ANALYSIS OF ALGORITHMS – COMP 4040

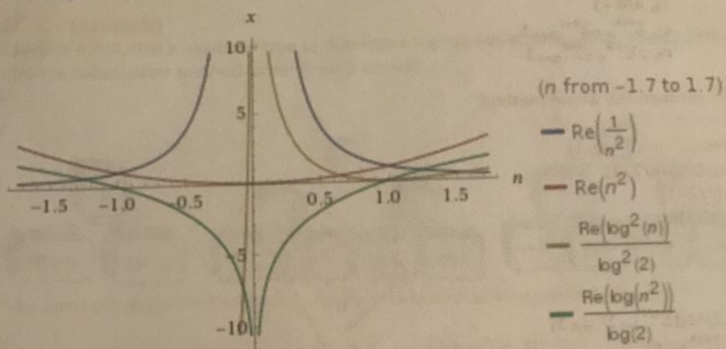
ASSIGNMENT-2 SOLUTIONS

1. Credits: Taylor M. Langlois

List the 4 functions below in non-decreasing asymptotic order of growth.

$(\log n)^2$ $\quad n^{-2} \quad \log_2(2^{\log(n^{\wedge}2)}) \quad n^2$

Justify your answer mathematically by showing values of c and n_0 for each pair of functions that are adja[cent in the] ordering.



(Graph courtesy of Wolfram Alpha)

Non-decreasing asymptotic order of growth: 1. n^{-2}, 2. $\log_2(2^{\log(n^{\wedge}2)})$, 3. $(\log n)^2$, 4. n^2

To prove this, we can take the first 2 values (1 and 2) we've ordered and plug them in c=1/2:

$n^{-2} \le c \log_2(2^{\log(n^{\wedge}2)})$

$1/n^2 \le c \log_2 n^2$

$1/n_0^2 = 2c \log_2 n_0$

$1/n_0^2 = \log_2 n_0$

We get $\sqrt{2}^{-2} = ½ = \log_2 \sqrt{2}$.

So, c= ½, $n_0 = \sqrt{2}$.

Now working with the second pair of values (2 and 3) we can plug in c=2:

$2 \log_2 n \le c (\log_2 n)^2$

$2 \le \log_2 n$

$1 = \log_2 n_0$

$2^1 = 2^{\log\_2 n\_0} = n_0$

Here we get 2 through division of $\log_2 n$.

So, c=2, n_0 = 2.

Finally, we get to the last pairs (3 and 4) and can plug in c=4:

$(\log_2 n)^2 \le cn^2$

$(\log_2 n_0)^2 = 4n_0^2$

Here we get ½ since $(\log_2 (1/2))^2 = 1 = 4(1/2)^2$.

So, c=4, n_0 = ½.

2. Credits: Ryan Cauble

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2. Pseudocode Analysis (25 points)
For the pseudocode below for procedure Mystery(n), derive tight upper and lower bounds on its asymptotic ***worst-case*** running time $f(n)$. That is, for the set of inputs including those that force Mystery to work its hardest, find $g(n)$ such that $f(n) \in \theta(g(n))$. Assume that the input n is a positive integer. Justify your answer.

Mystery (n)	
1. if n is an even number	$C_1, 1$
2. for $i = 1$ to n	$C_2, n+1$
3. for $j = n$ down to $n/2$	$C_3, \Sigma_{i=1}^{n} \Sigma_{j=\frac{n}{2}}^{n+1} 1$
4. print "even number"	$C_4, \Sigma_{i=1}^{n} \Sigma_{j=\frac{n}{2}}^{n+1} 1$
5. else	$C_5, 1$
6. for $k = 1$ to $n/4$	$C_6, \frac{n}{4} + 1$
7. for $m = 1$ to n	$C_7, \Sigma_{k=1}^{\frac{n}{4}} \Sigma_{m=1}^{n+1} 1$
8. print "odd number"	$C_7, \Sigma_{k=1}^{\frac{n}{4}} \Sigma_{m=1}^{n} 1$

When you sum up the even part lines 1-4 you get:
$T(n) = an^2 + bn + c \in \Theta(n^2)$

When you sum up the odd part lines 5-8 you get:
$T(n) = an^2 + bn + c \in \Theta(n^2)$

So the Worst case is $\Theta(n^2)$

3. Credits: Victoria Albanese

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Homework 2

3. **True or False**: 25 points

25

a. $n\lg^2 n \in O(n^2)$
b. $n\lg^2 n \in \Omega(n^{1.05})$
c. $n^3 \in o(n^3)$
d. The cost of the loop to the right is $O(n)$
e. The cost of the loop to the right is $\Omega(\lg n)$

```
for (i = 1; i <= n; i*=2) // n ≥ 1
{
        constant work;
}
```

a) $n\lg^2 n = (n)(\lg n)(\lg n) \approx (n^1)(n^{0.000\ldots1})(n^{0.000\ldots1}) \approx n^{1.000\ldots2} \in O(n^2)$

→ The power $1.000\ldots2 < 2$, so n^2 is a valid asymptotic upper bound

→ **TRUE**

b) $n\lg^2 n = (n)(\lg n)(\lg n) \approx (n^1)(n^{0.000\ldots1})(n^{0.000\ldots1}) \approx n^{1.000\ldots2} \notin \Omega(n^2)$

→ The power $1.000\ldots2 < 2$, so n^2 is NOT a valid asymptotic lower bound

→ **FALSE**

c) $n^3 \notin o(n^3)$

→ The o-notation requires the functions to be loosely upper bounded

→ This function would be tightly bound, which is not allowed

→ **FALSE**

d) loop cost = $\log n = n^{0.000\ldots1} \in O(n)$

→ The power $0.000\ldots1 < 1$, so n is a valid asymptotic upper bound

→ **TRUE**

e) loop cost = $\lg n \in \Omega(\lg n)$

→ The Ω allows for tight bounds, so this is a valid asymptotic lower bound

→ **TRUE**

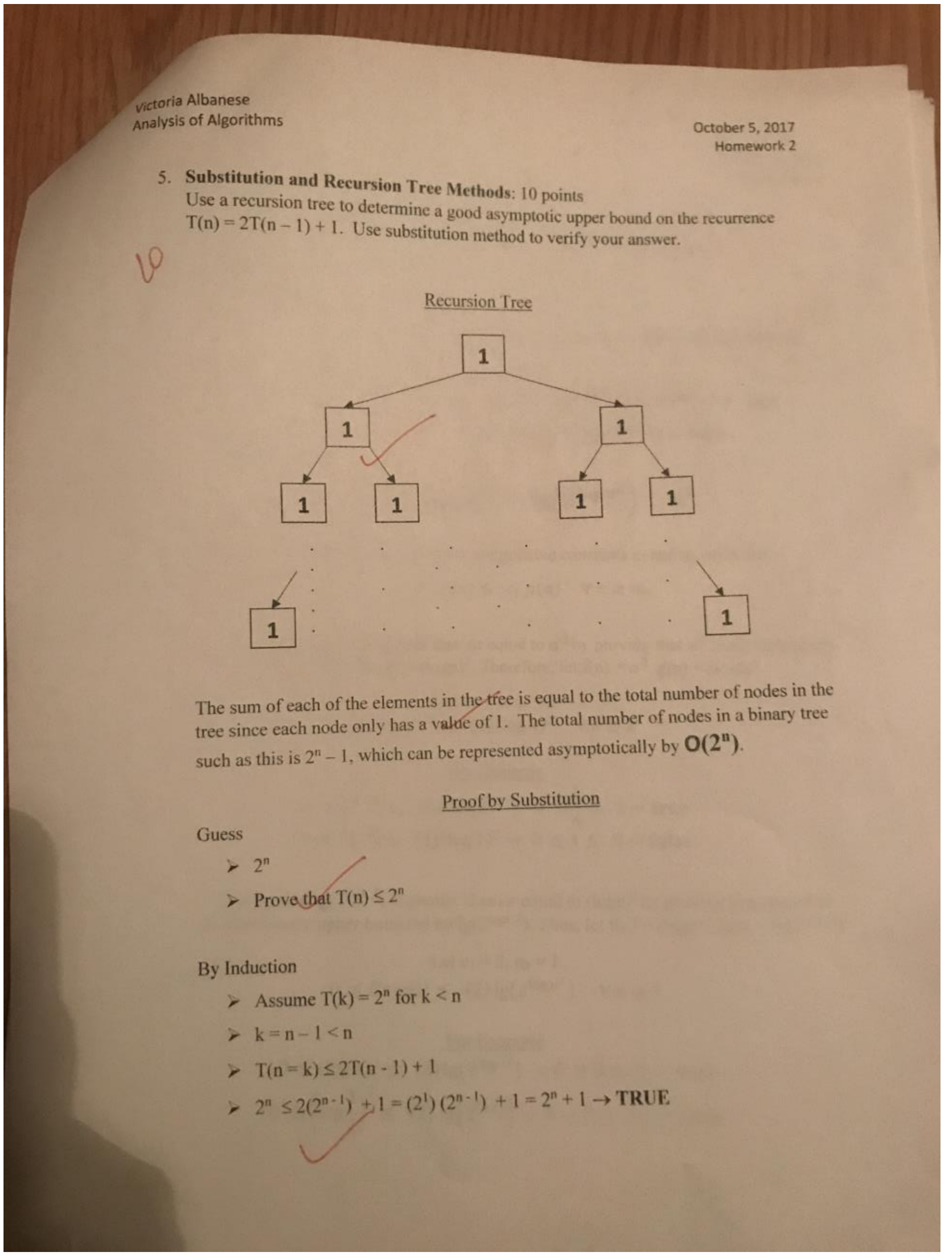
4. Credits: Taylor M. Langlois

4. (20 points)

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, explain why the Master Theorem does not apply. Justify your answer.

1. $T(n) = 3^n T(n/3) + n^3$ — NO, a is not a constant since its value is 3^n.
2. $T(n) = 5T(n/2) + \sqrt{10n^3}$ — $T(n) = \Theta(n^3)$ due to case 1 in the master theorem.
3. $T(n) = 1/4T(n/4) + n \log n$ — NO, the a value of ¼ for a doesn't work because it must be $a \geq 1$.
4. $T(n) = T(n-1) + 2n$ — NO, our value over b is n-1, not just n like the master theorem uses.
5. $T(n) = 16T(n/4) + n^2$ — $T(n) = \Theta(n^2 \log n)$ due to case 2 in the master theorem.

5. Credits: Victoria Albanese



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5. **Substitution and Recursion Tree Methods**: 10 points
Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = 2T(n-1) + 1$. Use substitution method to verify your answer.

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Recursion Tree

The sum of each of the elements in the tree is equal to the total number of nodes in the tree since each node only has a value of 1. The total number of nodes in a binary tree such as this is $2^n - 1$, which can be represented asymptotically by $\mathbf{O(2^n)}$.

Proof by Substitution

Guess

- 2^n
- Prove that $T(n) \le 2^n$

By Induction

- Assume $T(k) = 2^n$ for $k < n$
- $k = n - 1 < n$
- $T(n = k) \le 2T(n - 1) + 1$
- $2^n \le 2(2^{n-1}) + 1 = (2^1)(2^{n-1}) + 1 = 2^n + 1 \rightarrow$ **TRUE**