Section 4.3: Linearly Independent Sets; Bases:

<u>Note</u>: Here we identify & study the subsets that span a vector space V &/or a subspace H.

* Linear Independence vs. Linear Dependence *

① An indexed set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\} \in V$ is said to be **Linearly Independent** if the vector-equation

$$c_1\vec{v}_1 + \cdots c_p\vec{v}_p = \vec{0}$$

has ONLY the trivial solution.

*IOW: $c_1 = 0, \dots, c_p = 0$ ST $\{c_1, \dots, c_p\}$ = Scalars/Weights.

<u>Recall</u>: (Similarly to $\mathbb{R}^n$)

- A set containing a single vector $\vec{v}$ is <u>Linearly Independent</u> IFF $\vec{v} \neq \vec{0}$.

② An indexed set of vectors is said to be **Linearly Dependent** if the vector-equation has a Nontrivial Solution.

*IOW: $\exists$ a scalar/weight $= \{\vec{c}_1, \dots, \vec{c}_p\}$ (<u>NOT</u> all zero) ST the vector-eq. still holds true.

<u>Recall</u>: (Similarly to $\mathbb{R}^n$)

- A set of 2 vectors is <u>Linearly Dependent</u> IFF $\vec{v}_1 = c\vec{v}_2$ (<u>i.e.</u> Scalar Multiples).
- Any set containing $\vec{0}$ is <u>Linearly Dependent</u>.

*Theorem 4:

An indexed set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$, of 2 or more vectors ST $\vec{v}_1 \neq \vec{0}$, is Linearly Dependent IFF $\exists\ \vec{v}_j$ (with $j>1$) ST it's a Linear Combination of the proceeding vectors: $\vec{v}_1, \dots, \vec{v}_{j-1}$

*Note:

The main difference btw Linear Dependence in $\mathbb{R}^n$ & in a general vector space is that the vectors cannot always be made into the columns of a matrix A in order to study the eq: $A\vec{x} = \vec{0}$.

$\therefore$ We must rely on the general definition of Linear Dependence -AND- theorem 4 as we proceed.

Example' (Linear Dependence):

Determine if $\{p_1(t), p_2(t), p_3(t)\} \in \mathbb{P}$ ST $t \in \mathbb{R}$ is Linearly Independent. **Explain:**

$$p_1(t) = 1, \quad p_2(t) = t, \quad p_3(t) = 4 - t$$

Answer:

* Recall: "$\mathbb{P}$" = The set of all Polynomials w/ $\mathbb{R}$-coefficients, w/ operations in $\mathbb{P}$ defined as for functions.

* Check if the provided vectors are dependent:

(IOW: Do $\exists$ redundant vectors w/in the indexed set?)

- $p_3(t) = 4 - t = 4(1) - 1(t)$

$$= 4[p_1(t)] - 1[p_2(t)]$$

yes!

$\therefore$ Since $p_3(t) = 4p_1(t) - p_2(t)$, the set of vectors $\{p_1(t), p_2(t), p_3(t)\} \in \mathbb{P}$ is Linearly Dependent.

Answer.

Example² (Linear Independence / Dependence):

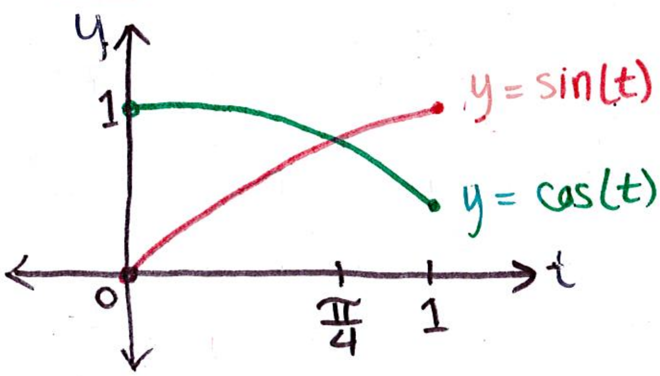
Determine if the following sets of vectors are Linearly Independent -OR- Linearly Dependent. Explain.

(a) $\{\sin(t), \cos(t)\}$ ST $0 \le t \le 1$

(b) $\{\sin(t)\cos(t), \sin(2t)\}$ ST $t \in (-\infty, \infty)$

Answer: Recall: A set of 2 vectors is Linearly Dep. IFF $\vec{V}_1 = c\vec{V}_2$.

*Part (a): $\{\sin(t), \cos(t)\}$ ST $t \in [0,1]$:



$\therefore$ Since $\sin(t) \neq c\cos(t)$, where c is any scalar, the vectors are NOT scalar multiples & so they are NOT dependent.

$\therefore \{\sin(t), \cos(t)\}$ ST $0 \le t \le 1$ is Linearly Independent.

Ans.

*Part (b): $\{\sin(t)\cos(t), \sin(2t)\}$ ST $t \in \mathbb{R}$:

Recall: (Double-Angle Formula) $\rightarrow \sin(2K\theta) = 2\sin(K\theta)\cos(K\theta)$ ST $K \in \mathbb{R}$

$\therefore$ Since $\sin(2t) = c \cdot \sin(t)\cos(t)$ ST $c = 2$, the vectors ARE scalar multiples & thus Linearly Dependent.

Ans.

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\} \in V$ is a **Basis** for H if:

(i) $\mathcal{B}$ is a Linearly Independent Set.

-AND-

(ii) The subspace spanned by $\mathcal{B}$ coincides w/ H

$\Rightarrow$ IOW: $H = \text{Span}\{\vec{b}_1, \dots, \vec{b}_p\}$

• Since any vector space is a subspace of itself, the def. of basis applies to the case when $H = V$:

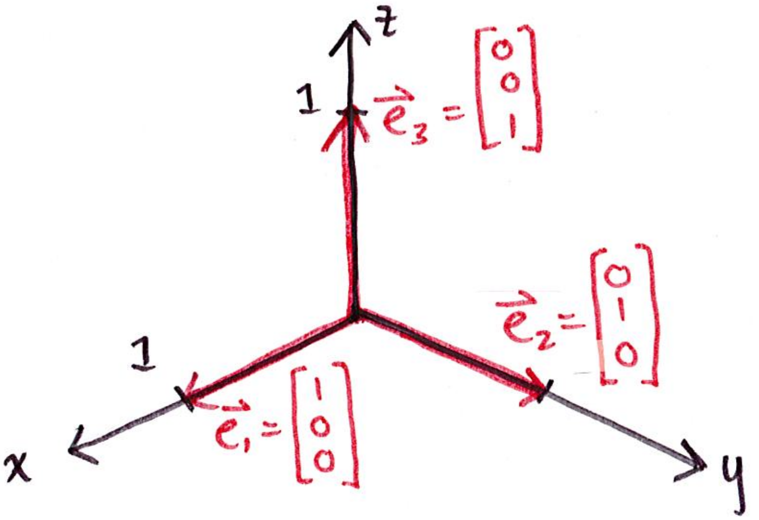
$\therefore$ A basis of V is a Linearly Independent set that spans V.

Note: (When $H \neq V$)
The 2nd condition of the Def of Basis includes the requirement that each vector $\{\vec{b}_1, \dots, \vec{b}_p\} \in V$ MUST belong to H b/c $H = \text{Span}\{\vec{b}_1, \dots, \vec{b}_p\}$ must contain $\vec{b}_1, \dots, \vec{b}_p$.

Illustration of a Basis:

Consider the column-vectors of the 3×3 Identity Matrix, $I_3 = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

***To verify that both conditions of the Def. are met, lets consider the Geometric Interpretation:**



- **Prop # 1:** Since $\nexists$ NO redundant vectors, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a <u>Linearly Indopendent set</u>
  ($A\vec{x} = \vec{0}$ has ONLY the trivial sol.)

  $\Updownarrow$ logically Equivalent :)

- **Prop. # 2:** <u>Span$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \mathbb{R}^3$</u>; The Columns of I_3 <u>fill</u> $\mathbb{R}^3$.

$$\boxed{\therefore \text{The set } \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ is the } \underline{\text{Standard Basis for } \mathbb{R}^3}.}$$

***General Conclusion:**

The set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called the "<u>Standard Basis of $\mathbb{R}^n$</u>" where $\vec{e}_1, \dots, \vec{e}_n$ = the Columns of an $n\times n$ Identity Matrix :)

Example 3 (Basis): Let A be an $n \times n$ invertible matrix. Do the columns of A form a basis in $\mathbb{R}^n$? Explain.

Answer:

- **Given:** A is an $n \times n$ invertible matrix

$$\Rightarrow A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$$

* **Goal:** Determine if the indexed set of vectors $B = \{\vec{a}_1, \ldots, \vec{a}_n\}$ in V is a Basis for H: ① B is a Linearly Independent set
② $H = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

* By the Invertible Matrix Theorem: (Section 2.3)

Since A is an $n \times n$, invertible matrix, then:

① The Columns of A Form a Linearly Independent Set ✓

$\Updownarrow$

② The Columns of A span $\mathbb{R}^n$ ✓

* Note: Remember that these 2 statements are logically equivalent.

$\therefore$ Since both conditions are met, the Columns of A Form a basis in $\mathbb{R}^n$.

Example (Basis): Determine if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

Answer:

*Recall: The columns of an $n \times n$ invertible matrix A form a basis for $\mathbb{R}^3$

• Let $A = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3]$

***Determine if A is an invertible matrix:**

$$\det(A) = \begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{vmatrix} \underset{N.R_3}{\overset{2R_1 + R_3}{\sim}} \begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \underset{N.R_3}{\overset{R_2 + R_3}{\sim}} \begin{vmatrix} -3 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$\Rightarrow$ $\boxed{\det(A) = -3(1)(2) = -6 \neq 0. \text{ So } A \text{ invertible.}}$

*Alternatively: Row-reducing $[A \,|\, \vec{0}]$ similarly shows that since A has $n=3$ pivots $\Rightarrow$ A invertible :)

These is NOT an exclusive solutions
$\Rightarrow$ verify these types of questions w/ the prop. you see first &/or you find the easiest

Ans.

∴ Since A is invertible, the Col. of A are Linearly Independent & $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3 \Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a Basis for $\mathbb{R}^3$.

Example: Determine whether the set $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\1\\-8\end{bmatrix}, \begin{bmatrix}-3\\-1\\11\end{bmatrix}\right\}$ is a Basis for $\mathbb{R}^3$. If the set is Not a Basis, determine whether the set is Linearly Independent & whether the set spans $\mathbb{R}^3$.

Answer:

* **Given:** $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\1\\-8\end{bmatrix}, \begin{bmatrix}-3\\-1\\11\end{bmatrix}\right\}$

$\Rightarrow$ Let matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ *We actually do NOT even need this here... Do you see why ¿

Recall: Let H be a subspace of a Vector Space V. An indexed set of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\} \in V$ is a Basis of H if:

① $\mathcal{B}$ is Linearly Independent

② The subspace spanned by $\mathcal{B}$ coincides w/ $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$.

* **The indexed set of vectors is Linearly Dependent:**

$$-\vec{v}_1 - \vec{v}_2 = \vec{v}_3 \Rightarrow -\begin{bmatrix}1\\0\\-3\end{bmatrix} - \begin{bmatrix}2\\1\\-8\end{bmatrix} = \begin{bmatrix}-1-2\\0-1\\3+8\end{bmatrix} = \begin{bmatrix}-3\\-1\\11\end{bmatrix}$$

$\therefore$ Since $\vec{v}_3$ is a redundant vector, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$:

- NOT Linearly Independent.
- Does NOT span $\mathbb{R}^3$

$\Rightarrow$ Is NOT a Basis for $\mathbb{R}^3$ (By Def.)

Answer ✓

Example: Determine if the following set of vectors is a Basis for $\mathbb{R}^3$. If the set is NOT a Basis, determine whether it is Linearly Independent & whether the set spans $\mathbb{R}^3$:

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -12 \\ 4 \\ 4 \end{bmatrix} \right\}$$

Answer:

Recall: Let H be a subspace of some Vector Space V. An indexed set of vectors $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\} \in V$ is a Basis of H if:

① $\mathcal{B}$ is Linearly Independent.

② The subspace spanned by $\mathcal{B}$ coincides w/ $H = \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$

My opinion: If you see one, Go for it! Multiple ways to find answer.

* Since no obvious dependence relations jump out, lets row-reduce $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ to Echelon Form:

$$[A \mid \vec{0}] = \left[\begin{array}{ccc|c} 3 & 3 & -12 & 0 \\ -1 & -2 & 4 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \overset{\frac{1}{3}R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ -1 & -2 & 4 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \underset{N.R_2}{\overset{R_1 + R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right]$$

$$\underset{N.R_3}{\overset{-R_1 + R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 8 & 0 \end{array}\right] \overset{\frac{1}{2}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \underset{N.R_3}{\overset{R_2 + R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

* Echelon Form *

$\therefore$ Since the Matrix has a pivot-position in each Column/Row:

* The vectors form a Basis for $\mathbb{R}^3$

$\Rightarrow$ The vectors are Linearly Independent

$\Rightarrow$ The vectors span $\mathbb{R}^3$.

Answer.

Example: Determine if the following set of vectors forms a Basis for $\mathbb{R}^3$. If the set is NOT a Basis, determine whether it is Linearly Independent -AND- whether the set of vectors spans $\mathbb{R}^3$:

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 16 \end{bmatrix} \right\}$$

Answer:

*Recall: Let H be a subspace of some vector space V. An indexed set of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\} \in V$ is a Basis of H if:

① $\mathcal{B}$ is Linearly Independent.

② The subspace spanned by $\mathcal{B}$ coincides w/ $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

*Given a set of 2 vectors:

From $\mathbb{R}^n$, we know that $\{\vec{v}_1, \vec{v}_2\}$ is Linearly Dependent IFF the vectors are scalar multiples (i.e. $\vec{v}_1 = c\vec{v}_2$ st $c \in \mathbb{R}$)

$\therefore$ Since $\vec{v}_1$ & $\vec{v}_2$ are NOT scalar multiples, the set of vectors is Linearly Independent

Ans✓

*To Determine if the Set Spans $\mathbb{R}^3$ (making it a Basis of $\mathbb{R}^3$), lets row-reduce $A = [\vec{v}_1 \ \vec{v}_2]$ to Echelon Form:

$$[A \mid \vec{0}] = \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 2 & 4 & 0 \\ -8 & 16 & 0 \end{array}\right]$$

STOP✓ *Since $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, A can have at most $n = 2$ pivots*

$\therefore$ Since matrix A CANNOT have a pivot in each row:

* The set of vectors does NOT span $\mathbb{R}^3$.

$\Rightarrow \therefore$ The set is NOT a Basis for $\mathbb{R}^3$.

Ans✓

The Spanning Set Theorem: Introduction

Note: A basis can be constructed from a spanning set by discarding unneeded/redundant vectors ∵

⇒ We explore this idea w/ the next example.

Example:

Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$ & $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Noticing that $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$, show that:

(a) The $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

(b) Then find a Basis for the subspace H.

Answer:

* Since $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is Linearly Dependent
  * $\vec{v}_3$ is redundant.

* (a) Show that: $\text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$:

Note: Every vector in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ belongs to $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

b/c: $c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3$

• Let $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ be any vector in H $(\vec{x} \in H)$

Goal: Show that $\vec{x}$ can be written as a Linear Combo of vectors $\vec{v}_1$ & $\vec{v}_2$.

• Since $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(5\vec{v}_1 + 3\vec{v}_2)$$

$$= c_1\vec{v}_1 + c_2\vec{v}_2 + 5c_3\vec{v}_1 + 3c_3\vec{v}_2$$

$$= \vec{v}_1(c_1 + 5c_3) + \vec{v}_2(c_2 + 3c_3) \Rightarrow$$

Ans. (a & b)

$\boxed{\therefore \vec{x} \in \text{Span}\{\vec{v}_1, \vec{v}_2\} = H}$

* Every vector in H already belongs $\text{Span}\{\vec{v}_1, \vec{v}_2\}$

(b) $\boxed{\therefore \{\vec{v}_1, \vec{v}_2\} \text{ is Linearly Independent \& thus a Basis for } H}$

*Theorem[5] (The Spanning Set Theorem):

Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a set in V, & let $H = \text{Span}\{\vec{v}_1, \ldots \vec{v}_p\}$.

(a) If one of the vectors in S (say $\vec{v}_k$) is a Linear Combination of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

(b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

Proof:

*Part (a):

- $ that $\vec{v}_p$ is a Linear Combination of the proceeding vectors (rearranging if needed):

$$\vec{v}_p = a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1} \quad \text{ST} \quad \{a_1, \ldots, a_{p-1}\} \rightarrow \text{Appropriate Scalars}$$

- Consider some arbitrary vector $\vec{x} \in H$:

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p \quad \text{ST} \quad \{c_1, \ldots c_p\} \rightarrow \text{Appropriate Scalars.}$$

- Substituting $\vec{v}_p$ into the Linear Combination for $\vec{x}$:

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_{p-1}\vec{v}_{p-1} + c_p(a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1})$$

$$= (c_1\vec{v}_1 + c_p a_1\vec{v}_1) + \cdots + (c_{p-1}\vec{v}_{p-1} + c_p a_{p-1}\vec{v}_{p-1})$$

* Applying the Distributive Prop. & regrouping terms

$$= \vec{v}_1(c_1 + c_p a_1) + \cdots + \vec{v}_{p-1}(c_{p-1} + c_p a_{p-1})$$

$\therefore$ $\vec{x}$ is a Linear Combination of the proceeding vectors $\{\vec{v}_1, \ldots \vec{v}_{p-1}\}$ & since $\vec{x} \in H$ is arbitrary $\Rightarrow \{\vec{v}_1, \ldots \vec{v}_{p-1}\}$ Spans H $\boxed{X}$

Proof of Th^m #5 Continued....

*Part (b):

Note: We verify (b) by different cases :)

- Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be the O.G. spanning set.

Case 1: IF S is Linearly Independent, then:

S is already a basis for H (done.) ✓

Case 2: IF S is NOT Linearly Independent (i.e. S is Lin Dep.), then:

A redundant vector $\exists$ in the spanning set & needs to be removed. 2 possibilities here ⤵

i) IF 2^+ vectors $\exists$ in the Spanning Set:

$\therefore$ Repeat (a) until S is Linearly Independent & thus a Basis for H. ✓

ii) IF 1 vector $\exists$ in the Spanning Set:

$\Rightarrow$ Let $\vec{v}$ be the one vector in S.

$\Rightarrow \vec{v} \neq \vec{0}$ because $H \neq 0$ (initial condition)

$\therefore$ Since $\vec{v} \neq \vec{0}$, then $\vec{v}$ is Linearly Independent & thus a Basis for H. ✓

$\boxed{X}$

Example: In the vector-space of all real-valued functions, find a Basis for the Subspace spanned by:

$$\{\sin(t),\ \sin(2t),\ \sin(t)\cos(t)\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

Answer:

* First, we need to identify if any redundant vectors $\exists$:

Recall (Double-Angle Formula): $\sin(2kt) = 2\sin(kt)\cos(kt)$

- Since $\vec{v}_2 = 2\vec{v}_3$ (or $\vec{v}_3 = \frac{1}{2}\vec{v}_2$:)), the given set is Linearly Dependent.

* Next, Apply the Spanning Theorem to remove the redundant vector ($\vec{v}_3$) & find a Basis for the subspace:

- Let $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$
- Since $\vec{v}_3 = \frac{1}{2}\vec{v}_2$, we can rewrite $\vec{x}$:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\left(\tfrac{1}{2}\vec{v}_2\right)$$

$$\vec{x} = c_1\vec{v}_1 + (c_2 + c_3/2)\vec{v}_2$$ * A Linear Combo. of $\{\vec{v}_1, \vec{v}_2\}$ *

$\Rightarrow \vec{x} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$ -AND- $\{\vec{v}_1, \vec{v}_2\}$ is Lin. Indep.

$\therefore$ Basis for the Subspace: $\boxed{\{\vec{v}_1, \vec{v}_2\} = \{\sin(t), \sin(2t)\}}$ Ans.

*Bases for the Nul(A) - AND - Col(A) *

* Basis For Nul(A):

Note: We have already established a method that produces a Basis for Nul(A) in the previous section :)

- To Find a Set of Vectors that Spans the Null Space of a Matrix, we must Find the General Solution of $A\vec{x} = \vec{0}$:

⇒ Row-reduce the augmented matrix, $[A \,\vdots\, \vec{0}]$, writing the solution in terms of the free variable(s).

⇒ Since the Free variable(s) act as "weights", this spanning set is automatically Linearly Independent.

Note: We use the following theorem & the Spanning Theorem to produce a Basis for Col(A) :)

*Theorem 6 (Basis for Col(A)):

The pivot columns of a matrix A form a Basis for Col(A).

- To Find a Set of Vectors that Spans the Column Space of a Matrix, we must reduce A to Echelon Form:

⇒ We can then construct a basis for the spanning set by removing any redundant vectors (ie: The non-pivot columns).

Example '(Bases for Col(B)): Find a basis for Col(B):

$$B = \left[\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4\ \vec{b}_5\right] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

*Note: Each nonpivot column of B is a Linear Combination of the pivot columns :)

$$* \vec{b}_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4\vec{b}_1$$

$$* \vec{b}_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2\vec{b}_1 - \vec{b}_2$$

$\Rightarrow$ $\boxed{\vec{b}_2 \ \& \ \vec{b}_4 \text{ are redundant vectors}}$

*By the Spanning Set Theorem:

Construct a basis from the spanning set by discarding the redundant vectors, $\vec{b}_2$ & $\vec{b}_4$

$\boxed{\text{So, } \{\vec{b}_1, \vec{b}_3, \vec{b}_5\} \text{ will still span Col(B).}}$

*Let $S = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$:

Since $\vec{b}_1 \neq 0$ & no vector is a Linear Combo. of the proceeding vectors, S is Linearly Indp. $\therefore$ S is a Basis for Col(B).

Example² (Bases for Col(A) & Col(B)):

It can be shown that the matrix A is row-equivalent to the matrix B. Find a Basis for Col(A):

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Answer:

Recall: In the previous example we found that:

- $\text{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\} = \text{Span}\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$

-AND-

- Since $\{\vec{b}_1, \vec{b}_3, \vec{b}_5\} = S$ is Linearly Independent, S is a Basis for the Col(B).

*When A is row-reduced to B, the columns will often appear completely different, BUT:

$A\vec{x} = \vec{0}$ & $B\vec{x} = \vec{0}$ still have exactly the same solution set :)

$\boxed{\text{The Columns of A have EXACTLY the same Linear Dependence Relation as the Columns of B}}$

∴ The Basis for Col(A) is: $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$

Example² Continued...

* We still need to check that our conclusions hold true by row-reducing A to echelon Form:

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 3 & 12 & 1 & 5 & 5 & 0 \\ 2 & 8 & 1 & 3 & 2 & 0 \\ 5 & 20 & 2 & 8 & 8 & 0 \end{array}\right] \underset{\sim}{\frac{-3R_1 + R_2}{N.R_2}} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 2 & 8 & 1 & 3 & 2 & 0 \\ 5 & 20 & 2 & 8 & 8 & 0 \end{array}\right]$$

$$\underset{\sim}{\frac{-2R_1 + R_3}{N.R_3}} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 0 & 0 & 1 & -1 & 4 & 0 \\ 5 & 20 & 2 & 8 & 8 & 0 \end{array}\right] \underset{\sim}{\frac{-5R_1 + R_4}{N.R_4}} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 0 & 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 2 & -2 & 13 & 0 \end{array}\right]$$

$$\underset{\sim}{\frac{-R_2 + R_3}{N.R_3}} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 2 & -2 & 13 & 0 \end{array}\right] \underset{\sim}{\frac{-2R_2 + R_4}{N.R_4}} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \end{array}\right]$$

$$\underset{\sim}{\frac{-\frac{3}{4}R_3 + R_4}{N.R_4}} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \boxed{\vec{a}_2 \text{ \& } \vec{a}_4 \text{ are still redundant \& will be discarded } \checkmark}$$

Echelon Form

$$\therefore \boxed{S = \{\vec{a}_1, \vec{a}_3, \vec{a}_5\} \text{ is a Basis for Col}(A)\ \checkmark}$$

Example: Find a Basis for the Null Space of the matrix:

$$A = \begin{bmatrix} 1 & 1 & -4 & -1 & 9 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & -9 & 0 & 18 \end{bmatrix}$$

Answer:

* **Note:** Finding the Basis for Nul(A) is the same procedure for finding a spanning set for Nul(A) seen in the last section ⋮ $\Rightarrow$ Find a Gen. Sol. to $A\vec{x} = \vec{0}$ in terms of free variable(s).

* **Row-reduce $[A \mid \vec{0}]$ to rref:**

$$\left[\begin{array}{ccccc|c} 1 & 1 & -4 & -1 & 9 & 0 \\ 0 & 1 & 0 & -3 & -2 & 0 \\ 0 & 0 & -9 & 0 & 18 & 0 \end{array}\right] \overset{-\frac{1}{9}R_3}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & -4 & -1 & 9 & 0 \\ 0 & 1 & 0 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{array}\right] \overset{\frac{-R_2 + R_1}{N.R_1}}{\sim}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -4 & 2 & 11 & 0 \\ 0 & 1 & 0 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{array}\right] \overset{\frac{4R_3 + R_1}{N.R_1}}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{array}\right] \checkmark$$

$$\Leftrightarrow \begin{cases} x_1 + 2x_4 + 3x_5 = 0 \\ x_2 - 3x_4 - 2x_5 = 0 \\ x_3 - 2x_5 = 0 \end{cases} \Rightarrow$$

* **General Solution:**

$$\begin{cases} \cdot\, x_1 = -2x_4 - 3x_5 \\ \cdot\, x_2 = 3x_4 + 2x_5 \\ \cdot\, x_3 = 2x_5 \\ \cdot\, x_4 \text{ is free} \\ \cdot\, x_5 \text{ is free} \end{cases}$$

$\Downarrow$

Example Continued...

***Decompose the vector giving the General Solution into a Linear Combination (ST weights/scalars = free variables):**

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_4 - 3x_5 \\ 3x_4 + 2x_5 \\ 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ 3x_4 \\ 0 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_5 \\ 2x_5 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{x} = x_4 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_4 \vec{v}_1 + x_5 \vec{v}_2}$$

* Every Linear Combination of $\vec{v}_1$ & $\vec{v}_2$ is an element of Nul(A) ∵ so,

∴ The Spanning Set & Basis for Nul(A) is:

$$\boxed{\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

Answer

Example: Find a Basis for the Set of Vectors in $\mathbb{R}^3$ in the plane $x-2y+4z=0$.

*Hint: Think of the eq. as a "system" of homogeneous eq. *

Answer:

* Think about the Eq. as a "System" of Homogeneous Eq.,

$A\vec{x}=\vec{0}$:

$$x-2y+4z=0 \iff \begin{bmatrix}1 & -2 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=0$$

* Find a Gen. Sol. to $A\vec{x}=\vec{0}$ by row-reducing $[A \,\vdots\, \vec{0}]$ to rref:

$$x-2y+4z=0 \Rightarrow \begin{cases} \cdot\, x = 2y-4z \\ \cdot\, y \text{ is free} \\ \cdot\, z \text{ is free} \end{cases}$$

* General Solution:

$$\boxed{\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2y-4z\\y\\z\end{bmatrix}=y\begin{bmatrix}2\\1\\0\end{bmatrix}+z\begin{bmatrix}-4\\0\\1\end{bmatrix}}$$

Note: Every Linear Combination of these vectors is in the Nul(A) ✓

∴ The Spanning Set -AND- Basis for Nul(A) is:

$$\boxed{\{\vec{v}_1,\vec{v}_2\}=\left\{\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}-4\\0\\1\end{bmatrix}\right\}}$$

Answer ✓

Example: Assume that matrix A is row-equivalent to matrix B. Find the Bases for $\text{Nul}(A)$ -AND- $\text{Col}(A)$:

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 & 6 \\ 1 & 2 & 0 & 2 & 6 \\ 4 & 8 & -4 & 12 & 5 \\ 2 & 4 & 0 & 4 & 5 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 2 & 0 & 2 & 7 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

* Since A is row-equiv. to B, it's Faster to row-reduce $B\vec{x} = \vec{0}$ ✓

*** Find the Basis For $\text{Nul}(A)$:**

$$[B \mid \vec{0}] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 7 & 0 \\ 0 & 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{-\frac{1}{6}R_3}{\overset{\frac{1}{2}R_2}{\sim}} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 7 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{\frac{-R_3 + R_2}{N.R_2}}{\sim}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 7 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{\frac{-7R_3 + R_1}{N.R_1}}{\sim} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 + 2x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -2x_2 - 2x_4 \\ x_2 \text{ is Free} \\ x_3 = x_4 \\ x_4 \text{ is Free} \\ x_5 = 0 \end{cases}$$

*** General Sol:**

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ **The Basis For $\text{Nul}(A)$:**

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$ Ans ✓

Example Continued...

*Find the Basis for Col(A):

Recall (Theorem): The pivot columns of A form the Basis for Col(A).

*Since A is row-equivalent to B, they have the same pivot columns: :)

$$\Rightarrow A \sim B = \begin{bmatrix} 1 & 2 & 0 & 2 & 7 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: Matrix B is in Echelon Form w/ pivot-columns $\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$

$\boxed{\therefore \text{The } 1^{st}, 3^{rd}, \& \ 5^{th} \text{ Columns of } A \text{ are its pivot-columns.}}$

$\Rightarrow$ Therefore, the Basis for Col(A) is:

$$\boxed{\{\vec{a}_1, \vec{a}_3, \vec{a}_5\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 6 \\ 5 \\ 5 \end{bmatrix} \right\}}$$

Answer.

Example: Find a basis for the space spanned by the given vectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 11 \\ -4 \\ 8 \\ -12 \end{bmatrix}, \begin{bmatrix} 15 \\ -2 \\ 4 \\ 13 \end{bmatrix} \right\}$$

Answer:

*Note: This problem is equivalent to the procedure for finding the Basis for the Col(A) :)

*Recall (Theorem 6): The pivot columns of A form the Basis for Col(A).

$$*\text{Let } A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \ \vec{v}_5] = \begin{bmatrix} 1 & 6 & 3 & 11 & 15 \\ 0 & 0 & -1 & -4 & -2 \\ 0 & 0 & 2 & 8 & 4 \\ 1 & -6 & -2 & -12 & 13 \end{bmatrix}$$

*Row-reduce $[A \mid \vec{0}]$ to Echelon Form to Identify the Pivot-Columns:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 3 & 11 & 15 & 0 \\ 0 & 0 & -1 & -4 & -2 & 0 \\ 0 & 0 & 2 & 8 & 4 & 0 \\ 1 & -6 & -2 & -12 & 13 & 0 \end{array}\right] \underset{\sim}{\frac{-R_1 + R_4}{N.R_4}} \left[\begin{array}{ccccc|c} 1 & 6 & 3 & 11 & 15 & 0 \\ 0 & 0 & -1 & -4 & -2 & 0 \\ 0 & 0 & 2 & 8 & 4 & 0 \\ 0 & -12 & -5 & -23 & -2 & 0 \end{array}\right]$$

$*-R_4$

$*\frac{1}{2}R_3$

$\sim$

*Interchange R_2, R_3, R_4

$$\left[\begin{array}{ccccc|c} 1 & 6 & 3 & 11 & 15 & 0 \\ 0 & 12 & 5 & 23 & -2 & 0 \\ 0 & 0 & -1 & -4 & -2 & 0 \\ 0 & 0 & 1 & 4 & 2 & 0 \end{array}\right] \underset{\sim}{\frac{R_3 + R_4}{N.R_4}} \left[\begin{array}{ccccc|c} 1 & 0 & 1/2 & -1/2 & 16 & 0 \\ 0 & 12 & 5 & 23 & -2 & 0 \\ 0 & 0 & -1 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

*Echelon Form of A

Example Continued...

∴ From the Echelon Form of A we see that $\vec{a}_1$, $\vec{a}_2$, & $\vec{a}_3$ ar the pivot columns

$\Downarrow$

∴ The Basis for the Space Spanned by the vectors:

$$\boxed{\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ -2 \end{bmatrix} \right\}}$$

Answer ✓

Note: The following provides us w/ an important connection b/w Linear Independence & Linear Transformations :)

*Ex // Prop:[1]

Let V & W be vector spaces.

Let $T: V \to W$ be a Linear Transformation.

Let $\{\vec{v}_1, \ldots, \vec{v}_p\}$ be a subset of V.

Show that if $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is Linearly Dependent in V, then the set of images, $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$, is Linearly Dependent in W.

PROOF:[1]

- $ that $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is Linearly Dependent.
- Then by Definition:

$\exists$ scalars/weights $\{c_1, \ldots, c_p\}$ NOT all zero, ST

$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$.

*Goal: Want to show that $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is also Linearly Dependent $\Rightarrow c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) = \vec{0}$

Recall: If T is a Linear Transformation, then:

$$\begin{cases} * \; T(\vec{0}) = \vec{0} \\ * \; T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) \end{cases}$$

- Since T is a Linear Transformation:

... *See next page* ...

Proof' Continued....

- **Since T is a Linear Transformation (By Def):**

$$\begin{cases} \bullet\, T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) \\ \text{-AND-} \\ \bullet\, T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = T(\vec{0}) = \vec{0} \end{cases}$$

- **Thus:**

$$c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) = \vec{0}$$

$\therefore$ Since the scalars/weights $\{c_1, \ldots, c_p\}$ are NOT all zero, then by definition:

$\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is Linearly Dependent. ✓

$\boxed{X}$

Important Conclusion From this Proof

This fact shows us that if a "Linear Transformation maps a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ onto a Linearly Independent set $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$, then the original set is Linearly Independent too" (because it CANNOT be Linearly Dependent :)).

Note: The following provides us w/ an important connection btw Linear Independence & Linear Transformations :)

*Ex.// Prop[2]:

Let V & W be vector spaces.

Let $T: V \to W$ be a Linear Transformation.

Let $\{\vec{v}_1, \dots, \vec{v}_p\}$ be a subset of V.

\$ that T is a 1-1 transformation ST $T(\vec{u}) = T(\vec{v})$ always implies $\vec{u} = \vec{v}$.

*Show that if the set of all images $\{T(\vec{v}_1), \dots, T(\vec{v}_p)\}$ is Linearly Dependent, then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is Linearly Dependent *

PROOF[2]:

- \$ that $\{T(\vec{v}_1), \dots, T(\vec{v}_p)\}$ is Linearly Dependent.
- Then by Definition:

  $\exists$ scalars/weights $\{c_1, \dots, c_p\}$, NOT all zero, ST

  $$c_1 T(\vec{v}_1) + \cdots + c_p T(\vec{v}_p) = \vec{0}$$

*Goal: Want to show that $\{\vec{v}_1, \dots, \vec{v}_p\}$ is also Linearly Dependent $\rightarrow$ $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ (same scalar conditions)

- Since T is a Linear Transformation: $\begin{cases} *\, T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v}) \\ *\, T(\vec{0}) = \vec{0} \quad \text{(Sect. 1.8)} \end{cases}$

$$T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = c_1 T(\vec{v}_1) + \cdots + c_p T(\vec{v}_p) = \vec{0} = T(\vec{0})$$

Proof² Continued...

- Since T is 1-1: $T(\vec{x}) = \vec{0}$ has only the Trivial Sol.; $\vec{x} = \vec{0}$.

$T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = T(\vec{0})$, which implies that

$\Rightarrow c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}.$

∴ Since the scalars/weights $\{c_1, \cdots, c_p\}$ cannot all be zero (initial condition), then by definition:

$\{\vec{v}_1, \ldots, \vec{v}_p\}$ is Linearly Dependent. ✓ $\boxed{\times}$

*Important Conclusion From this Proof:

This fact shows us that a "1-1 Linear Transformation maps a Linearly Independent Set onto a Linearly Independent Set" (b/c in this case, the set of images cannot be linearly dependent).