Name: Solution Spring 2020

Linear Algebra: Quiz 7

<u>Show ALL work, as unjustified answers may receive no credit</u>. Calculators are not allowed on any quiz or test paper. Make sure to exhibit skills discussed in class. Box all answers and clean up answers as much as possible.

1. ***<u>Vector Spaces & Subspaces (4.1)</u>***

[2pts] Find all values of h such that $\vec{y}$ will be in the subspace spanned by $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ if:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}, \quad \vec{v_1} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 4 \\ 2 \\ h \end{bmatrix}$$

* Row-reduce $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \,\vdots\, \vec{y}]$ to Echelon Form & solve for h:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & 4 & 0 & 2 \\ -4 & -8 & 0 & h \end{array}\right] \begin{array}{c} \tfrac{1}{2}R_2 \\ \sim \\ -\tfrac{1}{4}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & -\frac{h}{4} \end{array}\right] \begin{array}{c} -R_1+R_2 \to N.R_2 \\ \sim \\ -R_1+R_3 \to N.R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -1 & 1 & -3 \\ 0 & -1 & 1 & -4-\frac{h}{4} \end{array}\right] \begin{array}{c} -R_2+R_3 \to N.R_3 \\ \sim \\ -R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -1-\frac{h}{4} \end{array}\right]$$

* $\vec{y}$ will be in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ IFF

$$-1 - \frac{h}{4} = 0 \rightarrow -4 - h = 0 \rightarrow \boxed{h = -4}$$

Ans

2. ***Null Space, Column Space, & Linear Transformations (4.2) & Basis (4.3)***

Define a Linear Transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \\ x_1 + x_4 \end{bmatrix}$.

(a) [2pts] Find the Null Space of T

(b) [2pts] Find the Column Space of T

(c) [2pts] Find the Basis for the Null Space of T

(d) [2pts] Find the Basis for the Column Space of T

* *The Column Space of T is $Col(A)$ & the Null Space of T is the $Nul(A)$, where A is the Standard Matrix of T* *

*Given: $T: \mathbb{R}^4 \to \mathbb{R}^3$ ST $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \to \therefore A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Row-Reduce $[A \,\vdots\, \vec{0}]$ to RREF:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \underset{-R_1 + R_3 \to N.R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \underset{\substack{-R_2 + R_1 \to N.R_1 \\ R_2 + R_3 \to N.R_3}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \underset{\substack{R_3 + R_1 \to N.R_1 \\ -R_3 + R_2 \to N.R_2 \\ -R_3}}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Leftrightarrow \begin{cases} \cdot\, x_1 = -x_4 \\ \cdot\, x_2 = x_4 \\ \cdot\, x_3 = x_4 \\ x_4 \text{ is free} \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; x_4 \in \mathbb{R}$$

(a) $\boxed{Nul(A) = \left\{ x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; x_4 \in \mathbb{R} \right\}}$

(c) Basis For $Nul(A)$: $\mathcal{B}_N = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\boxed{Col(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$

ST $c_1, c_2, c_3, c_4 \to$ weights

(d) Basis For $Col(A)$: $\mathcal{B}_c = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

① Use the coordinate vectors to determine whether the given polynomials are Linearly Dependent in $\mathbb{P}_2$. Let $\mathcal{B}$ be the Standard Basis of the space $\mathbb{P}_2$ of polynomials, that is $\mathcal{B} = \{1, t, t^2\}$.

$$1+2t \;,\quad 3+6t^2 \;,\quad 1+3t+4t^2$$

* Find the Coordinate Vectors of the polynomials, relative to $\mathcal{B}$:

- $p_1(t) = 1+2t \longrightarrow [\vec{p}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

- $p_2(t) = 3+6t^2 \longrightarrow [\vec{p}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$

- $p_3(t) = 1+3t+4t^2 \longrightarrow [\vec{p}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

* Check if coord. vectors are Linearly Dependent:

- Row-reduce Augmented matrix, whose columns are the coord. vectors, to Echelon Form:

$$[P_{\mathcal{B}} \,\vdots\, \vec{0}] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 3 \\ 0 & 6 & 4 \end{bmatrix} \underset{N.R_2}{\overset{-2R_1 + R_2}{\sim}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -6 & 1 \\ 0 & 6 & 4 \end{bmatrix} \underset{N.R_3}{\overset{R_2 + R_3}{\sim}}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -6 & 1 \\ 0 & 0 & 5 \end{bmatrix} \Longrightarrow$$

* Echelon Form

Since matrix has $n = 3$ pivots, the columns of matrix are Linearly Independent.

$\therefore$ Polynomials are NOT Linearly Dependent in $\mathbb{P}_2$.

Answer.

② For the given matrix A, Find the:

(a) Basis and Dimension of the Column Space of A.

(b) Basis and Dimension of the Null Space of A.

$$A = \begin{bmatrix} \underline{1} & -2 & 3 & 1 & 0 & 5 & -4 \\ 0 & 0 & \underline{1} & -6 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

***Part (a):**

Note: Matrix A is already in Echelon Form! Pivot Columns: $\vec{a}_1, \vec{a}_3, \vec{a}_6$

$$\boxed{\begin{array}{l} \therefore \text{Basis for } Col(A): \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\} \\ \\ \therefore \text{Dimension of } Col(A): \quad \dim[Col(A)] = 3 \end{array}}$$

Ans

***Part (b):**

$$\begin{bmatrix} 1 & -2 & \underline{3} & 1 & 0 & 5 & -4 \\ 0 & 0 & 1 & -6 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{-3R_2+R_1}{\sim} \begin{bmatrix} 1 & -2 & 0 & 19 & -6 & \underline{11} & -4 \\ 0 & 0 & 1 & -6 & 2 & \underline{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} -11R_3+R_1 \\ \sim \\ 2R_3+R_2 \end{array}$$

$$\begin{bmatrix} \underline{1} & -2 & 0 & 19 & -6 & 0 & -37 \\ 0 & 0 & \underline{1} & -6 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} \cdot\, x_1 = 2x_2 - 19x_4 + 6x_5 + 37x_7 \\ \cdot\, x_3 = 6x_4 - 2x_5 - 6x_7 \\ \cdot\, x_6 = -3x_7 \end{cases}$$

RREF

* x_2, x_4, x_5, x_7 are free!

General Solution to $A\vec{x} = \vec{0}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2x_2 - 19x_4 + 6x_5 + 37x_7 \\ x_2 + 0 + 0 + 0 \\ 0 + 6x_4 - 2x_5 - 6x_7 \\ 0 + x_4 + 0 + 0 \\ 0 + 0 + x_5 + 0 \\ 0 + 0 + 0 - 3x_7 \\ 0 + 0 + 0 \quad x_7 \end{bmatrix}$$

← You do not need to write this part, but just in case you need/want a reminder.

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -19 \\ 0 \\ 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 37 \\ 0 \\ -6 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \quad \text{ST} \quad x_2, x_4, x_5, x_6 \in \mathbb{R}$$

$\boxed{\begin{array}{l} \text{*Basis for Nul(A): } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -19 \\ 0 \\ 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 37 \\ 0 \\ -6 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\} \\ \\ \text{*Dimension of Nul(A): } \dim[\text{Nul}(A)] = 4 \end{array}}$

Ans.

① Let $B = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$ & $C = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$ be 2 bases of $\mathbb{R}^3$

(a) Find the Change of Coordinates Matrix From B to C.

Answer

*To Find $\underset{C \leftarrow B}{P}$: Row-reduce $\left[\vec{c}_1 \; \vec{c}_2 \; \vec{c}_3 \;\vdots\; \vec{b}_1 \; \vec{b}_2 \; \vec{b}_3 \right]$ to RREF

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \underset{\sim}{\frac{-R_2 + R_3}{n.R_3}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \underset{\sim}{\frac{-R_3 + R_2}{N.R_2}}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Longrightarrow \boxed{\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

Ans

(b) Find the Change of Coordinates Matrix from C to B

Answer:

*To Find $\underset{B \leftarrow C}{P}$: Row-reduce $\left[\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3 \;\vdots\; \vec{c}_1 \; \vec{c}_2 \; \vec{c}_3 \right]$ to RREF

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\frac{-R_1 + R_3}{N.R_3}}{\overset{\frac{-R_1 + R_2}{N.R_2}}{\sim}} \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{array}\right] \overset{-R_2 \leftrightarrow -R_3}{\sim}$$

$$\underbrace{\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]}_{\text{Same augmented matrix as (a) :)}} \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\therefore \underset{B \leftarrow C}{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

Ans

(c) Let $\vec{x} \in \mathbb{R}^3$ ST $[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\vec{x}$ & $[\vec{x}]_c$

Ans.

Given: $\vec{x} \in \mathbb{R}^3$ ST $[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\Rightarrow \vec{x} = 1\vec{b}_1 + 2\vec{b}_2 + 3\vec{b}_3 = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

***Find $\vec{x}$**

$$\vec{x} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3 \\ 1+2+0 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} \qquad \boxed{\therefore \vec{x} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}}$$ Ans.

***Find $[\vec{x}]_c$:** (*Not an exclusive solution ∵ See next pg.)

$$\Rightarrow \vec{x} = [P_c][\vec{x}]_c = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{bmatrix} [\vec{x}]_c$$

$\Rightarrow$ Row-reduce $\left[\vec{c}_1\ \vec{c}_2\ \vec{c}_3 \,\vdots\, \vec{x}\right]$ to Find $[\vec{x}]_c$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{N.R_1}{\overset{-R_2+R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{N.R_2}{\overset{-R_3+R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{\therefore [\vec{x}]_c = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$ Ans

*Alternative Solution: A quicker method :)

* Find $[\vec{x}]_C$:

* Recall (By Def): $[\vec{x}]_C = \underset{C \leftarrow B}{P} [\vec{x}]_B$

$$[\vec{x}]_C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+0+3 \\ 0+2+0 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\therefore [\vec{x}]_C = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

Ans✓