Section 2.3: Characteristics of Invertible Matrices

Note: Here we explore the important concepts of Linear Systems of Equations (Ch.1) & how these concepts relate to $n \times n$ Square Matrices :

The Invertible Matrix Theorem

Let A be a square, $n \times n$, matrix. Then the following statements are logically equivalent (all true -or- all false):

(1) A is an invertible matrix.

(2) A is row-equivalent to the $n \times n$ Identity Matrix, I_n.

(3) A has n-pivot positions.

(4) The Equation $A\vec{x} = \vec{0}$ has only the Trivial Solution.

(5) The Columns of A form a Linearly Independent Set.

(6) The Linear Transformation $\vec{x} \mapsto A\vec{x}$ is 1-1.

(7) The Eq. $A\vec{x} = \vec{b}$ has @ least one solution $\forall\ \vec{b} \in \mathbb{R}^n$.

↳ *(equivalently: $A\vec{x} = \vec{b}$ has a unique solution $\forall\ \vec{b} \in \mathbb{R}^n$)

(8) The Columns of A span $\mathbb{R}^n$.

(9) The Linear Transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

*(10) $\exists$ an $n \times n$ matrix 'C' ST $CA = I$.

(11) $\exists$ an $n \times n$ matrix 'D' ST $AD = I$.

(12) A^T (the transpose of A) is an invertible matrix.

Note: The truth of 1 always implies that 10 is true : $1 \Rightarrow 10$

Notation for the Invertible Matrix Theorem

<u>Note</u>: Again, the truth of ① always implies the truth of ⑩ (<u>denoted</u>: $① \Rightarrow ⑩$). This implication establishes the following "Circle of Implications" & kick-starts the proof :)

"<u>Circle of Implications</u>":

*If any one of these 5 statements are true, then so are the others :)

*The proof helps to link the other statements of the thm to the statements in this circle.
→ See 4 additional 'links' below.

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \searrow & \\ 2 & & & & 10 \\ \Uparrow & & & & \Downarrow \\ 3 & & \Leftarrow & & 4 \end{array}$$

Proof for the Invertible Matrix Theorem

<u>Note</u>: Instead of attacking this proof w/ "brute-force", we will verify equivalence in several steps

① <u>Verify the 'circle of implications' is logically equivalent</u>
⇒ Here we use def./theorems/properties that we have previously verified (*detailed proofs in 2.1/2.2 notes :))

② <u>Verify the remaining statements using "links" to the implications in the circle:</u>

$$\begin{array}{ccc} & 1 & \\ \swarrow & & \nwarrow \\ 11 & \Rightarrow & 7 \end{array}, \quad 7 \Leftrightarrow 8 \Leftrightarrow 9, \quad 4 \Leftrightarrow 5 \Leftrightarrow 6, \quad \& \quad 1 \Leftrightarrow 12$$

Proof (of the Invertible Matrix Theorem):

Let A be some $n \times n$, square matrix.

(1) Verify the 5 statements of the 'Circle of Implications' are log. equiv & that A is an invertible matrix (*Statement ①).

*Since A is invertible, then by definition:

$\exists$ an $n \times n$ matrix C ST $CA = I_n$, where I_n = the $n \times n$ Identity Matrix & $C = A^{-1}$ = the Inverse of A (*Statement ⑩)

So, ① $\Rightarrow$ ⑩ ✓

*Since $\exists$ an $n \times n$ matrix C ST $CA = I_n$, then property (2.1):

$A\vec{x} = \vec{0}$ has only the Trivial Solution (*Statement ④)

So, ⑩ $\Rightarrow$ ④ ✓

*Since $A\vec{x} = \vec{0}$ has only the Trivial Solution, then by prop (2.2):

Matrix A has n-pivots (*Statement ③)

↳ The Columns of A Form a Linearly Independent Set (*Statement ⑤)

↳ $\vec{x} \mapsto A\vec{x}$ is 1-1 (*Statement ⑥)

So, ④ $\Rightarrow$ ③ ✓

$\boxed{④ \Leftrightarrow ⑤ \Leftrightarrow ⑥}$

*Since A has n-pivots & each pivot must $\exists$ in a different row:

The n-pivot positions must $\exists$/lie in the Main Diagonal, in which case: rref(A) is row-equiv. to I_n (*Statement ②)

So, ③ $\Rightarrow$ ② ✓

*Since rref(A) is row-equivalent to I_n, then by def:

Matrix A is invertible. Woohoo! Back @ statement ① & the circle of ~~life~~ implications is

So, ② $\Rightarrow$ ① ✓

Proof (Invertible Matrix Th^m) continued...

(11) Verify that the remaining 7 statements are logically equivalent using the four additional "links" to the circle:

*Since A is invertible, then by def:

$\exists$ an $n \times n$ matrix D ST $AD = I_n$, where I_n = Identity Matri:

& $D = A^{-1}$ = Inverse of A (*statement 11)

So, 1 $\Rightarrow$ 11 ✓

*Since $\exists$ a $n \times n$ matrix D ST $AD = I_n$, then by 2.1 Prop:

$A\vec{x} = \vec{b}$ has @ least one solution $\forall\ \vec{b} \in \mathbb{R}^n$ (*Statement 7)

(2.1)

So, 11 $\Rightarrow$ 7 ✓

*Since $A\vec{x} = \vec{b}$ has a solution $\forall\ \vec{b} \in \mathbb{R}^n$, then by 2.2 Prop:

Matrix A is invertible (*statement 1)

So, 7 $\Rightarrow$ 1 ✓

$\therefore$ properties 7 & 11 are "linked" through 1 ✓

*Since properties 7, 8, & 9 are equivalent statements (Th^m 1.4) & Th^m 1.9:

7 $\forall\ \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ is consistent

$\Updownarrow$

8 Columns of matrix A span $\mathbb{R}^m$

$\Updownarrow$

9 The Linear Transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

$\therefore$ properties 8 & 9 are "linked" through 7 ✓

Proof (Invertible Matrix Thm) Continued...

* Since we already confirmed 4 $\Leftrightarrow$ 5 $\Leftrightarrow$ 6 during part (i) of the proof:

$\therefore$ properties 5 & 6 are "linked" @ 4

* Since A is invertible, the by definition:

A^T is also invertible (* Statement 12)

So, 1 $\Leftrightarrow$ 12 ✓

Done! We have officially verified that the truth one 1 implication implies the truth of the other !!

Therefore: For an $n \times n$, square matrix A, all 12 properties are logically equivalent.

$\boxtimes$

* Conclusions from the Invertible Matrix Theorem *

- Let A & B be square matrices. If $AB = I$, then both A & B are invertible, with $B = A^{-1}$ & $A = B^{-1}$.

- The Invertible Matrix Th^m divides the set of all $n \times n$ matrices into 2 disjoint classes:

    ① Invertible Matrices:

    * Also called "Nonsingular Matrices"
    * Each statement in the th^m describes a property of every $n \times n$ invertible matrix.

    ② Noninvertible Matrices:

    * Also known as, "Singular Matrices"
    * The negation of each statement in the th^m describes a property of every $n \times n$ noninvertible matrix.

    Ex: (The Negation of Prop. 2)
    "A is NOT row equivalent to I_n"

Example: Determine if the given matrix is invertible. Explain:

$$A = \begin{bmatrix} -4 & -1 \\ 12 & 3 \end{bmatrix}$$

Answer:

* **Recall:** For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $\det(A) = (ad - bc) \neq 0$ then A is invertible.

* **Given:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 12 & 3 \end{bmatrix}$

* **Check the determinant:**

$ad - bc = -12 + 12 = 0$ ∴ Matrix is **NOT** invertible (Matrix is Singular)

Answer

Example: Determine if the following matrix is invertible

Explain:

$$A = \begin{bmatrix} \underline{4} & 5 & 8 & 5 \\ 0 & \underline{1} & 3 & 7 \\ 0 & 0 & \underline{2} & 9 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$$

Answer:

Note: This matrix is already in echelon-form.
How lucky!

$$\boxed{\begin{array}{l} \therefore \underline{\text{Matrix } A \text{ is invertible.}} \\ \text{Just by visual observation, we can} \\ \text{see that } \exists\ n=4 \text{ pivots (one per row).} \\ \iff A \text{ is invertible.} \end{array}}$$

Answer.

Example: Determine if the matrix below is invertible; Explain your answer: $\begin{bmatrix} 7 & 5 \\ -3 & -4 \end{bmatrix}$

Answer:

*Start row-reducing the given coefficient matrix (to observe the pivot positions :)):

$$\begin{array}{l} * \left(\frac{3}{7}\right)R_1 \\ + \quad R_2 \\ \hline \text{new } R_2 \end{array} \rightarrow \begin{bmatrix} 7 & 5 \\ 0 & -13/7 \end{bmatrix}$$

$\boxed{\text{The matrix is invertible b/c a pivot } \exists \text{ in each row.}}$

Alternative Solution

Note: Since the given matrix is 2×2, a faster/easier way to determine if it is invertible is to check the determinant :

$\Rightarrow$ For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $(ad - bc) \neq 0$, then A is invertible.

Given: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -3 & -4 \end{bmatrix}$

$\boxed{(ad - bc) = -28 + 15 = -13 \neq 0 \checkmark \quad \therefore \text{Matrix is invertible.}}$

Example: Determine if the following matrix is invertible. Explain.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -5 & -3 & 0 \\ 8 & 4 & -1 \end{bmatrix}$$

Answer: $\boxed{\because \text{Yes, } A \text{ is invertible!}}$

* By visual observation, we can see the 3 pivots will not be changed by any row operations.

Note: If this solution does not satisfy you &/or you want to verify A is invertible $\Longrightarrow$ Just row reduce A :)

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -5 & -3 & 0 \\ 8 & 4 & -1 \end{bmatrix} \overset{\frac{1}{4}R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -5 & -3 & 0 \\ 8 & 4 & -1 \end{bmatrix}$$

* $\begin{array}{r} 5R_1 \\ +\ R_2 \\ \hline \text{new } R_2 \end{array}$ $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 8 & 4 & -1 \end{bmatrix} \overset{-\frac{1}{3}R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 8 & 4 & -1 \end{bmatrix}$

* $\begin{array}{r} -8R_1 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array}$ $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & -1 \end{bmatrix}$

* $\begin{array}{r} -4R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array}$ $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Longrightarrow$ $\boxed{\therefore \text{Since the Matrix has } n = 3 \text{ pivots, the Matrix is invertible.}}$

Answer.

Example: Determine if the following matrix is invertible. Explain:

$$A = \begin{bmatrix} 3 & 0 & -4 \\ 2 & 0 & 3 \\ -5 & 0 & 5 \end{bmatrix}$$

Answer:

*Note: Column 2 is $\vec{0}$. No row-operations will change this column.

* Since Col #2 $= \vec{0}$, matrix A will have at most $n = 2$ pivots.

$\Rightarrow$ A free variable $\exists$ & $A\vec{x} = \vec{0}$ will have a nontrivial solution.

$\Rightarrow$ $\boxed{\therefore \text{ The Columns of } A \text{ are Linearly Dependent (NOT Linearly Independent) \& the matrix is NOT invertible.}}$

Answer.

Example: Determine if the given matrix is invertible.

Explain:

$$A = \begin{bmatrix} 1 & -2 & -7 \\ 0 & 3 & 4 \\ -3 & 4 & 0 \end{bmatrix}$$

Answer:

* Start row-reducing A:

- $\begin{array}{r} 3R_1 \\ + \; R_3 \\ \hline \text{new } R_3 \end{array}$

$$\sim \begin{bmatrix} 1 & -2 & -7 \\ 0 & 3 & 4 \\ 0 & -2 & -21 \end{bmatrix} \overset{\frac{1}{3}R_2}{\sim} \begin{bmatrix} 1 & -2 & -7 \\ 0 & 1 & 4/3 \\ 0 & -2 & -21 \end{bmatrix}$$

- $\begin{array}{r} 2R_2 \\ + \; R_3 \\ \hline \text{new } R_3 \end{array}$

$$\sim \begin{bmatrix} 1 & -2 & -7 \\ 0 & 1 & 4/3 \\ 0 & 0 & -53/3 \end{bmatrix}$$

$\therefore$ A pivot $\exists$ in every row $(n=3)$:

$\Longrightarrow$ Matrix A is invertible.

Ans.

Example: Determine if the following matrix is invertible. Explain:

$$A = \begin{bmatrix} -1 & -3 & 0 & 1 \\ 4 & 4 & 16 & -4 \\ -2 & -6 & 5 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

Answer:

*Start row-reducing A to observe the pivot positions:

$$\begin{array}{c} -R_1 \\ \sim \\ \frac{1}{4}R_2 \end{array} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 4 & -1 \\ -2 & -6 & 5 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c} * -R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & -2 & 4 & 0 \\ -2 & -6 & 5 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \underset{\sim}{-\tfrac{1}{2}R_2} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ -2 & -6 & 5 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c} * 2R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \rightarrow \begin{array}{c} * R_2 \\ +R_4 \\ \hline \text{new } R_4 \\ \\ * \frac{1}{5}R_3 \end{array} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{array}{l} \therefore \text{A pivot } \exists \text{ in each row } (n = 4 \text{ pivots}) \\ \Longrightarrow \text{Matrix } A \text{ is invertible.} \end{array}}$$

Ans.

Example: An mxn upper triangular matrix is one whose entries below the main diagonal are zeros, as demonstrated in the provided matrix. When is a square, upper triangular matrix invertible?

Explain:

$$A = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* The "main diagonal" of nonzero entries :)

Answer:

Note: When a square, upper triangular matrix has all nonzero entries in the main diagonal

$\Longrightarrow$ The matrix will have a pivot in every row (i.e. n-pivots)

$\therefore$ A square, nxn upper-triangular matrix is invertible when all entries that lie in the main diagonal are nonzero values

Example: Explain why the columns of A^2 span $\mathbb{R}^2$ whenever the columns of an $n \times n$ matrix A are Linearly Independent.

Answer:

* Let A be a square, $n \times n$ matrix
* & the columns of matrix A are Linearly Independent.

Then, by the Invertible Matrix Theorem:

Since matrix A is square & the columns of A are Linearly Independent, it follows directly that A is invertible.

* Since A is a square, $n \times n$ invertible matrix, then: the product $AA = A^2$ is also invertible. (by prop. of inverses 2.2)

* Again, by the Invertible Matrix Thm:

Since A^2 is square & an invertible matrix, it follows directly that the columns of A^2 span $\mathbb{R}^2$.

Example: Let A & B be $n \times n$ matrices.
Show that if AB is invertible, then so it B.

Answer:

* Let A & B be $n \times n$ matrices.

* $ that the product AB is invertible.

$\Rightarrow$ Let $C = AB$; C is invertible.

* Since $C = AB$ is invertible, then its inverse is also invertible. (by Def.)

$\Rightarrow$ Let C^{-1} be the inverse of $C = AB$:

So, $C^{-1}(C) = I$ & $C(C^{-1}) = I$

* Therefore, matrix B (& matrix A) is also invertible by the Invertible Matrix Thm (i.e. 10 $\rightarrow$ 1)

Example: Use the invertible matrix theorem to decide if the following matrix is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

Answer:

* **Note:** Since the 12 statements of the theorem are logically equivalent, which whichever one(s) is easiest for you.

* Lets start by row-reducing A: (b/c I want to see if a pivot $\exists$ in each row :))

$$\begin{array}{c} -3R_1 \\ +\ R_2 \\ \hline \text{new } R_2 \end{array} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ -5 & -1 & 9 \end{bmatrix}$$

$$\begin{array}{c} 5R_1 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{array}{c} R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Answer.

$\Rightarrow$ $\boxed{\therefore \text{ Since } A \text{ has } n=3 \text{ pivots (one per row), } A \text{ is invertible.}}$

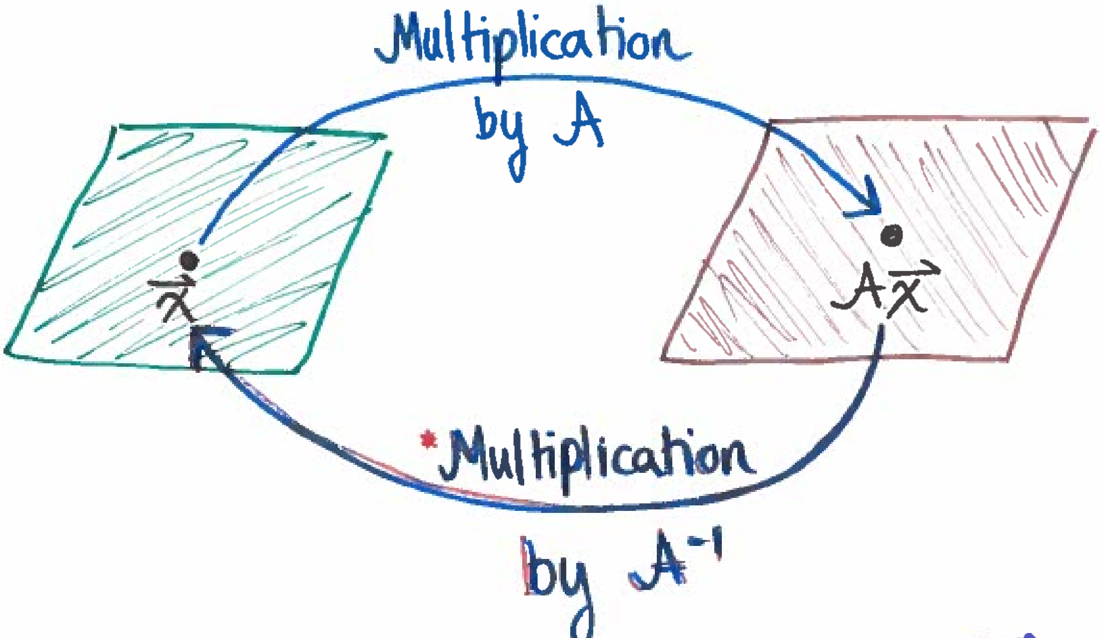
* **Note:** This is NOT an exclusive soluti

There are 11 other ways to attain the same conclusion (some easier than others!

Invertible Linear Transformations

<u>Note</u>: When a matrix A is invertible, the equation $A^{-1}A\vec{x} = \vec{x}$ can be viewed as a statement about linear transformations.

<u>Graphical Interpretation</u>: A^{-1} transforms "$A\vec{x}$" back to "$\vec{x}$"



A <u>Linear Transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible**</u> if $\exists$ a function $S: \mathbb{R}^n \to \mathbb{R}^n$

ST:

$$* \; S(T(\vec{x})) = \vec{x}, \; \forall \; \vec{x} \in \mathbb{R}^n$$

$$* \; T(S(\vec{x})) = \vec{x}, \; \forall \; \vec{x} \in \mathbb{R}^n$$

<u>Notes</u>: (About S)

- If 'S' $\exists$, it is <u>unique</u> & <u>must also be a Linear Trans.</u>
- Called: "The Inverse of T" & is <u>denoted</u>: " $S = T^{-1}$ "

Theorem (Invertible Linear Transformations):

*Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, & Let A be the Standard Matrix of T. Then:

$\boxed{T \text{ is invertible IFF } A \text{ is an invertible matrix.}}$

*In this Case, the Linear Transformation S given by $S(\vec{x}) = A^{-1}\vec{x}$ is the unique function satisfying the equations:

$$\begin{cases} \bullet S(T(\vec{x})) = \vec{x} \\ \bullet T(S(\vec{x})) = \vec{x} \end{cases}, \quad \forall \vec{x} \in \mathbb{R}^n$$

Proof:

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a Linear Transformation ST A is the Standard Matrix of T.

*Case 1: \$ T is invertible. (Show that: A is invertible)

Since $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then by def., $\exists$ a function

$$S: \mathbb{R}^n \to \mathbb{R}^n \text{ ST } S(T(\vec{x})) = \vec{x} \text{ \& } T(S(\vec{x})) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n.$$

*Recall: $T: \mathbb{R}^n \to \mathbb{R}^m$ is an onto mapping if $\forall \vec{b} \in \mathbb{R}^m$, $\exists$ @ least one $\vec{x} \in \mathbb{R}^n$ ST $T(\vec{x}) = \vec{b}$.

Then, by def: $T(S(\vec{x})) = \vec{x}$ $\forall \vec{x} \in \mathbb{R}^n$, maps T onto $\mathbb{R}^n$:

$\forall \vec{b} \in \mathbb{R}^n$, $\exists$ @ least one $\vec{x} = S(\vec{b}) \in \mathbb{R}^n$ ST: $T(\vec{x}) = T(S(\vec{b})) = \vec{b}$.

$\Rightarrow$ So, $\forall \vec{b} \in \mathbb{R}^n$ is in the Range of T $\therefore$ Property 9 $\Rightarrow$ 1

A is invertible ☒

Proof: (Invertible Linear Transformations Cont...)

***Case 2:** $\$$ A is invertible.

*Show that: T is invertible

By Definition: Since A is an $n \times n$ invertible matrix, then "S" is a Linear Transformation, $S(\vec{x}) = A^{-1}\vec{x}$, satisfying the equations:

$$\begin{cases} * \, S(T(\vec{x})) = \vec{x} \\ \& \\ * \, T(S(\vec{x})) = \vec{x} \end{cases}, \forall \ \vec{x} \in \mathbb{R}^n$$

Since $S(\vec{x}) = A^{-1}\vec{x}$:

$$S(T(\vec{x})) = S(A\vec{x}) = A^{-1}(A\vec{x}) = \vec{x}, \ \forall \ \vec{x} \in \mathbb{R}^n.$$

$\boxed{\therefore T \text{ is invertible}}$

Example: What can be said about a 1-1 linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$?

Answer:

* **Given:** $T: \mathbb{R}^n \to \mathbb{R}^n$ ST $T(\vec{x}) = A\vec{x}$ is 1-1

• Since T is 1-1:

* The Columns of A are Linearly Independent.
* The Equation $A\vec{x} = \vec{0}$ has only the Trivial Solution ($\vec{x} = \vec{0}$).

*Note: Those 2 observations come from section 1.9 :)

⋮

• Since $T: \mathbb{R}^n \to \mathbb{R}^n$

* Matrix A is an $n \times n$, square matrix
$\Updownarrow$
* Matrix A is invertible
$\Updownarrow$
* T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
$\Updownarrow$
* Col. of Matrix A span $\mathbb{R}^n$
⋮
etc. etc. (all 12 prop. hold true :))

*Note: 'All those properties come from the 'Invertible Matrix Theorem' :)

• Since Matrix A is invertible $\Longrightarrow$ T is invertible!

* Note: This last prop. comes from the Tall Linear Transformation Thm.

Example: The given 'T' is a Linear Transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that T is invertible & find a formula for T^{-1}.

$$T(x_1, x_2) = T(3x_1 - 5x_2,\ -3x_1 + 6x_2)$$

Answer: *Recall: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a Linear Transformation. Then T is invertible IFF A is invertible.

*Given: $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a Linear Transformation

ST $T(\vec{x}) = A\vec{x} \ \forall\ \vec{x} \in \mathbb{R}^2$, $A \to$ The Standard Matrix of T

$$\Rightarrow T(\vec{x}) = \underbrace{\begin{bmatrix} 3 & -5 \\ -3 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

So, the Standard Matrix of T is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -3 & 6 \end{bmatrix}$$

*Need to verify that the standard matrix of T (i.e. matrix A) is invertible:

- $\det(A) = (ad - bc) = 18 - 15 = 3 \neq 0$ ✓

$\boxed{\therefore \text{Since the } \det(A) = 3 \neq 0, \text{ the standard matrix of T is invertible.}}$

$\Downarrow$

$\Rightarrow$ By Def: $\boxed{\text{Since } A \text{ is invertible, then T is also invertible.}}$

Example: Continued...

* **Find a formula for T^{-1}:**

Note: Since T is invertible, the inverse of T $\exists$ & is defined: $T^{-1} = A^{-1}\vec{x}$, where:

A^{-1} = The Standard Matrix of T^{-1} :)

* **Find A^{-1}:**

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 & 5 \\ 3 & 3 \end{bmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} 2 & 5/3 \\ 1 & 1 \end{bmatrix}}$$

*Standard Matrix of T^{-1}

Ans

* **Find the inverse of T, T^{-1}:**

$$T^{-1}(x_1, x_2) = A\vec{x} = \begin{bmatrix} 2 & 5/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Inverse of T

$$\therefore \boxed{T^{-1}(x_1, x_2) = \left(2x_1 + \frac{5}{3}x_2,\ x_1 + x_2\right)}$$

Ans

Example: Define the Linear Transformations $T: \mathbb{R}^3 \to \mathbb{R}^3$ & $S: \mathbb{R}^3 \to \mathbb{R}^3$

ST $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_2 - x_3 \\ 2x_3 \end{bmatrix}$ & $S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_3 \\ -x_2 + 2x_3 \\ -x_1 + 2x_2 - x_3 \end{bmatrix}$.

(a) Find the Standard Matrix of $S \circ T$.

(b) Find the Standard Matrix of $T \circ S$.

(c) Find a vector $\vec{v}$ ST $(S \circ T)(\vec{v}) = \vec{b}$ where $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, if any exists.

Answer:

*Given:

- $T: \mathbb{R}^3 \to \mathbb{R}^3$ ST $T(\vec{x}) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow$ *Standard Matrix of T: $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

- $S: \mathbb{R}^3 \to \mathbb{R}^3$ ST $S(\vec{x}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow$ *Standard Matrix of S: $B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

*Part (a): Find the Standard Matrix of $S \circ T$:

Note: The Standard Matrix of $S \circ T = S(T(\vec{x}))$ is the product of the Standard Matrix of S (i.e. "B") & the standard matrix of T (i.e. "A) $\Rightarrow$ $\boxed{\therefore \text{Standard Matrix of } S \circ T: \ BA}$

$$BA = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0-2 \\ 0+0+0 & 0-1+0 & 0+1+4 \\ -1+0+0 & 2+2+0 & -1-2-2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & -2 \\ 0 & -1 & 5 \\ -1 & 4 & -5 \end{bmatrix}}$$

Answer ✓

Example Continued...

*Part (b): Find the Standard Matrix of $T \circ S$:

Note: The Standard Matrix of $T \circ S = T(S(\vec{x}))$ is the product of the Standard Matrix of T (i.e. "A") & the Standard Matrix of S ("B")

$\Rightarrow \therefore \boxed{\text{Standard Matrix of } T \circ S: \ AB}$

$$AB = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0+0-1 & 0+2+2 & -1-4-1 \\ 0+0+1 & 0-1-2 & 0+2+1 \\ 0+0-2 & 0+0+4 & 0+0-2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 4 & -6 \\ 1 & -3 & 3 \\ -2 & 4 & -2 \end{bmatrix}}$$

Ans ✓

*Part (c): Find a vector $\vec{v}$ ST $(S \circ T)(\vec{v}) = \vec{b}$ ST $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$:

Note: Here we are asked to find a vector $\vec{v} \in \mathbb{R}^3$ whose image under "$S \circ T$" is $\vec{b}$:)

$$* \ (S \circ T)\vec{v} = (BA)\vec{v} = \vec{b} \iff [BA \mid \vec{b}]$$

* Row-reduce the Augmented Matrix to find $\vec{v}$ (if it $\exists$):

$$[BA \mid \vec{b}] = \left[\begin{array}{ccc|c} 0 & 0 & -2 & 1 \\ 0 & -1 & 5 & 1 \\ -1 & 4 & -5 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} -1 & 4 & -5 & 0 \\ 0 & -1 & 5 & 1 \\ 0 & 0 & -2 & 1 \end{array}\right] \overset{\substack{-R_1 \\ -R_2 \\ -\frac{1}{2}R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -\underline{4} & 5 & 0 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

$$\overset{\frac{4R_2 + R_1}{N.R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -15 & -4 \\ 0 & 1 & -\underline{5} & -1 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \overset{\frac{5R_3 + R_2}{N.R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -\underline{15} & -4 \\ 0 & 1 & 0 & -7/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \overset{\frac{15R_3 + R_1}{N.R_1}}{\sim}$$

⇘

Example Continued...

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -23/2 \\ 0 & 1 & 0 & -7/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \Rightarrow \begin{cases} v_1 = -23/2 \\ v_2 = -7/2 \\ v_3 = -1/2 \end{cases}$$

$$\boxed{\therefore \vec{v} = \begin{bmatrix} -23/2 \\ -7/2 \\ -1/2 \end{bmatrix}}$$

Answer