Exam 2 Review - Discrete Structures II - Fall 2018

Exam 2 is on Tuesday, November 20 in class and will cover sections 7.1-7.3 and sections 8.1-8.2. No notes, textbook, calculator, or other electronic devices allowed! We will go over any questions you have from this sheet in class on Thursday, November 15.

Extra Office Hours: Monday, November 19 from noon to 3 pm in 428L Olney.

Review Problems:

1. Be able to use the methodology of Section 7.1 to compute probability (i.e. use the $p(E) = |E|/|S|$ formula).
2. Review the homework from sections 7.1-7.3 and 8.1-8.2. (Note there aren't any problems from 8.1 on this sheet, so you should definitely review those problems!)
3. A coin is flipped seven times. What's the probability that
   (a) exactly three tails occurs?
   (b) at most three heads occurs?
   (c) at most five heads occurs?
   (d) three heads occurs given that the first flip was heads?
4. Same question as the previous question, but assume the coin is biased so that probability of heads is 1/3. (You can leave your answers as a sum/difference/product/quotient of numbers.)
5. In a certain lottery, a subset of 4 different numbers is picked from the set $\{1, 2, \ldots, 10\}$ at random. A person wins \$100 if they pick all four numbers correctly, \$25 if they pick three out of four numbers correctly.
   (a) What's the probability of winning \$100?
   (b) What's the probability of winning \$25?
   (c) What's the probability that a person picks a subset that does not contain any of the winning numbers?
6. An urn contains 3 red marbles, 2 yellow marbles, and 5 green marbles. Suppose you are blindfolded, and you pick three marbles from the urn, one at a time, without replacement. Let E_1 be the event that the first marble selected is red, let E_2 be the event that the second marble selected is green, and let E_3 be the event that the third marble selected is green.
   (a) Compute the probabilities: $p(E_2|E_1)$, $p(E_1)$, $p(E_2)$, $p(E_1 \cap E_2)$, $p(E_3 \mid E_1 \cap E_2)$, and $p(E_1 \cup E_2)$.
   (b) What's the probability that all three marbles are green?
   (c) What's the probability that all three marbles are different colors?
7. Suppose the same urn from the previous problem is used. Four marbles are picked at random from the urn one at a time, with replacement (i.e. each marble chosen is put back into the urn before the next marble is selected at random). Let E_1 be the event that the first marble selected is red, and let E_2 be the event that the second marble selected is green.
   (a) Compute the following probabilities: $p(E_2|E_1)$, $p(E_1)$, $p(E_2)$, $p(E_1 \cap E_2)$, and $p(E_1 \cup E_2)$.
   (b) What's the probability all of the marbles are green?
   (c) What's the probability that exactly two of the four choices are yellow?
8. Suppose that E, F are events in a sample space which satisfy $p(E \cup F) = 0.5$, $p(E) = 0.3$, and $p(F) = 0.4$. Compute $p(E \cap F)$, $p(E \mid F)$, and $p(E \cap \overline{F})$.
9. Suppose a die is loaded so that the probability of rolling an even number is twice as likely as rolling an odd number. Also, the probabilities of rolling the even numbers are all equal, and the probabilities of rolling the odd numbers are all equal. (The sides of the die are labeled with the numbers 1 through 6, as usual.)
   (a) Find the probability distribution for the die.
   (b) A pair of the loaded dice is thrown. What is the probability that
      i. the numbers 2 and 3 are rolled on the first and second die, respectively?
      ii. the sum of the dice is 10?
      iii. one of the dice is even and the other is odd?

iv. the first die is less than 5?

v. the sum is 9 given that the first die is less than 5?

10. Suppose that 2% of people who do not use opium test positive for opium, and 5% of opium users test negative for opium. Furthermore, suppose that 1% of people actually use opium.

    Suppose that a person is selected at random. Let N be the event that the person tests negative, and let D be the event that the person uses opium. Compute the following probabilities. (You can leave your answers as a sum, difference, product, and/or quotient of numbers.)

    (a) $p(\overline{D})$ (b) $p(D \cap N)$ (c) $p(\overline{N} \cap D)$ (d) $p(D|N)$ (e) $p(N|D)$ (f) $p(\overline{N}|D)$

    (g) $p(N)$ (h) the probability the person uses opium, given that it's known they tested positive.

    (i) the probability that the person uses opium *and* tested positive.

    (j) the probability that the person tested negative or uses opium.

    (*Hint:* Do the parts in the order that makes sense, not necessarily in the given order!)

11. What's the characteristic equation of the recurrence relation $a_n = 3a_{n-2} + 4a_{n-3} - 7a_{n-6}$?

12. In each part, prove that the given sequence is a solution to the given recurrence relation using the method of section 2.4. Then find the general solution for the recurrence relation.

    (a) Sequence: $a_n = 4^{n+2}$
    Recurrence: $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$

    (b) Sequence: $a_n = 3n + 5$
    Recurrence: $a_n = 2a_{n-1} - 3n + 1$

13. Solve the following recurrence relations.

    (a) $a_n = 9a_{n-1}$, $a_0 = 18$

    (b) $a_n = 9a_{n-2}$, $a_0 = 2$, $a_1 = -3$

    (c) $a_n = a_{n-1} + 6a_{n-2}$, $a_0 = 9$, $a_1 = 7$

    (d) $a_n = -4a_{n-1} - 4a_{n-2}$, $a_0 = 5$, $a_1 = -6$

14. Suppose that the characteristic equation of a linear homogenous recurrence relation with constant coefficients for $\{a_n\}$ can be factored as $(x^2 - 9)^2(x^2 + x - 12)^3 = 0$. Find the general solution for the recurrence.

Answers:

3. (a) 35/128
   (b) 1/2
   (c) 15/16
   (d) 15/64

4. (a) $35 \cdot \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^4$
   (b) $\left(\frac{2}{3}\right)^7 + 7 \cdot \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right) + 21 \cdot \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^2 + 35 \cdot \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3$
   (c) $1 - 7 \cdot \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right) - \left(\frac{1}{3}\right)^7$
   (d) $15 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$

5. (a) 1/210
   (b) 4/35
   (c) 1/14

6. (a) $p(E_2|E_1) = 5/9$, $p(E_1) = 3/10$, $p(E_2) = 1/2$, $p(E_1 \cap E_2) = 1/6$, $p(E_3 \mid E_1 \cap E_2) = 1/2$, $p(E_1 \cup E_2) = 19/30$
   (b) 1/12
   (c) 1/4

7. (a) $p(E_2|E_1) = 1/2$, $p(E_1) = 3/10$, $p(E_2) = 1/2$, $p(E_1 \cap E_2) = 3/20$, $p(E_1 \cup E_2) = 13/20$
   (b) 1/16
   (c) 96/625

8. $p(E \cap F) = 0.2$, $p(E \mid F) = 0.2/0.4 = 0.5$,
   $p(E \cap \overline{F}) = 0.3 - 0.2 = 0.1$

9. (a) $p(1) = p(3) = p(5) = 1/9$,
   $p(2) = p(4) = p(6) = 2/9$
   (b) i. 2/81
   ii. 1/9
   iii. 4/9
   iv. 2/3
   v. 2/27

10. (a) 0.99
    (b) 0.0005
    (c) 0.0095
    (d) $\dfrac{(0.05)(0.01)}{(0.05)(0.01) + (0.98)(0.99)}$
    (e) 0.05
    (f) 0.95
    (g) $(0.05)(0.01) + (0.98)(0.99)$
    (h) $\dfrac{(0.95)(0.01)}{(0.95)(0.01) + (0.02)(0.99)}$
    (i) 0.0095
    (j) $p(N \cup D) =$
    $(0.05)(0.01) + (0.98)(0.99) + 0.01 - 0.0005$
    (using inclusion-exclusion with parts (b) and (g))

11. $x^6 - 3x^4 - 4x^3 + 7 = 0$ (*Note:* You can use r instead of x if you prefer the book's notation.)

12. (a) $a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$, where α, β are any real numbers
    (b) $a_n = 3n + 5 + \alpha \cdot 2^n$, where α is any real number.

13. (a) $a_n = 2 \cdot 9^{n+1}$
    (b) $a_n = \frac{1}{2} \cdot 3^n + \frac{3}{2} \cdot (-3)^n$
    (c) $a_n = 5 \cdot 3^n + 4(-2)^n$ or $a_n = 5 \cdot 3^n + (-2)^{n+2}$
    (d) $a_n = 5(-2)^n + n(-2)^{n+1}$

14. $a_n = (\alpha_0 + \alpha_1 n)(-3)^n + (\beta_0 + \beta_1 n + \beta_2 n^2 + \beta_3 n^3 + \beta_4 n^4)(3^n) + (\gamma_0 + \gamma_1 n + \gamma_2 n^2)(-4)^n$, where $\alpha_i, \beta_i, \gamma_i$ are any real numbers.