Examples for Mathematical Induction

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Mathematical Induction is an important proof technique which we will frequently use in this course. A typical induction proof consists of two steps, induction basis and induction step. We can also split the induction step into an induction hypothesis and a direct proof.

1 The First Principle of Induction

To show that $\forall n \geq a, P(n)$, we can show that $P(a)$ and $\forall k \geq a, (P(k) \to P(k+1))$ using the first principle of induction.

Example 1 Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

We want to prove $\forall n \geq 1, P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Induction basis: when n=1, the equation trivially holds, i.e., $P(1)$ is true.

Induction hypothesis: assume that $P(k)$ holds for $k \geq 1$. In other words, we assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Induction step: we want to show $P(k+1)$, i.e., $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \quad (\textit{induction hypothesis}) \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

2 The Second Principle of Induction

To show that $\forall n \geq a, P(n)$, we show that $P(a), P(a+1), ..., P(b)$ for a constant $b \geq a$ and $\forall k \geq b (\forall a \leq j \leq k, P(j) \to P(k+1))$, using the second principle .

Example 2 The Fibonacci sequence is defines as follows,

$$f_n = \begin{cases} n & n = 0, 1 \\ f_{n-1} + f_{n-2} & n \geq 2 \end{cases}$$

We can prove that

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n}), \phi = \frac{1+\sqrt{5}}{2},$$

using the second principle of mathematical induction.

Induction basis:

$f_0 = 0$ follows $\frac{1}{\sqrt{5}}(\phi^0 - (-\phi)^{-0}) = 0$.

$$\begin{aligned}
\frac{1}{\sqrt{5}}(\phi^1 - (-\phi)^{-1}) &= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} - (-\frac{1+\sqrt{5}}{2})^{-1}) \\
&= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}) \\
&= \frac{1}{\sqrt{5}}\frac{10+2\sqrt{5}}{2(1+\sqrt{5})} \\
&= \frac{10+2\sqrt{5}}{10+2\sqrt{5}} \\
&= 1 \\
&= f_1
\end{aligned}$$

Induction hypothesis: Assume that $f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n})$ for $0 \leq n \leq k$, and $k \geq 1$.

Induction step: We need to show that $f_{k+1} = \frac{1}{\sqrt{5}}(\phi^{k+1} - (-\phi)^{-(k+1)})$.

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \quad (\textit{definition of Fibonacci sequence}) \\
&= \frac{1}{\sqrt{5}}(\phi^k - (-\phi)^{-k}) + \frac{1}{\sqrt{5}}(\phi^{k-1} - (-\phi)^{-(k-1)}) \quad (\textit{induction hypothesis}) \\
&= \frac{1}{\sqrt{5}}(\phi^{k-1}(\phi+1) - (-\phi)^{-k}(1-\phi)) \\
&= \frac{1}{\sqrt{5}}(\phi^{k-1}\phi^2 - (-\phi)^{-k}(-\phi)^{-1}) \quad (\text{we prove later that } \phi+1 = \phi^2 \textit{ and } 1-\phi = (-\phi)^{-1}) \\
&= \frac{1}{\sqrt{5}}(\phi^{k+1} - (-\phi)^{-(k+1)})
\end{aligned}$$

It is easy to show $\phi + 1 = \phi^2$.

$\phi + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2}$.

$\phi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{(1+\sqrt{5})^2}{2^2} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = \phi + 1$.

Similarly,

$1 - \phi = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}$

$(-\phi)^{-1} = (-\frac{1+\sqrt{5}}{2})^{-1} = -\frac{2}{1+\sqrt{5}} = -\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = -\frac{2(1-\sqrt{5})}{1-5} = -\frac{2(1-\sqrt{5})}{-4} = \frac{1-\sqrt{5}}{2} = 1 - \phi$