Section 3.3: Cramer's Rule, Volume, & Linear Transformations:

*Here we use the theory of the previous sections to obtain important formulas & to develop a geometrical interpretation of the determinant.

Cramer's Rule

Cramer's Rule is used for a variety of theoretical calc.

- To Illustrate:

  Cramer's Rule can be used to study for the solution of $A\vec{x}=\vec{b}$ is affected by changes in the entries of $\vec{b}$

Note: The formula is inefficient for hand calculations, w/ the except of 2×2 matrices (sometimes 3×3) :)

For any $n \times n$ matrix A & $\forall\ \vec{b} \in \mathbb{R}^n$, let $A_i(\vec{b})$ be the matrix obtained from A by replacing Column i by $\vec{b}$:

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{b} & \cdots & \vec{a}_n \end{bmatrix}$$

* i^{th} Column

Theorem 7 (Cramer's Rule):

Let A be an invertible $n \times n$ matrix. $\forall\ \vec{b} \in \mathbb{R}^n$, the unique solution $A\vec{x}=\vec{b}$ has entries given by:

$$x_i = \frac{\det[A_i(\vec{b})]}{\det(A)}, \quad \text{where } i = 1,2,\ldots,n$$

Example (Cramer's Rule):

Use Cramer's Rule to solve the system of equations:

$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$$

Answer:

Recall: $\forall$ $n \times n$ matrix A & $\forall$ $\vec{b} \in \mathbb{R}^n$, let $A_i(\vec{b})$ be the matrix obtained by replacing the i^{th} Column of A w/ $\vec{b}$:

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n]$$

*View the given system as $A\vec{x} = \vec{b}$:

Since $A\vec{x} = \vec{b} \iff [A \mid \vec{b}] = \left[\begin{array}{cc|c} 3 & -2 & 6 \\ -5 & 4 & 8 \end{array}\right]$, then:

- $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \rightarrow \det(A) = 12 - 10 = 2$
- $A_1(\vec{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \rightarrow \det[A_1(\vec{b})] = 24 + 16 = 40$
- $A_2(\vec{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix} \rightarrow \det[A_2(\vec{b})] = 24 + 30 = 54$

*Note: Since $\det(A) = 2 \neq 0$, matrix A is invertible & thus has a unique solution!

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Example (Cramer's Rule) Continued...

***Use Cramer's Rule to compute the Unique Solution:**

<u>Recall</u>: For an $n \times n$ invertible matrix A, $\forall \vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries:

$$\vec{x}_i = \frac{\det[A_i(\vec{b})]}{\det(A)}, \quad i = 1, 2, \dots n$$

- $x_1 = \dfrac{\det[A_1(\vec{b})]}{\det(A)} = \dfrac{40}{2} = 20$

- $x_2 = \dfrac{\det[A_2(\vec{b})]}{\det(A)} = \dfrac{54}{2} = 27$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}}$$

Answer.

Example: Use Cramer's Rule to compute the solutions of the system: $\begin{cases} 8x_1 + 2x_2 = 12 \\ 3x_1 + 5x_2 = 13 \end{cases}$

Answer:

View the given system as the Nonhomogeneous Equation, $A\vec{x} = \vec{b}$:

Since $A\vec{x} = \vec{b} \Leftrightarrow [A \mid \vec{b}] = \left[\begin{array}{cc|c} 8 & 2 & 12 \\ 3 & 5 & 13 \end{array}\right]$, then:

- $A = \begin{bmatrix} 8 & 2 \\ 3 & 5 \end{bmatrix} \longrightarrow \boxed{\det(A) = 40 - 6 = 34}$

Note: Since $\det(A) \neq 0$, a unique solution $\exists$

- $A_1(\vec{b}) = \begin{bmatrix} 12 & 2 \\ 13 & 5 \end{bmatrix} \longrightarrow \boxed{\det[A_1(\vec{b})] = 60 - 26 = 34}$

- $A_2(\vec{b}) = \begin{bmatrix} 8 & 12 \\ 3 & 13 \end{bmatrix} \longrightarrow \boxed{\det[A_2(\vec{b})] = 104 - 36 = 68}$

Use Cramer's Rule to find the unique solution:

- $x_1 = \dfrac{\det[A_1(\vec{b})]}{\det(A)} = \dfrac{34}{34} = \boxed{1}$

- $x_2 = \dfrac{\det[A_2(\vec{b})]}{\det(A)} = \dfrac{68}{34} = \boxed{2}$

$\therefore \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ Answer.

Example: Use Cramer's Rule to compute the solutions of the system: $\begin{cases} 5x_1 + 3x_2 = -3 \\ 8x_1 + 7x_2 = 6 \end{cases}$

Answer:

View the system as the Nonhomogeneous Equation, $A\vec{x} = \vec{b}$:

Since $A\vec{x} = \vec{b} \Leftrightarrow [A \mid \vec{b}] = \left[\begin{array}{cc|c} 5 & 3 & -3 \\ 8 & 7 & 6 \end{array}\right]$, then:

- $A = \begin{bmatrix} 5 & 3 \\ 8 & 7 \end{bmatrix} \rightarrow \boxed{\det(A) = 35 - 24 = 11}$

*Note: Since $\det(A) \neq 0$, then a unique sol. $\exists$

- $A_1(\vec{b}) = \begin{bmatrix} -3 & 3 \\ 6 & 7 \end{bmatrix} \rightarrow \boxed{\det[A_1(\vec{b})] = -21 - 18 = -39}$

- $A_2(\vec{b}) = \begin{bmatrix} 5 & -3 \\ 8 & 6 \end{bmatrix} \rightarrow \boxed{\det[A_2(\vec{b})] = 30 + 24 = 54}$

Use Cramer's Rule to find the unique solution:

- $x_1 = \dfrac{\det[A_1(\vec{b})]}{\det(A)} = \dfrac{-39}{11}$

- $x_2 = \dfrac{\det[A_2(\vec{b})]}{\det(A)} = \dfrac{54}{11}$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -39/11 \\ 54/11 \end{bmatrix}}$$ Answer

Application to Engineering

A number of important engineering problems can be analyzed by "Laplace Transforms".

⇒ This approach converts an appropriate system of linear differential equations into a system of linear algebraic equations, whose coefficients involve a parameter "s".

Note: While these linear systems contain an unspecified parameter 's', we can still solve for the unique solution (if it $\exists$) using Cramer's Rule :)

Example (Laplace Transforms & Cramer's Rule):

Determine the values of s for which the system has a unique solution, and use Cramer's Rule to describe the solution: $\begin{cases} 3s x_1 - 2x_2 = 4 \\ -6x_1 + s x_2 = 1 \end{cases}$, for some parameter S.

Answer:

*View the given system as $A\vec{x} = \vec{b}$:

Since $A\vec{x} = \vec{b} \iff [A \,\vdots\, \vec{b}] = \left[\begin{array}{cc|c} 3s & -2 & 4 \\ -6 & s & 1 \end{array}\right]$, then:

- $A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix} \longrightarrow \boxed{\begin{aligned} \det(A) &= 3s^2 - 12 = 3(s^2-4) \\ &= 3(s-2)(s+2) \end{aligned}}$

- $A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \longrightarrow \boxed{\det[A_1(\vec{b})] = 4s + 2 = 2(2s+1)}$

- $A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix} \longrightarrow \boxed{\det[A_2(\vec{b})] = 3s + 24 = 3(s+8)}$

*Determine the values for which the system has a unique solution: * A is invertible IFF $\det(A) \neq 0$ *

$\det(A) = 3s^2 - 12 \neq 0 \longrightarrow s^2 - 4 \neq 0 \longrightarrow \boxed{s \neq \pm 2}$

$\therefore$ The system has a unique solution $\forall\, s$ except ⤴

Ans.

Example Continued (Laplace Transforms & Cramer's Rule)...

* Use Cramer's Rule to Find the Unique Solution:

For "s" ST $s \neq \pm 2$,

- $$x_1 = \frac{\det[A_1(\vec{b})]}{\det(A)} = \frac{2(2s+1)}{3(s-2)(s+2)}$$

- $$x_2 = \frac{\det[A_2(\vec{b})]}{\det(A)} = \frac{\cancel{3}(s+8)}{\cancel{3}(s-2)(s+2)} = \frac{(s+8)}{(s-2)(s+2)}$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{4s+2}{(3s^2-12)} \\ \\ \dfrac{s+8}{(s^2-4)} \end{bmatrix} \quad \text{ST} \quad s \neq \pm 2}$$

Answer.

Example: Determine the values of the parameter 'S' for which the system has a unique solution, and describe the solution:

$$\begin{cases} 4sx_1 + 5x_2 = 4 \\ 8x_1 + 2sx_2 = -2 \end{cases}$$

Answer:

***View the System as the Homogeneous Eq., $A\vec{x} = \vec{b}$:**

Since $A\vec{x} = \vec{b} \Leftrightarrow [A \,\vdots\, \vec{b}] = \left[\begin{array}{cc|c} 4s & 5 & 4 \\ 8 & 2s & -2 \end{array}\right]$, then:

- $A = \begin{bmatrix} 4s & 5 \\ 8 & 2s \end{bmatrix} \rightarrow \boxed{\begin{aligned} \det(A) &= 8s^2 - 40 = 8(s^2-5) \\ &= 8(s-\sqrt{5})(s+\sqrt{5}) \end{aligned}}$

- $A_1(\vec{b}) = \begin{bmatrix} 4 & 5 \\ -2 & 2s \end{bmatrix} \rightarrow \boxed{\det[A_1(\vec{b})] = 8s + 10 = 2(4s+5)}$

- $A_2(\vec{b}) = \begin{bmatrix} 4s & 4 \\ 8 & -2 \end{bmatrix} \rightarrow \boxed{\det[A_2(\vec{b})] = -8s - 32 = -8(s+4)}$

***Determine the value(s) for which the system is unique:**

Since a unique solution $\exists$ IFF $\det(A) \neq 0$

$$\Rightarrow 8(s-\sqrt{5})(s+\sqrt{5}) \neq 0 \quad \begin{cases} s \neq \sqrt{5} \\ s \neq -\sqrt{5} \end{cases}$$

$$\boxed{\therefore \text{A unique solution } \exists\ \forall \text{ values except } s = \pm\sqrt{5}}$$ Ans

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Example Continued...

*Use Cramer's Rule to Find the Unique Solution:

- $$x_1 = \frac{\det\left[A_1(\vec{b})\right]}{\det(A)} = \frac{2(4s+5)}{8(s^2-5)} = \boxed{\frac{(4s+5)}{4(s^2-5)}}$$

- $$x_2 = \frac{\det\left[A_2(\vec{b})\right]}{\det(A)} = \frac{-8(s+4)}{8(s^2-5)} = \boxed{-\frac{(s+4)}{(s^2-5)}}$$

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(4s+5)}{4(s^2-5)} \\[2ex] -\dfrac{(s+4)}{(s^2-5)} \end{bmatrix} \quad \text{ST} \quad s \neq \pm\sqrt{5}}$$

Answer

Example: Determine the values of the parameter 's' for which the system has a unique solution, and describe the solution:

$$\begin{cases} sx_1 - 6sx_2 = 3 \\ 4x_1 - 24sx_2 = 5 \end{cases}$$

Answer:

*View the System as the Nonhomogeneous Eq, $A\vec{x} = \vec{b}$:

Since $A\vec{x} = \vec{b} \Leftrightarrow [A \,\vdots\, \vec{b}] = \left[\begin{array}{cc|c} s & -6s & 3 \\ 4 & -24s & 5 \end{array}\right]$, then:

- $A = \begin{bmatrix} s & -6s \\ 4 & -24s \end{bmatrix} \longrightarrow \boxed{\det(A) = -24s^2 + 24s = -24s(s-1)}$

- $A_1(\vec{b}) = \begin{bmatrix} 3 & -6s \\ 5 & -24s \end{bmatrix} \longrightarrow \boxed{\det[A_1(\vec{b})] = -72s + 30s = -42s}$

- $A_2(\vec{b}) = \begin{bmatrix} s & 3 \\ 4 & 5 \end{bmatrix} \longrightarrow \boxed{\det[A_2(\vec{b})] = 5s - 12}$

*Determine the values for which the system is unique:

Since a unique solution $\exists$ IFF $\det(A) \neq 0$

$\Rightarrow -24s(s-1) \neq 0$ → $s \neq 0$, $s \neq 1$

$$\boxed{\therefore \text{A unique solution } \exists \ \forall \text{ values of } s \text{ except } s=0 \ \& \ s=1}$$

Example Continued...

***Use Cramer's Rule to Find the Unique Solution:**

$$\bullet\ x_1 = \frac{\det[A_1(\vec{b})]}{\det(A)} = \frac{\overset{6\cdot 7}{-42s}}{\underset{6\cdot 4}{-24s(s-1)}} = \boxed{\frac{7}{4(s-1)}}$$

$$\bullet\ x_2 = \frac{\det[A_2(\vec{b})]}{\det(A)} = \frac{5s-12}{-24s(s-1)} = \frac{-(12-5s)}{-24s(s-1)}$$

$$= \boxed{\frac{(12-5s)}{24s(s-1)}}$$

$$\boxed{\therefore\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{7}{4(s-1)} \\ \dfrac{(12-5s)}{24s(s-1)} \end{bmatrix} \quad \text{st} \quad \begin{matrix} s \neq 0 \\ s \neq 1 \end{matrix}}$$

Answer

*A Formula For A^{-1} *

Note: Cramer's Rule leads us to a general formula for the inverse of an $n \times n$ matrix A.

- The j^{th} Column of A^{-1} is a vector $\vec{x}$ that satisfies $A\vec{x} = \vec{e}_j$,

where:
- $\vec{e}_j \to$ The j^{th} Column of I_n
- i^{th} entry of $\vec{x}$ = $(i,j)^{th}$ entry of A^{-1}

- By Cramer's Rule:

$$\{(i,j)\text{-entry of } A^{-1}\} = x_i = \frac{\det[A_i(\vec{e}_j)]}{\det(A)}$$

Recall: A_{ji} denotes the submatrix of A formed by deleting row j & Column i

- A Cofactor Expansion down Column i of $A_i(\vec{e}_j)$ shows that:

$$\det[A_i(\vec{e}_j)] = (-1)^{i+j}\det(A_{ji}) = C_{ji}$$

* $C_{ji} \to$ Cofactor of A

- By Cramer's Rule above, the $(i,j)^{th}$ entry of A^{-1} is the Cofactor C_{ji} divided by $\det(A)$, and thus:

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{21} & & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Note: The Subscripts on C_{ji} are the reverse of (i,j).

*A Formula For A^{-1} Continued...

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

*The matrix of coefficients is called the <u>Adjugate of A</u> (or <u>Classical Adjoint of A</u>)

→ <u>Denoted By</u>: $\mathrm{adj}(A)$

<u>Note</u>: The following theorem simply restates this in compact terms :)

*Theorem 8 (An Inverse Formula):

Let A be an $n \times n$, invertible matrix. Then:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

*<u>Note</u>: The adjugate matrix, $\mathrm{adj}(A)$, is the <u>transpose</u> of the matrix of Cofactors :)

Example (An Inverse Formula):

Find the inverse of the matrix: $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

Answer:

Recall: If A is an $n \times n$ invertible matrix, then:

$A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$, where: $\operatorname{adj}(A)$ = the transpose of the matrix of cofactors.

1) First find the matrix of Cofactors:

Caution: Remember that the signs alternate b/w +/-

Row 1:

- $C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = 2-4 = \boxed{-2}$, • $C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-2-1) = \boxed{3}$

- $C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 4+1 = \boxed{5}$

Row 2:

- $C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -(-2-12) = \boxed{14}$, • $C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4-3 = \boxed{-7}$

- $C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -(8-1) = \boxed{-7}$

Row 3:

- $C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1+3 = \boxed{4}$, $C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2-3) = \boxed{1}$

- $C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -2-1 = \boxed{-3}$

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Example (An Inverse Formula) Continued...

So, the matrix of Cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{bmatrix}$$

$\therefore$ Since the Adjugate Matrix, $\text{adj}(A) = C^T$:

$$\boxed{\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}}$$

Note: There are several ways to proceed from here :)

① Compute $\det(A)$ directly & then find A^{-1}

② Compute $(\text{adj}(A))A = \det(A)I$ (this verifies $\text{adj}(A)$ AND finds A :))

② Compute $[\text{adj}(A)]A$ to find $\det(A)$:

$$\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -4+14+4 & -2-14+16 & -6+14-8 \\ 6-7+1 & 3+7+4 & 9-7-2 \\ 10-7-3 & 5+7-12 & 15-7+6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 14I_3$$

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Example (An Inverse Formula) Continued...

So, $[\mathrm{adj}(A)]A = 14 I_3 \Rightarrow \boxed{\therefore \det(A) = 14}$

③ Find the Inverse Formula:

$$A^{-1} = \frac{1}{\det(A)}[\mathrm{adj}(A)]$$

$$\boxed{A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{2} & \frac{1}{14} \\ \frac{5}{14} & -\frac{1}{2} & -\frac{3}{14} \end{bmatrix}}$$

Answer.

*Caution: This theorem & computation is used mainly for theoretical purposes $\Rightarrow$ The theorem seen in 2.2 is far more efficient & effective.

Example: Compute the adjugate of the given matrix, & then use the Inverse Formula to give the inverse of the matrix:

$$A = \begin{bmatrix} 0 & -5 & -1 \\ 5 & 0 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

Answer:

*Find the matrix of cofactors:

$\bullet\ C_{11} = +\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \boxed{0}$, $\bullet\ C_{12} = -\begin{bmatrix} 5 & 0 \\ -2 & 2 \end{bmatrix} = \boxed{-(10)}$, $\bullet\ C_{13} = +\begin{bmatrix} 5 & 0 \\ -2 & 1 \end{bmatrix} = \boxed{5}$

$\bullet\ C_{21} = -\begin{bmatrix} -5 & -1 \\ 1 & 2 \end{bmatrix} = -(\underbrace{-10+1}_{-9}) = \boxed{9}$, $\bullet\ C_{22} = +\begin{bmatrix} 0 & -1 \\ -2 & 2 \end{bmatrix} = \boxed{-2}$, $\bullet\ C_{23} = -\begin{bmatrix} 0 & -5 \\ -2 & 1 \end{bmatrix} = -(-10) = \boxed{10}$

$\bullet\ C_{31} = +\begin{bmatrix} -5 & -1 \\ 0 & 0 \end{bmatrix} = \boxed{0}$, $\bullet\ C_{32} = -\begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix} = \boxed{-(5)}$, $\bullet\ C_{33} = +\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = \boxed{+25}$

$\Rightarrow$ So, the Cofactor Matrix is:

$$C = \begin{bmatrix} 0 & -10 & 5 \\ 9 & -2 & 10 \\ 0 & -5 & 25 \end{bmatrix}$$

$\therefore$ Since $adj(A) = C^T$, then:

$$\boxed{adj(A) = \begin{bmatrix} 0 & 9 & 0 \\ -10 & -2 & -5 \\ 5 & 10 & 25 \end{bmatrix}}$$

Answer

Example Continued...

Recall: $A^{-1} = \frac{1}{\det(A)}[\text{adj}(A)]$

*Use the Inverse Formula to give A^{-1}:

(i) Compute the product, $\text{adj}(A)\cdot A$, to find $\det(A)$:

$$\text{adj}(A)\,A = \begin{bmatrix} 0 & 9 & 0 \\ -10 & -2 & -5 \\ 5 & 10 & 25 \end{bmatrix}\begin{bmatrix} 0 & -5 & -1 \\ 5 & 0 & 0 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0+45+0 & 0+0+0 & 0+0+0 \\ 0-10+10 & 50+0-5 & 10+0-10 \\ 0+50-50 & -25+0+25 & -5+0+50 \end{bmatrix}$$

$$= \begin{bmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix} = 45\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 45 I_3$$

$$\boxed{\therefore \det(A) = 45}$$

(ii) Find the Inverse Formula:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{45}\begin{bmatrix} 0 & 9 & 0 \\ -10 & -2 & -5 \\ 5 & 10 & 25 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & 0 \\ -\frac{2}{9} & -\frac{2}{45} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{5}{9} \end{bmatrix}}$$

Ans.

Example: Compute the adjugate of the given matrix, & then use the Inverse Formula to give the inverse of the matrix:

$$A = \begin{bmatrix} 3 & 6 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Answer:

* Find the matrix of Cofactors first:

- $C_{11} = + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = \boxed{-1}$, • $C_{12} = - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -\underbrace{(2-3)}_{-1} = \boxed{1}$, • $C_{13} = + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1-0 = \boxed{1}$

- $C_{21} = - \begin{vmatrix} 6 & 4 \\ 1 & 2 \end{vmatrix} = -\underbrace{(12-4)}_{8} = \boxed{-8}$, • $C_{22} = + \begin{vmatrix} 3 & 4 \\ 3 & 2 \end{vmatrix} = 6-12 = \boxed{-6}$, • $C_{23} = - \begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} = -\underbrace{(3-18)}_{-15} = \boxed{15}$

- $C_{31} = + \begin{vmatrix} 6 & 4 \\ 0 & 1 \end{vmatrix} = 6-0 = \boxed{6}$, • $C_{32} = - \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = -\underbrace{(3-4)}_{-1} = \boxed{1}$, • $C_{33} = + \begin{vmatrix} 3 & 6 \\ 1 & 0 \end{vmatrix} = 0-6 = \boxed{-6}$

So, the Cofactor Matrix is:

$$C = \begin{bmatrix} -1 & 1 & 1 \\ -8 & -6 & 15 \\ 6 & 1 & -6 \end{bmatrix}$$

∴ Since $adj(A) = C^T$, then:

$$\boxed{adj(A) = \begin{bmatrix} -1 & -8 & 6 \\ 1 & -6 & 1 \\ 1 & 15 & -6 \end{bmatrix}}$$

Answer.

Example Continued...

***Use the Inverse Formula to find A^{-1}:**

(i) Compute the product, adj(A) A, to find det(A):

$$\text{adj}(A)\cdot A = \begin{bmatrix} -1 & -8 & 6 \\ 1 & -6 & 1 \\ 1 & 15 & -6 \end{bmatrix}\begin{bmatrix} 3 & 6 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3-8+18 & 0 & 0 \\ 0 & 6+0+1 & 0 \\ 0 & 0 & 4+15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 7 I_3 \quad \checkmark$$

$$\boxed{\therefore \det(A) = 7}$$

(ii) Find A^{-1}:

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A) = \frac{1}{7}\begin{bmatrix} -1 & -8 & 6 \\ 1 & -6 & 1 \\ 1 & 15 & -6 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} -\frac{1}{7} & -\frac{8}{7} & \frac{6}{7} \\ \frac{1}{7} & \frac{-6}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{15}{7} & \frac{-6}{7} \end{bmatrix}}$$

Answer

Example: Compute the adjugate of the given matrix, & then use the Inverse Formula to give the Inverse of the matrix:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & 0 \\ -1 & 4 & 2 \end{bmatrix}$$

Answer:

*Find the matrix of Cofactors:

• $C_{11} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} = \boxed{4}$, • $C_{12} = -\begin{bmatrix} -3 & 0 \\ -1 & 2 \end{bmatrix} = -(-6) = \boxed{6}$, • $C_{13} = \begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix} = -12+2 = \boxed{-10}$

• $C_{21} = -\begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix} = \boxed{0}$, • $C_{22} = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} = \boxed{8}$, • $C_{23} = -\begin{bmatrix} 4 & 0 \\ -1 & 4 \end{bmatrix} = \boxed{-16}$

• $C_{31} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \boxed{0}$, • $C_{32} = -\begin{bmatrix} 4 & 0 \\ -3 & 0 \end{bmatrix} = \boxed{0}$, • $C_{33} = \begin{bmatrix} 4 & 0 \\ -3 & 2 \end{bmatrix} = \boxed{8}$

So, the Matrix of Cofactors:

$$C = \begin{bmatrix} 4 & 6 & -10 \\ 0 & 8 & -16 \\ 0 & 0 & 8 \end{bmatrix}$$

∴ Since $adj(A) = C^T$, then:

$$\boxed{adj(A) = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 8 & 0 \\ -10 & -16 & 8 \end{bmatrix}}$$

Ans ✓

Example Continued...

***Use the Inverse Formula to Find A^{-1}:**

(i) Compute the product, adj(A)·A, to find det(A):

$$\operatorname{adj}(A)\,A = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 8 & 0 \\ -10 & -16 & 8 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & 0 \\ -1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 16+0+0 & 0 & 0 \\ 0 & 0+16+0 & 0 \\ 0 & 0 & 0+0+16 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 16 I_3$$

$$\boxed{\therefore \det(A) = 16}$$

(ii) Find A^{-1}:

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{16}\begin{bmatrix} 4 & 0 & 0 \\ 6 & 8 & 0 \\ -10 & -16 & 8 \end{bmatrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{3}{8} & \frac{1}{2} & 0 \\ -\frac{5}{8} & -1 & \frac{1}{2} \end{bmatrix}}$$

Answer.

*Theorem 9 (Determinants As Area & Volume):

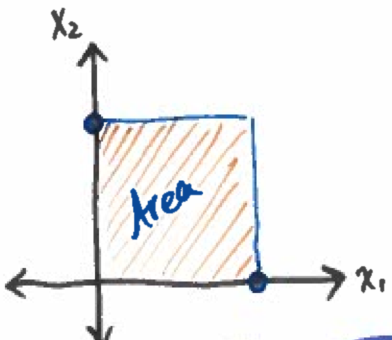
(i) If A is a 2×2 matrix, the area of the parallelogram determined by the Columns of A is: $|\det(A)|$

(ii) If A is a 3×3 matrix, the Volume of the paralleleopiped determined by the Columns of A is: $|\det(A)|$

PROOF of (i):

We can easily verify that this theorem holds true $\forall$ 2×2 diagonal matrices geometrically.

Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ be a 2×2 matrix ST a & d are any scalar.



*Length/Width: $L = |a-0| = a$

*Height: $H = |d-0| = d$

*Area: $\boxed{\text{Area} = LH = ad}$

$\therefore \left| \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad-0| = |ad| =$ Area of the Rectangle.

*It will suffice to show that $\forall$ 2×2 matrix $A = [\vec{a}_1 \ \vec{a}_2]$ can be transformed into a diagonal matrix in a way that changes neither the area nor $\det(A)$. As seen in 3.2: The Absolute Value of $\det(A)$ is NOT

*Geometric Observation (For vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$):

Let $\vec{a}_1$ & $\vec{a}_2$ be nonzero vectors.
Then for any scalar 'c', the area of the parallelogram determined by $\vec{a}_1$ & $\vec{a}_2$ equals the area of the parallelogram determined by $\vec{a}_1$ & $\vec{a}_2 + c\vec{a}_1$.

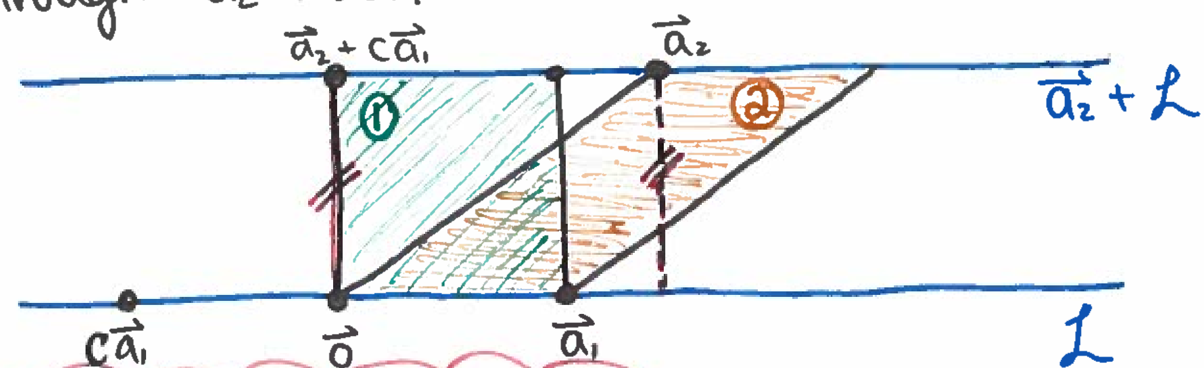
PROOF: (Continued Proof of (i))*

- Assume that $\vec{a}_2$ is NOT a multiple of $\vec{a}_1$.

*Note: If $\vec{a}_2 = c\vec{a}_1$, then the 2 parallelograms would degenerate & have zero area (DONE).

- Let "$\mathcal{L}$" be the line through $\vec{0}$ & $\vec{a}_1$.
- Let "$\vec{a}_2 + \mathcal{L}$" be the line through $\vec{a}_2$ & parallel to $\mathcal{L}$, passing through $\vec{a}_2 + c\vec{a}_1$.

Graphically:



*Goal: Show that $\begin{pmatrix}\text{area of} \\ 1\end{pmatrix} = \begin{pmatrix}\text{area of} \\ 2\end{pmatrix}$

- Notice that the points $(\vec{a}_2 + c\vec{a}_1)$ & $\vec{a}_2$ have the same $\perp$ dist. to the Line $\mathcal{L}$ $\Rightarrow$ IOW: They have the same height.
- Notice that the 2 parallelograms share a base, $b = |\vec{a}_1 - \vec{0}| = |\vec{a}_1|$

$\boxed{\therefore \text{Since the 2 parallelograms have the same height \& same base, they have the SAME area.}}$

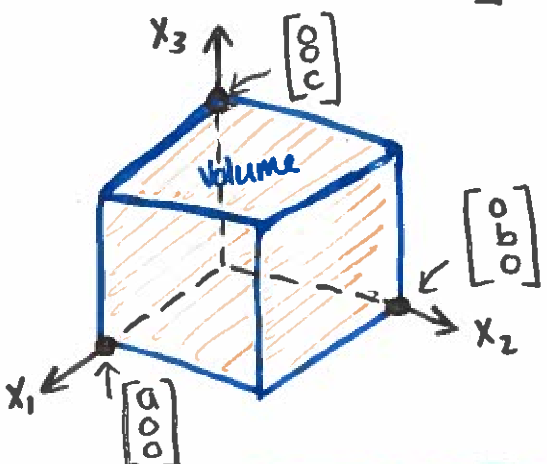
$\blacksquare$

Note: The proof for theorem 9 (ii) (for $\mathbb{R}^3$) is similar :)

PROOF of (ii):

Again, we can easily verify that this theorem holds true $\forall$ 3x3 diagonal matrices geometrically.

Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ be some 3x3 matrix st a,b,c are any scalars.



* **Length:** $L = |a|$
* **Width:** $W = |b|$
* **Height:** $H = |c|$
* $\boxed{\text{Volume: } V = LWH = |abc|}$

$$\therefore \left| \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right| = \left| a \det \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \right| = |a(bc-0)| = |abc|$$

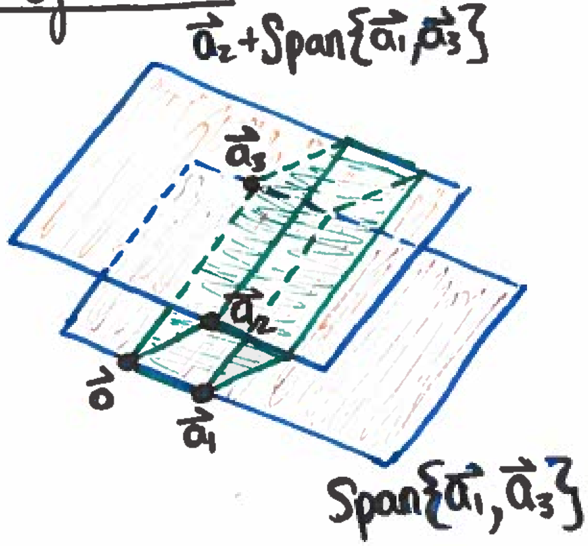
*Cofactor Expansion across Row 1

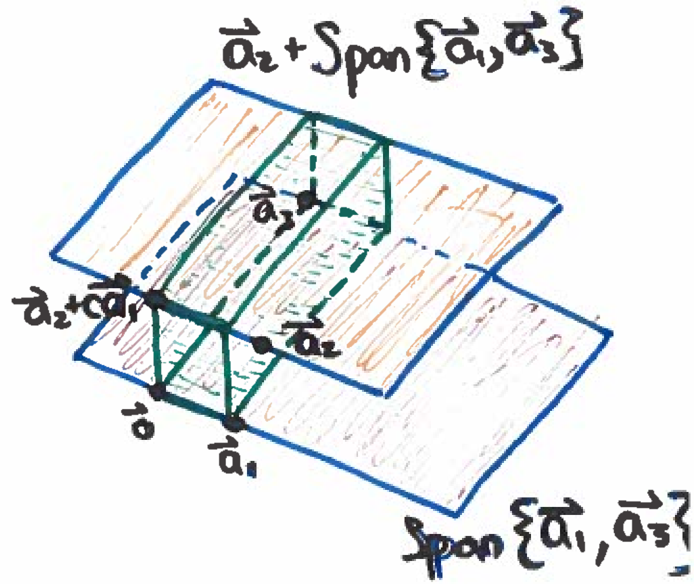
*Any 3 x 3 matrix can be transformed into a diagonal matrix using row operations that do NOT change $|\det(A)|$.
(*Consider row operations on A^T :))

$\therefore$ It suffices to show that these operations do NOT affect the volume of the parallelepiped determined by the columns of A.

Proof of (ii) Continued...

• A parallelepiped is shown below as a shaded box w/ two sloping sides:





• The Volume of the Parallelepiped is: The area of the base in the plane, $\text{Span}\{\vec{a}_1, \vec{a}_3\}$, times the altitude of $\vec{a}_2$ above $\text{Span}\{\vec{a}_1, \vec{a}_3\}$.

*Note: Any vector $\vec{a}_2 + c\vec{a}_1$ has the same altitude as $\vec{a}_2$ b/c it lies in the plane "$\text{Span}\{\vec{a}_1, \vec{a}_2\} + \vec{a}_2$", which is parallel to the plane "$\text{Span}\{\vec{a}_1, \vec{a}_2\}$".

Conclusion:

∴ The Volume of the Parallelepiped is unchanged when $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ is transformed to $[\vec{a}_1, \vec{a}_2 + c\vec{a}_1, \vec{a}_3]$

• Thus a Column replacement operation has No affect on the Volume of the Parallelepiped.

$\boxed{X}$

Example (Area By the Determinant):

Calculate the area of the parallelogram determined by the points $(-2,-2)$, $(0,3)$, $(4,-1)$ & $(6,4)$

Answer:

*Recall: If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of $A = |\det(A)|$

① First translate the parallelogram to one having the origin as a vertex:

*I.E.: Subtract $(-2,-2)$ from each vertice :)
⇒ This is one of 4 possible options.

*Original Vertices:

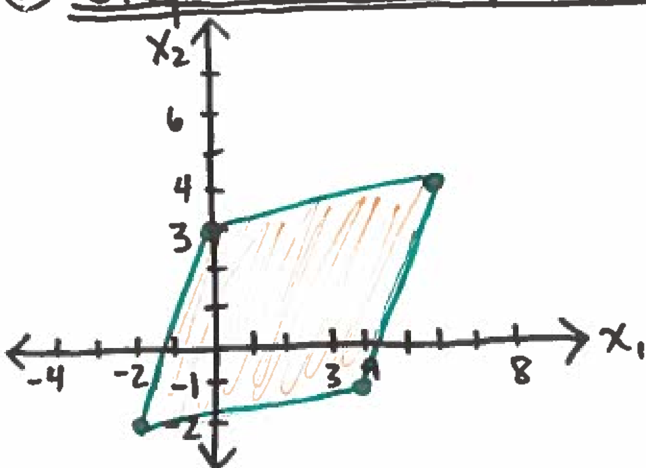
- $(-2,-2)$
- $(0,3)$
- $(4,-1)$
- $(6,4)$

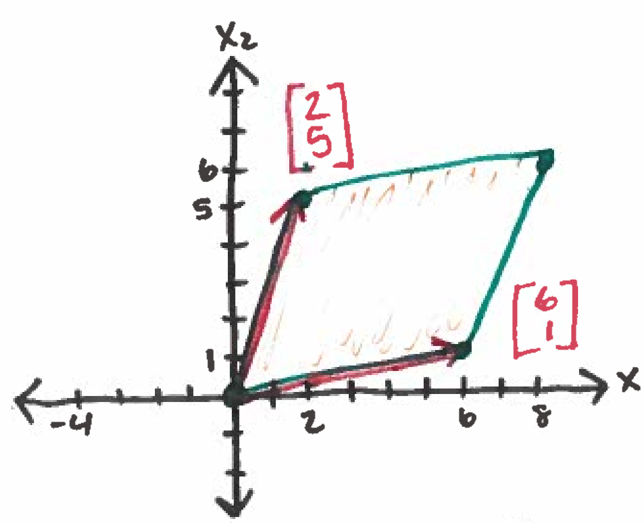
*Subtract $(-2,-2)$ →

*Transformed Vertices:

- $(-2,-2)-(-2,-2) = \boxed{(0,0)}$
- $(0,3)-(-2,-2) = \boxed{(2,5)}$
- $(4,-1)-(-2,-2) = \boxed{(6,1)}$
- $(6,4)-(-2,-2) = \boxed{(8,6)}$

② Graphical Interpretation:



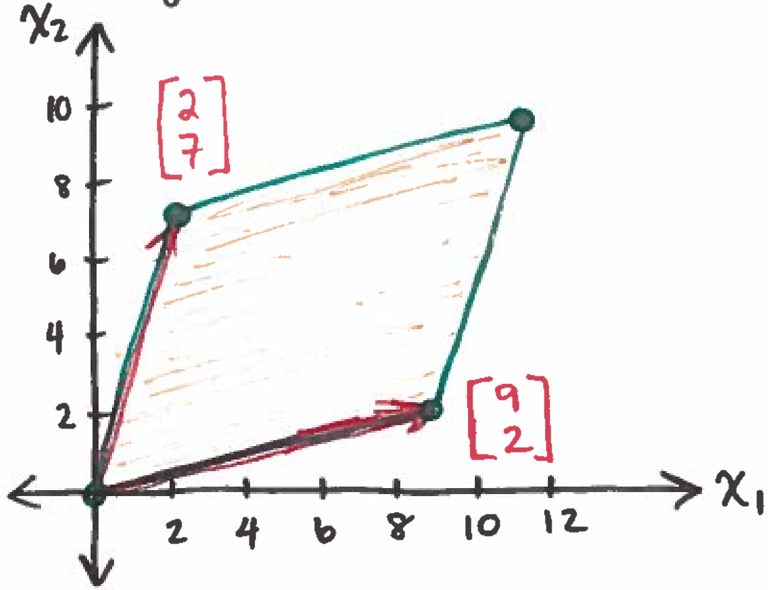


③ Find the Area: Since the parallelogram is determed by the columns of $A = \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}$ ⇒ ∴ Area: $|\det(A)| = |30-2| = 2$

Example: Find the area of the parallelogram whose vertices are listed: (0,0), (2,7), (9,2), (11,9)

Answer:

*Consider the graph of the Parallelogram:



So, the Parallelogram is determined by the columns of matrix:

$$A = \begin{bmatrix} 9 & 2 \\ 2 & 7 \end{bmatrix}$$

*Find the Area:

$$\text{Area} = |\det(A)| = |63-4| = |59| = 59$$

$\therefore$ Area: $\boxed{59 \text{ sq. units}}$ Ans.

Example: Find the area of the parallelogram whose vertices are: $(-1,-3), (3,4), (6,-7), (10,0)$

Answer:

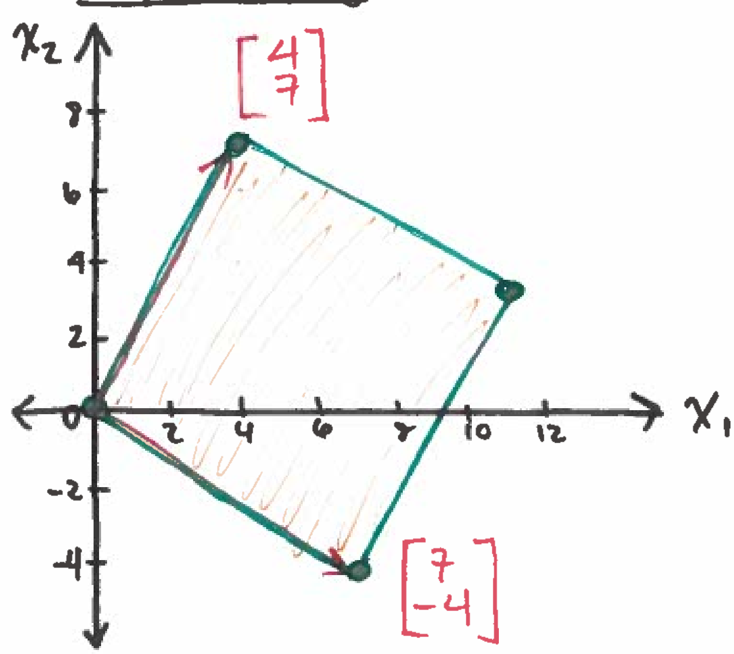
*Translate the Parallelogram to one having a vertex @ the origin $\Longrightarrow$ Subtract one of the given points from ea. vertice ✓

Note: Lets use the point $(-1,-3)$ here!

- $(-1,-3)-(-1,-3) = (0,0)$
- $(3,4)-(-1,-3) = (4,7)$
- $(6,-7)-(-1,-3) = (7,-4)$
- $(10,0)-(-1,-3) = (11,3)$

*Translated Vertices

*Graphically:



So, the parallelogram is determined by the columns of matrix: $A = \begin{bmatrix} 7 & 4 \\ -4 & 7 \end{bmatrix}$

*Find the Area:

$$\text{Area} = |\det(A)| = |49-(-16)| = |65| = 65$$

$\therefore$ Area: $\boxed{65 \text{ sq.units}}$ Ans✓

Example: Find the Volume of the Parallelepiped w/ one vertex at the origin & adjacent vertices at $(2,0,-5)$, $(1,3,4)$, & $(7,1,0)$.

Answer:

Note: The 3 adjacent vertices are the Columns of the matrix :)

* The Parallelepiped is determined by the Columns of matrix:

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 3 & 1 \\ -5 & 4 & 0 \end{bmatrix}$$

* Find the Volume:

$$\text{Volume} = |\det(A)| = \left| 2\det\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} - \det\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} + 7\det\begin{bmatrix} 0 & 3 \\ -5 & 4 \end{bmatrix} \right|$$

$$= \left| 2\underbrace{(0-4)}_{-4} - \underbrace{(0+5)}_{5} + 7\underbrace{(0+15)}_{15} \right| = \left| -8 - 5 + 105 \right|$$

$$= |92| = 92$$

$\therefore$ **Volume:** $\boxed{V = 92 \text{ cubic units}}$ Answer ✓

Linear Transformations

Note: The determinant can be used to describe an important geometric prop. of Linear Transformations in $\mathbb{R}^2$ & $\mathbb{R}^3$.

- If T is a Linear Transformation & S is in the set of the domain of T, let $T(S)$ = the set of images of pts in S

→ We are interested in comparing how the area/volume of $T(S)$ compares to the area/volume of the original set S.

*Theorem[10]:

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear Transformation determined by a 2×2 matrix A.

- If S is a parallelogram in $\mathbb{R}^2$, then:

$$\{\text{area of } T(S)\} = |\det(A)| \{\text{area of } S\}$$

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a Linear Transformation determined by a 3×3 matrix A.

- If S is a parallelepiped in $\mathbb{R}^3$, then:

$$\{\text{Volume of } T(S)\} = |\det(A)| \{\text{Volume of } S\}$$

Example: Let S be the parallelogram determined by vectors $\vec{b}_1 = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ & $\vec{b}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -7 \\ -3 & 7 \end{bmatrix}$.

Compute the area of the image of S under the mapping $\vec{x} \mapsto A\vec{x}$.

Answer:

*Recall: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear Transformation determined by a 2×2 matrix. If S is a paralleogram in $\mathbb{R}^2$, then:

$$\{\text{area of } T(S)\} = |\det(A)| \{\text{area of } S\}$$

*Since "S" is a parallelogram determined by vectors $\vec{b}_1$ & $\vec{b}_2$:

- The Parallelogram is determined by the columns of matrix: $B = [\vec{b}_1\ \vec{b}_2] = \begin{bmatrix} -5 & -5 \\ 5 & 7 \end{bmatrix}$

- So the Area of S: $|\det(B)| = |-35+25| = |10| = 10$

$\therefore$ Area of S: $\boxed{\therefore |\det(B)| = 10}$

*Find det(A): $\left|\det\begin{bmatrix} 6 & -7 \\ -3 & 7 \end{bmatrix}\right| = |42-21| = |21| = 21$

$\boxed{\therefore |\det(A)| = 21}$

*Find the Area of T(S):

$\boxed{\text{Area of } T(S) = (21)(10) = 210}$ Ans✓