Student: Phong Vo **Date:** 04/09/20	**Instructor:** Erica Yankowskas **Course:** Linear Algebra I (Spring 2020)	**Assignment:** Section 4.2 Homework

1. Determine if $\mathbf{w} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ is in Nul A, where $A = \begin{bmatrix} 3 & -4 & 7 \\ 3 & -1 & 13 \\ -7 & 5 & -25 \end{bmatrix}$.

Is **w** in Nul A? Select the correct choice below and fill in the answer box to complete your choice.

(Simplify your answer.)

A. Yes, because $A\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

B. No, because $A\mathbf{w} =$

2. Find an explicit description of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -2 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

A spanning set for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

3. Find an explicit description of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

A spanning set for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

4. Either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

$$W = \left\{ \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} : \begin{array}{c} p - 2q = -s \\ 3p = -s + r \end{array} \right\}$$

Rewrite the system of equations in the form $A\mathbf{x} = 0$.

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

What does the given set represent?

- **A.** The set of solutions to one of the homogeneous equations.
- **B.** The set of all solutions to the homogeneous system of equations.
- **C.** The set represents the values which are not solutions.

Therefore, the set W = Nul A.

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Which of the following is a true statement?

- **A.** The proof is complete since W is a subspace of $\mathbb{R}$. The given set W must be a vector space because a subspace itself is a vector space.
- **B.** The proof is complete since W is a subspace of $\mathbb{R}^4$. The given set W must be a vector space because a subspace itself is a vector space.
- **C.** The proof is complete since W is a subspace of $\mathbb{R}^3$. The given set W must be a vector space because a subspace itself is a vector space.
- **D.** The proof is complete since W is a subspace of $\mathbb{R}^2$. The given set W must be a vector space because a subspace itself is a vector space.

5. Either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

$$W = \left\{ \begin{bmatrix} s-2t \\ 3+2s \\ 3s+t \\ 3s \end{bmatrix} : s,t \text{ real} \right\}$$

The set W is a subset of $\mathbb{R}^4$. If W were a vector space, what else would be true about it?

- **A.** The set W would be a subspace of $\mathbb{R}^4$.
- **B.** The set W would be the null space of $\mathbb{R}^2$.
- **C.** The set W would be the null space of $\mathbb{R}^4$.
- **D.** The set W would be a subspace of $\mathbb{R}^2$.

Determine whether the zero vector is in W. Find values for t and s such that $\begin{bmatrix} s-2t \\ 3+2s \\ 3s+t \\ 3s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Select the correct choice below and, if necessary, fill in any answer boxes to complete your choice.

- **A.** The zero vector is in W. The vector equation is satisfied when t = ______ and s = ______ .
- **B.** The zero vector is not in W. There is no t and s such that the vector equation is satisfied.

Which of the following is a true statement?

- **A.** Since the zero vector is not in W, W is not the null space of $\mathbb{R}^2$. Thus W is not a vector space.
- **B.** Since the zero vector is in W, W is the null space of $\mathbb{R}^4$. Thus W is a vector space.
- **C.** Since the zero vector is not in W, W is not a subspace of $\mathbb{R}^4$. Thus W is not a vector space.
- **D.** Since the zero vector is in W, W is a subspace of $\mathbb{R}^2$. Thus W is a vector space.

6. Find A such that the given set is Col A.

$$\left\{ \begin{bmatrix} r+3t \\ -2r+2s-t \\ -3r-2s+2t \\ 3r-s-3t \end{bmatrix} : r, s, t \text{ real} \right\}$$

Choose the correct answer below.

A. $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -2 & -3 \\ -3 & -1 & 3 \end{bmatrix}$

B. $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -1 & 2 \\ -3 & 2 & -2 \\ 3 & -1 & -3 \end{bmatrix}$

C. $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & -1 \\ -3 & -2 & 2 \\ 3 & -1 & -3 \end{bmatrix}$

D. $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -2 & -1 \\ -2 & -3 & 2 \\ 3 & -1 & -3 \end{bmatrix}$

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7. Complete parts (a) and (b) for the matrix below.

$$A = \begin{bmatrix} 8 & 5 \\ -5 & 9 \\ -9 & 0 \\ -5 & 7 \\ -8 & -1 \end{bmatrix}$$

(a) Find k such that Nul(A) is a subspace of $\mathbb{R}^k$.

k = 2

(b) Find k such that Col(A) is a subspace of $\mathbb{R}^k$.

k = 5

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8. Complete parts (a) and (b) for the matrix below.

$$A = \begin{bmatrix} 3 & 0 & 6 \\ -6 & -9 & -8 \\ -2 & -6 & -6 \\ 2 & -2 & 8 \end{bmatrix}$$

(a) Find k such that Nul(A) is a subspace of $\mathbb{R}^k$.

k = 3

(b) Find k such that Col(A) is a subspace of $\mathbb{R}^k$.

k = 4

9. For the matrix A below, find a nonzero vector in Nul A and a nonzero vector in Col A.

$$A = \begin{bmatrix} -4 & -16 \\ 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}$$

A nonzero vector in Nul A is $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

A nonzero vector in Col A is $\begin{bmatrix} -4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$.

10. For parts a. through f., A denotes an $m \times n$ matrix. Determine whether each statement is true or false. Justify each answer.

a. A null space is a vector space.

Is this statement true or false?

- **A.** True because the null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$
- **B.** False, a vector space is a null space, but a null space is not necessarily a vector space
- **C.** True because the null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$
- **D.** False, a column space is a vector space, but a null space is not a vector space

b. The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.

Is this statement true or false?

- **A.** True because the column space of an $m \times n$ matrix A is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$
- **B.** True because the column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$
- **C.** False because the column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$
- **D.** False because the column space of an $m \times n$ matrix A is the set of all linear combinations of the columns of A

c. The column space of A, Col(A), is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.

Is this statement true or false?

- **A.** True because $\text{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$
- **B.** False because $\text{Col}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$
- **C.** False because Col(A) is the set of all solutions of $A\mathbf{x} = \mathbf{0}$
- **D.** True because Col(A) is the set of all solutions of $A\mathbf{x} = \mathbf{0}$

d. The null space of A, Nul(A), is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Is this statement true or false?

- **A.** False, the kernel of a linear transformation T, from a vector space V to a vector space W, is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. Thus, the kernel of a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is the column space of A, not the null space of A.
- **B.** False, the kernel of a linear transformation T, from a vector space V to a vector space W, is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. Thus, the kernel of a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is the column space of A, not the null space of A.
- **C.** True, the kernel of a linear transformation T, from a vector space V to a vector space W, is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. Thus, the kernel of a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is the null space of A.
- **D.** True, the kernel of a linear transformation T, from a vector space V to a vector space W, is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. Thus, the kernel of a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is the null space of A.

e. The range of a linear transformation is a vector space.

Is this statement true or false?

- **A.** False, the range of a linear transformation T, from a vector space V to a vector space W, is a subspace of W.
- **B.** True, the range of a linear transformation T, from a vector space V to a vector space W, is a subspace of W.
- **C.** False, the range of a linear transformation T, from a vector space V to a vector space W, is a subspace of V.
- **D.** True, the range of a linear transformation T, from a vector space V to a vector space W, is a subspace of V.

f. The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

Is this statement true or false?

- **A.** False, the set of all solutions of a homogeneous linear differential equation is the range of a linear transformation, rather than the kernal.
- **B.** False, a homogeneous linear differential equation cannot be modeled by a linear transformation.
- **C.** True, the linear transformation maps each function f to a linear combination of f and at least one of its derivatives, exactly as these appear in the homogeneous linear differential equation.

YOU ANSWERED: A.

11. Let V and W be vector spaces, and let T : V→W be a linear transformation. Given a subspace U of V, let T(U) denote the set of all images of the form T(**x**), where **x** is in U. Show that T(U) is a subspace of W.

To show that T(U) is a subspace of W, first show that the zero vector of W is in T(U). Choose the correct answer below.

- **A.** Since V is a subspace of U, the zero vector of U, $\mathbf{0}_U$, is in V. Since T is linear, $T(\mathbf{0}_U) = \mathbf{0}_W$, where $\mathbf{0}_W$ is the zero vector of W. So $\mathbf{0}_W$ is in T(U).
- **B.** Since V is a subspace of U, the zero vector of V, $\mathbf{0}_V$, is in U. Since T is linear, $T(\mathbf{0}_V) = \mathbf{0}_W$, where $\mathbf{0}_W$ is the zero vector of W. So $\mathbf{0}_W$ is in T(U).
- **C.** Since U is a subspace of V, the zero vector of V, $\mathbf{0}_V$, is in U. Since T is linear, $T(\mathbf{0}_V) = \mathbf{0}_W$, where $\mathbf{0}_W$ is the zero vector of W. So $\mathbf{0}_W$ is in T(U).
- **D.** Since U is a subspace of W, the zero vector of W, $\mathbf{0}_W$, is in U. Since T is linear, $T(\mathbf{0}_W) = \mathbf{0}_V$, where $\mathbf{0}_V$ is the zero vector of V. So $\mathbf{0}_W$ is in T(U).

Let **v** and **w** be in T(U). Relate **v** and **w** to vectors in U.

Since T(U) is the set of all images from U, there exist **x** and **y** in U such that $T(\mathbf{x}) = \mathbf{v}$ and $T(\mathbf{y}) = \mathbf{w}$.

Next, show that T(U) is closed under vector addition in W.

Let T(**x**) and T(**y**) be in T(U), for some **x** and **y** in U. Choose the correct answer below.

- **A.** Since **x** and **y** are in T(U) and U is a subspace of V, $\mathbf{x} + \mathbf{y}$ is also in U.
- **B.** Since **x** and **y** are in U and U is a subspace of V, $\mathbf{x} + \mathbf{y}$ is also in U.
- **C.** Since **x** and **y** are in T(U) and T(U) is a subspace of W, $\mathbf{x} + \mathbf{y}$ is also in W.
- **D.** Since **x** and **y** are in U and T(U) is a subspace of W, $\mathbf{x} + \mathbf{y}$ is also in W.

Use these results to explain why T(U) is closed under vector addition in W. Choose the correct answer below.

- **A.** Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x}) + T(\mathbf{y})$ is in W, and T(U) is closed under vector addition in W.
- **B.** Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x}) + T(\mathbf{y})$ is in V, and T(U) is closed under vector addition in W.
- **C.** Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x}) + T(\mathbf{y})$ is in T(U), and T(U) is closed under vector addition in W.
- **D.** Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x}) + T(\mathbf{y})$ is in U, and T(U) is closed under vector addition in W.

Next, show that T(U) is closed under multiplication by scalars.

Let c be any scalar and **x** be in U. Choose the correct answer below.

- **A.** Since **x** is in U and U is a subspace of W, c**x** is in U. Thus, T(c**x**) is in T(W).
- **B.** Since **x** is in U and U is a subspace of V, c**x** is in V. Thus, T(c**x**) is in T(U).
- **C.** Since **x** is in U and U is a subspace of V, c**x** is in W. Thus, T(c**x**) is in T(V).
- **D.** Since **x** is in U and U is a subspace of V, c**x** is in U. Thus, T(c**x**) is in T(U).

The preceding result helps to show why T(U) is closed under multiplication by scalars. Recall that every element of T(U) can be written as T(**x**) for some **x** in U. Choose the correct answer below.

- **A.** Since T is linear, $T(c\mathbf{x}) = cT(\mathbf{x})$ and $cT(\mathbf{x})$ is in V. Thus, T(U) is closed under multiplication by scalars.

- **B.** Since T is linear, T(c**x**) = cT(**x**) and cT(**x**) is in T(U). Thus, T(U) is closed under multiplication by scalars.
- **C.** Since T is linear, T(c**x**) = cT(**x**) and cT(**x**) is in W. Thus, T(U) is closed under multiplication by scalars.
- **D.** Since T is linear, T(c**x**) = cT(**x**) and cT(**x**) is in U. Thus, T(U) is closed under multiplication by scalars.

Use these results to explain why T(U) is a subspace of W. Choose the correct answer below.

- **A.** The image of the transformation T(U) is a subspace of W because T(U) contains the zero vector of V and is closed under vector addition and multiplication by a scalar.
- **B.** The image of the transformation T(U) is a subspace of W because T(U) is closed under vector addition and multiplication by a scalar.
- **C.** The image of the transformation T(U) is a subspace of W because U contains the zero vector of W and T(U) is closed under vector addition and multiplication by a scalar.
- **D.** The image of the transformation T(U) is a subspace of W because T(U) contains the zero vector of W and is closed under vector addition and multiplication by a scalar.