Solving homogeneous recurrences

Given $a_0 t_n + a_1 t_{n-1} + ... + a_k t_{n-k} = 0$. Guess $t_n = x^n$ for an unknown constant x.

We have $a_0 x^n + a_1 x^{n-1} + ... + a_k x^{n-k} = 0$

Ignoring solution $x = 0$, the equation is satisfied if and only if $p(x) = a_0 x^k + a_1 x^{k-1} + ... + a_k = 0$ It is called the *characteristic equation of the recurrence.*

Let $r_1, r_2, ..., r_k$ be the k roots of $p(x)$. We conclude that $t_n = \sum_{i=1}^{k} c_i r_i^n$ for any constants c_i.

This is the only solution when the k roots are distinct.

Example

$$f_n = \begin{cases} n, & n = 0,1 \\ f_{n-1} + f_{n-2}, & otherwise \end{cases}$$

We have $f_n - f_{n-1} - f_{n-2} = 0$. The characteristic equation is $x^2 - x - 1 = 0$

whose roots are $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.

So $f_n = c_1 * (\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$.

We know $f_0 = 0 = c_1 + c_2$ and $f_1 = 1 = c_1 * \frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2}$

We have $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$.

Multiple roots

In general, if $r_1, r_2, ..., r_l$ are the l distinct roots of the characteristic polynomial and their multiplicities are $m_1, m_2, ..., m_l$, then

$$t_n = \sum_{i=1}^{l} \sum_{j=0}^{m_i - 1} c_{ij} n^j r_i^n$$

.

Example: multiple roots

$$t_n = \begin{cases} n, & n = 0, 1, or\ 2 \\ 5t_{n-1} - 8t_{n-2} + 4t_{n-3} & \end{cases}$$

The characteristic polynomial is

$$x^3 - 5x^2 + 8x - 4 = (x-1)(x-2)^2$$

.

So $t_n = c_1 1^n + c_2 2^n + c_3 n 2^n$. Applying the initial conditions, we obtain $c_1 = -2, c_2 = 2$ and $c_3 = -1/2$. Therefore $t_n = 2^{n+1} - n2^{n-1} - 2$.

Inhomogeneous recurrences: a general form

Consider the following generalization

$$a_0 t_n + a_1 t_{n-1} + ... + a_k t_{n-k} = b_1^n p_1(n) + b_2^n p_2(n) + ...,$$

where b_i is a constant and $p_i(n)$ is a polynomial in n of degree d_i.

The characteristic polynomial is

$$(a_0 x^k + a_1 x^{k-1} + ... + a_k)\Pi_i (x - b_i)^{d_i+1}.$$

Inhomogeneous recurrences: an example

$$t_n = \begin{cases} 0, & n = 0 \\ 2t_{n-1} + n + 2^n & otherwise \end{cases}$$

Rewrite the recurrence as

$$t_n - 2t_{n-1} = 1^n n^1 + 2^n n^0$$

So $b_1 = 1, p_1(n) = n, b_2 = 2$, and $p_2(n) = 1$. The characteristic polynomial is

$$(x-2)(x-1)^2(x-2) = (x-1)^2(x-2)^2,$$

All solutions to the recurrence has the form

$$t_n = c_1 1^n + c_2 n 1^n + c_3 2^n + c_4 n 2^n$$

Substitute it into the original recurrence, which gives

$$n + 2^n = (2c_2 - c_1) - c_2 n + c_4 2^n$$

. We obtain $c_4 = 1$, and thus $t_n = \Theta(n2^n)$.

Master Theorem: a simple version

Let $T : N \rightarrow R^{+}$ be an eventually nondecreasing function such that

$$T(n) = lT(n/b) + cn^k, n > n_0$$

when n/n_0 is an exact power of b. The constants $n_0, l \geq 1, b \geq 2$, and $k \geq 0$ are all integers. c is a positive real number.

We have

$$T(n) \in \begin{cases} \Theta(n^k) & if \quad l < b^k \\ \Theta(n^k logn) & if \quad l = b^k \\ \Theta(n^{log_b l}) & if \quad l > b^k \end{cases}$$

Examples

$$T(n) = 9T(n/3) + n.$$