Section 1.2: Row Reduction & Echelon Forms

Note: Here we refine the previous section's method into a row reduction algorithm to help us analyze any system of linear equations. ⇒ This allows us to answer the fundamental existence & uniqueness questions.

A Variant of Gaussian Elimination

The following algorithm applies to any matrix, whether or not it is augmented.

- A nonzero row or column in a matrix contains @ least one non-zero entry
- A leading entry of a row refers to the leftmost nonzero entry (in a non-zero row).

*Definition:

A rectangular matrix is in Echelon Form -or- Row Echelon Form if it has the following 3 properties:

① All nonzero rows are above any rows of all zeros

② Each leading entry of a row is in a column to the right of the leading entry of the row above it.
*producing a "steplike" pattern.

③ All entries in a column below a leading entry are zeros.

* Definition Continued...

If a matrix in Echelon Form satisfies the following 2 conditions, then it is in **Reduced Echelon Form** -OR- **Reduced Row Echelon Form (RREF)**:

① The leading entry in each nonzero row is 1.

② Each leading 1 is the only nonzero entry in its column.

* Example of "Echelon Form":

$$\begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & 5/2 \end{bmatrix}$$

*Note: This is the triangular-form seen in 1.1

* Example of "Reduced Echelon Form":

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Notes:

(i) Any non-zero matrix may be row reduced (i.e transformed by elementary row operations) into more than one matrix in echelon form using sequences of row operations.

(ii) The reduced echelon form one obtains from a matrix is unique.

*__Theorem__: (Uniqueness of the Reduced Echelon Form)

Each matrix is row equivalent to one and only one reduced echelon matrix.

- If a matrix A is row equivalent to an echelon matrix U, we call "U" an: <u>Echelon Form -or- Row Echelon Form of A</u>
- If "U" is in reduced echelon form, we call "U" the: <u>Reduced Echelon Form of A</u>

Pivot Positions

When row operations on a matrix produce an Echelon Form further row operations to attain the REF do NOT change the positions of the leading entries.

- Since the R.E.F. is unique $\Rightarrow$ the leading entries are always in the same positions in any echelon form obtained from a given matrix :)

Note: These leading entries correspond to "1"'s in the Reduced Echelon Form.

*Definition:

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A pivot column is a column in A that contains a pivot position.

*pivot position

Example:

$$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

pivot column

* A pivot is a non-zer number in a pivot position that is used to create zeros via row operations.

Note: All circled entries represent that column's pivot :)

The Row Reduction Algorithm

Note: We are now ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon form!

Careful study/mastery of this procedure will help with entire coure

① **Step 1:** Begin with the leftmost nonzero column.

- *This is the pivot column.
- *The pivot position is the first entry of that colum (a_{11}).

② **Step 2:** Select a nonzero entry in the pivot column as a pivot.

- *If necessary, interchange rows to move this entry into the pivot position

③ **Step 3:** Use row replacement operations to create zeros in all positions below the pivot.

- *Same procedure as section 1.1

④ **Step 4:** Cover the row containing the pivot position & cover all row (if any) above it.

- *Apply steps 1-3 to the resulting submatrix.
- *Continue this process until $\exists$ no more nonzero rows to modify

⑤ **Step 5:** Starting w/ the Rightmost pivot & working upward to the left, create a zero above each pivot.

- *If pivot is not 1, use a scaling operation to make it 1.

*Notes on the Row Reduction Algorithm:

(i) Steps 1-4 are called the: "Forward Phase"

- The combination of these steps are just like what we saw in section 1.1 when altaring the "triangular form" of a system :

(ii) Step 5 is called the: "Backward Phase"

- This produces the unique row echelon form.
- This process is the same as "back-substitution" seen in section 1.1 :

Fun Side Note:

For Step 2 of the algorithm, a strategy know as "Partial Pivoting" chooses the entry in a column having the LARGEST absolute value as the pivot.

* Helps to reduce roundoff errors in the calculations
* Partial Pivoting is a technique used by many computer programs :

Solutions of Linear Systems

<u>Note</u>: When the row reduction algorithm is applied to the augmented matrix of the system, it leads directly to an explicit description of the solution set of a Linear System.

Consider the following Augmented Matrix that has been changed into its equivalent (& simpiler) R.E.F:

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array} \iff \begin{array}{rcll} x_1 - 5x_3 &=& 1 & \text{Eq 1} \\ x_2 + x_3 &=& 4 & \text{Eq(2)} \\ 0 &=& 0 & \text{Eq(3)} \end{array}$$

* The variables x_1 & x_2 correspond to the pivot columns of the matrix $\longrightarrow$ Called "<u>Basic Variables</u>"

* The variable x_3 is called a "<u>Free Variable</u>".
  $\Rightarrow$ We can choose <u>ANY</u> value for x_3 :)

* <u>Whenever a system is consistent</u>, the solution set can be described explicitly by solving the reduced system for the basic variables in terms of the free variables

$$\Rightarrow \begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

<u>Notes</u>:
(i) This is possible b/c each basic variable is in 1 & only 1 equation

* <u>Each Different choice of the free variable (x_3) determines a</u> different solution of the system; Every Solution is determined by the

* Parametric Descriptions of Solution Sets *

A description of a solution set in which the free variables act as a parameter are called: "**Parametric Descriptions**"

Ex:
$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

* x_3 acts as the parameter here for the solution set.

Solving a System amounts to:

(i) Finding a Parametric Description of the Solution Set.

-OR-

(ii) Determining that the Solution Set is Empty.

Notes:

(i) Whenever a system is consistent & has free variables, the solution set has many parametric descriptions

*We can use row operations to create equivalent systems as needed :)

(ii) Whenever a system is inconsistent, the solution set is empty, even if the system has free variables.

*The Solution Set has NO parametric descriptions

Back-Substitution

<u>Note</u>: Computer programs solve Systems of Linear Eq. using back-sub. <u>without</u> computing the reduced echelon form first...

*<u>Our matrix format for the backward phase of row reduction (Step 5 of the Algorithm) has the same number of arithmetic operations as back-sub used by computers</u> :)

- The Backward Phase substantially ↓ chance of error when performing hand computations
- Since we are <u>NOT</u> allowed calculators &/or computers in this class, we will only apply the Backward Phase (for our safety :))

<u>Note</u>: The process of back-substitution used in section 1.1 is different than the back-sub used by computer programs.

Existence & Uniqueness Questions

Note: Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering the 2 questions posed @ the end of 1.1

When a system is in echelon form & contains no equation of the form $0 = b$ (where $b \neq 0$), every nonzero equation contains a basic variable w/ a non-zero coefficient

2 Possible Outcomes: Either...

*Note: These 2 outcomes support the theorem below:

① The Basic Variables are completely determined, with NO free variables

* Here there is a unique solution.

② At least one of the Basic Variables may be expressed in terms of one or more free variables.

* Here there are infinitely many solutions; one for each choice of values for the free variables.

*Theorem: (Existence & Uniqueness Theorem)

A linear system is consistent IFF the rightmost column of the augmented matrix is NOT a pivot column

IOW: No row has the form $[0 \cdots 0 \; b]$ where $b \neq 0$

If a linear system is consistent, then the solution set contains either

(i) A unique solution, when $\exists$ No free variables

(ii) Infinitely many solutions, when $\exists$ at least one free variable.

Existence & Uniqueness Overview

A System of Linear Equations may have solution(s) & we can find them by performing matrix operations on a given system (rewriting it in reduced-echelon form)

3 Possible Outcomes: A system will have...

① One, Unique Solution:

A system will have a unique solution IFF the matrix row-reduces to the identify matrix.

② Infinitely Many Solutions:

A system will have infinitely many solutions IFF the augmented matrix row-reduces to a matrix w/ one (or more) row(s) of zeros → Free Variables $\exists$

③ No Solution:

A system will have NO solution IFF the aug. matrix row-reduces to a matrix w/ a row of zeros in all entries except the last → Contradiction produced

Using Row Reduction to Solve a Linear System

The following procedure outlines how to find and describe all solutions of a linear system:

① Write the augmented matrix of a system.

② Use the Row Reduction Algorithm to obtain an equivalent augmented matrix in echelon form.

* Decide if the system is consistent.
* If there is NO Solution, STOP.

③ Continue row reduction to obtain the reduced echelon form.

④ Write the system of equations corresponding to the matrix obtained in step 3.

⑤ Rewrite each non-zero equation from step 4 so that its one Basic Variable is expressed in terms of any Free Variables appearing in the equation.

Example: Find the general solution of the system whose augmented matrix is:

$$\begin{array}{c} \begin{bmatrix} 3 & -5 & 4 & 0 \\ 12 & -20 & 16 & 0 \\ 6 & -10 & 8 & 0 \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

Answer:

Note: Lets simplify R_2 & R_3 first $\therefore \Rightarrow \frac{1}{4}R_2$ & $\frac{1}{2}R_3$

$$\Rightarrow \begin{bmatrix} 3 & -5 & 4 & 0 \\ 3 & -5 & 4 & 0 \\ 3 & -5 & 4 & 0 \end{bmatrix}$$

$* R_1 = R_2 = R_3$ now! $\Rightarrow 3x_1 - 5x_2 + 4x_3 = 0$

$\therefore$ Since infinitely many solutions $\exists$, the system is consistent.

• Rewrite the augmented matrix in Reduced Echelon Form:

(i) $-R_1 + R_2$ / New R_2 $\Rightarrow$

$$\begin{array}{r} -3x_1 + 5x_2 - 4x_3 = 0 \\ + \quad 3x_1 - 5x_2 + 4x_3 = 0 \\ \hline 0x_1 + 0x_2 + 0x_3 = 0 \end{array} \checkmark$$

(ii) $-R_1 + R_3$ / New R_3 $\Rightarrow$

$$\begin{array}{r} -3x_1 + 5x_2 - 4x_3 = 0 \\ + \quad 3x_1 - 5x_2 + 4x_3 = 0 \\ \hline 0x_1 + 0x_2 + 0x_3 = 0 \end{array} \checkmark$$

So, $\begin{bmatrix} 3 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Ex. Continued...

(iii) $\frac{1}{3}R_1 = \text{New } R_1 \implies$ $\begin{bmatrix} 1 & -\frac{5}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

pivot

Reduced Echelon Form.

∴ The associated system B:

$$\begin{cases} x_1 - \frac{5}{3}x_2 + \frac{4}{3}x_3 = 0 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases} \implies \boxed{\begin{cases} x_1 = \frac{5}{3}x_2 - \frac{4}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}}$$

* General Solution

Example: Row reduce the matrix to reduced echelon form. Identify the pivot positions in the final matrix & in the original matrix, & list the pivot columns:

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 2 & 4 & 5 & -10 \\ 4 & 5 & 4 & -11 \end{bmatrix}$$

Answer:

Step 1: Begin w/ the left-most nonzero column:

pivot

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 2 & 4 & 5 & -10 \\ 4 & 5 & 4 & -11 \end{bmatrix}$$

↑

*Pivot Column

*Note: Step 2 is to pick the nonzero entry as the "pivot" position, but we have already done so!

*Step 3: Use row operations to create zeros in all positions below the pivot:

(i)
$$\begin{array}{r} -2R_1 \\ +\ R_2 \\ \hline \text{New } R_2 \end{array} \Rightarrow \begin{array}{rl} & 2x_1 - 4x_2 - 8x_3 + 16x_4 \\ + & 2x_1 + 4x_2 + 5x_3 - 10x_4 \\ \hline & 0x_1 + 0x_2 - 3x_3 + 6x_4 \end{array} \Longleftrightarrow 0x_1 + 0x_2 + x_3 - 2x_4$$

*divide by -3

(ii)
$$\begin{array}{r} -4R_1 \\ +\ R_3 \\ \hline \text{New } R_3 \end{array} \Rightarrow \begin{array}{rrrrr} & -4 & -8 & -16 & +32 \\ + & 4 & 5 & 4 & -11 \\ \hline \end{array}$$

$$0x_1 - 3x_2 - 12x_3 + 21x_4 \Longleftrightarrow 0x_1 + x_2 + 4x_3 - 7x$$

*divide by -3

So, the new matrix is:

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 4 & -7 \end{bmatrix}$$

Example Continued...

(iii) Interchange R_2 & R_3 $\Rightarrow \begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Step 4: Cover the row containing the original pivot (R_1) & repeat steps 1-3 on the resulting submatrix:

New pivot :)

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Pivot Column

✓ (2nd Pivot)

Again :)

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

New pivot

pivot colum

✓ (3rd Pivot)

Step 5: Starting w/ the right-most pivot & working up to the left, create a zero above each pivot

*Note: If a pivot is not already a "1", use a scaling operation.

Ex continued...

***Starting w/ R_3:**

(i) $\begin{array}{r} -4R_3 \\ +\ R_2 \\ \hline \text{New } R_2 \end{array} \Rightarrow + \begin{array}{r} 0x_1 + 0x_2 - 4x_3 + 8x_4 \\ 0x_1 + x_2 + 4x_3 - 7x_4 \\ \hline 0x_1 + x_2 + 0x_3 + x_4 \end{array}$

(ii) $\begin{array}{r} -4R_3 \\ +\ R_1 \\ \hline \text{New } R_1 \end{array} \Rightarrow + \begin{array}{r} 0x_1 + 0x_2 - 4x_3 + 8x_4 \\ x_1 + 2x_2 + 4x_3 - 8x_4 \\ \hline x_1 + 2x_2 + 0x_3 + 0x_4 \end{array}$

So, $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

***Moving to R_2.**

(i) $\begin{array}{r} -2R_2 \\ +\ R_1 \\ \hline \text{New } R_1 \end{array} \Rightarrow + \begin{array}{r} 0x_1 - 2x_2 + 0x_3 - 2x_4 \\ x_1 + 2x_2 + 0x_3 + 0x_4 \\ \hline x_1 + 0x_2 + 0x_3 - 2x_4 \end{array}$

So, $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Ex. Continued...

∴ Row reduced matrix in Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

*Pivot positions for each column are circled here

∴ Note: Pivot Positions in Original Matrix:

$$\begin{bmatrix} 1 & 2 & 4 & -8 \\ 2 & 4 & 5 & -10 \\ 4 & 5 & 4 & -11 \end{bmatrix}$$

Example: Find the general solution of the system whose augmented matrix is given below:

$$\begin{bmatrix} 1 & 0 & 7 & 0 & 9 \\ 0 & 1 & 8 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

Answer:

Note: The given augmented matrix is in reduced echelon form b/c:
(i) The pivot of each non-zero R is 1 (circled)
(ii) Each pivot is the only non-zero entry in that column

* Since the augmented matrix has 5 columns $\Rightarrow \exists$ 4 possible variables...

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 7 & 0 & 9 \\ 0 & 1 & 8 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \Longleftrightarrow \begin{cases} x_1 + 7x_3 = 9 & Eq(1) \\ x_2 + 8x_3 = 3 & Eq(2) \\ x_4 = -5 & Eq(3) \\ 0 = -6 & Eq(4) \end{cases}$$

Contradiction! $0 \neq -6$

$\boxed{\therefore \text{The system has } \underline{\text{NO}} \text{ solution}}$

Example: Find the general solution of the system whose augmented matrix is given below:

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & -8 & 0 & -4 & 3 \\ 0 & 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \end{array}$$

Answer:

Note: The augmented matrix is in Echelon Form here

⇒ The 3rd pivot column has a leading entry of 1, but is NOT the only non-zero entry in that pivot column (i.e. Entry $a_{15} = -4$)

* Rewrite the system in its equivalent, reduced echelon form:

$$\begin{array}{rl} & 4(R_3) \\ + & R_1 \\ \hline & \text{NEW } R_1 \end{array} \Rightarrow \begin{array}{rl} & 0x_1 + 0x_2 + 0x_3 + 0x_4 + 4x_5 = 0 \\ + & x_1 + 0x_2 - 8x_3 + 0x_4 - 4x_5 = 3 \\ \hline & x_1 + 0x_2 - 8x_3 + 0x_4 + 0x_5 = 3 \end{array}$$

So,

$$\left[\begin{array}{ccccc:c} \checkmark & \checkmark & & & \checkmark & \\ 1 & 0 & -8 & 0 & 0 & 3 \\ 0 & 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

* Reduced Echelon Form *

* There are 5 variables since the augmented matrix has 6 columns...

→

Example Continued...

*The Associated System of Equations is then:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -8 & 0 & 0 & 3 \\ 0 & 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 - 8x_3 = 3 & Eq(1) \\ x_2 + 2x_3 - x_4 = 5 & Eq(2) \\ x_5 = 0 & Eq(3) \\ 0 = 0 & Eq(4) \end{cases}$$

- $x_1, x_2,$ & x_5 → The Basic Variables (i.e. Related to the pivots)
- x_3 & x_4 → The Free Variables (i.e. We are free to choose any value)

So,

$$\boxed{\begin{cases} \bullet\ x_1 = 3 + 8x_3 \\ \bullet\ x_2 = 5 - 2x_3 + x_4 \\ \bullet\ x_3 \text{ is free} \\ \bullet\ x_4 \text{ is free} \\ \bullet\ x_5 = 0 \end{cases}}$$

Answer

*The General Solution of the given augmented matrix.

Example: \$ the coefficient matrix of a linear system of 4 equations in 4 variables has a pivot in each column.

(a) Explain why the system has a unique solution

Ans.

Let [A] be a 4x4 coefficient matrix in echelon form. Let $\{a_n\}_{n=1}^{4}$ be the pivot entries & $\{*\}$ be any number (including zero):

$$A = \begin{bmatrix} a_1 & * & * & * \\ 0 & a_2 & * & * \\ 0 & 0 & a_3 & * \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

Note: · No free variables $\nexists$
· No contradictions $\nexists$

* **Recall:** When a system is consistent (@ least one sol. $\exists$), the solution set can be described by solving the reduced system of equations for the basic variables in terms of the free variables.

$\Rightarrow$ **IOW:** (i) A consistent system w/ free variables has MANY possible parametric descriptions.

(ii) A consistent system w/ NO free variables has one, unique solution.

Note:
While coeff. matrix has a pivot in every row, the augmented matrix will

Example: (continued...)

* A 4x4 coefficient matrix in Echelon Form:

$$A = \begin{bmatrix} a_1 & * & * & * \\ 0 & a_2 & * & * \\ 0 & 0 & a_3 & * \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

- $\{a\}_{n=1}^{4} \rightarrow$ Pivots (Leading Entries) (non-zero #)
- $\{*\} \rightarrow$ any #

* This 4x4 matrix in Reduced Echelon Form:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* All pivots are now "1"
* All other entries in a pivot column are zero

* An Augmented 4x4 matrix in REF:

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$$

* 5^{th} column added (RHS of =)
ST a, b, c, d are any #.

$\Rightarrow$ $\boxed{\begin{cases} x_1 = a \\ x_2 = b \\ x_3 = c \\ x_4 = d \end{cases}}$

Basic Variables

$\therefore$ The Linear System is Consistent b/c one unique solution $\exists$. * No free variable

$\Rightarrow$ IOW: The Rightmost column of augmented matrix is NOT a pivot column.

Ex: \$ a system of linear equations has a 3×5 augmented matrix whose 5th column is not a pivot column. Is the system consistent? Explain.

Answer:

Let [A] be a 3×5 augmented matrix ST $\{a_n\}_{n=1}^{3}$ are pivots (any non-zero number) & $\{*\}, b, c, d$ be any #.

$$\Rightarrow A = \left[\begin{array}{cccc|c} a_1 & * & * & * & b \\ 0 & a_2 & * & * & c \\ 0 & 0 & a_3 & * & d \end{array}\right]$$

* pivots are circled.

Note: This is NOT a unique solution! Others can exist :)

⇒ For this example:

- x_1, x_2, x_3 are the Basic Variables
- x_4 is a Free Variable

*Caution: Solving a system amounts to
① Finding a Parametric Description of Solution Set.
-OR-
② Determining the Solution Set is empty ($\rightarrow\leftarrow$)

Recall:

(i) A Linear System is Consistent IFF the last column is not a pivot column ✓

* IOW: No contradictions can ∃; No rows of the form [0 0 0 0 ¦ b]

(ii) If a Linear System is Consistent, MANY parametric descriptions can ∃

(iii) If a Linear System is Inconsistent, the sol. set is empty (No parametric description?) ⇒ $\rightarrow\leftarrow$ ∃