Section 1.9: The Matrix of a Linear Transformation:

Note: Here we show that every linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation $\vec{x} \mapsto A\vec{x}$ - AND - the important properties of 'T' are related to the known/familiar properties of matrix A :)

***Hint:** The key to finding matrix A is to observe that T is completely determined by what it does to the columns of the $n \times n$ Identity Matrix I_n.

Illustration:

The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose "T" is the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T(\vec{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ & $T(\vec{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$

With no additional information, Find a formula for the image of an arbitrary $\vec{x}$ in $\mathbb{R}^2$.

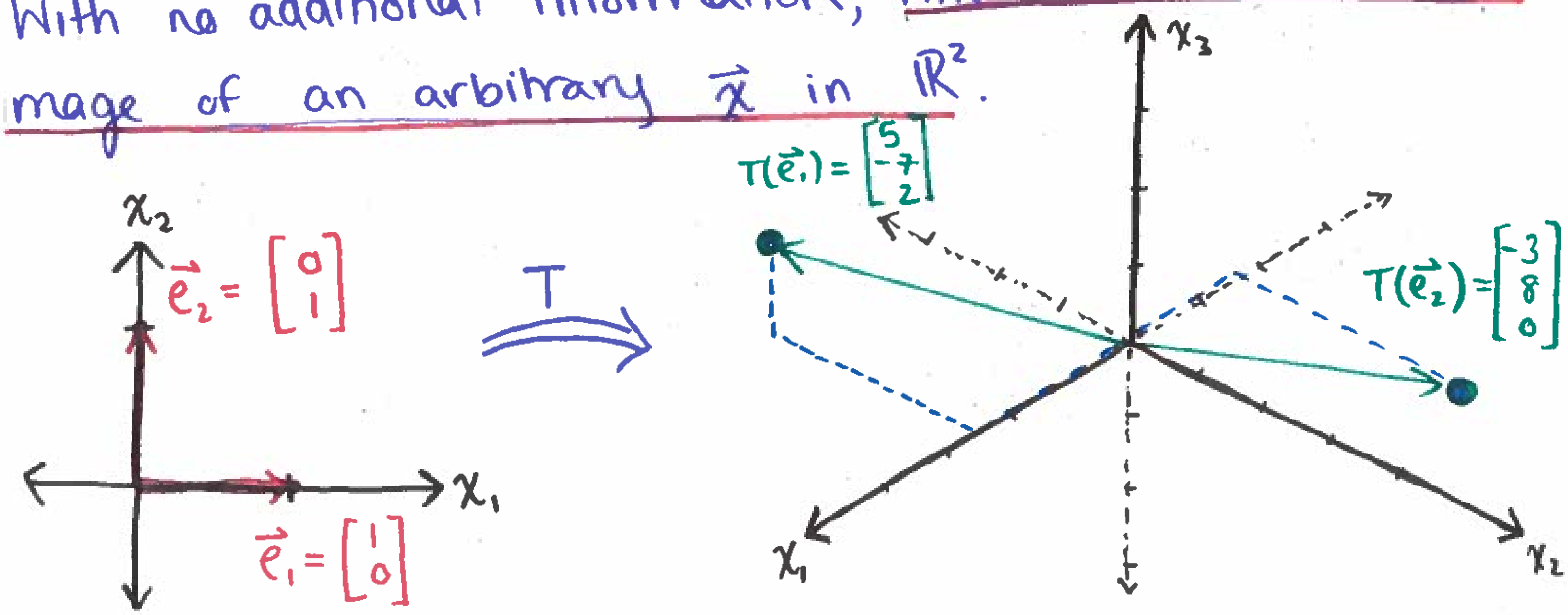


Illustration Continued...

* **Recall:** For any vector $\vec{x}$ in $\mathbb{R}^n$, the vector $T(\vec{x})$ in $\mathbb{R}^m$ is called the image of $\vec{x}$ under the transformation T.

* **Goal:**

Find a formula for the image of an arbitrary vector $\vec{x}$ in $\mathbb{R}^2$. $\Rightarrow$ IOW: Define a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$

① $\vec{x}$ can be defined as a linear combination of the column vectors of I_2:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$$

② Since T is a Linear Transformation, the image of $\vec{x}$ under the transformation T is defined by $T(\vec{x}) = A\vec{x}$:

$$T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0x_2 \end{bmatrix} \quad \boxtimes$$

* **Conclusion:** Since $T(\vec{x})$ expresses a linear combination of vectors, those vectors make up the columns of matrix A

$$\Rightarrow T(\vec{x}) = \begin{bmatrix} T(\vec{e}_1) & T(e_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

*Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation.

Then, $\exists$ a unique matrix A ST: $\boxed{T(\vec{x}) = A\vec{x} \quad \forall\, \vec{x} \in \mathbb{R}^n}$

In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(\vec{e}_j)$, where $\vec{e}_j$ is the j^{th} column of the Identity matrix in $\mathbb{R}^n$: $\boxed{A = [T(\vec{e}_1) \cdots T(\vec{e}_j) \cdots T(\vec{e}_n)]}$

Proof:

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation.

Let $\vec{x} \in \mathbb{R}^n$ be an arbitrary vector in $\mathbb{R}^n$.

Let $I_n = [\vec{e}_1 \cdots \vec{e}_j \cdots \vec{e}_n]$ be the Identity Matrix in $\mathbb{R}^n$.

*Goal: Show that $\exists$ a unique matrix A ST $T(\vec{x}) = A\vec{x}$, $\forall\, \vec{x} \in \mathbb{R}^n$

- Write $\vec{x} \in \mathbb{R}^n$ as a Linear Combo. of the column vectors of I_n:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} \iff \vec{x} = I_n \vec{x} = [\vec{e}_1 \cdots \vec{e}_j \cdots \vec{e}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$$

- Since T is a Linear Transformation:

$$\begin{aligned} T(\vec{x}) &= x_1 T(\vec{e}_1) + \cdots + x_j T(\vec{e}_j) + \cdots + x_n T(\vec{e}_n) \\ &= [T(\vec{e}_1) \cdots T(\vec{e}_j) \cdots T(\vec{e}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} \\ &= A\vec{x} \end{aligned}$$

$\therefore$ $\boxed{A = [T(\vec{e}_1) \cdots T(\vec{e}_j) \cdots T(\vec{e}_n)] \text{ is the standard matrix of } T.}$

Proof Continued... Note: We now need to verify the uniqueness of matrix A ∴

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation ST $T(\vec{x}) = B\vec{x}$ For some $m \times n$ matrix B.

* **Goal:** Show that if A is the standard matrix for T, then $A = B$.

Let A be the standard matrix for T.

- Then by definition:

$A = [T(\vec{e}_1) \cdots T(\vec{e}_j) \cdots T(\vec{e}_n)]$, where $\vec{e}_j$ in the j^{th} column of I_n

- Since $T(\vec{x}) = B\vec{x}$, then by matrix-vector multiplication:

$T(\vec{e}_j) = B\vec{e}_j = \vec{b}_j$, where: $\begin{cases} * \ \vec{e}_j \text{ is the } j^{th} \text{ col. of } I_n \\ * \ \vec{b}_j \text{ is the } j^{th} \text{ col. of } B \end{cases}$

T * But this also appears in the Standard Matrix of T defined by A!

- For BOTH statements to be true, the b_j must be the j^{th} column of A!

$\Rightarrow$ So, $A = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_j \cdots \vec{b}_n] = B$ ✓

∴ $\boxed{\text{Since } A = B, \text{ matrix } A \text{ is unique}}$ ☒

* Notes:

- Every linear transformation For $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation (& vice versa.)
- Linear Transformation → Focuses on the property of a mapping.
- Matrix Transformation → Describes how such a mapping is implemented.

Example: Assume that T is a linear transformation. Find the standard matrix of T.

$T: \mathbb{R}^2 \to \mathbb{R}^4$, $T(\vec{e}_1) = (9,1,9,1)$ & $T(\vec{e}_2) = (-2,3,0,0)$

where: $\vec{e}_1 = (1,0)$ & $\vec{e}_2 = (0,1)$

Answer:

* **Recall:** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\exists$ a unique matrix A st: $T(\vec{x}) = A\vec{x}$ $\forall$ $\vec{x}$ in $\mathbb{R}^n$.

• Since $T(\vec{x})$ expresses a linear combination of vectors, those vectors make up the columns of matrix A:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Given:

- $T(\vec{e}_1) = (9,1,9,1)$

- $T(\vec{e}_2) = (-2,3,0,0)$

Want: $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

$$\boxed{\therefore A = \begin{bmatrix} 9 & -2 \\ 1 & 3 \\ 9 & 0 \\ 1 & 0 \end{bmatrix}}$$

Answer.

Example: Assume that T is a linear transformation. Find the standard matrix of T.

$T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(\vec{e}_1) = (1,5)$, $T(\vec{e}_2) = (-3,7)$, & $T(\vec{e}_3) = (5,-9)$

where: $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the columns of the 3×3 identity matrix.

Answer:

* **Recall:** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation. Then $\exists$ a unique matrix A ST: $T(\vec{x}) = A\vec{x}$ $\forall$ $\vec{x}$ in $\mathbb{R}^n$ where $A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$ is the Standard Matrix of T

* **Given:**

- $T(\vec{e}_1) = (1,5)$
- $T(\vec{e}_2) = (-3,7)$, where $I_3 = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $T(\vec{e}_3) = (5,-9)$

* **Want:**

- $A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)] = ?$

By Definition: $\therefore \boxed{A = \begin{bmatrix} 1 & -3 & 5 \\ 5 & 7 & -9 \end{bmatrix}}$ Answer.

Example: Find the standard matrix A for the dilation transformation $T(\vec{x}) = 3\vec{x}$ for $\vec{x}$ in $\mathbb{R}^2$.

Answer:

* **Given:**
  - **Dilation Transformation:** $T(\vec{x}) = 3\vec{x}$ for $\vec{x} \in \mathbb{R}^2$

Recall: Since $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $\exists$ a unique matrix A ST: $T(\vec{x}) = A\vec{x}$ $\forall$ $\vec{x} \in \mathbb{R}^2$

where: $A = [T(\vec{e}_1) \;\; T(\vec{e}_2)]$

* **Want:** $A = [T(\vec{e}_1) \; T(\vec{e}_2)] = ?$

* **Since $T(\vec{x}) = 3\vec{x}$:**

  - $T(\vec{e}_1) = 3\vec{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

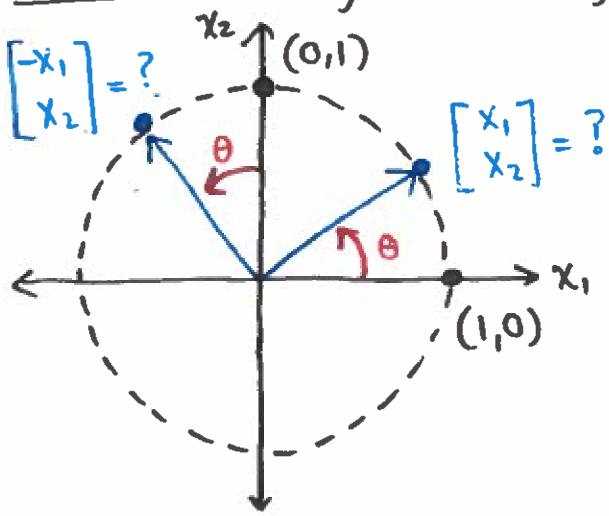
  - $T(\vec{e}_2) = 3\vec{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\Longrightarrow$ $\boxed{\therefore A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}$

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle θ, with CCW rotation for a $\oplus$ angle. Find the standard matrix A of the transformation.

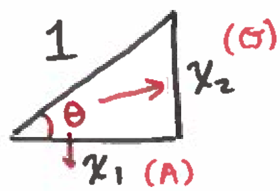
Answer:

* Lets start by sketching a graph to interpret geometrically:



Note: Here we can use right triangle trig. to determine what $\vec{e}_1$ & $\vec{e}_2$ rotate to :)

• Case 1 ($\vec{e}_1$; Quad.1):

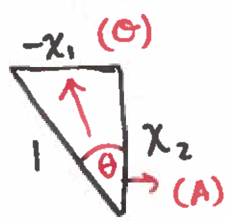


$* \cos(\theta) = \frac{x_1}{1} = x_1$

$* \sin(\theta) = \frac{x_2}{1} = x_2$

$\therefore \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates to $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$

• Case 2 ($\vec{e}_2$; Quad.2):



$* \cos(\theta) = \frac{x_2}{1} = x_2$

$* \sin(\theta) = \frac{-x_1}{1} = -x_1$

$\therefore \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates to $\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$

$$\boxed{\therefore A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}$$

Answer ✓

Example: Assume T is a Linear Transformation. Find the standard matrix of $T \Rightarrow A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$, rotates points (about the origin) through $\frac{3\pi}{2}$ radians.

Answer:

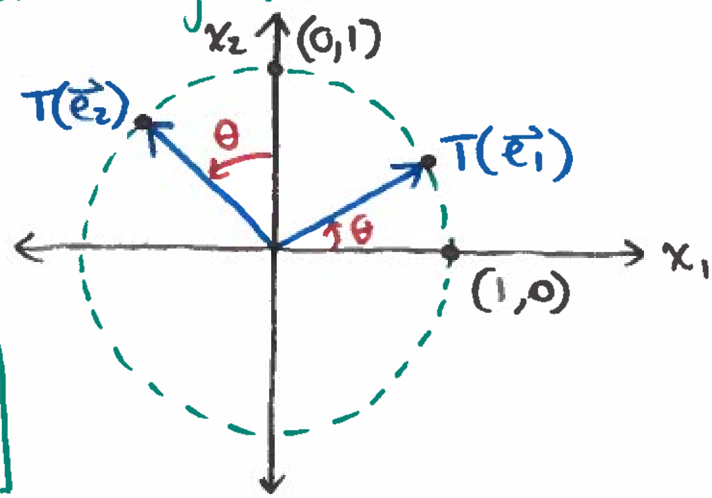
*Recall: (A rotation transformation of a pt. on the Unit Circle)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear Transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle θ (w/ CCW rotation for ⊕ angles).

• Using Right Triangle TRIG:

i) $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to T(\vec{e}_1) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$

ii) $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to T(\vec{e}_2) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$



*Given:

$\theta = \frac{3\pi}{2}$

*Want: $A = [T(\vec{e}_1) \;\; T(\vec{e}_2)] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = ?$

$\Rightarrow$ By Def: Since $\theta = \frac{3\pi}{2}$, then $A = \begin{bmatrix} \cos(\frac{3\pi}{2}) & -\sin(\frac{3\pi}{2}) \\ \sin(\frac{3\pi}{2}) & \cos(\frac{3\pi}{2}) \end{bmatrix}$

$\therefore \boxed{A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$ Ans.

Example: Fill in the missing entries of the matrix, assuming that the equation holds true $\forall$ values of the variables.

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ 3x_1 - 6x_3 \\ -2x_2 + 6x_3 \end{bmatrix}$$

Answer:

Note: The vector $\vec{b}$ on the RHS has 3 equations w/ 3 unknowns $\Rightarrow$ A is a 3×3 matrix.

*Rewrite the RHS as a vector equation:

$$\begin{bmatrix} 3x_1 - 2x_2 \\ 3x_1 \quad - 6x_3 \\ -2x_2 + 6x_3 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 0 \\ 3 & 0 & -6 \\ 0 & -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\vec{x} \checkmark$$

Answer:

$$\boxed{\therefore A = \begin{bmatrix} 3 & -2 & 0 \\ 3 & 0 & -6 \\ 0 & -2 & 6 \end{bmatrix}}$$

Example: Show that T is a Linear Transformation by finding a matrix that implements the mapping.
Note that x_1, x_2, x_3, x_4 are NOT vectors, but entries in vectors:

$$T(x_1, x_2, x_3, x_4) = (x_1 + 8x_2, 0, 6x_2 + x_4, x_2 - x_4)$$

Answer:

* Recall: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a Linear Transformation, then $\exists$ a unique matrix A ST: $T(\vec{x}) = A\vec{x} \quad \forall \; \vec{x} \in \mathbb{R}^n$.

$$\Rightarrow A = [T(\vec{e}_1) \cdots T(\vec{e}_2)]$$

* Show/Goal: A unique matrix "A" exists

\$ T is a Linear Transformation ....

Since T is linear: $T(\vec{x}) = x_1 T(\vec{e}_1) + \cdots x_4 T(\vec{e}_4)$

$$\Rightarrow \begin{bmatrix} x_1 + 8x_2 \\ 0 \\ 6x_2 + x_4 \\ x_2 - x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 0 \\ 6 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A\vec{x} \quad \checkmark$$

woohoo! Unique matrix $\exists$

$$\therefore \boxed{A = \begin{bmatrix} 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}}$$

Answer.

Example: Show that T is a linear transformation by finding a matrix that implements the mapping. Note that x_1, x_2, x_3 are NOT vectors, but entries in vectors:

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + 9x_3 \ , \ x_2 - 8x_3)$$

Answer:

*Recall: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a Linear Transformation, then $\exists$ a unique matrix A st: $T(\vec{x}) = A\vec{x} = [T(\vec{e}_1) \cdots T(\vec{e}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \ \forall \vec{x} \in \mathbb{R}^n$

Suppose T is a Linear Transformation $\Rightarrow$ (Find matrix A.)

Since T is Linear, $T(\vec{x}) = x_1 T(\vec{e}_1) + \cdots + x_3 T(\vec{e}_3)$

$$\Rightarrow \begin{bmatrix} x_1 - 2x_2 + 9x_3 \\ x_2 - 8x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 9 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 9 \\ 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underset{\uparrow}{A}\vec{x} \quad \checkmark$$

unique matrix $\exists$

Ansr

$$\boxed{\therefore A = \begin{bmatrix} 1 & -2 & 9 \\ 0 & 1 & -8 \end{bmatrix}}$$

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear Transformation ST $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 6x_2)$. Find $\vec{x}$ ST $T(\vec{x}) = (3, 2)$.

Answer:

$\boxed{\text{* Note: Since } T \text{ is Linear, then } \exists \text{ a unique matrix } A \text{ ST } T(\vec{x}) = A\vec{x} = [T(\vec{e}_1) \;\; T(\vec{e}_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\; \forall \; \vec{x} \text{ in } \mathbb{R}^2}$

Since T is a Linear Transformation:

$$T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) = [T(\vec{e}_1) \;\; T(\vec{e}_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\vec{b}} = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 6x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

Solve $A\vec{x} = \vec{b}$ by row-reducing the equiv. augmented matrix to its row-reduced echelon form:

$$[A \,|\, \vec{b}] = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 3 & 1 \end{array}\right]$$

$$\bullet \begin{array}{r} -2R_1 \\ +\;\; R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -5 \end{array}\right]$$

$$\bullet \begin{array}{r} -R_2 \\ +\;\; R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -5 \end{array}\right] \Longrightarrow$$

Answer

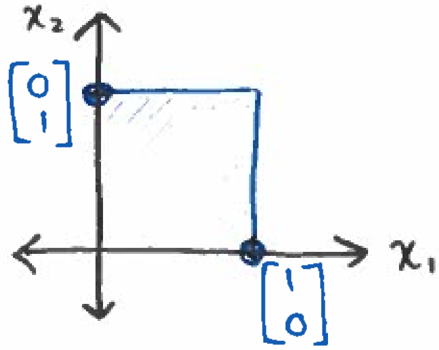
$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}}$$

* Geometric Linear Transformations of $\mathbb{R}^2$ *

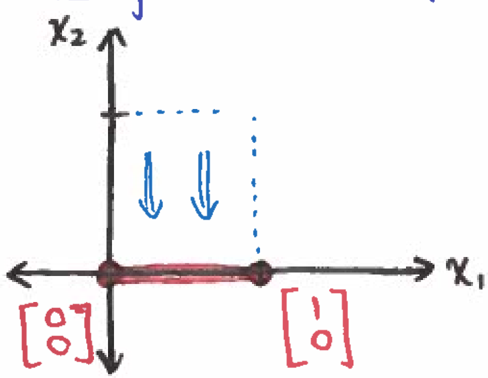
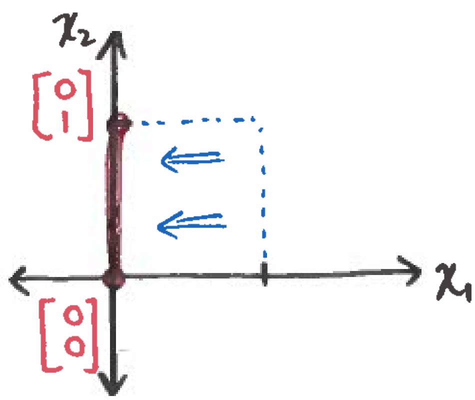
Since transformations are linear, they are determined completely by what they do to the columns of I_2.

Note: Instead of showing only the images of $\vec{e}_1$ & $\vec{e}_2$, the following shows what a transformation does to the unit square (I_2):

- The Unit Square: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



- **Projections: (2)**

Transformation:	Image of the Unit Square:	Standard Matrix:
① Projection onto the x_1-axis:		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
② Projection onto the x_2-axis:		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

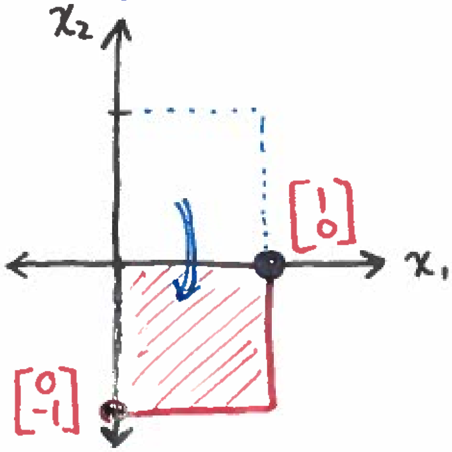
$$\begin{cases} * \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_1 \\ * \vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{0} \end{cases}$$

$$\begin{cases} * \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{0} \\ * \vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{e}_2 \end{cases}$$

Reflections: (5)

Transformation: | **Image of the Unit Square:** | **Standard Matrix:**

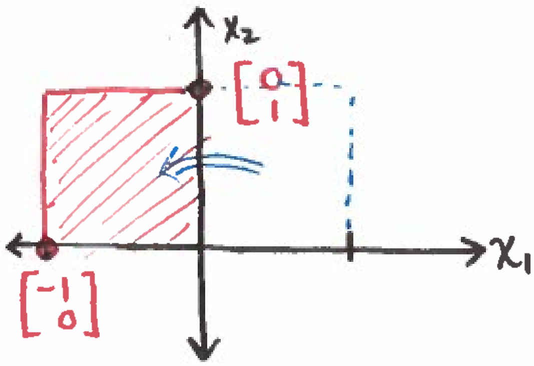
① Reflection through the x_1-axis:



$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

$$\begin{cases} * \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_1 \\ * \vec{e}_2 \longrightarrow T(\vec{e}_2) = -\vec{e}_2 \end{cases}$$

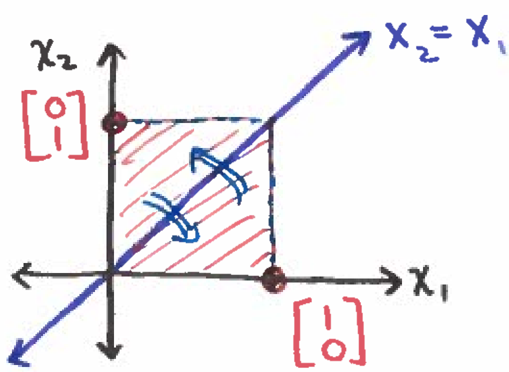
② Reflection through the x_2-axis:



$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{cases} * \vec{e}_1 \longrightarrow T(\vec{e}_1) = -\vec{e}_1 \\ * \vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{e}_2 \end{cases}$$

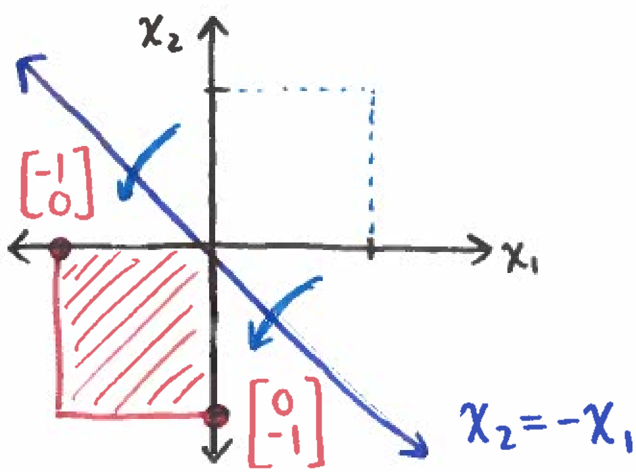
③ Reflection through the Line $x_2 = x_1$:



$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$\begin{cases} * \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_2 \\ * \vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{e}_1 \end{cases}$$

④ Reflection through the Line $x_2 = -x_1$:



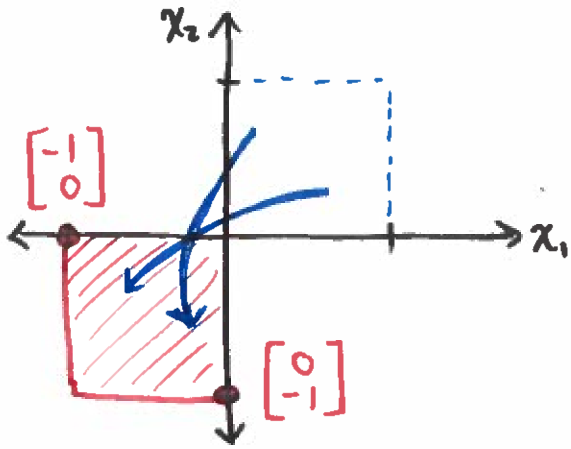
$$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

$$\begin{cases} * \vec{e}_1 \longrightarrow T(\vec{e}_1) = -\vec{e}_2 \\ * \vec{e}_2 \longrightarrow T(\vec{e}_2) = -\vec{e}_1 \end{cases}$$

Reflections Continued...

Transformation: | **Image of the Unit Square:** | **Standard Matrix:**

⑤ Reflection through the origin:



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{cases} * \vec{e}_1 \rightarrow T(\vec{e}_1) = -\vec{e}_1 \\ * \vec{e}_2 \rightarrow T(\vec{e}_2) = -\vec{e}_2 \end{cases}$$

• Contractions & Expansions: (4)

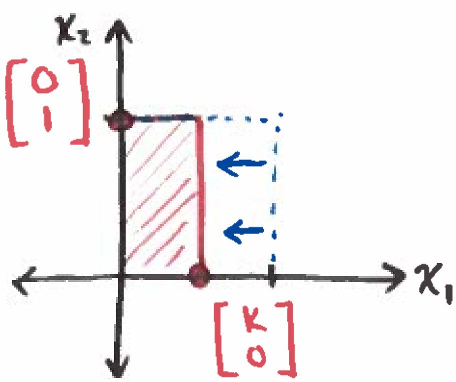
*2 types

Transformation: | **Image of the Unit Square:** | **Standard Matrix** *(same for 1a/1b)

$$\begin{cases} * \vec{e}_1 \rightarrow T(\vec{e}_1) = K\vec{e}_1 \\ * \vec{e}_2 \rightarrow T(\vec{e}_2) = \vec{e}_2 \end{cases}$$

(1a) Horizontal Contraction:

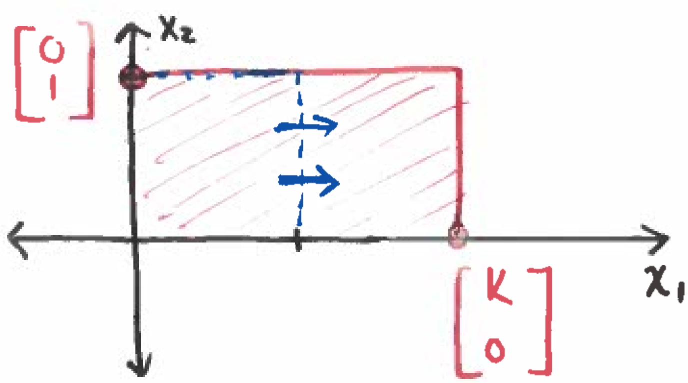
$(0 < K < 1)$



$$\begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$$

(1b) Horizontal Expansion:

$(K > 1)$

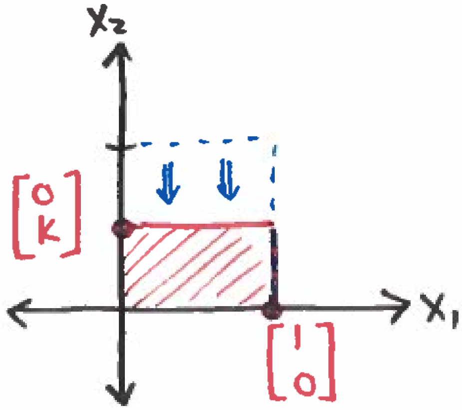


Contractions & Expansions Continued...

Transformation: | **Image of the Unit Square:** | **Standard Matrix:**

*(Same for 2a/2b)

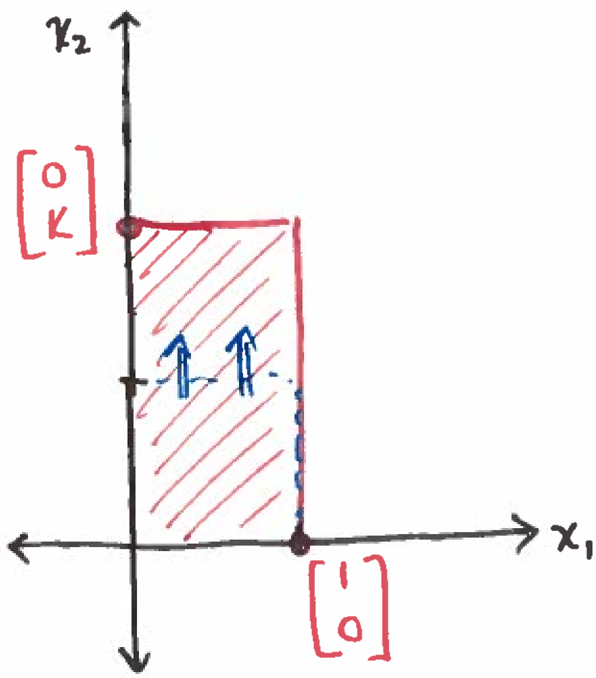
(2a) Vertical Contraction:
$(0 < k < 1)$



$$\begin{cases} *\vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_1 \\ *\vec{e}_2 \longrightarrow T(\vec{e}_2) = k\vec{e}_2 \end{cases}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

(2b) Vertical Expansion:
$(k > 1)$



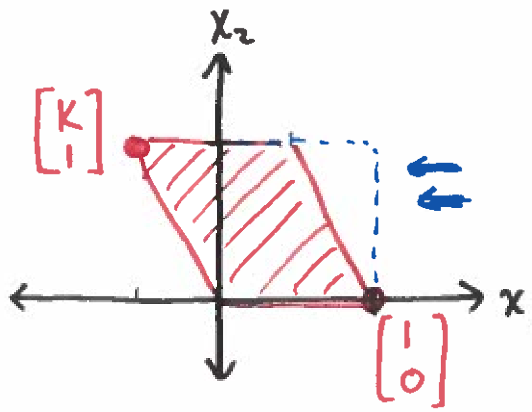
• Shears: (4)

2 Types

Transformation:	Image of the Unit Square:	Standard Matrix:

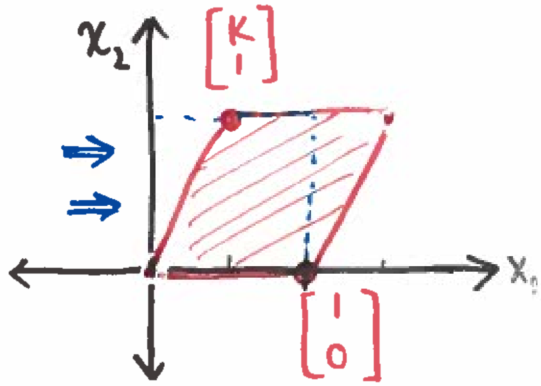
① Horizontal Shear

(i) IF $K<0$:



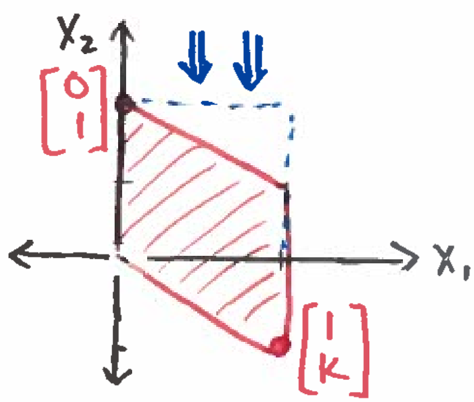
$$\begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}$$

(ii) IF $K>0$:



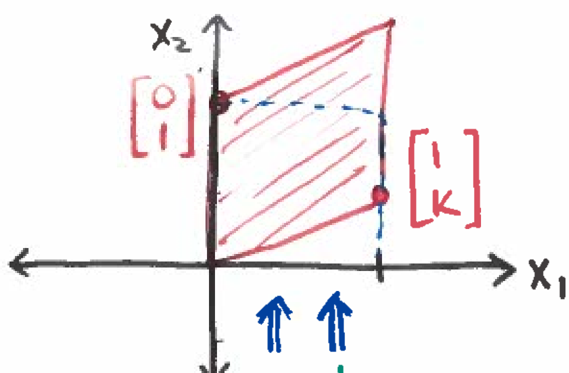
② Vertical Shear

(i) IF $K<0$:



$$\begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}$$

(ii) IF $K>0$:



Note: This is NOT an exclusive list! We can create new transformations by applying one transformation after another

Example: Assume that T is a linear transformation. Find the standard matrix of T : $A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$, first perform a horizontal shear that transforms $\vec{e}_2$ into $\vec{e}_2 + 18\vec{e}_1$ (leaving $\vec{e}_1$ unchanged) & then reflects points through the line $x_2 = -x_1$.

Answer:

Note: Here we construct a NEW transformation by applying one transformation after another :)

(i) Horizontal Shear
(ii) Reflection across the line $x_2 = -x_1$

*Here we perform the computations; Geometric Interpretation on the next page.

① Perform the Horizontal Shear:

- $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow T(\vec{e}_1) = \vec{e}_1$
- $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow T(\vec{e}_2) = \vec{e}_2 + 18\vec{e}_1$

② Reflect points through the line $x_2 = -x_1$:

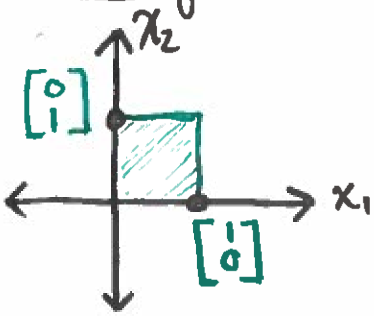
- $\vec{e}_1 \longrightarrow T(\vec{e}_1) = -\vec{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
- $\vec{e}_2 + 18\vec{e}_1 \longrightarrow T(\vec{e}_2 + 18\vec{e}_1) = -\vec{e}_1 + 18(-\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -18 \end{bmatrix} = \begin{bmatrix} -1 \\ -18 \end{bmatrix}$

$$\therefore \boxed{A = [T(\vec{e}_1) \quad T(\vec{e}_2 + 18\vec{e}_1)] = \begin{bmatrix} 0 & -1 \\ -1 & -18 \end{bmatrix}}$$ Ans.

Example Continued...

Note: Lets consider the geometric interpretations of this (double) transformation :)

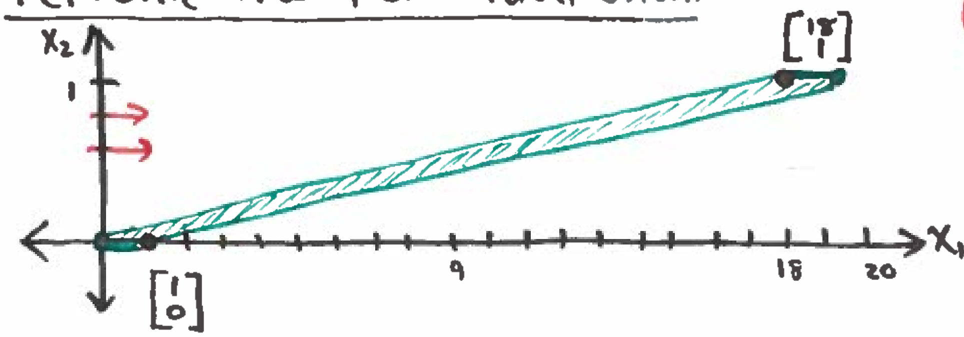
*** Starting with the Unit Square:**



$$I_2 = [\vec{e}_1 \ \ \vec{e}_2] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

① **Perform the Horizontal Shear:**

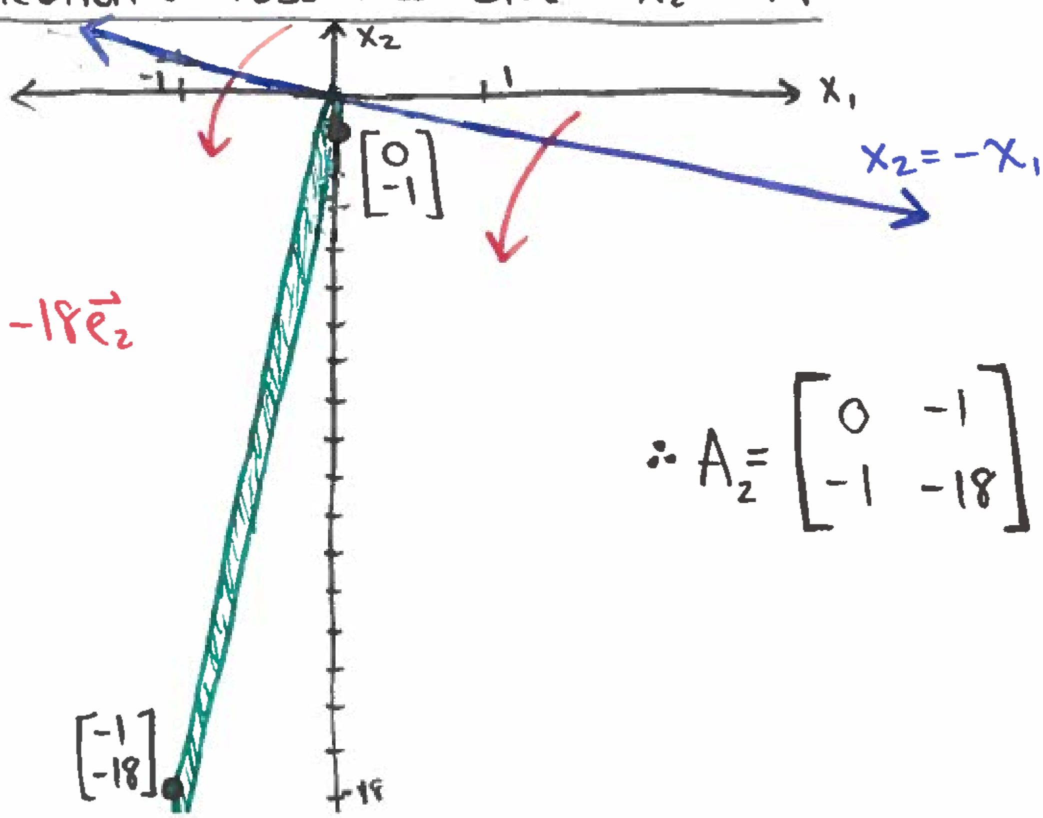
$$\begin{cases} * \ \vec{e}_1 \xrightarrow{T} \vec{e}_1 \\ * \ \vec{e}_2 \xrightarrow{T} \vec{e}_2 + 18\vec{e}_1 \end{cases}$$



$$A_1 = \begin{bmatrix}1 & 18\\0 & 1\end{bmatrix}$$

* $K = 18 > 0$ ($\vec{e}_2$ shifting Right ; add "$18\vec{e}_1$" :))

② **Perform the Reflection Across the Line $x_2 = -x_1$:**



$$\begin{cases} * \ \vec{e}_1 \xrightarrow{T} -\vec{e}_2 \\ * \ \vec{e}_2 + 18\vec{e}_1 \xrightarrow{T} -\vec{e}_1 - 18\vec{e}_2 \end{cases}$$

$$\therefore A_2 = \begin{bmatrix}0 & -1\\-1 & -18\end{bmatrix}$$

Example: Show that the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the horizontal x_1-axis & then reflects points through the line $x_2 = x_1$ is merely a rotation about the origin. What is the angle of rotation?

Answer:

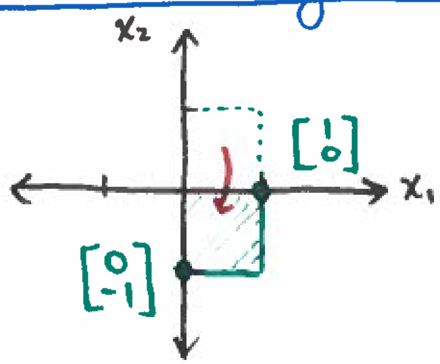
*Note: Here we construct a NEW transformation by considering one linear transformation after another.

① Reflect points through x_1-axis

② Reflect points through the line $x_2 = x_1$

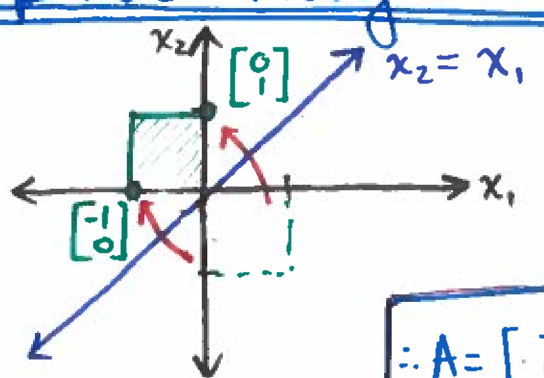
*Since $T: \mathbb{R}^2 \to \mathbb{R}^2$; Start w/ $I_2 = [\vec{e}_1 \;\; \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1) Reflect points through the x_1-axis:



$$\begin{cases} * \; \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_1 \\ * \; \vec{e}_2 \longrightarrow T(\vec{e}_2) = -\vec{e}_2 \end{cases}$$

2) Reflect points through the line $x_2 = x_1$:



$$\begin{cases} * \; \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_2 \\ * \; \vec{e}_2 \longrightarrow T(-\vec{e}_2) = -\vec{e}_1 \end{cases}$$

$$\boxed{\therefore A = [\vec{e}_2 \;\; -\vec{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

Example Continued...

* Verify that this transformation is merely a rotation about the origin & determine the angle θ:

Recall (Rotation Matrix):

$T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

* Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle $\theta = \frac{\pi}{2}$ $\Rightarrow$ $\boxed{A = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$

Answer ✓

$\therefore$ Angle of Rotation (in $\oplus$ direction) is:

$$\boxed{\theta = \pi/2}$$

Definition: (Onto Mappings)

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$

* Equivalently: T is onto $\mathbb{R}^m$ when the range of T is all of the codomain of $\mathbb{R}^m$

IOW: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if:

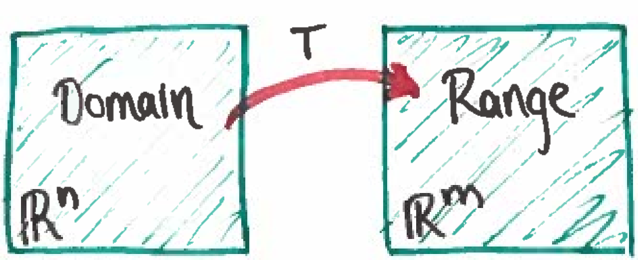
$\forall\ \vec{b}$ in the codomain $\mathbb{R}^m$, $\exists$ @ least one solution of $T(\vec{x}) = \vec{b}$

* Existence Questions:

- Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?
- Is the Range of T all of $\mathbb{R}^m$?

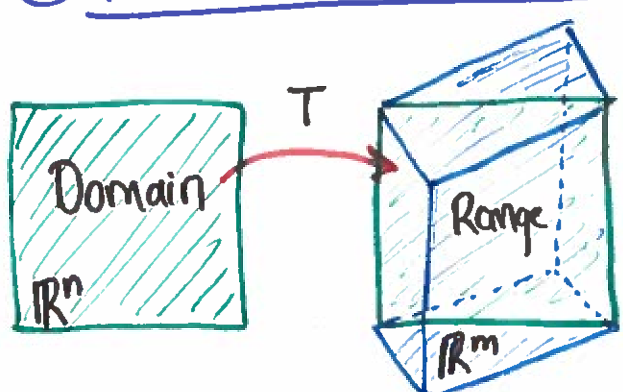
* Graphically:

① T is Onto $\mathbb{R}^m$:



* Range is equal to the Codomain.

② T is NOT onto $\mathbb{R}^m$:



* The mapping T is NOT onto when $\exists\ \vec{b}$ in $\mathbb{R}^m$ for which $T(\vec{x}) = \vec{b}$ has NO solution.

Definition: (One-to-One Mappings)

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be **1-1** if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$.

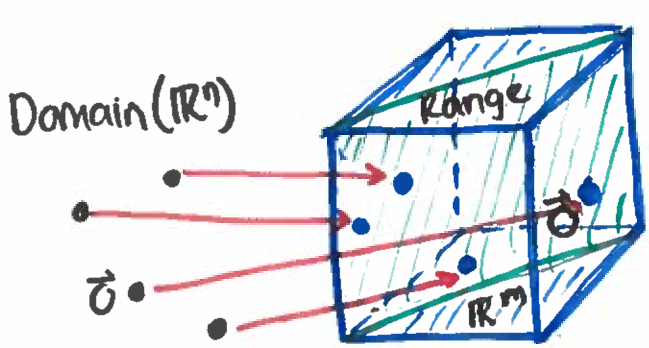
* Equivalently: T is 1-1 if $\forall \vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has either one unique solution or none at all.

* Uniqueness Questions:

- Is T 1-1?
- Is every $\vec{b}$ the image of @ most one vector?

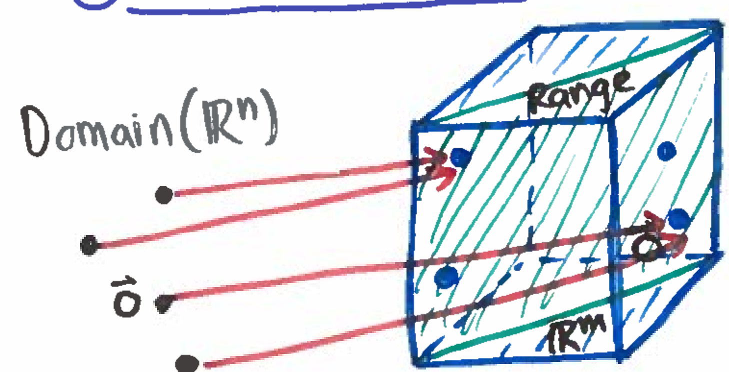
* Graphically:

① T is 1-1:



* $\forall \vec{b}$ in $\mathbb{R}^m$, $T(\vec{x}) = \vec{b}$ has one unique solution (or no solution).

② T is NOT 1-1:



* The mapping T is NOT 1-1 when some $\vec{b}$ in $\mathbb{R}^m$ is the image of more than one vector in $\mathbb{R}^n$

Example: Let T be the linear transformation whose standard matrix is:

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a 1-1 mapping?

Answer:

Note: Here we are given a 3×4 matrix in echelon form $\Rightarrow$ pivots in columns: 1, 2, & 4

- **Basic Variables:** $x_1, x_2,$ & x_4
- **Free Variable:** x_3

* **Does T map $\mathbb{R}^4$ ONTO $\mathbb{R}^3$?**

Check: $\forall \vec{b}$ in $\mathbb{R}^3$, does $\exists$ @ least one solution of $T(\vec{x}) = \vec{b}$?

Recall: $A\vec{x} = \vec{b}$ has at least one solution IFF the equation has at least one free variable.

∴ Since matrix A has a free variable:

$\forall \vec{b}$ in $\mathbb{R}^3$, $\exists$ at least one solution of $T(\vec{x}) = \vec{b}$ ✓

$\Rightarrow$ Yes, T maps $\mathbb{R}^4$ onto $\mathbb{R}^3$.

* **Is T a 1-1 mapping?**

Check: $\forall \vec{b}$ in $\mathbb{R}^3$, does the equation $T(\vec{x}) = \vec{b}$ have one, unique solution? (* Do not need to consider No sol. here)

Recall: IF # of unknowns > # of eq./vectors $\Rightarrow$ columns of A are linearly dependent.

∴ Since 4 unknowns > 3 equations, each $\vec{b}$ is the image of more than one $\vec{x}$ $\rightarrow$ ∴ T is NOT 1-1

<u>Note</u>: The following two theorems can be observed from the conclusions of the last example :

*<u>Theorem</u>:

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then, T is 1-1 IFF the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution.

*<u>Theorem</u>:

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Let A be the standard matrix for T.

Then:

(i) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ IFF the columns of A span $\mathbb{R}^m$.

*<u>Recall</u>: This is equivalent to the following 3 statements :

i) $\forall \vec{b}$ in $\mathbb{R}^m$, the eq. $A\vec{x} = \vec{b}$ has a solution

ii) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combo. of the columns of A

iii) A has a pivot in every row.

(ii) T is 1-1 IFF the columns of A are linearly independent.

*<u>Recall</u>: The columns of A are linearly independent IFF $A\vec{x} = \vec{0}$ has ONLY the trivial solution :

Example: Describe all possible echelon forms of the standard matrix for a linear transformation T where, $T: \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

Answer:

* **Recall:** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation.
Let $A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$ be the Standard Matrix of T.
Then: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ IFF the columns of A span $\mathbb{R}^m$

"*The Columns of A span $\mathbb{R}^m$" is logically equivalent to:
(i) $\forall$ $\vec{b}$ in $\mathbb{R}^m$, the matrix eq. $A\vec{x} = \vec{b}$ has a solution
(ii) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A
(iii) A has a pivot position in each row

Notes:
- Leading entries denoted "♥" may have ANY nonzero value.
- Entries denoted "*" may have ANY value (includes 0).

* **Find matrix A's:** Since $T: \mathbb{R}^4 \to \mathbb{R}^3$, A is a 3×4 matrix

4 Cases:

① $$A = \begin{bmatrix} ♥ & * & * & * \\ 0 & ♥ & * & * \\ 0 & 0 & ♥ & * \end{bmatrix}$$

* Pivot in every row
* x_4 is free ($A\vec{x} = \vec{b}$ consistent)

② $$A = \begin{bmatrix} ♥ & * & * & * \\ 0 & ♥ & * & * \\ 0 & 0 & 0 & ♥ \end{bmatrix}$$

* Pivot in every row
* x_3 is free ($A\vec{x} = \vec{b}$ consistent)

Example Continued...

③ $A = \begin{bmatrix} \heartsuit & * & * & * \\ 0 & 0 & \heartsuit & * \\ 0 & 0 & 0 & \heartsuit \end{bmatrix}$

* Pivot in every row
* x_2 is free ($A\vec{x} = \vec{b}$ consistent)

④ $A = \begin{bmatrix} 0 & \heartsuit & * & * \\ 0 & 0 & \heartsuit & * \\ 0 & 0 & 0 & \heartsuit \end{bmatrix}$

* Pivot in every row
* x_1 is free ($A\vec{x} = \vec{b}$ consistent)

Note: A matrix A w/ a row of zeros is NOT a valid solution here

$\Rightarrow$ The Equation $A\vec{x} = \vec{b}$ is not consistent for every $\vec{b} \in \mathbb{R}^m$ as some choices of $\vec{b}$ may be nonzero.

Example: Determine if the specified linear transformation is

(a) One-to-One (1-1)

(b) Onto

} Justify your answers.

$$T(x_1, x_2, x_3) = (x_1 - 4x_2 + 6x_3 \; , \; x_2 - 8x_3)$$

Answer:

* **Recall:** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation

(i) Then T is 1-1 IFF the eq. $T(\vec{x}) = \vec{0}$ has only the Trivial Sol. (1

Now Let $A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$ be the Standard Matrix of T.

(ii) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ IFF the columns of A span $\mathbb{R}^m$ (1.4

(iii) T is 1-1 IFF the columns of A are Linearly Independent (1.

Since T is a Linear Transformation

$$T(\vec{x}) = \begin{bmatrix} x_1 - 4x_2 + 6x_3 \\ x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 \\ 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, $A = \begin{bmatrix} 1 & -4 & 6 \\ 0 & 1 & -8 \end{bmatrix}$ is a 2×3 matrix (in echelon form)

$\therefore$ Linear Transformation is **NOT 1-1**: (1.7 Theorem 8) "1st special th"

Since # of unknowns > # of equations, the columns of A are Linearly Dependent.

$\therefore$ Linear Transformation **IS Onto**: (1.4 & 1.5)

Since a pivot position $\exists$ in every row of A, the columns of A span $\mathbb{R}^2$. *Also: $A\vec{x} = \vec{0}$ has a FREE VARIABLE, so ~~or~~ the system is consistent ☺