Section 4.6: Rank:

Note: Here we use our knowledge of vector spaces to look "inside" a matrix & explore the relationships btw rows & columns ∵

The Row Space

*If A is an $m \times n$ matrix, each row of A has 'n' entries & is thus identified w/ a vector in $\mathbb{R}^n$.

*The set of all Linear Combinations of the row-vectors is called, "The Row Space"

$\Rightarrow$ Denoted: $\text{Row}(A)$

Note: Since each row has n-entries, $\text{Row}(A)$ is a subspace of $\mathbb{R}^n$ ∵

*Since the rows of A = the columns of A^T

$\Rightarrow \text{Row}(A) = \text{Col}(A^T)$

*Theorem 13:

If two matrices (A & B) are row-equivalent, then their row spaces are the same.
If B is in echelon form, the nonzero rows of B form a basis for the row space of A, as well as, for B.

Example 1 (Row Space):

Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$ &

- $\vec{r}_1 = (-2, -5, 8, 0, -17)$
- $\vec{r}_2 = (1, 3, -5, 1, 5)$
- $\vec{r}_3 = (3, 11, -19, 7, 1)$
- $\vec{r}_4 = (1, 7, -13, 5, -3)$

Use the Definition of "Row Space" to explore the matrix.

Observations:

Note: A is an $m \times n = 4 \times 5$ matrix $\begin{cases} * \ m = 4 \text{ rows} \\ * \ n = 5 \text{ columns} \end{cases}$

- Since A is a 4×5 matrix:
  - i) Each row of A has $n = 5$ entries
  - ii) Each row can be identified w/ a vector in $\mathbb{R}^5$
  - iii) Since each row has 5 entries, Row(A) is a subspace of $\mathbb{R}^5$
- Since the Row(A) is the set of all Linear Combinations of the row-vectors:

$$\boxed{\therefore \text{Row}(A) = \text{Span}\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4\}}$$

Note: IF we knew the dependence-relation amongst the rows, we could use the spanning set thm to shrink the spanning set to a basis... BUT row-operations on A change row-dep. relations, so row-reducing A does NOT help here... Although other benefits $\exists$ as we shall see.

Example² [Row(A), Col(A), & Nul(A)]:

Find bases for the row-space, the column-space, and the null-space of the following matrix:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Answer:

* Recall (Thᵐ 13): If 2 Matrices (A & B) are row-equivalent, their row space is the same. If B is in Echelon-Form, the non-zero rows of B form a basis for the row-space of A&B.

* Recall (Thᵐ 6; Sect 4.3): The pivot columns of a matrix A form a basis for Col(A).

* Recall (4.2): Solving $A\vec{x}=\vec{0}$ to produce an explicit description of the Nul(A) = Spanning Set of the Nul(A); which is the basis for Nul(A)

• To find bases for Row(A) & Col(A), row-reduce $[A \mid \vec{0}]$ to echelon form:

$$R_1 \leftrightarrow R_2: \left[\begin{array}{ccccc|c} 1 & 3 & -5 & 1 & 5 & 0 \\ -2 & -5 & 8 & 0 & -17 & 0 \\ 3 & 11 & -19 & 7 & 1 & 0 \\ 1 & 7 & -13 & 5 & -3 & 0 \end{array}\right] \overset{\substack{2R_1 + R_2 \\ N.R_2}}{\sim} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \overset{\substack{-3R_1 + R_3 \\ N.R_3}}{\sim}$$

⇘

Example² [Row(A), Col(A), Nul(A)] Continued...

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \underset{\frac{1}{2}R_3}{\overset{\frac{-R_1 + R_4}{N.R_4}}{\sim}} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \underset{\frac{1}{4}R_4}{\overset{\frac{-R_2 + R_3}{N.R_3}}{\sim}}$$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -2 \end{bmatrix} \underset{\text{Then } R_3 \Leftrightarrow R_4}{\overset{\frac{-R_2 + R_4}{N.R_4}}{\sim}} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

* Echelon Form *

* By Theorem 13: The nonzero rows form a basis for ROW(A):
(Use the rows of the Echelon Form Matrix.)

$$\boxed{\therefore \text{Basis for Row}(A) = \{(1,3,-5,1,5), (0,1,-2,2,-7), (0,0,0,-1,5)\}}$$

Ans.

* By Theorem 6: The pivot columns of A form a basis for Col(A):

$$\boxed{\therefore \text{Basis for Col}(A) = \{\vec{a}_1, \vec{a}_2, \vec{a}_4\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}}$$

Ans.

Note: While the Echelon Form of A provides bases for Row(A) & Col(A), we need rref to find the Nul(A) ⋮

Example² [Row(A), Col(A), Nul(A)] Continued...

• To find a basis for Nul(A), continue row-reducing $[A \mid \vec{0}]$ to rref to find an 'explicit description':

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -3R_2 + R_1 \\ \text{N. } R_1 \\ \sim \\ -R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & -5 & 26 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 5R_3 + R_1 \\ \text{N. } R_1 \\ \sim \\ -2R_3 + R_2 \\ \text{N. } R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} \cdot\, x_1 = -x_3 - x_5 \\ \cdot\, x_2 = 2x_3 - 3x_5 \\ \cdot\, x_3 \text{ is free} \\ \cdot\, x_4 = 5x_5 \\ \cdot\, x_5 \text{ is free} \end{cases}$$

RREF :)

So,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \quad \text{ST } x_3, x_5 \in \mathbb{R}$$

*Since the Spanning Set of $A\vec{x} = \vec{0}$ Forms a basis for Nul(A):

$$\boxed{\therefore \text{Basis for Nul}(A) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}}$$

Ans.

Additional Observations About Row Space

- Unlike Col(A), the Row(A) & Nul(A) have **NO** simple connection w/ the entries of matrix A itself.

*Note: We can still find a basis for Row(A) uses the rows of A... but, it takes extra work :

1) Find A^T
2) Row-reduce $[A^T \mid \vec{0}]$ to echelon form to identify the pivot columns.
3) Since the 'Pivot Columns of A^T' = 'Rows of A', the pivot columns of A^T form a basis for Row(A).

As you can see, it is still advantageous to use the method previously defined in sections 4.2/4.3 :

- **WARNING**: It is important to keep in mind that row operations may change the Linear Dependence Relations amongst the rows of a Matrix!!

⇒ IOW: Linear Independence amongst the rows of a matrix in echelon-form does **NOT** imply that those rows are Linearly Independent for the original matrix!!

* The Rank Theorem *

Note: The following provides us w/ fundamental relations among the dimensions of Col(A), Row(A), & Nul(A).

* Definition:

The rank of matrix A is the dimension of the Column Space of A.

* $\boxed{\text{rank}(A) = \dim[\text{col}(A)]}$

• Since $\text{Row}(A) = \text{Col}(A^T) \Rightarrow \dim[\text{Row}(A)] = \text{rank}(A^T)$

Note: The dimension of the Nul(A) is sometimes called the "nullity of A", but we will avoid this terminology in this course.

* Theorem[14] (The Rank Theorem):

The dimensions of the column space & the row space of an $m \times n$ matrix A are equal.

This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation:

$$\text{rank}(A) + \dim[\text{Nul}(A)] = n$$

<u>Example</u>: Assume that the matrix A is row-equivalent to B. Without calculations, list rank(A) & dim[Nul(A)]. Then find bases for Col(A), Row(A), & Nul(A):

$$A = \begin{bmatrix} 1 & 3 & 4 & -7 & -3 \\ 2 & 6 & 10 & -16 & -7 \\ -2 & -6 & -17 & 23 & 15 \\ -2 & -6 & 0 & 6 & 0 \end{bmatrix} \sim B = \begin{bmatrix} \underline{1} & 3 & 4 & -7 & 3 \\ 0 & 0 & \underline{1} & -1 & -1 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Answer</u>:

*<u>Recall</u>: For some $m \times n$ matrix A

i) $\text{Rank}(A) = \dim[\text{Col}(A)] = \dim[\text{Row}(A)]$

ii) $\text{Rank}(A) + \dim[\text{Nul}(A)] = n$

*<u>Given</u>: A & B are $m \times n = 4 \times 5$ matrices $\Longrightarrow$ $\begin{cases} * \ m = 4 \text{ rows} \\ * \ n = 5 \text{ columns} \end{cases}$

- <u>Matrix B is in Echelon Form & has 3 pivot columns</u>

$\rightarrow \{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$

$\Longrightarrow$ <u>So Matrix A has 3 pivot columns</u>: $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$

*The pivot col.'s form a Basis for Col(A)

$$\boxed{\therefore \dim[\text{Col}(A)] = \text{rank}(A) = 3}$$ <u>Ans.</u>

$\Longrightarrow$ <u>By the Rank Th^m</u>: $\text{rank}(A) + \dim[\text{Nul}(A)] = n$

$\Longrightarrow 3 + \dim[\text{Nul}(A)] = 5$

$$\boxed{\therefore \dim[\text{Nul}(A)] = 2}$$ <u>Ans.</u>

Example Continued...

*** Find Bases for Row(A), Col(A), & Nul(A):**

Note: Here we use B (Echelon Form of A) :)

∴ By Thm 13, the nonzero rows form a basis for Row(A):

$$\boxed{\text{Basis for Row}(A) = \{(1,3,4,-7,3),\ (0,0,1,-1,-1),\ (0,0,0,0,1)\}}$$

Ans.

∴ By Thm 6, the pivot columns of A form a Basis for Col(A):

$$\boxed{\text{Basis for Col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ -17 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -7 \\ 15 \\ 0 \end{bmatrix} \right\}}$$

Ans.

Note: To find a Basis for Nul(A), we must continue row-reducing $[B \,\vdots\, \vec{0}]$ to rref.

$$\begin{bmatrix} 1 & 3 & 4 & -7 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{-4R_2 + R_1 \\ N.R_1}}{\sim} \begin{bmatrix} 1 & 3 & 0 & -3 & 7 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{-R_3 + R_1 \\ N.R_1 \\ R_3 + R_2 \\ N.R_2}}{\sim} \begin{bmatrix} 1 & 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \cdot\ x_1 = -3x_2 + 3x_4 \\ \cdot\ x_2 \text{ is free} \\ \cdot\ x_3 = x_4 \\ \cdot\ x_4 \text{ is free} \\ \cdot\ x_5 = 0 \end{cases} \iff \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$\ni$ $x_2, x_4 \in \mathbb{R}$

∴ Basis for Nul(A):

$$\boxed{\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

Ans.

Example: Assume Matrix A is row-equivalent to B. Without calculations, list the rank(A) & dim[Nul(A)]. Then, find bases for Row(A), Col(A), & Nul(A). :

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 2 & -3 \\ 1 & 2 & -3 & 0 & 0 & -6 \\ 1 & -2 & 1 & 0 & 14 & -6 \\ 1 & 3 & -3 & 1 & -7 & -6 \\ 1 & 3 & -4 & 0 & 0 & -8 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 1 & -2 & 0 & 2 & -3 \\ 0 & 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

*List the rank(A) & dim[Nul(A)], w/o calculations:

- A & B are $m \times n = 5 \times 6$ matrices
- B is in Echelon Form (of A) w/ pivot columns: $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_5, \vec{b}_6\}$
  - *Note: A has pivot columns in the same positions :
- Since the pivot columns of A form a Basis for Col(A)
  $\Rightarrow \dim[\text{Col}(A)] = 5$ (b/c $\exists$ 5-pivot columns)

$\therefore \boxed{\text{Rank}(A) = \dim[\text{Col}(A)] = 5}$ Ans✓

- By the Rank Th^m: $\text{rank}(A) + \dim[\text{Nul}(A)] = n$
  $5 + \dim[\text{Nul}(A)] = 6$
  $\therefore \boxed{\dim[\text{Nul}(A)] = 6-5 = 1}$ Ans✓

*Find Bases for Row(A), Col(A), & Nul(A):

- By Th^m 13: A Basis for Row(A) = Nonzero rows of Echelon Form (B)

Ans✓

$\therefore \boxed{\{(1,1,-2,0,2,-3), (0,1,-1,0,-2,-3), (0,0,1,1,-5,3), (0,0,0,0,1,-2), (0,0,0,0,0,1)\}}$ ⇓

Example Continued...

- By Thm 6: The pivot columns of A form a Basis for Col(A)

$$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 14 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ -6 \\ -6 \\ -8 \end{bmatrix} \right\}}$$

- To find a Basis for Nul(A), row-reduce $[B \,\vdots\, \vec{0}]$ to rref:

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 2 & -3 \\ 0 & 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\frac{-R_2 + R_1}{N.R_1}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\frac{R_3 + R_1}{N.R_1},\ \frac{R_3 + R_2}{N.R_2}}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 1 & -7 & 0 \\ 0 & 0 & 1 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\frac{R_4 + R_1}{N.R_1},\ \frac{7R_4 + R_2}{N.R_2},\ \frac{5R_4 + R_3}{N.R_3}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = -x_4 \\ x_4 \text{ is free} \\ x_5 = 0 \\ x_6 = 0 \end{cases} \longleftrightarrow \vec{x} = x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ st } x_4 \in \mathbb{R}$$

$$\boxed{\therefore \text{Basis for Nul(A)}: \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

Example 3 (Rank Th^m):

(a) If A is a 7×9 matrix, with a 2-dim. null space, what is the rank of A?

(b) Could a 6×9 matrix have a 2-dim. null space? Explain.

Answer:

* Recall (The Rank Theorem + Def.): For some $m\times n$ matrix A,
  - $\text{Rank}(A) = \dim[\text{Col}(A)] = \dim[\text{Row}(A)]$
  - $\text{Rank}(A) + \dim[\text{Nul}(A)] = n$

***Part (a):** \$ that A is a 7×9 matrix ST $\dim[\text{Nul}(A)] = 2$:

- Since A is $m\times n = 7\times 9$ $\Rightarrow$ A has $n=9$ columns.
- By the Rank Th^m: $\text{Rank}(A) + 2 = 9$

$$\Rightarrow \text{Rank}(A) = 9-2 = 7 \qquad \boxed{\therefore \text{Rank}(A) = 7}$$

Ans.

***Part (b):** \$ that A is a 6×9 matrix ST $\dim[\text{Nul}(A)] = 2$:

- By the Rank Th^m: $\text{Rank}(A) + \dim[\text{Nul}(A)] = n$

$$\Rightarrow \text{Rank}(A) + 2 = 9$$

$$\Rightarrow \text{Rank}(A) = 7$$

- By Def: $\boxed{\text{Rank}(A) = \dim[\text{Col}(A)] = 7}$ ... So the quick answer is NO... But why?
- Since each column has $m=6$ entries: A can have at most $n=6$ pivot positions
- Since the pivot columns of A form a Basis for $\text{Col}(A)$ (By Th^m 6):

$$\Rightarrow \dim[\text{Col}(A)] = 7 > 6 \rightarrow\leftarrow$$

Ans. $\boxed{\therefore \text{Since } \dim[\text{Col}(A)] \text{ CANNOT exceed } 6, \text{ a } 6\times 9 \text{ matrix CANNOT have } \dim[\text{Nul}(A)] = 2.}$

Example: If an 8×6 matrix A has rank 4, find the $\dim[\mathrm{Nul}(A)]$, $\dim[\mathrm{Row}(A)]$, & $\mathrm{rank}(A^T)$.

Answer:

* **Given:**
  - Matrix A is $m \times n = 8 \times 6$
  - $\mathrm{rank}(A) = 4$

* **Find $\dim[\mathrm{Nul}(A)]$:**

Recall (The Rank Th^m): $\mathrm{rank}(A) + \dim[\mathrm{Nul}(A)] = n$

$\Rightarrow 4 + \dim[\mathrm{Nul}(A)] = 6$

$$\boxed{\therefore \dim[\mathrm{Nul}(A)] = 6 - 4 = 2}$$ Ans.

* **Find $\dim[\mathrm{Row}(A)]$:**

Recall (The Rank Th^m): $\dim[\mathrm{col}(A)] = \dim[\mathrm{Row}(A)]$

$\Rightarrow$ Since $\mathrm{rank}(A) = \dim[\mathrm{col}(A)] = 4$

$$\boxed{\therefore \dim[\mathrm{Row}(A)] = 4}$$ Ans.

* **Find $\mathrm{rank}(A^T)$:**

Recall (Def. of Rank): $\mathrm{Row}(A) = \mathrm{Col}(A^T) \Longrightarrow \dim[\mathrm{Row}(A)] = \mathrm{rank}(A^T)$

$$\boxed{\therefore \mathrm{rank}(A^T) = 4}$$ Ans.

Example: $\exists$ a 6×8 matrix has 5 pivot columns.

a) Is $Col(A) = \mathbb{R}^5$? Explain.

b) Is $Nul(A) = \mathbb{R}^3$? Explain.

Answer:

*Given: A is an $m \times n = 6 \times 8$ matrix $\to m = 6$ rows, $\to n = 8$ columns

*Part (a): Is $Col(A) = \mathbb{R}^5$? No.

Note: The columns of matrix A each have $m=6$ entries $\Rightarrow$ $Col(A)$ is a subspace of $\mathbb{R}^6$.

$\boxed{\therefore Col(A) \text{ is a subspace of } \mathbb{R}^6 \ (\text{NOT } \mathbb{R}^5)}$ Ans

*Since A has 5 pivot columns $\Rightarrow$ $\dim[Col(A)] = 5$

*Part (b): Is $Nul(A) = \mathbb{R}^3$? No.

Note: A vector $\vec{x}$ that satisfies $A\vec{x}$ has $n=8$ entries $\Rightarrow$ $Nul(A)$ is a subspace of $\mathbb{R}^8$

$\boxed{\therefore Nul(A) \text{ is a subspace of } \mathbb{R}^8 \ (\text{NOT } \mathbb{R}^3)}$ Ans.

*Since A has 5 pivot columns $\Rightarrow$ $\dim[Nul(A)] = 8-5 = 3$

Example: If the null space of a 7×9 matrix A is 2-dimensional, what is the dimension of the row space of A?

Answer:

* **Given:**
  - Matrix A is $m \times n = 7 \times 9$
  - $\dim[\mathrm{Nul}(A)] = 2$

* A^T is $n \times m = 9 \times 7$:
  Since $\mathrm{Row}(A) = \mathrm{Col}(A^T)$,
  $\dim[\mathrm{Row}(A)] = \mathrm{rank}(A^T)$

* **Want:**
  - $\dim[\mathrm{Row}(A)] = ?$

Recall (The Rank Thm): $\boxed{\dim[\mathrm{Col}(A)] = \dim[\mathrm{Row}(A)]}$

* Since $\mathrm{rank}(A) = \dim[\mathrm{col}(A)]$, then:

$\Rightarrow \dim[\mathrm{col}(A)] + \dim[\mathrm{nul}(A)] = n$

So, $\dim[\mathrm{col}(A)] + 2 = 9$

$\Rightarrow \dim[\mathrm{col}(A)] = 9 - 2 = 7$

$$\boxed{\therefore \dim[\mathrm{Row}(A)] = 7}$$ Ansr

Example:

(a) If A is a 12×9 matrix, what is the largest possible rank(A)?

(b) If A is a 9×12 matrix, what is the largest possible rank(A)?

Explain:

Recall: $\text{rank}(A) = \dim[\text{col}(A)]$

Answer:

*Part (a): $ A is a 12×9 matrix:

- Since A has $n = 9$ columns $\Longrightarrow$ $\exists$ @ most $n = 9$ pivot columns

- Since the pivot columns of A form a Basis for col(A):

  $\Rightarrow$ $\dim[\text{col}(A)]$ can have @ most 9 (vectors in Basis)

$\therefore$ $\boxed{\text{Largest Possible rank}(A) \text{ is } = 9}$ Ans✓

*Part (b): $ A is a 9×12 matrix:

- Since A has $n = 12$ columns, BUT only $m = 9$ rows

  $\Rightarrow$ $\exists$ @ most "9" pivot positions (columns/rows)

$\therefore$ $\boxed{\text{For the same reasoning as (a), the largest possible rank}(A) \text{ is } = 9}$ Ans✓

Example:

(a) If A is a 6×3 matrix, what is the largest possible $\dim[\text{Row}(A)]$?

(b) If A is a 3×6 matrix, what is the largest possible $\dim[\text{Row}(A)]$?

Explain

Recall: $\dim[\text{Col}(A)] = \dim[\text{Row}(A)]$

Answer:

*Part (a): \$ A is a 6×3 matrix:

• Since A has $n=3$ columns $\Rightarrow \exists$ @ most 3 pivot columns.

• Since the pivot columns of A form a Basis for $\text{Col}(A)$

$\Rightarrow \dim[\text{col}(A)]$ can have @ most 3 (vectors)

$\boxed{\therefore \text{ Largest Possible } \dim[\text{row}(A)] = 3}$ Ans✓

*Part (b): \$ A is a 3×6 matrix:

• Since A has $n=6$ columns, BUT only $m=3$ rows

$\Rightarrow \exists$ @ most 3 pivot positions (columns/rows)

$\therefore$ For the same reasoning as (a), the largest possible $\dim[\text{row}(A)] = 3$ Ans✓

* Rank(A) & the Invertible Matrix Theorem *

Note: The various vector space concepts associated w/ a matrix provide several more equivalence statements to the "Invertible Matrix Th^m" seen in section 2.3 :

⇒ The following is added to the bottom of that list.

* The Invertible Matrix Theorem (Continued...):

Let A be an $m \times n$ matrix. Then the following are equivalent to the statement that:

1. "A is invertible"

⋮

13. The Columns of A form a Basis in $\mathbb{R}^n$.
14. The $Col(A) = \mathbb{R}^n$
15. The $\dim[Col(A)] = n$
16. The $\text{rank}(A) = n$
17. $Nul(A) = \{\vec{0}\}$
18. $\dim[Nul(A)] = 0$

Recall: (1) From the Invertible Matrix Th^m ⟺ (12) is true.

IOW: $\boxed{\text{"}A \text{ is invertible" IFF "}A^T \text{ is invertible"}}$

Conclusion: Since $row(A) = col(A^T)$, every statement in the IMT holds true for A^T as well. Producing a list of over 30 equivalence statements!!

Example 1 (Fundamental Subspaces Determined by A):

Consider an $m\times n$ matrix A.

(a) Which of the following subspaces are in $\mathbb{R}^n$?

(b) How many distinct subspaces are in this list?

$\Rightarrow$ Subspaces: $Row(A)$, $Col(A)$, $Nul(A)$, $Row(A^T)$, $Col(A^T)$, $Nul(A^T)$

Answer:

*Since A is an $m\times n$ matrix:

① $Nul(A)$ is a subspace of $\mathbb{R}^n$ *By Thm 2 (4.2)*

② $Col(A)$ is a subspace of $\mathbb{R}^m$ *By Thm 3 (4.2)*

*Since A^T is an $n\times m$ matrix:

③ $Nul(A^T)$ is a subspace of $\mathbb{R}^m$

④ $Col(A^T)$ is a subspace of $\mathbb{R}^n$

*Since rows of A = columns of A^T:

⑤ $Row(A)$ is a subspace of $\mathbb{R}^n$ *B/c $Row(A) = Col(A^T)$*

⑥ $Row(A^T)$ is a subspace of $\mathbb{R}^m$ *B/c $Row(A^T) = Col(A)$*

$\therefore$ Subspaces of $\mathbb{R}^m$: $\boxed{Col(A),\ Nul(A^T),\ \&\ Row(A^T)}$

$\therefore$ Subspaces of $\mathbb{R}^n$: $\boxed{Nul(A),\ Col(A^T),\ \&\ Row(A)}$

$\therefore$ Since $Row(A) = Col(A^T)$ & $Row(A^T) = Col(A)$: $\boxed{\exists\ 4 \text{ distinct subspaces}}$

Example[2] (Fundamental Subspaces Determined by A):

Let A be an $m \times n$ matrix.

Explain why the equation $A\vec{x} = \vec{b}$ has a solution $\forall\ \vec{b} \in \mathbb{R}^m$ IFF the equation $A^T\vec{x} = \vec{0}$ has only the trivial solution.

Answer:

* Let A be an $m \times n$ matrix.

* The equation $A\vec{x} = \vec{b}$ has a solution $\forall\ \vec{b} \in \mathbb{R}^m$ IFF:

  - The Columns of A span $\mathbb{R}^m$

  $\Updownarrow$

  - A has a pivot position in every row

    $\hookrightarrow$ Which happens IFF $\dim[\mathrm{Col}(A)] = m$

    * Need to determine when this occurs $\therefore$ Lets consider the transpose.

* Since A^T is an $n \times m$ matrix:

  - $\dim[\mathrm{col}(A)] = \dim[\mathrm{row}(A)] = \dim[\mathrm{col}(A^T)] = \mathrm{rank}(A^T)$

$$\Rightarrow \boxed{\mathrm{rank}(A^T) + \dim[\mathrm{Nul}(A^T)] = \dim[\mathrm{col}(A)] + \dim[\mathrm{Nul}(A^T)] = m}$$

* So, the $\dim[\mathrm{col}(A)] = m$ IFF:

  - $\dim[\mathrm{Nul}(A^T)] = 0$

  $\Updownarrow$

  - $\mathrm{Nul}(A) = \{\vec{0}\}$ $\rightarrow$ * Which only happens if: $A^T\vec{x} = \vec{0}$ has only the trivial solution!

$\boxed{X}$