CHAPTER 5

KARNAUGH MAPS

Truth table for NOT

X	F
0	1
1	0

x

0	1
1	0

F

x

1	0
0	1

F

x	
0	1
1	0

F

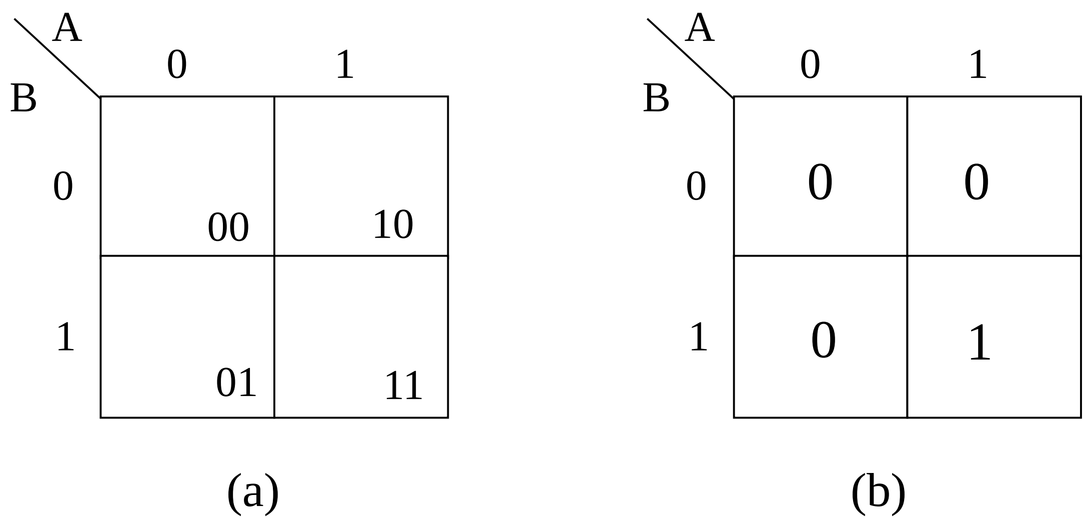


Figure 5.1 (a) Two-variable Karnaugh map.

(b) Karnaugh map for AB.

AB	
00	0_{0}
01	0_{1}
11	1_{3}
10	0_{2}

AB

00	01	11	10
0_{0}	0_{1}	1_{3}	0_{2}

AB

11	10	00	01
1_{0}	0_{1}	0_{3}	0_{2}

C \ AB	00	01	11	10
0	000	010	110	100
1	001	011	111	101

C \ AB	00	01	11	10
0	0	2	6	4
1	1	3	7	5

Figure 5.2 Three-variable Karnaugh maps.

C \ AB	00	01	11	10
0			110	
1	001		111	

A \ BC	00	01	11	10
0		001		
1			111	110

Figure 5.3 Two different 3-variable Karnaugh maps.

C \ AB	00	01	11	10
0	0_0	1_2	0_6	0_4
1	1_1	1_3	1_7	1_5

(a)

A \ BC	00	01	11	10
0	0_0	1_1	1_3	1_2
1	0_4	1_5	1_7	0_6

(b)

AB \ C	0	1
00	0_0	1_1
01	1_2	1_3
11	0_6	1_7
10	0_4	1_5

(c)

BC \ A	0	1
00	0_0	0_4
01	1_1	1_5
11	1_3	1_7
10	1_2	0_6

(d)

Figure 5.4 Karnaugh maps for the truth table in Table 4.3.

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Figure 5.5 Four-variable Karnaugh map.

CD \ AB	00	01	11	10
00	1	0	0	0
01	0	1	1	0
11	1	1	1	0
10	1	0	1	1

Figure 5.6 Karnaugh map for $F = \Sigma\, m(0, 2, 3, 5, 7, 10, 13, 14, 15)$

A = 0

DE \ BC	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

A = 1

BC \ DE	00	01	11	10
00	16	20	28	24
01	17	21	29	25
11	19	23	31	27
10	18	22	30	26

Figure 5.7 Five-variable Karnaugh map.

5.2 Prime Implicant

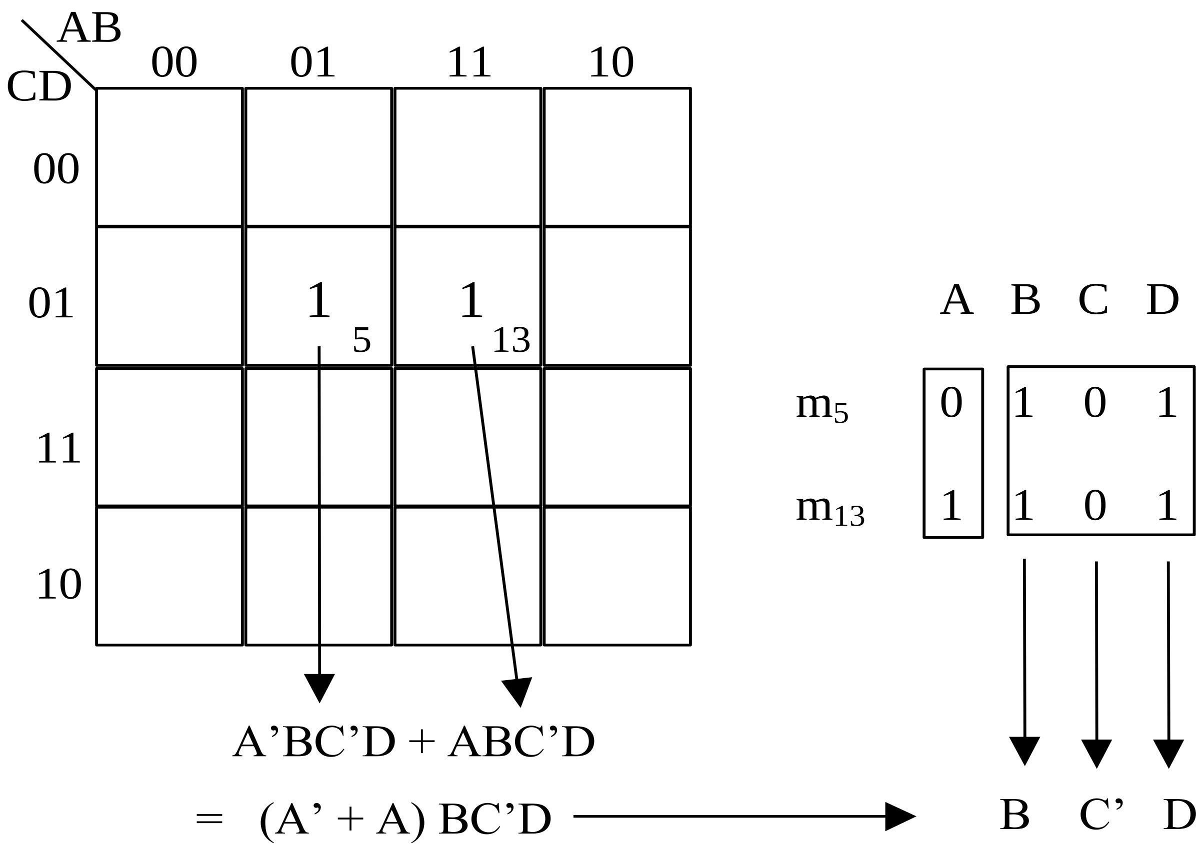


Figure 5.8 Two logically adjacent minterms.

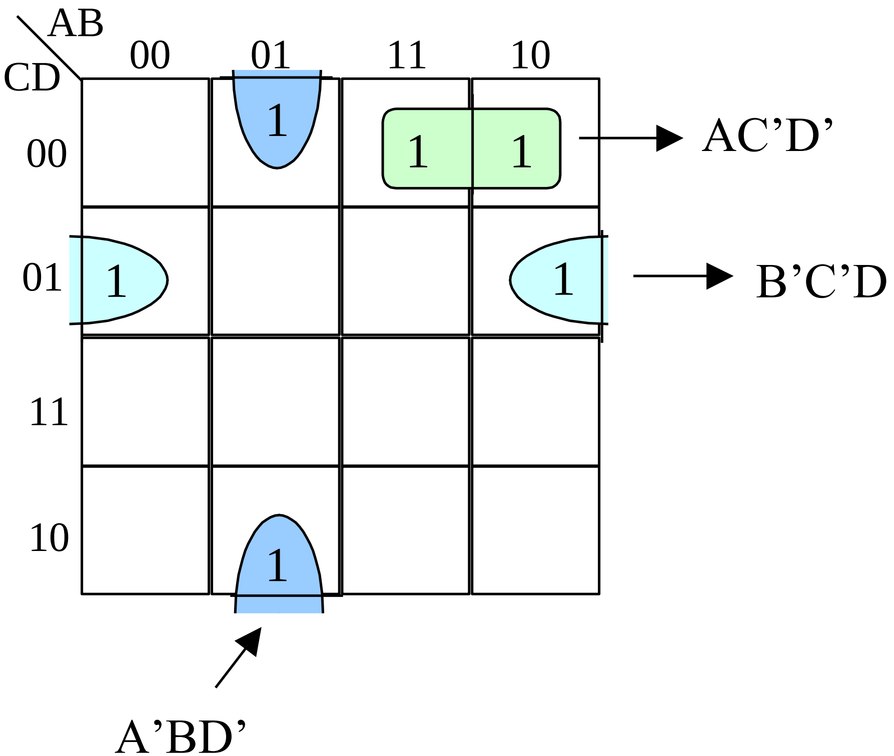
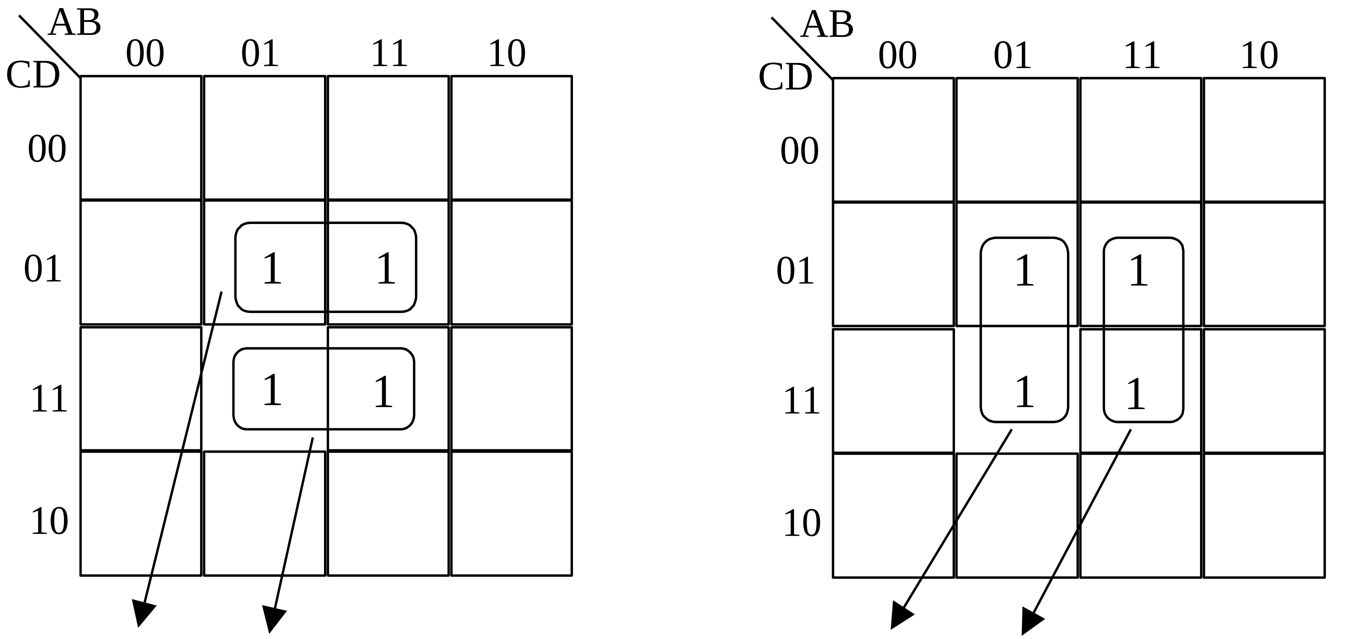


Figure 5.9 Examples of 1-cubes.



BC'D + BCD = (C'+C)BD = BD

(a)

A'BD + ABD = (A'+A)BD = BD

(b)

Figure 5.10 Formation of a 2-cube from two 1-cubes.

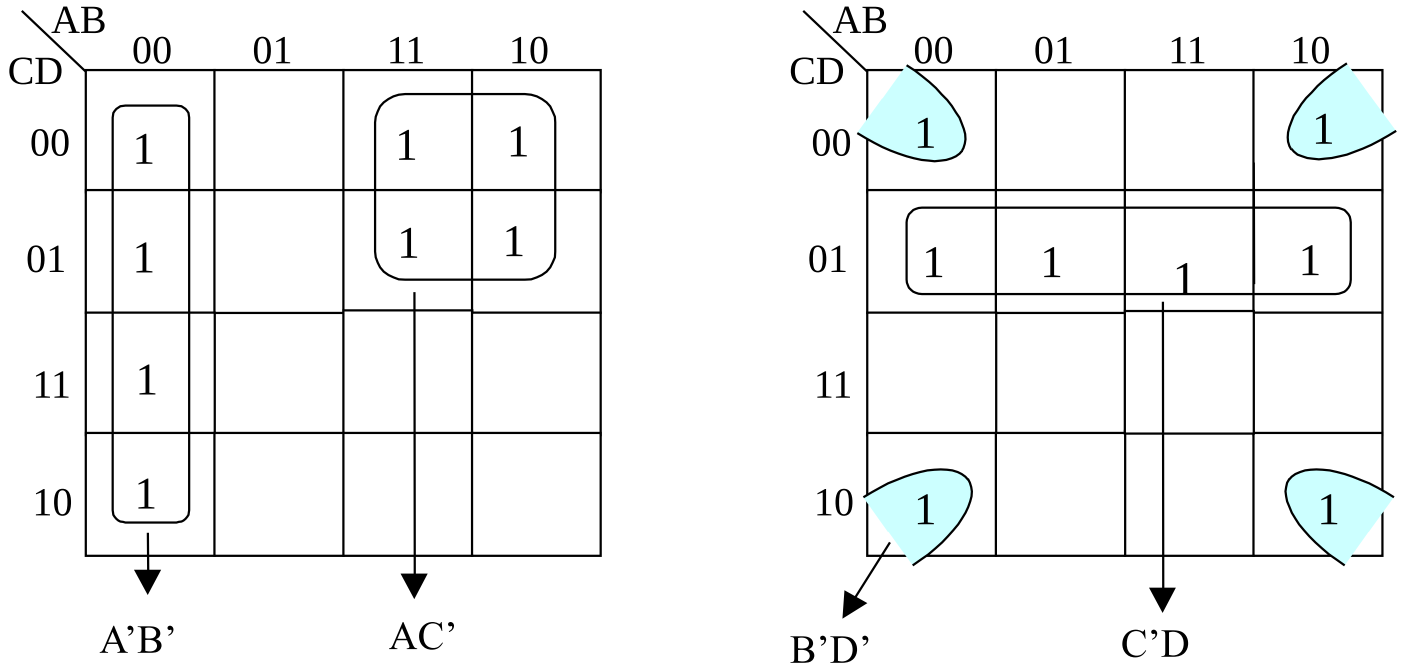


Figure 5.11 More examples of 2-cubes.

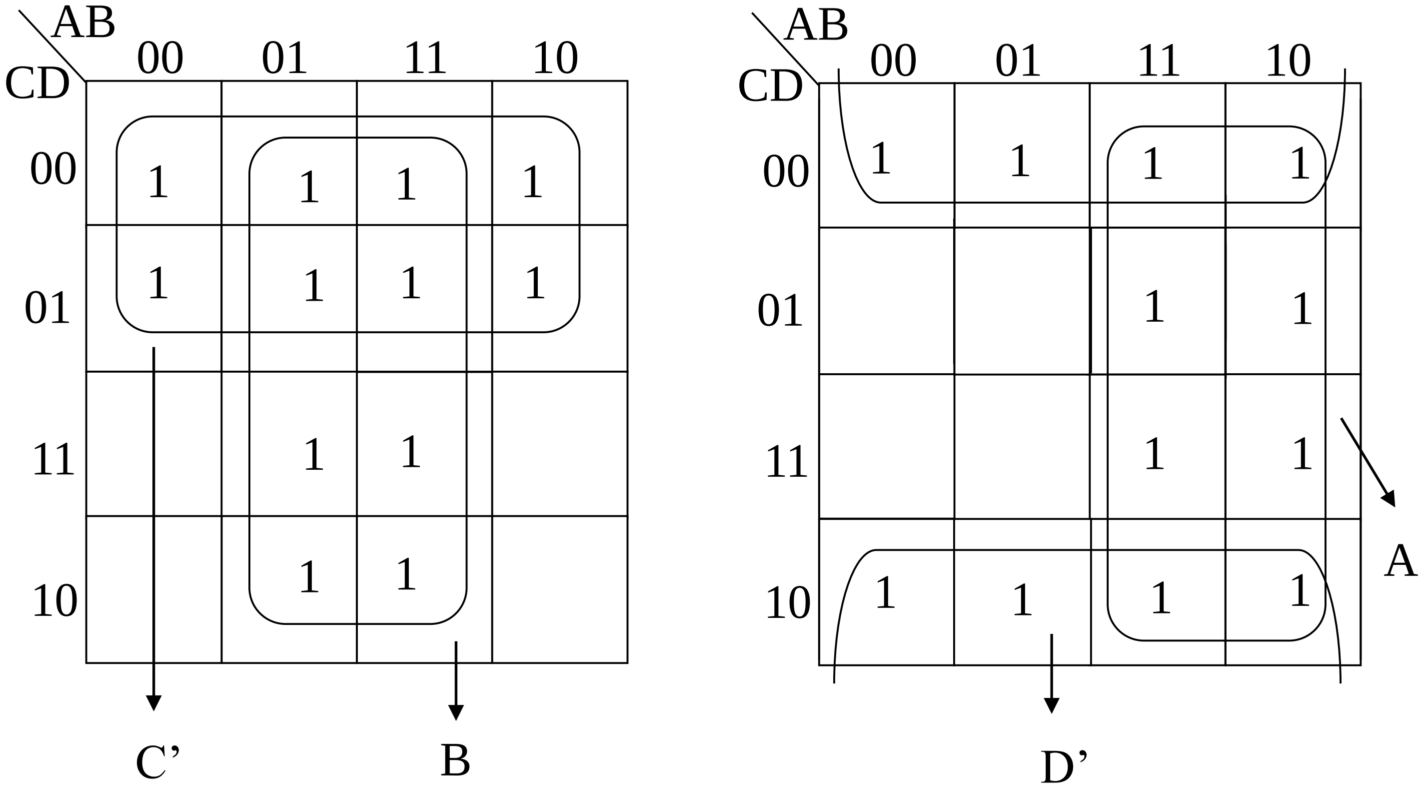


Figure 5.12 Examples of 3-cubes.

Definition 5.1: A j-cube is a grouping of 2^j logically adjacent 1-cells on a K-map for an n- variable function which can be combined to form a product of (n-j) literals. j is a positive integer, $0 \le j \le n$.

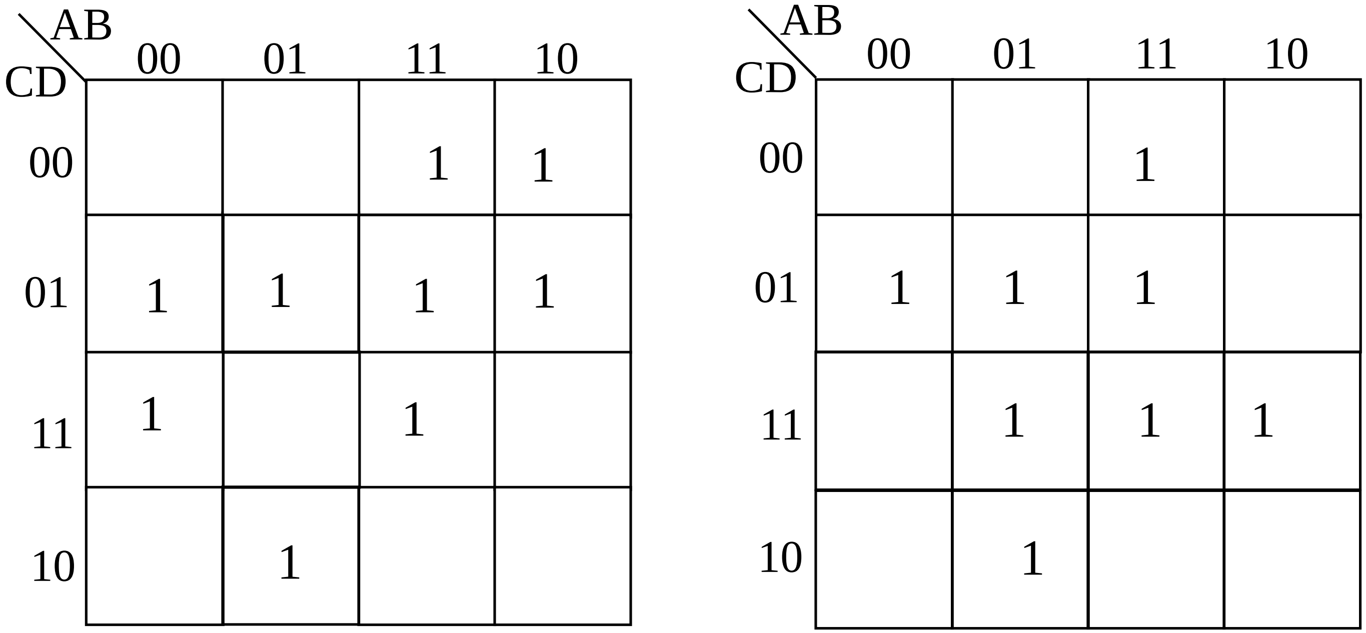
Note that a minterm is a 0-cube. A 0-cube is different from a 0-cell.

Definition 5.2: An implicant is a cube of any order.

Definition 5.3: A j-cube is called a prime implicant if it cannot combine with another j-cube to form a (j+1)-cube.

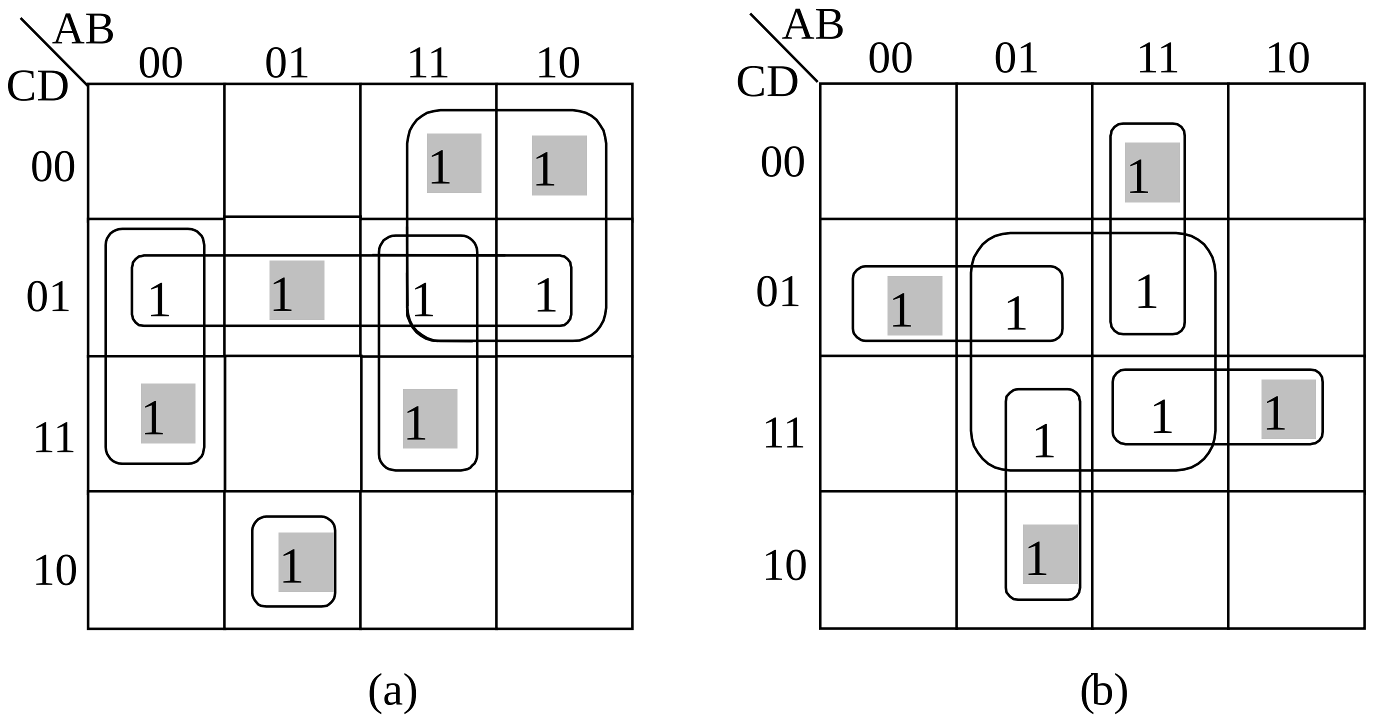
Definition 5.4: If a 1-cell can exist in one and only one prime implicant, it is called a distinguished 1-cell.

Definition 5.5: A prime implicant is called an essential prime implicant if it includes at least one distinguished 1-cell.



(a) (b)

Figure 5.13 Examples of prime implicants.



Prime implicants (a) A'BCD', A'B'D, ABD, C'D, AC'
(b) A'C'D, A'BC, ACD, ABC', BD

Essential Prime implicants (a) A'BCD', A'B'D, ABD, C'D, AC'
(b) A'C'D, A'BC, ACD, ABC'

Figure 5.13 Examples of prime implicants.

CD \ AB	00	01	11	10
00			1	1
01	1	1	1	
11	1	1	1	1
10		1		

(c)

CD \ AB	00	01	11	10
00	1		1	1
01	1	1	1	1
11		1		
10	1	1		1

(d)

Figure 5.13 Examples of prime implicants.

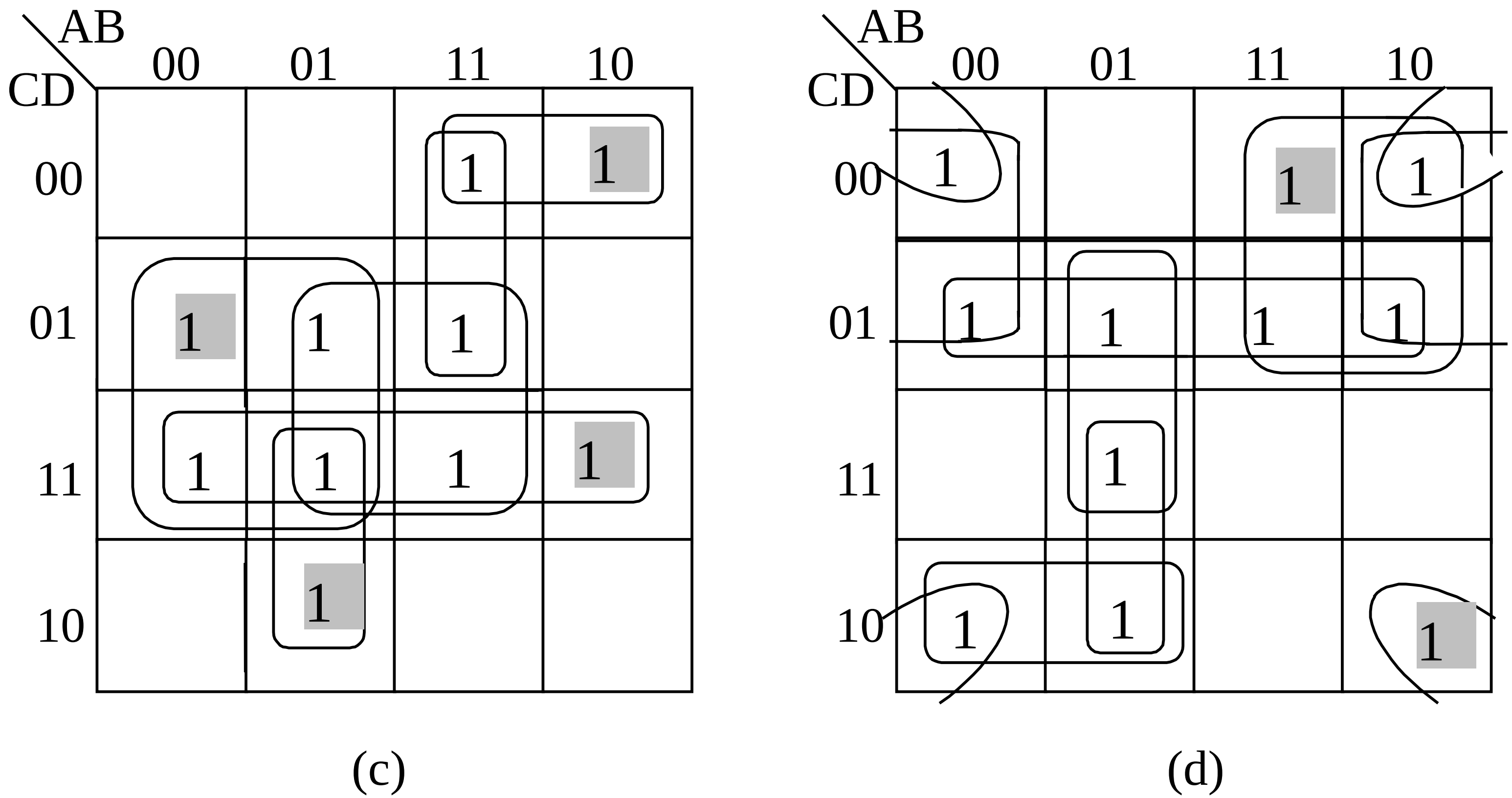


Figure 5.13 Examples of prime implicants.

Prime implicants (c) A'BC, ABC', AC'D', A'D, BD, CD
(d) A'CD', A'BC, A'BD, B'D', B'C', C'D, AC'

Essential prime implicants (c) A'BC, AC'D', A'D, CD
(d) B'D', AC'

A = 0

DE \ BC	00	01	11	10
00		1		
01		1		
11	1	1		
10		1	1	

A = 1

BC / DE	00	01	11	10
00	1	1	1	1
01		1		
11		1		
10		1	1	

Figure 5.14 Examples of prime implicants on a 5-variable K-map.

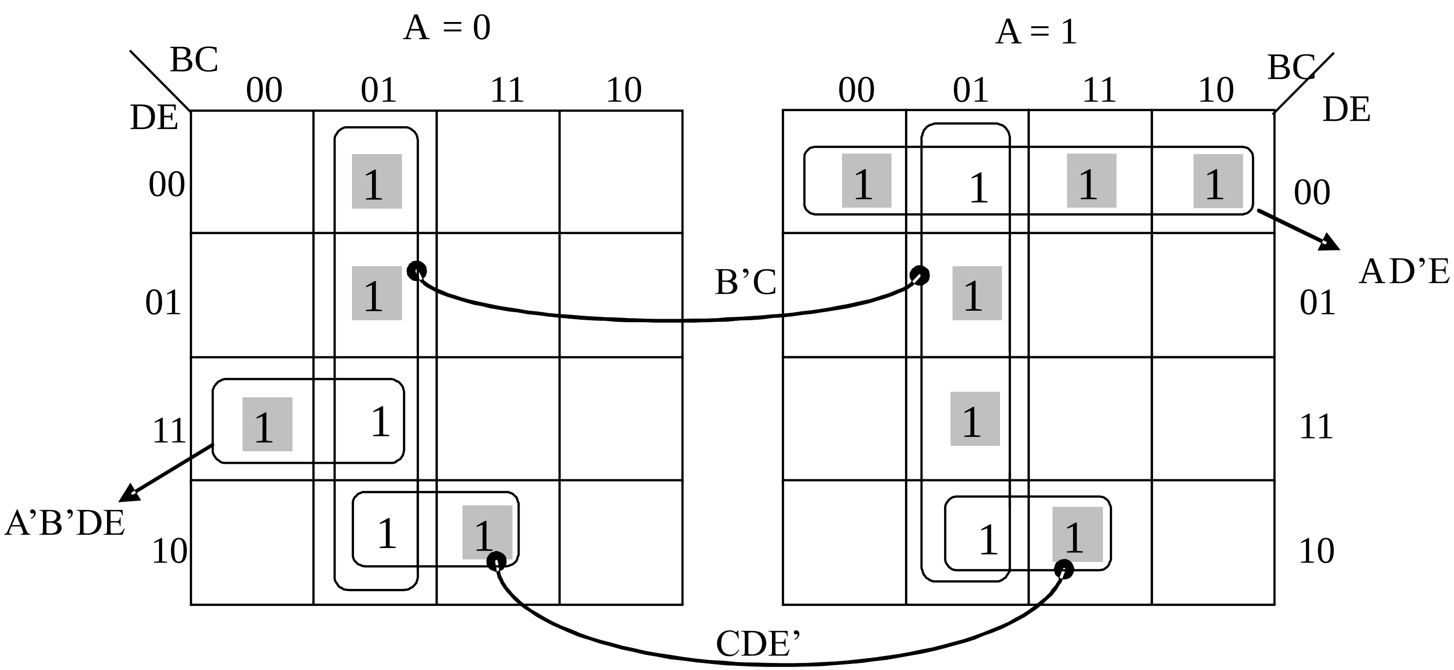
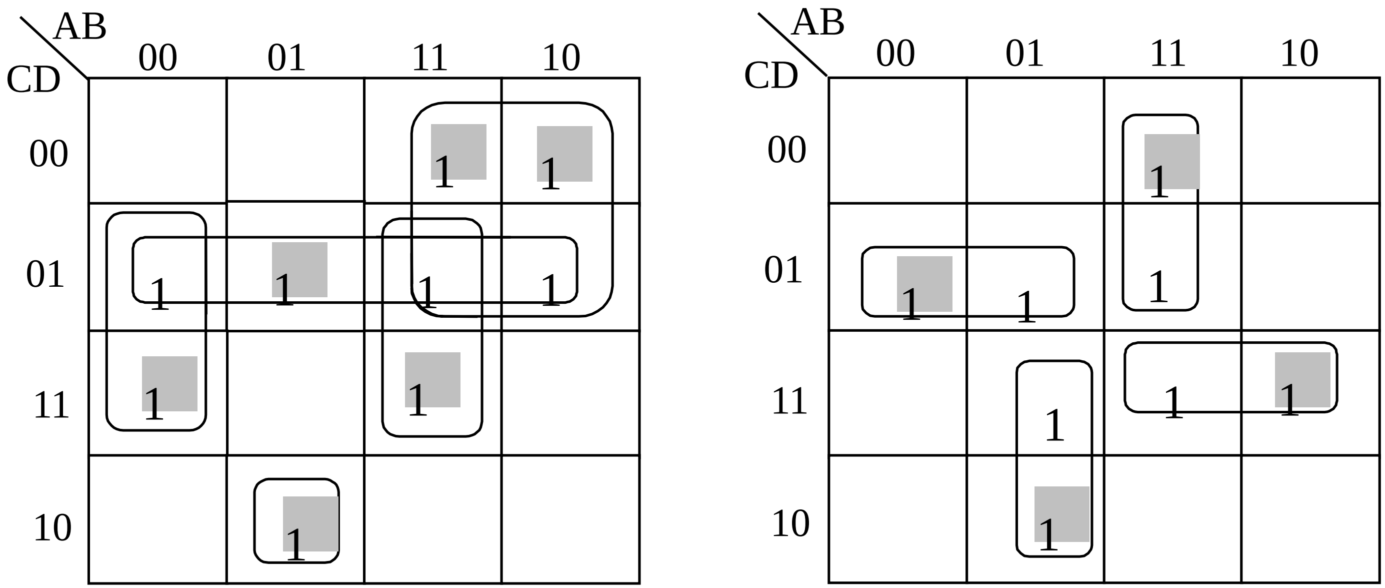


Figure 5.14 Examples of prime implicants on a 5-variable K-map.

5.3 Simplest Sum-of-Products Expression

(i) Select all the essential prime implicants.

(ii) Select a minimum number of secondary essential prime implicants with a minimum number of literals for all the 1-cells not covered by the essential prime implicants.



(a) (b)

Essential Prime implicants (a) A'BCD', A'B'D, ABD, C'D, AC'
(b) A'C'D, A'BC, ACD, ABC'

Simplest sum-of-proudcts

(a) F = A'BCD' + A'B'D + ABD + C'D + AC'

(b) F = A'C'D + A'BC + ACD + ABC'

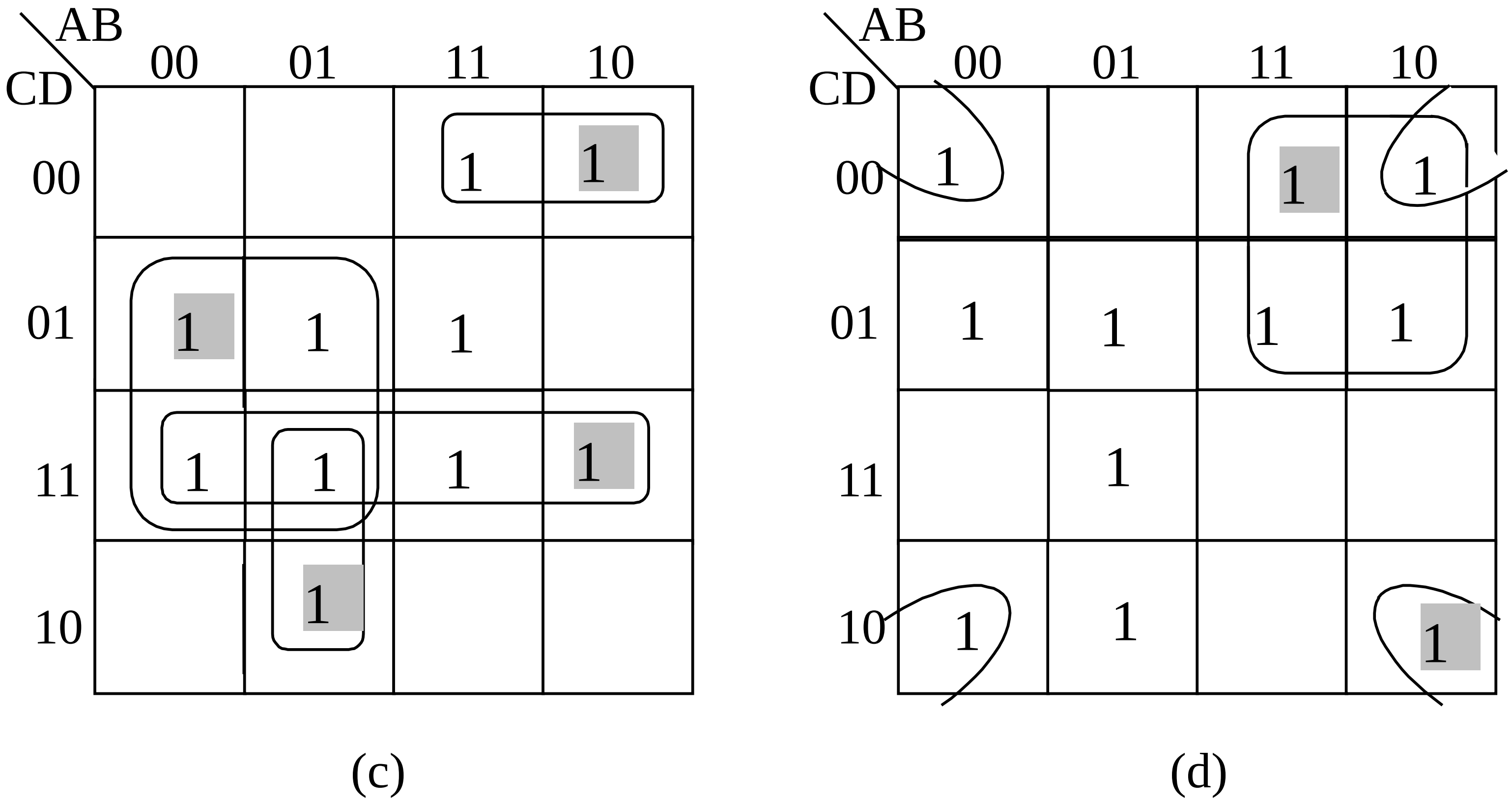


Figure 5.13 Examples of prime implicants.

Essential prime implicants (c) A'BC, AC'D', A'D, CD
(d) B'D', AC'

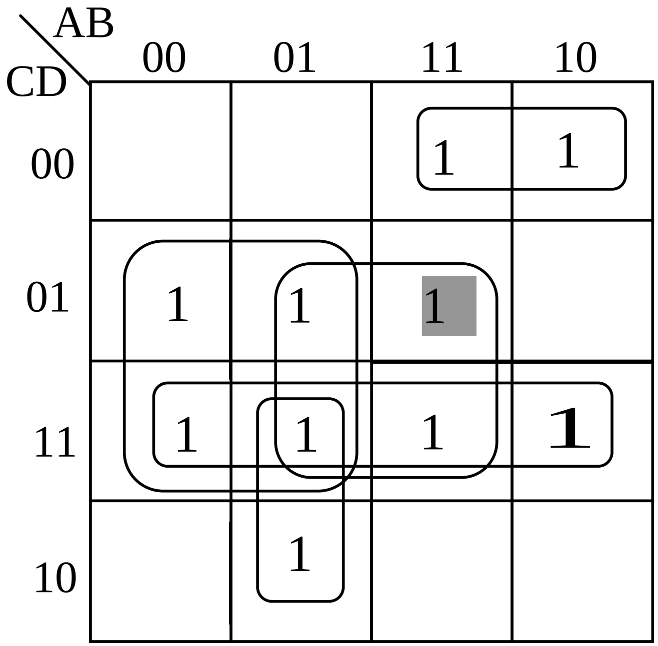


Figure 5.15(a) From Figure 5.13(c)

Essential prime implicants
A'BC, AC'D', A'D, CD

Simplest sum-of-products
$F = A'BC + AC'D' + A'D + CD + BD$

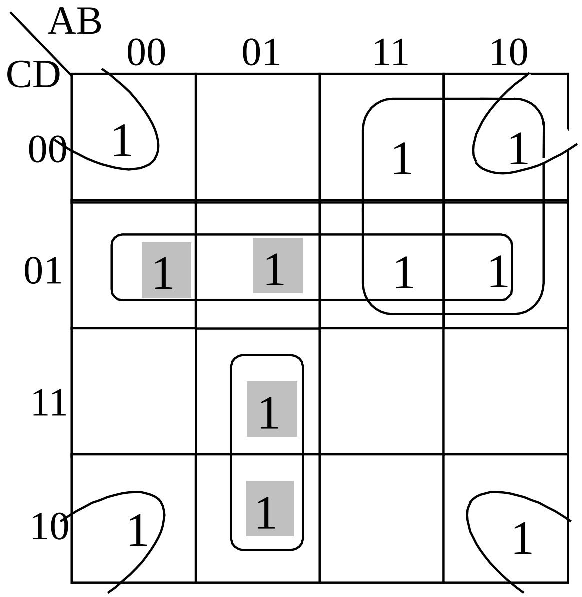


Figure 5.15 (b) From Figure 5.13 (d)

Essential prime implicants
B’D’, AC’

Simplest Sum-of-products
F = B’D’ + AC’ + C’D + A’BC

CD \ AB	00	01	11	10
00	1			1
01	1	1	1	1
11	1		1	1
10	1	1		1

Figure 5.16 K-map for Example 5.1.

CD \ AB	00	01	11	10
00	1	1		1
01			1	1
11		1	1	
10	1	1		

Figure 5.17 K-map for Example 5.2

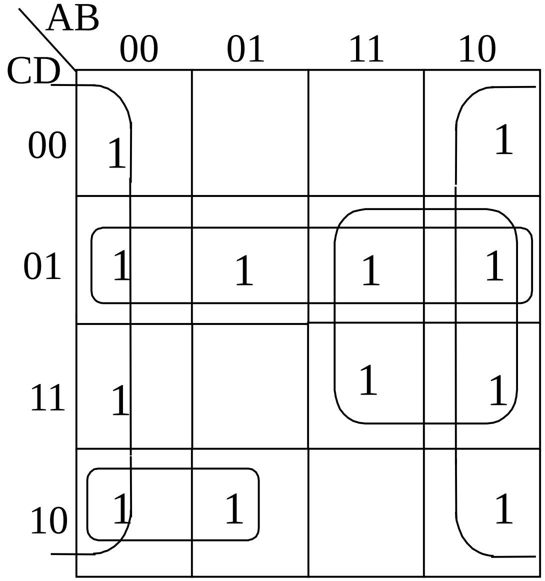


Figure 5.16 K-map for Example 5.1.

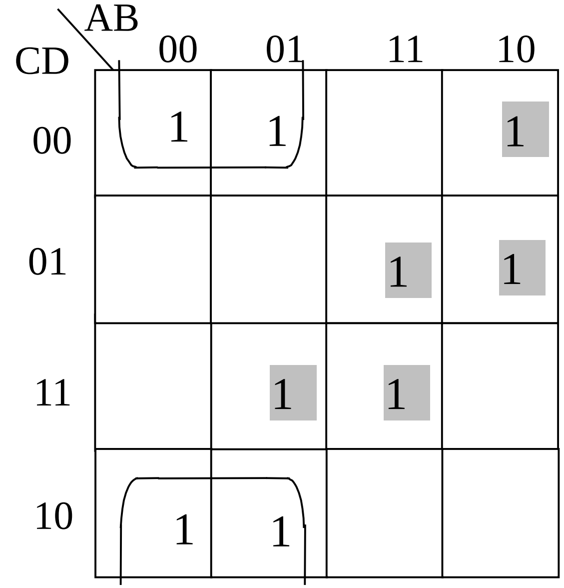


Figure 5.17 K-map for Example 5.2

Example 5.1

$F(A,B,C,D) = B' + C'D + A'CD' + AD$

Example 5.2

(a) $F(A,B,C,D) = A'D' + \underline{A'BC} + \underline{ABD} + \underline{AB'C'}$

(b) $F(A,B,C,D) = A'D' + \underline{BCD} + \underline{ABD} + \underline{AB'C'}$

(c) $F(A,B,C,D) = A'D' + \underline{BCD} + \underline{AC'D} + \underline{AB'C'}$

(d) $F(A,B,C,D) = A'D' + \underline{BCD} + \underline{AC'D} + \underline{B'C'D'}$

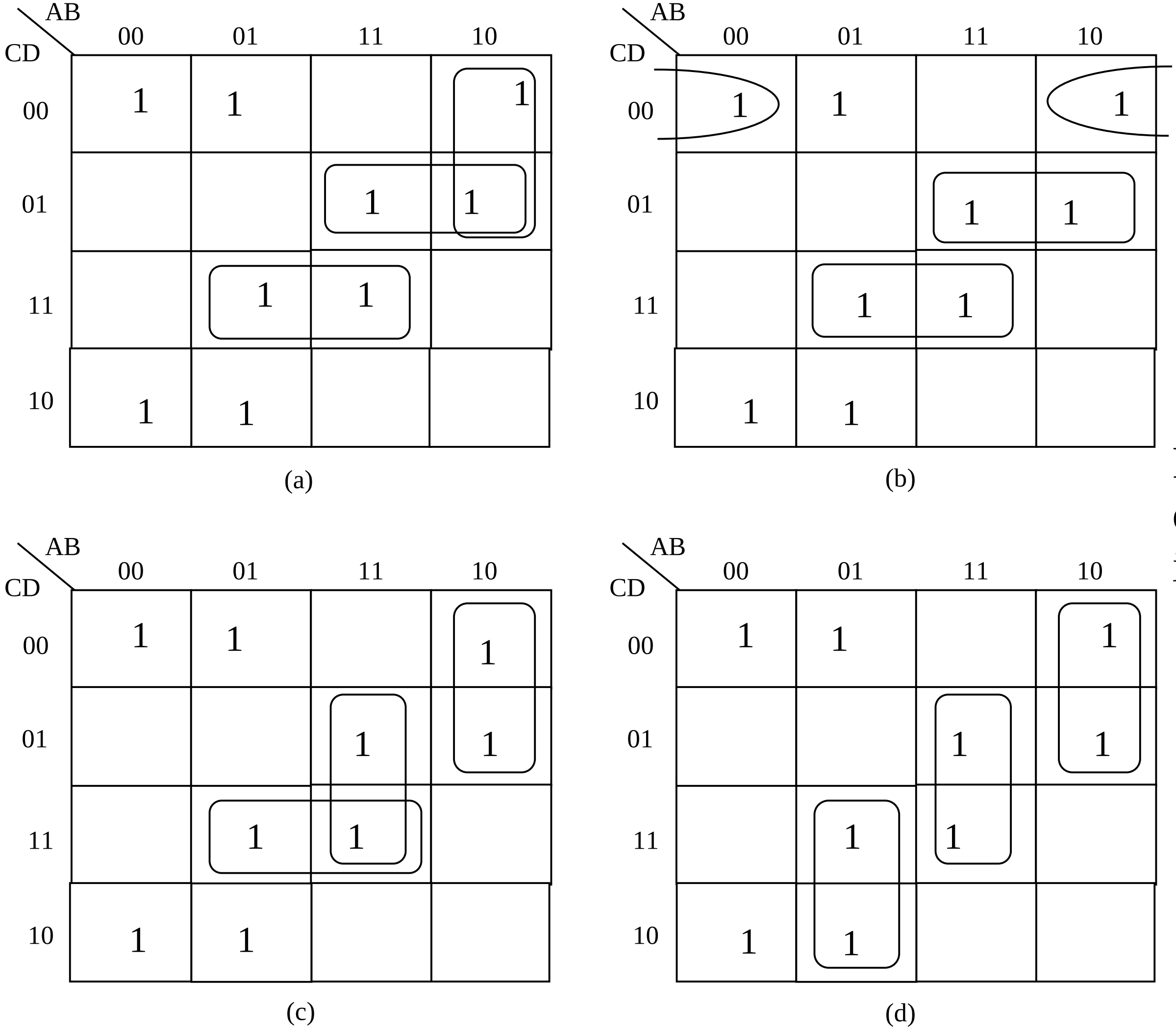


Figure 5.18 Selections of secondary essential prime implicants for Example 5.2

Example 5.2

(a) F(A,B,C,D) = A’D’ + A’BC + ABD + AB’C’

(b) F(A,B,C,D) = A’D’ + BCD + ABD + AB’C’

(c) F(A,B,C,D) = A’D’ + BCD + AC’D + AB’C’

(d) F(A,B,C,D) = A’D’ + BCD + AC’D + B’C’D’

❖ Example 5.3

A = 0

DE \ BC	00	01	11	10
00		1	1	
01	1	1	1	
11		1	1	
10				

A = 1

DE \ BC	00	01	11	10
00		1	1	
01		1		
11	1	1		1
10				

Figure 5.19 K-map for Example 5.3.

❖ Example 5.3

$$F(A,B,C,D,E) = A'B'D'E + A'CE + ACDE + B'CE' + B'CE$$

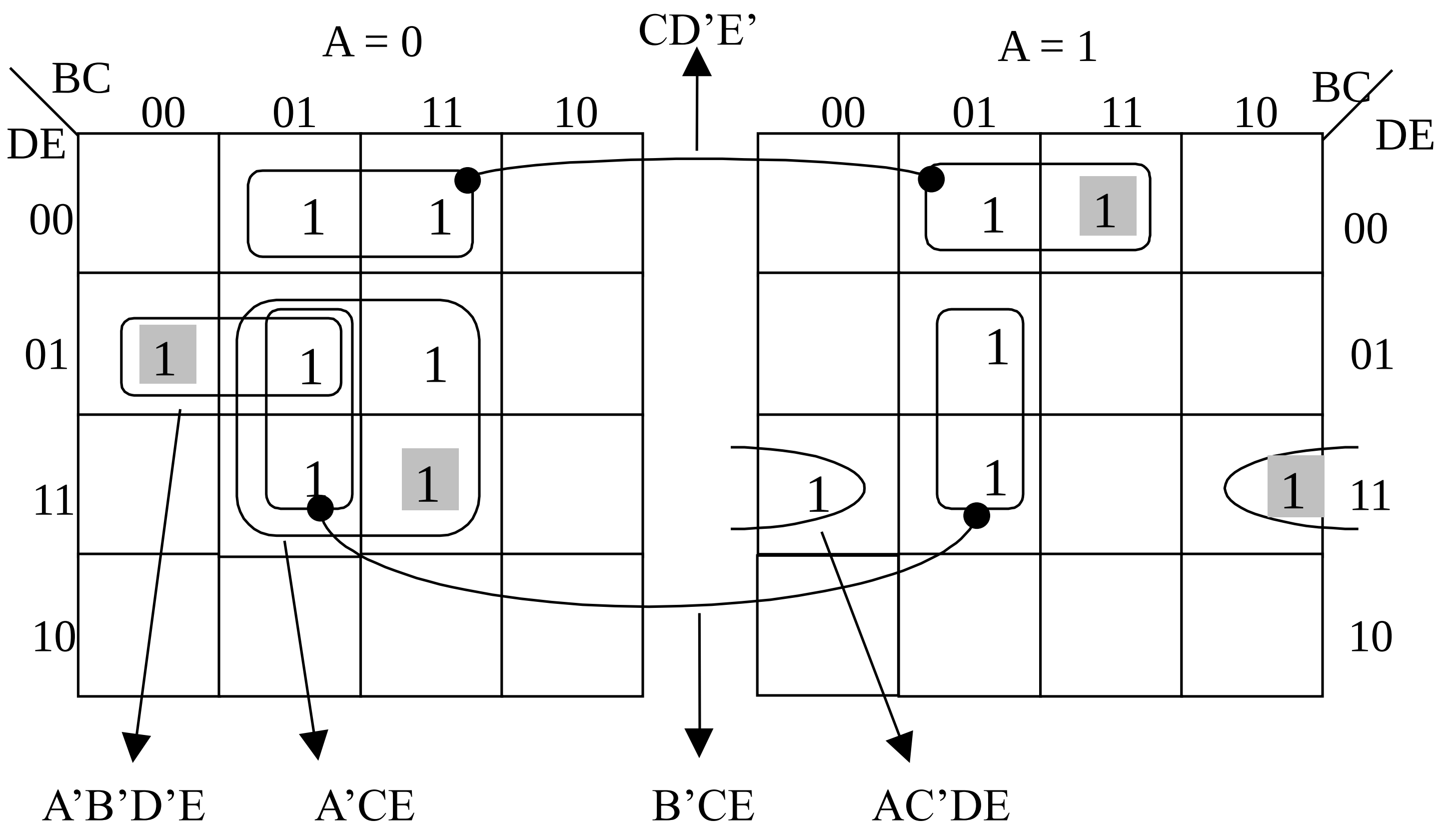


Figure 5.19 K-map for Example 5.3.

5.4 Simplest Product-of-Sums Expressions

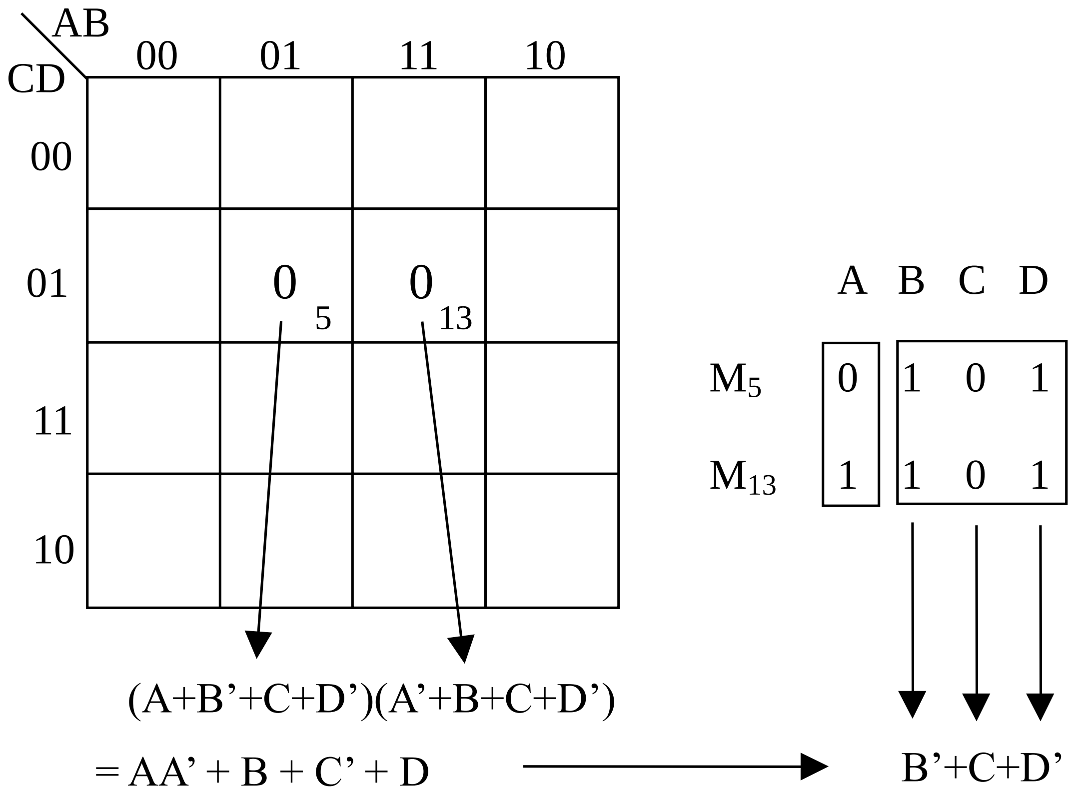


Figure 5.20 Logically adjacent maxterms.

Example 5.4

CD \ AB	00	01	11	10
00			0	
01		0	0	
11		0	0	
10	0	0	0	0

(a)

CD \ AB	00	01	11	10
00	0			0
01	0	0	0	0
11		0		
10		0	0	0

(b)

Figure 5.21 Karnaugh maps for Example 5.4.

❖ Example 5.4

(a) F(A,B,C,D) = (A’ + B’) (B’ + D’) (C’ + D)

(b) F(A,B,C,D) = (B + C) (C + D’) (A + B’ + C’) (A’ + C’ + D)

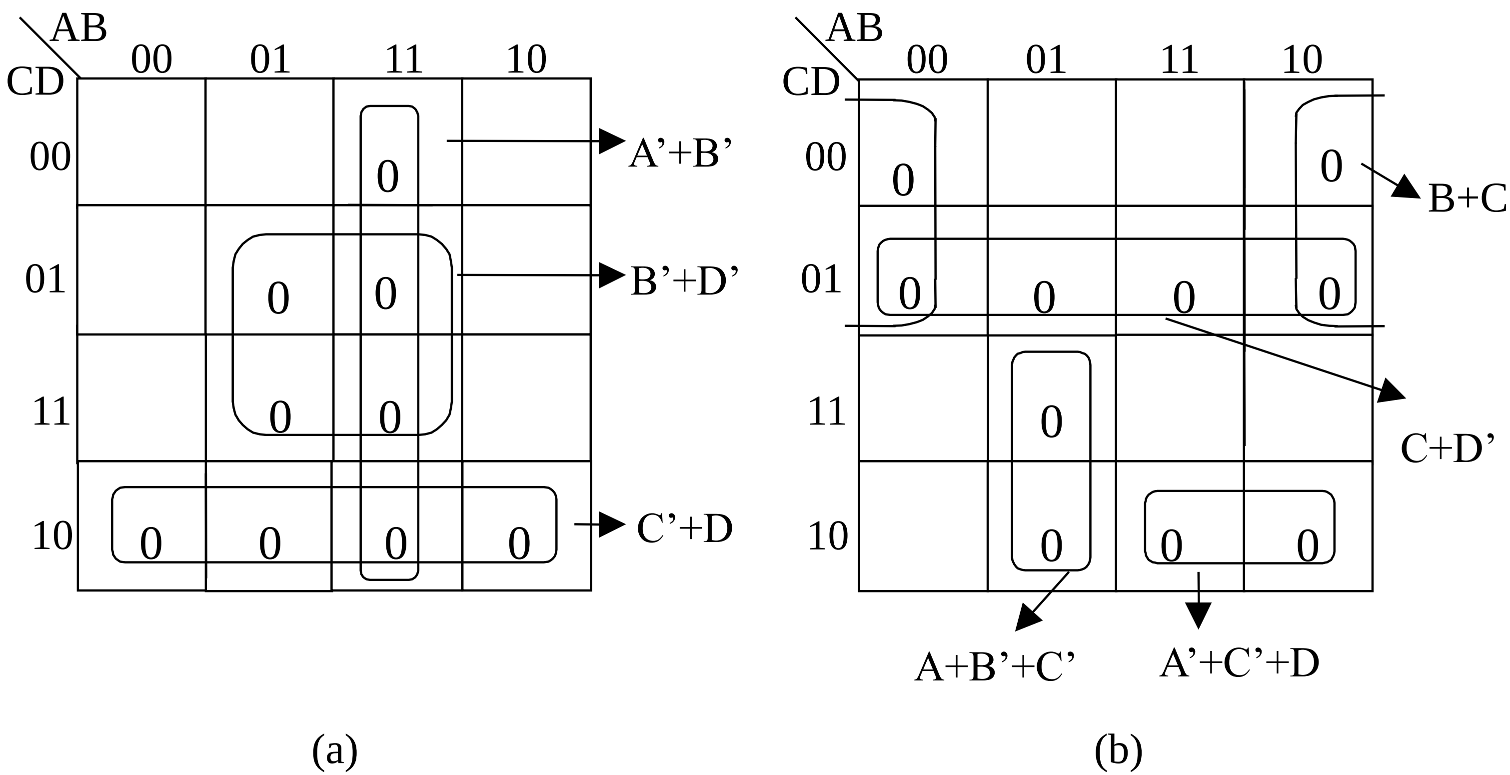


Figure 5.21 Karnaugh maps for Example 5.4.

- Example 5.5

A = 0

DE \ BC	00	01	11	10
00		0		
01		0		
11	0	0		0
10		0	0	0

A = 1

DE \ BC	00	01	11	10
00		0		
01		0		
11		0	0	0
10		0	0	0

Figure 5.22 Karnaugh map for Example 5.5.

❖ Example 5.5

$$F(A,B,C,D,E) = (B + C')\,(B' + D' + E)\,(A + C + D' + E')\,(A' + B' + D')$$

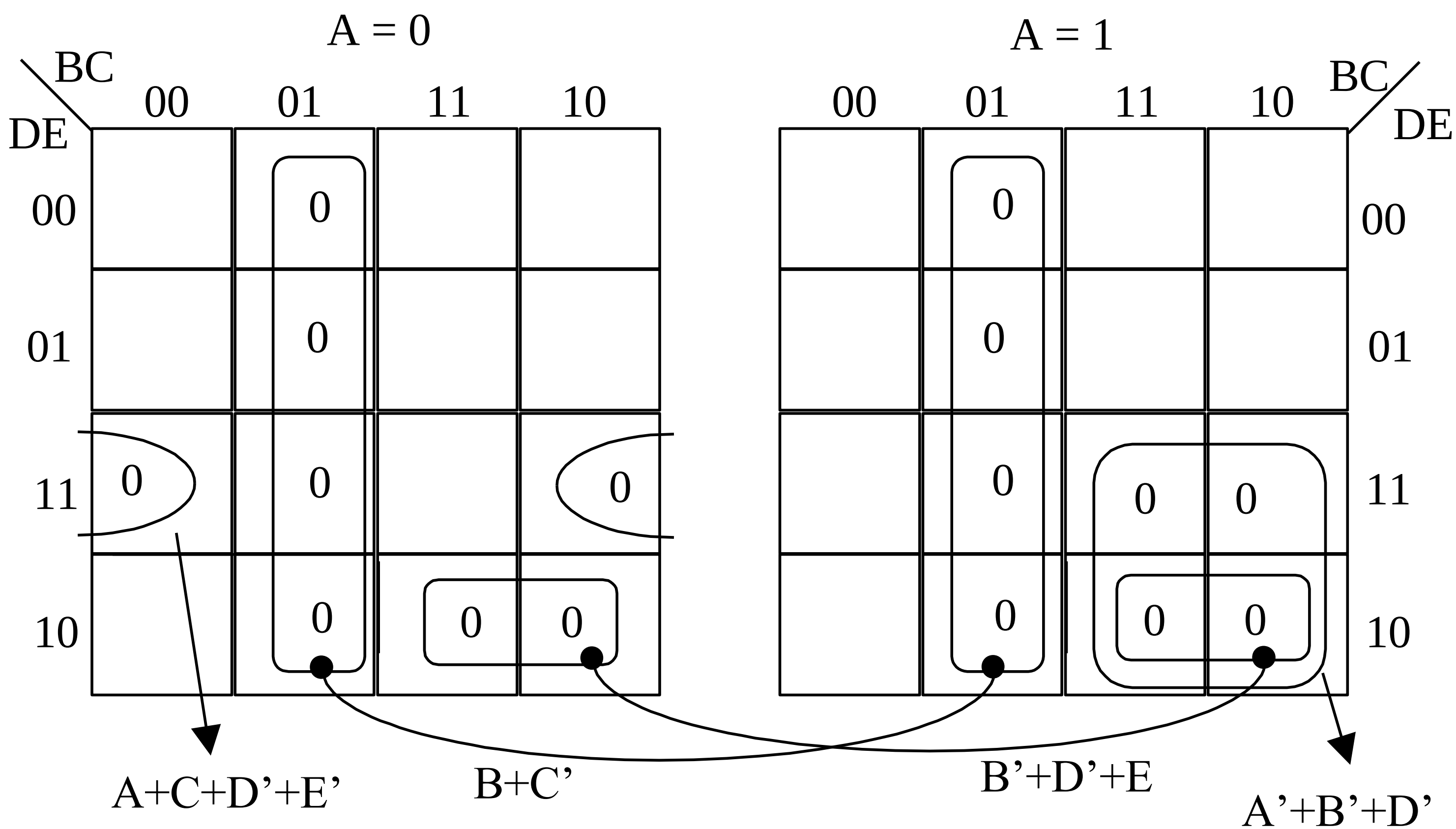


Figure 5.22 Karnaugh map for Example 5.5.

5.5 Minimization of Incompletely Specified Functions

(1) Do not consider a don't-care term to be a distinguished cell. A don’t care term is a distinguished cell if it can be included in only one prime implicant (or implicate).

(2) Always include don't-care terms with 1-cells or 0-cells to form a higher order cube or larger grouping.

yz \ wx	00	01	11	10
00			d	
01	1	1	d	
11	1	1	d	d
10	1		d	d

Figure 5.23 Karnaugh map for the verification of the function in Table 4.8.

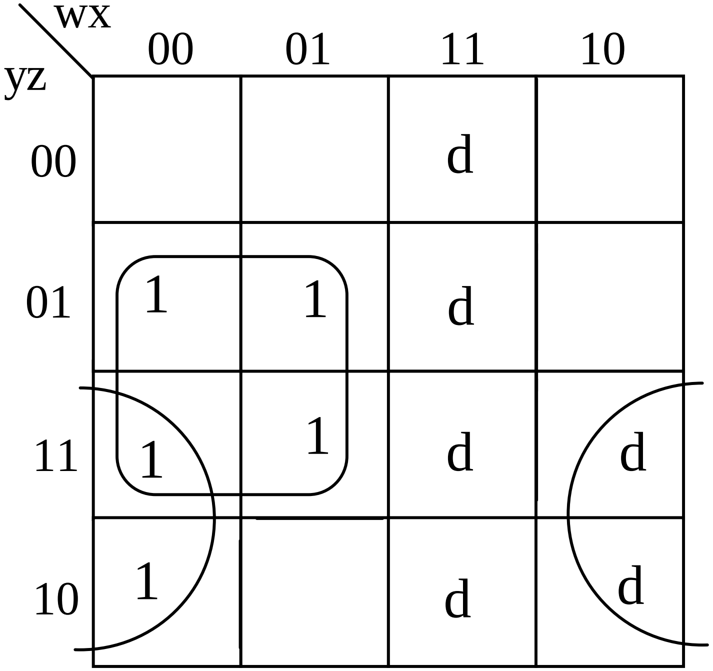


Figure 5.23 Karnaugh map for the verification of the function in Table 4.8.

Example 5.5

CD \ AB	00	01	11	10
00		1	1	1
01		1	d	
11	d	1		d
10	d	1	1	d

(a)

CD \ AB	00	01	11	10
00		d	0	0
01		0	0	
11	d	0	0	d
10	0		d	0

(b)

Figure 5.24 Karnaugh maps for Example 5.6.

Example 5.5

(a) $F(A,B,C,D) = A'B + AD'$

(b) $F(A,B,C,D) = (A' + D)(B + C')(B' + D')$

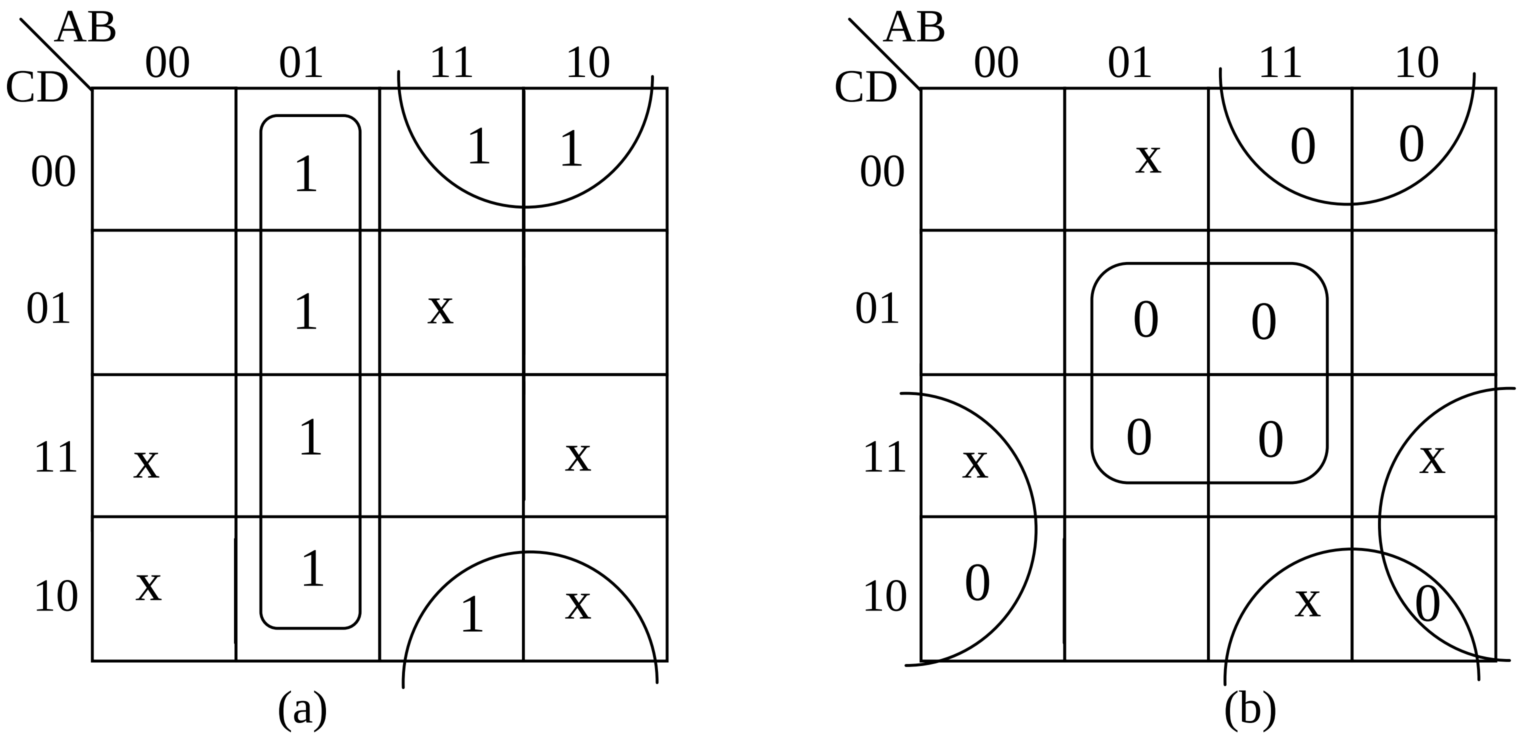


Figure 5.24 Karnaugh maps for Example 5.6.

5.6 Exclusive-OR and Exclusive-NOR Patterns

$m_0 + m_5 = A'B'C'D' + A'BC'D = A'C'(B'D' + BD) = A'C'(B \oplus D)'$ (Pattern 1)

$m_0 + m_{10} = A'B'C'D' + AB'CD' = B'D'(A'C' + AC) = B'D'(A \oplus C)'$ (Pattern 1)

$m_5 + m_9 = A'BC'D + AB'C'D = C'D(A'B + AB') = C'D(A \oplus B)$ (Pattern 2)

$m_9 + m_{10} = AB'C'D + AB'CD' = AB'(C'D + CD') = AB'(C \oplus D)$ (Pattern 3)

CD \ AB	00	01	11	10
00	1 0	4	12	8
01	1	1 5	13	1 9
11	3	7	15	11
10	2	6	14	1 10

Figure 5.25 Exclusive-OR patterns for 0-cubes.

$$A'B'C' + ABC' = C' (A'B' + AB) = C' (A \oplus B)' \qquad \text{(Pattern 2)}$$

$$A'B'C' + AB'C = B' (A'C' + AC) = B' (A \oplus C)' \qquad \text{(Pattern 1)}$$

$$ABC' + AB'C = A (BC' + B'C) = A (B \oplus C) \qquad \text{(Pattern 1)}$$

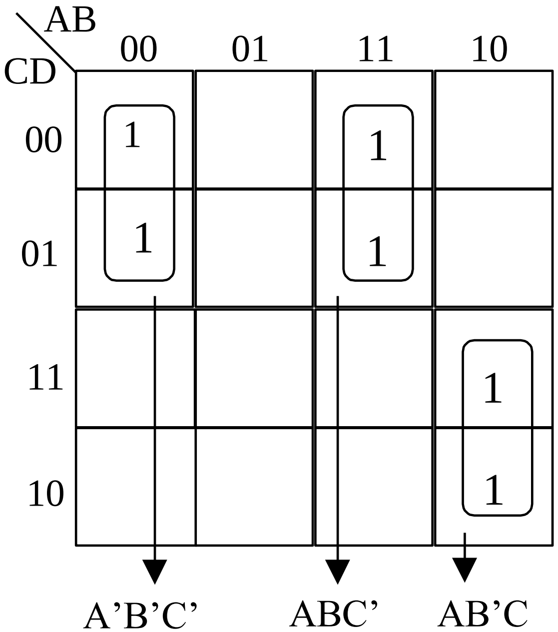


Figure 5.26 Exclusive-OR patterns for 1-cubes.

$A'B + AB' = (A \oplus B)$ (Pattern 2)

$B'D' + BD = (B \oplus D)'$ (Pattern 1)

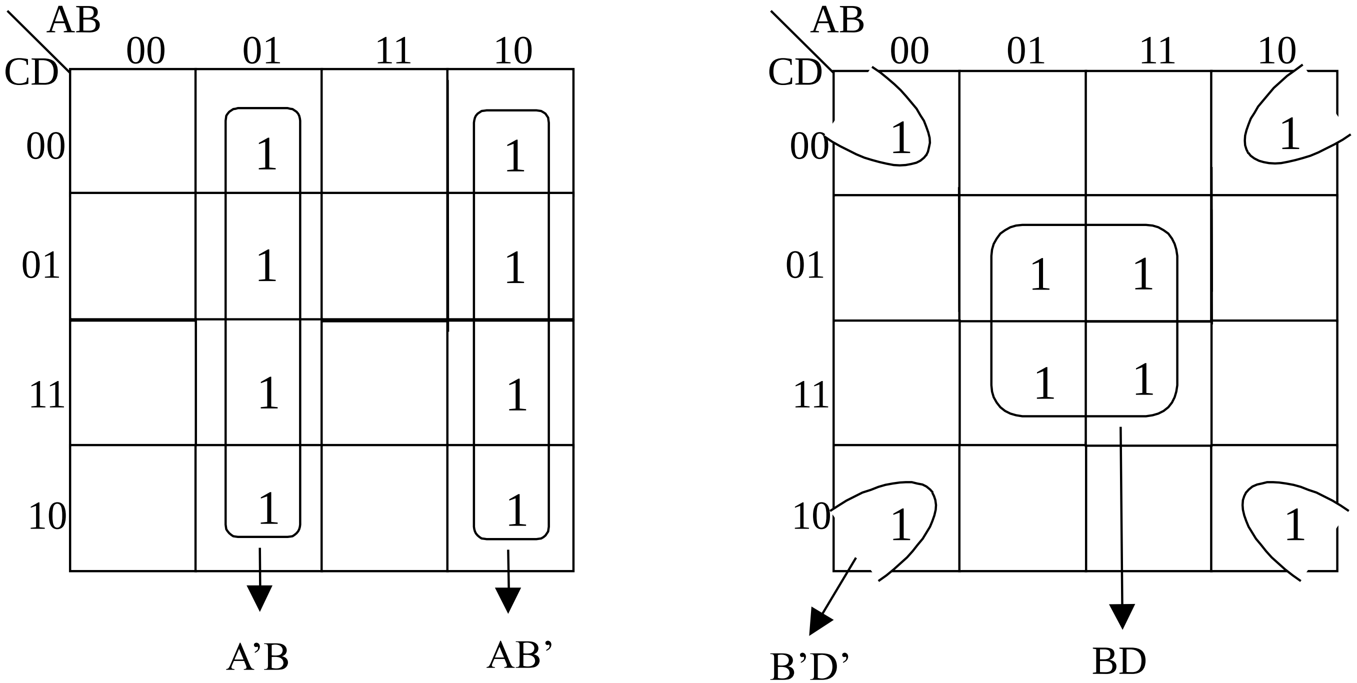


Figure 5.27 Exclusive-OR patterns for 2-cubes.

$(A' + C + D)(A' + C' + D') = A' + (C + D)(C' + D') = A' + (C \oplus D)$ (Pattern 3)

$(A + D')(A' + D) = (A \oplus D)'$ (Pattern 1)

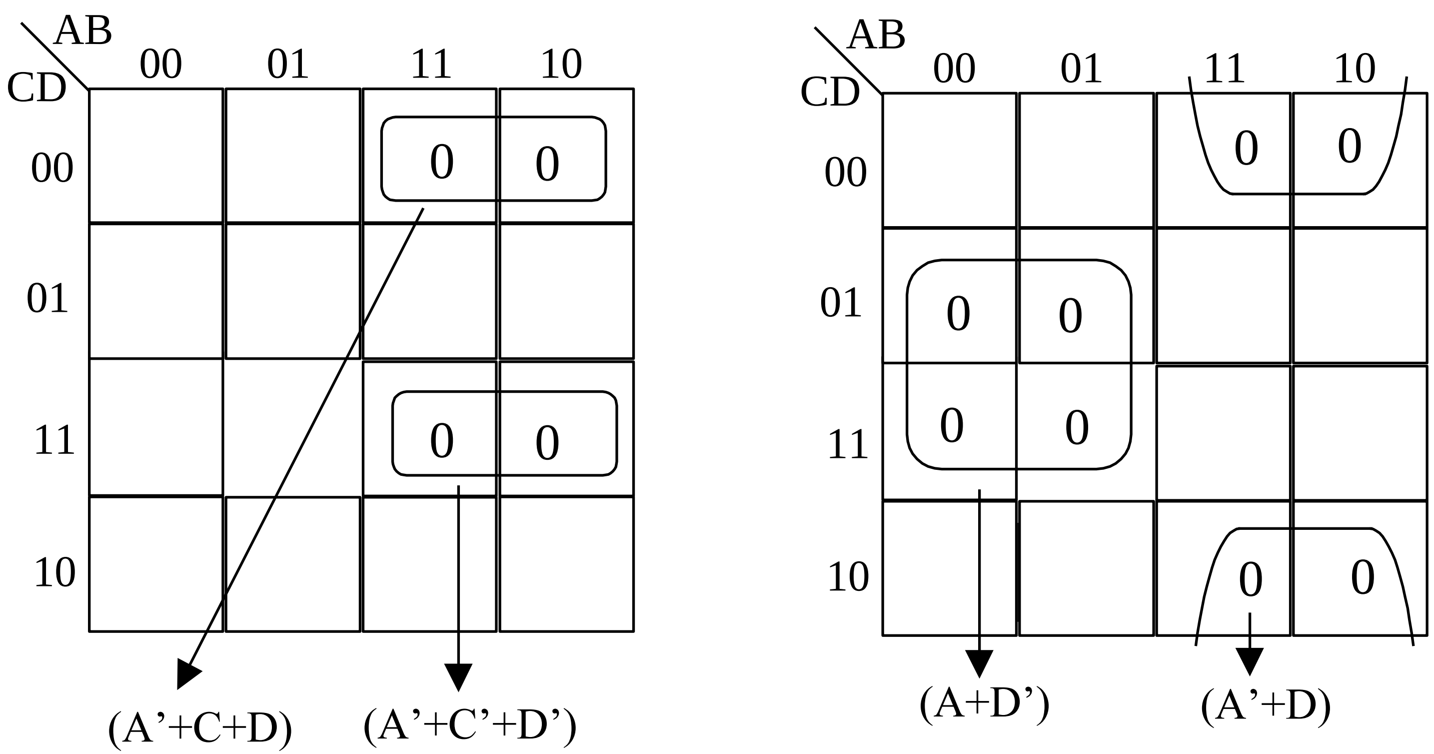


Figure 5.28 Exclusive-OR patterns from 0-cells.

❖ Example 5.7

$$y_0 = F(x_2, x_1, x_0) = \Sigma\, m(1, 2, 4,7)$$

$$F(x_2, x_1, x_0) = x_2'x_1'x_0 + x_2'x_1x_0' + x_2x_1'x_0' + x_2x_1x_0$$

$$\begin{aligned}
F(x_2, x_1, x_0) &= (m_1 + m_2) + (m_4 + m_7) \\
&= (x_2'x_1'x_0 + x_2'x_1x_0') + (x_2x_1'x_0' + x_2x_1x_0) \\
&= x_2'(x_1'x_0 + x_1x_0') + x_2(x_1'x0' + x_1x_0) \\
&= x_2'(x_1 \oplus x_0) + x_2(x_1 \oplus x_0)' \\
&= x_2 \oplus x_1 \oplus x_0
\end{aligned}$$

x_0 \ x_2x_1	00	01	11	10
0		1		1
1	1		1	

Figure 5.29 K-map for y_0 in Table 1.2.

4.3 Expansion of Boolean Functions

Shannon's expansion theorem

$$F(x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i, x_{i-1}, \ldots, x_2, x_1, x_0) = x_i' F_{x_i = 0} + x_i F_{x_i = 1} \qquad (4.2)$$

where $$F_{x_i = 0} = F(x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i = 0, x_{i-1}, \ldots, x_2, x_1, x_0) \qquad (4.3a)$$

and $$F_{x_i = 1} = F(x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i = 1, x_{i-1}, \ldots, x_2, x_1, x_0) \qquad (4.3b)$$

are called sub-functions of F and x_i is called an expansion variable.

x_i	Left-hand-side of (4.2)	Right-hand-side of (4.2)
0	$F(x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i = 0, x_{i-1}, \ldots, x_2, x_1, x_0) = F_{x_i = 0}$	$(0)'F_{x_i = 0} + (0)F_{x_i = 1} = F_{x_i = 0}$
1	$F(x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i = 1, x_{i-1}, \ldots, x_2, x_1, x_0) = F_{x_i = 1}$	$(1)'F_{x_i = 0} + (1)F_{x_i = 1} = F_{x_i = 1}$

Binary Tree

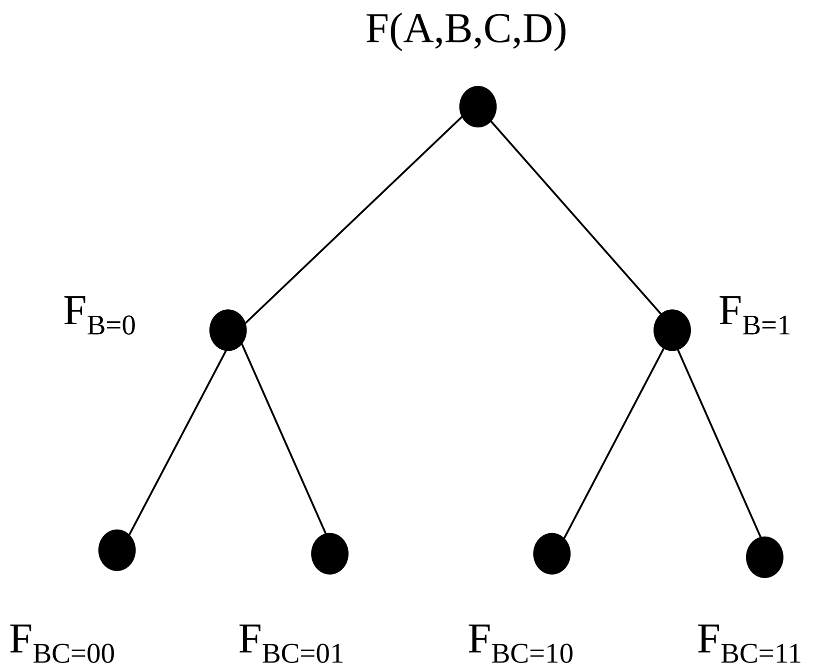


Figure 4.3 A binary tree for the expansion of a Boolean function.

Shannon's expansion theorem

$$F(x_{n-1}, x_{n-2}, ..., x_{i+1}, x_i, x_{i-1},......., x_2, x_1, x_0) = x_i' F_{x_i = 0} + x_i F_{x_i = 1} \quad (4.2)$$

$$F(A,B,C,D) = A'B'C + BC' + AC'D + ABD' \quad (4.4)$$

B : expansion variable

$$F_{B=0} = F(A, B = 0, C, D)$$
$$= A'(0)'C + (0)C' + AC'D + A(0)D' = A'C + AC'D \quad (4.5a)$$

$$F_{B=1} = F(A, B = 1, C, D)$$
$$= A'(1)'C + (1)C' + AC'D + A(1)D' = C' + AC'D + AD' = C' + AD \quad (4.5b)$$

$$F(A,B,C,D) = B'(\underline{A'C + AC'D}) + B(\underline{C' + AD'}) \quad (4.6)$$

$F(A,B,C,D) = B'(\underline{A'C + AC'D}) + B(\underline{C' + AD'})$ (4.6)

Sub-functions of $F_{B=0} = A'C + AC'D$ are

$F_{BC=00} = F(A, B = 0, C = 0, D) = A'(0) + A(0)'D = AD$ (4.7a)

$F_{BC=01} = F(A, B = 0, C = 1, D) = A'(1) + A(1)'D = A'$ (4.7b)

$F_{B=0} = A'C + AC'D = C'(\underline{AD}) + C(\underline{A'})$ (4.8)

Sub-functions of $F_{B=1} = C' + AD'$ are

$F_{BC=10} = F(A, B = 1, C = 0, D) = (0)' + AD' = 1$ (4.9a)

$F_{BC=11} = F(A, B = 1, C = 1, D) = (1)' + AD' = AD'$ (4.9b)

$F_{B=1} = C' + AD' = C'(\underline{1}) + C(\underline{AD'})$ (4.10)

$$\begin{aligned}
F(A,B,C,D) &= B'(\underline{A'C + AC'D}) + B(\underline{C' + AD'}) \\
&= B'[C'(\underline{AD}) + C(\underline{A'})] + B[C'(\underline{1}) + C(\underline{AD'})] \\
&= B'C'(\underline{AD}) + B'C(\underline{A'}) + BC'(\underline{1}) + BC(\underline{AD'}) \\
&= B'C'\,\underline{F_{BC=00}} + B'C\,\underline{F_{BC=01}} + BC'\,\underline{F_{BC=10}} + BC\,\underline{F_{BC=11}} \qquad (4.11)
\end{aligned}$$

Sub-Functions

$$F(A,B,C,D) = \Sigma\, m(0, 2, 3, 6, 8, 9, 12, 13, 15) + d(5, 7, 10) \qquad (5.1)$$

CD \ AB	00	01	11	10
00	1	0	1	1
01	0	d	1	1
11	1	d	1	0
10	1	1	0	d

(a)

$F_{A=0}$

CD \ B	0	1
00	1	0
01	0	d
11	1	d
10	1	1

$F_{A=1}$

CD \ B	1	0
00	1	1
01	1	1
11	1	0
10	0	d

(b)

Figure 5.30 (a) K-map of a 4-variable function F. (b) Partition of (a) into two sub-functions with A as expansion variable. (c) Partition of (a) into four sub-functions with A and B as expansion variables. (d) Partition of (a) into four sub-functions with C and D as expansion variables.

CD \ AB	00	01	11	10
00	1	0	1	1
01	0	d	1	1
11	1	d	1	0
10	1	1	0	d

(a)

$F_{A = 0}$

CD \ B	0	1
00	1	0
01	0	d
11	1	d
10	1	1

$F_{A = 1}$

CD \ B	1	0
00	1	1
01	1	1
11	1	0
10	0	d

(b)

CD	$F_{AB = 00}$	$F_{AB = 01}$	$F_{AB = 11}$	$F_{AB = 10}$
00	1	0	1	1
01	0	d	1	1
11	1	d	1	0
10	1	1	0	d

(c)

Figure 5.30 (a) K-map of a 4-variable function F. (b) Partition of (a) into two sub-functions with A as expansion variable. (c) Partition of (a) into four sub-functions with A and B as expansion variables. (d) Partition of (a) into four sub-functions with C and D as expansion variables.

CD \ AB	00	01	11	10
00	1	0	1	1
01	0	d	1	1
11	1	d	1	0
10	1	1	0	d

(a)

$$F(A,B,C,D) = A'C + BD + AC' + \cancel{A'}B'D' \qquad (5.2)$$

$$F_{AB=00} = F(A=0, B=0, C, D) = C + D' \qquad (5.3a)$$

$$F_{AB=01} = F(A=0, B=1, C, D) = C + D \qquad (5.3b)$$

$$F_{AB=10} = F(A=1, B=0, C, D) = C' + D' \qquad (5.3c)$$

$$F_{AB=11} = F(A=1, B=1, C, D) = C' + D \qquad (5.3d)$$

CD	$F_{AB=00}$	$F_{AB=01}$	$F_{AB=11}$	$F_{AB=10}$
00	1	0	1	1
01	0	d	1	1
11	1	d	1	0
10	1	1	0	d

(c)

$$F_{AB=00} = C + D' \qquad (5.4a)$$

$$F_{AB=01} = C \qquad (5.4b)$$

$$F_{AB=10} = C' \qquad (5.4c)$$

$$F_{AB=11} = C' + D \qquad (5.4d)$$

Figure 5.30 (a) K-map of a 4-variable function F. (b) Partition of (a) into two sub-functions with A as expansion variable. (c) Partition of (a) into four sub-functions with A and B as expansion variables. (d) Partition of (a) into four sub-functions with C and D as expansion variables.

CD \ AB	00	01	11	10
00	1	0	1	1
01	0	d	1	1
11	1	d	1	0
10	1	1	0	d

(a)

AB	00	01	11	10
$F_{CD = 00}$	1	0	1	1
$F_{CD = 01}$	0	d	1	1
$F_{CD = 11}$	1	d	1	0
$F_{CD = 10}$	1	1	0	1

(d)

Figure 5.30 (a) K-map of a 4-variable function F. (b) Partition of (a) into two sub-functions with A as expansion variable. (c) Partition of (a) into four sub-functions with A and B as expansion variables. (d) Partition of (a) into four sub-functions with C and D as expansion variables.

$$F(A,B,C,D) = \Sigma\, m(0, 2, 3, 6, 8, 9, 12, 13, 15) + d(5, 7, 10) \qquad (5.1)$$

A	$F_{BCD=000}$	$F_{BCD=001}$	$F_{BCD=010}$	$F_{BCD=011}$	$F_{BCD=100}$	$F_{BCD=101}$	$F_{BCD=110}$	$F_{BCD=111}$
0	1 $_{0}$	0 $_{1}$	1 $_{2}$	1 $_{3}$	0 $_{4}$	d $_{5}$	1 $_{6}$	d $_{7}$
1	1 $_{8}$	1 $_{9}$	d $_{10}$	0 $_{11}$	1 $_{12}$	1 $_{13}$	0 $_{14}$	1 $_{15}$

Figure 5.31 Construction of sub-functions K-maps from mintem list with B, C, and D as expansion variables.

$$F(A,B,C,D) = \Sigma\, m(0, 2, 3, 6, 8, 9, 12, 13, 15) + d(5, 7, 10) \qquad (5.1)$$

AD	$F_{BC=00}$	$F_{BC=01}$	$F_{BC=10}$	$F_{BC=11}$
00	1 $_{0}$	1 $_{2}$	0 $_{4}$	1 $_{6}$
01	0 $_{1}$	1 $_{3}$	d $_{5}$	d $_{7}$
11	1 $_{9}$	0 $_{11}$	1 $_{13}$	1 $_{15}$
10	1 $_{8}$	d $_{10}$	1 $_{12}$	0 $_{14}$

Figure 5.32 Construction of sub-function K-maps from mintem list with B and C as expansion variables.