Using Green's Theorem and the Fundamental Theorem of Line Integrals to Prove Clairaut's Theorem

Consider a function of two variables, $f(x,y)$. Now consider $f(x,y)$ as the potential function for the vector field, $\overline{F}(x,y)=\overline{\nabla} f(x,y)=< f_x(x,y), f_y(x,y) >$. Since $\overline{F}$ has a potential function (it is the gradient of a potential function), $\overline{F}$ is a conservative vector field.

For conservative vector fields, the Fundamental Theorem of line integrals states that the "work integral" over a path from $P(a,b)$ to $Q(c,d)$ can be computed from the values of the potential function at the end points as follows:

$$\int_P^Q \overline{F} \bullet \overline{dr} = f(c,d) - f(a,b)$$

If the path is any closed path (e.g., starting and ending at P), then this integral must equal zero:

$$\int_P^P \overline{F} \bullet \overline{dr} = \oint_{P\,to\,P} \overline{F} \bullet \overline{dr} = f(a,b) - f(a,b) = 0$$

Using the fact that $\overline{F} =< f_x, f_y >$ and that $\overline{dr} =< dx, dy >$, we can write

$$\oint_{Any\ Closed\ Path} f_x dx + f_y dy = 0$$

The circulation form of Green's Theorem states that this integral must be equal to the integral over the enclosed area of $f_{yx} - f_{xy}$ as follows:

$$\oint_{Any\ Closed\ Path} f_x dx + f_y dy = \iint_{Enclosed\ Area} (f_{yx} - f_{xy}) dA = 0$$

Since these must equal zero over any region, f_{yx} must equal f_{xy} at every point where they are continuous. QED

Note that this is easily proved by contradiction. E.g., assume that they are not equal at some point (a,b) and that the difference is positive. Since they are continuous at (a,b) there must be a region surrounding (a,b) where the difference is always positive. But if this were the case the second integral in the equation above over that region would be positive and therefore not equal to zero. Since that is impossible, they could not have been unequal at that point.