Section 4.1: Vector Spaces & Subspaces:

Note: Here we explore the simple & obvious algebraic properties of $\mathbb{R}^n$ (seen in chapters 1 & 2) applied to other mathematical systems.

*Definition (Vector Spaces):

A **vector space** is a nonempty set V of objects, called **vectors**, on which are defined 2 operations, called **addition** & **multiplication by scalars** ($\mathbb{R}$s), subject to the 10 axioms listed below.

These axioms must hold $\forall$ vectors $\vec{v}, \vec{u}, \& \vec{w} \in V$ & $\forall$ scalars $c, d \in \mathbb{R}$.

1. $\exists$ a sum of vectors $\vec{u}$ & $\vec{v}$ ST: $\vec{u} + \vec{v} \in V$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
4. $\exists \ \vec{0} \in V$ ST: $\vec{u} + \vec{0} = \vec{u}$. (* $\vec{0}$ is unique)
5. $\forall \ \vec{u} \in V$, $\exists \ -\vec{u} \in V$ ST: $\vec{u} + (-\vec{u}) = \vec{0}$. (* "$-\vec{u}$" called the Negative of $\vec{u}$)
6. $\exists$ a scalar multiple of $\vec{u}$ by c ST: $c\vec{u} \in V$.
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.
8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$.
9. $c(d\vec{u}) = (cd)\vec{u}$.
10. $1\vec{u} = \vec{u}$.

*Note: Technically, "V" is a REAL vector space, but all of the theory we explore here holds true for a Complex Vector Space as well.

*Prove that $-\vec{u}$ is the unique vector in V such that $\vec{u} + (-\vec{u}) = \vec{0}$:

*Note: Here we verify that $-\vec{u}$ is unique by showing that if $\vec{u}+\vec{w}=\vec{0}$, then $\vec{w}=-\vec{u}$.

Proof:

- Suppose that $\vec{w}$ satisfies $\vec{u}+\vec{w}=\vec{0}$.
- Adding "$-\vec{u}$" to both sides of the equation:

$$\vec{u}+\vec{w}=\vec{0}$$

$$-\vec{u}+(\vec{u}+\vec{w}) = -\vec{u}+\vec{0}$$

- By the 3rd axiom, we know: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

$$(-\vec{u}+\vec{u})+\vec{w} = -\vec{u}+\vec{0}$$

- By the 5th axiom, we know: $\vec{u}+(-\vec{u})=\vec{0}$

$$\vec{0}+\vec{w} = -\vec{u}+\vec{0}$$

- By the 4th axiom, we know: $\vec{u}+\vec{0}=\vec{u}$

$$\vec{w}=-\vec{u} \qquad \boxed{X}$$

Addition Properties of Vector Spaces

For each vector $\vec{u} \in V$ & $\forall$ scalars $c \in \mathbb{R}$,
the following holds true:

① $0\vec{u} = \vec{0}$.

② $c\vec{0} = \vec{0}$.

③ $-\vec{u} = (-1)\vec{u}$.

Note: Proofs for these properties can be found on the following pages :)

* Prove that $0\vec{u} = \vec{0} \quad \forall \ \vec{u} \in V$:

PROOF:

- By the 8th axiom, we know: $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$$0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}$$

- Adding "$-(0\vec{u})$" to both sides of the equation:

$$0\vec{u} = 0\vec{u} + 0\vec{u}$$

$$-(0\vec{u}) + 0\vec{u} = -(0\vec{u}) + (0\vec{u} + 0\vec{u})$$

- By the 3rd axiom, we know: $(\vec{u}+\vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$$-(0\vec{u}) + 0\vec{u} = \left[-(0\vec{u}) + 0\vec{u}\right] + 0\vec{u}$$

- By the 5th axiom, we know: $\vec{u} + (-\vec{u}) = \vec{0}$

$$\vec{0} = \vec{0} + 0\vec{u}$$

- By the 4th axiom, we know: $\vec{u} + \vec{0} = \vec{0}$

$$\vec{0} = 0\vec{u}$$

$\boxed{X}$

*Prove that $c\vec{0} = \vec{0}$ $\forall$ scalar c:

PROOF:

4) $\vec{u} + \vec{0} = \vec{u}$

7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

· By the 4th & 7th axioms, we know:

$$c\vec{0} = c(\vec{0} + \vec{0})$$

$$= c\vec{0} + c\vec{0}$$

· Adding "$-(c\vec{0})$" to both sides of the equation:

$$-(c\vec{0}) + c\vec{0} = -(c\vec{0}) + [c\vec{0} + c\vec{0}]$$

· By the 3rd axiom, we know: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$$[-(c\vec{0}) + c\vec{0}] = [-(c\vec{0}) + c\vec{0}] + c\vec{0}$$

· By the 5th axiom; we know: $\vec{u} + (-\vec{u}) = \vec{0}$

$$\vec{0} = \vec{0} + c\vec{0}$$

· By the 4th axiom, we know: $\vec{u} + \vec{0} = \vec{u}$

$$\vec{0} = c\vec{0}$$

$\boxed{X}$

Subspaces

Note: In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space.

*Definition (Subspace):

A subspace of a vector space V is a subset "H" of V that has the following 3 properties:

① The zero vector of V is in H.

② H is closed under vector addition.

*That is: $\forall\ \vec{u}, \vec{v} \in H$, the sum $\vec{u} + \vec{v} \in H$.

③ H is closed under multiplication by scalars.

*That is: $\forall\ \vec{u} \in H$ & $\forall$ scalars c, the vector $c\vec{u} \in H$

*Additional Conclusions:

- Every subspace is a vector space.
- Every vector space is a subspace of itself (or possibly of other larger spaces). *The axioms of a vector space include all the conditions for being a subspace.

Note: The term "subspace" is used when @ least 2^+ vector spaces are in mind.

Introductory Examples of Subspaces:

① The Zero Subspace: $\{\vec{0}\}$

⇒ The set consisting of only $\vec{0}$ in a vector space V is a subspace of V, called "the zero subspace".

② The Polynomial Subspace: $\mathbb{P}$

* Let $\mathbb{P}$ = the set of all polynomials w/ real coefficients, with operations in $\mathbb{P}$ defined by functions.

⇒ Then, $\mathbb{P}$ is a subspace of all $\mathbb{R}$-valued functions defined on $\mathbb{R}$.

* Also, $\forall\ n \geq 0$, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ b/c $\mathbb{P}_n$ is a subset of $\mathbb{P}$ that contains the zero polynomial, the sum of 2 polynomials in $\mathbb{P}_n$ is also in $\mathbb{P}_n$, -AND- a scalar multiple of $\mathbb{P}_n$ is also in $\mathbb{P}_n$:)

Introductory Examples Continued...

③ Planes in $\mathbb{R}^3$:

$\Rightarrow$ A plane in $\mathbb{R}^3$ NOT through the origin is NOT a subspace of $\mathbb{R}^3$!

(because it does NOT contain the $\vec{0} \in \mathbb{R}^3$)
↳ violates

④ Lines in $\mathbb{R}^2$:

$\Rightarrow$ A line in $\mathbb{R}^2$ NOT through the origin is NOT a subspace of $\mathbb{R}^2$!

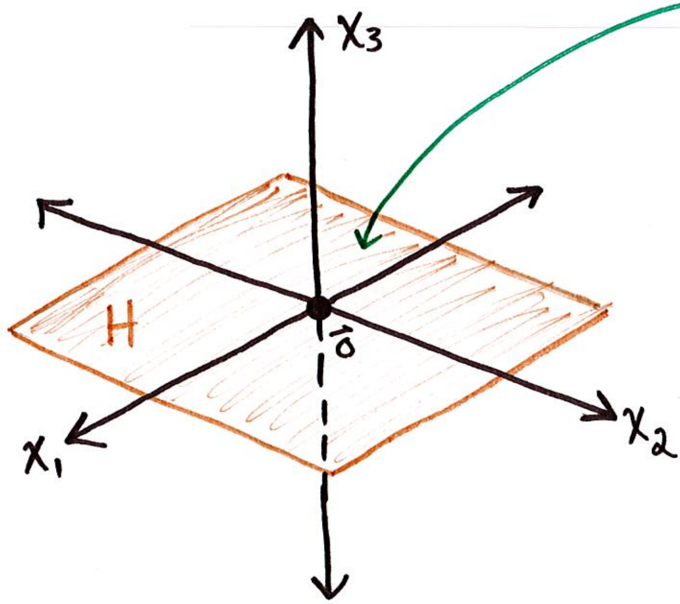
(because it does NOT contain $\vec{0} \in \mathbb{R}^2$)
↳ violates prop. # 1

Example: Show that the following set is a subspace of $\mathbb{R}^3$ (NOT $\mathbb{R}^2$):

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, \text{ where } s,t \in \mathbb{R} \right\}$$

Answer:

*Lets start by considering the Geometric Interpretation:



* Notes:

- The x_1x_2-plane is a subspace of $\mathbb{R}^3$ (NOT $\mathbb{R}^2$)
- While H may "look like" & "act like" $\mathbb{R}^2$, it is impertant to note that each vector in $\mathbb{R}^3$ has 3 entries (each vector in $\mathbb{R}^2$ only has 2 entries)

$\Rightarrow$ $\boxed{\mathbb{R}^2 \text{ \& } \mathbb{R}^3 \text{ are distinct!}}$

*Verify H is a Subset of $\mathbb{R}^3$:

(i) The zero vector of $\mathbb{R}^3$ is in H: $\Rightarrow \vec{0} \in H$ ✓

(ii) H is closed under vector addition & scalar multiplication:

- Addition: $\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2s \\ 2t \\ 0 \end{bmatrix} \in H$ ✓
- Scalar-Mult: $c\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} cs \\ ct \\ 0 \end{bmatrix} \in H$ ✓

* Both operations on H produce vectors whose 3rd entry is 0 $\Rightarrow$ So belong to H ✓

• H is a subset of $\mathbb{R}^3$

Example: Let W be the union of the 1^{st} & 3^{rd} quadrants in the xy-plane. That is, let

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}.$$

(a) If $\vec{u}$ is in W & c is any scalar, is $c\vec{u}$ in W? Explain your answer.

(b) Find specific vectors $\vec{u}$ & $\vec{v}$ in W ST $\vec{u}+\vec{v}$ is NOT in W (* This is enough to show that W is NOT a vector space *)

Answer:

*Part (a):

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in W$.

Then $\forall$ scalar c: $c\vec{u} = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$

Note: $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$

Check if $c\vec{u}$ is in W: $(cx)(cy) = c^2xy$

$\therefore$ $\boxed{\text{Since } c^2 > 0 \ \& \ xy \geq 0 \text{ (by def.), the } c^2xy \geq 0 \ \& \text{ so } c\vec{u} \in W.}$ Answer.

Example Continued...

***Part (b):**

$ $\vec{u}$ & $\vec{v} \in W$. $\Longrightarrow$ Let

$$\boxed{\begin{aligned} &* \vec{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix} \in W \quad (\text{Quad. } 3) \\ &* \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in W \quad (\text{quad. } 1) \end{aligned}}$$

Ans ✓

• **Find the sum:**

$$\vec{u} + \vec{v} = \begin{bmatrix} -1 \\ -7 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \notin W$$

***Note:** The vector $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ $\exists$ in Quadrant 4

-AND- $(1)(-4) = -4 < 0$

$\therefore$ $\vec{u}$ & $\vec{v}$ are both in W, **but** $\vec{u} + \vec{v}$ is **NOT** in W

Ans ✓

Example: Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 5x^2 + 2y^2 \leq 1 \right\}$, which represents the set of points on & inside an ellipse in the xy-plane. Find 2 specific examples - 2 vectors & a vector and a scalar - to show that H is NOT a subspace of $\mathbb{R}^2$.

Answer:

* Recall: A subspace of a vector space V (or $\mathbb{R}^2$ here :)) is a subset "H" of V st:
i) $\vec{0}$ of V is in H
ii) H is closed under addition
iii) H is closed under scalar-mult.

*Show that H is NOT closed under addition:

Goal: Show that while $\vec{u}$ & $\vec{v}$ are in H, $\vec{u}+\vec{v}$ is NOT

- $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/2 \end{bmatrix} \rightarrow 5\left(\frac{1}{25}\right) + 2\left(\frac{1}{4}\right) = \frac{7}{10} < 1$ ✓

- $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/2 \end{bmatrix} \rightarrow 5\left(\frac{1}{36}\right) + 2\left(\frac{1}{4}\right) = \frac{23}{36} < 1$ ✓

- $\vec{u}+\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/5 + 1/6 \\ 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 11/30 \\ 1 \end{bmatrix} \rightarrow 5\left(\frac{11}{30}\right)^2 + 2(1)^2 > 1$

$\rightarrow\leftarrow$

∴ While $\vec{u}$ & $\vec{v}$ are both in H, $\vec{u}+\vec{v}$ is NOT in H.
$\rightarrow$ H is NOT closed under addition.

Ans.

Example Continued...

***Show that H is NOT closed under scalar multiplication:**

Goal: Show that while $\vec{u}$ is in H, $c\vec{u}$ is NOT in H.

$$\boxed{\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/2 \end{bmatrix}} \rightarrow 5\left(\frac{1}{25}\right) + 2\left(\frac{1}{4}\right) = \frac{7}{10} < 1 \checkmark$$

Let $c = 3$:

$$c\vec{u} = 3\begin{bmatrix} 1/5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 3/2 \end{bmatrix} \longrightarrow 5\left(\frac{9}{25}\right) + 2\left(\frac{9}{4}\right) > 1 \rightarrow\leftarrow$$

$\therefore$ While $\vec{u}$ is in H, $c\vec{u}$ is NOT in H.
$\Rightarrow$ H is NOT closed under scalar multiplication

Answer

Example: Determine if the given set is a subspace of $\mathbb{P}_5$. Justify your answer.

$\Rightarrow$ The set of all polynomials of the form $p(t) = at^5$, where: a is in $\mathbb{R}$

Answer:

Recall: $\forall\, n \geq 0$, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ (ST $\mathbb{P}$ = Set of all Polynomial w/ $\mathbb{R}$ coeff.)

because:
(i) $\mathbb{P}_n$ contains the zero polynomial
(ii) The sum of 2 polynomials in $\mathbb{P}_n$ is also in $\mathbb{P}_n$.
(iii) A scalar multiple of $\mathbb{P}_n$ is also in $\mathbb{P}_n$.

Check that these 3 conditions are met: $p(t) = at^5$

① Let $a = 0$: $p(t) = 0t^5 = 0$ ✓ $\boxed{\therefore \text{ 0-polynomial } \exists}$

② Check the sum: $\cdot\, p(t) = at^5$, $\cdot\, p(t) = bt^5$ $\Rightarrow$ $p(t) = at^5 + bt^5 = (a+b)t^5$ ✓

*also $\exists$ in set

$\therefore$ Closed Under Addition

③ Check Scalar Mult.: Let c be any scalar.

$\Rightarrow p(t) = at^5$; $p(t) = c(at^5) = (ca)t^5$ ✓

*also $\exists$ in set

$\therefore$ Closed Under Scalar Multiplication.

$\therefore$ $\boxed{\text{Yes, the Set is a Subset of } \mathbb{P}_5.}$ Ans

<u>Example:</u> Determine if the given set is a subspace of $\mathbb{P}_7$. Justify your answer:

"All polynomials of degree @ most 7, with rational numbers as coefficients."

<u>Answer:</u>

*<u>Recall:</u> $\forall\ n \geq 0$, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ b/c:

(i) $\mathbb{P}_n$ contains the zero polynomial

(ii) The sum of 2 polynomials in $\mathbb{P}_n$ is also in $\mathbb{P}_n$.

(iii) A scalar multiple of $\mathbb{P}_n$ is also in $\mathbb{P}_n$.

*<u>To determine if the given set is a subspace of $\mathbb{P}_7$, we need to check that the set holds true w/ the 3 prop.:</u>

<u>Notes:</u> $*V \rightarrow \mathbb{P}_7$

$*H \rightarrow$ The described set

(i) <u>Is the zero vector of "V" in "H"?</u>

$\boxed{\text{Yes! Since 0 is a rational number, the zero vector } \vec{0} \text{ of } \mathbb{P}_7 \text{ is in the set.}}$

<u>Notes:</u> Let $p, q \in \mathbb{Z}$ ST $q \neq 0$.

$\cdot\ r = \frac{p}{q}$ is a rational #

$\cdot\ r + r = 2\left(\frac{p}{q}\right)$ ✓

(ii) <u>Is "H" closed under addition?</u>

$\boxed{\text{Yes! Since the sum of 2 rational numbers is rational, the given set is closed under addition}}$

(iii) <u>Is "H" closed under scalar multiplication?</u>

$\boxed{\text{No. Since "c" can be any } \mathbb{R}\text{-scalar (or complex), depending on the value of "c", the given set may or may not be closed under scalar-mult.}}$

[illegible]

*A Subspace Spanned By a Set *

The term Linear Combination refers to any sum of scalar multiples of vectors.

⇒ The Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\vec{v}_1, \dots, \vec{v}_p$.

Note: In section 4.5, we will see that every nonzero subspace of $\mathbb{R}^3$, other than $\mathbb{R}^3$ itself, is either Span$\{\vec{v}_1, \vec{v}_2\}$ for some linearly independent $\vec{v}_1$ & $\vec{v}_2$ -OR- Span$\{\vec{v}\}$ for $\vec{v} \neq \vec{0}$.

* If Span$\{\vec{v}_1, \vec{v}_2\}$ for some Linearly Independent $\vec{v}_1$ & $\vec{v}_2$, then: The subspace is a plane through the origin.

* If Span$\{\vec{v}\}$ for $\vec{v} \neq \vec{0}$, then: The subspace is a line through the origin.

*It can be helpful to keep these geometric pictures in mind, even for abstract vector spaces :)

Example (A Subspace Spanned By a Set):

Given $\vec{v}_1$ & $\vec{v}_2$ in a vector space V, let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Show that H is a subspace of V.

Answer:

(i) Show that the zero vector of V is in H:

$\boxed{\text{Since } \vec{0} = 0\vec{v}_1 + 0\vec{v}_2, \ \vec{0} \in H.}$ ✓

(ii) Show that H is closed under addition:

* Consider 2 arbitrary vectors in H:

$\vec{u} = s_1\vec{v}_1 + s_2\vec{v}_2$ & $\vec{w} = t_1\vec{v}_1 + t_2\vec{v}_2$, where $s_1, s_2, t_1, t_2 \in \mathbb{R}$

* By Axioms 2, 3, & 8, we know:

$$\vec{u} + \vec{w} = (s_1\vec{v}_1 + s_2\vec{v}_2) + (t_1\vec{v}_1 + t_2\vec{v}_2) \quad \text{*Axiom 2*}$$

$$= (s_1\vec{v}_1 + t_1\vec{v}_1) + (s_2\vec{v}_2 + t_2\vec{v}_2) \quad \text{*Axiom 3*}$$

$$= \vec{v}_1(s_1 + t_1) + \vec{v}_2(s_2 + t_2) \quad \text{*Axiom 8*}$$

$\therefore$ Since $\vec{u} + \vec{w}$ is a Linear Combination of $\vec{v}_1$ & $\vec{v}_2$, $\vec{u} + \vec{w}$ is in H ✓

Example (A subspace spanned by a set) Continued...

(iii) Show that H is closed under Scalar-Multiplication:

* Consider an arbitrary vector in H:

$$\vec{u} = s_1\vec{v}_1 + s_2\vec{v}_2 \quad \text{ST} \quad s_1, s_2 \in \mathbb{R}$$

* Let "c" be any scalar: $c \in \mathbb{R}$

* By Axioms 7 & 9, we know:

$$c\vec{u} = c\left(s_1\vec{v}_1 + s_2\vec{v}_2\right)$$

Axiom 7

$$= cs_1\vec{v}_1 + cs_2\vec{v}_2$$

$$= (cs_1)\vec{v}_1 + (cs_2)\vec{v}_2$$

Axiom 9

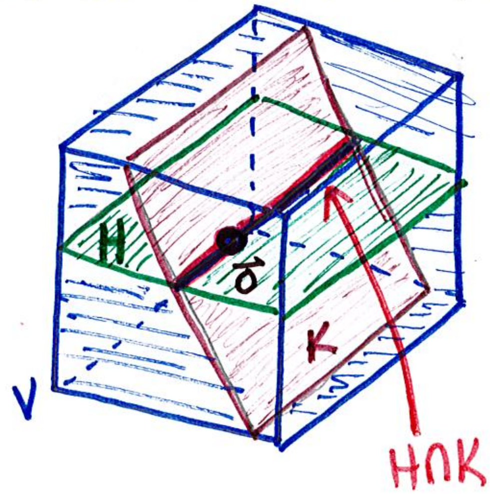
$\therefore$ Since $c\vec{u}$ is a Linear Combination of $\vec{v}_1$ & $\vec{v}_2$, $c\vec{u}$ is in H. ✓

*Therefore: Since all 3 properties hold true, H is a subspace of V!

Example: Let H & K be subspaces of a vector space V. The intersection of H & K, written as $H \cap K$, is the set of $\vec{v}$ in V that belong to both H & K.

* Show that $H \cap K$ is a subspace of V (See figure to the right).

* Give an example in $\mathbb{R}^2$ to show that the union of 2 subspaces is not, in general, a subspace.



Answer:

* Let H & K be subspaces of a vector space V.

* **Show that $H \cap K$ is a subspace of V:**

Note: We need to verify that the 3 prop. of a subspace hold true.

(i) Show that the $\vec{0}$ of V is in $H \cap K$:

- Since H & K are subspaces of V, the $\vec{0}$ of V is in H and the $\vec{0}$ of V is in K.

$\therefore$ $\boxed{\text{Since both H \& K contain } \vec{0} \text{ of V, then } H \cap K \text{ contains } \vec{0}}$ ✓

(ii) Show that $H \cap K$ is closed under addition:

- Let $\vec{u}$ & $\vec{v}$ be vectors in $H \cap K$. (*Goal: Show $\vec{u}+\vec{v}$ in $H \cap K$).
- Since $\vec{u}$ & $\vec{v}$ are in $H \cap K$, then $\vec{u}$ & $\vec{v}$ are in H.
  - $\Rightarrow$ Since H is a subspace, then $\vec{u}+\vec{v}$ is in H (by Def.)
- Since $\vec{u}$ & $\vec{v}$ are in $H \cap K$, then $\vec{u}$ & $\vec{v}$ are in K.
  - $\Rightarrow$ Since K is a subspace, then $\vec{u}+\vec{v}$ is in K (by Def.)
- $\boxed{\text{Since } \vec{u}+\vec{v} \text{ is in both H \& K, thus } \vec{u}+\vec{v} \text{ is in } H \cap K}$ ✓

Example Continued...

(iii) Show that $H \cap K$ is closed under scalar-multiplication:

- Let $\vec{u}$ be a vector in $H \cap K$. (*Goal: Show $c\vec{u}$ in $H \cap K$)
- Since $\vec{u}$ is in $H \cap K$, then $\vec{u}$ is in H.
  - $\Rightarrow$ Since H is a subspace, $c\vec{u}$ is in H ST c is any scalar (by Def.)
- Since $\vec{u}$ is in $H \cap K$, then $\vec{u}$ is in K.
  - $\Rightarrow$ Since K is a subspace, $c\vec{u}$ is in K (by Def.)

$\therefore$ Since $c\vec{u}$ is in both H & K, thus $c\vec{u}$ is in $H \cap K$ ✓

Therefore: Since all 3 conditions are met, $H \cap K$ is a subspace of V. ☒

*Give an example in $\mathbb{R}^2$ to show that $H \cup K$ is not, in general, a subspace:

- Let H & K be subspaces of $\mathbb{R}^2$ ST: $\begin{cases} \bullet\ H = x\text{-axis} \\ \bullet\ K = y\text{-axis} \end{cases}$
- While H & K are subspaces of $\mathbb{R}^2$, $H \cup K$ is NOT closed under vector addition $\Rightarrow$ $\therefore$ $H \cup K$ is NOT a subspace of $\mathbb{R}^2$

↑ violates (ii) on 1st page ∵

*Theorem # 1 :

If $\vec{v}_1, \dots, \vec{v}_p$ are a vector space of V, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of V.

*Notes:

- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_2\}$ is called "the Subspace Spanned by"

  -or- "the Subspace Generated by" $\{\vec{v}_1, \dots, \vec{v}_p\}$.

- Given any subspace H of V, a spanning/generating set for H is a set $\{\vec{v}_1, \dots, \vec{v}_p\}$ in H

  st:
$$H = \text{Span}\{\vec{v}_1, \dots, \vec{v}_2\}$$

Example (Applying theorem # 1):

Let H be the set of all vectors of the form $(a-3b,\ b-a,\ a,\ b)$, where a & b are arbitrary scalars. That is, let:

$$H = \{(a-3b, b-a, a, b) : a \text{ \& } b \text{ in } \mathbb{R}\}$$

Show that H is a subspace of $\mathbb{R}^4$.

Answer:

*Start by writing the vectors in H as column vectors:

$$H = \begin{bmatrix} a-3b \\ -a+b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ b \\ 0 \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

* Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ ST $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^4$

*Then:

$$\boxed{\begin{array}{l} H = a\vec{v}_1 + b\vec{v}_2 \text{ \& so } H = \text{Span}\{\vec{v}_1, \vec{v}_2\} \\ \therefore H \text{ is a subspace of } \mathbb{R}^4 \text{ (by Th}^{\text{m}}\text{ 1)} \end{array}}$$

Ans.

Example: Let W be the set of all vectors of the form: $\begin{bmatrix} 2s+2t \\ 2t \\ 4s-3t \\ 4s \end{bmatrix}$

Show that W is a subspace of $\mathbb{R}^4$ by finding vectors $\vec{u}$ & $\vec{v}$ ST $W = \text{Span}\{\vec{u}, \vec{v}\}$.

Answer:

*Recall:
- The $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ denotes the set of all vectors that can be written as a Linear Combination of vectors $\vec{v}_1, \dots, \vec{v}_p$.
- (Thm #1): If $\vec{v}_1, \dots, \vec{v}_p$ is a vector space of V, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of V.

***Write the vectors in W as Column-Vectors:**

$$W = \begin{bmatrix} 2s+2t \\ 2t \\ 4s-3t \\ 4s \end{bmatrix} = \begin{bmatrix} 2s \\ 0 \\ 4s \\ 4s \end{bmatrix} + \begin{bmatrix} 2t \\ 2t \\ -3t \\ 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 4 \\ 4 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ -3 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Let $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 4 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 0 \end{bmatrix}$

$\therefore W = s\vec{u} + t\vec{v}$ ST $\vec{u}$ & $\vec{v} \in \mathbb{R}^4$ (*Linear Combination*)

So, $W = \text{Span}\{\vec{u}, \vec{v}\}$ for $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 4 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 0 \end{bmatrix}$

Ans✓

***Why does this show W is a subspace of $\mathbb{R}^4$?**

By Thm #1: Since $\vec{u}$ & $\vec{v}$ are in $\mathbb{R}^4$ & $W = \text{Span}\{\vec{u}, \vec{v}\}$, then W is a subspace of $\mathbb{R}^4$.

Ans✓

Example: Let W be the set of vectors of the form $\begin{bmatrix} 3a+9b \\ 4b-7c \\ -2a+2c \\ 5b \end{bmatrix}$ where a, b, c represent arbitrary $\mathbb{R}$s.

Find a set S of vectors that spans W <u>or</u> give an example/explanation to show that W is <u>NOT</u> a vector space.

Answer:

* <u>Write the vectors in W as Column-Vectors:</u>

$$W = \begin{bmatrix} 3a+9b \\ 4b-7c \\ -2a+2c \\ 5b \end{bmatrix} = \begin{bmatrix} 3a \\ 0 \\ -2a \\ 0 \end{bmatrix} + \begin{bmatrix} 9b \\ 4b \\ 0 \\ 5b \end{bmatrix} + \begin{bmatrix} 0 \\ -7c \\ 2c \\ 0 \end{bmatrix} = a\begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 9 \\ 4 \\ 0 \\ 5 \end{bmatrix} + c\begin{bmatrix} 0 \\ -7 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ <u>Since a vector $\vec{w}$ in W can be written:</u>

$$\vec{w} = a\begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 9 \\ 4 \\ 0 \\ 5 \end{bmatrix} + c\begin{bmatrix} 0 \\ -7 \\ 2 \\ 0 \end{bmatrix}$$

<u>Then the set S defined:</u>

$$S = \left\{ \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 2 \\ 0 \end{bmatrix} \right\}$$ is a Set that <u>spans</u> W.

Answer.

Example: Let H be the set of all vectors of the form: $\begin{bmatrix} 5t \\ t \\ 7t \end{bmatrix}$.

Find a vector $\vec{v} \in \mathbb{R}^3$ ST $H = \text{Span}\{\vec{v}\}$.

Why does this show that H is a subspace of $\mathbb{R}^3$?

Answer:

*Recall:

- The $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ denotes the set of all vectors that can be written as a Linear Combination of vectors $\vec{v}_1, \ldots, \vec{v}_p$.
- (Thm 1): If $\vec{v}_1, \ldots, \vec{v}_p$ is a vector space of V, then $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of V.

*Write the vectors of H as Column-Vectors:

$$H = \begin{bmatrix} 5t \\ t \\ 7t \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} \Rightarrow \text{Let } \vec{v} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} \in \mathbb{R}^3$$

$\therefore$ $H = t\vec{v}$ ST $\vec{v} \in \mathbb{R}^3$ (*A Linear Combination of $\vec{v}$)

So, $H = \text{Span}\{\vec{v}\}$ for $\vec{v} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$

Answer.

*Why does this show H is a subspace of $\mathbb{R}^3$?

By theorem 1: Since $\vec{v}$ is in $V = \mathbb{R}^3$ and $H = \text{Span}\{\vec{v}\}$, then H is a subspace of $\mathbb{R}^3$.

Example: Let W be a set of all vectors of the form: $\begin{bmatrix} 8b-9c \\ -b \\ 6c \end{bmatrix}$

where b & c are arbitrary.

Find vectors $\vec{u}$ & $\vec{v}$ ST $W = \text{Span}\{\vec{u}, \vec{v}\}$.

Why does this show that W is a subspace of $\mathbb{R}^3$?

Answer:

* Recall:
  - The $\text{Span}\{\vec{v}_1, \dots \vec{v}_p\}$ denotes the set of all vectors that can be written as a Linear Combination of vectors $\vec{v}_1, \dots, \vec{v}_p$.
  - ($\underline{\text{Th}^m\ 1}$): If $\vec{v}_1, \dots, \vec{v}_p$ is a vector space of V, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of V.

* Write the vectors in W as Column-Vectors:

$$W = \begin{bmatrix} 8b-9c \\ -b \\ 6c \end{bmatrix} = \begin{bmatrix} 8b \\ -b \\ 0 \end{bmatrix} + \begin{bmatrix} -9c \\ 0 \\ 6c \end{bmatrix} = b\begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -9 \\ 0 \\ 6 \end{bmatrix}$$

$\Rightarrow$ Let $\vec{u} = \begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -9 \\ 0 \\ 6 \end{bmatrix}$

$\therefore W = b\vec{u} + c\vec{v}$ ST $\vec{u}, \vec{v} \in \mathbb{R}^3$ (* A Linear Combination of $\vec{u}$ & $\vec{v}$.)

So, $W = \text{Span}\{\vec{u}, \vec{v}\}$ for $\vec{u} = \begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -9 \\ 0 \\ 6 \end{bmatrix}$

Answer ✓

* Why does this show W is a subspace of $\mathbb{R}^3$?

By Th^m #1: Since $\vec{u}$ & $\vec{v}$ are in $V = \mathbb{R}^3$ & $W = \text{Span}\{\vec{u}, \vec{v}\}$, then W is a subspace of $\mathbb{R}^3$.

Example: Show that $\vec{w}$ is in the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$, where these vectors are defined as follows:

$$\vec{w} = \begin{bmatrix} 11 \\ -2 \\ -7 \\ 8 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 7 \\ -4 \\ -5 \\ 7 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -6 \\ 2 \\ -3 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -8 \\ 6 \\ -8 \\ -13 \end{bmatrix}$$

Answer:

*Note: To show that $\vec{w}$ is in the subspace of $\mathbb{R}^4$, we need to express $\vec{w}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$:

IVW: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{w} \longleftrightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{w}$

$\Rightarrow$ Solve: $\left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{array}\right]$

*Row-reduce the augmented matrix to find $\vec{x}$:

$$\left[\begin{array}{ccc|c} 7 & -6 & -8 & 11 \\ -4 & 2 & 6 & -2 \\ -5 & -3 & -8 & -7 \\ 7 & -7 & -13 & 8 \end{array}\right] \overset{\substack{-R_1 \\ +R_4 \\ \hline N.R_4}}{\sim} \left[\begin{array}{ccc|c} 7 & -6 & -8 & 11 \\ -4 & 2 & 6 & -2 \\ -5 & -3 & -8 & -7 \\ 0 & -1 & -5 & -3 \end{array}\right] \overset{\substack{\frac{1}{7}R_1 \\ -\frac{1}{2}R_2 \\ -R_4}}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{6}{7} & -\frac{8}{7} & \frac{11}{7} \\ 2 & -1 & -3 & 1 \\ -5 & -3 & -8 & -7 \\ 0 & 1 & 5 & 3 \end{array}\right] \overset{\substack{-2R_1 \\ +R_2 \\ \hline N.R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 5/7 & -5/7 & -15/7 \\ -5 & -3 & -8 & -7 \\ 0 & 1 & 5 & 3 \end{array}\right] \overset{\frac{7R_2}{5}}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{6}{7} & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ -5 & -3 & -8 & -7 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

Example Continued...

$$\left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ -5 & -3 & -8 & -7 \\ 0 & 1 & 5 & 3 \end{array}\right] \underset{\sim}{\frac{5R_1 + R_3}{N.R_3}} \left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ 0 & -51/7 & -96/7 & 6/7 \\ 0 & 1 & 5 & 3 \end{array}\right] \underset{\sim}{\tfrac{1}{7}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ 0 & -51 & -96 & 6 \\ 0 & 1 & 5 & 3 \end{array}\right] \underset{\sim}{\frac{51R_2 + R_3}{N.R_3}} \left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -147 & -147 \\ 0 & 1 & 5 & 3 \end{array}\right] \underset{\sim}{-\tfrac{1}{147}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 5 & 3 \end{array}\right] \underset{\sim}{\frac{-R_2 + R_3}{N.R_3}} \left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 6 & 6 \end{array}\right] \underset{\sim}{\tfrac{1}{6}R_4}$$

$$\left[\begin{array}{ccc|c} 1 & -6/7 & -8/7 & 11/7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\sim}{\frac{\frac{6}{7}R_2 + R_1}{N.R_1}} \left[\begin{array}{ccc|c} 1 & 0 & \overset{-2}{\cancel{-14/7}} & \overset{-1}{\cancel{-7/7}} \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\sim}{\frac{-R_3 + R_4}{N.R_4}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{\frac{R_3 + R_2}{N.R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{\frac{2R_3 + R_1}{N.R_1}}$$

Example Continued...

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 1 \end{cases} \Rightarrow \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

$\therefore \vec{W}$ IS in the subspace spanned by $\vec{V}_1, \vec{V}_2$ & $\vec{V}_3$

$$\Rightarrow \vec{W} = 1\vec{V}_1 - 2\vec{V}_2 + 1V_3 = \vec{V}_1 - 2\vec{V}_2 + \vec{V}_3$$

Ans.

Check:

$$1\begin{bmatrix} 7 \\ -4 \\ -5 \\ 7 \end{bmatrix} - 2\begin{bmatrix} -6 \\ 2 \\ -3 \\ -7 \end{bmatrix} + 1\begin{bmatrix} -8 \\ 6 \\ -8 \\ -13 \end{bmatrix} = \begin{bmatrix} 7+12-8 \\ -4-4+6 \\ -5+6-8 \\ 7+14-13 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ -7 \\ 8 \end{bmatrix} = \vec{W} \checkmark$$

Example (Thm #1 & Section 1.3):

For what values of "h" will $\vec{y}$ be in a subspace of $\mathbb{R}^3$ spanned by $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$ if:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \& \quad \vec{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Answer:

*Note: This is the same idea reviewed in section 1.3 BUT with the term "subspace" rather than "Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$".

*The vector $\vec{y}$ will be in a subspace of $\mathbb{R}^3$ IFF $\exists$ scalars x_1, x_2, x_3 ST: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{y}$

$$\underbrace{x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}}_{\text{Linear Combination}} \iff \underbrace{\begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}}_{\text{Matrix Eq.}} \vec{x} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

*Row-Reduce the Equivalent Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right] \underset{N.R_2}{\overset{R_1 + R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ -2 & -7 & 0 & h \end{array}\right] \underset{N.R_3}{\overset{2R_1 + R_3}{\sim}}$$

Example (Th^M #1 & Section 1.3): Continued...

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right] \underset{\sim}{\overset{-3R_2 + R_3}{\overline{N.\ R_3}}} \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right] *$$

* **CAUTION**: The system is consistent IFF $h-5=0$!
(IOW: NO pivot $\exists$ in Column 4)

$$\Rightarrow h-5=0$$

$$\boxed{h=5}$$

$\therefore$ Vector $\vec{y}$ will be in a subspace of $\mathbb{R}^3$ IFF $h=5$.

Answer.