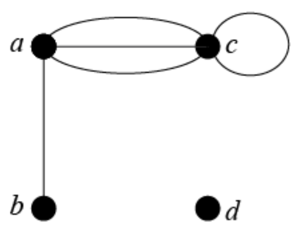
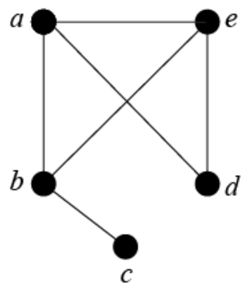
Sections 10.1-10.2 Homework

1. Find the degree of every vertex for the graph below. Then verify the Handshaking Theorem for this graph.

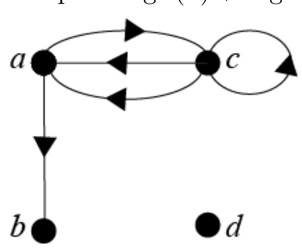


2. For the graph below:



   (a) Find $N(a)$ and $|N(a)|$.
   (b) Find the subgraph of G induced by the vertices a, c, d, e.
   (c) How many subgraphs of G have the vertex set $\{a, b, e\}$? How many subgraphs of G have the vertex set $\{a, b, c, e\}$?

3. For the following directed graph, compute the in-degree and out-degree of every vertex. For each vertex $x = a, b, c, d$, compare $\deg^+(x) + \deg^-(x)$ with the corresponding answers from problem 1.



4. Suppose that a graph has 5 vertices of degree 6, and 8 vertices of degree 7. Use the Handshaking Theorem to find the number of edges.

5. Use the Handshaking Theorem to explain why it's not possible for a graph to have 11 vertices which all have degree 7.

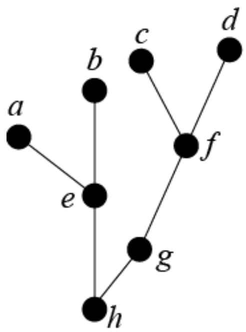
6. Suppose that a graph has 8 vertices and 18 edges. Use the Handshaking Theorem to explain why there must be at least one vertex of degree less than 5.

7. Draw the following graphs: K_6, $K_{3,4}$, C_6

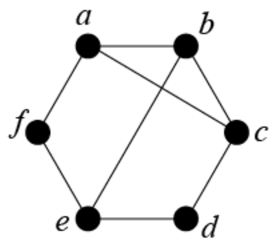
8. For $k = 1, 2, 3, 4$, determine the number of subgraphs of C_4 with exactly k vertices.

9. Which of the following graphs are bipartite? If the graph is bipartite, give a bipartition (V_1, V_2). If it's not bipartite, explain why not in terms of graph coloring.

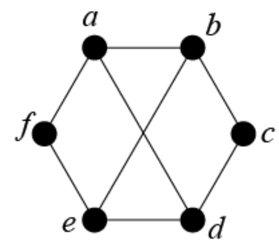
(a)



(b)



(c)



10. Suppose that there are five applicants (Abby, Ben, Caroline, Dan, and Ed) who are applying for five jobs labeled 1,2, 3, 4, 5.

- Abby is qualified for jobs 3 and 5.
- Ben is qualified for jobs 1, 3, and 4.
- Caroline is qualified for jobs 3, 4, and 5.
- Dan is qualified for jobs 1, 2, and 5.
- Ed is qualified for jobs 3 and 4.

Problems:

(a) Let V_1 be the set of applicants, and let V_2 be the set of jobs. Draw a bipartite graph G with bipartition (V_1, V_2) representing the employees and the jobs that they're qualified for.

(b) Find a matching in this graph.

(c) Remove an edge from the graph in (a) to get a graph G' that no longer has a matching. In G', find a subset of A of V_1 which violates the condition in Hall's Marriage Theorem. (Be able to show why this subset A works!)

11. Five professors (A, B, C, D, E) will give lectures at a conference. There are five possible time slots labeled 1, 2, 3, 4, 5. Each professor must be assigned to a different time slot. The professors are only available for certain time slots as given in the following table:

Professor	Time Slots
A	2, 4
B	2, 3, 4
C	1, 3, 4, 5
D	2, 4
E	1, 3, 5

Problems:

(a) Let V_1 be the set of professors, and let V_2 be the set of time slots. Draw a bipartite graph G with bipartition (V_1, V_2) representing the professors and the time slots they're available for.

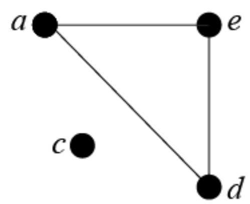
(b) Find a matching in this graph.

(c) How many different ways can the professors be assigned to time slots?

(d) Remove an edge from the graph in (a) to get a graph G' that no longer has a matching. In G', find a subset of A of V_1 which violates the condition in Hall's Marriage Theorem. (Be able to show why this subset A works!)

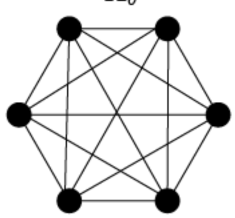
Answers:

1. $\deg a = 4, \quad \deg b = 1, \quad \deg c = 5, \quad \deg d = 0$
2. (a) $N(A) = \{b, d, e\}, \quad |N(A)| = 3$
   (b) Induced graph:

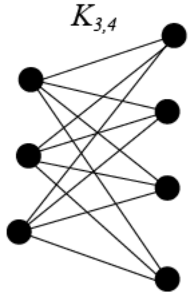


   (c) 8; 16
3. $\deg^+(a) = 2$, $\deg^-(a) = 2$
   $\deg^+(b) = 0$, $\deg^-(b) = 1$
   $\deg^+(c) = 3$, $\deg^-(c) = 2$
   $\deg^+(d) = 0$, $\deg^-(d) = 0$
4. 43
7. Graphs:

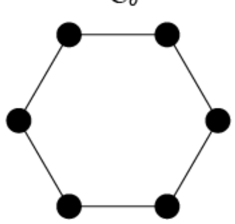
K_6



$K_{3,4}$

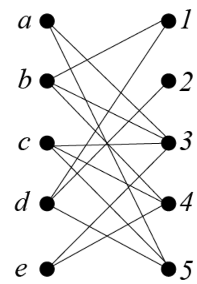


C_6

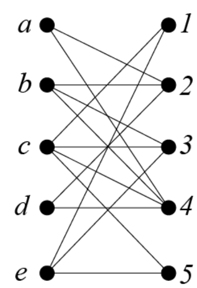


8. The number of subgraphs for $k = 1, 2, 3, 4$ is $4, 10, 16, 16$, respectively.
9. (a) The graph is bipartite.
   $V_1 = \{a, b, f, h\}$, $V_2 = \{c, d, e, g\}$
   (b) not bipartite
   (c) The graph is bipartite.
   $V_1 = \{a, c, e\}$, $V_2 = \{b, d, f\}$
10. Represent Abby, Ben, Caroline, Dan, and Ed using vertices a, b, c, d, e, respectively.

   (a)



   (b) Matching: $\{a, 5\}$, $\{b, 1\}$, $\{c, 3\}$, $\{d, 2\}$, $\{e, 4\}$
   (c) Remove $\{b, 1\}$. Let $A = \{a, b, c, e\}$. Then $N(A) = \{3, 4, 5\}$, which means that
   $|N(A)| < |A|$ since $|N(A)| = 3$ and $|A| = 4$.
11. (a)



   (b) Matching: $\{a, 2\}, \{b, 3\}, \{c, 5\}, \{d, 4\}, \{e, 1\}$
   (c) 4
   (d) Remove $\{b, 3\}$, and let $A = \{a, b, d\}$. (Explain why A works!)