Lengths of Curves in $\mathbb{R}^3$

Let $\vec{r}(t) = < f(t), g(t), h(t) >$ be a parameterized curve/trajectory in $\mathbb{R}^3$ for $t \in [a,b]$ and $\vec{v}(t)$ be the associated velocity vector composed of the derivatives of the components of $\vec{r}(t)$.

Similarly, let $|\vec{v}(t)|$ be a scalar function representing the speed at time t . We will assume that $|\vec{v}(t)|$ is a continuous function for $t \in [a,b]$ or that it has, at most, a finite number of jump discontinuities.

Regardless of the direction of $\vec{v}(t)$ at any given time, the distance travelled during a short interval of time Δt (say over the interval $[t_{k-1}, t_k]$ where $\Delta t_k = t_k - t_{k-1}$ and where these intervals do not span[1] any jump discontinuities in $|\vec{v}(t)|$) can be approximated as the product of the average value of the speed over that interval, $|\vec{v}(t_k)|_{AVG}$ and the duration of that interval, Δt_k .

Therefore, denoting the distance travelled during the interval $[t_{k-1}, t_k]$ by s_k , we can write, $s_k \approx |\vec{v}(t_k)|_{AVG}\, \Delta t_k$ where this approximation becomes exact as $\Delta t_k \to 0$.

Since $|\vec{v}(t)|$ is continuous for $t \in [t_{k-1}, t_k]$, there is some t_k^* in the interval $[t_{k-1}, t_k]$ where $|\vec{v}(t_k^*)| = |\vec{v}(t_k)|_{AVG}$. As a result, we can express s_k as $s_k \approx |\vec{v}(t_k^*)|\Delta t_k$ where, again, the approximation becomes exact as $\Delta t_k \to 0$.

Adding up these distances travelled over all subintervals yields an approximation of the total distance travelled given by $s \approx \sum_{k=1}^{n} |\vec{v}(t_k^*)|\Delta t_k$. The exact total distance is then given by the limit of this expression as the norm of the partition, $\|P\|$, goes to zero (yielding a Riemann sum and an integral representation).

Therefore,

$$s = \lim_{\|P\| \to 0} \sum_{k=1}^{n} |\vec{v}(t_k^*)|\Delta t_k = \int_a^b |\vec{v}(t)|dt = \int_a^b \left[\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 \right]^{1/2} dt$$

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[1] We will restrict partitions of $[a,b]$ to be composite partitions made up of unions of partitions of the continuous segments of $|\vec{v}(t)|$. Note that this restriction on the partition is levied only to make the explanation easier. If a sub-partition spans a discontinuity, its effect diminishes to zero as the norm of the partition is forced to zero and, since there are a finite number of such discontinuities, the overall effect of all discontinuities also goes to zero.