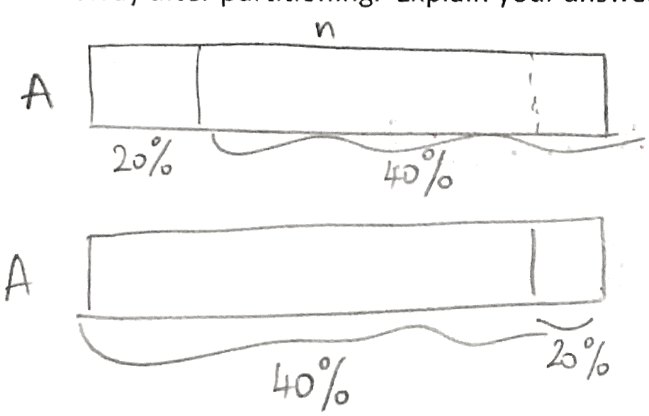
Name: (Print) PHONG VO

1. (25 points) **QuickSort.** Consider the quick sort algorithm in our textbook as shown below

```
QUICKSORT(A, p, r)
  if p < r
      q = PARTITION(A, p, r)
      QUICKSORT(A, p, q − 1)
      QUICKSORT(A, q + 1, r)
```

(1) (10 points) For an array *A* with *n* distinct elements, how often can we expect to see a split that's 4-to-1 (or 1-to-4) or better? Assume the pivot is equally likely to end up anywhere in the sub-array after partitioning. Explain your answer.



The best expect value to see a split that (4:1 or 1:4) is

$\underbrace{100\% - 20\%}_{\text{front-end}} - \underbrace{20\%}_{\text{back-end}} = 60\%$

(2) (15 points) Assume that the algorithm always produces a 4-to-1 split, provide a tight bound on the running time of quicksort in this case. Show your answer in recurrence and then solve the recurrence. You do not need to prove the answer with the substitution method.

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{4n}{5}\right) + \theta(1)$$

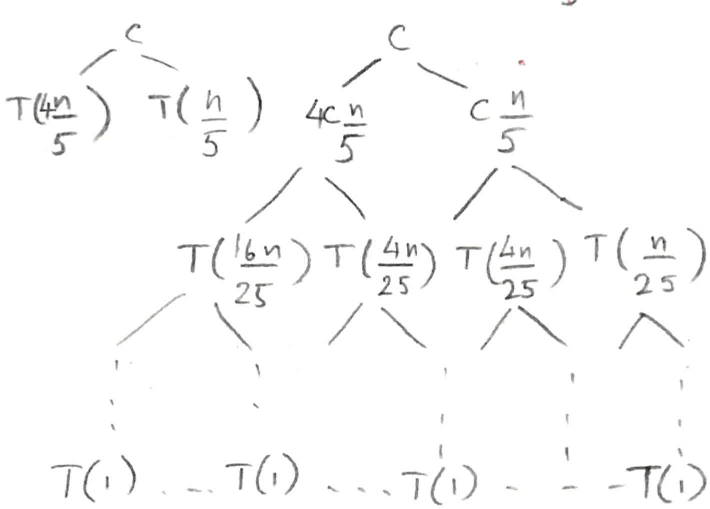
$$= T\left(\frac{4n}{5}\right) + T\left(\frac{n}{5}\right) + c$$

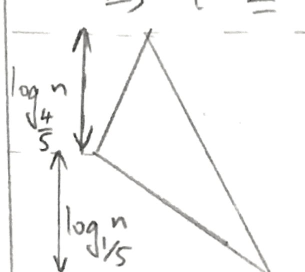
Left most leaf: $\left(\frac{4}{5}\right)^i = n$

$\Rightarrow i = \log_{4/5} n$

Right most leaf: $\left(\frac{1}{5}\right)^i = n$

$\Rightarrow i = \log_{1/5} n$





T(n) = ?

−6

$$T(n) = T(\alpha n) + T((1-\alpha)n) + \theta(n)$$
$$= \theta(n \lg n)$$

2. (20 points) Use **Indicator Random Variables** to solve the **hat-check problem**. Each of *n* customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the excepted number of customers who get back their own hat?

You must define the random variable and indicator random variables clearly, and show how that can help you to calculate the result. Also, be sure to show the detailed steps of calculating the final result. Specify the rule or lemma used when applicable.

n customers with n hats → Sample space = n X

event: { the customers get their own hats back } (2)

$= X = 1$ what is X, Xi ?

$Pr\{X\} = \frac{1}{n}$ (5)

Lemma: $E[X_i] = Pr\{X\} = \frac{1}{n}$ (expected value of each person gets his/her hat back.)

$E[X] = \sum_{i=1}^{m} E[X_i]$ (2) X (m: is the # of people who get their own hats back) that is what we try to calculate

$= \sum_{i=1}^{m} \frac{1}{n} = \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$ (in m times)

$= m \times \frac{1}{n} = \frac{m}{n}$ (−11)

$E[X] = \frac{m}{n}$ X

28/45

X: # of customers

X_i: $I\{i^{th}$ customer who gets own hat back$\}$

$$= \begin{cases} 1 \\ 0 \end{cases}$$

$$X = X_1 + X_2 + \dots + X_n$$

$$E[X] = E[X_1 + X_2 + \dots + X_n]$$

$$= E[X_1] + E[X_2] + \dots + E[X_n]$$

$$E[X_i] = Pr\{i^{th} \text{ customer} \dots\}$$

$$= \frac{1}{n}$$

$$E[X] = \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n} = 1$$