44

12

**Instructions:** No notes or calculators are allowed. Answers with little or no supporting work will get little or no credit. Work must be neat, organized and easily interpreted.

1. (6 Pts) Given the points P(-3,-2,5), Q(-2,2,4) and $R(1,1,1)$, find a **unit** vector ( call it $\vec{a}$ ) perpendicular to the plane containing P, Q and R.

$\overrightarrow{PQ} = \langle -2-(-3), 2-(-2), 4-5 \rangle = \langle 1, 4, -1 \rangle$ 1

$\overrightarrow{PR} = \langle 1-(-3), 1-(-2), 1-5 \rangle = \langle 4, 3, -4 \rangle$ 1

6

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 4 & -1 \\ 4 & 3 & -4 \end{vmatrix} = \langle -13, 0, -13 \rangle$$ 3

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-13)^2 + (-13)^2} = 13\sqrt{2}$

$\boxed{\vec{a} = \pm \langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle}$ 1

Unit vector $\vec{a} = \pm \dfrac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \pm \dfrac{\langle -13, 0, -13 \rangle}{13\sqrt{2}} = \pm \langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle$

2. Answer the following questions for $\vec{u} = \langle 1, -4, 1 \rangle$ and $\vec{v} = \langle -4, -2, 2 \rangle$

24

2.a (6 Pts) Compute $\vec{u} \bullet \vec{v}$

$\vec{u} \cdot \vec{v} = 1 \times (-4) + (-4) \cdot (-2) + 1 \times 2 = -4 + 8 + 2 = 6$ 6

$\boxed{\vec{u} \bullet \vec{v} = 6}$

2.b (6 Pts) Provide an expression for the angle, $\theta$, between $\vec{u}$ and $\vec{v}$

$|\vec{u}| = \sqrt{1^2 + (-4)^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$

$|\vec{v}| = \sqrt{(-4)^2 + (-2)^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$ 6

$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \dfrac{6}{3\sqrt{2} \cdot 2\sqrt{6}} = \dfrac{1}{\sqrt{12}} = \dfrac{1}{2\sqrt{3}}$ 4 1

$\theta = \cos^{-1}\left(\dfrac{1}{2\sqrt{3}}\right)$

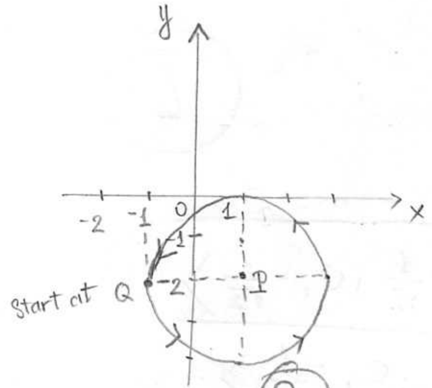
$\boxed{\theta = \cos^{-1}\left(\frac{1}{2\sqrt{3}}\right)}$ 1

2.c (6 Pts) Find the projection of $\vec{u}$ in the $\vec{v}$ direction.

$\text{proj}_{\vec{v}} \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v} = \dfrac{6}{24} \cdot \langle -4, -2, 2 \rangle = \langle -1, -\frac{1}{2}, \frac{1}{2} \rangle$ 1 3 1 1 6

$\boxed{proj_{\vec{v}}\vec{u} = \langle -1, -\frac{1}{2}, \frac{1}{2} \rangle}$

3.a (14 Pts) Provide parametric equations, $x(t)$ *and* $y(t)$, for a circle centered at $P(1,-2)$ having radius $r=2$ that starts at the point $Q(-1,-2)$ at time $t=0$, moves counterclockwise and completes one circuit in 3 minutes. Draw a sketch showing this curve, the starting point and the direction of travel. Be sure to indicate the values of $t$ that correspond to one complete circuit.




2

$0 \leq t \leq 3$

$0 \leq bt \leq 2\pi$

At $t=3$, $bt = 2\pi$, we have: $3b = 2\pi \Rightarrow b = \frac{2\pi}{3}$

(It moves counterclockwise, so $b > 0$)

2 0 2 3 3

$x(t) = 1 + 2\cos\left(\frac{2\pi}{3}t\right)$ for $0 \leq t \leq 3$

$-$

$y(t) = -2 + 2\sin\left(\frac{2\pi}{3}t\right)$

$-$

12

3.b (8 Pts) Determine if the parameterization $\vec{r}(t) = <3t, 2+4t>/5$ is an arc-length parameterization of a straight line? Your work should clearly indicate WHY you came to the conclusion you did.

$\vec{r}(t) = \frac{1}{5}\langle 3t, 2+4t\rangle = \left\langle \frac{3}{5}t, \frac{2}{5}+\frac{4}{5}t\right\rangle$

$\vec{r}'(t) = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$ 2

$|\vec{r}'(t)| = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$ 2

$s = \int_0^t |\vec{r}'(t)|\,dt = \int_0^t 1\,dt = t\Big|_0^t = t$

8

Therefore: $\vec{r}(s) = \frac{1}{5}\langle 3s, 2+4s\rangle$, for $-\infty < s < \infty$

The parameterization $\vec{r}(t) = \frac{1}{5}\langle 3t, 2+4t\rangle$ is an arc-length parameterization of a straight line.

4

22

4.a (10 Pts) Find the cross product of the vectors $\bar{u} = <1,3,-1>$ and $\bar{v} = <1,-3,2>$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 1 & -3 & 2 \end{vmatrix} = \langle 3\times(+2)-(-3)\times(-1), 1\times(-1)-1\times(+2), 1\times(-3)-1\times 3 \rangle$$

$$= \langle +6-3, -1-2, -3-3 \rangle = \langle 3, -3, -6 \rangle$$

$$\boxed{\vec{u}\times\vec{v} = \langle 3, -3, -6 \rangle}$$

10

5. Answer the following questions for the trajectory $\vec{r}(t) = <4t, e^{3t}, \sin(t)>$

a. (6 Pts) Find the **unit** tangent vector to this trajectory in the direction of travel at time $t = 0$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 4, 3e^{3t}, \cos(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4^2 + (3e^{3t})^2 + \cos^2(t)} = \sqrt{16 + 9e^{6t} + \cos^2(t)}$$

Unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle 4, 3e^{3t}, \cos(t)\rangle}{\sqrt{16 + 9(e^{6t}) + \cos^2(t)}}$

$$\vec{T}(0) = \frac{\langle 4, 3, 1 \rangle}{\sqrt{16+9+1}} = \frac{\langle 4, 3, 1\rangle}{\sqrt{26}}$$

$$\boxed{\vec{T}(0) = \frac{1}{\sqrt{26}} \langle 4, 3, 1 \rangle}$$

6

b. (6 Pts) Find the acceleration vector at time $t = 0$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle 0, 9e^{3t}, -\sin(t) \rangle$$

$$\vec{a}(0) = \langle 0, 9, 0 \rangle$$

$$\boxed{\vec{a}(0) = \langle 0, 9, 0 \rangle}$$

6

c. (6 Pts) Determine the magnitude of acceleration in the direction of travel at time $t = 0$.

$$|\vec{a}(0)| = \sqrt{9^2} = 9$$

0

Magnitude of acceleration in the direction of travel at $t = \cancel{1}\ 0$ is: $\boxed{9}$

26

6. (10 Pts) For the curve $\vec{r}(t) = < \tan(t), e^{3t}, t^2 >$ provide an expression for the arc length for $t \in [0, \pi/3]$. Do NOT attempt to perform the integration.

$$\vec{r}'(t) = \left\langle \frac{1}{\cos^2(t)}, 3e^{3t}, 2t \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{\frac{1}{\cos^4(t)} + 9e^{6t} + 4t^2}$$

$$\text{Arc length } L = \int_0^{\pi/3} |r'(t)|\,dt = \int_0^{\pi/3} \sqrt{\frac{1}{\cos^4(t)} + 9e^{6t} + 4t^2}\,dt$$

Use sec(t) instead of 1/cos(t)

10

7. (8 Pts) Use the triple scalar product (or any other way you know) to find the volume of the parallelepiped formed by the vectors $\vec{a} = < -1, 2, 1 >$, $\vec{b} = < 1, -3, 2 >$, and $\vec{c} = < 1, 1, -2 >$

Area of the parallelepiped:

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & J & k \\ -1 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = \langle 7, 3, 1 \rangle$$

Volume of the parallelepiped:

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \langle 1, 1, -2 \rangle \cdot \langle 7, 3, 1 \rangle$$

$$= 7 + 3 - 2 = \boxed{8}$$

8

8. (8 Pts) Write the formula for curvature, $\kappa$, and then evaluate it for the trajectory $\vec{r}(t) = < t^2, 2t, t^3/3 >$ at time $t = 1$.

The formula is $\kappa = \dfrac{|\vec{a}(t) \times \vec{v}(t)|}{|\vec{v}(t)|^3}$

$$\vec{v}(t) = \vec{r}'(t) = \langle 2t, 2, t^2 \rangle, \quad |\vec{v}(t)| = \sqrt{4t^2 + 4 + t^4}$$

$$\vec{a}(t) = \vec{v}'(t) = \langle 2, 0, 2t \rangle$$

$$\vec{a}(t) \times \vec{v}(t) = \begin{vmatrix} i & J & k \\ 2 & 0 & 2t \\ 2t & 2 & t^2 \end{vmatrix} = \langle -4t, 4t^2 - 2t^2, 4 \rangle = \langle -4t, 2t^2, 4 \rangle$$

8

$$|\vec{a}(t) \times \vec{v}(t)| = \sqrt{16t^2 + 4t^4 + 16}$$

$$K(t) = \frac{\sqrt{16t^2 + 4t^4 + 16}}{\left(\sqrt{4t^2 + 4 + t^4}\right)^3}$$

$$K(1) = \frac{\sqrt{16 + 4 + 16}}{\left(\sqrt{4 + 4 + 1}\right)^3} = \frac{\sqrt{36}}{(\sqrt{9})^3} = \frac{6}{3^3} = \frac{2}{9}$$

$$\boxed{\kappa(1) = \frac{2}{9}}$$

Calculus III Exam II (Ch 13) Pg1 Fall2017 COrmsby Name Dangnhi Ngoc Ngo

107 Awesome

**Instructions:** No notes or calculators are allowed. Answers with little or no supporting work will get little or no credit. Work must be neat, organized and easily interpreted.

47

1. (8 Points) Find the equation of the plane that contains the point P(4,-2,3) and that is parallel to the plane given by $2x - \pi y + 2z = 13$.

The plane is parallel to the plane $2x - \pi y + 2z = 13$ has the same normal vector with that plane $\vec{n} = \langle 2, -\pi, 2 \rangle$

48

Equation of the plane:

$2(x-4) - \pi(y+2) + 2(z-3) = 0$

$2x - 8 - \pi y - 2\pi + 2z - 6 = 0$

$\boxed{2x - \pi y + 2z = 14 + 2\pi}$

4 4 8

2. (8 Points) Use the two path test with $y = mx$ to prove that the limit $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$ does NOT exist.

Replacing $y = mx$: $\lim_{x \to 0} \frac{(x - mx)^2}{x^2 + (mx)^2} = \lim_{x \to 0} \frac{x^2(1-m)^2}{x^2(1+m^2)} = \lim_{x \to 0} \frac{(1-m)^2}{1+m^2}$

2 4

$= \frac{(1-m)^2}{1+m^2} \notin x$

Therefore $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$ does not exist

2 8

3. (8 Points, 4 Points each) Answer the following questions for the quadratic surfaces specified.

3.a $-4x^2 + y^2 + 9z^2 = 7$ Type: Hyperboloid 1 Axis of Symmetry: x - axis 1

Sub-type: One sheet 1 Does the surface intersect the axis of symmetry? No 1 If so, where? ______

3.b $-2x^2 - y + 4z^2 = 16$ Type: Paraboloid 1 Axis of Symmetry: y - axis 1

Sub-type: Hyperbolic 1 Does the surface intersect the axis of symmetry? Yes If so, where? $y = -16$

8 1

4.a. (8 Points) Find the partial derivatives listed below for the function $f(x,y) = y\sin(x)\cos(y)$

$f_x = y\cos(y)\cos(x)$ 2 $\qquad f_{xx} = -y\cos(y)\sin(x)$ 2 7

$f_y = \sin(x)[\cos(y) - y\sin(y)] = \sin(x)\cos(y) - y\sin(x)\sin(y)$

$f_{yx} = \cos(y)\cos(x) - y\sin(y)\cos(x)$

3

4.b (2 Points) What condition on a general function $g(x,y)$ would ensure that $g_{yx} = g_{xy}$ ?

$g_{yx}(x,y) = g_{xy}(x,y)$ with all values of x and y

gyx and gxy must be continuous

5. (8 Points) If $f(x,y) = x^3 - y^2$, $x(t) = 2 - t^3$ and $y(t) = te^{2t}$, find $\frac{df}{dt}$ using the chain rule. Drawing the dependency tree may be helpful. Do NOT just plug in x(t) and y(t) and then differentiate. If you do, you will receive NO credit. Please leave your final answer in terms of $x$, $y$, *and* $t$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$\frac{dx}{dt} = -3t^2$ 1 $\qquad \frac{dy}{dt} = e^{2t} + 2te^{2t}$ 1 8

$\frac{\partial f}{\partial x} = 3x^2$ 1 $\qquad \frac{\partial f}{\partial y} = -2y$ 1

$\frac{df}{dt} = 3x^2\cdot(-3t^2) - 2y(e^{2t} + 2te^{2t}) = -9x^2t^2 - 2ye^{2t}(1 + 2t)$ 2

6. (8 Points) $f(x,y) = xe^{xy} - y^2 = 1$ implicitly defines $y$ as a function of $x$. Find the slope, $dy/dx$, of the tangent line to the contour plot of $f(x,y)$ at any point $(x,y)$. What is slope of the curve at the point $P(1,0)$ ?

$f(x,y) = xe^{xy} - y^2 - 1 = 0$

$f_x(x,y) = e^{xy} + xye^{xy}$

$f_y(x,y) = x^2e^{xy} - 2y$

$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{e^{xy} + xye^{xy}}{x^2e^{xy} - 2y} = -\frac{e^{xy}(1+xy)}{x^2e^{xy} - 2y}$ 3 8

$\left.\frac{dy}{dx}\right|_{(1,0)} = -\left.\frac{e^{xy}(1+xy)}{x^2e^{xy} - 2y}\right|_{(1,0)} = -\frac{e^{1.0}(1+1.0)}{1^2e^{1.0} - 2.0} = -1$ 1 1

$\frac{dy}{dx} = -\frac{e^{xy}(1+xy)}{x^2e^{xy} - 2y}$ 2 $\qquad \left.\frac{dy}{dx}\right|_{(1,0)} = -1$

2 8

(... ts) Find the directional derivative of $f(x,y)=x^2y^3$ at the point $P(2,1,4)$ in the direction $\vec{u}=<3/5,4/5>$.

$f_x(x,y) = 2xy^3 \qquad f_y(x,y) = 3x^2y^2$

$\vec{\nabla}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 2xy^3, 3x^2y^2\rangle$

$\vec{\nabla}f(2,1,4) = \langle 2xy^3, 3x^2y^2\rangle\big|_{(2,1,4)} = \langle 2.2.1^3, 3.2^2.1^2\rangle = \langle 4, 12\rangle$

$D_{\vec{u}}f(2,1,4) = \vec{\nabla}f(2,1,4).\vec{u} = \langle 4,12\rangle.\langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{12}{5} + \frac{48}{5} = \frac{60}{5} = 12$

8. (14 Points total) Answer the following questions for the surface $f(x,y)=2\sin(x)e^y$ at the point $P(\pi/3,0,\sqrt{3})$

8.a (6 Points) Find a unit vector in the xy-plane in the direction of steepest ascent at P

$f_x(x,y) = 2\cos(x)e^y \qquad f_y(x,y) = 2\sin(x)e^y$

$\vec{\nabla}f(\frac{\pi}{3},0,\sqrt{3}) = \langle 2\cos(x)e^y, 2\sin(x)e^y\rangle\big|_{(\pi/3,0,\sqrt{3})} = \langle 2\cos(\frac{\pi}{3})e^0, 2\sin(\frac{\pi}{3})e^0\rangle = \langle 1, \sqrt{3}\rangle$

$|\vec{\nabla}f(\pi/3,0,\sqrt{3})| = \sqrt{1+(\sqrt{3})^2} = \sqrt{4} = 2$

The unit vector of steepest ascent at P,

$\frac{\vec{\nabla}f(\pi/3,0,\sqrt{3})}{|\vec{\nabla}f(\pi/3,0,\sqrt{3})|} = \frac{\langle 1,\sqrt{3}\rangle}{2}$

$\vec{u}_{SA} = \frac{\langle 1,\sqrt{3}\rangle}{2}$

8.b (4 Points) Find a unit vector in the direction of steepest descent at P

$\vec{u}_{SD} = \frac{\langle -1,-\sqrt{3}\rangle}{2}$

8.c (4 Points) Find a unit vector tangent to the level curve at $P$

$\vec{u}_{LVL} = \pm\frac{\langle -\sqrt{3}, 1\rangle}{2}$

9. (8 Points) Find the equation of the plane tangent to the hyperboloid $F(x,y,z)=z^2-12x^2-6y^2=-9$ at the point $P(1,1,3)$.

$F(x,y,z) = z^2 - 12x^2 - 6y^2 + 9 = 0$

$F_x = -24x \qquad F_y = -12y \qquad F_z = 2z$

The equation of the plane tangent at the point P(1,1,3)

$F_x(1,1,3)(x-1) + F_y(1,1,3)(y-1) + F_z(1,1,3)(z-3) = 0$

$-24(x-1) - 12(y-1) + 6(z-3) = 0$

$-24x - 12y + 6z = -18$

$-4x - 2y + z = -3$

$4x + 2y - z = 3$

10. Answer the following questions for the function $f(x,y)=xy+x^2-6x-y^2+2y+\pi$

10.a (10 Points) Find the critical point(s) of $f(x,y)$.

$f_x(x,y) = y + 2x - 6 = 0$
$f_y(x,y) = x - 2y + 2 = 0$
$\Leftrightarrow \begin{cases} y = 6-2x \\ x - 2(6-2x) + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 6-2x \\ 5x = 10 \end{cases} \Leftrightarrow \begin{cases} y = \\ x = 2 \end{cases}$

Critical point : $(2, 2)$

10.b (8 Points) Determine if the critical point(s) you found represent local maximums, local minimums or saddle points. You must explicitly carry out all portions of the required test using the discriminant, D.

$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = -2$
$f_{xy}(x,y) = 1 = f_{yx}(x,y)$
$D = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2 = 2 \cdot (-2) - 1^2 = -4 - 1 = -5$
$D < 0$; therefore, $f$ has saddle point

**BONUS (10 Points):** Consider the surface $f(x,y)=x^2-y^4$ If you start at the point $P(3,1,8)$ on this surface, find the path in the $xy-plane$ of the path of steepest descent (Note: Your answer should be $y(x) = some\ function\ of\ x$ and the starting point (3,1) should be on this curve).

$f_x(x,y) = 2x$ $\quad f_y(x,y) = -4y^3$

$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{-4y^3}{2x} = -2\frac{y^3}{x}$

$\frac{1}{y^3}\,dy = -\frac{2}{x}\,dx \Rightarrow \int \frac{1}{y^3}\,dy = -2\int \frac{1}{x}\,dx$

$-\frac{1}{2}\cdot\frac{1}{y^2} + C = -2\ln|x| + C$

$\frac{1}{y^2} = 4\ln|x| + C$

Start at point $P(3,1,8)$:

$\frac{1}{1} = 4\ln 3 + C \Rightarrow C = 1 - 4\ln 3$

$\frac{1}{y^2} = 4\ln|x| + 1 - 4\ln 3$

ok

$y = \ldots$

Calculus III____Exam III Pg1 Fall17 COrmsby Name Danghi Ngo (47½) 107½

**Instructions:** No notes or electronic devices are allowed. Answers with little or no supporting work will get little or no credit. Work must be neat, organized and easily interpreted.

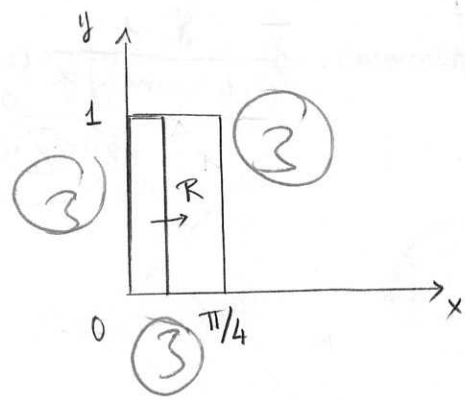
1. (12 Pts.) Find the volume under the surface $f(x,y)=4x\cos(2xy)$ within the region $R$ defined by $\{R:(x,y)\in 0\le x\le \pi/4, 0\le y\le 1\}$. Provide a **NEAT drawing** of your region of integration showing your intended order of integration with a **windshield wiper**.

$$\int_0^{\pi/4}\int_0^1 4x\cos(2xy)\,dy\,dx = \int_0^{\pi/4} 2\sin(2xy)\Big|_0^1 dx = 2\int_0^{\pi/4}[\sin(2x)-\sin 0]\,dx$$

$$= 2\int_0^{\pi/4}\sin(2x)\,dx = -\cos(2x)\Big|_0^{\pi/4} = -\cos\left(2\cdot\frac{\pi}{4}\right)+\cos(0) = \boxed{1}$$

1 1 1

12




2. (4 Points each) If you were asked to perform integrations of the following functions over the specified regions in $\mathbb{R}^3$, what coordinate system would you use and what would you use for the relevant differential volume, $dV$?

a. $f(x,y,z)=\cos(z)e^{x^2+y^2}$ over $\{D:\ 0\le x^2+y^2\le 7,\ -\pi/6\le z\le \pi/3\}$

Your selected coordinate system: Cylindrical (2) $dV =$ $dz\, r\, dr\, d\theta$ (2) 12

b. $f(x,y,z)=x+y-z$ over $\{D:\ 3\le x^2+y^2+z^2\le 7,\ z\ge 0,\ y\ge 0\}$

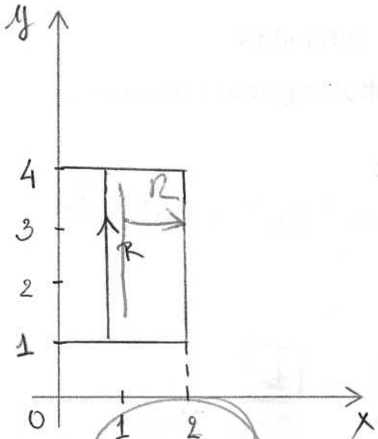
Your selected coordinate system: Spherical (2) $dV =$ $\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$ (2)

c. $f(x,y,z)=xy^2z\cos(x^2)$ over $\{d:\ 0\le x\le\sqrt{\pi},\ 1\le y\le 3,\ 0\le z\le \pi/2\}$

Your selected coordinate system: Cartesian (2) $dV =$ $dx\,dy\,dz$ (2)

3. (12 Points) Find the <u>average value</u> of the function $f(x,y)=6x^2y$ over the region $\{R: 0\le x\le 2,$ 1

Draw a **NEAT** sketch of the region $R$ showing your order of integration with a **windshield wiper**.




2½

$$\int_0^2\int_1^4 6x^2y\,dy\,dx = \int_0^2 3x^2y^2\Big|_1^4 dx$$ 2

$$= 3\int_0^2\left[x^2(4)^2 - x^2\right]dx$$ 2

$$= 45\int_0^2 x^2\,dx = 15x^3\Big|_0^2 = 15\times 2^3 = 120$$

Area of region R: $2\times(4-1) = 6$ 1

Average value of the function:

$$\bar{f} = \frac{1}{\text{area of } R}\int_0^2\int_1^4 6x^2y\,dy\,dx = \frac{1}{6}\times 120 = \boxed{20}$$

2 1 1

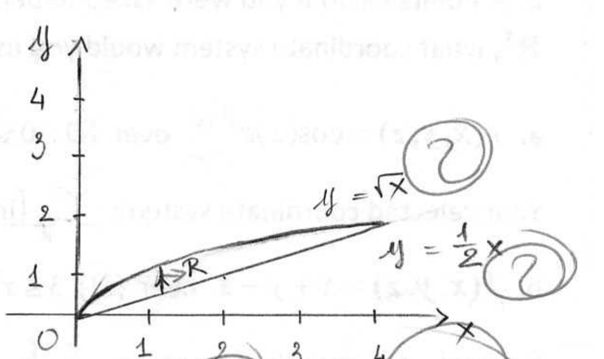
11½

4. (12 Points) For the double integral $\int_0^2\int_{y^2}^{2y} f(x,y)\,dx\,dy$ , create a <u>**NEAT**</u> diagram (with axes labelled) of the region of integration and then re-write the integral **reversing the order of integration**. Show your NEW windshield wiper on your diagram. Do <u>**NOT**</u> evaluate the resulting integral.

$$\begin{cases} x = y^2 \\ x = 2y \\ 0\le y\le 2\end{cases} \Rightarrow \begin{cases} y=\sqrt{x} \\ y=\frac{1}{2}x \\ 0\le x\le 4\end{cases}$$

$$\int_0^2\int_{y^2}^{2y} f(x,y)\,dx\,dy = \int_0^4\int_{\frac{1}{2}x}^{\sqrt{x}} f(x,y)\,dy\,dx$$

2 2 2 6




2 2 2 6

12

(30)

ate the integral $\iint_R \frac{4}{1+x^2+y^2} dA$ where the region $R$ is the region $R = \{1 \le x^2 + y^2 \le 4\}$. You NEAT diagram of the region showing your integration strategy with a windshield wiper.

$x^2 + y^2 = r^2 \Leftrightarrow 1 \le r^2 \le 4 \Leftrightarrow 1 \le r \le 2$

$dA = r\,dr\,d\theta$

$x = r\cos\theta$

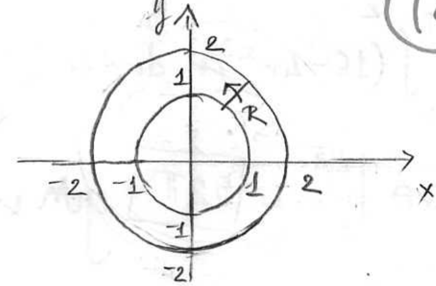
$y = r\sin\theta$

$$\int_0^{2\pi}\int_1^2 \frac{4}{1+r^2}\cdot r\,dr\,d\theta = 2\int_0^{2\pi} \ln(1+r^2)\Big|_1^2 d\theta = 2\int_0^{2\pi}(\ln 5 - \ln 2)\,d\theta$$

(12)

$$= 2(\ln 5 - \ln 2)\theta\Big|_0^{2\pi} = \boxed{4\pi(\ln 5 - \ln 2)}$$

$= 4\pi \ln(5/2)$




6. (18 Points) The density of electrons in an electron cloud is given by $D(x,y,z) = \dfrac{6}{3+(x^2+y^2+z^2)^{3/2}}$. Determine the number of electrons in the 3D region $D:\{where\ 0 \le x^2+y^2+z^2 \le 16$ **AND** $y \ge 0\}$.

$x^2 + y^2 + z^2 = p^2$

$dV = p^2 \sin\phi\,dp\,d\phi\,d\theta$

(18)

$0 \le p^2 \le 16 \Leftrightarrow 0 \le p \le 4$

$y \ge 0 \Rightarrow 0 \le \phi \le \pi, \quad 0 \le \theta \le \pi$

$$\int_0^{\pi}\int_0^{\pi}\int_0^4 \frac{6}{3+(p^2)^{3/2}} p^2\sin\phi\,dp\,d\phi\,d\theta = \int_0^{\pi}\int_0^{\pi}\int_0^4 \frac{6p^2}{3+p^3}\cdot\sin\phi\,dp\,d\phi\,d\theta$$

$$= 2\int_0^{\pi}\int_0^{\pi} \ln(3+p^3)\Big|_0^4 \sin\phi\,d\phi\,d\theta = 2\int_0^{\pi}\int_0^{\pi}[\ln(67)-\ln 3]\sin\phi\,d\phi\,d\theta$$

$$= 2\int_0^{\pi}(\ln 3 - \ln 67)\cos\phi\Big|_0^{\pi} d\theta = 2\int_0^{\pi}(\ln 3 - \ln 67)(-1-1)\,d\theta$$

$$= -4\int_0^{\pi}(\ln 3 - \ln 67)\,d\theta = 4(\ln 67 - \ln 3)\,\theta\Big|_0^{\pi} = \boxed{4\pi(\ln 67 - \ln 3)}$$

$= 4\pi \ln(67/3)$

30

7. (16 Pts) Determine the total quantity of gold having a density of $D(x,y,z)=3(16-4x^2-4y^2)z^2\ gm/cm^3$ in the volume $D$ defined by $\{D:\ 0\le x^2+y^2\le 4,\ 0\le z\le 1\}$ where $x$, $y$ and $z$ are measured in $cm$. Provide a NEAT diagram of the limits of your region of integration in the xy-plane along with your windshield wiper.

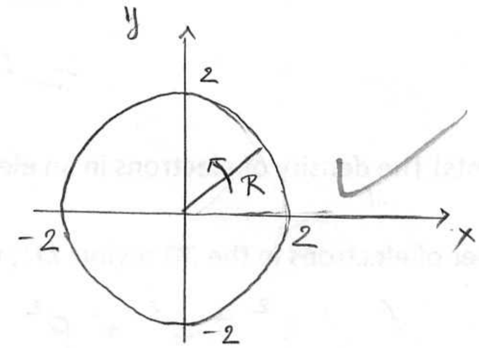
$x = r\cos\theta \qquad x^2+y^2=r^2 \iff 0\leq r^2 \leq 4 \iff 0\leq r\leq 2$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^2\int_0^1 3(16-4r^2)z^2\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (16-4r^2)z^3\Big|_0^1 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (16-4r^2)r\,dr\,d\theta = \int_0^{2\pi} 8r^2 - r^4\Big|_0^2 d\theta = \int_0^{2\pi}(8\cdot 2^2 - 2^4)\,d\theta = \int_0^{2\pi} 16\,d\theta$$

$$= 16\theta\Big|_0^{2\pi} = \boxed{32\pi}\ \text{gm}$$




16

8. (6 Points) If you are given the coordinates of a point in spherical coordinates, $(\rho,\emptyset,\theta)$, what are the formulas for converting these to Cartesian coordinates, $(x,y,z)$?

$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\cancel{\theta}\ \phi$

(-2) 4

**Bonus:** (10 Points) You are trying to identify the points $(x,y)$ in $\mathbb{R}^2$ that represent the absolute maximum and absolute minimum of the function $f(x,y)=x^2-2x+y^3+xy$ given the constraint $g(x,y)=x^2-y^2+2xy+7=0$. Create a set of equations that could be used to identify the locations of these extrema. Make sure the number of equations equals the number unknowns. DO NOT TRY TO SOLVE THE RESULTING EQUATIONS.

$\vec{\nabla} f(x,y) = \langle 2x-2+y,\ 3y^2+x\rangle$

$\vec{\nabla} g(x,y) = \langle 2x+2y,\ -2y+2x\rangle$

$$\begin{cases} 2x-2+y = \lambda(2x+2y) \\ 3y^2+x = \lambda(-2y+2x) \\ x^2-y^2+2xy+7=0 \end{cases}$$

10