Algorithms Fall 2016 Homework 6 Solution

October 25, 2016

PROBLEM 1

We can convert this problem to a maximum-flow problem by constructing a flow network $G = (V, E)$ by following steps:

1. For each corner in the map (including the corners where home and school are located), create a vertex v and add v to V.
2. For each pair of corners u and v, if there exists a block street between them, we add two directed edges (u, v) and (v, u) to E.
3. Set the capacity of each edge to 1.
4. Set the corner where Prof. Adam' house is located as source.
5. Set the corner where the school is located as sink.

Since the capacity of each edge is 1, there exist two edge-disjoint paths from the source to the sink only if we can find a flow of value 2. Therefore, the original problem can be converted to a maximum-flow problem: if the maximum-flow of G is greater than 1, then the two children can go to the same school; otherwise they can not go to the same school.

PROBLEM 2

We can assume that all the capacities of each edge are integer(otherwise we can scale up all the capacities to integer), so that the difference between two flows with different capacities is at least 1. We can solve this problem by constructing a new graph G' by adding $\delta = \frac{1}{2|E|}$ to the capacity of each edge. So we have $c' = c(u, v) + \delta$ for $(u, v) \in E$.
Then we will prove that the minimum cut in G' must be a minimum cut with the smallest number of edges in G. Suppose (S, T) is a minimum cut in G with the smallest number of edges, then we can prove that $c'(S, T) \leq c'(X, Y)$ for any cut (X, Y) in G. Let $N(X, Y)$ denote the number of edges in the cut (X, Y).
There are two cases:

- If (X, Y) is a minimum cut, we have

$$c'(X, Y) - c'(S, T) = |f_{max}| + N(X, Y)\delta - (|f_{max}| + N(S, T)\delta) = (N(X, Y) - N(S, T))\delta \geq 0$$

- If (X, Y) is not a minimum cut, we have

$$\begin{aligned} c'(X, Y) - c'(S, T) &= c(X, Y) + N(X, Y)\delta - (|f_{max}| + N(S, T)\delta) \\ &= (c(X, Y) - |f_{max}|) + (N(X, Y) - N(S, T))\delta \\ &\geq 1 + (N(X, Y) - N(S, T))\delta \\ &\geq 1 - \frac{N(S, T)}{2|E|} \\ &> 0 \end{aligned}$$

Therefore, we can find such (S, T) in G by finding the minimum cut (S, T) int G'.

PROBLEM 3

a) The capacity of a cut is the sum of the capacities of the edges crossing it. Since the number of such edges is at most $|E|$, and the capacity of each edge is at most C, the capacity of any cut of G is at most $C|E|$.

b) We use either breadth-first search or depth-first search to find a path from s to t, considering only edges with residual capacity at least k since the capacity of a augmenting path is the minimum capacity of any edge on the path. The search takes $O(V + E) = O(E) = O(E)$ times since $|V| = O(E)$ in a flow network.

c) MAX-FLOW-BY-SCALING uses the FORD-FULKERSON method. It repeatedly augments the flow along an augmenting path until there are no augmenting paths with capacity at least 1.Since all the capacities are integers, and the capacity of an augmenting path is positive, when there are no augmenting paths with capacity at least 1, there must be no augmenting paths in the residual network. Therefore with the max-flow min-cut theorem, MAX-FLOW-BY-SCALING returns a maximum flow.