*Section 4.4: Coordinate Systems:

Note: An important reason for specifying a Basis ($\mathcal{B}$) for a vector space V is to impose a coordinate system on V.

(i) If $\mathcal{B}$ contains n-vectors, then the coord. system will make V act like $\mathbb{R}^n$.

(ii) If V is already in $\mathbb{R}^n$, then $\mathcal{B}$ will determine a coord. system that gives a new "view" of V.

*Theorem 7 (The Unique Representation Th^m):

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis for a vector space V. Then $\forall \vec{x} \in V$, $\exists$ a unique set of scalars $c_1, \dots, c_n$ ST:

$$\vec{x} = c_1\vec{b}_1 + \cdots c_p\vec{b}_n$$

*PROOF:

- Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a Basis for V.
- Then by Definition: $\mathcal{B}$ is a Linearly Independent set that spans V $\Rightarrow$ $\forall \vec{x} \in V$, $\exists$ scalars $\{c_1, \dots, c_n\}$ ST

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

- \$ $\exists$ a vector $\vec{x} \in V$ ST: $\vec{x} = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n$ ST $\{d_1, \dots, d_n\} \rightarrow$ scalars.

*Goal: Show that the scalars are unique; IOW: $\{c_1, \dots, c_n\} = \{d_1, \dots, d_n\}$

- Since V is a vector space, then by def/axioms:

$$\forall \vec{x} \in V \Rightarrow \vec{x} + (-\vec{x}) = \vec{0} \in V$$

$$\Rightarrow \vec{x} + (-\vec{x}) = [c_1\vec{b}_1 + \cdots + c_n\vec{b}_n] - [d_1\vec{b}_1 + \cdots + d_n\vec{b}_n] = \vec{0}$$

*Grouping 'like terms'

$$= (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n = \vec{0}$$

$\Downarrow$

Proof Continued...

- Since β is Linearly Independent, the by def., the vector equation:

$$(c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n = \vec{0}$$ has ONLY the trivial solution ST $(c_1 - d_1) = 0, \ldots, (c_n - d_n) = 0$.

- Since all scalars must be zero, then:

$$c_j - d_j = 0 \implies c_j = d_j, \ \forall \text{ integers } j \text{ ST } 1 \leq j \leq n$$

$\boxed{\therefore \text{The set of scalars is unique!}}$ $\boxed{X}$

* **Definition:** $ $\beta = \{\vec{b}_1, \ldots, b_n\}$ is a Basis for V & that $\vec{x} \in V$. The coordinates of $\vec{x}$ relative to the Basis β -or- the β-coord. of $\vec{x}$ are the weights/scalars $c_1, \ldots, c_n$ ST: $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$

- If $c_1, \ldots, c_n$ are the β-coordinates of $\vec{x}$, then the vector

$$[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$ is the "β-coord. vector of $\vec{x}$" or "Coord. vector of $\vec{x}$ (relative to β)".

- The mapping $\vec{x} \mapsto [\vec{x}]_\beta$ is the Coord. Mapping (determined by β).

Example 1: Consider a basis $\beta = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$, where $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. \$ an $\vec{x} \in \mathbb{R}^2$ has the coordinate vector $[\vec{x}]_\beta = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\vec{x}$.

Answer:

* **Note:** The β-coordinates of $\vec{x}$ tell us how to build $\vec{x}$ from the vectors in β :)

* **Given:**

- **Basis for V:** $\beta = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
- **β-coord. vector of $\vec{x}$:** $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

* **Want to find:**

- **β-coord. of $\vec{x}$:** $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2$

$$\Rightarrow \vec{x} = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+3 \\ 0+6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\therefore \boxed{\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}}$$

Answer

* **Note:** The entries in $\vec{x}$ are the coord. of $\vec{x}$ relative to the Standard Basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\} \Rightarrow [\vec{x}]_{\mathcal{E}}$

$$\therefore \boxed{\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{e}_1 + 6\vec{e}_2}$$

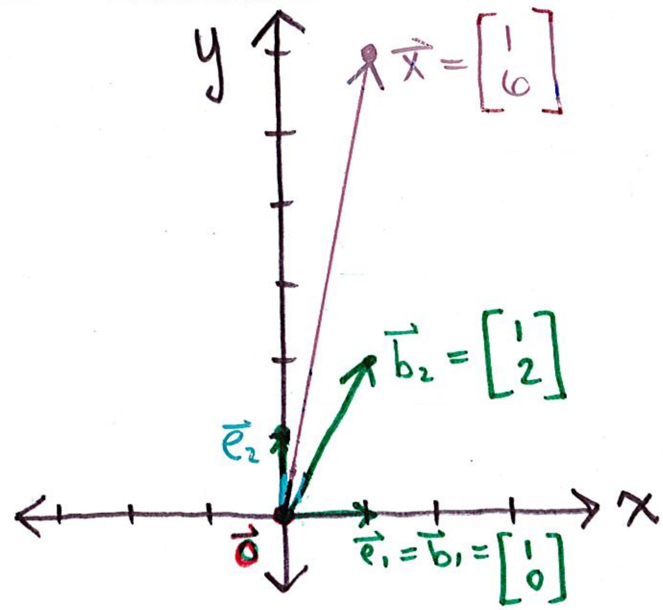
Answer.

A Graphical Interpretation of the Coordinates

A coordinate system on a set consists of a 1-1 mapping of the points in the set into $\mathbb{R}^n$.

Lets use the Standard Basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$, the vectors $\vec{b}_1 = \vec{e}_1$, $\vec{e}_2$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $[\vec{x}]_\beta = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, & $\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ to observe the differences btw Standard-Graphing & β-Graphing

① Standard-Graphing Paper:

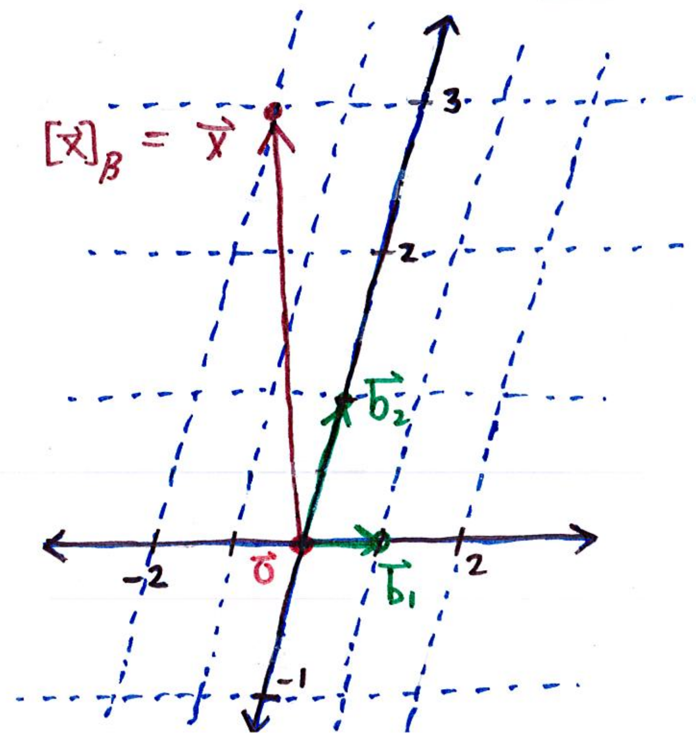


*Here we graph the vectors:
$\vec{b}_1 = \vec{e}_1$, $\vec{e}_2$, $\vec{b}_2$ & $\vec{x}$.

*The coord. of $\vec{x}$ (1 & 6) gives the location of $\vec{x}$ relative to the standard basis, $\{\vec{e}_1, \vec{e}_2\}$:

→ { • 1 unit in the $\vec{e}_1$ direction.
  • 6 units in the $\vec{e}_2$ direction.

② β-Graphing Paper:



*He we graph the vectors:

$\vec{b}_1$, $\vec{b}_2$, & $\vec{x}$ (adapted to β)

"x-axis" "y-axis" (based off Standard Graph)

*The coord. vector $[\vec{x}]_\beta = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, gives the location of $\vec{x}$ on this NEW coord. system:

→ { • -2 units in b_1 direction
  • 3 units in $\vec{b}_2$ direction

Example: Let $B = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a Basis For a Vector Space V. Explain why the β-coordinate vectors of $\vec{b}_1, \dots, \vec{b}_n$ are the elementary columns $\vec{e}_1, \dots, \vec{e}_n$ of an $n \times n$ Identity Matrix.

Answer:

*Recall (The Standard Basis For $\mathbb{R}^n$):

Let $\vec{e}_1, \dots, \vec{e}_n$ be the columns of the $n \times n$ Identity Matrix

$$\Rightarrow I_3 = [\vec{e}_1 \cdots \vec{e}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\therefore$ The Set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called the Standard Basis of $\mathbb{R}^n$.

*Let $B = \{\vec{b}_1, \dots, \vec{b}_K, \dots, \vec{b}_n\}$ be the Basis of some vector space V.

*Write the K^{th}-vector as a Linear Combination of $\{\vec{b}_1, \dots, \vec{b}_n\}$:

$$\vec{b}_K = 0\vec{b}_1 + 0\vec{b}_2 + \cdots + 1\vec{b}_K + \cdots + 0\vec{b}_n \quad , \text{ where } 1 \le K \le n.$$

$$= [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_K \cdots \vec{b}_n] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = P_\beta [\vec{b}_K]_\beta$$

Conclusion:

$\therefore$ The K^{th} β-coordinate vector, $[\vec{b}_K]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_K$, is the K^{th} elementary column vector of I_K.

$\Rightarrow$ Thus true $\forall$ β-coordinate vectors :)

Example: Find the vector $\vec{x}$ determined by the given coordinate vector $[\vec{x}]_{\beta}$ & the given Basis β:

$$\beta = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\} \quad ; \quad [\vec{x}]_{\beta} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Answer:

* Recall (The Unique Representation Th^m): Let $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a Basis for some Vector Space V. Then $\forall\ \vec{x} \in V$, $\exists$ a unique set of weights $c_1, \dots, c_n$ ST: $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$, where $[\vec{x}]_{\beta} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

* Given:

- Basis: $\beta = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$
- Coordinate Vector of $\vec{x}$: $[\vec{x}]_{\beta} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

* Want: $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 = ?$

$$\Rightarrow \vec{x} = 5\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix} + \begin{bmatrix} 6 \\ -9 \end{bmatrix} = \begin{bmatrix} -10+6 \\ 5-9 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}}$$

Answer ✓

Example: Find the vector $\vec{x}$ determined by the given coordinate vector & Basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix} \right\} \quad ; \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Answer:

* **Recall:** $ $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$ is the Basis for some vector space V & $ that $\vec{x} \in V$. Then, the Coordinates of $\vec{x}$ (relative to $\mathcal{B}$) are the weights $c_1, \dots, c_n$ ST: $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$, where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

* **Given:**

· **Basis:** $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix} \right\}$

· **Coordinate vector:** $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

* **Want:** $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \dots ?$

$$\vec{x} = 3\begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -10 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -12+1-10 \\ -6+0+4 \\ 6+2+2 \end{bmatrix} = \begin{bmatrix} -21 \\ -2 \\ 10 \end{bmatrix}$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} -21 \\ -2 \\ 10 \end{bmatrix}}$$

Answer

Example: Find the vector $\vec{x}$ determined by the given coordinate vector & Basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} -5 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\} \quad ; \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Answer:

* **Recall:** $ $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a Basis for some vector space V & that $\vec{x} \in V$. Then, the coordinates of $\vec{x}$ relative to $\mathcal{B}$ are weights $c_1, \dots, c_n$ ST: $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$, where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

* **Given:**

- **Basis:** $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} -5 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\}$

- **Coordinate Vector:** $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

* **Want:** $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = ?$

$$\vec{x} = 2\begin{bmatrix} -5 \\ -3 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -6 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -10+0-2 \\ -6+0+5 \\ -4+0-1 \end{bmatrix}$$

$$\boxed{\therefore \ \vec{x} = \begin{bmatrix} -12 \\ -1 \\ -5 \end{bmatrix}}$$

Ans✓

Example: Let $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, & $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$. Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ relative to $\mathcal{B}$.

Answer:

*Recall: The coordinate vector of $\vec{x}$ (relative to $\mathcal{B}$) is defined: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$

*The $\mathcal{B}$-coordinates c_1, c_2 of $\vec{x}$ must satisfy the vector-equation: $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$:

$$\Rightarrow \quad c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \text{(Vector-Equation)}$$

$$\Leftrightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \text{(Matrix-Equation)}$$

*To find $[\vec{x}]_{\mathcal{B}}$, we can now row-reduce the Augmented Matrix to rref:

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & -1 & 4 \end{array}\right] \underset{N \cdot R_2}{\overset{-2R_1 + R_2}{\sim}} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -3 & -6 \end{array}\right] \overset{-\frac{1}{3}R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

$$\underset{N \cdot R_1}{\overset{-R_2 + R_1}{\sim}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \boxed{\therefore [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{x} = 3\vec{b}_1 + 2\vec{b}_2}$$

Answer ✓

Example: Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ relative to the given Basis, $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$:

$$\vec{b}_1 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Answer:

*Recall: IF $c_1, \dots, c_n$ are the $\mathcal{B}$-coordinates of $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$, then the Coordinate Vector is: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$.

*Given:

- Basis: $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$
- Vector $\vec{x}$: $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

*The Coordinate Vector, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, must satisfy the Vector-Eq: $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$:

- The Vector-Eq: $c_1 \begin{bmatrix} -2 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is equivalent to

- The Matrix-Eq: $\begin{bmatrix} -2 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

*To find $[\vec{x}]_{\mathcal{B}}$, We can two options (@ least ☺):

1. Row-reduce the equivalent augmented matrix to rref
2. Check the matrix is invertible, & then LH-multiply $\vec{x}$ by the inverse (usually reserved for easy 2×2 matrices ☺)

→

Example Continued... (*Solving for $[\vec{x}]_\beta$ by Option 2*)

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & -3 \end{bmatrix} \Rightarrow \det(A) = 6-(-8) = 14$

$\boxed{\therefore \text{ Since } \det(A) = 14 \neq 0,\ A \text{ is invertible } \checkmark}$

- Find A^{-1}:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -3/14 & -1/7 \\ 2/7 & -1/7 \end{bmatrix}$$

- To Find $[\vec{x}]_\beta$, LH-Multiply the Matrix-Eq. by the Inverse:

$$\begin{bmatrix} -2 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3/14 & -1/7 \\ 2/7 & -1/7 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3/14 & -1/7 \\ 2/7 & -1/7 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$I_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 4 \begin{bmatrix} -3/14 \\ 2/7 \end{bmatrix} + 1 \begin{bmatrix} -1/7 \\ -1/7 \end{bmatrix} = \begin{bmatrix} -6/7 \\ 8/7 \end{bmatrix} + \begin{bmatrix} -1/7 \\ -1/7 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -6/7 - 1/7 \\ 8/7 - 1/7 \end{bmatrix} = \begin{bmatrix} -7/7 \\ 7/7 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ Woohoo!

$$\boxed{\therefore [\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$ Same answer $\checkmark$

Example Continued... *(Solving for $[\vec{x}]_\beta$ by option 1) *

$$\left[\begin{array}{cc|c} -2 & 2 & 4 \\ -4 & -3 & 1 \end{array}\right] \overset{-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ -4 & -3 & 1 \end{array}\right] \overset{\frac{4R_1 + R_2}{N.R_2}}{\sim} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & -7 & -7 \end{array}\right] \overset{-\frac{1}{7}R_2}{\sim}$$

$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 1 \end{array}\right] \overset{\frac{R_2 + R_1}{N.R_1}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases}$$

$$\boxed{\therefore [\vec{x}]_\beta = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

*See next page for alternative solution :)

Answer.

Check: $(-1)\begin{bmatrix} -2 \\ -4 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ -3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \checkmark$$

Optional, but encouraged.

*One of the most beautiful features of Linear Algebra is that when it is all said & done, it is just Algebra. Which means $\Rightarrow$ We can ALWAYS check!! :) (with more algebra.)

Example: Find the Coordinate Vector $[\vec{x}]_\beta$ of $\vec{x}$ Relative to the Basis $\beta = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} -3 \\ -2 \\ -15 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -8 \\ 1 \\ -30 \end{bmatrix}$$

Answer:

* Recall: If $c_1, \dots, c_n$ are the β-coordinates of $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$, then the Coordinate Vector is $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$.

* Given:

- Basis: $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -15 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} \right\}$

- Vector: $\vec{x} = \begin{bmatrix} -8 \\ 1 \\ -30 \end{bmatrix}$

• Goal: We want to find $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ ST : $c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{x}$

* The Vector Equation:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ -15 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ -30 \end{bmatrix}$$

is equivalent to

* The Matrix Equation:

$$\begin{bmatrix} 1 & -3 & 2 \\ 1 & -2 & -2 \\ 3 & -15 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ -30 \end{bmatrix}$$

* Row-Reduce the equivalent augmented matrix to rref:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -8 \\ 1 & -2 & -2 & 1 \\ 3 & -15 & 6 & -30 \end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -8 \\ 1 & -2 & -2 & 1 \\ 1 & -5 & 2 & -10 \end{array}\right] \underset{\frac{-R_1 + R_3}{N.R_3}}{\overset{\frac{-R_1 + R_2}{N.R_2}}{\sim}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -8 \\ 0 & 1 & -4 & 9 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

Example Continued...

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -8 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & -4 & 9 \end{array}\right] \overset{-\frac{1}{2}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -4 & 9 \end{array}\right] \underset{\frac{-R_2 + R_3}{N.R_3}}{\overset{\frac{3R_2 + R_1}{N.R_1}}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & 8 \end{array}\right] \overset{\frac{-1}{4}R_3}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \overset{\frac{-2R_3 + R_1}{N.R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \Leftrightarrow \begin{cases} c_1 = -1 \\ c_2 = 1 \\ c_3 = -2 \end{cases}$$

$$\boxed{\therefore [\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}}$$

Answer

Example: Find the coordinate vector $[\vec{x}]_\beta$ of $\vec{x}$, relative to the given Basis $\beta = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$: $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ -1 \\ -8 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$

& $\vec{x} = \begin{bmatrix} -2 \\ 3 \\ -19 \end{bmatrix}$

Answer:

* Recall: If $c_1, \dots, c_n$ are the β-coord. of $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$, then the Coordinate Vector $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^3$

* The Vector-Eq: $c_1\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -1 \\ -8 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -19 \end{bmatrix}$

is equivalent to,

* The Matrix-Eq: $\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ -4 & -8 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -19 \end{bmatrix}$

* Row-reduce the Augmented Matrix to rref:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ -1 & -1 & -1 & 3 \\ -4 & -8 & 5 & -19 \end{array}\right] \underset{\frac{4R_1 + R_3}{N.R_3}}{\overset{\frac{R_1 + R_2}{N.R_2}}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 9 & -27 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right] \overset{\frac{-2R_2 + R_1}{N.R_1}}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right] \overset{\frac{-R_3 + R_1}{N.R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right] \longrightarrow \boxed{\therefore [\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}}$$

Answer.

*Change of Coordinates in $\mathbb{R}^n$ *

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a Basis in $\mathbb{R}^n$.

Let $P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$

• Then the following Vector-Eq. & Matrix-Eq. are equivalent:

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n \iff \vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

* $P_{\mathcal{B}}$ is called the "Change of Coordinates Matrix" from $\mathcal{B}$ to the Standard Basis in $\mathbb{R}^n$:

- Left-Mult. by $P_{\mathcal{B}}$ transforms the coord. vector $[\vec{x}]_{\mathcal{B}}$ into $\vec{x}$ (shown above).

*Since the Columns of $P_{\mathcal{B}}$ form a basis in $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem):

- Left-Mult. by $P_{\mathcal{B}}^{-1}$ converts $\vec{x}$ into its $\mathcal{B}$-coordinate vector:
$$P_{\mathcal{B}}^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$$

- Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a 1-1 Linear Transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$ (By the Invertible Matrix Thm)

*Note: This ↑ prop. holds true in a general vector space that has a Basis (as we shall see soon :))

Example: Find the Change-of-Coordinate Matrix from $\mathcal{B}$ to the Standard Basis in $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

Answer:

* **Recall:** Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be the Basis of some Vector Space. Then, $P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$ is called the "Change of Coordinate" Matrix.

* **Given:**

- **Basis:** $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

∴ **The Change of Coordinate Matrix:**

$$\boxed{P_{\mathcal{B}} = [\vec{b}_1 \ \vec{b}_2] = \begin{bmatrix} 6 & 2 \\ 1 & -1 \end{bmatrix}}$$

Answer.

Example: Use an inverse matrix to find $[\vec{x}]_\beta$ for the given $\vec{x}$ & β:

$$\beta = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \end{bmatrix} \right\} \; ; \; \vec{x} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

Answer:

* Recall: Since the Change of Coordinate Matrix is invertible

$$\vec{x} = P_\beta [\vec{x}]_\beta \iff P_\beta^{-1} \vec{x} = [\vec{x}]_\beta$$

* The Vector-Equation:

$$c_1 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

is equivalent to,

* The Matrix Equation:

$$\underbrace{\begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}}_{P_\beta} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{[\vec{x}]_\beta} = \underbrace{\begin{bmatrix} 4 \\ -4 \end{bmatrix}}_{\vec{x}}$$

* To find $[\vec{x}]_\beta$, LH-Multiply the Matrix-Eq. by P_β^{-1}:

- Let $P_\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} \Rightarrow \det(P_\beta) = 21 - 20 = 1 \neq 0$ ✓

- $P_\beta^{-1} = \dfrac{1}{\det(P_\beta)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{1} \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$

- $[\vec{x}]_\beta = P_\beta^{-1} \vec{x} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4 \begin{bmatrix} 7 \\ 5 \end{bmatrix} - 4 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 28 & -16 \\ 20 & -12 \end{bmatrix}$

$$\boxed{\therefore [\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}}$$

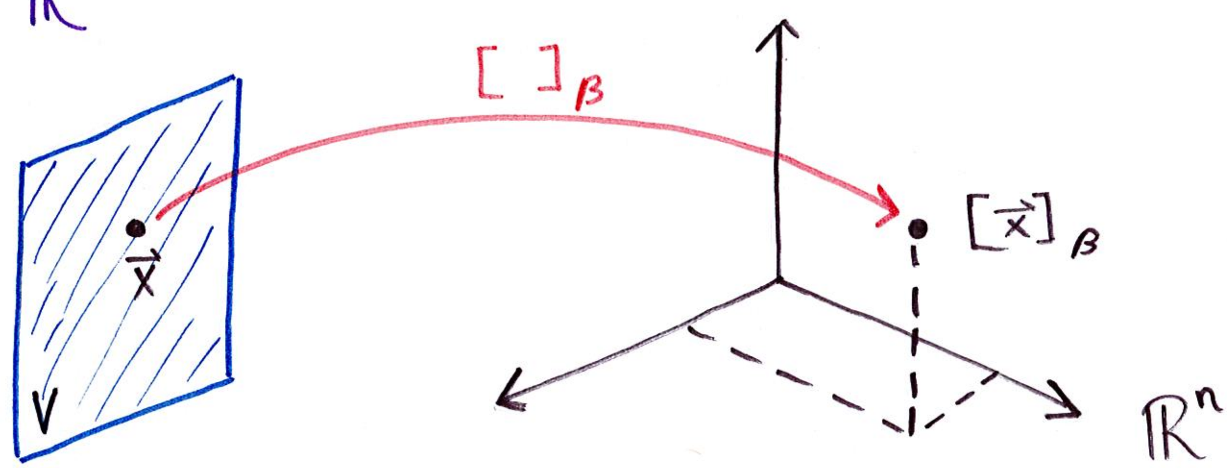
Answer ✓

The Coordinate Mapping

Note: Choosing a Basis $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for a vector space V introduces a coordinate system in V.

Illustration (The Coordinate Mapping from V onto $\mathbb{R}^n$):

The coordinate mapping $\vec{x} \mapsto [\vec{x}]_\beta$ connects the possibly unfamiliar space V to the familiar space $\mathbb{R}^n$



* Note: Points in V can now be identified by their new "names" :)

***Theorem 8:** Let $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a Basis for a vector space V. Then the coordinate mapping $\vec{x} \longmapsto [\vec{x}]_\beta$ is a 1-1 Linear Transformation from V onto $\mathbb{R}^n$.

* To Prove $\vec{x} \mapsto \vec{x}$ is a L.T. we need to show that $\forall\ \vec{u}, \vec{v} \in V$ & $\forall$ scalars c:

① $[\vec{u}+\vec{v}]_\beta = [\vec{u}]_\beta + [\vec{v}]_\beta$

② $[c\vec{u}]_\beta = c[\vec{u}]_\beta$

* Proof has 3 parts total

Proof of theorem 8:

Let $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a Basis for a vector space V.

*Goal: Show that $\vec{x} \mapsto [\vec{x}]_\beta$ is a 1-1 Linear Transformation

i) $[\vec{u} + \vec{v}]_\beta = [\vec{u}]_\beta + [\vec{v}]_\beta, \ \forall\ \vec{u}, \vec{v} \in V$

ii) $[c\vec{u}]_\beta = c[\vec{u}], \ \forall\ \vec{u} \in V$ & any scalar c.

Prop. of a Linear Transformation

① Show that the mapping is closed under addition:

- Since V is a vector space, then by def: $\vec{u} + \vec{v} \in V$:
  - $\forall\ \vec{u}, \vec{v} \in V$ ST $\vec{u} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ & $\vec{v} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$

$\Rightarrow \vec{u} + \vec{v} = (c_1 + d_1)\vec{b}_1 + \dots + (c_n + d_n)\vec{b}_n \in V$, ST

$(c_1 + d_1), \dots, (c_n + d_n)$ are the weights/scalars.

By Def $\Rightarrow$ $\boxed{\text{So, } (c_1 + d_1), \dots, (c_n + d_n) \text{ are the } \beta\text{-coordinates of } \vec{u} + \vec{v}.}$

- Then, the Coordinate vector of $\vec{u} + \vec{v}$ (relative to β) is:

$$[\vec{u} + \vec{v}]_\beta = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\vec{u}]_\beta + [\vec{v}]_\beta \ ✓$$

$\boxed{\therefore [\vec{u} + \vec{v}]_\beta = [\vec{u}]_\beta + [\vec{v}]_\beta \text{ \& the mapping is closed under addition.}}$ ✓

$\Downarrow$

Proof of theorem 8: Continued...

② Show that the mapping is closed under scalar-mult:

• Since V is a vector space, then by def: $c\vec{u} \in V$:

$\forall$ vectors $\vec{u} \in V$ ST $\vec{u} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$ & $\forall$ scalars "c"

$\Rightarrow c\vec{u} = (cc_1)\vec{b}_1 + \cdots + (cc_n)\vec{b}_n \in V$ ST

$(cc_1), \ldots, (cc_n)$ are the weights/scalars.

By Def $\Rightarrow$ $\boxed{\text{So, } (cc_1), \ldots, (cc_n) \text{ are the } \beta\text{-coordinates of } c\vec{u}}$

• Then, the coordinate vector of $c\vec{u}$ (relative to β) is:

$$[c\vec{u}]_\beta = \begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix} = c \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c \cdot [\vec{u}]_\beta \checkmark$$

∴ $[c\vec{u}]_\beta = c[\vec{u}]_\beta$ & so the mapping is closed under scalar-multiplication. ✓

*Therefore: Since both conditions are satisfied, the coordinate mapping is a 1-1 Linear Transformation that maps V onto $\mathbb{R}^n$. ☒

Note: To finish proving theorem 8, we need to show: ① $\vec{x} \mapsto [\vec{x}]_\beta$ is 1-1

-AND-

③ $\vec{x} \mapsto [\vec{x}]_\beta$ is onto

Proof[2] of theorem[8] continued....

* Show that the coordinate mapping is 1-1:

- \$ that $[\vec{u}]_\beta = [\vec{v}]_\beta$ for some vectors $\vec{u}, \vec{v} \in V$

ST : $$[\vec{u}]_\beta = [\vec{v}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c \end{bmatrix}$$

***Goal**: show that $\vec{u} = \vec{v}$ & thus the mapping is 1-1.

- Then, by definition of coordinate vectors:

$$\vec{u} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$
&
$$\vec{v} = c\vec{v}_1 + \cdots + c_n\vec{v}_n$$

$\Rightarrow$ So, $\vec{u} = \vec{v}$ ✓

∴ Since $\vec{u}, \vec{v} \in V$ are arbitrary vectors in V

ST $[\vec{u}]_\beta = [\vec{v}]_\beta$ when $\vec{u} = \vec{v}$:

∴ The coordinate mapping is 1-1.

☒

Proof[3] of theorem[8]: (ontinued...

***Show that the Coordinate Mapping is Onto $\mathbb{R}^n$:**

- Given any vector $\vec{y} \in \mathbb{R}^n$ ST: $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

- Let $\vec{u} \in V$ ST $\vec{u} = y_1\vec{b}_1 + \cdots + y_n\vec{b}_n$
  where $y_1, \dots, y_n$ are weights.

- Since $y_1, \dots, y_n$ are the $\mathcal{B}$-coordinates of $\vec{u}$,
  then the Coordinate Vector of $\vec{u}$ (relative to $\mathcal{B}$) in $\mathbb{R}^n$ is defined:

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{y} \in \mathbb{R}^n$$

$\therefore$ Since $\vec{y} \in \mathbb{R}^n$ is arbitrary & $[\vec{u}]_{\mathcal{B}} = \vec{y}$
$\Rightarrow$ The coordinate mapping is onto $\mathbb{R}^n$

$\boxtimes$

*Theorem 8 Proof is Complete :) *
- Linear Trans. ✓
- 1-1 ✓
- onto ✓

Note: The Linearity of the coordinate mapping seen in thm 8 extends to Linear Combinations :)

* **If $\vec{u}_1, \dots, \vec{u}_n \in V$ & if $c_1, \dots, c_n$ are scalars,**

Then:

$$\left[c_1\vec{u}_1 + \cdots + c_n\vec{u}_n\right]_\beta = c_1\left[\vec{u}_1\right]_\beta + \cdots + c_n\left[\vec{u}_n\right]_\beta$$

IOW: The β-coordinate vector of a Linear Combination of vectors $\vec{u}_1, \dots, \vec{u}_n$ is the SAME Linear Combination of their coordinate vectors :)

Isomorphism

Note: The coordinate mapping in theorem 8 is also an example of an Isomorphism from V onto $\mathbb{R}^n$.

• **In General:**

A 1-1 Linear Transformation from a vector space V onto a vector space W is called an "Isomorphism from V onto W"

* "iso" → 'SAME' in Greek
* "morph" → 'Form' or 'structure' in Greek

While the notation/terminology in V & W may differ, the two spaces are indistinguishable as vector spaces

⇒ Every vector space calculation in V is accurately reproduced in W (& vice versa :))

→ **Ex:** A $\mathbb{R}$ vector space w/ n-vectors is indistinguishable from $\mathbb{R}^n$.

Illustration of an Isomorphism:

Let $\beta = \{1, t, t^2, t^3\}$ be the Standard Basis of the space $\mathbb{P}_3$ of polynomials.

* Keeping the Standard Notation of a Polynomial $\mathbb{P}_n$ in mind, we know that some element "$\vec{p}$" $\in \mathbb{P}_3$ is defined:

$$\vec{p}(t) = a_0 + a_1 t^1 + a_2 t^2 + a_3 t^3$$

where: $\{a_0, a_1, a_2, a_3\} \to$ Scalars/Weights

* $\vec{p}$ is already displayed as a Linear Combination of the Standard Basis Vectors:

• Let $P_\beta = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4] = [1\ t\ t^2\ t^3]$

• The Vector-Eq. ($\vec{p} \in \mathbb{P}_3$): $\vec{p}(t) = a_0(\overset{\vec{b}_1}{1}) + a_1(\overset{\vec{b}_2}{t}) + a_2(\overset{\vec{b}_3}{t^2}) + a_3(\overset{\vec{b}_4}{t^3})$

• Is Equivalent to: $\vec{p}(t) = [1\ t\ t^2\ t^3]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = P_\beta[\vec{p}]_\beta$

$$\therefore [\vec{p}]_\beta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^4$$

* Conclusions:

$\boxed{\text{The coordinate mapping } \vec{p} \mapsto [\vec{p}]_\beta \text{ is an isomorphism from } \mathbb{P}_3 \text{ onto } \mathbb{R}^4 \text{ (all vector operations in } \mathbb{P}_3 \text{ correspond to } \mathbb{R}^4).}$

Example: The Set $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \{1-t^2, t-t^2, 1-2t-t^2\}$ is the Basis for $\mathbb{P}_2$. Find the coordinate vector of $\vec{p}(t) = -5+10t-3t^2$, relative to B.

Answer:

* **Recall:** \$ $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a Basis in a vector-space $\mathbb{P}_n$ & Let $\vec{p} \in \mathbb{P}_n$. If weights $a, \dots, a_n$ are the β-coordinates of $\vec{p}(t) = a_1\vec{b}_1 + \dots + a_n\vec{b}_n = [\vec{b}_1 \cdots \vec{b}_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then $[\vec{p}]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

* **Given:**

- **Basis:** $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \{1-t^2, t-t^2, 1-2t-t^2\}$
- **A vector-eq. for $\vec{p} \in \mathbb{P}_2$:** $\vec{p}(t) = -5+10t-3t^2$

* **Want to Find:** $[\vec{p}]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = ?$ ST

$$\vec{p}(t) = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 = -5+10t-3t^2$$

* **First we must find & simplify the gen. Vector-Eq:**

$$\vec{p}(t) = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 = a_1(1-t^2) + a_2(t-t^2) + a_3(1-2t-t^2)$$

Distribute $= a_1 - a_1t^2 + a_2t - a_2t^2 + a_3 - 2a_3t - a_3t^2$

Group Like Terms $= (a_1+a_3) + (a_2t - 2a_3t) + (-a_1t^2 - a_2t^2 - a_3t^2)$

$$= (a_1+a_3) + (a_2-2a_3)t + (-a_1-a_2-a_3)t^3$$

⇘

Example Continued...

So, $\begin{cases} \text{* General Vector-Equation: } \vec{p}(t) = (a_1+a_3) + (a_2-2a_3)t + (-a_1-a_2-a_3)t^2 \\ \text{* Provided Vector-Equation: } \vec{p}(t) = -5+10t-3t^2 \end{cases}$

*To find $[\vec{p}]_\beta$, equate the coefficients of the 2 vector equations to create a system of equations:

i) Constant-Terms: $a_1 + a_3 = -5$

ii) Linear-Terms: $a_2 - 2a_3 = 10$

iii) Quadratic-Terms: $-a_1 - a_2 - a_3 = -3$

$$\Longrightarrow \boxed{\begin{cases} a_1 + a_3 = -5 \\ a_2 - 2a_3 = 10 \\ a_1 + a_2 + a_3 = 3 \end{cases}}$$

* Row-reduce the equivalent augmented matrix to rref:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & -2 & 10 \\ 1 & 1 & 1 & 3 \end{array}\right] \underset{N.R_3}{\overset{-R_1+R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & -2 & 10 \\ 0 & 1 & 0 & 8 \end{array}\right] \underset{N.R_3}{\overset{-R_2+R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & -2 & 10 \\ 0 & 0 & 2 & -2 \end{array}\right] \overset{\frac{1}{2}R_3}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & -1 \end{array}\right] \underset{\substack{2R_3+R_2 \\ N.R_2}}{\overset{\substack{-R_3+R_1 \\ N.R_1}}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right] \Longrightarrow \boxed{\therefore [\vec{p}]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -1 \end{bmatrix}}$$

Answer✓

Example: Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, & $3+2t$ are Linearly Dependent in $\mathbb{P}_2$.

Answer:

* **Note:** A typical element $\vec{p} \in \mathbb{P}_2$ has the form

$$\vec{p}(t) = a_0 + a_1 t^1 + a_2 t^2 = \begin{bmatrix} 1 & t^1 & t^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = P_\beta [\vec{p}]_\beta$$

where:

* $P_\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & t^1 & t^2 \end{bmatrix} \rightarrow$ Change of Coordinate Matrix

* $\beta = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \{1, t^1, t^2\} \rightarrow$ Basis

* $[\vec{p}]_\beta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \rightarrow$ Coordinate Vector of $\vec{p}$

* **Find the Coordinate vector of $\vec{p}$ for each polynomial:**

- $\vec{p}_1(t) = 1+2t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \rightarrow [\vec{p}_1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

- $\vec{p}_2(t) = 4+t+5t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \rightarrow [\vec{p}_2]_\beta = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

- $\vec{p}_3(t) = 3+2t = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \rightarrow [\vec{p}_3]_\beta = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

* **Write the coord. vectors of $\vec{p}$ as column-vectors of a matrix:**

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$$

$\Downarrow$

Example Continued...

*To verify that the polynomials are Linearly Dependendent, row-reduce $[A \mid \vec{0}]$ to echelon form (checking for free-variables :)):

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right] \underset{N.R_3}{\overset{-2R_1 + R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \overset{-\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\underset{N.R_3}{\overset{-R_2 + R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

*STOP.

∴ Matrix A only has $n=2$ pivots ⇒ The Columns of A are Linearly Dependent (By Invertible Matrix Th^m)

Answer

*Therefore: Since the columns of A are Linearly Dependent, the corresponding polynomials are Linearly Dependent in $\mathbb{R}^2$.

Note: This is NOT an exclusive solution to this problem :)

Example: If $\mathcal{B}$ is the Standard Basis of the Space $\mathbb{P}_3$ of polynomials, then let $\mathcal{B} = \{1, t, t^2, t^3\}$. Use the coordinate vectors to test the Linear Independence of the set of polynomials below:

$$1 + 3t^2 - t^3, \quad t + 3t^3, \quad 1 + t + 3t^2$$

Answer: Let $P_{\mathcal{B}} = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \vec{b}_4] = [1 \ t \ t^2 \ t^3]$

*Find the Coordinate-Vector of $\vec{p} \in \mathbb{P}_3$ for each polynomial:

① $\vec{p}_1(t) = 1 + 3t^2 - t^3$: $\quad \vec{p}_1(t) = [1 \ t \ t^2 \ t^3]\begin{bmatrix}1\\0\\3\\-1\end{bmatrix} \Rightarrow \boxed{[\vec{p}_1]_{\beta} = \begin{bmatrix}1\\0\\3\\-1\end{bmatrix}}$

② $\vec{p}_2(t) = t + 3t^3$: $\quad \vec{p}_2(t) = [1 \ t \ t^2 \ t^3]\begin{bmatrix}0\\1\\0\\3\end{bmatrix} \Rightarrow \boxed{[\vec{p}_2]_{\beta} = \begin{bmatrix}0\\1\\0\\3\end{bmatrix}}$

③ $\vec{p}_3(t) = 1 + t + 3t^2$: $\quad \vec{p}_3(t) = [1 \ t \ t^2 \ t^3]\begin{bmatrix}1\\1\\3\\0\end{bmatrix} \Rightarrow \boxed{[\vec{p}_3]_{\beta} = \begin{bmatrix}1\\1\\3\\0\end{bmatrix}}$

*Writing the Coordinate-Vectors of $\vec{p}$ as a matrix:

$$A = \left[[\vec{p}_1]_{\beta} \ [\vec{p}_2]_{\beta} \ [\vec{p}_3]_{\beta}\right] = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\3 & 0 & 3\\-1 & 3 & 0\end{bmatrix}$$

*To test the Linear Independence, row-reduce the Aug. Matrix $[A \,\vdots\, \vec{0}]$ to rref:

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\3 & 0 & 3 & 0\\-1 & 3 & 0 & 0\end{array}\right] \underset{\frac{R_1 + R_4}{N.R_4}}{\overset{\frac{-3R_1 + R_3}{N.R_3}}{\sim}} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\\0 & 3 & 1 & 0\end{array}\right]$$

Example Continued...

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{N. } R_3]{-3R_2 + R_3} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \underline{1} & 0 & 1 & 0 \\ 0 & \underline{1} & 1 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon Form

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

* RREF *

Note: You can make the same conclusions using the echelon form :)

$\therefore$ Since matrix A has $n = 3$ pivots (i.e. a pivot per col.)
$\Rightarrow$ The Columns of A are Linearly Independent.

$\therefore$ The polynomials are also Linearly Independent.

Ans.

Example: Let B be the Standard Basis of the Space $\mathbb{P}_2$ of polynomials. Use the Coordinate Vectors of $\vec{p} \in \mathbb{P}_2$ to test whether the following set of polynomials spans $\mathbb{P}_2$.:

$$1-2t+3t^2, \quad -4+6t-8t^2, \quad -2+3t-4t^2, \quad 2-3t+4t^2$$

Answer:

* Let $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \{1, t, t^2\}$ be the Standard Basis of $\mathbb{P}_2$.

* Find the Coordinate Vector of $\vec{p}$, $[\vec{p}]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, for each of the given polynomials:

① $\vec{p}_1(t) = 1-2t+3t^2 \implies [\vec{p}_1]_\beta = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

② $\vec{p}_2(t) = -4+6t-8t^2 \implies [\vec{p}_2]_\beta = \begin{bmatrix} -4 \\ 6 \\ -8 \end{bmatrix}$

③ $\vec{p}_3(t) = -2+3t-4t^2 \implies [\vec{p}_3]_\beta = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$

④ $\vec{p}_4(t) = 2-3t+4t^2 \implies [\vec{p}_4]_\beta = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -4 & -2 & 2 \\ -2 & 6 & 3 & -3 \\ 3 & -8 & -4 & 4 \end{bmatrix}$

Note: Vectors $[\vec{p}_2]_\beta$, $[\vec{p}_3]_\beta$, $[\vec{p}_4]_\beta$ are redundant (Dependence Relation $\exists$) $\Longrightarrow$ The Set of Coord. Vectors is Linearly Dependent.

$\therefore$ By the Invertible Matrix Th^m (& isomorphism b/w $\mathbb{R}^3$ & $\mathbb{P}_2$): The Columns do NOT Span $\mathbb{R}^3$ & so the polynomials do NOT span $\mathbb{P}_2$

Example: Determine if $\vec{x}$ is in H, and if it is, find the coordinate vector of $\vec{x}$ relative to β:

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}, \quad \& \quad \beta = \{\vec{v}_1, \vec{v}_2\}.$$

*Then β is a Basis for $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Answer:

Note: If $\vec{x} \in H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, then the vector-eq. $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ is consistent (@ least 1 sol. $\exists$).

*Solve the nonhomogeneous eq. ($A\vec{x} = \vec{b}$):

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x} \iff \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{x} \iff \left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{array}\right]$$

*Row-reduce the augmented matrix to rref:

$$\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{x} \end{array}\right] = \left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -1 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{array}\right] \underset{\sim}{\overset{-3R_1 + R_2}{\overline{N.R_2}}}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 1 & 7 \end{array}\right] \underset{\sim}{\overset{-2R_1 + R_3}{\overline{N.R_3}}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \underset{\sim}{\overset{-R_2 + R_3}{\overline{N.R_3}}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \checkmark$$

*Consistent System ∴

$$\Rightarrow \boxed{[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

*Coordinate Vector of $\vec{x}$ *

*Check:

$$2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} \checkmark$$

Ex.// Prop: \$ that $\{\vec{v}_1, \ldots, \vec{v}_4\}$ is a linearly dependent spanning set for a vector space V. Show that each $\vec{w}$ in V can be expressed in more than one way as a Linear Combination of $\vec{v}_1, \ldots, \vec{v}_4$.

*Hint: Let $\vec{w} = k_1\vec{v}_1 + \cdots + k_4\vec{v}_4$ be an arbitrary vector in V. Use the Linear Dependence of $\{\vec{v}_1, \ldots, \vec{v}_4\}$ to produce another representation of $\vec{w}$ as a Linear Combination of $\vec{v}_1, \ldots, \vec{v}_4$.

Answer:

* \$ that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a Linearly Dependent Spanning Set of some Vector Space V:

① Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans V:

$\exists$ a vector $\vec{w} \in V$ ST $\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + k_4\vec{v}_4$

where k_1, k_2, k_3, k_4 are appropriate scalars.

② Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a Linearly Dep. set:

The vector-eq. $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$ has a Nontrivial solution (i.e. The scalars $c_1, \ldots, c_4$ NOT all zero ☺)

Note: (By the Spanning Set Th^m) We can rewrite the vector-eq. in ② & then substitute this into vector $\vec{w} \in V$ of ①
— or —
(By Def. of a Vector Space) Since V is Closed Under Add. $\Rightarrow$ We can sum ① & ②

Ex// Prop Continued...

*Note: Taking the sum of the 2 vectors in V will produce "cleaner" calculations ∵ (But both work ✓) "Scalar-Mult."

*Taking the sum:

Since V is a Vector Space $\to \vec{0} \in V$; $\to$ Closed Under Addition

So, $\vec{w} + \vec{0} = \left[k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 + k_4\vec{v}_4\right] + \left[c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4\right]$

$$\boxed{\vec{w} = (k_1+c_1)\vec{v}_1 + (k_2+c_2)\vec{v}_2 + (k_3+c_3)\vec{v}_3 + (k_4+c_4)\vec{v}_4}$$

∴ Since not all the weights from ②, $c_1, \dots, c_4$, can be zero (by def.) $\Rightarrow$ At least one coefficient $(k_1+c_1), \dots, (k_4+c_4)$ must be different than the original $\vec{w}$ ∵

Answer ✓

Example: The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find 2 different ways to express $\begin{bmatrix} -8 \\ 22 \end{bmatrix}$ as a Linear Combination of $\vec{v}_1, \vec{v}_2$, & $\vec{v}_3$.

Answer:

* **Note**: Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ do NOT form a Basis b/c they are linearly dependent.

* Notice the Dependence Relation amongst $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$:

$$-5\vec{v}_1 + \vec{v}_2 = -5\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 4 \\ -10 \end{bmatrix} = \begin{bmatrix} -5+4 \\ 10-10 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{v}_3$$

$$\Rightarrow \boxed{-5\vec{v}_1 + \vec{v}_2 = \vec{v}_3}$$

* This is NOT the only dependence relation here :)

This implies $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ does NOT form a Basis.

* Let $\vec{x} = \begin{bmatrix} -8 \\ 22 \end{bmatrix}$

* Want to Express $\vec{x}$ as a Linear Combo. of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$:

$\Rightarrow \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, ST c_1, c_2, c_3 are scalars.

① \$ $c_3 = 0$: $c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ -10 \end{bmatrix} = \begin{bmatrix} -8 \\ 22 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 4 & -8 \\ -2 & -10 & 22 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} 1 & 4 & -8 \\ 1 & 5 & -11 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & -8 \\ 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right] \Rightarrow \boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}}$$

Example Continued...

② $ $c_3 = 1$: Express $\vec{x}$ as a Linear Combo of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

* **Goal**: Write $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ ST

$$\begin{bmatrix} -8 \\ 22 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -10 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

*Simplify the vector-eq ^{n} & then row-reduce the Aug. Matrix:

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -10 \end{bmatrix} = \begin{bmatrix} -8 \\ 22 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -8+1 \\ 22+0 \end{bmatrix} = \begin{bmatrix} -7 \\ 22 \end{bmatrix}$$

So, $$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -10 \end{bmatrix} = \begin{bmatrix} -7 \\ 22 \end{bmatrix} \quad (\text{*Vector-Eq})$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 4 \\ -2 & -10 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 22 \end{bmatrix} \quad (\text{*Matrix-Eq.})$$

$$\Longleftrightarrow \left[\begin{array}{cc|c} 1 & 4 & -7 \\ -2 & -10 & 22 \end{array}\right] \overset{-\frac{1}{2}R_2}{\sim} \left[\begin{array}{cc|c} 1 & 4 & -7 \\ 1 & 5 & -11 \end{array}\right] \overset{\substack{-R_1 \\ +R_2 \\ \overline{N.R_2}}}{\sim} \left[\begin{array}{cc|c} 1 & 4 & -7 \\ 0 & 1 & -4 \end{array}\right] \overset{\substack{-4R_2 \\ +R_1 \\ \overline{N.R_1}}}{\sim}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -4 \end{array}\right] \Longrightarrow \boxed{\begin{cases} c_1 = 9 \\ c_2 = -4 \end{cases} \rightarrow \therefore \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 9\vec{v}_1 - 4\vec{v}_2 + 1\vec{v}_3}$$

Answer.