Divergence-Free Vector Fields

Finding their Vector Potentials and Determining their Flux via 1-D Integrals

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Vector fields that are curl-free ($\vec{\nabla} \times \vec{F} = \vec{0}$) are conservative and can be represented as the gradient of a scalar function, $\emptyset$, which we name the field's scalar potential function (note that $\emptyset$ can incorporate an arbitrary constant since $\vec{\nabla}(\emptyset + C) = \vec{\nabla}\emptyset$) . Knowing the scalar potential function for a conservative field permits one to calculate the line integral $\int_A^B \vec{F} \cdot \vec{n}\, ds$ between the points A and B without knowing the specific path taken. The value of the integral depends only on the values of the scalar potential at the end points; specifically, $\int_A^B \vec{F} \cdot \vec{n}\, ds = \emptyset(B) - \emptyset(A)$. A procedure is available to determine the function $\emptyset$ that involves integration of the components of $\vec{F}$ (see a standard text on vector calculus).

An analogous situation exists for vector fields that are divergence-free ($\vec{\nabla} \cdot \vec{F} = 0$). In this case the vector field can be written as the curl of a vector potential, $\vec{G}$, i.e., $\vec{F} = \vec{\nabla} \times \vec{G}$. Vector fields that are divergence-free are called solenoidal fields. Note that the vector potential $\vec{G}$ can incorporate an arbitrary conservative field, $\vec{\nabla}\emptyset$, since $\vec{\nabla} \times \left(\vec{G} + \vec{\nabla}\emptyset\right) = \vec{\nabla} \times \vec{G}$. Most undergraduate vector calculus textbooks do not discuss solenoidal fields or provide a procedure for determining their vector potential $\vec{G}$.

One reason for representing a divergence-free field as the curl of a vector potential field is that the flux of the divergence-free field through a designated surface (a 2D integral) can be written as 1D integral using the vector potential, $\vec{G}$. This is accomplished using Stokes' theorem as follows:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \left(\vec{\nabla} \times \vec{G}\right) \cdot \vec{n}\, dS = \int_C \vec{G} \cdot d\vec{r}$$

Given a complex surface S and/or a complex solenoidal field, $\vec{F}$, computing the flux through S can be very difficult forcing one to employ a 2D numeric integration procedure. Even if the curve integral on the right is very difficult to perform analytically, evaluating the 1D integral based on numeric integration procedure (e.g., Simpson's rule) is very simple.

To illustrate the procedure for determining the vector potential, $\vec{G}$, we will consider the solenoidal vector field, $\vec{F} = < -xy,\ \frac{1}{2}y^2 + 2z,\ x^2y >$ and consider calculating the flux of $\vec{F}$ through that portion of the surface $z = x^2 + y^3$ that is above the disk $x^2 + y^2 \le 9$.

For the vector potential field we will use $\vec{G} = \overrightarrow{G'} + \vec{\nabla}\emptyset$ where $G' = < g_1', \ g_2', \ g_3' >$ and we chose $\emptyset_z = -g_3'$. Therefore, $\vec{G} = < g_1' + \emptyset_x, \ g_2' + \emptyset_y, \ 0 > = < g_1, \ g_2, \ 0 >$. Note that we could have chosen $\emptyset$ to zero-out any of the three components of $\vec{G}$. $\vec{\nabla}\emptyset$ can be considered as a "constant of integration" for this "vector anti-differentiation" process.

Setting $\vec{F} = \vec{\nabla} \times \vec{G} = < -\frac{\partial g_2}{\partial z}, \ \frac{\partial g_1}{\partial z}, \ \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} >$ we obtain:

$$\frac{\partial g_2}{\partial z} = xy \qquad \text{or} \qquad g_2 = xyz + C_2(x,y) \quad \text{and}$$

$$\frac{\partial g_1}{\partial z} = \frac{1}{2}y^2 + 2z \qquad \text{or} \qquad g_1 = \frac{1}{2}y^2 z + z^2 + C_1(x,y)$$

Using these expressions for g_1 and g_2 to compute the z-component of $\vec{F}$ we obtain:

$$\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} = yz + \frac{\partial C_2}{\partial x} - yz - \frac{\partial C_1}{\partial x} \quad = \quad x^2 y$$

Therefore, $\frac{\partial C_2}{\partial x} - \frac{\partial C_1}{\partial x} = x^2 y$. Choosing to set $C_1 = 0$, we can solve for C_2 obtaining $C_2 = \frac{1}{3}x^3 y$. Combining these results, we obtain a vector potential:

$$\vec{G} = < g_1, g_2, 0 > \ = \ < \frac{1}{2}y^2 z + z^2, \ xyz + \frac{1}{3}x^3 y, \ 0 >$$

As noted previously, any conservative vector field could be added to $\vec{G}$ and $\vec{G}$ would continue to be a vector potential for the solenoidal field $\vec{F}$.

The flux of $\vec{F}$ through the surface S can now be computed using $\vec{G}$ and a 1D integral via:

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S \left(\vec{\nabla} \times \vec{G}\right) \cdot \vec{n} \, dS = \int_C \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \vec{G}(t) \cdot \vec{r}'(t) dt$$

where $\vec{r}(t) = < x(t), \ y(t), \ z(t) > = \ < 3\cos(t), \ 3sin(t), \ 9cos^2(t) + 27sin^3(t) >$ is a parameterization of the boundary curve C. In this case, if the integrand is expanded, the solution to this problem can be determined analytically. Given a more complex vector field, $\vec{F}$, and/or a more complex surface, S, this may not be possible. Yet, a very accurate value for the flux could still be easily obtained using numeric integration methods.