Linear Algebra I: Exam 1 (Summer 2019)

Name: Solutions:

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. ***Make sure to exhibit skills discussed in class.*** Box all answers and simplify answers as much as possible.

Good Luck! ☺

1. *Row-Reduction and Echelon Form*

6pts [pts] Determine when the augmented matrix below represents a consistent linear system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 2 & 1 & 5 & b \\ 1 & -1 & 1 & c \end{array}\right]$$

$*\ \dfrac{-2R_1 + R_2}{\text{new } R_2} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 1 & -1 & 1 & c \end{array}\right]$

$*\ \dfrac{-R_1 + R_3}{\text{new } R_3} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & -1 & -1 & c-a \end{array}\right] \overset{-R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 1 & 1 & a-c \end{array}\right]$

$*\ \dfrac{-R_2 + R_3}{\text{new } R_3} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & 3a-b-c \end{array}\right]$

Note: We want to prevent a row of zeros = constant (i.e. An inconsistent system)

$\boxed{\therefore \text{The Augmented matrix represents a consistent system IFF } 3a-b-c=0}$

equivalently: $-3a+b+c=0$

2. <u>Vector Equations</u>

(a) [~~pts~~ 9pt] Determine if $\vec{b}$ is a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ where:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix},\ \vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix},\ \vec{b} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$$

(b) [~~pts~~ 2pt] If $\vec{b}$ is a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, then express $\vec{b}$ as a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

*<u>Note</u>: We want to determine if $A\vec{x} = \vec{b}$ ST $A = [\vec{V}_1\ \vec{V}_2\ \vec{V}_3]$ is consistent.

*<u>Row-reduce $[A \mid \vec{b}]$ to rref</u> $\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ -1 & -1 & -1 & -4 \\ 0 & -1 & -3 & -7 \end{array}\right]$$

* $\dfrac{R_1 + R_2}{\text{new } R_2}$ $\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & -3 & 2 & 1 \\ 0 & -1 & -3 & -7 \end{array}\right]$ *interchange R_2 & $-R_3$ $\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & 1 & 3 & -7 \\ 0 & -3 & 2 & 1 \end{array}\right]$

* $\dfrac{2R_2 + R_1}{\text{new } R_1}$ $\sim \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & 2 & 1 \end{array}\right]$

* $\dfrac{3R_2 + R_3}{\text{new } R_3}$ $\sim \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 11 & 22 \end{array}\right]$ $\frac{1}{11}R_3$ $\sim \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$

* $\dfrac{-9R_3 + R_1}{\text{new } R_1}$ $\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$

* $\dfrac{-3R_3 + R_2}{\text{new } R_2}$ $\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

(a) $\therefore\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, so <u>yes</u>!

<u>$\vec{b}$ is a Linear Combination of $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$</u>:

(b) $\vec{V}_1 + \vec{V}_2 + 2\vec{V}_3 = \vec{b}$

3. **The Matrix Equation. $A\vec{x} = \vec{b}$**

9pt

(a) [pts] Solve the matrix equation $A\vec{x} = \vec{b}$ where:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2pt

(b) [pts] Is it possible to solve $A\vec{x} = \vec{b}$ for any vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, where A is the matrix given above?

Explain.

*Continued Explanation For Part (b):

(b) (i) $\forall \vec{b} \in \mathbb{R}^3$, $A\vec{x} = \vec{b}$ has a solution is equivalent to the statement,
(ii) A has $n = 3$ pivots.
$\therefore$ Since A has $n = 2$ pivots, we CANNOT solve $A\vec{x} = \vec{b}$ $\forall \vec{b} \in \mathbb{R}^3$

*Row-reduce $[A \,\vdots\, \vec{0}]$ to rref:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \quad * \begin{array}{r} -R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

$$* \begin{array}{r} -R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$* \begin{array}{r} R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad * \begin{array}{r} -2R_2 \\ +R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longleftrightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - x_3 = 0 \\ 0 = 0 \end{cases} \longrightarrow$$

(a)

$$\begin{cases} \cdot\, x_1 = -3x_3 \\ \cdot\, x_2 = x_3 \\ \cdot\, x_3 \text{ is free} \end{cases} \quad \text{-OR-} \quad \vec{x} = \begin{bmatrix} -3x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

where x_3 is any scalar.

(b) No. Row 3 could produce a contradiction, making the system inconsistent

4. Solution Sets of Linear Systems

Consider the linear system $A\vec{x} = \vec{b}$, where:

$$A = \begin{bmatrix} 1 & -1 & -2 & -2 & -2 \\ 3 & -2 & -2 & -2 & -2 \\ -3 & 2 & 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

9pt

(a) [pts] Solve the linear system. Write the general solution in parametric-vector form.

2pt

(b) [pts] Using your answer from (a), write the solution set for the homogeneous equation $A\vec{x} = \vec{0}$.

(b) Solution for the corresponding Homogeneous Eq, $A\vec{x} = \vec{0}$:

$$\boxed{\vec{x} = x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

where: x_4 & x_5 scalars.

* Row-reduce $[A \,\vdots\, \vec{b}]$ to rref:

$$\left[\begin{array}{ccccc|c} 1 & -1 & -2 & -2 & -2 & 3 \\ 3 & -2 & -2 & -2 & -2 & -1 \\ -3 & 2 & 1 & 1 & 1 & -1 \end{array}\right] \quad \begin{array}{c} * -3R_1 \\ + R_2 \\ \hline \text{new } R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & -1 & -2 & -2 & -2 & 3 \\ 0 & 1 & 4 & 4 & 4 & -10 \\ -3 & 2 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{c} * \ 3R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & -1 & -2 & -2 & -2 & 3 \\ 0 & 1 & 4 & 4 & 4 & -10 \\ 0 & -1 & -5 & -5 & -5 & 8 \end{array}\right] \quad \begin{array}{c} * \ R_2 \\ + R_1 \\ \hline \text{NEW } R_1 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 2 & -7 \\ 0 & 1 & 4 & 4 & 4 & -10 \\ 0 & -1 & -5 & -5 & -5 & 8 \end{array}\right]$$

$$\begin{array}{c} * \ R_2 \\ + R_3 \\ \hline \text{NEW } R_3 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 2 & -7 \\ 0 & 1 & 4 & 4 & 4 & -10 \\ 0 & 0 & -1 & -1 & -1 & -2 \end{array}\right] \quad \begin{array}{c} * \ 2R_3 \\ + R_1 \\ \hline \text{NEW } R_1 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -11 \\ 0 & 1 & 4 & 4 & 4 & -10 \\ 0 & 0 & -1 & -1 & -1 & -2 \end{array}\right]$$

$$\begin{array}{c} * \ 4R_3 \\ + R_2 \\ \hline \text{NEW } R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -11 \\ 0 & 1 & 0 & 0 & 0 & -18 \\ 0 & 0 & -1 & -1 & -1 & -2 \end{array}\right] \quad \begin{array}{c} -R_3 \\ \sim \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -11 \\ 0 & 1 & 0 & 0 & 0 & -18 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \checkmark$$

$$\Leftrightarrow \begin{cases} x_1 = -11 \\ x_2 = -18 \\ x_3 = 2 - x_4 - x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases} \Rightarrow$$

(a)

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 \\ -18 \\ 2 - x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

* where: x_4 & x_5 are any scalar.

5. <u>Linear Independence</u>

Determine if the following sets of vectors are linearly independent. Explain.

(a) ~~[pts]~~ 2pt $\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}$

* Since the set of vectors contains $\vec{0}$, the vectors are Linearly Dependent.

(b) ~~[pts]~~ 2pt $\left\{\begin{bmatrix}-5\\10\end{bmatrix},\begin{bmatrix}-4\\-2\end{bmatrix},\begin{bmatrix}36\\12\end{bmatrix},\begin{bmatrix}-3\\0\end{bmatrix}\right\}$

· 4 vectors (Columns)
· 2 entries/vector (Rows)

* Since (# of unknowns, 4) > (# of eq., 2), the set of vectors is Linearly Dependent.

(c) ~~[pts]~~ 2pt $\left\{\begin{bmatrix}-1\\2\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix}\right\}$

* Since neither vector is a scalar multiple of the other, the set of vectors is Linearly Independent.

(d) ~~[pts]~~ 2pt $\left\{\underset{\vec{V}_1}{\begin{bmatrix}1\\2\\-4\end{bmatrix}},\underset{\vec{V}_2}{\begin{bmatrix}3\\3\\-2\end{bmatrix}},\underset{\vec{V}_3}{\begin{bmatrix}4\\5\\-6\end{bmatrix}}\right\}$

$* \ \vec{V}_1+\vec{V}_2=\begin{bmatrix}1\\2\\-4\end{bmatrix}+\begin{bmatrix}3\\3\\-2\end{bmatrix}=\begin{bmatrix}4\\5\\-6\end{bmatrix}=\vec{V}_3$!

* Since $\vec{V}_1+\vec{V}_2=\vec{V}_3$, the set of vectors is Linearly Dependent.

(e) ~~[pts]~~ 3pt $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\3\\-4\end{bmatrix},\begin{bmatrix}-4\\2\\-1\end{bmatrix}\right\}$

* Here we need to check if $A\vec{x}=\vec{0}$ has only the trivial solution :)

$$\left[\begin{array}{ccc|c}1&-1&-4&0\\0&3&2&0\\-1&4&-1&0\end{array}\right] \xrightarrow[\text{new } R_3]{*\ R_1+R_3} \left[\begin{array}{ccc|c}1&-1&-4&0\\0&3&2&0\\0&3&-5&0\end{array}\right] \xrightarrow[\text{new } R_3]{*\ -R_2+R_3} \left[\begin{array}{ccc|c}1&-1&-4&0\\0&3&2&0\\0&0&-7&0\end{array}\right]$$

∴ Since $[A \mid \vec{0}]$ has a pivot in each row, $n=3$, $A\vec{x}=\vec{0}$ has a solution ($\vec{x}=\vec{0}$, Trivial Sol.) ⇒ ∴ Vectors are Linearly Independent

Bonus Question:

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by:

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3 \,,\, 2x_1 + x_3 \,,\, -x_1 - 2x_2 + 2x_3)$$

5pt **Find the standard matrix of T.**

* Given: $T: \mathbb{R}^3 \to \mathbb{R}^3$ ST $T(\vec{x}) = A\vec{x}$, where $\vec{x} \in \mathbb{R}^3$

$\Rightarrow$ Want: $A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)] = ?$

* $T(\vec{e}_1) = T(1,0,0) = (1, 2, -1)$
* $T(\vec{e}_2) = T(0,1,0) = (-1, 0, -2)$
* $T(\vec{e}_3) = T(0,0,1) = (2, 1, 2)$

$$\boxed{\therefore A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ -1 & -2 & 2 \end{bmatrix}}$$