Section 10.2: Complete Matchings and Hall's Marriage Theorem

1. Suppose that Bob, Carl, Dan, and Ernest are workers at a company. The company wants to assign four different tasks to the four workers so that each worker gets one task that he is qualified to do. Suppose that Bob, Dan, and Ernest can do tasks 2 and 3, and Carl can do tasks 1, 3, and 4.

   (a) Model the tasks the workers are qualified to do using a bipartite graph G with bipartition (V_1, V_2), where V_1 is the set of workers, and V_2 is the set of tasks.

   (b) There's no complete matching of the graph G. Find a subset A that violates the condition in Hall's Marriage Theorem. Show why your subset works.

2. Four women named Alice, Barbara, Candy, and Darla are contemplating marriage with four men named Quentin, Russell, Stan, and Tom. The following is known:

   - Alice is compatible with Russell and Tom.
   - Barbara is compatible with Quentin and Russell.
   - Candy is compatible with Quentin and Tom.
   - Darla is compatible with Russell and Stan.

   (a) Model the compatibility of the women with the men using a bipartite graph G with bipartition (V_1, V_2), where V_1 is the set of women, and V_2 is the set of men.

   (b) Find a complete matching. (Are there other complete matchings? If so, how many?)

★★

Sections 8.5 and 8.6: Inclusion-Exclusion Problems

1. Suppose that there are 5 freshmen, 6 seniors, and 3 juniors in a class. How many groups of five students ...

   (a) ... contain exactly two freshmen? $C(5,2) \times C(9,3)$

   (b) ... contain exactly two freshmen or two ~~sophomores~~ seniors? $C(5,2) \times C(9,3) + C(6,2) \times C(8,3) - C(5,2) \times C(6,2) \times 3$

2. Suppose that a bakery sells five types of muffins (apple, blueberry, carrot, pineapple, and strawberry). How many ways can ten muffins be selected if ...

   (a) ... there's no restrictions? $C(5+10-1, 5)$

   (b) ... at least three apple and four blueberry muffins are picked?

   (c) ... [at least [three $\overset{P_1}{\text{apple}}$ muffins] or [at least four blueberry $\overset{P_2}{\text{muffins}}$] are picked?

   (d) ... at most two apple muffins and three blueberry muffins are picked?

3. Alice has ten identical marbles she wants to distribute between four friends (Bob, Candy, Darren, and Ernest). How many ways can she do this if ...

   (a) ... there's no restrictions?

   (b) ... Bob gets at least two marbles and Candy gets at least four marbles?

   (c) ... Bob gets at least two marbles or Candy gets at least four marbles?

   (d) ... Bob gets at most one marble and Candy gets at most three marbles? $= a - c$

4. A company wants to assign six different tasks to three employees (Alice, Bob, and Candy). How many ways can this be done if ...

   (a) ... there's no restrictions?

   (b) ... each employee is assigned at least one task?

Section 8.5: Inclusion-Exclusion Problems

1. A class of 20 students contains only students majoring in math or physics (or both). There are 12 students majoring in math and 15 majoring in physics.

   (a) How many students are majoring in both areas?

   (b) How many students are majoring in only math?

2. How many 3-permutations of the set $\{1, 2, \ldots, 11\}$ contain only even digits or consist of all different digits?

   $A_1 = \{$3-terms that contain only 3 even digits$\}$

   $A_2 = \{$ ______ consist of all different digits$\}$

   $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

   $= 5^3 + (11 \times 10 \times 9) - (5 \times 4 \times 3) = 1055$

3. In a survey of 270 students, it's found that

   - 64 like asparagus, $|A| = 64$
   - 94 like broccoli, $|B| = 9$
   - 58 like cauliflower, $|C| = 58$
   - 26 like both asparagus and broccoli, $|A \cap B| = 26$
   - 28 like both asparagus and cauliflower, $|A \cap C| = 28$
   - 22 like both broccoli and cauliflower, and $|B \cap C| = 22$
   - 14 like all three vegetables. $|A \cap B \cap C| = 14$

   Questions: How many of the survey's participants like ...

   (a) at least one of the vegetables? $|A \cup B \cup C| = 154$

   (b) none of the vegetables?

   (c) asparagus and broccoli but not cauliflower?

Notes on the makeup for Quiz 5

- The makeup will be on **Wednesday, March 6** in class.
- The makeup will cover only section 7.1. To prepare, do the assigned homework from this section and review the problems from lecture.
- Your grade on the makeup will replace your Quiz 5 grade (unless your makeup grade is lower).
- The makeup is optional!
- For every problem, you must use the formula $p(E) = |E|/|S|$.
  - Be able to describe in words what the sample space S is.
  - Also, be able to describe in words the event E as a subset of S.
  - Use your descriptions to compute $|S|$ and $|E|$, then use these numbers to compute $p(E)$.

Homework problems:

1. A password consists of a string of seven letters taken from the set $\{a, b, c, d\}$. How many passwords ...
   (a) ... contain exactly three a's or exactly four b's?
   (b) ... start with a string of four different letters or end with a string of four different letters? (e.g. *bacdaab* starts with a string of four different letters)
2. Suppose that there are five freshmen, three sophomores, and four juniors in a class.
   (a) Suppose that 1st, 2nd, and 3rd prizes will be awarded to three of the students (no ties). How many ways are there to award these prizes if a freshmen wins first or a junior wins second?
   (b) Suppose that five students will selected to win identical prizes. How many ways are there to do this if exactly two freshmen are winners or exactly two juniors are winners?
3. A fruit stand sells four different types of fruit (apples, bananas, pears, and oranges). How many different ways are there to select nine pieces of fruit if ...
   (a) ... at least two apples or at least four bananas are picked?
   (b) ... exactly three apples or exactly two pears are picked?

   (Assume that fruit pieces are identical if they're of the same type – only the numbers of each type of fruit matters.)
4. Alice has a dozen identical kiwis she wants to distribute among five friends who include Bob and Candice. How many ways can she do this if ...
   (a) ... Bob gets at least three or Candice gets at least four?
   (b) ... Bob gets at most three and Candice gets at most four?
5. How many ways are there to distribute five different toys among three children if ...
   (a) ... at least one child gets no toys?
   (b) ... each child gets at least one toy?

Answers:

1. (a) $2835 + 945 - 35 = \boxed{3745}$
   (b) $1536 + 1536 - 144 = \boxed{2928}$
2. (a) $550 + 440 - 200 = \boxed{790}$
   (b) $350 + 336 - 180 = \boxed{506}$
3. (a) $120 + 56 - 20 = \boxed{156}$
   (b) $28 + 36 - 5 = \boxed{59}$
4. (a) $715 + 495 - 126 = \boxed{1084}$
   (b) $1820 - 495 - 330 + 35 = \boxed{1030}$
5. (a) $32 + 32 + 32 - 1 - 1 - 1 + 0 = \boxed{93}$
   (b) $243 - 93 = \boxed{150}$

For $|A_1|$, note that if no tasks are assigned to employee 1, then the tasks must all be assigned to the other two employees. There are $2^8 = 256$ ways to do this. By similar reasoning, $|A_2| = |A_3| = 256$.

For $|A_1 \cap A_2|$, note that $A_1 \cap A_2$ is the set of all ways of assigning the tasks so that employees 1 and 2 both get no tasks. This means employee 3 gets all the tasks, so $|A_1 \cap A_2| = 1$. Likewise, $|A_1 \cap A_3| = 1$ and $|A_2 \cap A_3| = 1$.

For $|A_1 \cap A_2 \cap A_3|$, note that $A_1 \cap A_2 \cap 3$ is the set of all ways of assigning the tasks so that employees 1,2, and 3 all get no tasks. This can't happen, so $|A_1 \cap A_2 \cap A_3| = 0$.

Putting it all together using the principle of inclusion-exclusion, we get

$$|A| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 256 + 256 + 256 - 1 - 1 - 1 + 0 = \boxed{765}$$

Example 5: How many ways can 8 different tasks be assigned to three different people if every employee has at least one task?

Solution: Use "counting the complement":

- The total number of ways to assign the tasks is $3^8 = 6561$.
- The number of ways to assign tasks where it's *not* true each employee has at least one task is 765 (by the previous example).

Therefore, the answer is $6561 - 765 = \boxed{5796}$

Next, let's consider a different type of problem: Let U be a universal set, and let B be the set of elements of U which satisfies all of the properties $P_1, P_2, \ldots, P_n$. Suppose we're interested in counting how many elements the set B has. Using "counting the complement", we can do this by computing $|U|$ minus the number of elements of U which don't satisfy at least one of the properties $P_1, P_2, \ldots,$ or P_n. Equivalently, we can compute $|U| - |\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n|$. The cardinality of $\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n$ can be computed using the principle of inclusion-exclusion.

For $n = 2$, we get

$$|B| = |U| - |\overline{A_1}| - |\overline{A_2}| + |\overline{A}_1 \cap \overline{A}_2|,$$

where $|\overline{A}_1|$ is the number of elements in U that don't satisfy property P_1, $|\overline{A}_2|$ is the number of elements in U that don't satisfy property P_2, and $|\overline{A}_1 \cap \overline{A}_2|$ is the number of elements that don't satisfy both properties P_1 and P_2.

Example 6: A bagel shop sells five different types of bagels (poppy, onion, everything, raisin, and cheddar). How many ways can ten bagels be selected with at most two onion bagels and at most two poppy seed bagels?

Solution: Let U be the set of ways to pick ten bagels with no restrictions, let A_1 be the set of ways to pick ten bagels at most two onion bagels, and let A_2 be the set of ways to pick ten bagels with at most two poppy seed bagels. We want to compute the cardinality of $B = A_1 \cap A_2$.

Note that $\overline{A}_1$ is the set of ways to pick ten bagels with at least three onion bagels, and $\overline{A}_2$ is the number of ways to pick ten bagels with at least three poppy seed bagels.

You can verify that:

$|U| = C(10 + 5 - 1, 5 - 1) = C(14, 4) = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2 \cdot 1} = 1001$

$|\overline{A}_1| = C(7 + 5 - 1, 5 - 1) = C(11, 4) = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} = 330, \quad |\overline{A}_2| = C(7 + 5 - 1, 5 - 1) = C(11, 4) = 330$

$|\overline{A}_1 \cap \overline{A}_2| = C(4 + 5 - 1, 5 - 1) = C(8, 4) = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70.$

So $|A_1 \cap A_2| = 1001 - 330 - 330 + 70 = \boxed{411}$

Counting Using the Principle of Inclusion-Exclusion

Let U be a universal set, and let A be the set of elements of U which satisfies at least one property on the list of properties $P_1, P_2, \ldots, P_n$. Suppose we're interested in counting how many elements the set A has.

For $i = 1, \ldots, n$, let A_i be the set of elements in U that satisfy property P_i. Then

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

Roughly speaking, the principle of inclusion-exclusion implies that $|A|$ is equal to a certain sum / difference of cardinalities $|A_{i_1} \cap \ldots \cap A_{i_k}|$.

Each intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ is the set of elements of U which satisfy all of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$. If we can figure out how to use this description to calculate the cardinality $|A_{i_1} \cap \ldots \cap A_{i_k}|$, then we can compute $|A|$ using the principle of inclusion-exclusion.

Counting problems with two sets A_1 and A_2:

Using the notation above, A is the set of elements of U that satisfy property P_1 or property P_2. The law of inclusion-exclusion for two sets states that $|A| = |A_1| + |A_2| - |A_1 \cap A_2|$

This implies the following: The number of elements in U satisfying property P_1 or P_2 equals ...
[# elements in U satisfying property P_1] plus
[# elements in U satisfying property P_2] minus
[# elements in U satisfying both properties]

Example 1: Find the number of solutions in nonnegative integers for the equation

$$x_1 + x_2 + x_3 + x_4 = 10 \tag{1}$$

if $x_1 \geq 2$ or $x_2 \geq 3$.

Solution: Let U be the set of solutions (x_1, x_2, x_3, x_4) in nonnegative integers for equation (1). Also, let A_1 be the set of elements (x_1, x_2, x_3, x_4) in U so that $x_1 \geq 2$, and let A_2 be the set of elements (x_1, x_2, x_3, x_4) in U so that $x_2 \geq 3$. It follows that $A = A_1 \cup A_2$ is the set of elements in U so that $x_1 \geq 2$ or $x_2 \geq 3$.

Next, we compute $|A|$ using the principle of inclusion-exclusion. For $|A_1|$, note that finding all nonnegative integer solutions for (1) satisfying $x_1 \geq 2$ is equivalent to finding all nonnegative integer solutions (x_1', x_2, x_3, x_4) to

$$x_1' + x_2 + x_3 + x_4 = 8$$

$C(n+r-1, n-1)$
$r = 8$
$n = 4$

where $\boxed{x_1 = x_1' - 2}$ Therefore, $|A_1| = C(8+4-1, 4-1) = C(11,3) = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$.

By similar reasoning, $|A_2| = C(7+4-1, 4-1) = C(10,3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$.

Finally, to compute $|A_1 \cap A_2|$, note that finding all nonnegative integer solutions for (1) satisfying $x_1 \geq 2$ and $x_2 \geq 3$ is equivalent to finding all nonnegative integer solutions (x_1', x_2', x_3, x_4) to

$$x_1' + x_2' + x_3 + x_4 = 5$$

where $x_1 = x_1' - 2$ and $x_2 = x_2' - 3$. Therefore, $|A_1 \cap A_2| = C(5+4-1, 4-1) = C(8,3) = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$.

Using the principle of inclusion-exclusion, we get $|A| = 165 + 120 - 56 = \boxed{229}$

Example 2: How many ways are there to distribute eight identical cookies to three different children (Alice, Bob, and Candice) if Alice gets at least two cookies or Bob gets at least three cookies?

Solution: Let U be the set of all ways of distributing the cookies to the children with no restrictions. Also, let A_1 be the set of all ways to distribute the cookies so that Alice gets at least two, and let A_2 be the set of of all ways to distribute the cookies so that Bob gets at least three. We want to compute the cardinality of $A = A_1 \cup A_2$ using the principle of inclusion-exclusion.

To compute $|A_1|$, note that if we are distributing the cookies so that Alice gets at least two, then that means we can distribute the cookies by automatically giving Alice two cookies, then taking the remaining 6 cookies and distributing to the three children in any way. (Note that Alice may get additional cookies beyond the two she initially received.) There are $C(6+3-1, 3-1) = C(8,2) = \frac{8\cdot 7}{2\cdot 1} = 28$ ways to do this.

Similar, we can show that $|A_2| = C(5+3-1, 3-1) = C(7,2) = \frac{7\cdot 6}{2\cdot 1} = 21$.

Finally, to compute $|A_1 \cap A_2|$, note that if we are distributing the cookies so that Alice gets at least two and Bob gets at least three, then we can automatically give Alice two cookies and Bob three cookies, which means there are three cookies left to distribute among the three children in any way. There are $C(3+3-1, 3-1) = C(5,2) = 10$ ways to do this.

Therefore, $|A| = 28 + 21 - 10 = \boxed{39}$

Note: We could reframe the problem in Example 2 as follows:
Let x_1, x_2, x_3 be the number of cookies that Alice, Bob, and Candice get, respectively. How many solutions in nonnegative integers does the equation $x_1 + x_2 + x_3 = 8$ have if $x_1 \geq 2$ and $x_2 \geq 3$?

Example 3: How many ways are there to distribute eight identical cookies to three different children (Alice, Bob, and Candice) if Alice gets at most one cookie and Bob gets at most two cookies?

Solution: Let U be the same set as in the solution to the previous example. Then $|U| = C(8+3-1, 3-1) = \frac{10\cdot 9}{2\cdot 1} = 45$.

Note that the following statements are equivalent by DeMorgan's Law for logic:

- It's not true that Alice gets at most one cookie and Bob gets at most two cookies.
- Alice gets at least two cookies or Bob gets at least three cookies.

Using "counting the complement", we want to compute $|U|$ minus the number of ways of distributing the cookies so that Alice gets at least two cookies or Bob gets at least three cookies. Using the previous example, this is equal to $45 - 39 = \boxed{6}$

Note: It's pretty easy to list out all the possibilities in this case – try it!

Counting problems with three or more sets:

Naturally, things are more complicated in this situation, but we can still use the same idea outlined above. Let's see how this works by example.

Example 4: How many ways can 8 different tasks be assigned to three different people so at least one employee is assigned no tasks?

Note: There are $3^8 = 6561$ ways to assign the tasks with no restrictions. (Use the product rule: There's three ways to assign an employee to task 1, three ways to assign an employee to task 2, etc.)

Solution: For $i = 1, 2, 3$, let A_i be the set of all ways to assign the tasks so that the ith employee is assigned no tasks. Therefore, $A_1 \cup A_2 \cup A_3$ is the set of all ways to assign the tasks so that at least one employee is assigned no tasks.

We want to compute $|A_1 \cup A_2 \cup A_3|$ using the principle of inclusion-exclusion.

- Handshaking Thm
- Bipartite
- Thm 4
- Neighborhood
- Complete matching
- Hall's Marriage Thm

$\alpha . 2^n$ $\quad a_n = \alpha . 9^n =$ $\quad x^2 - 9x = 0$

$a_0 = \alpha . 9^0 = 18$ $\quad \alpha . 3^n + -3$

$a_1 = \alpha = 18$ $\quad x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2 \quad x = 3$

$x = \pm 3$ $\quad (x+4)(x-3)$ $\quad \alpha(-2)^n + \beta(3)^n$

$x = -4$

$x = 3$

$\alpha(-3)^n + \beta(3)^n + \gamma(-4)^n$

$(\alpha_0 + \alpha_1 n)(-3)^n + (\beta_0 + \beta_1 n + \beta_2 n^2 + \beta_3 n^3 + \beta_4 n^4)(3)^n + (\gamma_0 + \gamma_1 n + \gamma_2 n^2)(4)^n$

$30 + 12 = 2x$

$42 = 2x \quad x = 21$

Exam 3 Topics Outline / Discrete Structures II / Spring 2018

Section 2.4:

- Given a recurrence relation with initial conditions for a sequence, be able to compute terms in the sequence.
- Given a recurrence relation and a sequence $\{a_n\}$, be able to prove that the sequence is a solution, or demonstrate why the sequence is not a solution.

Section 8.1

- Model counting problems by setting up recurrence relations with initial conditions. Justify why your recurrence works by describing what each term represents.

Section 8.2

- Find the general solution for any linear homogeneous recurrence relation with constant coefficients by using the roots of the characteristic equation. (See Theorem 4.)
- If initial conditions are given, use the general solution to find the unique solution which satisfies these initial conditions.
- Find the general solution for a nonhomogeneous recurrence relation with constant coefficients by finding a particular solution, then adding this solution to the general solution for the corresponding homogeneous recurrence. (See Theorem 5.)

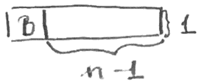
Section 10.1-10.2

- Understand and know how to apply the definitions on pages 651-654.
- Know how to apply the Handshaking Theorem (Theorem 1).
- Special simple graphs: Complete graph (K_n); complete bipartite graph ($K_{n,m}$); cycle (C_n).
- Degree sequence: Given a degree sequence, construct a simple graph with the degree sequence when possible. Explain why a simple graph can't exist if not possible.
- Bipartite graphs: Find a complete matching when one exists. If not, be able to find a subset that violates the condition in Hall's Marriage Theorem.
- Know how to apply Theorem 4 to identify bipartite graphs. Be able to find a bipartition (V_1, V_2) for a graph that is bipartite.
- Model job assignments using bipartite graphs.

1/c/ a_n = # tilings of $1 \times n$ rect. using only 1×1 tiles and w/ consecutive red tiles

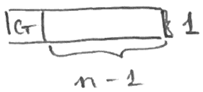
Tiles: R, G, B

Case 1:



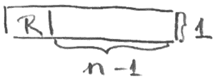
# of tilings of $1 \times n$ rectangle where blue tile 1^{st} : a_{n-1}

Case 2:



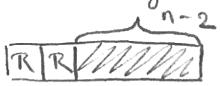
a_{n-1}

Case 3:



Can't fill w/any tiling w/ the prob of having consecutive red tiles

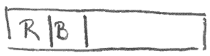
Subcase: 3(a)



3^{n-2}

Subcase 3(b)

a_{n-2}



a_{n-2}

Recurrence: $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$

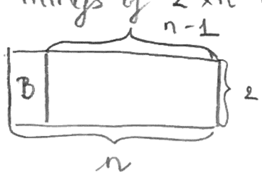
$a_0 = 0$

$a_1 = 0$

$a_2 = 1$

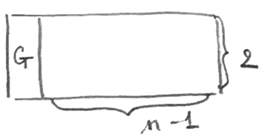
e/ a_n = # tilings of $2 \times n$ rect. using B & G 1×2 tiles

Case 1:



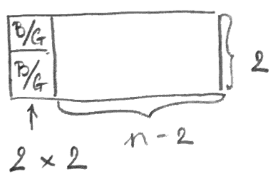
a_{n-1}

Case 2:



a_{n-1}

Case 3:



$4a_{n-2}$ ways to fill

Recurrence: $a_n = 2a_{n-1} + 4a_{n-2}$

Section 8.1: Example from Wednesday

Let a_n be the number of ways to insert n dollars into a vending machine using \$1 and \$2 tokens so that consecutive \$2 are not used.

On Wednesday, we saw the sequence $\{a_n\}$ satisfies the recurrence relation:

$$a_n = a_{n-1} + a_{n-3}$$

Call a string of 1s and 2s a *good n-string* if the sum of the numbers in the string is n, and the string doesn't contain consecutive 2's. Then a_n is the number of good n-strings.

A recursive procedure for generating the list of good n-strings (where $n \geq 3$)

- Append 1 to the beginning of all good $(n-1)$-strings. (There are a_{n-1} such strings.)
- Append 21 to the beginning of all good $(n-3)$-strings. (There are a_{n-3} such strings.)

Good n-string is encoding a valid way of inserting \$n into machine.

Lists of good n-strings

Good 0-strings: The empty string is the only good 0-string. So $a_0 = 1$.

Good 1-strings
($a_1 = 1$)

Start with 1:	1
Start with 21:	

Good 2-strings
($a_2 = 2$)

Start with 1:	11
Start with 21:	2

Good 3-strings
($a_3 = 3$)

Start with 1:	111	12
Start with 21:	21	

Good 4-strings
($a_4 = 4$)

Start with 1:	1111	112	121
Start with 21:	211		

Good 5-strings
($a_5 = 6$)

Start with 1:	11111	1112	1121	1211
Start with 21:	2111	212		

Good 6-strings
($a_6 = 9$)

Start with 1:	111111	11112	11121	11211	12111	1212
Start with 21:	21111	2112	2121			

Good 7-strings
($a_7 = 13$)

Start with 1:	1111111	111112	111121	111211	112111	11212	121111	12112	12121
Start with 21:	211111	21112	21121	21211					

1/ $p(F|E) = \dfrac{p(E|F)\,p(F)}{\underbrace{p(E|F).p(F) + p(E|\bar{F}).p(F)}_{p(E)}} = \dfrac{2/5 \times 1/3}{1/3} = 3/5$

3/ 1st box: 2W, 3B

2nd box: 4W, 1B

E = {select blue ball}

$\bar{E}$ = {select white ball}

F = {select a ball from 1^{st} box}

$\bar{F}$ = {select ______ 2^{nd} box}

[?] p(pick a ball from 1^{st} box if she has selected a blue ball}

$$p(F|E) = \frac{p(E|F) \times p(F)}{p(E|F) \times p(F) + p(E|\bar{F})\,p(\bar{F})}$$

$$p(E|F) = \cancel{\frac{p(E \cap F)}{p(F)}} = \frac{C(3,1)}{C(5,1)} = 3/5$$

$$p(F) = 1/2$$

$$p(E|\bar{F}) = 1/5 = \frac{C(1,1)}{C(5,1)}$$

$$p(F|E) = \frac{3/5 \times \frac{1}{2}}{3/5 \times 1/2 + 1/5 \times 1/2} = \frac{3}{4}$$

7/ O = {randomly chosen person uses opium}

I = {randomly chosen person is tested positive in opium test}

1% of ppl actually use opium: $p(O) = 0.01$

A test for opium use has a 2% false positive rate $\Rightarrow$ ~~p(I|O) = 0.02~~

5% false negative rate $\Rightarrow p(\bar{I}|O) = 0.05$ ✓

No using opium 2% positive $\Rightarrow p(I|\bar{O}) = 0.02$ ✓

5% negative $\Rightarrow$ ~~p(Ī|Ō) = 0.05~~

$\Rightarrow p(I|O) = 1 - p(\bar{I}|O) = 0.95$

$p(\bar{I}|\bar{O}) = 1 - p(I|\bar{O}) = 0.98$

a/ $p(\bar{O}|\bar{I}) = \dfrac{p(\bar{I}|\bar{O})\,p(\bar{O})}{p(\bar{I}|\bar{O})\,p(\bar{O}) + p(\bar{I}|O)\,p(O)} = \dfrac{0.98 \times 0.99}{\text{— " —} + 0.05 \times 0.01} = 0.999$

b/ $p(O|I) = \dfrac{p(I|O)\,p(O)}{p(I|O).p(O) + p(I|\bar{O}).p(\bar{O})} =$

9/ ~~$I = \{$infected with HIV$\}$~~

~~$P = \{$infected with HIV test positive$\}$~~

~~$= \{$not infected with$\}$~~

$E = \{$ infected with HIV $\} \Rightarrow p(E) = 0.08$

$F = \{$ not infected with HIV $\} \Rightarrow p(F) = 0.92$

$E_1 = \{$ infected with HIV tests positive $\} \Rightarrow p(E_1 | E) = 0.9$

$E_2 = \{$ not infected with HIV test positive $\} \Rightarrow p(E_2 | F) = 0.0$

a/ ~~$p(E_2|E)$~~ Prob. that a patient testing positive for HIV with this test is infected with it

$$p(E | E_1) = \frac{p(E_1 | E)\, p(E)}{-//- + p(E_2 | F)\, p(F)}$$

b/ Prob. that a patient testing positive for HIV with this test is not infected with it ?

$$p(F | E_2) = \frac{p(E_2 | F)\, p(F)}{-//- + p(E_1 | E)\, p(E)}$$

c/ Prob. that a patient testing negative for HIV with this test is infected with it ?

$$p(E | \overline{E}_1) = \frac{p(\overline{E}_1 | E)\, p(E)}{-//- + p(\overline{E}_2 | F)\, p(F)}$$

d/ $$p(F | \overline{E}_2) = \frac{p(\overline{E}_2 | F)\, p(F)}{p(\overline{E}_2 | F)\, p(F) + p(\overline{E}_1 | E)\, p(E)}$$

11/ $P = \{$ new camera phone was predicted to be successful $\}$ ~~$\Rightarrow p$~~

$S = \{$ new camera phone was successful $\} \Rightarrow p(S) = 0.6$

$\overline{S} = \{$ new camera phone was failed $\} \Rightarrow p(\overline{S}) = 0.4$

$p(P | S) = 0.7$

$p(P | \overline{S}) = 0.4$

The prob. that this new camera phone will be successful if its success has been predicted

$$p(S | P) = \frac{p(P|S) \times p(S)}{p(P|S)\, p(S) + p(P|\overline{S})\, p(\overline{S})} \approx 0.724$$

Section 7.1 homework

Note: Use the formula $p(E) = |E|/|S|$ for all problems below. (See Definition 1 on page 446 of the textbook.)

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1. What's the probability that a card selected at random from a deck of cards is ...
   (a) an ace?
   (b) a heart?
   (c) an ace or a heart?
2. A pair of dice is tossed. What's the probability that the sum on the dice is a multiple of 4?
3. Suppose a class contains five women and six men (including Bob).
   (a) What's the probability that a person selected at random is a woman?
   (b) A group of four people is selected at random. What's the probability that ...
      i. ... the group contains an equal number of men and women?
      ii. ... the group contains at least one woman?
      iii. ... Bob is in the group?
4. A coin is tossed six times.
   (a) What's the probability that heads occurs an equal number of times as tails in the sequence of flips?
   (b) What's the probability that heads occurs more times than tails in the sequence of flips?
5. An urn contains 7 blue marbles (labeled $1, 2, \ldots, 7$) and 5 red marbles (labeled A, B, C, D, E). Suppose that three marbles are selected at random, one at a time, without replacement.
   (a) What's the probability exactly two of the marbles are blue?
   (b) What's the probability that one of the marbles is the red "C" marble?

   Note: "Without replacement" means that the marbles aren't put back into the urn after being selected. That is, after the first marble is removed, the second marble is picked from the remaining 11 marbles. After that, the third marble is picked from the remaining 10 marbles.

Answers

1. (a) 1/13
   (b) 1/4
   (c) 4/13
2. 1/4
3. (a) 5/11
   (b) i. 5/11
       ii. 21/22
       iii. 4/11
4. (a) 5/16
   (b) 11/32
5. (a) 21/44
   (b) 1/4

Section 7.2 Homework / Discrete Structures II / Spring 2018

1. An unfair die has the following properties:

   - The probability of rolling a 2 is twice as likely as rolling a 1. $p(2) = 2p(1)$
   - The probability of rolling a 3 is three times as likely as rolling a 2. $p(3) = 3p(2)$
   - Rolling the numbers 1, 4, 5, 6 are all equally likely. $p(4) = p(5) = p(6) = p(1)$

   Questions:

   $\to p(1) = 1/12$

   (a) Find the probability of each outcome (i.e. find $p(k)$ for $k = 1, 2, \ldots, 6$). $p(1) + 2p(1) + 6p(1) + 3p(1) = 1$

   (b) The die is rolled. What's the probability of getting an even number? $p(2) + p(4) + p(6) = 1/6 + 1/12 + 1/12 = 1/3$

   (c) Suppose a pair of the unfair dice is rolled.

      i. What's the probability that 4 occurs on the first die $\boxed{\text{and}}$ 5 occurs on the second die? $p(4) \times p(5) = 1/12 \times 1/12 = 1/144$

      ii. What's the probability that 4 occurs on the first die $\boxed{\text{or}}$ 5 occurs on the second die? $p(4 \cup 5) = 1/12 + 1/12 - 1/144 = 23/144$

      iii. What's the probability that the sum is equal to 4? $\{(1,3),(2,2),(3,1)\}$ $1/12 \times 1/2 + 1/6 \times 1/6 + 1/2 \times 1/12 = 1/9$

2. Suppose that a number is picked from the set $S = \{0, 1, 2, 3, \ldots\}$, and that the probability that the number n is picked is $p(n) = 0.3(0.7)^n$.

   (a) Find the probability that

      i. a number less than 3 is picked. $p(0) + p(1) + p(2) = 0.3 + 0.21 + 0.147 = 0.657$

      ii. a number greater than 2 is picked. $1 - 0.657 = 0.343$

   (b) Find the probability that an even number ($\to E$) is picked given that it's known that a number less than 3 ($\to F$) is picked. $p(E|F) = p(E \cap F)/p(F) = [p(0) + p(2)]/p(F) = 0.447/0.657$

3. A pair of dice is rolled. What is the probability that the sum of the faces is greater than 7 (E), given that

   (a) the first roll was a 4 (F)? $p(E|F) = \dfrac{p(E \cap F)}{p(F)} = \dfrac{3}{6} = 1/2$

   (b) the first roll was greater than 4? $9/12 = 3/4$

4. Three cards are drawn from a deck of cards, one at a time, without replacement. Use the "multiplication rule" to compute the probability that

   (a) the cards are all hearts? $13/52 \times 12/51 \times 11/50$

   (b) the first card is a jack, the second card is a queen, and the third card is a king? $4/52 \times 4/51 \times 4/50$

   (c) the cards are a jack, queen, king, in <u>some order</u>? $6 \times 4/52 \times 4/51 \times 4/50$

5. A coin is flipped seven times.

   (a) What's the probability that heads occurs exactly three times? $\dfrac{C(7,3)}{2^7} = \dfrac{35}{128}$

   ✓ (b) What's the probability that heads occurs exactly three times, given that the first flip was heads? $\dfrac{\quad}{2^6}$

6. A class contains 12 students who are taking calculus (C), 17 students who are taking physics (P), and 5 students who are taking both. Every student in the class is taking calculus or physics. Suppose a student is selected at random from the class. $|C| = 12$, $|P| = 17$, $|C \cap P| = 5$

   (a) What's the probability that the student is taking calculus but not physics? $p(C) = \dfrac{12 - 5}{12 + 17 - 5} = 7/24$

   (b) What's the probability that the student is taking calculus given that it's known they're taking physics?

   $p(C|P) = \dfrac{p(C \cap P)}{p(P)} = \dfrac{|C \cap P|}{|P|} = \dfrac{5}{17}$

7. Let E, F be events so that $p(E \cap F) = 0.2$, $p(E \cup F) = 0.9$, and $p(E|F) = 0.5$. Compute the following probabilities:

   (a) $p(F) = \dfrac{p(E \cap F)}{p(E|F)} = \dfrac{0.2}{0.5} = 0.4$
   (*Hint:* Use the definition of $p(E|F)$.)

   (b) $p(E) = p(E \cup F) - p(F) + p(E \cap F) = 0.9 - 0.4 + 0.2 = 0.7$
   (*Hint:* Inclusion-exclusion.)

   (c) $p(F|E) = \dfrac{p(F \cap E)}{p(E)} = 0.2/0.7 = 2/7$

   (d) $p(E \cap \overline{F}) = p(E) - p(E \cap F) = 0.7 - 0.2 = 0.5$
   (*Hint:* How are the sets E, $E \cap F$, and $E \cap \overline{F}$ related? Sketch a Venn diagram if you're not sure.)

8. It's known that 10% of the widgets produced by a certain factory are defective. Suppose that five widgets are selected at random from the factory. What's the probability that

   (a) all the widgets are defective? $(0.1)^5 = 0.00001$

   (b) exactly two are defective? $(0.1)^2 (0.9)^3 = 0.00729$

   (c) at least three are defective? $(0.1)^3 . (0.9)^2 + (0.1)^4 . (0.9)^1 + (0.1)^5 = 9.1 \times 10^{-4}$

   NOTE: Assume that picking one widget is a Bernoulli trial with 10% of being defective, and 90% of not being defective. (This is a reasonable assumption if the total number of widgets produced by the factory is large, and the number selected is small.)

9. A true-false quiz has ten questions. (Note: You can use a calculator on this problem, but you should be able to set up an expression for what you need to compute.)

   (a) Assume that the student randomly guesses on the entire quiz. What's the probability that the student gets at least 80% on the quiz?

   (b) Suppose that the student believes he has a 70% probability of getting each question right. Supposing this assumption is correct, what's the probability that the student gets at least 80% on the quiz?

Answers

1. (a) $p(1) = p(4) = p(5) = p(6) = 1/12, \quad p(2) = 1/6, \quad p(3) = 1/2$
   (b) 1/3
   (c) i. 1/144
       ii. 23/144
       iii. 1/9

2. (a) i. 0.657
       ii. 0.343 (*Hint:* Use the complement of the event.)
   (b) $0.447/0.657 \approx 0.6804$

3. (a) 1/2
   (b) 3/4

4. (a) $\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \approx 0.012941$
   (b) $\frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \approx 0.00048265$
   (c) $6 \cdot \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \approx 0.0028959$

5. (a) 35/128
   (b) 15/64

6. (a) 7/24
   (b) 5/17

7. (a) 0.4
   (b) 0.7
   (c) 2/7
   (d) 0.5

8. (a) 0.00001
   (b) 0.0729
   (c) 0.00856

9. (a) $C(10,8) \cdot (0.5)^{10} + C(10,9) \cdot 0.5^{10} + C(10,10) \cdot 0.5^{10} \approx 0.0547$
   (b) $C(10,8) \cdot (0.7)^8 (0.3)^2 + C(10,9) \cdot (0.7)^9 (0.3) + C(10,10) \cdot 0.7^{10} \approx 0.3828$

   Hint: You can use the binomial distribution from problems 8 and 9.

Nhe

Section 7.3: Bayes' Theorem

Geometry — 7F, 3M
Algebra — 8F, 12M

Warm-up Problem

A geometry class contains 7 females and 3 males, and an algebra class contains 8 females and 12 males. A random student is selected in the following way: First, one of the classes is selected at random (50% chance for each class). Next, a student is selected at random from the chosen class.

Let A be the event that the chosen student is from the algebra class, and let M be the event that the chosen student is male.

Compute the following probabilities. (Leave your answers as reduced fractions.)

1. $p(A) = 1/2$
2. $p(\overline{A}) = 1/2$
3. $p(M \mid A) = p(\text{male}/\text{algebra}) = \frac{12}{20} = \frac{3}{5}$
4. $p(M \mid \overline{A}) = p(\text{male}/\text{geometry}) = 3/10$
5. $p(A \cap M) = p(A)\,p(M|A) = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$
(*Hint:* Use the multiplication rule.)

Examples

1. Use the same information as the warm-up problem.

Questions:

(a) What's the probability that the student is from the algebra class given that the student is male?

(b) What's the probability that the student is from the geometry class given that the student is male?

Beginning of solution for (a):

Goal: Compute $p(A|M)$. Bayer's Theorem

a/ $$p(A|M) = \frac{p(M|A)\,p(A)}{p(M|A)\,p(A) + p(M|\bar{A})\,p(\bar{A})}$$

$$= \frac{3/5 \times 1/2}{3/5 \times 1/2 + 3/10 \times 1/2} = \frac{2}{3}$$

b/ $$p(\bar{A}|M) = \frac{p(M|\bar{A})\,p(\bar{A})}{p(M|\bar{A})\,p(\bar{A}) + p(M|A)\,p(A)}$$

or $= 1 - \frac{2}{3} = 1/3$

2. When a test for steroids is given to soccer players, 98% of the players taking steroids test positive and 12% of the players not taking steroids test positive. Suppose that 5% of the soccer players take steroids.

A soccer player is selected at random.

(a) What is the probability that the player takes steroids given that the the player tests positive?

(b) What is the probability that the player does not take steroids given that the player tests positive?

(c) What's the probability that the player does not take steroids given that the player tests negative?

(d) What's the probability that a player not on steroids tests positive twice? (Assume that the tests are independent, and the player never takes steroids.)

Beginning of solution for (a): Let P be the event that the player test positive, and let S be the event that the player takes steroids.

Goal: Compute $p(S|P)$.

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3. Suppose that the percentage of spam emails that contain the word "Rolex" is 20%, and the percentage of non-spam emails that contain the word "Rolex" is 1%. The percentage of all email messages that are spam is 30%.

An email message is selected at random. Let R be the event that the message contains the word "Rolex," and let S be the event the message is spam.

(a) What's the probability that a random email message is spam given the fact that the email contains the word "Rolex"?

(b) What's the probability that a random email message is spam given the fact that the email does not contain the word "Rolex"?

Beginning of solution for (a):

Goal: Compute $p(S|R)$.