Finding the (x,y)-path of steepest descent

EXAMPLE #1

Consider the upper sheet of a 2-sheet hyperboloid $z = \left(\frac{1}{4}x^2 + \frac{1}{9}y^2 + 1\right)^{1/2}$ starting @ $P(2, 3, \sqrt{3})$

For any point (x, y, z) on this surface $-\vec{\nabla} z$ points in the direction of steepest descent.

$$-\vec{\nabla} z = \underbrace{\frac{-1}{2}\left(\frac{1}{4}x^2 + \frac{1}{9}y^2 + 1\right)^{-1/2}} \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle$$

so $\dfrac{dy}{dx} = \dfrac{\left(\frac{2y}{9}\right)}{\left(\frac{x}{2}\right)}$ since the common leading factor cancels

$$\Rightarrow \frac{dy}{dx} = \frac{4y}{9x} \quad \text{or} \quad \frac{dy}{y} = \frac{4}{9}\,\frac{dx}{x}$$

{separable differential equation

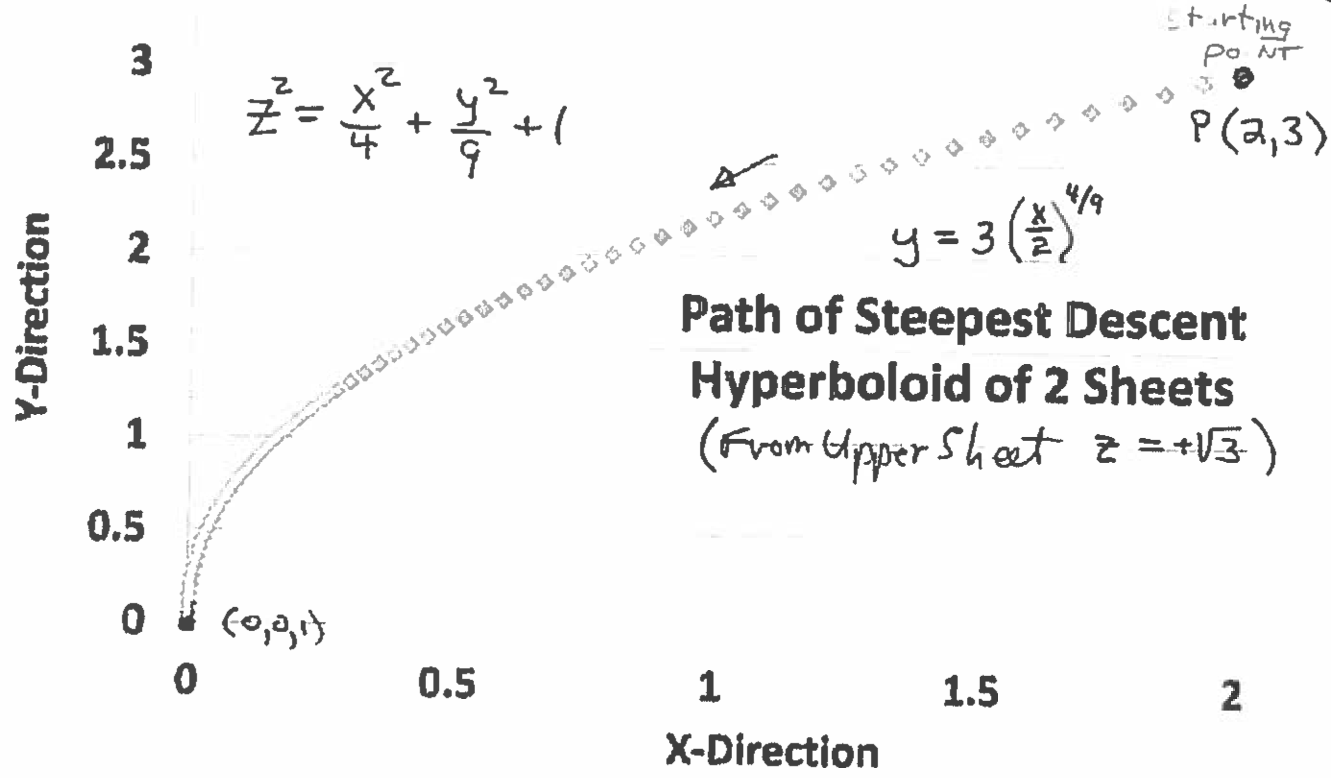
Integrating both sides, we get

$$\int \frac{dy}{y} = \frac{4}{9}\int \frac{dx}{x} \Rightarrow \ln(|y|) = \frac{4}{9}\ln(|x|) + C$$

Letting $C = \ln(|K|)$ yields $\ln(|y|) = \ln\left(K x^{4/9}\right)$

$\Rightarrow y = Kx^{4/9}$ At $(2, 3, \sqrt{3})$ $K = 3/2^{4/9}$

so $\boxed{y = 3\left(\frac{x}{2}\right)^{4/9}}$ Path of steepest descent



EXAMPLE TWO

Steepest Descent Example #2

$f(x,y) = \ln(x^2+y^3)$ $\quad P(3,2,\ln(17))$ starting point

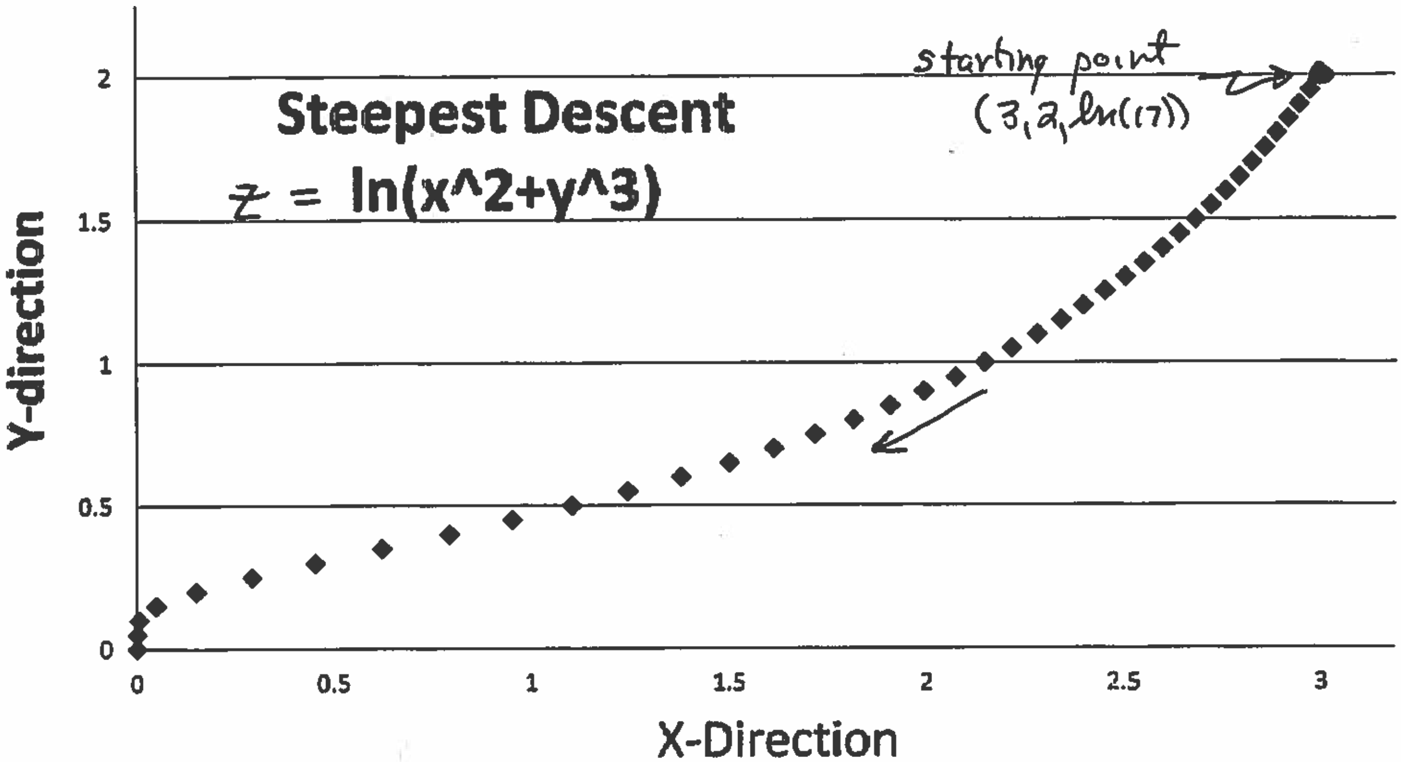
$\vec{\nabla} f = \frac{1}{x^2+y^3}\langle 2x, 3y^2\rangle$ easiest way

FOR Level Curve $\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x}{3y^2} \Rightarrow \perp$ slope $= \boxed{m_{\text{steepest}} = \frac{3y^2}{2x}}$

$$\left(\frac{dy}{dx}\right)_{\text{STEEPEST}} = \frac{3y^2}{2x} \Rightarrow \int \frac{dy}{y^2} = \frac{3}{2}\int \frac{dx}{x} \Rightarrow -\frac{1}{y} = \frac{3}{2}\ln(x) + C$$

Plugging in $(3,2) \Rightarrow C = -\frac{1}{2}(1+3\ln(3))$

$$\Rightarrow y = \frac{-2}{3\ln(x) - 1 - 3\ln(3)} \quad \text{OR} \quad x = e^{\left[-\frac{2}{3y} + \frac{1}{3} + \ln(3)\right]}$$



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CONTOUR PLOT $f(x,y) = \ln(x^2+y^3)$