Solution

Name:

Linear Algebra: Quiz 5

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. Make sure to exhibit skills discussed in class. Box all answers and clean up answers as much as possible.

1. ***The Inverse of a Matrix (2.2) & Characteristics of Invertible Matrices (2.3)***

[10pts] Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a Linear Transformation defined by:

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ x_1 - x_2 - x_3,\ x_1 - x_2 + x_3)$$

Is T an invertible transformation? If it is, find a formula for T^{-1}.

Recall: A Linear Transformation is invertible when the Standard Matrix A of T is invertible.

*Standard Matrix of T: $\vec{x} \mapsto \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ x_1 - x_2 + x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{\text{Standard Matrix of } T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

*Use the Inverse Algorithm to check if A^{-1} $\exists$:

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{-R_1+R_3 \to N.R_3}{\overset{-R_1+R_2 \to N.R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \end{array}\right] \underset{-R_2+R_3 \to N.R_3}{\overset{-R_2+R_1 \to N.R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1 & 0 & 1/2 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right] \overset{-R_3+R_2 \to N.R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right] = [I_3 \mid A^{-1}]$$

$\therefore$ $\boxed{A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \text{ (A is invertible)}}$ $\Longrightarrow$

$\therefore$ T is also invertible & $T^{-1}(\vec{x}) = A^{-1}\vec{x}$

$$\boxed{T^{-1}(\vec{x}) = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix}}$$