/ Bonus: +5

54/50 Excellent!

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Linear Algebra I: Exam 1 (Spring 2020)

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. *Make sure to exhibit skills discussed in class*. Box all answers and simplify answers as much as possible.

Good Luck! ☺

1. ***Systems of Linear Equations***

[6pts] Determine the value(s) of h for which the following linear system is consistent:

$$\begin{cases} 9x_1 + hx_2 = 9 \\ hx_1 + x_2 = -3 \end{cases}$$

$$[A \,\vdots\, b] = \left[\begin{array}{cc|c} 9 & h & 9 \\ h & 1 & -3 \end{array}\right] \xrightarrow[R_2/h]{R_1/9} \left[\begin{array}{cc|c} 1 & \frac{h}{9} & 1 \\ 1 & \frac{1}{h} & \frac{-3}{h} \end{array}\right] \quad (h \neq 0)$$

$$\xrightarrow[=nR_2]{R_1 - R_2} \left[\begin{array}{cc|c} 1 & \frac{h}{9} & 1 \\ 0 & \frac{h}{9} - \frac{1}{h} & 1 + \frac{3}{h} \end{array}\right]$$

✓

The linear system is consistent if: $\begin{cases} \frac{h}{9} - \frac{1}{h} \neq 0 \\ h \neq 0 \end{cases}$ (to let the system be defined)

$$\Rightarrow \begin{cases} h^2 - 9 \neq 0 \\ h \neq 0 \end{cases} \Rightarrow \begin{cases} h \neq 3 \\ \cancel{h \neq -3} \\ \cancel{h \neq 0} \end{cases}$$

✓

these produce a consistent system :)

-1

2. ***The Matrix Equation, $A\vec{x} = \vec{b}$***

Consider the following matrix equation:

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 13 \\ 1 & -1 & -2 \\ 2 & 4 & 26 \\ 2 & 1 & 11 \\ 3 & 3 & 24 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 2 \\ -26 \\ -11 \\ -24 \end{bmatrix}$$

(a) [3pts] Write the given Matrix Equation as a System of Linear Equations.

(b) [9pts] Solve the system and write the general solution in a parametric vector form.

(a)
$$\begin{cases} x_1 + 2x_2 + 13x_3 = -13 \\ x_1 - x_2 - 2x_3 = 2 \\ 2x_1 + 4x_2 + 26x_3 = -26 \\ 2x_1 + x_2 + 11x_3 = -11 \\ 3x_1 + 3x_2 + 24x_3 = -24 \end{cases}$$ ✓

(b)
$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 1 & -1 & -2 & 2 \\ 2 & 4 & 26 & -26 \\ 2 & 1 & 11 & -11 \\ 3 & 3 & 24 & -24 \end{array}\right] \xrightarrow[R_5/3]{R_3/2} \left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 1 & -1 & -2 & 2 \\ 1 & 2 & 13 & -13 \\ 2 & 1 & 11 & -11 \\ 1 & 1 & 8 & -8 \end{array}\right] \xrightarrow[\substack{R_1 - R_2 = nR_2 \\ 2R_1 - R_4 = nR_4}]{R_1 - R_3 = nR_3}$$ ✓

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 0 & 3 & 15 & -15 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 15 & -15 \\ 1 & 1 & 8 & -8 \end{array}\right] \xrightarrow[\substack{R_2/3 \\ R_1 - R_5 = nR_5}]{R_2 - R_4 = nR_4} \left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 \end{array}\right] \xrightarrow{R_2 - R_5 = nR_5}$$ ✓

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ ✓

x_3 is a free variable

$x_2 = -5 - 5x_3$

$x_1 = -13 - 2x_2 - 13x_3 = -13 - 2(-5 - 5x_3) - 13x_3$

$= -3 - 3x_3$

Beautiful!

Solution:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 - 3x_3 \\ -5 - 5x_3 \\ 1x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ -5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}}$$ ✓

Beautiful!

3. ***Solution Sets of Linear Systems***

Consider the following:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix}$$

(a) [9pts] Solve the Nonhomogeneous System $A\vec{x} = \vec{b}$ and write the solution in parametric-vector form.

(b) [3pts] Using the parametric vector form of the solution in part (a), determine a particular solution.

(c) [3pts] Write the general solution for the Homogeneous System, $A\vec{x} = \vec{0}$, in parametric vector form.

a) $[A\vec{x} \,\vdots\, \vec{b}\,] \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 6 & -4 \\ 1 & 2 & 3 & -2 \\ -1 & -2 & -3 & 2 \end{array}\right] \xrightarrow[\frac{R_3}{-1}]{\frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 1 & 2 & 3 & -2 \\ 1 & 2 & 3 & -2 \end{array}\right]$

$\xrightarrow[R_1 - R_3 = nR_3]{R_1 - R_2 = nR_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- x_2, x_3 are free vars.
- x_1 is a basic var.

$x_1 = -2 - 2x_2 - 3x_3$

$$\Rightarrow \boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}$$

b) Make $\boxed{\begin{cases} x_2 = 1 \\ x_3 = 1 \end{cases}}$ ok. $\Rightarrow x_1 = -2 - 2 - 3 = -7 \Rightarrow \boxed{x = \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}}$ Nice

c) $A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cccc} 2 & 4 & 6 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 = -2x_2 - 3x_3$

$$\Rightarrow \boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}$$

Check for (b):

$$1x = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2-2-3 \\ 0+1+0 \\ 0+0+1 \end{bmatrix}$$

Free

4. ***Linear Independence***

Consider the following vectors:

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

~~(a) [3pts] Show that the following set of vectors is Linearly Dependent: $\{\vec{v_1}, \vec{v_2}\}$~~

(b) [7pts] Show that the following set of vectors is Linearly Independent: $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

(c) [7pts] Write $\vec{v_4}$ as a Linear Combination of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, if possible.

(a)
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - R_3 = nR_3]{R_1 + R_2 = nR_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{R_2}{3}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_3 = nR_3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \{\vec{v_1}, \vec{v_2}\}$ $\boxed{\text{Independent}}$

(b)
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - R_3 = nR_3]{R_1 + R_2 = nR_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2 = nR_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

pivot $\exists$ in each row

$\Rightarrow$ $\boxed{\text{Linearly Independent}}$

(c)
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ -1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - R_3 = nR_3]{R_1 + R_2 = nR_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & -1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2 = nR_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 5 & 9 \end{array}\right] \xrightarrow[R_3/5]{5R_2 + R_3 = nR_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 5 & 0 & -6 \\ 0 & 0 & 1 & 9/5 \end{array}\right] \xrightarrow{R_2/5}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -6/5 \\ 0 & 0 & 1 & 9/5 \end{array}\right] \xrightarrow{R_1 - 2R_2 = nR_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & -6/5 \\ 0 & 0 & 1 & 9/5 \end{array}\right]$$

OR:

$$\boxed{\vec{v_4} = \frac{2}{5}\vec{v_1} - \frac{6}{5}\vec{v_2} + \frac{9}{5}\vec{v_3}}$$

Excellent!

Bonus Question [5pts]:

Let $\vec{e_1}, \vec{e_2}, \vec{e_3} \in \mathbb{R}^3$ be the elementary vectors in $\mathbb{R}^3$, and let $\vec{y_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{y_2} = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix}$, & $\vec{y_3} = \begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix}$.

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a Linear Transformation that maps $\vec{e_1}$ to $\vec{y_1}$, maps $\vec{e_2}$ to $\vec{y_2}$, and maps $\vec{e_3}$ to $\vec{y_3}$.

Find the image under T of $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow T(\vec{e_1}) = \vec{y_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

$$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to T(\vec{e_2}) = \vec{y_2} = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix} \checkmark$$

$$\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to T(\vec{e_3}) = \vec{y_3} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \checkmark$$

So:

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 3\vec{e_1} + 6\vec{e_2} + 9\vec{e_3} \checkmark$$

$$\Rightarrow T\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = 3\vec{y_1} + 6\vec{y_2} + 9\vec{y_3} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 6\begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix} + 9\begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} 3 - 24 + 63 \\ 6 + 30 + 72 \\ 9 + 36 - 81 \end{bmatrix} = \boxed{\begin{bmatrix} 42 \checkmark \\ 108 \checkmark \\ -36 \checkmark \end{bmatrix}}$$

+5

Awesome!