Discrete Structures II

01/22

Chapter 6 : Counting Techniques

§ 6.1 Basics of counting

* Product Rule : Suppose that a certain procedure generates the list of desired posibilities so that each possibility occurs once & exactly once
If the procedure consists of m tasks, where there are
n_1 way to do the 1^{st} task
n_2 way to do the 2^{nd} task, etc.
then the total # of desired possibilities is $n_1 n_2 \dots n_m$

Ex1 Mailboxes in a building are labeled with an uppercase letter followed by 2 digits (0-9)
How many different labels are possible ? (eg: A05, N76)
Soln : A-Z 0-9 diff 0-9
$26 \times 10 \times 9 = 2340$ labels

Ex2 There are 3 women and 2 men.
How many ways are there to arrange the 5 people in a row if
a/ there are no restrictions ?
b/ men and women are alternating?
Soln :
a/ $5 \times 4 \times 3 \times 2 \times 1 = 120$
b/ W M W M W
$3 \times 2 \times 2 \times 1 \times 1 = 12$

Ex3 : PIN is a string of 4 digits (0-9)
Ex : 0013, 4514
Note : 0-9 0-9 0-9 0-9
$10 \times 10 \times 10 \times 10 = 1000$ (PINS)

Ex4 : How many PINS contain exactly one 3 ?
eg : 0013
Soln : Procedure : Pick one position for 3 : 4 ways
Fill 1^{st} remaining blank : 9 ways
2^{nd} : 9 ways
3^{rd} : 9 ways

$4 \times 9 \times 9 \times 9 = 2916$

* Case Analysis ("Sum Rule" in book)

- Break into cases with no overlap
- Calculate the # of possibilities in each case
- Add the #'s together

Ex: How many PINs start with a digit less than 4, and the remaining digits are less than the 1^{st} digit?

eg: 3112, 2100, ...

Case 1: PIN starts with 3: 3 0-2 0-2 0-2

$1 \times 3 \times 3 \times 3 = 27$

Case 2: PIN starts with 2: 2 0-1 0-1 0-1

$1 \times 2 \times 2 \times 2 = 8$

Case 3: PIN starts with 1: Only 1 possibility: 1000

Case 4: PIN starts with 0: No possibilities

Sum: $27 + 8 + 1 + 0 = 36$

Ex: There are 7 juniors & 4 sophomores competing for 1^{st} and 2^{nd} prize. How many ways are there to assign prizes if a junior must win at least one of the prizes?

Soln:

Case 1: Juniors win both: $7 \times 6 = 42$

Case 2: Junior wins 1^{st}, Senior wins 2^{nd} } $7 \times 4 = 28$

Case 3: Senior wins 1^{st}, Junior wins 2^{nd} } $4 \times 7 = 28$

Sum: $42 + 28 + 28 = 98$

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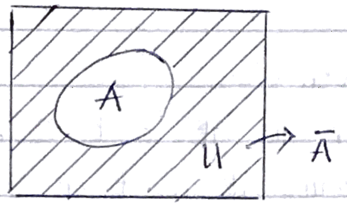
* Def: Let A be a set contained in a universal ~~of~~ set U

The component of A is the set: $\boxed{\bar{A} = U - A}$

$\bar{A}$ is the set of all elements in U that aren't in A

Alt. notation: $\bar{A}$



Fact: $\boxed{|A| = |U| - |\bar{A}|}$

"Counting the complement" technique:

Suppose we want to compute $|A|$, where A is the set of all elements in U satisfying a certain property P

Note that $\bar{A}$ is the set of elements in U that don't satisfy property P

Compute: $|A| = |U| - |\bar{A}|$

Ex: How many PINs have at least one consecutive repeated digit?

eg: 3447 or 2111, but not 7174

Soln: Let $U = \{$all PINs$\}$

$A = \{$all PINs that have at least 1 consecutive digit$\}$

(Want: $|A| = ?$)

$\bar{A} = \{$all PINs with no repeated consecutive digits$\}$

$|U| = 10^4 = 10{,}000$

$|\bar{A}| = 10 \times 9 \times 9 \times 9 = 7{,}290$

$|A| = 10{,}000 - 7{,}290 = 2{,}710$

Ex: A class has 5 students.

How many ways can the prof assign grades of A, B, C if at least one students get an A?

Soln: Encode the possibilities as strings like:

CABCB

Let $U = \{$all possible strings of length 5 involving A, B, C$\}$

$A = \{$all strings in U with at least 1 A$\}$

$\bar{A} = \{$all strings in U without at least no A$\}$

$|U| = 3^5 = 243$

$|\bar{A}| = 2^5 = 32$

$\Rightarrow |A| = 243 - 32 = 211$

§ 6.2 Pigeonhole Principle

* **Pigeonhole Principle (Thm. 1)** PP

Let k be a positive integer

If k+1 or more objects are placed into k boxes, then there's at least one box which contains at least 2 objects

Ex: Suppose 7 pigeons fly into 5 pigeonholes

Then at least one pigeonhole must contain at least 2 pigeons.

If not: Each pigeonhole would contain at most one pigeon.

This would mean there's at most 5 pigeons in the holes, contradiction.

Ex: In a group of 10 people, at least two must be born on the same day of the week (Sunday → Saturday)

Apply PP,

- 7 boxes, one for each day of the week
- Place 10 ppl into boxes according to birth day.
- By PP, there must be a box with more than 1 person

e.g: there must be 2 people who are born on the same day of the week

* **Ceiling function**: Let $x \in \mathbb{R}$

The ceiling of x is the smallest integer m so that $m \geq x$

("Round up" to the ~~nearest~~ nearest integer)

Notation: $\lceil x \rceil$

Ex: $\lceil \frac{10}{3} \rceil = \lceil 3\frac{1}{3} \rceil = 4$

$\lceil -3 \rceil = -3$

In general: $\lceil m \rceil = m$ for any $m \in \mathbb{Z}$

Property: Let $m = \lceil x \rceil$

Then: $m - 1 < x \leq m$

* **Generalized pigeonhole principle** GPP

Suppose N objects are placed in k boxes. Then there must be at least one box with at least $\lceil \frac{N}{k} \rceil$ objects

Ex: Suppose that 20 objects are placed in 6 boxes.

Then, by the GPP, there's at least 1 box that contains at least

$\lceil \frac{20}{6} \rceil = \lceil \frac{10}{3} \rceil = 4$ objects

Why? If not, every box has at most 3 objects. Then, there at most $3 \times 6 = 18$ objects, contradiction.

Pf of GPP:

Let $m = \lceil \frac{N}{k} \rceil$. Then $m - 1 < \frac{N}{k} \le m$ (*)

Assume for the sake of contradiction that all boxes have at most $m-1$ objects. Then there are:

$k(m-1)$ total objects in the boxes.

But by (*), $k(m-1) < N$

This is a contradiction. Therefore, there must be at least one box with at least m objects.

01/26 Warm-up Exercise:

Suppose that there are 5 ppl in a group, including Abby & Ben. How many ways are there to seat the ppl in a row if

1/ There are no restrictions: $5! = 120$

2/ Abby & Ben must be sitting next to each other?

3/ Abby & Ben are not sitting next to each other?

$120 - 48 = 72$ (ways)

Procedure:

+ Seat A, B first: AB _ _ _, _ AB _ _, _ _ AB _, _ _ _ AB

4 possible ways plus another 4 with AB replaced by BA

$\Rightarrow$ 8 ways to seat A, B first

+ Fill 1^{st} remaining seat: 3 ways
+ — 2^{nd} : 2 ways
+ — last : 1 way

Product rule: $8 \times 3 \times 2 \times 1 = 48$ (ways)

GPP:

Ex: Suppose that 60 words are selected at random. Show that there's at least 3 words that start with the same letter.

Soln: 26 boxes, one for each letter.

Put 60 words in boxes using the 1st letter of the word

Since, $\lceil \frac{60}{26} \rceil = \lceil 2\frac{8}{26} \rceil = 3$, the GPP implies that there's at least 3 words

ie: there's at least 3 words starting with the same letter

Ex: What's the smallest #N of words ~~..~~ that could picked at random so that at least 5 words start with the same letter?

Soln: By the GPP, we need to pick the smallest integer N so that

$$\lceil \frac{N}{26} \rceil = 5$$

Equivalently, $4 < \frac{N}{26} \leq 5$

ie $104 < N \leq 130$

The smallest N that works is $N = 105$

Note that $N = 104$ wouldn't work b/c we might pick 4 words starting with each letter, and there would be $4 \cdot 26 = 104$ words

§6.3 Permutations & Combinations

* Def: A permutation of a set S is an ordered arrangement of the objects in S.

An r-permutation of S is an ordered arrangement of r of the objects in S

Ex: Let $S = \{a, b, c, d, e\}$

Some permutations of S: abcde, cadbe

Some 3-permuations of S: dac, acb, bec

* Let $P(n,r)$ denote the # of r-permutations of an n-element set

Thm 1:

$$P(n,r) = \underbrace{n(n-1)(n-2)\ldots(n-r+1)}_{r \text{ factors}}$$

Ex: There are 8 runners in a race. How many ways are there to award 1^{st}, 2^{nd}, 3^{rd} prizes if there's no ties?

Soln: Count # of 3-permutations of 8 runners

$P(8,3) = 8 \cdot 7 \cdot 6 = 336$

* Notation:

Let n be a nonnegative integer.

n factorial is defined by: $n! = n(n-1)(n-2) \ldots 2 \cdot 1$ for $n > 0$

$0! = 1$

Fact: The # of permutations of an n-element set is

$n! = P(n,n)$

Ex: In exercise #1, count permutations of 5 ppl

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

* Def: An r-combination of a set S is an unordered selection of exactly r elements from S

i.e: it's an r-element subset of S

Ex: Let $S = \{a, b, c, d, e\}$

Some 3-combination of S: $\{a, b, e\}$, $\{b, d, e\}$

Note: $\{a, b, e\}$ is the same 3-combination as $\{e, b, a\}$, $\{b, a, e\}$

* Let $C(n,r)$ be the # of r-combinations of an n-element set S

$C(n,r)$ is called a binomial coefficient

Alt. notation: $\binom{n}{r}$

Thm 2: $\boxed{C(n,r) = P(n,r)/r!}$

Ex: There are 8 contestant in a contest. How many ways are there to award 3 identical prizes?

Soln: Pick 3 ppl from 8 contestants (order doesn't matter!)

$C(8,3) = \frac{P(8,3)}{3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$

01/29 Ex. A class has 5 seniors and 7 juniors (12 people total)
How many different groups of 5 people are there?
a/ with no restrictions?
b/ which consist of 3 seniors & 2 juniors?

Soln:
a/ Count the # of 5-combinations of 12 people:
$C(12, 5) = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$

b/ Procedure:
- Picking 3 seniors: $C(5,3) = 10$
- Picking 2 juniors: $C(7, 2) = 21$

Product Rule: $10 \times 21 = 210$

* A big string is string consisting of 0's and 1's
eg: 01011 and 11001 are bit strings of length 5
Ex: How many bit strings are there of length 5? $2^5 = 32$

☆ * FACT: The # of bit strings of length n with exactly r 1's is $C(n,r)$
Ex: What's the # of bit strings of length 7 with exactly three 1's?
Soln: $C(7,3) = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$
$\Rightarrow$ Idea why it works?
Ex: How many sequences of 5 coin flips certain exactly 2 heads?
Soln: $C(5, 2) = 0$

* Useful fact: $C(n,r) = C(n, n-r)$
Why? # of bitstrings ~~of length~~ of length n with r 1's is $C(n,r)$
This is equal to the # of bitstrings of length n with exactly $(n-r)$ 0's, which is $C(n, n-r)$
eg: $C(20, 18) = C(20, 2) = 190$

$$\boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$$ $\rightarrow$ Useful in Algebratic manipulations

Ex: What's the # of bitstrings of length 7 & at most 1's?

Soln:

Case 1: # bitstrings with exactly two 1's $C(7,2) = 21$

Case 2: one 1, $C(7,1) = 7$

Case 3: zero 1's $C(7,0) = 1$

Sum 29

1/31 §6.5 Generalized Combinations / Permutations

Ex: How many solutions does the equation

$$x_1 + x_2 + x_3 + x_4 = 7$$

have in nonnegative integers?

e.g.: solutions (x_1, x_2, x_3, x_4)

$(1, 3, 1, 2)$

$(5, 0, 2, 0)$

$(0, 0, 2, 5)$

Soln: 7 stars: ★★★★★★

Each soln. to ★ corresponds to a ways of dividing 7 stars into 4 portions using 3 bars

$(1, 3, 1, 2) \leftrightarrow$ ★|★★★|★|★★

$(5, 0, 2, 0) \leftrightarrow$ ★★★★★||★★|

Thus, each soln. corr. to a string of length 10 consisting of 7 stars and 3 bars

eg: |★★★||★★★★ $\leftrightarrow (0, 3, 0, 4)$

The # of solns to ★ is thus:

$$C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \boxed{120}$$

FACT: The # of solutions for: $x_1 + x_2 + x_3 + \dots + x_n = r$ in nonnegative integers is:

$$\boxed{C(n + r - 1, n - 1) = C(n + r - 1, r)}$$

Idea: Each soln. to ★ corr. to a string of length $n + r - 1$ consisting of r stars and $(n-1)$ bars: $\underset{a_1 \text{ stars}}{★}|\underset{a_2 \text{ stars}}{★}|★★||\dots|★$

This would corr. to the soln. $(a_1, a_2, \dots, a_n)$ for ★

Ex: How many ways are there to distribute 8 identical cookies among 3 ppl?
↳ A B C

eg: possibilities

#A #B #C
(3, 2, 3)

Soln: Count the # of solutions for

$x_1 + x_2 + x_3 = 8$

in nonnegative integers, where

x_1 = # cookies A gets
x_2 = # cookies B gets
x_3 = # cookies C gets

Using $n = 3$, $r = 8$

The # of solutions is: $C(10, 2) = 45$

⇒ This is an example of distributing indistinguishable objects (cookies) in distinguishable boxes (people A, B, C) (See book)

Ex: A bakery sells apple, banana, and chocolate muffins. How many ways are there to select 6 muffins?

eg: # apple # banana # chocolate
(3, 1, 2)
(5, 1, 0)

Soln: Count the # of solutions for: $x_1 + x_2 + x_3 = 6$

where: x_1 = # apple
x_2 = # banana
x_3 = # chocolate

Using: $n = 3$, $r = 6$

The # of solutions is: $C(8, 2) = \boxed{28}$

* An r-combination of a set with n objects with repetition allowed is an unordered selected of r objects, where each object may be selected more than once

⇒ Only the # of each object matters

⇒ The # of such r-combination is: $\boxed{C(n+r-1, n-1) = C(n+r-1, r)}$

In this example:

We're counting the # of 6-combinations of a set of 3 objects (A, B, C) with repetition allowed

So you can compute $C(8, 2)$

Ex: Suppose there are 100 identical marbles of each of the following colors: red, blue, yellow, green.

How many ways are there to pick 7 marbles?

eg: $\overset{\#R}{(2}, \overset{\#B}{3}, \overset{\#Y}{1}, \overset{\#G}{1})$

$(7, 0, 0, 0)$

Soln: Count the # of 7-combinations of 4 objects (R, B, Y, G marbles) where repetition is allowed

Using $n = 4$, $r = 7$: $C(10, \cancel{4}\,3) = \cancel{210}\ 120$

* FACT: Let $n_1, n_2, \dots, n_k$ be positive integers whose sum is n

The # of strings of length n with exactly n_1 of 1 letter, n_2 of a 2^{nd} letter, etc

is:

$$\boxed{\frac{n!}{n_1!\, n_2! \cdots n_k!}}$$ ← multinomial coefficient

Ex: How many strings of length 9 contain exactly four a's, three b's and two c's?

eg: abbaccbaa

Soln: $\dfrac{9!}{4!\ 3!\ 2!} = \boxed{1260}$

?] Why this works?

$\boxed{b}\ \underset{\uparrow}{\boxed{a}}\ \boxed{c}\ \boxed{c}\ \underset{\uparrow}{\boxed{a}}\ \underset{\uparrow}{\boxed{a}}\ \boxed{b}\ \boxed{b}\ \underset{\uparrow}{\boxed{a}}$

Procedure:

+ Pick 4 out of 9 blanks for a's: $C(9, 4)$
+ Pick 3 blanks for b's: $C(5, 3)$
+ Pick 2 blanks for c's: $C(2, 2)$

Product Rule: # strings

$C(9, 4) \times C(5, 3) \times C(2, 2) =$

02/02 * FACT: Let $n_1, n_2, \dots, n_k$ be positive integers whose sum is n
Suppose that n distinguishable objects are distributed into k distinguishable boxes, where the i^{th} box has n_i objects for each i
Then there are $\dfrac{n!}{n_1!\, n_2! \dots n_k!}$ ways to distribute the n objects

Ex: 10 students are to be divided into groups:
→ group A will have 3 students
B 2
C 5

How many ways can this be done?
Soln: $\dfrac{10!}{3!\,2!\,5!} = 2520$

Why this works?
Procedure:
+ Pick 3 students for group A: $C(10,3)$
+ 2 B: $C(7,2)$
+ 5 : $C(5,5)$
Product Rule: $C(10,3) \times C(7,2) \times C(5,5) = \dfrac{10!}{3!\,2!\,5!}$

* Method 3:
Each way or dividing the class into groups corr. to a string of length 10
with exactly 3A's, 2B's, 5C's
eg: Student 1, 2, ..., 10
A = {1, 3, 7}
B = {2, 5}
C = {3, 4, 6, 8, 9, 10}
The # of strings is: $\dfrac{10!}{3!\,2!\,5!}$

Ex: A bakery sells apple, banana, chocolate muffins.
How many ways are there to pick 7 muffins if...
a/ there's at least 1 muffin of each type?
b/ there's at most 2 apple muffins?

Soln a/ I know I must pick
1A, 1B, 1C plus 4 more muffins
$\Rightarrow$ Count # of 4-combinations of 3 objects (A, B, C muffins) where repetition is allowed
Use ~~formula~~ formula with $r = 4$ and $n = 3$
$C(n+r-1, n-1) = C(6,2) = 15$

b/ $\underbrace{\begin{bmatrix} \text{Total \# of ways} \\ \text{to pick 7 muffins} \end{bmatrix}}_{N} - \underbrace{\begin{bmatrix} \text{\# of ways with at least} \\ \text{3 apple muffins} \end{bmatrix}}_{M}$

Then N = # of 7-combin. of 3 objects (repetition allowed)
$= C(9,2) = 36$
M = # of 4-combin. of 3 objects (repetition allowed)
$= C(6,2) = 15$
Ans: $N - M = 21$

Ex: How many solutions in nonnegative integers are there for the equation: $x_1 + x_2 + x_3 + x_4 = 10 \quad (*)$
where $x_1 \geq 2$?
Soln: Let $x_1' = x_1 - 2$
Then $(*)$ is equivalent to $x_1' + x_2 + x_3 + x_4 = 8 \quad (**)$
Finding all solutions to $(*)$ which satisfy $x_1 \geq 2$
is equivalent to finding all solns to $(**)$ which satisfy $x_1' \geq 0$
So the desired # of solns is: $(r = 8, n = 4)$
$C(n+r-1, n-1) = C(11,3) = 165$

02/05 §6.6 Generating Permutations & Combinations
Consider permutation of the set $S = \{1, 2, \dots, n\}$
$\Rightarrow$ Lexicographic (or dictionary) order:
Let $A = a_1 a_2 \dots a_n$ and $B = b_1 b_2 \dots b_n$ be two different permutations of S
Let k be the largest index so that $a_i = b_i$ for all $i < k$
If $a_k < b_k$, then $A < B$ in lex order.
Otherwise, $B < A$
Ex: 52143 is less than 52314 in lex order
Also, 31425 is less than 42135

* $\boxed{\text{FACT}}$

The smallest & largest permutations of $S = \{1, \dots, n\}$ in lex order are

$1, 2, 3, \dots, n$

and

$n, n-1, n-2, \dots, 1$

respectively

* Let $a_1 a_2 \dots a_n$ be a permutation of S which is not $n, n-1, n-2, \dots, 2, 1$

Procedure for finding the next permutation in lex order

- Step 1: Find the smallest index J so that

$$a_{J+1} > a_{J+2} > \dots > a_n \quad \text{and} \quad a_J < a_{J+1}$$

i.e.: $\underbrace{a_1 a_2 \dots a_J}\, \underbrace{a_{J+1} \dots a_n}$ $\hookrightarrow$ decreasing from left to right

- Step 2: Find the largest index k so that $a_k > a_J$

$$\underbrace{a_1 \dots a_J}\,\underbrace{a_{J+1} \dots a_k \dots a_n}$$

- Step 3: Switch a_J and a_k
- Step 4: Reorder the part between a_{J+1} and a_n in increasing order

Ex: Find the next 3 permutations after 542631

	J k	J k	J k
	$54\boxed{2}6\boxed{3}1$	$5431\boxed{2}\boxed{6}$	$543\boxed{1}6\boxed{2}$
Switch:	543621	543162	543261
Reorder:	543126 (*)	543162 (**)	543216 (***)

* $\boxed{\text{FACT}}$

In Step 1 of procedure:

$a_1 a_2 \dots a_n$ is the largest permutation starting with $a_1 \dots a_J$

eg. 542631 is the largest permutation starting with 542

* Consider r-combinations of the set $S = \{1, 2, \dots, n\}$

The smallest / largest r-combinations of S are: $\{1, 2, \dots, r\}$

and $\cancel{\{n, n-1, \dots\}}$ $\{n-r+1, n-r+2, \dots, n\}$

(in lex order)

* Suppose $\{a_1, a_2, \dots, a_r\}$ is an r-combinations of S which is not the largest (Suppose $a_1 < a_2 < \dots < a_n$)

Procedure to find the next r-combination in lex. order (Algorithm 3

- Step 1: Find the largest index i so that $a_i \neq n - r + i$

$$\{a_1, a_2, \dots, \underset{n-r+i}{\underset{\neq}{a_i}}, a_{i+1}, \dots, \underset{n-1}{\underset{\uparrow}{a_{r-1}}}, \underset{n}{\underset{\uparrow}{a_r}}\}$$

- Step 2: Replace a_i by $a_i + 1$
- Step 3: Replace everything after this element by $a_i + 2, a_i + 3, \dots$ until we have r elements in total

Ex: Let $S = \{1, 2, \dots, 7\}$

Find the next 3 4-combinations of S after $\{1, \boxed{3}, 6, 7\}$

Next set: $\{1, 4, 5, \boxed{6}\}$

After that $\{1, 4, \boxed{5}, 7\}$

$\{1, 4, 6, 7\}$

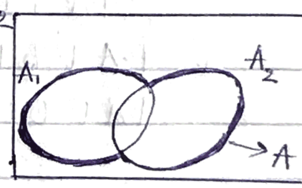
Quiz 2

02/09 § 8.5 Principle of Inclusion - Exclusion (PIE)

* PIE for 2 sets

Let A_1, A_2 be two sets, and let $A = A_1 \cup A_2$

Then: $|A| = |A_1| + |A_2| - |A_1 \cap A_2|$



Problems:

$\boxed{A}$ Prob 1/ $M = \{\text{Math major}\}$

$P = \{\text{Physics major}\}$

a/ $M \cap P =$

$M \cup P =$

By PIE, $|M \cup P| = |M| + |P| - |M \cap P|$

$20 = 12 + 15 - |M \cap P|$

$|M \cap P| = 7$

b/ $12 - 7 = 5$

$\boxed{A}$ Prob 2/ Let $A_1 = \{$ 3-perms that contain only 3 even digits $\}$ $\leftarrow P_1$

$A_2 = \{$ 3-perms that consist of all different digits $\}$ $\leftarrow P_2$

Then $A_1 \cup A_2 = \{$ 3-perms with at least one property P_1 or $P_2 \}$

Want: $|A_1 \cup A_2| = ?$

$\rightarrow |A_1| = 5^3 = 125$

$|A_2| = 11 \times 10 \times 9 = 990$

$|A_1 \cap A_2| = 5 \times 4 \times 3 = 60$

$A_1 \cap A_2 = \{$ 3-terms that have both properties P_1 and P_2)

$\Rightarrow |A_1 \cup A_2| = 125 + 990 - 60 = 1055$

* PIE for 3 sets

Let A_1, A_2, A_3 be sets and let $A = A_1 \cup A_2 \cup A_3$

Then: $|A| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$
$+ |A_1 \cap A_2 \cap A_3|$

$\boxed{A}$ Prob 3/ Let $A = \{$ students who like asparagus $\}$ $\quad |A| = 64$

$B = \{$ broccoli $\}$ $\quad |B| = 94$

$C = \{$ cauliflower $\}$ $\quad |C| = 58$

$|A \cap B| = 26$

$|A \cap C| = 28$

$|B \cap C| = 22$

$|A \cap B \cap C| = 14$

a/ $A \cup B \cup C = \{$ students who like at least 1 vegetable $\}$

$|A \cup B \cup C| = 64 + 94 + 58 - 26 - 28 - 22 + 14 = 154$

2/14 Problems

☆☆

Prob 1/a/ $C(5,2) \times C(9,3) = \frac{5 \times 4}{2 \times 1} \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 840$

2/b/ Count # of 3-combinations of 5 objects (repetition allowed)

$n = 5$ $\quad C(7,4) = \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35$

$r = 3$

3/b/ $n = 4$ $\quad C(7,3) = 35$

$r = 4$

02/20 §8.5 Principle of Inclusion - Exclusion (General Statement)

Let $A_1, A_2, \ldots, A_n$ be sets, and let $A = A_1 \cup A_2 \cup \ldots \cup A_n$

For $k = 1, 2, \ldots, n$ let

$N_k = \Sigma |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$, where the sum is over all k-element subsets $\{i_1, i_2, \ldots, i_k\}$ of the set $\{1, 2, \ldots, n\}$

PIE:

$$\boxed{|A| = N_1 - N_2 + N_3 - N_4 \ldots + (-1)^{n+1} N_n}$$

Ex: PIE for 3 sets: A_1, A_2, A_3

Let $A = A_1 \cup A_2 \cup A_3$

Then: $|A| = N_1 - N_2 + N_3$

$$= \underbrace{|A_1| + |A_2| + |A_3|}_{N_1} \underbrace{- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|}_{-N_2} + \underbrace{|A_1 \cap A_2 \cap A_3|}_{+N_3}$$

Ex: Let A_1, A_2, A_3, A_4 be sets, and let $A = A_1 \cup A_2 \cup A_3 \cup A_4$

Suppose: $|A_i| = 20$ for all i

$|A_i \cap A_j| = 3$ for all $i \neq j$

The intersection of any 3 sets is $\emptyset$

? Compute $|A|$

Soln: Use PIE. Note:

$N_3 = 0$

$N_4 = 0$

PIE for 4 sets:

$|A| = N_1 - N_2 + N_3 - N_4$

$$= \underbrace{|A_1| + |A_2| + |A_3| + |A_4|}_{N_1} \underbrace{- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4|}_{-N_2}$$

$= 20 + 20 + 20 + 20 - 3 \times 6$

$= 80 - 18 = \boxed{62}$

Counting using PIE:

Let U be a universal set

Suppose we want to count the # of elements in U which satisfy property P_1 or P_2

Let $A_1 = \{$ elements in U satisfying property $P_1 \}$

$A_2 = \{$ ______ // ______ $P_2 \}$

By PIE:

$$\underbrace{|A_1 \cup A_2|}_{\substack{\#\text{ objects in } U \\ \text{with property } P_1 \\ \text{or } P_2}} = \underbrace{|A_1|}_{\substack{\#\text{ objects} \\ \text{with} \\ \text{prop } P_1}} + \underbrace{|A_2|}_{\substack{\#\text{ objects} \\ \text{with} \\ \text{prop } P_2}} - \underbrace{|A_1 \cap A_2|}_{\substack{\#\text{ objects in } U \\ \text{with both properties} \\ P_1 \text{ and } P_2}}$$

★★ Problems:

1/b/ ~~C(8,2)×C~~

groups with 2F: 840

groups with 2S: $C(6,2) \times C(8,3) = 840$

groups with both 2F and 2S: $C(5,2) \times C(6,2) \times 3 = 10 \times 15 \times 3 = 450$

Ans: $840 + 840 - 450 = \boxed{1230}$

2/a/ $r = 10$, $n = 5$

$C(n+r-1, n-1) = C(14,4) = 1001$

c/ #ways with at least 3 apples: $n = 5, r = 7$ $C(11,4) = 330$

#ways with at least 4 blueberries: $C(10,4) = 210$

______ // ______ 3 apples ~~or~~ and at least 4 blueberries: $C(7,4) = 35$
$n = 5, r = 3$

Ans: $330 + 210 - 35 = \boxed{505}$

d/ [total # of ways] − [# ways where it's not true that at most 2 apples or 3 blueberries are picked]

DeMorgan law logic

= [total] − [# ways where at least 3 apples & at least 4 blueberries]

$= \underset{\text{(part a)}}{1001} - 505 = \boxed{496}$

3/b/ Give Bob 2 marbles and give Candy 4 marbles, then distribute 4 marbles between 4 ppl

Count the # of solutions to the equa:

$$x_1 + x_2 + x_3 + x_4 = 4 \qquad r = 4$$

extra marble for Bob, for Candy, for D, for E $\qquad n = 4$

Solns: $C(7,3) = 35$

02/21 c/ * # ways if Bob gets at least 2: Give Bob 2 marbles, then distribute the remaining 8 marbles

Count the # of nonnegative integer solutions to: $x_1 + x_2 + x_3 + x_4 = 8$

$C(n+r-1, n-1) = C(11,3) = 165$

$(r = 8, n = 4)$

* # ways if Candy gets at least 4: $C(9,3) = 84$

(b) * # ways if Bob gets at least 2 <u>and</u> Candy gets at least 4: 35

Ans: $165 + 84 - 35 = 214$

Counting using PIE (more general situation)

Let U be a universal set.

Suppose we want to count the # of elements in U that satisfy at least one of the properties $P_1, P_2, \dots, P_n$

Let $A_i = \{$elements in U which satisfy property $P_i\}$

Let $A = A_1 \cup A_2 \cup \dots \cup A_n$

Goal: Compute $|A|$

⇒ To do this, we can try to use PIE which (roughly) states that $|A|$ is the sum / difference of terms like $|A_{i1} \cap A_{i2} \cap \dots \cap A_{id}|$

This cardinality is the # of elements in U which satisfy all properties $P_{i1}, P_{i2}, \dots, P_{ik}$

★★ Prob 4/a/ $3^6 = 729$

Extra questions: How many ways are there to assign the 6 tasks if at least 1 employee has no tasks assigned?

Let $A_1 = \{$ways of assigning tasks so that Alice doesn't get any task$\}$

$A_2 = \{$ Bob $\}$

$A_3 = \{$ Candy $\}$

Goal: Compute $|A|$ where $A = A_1 \cup A_2 \cup A_3$

Using PIE: $|A_1| = 2^6 = 64$ (b/c we assign Bob / Candy to each task)

$|A_2| = 64$

$|A_3| = 64$

$\Rightarrow |A_1 \cap A_2| = 1$ (Candy must take all tasks)

$|A_1 \cap A_3| = 1$

$|A_2 \cap A_3| = 1$

$\Rightarrow |A_1 \cap A_2 \cap A_3| = 0$ (not possible)

PIE for A_1, A_2, A_3:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 64 + 64 + 64 - 1 - 1 - 1 + 0$$

$$= 189$$

b/ $|$Total # of ways$| - |$# ways where at least 1 employee has no tasks$|$

$= 729 - 189$

$= 540$

Chapter 7

§7.1 Intro to discrete probability

Terminology

- An experiment is any process involving chance.

Ex1: Throw a pair of dice, record sum

Ex2: Flip a coin 4 times, record the sequence of flips

- An experiment results in an outcome

Ex1: Possible dice sums: 2, 3, ..., 12

Ex2: All sequences of 4 flips, eg: HTHH, TTHH

The total set of all possible outcomes is called the sample space

- An event is a subset of the sample space

Ex1, Event: The sum is greater than 9. $E = \{10, 11, 12\}$

Ex2, Event: Head occurs exactly one $E = \{HTTT, THTT, TTHT, TTTH\}$

* Def: Let S be a finite, non empty sample space of equally likely outcomes, and let E be an event. Then probability of E is

$$P(E) = \frac{|E|}{|S|} \rightarrow \frac{\#\text{ favorable outcomes}}{\text{total }\#\text{ of outcomes}}$$

Ex: An integer is picked at random from the set $\{1, 2, ..., 30\}$

What's the probability that a perfect square is picked?

Soln, Sample space: $S = \{1, 2, ..., 30\}$, so $|S| = 30$

Event $E = \{1, 4, 9, 16, 25\}$, so $|E| = 5$

Therefore, $p(E) = \frac{|E|}{|S|} = \frac{5}{30} = 1/6$

Ex: Two dices are rolled.

What's the probability that the sum is 8?

Soln: Sample space: The set of all possible ordered pairs 1^{st} roll, 2^{nd} roll

$|S| = 6 \times 6 = 36$

Event $E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$$p(E) = \frac{|E|}{|S|} = \frac{5}{36}$$

02/23 Ex: A subset of 3 #'s is selected at random from the set $\{1, 2, ..., 10\}$

? What's the probability all 3 #'s are less than 6?

Soln:

- Sample space S: The set of all possible subsets of $\{1, 2, ..., 10\}$ with 3 elements.

$$|S| = C(10, 3) = 120$$

- Event E: E is the collection of all subsets of $\{1, 2, 3, 4, 5\}$ with 3 elements

$$|E| = C(5, 3) = 10$$

So $p(E) = \frac{|E|}{|S|} = \frac{10}{120} = \frac{1}{12}$

Ex: Suppose 10 people enter a contest, and 3 different winners are selected at random for 1st, 2nd, 3rd prizes

A, B, C are among the contestants

What's the probability ...

a/ A, B, C win 1st, 2nd, 3rd respectively?

b/ A, B, C each win a prize?

c/ B wins a prize?

Soln:

- Sample space S: The set of triples (1st winner, 2nd winner, 3rd winner)

$$|S| = 10 \times 9 \times 8 = 720$$

a/ $|E| = 1 \Rightarrow p(E) = 1/720$

b/ $|E| = 3 \times 2 \times 1 = 6 \Rightarrow p(E) = 6/720 = 1/120$

c/ $|E| = 3 \times 9 \times 8 = 216$

Picking which prize B wins: 3 ways

Assign the 1st remaining prize: 9 ways

Assign the last prize: 8 ways

$\Rightarrow p(E) = 216/720 = 3/10$

02/26 §7.2 Probability Theory

* Def: Let S be a sample space with a finite or countable number of outcomes

A probability distribution for S is a function $p(s)$ which assigns a probability $p(s)$ to every outcome S

The following properties must be satisfied:

(i) $0 \leq p(s) \leq 1$ for all $s \in S$

(ii) $\sum_{s \in S} p(s) = 1$ (sum of all probabilities is 1)

* Def: Suppose S is asset with n elements

The uniform distribution assigns the prob. of $1/n$ to every outcome in S

Ex: Experiment: Pick a # at random from $\{1, 2, 3, 4\}$

Uniform distribution: $p(s) = 1/4$ for all $s = 1, 2, 3, 4$

Ex: Pick a # at random from the set $\{1, 2, 3, 4\}$

Suppose the prob. 2 is picked is twice the prob. that 1 is picked

Suppose — " — 3 — " —

Suppose — " — 4 — " —

[?] Find the prob. distribution of ~~3~~ S.

Soln:
$$p(2) = 2 \cdot p(1)$$
$$p(3) = 2 \cdot p(1)$$
$$p(4) = 2 \cdot p(1)$$

Let $x = p(1)$. Since $p(1) + p(2) + p(3) + p(4) = 1$

$$x + 2x + 2x + 2x = 1$$

This implies $7x = 1$

So $x = 1/7$

Ans: $p(1) = 1/7$, $p(2) = p(3) = p(4) = 2/7$

* For all definitions / general properties below:

Let S be a sample space with associated probability distribution

Def: Let E be an event. The probability of E is

$$p(E) = \sum_{s \in E} p(s) \quad \text{(sum of probabilities of elements in E)}$$

Ex: Using prev. probability distrib:

Compute the probability that an odd # is picked

Soln: $E = \{1, 3\} \Rightarrow p(E) = p(1) + p(3) = 1/7 + 2/7 = 3/7$

General Properties

Sum Rule: Let $E_1, E_2, \ldots, E_n$ be events, and suppose E is the disjoint union of $E_1, E_2, \ldots, E_n$

i.e. $E = \bigcup_{i=1}^{n} E_i$, and

$$E_i \cap E_j = \emptyset \text{ if } i \neq j$$

Then: $p(E) = p(E_1) + p(E_2) + \ldots + p(E_n)$

$\boxed{\text{Why?}}$ $p(E) = p(E_1) + p(E_2) + \ldots + p(E_n)$

Recall example from Fri

10 contestants: 1^{st}, 2^{nd}, 3^{rd} prizes awarded

$\boxed{?}$ What's the prob that B wins a prize?

Soln. Break event E into cases with no overlap:

E_1: B wins 1^{st}

E_2: B wins 2^{nd}

E_3: B wins 3^{rd}

Then E is a disjoint union of E_1, E_2, E_3

Compute probabilities:

$p(E_1) = 1/10$

$p(E_2) = 1/10$

$p(E_3) = 1/10$

So $p(E) = 3/10$

Property: Let E be an event

Then $\boxed{p(E) = 1 - p(\bar{E})}$

Why? S is the disjoint union of E and $\bar{E}$, which means $p(E) + p(\bar{E}) = 1 = p($

$$\begin{bmatrix}\text{Prob. of} \\ \text{event E}\end{bmatrix} = 1 - \begin{bmatrix}\text{Prob. of event E} \\ \text{doesn't occur}\end{bmatrix}$$

Ex: (Contest above)

What is the probability that B doesn't win a prize?

Soln: $1 - 3/10 = 7/10$

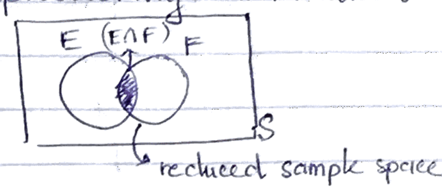
* ## Inclusion – Exclusion property for probability

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

[prob. event E or F occurs]   [prob. that both E and F occur]

* ## For all definitions/general properties below:

Let S be a ~~small~~ sample space with associated probability distribution



Def: Let E, F be events with $p(F) > 0$

The conditional probability of E given F is

$$\boxed{p(E|F) = \frac{p(E \cap F)}{p(F)}}$$

$\Rightarrow$ $p(E|F)$ is the prob. that E occurs if it's known that F occurs

Ex: A class has the following #'s of students

	Freshman	Sophomore
Female	4	3
Male	3	5

} 15 total

[?] What is the prob. that a randomly selected students is female (E) given that they're a sophomore? (F)

Soln: $p(E|F) = \dfrac{p(E \cap F)}{p(F)}$

$$= \frac{3/15}{8/15} = 3/8$$

Note:

If the prob. distrib. for S is uniform, then

$$\boxed{p(E|F) = \frac{|E \cap F|}{|F|}}$$

Ex: A PIN is selected at random. What the probability that the PIN contai

1/ exactly two 3's (eg: 5303)

2/ exactly two 3's given that the first digit is a 3?

(Recall that a PIN is a string of 4 digits (0-9)).

Soln:

1/

2/

1/ $\frac{C(4,2)\times 9^2}{10^4}$

2/ $\frac{3\times 9^2}{10^4} \times \frac{10^4}{\frac{\cancel{C(4,2)\times 9^2}}{10^3}} = \frac{3}{\frac{C(4,2)}{10^3}}$

Ans

1/ $|S| = 10^4 = 10,000$ $\quad \boxed{}\ \boxed{3}\ \boxed{}\ \boxed{3}$

$|E| = ?$

Pick 2 out of 4 positions for the 3's : $C(4,2) = 6$

There are $9 \times 9 = 81$ ways to fill the remaining positions.

Product Rule : $6 \times 81 = 486$

$p(E) = \frac{486}{10,000} = 4.86\%$

2/ $p(E|F) = \frac{|E \cap F|}{|F|}$

$|E \cap F|$ = # PINs that contain two 3's and start with 3 $\quad \boxed{3}\ \boxed{}\ \boxed{}\ \boxed{3}$

$= 9^2 \times 3$ (ways) $= 243$

$|F|$ = # PINs where 1^{st} digit is 3 $\quad \boxed{3}\ \boxed{}\ \boxed{}\ \boxed{}$

$= 10^3 = 1000$

Ans : $p(E|F) = \frac{243}{1000} = 24.3\%$

✶ **Multiplication Rule :**

Let E, F be events

Then : $\underbrace{p(E \cap F)}_{\text{prob that } E \,\&\, F \text{ occur}} = p(E)\,p(F/E)$

Why? : From the def. of conditional prob,

$p(F|E) = \frac{p(E \cap F)}{p(E)}$

Ex. An urn contains 5 blue marbles and 7 green marbles (Total = 12)

Two marbles are selected at random one at time, w/o replacement

[?] What's the prob. that both marbles are blue?

Soln : E : 1^{st} marble is blue

F : 2^{nd} marble is blue

Want : $p(E \cap F) = p(E)\,p(F|E)$

$$p(E \cap F) = p(E)\,p(F|E)$$
$$= \frac{5}{12} \times \frac{4}{11} \rightarrow \left(\begin{array}{l}11 \text{ marbles left}\\ 4 \text{ blue}\end{array}\right)$$
$$= \frac{5}{33}$$

✱ Extended multiplication Rule

Let $E_1, E_2, \ldots, E_n$ be events. Then the probability that all the events occur is: $p(E_1 \cap E_2 \cap \ldots \cap E_n) = p(E_i) \cdot \underbrace{\prod_{k=2}^{n} p(E_k \mid E_1 \cap E_2 \cap \ldots \cap E_{k-1})}_{\text{prob, Ex occurs given that all previous events } E_1, \ldots, E_{k-1} \text{ occur}}$

$$= \underbrace{\underbrace{\underbrace{p(E_1)\,p(E_2|E_1)}_{p(E_1 \cap E_2)}\,p(E_3|E_1 \cap E_2)}_{p(E_1 \cap E_2 \cap E_3)}\,p(E_4|E_1 \cap E_2 \cap E_3)}_{p(E_1 \cap E_2 \cap E_3 \cap E_4)} \ldots p(E_n|E_1 \cap \ldots \cap E_{n-1})$$

Ex:

10 contestants: 1^{st}, 2^{nd}, 3^{rd} prizes awarded

[?] What's the prob that A, B, C win 1^{st}, 2^{nd}, 3^{rd} prize respectively?

Soln: E_1: A win 1^{st}

E_2: B win 2^{nd} $\qquad p(E_1 \cap E_2 \cap E_3) = ?$

E_3: C win 3^{rd}

$$p(E_1 \cap E_2 \cap E_3) = p(E_1)\,p(E_2|E_1)\,p(E_3|E_1 \cap E_2)$$
$$\frac{1}{10} \times \frac{1}{9} \times \frac{1}{8} = 1/720$$

Ex: an urn contains 5 blue and 4 green marbles (9 total)

3 marbles selected at random, one at a time w/o replacement

① What's the probability that the 1^{st} two marbles are green, & the last is blue?

Soln: E_1: 1^{st} marble green

E_2: 2^{nd} marble green

E_3: 3^{rd} marble blue

Want: $p(E_1 \cap E_2 \cap E_3) = p(E_1) \cdot p(E_2|E_1)\,p(E_3|E_1 \cap E_2)$

$$= \frac{4}{9} \times \frac{3}{8} \times \frac{5}{7} = \frac{5}{42}$$

② What's the probability that you draw 2 green & 1 blue marble

Soln: 3 disjoint cases: GGB, GBG, BGG

$p(GGB) = 5/42$

$p(GBG) = 5/42$

$p(BGG) = 5/42$

Sum: $\frac{15}{42} = \frac{5}{14}$

Independence

Def: Two events E and F are independent if $\boxed{p(E \cap F) = p(E)\,p(F)}$

Let S be a sample space with probability distribution.

Fact: Let $p(F) > 0$

Then E, F are independent if and only if $p(E|F) = p(E)$

(i.e: Knowing F occurs doesn't affect the prob. of E occuring)

Roughly why: if know E and F are independent, then

$$\underbrace{\cancel{p(F)}\,p(E|F) = p(E \cap F)}_{\text{mult. rule}} = \underbrace{p(E)\,\cancel{p(F)}}_{\text{def}}$$

Ex: Two dices are rolled

What's the prob. an even # occurs on the 1^{st} ~~dice~~ die, and 5 occurs on the 2^{nd} die?

Soln: E: even # occurs on 1^{st} die

F: 5 occurs on 2^{nd} die

$$p(E \cap F) = p(E)\,p(F) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

* If $E_1, \ldots, E_n$ are events that are "mutually independent" (see book), then: $\boxed{p(E_1 \cap E_2 \cap \ldots \cap E_n) = p(E_1)\,p(E_2) \ldots p(E_n)}$

* Bernoulli trial:

Two outcomes: "success" with probability p

"failure" with probability $q = 1 - p$

Ex: A biased coin has prob. 0.1 of heads.
What's the prob. you get THT if the coin is flipped 3 times?
Soln: E_1: tails on 1^{st} flip
E_2: H on 2^{nd} flip
E_3: T on 3^{rd} flip
$p(E_1 \cap E_2 \cap E_3) = p(E_1)\,p(E_2)\,p(E_3)$ (mutually independent)
$= 0.9 \times 0.1 \times 0.9 = 0.081$

02/05 Ex: Biased coin where head turns up with prob. 0.1
- What's the prob. you get THHHT if the coin is flipped 5 times?
Soln: $p(THHHT) = (0.9)(0.1)(0.1)(0.1)(0.9) = 0.00081$
- What's the prob. you get heads exactly 3 times if the coin is flipped 5 times?
Soln: The prob. that are particular sequence of 5 flips w/ exactly 3 heads occuring is $(0.1)^3(0.9)^2 = 0.00081$
eg: $p(HTHHT) = (0.1)^3(0.9)^2$
The # of sequences w/ 3 heads is $C(5,3) = 10$
So $p(3 \text{ heads}) = 0.00081 \times 10 = 0.0081$
Note: We're basically using sum rule:

$$10 \text{ cases} \begin{cases} E_1: \text{ HHHTT occurs}: 0.00081 = \text{prob.} \\ E_2: \text{ HHTHT occurs}: 0.00081 = \text{prob.} \\ \vdots \end{cases}$$

+ Binominal distribution
Experiment: Perform n independent Bernoulli trials, where the prob. of success is p, and the prob. of failure is $q = 1-p$.
Record the number of successes
The prob. of exactly k successes is
$$b(k,n,p) = C(n,k)\,p^k q^{n-k}$$
The function $b(k,n,p)$ is called the binomial distribution

Ex: The biased coin above is flipped 6 times

1/ What's the prob. of getting exactly 4 heads?

H → success → 0,1

T → failure → 0.9

$$b(4, 6, 0.1) = \underbrace{C(6,4)}_{\text{number of such sequences}} \cdot \underbrace{(0.1)^4 \cdot (0.9)^2}_{\text{prob. of getting exactly 4 heads in a seq. of 6 times}}$$

$$= 15 \cdot (0.1)^4 \cdot (0.9)^2$$

$$= 0.001215$$

2/ What's the prob of getting at least 4 heads if the coin is flipped 6 times

Soln: $p(4H) + p(5H) + p(6H)$

$$= C(6,4) \cdot (0.1)^4 \cdot (0.9)^2 + C(6,5) \cdot (0,1)^5 \cdot (0,9)^1 + C(6,6) \cdot (0.1)^6$$

$$= 0,00127$$

§7.3 Bayer's Theorem

Let S be a sample space w/ prob. distribution

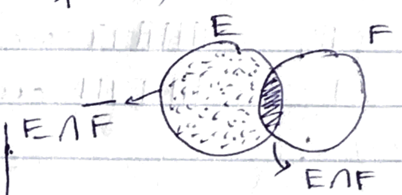
Let E, F be events in S so that $p(E), p(F) > 0$

? If I know $p(E|F)$ (and some other prob.)

Can I "reverse" the conditional, and compute $p(F|E)$?

$$p(F|E) = \frac{p(E \cap F)}{p(E)} \quad (\text{def. of conditional prob.})$$

$$= \frac{p(E \cap F)}{p(E \cap F) + p(E \cap \bar{F})}$$



Bayer's Theorem

$$p(F|E) = \frac{p(F)\, p(E|F) \leftarrow \text{mult. rule}}{p(F)\, p(E|F) + p(\bar{F})\, p(E|\bar{F})}$$

* General fact: $p(E|F) = 1 - p(\bar{E}|F)$

03/19 Bayes's Theorem (Examples)

2/ p : positive $\quad$ s : steroids

$\bar{p}$: negative $\quad$ $\bar{s}$: no steroids

steroids $p(s) = 0.05$
- positive $p(p|s) = 0.98$
- negative $p(\bar{p}|s) = 0.02$

no steroids $p(\bar{s}) = 0.95$
- positive $p(p|\bar{s}) = 0.12$
- negative $p(\bar{p}|\bar{s}) = 0.88$

a/ $p(s|p) = \dfrac{p(s \cap p)}{p(p)}$

$= \dfrac{p(\text{steroids \& positive})}{p(\text{positive})} = \dfrac{(0.05).(0.98)}{(0.05)(0.98)+(0.95)(0.12)} \approx 0.3006$

$\approx 30.1\%$

b/ $p(\bar{s}|p) = 1 - p(s|p) = 70\%$

c/ $p(\bar{s}|\bar{p}) = \dfrac{p(\bar{s} \cap \bar{p})}{p(\bar{p})}$

$= \dfrac{p(\text{no steroids and negative})}{p(\text{negative})} = \dfrac{(0.95)\times(0.88)}{(0.05)\times(0.02)+(0.95)\times(0.88)} \approx 99.9\%$

when the test returns negative, the test is very accurate

3/ S : spam $\quad$ R : rolex

$\bar{S}$: no spam $\quad$ $\bar{R}$: no rolex

spam $p(s) = 0.3$
- Rolex $p(R|s) = 0.2$
- no rolex $p(\bar{R}|s) = 0.8$

no spam $p(\bar{s}) = 0.7$
- rolex $p(R|\bar{S}) = 0.01$
- no rolex $p(\bar{R}|\bar{S}) = 0.99$

a/ $p(S|R) = \dfrac{p(S \cap R)}{p(R)} = \dfrac{p(\text{spam \& rolex})}{p(\text{rolex})} = \dfrac{(0.3)\times(0.2)}{(0.8)\times(0.3)+(0.7)\times(0.99)} = 7.97\%$

b/ ~~$p(\bar{S}|\bar{R}) = 1 - p(S|\bar{R}) =$~~ 25.7%

$p(S|\bar{R}) = \dfrac{p(\bar{R}|S)\,p(S)}{p(\bar{R}|S)\,p(S) + p(\bar{R}|\bar{S})\,p(\bar{S})}$

§2.4 Sequences / Recurrence Relations

* Def: A sequence is a list of numbers $a_0, a_1, a_2, \dots$ which are called terms

Notation: $\{a_n\}$

Sequence can be given by formulas: $a_n = f(n)$ for some function

Ex: Suppose the sequence $\{a_n\}$ is defined by $a_n = 2^n + n$

Compute a_0, a_1, a_2, a_3

Soln:
$$a_0 = 2^0 + 0 = 1$$
$$a_1 = 2^1 + 1 = 3$$
$$a_2 = 2^2 + 2 = 6$$
$$a_3 = 11$$

* Def: A recurrence relation for a sequence $\{a_n\}$ is an equation that express a_n in terms of $a_0, a_1, a_2, \dots, a_{n-1}$

Ex: Let $\{a_n\}$ be the sequence defined by $a_n = 2a_{n-1} + n a_{n-2}$ (the recurrence relation)

$a_0 = 1,\ a_1 = 3$ (initial conditions)

[?] Compute a_2, a_3, etc

Soln:
$$a_2 = 2a_1 + 2a_0 = 8$$
$$a_3 = 2a_2 + 3a_1 = 25$$
$$a_4 = 2a_3 + 4a_2 = 50 + 32 = 82$$

* Def: A sequence is a solution for a recurrence relation if its terms satisfy the recurrence.

Ex: Given the recurrence,
$$a_n = 2a_{n-1} + 3a_{n-2} \quad (\text{for } n \geq 2)$$
Check if the following sequences are solutions

1/ $a_n = 5n$

2/ $a_n = 3^n$

Soln:

1/ 1st terms: $a_0 = 0$
$a_1 = 5$
$a_2 = 10$
$a_3 = 15$

a_n in pt(1) is not a solution to the recurrence

Check $n = 2$: $a_2 \stackrel{?}{=} 2a_1 + 3a_0$

$10 \stackrel{?}{=} 2 \times 5 + 3 \times 0$ ✓

Check $n = 3$: $a_3 \stackrel{?}{=} 2a_2 + 3a_1$

$15 \stackrel{?}{=} 2 \times 10 + 3 \times 5$

$15 \stackrel{?}{=} 35$ ✗

Note:

03/21 If a sequence $\{a_n\}$ satisfies the recurrence then all of the equations below should be satisfied:

$$a_2 = 2a_1 + 3a_0$$
$$a_3 = 2a_2 + 3a_1$$
$$a_4 = 2a_3 + 3a_2$$

Optional

$\underbrace{\hphantom{x}}$

2/ 1st terms: $a_0 = 1$
$a_1 = 3$
$a_2 = 9$
$a_3 = 27$

Check: $a_2 \overset{?}{=} 2a_1 + 3a_0$
$9 \overset{?}{=} 2(3) + 3(1)$
$9 \overset{\checkmark}{=} 9$

Check: $a_3 \overset{?}{=} 2a_2 + 3a_1$
$27 = 2(9) + 3(3)$
$27 \overset{\checkmark}{=} 27$

Ex: Prove the recurrence works for all $n \geq 2$

Verify $a_n = 2a_{n-1} + 3a_{n-2}$ is true for all $n \geq 2$

Plug $a_n = 3^n$

$$3^n \overset{?}{=} 2(3^{n-1}) + 3(3^{n-2})$$
$$3^n \overset{?}{=} 2(3^{n-1}) + 3^{n-1}$$
$$3^n \overset{?}{=} 3^{n-1}(2+1)$$
$$3^n \overset{?}{=} 3^{n-1} \cdot 3$$
$$3^n \overset{\checkmark}{=} 3^n$$

a_n ~~is~~ 3^n is a solution for the recurrence

Ex: Verify that $a_n = 2^n + n$ is a solution for the recurrence:

$$a_n = a_{n-1} + 2a_{n-2} - 2n + 5$$

Proof:

$$2^n + n \overset{?}{=} [2^{n-1} + n - 1] + 2[2^{n-2} + n - 2] - \cancel{2[2^n + n]}^{\,2n} + 5$$
$$2^n + n \overset{?}{=} 2^{n-1} + n \cancel{- 1} + 2 \cdot 2^{n-2} + \cancel{2n} - 4 - \cancel{2n} + \cancel{5}$$
$$2^n + n \overset{?}{=} n + 2^{n-1} + 2 \cdot 2^{n-2}$$
$$2^n + n \overset{?}{=} n + 2^{n-1} + 2^{n-1}$$
$$2^n + n \overset{?}{=} n + 2^{n-1} \cdot 2$$
$$2^n + n \overset{\checkmark}{=} 2^n + n$$

03/23 Exam 2 Review:

4/ PIE for 4 sets A_1, A_2, A_3, A_4

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \underbrace{|A_1| + |A_2|}_{4} + |A_3| + |A_4| - \overbrace{|A_1 \cap A_2|}^{6} - \overbrace{|A_1 \cap A_3|}^{9} - |A_1 \cap$$
$$- \underbrace{|A_2 \cap A_3|}_{3} - |A_2 \cap A_4| - |A_3 \cap A_4|$$
$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_3$$
$$- \underbrace{|A_1 \cap A_2 \cap A_3 \cap A_4|}_{0}$$

Let $x = |A_1|$. Then $100 = 4x - 60 + 12 + 0$

$148 = 4x$

$x = 37$

8/a/ $p(E_2 | E_1) = p(\text{2nd marble green} \mid \text{1st marble green}) = 5/9$

c/ p(all 3 are diff. colors)

= p(1R, 1Y, 1G is some order)

Start with $p(R, Y, G) = \frac{3}{10} \times \frac{2}{9} \times \frac{5}{8} = 1/24$

Note: there's $3! = 6$ possible orders for the colors

So final ans: $6 \times 1/24 = 1/4$

7/b/ S = {all subsets of {1, 2, ..., 10} with cardinality 4}

E = {all subsets of {1, 2, ..., 10} which have exactly 3 elements from {1, 3, 5, 7}}

$|S| = C(10, 4)$

$|E| = C(4, 3) . 6 = 24$

03/28 §8.1 Modeling using recurrence relation

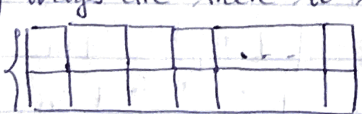
i.e using recurrence on counting problems

Ex: Let n be a positive integer

[?] How many ways are there to tile $2 \times n$ grid with dominos?

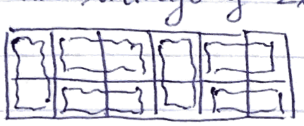
$2 \times n$ grid

2

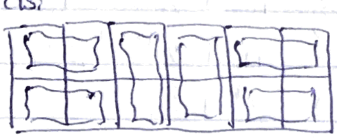


n

Two tilings of $2 \times n$ grids:



or



- Let a_n be the number of ways to tile a $2\times n$ grid
(Then $\{a_n\}$ is a sequence of #'s)
- Find a recurrence relation for $\{a_n\}$ and set up initial conditions:
Let $n \geq 2$, then

$$a_n = \begin{bmatrix} \text{\# of ways to tile} \\ \text{a } 2\times n \text{ grid, using} \\ \text{in top left} \end{bmatrix} + \begin{bmatrix} \text{\# of ways to tile} \\ \text{a } 2\times n \text{ grid using} \\ \text{in top left} \end{bmatrix}$$

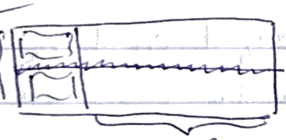
Case 1:

2 

$n-1$

The # of ways to fill in the shaded area is a_{n-1}

Case 2:

2 

$n-2$

The # of ways to tile is a_{n-2}

Initial conditions: $a_0 = 1$ ← # ways to tile a 2×0 grid (only 1 way: use 0 tiles)
$a_1 = 1$ ← # ways to tile a 2×1 grid

Compute more terms:

$a_2 = a_1 + a_0 \Rightarrow a_2 = 1 + 1 = 2$ ← Double check

2 ways!

$a_3 = a_2 + a_1 \Rightarrow a_3 = 2 + 1 = 3$

$a_4 = 5$

$a_5 = 8$

$\vdots$

You get the Fibonacci sequence!

Ex: A certain rending machine accepts \$1, \$2 and \$5 tokens
Let ~~an~~ a_n be the # of ways to insert \$n worth of tokens.
(Order matters: eg. 1125 is different from inserting 5121)
$\Rightarrow$ Set up recurrence: Let $n \geq 5$

$$a_n = \begin{bmatrix} \text{\#ways to insert \$n} \\ \text{start with \$1 token} \end{bmatrix} + \begin{bmatrix} \text{---\ \ // \ \ ---} \\ \text{--- // \$2 ---} \end{bmatrix} + \begin{bmatrix} \text{---\ \ // \ \ ---} \\ \text{--- \ \$5 ---} \end{bmatrix}$$

Case 1: # way start with \$1

\$1 + [insert \$(n-1)]

ways to insert \$(n-1) is a_{n-1}

Case 2: # ways start with \$2

\$2 + [insert \$(n-2)]

ways to insert \$(n-2) is a_{n-2}

Case 3: # ways start with \$5

ways to do then is a_{n-5}

⇒ Recurrence relations:

$$a_n = a_{n-1} + a_{n-2} + a_{n-5}$$

Initial values: $a_0 = 1$ $\quad a_4 = 5$

$a_1 = 1$ $\quad a_5 = a_4 + a_3 + a_0 = 9$

$a_2 = 2$ $\quad a_6 = 15$

$a_3 = 3$

$a_n = 0$ for n negative

Ex: Let a_n = # ways to insert \$n in token uses only \$1 & \$2 and you don't use consecutive \$2 token

eg. For \$5, inserting 212, but not 122

$$a_n = \left[\begin{array}{c}\text{\# way of inserting} \\ \text{\$n start with \$1}\end{array}\right] + \left[\begin{array}{c}\text{—''—} \\ \text{—''— \$2}\end{array}\right]$$

Case 1:

\$1 + (n-1) dollar

ways is a_{n-1}

Case 2:

\$2 + \$1 + (n-3) dollar

ways is a_{n-3}

$$\Rightarrow a_n = a_{n-1} + a_{n-3}$$

03/30 * Def: A ternary string is a string consisting of 0's, 1's and 2's (eg: 0121112)

Ex: Let a_n be the # of ternary strings of length n which [contain consecutive 2's] (eg: 1022210)

property P

Recurrence:

$a_n = [\ ] + [\ ] + [\ ] + \dots$

Case 1: Start with 0

$\boxed{0}$ any string of length $(n-1)$ with property P

→ $(n-1)$ positions

There are a_{n-1} ways to fill in this blank

Case 2: Start with 1

$\boxed{1}$ any string of length $(n-1)$ with property P

→ $(n-1)$ blanks

a_{n-1} ways!

Case 3: Start with 2

+ Subcase 3a: Start with 22:

$\boxed{22}$ any string of length $(n-2)$

→ $(n-2)$ blanks

→ 3^{n-2} ways to fill blanks

+ Subcase 3b: Start with 21

$\boxed{21}$ any string of length $(n-1)$ with property P

→ $(n-2)$ blanks

→ a_{n-2} ways!

+ Subcase 3c: Start with 20

$\boxed{20}$

→ a_{n-2} ways!

Simplify:

$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$

or $a_n = a_{n-1} + a_{n-1} + 3^{n-2} + a_{n-2} + a_{n-2}$

n-strings start with 0 | # n-strings start w/ 0 | — // — // w/ 22 | w/ 21 | w/ 20

04/02 Initial values: $a_0 = 0$

$a_1 = 0$

More values: $a_2 = 2a_0 + 2a_1 + 3^0 = 1$

$a_3 = 5$

$a_4 = 21$

⇒ The fact that $a_4 = 2$ means there's 21 ternary strings w/ consecutive 2's

⇒ Using procedure to list all nice 3-strings:

+ Append 0 to all nice 2-strings:

022

+ Append 1 ——— // ———:

122

+ Append 22 to any ternary string of length 1:

220 221 222

+ Append 21 to any nice 1-string: no extra strings

+ 20 : no extra strings

⇒ Total # nice 3-strings: 3

$a_3 = 2a_2 + 2a_1 + 3^1$

§8.2 Solving recurrence relations

* Def: A linear homogenous recurrence relation with constant coefficients is a recurrence of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

where $c_1, c_2, \dots, c_k$ are real numbers

⇒ If $c_k \neq 0$, the degree is k

Ex: $a_n = 2a_{n-1} - 3a_{n-2} + 5a_{n-3}$ is a LHRC of degree 3

⇒ $a_n = 7a_{n-2}$ is a LHRC of degree 2

Ex: $a_n = 3a_{n-1} + n a_{n-2}$

Not a LHRC

coefficient is not a constant

Ex: $a_n = a_{n-1} * a_{n-2} + a_{n-3}$

neither can occur in a linear recurrence

★ **Degree 1 LHRC**

$$a_n = c a_{n-1}$$

[?] All solns of recurrence ?

Ex: $a_n = 5a_{n-1}$

Calculate terms: $a_0 = \alpha$

$a_1 = 5\alpha$

$a_2 = 5a_1 = 5.5\alpha = 5^2\alpha$

$a_3 = 5a_2 = 5^3\alpha$

$\Rightarrow a_n = \alpha 5^n$

★ FACT: The general soln. for the recurrence

$$a_n = c a_{n-1}$$

is $\boxed{a_n = \alpha c^n}$

where α is a real number.

Ex. Suppose a sequence $\{a_n\}$ satisfies $a_n = 3a_{n-1}$ and $a_2 = 6$

[?] Find a closed formula for a_n

Soln. General solution for recurrence

$$a_n = \alpha 3^n$$

Plug in $a_2 = 6$: $\alpha 3^2 = 6$

$9\alpha = 6$

$\alpha = 2/3$

Plug α into gen. soln: $a_n = 2/3 . 3^n \Rightarrow a_n = 2.3^{n-1}$

★ **Degree 2 LHRC**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad (*)$$

The characteristic equation of $(*)$ is:

$$\boxed{x^2 - c_1 x - c_2 = 0} \quad (**)$$

★ FACT: If r is a root of $(**)$, then the sequence:

$$\boxed{a_n = r^n}$$

is a solution for the LHRC $(*)$

Check fact: $r^n \stackrel{?}{=} c_1(r^{n-1}) + c_2(r^{n-2})$

$r^n - c_1 r^{n-1} - c_2 r^{n-2} \stackrel{?}{=} 0$

$r^{n-2}\underbrace{(r^2 - c_1 r - c_2)}_{0} \stackrel{?}{=} 0$ ✓

Ex: $a_n = 7a_{n-1} - 10a_{n-2}$

The characteristic equation: $x^2 - 7x + 10 = 0$

$(x - 2)(x - 5) = 0$

Roots: $x = 5$, $x = 2$

By fact, the sequences: $a_n = 5^n$

and $a_n = 2^n$

are solutions to the recurrence

* FACT: Suppose that $\{b_n\}$ and $\{d_n\}$ are sequences which are solutions for the LHRC (*)

Let α be any real #:

Then $\{b_n + d_n\}$ and $\{\alpha b_n\}$ are sequences that are solns' for (*)

Ex: as above:

The sequences: $a_n = 2^n + 5^n$

and $a_n = 3 \cdot 5^n$

are both solutions to the recurrence.

04/04 Ex: Recurrence: $a_n = a_{n-1} + 6a_{n-2}$

Char. equ: $x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$$\left[\begin{array}{l} x = 3 \\ x = -2 \end{array}\right.$$

Some solutions to recurrence:

$a_n = 3^n$

$a_n = (-2)^n$

$a_n = 3^n + (-2)^n$

$a_n = \alpha \cdot 3^n$

$a_n = \alpha \cdot 3^n + \beta(-2)^n$ for any α, β reals

(This is the general solutions to recurrence!)

Ex: Suppose $\{b_n\}, \{d_n\}$ are solutions to the recurrence:

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Show: $\{b_n + d_n\}$ is also a solution

Pf: Since $\{b_n\}, \{d_n\}$ are solutions, we know,

$b_n = c_1 b_{n-1} + c_2 b_{n-2}$

$d_n = c_1 d_{n-1} + d_2 b_{n-2}$

Need to verify: $b_n + d_n \stackrel{?}{=} c_1(b_{n-1} + d_{n-1}) + c_2(b_{n-2} + d_{n-2})$

$b_n + d_n \stackrel{?}{=} (c_1 b_{n-1} + c_2 b_{n-2}) + (c_1 d_{n-1} + c_2 d_{n-2})$

$b_n + d_n \stackrel{?}{=} b_n + d_n$ ✓

* Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ $(*)$

Suppose r_1, r_2 are the roots of the characteristic equations

✓ If $r_1 \neq r_2$, then the general solution for $(*)$

$$\boxed{a_1 = \alpha_1 r_1^n + \alpha_2 r_2^n}$$

where α_1, α_2 are any real #'s

✓ If $r_1 = r_2$, then the general solution for $(*)$:

$$\boxed{a_n = \alpha_1 r_1^n + \underbrace{\alpha_2 n r_1^n}_{\text{solution to recurrence}}}$$

Ex: Find the solution for the recurrence:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 2$

Soln: Char. equ: $x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, \ x = -1$

General soln to the recurrence: $a_n = \alpha_1(3^n) + \alpha_2(-1)^n$

Find α_1, α_2

$$\left.\begin{aligned} a_0 = 1 &\Rightarrow \alpha_1 + \alpha_2 = 1 \\ a_1 = 2 &\Rightarrow 3\alpha_1 - \alpha_2 = 2 \end{aligned}\right\} \text{add}$$

$4\alpha_1 = 3 \rightarrow \alpha_1 = 3/4$

Using $\alpha_1 + \alpha_2 = 1$, we get $\alpha_2 = 1/4$

Final answer: $a_n = \frac{3}{4} 3^n + \frac{1}{4}(-1)^n$

$$a_n = \frac{3^{n+1} + (-1)^n}{4}$$

Ex: Solve: $a_n = 6a_{n-1} - 9a_{n-2}$

w/ initial conditions: $a_0 = 2$ and $a_1 = 3$

Soln: Char. equ: $x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

Roots: $3, 3$

(or say: 3 is a root of multiplicity 2)

General soln: $a_n = \alpha_1 3^n + \alpha_2 n 3^n$

Solve for α_1, α_2:

$a_0 = 2 \Rightarrow \alpha_1 = 2$

$a_1 = 3 \Rightarrow 3\alpha_1 + 3\alpha_2 = 3$

$6 + 3\alpha_2 = 3$

$\alpha_2 = -1$

Final ans: $a_n = 2 \cdot 3^n - n \cdot 3^n$

* Def: A LHRC of degree k is a recurrence of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + a_k n_{n-k}$$

$\neq 0$

where the c_i are real #'s with $a_k \neq 0$

Characteristic equation:

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_k = 0$$

FACTS: about degree 2 LHRC also true about degree k LHRCs!

* Solving in general:

Suppose the distinct roots of the char. equation are $r_1, r_2, \dots, r_p$ and the root r_i occurs with multiplicity m_i:

Then any solution to the recurrence is of the form:

$$\boxed{a_n = \sum_{i=1}^{p} f_i(n) r_i^n}$$

where $f_i(n)$ is polynomial in the variable n of degree less than m_i

Also, any sequence of this form is a soln. to the recurrence.

04/06 Ex. Given the recurrence: $a_n = 18a_{n-2} - 81a_{n-4}$ (Degree 4)

[?] Find the general soln.

Soln: Characteristic equation:

$$a_n = 0a_{n-1} + 18a_{n-2} + 0a_{n-3} - 81a_{n-4}$$

$$\Rightarrow x^4 - 0x^3 - 18x^2 - 0x + 81 = 0$$

$$x^4 - 18x^2 + 81 = 0$$

Find the roots: $(x^2-9)(x^2-9) = 0$

$(x+3)^2(x-3)^2 = 0$

Roots: -3 w/ mult. 2

3 w/ mult. 2

General soln:

$$a_n = \underbrace{(\alpha_0 + \alpha_1 n)}_{\text{degree} < 2}(-3)^n + \underbrace{(\beta_0 + \beta_1 n)}_{\text{degree} < 2}(3^n)$$, where $\alpha_0, \alpha_1, \beta_0, \beta_1$ are any real #'s

Specific soln: $a_n = (5 + 7n)(-3)^n + n3^n$

Ex: Suppose you're given an LHRC whose characteristic equ. is:

$(x-5)^3(x-7) = 0 \rightsquigarrow$ Roots: $r = 5$ w/ mult. 3, $r = 7$ with mult. 1

General soln to LHRC: $a_n = \underbrace{(\alpha_0 + \alpha_1 n + \alpha_2 n^2)}_{\deg < 3} 5^n + \underbrace{(\beta_0)}_{\deg < 1} 7^n$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0$ are any real #'s

★★★

✱ Def: A <u>linear nonhomogeneus recurrence</u> relation w/ constant coefficients is one of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

where $F(n)$ is a function of n, and the c_i are constants

→ To find the general soln:

- Find a particular solution $\{a_n^{(P)}\}$ for the recurrence (Note: You will be given this!)
- Find the general solution $\{a_n^{(h)}\}$ to the <u>associated homogenous recurrence</u> $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$
- Add these solutions together for the general soln to the original recurrence

Ex: Suppose that it's known that $a_n = \dfrac{-3^{n+1}}{2}$ is a solution for the recurrence: $a_n = 5a_{n-1} + 3^n$

[?] Find the general soln:

Associated homogenous recurrence: $a_n = 5a_{n-1} + 3^n$

General solution for homogenous: $a_n^{(h)} = \alpha . 5^n$

General solution for recurrence: $a_n = \dfrac{-3^{n+1}}{2} + \alpha . 5^n$

where α is any real #

04/09 Ex: Recurrence

$$\underbrace{a_n = a_{n-1} + 6a_{n-2}}_{\text{LHRC}}, \underbrace{-6n+3}_{F(n)} \quad (*)$$

$\boxed{?}$ Find the general soln. if a particular ~~stuff~~ soln. $a_n = n$ is known.

Soln: Find the general soln for associated LHRC: $a_n = a_{n-1} + 6a_{n-2}$ (degree 2)

Char. equation: $x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

Roots: $x = 3$, $x = -2$ (both w/ multiplicity 1)

$\Rightarrow a_n^{(h)} = \alpha(3^n) + \beta(-2)^n$

$\Rightarrow$ General soln for $(*)$

$a_n = n + \alpha(3^n) + \beta(-2)^n$

* Note: To find a particular soln to $(*)$ in the first place, we would "guess" that such a soln should be of a similar form to $F(n)$, namely, $a_n^{(p)} = cn + d$ for some constants c, d

$\Rightarrow$ Plug this into recurrence, solve for c, d.

Get $c = 1$, $d = 0$ $\Rightarrow a_n^{(p)} = n$

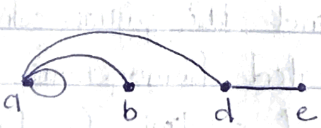
§10.1 and 10.2 Graphs

* Exercises:

1/ deg a = 5
deg b = 2
deg c = 3
deg d = 3
deg e = 1
(e is a pendant vertex)

2/ (5, 3, 3, 2, 1)

3/



4/ Sum of degrees = 5 + 2 + 3 + 3 + 1 = 14

\# edges: 7

Sum of degrees = 2 × # edges

4/11

§ 10.2 Graph = Undirected Graph

* For all general facts / defs:

Let $G = (V, E)$ be an undirected graph.

* **Thm 1:** (Handshaking Thm)

Let m be the # of edges in G

Then $$\boxed{\sum_{u \in V} \deg(u) = 2m}$$

sum of degrees = 2 x (# edges)

Corollary: The sum of the degrees is even

Rough pf. of thm: For simplicity, assume no loops

- Vertices : People at a party
- Edges : Hand shakes between 2 people

3 handshakes

$\Rightarrow \deg(u) = $ # of handshakes that person u makes

So $\sum_{u \in V} \deg(u) = 2 \times \begin{pmatrix} \text{total \# of} \\ \text{handshakes} \\ \text{at party} \end{pmatrix} = 2m$

(since every handshake is counted twice)

Ex: Suppose that the degree sequence of a graph is $(4, 3, 3, 3, 1)$

$\boxed{?}$ # edges

Soln: The sum of degrees is: $4 + 3 + 3 + 3 + 1 = 14$

so the # of edges is $\frac{14}{2} = 7$

Ex: Show there is no graph with degree sequences: $(4, 3, 2, 2, 2)$

Soln: Sum of degrees: $4 + 3 + 2 + 2 + 2 = 13$, which is odd

But by the ~~contrary~~, corollary, the sum of the degrees of a graph must be ~~equal~~ even.

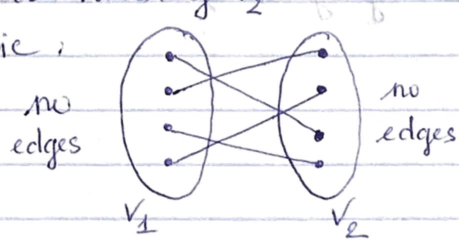
Bipartite graphs:

* Let $G = (V, E)$ be a simple graph (no mult. edges / loops)

* Def: G is bipartite if there exists ~~subjects~~ subsets V_1, V_2 of V so that V is a disjoint union of V_1, V_2, and any edge in G connects a vertex of V_1 with a vertex of V_2

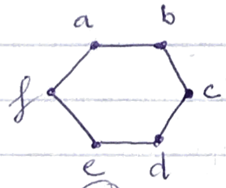
$V = V_1 \cup V_2$ & $V_1 \cap V_2 = \emptyset$

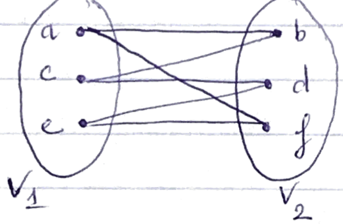
Rough pic:



$\Rightarrow$ We call (V_1, V_2) a bipartition of G

Ex: C_6





Let $V_1 = \{a, c, e\}$

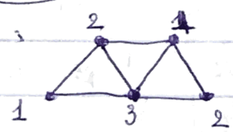
$V_2 = \{b, d, f\}$

Then (V_1, V_2) is a bipartition of C_6

✱ **Def (§10.8):** A <u>coloring</u> of G is an assignment of color to each vertex of G so that no two adjacent vertices have the same color

Colors: 1, 2, 3, ...

Ex:



← Coloring using colors 1, 2, 3, ...

✱ **Thm 4: (in §10.2)**

G is bipartite if and only if there's a coloring of G with at most 2 colors

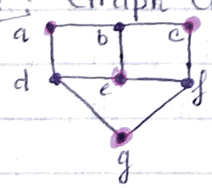
Proof: ($\Leftarrow$) Suppose that there is coloring of G using colors from $\{1, 2\}$

For $k = 1, 2$: Let V_k be the set of vertices with color k

There's no edges between any two vertices in V_1, and similarly with V_2

Therefore, (V_1, V_2) is a bipartition of G

Ex: Graph G:

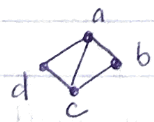


$V_1 = \{a, c, e, g\}$

$V_2 = \{b, d, f\}$

Then (V_1, V_2) is a bipartition of G

Ex: Let G =



Show G is not bipartite

Soln: In a valid coloring of G, the vertices a, b, c must all receive different colors b/c there's an edge between any pair of vertices

So there can't be a valid coloring of G w/ at most 2 colors

04/13 * Notation: Let v be a vertex in G. $G = (V, E)$ is a graph.

The neighborhood of v is the set of all vertices adjacent to v

Notation: $N(v)$

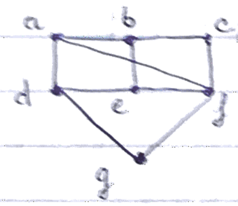
Let A be a subset of vertices.

Then the neighborhood of A is

$$N(A) = \bigcup_{v \in A} N(v)$$

(set of all vertices that are adjacent to at least 1 vertex in A)

Ex:



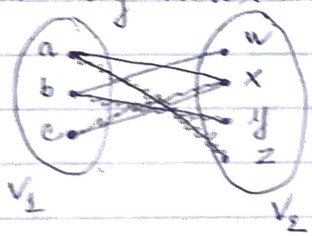
$N(b) = \{a, c, e\}$

$N(\{a, c\}) = \{b, d, f\}$

* Let $G = (V, E)$ be a bipartite graph w/ bipartition (V_1, V_2)

A complete matching of G from V_1 to V_2 is a subset of S of edges so that every vertex in V_1 is the endpoint of exactly one edge in S

Ex:



* Hall's Marriage Thm:

G has a complete matching if and only if

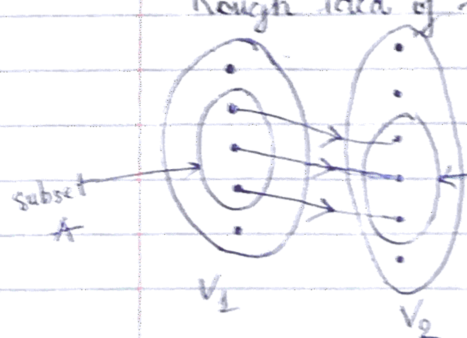
$|N(A)| \geq |A|$ for any subset A of the vertex set V_1

Rough idea of pf of $\Rightarrow$ Suppose G has a complete matching

Let A be a subset of V_1, and suppose $|A| = k$

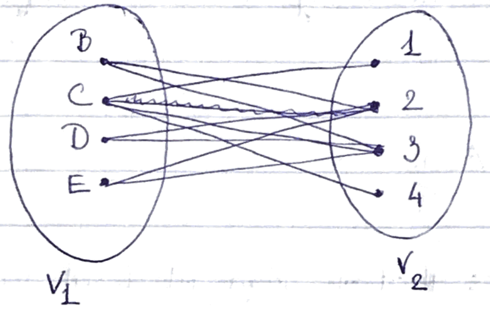
Then A is "matched" w/ k elements in V_2, so

$|N(A)| \geq k = |A|$



Exercises : (paper 10.2)

1/a/



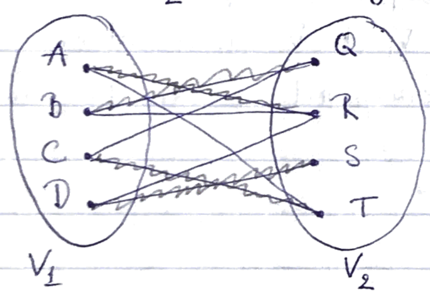
b/ [?] Find a subset A of V_1 so that $|N(A)| < |A|$

Soln: $A = \{B, D, E\}$

$N(A) = \{2, 3\}$

Then $\underset{2}{|N(A)|} < \underset{3}{|A|}$

2/



→ Complete matching

04/23 §10.3 Adjacency Matrix

* Def: Let G be a graph (undirected / directed)

Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set

The adjacency matrix $A = (a_{ij})$ for G is the $n \times n$ matrix whose entries are given below:

+ If G is undirected:

a_{ij} = # edges between v_i and v_j

+ If G is directed

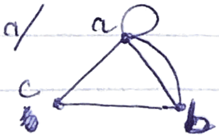
a_{ij} = # edges from v_i to v_j

Note: For undirected graph, A is symmetric

(i.e: $a_{ij} = a_{ji}$ for all i, j)

Ex: Find the adjacency matrices for the following graphs:

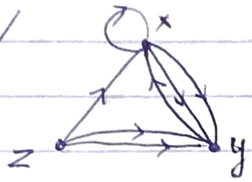
a/



Let $V = \{a, b, c\}$

$$A = \begin{array}{c|ccc} & a & b & c \\ \hline a & & & \\ b & & & \\ c & & & \end{array} \quad \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

b/



Let $V = \{x, y, z\}$

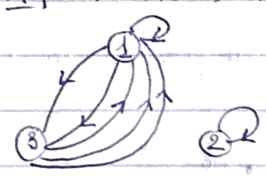
$$A = \begin{array}{c} \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$$

Ex:

$$\text{Let } A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \end{array}$$

Suppose that the adjacency matrix of undirected graph with vertices 1, 2, 3 is given by A

[?] Sketch the graph

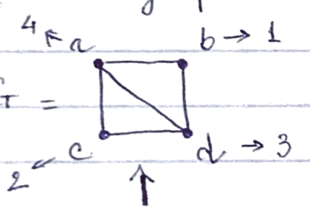


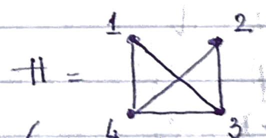
Isomorphic graphs

Assume graphs are simple undirected

Roughly: Two graphs are isomorphic if they're "the same" after relabeling.

Ex:

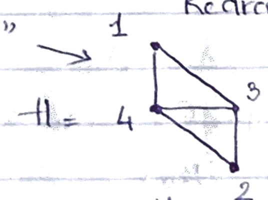




↑

"the same"

Redraw:



G and H are isomorphic!

G and H are isomorphic!!

Def: Let G, H be graphs with vertex set V, W respectively.

A function $f: V \to W$ is a graph isomorphism if it's one-to-one and onto and satisfies the following property.

For every $a, b \in V$, $\{a, b\}$ is an edge in G $\Leftrightarrow$ $\{f(a), f(b)\}$ is an edge in H

Note: f is basically an appropriate "relabeling function" on the vertices of G.

Graph isomorphism:

$$f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$$

Given by: $f(a) = 4$ $\quad f(c) = 2$

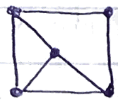
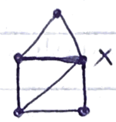
$f(b) = 1$ $\quad f(d) = 3$

* FACT:

If two graphs are isomorphic, then they must have the same:

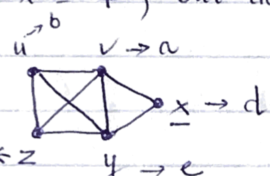
+ # of vertexes
+ # of edges
+ degree sequence
+ subgraphs

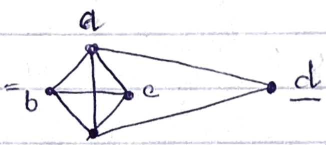
Ex:

G =    H = 

Can show: G and H aren't isomorphic!

Why? deg x = 4, but all vertices in G have degree at most 3

Ex: G = 

H = 

[?] Find an isomorphism f from G to H

Isomorphism: $f(u) = b$
$f(v) = a$
$f(x) = d$
$f(y) = e$
$f(z) = c$

§10.4 Connectivity

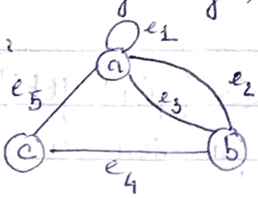
A path in an undirected graph G is a sequence of vertices and edges

$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \longrightarrow \cdots \xrightarrow{e_n} v_n$

A path in a directed graph:

$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \longrightarrow \cdots \longrightarrow v_n$

The length of the path is n: (# of edges)

Ex: 

[?] Paths ~~between~~ from vertex a ~~and~~ to vertex b?

Length 1: e_2, e_3 } 2 paths

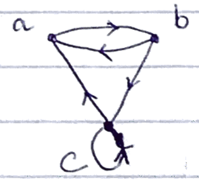
Length 2: e_5, e_4 ; e_1, e_2 ; e_1, e_3 } 3 paths

Length 3: e_1, e_5, e_4 ; e_1, e_1, e_2 ; e_1, e_1, e_3 ... more paths (15 total)

* Thm 2: Let A be the adjacency matrix for a graph (directed / undirected) with vertex set $V = \{v_1, v_2, \dots, v_n\}$

Then the # of paths from v_i to v_j is the $(i,j)^{th}$ entry of A^n

Ex:



Adjacency Matrix: $A = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 1 & 0 & 1 \end{array}$ $\quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$A^2 = AA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$A^3 = A^2A = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 1 & 1 \\ b & 2 & 1 & 2 \\ c & 2 & 1 & 2 \end{array}$

$\boxed{?}$ # of paths of length 3 from...

+ ... b to c? 2 paths $\to$ babc, bccc
+ ... c to b? 1 path $\to$ ccab

* FUNFACT: # of paths of length $(n+1)$ from b to c is the n^{th} Fibonacci #

Can show: $A^n \overset{\nearrow}{=} A^{n-1} + A^{n-2}$

induction / diagonalization

* Rough idea of Thm 2: Let $A = (a_{ij})$

The $(i,j)^{th}$ entry of A^2 is $\sum_{k=1}^{n} a_{ik} a_{kj}$

$a_{i1} a_{1j}$ = # paths $i \to 1 \to j$

$a_{i2} a_{2j}$ = # paths $i \to 2 \to j$, etc.

$\Rightarrow$ Sum = total # of paths of the form $i \to \square \to j$

i.e: # paths of length 2

Can show: The $(i,j)^{th}$ entry of A^3 is:

$\sum a_{ik} a_{k1} a_{k2} a_{k_2 j}$ $\leftarrow$ Each term represents # paths $i - k_1 - k_2 - j$

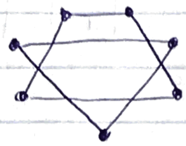
And so on! $(ij)^{th}$ entry of A^n is

$\underbrace{\sum a_{ik_1} a_{k_1 k_2} \cdots a_{(k-n-1)j}}_{\text{term = \# paths}}$

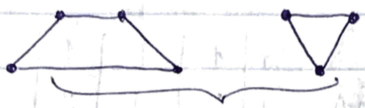
$i - k_1 - k_2 - \cdots - k_{n-1} - j$

* Def: Let G be an undirected graph. Then G is connected if there's a path from x to y for all vertices x, y

Ex: 
Graph is connected

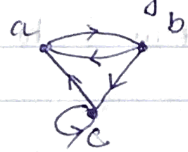
Ex: 

not connected
No path from a to b!

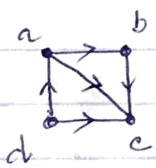
Redraw: 

2 connected components

* Def: Let G be a directed graph
Then G is strongly connected if there's a path from any vertex to any other vertex
G is weakly connected if the underlying (undirected) graph is connected

Ex: 

G is strongly connected

Ex: 

G is not strongly connected
There's no path from a to d

Ex. 
G is weakly connected
This graph is connected

Note: Strongly connected implies weakly connected

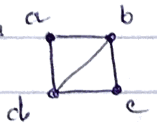
* Def: Let G be any graph.
A <u>circuit</u> is a path that starts and ends at the same vertex, and has at least 1 edge.

A path is simple if it doesn't contain the same edge more than once.

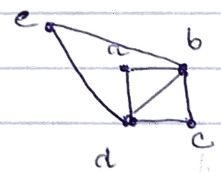
§ 10.5 Euler paths / circuits

* **Def:** Let G be a graph.
An Euler path/circuit in G is a simple path/circuit which contains every edge in G

Ex:



Euler path: dabdcb
No Euler circuit

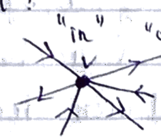


← Euler circuit
dabdcbed

* **Thm 1:** Let G be a connected, undirected graph.
Then G has an Euler circuit $\Leftrightarrow$ every vertex has even degree

Rough idea of $\Rightarrow$: Suppose G has an Euler circuit.
Trace this circuit!

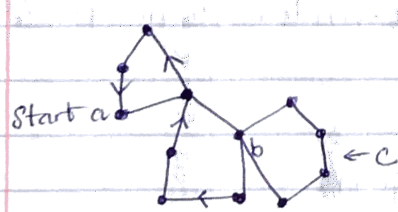
At vertex v:



→ Every edge "in" corresponds to an edge "out"
→ deg(v) must be even

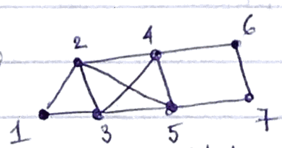
~~§10.5 Euler paths~~

Suppose every vertex has even degree
Algorithm for building an Euler circuit C
Start at any vertex, and add edges until not possible
Get a circuit C w/o repetition



Remove C from G (delete all edges/isolated vertices)
+ Get a new graph G' where all vertices have even degree
+ Pick vertex b from C, and add edges from G' to get a circuit C'
+ "Glue" C and C' together. Call the new circuit C
+ Repeat until all edges are used

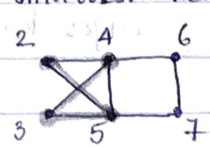
Ex:



☑ All degrees are even, so there is an Euler circuit

Note: We will pick smallest vertex possible

→ C = 1231
↑
insert



Remove C to get

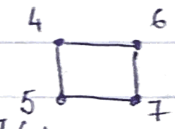
→ C' = 24352

Glue C and C'

$\rightarrow C = 1\,2\,4\,3\,5\,2\,3\,1$

insert C'

Remove C':



New $C' = \underline{4}\,5\,7\,6\,\underline{4}$

Glue C and C' together

$C = 1\,2\,\underline{4\,5\,7\,6\,4}\,3\,5\,2\,3\,1$

insert C' into C

* **Thm 2:** Let G be a connected, undirected graph.

G has an Euler path but not an Euler circuit

$\Leftrightarrow$ it has exactly two vertices of odd degree

Idea: Suppose G has two vertices of odd degree, say a, b

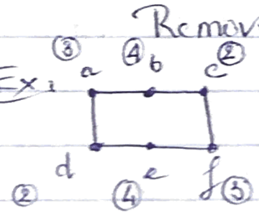
Add degree $\{a, b\}$

In the new graph H, every vertex has even degree

So H has an Euler circuit

Remove $\{a, b\}$ from the circuit to get an Euler path

Ex:



By Thm 2, there's an Euler path but no Euler circuit

a d e b e f b c f

$\rightarrow$ Add edge $\{a, f\}$

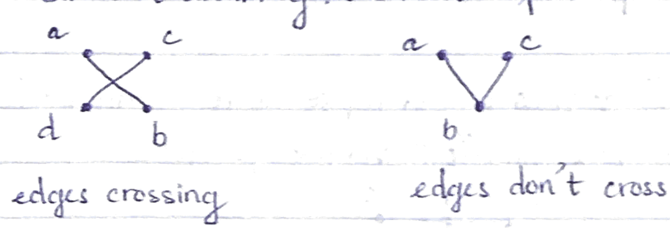
$\rightarrow$ Then the path can be extended to an Euler path

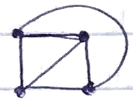
§10.7 Planar Graphs

Assume G is a connected simple undirected graph

* **Def:** G is called planar if it can be drawn in the plane without any edges crossing

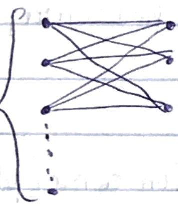
Such a drawing is called a planar representation

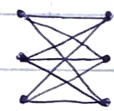


Ex: K_4  Redraw: 

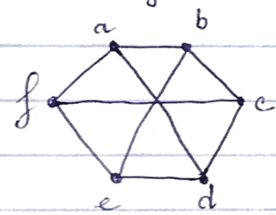
planar representation of K_4

✱ Note (for 10.2) $K_{m,n}$ is a complete bipartite graph
Connect all "left" vertices with all "righ" vertices



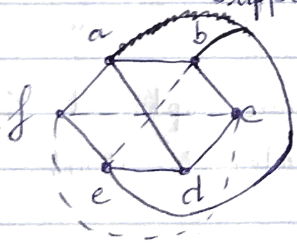
$K_{3,3}$  $3 \times 3 = 9$ edges

✱ FACT: $K_{3,3}$ is not planar
Rough idea: Try to draw the planar rep. of $K_{3,3}$



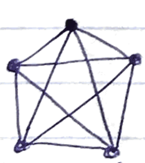
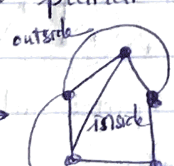
redraw of $K_{3,3}$
Note: abcdefa is a circuit
In any redrawing of $K_{3,3}$, this circuit will split the plane into 2 regions ("inside/outside" regions)

Suppose $\{a, d\}$ is "inside"



$\{b, e\}$ must be "outside"
Nowhere for $\{c, f\}$ to go

✱ FACT: K_5 is not planar

 ⟶ 

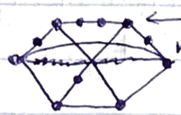
You check: Any proper subgraph of $K_{3,3}$ and K_5 is planar
not equal to graph itself

✱ FACT: If H is a subgraph of G, and H is not planar, then G is also <u>not</u> planar

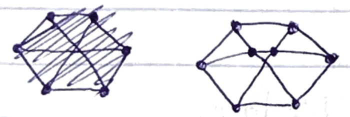
* (Rough) def: A graph H is homeomorphic to $K_{3,3}$ if H can be obtained from $K_{3,3}$ by "adding" vertices on edges at noncrossing points

(Similar def. for "homeomorphic to K_5")

Ex:

not planar: ← H_1 =  H_1 is homeomorphic to $K_{3,3}$

If there were a planar rep. for H_1, then there would be also a planar rep. for $K_{3,3}$

H_2 =  not homeomorphic to $K_{3,3}$ (vertices added at "crossing points")

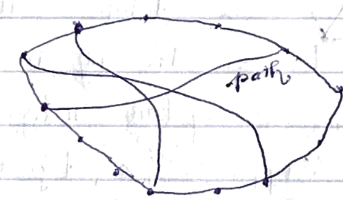
* Thm 2: (Kuratowski's Thm)

A graph is non planar $\Leftrightarrow$ it contains a subgraph homeomorphic to $K_{3,3}$ or K

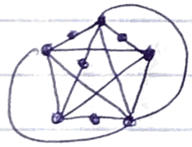
Idea of pf. of $\Leftarrow$: ~~Such~~ Same as above

$\Rightarrow$ To find a subgraph homeomorphic to $K_{3,3}$:

→ find a simple circuit:

homeomorphic to $K_{3,3}$ →  → no common vertices below the 3 paths

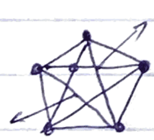
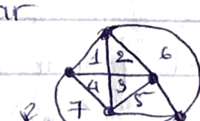
Ex:



→ contains a subgraph homeomorphic to K_5

Graph is not planar by Kuratowski's Thm!

Ex:

 → Planar 

* Note: A planar rep. divides the plane into regions (includes an oute unbounded region)

eg: 8 regions above

Thm 1. (Euler's Formula)

Let G be a connected planar simple graph with e edges and v vertices.

Let r be the # of regions in a planar rep. of G

Then: $\boxed{r = e - v + 2}$

Ex. above

$r = 8$

$e = 12 \quad \Rightarrow \quad 8 \overset{?}{=} 12 - 6 + 2$ ✓

$v = 6$

Ex: Suppose a planar graph has 10 vertices of degree 3

[?] How many regions are there in any planar rep. of G?

Soln: Know $v = 10$. Want $r = ?$

By Handshaking Thm, the # of edges is half of sum of degrees, which is $10 \times 3 = 30$

So $e = \frac{30}{2} = 15$

By Euler's formula, $r = 15 - 10 + 2 = 7$ regions

* (Rough) fact: $\boxed{G \text{ planar} \Rightarrow e \leq 3v - 6}$

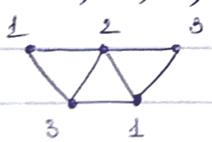
§10.8 Graph coloring

Review of def.: Let G be a simple, undirected graph

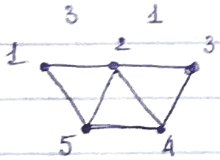
A coloring of G is an assignment of colors to the vertices of G so that no two adjacent vertices receive the same color

Colors: 1, 2, 3, ...

Ex:



→ valid coloring using 3 colors!

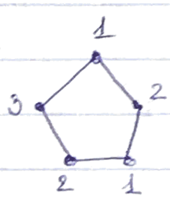


→ coloring using 5 colors

* Def: The chromatic number of G is the minimum # of colors needed for a coloring of G

Notation: $\chi(G)$

Ex. C_5

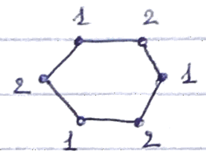


Try to color w/ 1, 2!

Doesn't work

$\chi(C_5) = 3$

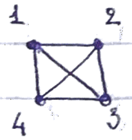
Ex: C_6



$\chi(C_6) = 2$

$*$ FACT: $\chi(C_n) = \begin{cases} 3 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$

Ex: K_4



4 is the min # colors needed b/c all vertices must receive diff-colors

$\chi(K_4) = 4$

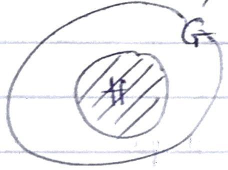
$*$ FACT: $\chi(K_n) = n$

$*$ FACT: Let H be a subgraph of G. Then

$\chi(H) \leq \chi(G)$

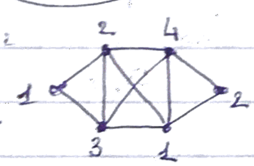
Why? Let $k = \chi(H)$

To color H, we must use at least k colors



G must use at least k colors as well

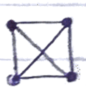
Ex:



G

This is a valid coloring

So $\chi(G) \leq 4$

$K_4 =$  is a subgraph so

$4 = \chi(K_4) \leq \chi(G)$

FACT

Therefore, $\chi(G) = 4$

$*$ $\chi(G)$ = min # of colors needed to color G

↳ chromatic number of G

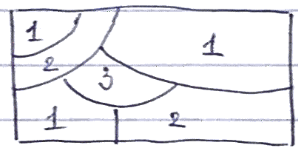
Note: To show $\chi(G) = k$, you must show:

$\chi(G) \leq k$ + There's a valid coloring of G w/ exactly k colors

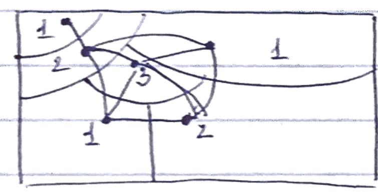
$\chi(G) \geq k$ + G can't be colored using fewer than k colors

Map coloring:

Map: Collection of regions



- A coloring of map is an assignment of colors to each region so that regions that share a border have different colors
- Coloring the map is equivalent to coloring the dual graph
- Dual graph of map:
  - Vertices: Regions
  - Edges: Connect two vertices if their corresponding regions share a border



Dual graph

* Note: The dual graph is always planar

Four Color Thm: Any planar graph can be colored using at most 4 colors

App. to scheduling exams:

Problem: Find the min # of time slots needed to schedule final exams so that there are no time conflicts

→ Model using graph G:

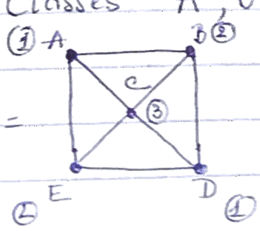
- Vertices: classes
- Edges: connect two classes if there's a student in both classes

Note: HW 10.2 (Intersection graphs)

Then, Any coloring of G is a way of assigning time slots to the classes so that there's no time conflicts.

$\chi(G)$ = min # of time slots required

Ex: Classes A, B, C, D, E

G = 

This is a valid coloring of G, so

$\chi(G) \leq 3$

Also, $K_3 = \triangle$ is a subgraph of G, so

$3 = \chi(K_3) \leq \chi(G)$

So $\chi(G) = 3$

Time slots	Classes
1	A, D
2	B, E
3	C