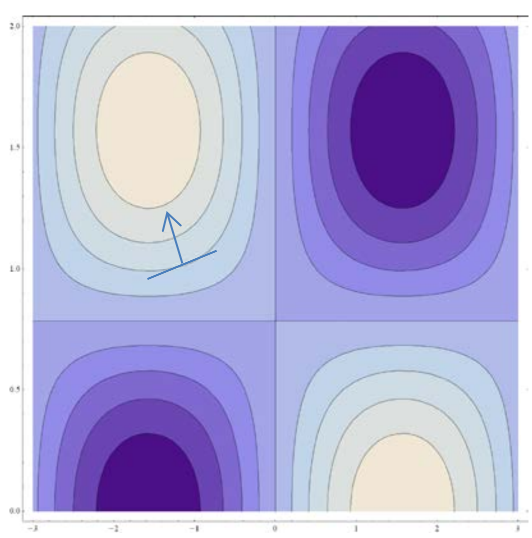
Taking the Mystery Out of dy/dx For Surfaces $z = f(x, y)$

When asked for the expression for dy/dx when solving a steepest descent problem for the surface $z = f(x, y)$, we take the gradient of f $\left(\vec{\nabla} f = < f_x, f_y >\right)$ and use its components to write: $dy/dx = \frac{f_y}{f_x}$ which we then solve to determine y as a function of x (either explicitly or implicitly). This is the correct expression because we want the change in y due to a change in x when moving in a direction parallel to the gradient vector.

At other times you might be asked for the slope of a line tangent to the level curves of $z = f(x, y)$. Note that this direction is perpendicular to the gradient of f. Therefore, since $\vec{\nabla} f = < f_x, f_y >$, a vector perpendicular to $\vec{\nabla} f$ would be in the direction $\pm < f_y, -f_x >$ {i.e., reverse the components and negate either of them}. The change in y due to a change in x when moving in this direction would therefore be $dy/dx = -\frac{f_x}{f_y}$ which is the relationship we use for the slope (in the $xy-plane$) of the level curves.

Remember that in Algebra you were taught that to get the slope of a line perpendicular to a given line, you took the negative reciprocal of the given line's slope. That is all that is going on here. dy/dx for the slope of a level curve is the negative reciprocal of the slope of a curve (e.g., the path of steepest ascent OR descent) in the direction of the gradient since these curves are always perpendicular (see the example below).



The gradient vector (shown perpendicular to the level curves) is perpendicular to the level curve's tangent line and therefore their slopes are negative reciprocals.

Mystery solved!!