Section 3.1: Introduction to Determinants:

Recall: A 2×2 matrix is invertible IFF its determinant $\neq 0$.

$\Rightarrow$ To extend this useful fact to larger matrices, we must establish a definition for the determinant of an $n \times n$ matrix :

Intro to Determinants of Larger Matrices

In general, an $n \times n$ determinant is defined by the determinants of $(n-1)(n-1)$ submatrices.

Definition:

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det(A_{1j})$, with plus & minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are the first row of A.

*Recursive Def. of the Determinant:

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1} a_{1n} \det(A_{1n})$$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j})$$

Note:
For any square matrix, let A_{ij} denote the submatrix formed by deleting the i^{th} row & j^{th} column of A.

Example: For the following 4×4 matrix, find the submatrix A_{32} & compute its determinant:

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

Answer:

* **Note:** The submatrix "A_{32}" is formed by deleting the 3rd-Row & 2nd-Column of A.

* **Find the Submatrix, A_{32}:**

$$A = \begin{array}{c c} & \text{2nd Column} \\ & \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \\ \text{3rd Row} & \end{array} \Longrightarrow \boxed{\therefore A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}}$$

Ans.

* **Note:** To generalize the def. of the determinant of larger matrices we use 2×2 determinants to rewrite the 3×3 determinant

$$\det(A_{32}) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$$

where: A_{11}, A_{12}, & A_{13} are obtained from A_{32} by deleting the first row & one of the 3 columns

Example Continued...

*Compute the determinant of the submatrix, A_{32}:

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

*Det(A) st A is 3x3:

$$\det(A_{32}) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$$

$$= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

So,

$$\det(A_{32}) = 1\det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5\det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0\det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0-2) - 5(0+0) + 0(-4+0)$$

$$= -2 + 0 + 0$$

$$= -2$$

$$\boxed{\therefore \det(A_{32}) = -2}$$

Ans.

Cofactor Expansion

Note: Conveniently, we can rewrite the definition of the det(A) as follows; which, leads us to our first theorem :)

Cofactor Expansion Across the First Row of A:

Given $A = [a_{ij}]$, the $(i,j)^{th}$ - Cofactor of A is the number C_{ij} given by:

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

*Theorem # 1:

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

(i) The Expansion across the i^{th} row:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(ii) The Expansion down the j^{th} Column:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Note:

he +/- sign in the $(i,j)^{th}$ - Cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} itself!

he Factor $(-1)^{i+j}$ determines the following pattern of signs →
$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & & & \ddots \end{bmatrix}$$

Example (Cofactor Expansion):

Use cofactor expansion across the 3^{rd} Row to compute $\det(A)$, where:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

*Fun/Helpful Note: Thm #1 is particularly helpful for computing the determinant of a matrix that contains many zeros, as it eliminates extra calculations :)

Answer:

Recall:

- Given $A = [a_{ij}]$, the $(i,j)^{th}$ - Cofactor of A is the number C_{ij} st: $C_{ij} = (-1)^{i+j} \det(A_{ij})$.
- Then the expansion across the i^{th} - Row using Cofactors is: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

* A is a 3×3 matrix.

* Compute the determinant across the 3^{rd} Row:

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= a_{31}\overset{1}{\cancel{(-1)^{3+1}}}\det(A_{31}) + a_{32}\overset{-1}{\cancel{(-1)^{3+2}}}\det(A_{32}) + a_{33}\overset{1}{\cancel{(-1)^{3+3}}}\det(A_{33})$$

$$= 0\det\begin{bmatrix} 5 & 0 \\ 4 & -1 \end{bmatrix} - 2(-1)\det\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + 0\det\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = 2(-1-0) = -2$$

$$\boxed{\therefore \det(A) = -2}$$

Ans ✓

Example (Cofactor Expansion): Continued...

Note: Using cofactor expansion across the 3rd row is NOT an exclusive solution! Any row &/or column will work ∵

*Compute the determinant across the first row:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1}\det(A_{11}) + a_{12}(-1)^{1+2}\det(A_{12}) + a_{13}(-1)^{1+3}\det(A_{13})$$

$$= 1(1)\det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} + 5(-1)\det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0(1)\det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0-2) + 0 + 0$$

$$= \boxed{-2} \checkmark$$

*Compute the determinant across the 3rd Column:

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

$$= a_{13}(-1)^{4}\det(A_{13}) + a_{23}(-1)^{5}\det(A_{23}) + a_{33}(-1)^{6}\det(A_{33})$$

$$= 0(1)\det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} + (-1)(-1)\det\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} + 0(1)\det\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$$

$$= 0 + (-2-0) + 0$$

$$= \boxed{-2} \checkmark$$

*Again, cofactor expansion is helpful in computing the determinant of a matrix containing many zeros, as the cofactors of these terms need NOT be calculated ∵

Example: Compute the determinant using a cofactor expansion:

(a) Across the 1^{st} Row.

(b) Down the 2^{nd} Column.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 2 & 4 & 2 \\ 0 & 4 & -1 \end{bmatrix}$$

Answer:

*Recall: Given $A = [a_{ij}]$, the $(i,j)^{th}$ - Cofactor Expansion of A is the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Then: $\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

*Part (a): Cofactor-Expansion across Row 1

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 2(-1)^2 \det\begin{bmatrix} 4 & 2 \\ 4 & -1 \end{bmatrix} + \cancel{0(-1)^3 \det\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}} + 3(-1)^4 \det\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}$$

0

$$= 2(-4-8) + 0 + 3(8-0)$$

$$= 2(-12) + 3(8)$$

$$= -24 + 24$$

$$= 0$$

$$\boxed{\therefore \det(A) = 0}$$

Answer

Example Continued...

*Part (b): Cofactor Expansion Down Column # 2:

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= 0(-1)^3 \det\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} + 4(-1)^4 \det\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + 4(-1)^5 \det\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$$

$$= 0 + 4(-2+0) - 4(4-6)$$

$$= -8 + 8$$

$$= 0$$

$$\boxed{\therefore \det(A) = 0}$$ Ans.

Example: Compute the determinant using a cofactor expansion, (a) Across the 1st Row
(b) Down the 2nd Column

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 3 & 1 & 3 \\ 1 & 5 & -1 \end{bmatrix}$$

Answer:

***Part (a): Across the 1st Row:**

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 2(-1)^2 \det\begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} - 3(-1)^3 \det\begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} + 3(-1)^4 \det\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$$

$$= 2(-1-15) + 3(-3-3) + 3(15-1)$$

$$= 2(-16) + 3(-6) + 3(14)$$

$$= -32 - 18 + 42$$

$$= -8$$

$$\boxed{\therefore \det(A) = -8}$$ Ans.

Part (b): Down the 2nd Column:

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= -3(-1)^3 \det\begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} + 1(-1)^4 \det\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} + 5(-1)^5 \det\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$$

$$= 3(-3-3) + (-2-3) - 5(6-9)$$

$$= 3(-6) - 5 - 5(-3) = -18 - 5 + 15 = -8 \checkmark$$

$$\boxed{\therefore \det(A) = -8}$$ Ans.

Example: Compute the determinant of the following matrix using a cofactor expansion across the 1^{st} Row:

$$A = \begin{bmatrix} 3 & 6 & -5 \\ 5 & 0 & 4 \\ 4 & 5 & 2 \end{bmatrix}$$

Answer:

*Since A is a 3×3, the cofactor expansion across Row 1 gives us:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 3(-1)^2 \det\begin{bmatrix} 0 & 4 \\ 5 & 2 \end{bmatrix} + 6(-1)^3 \det\begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} - 5(-1)^4 \det\begin{bmatrix} 5 & 0 \\ 4 & 5 \end{bmatrix}$$

$$= 3(0-20) - 6(10-16) - 5(25-0)$$

$$= 3(-20) - 6(-6) - 5(25)$$

$$= -60 + 36 - 125$$

$$= -149$$

$$\boxed{\therefore \det(A) = -149}$$

Ans.

Example: Compute the determinant using a cofactor expansion down the 1st Column:

$$A = \begin{bmatrix} 4 & -5 & 2 \\ 8 & 1 & 3 \\ 0 & 4 & -2 \end{bmatrix}$$

Answer:

*Since A is 3x3, the Cofactor Expansion down the 1st column is:

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 4(-1)^2 \det\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} + 8(-1)^3 \det\begin{bmatrix} -5 & 2 \\ 4 & -2 \end{bmatrix} + 0(-1)^4 \det\begin{bmatrix} -5 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= 4(-2-12) - 8(10-8) + 0$$

$$= 4(-14) - 8(2)$$

$$= -56 - 16$$

$$= -72$$

$$\boxed{\therefore \det(A) = -72}$$

Ans.

Example (Cofactor Expansion of a 5×5 matrix):

Compute the determinant of A where:

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Answer:

Note: While any row/column can be used in the cofactor expansion using the one containing the most zeros produces the easiest calculations ⋮

* Column 1 contains all zeros except entry 1 (Row 5 would also be a good choice)

* Use Cofactor Expansion across the 1st Column:

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} + a_{51}C_{51}$$

$$= 3C_{11} + \cancel{0C_{21}}^{0} + \cancel{0C_{31}}^{0} + \cancel{0C_{41}}^{0} + \cancel{0C_{51}}^{0}$$

$$= 3(-1)^{1+1}\det(A_{11})$$

$$= 3(1)\det\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

* Note: So we have reduced the 5×5 matrix to a 4×4

⇒ We need to apply cofactor expansion again ⋮

Example (Cofactor Expansion w/ a 5×5) Continued...

Thus Far:

$$\det(A) = 3\det\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

<u>Note:</u> Again, cofactor expansion of <u>any</u> row/column will work, <u>BUT</u> the one containing the most zeros is easiest.

*<u>Lets use Cofactor Expansion Across the 1st Column again:</u>

$$\det(A) = 3\left[a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}\right]$$

$$= 3\left[2C_{11} + 0\cancel{C_{21}}^{0} + 0\cancel{C_{31}}^{0} + 0\cancel{C_{41}}^{0}\right]$$

$$= 6(-1)^2\det\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$= 6\left\{1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}\right\}$$

$$= 6\left\{1(0-2) - 5(0+0) + 0(-4+0)\right\}$$

$$= 6(-2)$$

$$= -12$$

$$\boxed{\therefore \det(A) = -12}$$

Ans.

Example: Compute the determinant by Cofactor Expansion. At each step, choose a row or column that involves the least amount of computation:

$$A = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 9 & 8 & 3 & -7 \\ 3 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 \end{bmatrix}$$

Answer:

Note: Row 3 contains only one nonzero entry!

*Compute the det(A) by Cofactor Expansion Across the 3rd Row:

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$$

$$= 3(-1)^4 \det\begin{bmatrix} 0 & 0 & 4 \\ 8 & 3 & -7 \\ 2 & 1 & 1 \end{bmatrix} + 0\cancel{C_{32}}^{0} + 0\cancel{C_{33}}^{0} + 0\cancel{C_{34}}^{0}$$

$$= 3\det\begin{bmatrix} 0 & 0 & 4 \\ 8 & 3 & -7 \\ 2 & 1 & 1 \end{bmatrix}$$

* **Note:** Since Row 1 contains only 1 nonzero entry, perform the 2nd Cofactor Expansion Across 1st Row

$$= 3\left(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\right)$$

$$= 3\left(0\cancel{C_{11}}^{0} + 0\cancel{C_{12}}^{0} + 4(-1)^4 \det\begin{bmatrix} 8 & 3 \\ 2 & 1 \end{bmatrix}\right)$$

$$= 3\left[4(8-6)\right] = 3(8) = 24 \checkmark$$

$$\boxed{\therefore \det(A) = 24}$$

Example: Compute the following determinant by Cofactor Expansion. At each step, choose the row or column that involves the least amount of computation:

$$A = \begin{bmatrix} 2 & -3 & 5 & 1 \\ 0 & 0 & 3 & 0 \\ 3 & -5 & -9 & 6 \\ 4 & 0 & 4 & 2 \end{bmatrix}$$

Answer:

Note: Row 2 has only 1 nonzero entry :)

*Compute det(A) using Cofactor Expansion Across the 2nd Row:

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + C_{24}C_{24}$$

$$= 0C_{21} + 0C_{22} + 3(-1)^5 \det \begin{bmatrix} 2 & -3 & 1 \\ 3 & -5 & 6 \\ 4 & 0 & 2 \end{bmatrix} + 0C_{24}$$

$$= -3\det \begin{bmatrix} 2 & -3 & 1 \\ 3 & -5 & 6 \\ 4 & 0 & 2 \end{bmatrix}$$

* Note: Since Row 3 has a zero, compute the next cofactor expansion across the 3rd Row

$$= -3\left(a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \right)$$

$$= -3\left(4(-1)^4 \det \begin{bmatrix} -3 & 1 \\ -5 & 6 \end{bmatrix} + 0 + 2(-1)^6 \det \begin{bmatrix} 2 & -3 \\ 3 & -5 \end{bmatrix} \right)$$

$$= -3\left[4(\underbrace{-18+5}_{-13}) + 2(\underbrace{-10+9}_{-1}) \right]$$

$$= -3[52-2] = -3(-54) = 162 ✓$$

$$\boxed{\therefore \det(A) = 162}$$

Ans✓

Example: Compute the determinant by Cofactor Expansion. At each step, choose a row or column that involves the least amount of computation:

$$A = \begin{bmatrix} 8 & 2 & 3 & 4 & 0 \\ 4 & 0 & -4 & 1 & 0 \\ 9 & -5 & 7 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 2 & 0 \end{bmatrix}$$

Answer:

Note: There are 2 viable options here i) Across Row 4 / ii) Down Column 5

↳ Other options ∃, but these 2 are the easiest to start with

*Compute the det(A) by Cofactor Expansion Down Column 5:

$$\det(A) = a_{15}C_{15} + a_{25}C_{25} + a_{35}C_{35} + a_{45}C_{45} + a_{55}C_{55}$$

$$= 0\cancel{C_{15}} + 0\cancel{C_{25}} + 1\cancel{(-1)}^{8}\det\begin{bmatrix} 8 & 2 & 3 & 4 \\ 4 & 0 & -4 & 1 \\ 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 2 \end{bmatrix} + 0\cancel{C_{55}}$$

$$= \det\begin{bmatrix} 8 & 2 & 3 & 4 \\ 4 & 0 & -4 & 1 \\ 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 2 \end{bmatrix}$$ *Row 3 contains the most zeros :)

:Compute the next cofactor expansion across Row 3:

$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$$

$$= 3(-1)^4\det\begin{bmatrix} 2 & 3 & 4 \\ 0 & -4 & 1 \\ 2 & 4 & 2 \end{bmatrix} + 0\cancel{C_{32}} + 0\cancel{C_{33}} + 0\cancel{C_{34}}$$

Example Continued...

$$\det(A) = 3\det\begin{bmatrix} 2 & 3 & 4 \\ 0 & -4 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

$$= 3\left(2\det\begin{vmatrix} -4 & 1 \\ 4 & 2 \end{vmatrix} - 3\det\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} + 4\det\begin{vmatrix} 0 & -4 \\ 2 & 4 \end{vmatrix} \right)$$

$$= 3\left[2(-8-4) - 3(0-2) + 4(0+8) \right]$$

$$= 3\left[2(-12) - 3(-2) + 4(8) \right]$$

$$= 3\left[-24 + 6 + 32 \right]$$

$$= 3(14)$$

$$= 42 \checkmark$$

$$\boxed{\therefore \det(A) = 42}$$

Ans.

*Theorem #2:

IF A is a triangular matrix, then the $\det(A)$ is the product of the entries on the main diagonal of A.

*Additional Notes on Cofactor Expansion:

- Cofactor Expansion works well for on matrices containing entire rows/columns of zeros (only)

*Caution: Most cofactor expansion is NOT evaluated so quickly!

→ In General: For an $n \times n$ matrix, cofactor expansion requires more than $n!$ multiplications

*Perspective: A computer performs 1 trillion multiplications per second... So, it would take 500,000 years to compute the determinant of a 25×25 matrix using cofactor expansion!!
(PS. A 25×25 matrix is considered small)

⇒ Fortunately other methods ∃ as we will soon see :)

Example: The expansion of a 3×3 determinant can be remembered with the following device, "Write a 2^{nd} copy of the 1^{st} 2 Columns to the right of the matrix, & compute the determinant by multiplying entries on the 6 diagonals. Add the downward products & subtract the upward products."

Use this ↑ method to compute the following determinant.

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \quad ; \quad A = \begin{bmatrix} 3 & -5 & 2 \\ -2 & -5 & 0 \\ 0 & -4 & 2 \end{bmatrix}$$

$- \quad - \quad -$

$+ \quad + \quad +$

Answer:

$$A = \left[\begin{array}{ccc} 3 & -5 & 2 \\ -2 & -5 & 0 \\ 0 & -4 & 2 \end{array}\right] \begin{array}{cc} 3 & -5 \\ -2 & -5 \\ 0 & -4 \end{array}$$

Upward: $-0 \qquad -0 \qquad -(2)(-2)(-5) = -20$

Downward: $+(3)(-5)(2) = -30 \qquad +0 \qquad +2(-2)(-4) = 16$

$$^{*}\det(A) = \left(\begin{array}{c}\text{sum of downward} \\ \text{products}\end{array}\right) - \left(\begin{array}{c}\text{sum of upward} \\ \text{product}\end{array}\right)$$

$$= (-30 + 0 + 16) - (0 + 0 + 20)$$

*Factored ⊖ out b/c we are subtracting ∵

$$= -14 - 20$$

$$= -34 \checkmark$$

$$\boxed{\therefore \det(A) = -34}$$

Ans.

Example: State the row operation performed below & describe how it affects the determinant:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Answer:

* Given:

$\cdot A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\cdot A \longmapsto B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

* Row operation Performed: $\boxed{R_1 \text{ \& } R_2 \text{ are interchanged}}$

Ans.

* How does this affect the determinant?

$\cdot A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow \det(A) = ab - cd$

$\cdot B = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \Longrightarrow \det(B) = cb - ad = -(ab - cd)$

$= -\det(A)$

$\boxed{\therefore \det(B) = -\det(A) \text{ ; Sign is changed.}}$

Ans.

Example: State the row operation performed below & describe how it affects the determinant:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ; \quad B = \begin{bmatrix} a & b \\ 5c & 5d \end{bmatrix}$$

Answer:

* **Describe the Elementary Row Operation:**

  - Transformation: $\boxed{R_2 \mapsto 5R_2}$
  - Description: $\boxed{\text{Scale Row 2 by a factor of } 5}$
  - Elementary Matrix: $\boxed{E = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}}$

*Check: $EA \stackrel{?}{=} B$

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+0 & b+0 \\ 0+5c & 0+5d \end{bmatrix} = \begin{bmatrix} a & b \\ 5c & 5d \end{bmatrix} \checkmark$$

* **Describe how this affects the determinant:**

  - $\det(A) = ad - bc$
  - $\det(B) = 5ad - 5bc = 5(ad-bc) = 5\det(A)$

$\boxed{\therefore \det(B) = 5\det(A)}$; The determinant is scaled by a factor of 5.

Example: State the row operation below & describe how it affects the determinant:

$$A = \begin{bmatrix} 4 & 5 \\ 8 & 9 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 4 & 5 \\ 8+4K & 9+5K \end{bmatrix}$$

Answer:

* **Describe the elementary row operation:**

- **Transformation:** $\boxed{R_2 \longmapsto KR_1 + R_2}$
- **Description:** Scale Row 1 by a factor of K & add to Row 2.
- **Elementary Matrix:** $\boxed{E = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}}$

* **Check:** $EA \stackrel{?}{=} B$

$$\begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 4+0 & 5+0 \\ 4K+8 & 5K+9 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 4K+8 & 5K+9 \end{bmatrix} \checkmark$$

* **Describe how this affects the Determinant:**

- $\det(A) = 4(9) - 5(8) = 36 - 40 = -4$
- $\det(B) = 4(9+5K) - 5(8+4K) = 36 + 20K - 40 - 20K = -4 = \det(A)$

$\therefore \det(B) = \det(A)$; The determinant is the same (unchanged.)

Example: State the elementary row operation & describe how it affects the determinant:

$$A = \begin{bmatrix} -2 & 5 & -3 \\ 1 & 1 & 1 \\ 3 & -3 & 3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} -2 & 5 & -3 \\ K & K & K \\ 3 & -3 & 3 \end{bmatrix}$$

Answer:

Note: Row 2 is being scaled by a factor of K

* Transformation: $\boxed{R_2 \mapsto KR_2}$
* Elementary Matrix: $\boxed{E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

$\therefore$ Elementary Row Operation: Replace R_2 with KR_2 Ans

*Describe how this affects the determinant:

- $\det(A) = (-2)\det\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} - (5)\det\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + (-3)\det\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix}$

$$= -2(3+3) - 5(3-3) - 3(-3-3)$$
$$= -2(6) - 5(0) - 3(-6)$$
$$= -12 + 18$$
$$= 6$$

- $\det(B) = (-2)\det\begin{vmatrix} K & K \\ -3 & 3 \end{vmatrix} - (5)\det\begin{vmatrix} K & K \\ 3 & 3 \end{vmatrix} + (-3)\det\begin{vmatrix} K & K \\ 3 & -3 \end{vmatrix}$

$$= -2K(6) - 5K(0) - 3K(-6)$$
$$= 6K$$
$$= K\det(A)$$

$\therefore \det(B) = K\det(A)$; The determinant is scaled by a factor of K.

Ans.

Example: Compute the determinant of the following elementary matrix:

$$A = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

Recall (Thm #2): If matrix A is a triangular matrix, then the $\det(A)$ = the product of the entries along the main diagonal.

* Since A is a triangular matrix, then:

$\therefore$ $\boxed{\det(A) = (1)(1)(1) = 1}$ Ans.

* We observed in a previous ex. that the elementary row operation of "Combining" does not affect the determinant :)

Example: Compute the determinant of the following elementary matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

Note: Matrix A is NOT triangular $\Rightarrow$ Need to compute $\det(A)$ using the Def. &/or Cofactor expansion :)

*Compute the Determinant:

$$\det(A) = 0\det\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - (1)\det\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\det\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= 0 - 1(1-0) + 0$$

$$= -1$$

$$\boxed{\therefore \det(A) = -1}$$ Answer.

*The elementary row operation of 'Scaling' changes the sign of the determinant (as seen in previous example :)).

*General Conclusions: Elementary Row Operations & the Determinant:

① Interchanging:

When two rows are interchanged, the sign of the determinant changes (+/-).

② Scaling:

When a row is scaled by a factor of "K", then the determinant is scaled by a factor of "K" (where "K" is any scalar).

③ Combining:

When two rows are combined, the determinant does NOT change.