MATH.3860

Prob and Stats – Test 2 Spring 19   Name: Phong Vo

√ 1. (10 points) A bleary eyed student awakens one morning late for an 8:00 class, and pulls out two socks out of a drawer that contains two black, six brown and two blue socks, all randomly arranged. Compute the probability that the two he draws are a matched pair.

√ 2. (10 points) A manufacturer of electrical equipment markets a lightbulb that has an average life expectancy Y of 3000 hours, and pdf

$$f_Y(y) = \frac{1}{3000}e^{-y/3000} \quad y > 0$$

He offers a moneyback guarantee on bulbs that fail to last 300 hours. For what proportion of his sales will he need to make a refund?

√ 3. (10 points) Five cards are dealt from a standard poker deck. Let X be the number of aces received, and Y the number of kings. Compute the conditional probability $P(X = 2|Y = 2)$

√ 4. (15 points) A random variable X has the pdf

$$f_X(x) = 2x \quad 0 < x < 1$$

What is the variance of $Y = 3X + 2$

5. (15 points) (a) Urn I contains 5 red chips and 4 white chips. Two chips are drawn from Urn I without replacement. Consider the number of white chips in the sample of two drawn. Compute the probability distribution of the number of white chips in the sample: I want the probabilities of the three events W_0, W_1 and W_2 of drawing zero, one or two white chips.

   (b) Urn II has 4 red and 5 white chips. The sample of two drawn from urn I are put into urn II. Then a single chip is drawn from urn II. What is the probability that the chip drawn from urn II is white? (Hint: condition on the three events W_0, W_1 and W_2.)

√ 6. (20 points) A continuous random variable Y has pdf $f_Y(y) = 3y^2$ for $0 \leq y \leq 1$

   √ (a) What is the probability that Y takes a value in the interval $(1/2, 1)$?

   √ (b) Suppose that 15 observations are chosen of the random variable Y. Let X denote the number of these observations that lie in $(1/2, 1)$. What kind of random variable is X?

   √ (c) Determine $E(X)$

√ 7. (20 points) On planet Alpha, the prison sentence X (in years) of persons convicted of cheating on probability exams has the pdf

$$f_X(x) = \frac{1}{9}x^2 \quad 0 < x < 3$$

   (a) What is the *average* length of time these cheaters spend in jail?

   (b) What is the *median* time in jail (I want the number m so that $P(X < m) = P(X > m)$).

① 1^{st} Sock 2^{nd} Sock $\square$ $\square$ 2 Blk + 6 Br + 2 Blue = 10 Socks

$P(\text{1 pair of Blk}) = \frac{2}{10} \cdot \frac{1}{9} = \frac{2}{90} = \frac{1}{45}$

$P(\text{1 pair of Brown}) = \frac{6}{10} \cdot \frac{5}{9} = \frac{1}{3}$

$P(\text{1 pair of Blue}) = \frac{2}{10} \cdot \frac{1}{9} = \frac{1}{45}$

20/20

$P(\text{a matched pair}) = \frac{1}{45} + \frac{1}{3} + \frac{1}{45} = \boxed{.378}$

③ $P(X=2 \mid Y=2) = \frac{P(X=2 \text{ AND } Y=2)}{P(Y=2)}$

X: Aces, Y: Kings

$= \frac{\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{4}{2}\binom{48}{3}} = \frac{\left(\frac{4\times3}{2.1}\right)\left(\frac{4\times3}{2.1}\right)(44)}{\left(\frac{4.3}{2.1}\right)\frac{48.47.46}{3.2.1}}$

$= \frac{6 \cdot 6 \cdot 44}{6 \cdot 17296} = \boxed{0.0153}$

10/10

④ $f_X(x) = 2x \qquad 0 < x < 1$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$E(Y^2) = \int_0^1 x^2 \cdot 2x \, dx = \int_0^1 2x^3 \, dx$$

$$= \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}$$

$$E(Y) = \int_0^1 x \, 2x \, dx = \int_0^1 2x^2 \, dx$$

$$= \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$Var(3x+2) = 3^2 Var(x)$

$= 9 \cdot \frac{1}{18}$

$= \frac{1}{2}$

$Y = 3X+2$

$Var(Y) = 9Var(X)$

10/15

$$\Rightarrow Var(\cancel{Y}) = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$\Rightarrow Var(Y) = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \boxed{\frac{1}{18}}$$

⑥ $f_Y(y) = 3y^2 \quad 0 \leq y \leq 1$

a)

$$P(\tfrac{1}{2} < y < 1) = \int_{1/2}^{1} 3y^2\,dy = y^3 \Big|_{1/2}^{1}$$

$$\Rightarrow \quad 7/7 \quad p = 1^3 - \left(\tfrac{1}{2}\right)^3 = 1 - \tfrac{1}{8} = \boxed{7/8}$$

c) $E(x) = \int_{1/2}^{1} y \cdot f_Y(y) = \int_{1/2}^{1} y \cdot 3y^2\,dy$

$E(x) = n \cdot p$

$= 15 \cdot \frac{7}{8}$

$= \frac{105}{8}$

$$= \int_{1/2}^{1} 3y^3\,dy = \frac{3}{4} y^4 \Big|_{1/2}^{1}$$

$$= \frac{3}{4}\left(1^4 - \left(\tfrac{1}{2}\right)^4\right) = \frac{3}{4}\left(1 - \frac{1}{16}\right)$$

$$= \frac{3}{4} \cdot \frac{15}{16} = \frac{45}{64} \approx \boxed{.7031}$$

13/20

b) X is binomial $n = 15,\ p = 7/8$

6/6

⑦ $f_X(x) = \frac{1}{9}x^2 \quad 0 < x < 3$

a) $\text{average} = \int_0^3 x \cdot f_X(x)\,dx = E(x)$

$$= \int_0^3 x \cdot \frac{1}{9}x^2\,dx$$

$$= \int_0^3 \frac{1}{9}x^3\,dx = \frac{1}{36}x^4 \Big|_0^3$$

$$= \frac{1}{36} \times (3^4 - 0) = \frac{81}{36}$$

20/20

$\Rightarrow \text{average} = \frac{81}{36} = 2.25 \text{ (years)}$

b) Median $m \Leftrightarrow \int_0^m f_X(x)\,dx = \frac{1}{2}$

$\Leftrightarrow \int_0^m \frac{1}{9}x^2\,dx = \frac{1}{2} \Leftrightarrow \frac{1}{27}x^3 \Big|_0^m = \frac{1}{2}$

$\Leftrightarrow \frac{m^3}{27} = \frac{1}{2} \Rightarrow m^3 = \frac{27}{2} = 13.5$

$\Rightarrow m = \sqrt[3]{13.5} = \boxed{2.381}$ years

(2) $f_Y(y) = \frac{1}{3000} e^{-y/3000} \qquad y > 0$

prob. $p(0 < y < 300) = \int_0^{300} f_Y(y)\, dy$

$$= \int_0^{300} \frac{1}{3000} e^{-y/3000}\, dy$$

$$= \left(-e^{-y/3000}\right)\Big|_0^{300}$$

$$= -e^{-300/3000} + \underbrace{e^{0}}_{=1}$$

$$= 1 - e^{-0.1} = 1 - .9048 = \boxed{.0952}$$

10/10

Hence, the proportion of his salces will need to refund

$$= \boxed{.0952 \text{ or } 9.52\%}$$

⑤ $4W + 5R$ 9 clips total

$$W_0 = p(w=0) = \overset{1^{st}R}{\frac{5}{9}} \cdot \overset{2^{nd}R}{\frac{1}{4}} = \frac{5}{}$$

$$W_1 = p(w=1) = \overset{W}{\frac{4}{9}} \cdot \overset{R}{\frac{5}{8}} = \frac{}{8}$$

$$W_2 = p(w=2) = \overset{W}{\frac{4}{9}} \cdot \overset{W}{\frac{1}{3}} =$$

$$W_0 = p(w=0, R=2) = \frac{\binom{4}{0}\binom{5}{2}}{\binom{9}{2}}$$

$$= \frac{1 \cdot \frac{5 \cdot 4}{2 \cdot 1}}{\frac{9 \cdot 8}{2 \cdot 1}} = \frac{5 \cdot 4}{9 \cdot 8} = \boxed{\frac{5}{18}}$$

$$W_1 = p(w=1, R=1) = \frac{\binom{4}{1}\binom{5}{1}}{\binom{9}{2}}$$

$$= \frac{4 \cdot 5}{\frac{9 \cdot 8}{2 \cdot 1}} = \frac{(4)(5)}{36} = \frac{20}{36} = \boxed{\frac{5}{9}}$$

$$w_2 = p(w=2, R=0) = \frac{\binom{4}{2}\binom{5}{0}}{\binom{9}{2}}$$

$$= \frac{\frac{4\cdot 3}{2\cdot 1}\cdot 1}{\frac{9\cdot 8}{2\cdot 1}} = \frac{4\times 3}{9\times 8} = \frac{1}{6}$$

6/6

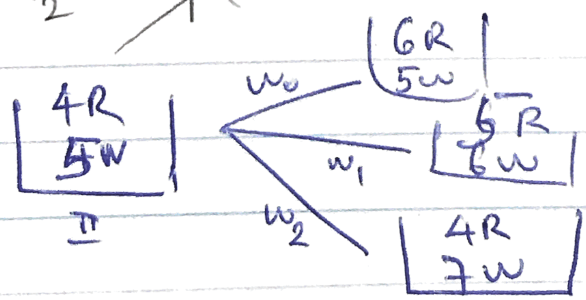
(b)

6/15

a) $w_0 = p(w=0, R=2) = \frac{5}{9}\cdot\frac{4}{8} = \frac{5}{18}$

$w_1 = p(w=1, R=1) = \frac{4}{9}\,\frac{5}{8} = \frac{5}{9}$

$w_2 = p(w=2, R=0) = \frac{4}{9}\,\frac{3}{8} = \frac{1}{6}$

b) 

$$P(w) = P(w\cap w_0) + P(w\cap w_1) + P(w\cap w_2)$$

$$= P(w_0)\,p(w|w_0) + p(w_1)\,p(w|w_1) + p(w_2)\,p(w|w_2)$$

$$= \frac{5}{18}\cdot\frac{5}{11} + \frac{5}{9}\cdot\frac{6}{11} + \frac{1}{6}\cdot\frac{7}{11} \approx \boxed{\frac{53}{99}}$$