Spring 2020

Solution

Name:

Linear Algebra: Quiz 4

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. Make sure to exhibit skills discussed in class. Box all answers and clean up answers as much as possible.

1. [4pts] Assume that T is a Linear Transformation. Find the Standard Matrix of T:

$T: \mathbb{R}^2 \to \mathbb{R}^2$ *first performs a horizontal shear that transforms* $\vec{e_1}$ *to* $\vec{e_1} - 2\vec{e_2}$ *(leaving* $\vec{e_2}$ *unchanged), and then reflects points through the line* $x_2 = -x_1$

*Mapping:

- $\vec{e}_1 - 2\vec{e}_2 \mapsto T(\vec{e}_1 - 2\vec{e}_2) = -\vec{e}_2 - 2(-\vec{e}_1) = -\vec{e}_2 + 2\vec{e}_1 = \begin{bmatrix}0\\-1\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$
- $\vec{e}_2 \longmapsto T(\vec{e}_2) = -\vec{e}_1 = \begin{bmatrix}-1\\0\end{bmatrix}$

∴ Standard Matrix of T: $\boxed{A = \begin{bmatrix}T(\vec{e}_1 - 2\vec{e}_2) & T(\vec{e}_2)\end{bmatrix} = \begin{bmatrix}2 & -1\\-1 & 0\end{bmatrix}}$ Ans.

2. Define a Linear Transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}x_1 - x_2\\-x_1 + 2x_2\\x_1 + x_2\end{bmatrix}$$

(a) [2pts] Find the Standard Matrix of T.

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto x_1\begin{bmatrix}1\\-1\\1\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\1\end{bmatrix} = \underbrace{\begin{bmatrix}1 & -1\\-1 & 2\\1 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\vec{x}}$$

Standard Matrix of T: ∴ $\boxed{A = \begin{bmatrix}1 & -1\\-1 & 2\\1 & 1\end{bmatrix}}$ Ans✓

(b) [2pts] Is T one-to-one? Explain.

Recall: T is a 1-1 mapping when $T(\vec{x}) = \vec{0}$ has only the trivial sol✓

$$[A \,\vdots\, \vec{0}] = \begin{bmatrix}1 & -1\\-1 & 2\\1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -1\\0 & 1\\0 & 2\end{bmatrix} \sim \begin{bmatrix}\underline{1} & -1\\0 & \underline{1}\\0 & 0\end{bmatrix}$$

$\boxed{\text{∴ Since NO free variables ∃, } A\vec{x} = \vec{0} \text{ has only the trivial solution✓} \Rightarrow T \text{ is 1-1}}$ Ans✓

(c) [2pts] Is T onto? Explain.

Recall: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ when columns of A span $\mathbb{R}^m$.

⟺ A pivot position ∃ in each row (By Equivalence Th^m 1.4)

∴ $\boxed{\text{Since the \# of pivot positions} < \text{the \# of rows} \Rightarrow T \text{ is NOT onto.}}$ Ans✓