Section 1.3: Vector Equations

Note: In this section, we will connect equations involving vectors to ordinary systems of equations.

*For now, vectors will mean an "Ordered List of Numbers".

* Vectors in $\mathbb{R}^2$ *

A matrix with only one column is called a "Column Vector", or simply a "vector"

- Denoted: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where $v_1, v_2 \in \mathbb{R}$

The set of all vectors with two entries in denoted "$\mathbb{R}^2$", where:

- $\mathbb{R} \rightarrow$ The Set of all Real #s
- $2 \rightarrow$ Each vector has two entries
  (exponent)
- read: "r-two"

*Note: While we will be mostly concerned w/ vectors & matrices consisting of $\mathbb{R}$-valued entries only, it is important to note that all definitions & theorems remain valid for entries that are complex #s as well :)

Vector Operations: Arithmetic

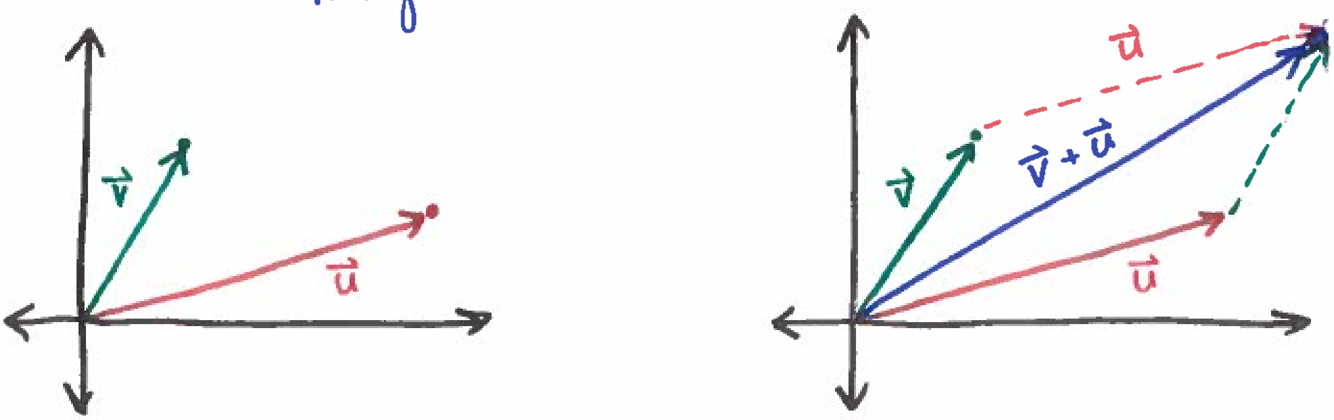
Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ & $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$.

Let $c \in \mathbb{R}$ be some Real-valued constant #.

① The Sum of 2 Vectors: $\vec{v} + \vec{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \end{bmatrix}$

*operation: the sum of 2 vectors is attained by adding their corresponding components.

*graphically: to visualize the sum, position the vectors st the initial pt. of one coincides w/ the terminal pt. of the other (w/o changing the magnitude or direction)



Note: The following rule can be verified by analytic geometry.

*The Parallelogram Rule for Addition:

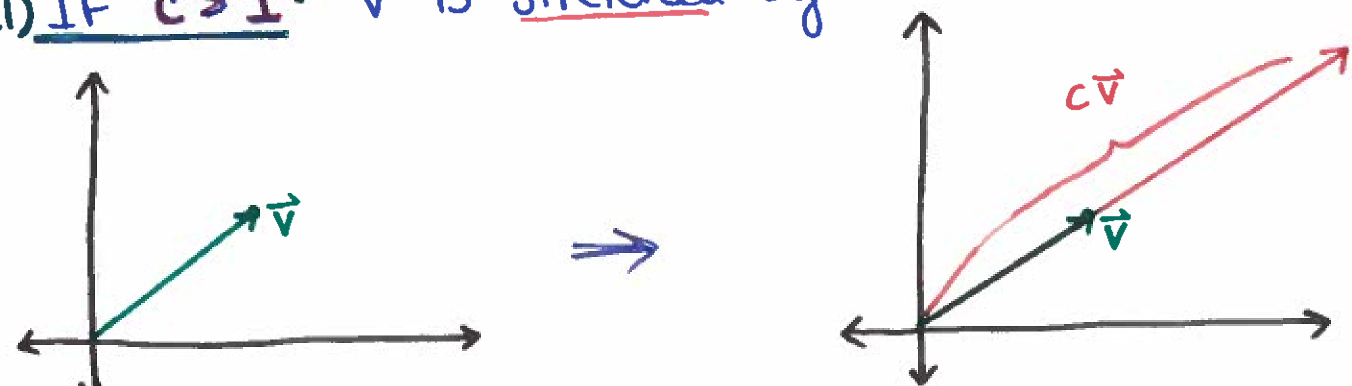
If $\vec{u}$ & $\vec{v}$ are vectors in $\mathbb{R}^2$, represented by points in the plane, the $\vec{u}+\vec{v}$ corresponds to the 4^{th} vertex of the parallelogram whose other vertices are $\vec{u}$, $\vec{v}$, & $\vec{0}$ (see graph above :))

② Scalar Multiples of Vectors: $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$

***operation:** A scalar multiple is attained by multiplying each entry by some $\mathbb{R}$-valued constant "c"

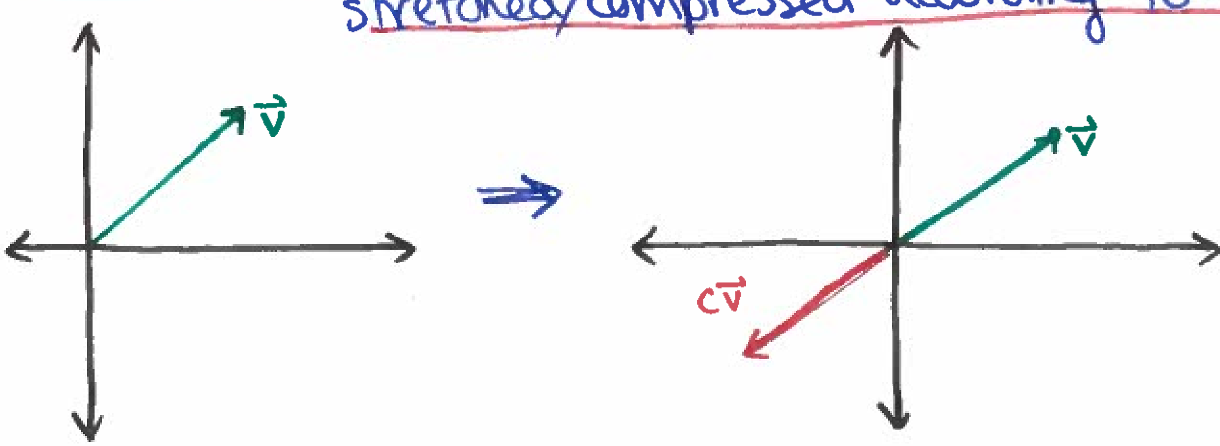
***graphically:** There are 3 possible graphical interpretation:

(i) IF $c > 1$: $\vec{v}$ is stretched by a factor "c".



(ii) IF $0 < c < 1$: $\vec{v}$ is compressed by a factor of "c".

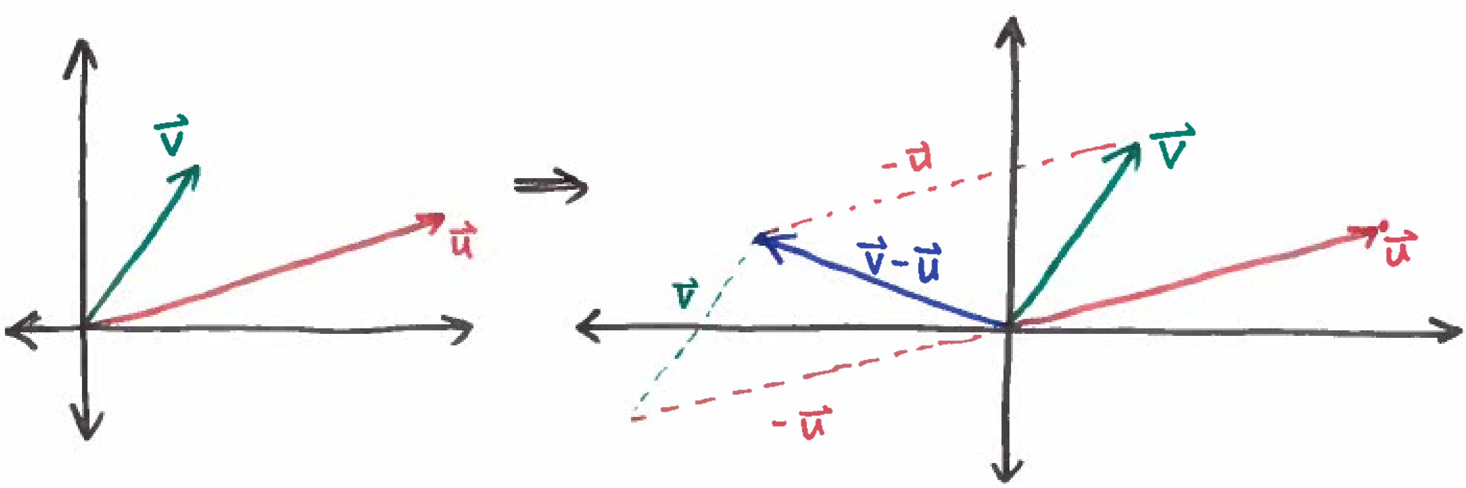
(iii) IF $c < 0$: $\vec{v}$ moves in the opposite direction & is then stretched/compressed according to (i) & (ii)



③ The Difference of 2 Vectors: $\vec{v}-\vec{u} = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \end{bmatrix}$

***operation:** the difference of 2 vectors is attained by subtracting their corresponding entries respectively

***graphically:** it is easier to visualize the difference here if we consider "$\vec{v}-\vec{u} = \vec{v} + (-\vec{u})$", keeping in mind that "$-\vec{u}$" moves in the opposite direction as "$\vec{u}$", & then apply the parallelogram rule.



④ Equal Vectors:

***operation/prop:** 2 vectors in $\mathbb{R}^2$ are equal IFF their entries are equal.

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ & $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then $\vec{v} = \vec{u}$ IFF $v_1 = u_1$ AND $v_2 = u_2$

Because $(a,b) \Leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$

:)

*Geometric Descriptions of $\mathbb{R}^2$ *

Consider a rectangular coordinate system (Cartesian Coord. System) in the plane (2-dimensions).

⇒ Since each point in the plane is determined by an ordered pair of numbers, we can identify:

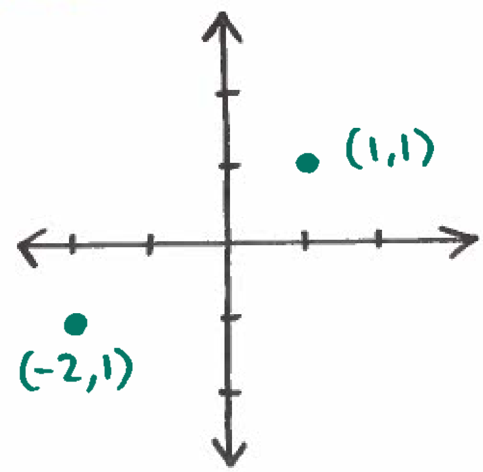
$$(a, b) \iff \begin{bmatrix} a \\ b \end{bmatrix}, \text{ where } a,b \in \mathbb{R}$$

*Geometric Point   *vector

⇒ Therefore, we can think of $\mathbb{R}^2$ as the set of all pts. in 2-D.

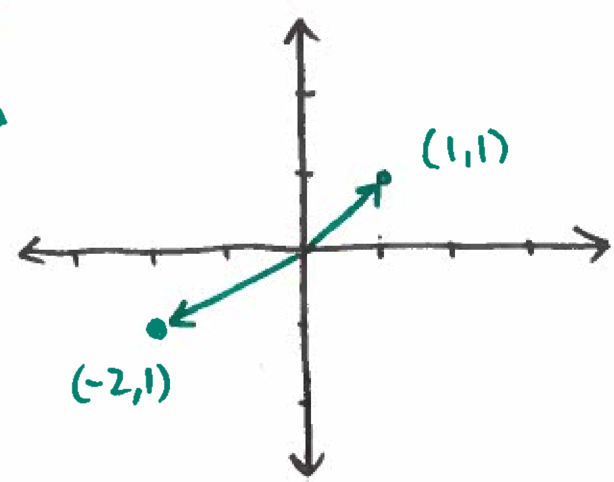
*2 Graphical Representations of Vectors in $\mathbb{R}^2$ *

① Vectors as Points, (a,b):



② Vectors as directed line segments:

*Note: Vectors depicted w/ arrows → have an initial pt. @ the origin (0,0) & terminal pt. @ (a,b).



Example: Let $\vec{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

Find the Following: (a) $\vec{u} + \vec{v}$

(b) $\vec{u} - 2\vec{v}$

Answer:

Part (a):

$$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 \\ -7 \end{bmatrix} = \begin{bmatrix} 3-4 \\ -2-7 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$$

$$\boxed{\therefore \vec{u} + \vec{v} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}}$$

Part (b):

$$\vec{u} - 2\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -4 \\ -7 \end{bmatrix} = \begin{bmatrix} 3-2(-4) \\ -2-2(-7) \end{bmatrix}$$

$$= \begin{bmatrix} 3+8 \\ -2+14 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$$

$$\boxed{\therefore \vec{u} - 2\vec{v} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}}$$

Example: Use the accompanying figure & vector arithmetic to write each of the following vectors as a linear combination of $\vec{u}$ & $\vec{v}$.

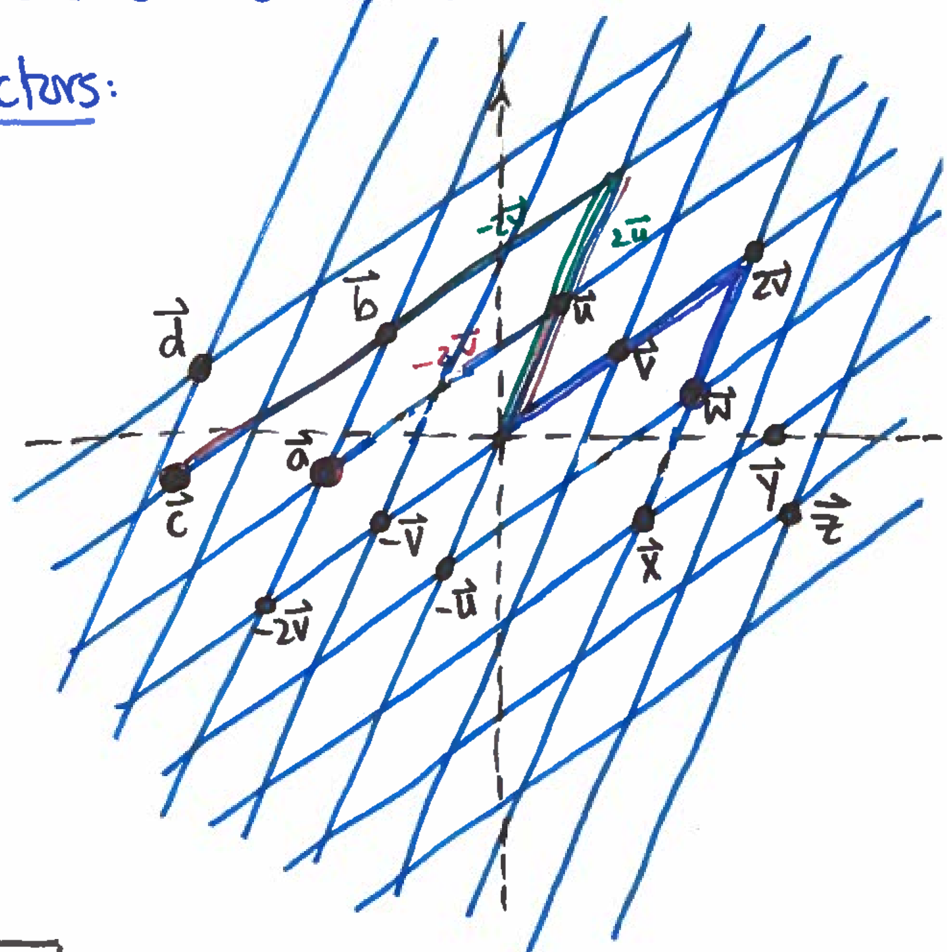
*Find the following vectors:

(a) $\vec{a}$

(b) $\vec{b}$

(c) $\vec{c}$

(d) $\vec{w}$



Recall:
$\vec{u} - \vec{v} = \vec{u} - (\vec{v})$:)

Answer:

Part (a): $\boxed{\vec{a} = \vec{u} - 2\vec{v}}$

Part (b): $\boxed{\vec{b} = 2\vec{u} - 2\vec{v}}$

Part (c): $\boxed{\vec{c} = 2\vec{u} - 3.5\vec{v}}$

Note: It might be safer to use the scale on MML &/or your text (hard when drawn by hand)

Part (d): $\boxed{\vec{w} = -\vec{u} + 2\vec{v}}$

*Vectors in $\mathbb{R}^n$ *

If n is a ⊕ Integer, $n \in \mathbb{Z}$ ST $n > 0$, then "$\mathbb{R}^n$" denotes the collection of all lists or ordered n-tuples of n-Real Numbers :

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$ where $\vec{u}$ is an $n \times 1$ matrix :)

*Algebraic Properties of $\mathbb{R}^n$ *

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$. Let $c, d \in \mathbb{R}$ be scalars.

Note: Before proceeding, please see def. of "zero Vector, $\vec{0}$" below :)

① Commutative Prop: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

② Associative Prop. (vectors): $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

③ Additive Identity Prop: $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

④ Additive Inverse Prop: $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$, where $-\vec{u} = (-1)\vec{u}$

⑤ Distributive Prop. 1 (vectors): $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

⑥ Distributive Prop. 2 (scalars): $(c + d)\vec{u} = c\vec{u} + d\vec{u}$

⑦ Associative-Scalar Prop.: $c(d\vec{u}) = (cd)\vec{u}$

⑧ Multiplicative Identity Prop: $1(\vec{u}) = \vec{u}$

*Zero Vector: A vector whose entries are ALL zero; $\vec{0}$.

Linear Combinations

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ be vectors in $\mathbb{R}^n$.

Let $c_1, c_2, \dots, c_p \in \mathbb{R}$ be scalars.

A <u>Linear Combination</u> of vectors $\vec{v}_1, \dots, \vec{v}_p$ with weights $c_1, \dots, c_p \in \mathbb{R}$ is defined by some vector $\vec{y}$ ST :

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$$

<u>Example</u>: \$ that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$. Write the following as a linear combination of $\vec{v}_1$ & $\vec{v}_2$.

(a) $\frac{1}{7}\vec{v}_2$, (b) $\vec{0}$

<u>Answer</u>:

<u>Part (a)</u>: $\frac{1}{7}\vec{v}_2 = \boxed{0\vec{v}_1 + \frac{1}{7}\vec{v}_2}$ Ans.

<u>Note</u>: Since only $c_2 = \frac{1}{7}$ is defined here, we let $c_1 = 0 \in \mathbb{R}$.

*"0" is the Real # zero, NOT the zero vector ⁝)

<u>Part (b)</u>: $\vec{0} = \boxed{0\vec{v}_1 + 0\vec{v}_2}$ Ans.

<u>The Zero Vector</u> ⁝) $\Rightarrow$ <u>ALL</u> entries of $\vec{0}$ are $0 \in \mathbb{R}$

* Fundamental Facts (Vectors & Linear Combinations):

- A **vector equation**, defined:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has the SAME solution set as the **linear system whose augmented matrix** is defined:

$$[\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n \quad \vec{b}]$$

- In particular, $\vec{b}$ can be generated as a linear combination of $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ IFF $\exists$ a solution to the linear system corresponding to the above matrix!

  * <u>Consistent System</u> $\Rightarrow$ At least one solution $\exists$

    - If there is a pivot position in every row of matrix A, then the system is consistent.

      (i) <u>No Free Variables</u> $\Rightarrow$ One, unique solution (trivial solution)

      (ii) <u>Free Variable(s) $\exists$</u> $\Rightarrow$ many solutions $\exists$ (nontrivial sol.)

Example: Write a system of equations that is equivalent to the given vector equation:

$$x_1\begin{bmatrix}3\\-2\\8\end{bmatrix} + x_2\begin{bmatrix}4\\0\\-6\end{bmatrix} = \begin{bmatrix}5\\-3\\7\end{bmatrix}$$

Answer:

* By Scalar Multiples:

$$\begin{bmatrix}3x_1\\-2x_1\\8x_1\end{bmatrix} + \begin{bmatrix}4x_2\\0x_2\\-6x_2\end{bmatrix} = \begin{bmatrix}5\\-3\\7\end{bmatrix}$$

* By Vector Addition:

$$\begin{bmatrix}3x_1+4x_2\\-2x_1+0x_2\\8x_1-6x_2\end{bmatrix} = \begin{bmatrix}5\\-3\\7\end{bmatrix}$$

* By Vector Equality:

$$\boxed{\begin{cases}3x_1+4x_2=5\\-2x_1+0x_2=-3\\8x_1-6x_2=7\end{cases}}$$

Recall: 2 vectors are equal IFF their corresponding entries are equal.

Example: Write a system of equations that is equivalent to the given vector equation:

$$x_1\begin{bmatrix}4\\-5\end{bmatrix}+x_2\begin{bmatrix}6\\2\end{bmatrix}+x_3\begin{bmatrix}-4\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Answer:

* **By Scalar Multiples:**

$$\begin{bmatrix}4x_1\\-5x_1\end{bmatrix}+\begin{bmatrix}6x_2\\2x_2\end{bmatrix}+\begin{bmatrix}-4x_3\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

* **By Vector Addition:**

$$\begin{bmatrix}4x_1+6x_2-4x_3\\-5x_1+2x_2+x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

* **By Vector Equality:**

$$\boxed{\begin{aligned}4x_1+6x_2-4x_3&=0\\-5x_1+2x_2+x_3&=0\end{aligned}}$$

Answer

Example: Write a vector equation that is equivalent to the given system of equations:

$$\begin{cases} x_2 + 4x_3 = 0 \\ 3x_1 + 6x_2 - x_3 = 0 \\ -x_1 + 4x_2 - 6x_3 = 0 \end{cases}$$

Answer:

Note: Here we work in the reverse direction of the previous examples :)

*Group the corresponding entries into column vectors:

$$\begin{bmatrix} 0x_1 + x_2 + 4x_3 \\ 3x_1 + 6x_2 - x_3 \\ -x_1 + 4x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0x_1 \\ 3x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -x_3 \\ -6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{x_1 \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$ Answer

Example: Determine if $\vec{b}$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, & $\vec{a}_3$.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 5 \\ -4 \\ 18 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix}$$

Answer:

Note: Here we want to determine whether the weights $x_1, x_2, \& \ x_3 \ \exists$ ST $\Rightarrow$ $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$

* First write the vector equation $\uparrow$, and the row reduce the augmented matrix of the system to see if a solution $\exists$.

Use vector arithmetic to write the vector equation:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

$$x_1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -4 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix}$$

By Scalar Mult:

$$\begin{bmatrix} x_1 \\ -2x_1 \\ 0x_1 \end{bmatrix} + \begin{bmatrix} 0x_2 \\ x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -4x_3 \\ 18x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix}$$

By Vector Add:

$$\begin{bmatrix} x_1 + 0x_2 + 5x_3 \\ -2x_1 + x_2 - 4x_3 \\ 0x_1 + 3x_2 + 18x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix}$$

Ex. Continued...

So, the Vector Equation is:
$$\begin{cases} x_1 + 5x_3 = 2 \\ -2x_1 + x_2 - 4x_3 = -1 \\ 3x_2 + 18_3 = 9 \end{cases}$$
*Simplify

$$\Rightarrow \boxed{\begin{cases} x_1 + 5x_3 = 2 \\ -2x_1 + x_2 - 4x_3 = -1 \\ x_2 + 6x_3 = 3 \end{cases}}$$

So, the augmented matrix is:
$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -4 & -1 \\ 0 & 1 & 6 & 3 \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

*Note: Now we row reduce to solve/see if a solution ∃

- Switch R_2 & R_3:
$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 6 & 3 \\ -2 & 1 & -4 & -1 \end{array}\right]$$

- Add
$$\begin{array}{r} 2R_1 \\ + \; R_3 \\ \hline \text{New } R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 6 & 3 \\ 0 & 1 & 6 & 3 \end{array}\right]$$

- Add
$$\begin{array}{r} -R_2 \\ + \; R_3 \\ \hline \text{New } R_3 \end{array} \Rightarrow \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

*Reduced Echelon Matrix!

* x_3 is a free variable
⇒ consisent system w/ ∴ ~~many~~ possible solutions.

Pivots are circled above

$$\boxed{\therefore \vec{b} \text{ is a linear combination of } \vec{a_1}, \vec{a_2} \;\&\; \vec{a_3}}$$

Example: Determine if $\vec{b}$ is a linear combination of $\vec{a}_1$, $\vec{a}_2$, & $\vec{a}_3$:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} -3 \\ 5 \\ -3 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 4 \end{bmatrix}, \quad \& \quad \vec{b} = \begin{bmatrix} 13 \\ -2 \\ 9 \end{bmatrix}$$

Answer:

*Recall: $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \& \vec{a}_3$ IFF $\exists$ a solution to the augmented matrix $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{b}]$.

*Rewrite the vectors in the aug. matrix form:

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \,\vdots\, \vec{b}] = \left[\begin{array}{ccc:c} 1 & -3 & -6 & 13 \\ 0 & 5 & 7 & -2 \\ 1 & -3 & 4 & 9 \end{array}\right]$$

*Row-reduce the matrix to determine if a solution $\exists$:

$$\begin{array}{l} -R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc:c} 1 & -3 & -6 & 13 \\ 0 & 5 & 7 & -2 \\ 0 & 0 & 10 & -4 \end{array}\right] \underset{\frac{1}{2}R_3}{\sim} \left[\begin{array}{ccc:c} 1 & -3 & -6 & 13 \\ 0 & 5 & 7 & -2 \\ 0 & 0 & 5 & -2 \end{array}\right] \checkmark$$

* pivot position in every row $\Rightarrow$ one, unique sol. $\exists$

$\boxed{\therefore \text{ Vector } \vec{b} \text{ \underline{is} a linear combination of } \vec{a}_1, \vec{a}_2, \& \vec{a}_3}$

Example: Determine if $\vec{b}$ is a linear combination of the vectors formed from the columns of the matrix A:

$$A = \begin{bmatrix} 1 & -6 & 4 \\ 0 & 4 & 7 \\ -2 & 12 & -8 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} 2 \\ -7 \\ -2 \end{bmatrix}$$

Answer:

Recall: The vector $\vec{b}$ can be generated by a linear combination of $\vec{a}_1, \vec{a}_2$ & $\vec{a}_3$ IFF $\exists$ a solution to the linear system corresponding to the augmented matrix $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{b}]$

*Use matrix A & $\vec{b}$ to create the aug. matrix:

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & -6 & 4 & 2 \\ 0 & 4 & 7 & -7 \\ -2 & 12 & -8 & -2 \end{array}\right] \overset{\frac{1}{2}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -6 & 4 & 2 \\ 0 & 4 & 7 & -7 \\ -1 & 6 & -4 & -1 \end{array}\right]$$

*Row reduce the augmented matrix to see if a sol. $\exists$:

- Add $\begin{array}{r} R_1 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array}$ $\sim \left[\begin{array}{ccc|c} 1 & -6 & 4 & 2 \\ 0 & 4 & 7 & -7 \\ 0 & 0 & 0 & 1 \end{array}\right]$ * $\rightarrow\leftarrow$

Note: R_3 produces the equation, "$0 = 1$", which is a contradiction!

$\therefore R_3$ has NO Solution $\Rightarrow$ Linear System has NO Solution! $\Rightarrow \vec{b}$ is NOT a linear combination

__Note:__ One of the key ideas in this course is to study the set of all vectors that can be generated or written as a linear combo of a fixed set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ of vectors :)

*__Definition:__

If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combos of $\vec{v}_1, \dots, \vec{v}_p$ is denoted by $\text{Span}\{v_1, \dots, v_p\}$ & is called the "__Subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \dots, \vec{v}_p$.__"

*__IOW:__ $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is the collection of all vectors that can be written in the form:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p \quad , \text{ where } c_1, \dots, c_p \in \mathbb{R} \text{ (scalars)}$$

*__Asking whether a vector $\vec{b}$ is in the $\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$ amounts to asking whether the:__

① Vector Eq., $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{b}$, has a solution

-AND/OR-

② Linear System w/ augmented matrix $[\vec{v}_1 \; \vec{v}_2 \dots \vec{v}_p \; b]$ has a solution

__Note:__ $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ contains every scalar multiple of $\vec{v}_1$ since

*Notes (on the Span$\{\vec{v}_1, \ldots, \vec{v}_p\}$):

(i) The Span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ contains every scalar multiple of $\vec{v}_1$ since $c\vec{v}_1 = c\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p$.

this implies that

(ii) The zero vector, $\vec{0}$, must be in the Span$\{\vec{v}_1, \ldots, \vec{v}_p\}$.

A Geometric Description of the Span

① Case 1: Span$\{\vec{v}\}$ as a Line Through the Origin:

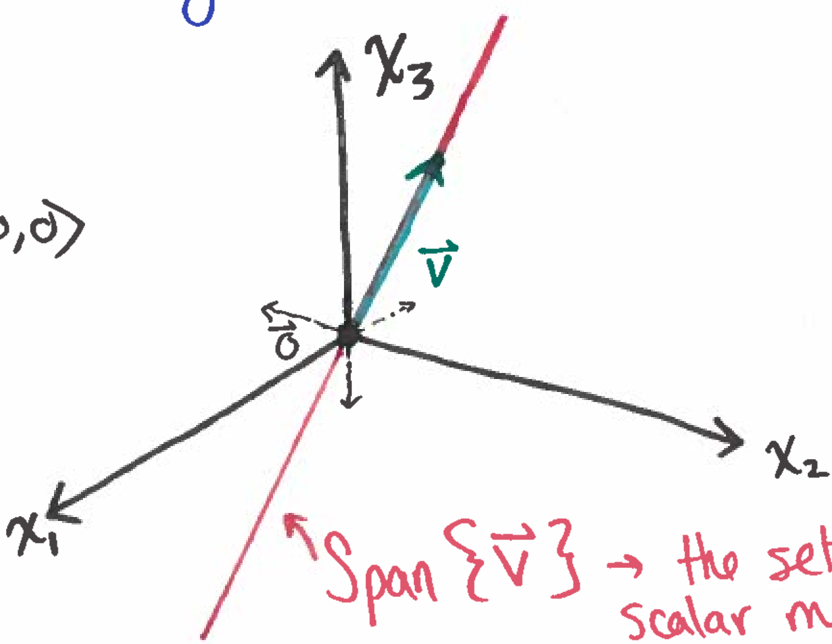
Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^3$.

Then the **Span$\{\vec{v}\}$** is the set of all scalar multiples of $\vec{v}$, which is the set of ALL points on the line in $\mathbb{R}^3$ through $\vec{v}$ & $\vec{0}$

*graphically:

* $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \langle 0,0,0 \rangle$

(the Origin)



Span$\{\vec{v}\}$ → the set of ALL scalar mult. of $\vec{v}$

② Case 2: Span$\{\vec{u},\vec{v}\}$ as a Plane through the Origin:

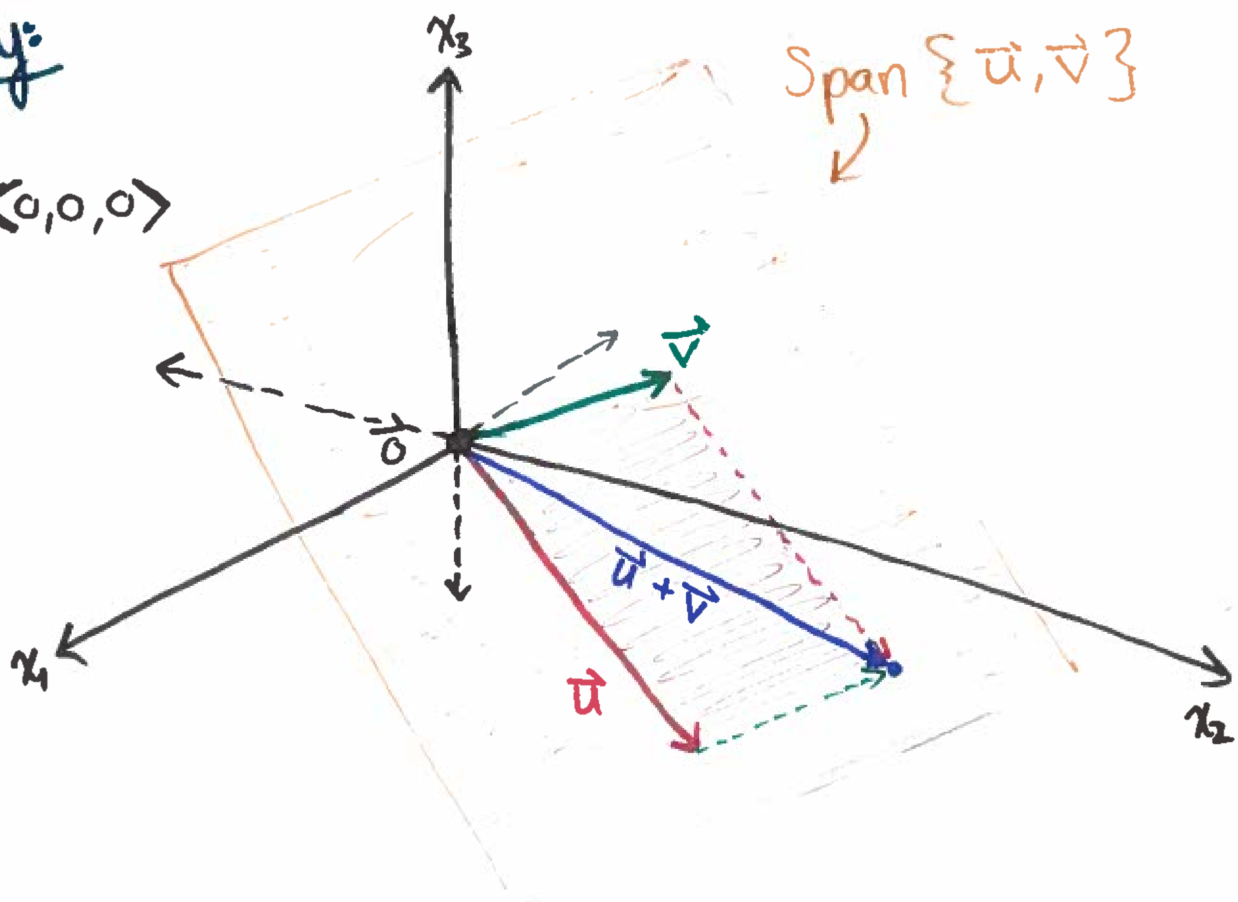
Let $\vec{u}$ & $\vec{v}$ be non-zero vectors in $\mathbb{R}^3$ ST $\vec{v}$ is NOT a multiple of $\vec{u}$.

Then, the Span$\{\vec{u},\vec{v}\}$ is the plane in $\mathbb{R}^3$ containing the vectors $\vec{u}$, $\vec{v}$, & $\vec{0}$.

⇒ IOW: Span$\{\vec{u},\vec{v}\}$ contains the Line in $\mathbb{R}^3$ through $\vec{u}$ & $\vec{0}$ and the Line in $\mathbb{R}^3$ through $\vec{v}$ & $\vec{0}$.

*graphically:

$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \langle 0,0,0 \rangle$



Example: Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -6 \\ -14 \\ 3 \end{bmatrix}$, & $\vec{b} = \begin{bmatrix} 5 \\ -1 \\ h \end{bmatrix}$

For what value(s) of h is $\vec{b}$ in the plane spanned by $\vec{a}_1$ & $\vec{a}_2$?

Answer:

Note: A vector $\vec{b}$ is in the span$\{\vec{a}_1, \vec{a}_2\}$ IFF $\exists$ a solution to the linear system w/ augmented matrix $[\vec{a}_1 \ \vec{a}_2 \ \vec{b}]$.

*A consistent linear system has @ least one solution; the entry in the right-most column is NOT a pivot

*Assume $\vec{b}$ is in the plane spanned by $\vec{a}_1$ & $\vec{a}$

*Want: Find "h" ST $[\vec{a}_1 . \vec{a}_2 \ \vec{b}]$ is a consistent, linear system :)

① Defined the augmented matrix:

$$[\vec{a}_1 \ \vec{a}_2 \mid \vec{b}] = \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 3 & -14 & -1 \\ -1 & 3 & h \end{array}\right]$$

② Row Reduce the augmented matrix:

- $\begin{array}{r} -3R_1 \\ + \ R_2 \\ \hline \end{array}$ $\rightarrow \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 4 & -16 \\ -1 & 3 & h \end{array}\right]$

Ex Continued...

- $\frac{1}{4}R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 4 & -16 \\ -1 & 3 & h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 1 & -4 \\ -1 & 3 & h \end{array}\right]$

- $\begin{array}{r} R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 1 & -4 \\ -1 & 3 & h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 1 & -4 \\ 0 & -3 & 5+h \end{array}\right]$

- $\begin{array}{r} 3R_2 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 1 & -4 \\ 0 & -3 & 5+h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & -7+h \end{array}\right]$

Reduced Aug. Matrix

*Note: Need to find h-value(s) ST equation 3 holds true (A solution will NOT ∃ if a →← occurs)

*Solve for h:

$$0 = -7 + h$$

$$7 = h$$

$\boxed{\therefore \text{ If: } h = 7, \text{ then } \vec{b} \text{ is in the plane spanned by } \vec{a}_1 \text{ \& } \vec{a}_2}$

Example: Construct a 3×3 matrix A w/ nonzero entries & a vector $\vec{b}$ in $\mathbb{R}^3$ such that $\vec{b}$ is NOT in the set spanned by the columns of matrix A.

Answer:

- Let A be a 3×3 matrix: $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$

  ST $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are nonzero.

*Recall: Asking if "$\vec{b}$ is in the span$\{A\}$" amounts to determining if $\vec{b}$ is a linear combination of A;

IOW: A solution $\exists$ to $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{b}]$:)

*Want: To construct a matrix A ST $\vec{b}$ is NOT in span$\{A\}$ $\Rightarrow$ $\vec{b}$ is NOT a linear combination of $\vec{a}_1$, $\vec{a}_2$, & $\vec{a}_3$.

*Define a matrix A & vector $\vec{b}$ ST row reducing the augmented matrix will produce a contradiction

$\Rightarrow$ Show NO solution $\exists$ for $[a_1\ a_2\ a_3\ b]$

- Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

↘

Example Continued...

*Row Reduce the augmented matrix $[a_1\ a_2\ a_3\ b]$

$$[a_1\ a_2\ a_3 \vdots b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 5 \\ 3 & 3 & 3 & 6 \end{array}\right]$$

$$\begin{array}{l} \bullet -2R_1 \\ \underline{+\ R_2} \\ \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 3 & 3 & 3 & 6 \end{array}\right]$$

STOP! $\rightarrow\leftarrow$ (contradiction)

$\therefore$ Since R_2 produces a contradiction, NO solution $\exists$ and $\vec{b}$ is NOT a linear combination of matrix A.

$\Downarrow$

$\therefore$ $\vec{b}$ is NOT in the span$\{A\}$

if: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

One Possible Answer

Note: This is NOT an exclusive solution :)

Example: Let $A = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 3 & -3 \\ -2 & 6 & 3 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$.

Denote the columns of A by $\vec{a_1}, \vec{a_2}$ & $\vec{a_3}$. Let

Let $W = \text{Span}\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$. Find the following:

(a) Is $\vec{b}$ in $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$? How many vectors are in $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$?

(b) Is $\vec{b}$ in W? How many vectors are in W?

(c) Show that $\vec{a_2}$ is in W. (Hint: Row operations are not necessary)

Answer:

Part (a):

$\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ is a set of 3 vectors.

Since $\vec{b} \neq \vec{a_1}$, nor $\vec{a_2}$, nor $\vec{a_3}$

$\Rightarrow \vec{b}$ is NOT in $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$.

Part (b): *Note: The vector $\vec{b}$ is in W if the linear system w/ augmented matrix $[\vec{a_1}\ \vec{a_2}\ \vec{a_3}\ b]$ has a solution

So, check if a solution $\exists$:

$$[\vec{a_1}\ \vec{a_2}\ \vec{a_3} \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 3 & -3 & 1 \\ -2 & 6 & 3 & -3 \end{array}\right]$$

Now Row Reduce the aug. matrix.

Ex Continued...

- $\begin{array}{l} 2R_1 \\ + \underline{\ R_3\ } \\ \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 3 & -3 & 1 \\ -2 & 6 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 3 & -3 & 1 \\ 0 & 6 & -9 & 3 \end{array}\right]$

- $\begin{array}{l} -2R_2 \\ + \underline{\ R_3\ } \\ \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 3 & -3 & 1 \\ 0 & 6 & -9 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right]$

*pivot position in every row ✓ $\Rightarrow$ Solution $\exists$!

$\boxed{\therefore \vec{b} \text{ is in } W}$

Note: There are infinitely many vectors in W

$\Rightarrow W = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is the plane in $\mathbb{R}^3$ that contains $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, & $\vec{0}$

Part (c):

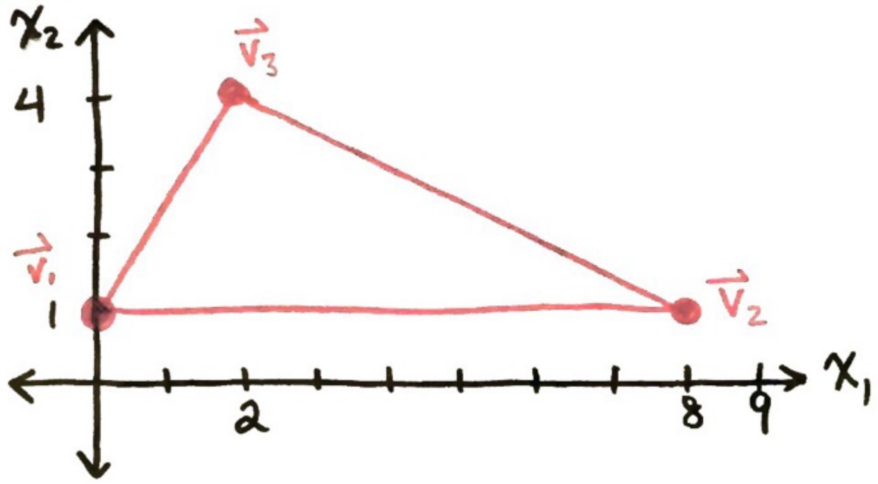
$W = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is the collection of all vectors that can be written in the form

$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3$, where c_1, c_2, c_3 are all scalars.

Example: A thin triangular plate of uniform density & thickness has vertices at $\vec{v}_1 = (0,1)$, $\vec{v}_2 = (8,1)$, $\vec{v}_3 = (2,4)$, as in the figure below. The mass of the plate is 3g.

(a) Find the (x,y) coordinates of the center of mass of the plate. This "balance point" of the plate coincides w/ the center of mass of the system consisting of three 1-gram point masses located at the vertices.

(b) Determine how to distribute an additional mass of 6gram at the vertices of the plate to move the balance point of the plate to $(2,2)$. (Hint: Let w_1, w_2, & w_3 denote the masses added at the three vertices, ST $w_1 + w_2 + w_3 = 6$).



Note: Let $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ be points in $\mathbb{R}^n$ & $ that for $j = 1, 2, \dots, k$ an object with mass m_j is located at point $\vec{v}_j$. $\Rightarrow$ Such objects are called "point masses".

* The total mass of the system of point masses is:

$$m = m_1 + m_2 + \cdots + m_k$$

* The Center of Gravity (or Center of Mass) of the System is:

$$\vec{v} = \frac{1}{m}\left[m_1\vec{v}_1 + \cdots + m_k\vec{v}_k\right] = \frac{m_1}{m}\vec{v}_1 + \frac{m_2}{m}\vec{v}_2 + \cdots + \frac{m_k}{m}\vec{v}_k$$

Answer:

*Part (a): Find the coordinates (x,y) of the Center of Mass of the Plate:

• Mass of the Plate: $m = 3\text{grams}$

* The point mass of the plate: $m = (m_1, m_2, m_3) = (1,1,1)$

* Total mass of the system of point masses: $m = m_1 + m_2 + m_3$

$\Rightarrow m = 1+1+1 = 3$ ✓

• Three vertices: $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

• The Center of Mass of the System:

$$\vec{v} = \frac{1}{m}\left[m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3\right] = \frac{1}{3}\left(1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 8 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} 0+8+2 \\ 1+1+4 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 6/3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2 \end{bmatrix}$$

$$\boxed{\therefore \vec{v} = \begin{bmatrix} 10/3 \\ 2 \end{bmatrix}}$$

*Part (b): Determine how to distribute an additional mass of 6grams at the vertices of the plate to move the balance point to $(2,2)$:

• Total Mass of the New System: $m = 3g + 6g = 9\text{grams}$

*The point mass for the new system $m = (m_1 + w_1, m_2 + w_2, m_3 + w_3)$

$= (1 + w_1, 1 + w_2, 1 + w_3)$

Answer Continued...

Recall: The Center of Mass of the original system

$$\Rightarrow \vec{v} = \frac{1}{m}\left[m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3\right]$$

*The Center of Mass of the New System (w/ three new masses w_1, w_2, w_3 added):

Since we want the NEW balance point to be, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{9}\left\{(1+w_1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (1+w_2)\begin{bmatrix} 8 \\ 1 \end{bmatrix} + (1+w_3)\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$

$$= \frac{1}{9}\left\{\underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + w_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \underline{\begin{bmatrix} 8 \\ 1 \end{bmatrix}} + w_2\begin{bmatrix} 8 \\ 1 \end{bmatrix} + \underline{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} + w_3\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$

$$= \frac{1}{9}\left\{\begin{bmatrix} 0+8+2 \\ 1+1+4 \end{bmatrix} + w_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2\begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$

$$9\cdot\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \cancel{9}\,\cancel{\frac{1}{9}}\left\{\begin{bmatrix} 10 \\ 6 \end{bmatrix} + w_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2\begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$

$$\Rightarrow \begin{bmatrix} 18 \\ 18 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix} + w_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2\begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 18 \\ 18 \end{bmatrix} + \begin{bmatrix} -10 \\ -6 \end{bmatrix} = w_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2\begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 \\ 12 \end{bmatrix} = w_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2\begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

← A vector-equation (a linear combination of w_1, w_2, & w_3)

Answer Continued...

* **The resulting Vector Equation:** $W_1\begin{bmatrix}0\\1\end{bmatrix} + W_2\begin{bmatrix}8\\1\end{bmatrix} + W_3\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}8\\12\end{bmatrix}$

Recall (the provided 'hint' w/ part b):

The masses added @ the 3 vertices $\Rightarrow$ $W_1 + W_2 + W_3 = 6$

* **Adding this condition to the resulting vector equation:**

$$W_1\begin{bmatrix}1\\0\\1\end{bmatrix} + W_2\begin{bmatrix}1\\8\\1\end{bmatrix} + W_3\begin{bmatrix}1\\2\\4\end{bmatrix} = \begin{bmatrix}6\\8\\12\end{bmatrix}$$

The vector eq.

$$\Longleftrightarrow \begin{bmatrix}1&1&1\\0&8&2\\1&1&4\end{bmatrix}\begin{bmatrix}W_1\\W_2\\W_3\end{bmatrix} = \begin{bmatrix}6\\8\\12\end{bmatrix}$$

The Matrix Eq. ($A\vec{x} = \vec{b}$)

* **Solve for the Solution vector $\vec{w}$: Row-reduce the equivalent augmented matrix to row-reduced echelon form:**

$$[A \,\vdots\, \vec{b}] = \left[\begin{array}{ccc|c}1&1&1&6\\0&8&2&8\\1&1&4&12\end{array}\right] \overset{\frac{1}{8}R_2}{\sim} \left[\begin{array}{ccc|c}1&1&1&6\\0&1&\frac{1}{4}&1\\1&1&4&12\end{array}\right]$$

*Use the 1st pivot to eliminate of entries in Col. 1

$$\frac{\begin{array}{r}-R_1\\+R_3\end{array}}{\text{new}R_3} \rightarrow \left[\begin{array}{ccc|c}1&1&1&6\\0&1&1/4&1\\0&0&3&6\end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c}1&1&1&6\\0&1&1/4&1\\0&0&1&2\end{array}\right]$$

*Use the 2nd pivot to eliminate the other entries in Col. 2

$$\frac{\begin{array}{r}-R_2\\+R_1\end{array}}{\text{new}R_1} \rightarrow \left[\begin{array}{ccc|c}1&0&3/4&5\\0&1&1/4&1\\0&0&1&2\end{array}\right]$$

*Use the 3rd pivot to eliminate the other entries in Col. 3

Answer Continued...

* $\dfrac{-\frac{3}{4}R_3 + R_1}{\text{new } R_1} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/2 \\ 0 & 1 & 1/4 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

* $\dfrac{-\frac{1}{4}R_3 + R_2}{\text{new } R_2} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longleftrightarrow \boxed{\begin{cases} \bullet\ w_1 = 7/2 \\ \bullet\ w_2 = 1/2 \\ \bullet\ w_3 = 2 \end{cases}}$

∴ We need to add an additional mass of:

Ans✓

* $w_1 = \frac{7}{2}$ = 3.5 grams at the 1st vertice, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

* $w_2 = \frac{1}{2}$ = 0.5 grams at the 2nd vertice, $\vec{v}_2 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

* $w_3 =$ 2 grams at the 3rd vertice, $\vec{v}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(to move the balance point to (2,2) :))