1.8: INTRO. TO LINEAR TRANSFORMATION

Def: Let T be a special rule / function that assigns / maps each $\vec{x} \in \mathbb{R}^n$ to $T(\vec{x}) \in \mathbb{R}^m$

* Def.: $\boxed{T: \mathbb{R}^n \to \mathbb{R}^m \text{ ST } T(\vec{x}) = A\vec{x}}$

* Domain: $\mathbb{R}^n$ * Codomain: $\mathbb{R}^m$

* The image of $\vec{x}$ under the action of T: $T(\vec{x})$

→ The set of all images, $T(\vec{x})$ is the Range.

* Example: $T: \mathbb{R}^2 \to \mathbb{R}^3$ ST $T(\vec{x}) = A\vec{x}$, where:

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

* (a) Find $T(\vec{u})$
* (b) Find $\vec{x}$ whose image under T is $\vec{b}$.

(a) * Since $T(\vec{x}) = A\vec{x}$, then:

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Apply Row-Column Rule (i.e: Dot Product)

$$= \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

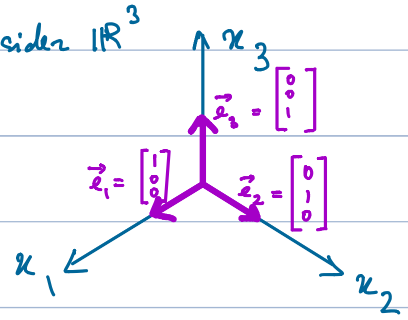
Recall: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a transformation ST: $T(\vec{x}) = A\vec{x}$ where A is an $m \times n$ matrix & $\vec{x} \in \mathbb{R}^n$. T is linear if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(c\vec{u}) = cT(\vec{u})$$

ST $\vec{u}, \vec{v} \in \mathbb{R}^n$
c is any scalar

* <u>Elementary vectors</u>: An elementary vector is a unit vector (length 1) pointing in the direction of the coordinate axes. (+ direction)

$\Rightarrow$ Consider $\mathbb{R}^3$



* <u>Notes</u>: we can write any $\vec{x} \in \mathbb{R}^3$ as a linear combination of the elementary vectors.

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 \quad (\text{* Vector Eq.})$$

* Matrix Eq.: $A\vec{x} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

<u>Example</u>: Let $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$ (elementary vectors in $\mathbb{R}^2$). Let $\vec{y}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ $\vec{y}_2 = \begin{bmatrix}3\\4\end{bmatrix}$

Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a Linear Transformation that maps $\vec{e}_1 \to \vec{y}_1$ and $\vec{e}_2 \to \vec{y}_2$.

Find the image under T of $\begin{bmatrix}5\\6\end{bmatrix}$?

<u>Ans</u>: want $T\left(\begin{bmatrix}5\\6\end{bmatrix}\right) = ?$

Given: $\vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{y}_1 = \begin{bmatrix}1\\2\end{bmatrix}$

$\vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{y}_2 = \begin{bmatrix}3\\4\end{bmatrix}$

Rewrite $\begin{bmatrix}5\\6\end{bmatrix}$ as a linear combination of $\vec{e}_1$ & $\vec{e}_2$

$$\begin{bmatrix}5\\6\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix} = 5\vec{e}_1 + 6\vec{e}_2$$

Sub in:

$$T\left(\begin{bmatrix}5\\6\end{bmatrix}\right) = T(5\vec{e}_1 + 6\vec{e}_2) = 5\,T(\vec{e}_1) + 6\,T(\vec{e}_2) = 5\vec{y}_1 + 6\vec{y}_2 = 5\begin{bmatrix}1\\2\end{bmatrix} + 6\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}23\\34\end{bmatrix}$$