Section 1.7: Linear Independence:

Note: Here our focus shifts from the unknown solutions of $A\vec{x}=\vec{0}$, to the vectors that appear in the vector equations.

* Again, the main issue is to determine whether the non-trivial solution is the only solution.

*Definitions:

(i) An indexed set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be Linearly Independent if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has ONLY the trivial solution.

(ii) An indexed set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be Linearly Dependent if $\exists$ weights $c_1, c_2, \dots, c_p$ (not all zero) such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$$

Note: The second def. is called a 'linear dependent relation'

*To Determine if a set is Linearly Independent/Dependent:

Recalling that the homogeneous eq. $A\vec{x}=\vec{0}$ has a nontrivial solution IFF the eq. has @ least one free variable

$\Rightarrow$ If free variable $\exists$, then set is linearly dependent ::

Example (Linear Independent/Dependent Sets):

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, & $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(a) Determine if the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

(b) If possible, find a linear dependent relation among v_1, v_2, & v_3.

Answer:

Note: Here we must determine if a nontrivial solution $\exists$

i) If $A\vec{x} = \vec{0}$ has a nontrivial solution $\Longrightarrow$ Linearly Dependent

ii) If $A\vec{x} = \vec{0}$ has only the nontrivial sol. $\Longrightarrow$ Linearly Independent

Part (a):

*Use row reduction operations on the augmented matrix to check if the echelon form has a free variable:

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right]$$

- $\begin{array}{r} -2R_1 \\ + R_2 \\ \hline \text{new } R_2 \end{array}$ $\longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \overset{-\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right]$

- $\begin{array}{r} -R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array}$ $\longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \overset{-\frac{1}{2}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

Example (Linearly Dependent/Independent Sets) Cont...

• $-R_2 + R_3$ / new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ * x_3 is a free variable

$\Rightarrow$ A nontrivial solution $\exists$

$\boxed{\therefore \text{Since a nontrivial solution } \exists \text{ for } A\vec{x} = \vec{0}, \text{ the set is Linearly Dependent}}$ Ans.

Notes on Conclusion: Since x_1 & x_2 are Basic Variables & x_3 is a Free Variable, each nonzero value of x_3 determines a nontrivial solution of $A\vec{x} = \vec{0}$ (same eq. as def. 1)

$\Rightarrow$ $\vec{v}_1, \vec{v}_2$, & $\vec{v}_3$ are NOT linearly independent

*Part (b):

To find a linear dependence relation for $\vec{v}_1, \vec{v}_2$, & $\vec{v}_3$

① Solve the aug. matrix to row-reduced echelon form & write the new system:

• $-4R_2 + R_1$ / new R_1 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\Longleftrightarrow$ $\begin{cases} \bullet\ x_1 = 2x_3 \\ \bullet\ x_2 = -x_3 \\ \bullet\ x_3 \text{ is free} \end{cases}$

reduced-echelon form — general solution

② Choose any nonzero value for x_3 & use the general solution to find the Basic Variable values & thus a linear independence relation (infinitely many solutions)

• $ $x_3 = 17$ $\Longrightarrow$ • $x_1 = 2(17) = 34$

• $x_2 = -(17) = -17$

• Substitute these values into $A\vec{x} = \vec{0}$ to obtain a Linear Dependence Relation among $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$:

Note: $A\vec{x} = \vec{0} \iff x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$

$$\therefore \boxed{34\vec{v}_1 - 17\vec{v}_2 + 17\vec{v}_3 = \vec{0}}$$

*Again, this is only ONE of infinitely many possible linear dependence relations amongst $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$:)

Example: Determine if the vectors are linearly independent. Justify your answer: $\begin{bmatrix}4\\0\\0\end{bmatrix}, \begin{bmatrix}9\\2\\-4\end{bmatrix}, \begin{bmatrix}8\\8\\-24\end{bmatrix}$

Answer:

*Note: To determine if the vectors are linearly independent, solve the corresponding augmented matrix $A\vec{x}=\vec{0} \Leftrightarrow [A \mid 0]$ to echelon form to see if a free variable(s) $\exists$.

$$[A \mid 0] = \left[\begin{array}{ccc|c} 4 & 9 & 8 & 0 \\ 0 & 2 & 8 & 0 \\ 0 & -4 & -24 & 0 \end{array}\right] \underset{-\frac{1}{4}R_3}{\overset{\frac{1}{2}R_2}{\sim}} \left[\begin{array}{ccc|c} 4 & 9 & 8 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 6 & 0 \end{array}\right]$$

- $\begin{array}{r} -R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 4 & 9 & 8 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \underset{\frac{1}{2}R_3}{\overset{\frac{1}{4}R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 9/4 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

*Basic Variable $\exists$ in each column.

$\boxed{\therefore \text{ Since NO free variable } \exists\text{, the vectors are linearly independent.}}$ Ans.

IOW: The vector equation only has a trivial solution ⸬

Example: Find the value(s) of h for which the vectors are linearly dependent. Justify your answer. L!

$$\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ h \end{bmatrix}$$

Answer:

Recall: A set of vectors is linearly dependent if $A\vec{x} = \vec{0}$ has a nontrivial solution.

Solve the corresponding aug matrix $[A \,\vdots\, \vec{0}]$ to find h:

$$[A \,\vdots\, 0] = \left[\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ -2 & -6 & 3 & 0 \\ 4 & 7 & h & 0 \end{array}\right]$$

- $\dfrac{R_1 + R_2}{\text{new } R_2} \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 7 & h & 0 \end{array}\right] \overset{-\frac{1}{2}R_2}{\sim} \left[\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 7 & h & 0 \end{array}\right]$

- $\dfrac{-2R_1 + R_3}{\text{new } R_3} \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 6+h & 0 \end{array}\right]$

Example Continued...

- $R_2 + R_3$ new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6+h & 0 \end{array}\right]$

Note: $A\vec{x} = \vec{0}$ must have @ least one free variable for a nontrivial solution to $\exists$.

So, $\boxed{6+h=0 \longrightarrow h=-6}$

Conclusion:

$\therefore h=-6$ so that x_3 is a free variable $\Rightarrow$ Nontrivial Sol. $\exists$

$\Rightarrow$ Vectors are linearly dependent

Ans.

Example: Find the value(s) of h for which the vectors are linearly dependent. Justify your answer.

$$\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -4 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ h \end{bmatrix}$$

Answer:

* Solve the corresponding augmented matrix $A\vec{x} = \vec{0}$ to reduced echelon form to find h-value(s) ST a nontrivial solution $\exists$ (i.e. Free Variable(s) $\exists$ ✓):

$$[A \mid \vec{0}] = \left[\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ -2 & 9 & 1 & 0 \\ -4 & 8 & h & 0 \end{array}\right]$$

• $\dfrac{2R_1 + R_2}{\text{new } R_2} \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & 1 & 7 & 0 \\ -4 & 8 & h & 0 \end{array}\right]$

• $\dfrac{4R_1 + R_3}{\text{new } R_3} \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & -8 & 12+h & 0 \end{array}\right]$

• $\dfrac{8R_2 + R_3}{\text{new } R_3} \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 68+h & 0 \end{array}\right]$

So, $68 + h = 0$

$h = -68$

$$\boxed{\therefore \underline{h=-68} \text{ makes } x_3 \text{ a free variable \& the vectors linearly dependent}}$$ ☺

Answer ✓

Linear Independence of Matrix Columns

$ we begin with a matrix $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$.

We know that the matrix equation $A\vec{x} = \vec{0}$

$\Updownarrow$

can be written as the vector equation $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$.

→ Important Conclusion:

Since each linear dependent relation amongst the columns of A correspond to a nontrivial solution of $A\vec{x} = \vec{0}$:

The columns of A are linearly independent IFF the equation $A\vec{x} = \vec{0}$ has ONLY the trivial solution $(\vec{0})$.

*To Determine if the matrix columns are linearly independent

(Again.:)

→ Row reduce the augmented matrix to echelon form to check if @ least one free variable exists :)

Example (Linear Independence of Matrix Columns):

Determine if the columns of the following matrix are linearly independent: $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$

Answer:

*To check the matrix columns, row reduce the corresponding augmented matrix $A\vec{x} = \vec{0}$ to echelon form to see if a free variable(s) $\exists$:

$$[A \mid 0] = \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right] \underset{R_1 \,\&\, R_2}{\overset{\text{Inter-change}}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right]$$

• $-5R_1 + R_3$ / new R_3 $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right]$$

• $2R_2 + R_3$ / new R_3 $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right] \overset{\frac{1}{13}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

*echelon form has a Basic Variable in each column.

$\therefore$ Since NO free variables $\exists$, the equation $A\vec{x} = \vec{0}$ has ONLY the trivial solution. $\rightarrow$ Columns are Linearly Independent.

Example: Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -7 \\ 1 & -4 & -2 \end{bmatrix}$$

Answer:

*Note: To check if the columns form a linearly independent set, solve the corresponding aug. matrix ($A\vec{x}=\vec{0} \leftrightarrow [A \,|\, 0]$) for echelon form to see if a free variable $\exists$.

$$[A \,|\, 0] = \left[\begin{array}{ccc|c} 0 & -3 & 9 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -7 & 0 \\ 1 & -4 & -2 & 0 \end{array}\right] \overset{-\frac{1}{3}R_1}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -7 & 0 \\ 1 & -4 & -2 & 0 \end{array}\right]$$

*interchanging the rows $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -7 & 0 \end{array}\right]$$

• $-2R_1 + R_3$ / new R_3 $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 9 & -3 & 0 \\ -1 & 4 & -7 & 0 \end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & 4 & -7 & 0 \end{array}\right]$$

• $R_1 + R_4$ / new R_4 $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right] \overset{-\frac{1}{9}R_4}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example Continued...

- $\begin{array}{r} -3R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{\frac{1}{8}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

- $\begin{array}{r} -R_3 \\ +R_4 \\ \hline \text{new } R_4 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

*Basic Variable in each column → NO free variable(s)

$\therefore$ The reduced echelon form of the augment matrix has NO free variables $\Rightarrow$ Only a trivial solution $\exists$.

$\Rightarrow$ Columns of the matrix are linearly independent.

Answer ✓

Example: Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 1 & -5 \\ 2 & 1 & -10 \end{bmatrix}$$

Answer:

Note: Solve the corresponding augment matrix to reduced echelon form to see if a free variable(s) $\exists$.

$* A\vec{x} = \vec{0} \leftrightarrow [A \mid 0]$:

$$[A \mid 0] = \left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{array}\right] \underset{*-1R_2}{\overset{*\text{interchanging } R_1 \& R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ -4 & -3 & 0 & 0 \\ 2 & 1 & -10 & 0 \end{array}\right]$$

- $\begin{array}{r} 4R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 1 & -20 & 0 \\ 2 & 1 & -10 & 0 \end{array}\right]$

- $\begin{array}{r} -2R_1 \\ + R_4 \\ \hline \text{new } R_4 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 1 & -20 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

Example Continued...

- $\begin{array}{r} -R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & -15 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \underset{\sim}{-\frac{1}{15}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

- $\begin{array}{r} R_2 \\ +R_4 \\ \hline \text{new } R_4 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$

- $\begin{array}{r} 5R_3 \\ +R_4 \\ \hline \text{new } R_4 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

*Basic Variable $\exists$ in each column
$\Rightarrow$ NO free variable(s)

$\therefore$ Since NO free variable $\exists$, the reduced echelon form of $A\vec{x} = \vec{0}$ indicates that ONLY a trivial solution $\exists$.

$\rightarrow$ Columns of matrix are Linearly Independent.

Sets of One -OR- Two Vectors

Note: When considering sets of one or two vectors, row operations are unnecessary!

$\Rightarrow$ We can simply check for linear independence/dependence using 'visual inspection' :)

(i) A set containing ONE vector $\{\vec{v}\}$ is:

- Linearly Independent IFF $\vec{v}$ is NOT the zero vector

  $\Rightarrow$ The vector eq. $x_1\vec{v} = \vec{0}$ has ONLY the trivial solution when $\vec{v} \neq \vec{0}$

- The zero vector is Linearly Dependent b/c $x_1\vec{0} = \vec{0}$ has many nontrivial solutions.

(ii) A set containing TWO vectors $\{\vec{v}_1, \vec{v}_2\}$ is:

- Linearly Dependent if @ least one of the vectors is a scalar multiple of the other

  $\Rightarrow$ Geometrically, 2 vectors are linearly dependent IFF they lie on the same line through the origin.

- Linearly Independent if neither vector is a multiple of the other.

Example: Determine if the following sets of vectors are linearly independent:

(a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(b) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Answer:

Note: While row operations work here, since each part contains a set w/ 2 vectors, we will use visual inspection :)

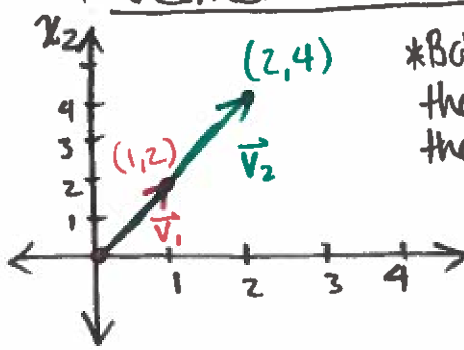
Part (a):

$\boxed{\text{Since } \vec{v}_2 = 2\vec{v}_1 \text{ the vectors are linearly dependent.}}$

*Algebraic Check:

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\vec{v}_1 \checkmark$$

* Geometric Check:

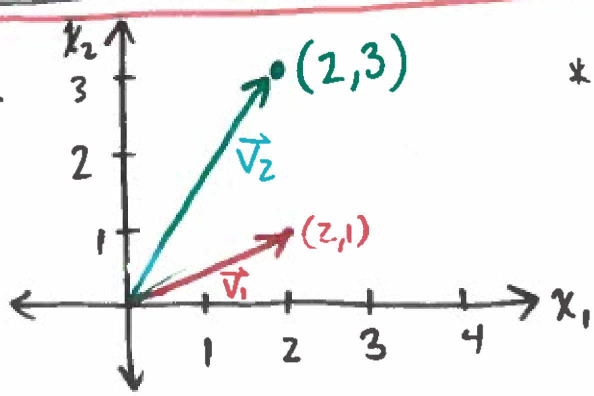


*Both vectors lie on the same line, through the origin.

Part (b):

$\boxed{\text{NO scalar multiples exist b/w } \vec{v}_1 \text{ \& } \vec{v}_2 \Rightarrow \therefore \text{Linearly Indendendent}}$

*Geometric Check:



*Vectors are NOT on the same line.

Example: Determine by inspection whether the vectors are linearly independent. Justify.

$$\begin{bmatrix} 8 \\ -16 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

Ans.

> The 2 vectors are <u>linearly independent</u> b/c they are <u>NOT scalar multiples</u>.

Sets of Two -or- More Vectors

Theorem: (Characterization of Linearly Dependent Sets)

* An indexed set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ of 2 or more vectors is linearly dependent IFF @ least one of the vectors in S is a linear combination of the others.

* IF S is linearly dependent & $\vec{v}_1 \neq \vec{0}$, then some $\vec{v}_j$ (w/ $j>1$) is a linear combination of the proceeding vectors, $v_1, v_2, \dots, v_{j-1}$.

Caution: This does NOT say that every vector in a linear dependent set is a linear combo of the proceeding vectors!

$\Rightarrow$ A vector in a linear dependent set may fail to be a linear combination of the others.

Recall:

Let $\vec{v}$ & $\vec{u}$ be nonzero vectors in $\mathbb{R}^3$.

(i) The Span$\{\vec{v}\}$ is the set of all scalar multiples of $\vec{v}$

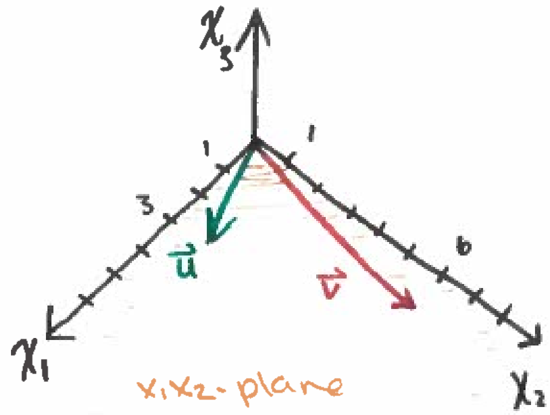
$\Rightarrow$ The set of all points on the line in $\mathbb{R}^3$ through $\vec{v}$ & $\vec{0}$.

(ii) IF $\vec{v}$ is NOT a multiple of $\vec{u}$, then Span$\{\vec{v}, \vec{u}\}$ is the plane in $\mathbb{R}^3$ containing $\vec{v}, \vec{u}$ & $\vec{0}$.

$\Rightarrow$ The Span$\{\vec{v}, \vec{u}\}$ contains the line through $\vec{u}$ & $\vec{0}$ -and- $\vec{v}$ & $\vec{0}$

Example (Sets of 2 or more vectors):

Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$



Describe the set spanned by $\vec{u}$ & $\vec{v}$.

Explain why $\vec{w}$ is in Span$\{\vec{u},\vec{v}\}$ IFF $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent.

Answer:

* $\vec{u}$ & $\vec{v}$ are nonzero vectors in $\mathbb{R}^3$.
* $\vec{u}$ & $\vec{v}$ are NOT scalar multiples $\Rightarrow$ Linearly Independent!

$\therefore$ Since $\vec{u}$ is NOT a multiple of $\vec{v}$, then Span$\{\vec{u},\vec{v}\}$ is the plane in $\mathbb{R}^3$ containing $\vec{u}$, $\vec{v}$, & $\vec{0}$.

Note: The Span$\{\vec{u},\vec{v}\}$ is the x_1x_2-plane (in $\mathbb{R}^3$) :)

*Explain: $\vec{w}$ is in Span$\{\vec{u},\vec{v}\}$ $\underset{\Longleftrightarrow}{\text{IFF}}$ $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependen

(i) IF $\vec{w}$ is in the Span$\{\vec{u},\vec{v}\}$:

Then $\vec{w}$ is a linear combo of $\vec{u}$ & $\vec{v}$ $\underset{(1.3)}{\Longrightarrow}$ Then $\{\vec{u},\vec{v},\vec{w}\}$ are linearly dependent ✓
(Thm)

(ii) IF $\{\vec{u},\vec{v},\vec{w}\}$ are linearly dependent:

Then @ least one of the vectors is a linear combo. of the others. $\Longrightarrow$ Since $\vec{v}$ is NOT a multiple of $\vec{u}$ & $\vec{u} \neq \vec{0}$, $\vec{w}$ is a linear combo of $\vec{u}$ & $\vec{v}$. $\therefore \vec{w}$ is in Span$\{\vec{u},\vec{v}\}$ ✓

Generalization For Previous Example

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be any set in $\mathbb{R}^3$ ST $\vec{u}$ & $\vec{v}$ are linearly independent.

→ The set $\{\vec{u}, \vec{v}, \vec{w}\}$ will be linearly dependent IFF $\vec{w}$ is in the plane spanned by $\vec{u}$ & $\vec{v}$.

Special Cases of Linear Dependence

Note: The following theorems are special cases where linear dependence of a set is automatic :)

*(Special) Theorem # 2

* This theorem will continue to be important in later chapters :)

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

→ IOW: Any set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof: \$ $A = [\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_p]$ in $\mathbb{R}^n$. So, A is an $n \times p$ matrix.

Then $A\vec{x} = \vec{0}$ corresponds to the linear system with n-equations (rows) & p-unknowns (columns).

If the # of columns > the # of rows (i.e. $p > n$), then at least one free variable MUST exist.

If a free variable ∃, the $A\vec{x} = \vec{0}$ has a nontrivial solution & thus is linearly dependent $\boxed{X}$

*(Special) Theorem #3:

IF a set $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly independent.

Pf: \$ $\vec{v}_1 = \vec{0}$.

Since $A\vec{x} = \vec{0} \iff x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$,

then: $1\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p = \vec{0} \overset{b/c}{\iff} \vec{v}_1 = \vec{0}$

$\Rightarrow$ So a nontrivial solution $\exists$ & S is linearly dependent $\boxed{X}$

Example (Special Case Theorems): Determine (by inspection) if the following sets are linearly dependent:

(a) $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$ (b) $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$ (c) $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$

Answer:

* **Part (a):** The matrix created by these vectors is 3×4 (3-equations w/ 4-unknowns) $\Rightarrow$ Free variable $\exists$

  $\boxed{\therefore \text{Set is linearly dependent (Th}^{m}\text{ 2)}}$

* **Part (b):** Since the Set contains the zero vector $\vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

  $\Rightarrow \boxed{\therefore \text{Set is linearly dependent (Th}^{m}\text{ 3)}}$

* **Part (c):** Since the 2 vectors are NOT scalar multiples

  $\Rightarrow \boxed{\therefore \text{Set is linearly independent}}$

Example: Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -2 & 4 & -6 & 2 \end{bmatrix}$$

Answer:

* Given a 2×4 matrix $\rightarrow n = 2$ rows (equations), $\rightarrow p = 4$ columns (unknowns)

Since 4 unknowns > 2 equations, $A\vec{x} = \vec{0}$ has @ least one free variable!

$\Rightarrow$ A nontrivial solution $\exists$ & so the columns of the matrix are linearly dependent.

Ans.

* Uses 'special' theorem# 2 $\Rightarrow$ matrix contains more vectors than entries in each vector ⁚

Example: Consider the vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -4 \\ 16 \\ -8 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 8 \\ h \end{bmatrix}$$

(a) For what values of "h" is $\vec{v}_3$ in the span$\{\vec{v}_1, \vec{v}_2\}$

(b) For what values of "h" is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent?

Answer:

Recall: $\vec{v}_3$ is in the span$\{\vec{v}_1, \vec{v}_2\}$ IFF $\exists$ a solution to $A\vec{x} = \vec{v}_3 \leftrightarrow [A \mid \vec{v}_3] = [\vec{v}_1 \ \vec{v}_2 \mid \vec{v}_3]$

(a)

Find the row-reduced echelon form for the corresponding augmented matrix, $[\vec{v}_1 \ \vec{v}_2 \mid \vec{v}_3]$

$$\left[\begin{array}{cc|c} 1 & -4 & 5 \\ -4 & 16 & 8 \\ 2 & -8 & h \end{array}\right] \begin{array}{c} \\ -\frac{1}{4}R_2 \\ \frac{1}{2}R_3 \end{array} \sim \left[\begin{array}{cc|c} 1 & -4 & 5 \\ 1 & -4 & -2 \\ 1 & -4 & h/2 \end{array}\right]$$

• $\begin{array}{r} -R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 5 \\ 0 & 0 & -7 \\ 1 & -4 & h/2 \end{array}\right] \rightarrow\leftarrow$ Contradiction!

$\boxed{\therefore \text{NO values of } h \text{ are in the Span}\{\vec{v}_1, \vec{v}_2\}}$

Example Continued...

Part (b):

*Recall: For $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to be linearly dependent, @ least one of the vectors must be a linear combination of the others

$\Rightarrow x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0} \iff \left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{array}\right]$ must have a nontrivial solution (i.e. free variable(s) $\exists$)

*Solve the corresponding aug. matrix to echelon form to find the h-value(s) ST a free variable(s) $\exists$:

$$\left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ -4 & 16 & 8 & 0 \\ 2 & -8 & h & 0 \end{array}\right]$$

- $4R_1 + R_2$ = new R_2 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 0 & 28 & 0 \\ 2 & -8 & h & 0 \end{array}\right] \overset{\frac{1}{28}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -8 & h & 0 \end{array}\right]$

- $-2R_1 + R_3$ = new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h-10 & 0 \end{array}\right]$

↘

Example Continued...

- $-(h-10)R_2 + R_3$ new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

* x_2 is a free variable
$\Rightarrow$ Nontrivial Sol. $\exists$

$\because$

$$\boxed{\therefore \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is linearly dependent } \forall \text{ possible } h\text{-values!}}$$

Example: Determine (by inspection) whether the vectors are linearly independent. Justify your answer:

$$\begin{bmatrix}5\\1\end{bmatrix}, \begin{bmatrix}3\\9\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-1\\8\end{bmatrix}$$

Answer:

* Since $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0} \leftrightarrow [\vec{v}_1 \cdots \vec{v}_p \mid 0]$

$\Rightarrow$ The corresponding matrix is a 2×4

- $n = 2$ rows (# of eq.)
- $p = 4$ columns (# of unknowns)

$\boxed{\text{∴ Since \# of unknowns > \# of eq., @ least one free variable ∃ \& thus } [\vec{v}_1 \cdots \vec{v}_p \mid 0] \text{ has a nontrivial solution.}}$

$\downarrow$

$\boxed{\text{∴ Vectors are Linearly Dependent.}}$

Example: For the following matrix, observe that the 3rd column is the sum of the first & second columns. Find a nontrivial solution of $A\vec{x} = \vec{0}$ w/o performing row operations. *Hint: Write $A\vec{x} = \vec{0}$ as a vector eq.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -7 & 1 & -6 \\ -3 & -2 & -5 \\ 1 & 0 & 1 \end{bmatrix}$$

Answer:

*Write $A\vec{x} = \vec{0}$ as a vector eq:

$$\vec{x}_1 \begin{bmatrix} 2 \\ -7 \\ -3 \\ 1 \end{bmatrix} + \vec{x}_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} + \vec{x}_3 \begin{bmatrix} 3 \\ -6 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 \begin{bmatrix} 2 \\ -7 \\ -3 \\ 1 \end{bmatrix} + \vec{x}_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \vec{x}_3 \begin{bmatrix} -3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$ Ans.