CHAPTER 3

BOOLEAN ALGEBRA

Basic Logical Operations

NOT $\qquad F = x' = \overline{x}$

Table 3.1 Truth table for NOT.

x	F
0	1
1	0

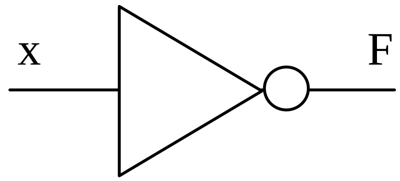


Figure 3.1 Logic symbol for inverter.

$F(x) = x'$

$(x')' = x$

F : logic function, Boolean function, switching function, or in short a function of x.
x : Boolean variable, switching variable, or in short, a variable.

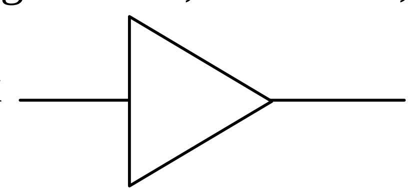


Figure 3.2 Logic symbol for buffer.

AND $F(x,y) = x \bullet y = xy$

Table 3.2 Truth table for AND.

x	y	F(x,y)
0	0	0
0	1	0
1	0	0
1	1	1

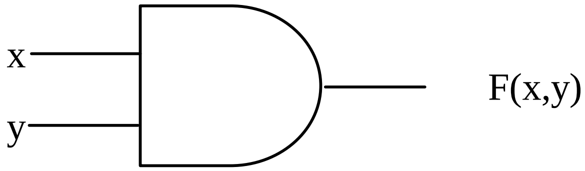


Figure 3.3 Logic symbol for AND gate.

$A \bullet A = A$

$A \bullet 1 = A$

$A \bullet 0 = 0$

$A \bullet A' = 0$

$A \bullet B = B \bullet A$

OR $\quad F(x,y) = x + y$

Table 3.3 Truth table for OR.

X	Y	F(x,y)
0	0	0
0	1	1
1	0	1
1	1	1

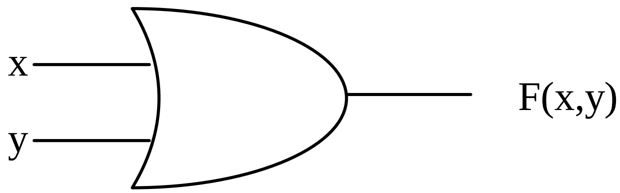


Figure 3.4 Logic symbol for OR.

$A + A = A$

$A + 0 = A$

$A + 1 = 1$

$A + A' = 1$

$A + B = B + A$

$$(A \cdot C) + (D \cdot ((A + B')' + C))$$

1 1 4 3 2 2 3 4

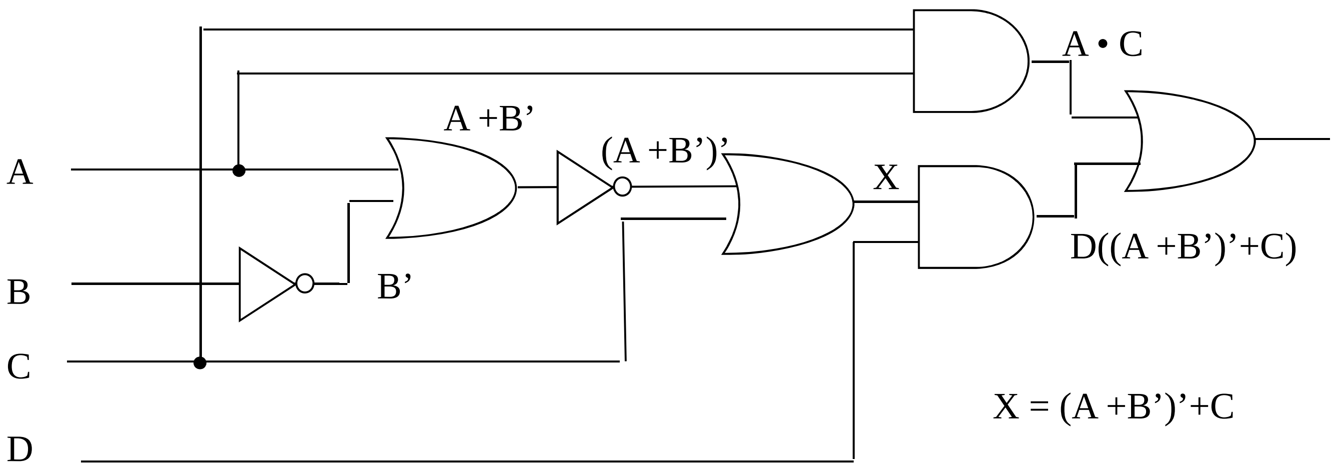


Figure 3.5 Logic circuit.

Basic Laws

(1) Involution law

$(A')' = A$

(2) Idempotency law

(a) $A \bullet A = A$

(b) $A + A = A$

(3) Laws of 0 and 1

(a) $A \bullet 1 = A$

(b) $A + 0 = A$

(a') $A \bullet 0 = 0$

(b') $A + 1 = 1$

(4) Complementary law

(a) $A \bullet A' = 0$

(b) $A + A' = 1$

(5) Commutative law

(a) $A \bullet B = B \bullet A$

(b) $A + B = B + A$

(6) Associative law

(a) $(A \bullet B) \bullet C = A \bullet (B \bullet C)$

(b) $(A + B) + C = A + (B + C)$

Table 3.4 Proof of associative law (6a)

A B C	A B	Left-hand-side of (6a) (A B) C	B C	Right-hand-side of (6a) A (B C)
0 0 0	0	0	0	0
0 0 1	0	0	0	0
0 1 0	0	0	0	0
0 1 1	0	0	1	0
1 0 0	0	0	0	0
1 0 1	0	0	0	0
1 1 0	1	0	0	0
1 1 1	1	1	1	1

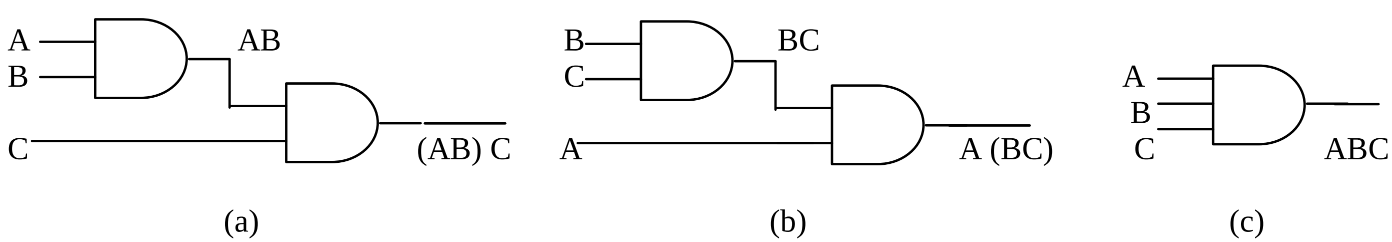


Figure 3.6 (a) Logic circuit for (AB) C. (b) Logic circuit for A (BC). (c) 3-input AND gate.

(6) Associative law

(a) $(A \bullet B) \bullet C = A \bullet (B \bullet C)$

(b) $(A + B) + C = A + (B + C)$

Table 3.4 Proof of associative law (6a)

A B C	A B	Left-hand-side of (6a) (A B) C	B C	Right-hand-side of (6a) A (B C)
0 0 0		0		0
0 0 1		AB = 0		0
0 1 0		0		0
0 1 1		AB = 0		0
1 0 0		0		BC = 0
1 0 1		AB = 0		BC = 0
1 1 0		0		BC = 0
1 1 1		AB = 1		BC =1

(7) Distributive law

(a) $A (B + C) = A B + A C$

(b) $A + B C = (A + B) (A + C)$

Table 3.5 Proof of distributive law (7a).

A B C	B + C	Left-hand-side of (7a) A (B + C)	A B	A C	Right-hand-side of (7a) AB + AC
0 0 0	0	0	0	0	0
0 0 1	1	0	0	0	0
0 1 0	1	0	0	0	0
0 1 1	1	0	0	0	0
1 0 0	0	0	0	0	0
1 0 1	1	1	0	1	1
1 1 0	1	1	1	0	1
1 1 1	1	1	1	1	1

Table 3.6 Proof of distributive law (7b).

A	Left-hand-side of (7b) $A + B C$	Right-hand-side of (7b) $(A + B)(A + C)$
0	$0 + B C = B C$	$(0 + B)(0 + C) = B C$
1	$1 + B C = 1$	$(1 + B)(1 + C) = 1 \bullet 1 = 1$

Distributive law (7a)

$$A\ (B + C) \xrightarrow[\text{Distribution}]{} A\,B + A\,C$$

$$A\ (B + C) \xleftarrow[\text{Collection}]{} A\,B + A\,C$$

Distributive law (7b)

$$A + B\,C \xrightarrow[\text{Distribution}]{} (A + B)(A + C)$$

$$A + B\,C \xleftarrow[\text{Collection}]{} (A + B)(A + C)$$

3.3 Sum-of-Products and Product-of-sums Expressions

Literal : a variable appears unprimed or primed in a switching expression.

Sum-of-product (SOP)

AB’ + BC + A’BD’

B’ + CD + A’C’D’ + AE’

Product-of-sums (POS)

(A’ + C’) (A + C + D’) (B + D’)

C’ (B’ + D’) (A + B + D)

(7) Distributive law

(a) $A(B + C) = A B + A C$

POS SOP

(b) $A + B C = (A + B)(A + C)$

SOP POS

Simplest (Minimal) Sum-of-Products and Product-of-Sums Expressions

When a literal or a product is deleted from a sum-of-products expression for a switching function, the expression with deleted literal/product is no longer correct for the function. Then the sum-of-products expression is said to be simplest or minimal. In other words, a sum-of-products expression is simplest if and only if no literal or product can be deleted from the expression. Thus a simplest sum-of-products expression for a function consists of a minimum number of product terms and the total number of literals in all the number of product terms is also a minimum.

❖ Example 3.1

Show that the sum-of-product expression (AB’ + BCD + A’B’D’) is not minimal and can be simplified by removing the literal A’ from the third product. In other words, show that the following equation is valid.

$$AB' + BCD + A'B'D' = AB' + BCD + B'D' \tag{3.1}$$

Table 3.7 Proof of Equation (3.1).

A B	Left-hand-side of Equation (3.1) AB’ + BCD + A’B’D’	Right-hand-side of Equation (3.1) AB’ + BCD + B’D’
0 0	0•0’ + 0•C•D + 0’•0’•D’ = D’	0•0’ + 0•C•D + 0’•D’ = D’
0 1	0•1’ + 1•C•D + 0’•1’•D’ = CD	0•1’ + 1•C•D + 1’•D’ = CD
1 0	1•0’ + 0•C•D + 1’•0’•D’ = 1	1•0’ + 0•C•D + 0’•D’ = 1 + D’ = 1
1 1	1•1’ + 1•C•D + 1’•1’•D’ = CD	1•1’ + 1•C•D + 1’•D’ = CD

❖ Example 3.2

Show that the sum-of-products expression (AB’ + BCD + B’D’) on the right-hand-side of Equation (3.1) is minimal.

Table 3.8 Proof for simplest sum-of-products expression.

A B C D	Right-hand-side of Equation (3.1)	Expression after removing either a literal or a product from right-hand-side of Equation (3.1)	Literal or product removed from right-hand-side of Equation (3.1)
0 0 0 1 0 0 1 1	AB’ + BCD + B’D’ = 0	B’ + BCD + B’D’ = 1	A in 1st product
1 1 0 0 1 1 0 1 1 1 1 0	AB’ + BCD + B’D’ = 0	A + BCD + B’D’ = 1	B’ in 1st product
0 0 1 1	AB’ + BCD + B’D’ = 0	AB’ + CD + B’D’ = 1	B in 2nd product
0 1 0 1 1 1 0 1	AB’ + BCD + B’D’ = 0	AB’ + BD + B’D’ = 1	C in 2nd product
0 1 1 0 1 1 1 0	AB’ + BCD + B’D’ = 0	AB’ + BC + B’D’ = 1	D in 2nd product
0 1 0 0 0 1 1 0 1 1 0 0 1 1 1 0	AB’ + BCD + B’D’ = 0	AB’ + BCD + D’ = 1	B’ in 3rd product
0 0 0 1 0 0 1 1	AB’ + BCD + B’D’ = 0	AB’ + BCD + B’ = 1	D’ in 3rd product
1 0 0 1 1 0 1 1	AB’ + BCD + B’D’ = 1	BCD + B’D’ = 0	Product AB’
0 1 1 1 1 1 1 1	AB’ + BCD + B’D’ = 1	AB’ + B’D’ = 0	Product BCD
0 0 0 0 0 0 1 0	AB’ + BCD + B’D’ = 1	AB’ + BCD = 0	Product B’D’

3.4 Theorems

(1) Combination theorem

(a) $A B + A B' = A$

(b) $(A + B)(A + B') = A$

Proof: (a)

$$
\begin{aligned}
\text{LHS} &= A B + A B' \\
&= A (B + B') \\
&= A \bullet 1 \\
&= A = \text{RHS}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\text{LHS} &= (A + B)(A + B') \\
&= A + B B' \\
&= A + 0 \\
&= A = \text{RHS}
\end{aligned}
$$

(2) Absorption theorem

(a) $A + A\,B = A$

(b) $A\,(A + B) = A$

Proof: (a) $LHS = A + A\,B$

$= A \bullet 1 + A\,B$

$= A\,(1 + B)$

$= A \bullet 1$

$= A = RHS$

(b) $LHS = A\,(A + B)$

$= A\,A + A\,B$

$= A + A\,B$

$= A = RHS$

❖ Example 3.3

(a) $AC + AB'CDE = (AC) + (AC)(B'DE) = AC$

(b) $B'(A + B')(B' + CD') = B'(B' + CD') = B'$

(3) Elimination theorem

(a) $A + A'B = A + B$

(b) $A(A' + B) = AB$

Proof: (a) LHS $= A + A'B$

$= (A + A')(A + B)$

$= 1 \cdot (A + B)$

$= A + B = \text{RHS}$

(b) LHS $= A(A' + B)$

$= AA' + AB$

$= 0 + AB$

$= AB = \text{RHS}$

❖ Example 3.4

(a) $AC' + AB'CDE' = A(C' + B'CDE') = A[C' + C(B'DE')]$

$= A(C' + B'DE') = AC' + AB'DE'$

(b) $(B + C')(A + B + C' + D + E)$

$= (B + C')[(B + C') + (A + D + E)]$

$= B + C'$

❖ Example 3.5

Simplify the sum-of-products expression $(AB' + BCD + A'B'D')$.

This example is the revisit of Example 3.1. By applying the elimination theorem

$$
\begin{aligned}
& AB' + BCD + A'B'D' \\
&= BCD + B'(A + A'D') \\
&= BCD + B'(A + D') \\
&= AB' + BCD + B'D'
\end{aligned}
$$

(4) Consensus theorem

(a) $A B + A' C + B C = A B + A' C$

(b) $(A + B)(A' + C)(B + C) = (A + B)(A' + C)$

Proof: (a) LHS

$$
\begin{aligned}
&= A B + A' C + B C \\
&= A B + A' C + 1 \bullet B \bullet C \\
&= A B + A' C + (A + A') B C \\
&= A B + A' C + A B C + A' B C \\
&= (A B) + (A B) C + (A' C) + (A' C) B \\
&= A B + A' C = \text{RHS}
\end{aligned}
$$

❖ Example 3.6

Variable in true form

Variable in complemented form

$$A B D' + A B C' + C D'$$

$$= A B D' + C'(AB) + C(D')$$

$$(AB)(D') = ABD'$$ ← Consensus term

$$= ABC' + CD'$$

❖ Example 3.7

B D’ + A B’ C’ + A’ C’ D’

Consensus term from A and A’ → (B’C’) (C’D’) = B’C’D’

B D’ + A B’ C’ + A’ C’ D’

Consensus term from B and B’ → D’ (A C’) = AC’D’

BD’ + AB’C’ + A’C’D’ = BD’ + AB’C’ + A’C’D’ + AC’D’

A’C’D’ + AC’D’ = (A’ + A) C’D’ = C’D’

BD’ + AB’C’ + C’D’

- Example 3.8

$$A'C + BCD + AC'D + AB'C'$$

$$A'C + B\underline{C}D + A\underline{C'}D + AB'C' = A'C + BCD + AC'D + AB'C' + ABD$$

$$\underline{A'}C + BCD + AC'D + AB'C' + \underline{A}BD = A'C + AC'D + AB'C' + ABD$$

$$A'C + AC'D + A\underline{B'}C' + A\underline{B}D = A'C + AB'C' + ABD$$

(5) Interchange Theorem

$$A B + A' C = (A + C)(A' + B)$$

Proof: RHS $= (A + C)(A' + B)$

$= A A' + A B + A' C + B C$

$= 0 + A B + A' C + B C$

$= A B + A' C + B C$

$= A B + A' C = \text{LHS}$

POS to SOP

$$(A + B + C)(A + B + D)$$

$$= AA + AB + AD + AB + BB + BD + AC + BC + CD$$

$$= A + AB + AD + B + BD + AC + BC + CD$$

$$= A + B + BD + BC + CD \qquad = A + B + CD$$

POS to SOP

$$(A + B + C)(A + B + D)$$
$$= AA + AB + AD + AB + BB + BD + AC + BC + CD$$

$$AA + AB + AD + AB + BB + BD + AC + BC + CD$$
$$= A + AB + AD + B + BD + AC + BC + CD$$
$$= A + B + BD + BC + CD \quad = A + B + CD$$

$$(A + B + C)(A + B + D)$$
$$= [\,\underline{(A + B) +}\, C]\,[\,\underline{(A + B) +}\, D]$$

$$= \underline{(A + B) +}\, CD$$

Sandwich Algorithm

Distributive law (collections)

Interchange Theorem

Distributive law (distribution)

- Example 3.9

$$(\underline{A + B +} C)(\underline{A + B +} D)(\underline{A + B +} E)(B' + D')$$

$$= (\underline{A + B +} CDE)(B' + D')$$

$$= [B + (A + CDE)](B' + D')$$

$$= BD' + B'(A + CDE)$$

$$= BD' + AB' + B'CDE$$

- Example 3.10

Convert (BCD’ + B’D + AB) to a product-of-sums expression.

$$BCD' + B'D + AB = B(A + CD') + B'D$$

$$(B + D)(B' + A + CD')$$

$$A + B' + CD' = (A + B' + C)(A + B' + D')$$

$$BCD' + B'D + A'B = (B + D)(A + B' + C)(A + B' + D')$$

❖ Example 3.11 POS to SOP

$$(A + B)(A' + C)(C' + D)$$

$$= (AC + A'B)(C' + D)$$

$$= ACC' + ACD + A'BC' + A'BD$$

$$= ACD + A'BC' + A'BD$$

- Example 3.12

Convert the SOP expression $(A'B + CD)$ to a POS.

$$A'B + CD = (A'B + C)\,(A'B + D)$$

$$C + A'B = (C + A')\,(C + B)$$

$$D + A'B = (D + A')\,(D + B)$$

$$A'B + CD = (A' + C)\,(B + C)\,(A' + D)\,(B + D)$$

(6) DeMorgan's theorem

(a) $(A \bullet B)' = A' + B'$

(b) $(A + B)' = A' \bullet B'$

Distribute the prime, Change the sign

Collect the primes, change the sign

(6) DeMorgan's theorem

(a) $(A \bullet B)' = A' + B'$

(b) $(A + B)' = A' \bullet B'$

Table 3.9 Proof of DeMorgan's theorem (6a).

A B	A B	Left-hand-side of (6a) (A B)'	A' B'	Right-hand-side of (6a) A' + B'	A'•B'
0 0	0	1	1 1	1	1
0 1	0	1	1 0	1	0
1 0	0	1	0 1	1	0
1 1	1	0	0 0	0	0

$(A\,B)' \neq A'\,B'$

$(A + B)' \neq A' + B'$

(a) $(x_1 \bullet x_2 \bullet x_3 \bullet \dots\dots\dots \bullet x_{n-1} \bullet x_n)' = x_1' + x_2' + x_3' + \dots\dots\dots\dots + x_{n-1}' + x_n'$

(b) $(x_1 + x_2 + x_3 + \dots\dots\dots\dots + x_{n-1} + x_n)' = x_1' \bullet x_2' \bullet x_3' + \dots\dots\dots\dots \bullet x_{n-1}' \bullet x_n'$

$$(x_1 \bullet x_2 \bullet x_3)' = ((x_1 \bullet x_2) \bullet x_3)' = (x_1 \bullet x_2)' + x_3'$$

$$= (x_1' + x_2') + x_3' = x_1' + x_2' + x_3'$$

$$(x_1 \bullet x_2 \bullet x_3 \bullet \dots\dots\dots \bullet x_{n-1})' = x_1' + x_2' + x_3' + \dots\dots\dots + x_{n-1}'$$

$$(x_1 \bullet x_2 \bullet x_3 \bullet \dots\dots\dots \bullet x_{n-1} \bullet x_n)'$$

$$= ((x_1 \bullet x_2 \bullet x_3 \bullet \dots\dots\dots \bullet x_{n-1}) \bullet x_n)'$$

$$= (x_1 \bullet x_2 \bullet x_3 \bullet \dots\dots\dots \bullet x_{n-1})' + x_n'$$

$$= (x_1' + x_2' + x_3' + \dots\dots\dots\dots + x_{n-1}') + x_n'$$

$$= x_1' + x_2' + x_3' + \dots\dots\dots\dots + x_{n-1}' + x_n'$$

(a) $(x_1 \bullet x_2 \bullet x_3 \bullet \text{ .... } \bullet x_{n-1} \bullet x_n)' = x_1' + x_2' + x_3' + \text{ .... } + x_{n-1}' + x_n'$

(b) $(x_1 + x_2 + x_3 + \text{ .... } + x_{n-1} + x_n)' = x_1' \bullet x_2' \bullet x_3' + \text{ .... } \bullet x_{n-1}' \bullet x_n'$

Distribute the prime, Change the sign
Collect the primes, change the sign

Example 3.13

$$[A' + B(C + D') + E]'$$

$$= A \bullet [B(C + D')]' \bullet E'$$

$$= A \bullet [B' + (C + D')'] \bullet E'$$

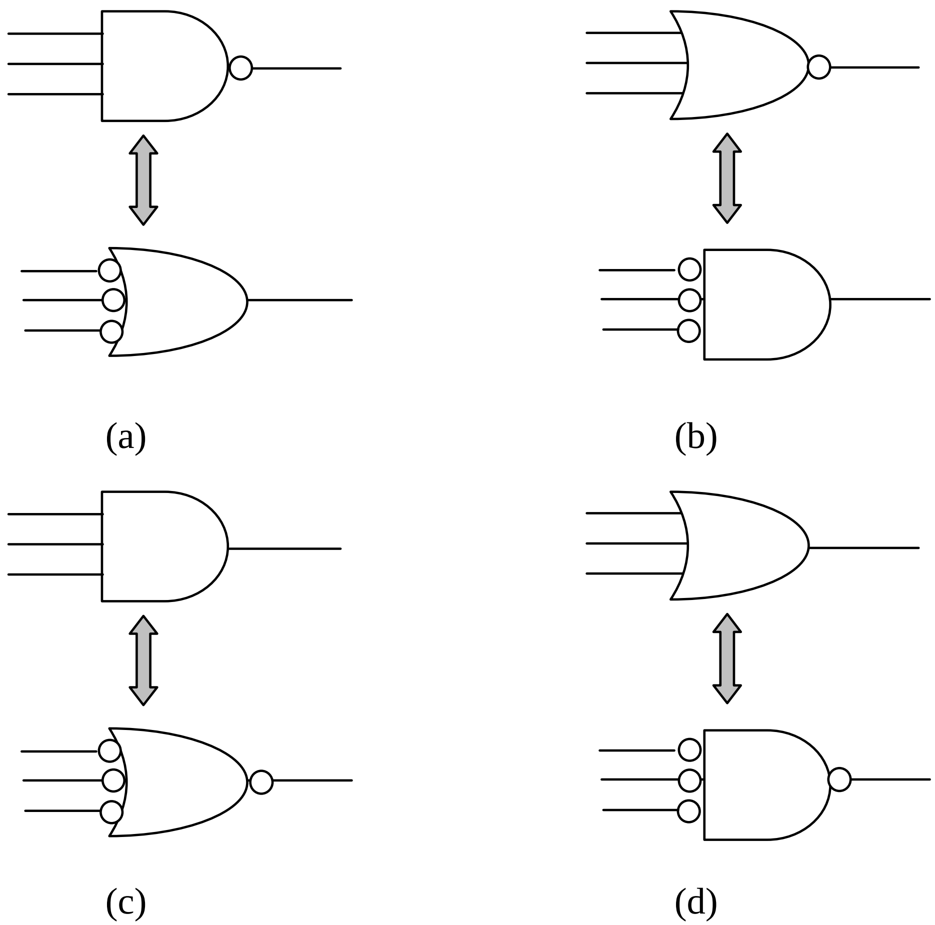
$$= A \bullet (B' + C'D) \bullet E'$$

$$= AB'E' + AC'DE'$$

Eliminating of internal inversions by gate equivalencies

(a) $(x_1 \bullet x_2 \bullet x_3 \bullet \ldots\ldots\ldots \bullet x_{n-1} \bullet x_n)' = x_1' + x_2' + x_3' + \ldots\ldots\ldots + x_{n-1}' + x_n'$

(b) $(x_1 + x_2 + x_3 + \ldots\ldots\ldots + x_{n-1} + x_n)' = x_1' \bullet x_2' \bullet x_3' + \ldots\ldots\ldots \bullet x_{n-1}' \bullet x_n'$



Move bubble(s),

change symbol.

Add bubble(s),

change symbol.

Figure 4.13 Gate equivalencies using DeMorgan’s theorem.

Minimization of Literals

- Example 3.14

$$F(A,B,C,D) = BD + CD + A'BC + ABC'$$

$$D(B + C) + B(A'C + AC')$$

$$CD + B(D + A'C + AC')$$

$$B(D + AC') + C(D + A'B)$$

- Example 3.15

$$F(A,B,C,D) = (A + C')(B + D)(A' + C + D)$$

$$\begin{aligned} F(A,B,C,D) &= (A + C')(B + D)(A' + C + D) \\ &= (A + C')[D + B(A' + C)] \end{aligned}$$

.

$$\begin{aligned} F(A,B,C,D) &= (A + C')(B + D)(A' + C + D) \\ &= (B + D)(A + C')[A' + (C + D)] \\ &= (B + D)[A'C' + A(C + D)] \end{aligned}$$

Duality

AND $\longrightarrow$ OR

OR $\longrightarrow$ AND

0 $\longrightarrow$ 1

1 $\longrightarrow$ 0

$$F^D(x_{n-1}, x_{n-2}, \ldots\ldots, x_2, x_1, x_0, 0, 1, \bullet, +)$$
$$= F(x_{n-1}, x_{n-2}, \ldots\ldots, x_2, x_1, x_0, 1, 0, +, \bullet)$$

Laws of 0 and 1: (L3a') $\quad A \bullet 0 = 0$

Laws of 0 and 1: (L3b') $\quad A + 1 = 1$

Distributive law: (7a) $\quad A \bullet (B + C) = (A \bullet B) + (A \bullet C)$

Distributive law: (7b) $\quad A + (B \bullet C) = (A + B) \bullet (A + C)$

❖ Example 3.11

$$F = [\,A' + B\,(\,C + D'\,) + E \bullet 0\,]' \bullet B'$$

F fully parenthesized: $F = \{\,A' + [\,B \bullet (\,C + D'\,)\,] + (\,E \bullet 0\,)\,\}' \bullet B'$

Transformation: $F^{D} = \{\,A' \bullet [\,B + (\,C \bullet D'\,)\,] \bullet (\,E + 1\,)\,\}' + B'$

$$= [\,A'\,(\,B + C\,D'\,)\,(\,E + 1\,)\,]' + B'$$

Positive Logic and
Negative Logic

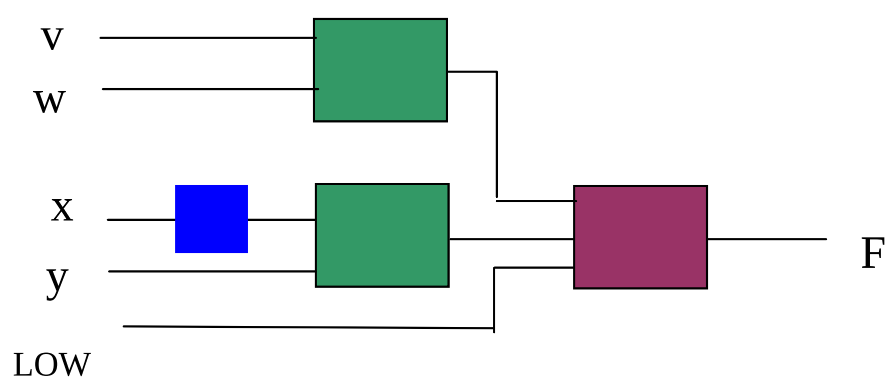


Table 3.10 Truth tables for three types of gates

(a)

Input	Output
L	H
H	L

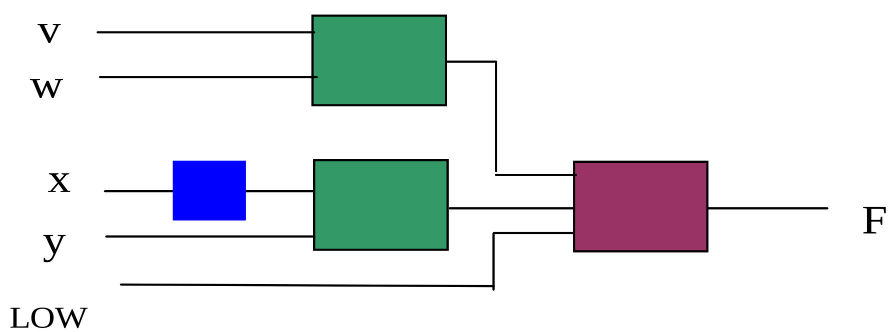
(b)

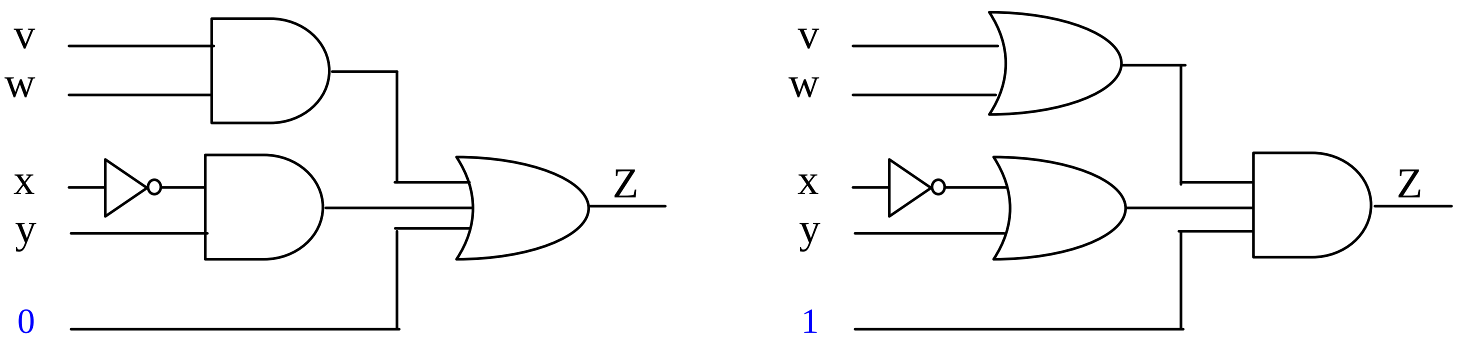
Inputs	Output
L L	L
L H	L
H L	L
H H	H

(c)

Inputs	Output
L L L	L
L L H	H
L H L	H
L H H	H
H L L	H
H L H	H
H H L	H
H H H	H

Positive Logic and Negative Logic





(a) (b)

Figure 3.7 A digital circuit with different logic. (a) Positive logic. (b) Negative logic.

Positive Logic $Z = vw + x'y + 0$

Negative Logic $Z = (v + w) \bullet (x' + y) \bullet 1$