 Name SOLUTION

Instructions: No notes or calculators are allowed. Answers with little or no supporting work will get little or no credit. Work must be neat, organized and easily interpreted.

1. A plane contains the two points P(1,3,2), Q(2,2,3) and is parallel to the vector $\bar{u} = <-2,1,1>$.

1.a (4 Points) Find a normal vector to the plane.

$$\vec{v} = \vec{PQ} = \langle 1,-1,1\rangle$$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 1+1, 1+2, 2-1\rangle = \langle 2,3,1\rangle$$

1.b (4 Points) Find an equation of the plane and determine its distance from the origin.

$$2x+3y+z = 2+9+2 = 13$$

$$|\vec{n}| = (4+9+1)^{1/2} = \sqrt{14} \qquad d = 13/\sqrt{14}$$

Equation: $2x+3y+z=13$ Distance: $13/\sqrt{14}$

2. (6 Points, 3 points each) Answer the following questions for the quadratic surfaces specified.

2.a $-4x^2+y^2+9z^2=7$ Type: Hyperboloid Axis of Symmetry: x-axis

Sub-type: one-sheet

2.b $-x+y^2-4z^2=16$ Type: Paraboloid Axis of Symmetry: x-axis

Sub-type: Hyperbolic

3. (8 Points) Find the below-requested partial derivatives of $f(x,y)=xe^{xy^2}$

$f_x = e^{xy^2} + xy^2e^{xy^2}$ $f_y = 2x^2ye^{xy^2}$

$f_{yy} = 2x^2e^{xy^2} + 4x^3y^2e^{xy^2}$ $f_{yx} = 4xye^{xy^2} + 2x^2y^3e^{xy^2}$

4. (6 points) If $f(x,y) = x\sin(y)$, $x(t) = t^2 - t + 1$ and $y(t) = e^{2t} - 1$, find $\frac{df}{dt}$ <u>using the chain rule.</u>

Please leave your answer in terms of $x,\ y,\ and\ t$. Finally, what is the value of $\frac{df}{dt}$ when $t = 0$?

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \sin(y)\,(2t-1) + x\cos(y)\,2e^{2t}$$

$x(0) = 1$

$y(0) = 0$

$\frac{df}{dt} =$ $\sin(y)\,(2t-1) + x\cos(y)2e^{2t}$

$\left.\frac{df}{dt}\right|_{t=0} =$ 2

5. (8 Points) $f(x,y) = x^3 - 3xy + 5 = 0$ implicitly defines y as a function of x. Find the slope, $\frac{dy}{dx}$, of the tangent line to the level curves of $f(x,y)$ at any point (x,y) using implicit differentiation. What is slope of the level curve at the point $P(1,2)$?

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 - 3y}{-3x}$$

$$= (x - y/x)\Big|_{(1,2)} = -1$$

$\frac{dy}{dx} =$ $x - y/x$

$\left.\frac{dy}{dx}\right|_{\cancel{(0,1)}\ (1,2)} =$ -1

6. (10 Points) Find the directional derivative of $f(x,y) = xe^y$ at the point $P(2,0,2)$ in the direction $\vec{u} = \langle 1,3 \rangle$.

$$D_{\vec{u}}f = \vec{\nabla} f \cdot \vec{u} / |\vec{u}|$$

Note: the direction vector must be normalized

$$= \langle e^y, xe^y \rangle\Big|_{(2,0)} \cdot \langle 1,3 \rangle / \sqrt{10}$$

$$= \langle 1,2 \rangle \cdot \langle 1,3 \rangle / \sqrt{10} = 7/\sqrt{10}$$

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7. Answer the following questions for the surface $f(x,y)=3e^{xy}-2x^3$ at the point $P(1,0,1)$

7.a (8 points) Find a unit vector in the xy-plane in the direction of steepest ascent at P. Simplify your answer.

$$\vec{\nabla} f = \langle 3ye^{xy}-6x^2, 3xe^{xy}\rangle\Big|_{(1,0)} = \langle -6, 3\rangle$$

$$\vec{u}_{SA} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{3\langle -2,1\rangle}{3\sqrt{5}} = \langle -2,1\rangle/\sqrt{5}$$

$\vec{u}_{SA} =$ $\langle -2,1\rangle/\sqrt{5}$

7.b (4 Points) Find a unit vector in the direction of steepest descent at P

$\vec{u}_{SD} = -\vec{u}_{SA}$

$\vec{u}_{SD} =$ $\langle 2,-1\rangle/\sqrt{5}$

7.c (6 Points) Find a unit vector tangent to the level curve at P

Reverse components and negate one of them. $\vec{u}_{LVL} =$ $\pm\langle 1,2\rangle/\sqrt{5}$

8. (12 Points) Find the equation of the plane tangent to the one-sheet hyperboloid $F(x,y,z)=10x^2+6y^2-z^2=7$ at the point $P(1,1,3)$.

$$\vec{\nabla} F = \langle 20x, 12y, -2z\rangle\Big|_{(1,1,3)} = \langle 20,12,-6\rangle$$

Use the normal vector $\vec{n} = \langle 10, 6, -3\rangle$

$$10x+6y-3z = 10+6-9 = 7$$

$$\boxed{10x+6y-3z=7}$$

This method was used because the surface was defined implicitly. If you solve for z you can obtain the same result using a linear approx. of the resulting function. See page 5 for this solution.

9. (12 Points) Find the critical point(s) of the function $f(x,y)=x^4-2x^2+y^2-4y+5$

$f_x = 4x^3-4x = 0 \Rightarrow x(x^2-1)=0 \Rightarrow \boxed{x=0 \text{ or } x=\pm 1}$

$f_y = 2y-4=0 \Rightarrow \boxed{y=2}$

THREE CRITICAL POINTS: $(0,2)$, $(-1,2)$, $(1,2)$

10. (12 Points) Assume you found a critical point of the function $g(x,y)$ and at this critical point the second partial derivatives were $g_{xx}=-2$, $g_{yy}=-5$, *and* $g_{xy}=g_{yx}=-3$. Determine if this critical point represents a local maximum, a local minimum, or a saddle point. NOTE: Provide the formula for the discriminant, D, and then explicitly carry out all portions of the test using D.

$D = f_{xx}f_{yy} - f_{xy}^2 = (-2)(-5)-(-3)^2$

$= 10-9 = 1 > 0$

Since f_{xx} and $f_{yy} < 0 \Rightarrow$ The point is a local maximum

Bonus (10 Points): Consider the surface $f(x,y)=2x^4+4y^3$. If you start at the point $P(1,1,6)$ on this surface, find the path in the $xy-plane$ of the path of steepest descent (Note: Your answer should be $y(x) = \textit{some function of } x$ and the starting point ~~(1,2)~~ (1,1) should be on this curve).

$\vec{\nabla} f = \langle 8x^3, 12y^2 \rangle \Rightarrow \frac{dy}{dx} = \frac{12y^2}{8x^3} = \frac{3y^2}{2x^3}$

Therefore $y^{-2}dy = \frac{3}{2}x^{-3}dx$

$-y^{-1} = -\frac{3}{4}x^{-2} + C$

plugging in $(1,1)$ yields $-\frac{1}{1} = -\frac{3}{4} + C \Rightarrow \boxed{C=-\frac{1}{4}}$

$y^{-1} = \frac{3}{4}x^{-2} + \frac{1}{4}$ $\boxed{y = \frac{4x^2}{3+x^2}}$

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Problem #8: Alternative Solution

Solving for z yields $z(x,y) = \sqrt{10x^2+6y^2-7}$

where positive square root was selected since we are at the point $(1,1,3)$. The linear approximation for $z(x,y)$ at $(1,1,3)$ is given by

$$z \approx 3 + \left.\frac{\partial z}{\partial x}\right|_{(1,1,3)} (x-1) + \left.\frac{\partial z}{\partial y}\right|_{(1,1,3)} (y-1)$$

$$= 3 + \left.\frac{20x}{2(10x^2+6y^2-7)^{1/2}}\right|_{(1,1,3)} (x-1) + \left.\frac{12y}{2(10x^2+6y^2-7)^{1/2}}\right|_{(1,1,3)} (y-1)$$

$$= 3 + \frac{10}{3}(x-1) + \frac{6}{3}(y-1)$$

Multiplying by three and rearranging yields

$$\boxed{10x + 6y - 3z = 7}$$

... the same plane derived using $\vec{\nabla} F$.

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