solve $U\vec{x} = \vec{y}$: Row-reduce $[U \mid \vec{y}]$ to $[I \mid \vec{x}]$:

Here we 'backwards' row reduce :

$$[U \mid \vec{y}] = \left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ -R_2 \\ -R_3 \\ -R_4 \end{array} \sim \left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & 2 & 1 & -2 & 4 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{\substack{R_4 \\ +R_3 \\ \overline{N.R_3}}}{\sim} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & 2 & 1 & -2 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \overset{\substack{2R_4 \\ +R_2 \\ \overline{N.R_2}}}{\sim} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \overset{\substack{-2R_4 \\ +R_1 \\ \overline{N.R_1}}}{\sim} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 0 & -7 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{\substack{-R_3 \\ +R_2 \\ \overline{N.R_2}}}{\sim} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 0 & -7 \\ 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \overset{\frac{1}{2}R_2}{\sim} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 0 & -7 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \overset{\substack{2R_3 \\ +R_1 \\ \overline{N.R_1}}}{\sim} \left[\begin{array}{cccc|c} 3 & -7 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{\substack{7R_2 \\ +R_1 \\ \overline{N.R_1}}}{\sim} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \overset{\frac{1}{3}R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow$$

Ans.

$$\boxed{\vec{x} = \begin{bmatrix} -3 \\ 4 \\ -6 \\ -1 \end{bmatrix}}$$

Example: Solve the equation $A\vec{x}=\vec{b}$ using the LU Factorization for A:

$$A = \begin{bmatrix} 3 & -5 & 3 \\ -9 & 12 & -3 \\ 6 & -7 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & -5 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \; ; \; \vec{b} = \begin{bmatrix} -1 \\ 15 \\ -14 \end{bmatrix}$$

Answer:

Recall: To solve $A\vec{x}=\vec{b}$ using the LU Factorization of A, first solve $L\vec{y}=\vec{b}$ & then solve $U\vec{x}=\vec{b}$.

*Solve $L\vec{y}=\vec{b}$: Row-Reduce $[L \mid \vec{b}]$ to $[I \mid \vec{y}]$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -3 & 1 & 0 & 15 \\ 2 & -1 & 1 & -14 \end{array}\right] \underset{\sim}{\frac{3R_1 + R_2}{N.R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 12 \\ 2 & -1 & 1 & -14 \end{array}\right] \underset{\sim}{\frac{-2R_1 + R_3}{N.R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 12 \\ 0 & -1 & 1 & -12 \end{array}\right] \underset{\sim}{\frac{R_2 + R_3}{N.R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \boxed{\vec{y} = \begin{bmatrix} -1 \\ 12 \\ 0 \end{bmatrix}}$$

Answer

Note: Since the $\vec{x}$ relies on the accuracy of $\vec{y}$, it may be beneficial to check $\vec{y}$ is correct first

$$-1\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 12\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0+0 \\ 3+12+0 \\ -2-12+0 \end{bmatrix} = \begin{bmatrix} -1 \\ 15 \\ -14 \end{bmatrix} = \vec{b} \;\checkmark$$

*Solve $U\vec{x}=\vec{y}$: Row-Reduce $[U \mid \vec{y}]$ to $[I \mid \vec{x}]$:

$$\left[\begin{array}{ccc|c} 3 & -5 & 3 & -1 \\ 0 & -3 & 6 & 12 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -5 & 3 & -1 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{\sim}{\frac{2R_3 + R_2}{N.R_2}} \left[\begin{array}{ccc|c} 3 & -5 & 3 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{\sim}{\frac{-3R_1 + R_3}{N.R_1}} \left[\begin{array}{ccc|c} 3 & -5 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{\frac{5R_2 + R_1}{N.R_1}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & -21 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \boxed{\vec{x} = \begin{bmatrix} -7 \\ -4 \\ 0 \end{bmatrix}}$$

Answer

Example: Solve the equation $A\vec{x}=\vec{b}$ by using the LU Factorization given for A:

$$A=\begin{bmatrix}2&-5&-3\\-2&2&2\\6&0&-5\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\3&-5&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}2&-5&-3\\0&-3&-1\\0&0&-1\end{bmatrix}}_{U}\ ;\ \vec{b}=\begin{bmatrix}-4\\-6\\43\end{bmatrix}$$

Answer:

Recall: To solve $A\vec{x}=\vec{b}$ using the LU Factorization of A, first solve $L\vec{y}=\vec{b}$ & then solve $U\vec{x}=\vec{y}$.

*Solve $L\vec{y}=\vec{b}$: Row-Reduce $[L \mid \vec{b}]$ to $[I \mid \vec{y}]$:

$$[L \mid \vec{b}]=\left[\begin{array}{ccc|c}1&0&0&-4\\-1&1&0&-6\\3&-5&1&43\end{array}\right]\overset{\frac{R_1+R_2}{N.R_2}}{\sim}\left[\begin{array}{ccc|c}1&0&0&-4\\0&1&0&-10\\3&-5&1&43\end{array}\right]\overset{\frac{-3R_1+R_3}{N.R_3}}{\sim}\left[\begin{array}{ccc|c}1&0&0&-4\\0&1&0&-10\\0&-5&1&55\end{array}\right]$$

$$\overset{\frac{5R_2+R_3}{N.R_3}}{\sim}\left[\begin{array}{ccc|c}1&0&0&-4\\0&1&0&-10\\0&0&1&5\end{array}\right]\Rightarrow\boxed{\therefore\ \vec{y}=\begin{bmatrix}-4\\-10\\5\end{bmatrix}}$$

Ans.

*Solve $U\vec{x}=\vec{y}$: Row-Reduce $[U \mid \vec{y}]$ to $[I \mid \vec{x}]$:

$$[U \mid \vec{y}]=\left[\begin{array}{ccc|c}2&-5&-3&-4\\0&-3&-1&-10\\0&0&-1&5\end{array}\right]\overset{-R_3}{\sim}\left[\begin{array}{ccc|c}2&-5&-3&-4\\0&-3&-1&-10\\0&0&1&-5\end{array}\right]\overset{\frac{R_3+R_2}{N.R_2}}{\sim}\left[\begin{array}{ccc|c}2&-5&-3&-4\\0&-3&0&-15\\0&0&1&-5\end{array}\right]\overset{\frac{*3R_3+R_1}{N.R_3}}{\sim}$$

$*-\frac{1}{3}R_2$

$$\left[\begin{array}{ccc|c}2&-5&0&-19\\0&1&0&5\\0&0&1&-5\end{array}\right]\overset{\frac{5R_2+R_1}{N.R_1}}{\sim}\left[\begin{array}{ccc|c}2&0&0&6\\0&1&0&5\\0&0&1&-5\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&5\\0&0&1&-5\end{array}\right]\Rightarrow\boxed{\vec{x}\begin{bmatrix}3\\5\\-5\end{bmatrix}}$$

✓

Example: Solve the equation $A\vec{x} = \vec{b}$ by using the LU Factorization given for A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -2 & -2 & -6 & 10 \\ 2 & 0 & 3 & -20 \\ -5 & -2 & -12 & 41 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 10 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad \vec{b} = \begin{bmatrix} 9 \\ -20 \\ 13 \\ -42 \end{bmatrix}$$

Answer:

*Recall: To solve $A\vec{x} = \vec{b}$ using the LU Factorization, first solve $L\vec{y} = \vec{b}$ & then solve $U\vec{x} = \vec{y}$:)

*Solve $L\vec{y} = \vec{b}$: Row-Reduce $[L \,\vdots\, \vec{b}]$ to $[I \,\vdots\, \vec{y}]$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ -2 & 1 & 0 & 0 & -20 \\ 2 & -2 & 1 & 0 & 13 \\ -5 & 4 & -1 & 1 & -42 \end{array}\right] \underset{N.R_2}{\overset{2R_1 + R_2}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -2 \\ 2 & -2 & 1 & 0 & 13 \\ -5 & 4 & -1 & 1 & -42 \end{array}\right] \underset{N.R_3}{\overset{-2R_1 + R_3}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & -2 & 1 & 0 & -5 \\ -5 & 4 & -1 & 1 & -42 \end{array}\right] \underset{N.R_4}{\overset{5R_1 + R_4}{\sim}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & -2 & 1 & 0 & -5 \\ 0 & 4 & -1 & 1 & 3 \end{array}\right] \underset{N.R_3}{\overset{2R_2 + R_3}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -9 \\ 0 & 4 & -1 & 1 & 3 \end{array}\right] \underset{N.R_4}{\overset{-4R_2 + R_4}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & -1 & 1 & 11 \end{array}\right] \underset{N.R_4}{\overset{R_3 + R_4}{\sim}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \boxed{\vec{y} = \begin{bmatrix} 9 \\ -2 \\ -9 \\ 2 \end{bmatrix}}$$

Answer

Check:

$$9\begin{bmatrix} 1 \\ -2 \\ 2 \\ -5 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \; 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 13 \\ -42 \end{bmatrix} = \vec{b} \checkmark$$

Example Continued...

*Solve $U\vec{x}=\vec{y}$: Since U is already in echelon form, an alternative to solving by row-reduction is straight-up back-substitution:

$$[U \mid \vec{y}] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 9 \\ 0 & 2 & 0 & 10 & -2 \\ 0 & 0 & -3 & 0 & -9 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 9 \\ 0 & 1 & 0 & 5 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 9 & \longrightarrow \quad x_1 = 9 - 2(-11) - 3(3) = 9 + 22 - 9 = 22 \\ x_2 + 5x_4 = -1 & \longrightarrow \quad x_2 = -1 - 5(2) = -11 \\ x_3 = 3 \\ x_4 = 2 \end{cases}$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 22 \\ -11 \\ 3 \\ 2 \end{bmatrix}}$$

Ans ✓

*An LU Factorization Algorithm *

<u>Note</u>: While using an LU Factorization is useful/quick, this computational efficiency depends on knowing L & U!

⇒ This algorithm shows that the row-reduction of A to echelon form 'U' amounts to an LU factorization b/c it produces $\mathcal{L}$ w/ essentially no extra work ∴

* After this first row-reduction, $\mathcal{L}$ & U are available to solve other equation w/ the same coefficient matrix.

*$ that A can be row-reduced to echelon form of U using only row replacements that add a multiple of one row to another row <u>below</u> it.

* <u>∃ unit lower triangular matrices $E_1, \dots, E_p$ ST:</u>

$$E_p \cdots E_1 A = U$$

* <u>LH Multiplying by the inverses of these elementary matrices & applying the associative property of matrix multiplication gives us:</u>

$$\underbrace{(E_p \cdots E_1)^{-1}(E_p \cdots E_1)}_{I} A = (E_p \cdots E_1)^{-1} U$$

$$A = (E_p \cdots E_1)^{-1} U = \mathcal{L} U$$

<u>where</u>: $\mathcal{L} = (E_p \cdots E_1)^{-1}$

<u>Note</u>: This same row operation that reduces A to U, also reduces $\mathcal{L}$ to I (the Identity Matrix).

$$\mathcal{L} = (E_p \cdots E_1)^{-1} \rightarrow (E_p \cdots E_1)\mathcal{L} = (E_p \cdots E_1)(E_p \cdots E_1)^{-1} = I$$

Algorithm for an LU Factorization

① Reduce A to an echelon form U by a sequence of row replacement operations, if possible.

② Place entries in L such that the same sequence of row operations reduces L to I.

Note: Step 1 is not always possible, but when it is, this shows that an LU factorization exists :

Additional Notes on the Above Algorithm:

*By construction, L will satisfy: $(E_p \cdots E_1) L = I$ (as previously seen).

*So, by definition & the Invertible Matrix Th^m:

$\Rightarrow L$ is invertible & $L^{-1} = (E_p \cdots E_1)$

$\Rightarrow \underline{(E_p \cdots E_1) A = U}$:

$L^{-1} A = U$ & $A = LU$

*Thus confirming that step ② of the above algorithm produces an acceptable L :

Example (Finding an LU Factorization):

Find an LU Factorization of:

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Answer:

Note: Since A is a 4×5 matrix $\Rightarrow L$ is a 4×4 matrix :

1) The first column of L is the first column of A divided by the top pivot entry:

Let $A = [\vec{a_1}\ \vec{a_2}\ \vec{a_3}\ \vec{a_4}\ \vec{a_5}]$ $\Rightarrow$ $\frac{1}{2}\vec{a_1} = \vec{L_1}$

$L = [\vec{L_1}\ \vec{L_2}\ \vec{L_3}\ \vec{L_4}]$

So,
$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ -3 & * & * & 1 \end{bmatrix}}$$

*Important Note/Observation:
The row operations that create zeros in the 1st column of A will also create zeros in the first column of L

$\Rightarrow$ By row-reducing A to an echelon form of U, we will be able to produce the remaining columns of L using the same technique as used in ① to find $\vec{L_1}$:

Ex (Finding an LU Factorization) Continued...

*Helpful Tip When Getting Started:
As you row-reduce A to U, highlight the entries in each matrix that are used to determine the sequence of row-operations that transform A to U :: (*Below in this Color)

② Row-Reduce A to an Echelon Form of U:

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \underset{N.R_2}{\overset{2R_1 + R_2}{\sim}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \underset{N.R_2}{\overset{-R_1 + R_2}{\sim}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}.$$

$$\underset{N.R_4}{\overset{3R_1 + R_4}{\sim}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \underset{N.R_3}{\overset{3R_2 + R_3}{\sim}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \underset{N.R_4}{\overset{-4R_2 + R_4}{\sim}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}.$$

$$\underset{N.R_4}{\overset{-2R_3 + R_4}{\sim}} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \checkmark$$

$$\boxed{\therefore U = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}}$$

③ Use the highlighted entries above (that determined row-reduction of A to U) to find the remaining columns of L:

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} \Longrightarrow$$

So the resulting columns of L are:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

*Divide each column by the circled pivot.

Ex (Finding an LU Factorization) Continued...

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ -3 & * & * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Therefore, the LU Factorization of A:

$$\underset{A}{\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 6 & 0 & 7 & -3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 4 & -1 & 5 & 2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}}$$

Example: Find an LU Factorization of the matrix A:

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 6 & 2 & -5 \\ 6 & 12 & 23 \end{bmatrix}$$

Answer:

① The first column of L is the 1^{st} column of A, divided by the top pivot (-2): { IOW: $\vec{L}_1 = -\frac{1}{2}\vec{a}_1$ }

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & * & * \end{bmatrix}$$

② Row-Reduce A to an echelon form of U:

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 6 & 2 & -5 \\ 6 & 12 & 23 \end{bmatrix} \underset{}{\overset{\frac{3R_1 + R_2}{N.R_2}}{\sim}} \begin{bmatrix} -2 & 0 & 3 \\ 0 & 2 & 4 \\ 6 & 12 & 23 \end{bmatrix} \overset{\frac{3R_1 + R_3}{N.R_3}}{\sim} \begin{bmatrix} -2 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 12 & 32 \end{bmatrix} \overset{\frac{-6R_2 + R_3}{N.R_3}}{\sim}$$

$$\begin{bmatrix} -2 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix} \checkmark \implies \boxed{\therefore U = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix}}$$

Ans✓

* Echelon Form *

③ Use the Highlighted Columns to find the remaining columns of L:

$$\begin{bmatrix} -2 \\ 6 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ 12 \end{bmatrix} \begin{bmatrix} 8 \end{bmatrix} \implies \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$\boxed{\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 6 & 1 \end{bmatrix}}$$

* Divide by the pivots *

Ans✓

Example Continued...

Note: Using the highlighted columns of A to find the columns of L is a short cut to LH multiplying both sides by the inverse of the sequence of elementary matrices :)

* Sequence of Elementary Matrices:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

* By the LU-Factorization Algorithm, we know:

$$(E_3 \cdot E_2 \cdot E_1) A = U$$

$$A = (E_3 E_2 E_1)^{-1} U$$

$$A = LU$$

where: $L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$

Lets Check:

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 6 & 1 \end{bmatrix} \checkmark$$

Example: Find an LU Factorization of the matrix A:

$$A = \begin{bmatrix} 4 & 6 \\ 12 & 16 \end{bmatrix}$$

Answer:

① The first column of L is the 1^{st} Column of A, divided by the top pivot (4): $\{ * \ \vec{l}_1 = \frac{1}{4}\vec{a}_1 \}$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & * \end{bmatrix}$$

*We know $* = 1$, but need to verify w/ computation.

② Row-Reduce A to echelon form of U:

$$A = \begin{bmatrix} 4 & 6 \\ 12 & 16 \end{bmatrix} \underset{}{\overset{-3R_1 + R_2}{\sim}} \begin{bmatrix} 4 & 6 \\ 0 & -2 \end{bmatrix} \checkmark \implies \boxed{\therefore\ U = \begin{bmatrix} 4 & 6 \\ 0 & -2 \end{bmatrix}}$$

echelon form

③ Use the highlighted entries to find the remaining columns of A:

$$\begin{bmatrix} 4 \\ 12 \end{bmatrix} \begin{bmatrix} -2 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \quad \& \quad \boxed{\therefore\ L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

*Divide by the pivots

*Check: $LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4+0 & 6+0 \\ 12+0 & 18-2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 12 & 16 \end{bmatrix} = A \checkmark$

Example: Find an LU Factorization of the matrix A:

$$A = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$$

Answer:

Note: Since A is $2\times2 \Rightarrow L$ is 2×2

① The first column of L is the first column of A divided by the top pivot (3):

$$\boxed{L = \begin{bmatrix} 1 & 0 \\ -4/3 & * \end{bmatrix}}$$

Note: The missing entry "$*$" should be 1, but lets verify computationally
$\Rightarrow$ Not all matrices are cute 2×2s :'(

② Row-Reduce A to echelon form of U:

$$A = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix} \xrightarrow[\text{N. } R_2]{\frac{4}{3}R_1 + R_2} \sim \begin{bmatrix} 3 & 4 \\ 0 & 7/3 \end{bmatrix} \checkmark \qquad \boxed{\therefore U \begin{bmatrix} 3 & 4 \\ 0 & 7/3 \end{bmatrix}} \quad \text{Ans.}$$

echelon form

③ Use the highlighted Entries to Determine remaining columns of L:

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} \begin{bmatrix} 7/3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ -4/3 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} ; \qquad \boxed{\therefore L = \begin{bmatrix} 1 & 0 \\ -4/3 & 1 \end{bmatrix}} \quad \text{Ans.}$$

Divide by pivot

Check (For Good Luck):

$$LU = \begin{bmatrix} 1 & 0 \\ -4/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 7/3 \end{bmatrix} = \begin{bmatrix} 3+0 & 4+0 \\ -4+0 & \underbrace{-\frac{16}{3} + \frac{7}{3}}_{-\frac{9}{3}} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix} = A \checkmark$$

woohoo!

Example: When A is invertible, MATLAB finds A^{-1} by factoring LU (where L may be permuted lower triangle), inverting L & U, and then computing $U^{-1}L^{-1}$. Use this method to compute the inverse of the given matrix:

$$A = \begin{bmatrix} 4 & -12 & 4 \\ -20 & 56 & -12 \\ 0 & -4 & 4 \end{bmatrix}, \text{ where: } A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & -12 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -4 \end{bmatrix}}_{U}$$

Answer:

*Find L^{-1}: Row-Reduce the Augmented Matrix $[L \,\vdots\, I_3]$ to $[I_3 \,\vdots\, L^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{N.R_2}{\overset{5R_1+R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{N.R_3}{\overset{-R_2+R_3}{\sim}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] \Rightarrow \boxed{\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -5 & -1 & 1 \end{bmatrix}}$$ Answer

*Find U^{-1}: Row-Reduce the Aug. Matrix $[U \,\vdots\, I_3]$ to $[I_3 \,\vdots\, U^{-1}]$:

$$\left[\begin{array}{ccc|ccc} 4 & -12 & 4 & 1 & 0 & 0 \\ 0 & -4 & 8 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & -2 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right] \underset{N.R_1}{\overset{3R_2+R_1}{\sim}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & \frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 1 & -2 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right] \underset{N.R_2}{\overset{2R_3+R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1/4 & -3/4 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/4 \end{array}\right] \underset{N.R_1}{\overset{5R_3+R_1}{\sim}}$$

Example Continued...

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -3/4 & -5/4 \\ 0 & 1 & 0 & 0 & -1/4 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/4 \end{array}\right] \Rightarrow$$

$$\boxed{U^{-1} = \begin{bmatrix} 1/4 & -3/4 & -5/4 \\ 0 & -1/4 & -1/2 \\ 0 & 0 & -1/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -3 & -5 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}}$$

Ans✓

*Find $A^{-1} = U^{-1} L^{-1}$:

$$\frac{1}{4}\begin{bmatrix} 1 & -3 & -5 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -5 & -1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1-15+25 & 0-3+5 & 0+0-5 \\ 0-5+10 & 0-1+2 & 0+0-2 \\ 0+0+5 & 0+0+1 & 0+0-1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 11 & 2 & -5 \\ 5 & 1 & -2 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 11 & 2 & -5 \\ 5 & 1 & -2 \\ 5 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 11/4 & 1/2 & -5/4 \\ 5/4 & 1/4 & -1/2 \\ 5/4 & 1/4 & -1/4 \end{bmatrix}}$$

Ans✓

Example/Property: \$ that $A = BC$, where B is invertible. Show that any sequence of row operations that reduces B to I also reduces A to C, where I = Identity Matrix.

Note: The converse is NOT true, since the zero matrix may be factored as $0 = B \cdot 0$

Answer:

\$ that $A = BC$, where: $\begin{cases} * B \text{ is invertible} \\ * A, B, C \text{ are size appropriate matrices.} \end{cases}$

- By Definition: Since B is invertible, $\exists$ elementary matrices $E_1, E_2, \dots, E_p$ (corresponding to elementary row-operations), that reduce B to the Identity Matrix I

$$\Rightarrow (E_p \cdots E_1) B = I$$

* Goal: Show that this same sequence of elementary row operations ALSO reduces A to C.

* LH Multiply $A = BC$ by the same sequence of elementary matrices:

$$A = BC$$

$$(E_p \cdots E_1) A = (E_p \cdots E_1) BC$$

* By the Associative Prop:

$$(E_p \cdots E_1) A = [(E_p \cdots E_1) B] C$$

$$(E_p \cdots E_1) A = I C$$

$(E_p \cdots E_1) A = C$ ✓ $\rightarrow$ ∴ The same seq. reduces A to C ☒