55/60

16/20

Probability and Statistics

Week 5-6 Exam

Name: Giang Tran

A single ONE-SIDED 4×6 *notecard may be used.*

A calculator may be used for basic calculations only (addition, multiplication, etc).

Show ALL work. If you're taking $2 \times 3 = 6$ **then WRITE** $2 \times 3 = 6$**.**

Answers without work will receive no credit!

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1 (Discrete Random Variables) Suppose that a discrete random variable X has PDF given by:

$$\wp_X(k) = \begin{cases} 0.1 & \text{if } k = -2 \\ 0.6 & \text{if } k = 0 \\ 0.3 & \text{if } k = 1 \\ 0 & \text{everywhere else} \end{cases}$$

(a) Use the PDF to find the CDF $F_X(k)$.

$F_X(k)$	k
0.1	-2
0.7	0
1	1

$F_X(-2) = \sum_{k \le -2} p_X(k) = p_X(-2) = 0.1$

$F_X(0) = \sum_{k \le 0} p_X(k) = p_X(-2) + p_X(0) = 0.1 + 0.6 = 0.7$

$F_X(1) = \sum_{k \le 1} p_X(k) = p_X(-2) + p_X(0) + p_X(1) = 0.1 + 0.6 + 0.3 = 1$

0 if $k < -2$

if $-2 \le k < 0$

if $0 \le k <$

if $k > 0$

6/10

(b) Use the PDF to find $P(-2 < X < 1)$. And then use the CDF to find the probability again.

Using PDF: $P(-2 < X < 1) = \sum_{-2 < k < 1} p_X(k)$

$= p_X(0) = 0.6$

10/10

Using CDF: $P(-2 < X < 1) = P(X < 1) - P(X \le -2)$

$= P(X \le 0) - P(X \le -2)$

$= F_X(0) - F_X(-2)$

$= 0.7 - 0.1$

$= 0.6$

$\frac{39}{40}$

$\boxed{2}$ (Continuous Random Variables) A continuous random variable Y has PDF given by

$$f_Y(y) = \begin{cases} 1/4 & \text{if } -3 \le y < -1 \\ c\,y^3 & \text{if } 0 \le y < 2 \\ 0 & \text{everywhere else} \end{cases}$$

(a) Find the correct value of c to make this into a probability density function (PDF).

$\frac{10}{10}$

$\int_{-\infty}^{\infty} f_Y(y)dy = 1$

$\Leftrightarrow \int_{-\infty}^{-3} 0\,dy + \int_{-3}^{-1} \frac{1}{4}dy + \int_{-1}^{0} 0\,dy + \int_{0}^{2} cy^3\,dy + \int_2^{\infty} 0\,dy = 1$

$\Leftrightarrow 0 + \frac{1}{4}y\Big|_{-3}^{-1} + 0 + \frac{cy^4}{4}\Big|_0^2 + 0 = 1 \Leftrightarrow \frac{1}{4}(-1+3) + c(4-0) = 1$

$\Leftrightarrow \frac{1}{2} + 4c = 1 \Leftrightarrow \boxed{c = \frac{1}{8}}$

(b) Find the CDF $F_Y(y)$.

$\frac{10}{10}$

When $y < -3$: $F_Y(y) = \int_{-\infty}^{y} 0\,dy = 0$

When $-3 \le y < -1$: $F_Y(y) = \int_{-\infty}^{-3} 0\,dy + \int_{-3}^{y} \frac{1}{4}dy = 0 + \frac{y}{4}\Big|_{-3}^{y} = \frac{y}{4} + \frac{3}{4} = \frac{y+3}{4}$

When $-1 \le y < 0$: $F_Y(y) = \int_{-\infty}^{-3} 0\,dy + \int_{-3}^{-1} \frac{1}{4}dy + \int_{-1}^{y} 0\,dy = 0 + \frac{y}{4}\Big|_{-3}^{-1} + 0 = \frac{-1}{4} + \frac{3}{4} = \frac{1}{2}$

When $0 \le y < 2$: $F_Y(y) = \int_{-\infty}^{-3} 0\,dy + \int_{-3}^{-1} \frac{1}{4}dy + \int_{-1}^{0} 0\,dy + \int_0^y \frac{1}{8}y^3 dy = 0 + \frac{1}{2} + 0 + \frac{1}{8}\left(\frac{y^4}{4}\Big|_0^y\right)$
$= \frac{1}{2} + \frac{y^4}{32}$

When $y \ge 2$: $F_Y(y) = 1$

(c) Use the PDF to find $P(-2 < Y < 1)$. And then use the CDF to find the probability again.

Using PDF: $P(-2 < Y < 1) = \int_{-2}^{1} f_Y(y)dy = \int_{-2}^{-1} \frac{1}{4}dy + \int_{-1}^{0} 0\,dy + \int_0^1 \frac{1}{8}y^3 dy$

$= \frac{y}{4}\Big|_{-2}^{-1} + \frac{1}{8}\left(\frac{y^4}{4}\Big|_0^1\right)$

$= \frac{-1}{4} + \frac{2}{4} + \frac{1}{32} = \frac{1}{4} + \frac{1}{32} = \frac{9}{32}$

Using CDF: $P(-2 < Y < 1) = P(Y < 1) - P(Y \le -2)$

$= P(Y \le 0) - P(Y \le -2)$

$= F_Y(1) - F_Y(-2)$

$= \frac{1}{2} + \frac{1^4}{32} - \frac{-2+3}{4} = \frac{1}{2} + \frac{1}{32} - \frac{1}{4} = \frac{9}{32}$

$\frac{10}{10}$

(d) Use the CDF to "find" the PDF $f_Y(y)$. Make sure to show work or I will assume you just copied $f_Y(y)$ from the question.

$9/10$

When $y < -3$: $F_Y(y) = 0 \Rightarrow f_Y(y) = \frac{d}{dy}(0) = 0$

When $-3 \le y < -1$: $F_Y(y) = \frac{y+3}{4} = \frac{y}{4} + \frac{3}{4} \Rightarrow f_Y(y) = \frac{d}{dy}\left(\frac{y}{4} + \frac{3}{4}\right) = \frac{1}{4} + 0 = \frac{1}{4}$

When $-1 \le y < 0$: $F_Y(y) = \frac{1}{2} \Rightarrow f_Y(y) = \frac{d}{dy}\left(\frac{1}{2}\right) = 0$

When $0 \le y < 2$: $F_Y(y) = \frac{1}{2} + \frac{y^4}{32} \Rightarrow f_Y(y) = \frac{d}{dy}\left(\frac{1}{2} + \frac{y^4}{32}\right) = 0 + \frac{4y^3}{32} = \frac{y^3}{8}$

When $y \ge 2$: $F_Y(y) = 1 \Rightarrow f_Y(y) = \frac{d}{dy}(1) = 0$

89/90

30/30

Probability and Statistics
Week 7-8 Exam

Name: Giang Tran

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Show ALL work. If you're taking $2 \times 3 = 6$ **then WRITE** $2 \times 3 = 6$**.**

Answers without work will receive no credit!

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1 (Empirical Rule)
Suppose UML has a freshman class of 2,352 new students. Based on (actual) past data each student has a 75% chance of returning, independent of other students. The administration would like an estimate of the number of students who will return next year.

(a) What is the expected number of students who will return?
What is the standard deviation in the number that will return?

~~(a)~~ ~~(Binom~~) Let X be the binomial random variable of students who will return with $n = \cancel{25}\ 2352$, $p = 75\% = 0.75$

$E(X) = np = 2352 \times 0.75 = \boxed{1764 \text{ students}}$

$\sigma(X) = \sqrt{np(1-p)} = \sqrt{2352 \times 0.75 \times 0.25} = \boxed{21 \text{ students}}$

10/10

(b) Use the Empirical Rule to approximate the probability that more than 76% of students return.

$76\% \times 2352 =$ (~~1788 students~~) $1787.5 \approx 1788$ students

We have: $E(X) + \sigma(X) = \mu + \cancel{1}\sigma(X) = 1764 + 21 = 1785 \approx 1788$

Apply Empirical Rule, there ~~(are)~~ is 68% chance that number of students who will return is within 1743 to 1788

$\Rightarrow$ $\boxed{16\%}$ chance that more than 76% (1788 students) of students return.

10/10

(c) Use Binomial Probabilities to find the exact probability of this occurring.

$$P(X > 1788) = \sum_{k=1788}^{2352} \binom{2352}{k} 0.75^k \cdot 0.25^{2352-k}$$

10/10

$\frac{29}{30}$

2 (Discrete Random Variables)
Parking meter X costs \$0.25. The chance I get a \$20 parking ticket when I park and pay (and sometimes don't come back on time) is about 3%. Let X be a random variable representing the cost of parking (so $X = \$0.25$ or $X = \$20.25$ depending on whether I return on time).

(a) What is the average cost of parking here, i.e. the expected cost to me of parking here? Based on your answer, would I be better off paying \$0.50 (and getting no tickets)?

$E(X) = \sum_k k p_X(k) = \$0.25 \times 0.97 + \$20.25 \times 0.03 = \boxed{\$0.85}$

$\Rightarrow$ I should pay \$0.50 because \$0.50 < \$0.85.

$\frac{10}{10}$

(b) What is the standard deviation in the cost of parking here.

$E(X^2) = \sum_k k^2 p_X(k) = \$0.25^2 \times 0.97 + \$20.25^2 \times 0.03 \approx \$12.36$

$Var(X) = E(X^2) - [E(X)]^2 = \$12.36 - \$0.85^2 \approx \$11.64$

$\sigma(X) = \sqrt{Var(X)} \approx \boxed{\$3.41}$

$\frac{10}{10}$

(c) Parking meter Y isn't as convenient but also isn't monitored as carefully, so I get fewer tickets there. It has average parking cost of \$0.45 and standard deviation of \$2.10. This month I park 10 times at meter X and 7 times at meter Y. What is the expected total parking cost this month? What is the standard deviation of it?

Note: You may write your answer in terms of the answers in parts (a) and (b), such as

$$\textit{expected cost} = \$17.32 + 3 \cdot (a)$$

$E(Y) = \$0.45$, $\sigma(Y) = \$2.10 \Rightarrow Var(Y) = \$2.10^2 = \$4.41$

$E(10X + 7Y) = 10E(X) + 7E(Y) = 10 \times \$0.85 + 7 \times \$0.45 = \boxed{\$11.65}$

$\frac{6}{6}$

X and Y are independent $\Rightarrow Cov(X, Y) = 0$

$Var(\cancel{10X + 7Y}) = 10^2 Var(X) + 7^2 Var(Y)$
$= 100 \times \$11.64 + 49 \times \$4.41$
$\approx \$1380$

$\Rightarrow \sigma(10X + 7Y) = \sqrt{Var(10X + 7Y)} = \sqrt{\$1380} \approx \boxed{\$\cancel{37}.15}$

\$12.13

$\frac{3}{4}$

$\frac{30}{30}$

3 (Continuous Random Variables)
A continuous random variable Y has PDF given by

$$f_Y(y) = \begin{cases} \frac{1}{4} & \text{if } -3 \le y < -1 \\ \frac{1}{8}y^3 & \text{if } 0 \le y < 2 \\ 0 & \text{everywhere else} \end{cases}$$

(a) Find the expectation of Y.

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$
$$= \int_{-\infty}^{-3} 0\,dy + \int_{-3}^{-1} \frac{y}{4}\,dy + \int_{-1}^{0} 0\,dy + \int_0^2 \frac{y^4}{8}\,dy + \int_2^\infty 0\,dy$$
$$= 0 + \frac{y^2}{8}\Big|_{-3}^{-1} + 0 + \frac{y^5}{40}\Big|_0^2 + 0$$
$$= \left(\frac{1}{8} - \frac{9}{8}\right) + \left(\frac{32}{40} - 0\right) = \frac{-1}{5} = \boxed{-0.2}$$

$\frac{10}{10}$

(b) Find the variance of Y.

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy$$
$$= \int_{-\infty}^{-3} 0\,dy + \int_{-3}^{-1} \frac{y^2}{4}\,dy + \int_{-1}^{0} 0\,dy + \int_0^2 \frac{y^5}{8}\,dy + \int_2^\infty 0\,dy$$
$$= 0 + \frac{y^3}{12}\Big|_{-3}^{-1} + 0 + \frac{y^6}{48}\Big|_0^2 + 0$$
$$= \frac{-1}{12} - \left(\frac{-27}{12}\right) + \left(\frac{64}{48} - 0\right) = \frac{7}{2}$$

$$\Rightarrow Var(Y) = E(Y^2) - (E(Y))^2 = \frac{7}{2} - \left(\frac{-1}{5}\right)^2 = \frac{173}{50} = \boxed{3.46}$$

$\frac{10}{10}$

(c) Suppose I take 50 i.i.d. measurements from this distribution, and look at the average measurement $\overline{Y}$. What is the expected value and standard deviation of $\overline{Y}$?

Note: You may write your answer in terms of the answers in parts (a) and (b), such as $\sigma(\overline{Y}) = (b)/3$.

$$\overline{Y} = \frac{Y_1 + Y_2 + \ldots + Y_{50}}{50}$$
$$E(\overline{Y}) = E(Y) = \mu = \frac{-1}{5} = \boxed{-0.2}$$
$$\sigma(\overline{Y}) = \frac{1}{\sqrt{n}} \cdot \sigma(Y) = \frac{1}{\sqrt{50}} \times \sqrt{Var(Y)} = \frac{1}{\sqrt{50}} \times \sqrt{3.46}$$
$$\simeq \boxed{0.26}$$

$\frac{10}{10}$

50/50

60/60

Probability and Statistics

Week 9-11 Exam

Name: Giang Tran

You will not receive credit if you don't show work

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1 Suppose X and Y are discrete random values with joint PDF $p_{X,Y}(x,y)$ given by

$$p_{X,Y}(x,y) = \frac{1+xy}{6} \qquad \text{when } x \in \{-1, 0, 2\} \text{ and } y \in \{-0.5, 0.5\}$$

(a) What is the marginal probability $p_Y(y)$?

$$P_Y(-0.5) = P_{X,Y}(-1,-0.5) + P_{X,Y}(0,-0.5) + P_{X,Y}(2,-0.5)$$

$$= \frac{1+(-1)\times(-0.5)}{6} + \frac{1+0\times(-0.5)}{6} + \frac{1+2\times(-0.5)}{6}$$

$$= \frac{5}{12}$$

5/5

$$P_Y(0.5) = P_{X,Y}(-1,0.5) + P_{X,Y}(0,0.5) + P_{X,Y}(2,0.5)$$

$$= \frac{1+(-1)\times 0.5}{6} + \frac{1+0\times 0.5}{6} + \frac{1+2\times 0.5}{6}$$

$$= \frac{7}{12}$$

(b) What is the probability $P(Y > X)$?

$$P(Y>X) = P_{X,Y}(-1,-0.5) + P_{X,Y}(-1,0.5) + P_{X,Y}(0,0.5)$$

$$= \frac{1+(-1)\times(-0.5)}{6} + \frac{1+(-1)\times 0.5}{6} + \frac{1+0\times 0.5}{6}$$

$$= \frac{1}{2}$$

5/5

15/15

2 Consider the continuous probability density function

$$f_Y(y) = \begin{cases} y & \text{if } 0 \le y \le 1 \\ 2-y & \text{if } 1 \le y \le 2 \end{cases}$$

(a) Find the moment generating function $M_Y(t)$. It suffices to write out a full integral, but you do not need to integrate.

$$M_Y(t) = E(e^{tY})$$
$$= \int_{-\infty}^{\infty} e^{tY} f_Y(y)\,dy$$
$$= \int_0^1 e^{tY} \cdot y\,dy + \int_1^2 e^{tY}(2-y)\,dy$$

5/5

(b) Suppose (incorrectly) that the MGF is $M_Y(t) = \frac{e^t - 1}{t}$.
Use the MGF to find the expectation of Y.

$$\frac{d}{dt} M_Y(t) = \frac{d}{dt}\left(\frac{e^t-1}{t}\right) = \frac{te^t - (e^t - 1)}{t^2} = \frac{te^t - e^t + 1}{t^2} \quad (*)$$

$$E(Y) = M_Y'(0) = \frac{0.e^0 - e^0 + 1}{0^2} = \frac{0}{0} \Rightarrow \text{Use L'Hopital Rule 2 times}$$

$$(*) \overset{(1)}{=} \frac{e^t + te^t - e^t}{2t} \overset{(2)}{=} \frac{\cancel{2}e^t + te^t - \cancel{e^t}}{2t} = \frac{e^t \cancel{+ te^t}}{2}$$

7.5/7.5

$$\Rightarrow E(Y) = M_Y'(0) = \frac{e^0 + \cancel{0.e^0}}{2} = \frac{1}{2}$$

(c) You should find that $E(Y) = 1/2$. Suppose that another random variable X is defined by $X = 5 - 4Y$. What is $E(X)$?

$$E(X) = E(5-4Y)$$
$$= -4E(Y) + 5$$
$$= 3$$

2.5/2.5

$\frac{12.5}{12.5}$

3 A study reported that the average heart rate of an adult is 75bpm, normally distributed, with standard deviation of 5bpm.

(a) Professor Montenegro's average heart rate is quite low, about 63bpm. What percent of adults have a lower heart rate than him?

$\mu = 75\ bpm$, $\sigma = 5\ bpm$

Let X be heart rate of adults.

$P(X \leq 63) = P(Z \leq Z_{63})$

By De Moivre-Laplace: $Z_{63} = \frac{63 - \mu}{\sigma} = \frac{63-75}{5} = -2.4$

$P(X \leq 63) = F_Z(-2.4)$
$= 0.00820$
$= 0.82\%$

$\frac{5}{5}$

(b) You are confident that the average heart rate is more than than 75 bpm. You collect a random sample of some adults and find heart rates of:

70, 82, 75, 65, 74, 81, 85, 68, 81, 79, 73 bpm

Write a hypothesis to test if the average heart rate is more than 75bpm, find the p-value of the test, and determine if this is / isn't statistically significant to conclude that the average heart rate is more than 75bpm.

Note: In a hypothesis test we assume that the status quo is valid, so you should assume that the standard deviation of 5bpm is correct.

Average heart rate $\overline{W} = \frac{70+82+75+65+74+81+85+68+81+79+73}{11}$

≈ 75.73

$n = 11$

Hypothesis:

$H_0 : \mu_0 = 75\ bpm$

$H_1 : \mu > 75\ bpm$

By De Moivre-Laplace:

$Z_{75.73} = \frac{75.73 - \mu_0}{\sigma/\sqrt{n}} = \frac{75.73 - 75}{5/\sqrt{11}} \simeq 0.48$

$\frac{7.5}{7.5}$

p-value $= P(\overline{W} \geq 75.73) = P(Z \geq Z_{75.73}) = 1 - P(Z \leq Z_{75.73})$
$= 1 - F_Z(0.48)$
$= 1 - 0.68439$
$= 0.31561$

p-value $= 0.31561 > 0.05 \Rightarrow$ this isn't statistically significant to conclude that average heart rate is larger than 75 bpm.

$\frac{12.5}{12.5}$

(c) Suppose it is said that 35% of adults have a heart rate of 73 – 77 *bpm*. What is the probability that a sample of size 50 would find 10 or fewer with heart rates in this range? Use the DeMoivre-Laplace Theorem with Continuity Correction.

$p = 0.35$, $n = 50$, Let Y be the number of people whose heart rates range 73-77 bpm.

By DeMoivre-Laplace:

$P(Y \leq 10) = P(Y \leq 10.5)$ (continuity Correction)

$$Z_{10.5} = \frac{10.5 - np}{\sqrt{np(1-p)}} = \frac{10.5 - 50 \times 0.35}{\sqrt{50 \times 0.35 \times 0.65}} \approx -2.08$$

$$\Rightarrow P(Y \leq 10) = P(Z \leq Z_{10.5}) = F_Z(-2.08)$$
$$= \cancel{0.01876}\ 0.01876$$
$$= 1.876\%$$

$\frac{7.5}{7.5}$

(d) Consider the sample data from the previous question. Write a hypothesis to test the 35% claim. Then use your answer to the previous part to find the *p*-value and determine if this is / isn't statistically significant to conclude that the 35% claim is incorrect.

Hypothesis:

$H_0 : p_0 = 35\%$

$H_1 : p \neq 35\%$

$$p\text{-value} = 2 \times P(Y \leq 10)$$
$$= 2 \times 0.01876$$
$$= 0.03752$$

$\frac{5}{5}$ $\Rightarrow$ p-value $= 0.03752 < 0.05$

$\Rightarrow$ this is statistically significant to conclude that the 35% claim is incorrect.

41/55

15/20

Probability and Statistics

Exam #2

Name: Giang Tran

A single ONE-SIDED 4 × 6 *notecard may be used.*

A calculator may be used for basic calculations only (addition, multiplication, etc).

Show ALL work. If you're taking $2 \times 3 = 6$ **then WRITE** $2 \times 3 = 6$.

Answers without work will receive no credit!

$\boxed{1}$ A small firm has 3 web servers: Schumacher, Andretti, and Patrick. We'll call them S, A, and P for short. The probability that S crashes this week is 10%, for A it's 6%, and for P it's 8%.

(a) If crashes are independent then what is the probability that exactly 1 server crashes this week?

$P(S) = 0.10$, $P(A) = 0.06$, $P(P) = 0.08$

Probability that exactly 1 ~~sever~~ server crashes

$= \cancel{P(S \cup A \cup P)}\ 1 - P(A \cap S) - P(A \cap P) - P(S \cap P) - P(A \cap S \cap P)$

$= 1 - 2P(A) \times P(S) - 2P(A) \times P(P) - 2P(S) \times P(P) + 6P(A) \times P(S) \times P(P)$

$= 1 - 0.06 \times 0.10 - 0.06 \times 0.08 - 0.10 \times 0.08 - 0.06 \times 0.10 \times 0.08$

$\approx 0.98 \approx 98\%$

10/15

(b) There is a 1% chance that S and A both crash this week. Are crashes of S and A independent?

If those are independent $\Rightarrow P(S \cap A) = P(S) \times P(A)$

$= 0.10 \times 0.06$

$= 0.006$

$\Rightarrow P(S \cap A) \neq 0.01$

or $P(S \cap A) \neq P(S) \times P(A)$

$\Rightarrow$ These crashes are dependent.

5/5

26/35

2 Suppose that 45 students, 20 female and 25 male, are applying to a company. There are 5 job openings.

Important: You will receive no credit on any part if you do not explain. Say something like "Use a permutation because ______."

(a) Interview panel members rank students from 1^{st} choice to 30^{th}, leaving 15 students unranked. If Tammy is a panelist then how many ways can she rank the students?

We choose 30 students from 45 students to rank

5/5 $\Rightarrow P(45,30) = \dfrac{45!}{(45-30)!} \approx 9.148 \times 10^{43}$ ways to rank

Use a permutation because it's ranking $\Rightarrow$ order matters.

(b) The company knows that some students will turn them down, so they decide to make 7 or 8 offers. How many ways can they select students to make offers to?

A: Choose 7 from 45 students to offer $\Rightarrow |A| = C(45,7)$

10/10 B: Choose 8 from 45 students to offer $\Rightarrow |B| = C(45,8)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= C(45,7) + C(45,8)$ ways ($P(A \cap B) = 0$ because cannot choose both 7 and 8 offers)

(c) Suppose that 8 offers are made. If the offers are made randomly then we would expect 3 or 4 of the 8 offers would be made to women. The company makes 6 offers to women. What is the probability that 6 or more of the offers would go to women by chance if there is no gender bias and both men and women are equally qualified for the work?

Bionomial probability: $P(k \geq 6) = P(k=6) + P(k=7) + P(k=8)$

$= \binom{8}{6} p^6 (1-p)^2 + \binom{8}{7} p^7 (1-p)^1 + \binom{8}{8} p^8 (1-p)^0$

3/10 With p is the probability that an offer made to a woman.

Hypergeometric +0/7

Add cases +3/3

(d) Again suppose that 8 offers are made. Students have a 75% chance of accepting a job offer, independent of the other student's decisions. What is the chance the company fails to fill all the positions, i.e. 4 or fewer students accept?

p: probability a student accepts a job

$p = 0.75$

k is the number of students who accept the offers.

Bionomial probability

8/10 $P(k \leq 4) = P(k=0) + P(k=1) + P(k=2) + P(k=3) + P(k=4)$

$= \binom{45}{0} p^0 (1-p)^{45} + \binom{45}{1} p^1 (1-p)^{44} + \binom{45}{2} p^2 (1-p)^{43} + \binom{45}{3} p^3 (1-p)^{42} + \binom{45}{4} p^4 (1-p)^{41}$

8 8 8

$\frac{10}{10}$ G.T

Name: Giang Tran

Homework #2

$\frac{6}{6}$ 2.4.11

A: voters like A

B: voters like B

$A \cap B$: voters like both

$A \cup B$: voters like either A or B

$\neg(A \cup B)$: voters like neither A nor B

a) $$\begin{aligned} P(\neg(A \cup B)) &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - (0.65 + 0.55 - 0.25) \\ &= 1 - 0.95 \\ &= 0.05 = \frac{5}{100} \end{aligned}$$

b) Probability that someone likes exactly one:

$$\begin{aligned} P(\text{like exactly one}) &= 1 - P(A \cap B) - P(\neg(A \cup B)) \\ &= 1 - 0.25 - 0.05 \\ &= 0.70 = \frac{70}{100} \end{aligned}$$

c) $$\begin{aligned} P(\text{like at least one}) = P(A \cup B) &= 1 - P(\neg(A \cup B)) \\ &= 1 - 0.05 \\ &= 0.95 = \frac{95}{100} \end{aligned}$$

d) $$\begin{aligned} P(\text{like at most one}) &= P(\text{like exactly one}) + P(\neg(A \cup B)) \\ &= 0.70 + 0.05 \\ &= 0.75 = \frac{75}{100} \end{aligned}$$

G.T

e) $P(\text{like exactly one} \mid \text{like at least one})$

$$= \frac{P(\text{like exactly one and like at least one})}{P(\text{like at least one})}$$

$$= \frac{0.70}{0.95} = \frac{70}{95}$$

f) $P(A \cap B \mid \text{like at least one})$

$$= \frac{P(\text{like both and like at least one})}{P(\text{like at least one})}$$

$$= \frac{0.25}{0.95} = \frac{25}{95}$$

g) $$P(B \mid \bar{A}) = \frac{P(B \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(B) - P(B \cap A)}{P(\bar{A})} = \frac{P(B) - P(B \cap A)}{1 - P(A)}$$

$$= \frac{0.55 - 0.25}{1 - 0.65}$$

$$= \frac{0.30}{0.35} = \frac{30}{35}$$

2.4.13

A: number on the first die was at least as large as 4.

B: Sum of the two dice was 8.

$B = \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$

$A \mid B = \{(6,2), (5,3), (4,4)\}$

$\Rightarrow P(A \mid B) = \frac{3}{5}$

2.4.19

A: Austialian Doll wins

B: Dusty Stake wins

C: Outandout wins

$$P(C \mid \bar{A} \cap \bar{B}) = \frac{P(C \cap \bar{A} \cap \bar{B})}{P(\bar{A} \cap \bar{B})}$$

$$= \frac{0.20}{1-0.15-0.30} = \frac{0.20}{0.55} = \frac{20}{55}.$$

2.4.20

A: Andy gets released

B: Bob gets released

C: Charlie gets released

X: The guard says B

$$P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{3}$$

If X happens, Andy knows $A \cap C$ won't happen

$\Rightarrow$ according to Bayes Theorem:

$$P(A \cap B \mid X) = \frac{P(A \cap B) \times P(X \mid A \cap B)}{P(A \cap B) \times P(X \mid A \cap B) + P(B \cap C) \times P(X \mid B \cap C)}$$

$$= \frac{P(X \mid A \cap B)}{P(X \mid A \cap B) + P(X \mid B \cap C)} \quad (1)$$

The guard just says a name other than "Andy"

$\Rightarrow P(X \mid A \cap B) = 1, \quad P(X \mid B \cap C) = \frac{1}{2}$

$(1) \Rightarrow \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \Rightarrow$ Andy's concern isn't justified.

G.T

~~2.4.20~~ 2.4.22

If he wants to get to the third key, the first and second ones have to be the wrong keys.

A: first key is successful
B: second key is successful
C: third key is successful

$$P(C \cap \bar{B} \cap \bar{A}) = P(C|\bar{A} \cap \bar{B}) \times P(\bar{B}|\bar{A}) \times P(\bar{A})$$
$$= \left(\frac{1}{n-2}\right) \times \left(1 - \frac{1}{n-1}\right) \times \left(1 - \frac{1}{n}\right)$$
$$= \left(\frac{1}{n-2}\right) \times \left(\frac{n-2}{n-1}\right) \times \left(\frac{n-1}{n}\right)$$
$$= \frac{1}{n}$$

=> Probability of the third key to open the door is $\frac{1}{n}$.

2.4.23

A: draw 7 of diamonds
B: draw jack of spades
C: draw 10 of diamonds
D: draw 5 of hearts

$$P(A \cap B \cap C \cap D) = P(D|A \cap B \cap C) \times P(C|A \cap B) \times P(B|A) \times P(A)$$
$$= \frac{1}{49} \times \frac{1}{50} \times \frac{1}{51} \times \frac{1}{52}$$
$$= \frac{1}{6497400}$$

G.T

2.4.48

A: child is abused

B~~[illegible]~~: detected as abused

~~[illegible]~~ $P(A) = \frac{1}{90} = 0.011$

$P(B \mid \text{Abused}) = 90\% = P(B \mid A)$

$P(B \mid \text{Nonabused}) = 3\% = P(B \mid \bar{A})$

Bayes Theorem:

$$P(A \mid B) = \frac{P(A) \times P(B \mid A)}{P(A) \times P(B \mid A) + P(\bar{A}) \times P(B \mid \bar{A})}$$

$$= \frac{0.011 \times 0.90}{0.011 \times 0.90 + (1-0.011) \times 0.03}$$

$$= 0.25 = 25\%$$

If $P(A) = \frac{1}{1000} = 0.001$

$$P(A \mid B) = \frac{0.001 \times 0.90}{0.001 \times 0.90 + (1-0.001) \times 0.03}$$

$$= 0.029 = 2.9\%$$

If $P(A) = \frac{1}{50} = 0.02$

$$P(A \mid B) = \frac{0.02 \times 0.90}{0.02 \times 0.90 + (1-0.02) \times 0.03}$$

$$= 0.38 = 38\%$$

% of children that the screening will identify as abused.

~~[illegible]~~

1/1

$P(B) = P(A) \times P(B \mid A) + P(\bar{A}) \times P(B \mid \bar{A})$

$= 0.011 \times 0.90 + (1-0.011) \times 0.03 = 0.040 = 4.0\%$

G.T

Bonus:

A: Individuals have HIV

~~B~~ B: Test will be positive

a) $P(B) = P(A) \times P(B|A) + P(\bar{A}) \times P(B|\bar{A})$

$= 0.0002 \times \left(1 - \frac{1}{12}\right) + (1 - 0.0002) \times \frac{1}{4500}$

$= 0.0004 = 0.04\%$

b) $P(A|B) = \dfrac{P(A) \times P(B|A)}{P(A) \times P(B|A) + P(\bar{A}) \times P(B|\bar{A})}$

$= \dfrac{0.0002 \times \left(1 - \frac{1}{12}\right)}{0.0002 \times \left(1 - \frac{1}{12}\right) + (1 - 0.0002) \times \frac{1}{4500}}$

$= 0.458 = 45.8\%$

c) If $P(A) = 0.05$

$\Rightarrow P(A|B) = \dfrac{P(A) \times P(B|A)}{P(A) \times P(B|A) + P(\bar{A}) \times P(B|\bar{A})}$

$= \dfrac{0.05 \times \left(1 - \frac{1}{12}\right)}{0.05 \times \left(1 - \frac{1}{12}\right) + (1 - 0.05) \times \frac{1}{4500}}$

$= 0.995 = 99.5\%$

G.T

10/10

Name : Giang Tran

Homework #3

2.4.13

Two events are not ~~indepents~~ independent because

$P(A) \neq P(A|B)$

5/5

2.5.2

A: Passing Chemistry

B: Passing Mathematics

$A \cap B$: Passing both

$\neg(A \cup B)$: Failing both

$$\begin{aligned}
P(\neg(A \cup B)) &= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 - (0.35 + 0.40 - 0.12) \\
&= 0.37
\end{aligned}$$

These 2 events are not independent because

$P(A \cap B) \neq P(A) \times P(B)$

$\Leftrightarrow 0.12 \neq 0.35 \times 0.40$

$\Leftrightarrow 0.12 \neq 0.14$

2.5.7 $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{8}$

a) 1. If A and B are mutually exclusive:

$P(A \cup B) = P(A) + P(B) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375$

G.T

2. If A and B are independent:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{4} + \frac{1}{8} - P(A) \times P(B)$$
$$= \frac{3}{8} - \frac{1}{4} \times \frac{1}{8}$$
$$= \frac{3}{8} - \frac{1}{32} = \frac{11}{32}$$

b) 1. If A and B are mutually exclusive:

$$P(A|B) = \qquad \frac{P(A \cap B)}{P(B)}$$
$$= \frac{0}{\frac{1}{8}} = 0 \quad (A \cap B = \emptyset)$$

2. If A and B are independent:

$$P(A|B) = P(A) = \frac{1}{4}$$

2.5.17

A: students fails mathematics

B: Student fails language skills

C: Student fails general knowledge

$$P(A) = \frac{3325}{9500} = 0.35$$

$$P(B) = \frac{1900}{9500} = 0.20$$

$$P(C) = \frac{1425}{9500} = 0.15$$

G.T

Student fails to qualify for a diploma

$\Rightarrow$ Student fails at least one of three subjects

$\Rightarrow$ $A \cup B \cup C$ is that event

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= 0.35 + 0.20 + 0.15 - 0.35 \times 0.20 - 0.35 \times 0.15 - 0.20 \times 0.15 + 0.35 \times 0.20 \times 0.15$$

$$= 0.558 \approx 0.56$$

The independence seems a reasonable assumption in this situation.

25.9

a) Probability of heads appearing in 1 toss: $\frac{1}{2}$

$|S| = 2 \times 2 \times 2 \times 2 = 16$

A: events that get numbers of heads appearing in first two tosses is equal to number of heads appearing in second two tosses

$A = \{(T,T,T,T), (T,H,H,T), (T,H,T,H), (H,T,T,H), (H,T,H,T), (H,H,H,H)\}$

$\Rightarrow |A| = 6$

$\Rightarrow P(A) = \frac{6}{16} = 0.375$

G.T

b) ~~$S = \{(H,H,H,H), (H,T,H,T), (T,H,T,H), (T,T,T,T)$~~

$A = \{(H,H,H,H), (H,T,H,T), (T,H,T,H), (H,T,T,H),$
$(T,H,H,T), (T,T,T,T)\}$

B: events that the second two rolls are identical to the first two rolls.

$A|B = \{(H,H,H,H), (H,T,H,T), (T,H,T,H), (T,T,T,T)\}$

$P(A|B) = \frac{4}{16} = 0.25$

c) $P(A|B) \neq P(A)$ $(0.25 \neq 0.375)$
because A and B are not independent (B forces the second two tosses in A ~~be~~ to be the same as the first two tosses even in order)

2.6.2

First part: 20 ways (20 consonants)
Second part: 9 ways (9 numbers from $1 \to 9$)
Third part: 6 ways (6 vowels)
Last part: 20 ways (20 consonants)

=> There are $20 \times 9 \times 6 \times 20 = 21600$ different ciphers can be transmitted.

2.6.9

A. appetizers
E. entrees
D: desserts
B: Berverages

G.T

Possible courses: (A, E, D) or (A, E, B) or (A, D, B) or (E, D, B)

$\Rightarrow$ There are $4 \times 14 \times 6 + 4 \times 14 \times 5 + 4 \times 6 \times 5 + 14 \times 6 \times 5 = 1156$ different meals if order only three courses.

Bonus:

a) Go home and hang out: 3 ways (3 friends)
Stay at UML and hang out: 12 ways (12 friends)

$\Rightarrow$ There are $12 + 3 = 15$ ways to hang out if you stay at UML or go home.

b) Go home and hang out: 3 ways (3 friends)
Come back to UML and hang out with someone different:

1. I already hung out with a friend who is not my schoolmate
12 ways (12 friends)
2. I already hung out with one of my schoolmates at home:
11 ways

$\Rightarrow$ There are $1 \times 12 + 1 \times 11 + 1 \times 11 = 34$ ways to hang out with some one at home and hang out with some one different at UML.

G.T

$\frac{9}{10}$

Name: Giang Tran

Homework #5

① $P(S) = 0.10$, S, A, P are independent
$P(A) = 0.06$, $P(X=k)$ or $p_X(k)$
$P(P) = 0.08$

a) $P(X=0) = p_X(0)$
$= 1 - P(S \cup A \cup P)$
$= 1 - (P(S) + P(A) + P(P) - P(S \cap A) - P(S \cap P) - P(A \cap P) + P(S \cap A \cap P))$
$= 1 - (0.10 + 0.06 + 0.08 - 0.10 \times 0.06 - 0.10 \times 0.08 - 0.06 \times 0.08 + 0.10 \times 0.06 \times 0.08)$
$= 0.778$

$P(X=2) = p_X(2)$
$= \cancel{P(S \cap A) + P(S \cap P) + P(A \cap P)}$
$= P(S) \times P(A) \times (1 - P(P)) + P(S) \times (1 - P(A)) \times P(P) + (1 - P(S)) \times P(A) \times P(P)$
$= 0.10 \times 0.06 \times 0.92 + 0.10 \times 0.94 \times 0.08 + 0.90 \times 0.06 \times 0.08$
$= 0.0174$

$P(X=3) = p_X(3)$
$= P(S \cap A \cap P)$
$= P(S) \times P(A) \times P(P)$
$= 0.10 \times 0.06 \times 0.08$
$= 0.00048$

$P(X=1) = 1 - P(X=0) - P(X=2) - P(X=3)$
$= 1 - 0.778 - 0.0174 - 0.00048$
$= 0.204$

G.T

k	$p_x(k)$
0	0.778
1	0.204
2	0.0174
3	0.00048

b) $F_x(0) = p_x(0) = 0.778$

$F_x(1) = p_x(0) + p_x(1) = 0.778 + 0.204 = 0.982$

$F_x(2) = \cancel{(F_x(0) + p)}\ p_x(0) + p_x(1) + p_x(2)$

$= 0.778 + 0.204 + 0.0174 = 0.9994$

$F_x(3) = \underbrace{p_x(0) + p_x(1) + p_x(2)}\ + p_x(3)$

$= 0.9994 + 0.00048$

$= 0.99988 \approx 1$

k	$F_x(k)$
0	0.778
1	0.982
2	0.9994
3	1

② Let X be the random variable that tells the number of women who are offered jobs

$P(X=k)$ or $p_x(k)$

N is the number of students

n is the number of offers made

r is the number of ~~(women)~~ female students
w is the number of male students

$$P(X=k) \; (\cancel{P(k)}) = \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}}$$ compute values

$$\Rightarrow p_X(k) = \frac{\binom{20}{k}\binom{25}{8-k}}{\binom{45}{8}}$$

b) p is the chance students accept the job offers.
Let X be the random variable that tells the number of students who accept job offers.
~~N is the number of students N is the number of students~~
~~k is the number of offers made students who accept offers~~

$P(X=k)$ or $p_X(k)$

$$p_X(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\Rightarrow F_X(k) = \sum_{i=0}^{k} p_X(i) = \sum_{i=0}^{k} \binom{N}{i} p^i (1-p)^{N-i}$$

with N is the number of offers made

$$\cancel{\Rightarrow F_X(k) = \sum_{k=0}^{8} \binom{45}{k} 0.75^i \times 0.25^{(45-i)}}$$

$$\cancel{\Rightarrow F_X(k) = \sum_{k=0}^{8} \binom{45}{k} \times 0.75^k \times 0.25^{(45-k)}}$$

~~Choose 8 students from 45 students to receive offers. $\binom{45}{8}$ ways~~

G.T

$$\Rightarrow F_X(k) = \sum_{k=0}^{8} \binom{8}{k} \times 0.75^k \times 0.25^{(8-k)}$$

compute values

G.T

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Name: Giang Tran

Homework #6

(1)

$$F_X(k) = \begin{cases} 0 & \text{if } k < -3 \\ 0.25 & \text{if } -3 \le k < 1 \\ 0.5 & \text{if } 1 \le k < 3 \\ 0.6 & \text{if } 3 \le k < 5 \\ 1 & \text{if } k \geq 5 \end{cases}$$

a) ~~Case 1: If $k<-3$ then $F_X(k) = 0$~~
~~so $f_X(k) = \frac{d}{dx} 0 = 0$~~

~~Case 2: If $-3 \le k < 1$ then $F_X(k) = 0.25$~~
~~so $F_X(k) = \frac{d}{dx} 0.25$~~

$$P(-3 \le X \le 3) = P(X \le 3) - P(X < -3)$$
$$= F_X(3) - F_X(-4)$$
$$= 0.6 - 0$$
$$= 0.6$$

b) $$P(-3 < X < 3) = P(X < 3) - P(X \le -3)$$
$$= F_X(2) - F_X(-3)$$
$$= 0.5 - 0.25$$
$$= 0.25$$

c) $$p_X(-3) = P(X \le -3) - P(X < -3)$$
$$= F_X(-3) - F_X(-4)$$
$$= 0.25 - 0 = 0.25$$

$$p_X(1) = P(X \le 1) - P(X < 1)$$
$$= F_X(1) - F_X(0)$$
$$= 0.5 - 0.25 = 0.25$$

G.T

$$p_X(3) = P(X \leq 3) - P(X < 3)$$
$$= F_X(3) - F_X(2)$$
$$= 0.6 - 0.5$$
$$= 0.1$$

$$p_X(5) = P(X \leq 5) - P(X < 5)$$
$$= F_X(5) - F_X(4)$$
$$= 1 - 0.6$$
$$= 0.4$$

k	$p_X(k)$
-3	0.25
1	0.25
3	0.1
5	0.4

d) $P(-3 \leq X \leq 3) = \sum_{-3 \leq k \leq 3} p_X(k)$

$$= p_X(-3) + \cancel{p_X(3)}\ p_X(1) + p_X(3)$$
$$= 0.25 + 0.25 + 0.1$$
$$= 0.6$$

$P(-3 < X < 3) = \sum_{-3 < k < 3} p_X(k)$

$$= p_X(1) = 0.25$$

e) $F_X(-3) = P(X \leq -3) = \cancel{\sum_{k \leq -3} p_X(k)}\ \sum_{k \leq -3} p_X(k)$

$$= p_X(-3) = 0.25$$

G.T

$$F_X(1) = P(X \leq 1) = \sum_{k \leq 1} p_X(k)$$

$$= p_X(-3) + p_X(1)$$

$$= 0.25 + 0.25 = 0.5$$

$$F_X(3) = P(X \leq 3) = \sum_{k \leq 3} p_X(k)$$

$$= p_X(-3) + p_X(1) + p_X(3)$$

$$= 0.25 + 0.25 + 0.1$$

$$= 0.6$$

$$\Rightarrow F_X(k) = \begin{cases} 0 & \text{if } k < -3 \\ 0.25 & \text{if } -3 \leq k < 1 \\ 0.5 & \text{if } 1 \leq k < 3 \\ 0.6 & \text{if } 3 \leq k < 5 \\ 1 & \text{if } k \geq 5 \end{cases}$$

$$F_X(5) = P(X \leq 5) = \sum_{k \leq 5} p_X(k)$$

$$= p_X(-3) + p_X(1) + p_X(3) + p_X(5)$$

$$= 0.25 + 0.25 + 0.1 + 0.4$$

$$= 1$$

G.T

(2)

$$f_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ 1+y & \text{if } -1 \leq y < 0 \\ 2-y & \text{if } 0 \leq y \leq c \\ 0 & \text{if } y > c \end{cases}$$

a) Find c:

$$\int_{-\infty}^{\infty} f(y)\,dy = 1$$

$$\Leftrightarrow \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{0} (1+y)\,dy + \int_{0}^{c} (2-y)\,dy + \int_{c}^{\infty} 0\,dy = 1$$

$$\Leftrightarrow 0 + \left(y + \frac{y^2}{2}\right)\Big|_{-1}^{0} + \left(2y - \frac{y^2}{2}\right)\Big|_{0}^{c} + 0 = 1$$

$$\cancel{+ \lim_{a \to \infty} \int_c^a x\,dy = 1}$$

$$\Leftrightarrow -\left(-1 + \frac{1}{2}\right) + \left(2c - \frac{c^2}{2}\right) \cancel{+ \lim_{a \to \infty} xy \Big|_c^a} = 1$$

$$\Leftrightarrow \frac{1}{2} + 2c - \frac{c^2}{2} = 1$$

$$\Leftrightarrow 1 + 4c - c^2 = 2$$

$$\Leftrightarrow c^2 - 4c + 1 = 0$$

$$\Leftrightarrow \left[\begin{array}{l} c = 2 + \sqrt{3} \approx 3.73 \\ c = 2 - \sqrt{3} \approx 0.27 \text{ (Choose this because } < \tfrac{1}{2}\text{)} \end{array}\right.$$

From now, replace all $c = 2 - \sqrt{3}$

G.T

b) $P(Y=0) = p_Y(0) = P(Y \leq 0) - P(Y<0)$

$$= (2-y) - (1+y)$$

$$= 1-2y$$

c) $\cancel{P(0 \leq Y \leq 1/2)}$

$$\cancel{= P(Y \leq 1/2) - P(Y<0)}$$

$$\cancel{= (2-y) - (1+y)} \quad \cancel{(\text{when } c = \cancel{(0.27)}\ 3.73)}$$

$$\cancel{= 1-2y}$$

$$\cancel{P(0 \leq Y \leq 1/2)}$$

$$\cancel{= P(Y \leq 1/2) - P(Y<0)}$$

$$\cancel{= 0 - (1+y)}$$

$$P(0 \leq Y \leq 1/2) = \int_0^{1/2} f_Y(y)\,dy$$

$$= \int_0^{2-\sqrt{3}} (2-y)\,dy + \int_{2-\sqrt{3}}^{1/2} 0\,dy$$

$$= \left(2y - \frac{y^2}{2}\right)\Big|_0^{2-\sqrt{3}}$$

$$= \cancel{(2 \cdot 2\sqrt{2})}\ 2 \times (2-\sqrt{3}) - \frac{(2\sqrt{3})^2}{2}$$

$= \frac{1}{2}$ ((~~not~~) true because not $> \frac{1}{2}$)

d) $P(-1/2 \leq Y \leq 1/2) = \int_{-1/2}^{1/2} f_Y(y)\,dy$

$$= \int_{-1/2}^{0} (1+y)\,dy + \int_0^{2-\sqrt{3}} (2-y)\,dy + \int_{2-\sqrt{3}}^{1/2} 0\,dy$$

$$= \left(y + \frac{y^2}{2}\right)\Big|_{-1/2}^{0} + \left(2y - \frac{y^2}{2}\right)\Big|_0^{2-\sqrt{3}} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

G.T

We choose $\frac{7}{8}$ because ~~(b)~~ $\frac{7}{8} \leq 1$.

e) Find CDF:

Case 1: If $y < -1$ then:

$$F_Y(y) = \int_{-\infty}^{y} 0\,dy = 0$$

Case 2: If $-1 \leq y < 0$ then:

$$F_Y(y) = \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{y} (1+y)\,dy$$

$$= 0 + \left(y + \frac{y^2}{2}\right)\Big|_{-1}^{y}$$

$$= y + \frac{y^2}{2} - \left(-1 + \frac{1}{2}\right)$$

$$= \frac{y^2}{2} + y + \frac{1}{2}$$

Case 3: If $0 \leq y \leq c$ then:

$$F_Y(y) = \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{0} (1+y)\,dy + \int_{0}^{y} (2-y)\,dy$$

$$= 0 + \left(y + \frac{y^2}{2}\right)\Big|_{-1}^{0} + \left(2y - \frac{y^2}{2}\right)\Big|_{0}^{y}$$

$$= \frac{1}{2} + 2y - \frac{y^2}{2}$$

$$= \frac{-y^2}{2} + 2y + \frac{1}{2}$$

Case 4: If $y > c$ then: $F_Y(y) = 1$

G.T

$\Rightarrow$ CDF is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ \frac{y^2}{2} + y + \frac{1}{2} & \text{if } -1 \leq y < 0 \\ \frac{-y^2}{2} + 2y + \frac{1}{2} & \text{if } 0 \leq y \leq c \\ 1 & \text{if } y > c \end{cases}$$

f) $P(0 \leq Y \leq \frac{1}{2}) = P(Y \leq \frac{1}{2}) - P(Y \leq 0)$

$$= F_Y\left(\frac{1}{2}\right) - F_Y(0)$$

$$= \cancel{\frac{1}{8}}\ 1 - \frac{1}{2} \quad \text{when } c = 2-\sqrt{3}$$

$$= \frac{1}{2}$$

$P(-1/2 \leq Y \leq 1/2) = P(Y \leq 1/2) - P(Y \leq -1/2)$

$$= F_Y(1/2) - F_Y(-1/2)$$

$$= 1 - \left(\frac{(-1/2)^2}{2} + \frac{-1}{2} + \frac{1}{2}\right)$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

g) Case 1: If $y < -1$ then $F_Y(y) = 0$

so $f_Y(y) = \frac{d}{dy} 0 = 0$

G.T

Case 2: If $-1 \leq y < 0$ then $F_Y(y) = \frac{y^2}{2} + y + \frac{1}{2}$

$$\text{so } f_Y(y) = \frac{d}{dy}\left(\frac{y^2}{2} + y + \frac{1}{2}\right)$$

$$= y + 1$$

Case 3: If $0 \leq y \leq c$ then $F_Y(y) = \frac{-y^2}{2} + 2y + \frac{1}{2}$

$$\text{so } f_Y(y) = \frac{d}{dy}\left(\frac{-y^2}{2} + 2y + \frac{1}{2}\right)$$

$$= -y + 2$$

Case 4: If $y > c$ then $F_Y(y) = 1$

$$\text{so } f_Y(y) = \frac{d}{dy}(1)$$

$$= 0$$

Combine cases to get:

$$f_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ 1+y & \text{if } -1 \leq y < 0 \\ 2-y & \text{if } 0 \leq y \leq c \\ 0 & \text{if } y > c \end{cases}$$

$\frac{10}{10}$ G.T

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Homework #7

(1)
$$p_X(k) = \begin{cases} 0.25 & \text{if } k=-3 \\ 0.25 & \text{if } k=1 \\ 0.1 & \text{if } k=3 \\ 0.4 & \text{if } k=5 \end{cases}$$

$$E(X) = \sum_k k \cdot p_X(k)$$
$$= -3\times 0.25 + 1\times 0.25 + 3\times 0.1 + 5\times 0.4$$
$$= 1.8 \quad = \mu$$
$$E(X^2) = \sum_k k^2 \cdot p_X(k)$$
$$= (-3)^2\times 0.25 + 1^2\times 0.25 + 3^2\times 0.1 + 5^2\times 0.4$$
$$= 13.4$$
$$Var(X) = E(X^2) - \mu^2$$
$$= 13.4 - 1.8^2$$
$$= 10.16$$

(2)
$$f_Y(y) = \begin{cases} 0 & \text{if } y<-1 \\ 1+y & \text{if } -1\le y<0 \\ 2-y & \text{if } 0\le y\le c \\ 0 & \text{if } y>c \end{cases}$$

. Find c:
$$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$$
$$\Leftrightarrow \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{0}(1+y)\,dy + \int_{0}^{c}(2-y)\,dy + \int_{c}^{\infty} 0\,dy = 1$$

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$$\Leftrightarrow 0 + \left(y + \frac{y^2}{2}\right)\Big|_{-1}^{0} + \left(2y - \frac{y^2}{2}\right)\Big|_{0}^{c} + 0 = 1$$

$$\Leftrightarrow -\left(-1 + \frac{1}{2}\right) + \left(2c - \frac{c^2}{2}\right) = 1$$

$$\Leftrightarrow \frac{1}{2} + 2c - \frac{c^2}{2} = 1$$

$$\Leftrightarrow c^2 - 4c + 1 = 0$$

$$\Leftrightarrow \left[\begin{array}{l} c = 2+\sqrt{3} \\ c = 2-\sqrt{3} \text{ (choose this)} \end{array}\right.$$

as previous homework

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy$$

$$= \int_{-1}^{0} y(1+y)\,dy + \int_{0}^{2-\sqrt{3}} y(2-y)\,dy$$

$$= \int_{-1}^{0} (y+y^2)\,dy + \int_{0}^{2-\sqrt{3}} (2y-y^2)\,dy$$

$$= \left(\frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_{-1}^{0} + \left(\frac{2y^2}{2} - \frac{y^3}{3}\right)\Big|_{0}^{2-\sqrt{3}}$$

$$= -\left(\frac{1}{2} - \frac{1}{3}\right) + \left((2-\sqrt{3})^2 - \frac{(2-\sqrt{3})^3}{3}\right)$$

$$\simeq -0.101 = \mu$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 \cdot f_Y(y)\,dy$$

$$= \int_{-1}^{0} y^2(1+y)\,dy + \int_{0}^{2-\sqrt{3}} y^2(2-y)\,dy$$

$$= \int_{-1}^{0} (y^2+y^3)\,dy + \int_{0}^{2-\sqrt{3}} (2y^2 - y^3)\,dy$$

$$= \left(\frac{y^3}{3} + \frac{y^4}{4}\right)\Bigg|_{-1}^{0} + \left(\frac{2y^3}{3} - \frac{y^4}{4}\right)\Bigg|_{0}^{2-\sqrt{3}}$$

$$= -\left(\frac{-1}{3} + \frac{1}{4}\right) + \left(\frac{2\times(2-\sqrt{3})^3}{3} - \frac{(2-\sqrt{3})^4}{4}\right)$$

$$= 0.095$$

$$Var(Y) = E(Y^2) - \mu^2$$
$$= 0.095 - (-0.101)^2$$
$$= 0.085$$

③ $\mu = 80\,bpm$, $\sigma = 9\,bpm$

a) $\frac{2}{3} = 66.67\% \approx 68\%$

Apply the Empirical Rule:

$\Rightarrow$ $\frac{2}{3}$ of ~~(w)~~ all women's heart rates range from

$\mu - \sigma = 80 - 9 = \boxed{71\,bpm}$ to $\mu + \sigma = 80 + 9 = \boxed{89\,bpm}$

b) $98\,bpm = 80 + 18 = \mu + 2\sigma$

Apply the Empirical Rule

$\Rightarrow$ 95% of all women's heart rates range from

$\mu - 2\sigma = 80 - 2\times 9 = 62\,bpm$ to $98\,bpm$

$\Rightarrow$ 5% of all women's heart rates are outside this range

$\Rightarrow$ $5/2 = 2.5\%$ of all women's heart rates are over 98bp

G.T

c) I take a sample of size $n = 50$ women ~~(for)~~ from distribution Y.

Write the measurements of heart rates as: $y_1, y_2, \dots, y_{50}$

and $\bar{y} = \dfrac{y_1 + y_2 + \dots + y_{50}}{50}$ ($\bar{y}$ is estimate of heart rate)

Then $E(\bar{y}) = E(Y) = \mu$

Standard error: $\sigma(\bar{y}) = \dfrac{1}{\sqrt{n}} \cdot \sigma(Y)$

$$= \frac{1}{\sqrt{50}} \cdot 9$$

$$\approx 1.27 \text{ bpm}$$

By Empirical Rule there is a 95% chance that $\bar{y}$ is within $2\sigma(\bar{y}) = 2.54$ bpm of $\mu(\bar{y})$.

Therefore, there is a 95% chance that my estimate of heart rate will be within 2.54 bpm of the true average.