Linear I Exam 3 : Spring 2020

① Suppose that $T: \mathbb{R}^4 \to \mathbb{R}^6$ is a Linear Transformation.

(a) [2pts] If $\dim[\mathrm{Nul}(T)] = 2$, find $\dim[\mathrm{range}(T)]$.

(b) [2pts] If $\dim[\mathrm{range}(T)] = 3$, find $\dim[\mathrm{Nul}(T)]$.

Answer:

* $T: \mathbb{R}^4 \to \mathbb{R}^6 \Rightarrow$ The Standard Matrix of T is a 6×4 matrix $\to$ 6 Rows, $\to$ 4 Columns

(a) $\dim[\mathrm{Nul}(T)] = \#$ of non-pivot columns $= 2$:

$\therefore \dim[\mathrm{range}(T)] = \begin{pmatrix}\text{Total}\\ \text{Columns}\end{pmatrix} - \begin{pmatrix}\text{Non-Pivot}\\ \text{Columns}\end{pmatrix}$

$= 4 - 2$

2pts $= \boxed{2}$

(b) $\dim[\mathrm{range}(T)] = \#$ of pivot columns $= 3$:

$\therefore \dim[\mathrm{range}(T)] = \begin{pmatrix}\text{Total}\\ \text{Columns}\end{pmatrix} - \begin{pmatrix}\text{Pivot}\\ \text{Columns}\end{pmatrix}$

$= 4 - 3$

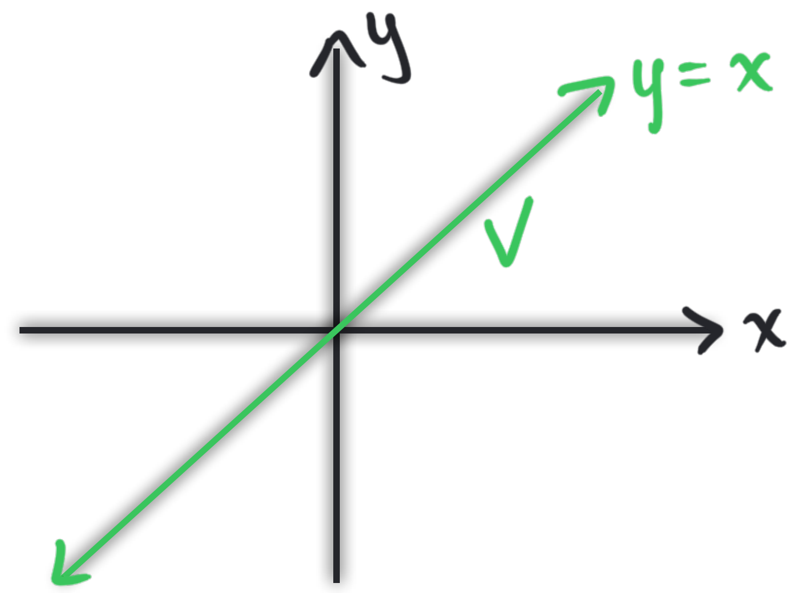
2pts $= \boxed{1}$

② Determine which set is a Vector Space.
Create Geometric Figure for each.
[9pts] For 2 that are NOT Vector Spaces, provide specific example as to why NOT.

Ans.

3pts

* $E = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x \right\}$



i) $\vec{0} \in E$ ✓

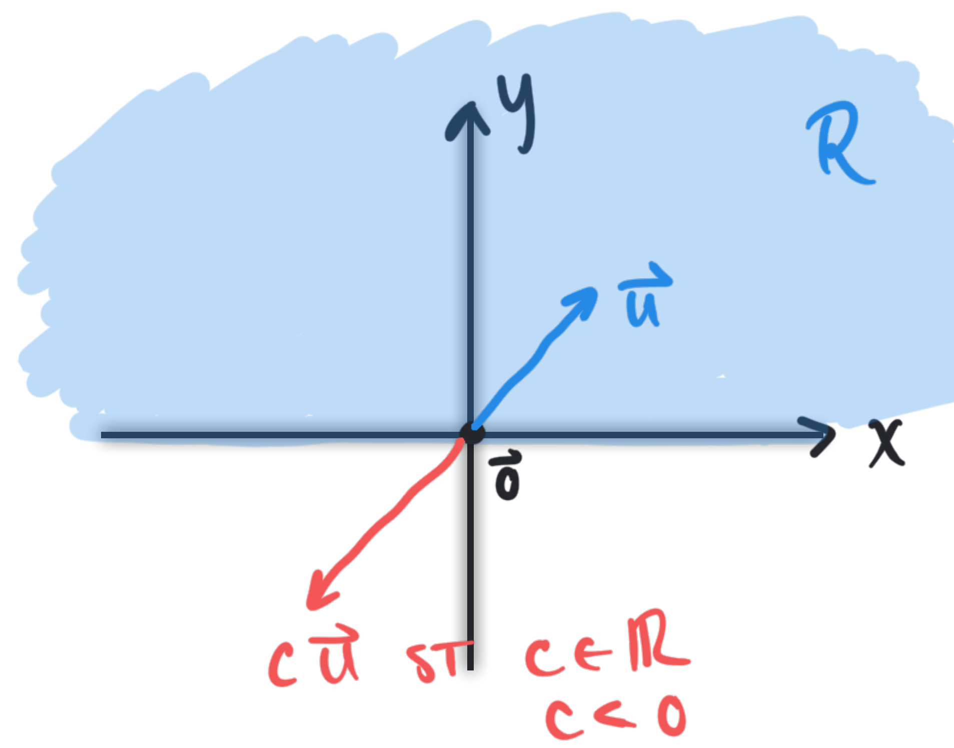
ii) $\forall\, \vec{u}, \vec{v} \in E$, $\vec{u} + \vec{v} \in E$ ✓

iii) $\forall\, \vec{u} \in E$ & $\forall\, c \in \mathbb{R}$, $c\vec{u} \in E$ ✓

$\therefore$ E is a Vector Space.

3pts

* $R = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y \geq 0 \right\}$



i) $\vec{0} \in R$ ✓

ii) $\forall\, \vec{u}, \vec{v} \in R$, $\vec{u} + \vec{v} \in R$ ✓

* iii) $\forall\, \vec{u} \in R$ & $\forall\, c \in \mathbb{R}$, $c\vec{u} \notin R$ X

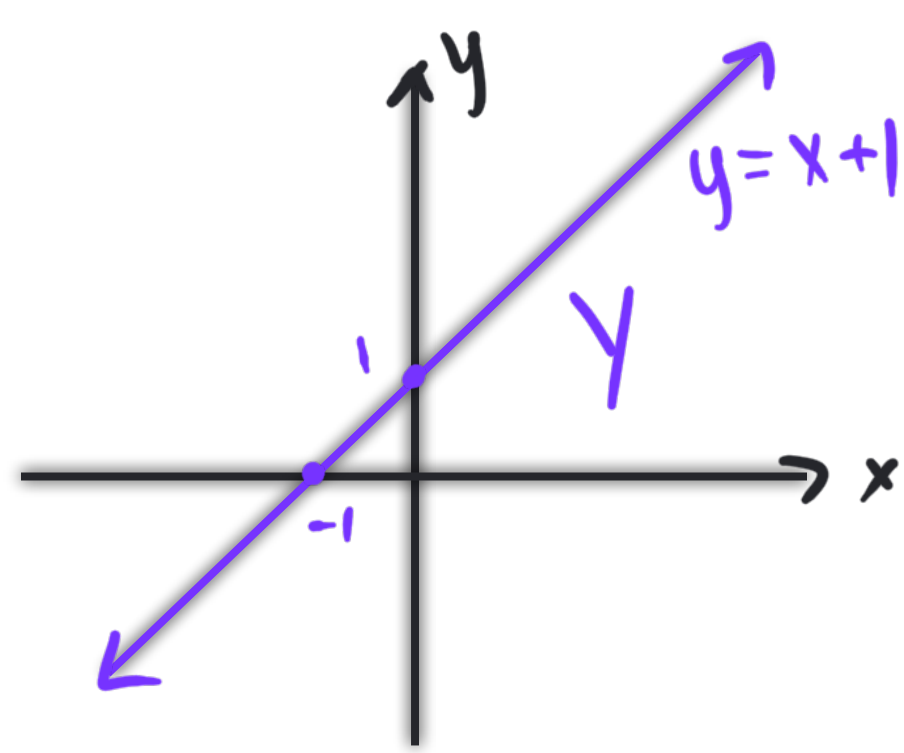
Ex: $\exists\, \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in R$ & Let $c = -1$

Then: $c\vec{u} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin R$

$\therefore$ R is NOT a Vector Space

3pts

* $Y = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x + 1 \right\}$



i) $\vec{0} \notin Y$ X

$\therefore$ Y is NOT a Vector Space

Ex: $\exists\, \vec{u} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin R$.

5pts ③ Linear Independent or Linear Dependent?

$\vec{p}_1(t) = 1,\ \vec{p}_2(t) = -2+4t^2,\ \vec{p}_3(t) = 2t,\ \vec{p}_4(t) = -12t+8t^3$

Ans.

* Convert polynomials in $\mathbb{P}_3$:

$\cdot\ \vec{p}_1(t) = 1 \longrightarrow \vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\cdot\ \vec{p}_3(t) = 2t \longrightarrow \vec{p}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$\cdot\ \vec{p}_2(t) = -2+4t^2 \longrightarrow \vec{p}_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$

$\cdot\ \vec{p}_4(t) = -12t+8t^3 \longrightarrow \vec{p}_4 = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$

* Check if a Free Variable $\exists$:

$$[\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4 \mid \vec{0}] = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -12 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & -12 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\frac{\frac{3}{2}R_4 + R_3}{N.R_3} \sim \begin{bmatrix} \underline{1} & -2 & 0 & 0 \\ 0 & \underline{4} & 0 & 0 \\ 0 & 0 & \underline{2} & 0 \\ 0 & 0 & 0 & \underline{8} \end{bmatrix}$$

* Echelon Form ✓

Note: A pivot position $\exists$ in each column $\Rightarrow$ No free variables.

$$\boxed{\therefore \{\vec{p}_1, \vec{p}_2, \vec{p}_3, \ \&\ \vec{p}_4\} \text{ are Linearly Independent}}$$

④ $V = P_2$; $\mathcal{B} = \{t-1, t+1, t^2-1\}$ & $\mathcal{C} = \{1, t+1, t^2+t\}$

6pts (a) Find Change of Coord. Matrix From $\mathcal{B}$ to $\mathcal{C}$.

6pts (b) Express $\vec{p}(t) = t^2 - t + 5$ relative to Basis $\mathcal{B}$.

Answer:

$$* \ \mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$* \ \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$*$ Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$: Note: Row-reduce $[\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3 \ \vdots \ \vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\overset{-R_2 + R_1}{\overline{N.R_1}}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_3 + R_1 \to N.R_1 \\ \sim \\ -R_3 + R_2 \to N.R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \boxed{\therefore \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$*$ Express $\vec{p}(t) = t^2 - t + 5$, relative to $\mathcal{B}$:

$$\vec{p}(t) = t^2 - t + 5 \implies \vec{p} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

$*$ Find $[\vec{p}]_{\mathcal{B}}$ by row-reducing $[\mathcal{B} \ \vdots \ \vec{p}]$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 5 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} R_1 + R_2 \to N.R_2 \\ \sim \\ -R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -5 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \frac{1}{2}R_2 + R_1 \to N.R_1 \\ \sim \\ \frac{1}{2}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & -3 \\ 0 & 1 & -1/2 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} -\frac{1}{2}R_3 + R_1 \\ \sim \\ \frac{1}{2}R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \implies \therefore [p]_{\mathcal{B}} = \begin{bmatrix} -7/2 \\ 5/2 \\ 1 \end{bmatrix}$$

$$\boxed{\therefore \vec{p}(t) = -\frac{7}{2}\vec{b}_1 + \frac{5}{2}\vec{b}_2 + 1\vec{b}_3 = -\frac{7}{2}(t-1) + \frac{5}{2}(t+1) + 1(t^2-1)}$$

⑤ Define a Linear Transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ ST

$$T(x_1,x_2,x_3,x_4) = \begin{bmatrix} x_1 + x_2 - x_3 + x_4 \\ 2x_1 + x_2 + 4x_3 + x_4 \\ 3x_1 + x_2 + 9x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Answer:

Row-reduce $[A \,\vdots\, \vec{c}]$ to RREF:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix} \underset{-3R_1+R_3}{\overset{-2R_1+R_2}{\sim}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 6 & -1 \\ 0 & -2 & 12 & -3 \end{bmatrix} \begin{matrix} R_2+R_1 \to N.R_1 \\ -2R_2+R_3 \to N.R_3 \\ -R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* RREF :)

* Note:
  - Columns 1,2,3 are pivot col.
  - x_3 is free

$\therefore$ Basis for Col(A): $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$ 6pt

$\therefore$ Basis for Row(A): $\boxed{\{(1,0,5,0), (0,1,-6,0), (0,0,0,1)\}}$ 6pt

*General Solution (in Parametric Vector Form):

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix} \quad \text{ST} \quad x_3 \in \mathbb{R}$$

$\therefore$ Basis for Nul(A) $\boxed{\left\{ \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix} \right\}}$ 6pt

* $\boxed{\text{Rank}(T) = 3 \quad \& \quad \dim[\text{Nul}(T)] = 1}$ 2pt