4.3 Expansion of Boolean Functions

Definition 4.1

If a variable is selected from an n-variable function $F(x_{n-1}, x_{n-2}, ..., x_{i+1}, x_i, x_{i-1},......., x_2, x_1, x_0)$ and substituted by either 0 or1, it is called an expansion variable of F.

Definition 4.2

When k expansion variables, where $1 \leqq k \leqq n$, are selected from an n-variable function $F(x_{n-1}, x_{n-2}, ..., x_{i+1}, x_i, x_{i-1},......., x_2, x_1, x_0)$, for each set of values for the k expansion variables, F is reduced to an (n-k)-variable function and is called a sub-function of F.

The following 4-variable function in Example 4.2 is used as an illustration for sub-functions

$$F(A,B,C,D) = A'B'C + BC' + AC'D + ABCD'$$

By selecting B as an expansion variable, it results in two sub-functions $F_{B = 0}$ and $F_{B=1}$. They are

$$F_{B = 0} = F(A, B = 0, C, D)$$

$$= A'(0)'C + (0)C' + AC'D + A(0)D'$$

$$= A'C + AC'D \qquad (4.2a)$$

and $$F_{B =1} = F(A, B = 1, C, D)$$

$$= A'(1)'C + (1)C' + AC'D + A(1)D'$$

$$= C' + AC'D + AD' = C' + AD' \qquad (4.2b)$$

If B and C are selected as expansion variables, the four sub-functions are

$$F_{BC = 00} = F(A, B = 0, C = 0, D) = AD \qquad (4.3a)$$
$$F_{BC = 01} = F(A, B = 0, C = 1, D) = A' \qquad (4.3b)$$
$$F_{BC = 10} = F(A, B = 1, C = 0, D) = 1 \qquad (4.3c)$$
$$F_{BC = 11} = F(A, B = 1, C = 1, D) = AD' \qquad (4.3d)$$

The expansion of a function into sub-functions can be shown by a binary tree in Figure 4.3. The n-variable function F, represented by a note called the root, is expanded with B into two sub-functions, each represented by a dot called a node in the second level: $F_{B = 0}$ and $F_{B = 1}$. A line that connects two nodes is called a branch. If the expansion continues with a second variable, shown in Figure 4.3 as C, $F_{B=0}$ and $F_{B=1}$ each will expand into two sub-functions, which are shown in the third level as $F_{BC=00}$, $F_{BC=01}$, $F_{BC=10}$, and $F_{BC=11}$. The expansion with k variables generates 2^k sub-functions. Each sub-function is a function of (n – k) variables. If an n-variable function is expanded with all the variables, each of the 2^n sub-functions is a minterm coefficient, a constant of either 0 or 1.

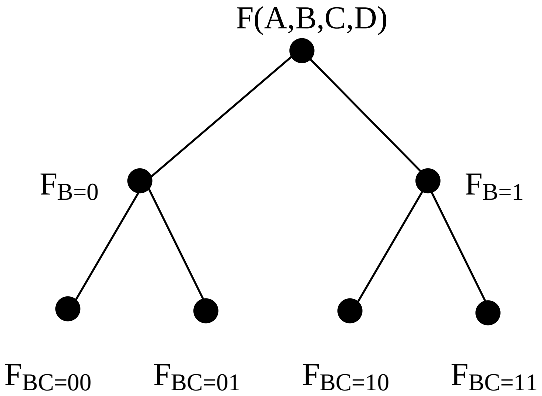

Figure 4.3 A binary tree for the expansion of a Boolean function.

Shannon's Expansion Theorem

A Boolean function can be expanded to or developed into two terms with respect to an expansion variable using the following theorem known as Shannon's expansion theorem.

$$F(x_{n-1}, x_{n-2}, ..., x_{i+1}, x_i, x_{i-1},......., x_2, x_1, x_0) = x_i' F_{x_i=0} + x_i F_{x_i=1} \qquad (4.4)$$

The complemented form of the expansion variable is associated with the sub-function $F_{x_i=0}$ and the true form of the expansion variable with $F_{x_i=1}$. The proof is show in Table 4.9.

Table 4.9 Proof of Shannon's expansion theorem.

x_i	Left-hand-side of (4.2)	Right-hand-side of (4.2)
0	$F(x_{n-1}, x_{n-2}, ..., x_{i+1}, x_i = 0, x_{i-1},......., x_2, x_1, x_0) = F_{x_i=0}$	$(0)'F_{x_i=0} + (0)F_{x_i=1} = F_{x_i=0}$
1	$F(x_{n-1}, x_{n-2}, ..., x_{i+1}, x_i = 1, x_{i-1},......., x_2, x_1, x_0) = F_{x_i=1}$	$(1)'F_{x_i=0} + (1)F_{x_i=1} = F_{x_i=1}$

The following equation shows the reconstruction of a 4-varaible function F(A,B,C,D) from the two sub-functions in Equation (4.2) using Shannon's expansion theorem..

$$F(A,B,C,D) = B'\underline{F_{B=0}} + B\,\underline{F_{B=1}}$$
$$= B'(\underline{A'C + AC'D}) + B(\underline{C' + AD'}) \qquad (4.6)$$

where the sub-functions are underlined. F(A,B,C,D) can be re-constructed from the nodes (sub-functions) of any level of a binary tree. The following equation shows how to obtain F from the four sub-functions in the third level of the binary tree in Figure 4.3,

$$F(A,B,C,D) = (B'C')\,\underline{F_{BC=00}} + (B'C)\,\underline{F_{BC=01}} + (BC')\,\underline{F_{BC=10}} + (BC)\,\underline{F_{BC=11}} \qquad (4.11a)$$

$$F(A,B,C,D) = B'C'\,(\underline{AD}) + B'C\,(\underline{A'}) + BC'(\underline{1}) + BC(\underline{AD'}) \qquad (4.11b)$$

In equation (4.11a), F is obtained as the sum of four terms. Each term is obtained by the AND of a sub-function and a canonical product based on the binary values of the expansion variables. The canonical products for BC = 00, 01, 10, and 11 are B'C', B'C, BC', and BC respectively.