Exam 1 / Discrete Structures I / Fall 2017

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SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! No notes, calculator, textbook, etc.! This exam is worth 100 points.

101

Excellent

23

1. (23 points) Let $A = \{1, 2, 4\}$, $B = \{1, 4, 7, 8\}$, $C = \{4, 5, 6, 7\}$, and $U = \{1, 2, \ldots, 8\}$. Compute the following:

(a) $(A \cap B)^c$

$A \cap B = \{1, 4\}$

$(A \cap B)^c = \{2, 3, 5, 6, 7, 8\}$

(b) $(A \cup B) - C$

$A \cup B = \{1, 2, 4, 7, 8\}$

$(A \cup B) - C = \{1, 2, 8\}$

(c) A^2

$A^2 = A \times A = \{(1,1), (1,2), (1,4),$
$(2,1), (2,2), (2,4),$
$(4,1), (4,2), (4,4)\}$

(d) The power set $\mathcal{P}(A)$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$

(e) Compute the set $S = \{(x, y) \in A \times C \,|\, 6 \leq xy \leq 16\}$. (List all of the elements.)

$S = \{(1,6), (1,7), (2,4), (2,5), (2,6), (2,7), (4,4)\}$

4

2. (4 points) Expand and simplify: $\prod_{k=2}^{5}(3k-7)$

$$\prod_{k=2}^{5}(3k-7) = (3\,2-7)\times(3.3-7)\times(3.4-7)\times(3.5-7)$$
$$= (6-7)\times(9-7)\times(12-7)\times(15-7)$$
$$= -1\times 2\times 5\times 8$$
$$= -80$$

6

3. (6 points) Let $S_k = \{x \in \mathbb{Z} \mid k < x < 2k\}$ for $k = 1, 2, 3, \ldots$

Compute $\bigcap_{k=6}^{8} S_k$.

$$S_6 = \{7, 8, 9, 10, 11\}$$
$$S_7 = \{8, 9, 10, 11, 12, 13\}$$
$$S_8 = \{9, 10, 11, 12, 13, 14, 15\}$$
$$\bigcap_{k=6}^{8} S_k = S_6 \cap S_7 \cap S_8$$
$$= \{9, 10, 11\}$$

7

4. (7 points) Use the algorithm from Section 1.4 to compute the binary representation of 57.

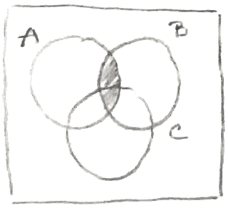
$57 = 2\times 28 + 1$	List	$= 1$
$28 = 2\times 14 + 0$	List	$= 01$
$14 = 2\times 7 + 0$	List	$= 001$
$7 = 2\times 3 + 1$	List	$= 1001$
$3 = 2\times 1 + 1$	List	$= 11001$
$1 = 2\times 0 + 1$	List	$= 111001$

$$57 = (111001)_2$$

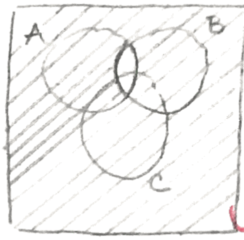
17

7

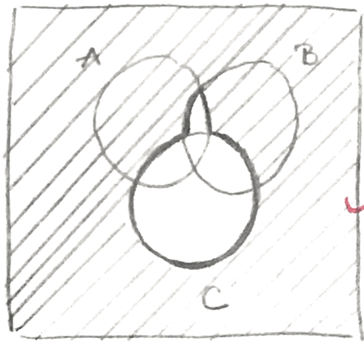
5. (7 points) Let A, B, C be any sets. Sketch the Venn diagram for $(A \cap B)^c - C$, showing at least one intermediate diagram.



$A \cap B$



$(A \cap B)^c$



$(A \cap B)^c - C$

8

6. (8 points) How many bit strings of length 10 contain of at least eight 0's? (e.g. 0010010000)

Case 1 Bit strings of length 10 contain of 8 0's : $\binom{10}{8} = \frac{10!}{8!\ 2!} = \frac{10 \times 9}{2 \times 1} = \frac{90}{2} = 45$

Case 2 Bit strings of length 10 contain of 9 0's : $\binom{10}{9} = \frac{10!}{9!\ 1!} = 10$

Case 3. Bit strings of length 10 contain of 10 0's : $\binom{10}{10} = 1$

Sum $45 + 10 + 1 = 56$

Therefore, there are 56 bit strings of length 10 contain of at least 8 0's

8

7. (8 points) Let $S = \{1, 2, \ldots, 12\}$. Consider all subsets A of S which satisfy the following property:

[$\star$] $|A| = 3$ and $A \cap \{2, 5\} = \varnothing$

(a) Give an example of a subset A of S which satisfies property [$\star$].

$A = \{1, 3, 4\}$

(b) Determine the number of different subsets A of S which satisfy property [$\star$].

$\binom{10}{3} = \frac{10!}{3!\ (10-3)!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$

23

26

8. (30 points) A class consists of six seniors, three juniors, and two sophomores.

(a) How many different groups of three students consist of exactly one senior, one junior, and one sophomore?

5/5

$6 \times 3 \times 2 = 18 \times 2 = 36$

(b) How many different groups of four students don't contain any juniors?

5/5

There are $6 + 2 = 8$ seniors and sophomores

Total different groups of 4 students don't contain any juniors:

$$\binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$$

(c) How many different groups of three students consist of at most two seniors?

8/20

Case 1. Groups of 3 students consist of 2 seniors: $\binom{6}{2} \times \binom{5}{1} = \frac{6!}{2!\,4!} \times 5 = \frac{6 \times 5}{2 \times 1} \times 5 = 75$

Case 2. Groups of 3 students consist of 1 senior: $\binom{6}{1} \times \binom{5}{2} = 6 \times \frac{5!}{2!\,3!} = 6 \times \frac{5 \times 4}{2 \times 1} = 60$

Case 3. Groups of 3 students consist of no senior: $\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \times 4}{2 \times 1} = 10$

Sum: $75 + 60 + 10 = 145$

There are 175 different groups of 3 students consist of at most 2 seniors

2/5 (d) How many different subsets of the class contain all of the seniors?

$2^6 = 64$ X

3 juniors + 2 sophomores

$2^5 = 32$

2/2 (e) The professor for the class is planning to give exactly six students A's; each of the remaining students will receive either a B or C. In how many different ways can the grades be assigned to the students?

There are $6 + 3 + 2 = 11$ students in the class:

6 students receive exactly A's : $\binom{11}{6}$

Therefore, 5 students will receive either a B or C: 2^5

Total number of different ways the grades can be assigned to the students:

$$\binom{11}{6} \times 2^5 = \frac{11!}{6!(11-6)!} = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} \times 32$$

$$= 462 \times 32$$

$$= 14,784$$

9. (7 points) Compute the coefficient of x^4y^3 in the expansion of $(10x - 3y)^7$.

$$(10x - 3y)^7 = \sum_{k=0}^{7} \binom{7}{k} (10x)^{7-k} \cdot (-3y)^k = \sum_{k=0}^{7} \binom{7}{k} 10^{(7-k)} \cdot (-3)^k \cdot x^{(7-k)} \cdot y^k$$

$k = 3$ The coefficient of x^4y^3 is

$$\binom{7}{3} \times 10^{(7-3)} \times (-3)^3 = \frac{7!}{3!(7-3)!} \times (10{,}000) \times (-27)$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times (-270{,}000)$$

$$= 35 \times (-270{,}000)$$

$$= -9{,}450{,}000$$

12 pt

EXTRA CREDIT (5 POINTS): A password for a certain website must consist of five digits (e.g. 09223). How many different passwords contain at least one even digit?

Let A = { passwords contain at least 1 even digit }

A^c = { passwords contain no even digit }

U = { all passwords }

$|U| = 10^5 = 100{,}000$

$|A^c| = 5^5 = 3{,}125$ (There are 5 odd digits)

$|A| = |U| - |A^c| = 100{,}000 - 3{,}125 = 96{,}875$