Divide-and-conquer approach

Text

Chapters 2

Divide and Conquer

- What's divide-and-conquer
- How to analyze a divide-and-conquer algorithm
- Examples: merge sort, binary search

What is divide and conquer

- A technique for designing algorithms that decompose instance into smaller sub-instance of the same problem
  - Solving the sub-instances independently
  - Combining the sub-solutions to obtain the solution of the original instance

Divide-and-conquer

- basic steps:
  - **divide** the problem into sub-problems similar to original problem but smaller in size
  - **conquer** the sub-problems recursively
  - **combine** solutions to create solution to original problem

Merge Sort Algorithm

- **Divide**: divide *n*-element sequence into two sub-sequences of *n/2* elements each
- **Conquer**: sort two sub-sequences recursively using merge sort
- **Combine**: Merge the two sorted sub-sequences to produce the sorted answer

Merge Sort Algorithm

Merge-Sort A[1..n]

1. if n =1, done.
2. Recursively sort $A[1..\lceil n/2 \rceil]$ and $A[\lceil n/2 \rceil+1..n]$
3. Merge the two sorted lists

Merging Two Sorted Lists

- choose the smaller element of the two lists
- remove it from list and put it into a list
- repeat previous steps

Merge sort: top down

divide

3	1	4	1	5	9	2	6	5	5	3	8	9
3	1	4	1	5	9	2	6	5	5	3	8	9
3	1	4	1	5	9	2	6	5	5	3	8	9
3	1	4	1	5	9	2	6	5	5	3	8	9
1	3	4	1	5	9	2	5	6	3	5	8	9

Ad-hoc sort for each small instance

Merge sort: top down

merge

1	3	4	1	5	9	2	5	6	3	5	8	9

1	1	3	4	5	9	2	5	6	3	5	8	9

1	1	3	4	5	9	2	3	5	5	6	8	9

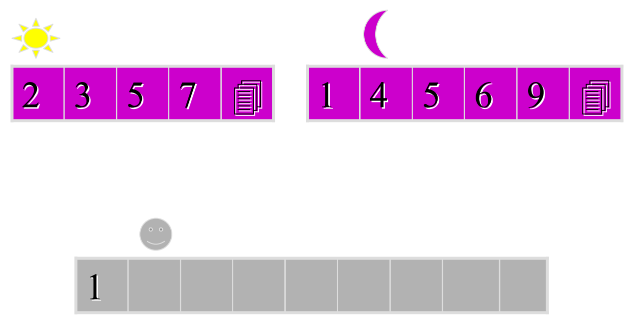
1	1	2	3	3	4	5	5	5	6	8	9	9

Merge two arrays

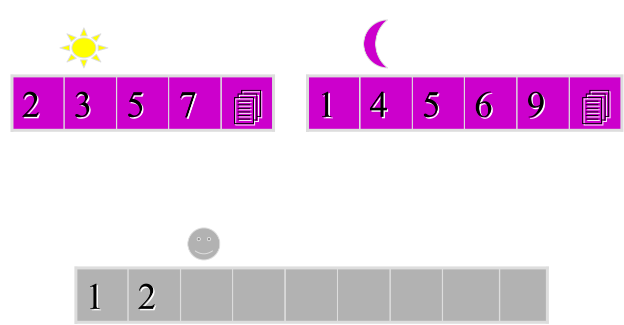
2	3	5	7	

1	4	5	6	9	

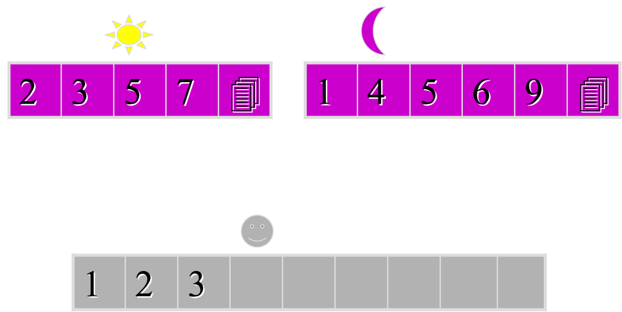
Merge two arrays



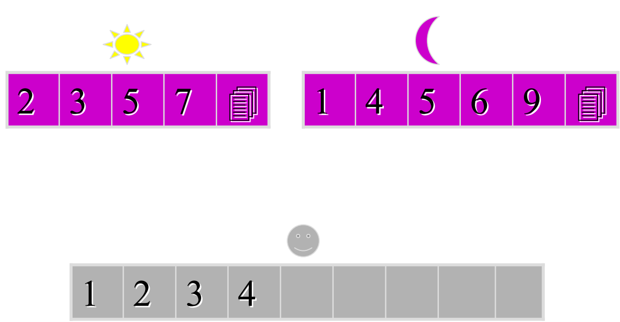
Merge two arrays



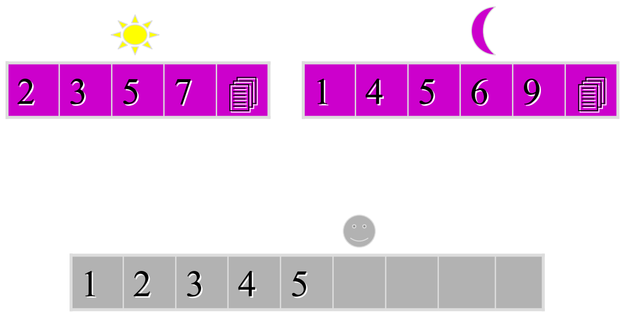
Merge two arrays



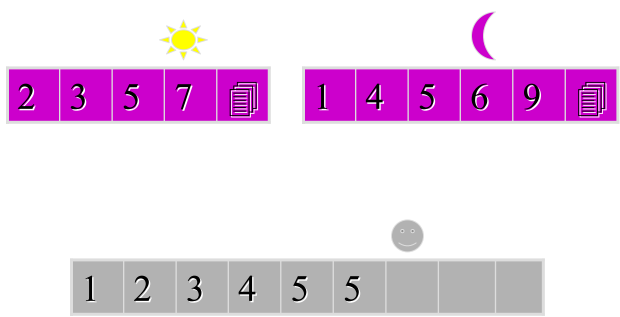
Merge two arrays



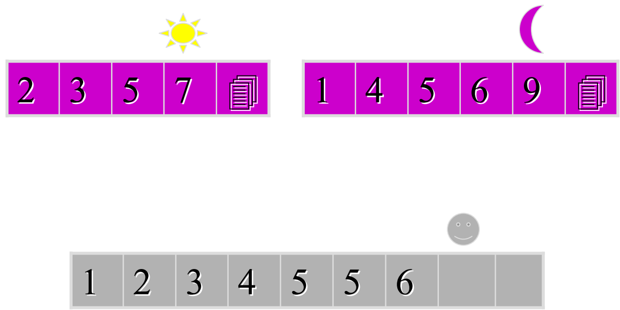
Merge two arrays



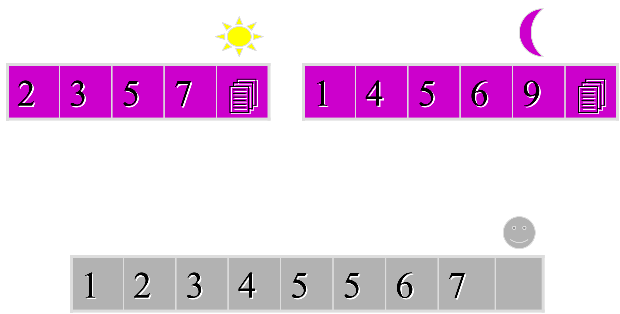
Merge two arrays



Merge two arrays



Merge two arrays

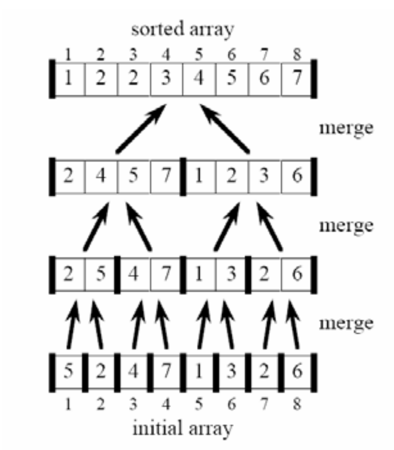


Merge two arrays

2	3	5	7			1	4	5	6	9

1	2	3	4	5	5	6	7	9

Merge Sort



Analyzing Merge Sort

T(n)	Merge-Sort A[1..n]
Θ (1) ⟵	1. if n =1, done.
Θ ($\lceil n/2 \rceil$)+ Θ ($\lfloor n/2 \rfloor$) ⟵ ~ 2Θ (n/2)	2. Recursively sort A[1..$\lceil n/2 \rceil$] and A[$\lceil n/2 \rceil$+1.. n]
Θ (n) ⟵	3. Merge the two sorted lists

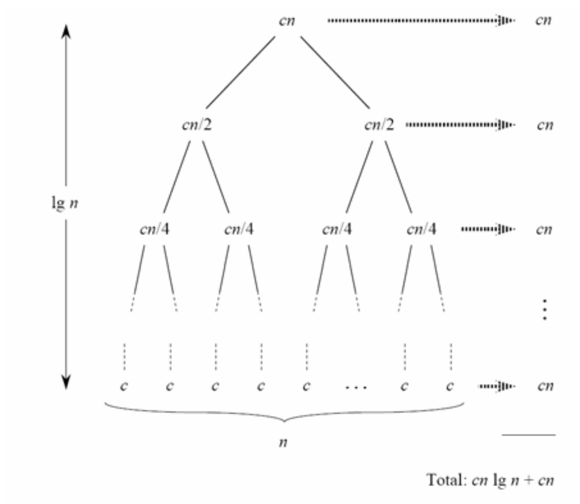
Recurrence: $$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1 \\ 2T(n/2) + \Theta(n), & \text{if } n > 1 \end{cases}$$

Recursion Tree

$$T(n) = \begin{cases} c, & \text{if } n = 1 \\ 2T(n/2) + cn, & \text{if } n > 1 \end{cases}$$

T(n) = cn lg n + cn = Θ (n lg n)

Recursion Tree



A general template

```
DC(x)
{
    if (x is sufficiently small or simple)
            adhoc(x);   // use a basic sub-algorithm

    decompose x into small instances x[0],..,x[l-1];  // divide

    for (i=0;  i<l; i++)                                   //conquer
            s[i] = DC(x[i]);

    combine s[0],.., s[l-1] to obtain solution s for x;  // combine
    return s;
}
```

- Three conditions to be considered
  - When to use the basic sub-algorithm
  - Efficient decomposition and recombination
  - The sub-instances must be roughly the same size

Sequential Search from a sorted sequence

- T[] is a sequence in nondecreasing order
- Find an element in T[]

```
sequentialSearch(T[], x)
{
    for (i=0; i<n; i++) {
        if (T[i] >= x) // T[i-1]<x<=T[i]
            return i;
    }

}
```

Cost: best, worst, average?

Binary Search

- **Divide**: check middle element
- **Conquer**: recursively search 1 subarray
- **Combine**: trivial

Binary Search

```
binarySearch(T[], x)
{
    if (n==0 || x>T[n])
        return n;
    else
        return binaryRecursive(T, 1, n, x);
}
```

```
binaryRecursive(T[], i, j, x)
{
    // we know T[i-1] < x <= T[j]
    if (i==j)
        return i;
    k = (i+j)/2;
    if (x <= T[k])
        return binaryRecursive(T, i, k, x);
    else
        return binaryRecursive(T, k+1, j, x);
}
```

Cost?

Binary Search: Example

3	5	7	8	9	12	15
3	5	7	8	9	12	15
3	5	7	8	9	12	15
3	5	7	8	9	12	15

Cost of binary search

- $T(n) = 1\ T(n/2) + \Theta(1)$

work dividing and combining

number of sub-problem   size of sub-problem

$T(n) = \Theta(\lg n)$