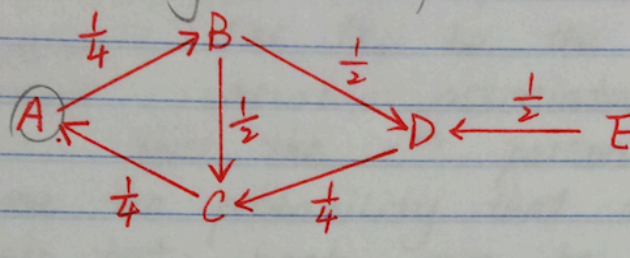
COMP.4040 HW5 Solution

1. **Solution**(credit from GaoGao):

1. Probability with Graph

a) A message sent from A can reach node B, C, D



b) The probability that a message can travel from A to B then to D and then to C

$$A \to B \to D \to C$$

$$P_{A-B-D-C} = \frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{32}$$

The probability that a message can travel from A to B then to C

$$A \to B \to C$$

$$P_{A-B-C} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

So the probability that a message sent from A can travel to C

☆ $$P_{A\to C} = \frac{1}{8} + \frac{1}{32} = \frac{5}{32}$$

The probability that a message sent from A can travel to C and then a responding message can travel from C to A

$$P = \frac{5}{32} \times \frac{1}{4} = \frac{5}{128}$$

2. **Solution**(credit from Gao Gao):

Solution: Let X_i be the indicator random variable associated with the event that the i^{th} person gets their hat back.

Since the probability that each person gets their hat back can be $1/n$. So

$$E[X] = E[X_1 + X_2 + X_3 + \cdots + X_n]$$

$$= \sum_{i=1}^{n} E[X_i]$$

E[Xi] = Pr{Xi = 1} = 1/n

$$= \sum_{i=1}^{n} \frac{1}{n} = 1$$

Therefore the expected number of customers who get back their own hat is 1.

Add the following two lines after line 3 above.

X_i = I {customer I gets back his own hat}

Then $X = X_1 + X_2 + \ldots + X_n$.

3. **Solution (**From Denzel Pierre**)**:

Let X_{ij} be the indicator random variable that represents whether the i^{th} and j^{th} element is an inversion.

Therefore, $X_{ij} = \begin{cases} 1, & i^{th} \text{ and } j^{th} \text{ element is an inversion.} \\ 0, & i^{th} \text{ and } j^{th} \text{ element is not an inversion.} \end{cases}$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

$$\Pr(i < j) = \frac{1}{2}$$

$$E[X_{ij}] = \frac{1}{2}$$

$$\begin{aligned} E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{2} \\ &= \binom{n}{2} * \left(\frac{1}{2}\right) \\ &= \frac{n(n-1)}{2} * \frac{1}{2} \end{aligned}$$

4. **Solution**(credit from Denzel Pierre):

a. Quicksort Runtime: $T(n) = T(q-1) + T(n-q) = \Theta(n)$

f every element in the array was equal, after Partition is called, $q = r$, and every element in the subarray $A[p \dots q-1]$ are equal.

Quicksort then becomes $T(n) = T(n-1) + T(0) + \Theta(n)$
$= \Theta(n^2)$

b. f each element of array $A[p \dots q-1] \le A[q]$, and $A[q+1 \dots r] > A[q]$

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PARTITION'(A, p, r)
x = A[p]
i = p
t = p
for j = p + 1 to r
    if A[j] < x
        y = A[j]
        A[j] = A[t + 1]
        A[t + 1] = A[i]
        A[i] = y
        i = i + 1 //i++
        t = t + 1 //t++
    else if A[j] == x
        exchange A[t + 1] and A[j]
        t = t + 1
return (i, t)
```

c.

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RANDOMIZED-QUICKSORT'(A, p, r)
if p < r
    (q, t) = PARTITION'(A, p, r)
    RANDOMIZED-QUICKSORT'(A, p, q - 1)
    RANDOMIZED-QUICKSORT'(A, t + 1, r)
```

```
QUICKSORT'(A, p, r)
if p < r
    (q, t) = RANDOMIZED-PARTITION'(A, p, r)
    QUICKSORT'(A, p, q - 1)
    QUICKSORT'(A, t + 1, r)
```