Section 4.2: Null Spaces, Column Spaces, & Linear Transformations

Note: Subspaces of $\mathbb{R}^n$ usually arise 1 of 2 ways:

① As a set of all solutions to a system of Homogeneous Linear Eq.

② As the set of all linear combinations of certain specified vectors.

*In this section, we compare & contrast these ↑ 2 descriptions, quickly realizing that we have been working w/ subspaces since Day 1 *

The Null Space of a Matrix

Consider the following system of homogeneous eq:

$$\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 4x_2 = 0 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

Recall: The set of all $\vec{x}$ that satisfy ↑ is called: "The Solution Set" of the system.

We will now call the set of $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$ the "Null Space" of the Matrix A.

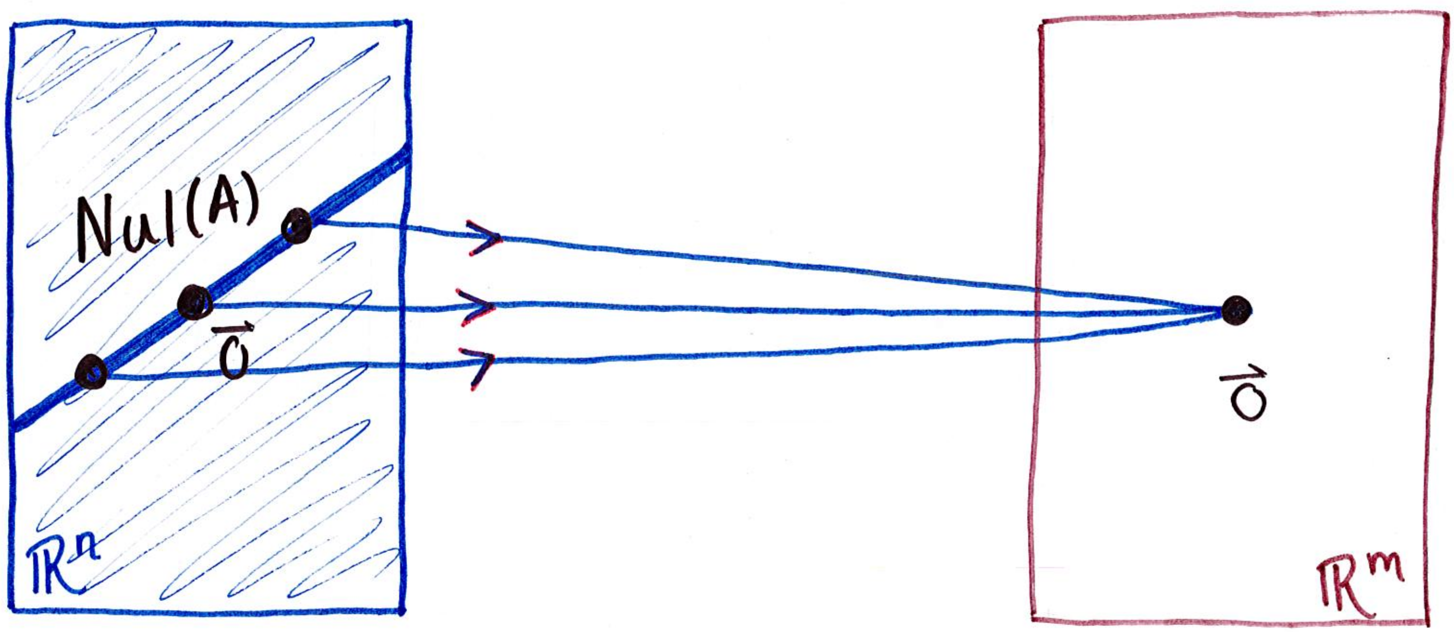
*Definition: The **Null Space** of an $m \times n$ matrix A, written as "Nul(A)", is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$.

- In Set Notation: $\text{Nul}(A) = \{\vec{x} : \vec{x} \in \mathbb{R}^n \ \& \ A\vec{x} = \vec{0}\}$

The Null Space as a Linear Transformation

A more dynamic description of the Nul(A) is the set of all $\vec{x} \in \mathbb{R}^n$ that are mapped onto $\vec{0} \in \mathbb{R}^m$ via the Linear Transformation $\vec{x} \mapsto A\vec{x}$

Graphical Interpretation:



Example (Null Space): Determine if $\vec{u}$ belongs to the null space of A:

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \quad \& \quad \vec{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Answer:

*Note: To test if $\vec{u}$ satisfies $A\vec{u} = \vec{0}$, simply compute "$A\vec{u}$".

* Compute the Matrix Equation, $A\vec{u}$:

$$A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 9 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -25 \end{bmatrix} + \begin{bmatrix} -9 \\ 27 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

$\therefore$ Since $A\vec{u} = \vec{0}$, $\vec{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ is in the Null Space of A.

Answer.

Example: Determine if $\vec{w}$ is in the Nul(A):

$$\vec{w} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & -5 & -10 \\ 4 & -2 & -9 \\ -2 & 2 & 4 \end{bmatrix}$$

Answer:

*Recall: To determine if $\vec{w}$ is in the Nul(A), compute: $A\vec{w}$

*Compute $A\vec{w}$:

$$\begin{bmatrix} 5 & -5 & -10 \\ 4 & -2 & -9 \\ -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} -5 \\ -2 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -10 \\ -9 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -5 & -20 \\ 20 & -2 & -18 \\ -10 & +2 & +8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$$\boxed{\therefore \text{ Since } A\vec{w} = \vec{0}, \ \vec{w} \in \text{Nul}(A)}$$

Answer

Theorem 2:

* The Null Space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

-Equivalently-

* The set of all solutions to a system $A\vec{x} = \vec{0}$ of m-homogeneous Linear Equations in n-unknowns is a subspace of $\mathbb{R}^n$.

Note: The Null Space of A is certainly a subset of $\mathbb{R}^n$ (b/c A has n-Col.) BUT we must still show that the $Nul(A)$ satisfies the 3 properties of a subset ∵

Proof:

- Let A be an $m \times n$ matrix.

① Show that $\vec{0} \in \mathbb{R}^n$ is in $Nul(A)$: $\boxed{\text{Of course } \vec{0} \in Nul(A) \checkmark}$

② Show that the $Nul(A)$ is closed under addition:

Goal: To show $\vec{u} + \vec{v} \in Nul(A)$, we must show: $A(\vec{u} + \vec{v}) = \vec{0}$.

- Let $\vec{u}$ & $\vec{v}$ be vectors in $Nul(A)$, $\vec{u}, \vec{v} \in Nul(A)$.
- Then by Def: $A\vec{u} = \vec{0}$ & $A\vec{v} = \vec{0}$.
- By Prop. of Matrix Multiplication: $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$ ✓

∴ Since $A(\vec{u} + \vec{v}) = \vec{0}$, then $\vec{u} + \vec{v} \in Nul(A)$ & so $Nul(A)$ is closed under addition.

③ Show that the $Nul(A)$ is closed under scalar multiplication:

Goal: To show $c\vec{u} \in Nul(A)$, we must show: $A(c\vec{u}) = \vec{0}$.

- Let c be any scalar.
- By Prop. of Matrix Multiplication: $A(c\vec{u}) = c(A\vec{u}) = c(\vec{0}) = \vec{0}$ ✓

∴ Since $A(c\vec{u}) = \vec{0}$, then $c\vec{u} \in Nul(A)$ & so $Nul(A)$ is closed under scalar-multiplication.

∴ Therefore: $Nul(A)$ is a subspace of $\mathbb{R}^n$ $\boxed{X}$

Example: (Nul(A) as a subspace of $\mathbb{R}^n$):

Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a,b,c,d satisfy the equations $a-2b+5c=d$ & $c-a=b$. Show that H is a subspace of $\mathbb{R}^4$.

Answer:

*Note: It is important that the Linear Equations defining the set H are homogeneous ($A\vec{x}=\vec{0}$)

→ Otherwise, the set of solutions will NOT be a subspace (b/c $\vec{0}$ is NOT a solution of a nonhomogeneous system)

*Also, in some cases, the set of solution could be empty.

*Rearrange the equations that describe the elements of H:

$$\begin{cases} a-2b+5c-d=0 \\ -a-b+c=0 \end{cases} \Longleftrightarrow \underbrace{\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

So, H is the set of all solutions that satisfy the system $A\vec{x}=\vec{0}$ of $m=2$ Homogeneous Linear Equations, in $n=4$ unknowns.

$\Rightarrow \therefore$ $\boxed{\text{(By Theorem \#2): H is a subspace of } \mathbb{R}^4}$

Answer.

*An Explicit Description of Nul(A) *

- There is NO obvious relation btw Nul(A) & the entries of A.

$\Rightarrow$ Nul(A) is defined implicitly b/c it is defined by a condition that must be checked.

$\Rightarrow$ NO explicit list/description of the elements in Nul(A) is provided/given.

*

- **To produce an explicit description of Nul(A), we solve $A\vec{x}=\vec{0}$:** *

① Find a General Solution of $A\vec{x}=\vec{0}$ in terms of any free variables

* Row-reduce $[A \,\vdots\, \vec{0}]$ to rref & write the basic variable(s) in terms of the free variable(s).

② Decompose the vector giving the General Solution into a Linear Combination of vectors

* The free variable(s) are the weights.

*Important Notes: (When Nul(A) contains nonzero vectors) *

① The spanning set produced by solving $A\vec{x}=\vec{0}$ is automatically Linearly Independent b/c the Free Variable(s) are the weights of the spanning set. (By Def.)

② The number of vectors in the spanning set for Nul(A) equals the number of free variables in the equation $A\vec{x}=\vec{0}$.

Example (Explicit Descriptions of Nul(A)):

Find a spanning set for the null space of the matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Answer:

Note: Here we need to find a general solution of $A\vec{x} = \vec{b}$ ∵

*Row-Reduce $[A \,\vdots\, \vec{0}]$ to rref:

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \overset{\frac{3R_1 + R_2}{N.R_2}}{\sim}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \overset{\frac{1}{5}R_2}{\sim} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \overset{\frac{-2R_1 + R_3}{N.R_3}}{\sim} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\overset{\frac{-R_2 + R_3}{N.R_3}}{\sim} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{\frac{-2R_2 + R_1}{N.R_1}}{\sim} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

* Echelon Form *

$$\Rightarrow \begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} \cdot\, x_1 = 2x_2 + x_4 - 3x_5 \\ \cdot\, x_2 \text{ is free} \\ \cdot\, x_3 = -2x_4 + 2x_5 \\ \cdot\, x_4 \text{ is free} \\ \cdot\, x_5 \text{ is free} \end{cases}$$

Notes:

- Basic Variables: x_1 & x_3
- Free Variables: $x_2, x_4,$ & x_5

Example (Explicit Description) Continued...

* **Decompose the General Solution Vector into a Linear Combination of Vectors (ST 'Weights' = Free Variables):**

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_5 \\ 0 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix}$$

$$\Longrightarrow \boxed{\vec{x} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}}}$$ Ans.

So, every Linear Combination of $\vec{u}, \vec{v}, \& \vec{w}$ is an element of Nul(A) (& vice versa).

$\boxed{\therefore \{\vec{u}, \vec{v}, \vec{w}\} \text{ is a spanning set of } \mathrm{Nul}(A).}$ Ans.

* **Additional Notes/ Observations**: Since Nul(A) contains nonzero vectors

① The set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ is Linearly Independent!

$\Rightarrow$ $\boxed{\vec{x} = x_2\vec{u} + x_4\vec{v} + x_5\vec{w} \text{ has only the trivial sol., } \vec{x} = \vec{0}, \text{ when } x_2 = x_4 = x_5 = 0}$

② (# of vectors in the Spanning Set of Nul(A)) = (# of free variables in $A\vec{x} = \vec{0}$)

$$\boxed{3 = 3}$$

Example: Find an explicit description of Nul(A) by listing vectors that span the null space:

$$A = \begin{bmatrix} 1 & -5 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Answer:

* Row-reduce the augmented matrix $[A \,\vdots\, \vec{0}]$ to rref to write the basic variables in terms of the free variables:

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{cccc|c} 1 & -5 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}R_2}{\sim} \left[\begin{array}{cccc|c} 1 & -5 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\overset{\frac{2R_2 + R_1}{N.R_1}}{\sim} \left[\begin{array}{cccc|c} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 - 5x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 5x_2 \\ x_2 \text{ is free} \\ x_3 = 0 \\ x_4 \text{ is free} \end{cases}$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} \quad \text{where } x_2 \,\&\, x_4 \text{ are any scalar.}}$$

$\therefore$ Every Linear Combination of $\vec{u}$ & $\vec{v}$ is an element of Nul(A):

$$\boxed{\text{So, } \left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a spanning set for Nul(A)}}$$

Ans✓

Example: Find an explicit description of Nul(A) by listing vectors that span the null space:

$$A = \begin{bmatrix} 1 & 4 & 0 & -2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Answer:

*Row-reduce the Augmented Matrix $[A \,\vdots\, \vec{0}]$ to rref to write the basic variables in terms of the free-variables:

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{ccccc|c} 1 & 4 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{array}\right] \overset{-\frac{1}{5}R_3}{\sim} \left[\begin{array}{ccccc|c} 1 & 4 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 4x_2 - 2x_4 = 0 \\ x_3 - 3x_4 = 0 \\ x_5 = 0 \end{cases} \iff \begin{cases} x_1 = -4x_2 + 2x_4 \\ x_2 \text{ is free} \\ x_3 = 3x_4 \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

$\therefore$ General Solution:

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 + 2x_4 \\ x_2 \\ 3x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad \text{where } x_2 \text{ \& } x_4 \text{ are any scalar}}$$

Example Continued...

Note: Every Linear Combination of the vectors

$\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ is an element of Nul(A) :)

$\therefore$ A Spanning Set for the Nul(A):

$$\boxed{\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

Answer

Example: Either use an appropriate theorem to show that the given set, W, is a vector space -OR- Find a specific example to the contrary:

$$W = \left\{ \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}, \begin{array}{l} -p+q = 2s \\ -2p = -s+3r \end{array} \right\}$$

Answer:

* **Recall (Thm #2):** The null space of an $m \times n$ matrix A is a supspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\vec{x} = \vec{0}$ of m-homogeneous linear eq. in n-unknowns is a subspace of $\mathbb{R}^n$.

* **Rearrange** the terms that define "W" ST $A\vec{x} = 0$:

$$W = \begin{cases} -p+q-2s = 0 \\ -2p-3r+s = 0 \end{cases} \iff \boxed{\underbrace{\begin{bmatrix} -1 & 1 & 0 & -2 \\ -2 & 0 & -3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}}_{\vec{x}} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}}$$

'W' is the set of all solutions of the system of Homogeneous Eq, $A\vec{x} = \vec{0}$.

$\therefore$ By Theorem 2, $W = \text{Null}(A)$ is a subspace of $\mathbb{R}^4$

Note: W must be a vector space b/c a subspace IS a vector space :)

Example: Either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary:

$$W = \left\{ \begin{bmatrix} s-2t \\ 3+3s \\ 3s+t \\ 3s \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

Answer:

*Recall (Thm #2): The null space of an m×n matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\vec{x}=\vec{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

* Rearrange the terms that define "W" ST $A\vec{x}=\vec{0}$

$$W = \left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 3 \\ 3 \\ 3 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right. \Rightarrow \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 0 \\ 3 & 1 \\ 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s \\ t \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}} \rightarrow\leftarrow$$

* The zero vector $\vec{0} \in \mathbb{R}^4$ is NOT in W

$$\boxed{\begin{array}{l} \therefore \text{W is NOT a subspace of } \mathbb{R}^4 \text{ \& so} \\ \text{W is NOT a vectorspace} \end{array}}$$

Ans.

The Column Space of a Matrix

*Definition: The **Column Space** of an $m \times n$ matrix A, written as "$\text{Col}(A)$", is the set of all linear combinations of the columns of A.

- If $A = [\vec{a}_1 \cdots \vec{a}_n]$, then: $\text{Col}(A) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

- In Set Notation: $\boxed{\text{Col}(A) = \{\vec{b} : \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}}$

*Notes:
- "$A\vec{x}$" stands for a Linear Combination of the Columns of A.
- "$A\vec{x}$" also shows that $\text{Col}(A)$ is in the range of the Linear Transformation $\vec{x} \mapsto A\vec{x}$.

- Since $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$ is a subspace (by Th^m 1, sect. 4.1), the next theorem is a direct result of the Def. of $\text{Col}(A)$ -AND- the fact that the columns of A are in $\mathbb{R}^m$.

*Theorem 3:
The Column Space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$

*Important Note/Connection w/ previous section:
The Column Space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ IFF the equation $A\vec{x} = \vec{b}$ has a solution $\forall\ \vec{b} \in \mathbb{R}^m$.

(*Same Equivalence Prop. seen in chapter 1 :))

Example (Column Space): Find a matrix A st $W = \text{Col}(A)$.

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a,b \text{ in } \mathbb{R} \right\}$$

Answer:

*First we must rewrite W as a Linear Combination: $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$

*IOW: Decompose W :)

$$W = \left\{ \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} = a\underbrace{\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}}_{\vec{a}_1} + b\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}_2} : a,b \in \mathbb{R} \right\}$$

So, $\boxed{W = \text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}}$

*Next, use the vectors of the spanning set to define the columns of matrix A:

$$\boxed{A = [\vec{a}_1 \ \vec{a}_2] = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}}$$

*Therefore: (By Def.)

$$\boxed{\text{Since } A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}, \text{ then } W = \text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix} \right\}}$$

Ans.

Example: Find A ST the given set is $\text{Col}(A)$:

$$\left\{ \begin{bmatrix} 2r+3s+t \\ -r-2s-3t \\ -2r+s+2t \\ -s+3t \end{bmatrix} : r,s,t \in \mathbb{R} \right\}$$

Answer:

Write as a set of Linear Combinations:

$$\left\{ r\begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix} : r,s,t \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Use the vectors in the spanning set as the columns of A:

$$\boxed{A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & -3 \\ -2 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}}$$

Ans

The Contrast Btw Nul(A) & Col(A)

Nul(A):	vs.	Col(A):
① Nul(A) is a subspace of $\mathbb{R}^n$.		① Col(A) is a subspace of $\mathbb{R}^m$.
② Nul(A) is implicitly defined; *IOW: we are only given a condition ($A\vec{x}=\vec{0}$) that vectors in Nul(A) must satisfy.		② Col(A) is explicitly defined. *IOW: We are told how to build vectors in Col(A).
③ It takes time to find vectors in Nul(A): $\Rightarrow$ row-reduce: $[A \vdots \vec{0}]$		③ It is easy/quick to find vectors in Col(A) $\Rightarrow$ Columns of A are displayed; others can be formed from them.
④ NO obvious relation btw Nul(A) & the entries in A.		④ Obvious relation btw Col(A) & entries in A $\Rightarrow$ Each Column of A is in Col(A)
⑤ A typical vector $\vec{v}$ in Nul(A) has the prop: $\boxed{A\vec{v}=\vec{0}}$		⑤ A typical vector $\vec{v}$ in Col(A) has the prop: $\boxed{A\vec{x}=\vec{v} \text{ is consistent}}$
⑥ Given a specific vector $\vec{v}$, it is easy to tell if $\vec{v} \in$ Nul(A) $\Longrightarrow$ $\boxed{\text{Compute } A\vec{v}}$		⑥ Given a specific vector $\vec{v}$, it may take time to tell if $\vec{v} \in$ Col(A) $\Rightarrow$ $\boxed{\text{Row-Reduce } [A \vdots \vec{v}]}$
⑦ Nul(A) = $\{\vec{0}\}$ IFF $A\vec{x}=\vec{0}$ has ONLY the Trivial Solution.		⑦ Col(A) = $\mathbb{R}^m$ IFF $A\vec{x}=\vec{b}$ has a solution $\forall\, \vec{b} \in \mathbb{R}^m$.
⑧ Nul(A) = $\{\vec{0}\}$ IFF the Linear Transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one		⑧ Col(A) = $\mathbb{R}^m$ IFF the Linear Transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Example (Contrast #1): Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

(a) If the Column Space of A is a subspace of $\mathbb{R}^K$, what is K?

(b) If the Null Space of A is a subspace of $\mathbb{R}^K$, what is K?

Answer:

* **Part (a): \$ Col(A) is a subspace of $\mathbb{R}^K$, find K:**

• **Recall (Theorem 3):** The Column Space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$

• Given: 3×4 matrix $A \Rightarrow \begin{cases} \cdot\ 3\text{-Rows (Eq.)} \\ \cdot\ 4\text{-Columns (Unknowns)} \end{cases}$

* The Columns of A each have 3 entries.

∴ Since A has $m = 3$ rows, then the Column Space of A is a subspace of $\mathbb{R}^3$. $\Rightarrow \boxed{K = 3}$

* **Part (b): \$ Nul(A) is a subspace of $\mathbb{R}^K$, find K:**

• **Recall (Thm #2):** The Null Space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

* The vector $\vec{x}$ ST $A\vec{x}$ must have 4 entries.

∴ Since A has $n = 4$ columns, then the Null Space of A is a subspace of $\mathbb{R}^4$. $\Rightarrow \boxed{K = 4}$

<u>Example (Contrast #2)</u>: Find a nonzero vector in Col(A) <u>and</u> find a nonzero vector in Nul(A):

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

<u>Answer:</u>

* <u>Find a nonzero vector in Col(A):</u>

<u>Note</u>: Each column of A is a vector in Col(A) :)

$\Rightarrow$ Any one of the 4 columns will suffice ✓

$$\boxed{\therefore \vec{a_4} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \text{ is a nonzero vector ST } \vec{a_4} \in Col(A)}$$

<u>Ans</u> ✓

* <u>Find a nonzero vector in Nul(A):</u>

<u>Note</u>: Here we must row-reduce $[A \,\vdots\, \vec{0}]$ to find a nonzero vector in Nul(A).

$$[A \,\vdots\, \vec{0}] = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \underset{\sim}{\frac{R_1 + R_2}{N.R_2}} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 3 & 7 & -8 & 6 \end{bmatrix} \underset{\sim}{\overset{\frac{1}{2}R_1}{-R_2}} \begin{bmatrix} 1 & 2 & -1 & 1/2 \\ 0 & 1 & -5 & -4 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

$$\underset{\sim}{\frac{-3R_1 + R_3}{N.R_3}} \begin{bmatrix} 1 & 2 & -1 & 1/2 \\ 0 & 1 & -5 & -4 \\ 0 & 1 & -5 & 9/2 \end{bmatrix} \underset{\sim}{\frac{-R_2 + R_3}{N.R_3}} \begin{bmatrix} 1 & 2 & -1 & 1/2 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 17/2 \end{bmatrix} \underset{\sim}{\frac{-2R_2 + R_1}{N.R_1}} \begin{bmatrix} 1 & 0 & 9 & 17/2 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 17/2 \end{bmatrix}$$

↘

Example (Contrast #2) Continued...

$$\begin{bmatrix} 1 & 0 & 9 & 17/2 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 17/2 \end{bmatrix} \underset{N.R_1}{\overset{-R_3+R_1}{\sim}} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 17/2 \end{bmatrix} \overset{\frac{2}{17}R_3}{\sim} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{N.R_2}{\overset{4R_3+R_2}{\sim}} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow \begin{cases} x_1 + 9x_3 = 0 \\ x_2 - 5x_3 = 0 \\ x_4 = 0 \end{cases} \Longrightarrow \boxed{\begin{cases} x_1 = -9x_3 \\ x_2 = 5x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}}$$

A nonzero vector in Nul(A):

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix} \quad \text{where } x_3 \text{ is any scalar}}$$

Ans.

Example (Contrasts^3): Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \& \quad \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(a) Determine if $\vec{u}$ is in Nul(A). Could $\vec{u}$ be in Col(A)?

(b) Determine if $\vec{v}$ is in Col(A). Could $\vec{v}$ be in Nul(A)?

Answer:

* **Part(a):** To determine if $\vec{u} \in \text{Nul}(A)$, compute $A\vec{u}$:

$$A\vec{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} - \begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 - 8 + 2 + 0 \\ -6 + 10 - 7 + 0 \\ 9 - 14 + 8 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $\boxed{\vec{u} \text{ is } \underline{\text{NOT}} \text{ in Nul}(A)}$

* Since $\vec{u}$ is NOT a solution of $A\vec{x} = \vec{0}$, $\vec{u} \notin \text{Nul}(A)$.

* Since Col(A) is a subspace of $\mathbb{R}^3$ & $\vec{u} \in \mathbb{R}^4$
$\Rightarrow \vec{u} \notin \text{Col}(A)$.

Similarly:

* Since Nul(A) is a subspace of $\mathbb{R}^4$ & $\vec{v} \in \mathbb{R}^3$
$\Rightarrow \vec{v} \notin \text{Nul}(A)$.

Example (Contrasts³) continued...

*Note: Stopping once you reach echelon form is sufficient ∴

***Part (b): To Determine if $\vec{v} \in \text{Col}(A)$, row-reduce $[A \,\vdots\, \vec{v}]$**

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \overset{\frac{R_1 + R_2}{N.R_2}}{\sim} \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \overset{\frac{1}{2}R_1}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1/2 & 3/2 \\ 0 & -1 & 5 & 4 & 2 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \overset{\frac{-3R_1 + R_3}{N.R_3}}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1/2 & 3/2 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 1 & -5 & 9/2 & -3/2 \end{array}\right] \overset{\frac{R_2 + R_3}{N.R_3}}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1/2 & 3/2 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17/2 & 1/2 \end{array}\right] \overset{\frac{2R_2 + R_1}{N.R_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 9 & 17/2 & 11/2 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17/2 & 1/2 \end{array}\right] \overset{-R_2}{\sim}$$

*We can stop here ↑ echelon-form shows us system is consistent.

$$\left[\begin{array}{cccc|c} 1 & 0 & 9 & 17/2 & 11/2 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17/2 & 1/2 \end{array}\right] \overset{\frac{-R_3 + R_1}{N.R_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17/2 & 1/2 \end{array}\right] \overset{\frac{2}{17}R_3}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 1 & 1/17 \end{array}\right] \overset{\frac{4R_3 + R_2}{N.R_2}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right]$$

rref

Ans: $\boxed{\begin{array}{l} \therefore \text{Since } A\vec{x} = \vec{v} \text{ is consisent (@ least one solution } \exists) \\ \implies \text{Yes, } \vec{v} \in \text{Col}(A). \end{array}}$

Example: Find the following for the provided matrix:

$$A = \begin{bmatrix} 0 & 9 \\ 5 & -3 \\ -7 & -8 \\ -7 & -5 \end{bmatrix}$$

(a) Find K ST Nul(A) is a subspace for $\mathbb{R}^K$.

(b) Find K ST Col(A) is a subspace for $\mathbb{R}^K$.

Answer:

* Recall: For an m×n matrix A:
  - Nul(A) is a subspace of $\mathbb{R}^n$
  - Col(A) is a subspace of $\mathbb{R}^m$

* Given: $m \times n = 4 \times 2$ matrix A

(a) A vector $\vec{x}$ ST $A\vec{x}$ is defined must have 2 entries:

$\therefore$ Nul(A) is a subspace of $\mathbb{R}^2 \Rightarrow \boxed{K = 2}$

Ans.

(b) The Columns of A each have 4 entries:

$\therefore$ Col(A) is a subspace of $\mathbb{R}^4 \Rightarrow \boxed{K = 4}$

Ans.

Example: Find the following for the provided matrix

$$A = \begin{bmatrix} 5 & -2 & 9 & 6 \\ 4 & 6 & 6 & 0 \\ -5 & 1 & -1 & -6 \\ -8 & 2 & 4 & 5 \\ 6 & -2 & 5 & -5 \end{bmatrix}$$

(a) Find K ST $Nul(A)$ is a subspace of $\mathbb{R}^K$.

(b) Find K ST $Col(A)$ is a subspace of $\mathbb{R}^K$.

Answer:

*Recall: For some $m \times n$ matrix A
- Nul(A) is a subspace of $\mathbb{R}^n$. (# of entries in $\vec{x}$ ST $A\vec{x}$)
- Col(A) is a subspace of $\mathbb{R}^m$. (# of entries per column)

*Given: $m \times n = 5 \times 4$ matrix A

(a) A vector $\vec{x}$ (ST $A\vec{x}$ is defined) must have 4 entries:

$\therefore$ $\boxed{Nul(A) \text{ is a subspace of } \mathbb{R}^4 \implies K = 4}$

(b) The Columns of A each have 5 entries:

$\therefore$ $\boxed{Col(A) \text{ is a subspace of } \mathbb{R}^5 \rightarrow K = 5}$

Example: For the matrix A below, Find the Following:

$$A = \begin{bmatrix} 6 & 8 \\ 3 & 4 \\ -12 & -16 \\ 9 & 12 \end{bmatrix}$$

(a) Find a nonzero vector for Nul(A).

(b) Find a nonzero vector for Col(A).

Answer:

Recall: To find a nonzero vector in:
- Nul(A) → row-reduce $[A \mid \vec{0}]$
- Col(A) → choose any column-vector

***Part(a): Find a nonzero vector in Nul(A):**

$$[A \mid \vec{0}] = \left[\begin{array}{cc|c} 6 & 8 & 0 \\ 3 & 4 & 0 \\ -12 & -16 & 0 \\ 9 & 12 & 0 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ \\ -\frac{1}{4}R_3 \\ \frac{1}{3}R_4 \end{array} \sim \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 3 & 4 & 0 \\ 3 & 4 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

Whoa!

$\frac{-R_1 + R_2}{N.R_2}, \frac{-R_1 + R_3}{N.R_3}, \frac{-R_1 + R_4}{N.R_4}$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$$

∴ If $\vec{x}$ satisfies $A\vec{x} = \vec{0}$, then:

$$\begin{cases} x_1 = -\frac{4}{3}x_2 \\ x_2 \text{ is free} \end{cases} \Rightarrow \boxed{\vec{x} = x_2 \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix} \text{ where } x_2 \in \mathbb{R}}$$

Ans.

***Part(b): Find a nonzero vector in Col(A):**

$A = [\vec{a}_1 \ \vec{a}_2] \Rightarrow \vec{a}_1, \vec{a}_2 \in Col(A)$

$$\therefore \boxed{\begin{bmatrix} 6 \\ 3 \\ -12 \\ 9 \end{bmatrix} \in Col(A)}$$

Ans.

Kernel & Range of a Linear Transformation

Note: Subspaces of vector spaces other than $\mathbb{R}^n$ are often described in terms of a Linear Transformation instead of a matrix.

*Definition: A Linear Transformation, T, from a vector space V into a vector space W is a rule that assigns to each vector $\vec{x} \in V$ a unique vector $T(\vec{x}) \in W$ ST:

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}), \quad \forall \ \vec{u},\vec{v} \in V.$

-AND-

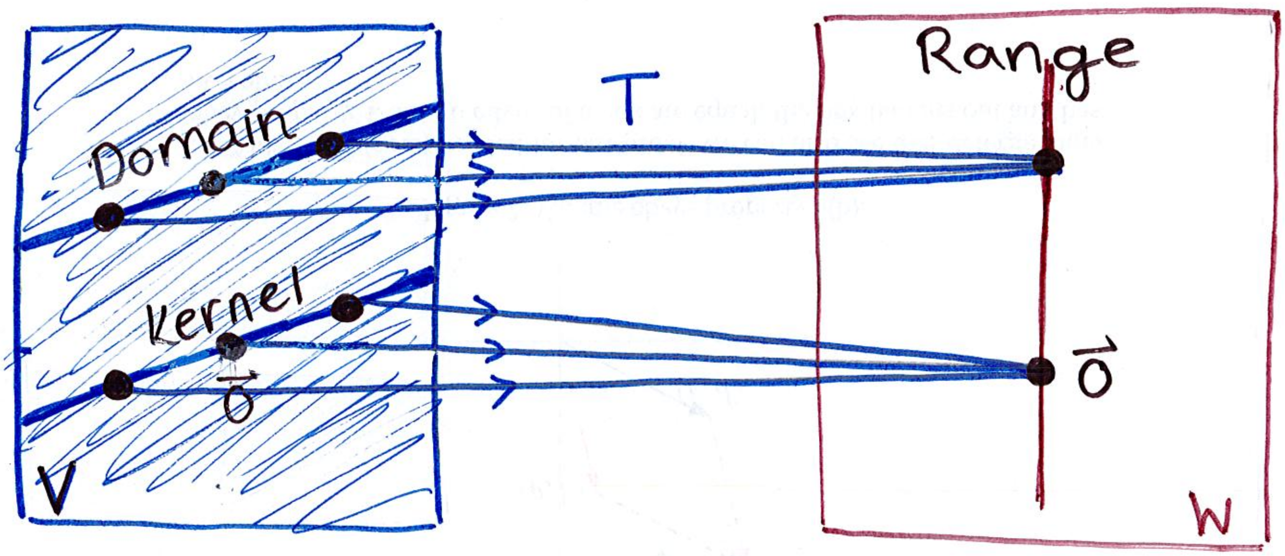
(ii) $T(c\vec{u}) = cT(\vec{u}), \quad \forall \ \vec{u} \in V \ \& \ \forall$ scalars c.

- The Kernel (or Null Space) of such a T is the set of all $\vec{u} \in V$ ST $T(\vec{u}) = \vec{0}$ (the zero vector in W).

- The range of T is the set of all vectors in W of the form $T(\vec{x})$ for some $\vec{x} \in V$.

Note:
*If T happens to arise as a matrix transformation (i.e. $T(\vec{x}) = A\vec{x}$), then the kernel & the range of T are just the Null Space & Column Space of A :)

Subspaces Associated with a Linear Transformation



*The Kernel is a subspace of V.

*The range is a Subspace of W

*Proofs on next page :)

Property: Let $T: V \to W$ be a Linear Transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W.

*Hint: Typical elements in the range have the form $T(\vec{x})$ & $T(\vec{w})$ for some $\vec{x}, \vec{w} \in V$.

Proof:

Let $T: V \to W$ be a Linear Transformation from a vector space V into a vector space W.

Then by Definition: T is a rule that assigns each $\vec{x} \in V$ to a unique vector $T(\vec{x}) \in W$ ST: $\begin{cases} \cdot\, T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \\ \cdot\, T(c\vec{u}) = cT(\vec{u}) \end{cases}$

$\forall\, \vec{u}, \vec{v} \in V$ & scalars c.

*Goal: Show that the range of T is a subspace of W

i) Show that the $\vec{0} \in W$ is in W ($\vec{0}_W \in W$):

Let $\vec{x} = \vec{0} \in V \implies$ Then $T(\vec{0}) = \vec{0}_W$.

$\Rightarrow$ $\boxed{\therefore \text{The } \vec{0}_W \text{ of } W \text{ is in the Range of } T.}$ ✓

*Let $T(\vec{x})$ & $T(\vec{w})$ be typical elements in the range of T.

ii) Show that the Range of T is closed under addition:

Since $T(\vec{x}) + T(\vec{w}) = T(\vec{x} + \vec{w})$ by definition

$\Rightarrow$ $T(\vec{x}) + T(\vec{w})$ is in the Range of T

– AND –

The Range of T is closed under addition. ✓

ℕ

Proof Continued...

iii) Show that the Range of T is closed under scalar multiplication:

* Let c be any scalar.

* Since $cT(\vec{x}) = T(c\vec{x})$ By Definition,

$\Rightarrow$ $cT(\vec{x})$ is in the Range of T

-AND-

The Range of T is closed under scalar mult.

✓

Therefore:

Since all 3 conditions are met, the Range of T is a subspace of W.

$\boxed{X}$

Ex/Property: Let V & W be vector spaces.

Let $T: V \to W$ be a Linear Transformation.

Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(\vec{x})$, where $\vec{x} \in U$.

Show that $T(U)$ is a subspace of W

Proof:

*Recall: A subspace of a Vector Space V is a subset H ST

i) The zero vector of V is in H: $\vec{0}_V \in H$

ii) H is closed under addition: $\forall\ \vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$

iii) H is closed under scalar multiplication: $\forall\ \vec{u} \in H$, $c\vec{u} \in H$ $\forall$ scalars c

Prop 1:

- Since U is a subspace of V, then by definition:

$\vec{0}_V \in U$.

- Since T is a Linear Transformation, then by definition:

$T(\vec{0}_V) = \vec{0}_W$. $\Rightarrow$ $\boxed{\text{So } \vec{0}_W \in T(U)}$ *The zero vector of W is in $T(U)$ ✓

Prop 2:

- Let $T(\vec{x})$ & $T(\vec{y})$ be typical elements in $T(U)$:

$\Rightarrow$ Then $\vec{x}, \vec{y} \in U$

*Since U is a subspace of V: $\vec{x} + \vec{y} \in U$

*Since T is a Linear Transformation: $T(\vec{x}) + T(\vec{y}) = T(\vec{x} + \vec{y})$

$\therefore$ $\boxed{T(\vec{x}) + T(\vec{y}) \in T(U) \text{ \& so } T(U) \text{ is closed under addition}}$ ✓

⇓

Proof. Continued...

Prop 3:

- Let $T(\vec{x})$ be a Typical Element in $T(U)$ & let c be any scalar:

$\Rightarrow$ Then $\vec{x} \in U$.

* Since U is a subspace of V: $c\vec{u} \in U$
* Since T is a Linear Transformation: $cT(\vec{u}) = T(c\vec{u})$

$\therefore$ $\boxed{cT(\vec{u}) \in T(U) \text{ \& so } T(U) \text{ is closed under scalar multiplication.}}$ ✓

* Therefore: Since all 3 conditions are met, $\boxed{T(U) \text{ is a subspace of } W.}$