Exam 3 Solution

Name:

Linear Algebra I: Exam 3 (Summer 2019)

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. _Make sure to exhibit skills discussed in class._ Box all answers and simplify answers as much as possible.

Good Luck! ☺

1. Linearly Independent Sets; Bases

Let $\vec{v}_1 = (1,1,1)$, $\vec{v}_2 = (1,2,3)$, $\vec{v}_3 = (1,1,2)$.

(a) [5 pts] Show that the vectors are Linearly Independent.

(b) [5 pts] Find the unique weights(scalars) c_1, c_2, c_3 such that $\vec{v} = (2,1,3)$ can be written as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

(a) $$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \mid \vec{0}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \underset{\substack{-R_1+R_3 \\ \text{N.}R_3}}{\overset{\substack{-R_1+R_2 \\ \text{N }R_2}}{\sim}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \overset{\substack{-2R_2+R_3 \\ \text{N.}R_3}}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{∴ Since a pivot position ∃ in row/column, NO free variables ∃} \Rightarrow A\vec{x} = \vec{0} \text{ has only the trivial solution \& thus the columns of } A, \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}, \text{ are Linearly Independent.}}$

(b) $$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{v} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Row-reduce to rref:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\Rightarrow \boxed{\begin{cases} \cdot\, c_1 = 0 \\ \cdot\, c_2 = -1 \\ \cdot\, c_3 = 3 \end{cases} \quad \& \quad \vec{v} = 0\vec{v}_1 + (-1)\vec{v}_2 + 3(\vec{v}_3)}$$

2. **Null Spaces, Column Spaces, and Linear Transformations**

Define the Linear Transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ *by* $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \\ x_1 + x_4 \end{bmatrix}$. $\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\Rightarrow [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]\vec{x}$

$\to A\vec{x}$

(a) [5 pts] Find the column space of *T*.
(b) [5 pts] Find the null space of *T*.
(c) [2 pts] Find a basis for the column space of *T*.
(d) [2 pts] Find the basis for the null space of *T*.

Hint: The column space of T is Col(A) and the null space of T is Nul(A), where A is the standard matrix of T ☺

(a) $\boxed{\text{Col}(A) = (\underline{\text{all}} \text{ columns of } A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$

(b) $[A \,|\, \vec{0}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -x_4 \\ x_2 = x_4 \\ x_3 = x_4 \\ x_4 \text{ is free} \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

ST $x_4 \in \mathbb{R}$

$\boxed{\therefore \text{Nul}(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$

(c) Basis for Col(A) = The pivot columns of A

$= \boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}}$

(d) Basis for Nul(A) = $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$

Therefore: Since all 3 conditions are satisfied, $H+K$ is a subspace of V.

3. **Vector Spaces and Subspaces**

Let H and K be subspaces of a vector space V. Let $H+K = \{\vec{w} : \vec{w} = \vec{u} + \vec{v}\ ,\ \vec{u} \in H \ \ and \ \ \vec{v} \in K\}$.

[9 pts] Show that $H+K$ is a subspace of V.

Initial Conditions:
① H & K are subspaces of vector space V
② Let $H+K = \{\vec{w} : \vec{w} = \vec{u} + \vec{v}\ ,\ \vec{u} \in H \ \& \ \vec{v} \in K$

① Show that $\vec{0}_V \in H+K$:

- Let $\vec{w} \in H+K$ ST $\vec{w} = \vec{u} + \vec{v}$, where $\vec{u} \in H$ & $\vec{v} \in K$
- Since H & K are subspaces: $\vec{u} = \vec{0} \in H$ & $\vec{v} = \vec{0} \in K$
- Then: $\boxed{\vec{w} = \vec{0} + \vec{0} = \vec{0} \in H+K\ \checkmark}$ ✓

② Show that $H+K$ is closed under addition:

- Let $\vec{w}_1, \vec{w}_2 \in H+K$ ST $\begin{cases} \vec{w}_1 = \vec{u}_1 + \vec{v}_1 \\ \vec{w}_2 = \vec{u}_2 + \vec{v}_2 \end{cases}$, where $\cdot\ \vec{u}_1, \vec{u}_2 \in H$ $\cdot\ \vec{v}_1, \vec{v}_2 \in K$
- Take the Sum: $\vec{w}_1 + \vec{w}_2 = (\vec{u}_1 + \vec{v}_1) + (\vec{u}_2 + \vec{v}_2) = \underline{\vec{u}_1} + \underline{\underline{\vec{v}_1}} + \underline{\vec{u}_2} + \underline{\underline{\vec{v}_2}}$

→ $\vec{w}_1 + \vec{w}_2 = (\vec{u}_1 + \vec{u}_2) + (\vec{v}_1 + \vec{v}_2)$, where $\begin{cases} \cdot\ \vec{u}_1 + \vec{u}_2 \in H \\ \cdot\ \vec{v}_1 + \vec{v}_2 \in K \end{cases}$ *b/c H & K are subspaces :)

$\therefore \vec{w}_1 + \vec{w}_2 \in H+K$ & Closed Under $\oplus$. ✓

③ Show that $H+K$ is closed under scalar-multiplication:

- Let $\vec{w} \in H+K$ ST $\vec{w} = \vec{u} + \vec{v}$, where $\vec{u} \in H$ & $\vec{v} \in K$
  Let c be any scalar, $c \in \mathbb{R}$.
- Take the Product:

$c\vec{w} = c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$, where $\cdot\ c\vec{u} \in H$ $\cdot\ c\vec{v} \in K$ *b/c H & K are subspaces :)

$\therefore c\vec{w} \in H+K$ & Closed Under $\odot$ ✓

4. **Null Spaces, Column Spaces, and Linear Transformations**

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation and $\mathcal{B} = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ be a basis for $\mathbb{R}^3$. Suppose that $T(\vec{v}_1) = (-2,1,1)$, $T(\vec{v}_2) = (0,1,-1)$, $T(\vec{v}_3) = (-2,2,0)$.

(a) [5 pts] Determine whether $\vec{w} = (-6,5,0)$ is in the range of T.

(b) [5 pts] Find a basis for the kernel of T.

$$* \; T: \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{ST} \quad T(\vec{v}) = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\vec{x} = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a)
$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & -6 \\ 1 & 1 & 2 & 5 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 2 & 5 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right] \quad \rightarrow\leftarrow$$

$\therefore$ System is inconsistent $\Rightarrow \vec{w}$ is **NOT** in the range.

(b)
$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{ST} \quad x_3 \in \mathbb{R}$$

$\therefore$ Basis for the Kernel: $\boxed{B = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}}$

5. ***Coordinate Systems***

The set $B = \{1 + t^2,\ 2t - t^2, 1 - t + t^2\}$ be a basis for $\mathbb{P}_2$.

[5pts] Find the coordinate vector $p(t) = 1 + 16t - 6t^2$ relative to B.

$$* \mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$* \vec{p}(t) = 1 + 16t - 6t^2 = \begin{bmatrix} 1 \\ 16 \\ -6 \end{bmatrix}$$

$$* \vec{p}(t) = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = [\vec{b}_1\, \vec{b}_2\, \vec{b}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \text{ST} \quad [\vec{p}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 16 \\ 1 & -1 & 1 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 16 \\ 0 & -1 & 0 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 7 \\ 0 & 2 & -1 & 16 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\Rightarrow \begin{cases} c_1 = 3 \\ c_2 = 7 \\ c_3 = -2 \end{cases} \qquad \boxed{\therefore [\vec{p}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}}$$

Bonus Question: ***Coordinate Systems***

Let $B = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since the coordinate mapping determined by B is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, this mapping must be implemented by some 2 x 2 matrix A.

[5pts] Find it.

*Hint: Multiplication by "A" should transform a vector into its coordinate vector, $[\vec{x}]_\beta$:)

Recall: $\vec{x} = P_\beta [\vec{x}]_\beta$ ST $P_\beta = [\vec{b}_1 \cdots \vec{b}_n]$

& $P_\beta^{-1} \vec{x} = [\vec{x}]_\beta$

... So $A = P_\beta^{-1}$!!

Find P_β^{-1}:

$$* \; P_\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$* \; P_\beta^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(9-8)} \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$$

$$\boxed{\therefore A = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}}$$

Ans