Analysis of Algorithms 91.404, Fall, 2012

Homework #2

1. (25 points) **Rank the following three functions by order of asymptotic growth.**
Ans:
The order can be: *g1(n), g3(n), g2(n)*, where

$$g_1(n) = (\frac{1}{2})^{n^3}$$

$$g_2(n) = 3^{4\log_3 n} = n^4$$

$$g_3(n) = 5lgn + n^2 lglgn$$

1) $g_1(n) \in O(g_2(n))$:
Since$g_1(n) < 1$and$g_3(n) \geq 1$for$n \geq 2$, we have$g_1(n) \leq c \cdot g_2(n) \forall n \geq n_0$where $c = 1$and$n_0 = 2$.
2) $g_3(n) \in O(g_2(n))$:
Let $G(n) = g_3(n) - g_2(n) = n^4 - 5lgn - n^2 lglgn$, we assume $n \geq 3$then

$$G(n) = n^2(n^2 - \frac{5}{n^2} lgn - lglgn)$$

$$\geq n^2(n^2 - lgn - lgn)(\forall n \geq 3, \frac{5}{n^2} \leq 1 and lglgn \leq lgn)$$

$$= n^2(n^2 - 2lgn)$$

$$\geq n^2(n^2 - 2n)(\forall n \geq 3, lgn \leq n)$$

$$\geq 0$$

So, we have $g_3(n) \leq c \cdot g_2(n) \forall n \geq n_{0,}$where$c = 1andn_0 = 3$

2. (25 points) i) $f_1(n) \in \Omega((\frac{1}{2})^n)$**ii)** $f_2(n) \in \Theta(n^2 lgn)$**iii)** $f_3(n) \in O(lg^3 n)$

a) If statements i) – iii) are true, can we conclude that $f_3(n) \in O(f_2(n))$**?**

b) If statements i) – iii) are true, can we conclude that $f_2(n) \in \Omega(f_1(n))$**?**

Ans:

Since i) – iii) are true, we can have following statements:

(1) There exist positive constants *c1* and *n1* such that $0 \leq c_1(\frac{1}{2})^n \leq f_1(n), \forall n \geq n_1$.

(2) There exist positive constants a, b and *n2* such that$0 \leq an^2 lgn \leq f_2(n) \leq bn^2 lgn, \forall n \geq n_2$.

(3) There exist positive constant c_3 and n3 such that $0 \leq f_3(n) \leq c_3(lg^3 n), \forall n \geq n_3$.

a) True. Let $c_{23} = \frac{c3}{a}$and $n_{23} = max(1, n_{2,} n_3)$, from (2) and (3), and we already know that $lg^3 n \leq n^2 lgn \forall n \geq 1$, we have:

$0 \leq f_3(n) \leq c_3(lg^3 n) \leq c_3 n^{2\lg} n = \frac{c_3}{a} an^2 lgn \leq c_{23} f_2(n), \forall n \geq n_{23}$.

So we can conclude that $f_3(n) \in O(f_2(n))$.

b) False. Because i) only shows that the lower bound of $f_1(n)$, but we do not know its exact upper bound, we can not say that $f_2(n)$ would be the upper bound of $f_1(n)$ even if $n^2 lgn$ is the upper bound of $(1/2)^n$. It is possible that $f_1(n)$ is the upper bound of $f_2(n)$, for example, $f_1(n) = n^4$and $f_2(n) = n^2 lgn$. That satisfies statements i) and ii), but it is obvious that $f_1(n) \in \Omega(f_2(n))$.

So we cannot conclude that $f_2(n) \in \Omega(f_1(n))$.

3. (25 points) **True or False.**

Ans: a) True; b) False; c) False; d) True; e) True

For b), use the limit rule and we get 0, which means $nlg^2 n \in O(n^{1.05})$.

For d) and e), the cost function T(n) could be: $T(n) = c_1(\lfloor \log_2 n \rfloor + 1) + c_2 \lfloor \log_2 n \rfloor$. d) is true since $T(n) \leq c_1(\log_2 n + 1) + c_2 \log_2 n = O(n)$, and e) is true since $T(n) \geq c_1 \log_2 n + c_2(\log_2 n - 1) = \Omega(n)$.

4. (25 points) **Pseudocode Analysis: find the tight upper-and-lower bounds on the asymptotic worst-case running time *f(n)*.**

Ans:

Mystery(n)	**Cost**	**Times**
1. c ← 1	c_1	1

2. for i ← 1 to n	c_2	n+1
3. do for j ← i to n	c_3	$\sum_{i=n+1}^{1} i$
4. do for k ← n down to $\lfloor\frac{n}{2}\rfloor$	c_4	$(n-\lfloor\frac{n}{2}\rfloor+1)\sum_{i=n+1}^{1}(i-1)$
5. do c ← c + 1		
6. print c	c_5	$(n-\lfloor\frac{n}{2}\rfloor)\sum_{i=n+1}^{1}(i-1)$
	c_6	1

The procedure Mystery(n) is a 3-level loop, and the worst-case running time is:

$$f(n) = c_1 + c_2(n+1) + c_3\frac{(n+1)(n+2)}{2} + c_4(n-\lfloor\frac{n}{2}\rfloor+1)\frac{n(n+1)}{2} + c_5(n-\lfloor\frac{n}{2}\rfloor)\frac{n(n+1)}{2} + c_6$$

Since $\frac{n}{2}-1 \le \lfloor\frac{n}{2}\rfloor \le \frac{n}{2}$, we have:

$$f(n) \ge c_1 + c_2(n+1) + c_3\frac{(n+1)(n+2)}{2} + c_4(\frac{n}{2}+1)\frac{n(n+1)}{2} + c_5(\frac{n}{2})\frac{n(n+1)}{2} + c_6$$
$$= (\frac{c_4}{4}+\frac{c_5}{4})n^3 + (\frac{c_3}{2}+\frac{3c_4}{4}+\frac{c_5}{4})n^2 + (c_2+\frac{3c_3}{2}+\frac{c_4}{2})n + (c_1+c_2+c6)$$, and

$$f(n) \le c_1 + c_2(n+1) + c_3\frac{(n+1)(n+2)}{2} + c_4(\frac{n}{2}+2)\frac{n(n+1)}{2} + c_5(\frac{n}{2}+1)\frac{n(n+1)}{2} + c_6$$
$$= (\frac{c_4}{4}+\frac{c_5}{4})n^3 + (\frac{c_3}{2}+\frac{5c_4}{4}+\frac{3c_5}{4})n^2 + (c_2+\frac{3c_3}{2}+c_4+\frac{c_5}{2})n + (c_1+c_2+c6)$$

Then, let $g(n) = n^3$, $a = (\frac{c_4}{4}+\frac{c_5}{4}) + (\frac{c_3}{2}+\frac{3c_4}{4}+\frac{c_5}{4}) + (c_2+\frac{3c_3}{2}+\frac{c_4}{2}) + (c_1+c_2+c6)$, $b = (\frac{c_4}{4}+\frac{c_5}{4}) + (\frac{c_3}{2}+\frac{5c_4}{4}+\frac{3c_5}{4}) + (c_2+\frac{3c_3}{2}+c_4+\frac{c_5}{2}) + (c_1+c_2+c6)$, and $n_0 = 1$. We can conclude that:$0 \le ag(n) \le f(n) \le bg(n), \forall n \ge n_0$. That is $f(n) \in \Theta(g(n))$.