Student: Phong Vo **Date:** 02/19/20	**Instructor:** Erica Yankowskas **Course:** Linear Algebra I (Spring 2020)	**Assignment:** Section 1.2 Homework

1. Determine which matrices are in reduced echelon form and which others are only in echelon form.

a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Is matrix **a** in reduced echelon form, echelon form only, or neither?

- [x] echelon form only
- [ ] reduced echelon form
- [ ] neither

Is matrix **b** in reduced echelon form, echelon form only, or neither?

- [ ] echelon form only
- [x] reduced echelon form
- [ ] neither

Is matrix **c** in reduced echelon form, echelon form only, or neither?

- [ ] reduced echelon form
- [ ] echelon form only
- [x] neither

2. Row reduce the matrix to reduced echelon form. Identify the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

$$\begin{bmatrix} 1 & 2 & 4 & -10 \\ 2 & 4 & 6 & -16 \\ 4 & 6 & 8 & -22 \end{bmatrix}$$

Row reduce the matrix to reduced echelon form and identify the pivot positions in the final matrix. The pivot positions are indicated by bold values. Choose the correct answer below.

A. $\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} \end{bmatrix}$

B. $\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & -2 \end{bmatrix}$

C. $\begin{bmatrix} \mathbf{1} & 0 & 0 & -1 \\ 0 & \mathbf{1} & 0 & -2 \\ 0 & 0 & \mathbf{1} & -1 \end{bmatrix}$

D. $\begin{bmatrix} \mathbf{1} & 0 & 0 & -1 \\ 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix}$

Identify the pivot positions in the original matrix. The pivot positions are indicated by bold values. Choose the correct answer below.

A. $\begin{bmatrix} \mathbf{1} & 2 & 4 & -10 \\ 2 & \mathbf{4} & 6 & -16 \\ 4 & 6 & \mathbf{8} & \mathbf{-22} \end{bmatrix}$

B. $\begin{bmatrix} \mathbf{1} & 2 & 4 & -10 \\ \mathbf{2} & 4 & 6 & -16 \\ \mathbf{4} & 6 & 8 & -22 \end{bmatrix}$

C. $\begin{bmatrix} \mathbf{1} & 2 & 4 & -10 \\ 2 & \mathbf{4} & 6 & -16 \\ 4 & 6 & \mathbf{8} & -22 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 & 4 & \mathbf{-10} \\ 2 & 4 & \mathbf{6} & -16 \\ 4 & \mathbf{6} & 8 & -22 \end{bmatrix}$

List the pivot columns. Select all that apply.

- [x] **A.** Column 3
- [x] **B.** Column 2
- [x] **C.** Column 1
- [ ] **D.** Column 4

3. Find the general solution of the system whose augmented matrix is given below.

$$\begin{bmatrix} 5 & -3 & 7 & 0 \\ 20 & -12 & 28 & 0 \\ 15 & -9 & 21 & 0 \end{bmatrix}$$

Choose the correct answer below.

A. $\begin{cases} x_1 = -5x_2 \\ x_2 = 3x_3 \\ x_3 \text{ is free} \end{cases}$

B. $\begin{cases} x_1 = \frac{3}{5}x_2 - \frac{7}{5}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

C. $\begin{cases} x_1 = 5 \\ x_2 = -3 \\ x_3 = 7 \end{cases}$

D. The system has no solutions.

4. Find the general solution of the system whose augmented matrix is given below.

$$\begin{bmatrix} 1 & 0 & 6 & 0 & -2 \\ 0 & 1 & 7 & 0 & 9 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & -7 \end{bmatrix}$$

Select the correct choice below and, if necessary, fill in any answer boxes to complete your answer.

A.
$x_1 =$ ______
$x_2 =$ ______
$x_3 =$ ______
x_4 is free

B.
$x_1 =$ ______
$x_2 =$ ______
x_3 is free
$x_4 =$ ______

C.
$x_1 =$ ______
$x_2 =$ ______
x_3 is free
x_4 is free

D. The system has no solution.

5. Find the general solution of the system whose augmented matrix is given below.

$$\begin{bmatrix} 1 & 0 & -7 & 0 & -2 & 3 \\ 0 & 1 & 5 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Select the correct choice below and, if necessary, fill in any answer boxes to complete your answer.

A.
$x_1 =$ ______
$x_2 =$ ______
$x_3 =$ ______
$x_4 =$ ______
x_5 is free

B.
$x_1 =$ ______
$x_2 =$ ______
x_3 is free
$x_4 =$ ______
$x_5 =$ ______

C.
$x_1 = 7x_3 + 3$
$x_2 = -5x_3 + x_4 + 6$
x_3 is free
x_4 is free
$x_5 = 0$

D. The system is inconsistent.

6. In parts **(a)** through **(e)** below, mark the statement True or False. Justify each answer.

(a) In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

Is this statement true or false?

- **A.** The statement is false. Each matrix is row equivalent to one and only one reduced echelon matrix.
- **B.** The statement is true. The echelon form of a matrix is always unique, but the reduced echelon form of a matrix might not be unique.
- **C.** The statement is false. For each matrix, there is only one sequence of row operations that row reduces it.
- **D.** The statement is true. It is possible for there to be several different sequences of row operations that row reduces a matrix.

(b) The row reduction algorithm applies only to augmented matrices for a linear system.

Is this statement true or false?

- **A.** The statement is true. The row reduction algorithm is only useful when it is used to find the solution of a linear system.
- **B.** The statement is true. Every matrix with at least two columns can be interpreted as the augmented matrix of a linear system.
- **C.** The statement is false. It is possible to create a linear system such that the row reduction algorithm does not apply to the corresponding augmented matrix.
- **D.** The statement is false. The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system.

(c) A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

Is this statement true or false?

- **A.** The statement is true. If a linear system has both basic and free variables, then each basic variable can be expressed in terms of the free variables.
- **B.** The statement is true. It is the definition of a basic variable.
- **C.** The statement is false. Not every linear system has basic variables.
- **D.** The statement is false. A variable that corresponds to a pivot column in the coefficient matrix is called a free variable, not a basic variable.

(d) Finding a parametric description of the solution set of a linear system is the same as solving the system.

Is this statement true or false?

- **A.** The statement is false. The solution set of a linear system can only be expressed using a parametric description if the system has no more than one solution.
- **B.** The statement is true. Solving a linear system is the same as finding the solution set of the system. The solution set of a linear system can always be expressed using a parametric description.
- **C.** The statement is true. Regardless of whether a linear system has free variables, the solution set of the system can be expressed using a parametric description.
- **D.** The statement is false. The solution set of a linear system can only be expressed using a parametric description if the system has at least one solution.

(e) If one row in an echelon form of an augmented matrix is $\begin{bmatrix} 0 & 0 & 0 & 5 & 0 \end{bmatrix}$, then the associated linear system is inconsistent.

Is this statement true or false?

- **A.** The statement is true. The indicated row corresponds to the equation $5 = 0$. This equation is a contradiction, so the linear system is inconsistent.
- **B.** The statement is true. The indicated row corresponds to the equation $5x_4 = 0$. This equation is not a contradiction, so the linear system is inconsistent.
- **C.** The statement is false. The indicated row corresponds to the equation $5x_4 = 0$, which does not by itself make the system inconsistent.
- **D.** The statement is false. The indicated row corresponds to the equation $5x_4 = 0$, which means the system is consistent.

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7. Suppose the coefficient matrix of a linear system of four equations in four variables has a pivot in each column. Explain why the system has a unique solution.

What must be true of a linear system for it to have a unique solution? Select all that apply.

- [x] **A.** The system has no free variables.
- [ ] **B.** The system has at least one free variable.
- [x] **C.** The system is consistent.
- [ ] **D.** The system has exactly one free variable.
- [ ] **E.** The system is inconsistent.
- [ ] **F.** The system has one more equation than free variable.

Use the given assumption that the coefficient matrix of the linear system of four equations in four variables has a pivot in each column to determine the dimensions of the coefficient matrix.

The coefficient matrix has four rows and four columns.

Let the coefficient matrix be in reduced echelon form with a pivot in each column, since each matrix is equivalent to one and only one reduced echelon matrix. Construct a matrix with the dimensions determined in the previous step that is in reduced echelon form and has a pivot in each column.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now find an augmented matrix in reduced echelon form that represents a linear system of four equations in four variables for which the corresponding coefficient matrix has a pivot in each column. Choose the correct answer below.

○ **A.**
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

○ **B.**
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & a \\ 0 & \mathbf{1} & 0 & b \\ 0 & 0 & \mathbf{1} & c \\ 0 & 0 & 0 & \mathbf{d} \end{bmatrix}$$

● **C.**
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & a \\ 0 & \mathbf{1} & 0 & 0 & b \\ 0 & 0 & \mathbf{1} & 0 & c \\ 0 & 0 & 0 & \mathbf{1} & d \end{bmatrix}$$

○ **D.**
$$\begin{bmatrix} \mathbf{a} & 0 & 0 & 0 & 1 \\ 0 & \mathbf{b} & 0 & 0 & 1 \\ 0 & 0 & \mathbf{c} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{d} & 1 \end{bmatrix}$$

Use the augmented matrix to determine if the linear system is consistent. Is the linear system represented by the augmented matrix consistent?

○ **A.** Yes, because the rightmost column of the augmented matrix is a pivot column.
● **B.** Yes, because the rightmost column of the augmented matrix is not a pivot column.
○ **C.** No, because the rightmost column of the augmented matrix is a pivot column.
○ **D.** No, because the rightmost column of the augmented matrix is not a pivot column.

Write the system of equations corresponding to the augmented matrix found above to determine the number of free variables.

$$\begin{aligned} x_1 &= a \\ x_2 &= b \\ x_3 &= c \\ x_4 &= d \end{aligned}$$

Free variables are variables that can take on any value. How many free variables are in the system?

- **A.** One, because x_1 can take on any value, but x_2, x_3, and x_4 depend on x_1.
- **B.** None, because x_1, x_2, x_3, and x_4 are all fixed values.
- **C.** Four, because x_1, x_2, x_3, and x_4 can all take on any value.
- **D.** Three, because x_1, x_2, and x_3 can take on any values, but x_4 depends on x_1, x_2, and x_3.

Why does the system have a unique solution?

The system is consistent and has no free variables.

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8. Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is not a pivot column. Is the system consistent? Why or why not?

To determine if the linear system is consistent, use the portion of the Existence and Uniqueness Theorem, shown below.

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. That is, if and only if an echelon form of the augmented matrix has no row of the form [0 ... 0 b] with b nonzero.

In the augmented matrix described above, is the rightmost column a pivot column?

- No
- Yes

In the echelon form of the augmented matrix, is there a row of the form [0 0 0 0 b] with b nonzero?

- No
- Yes

Therefore, by the Existence and Uniqueness Theorem, the linear system is consistent.

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9. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

Choose the correct answer below.

- **A.** The system is consistent because the rightmost column of the augmented matrix is not a pivot column.
- **B.** The system is consistent because the augmented matrix will contain a row of the form $\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix}$ with b nonzero.
- **C.** The system is consistent because the augmented matrix is row equivalent to one and only one reduced echelon matrix.
- **D.** The system is consistent because all the columns in the augmented matrix will have a pivot position.

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10. Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

Choose the correct answer below.

- **A.** There is at least one row of the coefficient matrix that does not have a pivot position. This means the augmented matrix, which will have six columns, could have a row of the form $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, so the system could be inconsistent.
- **B.** There is a pivot position in each row of the coefficient matrix. The augmented matrix will have four columns and will not have a row of the form $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is consistent.
- **C.** There is at least one row of the coefficient matrix that does not have a pivot position. This means the augmented matrix, which will have six columns, must have a row of the form $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is inconsistent.
- **D.** There is a pivot position in each row of the coefficient matrix. The augmented matrix will have six columns and will not have a row of the form $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, so the system is consistent.

11. A system of linear equations with fewer equations than unknowns is sometimes called an underdetermined system. Can such a system have a unique solution? Explain.

Choose the correct answer below.

- **A.** Yes, it can have a unique solution. Because there are more variables than equations, there must be at least one free variable. If the linear system is consistent and there is at least one free variable, the solution set contains either a unique solution or infinitely many solutions. If the linear system is inconsistent, there is no solution.
- **B.** No, it cannot have a unique solution. Because there are more variables than equations, there must be at least one free variable. If the linear system is consistent and there is at least one free variable, the solution set contains infinitely many solutions. If the linear system is inconsistent, there is no solution.
- **C.** No, it cannot have a unique solution. Because there are more variables than equations, there must be at least one free variable. If there is a free variable, the solution set contains a unique solution.
- **D.** Yes, it can have a unique solution. Because there are more equations than variables, there are no free variables. If the system is consistent and there are no free variables, the solution set contains a unique solution. If the system is inconsistent, there is no solution.