'ution Spring 2020

unjustified answers may receive no credit. Calculators are not allowed on any
r. *Make sure to exhibit skills discussed in class*. Box all answers and simplify answers as
ble.

Good Luck! ☺

Systems of Linear Equations

pts] Determine the value(s) of h for which the following linear system is consistent:

$$\begin{cases} 9x_1 + hx_2 = 9 \\ hx_1 + x_2 = -3 \end{cases}$$

*Apply the row-reduction algorithm to the corresponding Aug. Matrix:

$$[A \,\vdots\, \vec{b}] = \left[\begin{array}{cc|c} 9 & h & 9 \\ h & 1 & -3 \end{array}\right] \underset{\sim}{\overset{-\frac{h}{9}R_1 + R_2}{N.R_2}} \left[\begin{array}{cc|c} 9 & h & 9 \\ 0 & 1-\frac{h^2}{9} & -3-h \end{array}\right]$$

Echelon Form

*We know the system is consistent when $1 - \frac{h^2}{9} \neq 0$:

$$1 - \frac{h^2}{9} = 0 \rightarrow 9 - h^2 = 0 \rightarrow (3-h)(3+h) = 0 \quad h = 3, \; h = -3$$

*Check:

i) When $h = 3$: $1 - \frac{(3)^2}{9} \overset{?}{=} -3-3 \rightarrow 1-1 \overset{?}{=} -6 \rightarrow 0 = -6$

$* \therefore h \neq 3$

ii) When $h = -3$: $1 - \frac{(-3)^2}{9} \overset{?}{=} -3-(-3) \rightarrow 1-1 = -3+3 \rightarrow 0 = 0$ ✓

$* \therefore h = -3$

$\therefore$ $\boxed{\text{The system is consistent } \forall\, h \in \mathbb{R}, \text{ except } h = 3}$

Ans.

2. ***The Matrix Equation, $A\vec{x} = \vec{b}$***

Consider the following matrix equation:

$$\begin{bmatrix} 1 & 2 & 13 \\ 1 & -1 & -2 \\ 2 & 4 & 26 \\ 2 & 1 & 11 \\ 3 & 3 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 2 \\ -26 \\ -11 \\ -24 \end{bmatrix}$$

(a) [3pts] Write the given Matrix Equation as a System of Linear Equations.

(b) [9pts] Solve the system and write the general solution in a parametric vector form.

Part(a):

$$\boxed{\begin{cases} x_1 + 2x_2 + 13x_3 = -13 \\ x_1 - x_2 - 2x_3 = 2 \\ 2x_1 + 4x_2 + 26x_3 = -26 \\ 2x_1 + x_2 + 11x_3 = -11 \\ 3x_1 + 3x_2 + 24x_3 = -24 \end{cases}}$$

Ans✓

Part (b): Row-reduce $[A \,\vdots\, \vec{b}]$ to R.R.E.F.:

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 1 & -1 & -2 & 2 \\ 2 & 4 & 26 & -26 \\ 2 & 1 & 11 & -11 \\ 3 & 3 & 24 & -24 \end{array}\right] \begin{array}{c} \\ \\ \tfrac{1}{2}R_3 \\ \\ \tfrac{1}{3}R_5 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 1 & -1 & -2 & 2 \\ 1 & 2 & 13 & -13 \\ 2 & 1 & 11 & -11 \\ 1 & 1 & 8 & -8 \end{array}\right] \begin{array}{c} -R_1+R_2=N.R_2 \\ -R_1+R_3=N.R_3 \\ \sim \\ -2R_1+R_4=N.R_4 \\ -R_1+R_5=N.R_5 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 0 & -3 & -15 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -15 & 15 \\ 0 & -1 & -5 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 13 & -13 \\ 0 & 1 & 5 & -5 \\ 0 & -1 & -5 & 5 \\ 0 & -1 & -5 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -2R_2+R_1=N.R_1 \\ \sim \\ R_2+R_3=N.R_3 \\ R_2+R_4=N.R_4 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -3 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} \cdot\, x_1 + 3x_3 = -3 \\ \cdot\, x_2 + 5x_3 = -5 \end{cases}$$

R.R.E.F :)

$$\Leftrightarrow \begin{cases} \cdot\, x_1 = -3 - 3x_3 \\ \cdot\, x_2 = -5 - 5x_3 \\ \cdot\, x_3 \text{ is free} \end{cases}$$

$$\therefore \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, \; x_3 \in \mathbb{R}}$$

Ans✓

3. *Solution Sets of Linear Systems*

Consider the following:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix}$$

(a) [9pts] Solve the Nonhomogeneous System $A\vec{x} = \vec{b}$ and write the solution in parametric-vector form.

(b) [3pts] Using the parametric vector form of the solution in part (a), determine a particular solution.

(c) [3pts] Write the general solution for the Homogeneous System, $A\vec{x} = \vec{0}$, in parametric vector form.

*Part (a): Row-Reduce $[A \mid \vec{b}]$ to RREF

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & -4 \\ 1 & 2 & 3 & -2 \\ -1 & -2 & -3 & 2 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 1 & 2 & 3 & -2 \\ -1 & -2 & -3 & 2 \end{array}\right] \underset{\substack{R_1 + R_3 \\ \overline{N.R_3}}}{\overset{\substack{-R_1 + R_2 \\ \overline{N.R_2}}}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \cdot x_1 + 2x_2 + 3x_3 = -2 \Leftrightarrow \begin{cases} \cdot x_1 = -2 - 2x_2 - 3x_3 \\ \cdot x_2 \text{ is free} \\ \cdot x_3 \text{ is free} \end{cases} ; \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 - 2x_2 - 3x_3 \\ 0 + x_2 + 0 \\ 0 + 0 + x_3 \end{bmatrix}$$

$$\therefore \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}$$ Ans✓

*Part (b): Let $x_2 = x_3 = 0 \rightarrow \therefore \boxed{\vec{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}}$ Ans✓

*Note: This is NOT a unique solution✓ This is one of infinitely many!✓

*Part (c): To find the Gen.Sol. for $A\vec{x} = \vec{0}$, simply remove $\vec{p}$:)

$$\therefore \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}}$$ Ans✓

4. ***Linear Independence***

Consider the following vectors:

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},\ \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},\ \vec{v_4} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

~~(a) [3pts] Show that the following set of vectors is Linearly Dependent: $\{\vec{v_1}, \vec{v_2}\}$~~ *(-3) on my test; (+3) on your test ☺*

(b) [7pts] Show that the following set of vectors is Linearly Independent: $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

(c) [7pts] Write $\vec{v_4}$ as a Linear Combination of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, if possible.

*Part(b): Let $A = [\vec{V_1}\ \vec{V_2}\ \vec{V_3}]$ & show that $A\vec{x}=\vec{0}$ has only trivial sol.

$$[A \mid \vec{0}] = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \underset{\substack{*-R_1+R_3 \\ N.R_3}}{\overset{\substack{*R_1+R_2 \\ N.R_2}}{\sim}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} \overset{-R_3 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} \overset{\substack{-3R_2+R_3 \\ N.R_3}}{}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

Echelon Form

∴ Since NO free variables ∃, $A\vec{x}=\vec{0}$ has only the trivial solution $(\vec{x}=\vec{0})$:

$\Rightarrow \boxed{\{\vec{V_1}, \vec{V_2}, \vec{V_3}\} \text{ are Linearly Independent}}$

Ans✓

*Part(c): Let $A = [\vec{V_1}\ \vec{V_2}\ \vec{V_3}]$ & solve $A\vec{x} = \vec{V_4}$

*Same row-operations as (a); Additional operation indicated ::

$$[A \mid \vec{V_4}] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ -1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{\substack{*-2R_2+R_1 \\ N.R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 5 & 9 \end{array}\right] \overset{*\frac{1}{5}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 9/5 \end{array}\right] \underset{\substack{-2R_3+R_1 \\ N.R_1}}{\overset{\substack{R_3+R_2 \\ N.R_2}}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & -6/5 \\ 0 & 0 & 1 & 9/5 \end{array}\right]$$

$$\Rightarrow \begin{cases} \cdot\ x_1 = 2/5 \\ \cdot\ x_2 = -6/5 \\ \cdot\ x_3 = 9/5 \end{cases}$$

∴ $\boxed{\vec{V_4} = \frac{2}{5}\vec{V_1} - \frac{6}{5}\vec{V_2} + \frac{9}{5}\vec{V_3}}$ Answer✓

Bonus Question [5pts]:

Let $\vec{e_1}, \vec{e_2}, \vec{e_3} \in \mathbb{R}^3$ be the elementary vectors in $\mathbb{R}^3$, and let $\vec{y_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{y_2} = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix}$, & $\vec{y_3} = \begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix}$.

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a Linear Transformation that maps $\vec{e_1}$ to $\vec{y_1}$, maps $\vec{e_2}$ to $\vec{y_2}$, and maps $\vec{e_3}$ to $\vec{y_3}$.

Find the image under T of $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

$$* \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 3\vec{e_1} + 6\vec{e_2} + 9\vec{e_3}$$

$$* T\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = T\left(3\vec{e_1} + 6\vec{e_2} + 9\vec{e_3}\right) = 3T(\vec{e_1}) + 6T(\vec{e_2}) + 9T(\vec{e_3})$$

$$= 3\vec{y}_1 + 6\vec{y}_2 + 9\vec{y}_3 = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 6\begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix} + 9\begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} -24 \\ 30 \\ 36 \end{bmatrix} + \begin{bmatrix} 63 \\ 72 \\ -81 \end{bmatrix} = \begin{bmatrix} 42 \\ 108 \\ -36 \end{bmatrix}$$

$$\therefore \boxed{T\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 42 \\ 108 \\ -36 \end{bmatrix}}$$

Ansr

Scratch Work (Not Graded)