Solutions

Spring 2020

Name:

Linear Algebra 2: Exam 1 (Spring 2020)

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. *Make sure to exhibit skills discussed in class.* Box all answers and simplify answers as much as possible.

Good Luck! ☺

1. The Properties of Determinants (3.2)

[10 pts] Let A and B be 4×4 matrices with $det(A) = -1$ and $det(B) = 2$. Find the following:

$$det(B^{-1}AB)$$

* By the Multiplicative Property:

$$det(B^{-1}AB) = det(B^{-1}) \cdot det(A) \cdot det(B)$$

* By the Inverse Property:

$$= \frac{1}{\cancel{det(B)}_{1}} \cdot det(A) \cdot \cancel{det(B)}^{1}$$

$$= det(A)$$

$$= -1$$

$$\therefore \boxed{det(B^{-1}AB) = -1}$$

Ans

2. ***The Inverse of a Matrix (2.2) & Characteristics of Invertible Matrices (2.3)***

[10 pts] Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a Linear Transformation defined by:

$$T(x_1, x_2, x_3) = (x_1,\ x_1 + 2x_2,\ x_1 + 2x_2 + 3x_3)$$

Is T an invertible transformation? If it is, find a formula for T^{-1}.

*Rewrite: $T: \mathbb{R}^3 \to \mathbb{R}^3$ ST $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_1 + 2x_2 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

∴ Standard Matrix of A: $\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}}$

*Find A^{-1}:

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \underset{-R_1+R_3 \to N.R_3}{\overset{-R_1+R_2 \to N.R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right] \underset{\frac{1}{2}R_2}{\overset{-R_2+R_3}{\sim}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 3 & 0 & -1 & 1 \end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}}$$

*Find $T^{-1}(\vec{x}) = A^{-1}\vec{x}$:

$$T^{-1}(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \boxed{\therefore T^{-1}(\vec{x}) = \begin{bmatrix} x_1 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 \\ -\frac{1}{3}x_2 + \frac{1}{3}x_3 \end{bmatrix}}$$

Ans

Alternative Solution to # 3 (Others ∃ as well!) at the end of the test :)

3. Cramer's Rule, Volume, and Linear Transformations (3.3)

[8pts] Solve the linear system using *Cramer's Rule*:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_2 + 3x_3 &= 5 \\ x_3 &= 1 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}}_{\vec{b}}$$

* Find det(A): By Cofactor Expansion Down C_1

$$\det(A) = 1(-1)^2 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + 0 + 0 = (2-0) = \boxed{2}$$

* Find $\det[A_1(\vec{b})]$: By Recursive Def.

$$\det[A_1(\vec{b})] = \det \begin{bmatrix} 6 & 2 & 3 \\ 5 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} = 6 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 5 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= 6(2-0) - 2(5-3) + 3(0-2) = 12 - 4 - 6 = \boxed{2}$$

* Find $\det[A_2(\vec{b})]$: By Cofactor Expansion Down C_1

$$\det[A_2(\vec{b})] = \det \begin{bmatrix} 1 & 6 & 3 \\ 0 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} = 1(-1)^2 \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} + 0 + 0 = (5-3) = \boxed{2}$$

* Find $\det[A_3(\vec{b})]$: By Cofactor Expansion Down C_1

$$\det[A_3(\vec{b})] = \det \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix} = 1(-1)^2 \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} + 0 + 0 = (2-0) = \boxed{2}$$

* By Cramers Rule:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{\det[A_1(\vec{b})]}{\det(A)} \\ \frac{\det[A_2(\vec{b})]}{\det(A)} \\ \frac{\det[A_3(\vec{b})]}{\det(A)} \end{bmatrix} = \begin{bmatrix} \frac{2}{2} \\ \frac{2}{2} \\ \frac{2}{2} \end{bmatrix} \longrightarrow \boxed{\therefore \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Ansr

4. ***Characteristics of Invertible Matrices (2.3)***

[10 pts] Use the *Invertible Matrix Theorem* to find the value(s) of x so that the matrix is invertible:

$$A = \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{bmatrix}$$

Note: By the Invertible Matrix Theorem, we want 3 pivots positions along the main diagonal

* $ that $x \neq 0$

* Row-Reduce $[A \,\vdots\, \vec{0}]$ to Echelon Form:

$$\begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{bmatrix} \overset{*\, -\frac{1}{x}R_2 + R_3 \to N.R_3}{\sim} \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x - \frac{1}{x} \end{bmatrix}$$

* Echelon Form :)

* Since we want a pivot in each row/colum:

• R_1 & R_2: $\boxed{x \neq 0}$ ✓

• R_3: $x - \frac{1}{x} \neq 0 \longrightarrow \frac{x^2 - 1}{x} \neq 0 \longrightarrow x^2 - 1 \neq 0$

$(x-1)(x+1) \neq 0 \Longrightarrow \boxed{x \neq -1 \text{ \& } x \neq 1}$

∴ $\boxed{\text{Matrix } A \text{ is invertible } \forall\, x \text{ except}: -1, 0, 1}$

Ans ✓

5. **Cramer's Rule, Volume, and Linear Transformations (3.3)**

[10 pts] Find the volume of the box formed by the triple of vectors in $\mathbb{R}^3$:

$$\vec{x} = (1,1,1)\,, \qquad \vec{y} = (2,3,4)\,, \qquad \vec{z} = (1,1,5)$$

* Let $A = [\vec{x}\ \vec{y}\ \vec{z}]$

Recall: Volume of a Box $= |\det(A)|$

* Compute $\det[\vec{x}\ \vec{y}\ \vec{z}]$:

$$\det\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix} = 1\begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= (15-4) - 2(5-1) + (4-3)$$

$$= 11 - 8 + 1$$

$$= 4$$

$\therefore$ Volume of Box: $\boxed{4 \text{ cubic units}}$

Ans

3. ***Cramer's Rule, Volume, and Linear Transformations (3.3)***

[10 pts] Solve the linear system using *Cramer's Rule*:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{\det[A_1(\vec{b})]}{\det(A)} \\ \frac{\det[A_2(\vec{b})]}{\det(A)} \\ \frac{\det[A_3(\vec{b})]}{\det(A)} \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_2 + 3x_3 &= 5 \\ x_3 &= 1 \end{aligned}$$

*Rewrite as a Matrix Eq:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

*Find det(A) → By Cofactor Factor Expansion Down C_1

$$\det(A) = 1(-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + 0 + 0 = 1(2-0) = \boxed{2}$$

*Find $\det[A_1(\vec{b})]$: By Recursive Def.

$$\det \begin{bmatrix} 6 & 2 & 3 \\ 5 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} = 6\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 5 & 2 \\ 1 & 0 \end{vmatrix} = 6(2-0) - 2(5-3) + 3(0-2)$$

$$= 12 - 4 - 6$$

$$= \boxed{2}$$

*Find $\det[A_2(\vec{b})]$: By Recursive Def.

$$\det \begin{bmatrix} 1 & 6 & 3 \\ 0 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} = 1\begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} - 6\begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 5 \\ 0 & 1 \end{vmatrix} = 1(5-3) - 0 + 0 = \boxed{2}$$

*Find $\det[A_3(\vec{b})]$: By Recursive Def.

$$\det \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix} = 1\begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 0 & 1 \end{vmatrix} + 6\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 1(2-0) - 0 + 0 = \boxed{2}$$

*Find Compenents of $\vec{x}$:

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/2 \\ 2/2 \\ 2/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Answ