Instructions: No notes or calculators are allowed. Answers with little or no supporting work will get little or no credit. Work must be neat, organized and easily interpreted.

1. Answer the following questions for $\vec{u} = \langle -2,1,1 \rangle$ and $\vec{v} = \langle 1,2,-3 \rangle$

1.a (6 Pts) Compute $\vec{u} \bullet \vec{v}$

$$\vec{u}\cdot\vec{v} = -2+2-3 = -3$$

$\vec{u} \bullet \vec{v} = -3$

1.b (6 Pts) Provide an expression for the angle, θ, between $\vec{u}$ and $\vec{v}$

$$|\vec{u}| = \sqrt{6} \qquad |\vec{v}| = \sqrt{14} \qquad |\vec{u}||\vec{v}| = 2\sqrt{21}$$

$$|\vec{u}||\vec{v}|\cos(\theta) = \vec{u}\cdot\vec{v} = -3$$

$$\cos(\theta) = -3/2\sqrt{21}$$

$\theta = \cos^{-1}\left(-\frac{3}{2\sqrt{21}}\right)$

1.c (7 Pts) Find the scalar projection of $\vec{u}$ in the direction of $\vec{v}$

$$scal_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{-3}{\sqrt{14}}$$

$scal_{\vec{v}}\vec{u} = -3/\sqrt{14}$

1.d (6 Pts) Determine the area of the triangle formed by $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \langle -5, -5, -5 \rangle$$

$$= -5\langle 1,1,1 \rangle$$

$$\text{Area of triangle} = |\vec{u}\times\vec{v}|/2 = \frac{5\sqrt{3}}{2}$$

Area of triangle = $5\sqrt{3}/2$

2. (14 Pts) If the velocity of a particle is given by $\vec{v}(t) =< -2t, 4\cos(2t), 3e^{3t} >$ and its initial position at time $t = 0$ is given by $\vec{r}(0) =< 1, 2, 3 >$, determine the particle's position, $\vec{r}(t)$, for all $t \geq 0$.

$$\vec{r}(t) = \int \langle -2t, 4\cos(2t), 3e^{3t} \rangle \, dt$$

$$= \langle -t^2, 2\sin(2t), e^{3t} \rangle + \langle c_1, c_2, c_3 \rangle$$

$$\vec{r}(0) = \langle 0 + c_1, 0 + c_2, 1 + c_3 \rangle = \langle 1, 2, 3 \rangle$$

$$\Rightarrow c_1 = 1, \quad c_2 = 2, \quad c_3 = 2$$

$$\vec{r}(t) = \langle 1 - t^2, 2\sin(2t) + 2, e^{3t} + 2 \rangle$$

3. (10 Pts) For the curve $\vec{r}(t) =< 2\sin(\pi t), e^{2t}, 8 - t^3 >$ provide an expression for the arc length for $t \in [0, 2]$. Do NOT attempt to perform the integration.

$$L = \int_a^b |\vec{v}(t)| \, dt$$

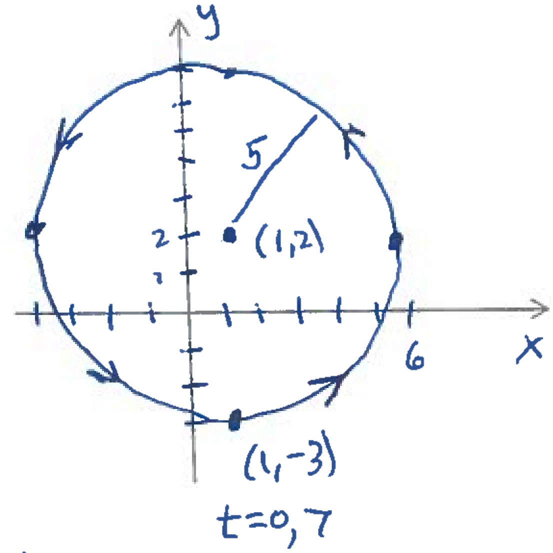
$$\vec{v}(t) = \langle 2\pi\cos(\pi t), 2e^{2t}, -3t^2 \rangle$$

$$L = \int_0^2 \left[4\pi^2\cos^2(\pi t) + 4e^{4t} + 9t^4 \right]^{1/2} dt$$

4.a (16 Pts) Provide parametric equations, $x(t)\, and\, y(t)$, for a circle centered at $(1,2)$ having radius $r=5$ that starts at the point $(1,-3)$ at time $t=0$ and moves counterclockwise with one full circuit completed every 7 seconds. Draw a sketch showing this curve, the starting point and the direction of travel. Check that your solution begins at (1,-3) and proceeds in the correct direction. Express your final solution in vector form, $\vec{r}(t) = < x(t), y(t) >$.

$$x(t) = 1 + 5\sin\left(\frac{2\pi t}{7}\right)$$

$$y(t) = 2 - 5\cos\left(\frac{2\pi t}{7}\right)$$



$$\vec{r}(t) = \left< 1 + 5\sin\left(\frac{2\pi t}{7}\right), 2 - 5\cos\left(\frac{2\pi t}{7}\right) \right>$$

4.b (6 Pts) Determine the speed associated with the parameterization in problem 6a. Revise this parameterization so it becomes an arc-length parameterization [i.e., $|\vec{v}(t)| = 1$ at all times].

Distance travelled in 7 sec $= 2\pi 5 = 10\pi$

$\Rightarrow$ speed $= 10\pi/7$

or $\vec{v} = \left< \frac{10\pi}{7}\cos\left(\frac{2\pi t}{7}\right), \frac{10\pi}{7}\sin\left(\frac{2\pi t}{7}\right) \right>$

$|\vec{v}| = 10\pi/7$ units of distance/sec

Revised Parameterization $\vec{r}(t) = \left< 1 + 5\sin\left(\frac{t}{5}\right), 2 - 5\cos\left(\frac{t}{5}\right) \right>$

$0 \le t \le 10\pi$

Speed of original parameterization = $10\pi/7$

New parameterization: $\vec{r}(t) = \left< 1 + 5\sin\left(\frac{t}{5}\right), 2 - 5\cos(t/5) \right>$

5.a (16 Pts) For the trajectory $\bar{r}(t) = \langle e^{2t}, 3t, t+\sin(t) \rangle$ find the unit tangent vector, $\bar{T}(t)$, and then evaluate it at $t = 0$.

$$\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)| = \frac{\langle 2e^{2t}, 3, 1+\cos(t)\rangle}{[4e^{4t}+9+(1+\cos(t))^2]^{1/2}}$$

$$\bar{T}(t) = \frac{\langle 2e^{2t}, 3, 1+\cos(t)\rangle}{[4e^{4t}+10+2\cos(t)+\cos^2(t)]^{1/2}}$$

$$\bar{T}(0) = \frac{\langle 2, 3, 2\rangle}{\sqrt{17}}$$

5.b (7 Pts) Determine the scalar projection of acceleration <u>in the direction of travel</u> at $t = 0$.

$$\vec{a}(t) = \vec{r}\,''(t) = \langle 4e^{2t}, 0, -\sin(t)\rangle$$

$$\vec{a}(0) = \langle 4, 0, 0\rangle$$

$$scal_{\vec{v}}\,\vec{a} = \vec{a}(0)\cdot\vec{T}(0) = 8/\sqrt{17}$$

$scal_{\bar{v}}\,\bar{a} =$ $8/\sqrt{17}$

6. BONUS (10 Pts) Assume that you are moving along a trajectory and at time $t = 3\,\text{sec}$ your velocity and acceleration are given by: $\bar{v} = \langle 1, -2, 2\rangle$ and $\bar{a} = \langle 0, 1, 0\rangle$. Place your answer to the following question in the box provided. Find the unit vectors $\bar{T}, \bar{B},$ and $\bar{N}$ <u>and</u> the curvature, K, at this time.

$$\vec{T} = \vec{v}/|\vec{v}| = \langle 1, -2, 2\rangle/3 \qquad |\vec{v}| = 3$$

$$\vec{B} = (\vec{v}\times\vec{a})_{\text{Normalized}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 0 & 1 & 0 \end{vmatrix}_N = \langle -2, 0, 1\rangle/\sqrt{5}$$

$$\vec{N} = B\times T = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 1 \\ 1 & -2 & 2 \end{vmatrix}_N = \langle 2, 5, 4\rangle/3\sqrt{5}$$

$$K = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{5}}{3^3} = \frac{\sqrt{5}}{27}$$

$\bar{T} = \langle 1, -2, 2\rangle/3$

$\bar{B} = \langle -2, 0, 1\rangle/\sqrt{5}$

$\bar{N} = \langle 2, 5, 4\rangle/3\sqrt{5}$

$\kappa = \sqrt{5}/27$