Section 4.5: The Dimension of a Vector Space:

Recall: In the last section, we saw that a Vector Space V w/ a Basis β containing n-vectors is isomorphic to $\mathbb{R}^n$.

*In this section: We see that "n" is an intrinsic property (the dimension) of V that does NOT depend on the basis.

***Theorem 9:** If a vector space V has a Basis $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$, then any set in V containing more than n-vectors must be Linearly Dependent.

***PROOF:**

- Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be some set in V w/ more than n-vectors.
- Since there are more vectors (p) than entries per vector (n):

$\Rightarrow$ The coordination vectors $[\vec{u}_1]_\beta, \ldots, [\vec{u}_p]_\beta$ are Linearly Dependent in $\mathbb{R}^n$. (*Sect. 1.7, Thm 8)

- By Definition of Linear Dependence:

$$\Rightarrow c_1[\vec{u}_1]_\beta + \cdots + c_p[\vec{u}_p]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ where: } \begin{cases} \bullet\ c_1, \ldots, c_p \rightarrow \text{Scalars/weights} \\ \bullet\ \text{NOT all scalars} = 0. \end{cases}$$

- Since the Coordinate Mapping is a Linear Transformation:

$$\Rightarrow [c_1\vec{u}_1 + \cdots + c_p\vec{u}_p]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- The $\vec{0}$ on the RHS displays the n-weights needed to build the vector-eq from the Basis vectors in β:

$$\Rightarrow c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = 0 \cdot \vec{b}_1 + \cdots + 0\vec{b}_n = \vec{0}$$

$\therefore$ $\boxed{\text{Since NOT all scalars are zero: } \{\vec{u}_1, \ldots, \vec{u}_p\} \text{ are Linearly Dependent.}}$ $\boxtimes$

Additional Notes on Theorem 9:

① This implies that if a vector space V has a Basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$, then each Linearly Independent set in V has <u>NO More</u> than n-vectors.

② <u>This theorem also applies to infinite sets in V:</u>

* An infinite set is said to be Linearly Dependent if some finite subset is Linearly Dependent; <u>otherwise</u>, the set is Linearly Independent :)

* <u>To Verify / Prove</u>: If S is an infinite set in V, take any subset of S, $\{\vec{u}_1, \dots, \vec{u}_p\}$, with $p > n$.
  → The same proof as theorem 9 shows that if a subset is Linearly Dependent, so is S!

***Theorem 10:** If a Vector Space V has a Basis of n-vectors, then every basis of V must consist of exactly n-vectors.

***Proof:**

- Let B_1 be a Basis of n-vectors, for some vector space V.
- Let B_2 be any other Basis, for some vector space V.
- <u>By Thm 9: Since B_1 is a Basis & B_2 is Lin. Independent:</u>
  → B_2 has NO more than n-vectors.
- <u>By Thm 9: Since B_2 is a Basis & B_1 is Lin. Independent:</u>
  → B_2 has has at least n-vectors.

∴ $\boxed{B_2 \text{ must have EXACTLY n-vectors}}$:)

☒

*Note: If a nonzero vector space V is spanned by a finite set S, then a subset of S is a Basis for V (By the 'Spanning Set Thm').

⇒ In this case, Thm 10 helps ensure the validity of the following definition :)

*Definition:

- If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **Dimension of V**, written as "dim (V)", is the number of vectors in a Basis for V.
- The dimension of the zero vector space $\{\vec{0}\}$ is defined to be zero.
- If V is NOT spanned by a finite set, then V is said to be **infinite-dimensional**.

Intro. Examples of the Dimension of V

① The Standard Basis for $\mathbb{R}^n$ contains n-vectors: ⇒ $\boxed{\dim(\mathbb{R}^n) = n}$

② The Standard Basis for $\mathbb{P}_2$ $\{1, t, t^2\}$:) contains 3-vectors: ⇒ $\boxed{\dim(\mathbb{P}_2) = 3}$

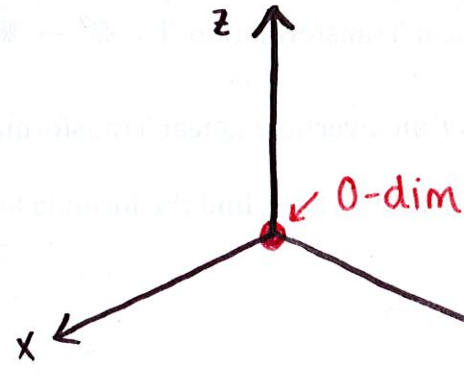
· Note: In general, $\dim(\mathbb{P}_n) = n+1$

The Subspaces of $\mathbb{R}^3$ can be classified by Dimension

Note: The following graphs are samples of possible subspaces in $\mathbb{R}^3$.

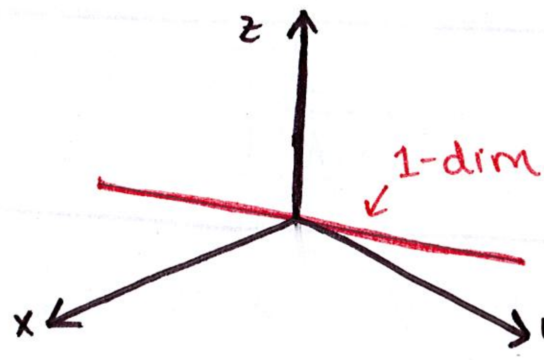
① 0-dimensional subspace:

- Only the zero subspace.



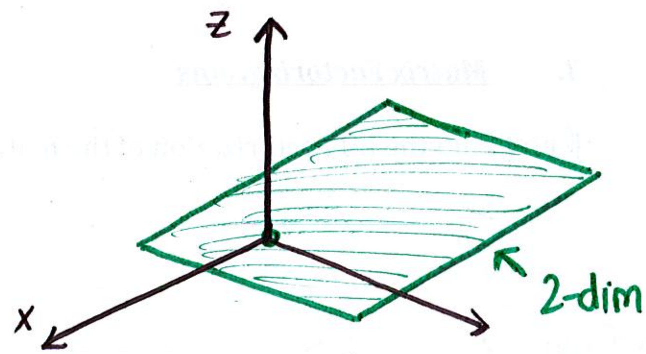
② 1-dimensional subspaces:

- Any subspace spanned by a single, nonzero vector.
- Such subspaces are <u>lines</u> through the origin.



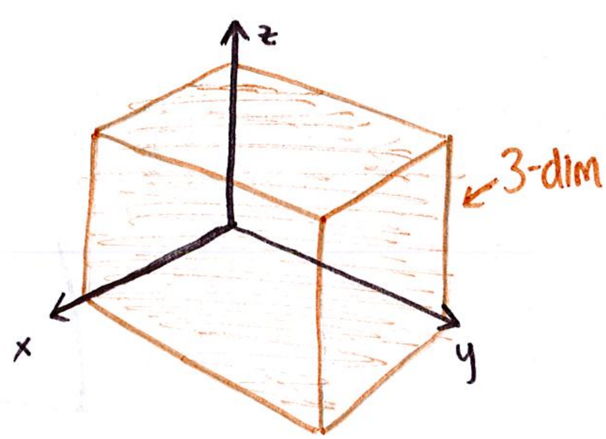
③ 2-dimensional subspaces:

- Any subspace spanned by 2, linearly independent vectors.
- Such subspaces are <u>planes</u> through the origin.



④ 3-dimensional subspaces:

- Only $\mathbb{R}^3$ itself.
- Any 3 Linearly Independent Vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$ (by: Invertible Matrix Thm)



Example[1] (Intro. to $\dim(V)$):

Let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ ST: $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Find the dimension of the subspace H.

Answer:

Recall: If V is spanned by a finite set, then V is said to be "finite-dimensional" & the $\dim(V)$ is the # of vectors in the Basis for V.

* Given: $H = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

* To Find $\dim(H)$, we need the number of vectors in the Basis for H:

Note: Since $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, we just need to check that the vectors are Linearly Independent:

* Since $\vec{v}_1 \neq c\vec{v}_2$ (NO scalar-Multiples $\exists$), $\{\vec{v}_1, \vec{v}_2\}$ in Linearly Independent ✓

$\Rightarrow \therefore$ Basis for H: $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ Since $\exists$ 2-vectors in the Basis for H: $\boxed{\dim(H) = 2}$

Ans✓

Example² (Intro to dim(V)): Find the dimension of the subspace:

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$$

Answer:

Note: To find dim(H), we need to first identify the # of vectors in the Basis of H :)

*Rewrite H as a Linear Combination of Column-Vectors:

$$H = \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} = \begin{bmatrix} a \\ 5a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ 0 \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 6c \\ 0 \\ -2c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4d \\ -d \\ 5d \end{bmatrix}$$

$$= a\underbrace{\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} + c\underbrace{\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}}_{\vec{v}_3} + d\underbrace{\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}}_{\vec{v}_4}$$

, where: $a,b,c,d \in \mathbb{R}$ (weights/scalars)

*To find the Basis for H, we need to determine if any Dependence Relations ∃ for $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$:

- $\vec{v}_3 = -2\vec{v}_2 = -2\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} \Longrightarrow \boxed{\vec{v}_3 \text{ is redundant}}$

So, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, v_4\}$ is Linearly Dependent....

Example² Continued...

• **By the Spanning Set Theorem:**

Since $\vec{v}_3 = -2\vec{v}_2$, we can remove $\vec{v}_3$ from the set of vectors & still have a set that spans H!

$$\Rightarrow \boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}}$$

∴ Since NO other dependence-relations ∃, $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is Linearly Independent ✓

$\Rightarrow$ A Basis for H: $\boxed{\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}}$

Therefore: Since ∃ 3-vectors in the Basis for H:

$$\boxed{\therefore \dim(H) = 3}$$ Ans.

Example: For the subspace below, find the following:

(a) A Basis For the Subspace.

(b) State the Dimension.

$$\left\{ \begin{bmatrix} s-5t \\ s+t \\ 5t \end{bmatrix} : t,s \in \mathbb{R} \right\}$$

Answer:

* Recall: If V is spanned by a finite set, then V is said to be finite-demsional & the dim(V) = # vectors in Basis.

*Part (a): Find the Basis of the Subspace:

$$\begin{bmatrix} s-5t \\ s+t \\ 5t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} = s\vec{V}_1 + t\vec{V}_2 \quad , \quad s,t \in \mathbb{R}$$

$\Rightarrow$ Since $\vec{V}_1 \neq c\vec{V}_2 \Rightarrow \{\vec{V}_1, \vec{V}_2\}$ is Linearly Independent.

$\therefore$ A Basis for the Subspace: $\boxed{\{\vec{V}_1, \vec{V}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} \right\}}$

Ans.

*Part (b): State the Dimension:

Since the Basis has 2 vectors $\Rightarrow$ $\boxed{\text{dimension is } 2}$

Ans.

Example: Find the Following for the provided subspace:

(a) Find the Basis.

(b) State the Dimension.

$$H = \left\{ \begin{bmatrix} p-5q \\ 7p+7r \\ -5q+8r \\ -3p+6r \end{bmatrix} : p,q,r \in \mathbb{R} \right\}$$

Answer:

* Part (a): Find the Basis of the Subspace H:

$$H = \begin{bmatrix} p-5q \\ 7p+7r \\ -5q+8r \\ -3p+6r \end{bmatrix} = p \underbrace{\begin{bmatrix} 1 \\ 7 \\ 0 \\ -3 \end{bmatrix}}_{\vec{v}_1} + q \underbrace{\begin{bmatrix} -5 \\ 0 \\ -5 \\ 0 \end{bmatrix}}_{\vec{v}_2} + r \underbrace{\begin{bmatrix} 0 \\ 7 \\ 8 \\ 6 \end{bmatrix}}_{\vec{v}_3}, \quad p,q,r \in \mathbb{R}$$

Check for any dependence relations amongs the vectors

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \mid \vec{0}] = \left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 7 & 0 & 7 & 0 \\ 0 & -5 & 8 & 0 \\ -3 & 0 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -5 & 8 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & -5 & 8 & 0 \\ 0 & 5 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* Echelon Form *

∴ Since NO Free-variables ∃ (i.e. n=3 pivots)

⇒ vectors are Linearly Independent

* Basis for H: $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}$ & $\boxed{\text{Dim}(H) = 3}$ Ans.

Example: Find the following for the provided subspace:

(a) Basis for H.

(b) State the Dimension.

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 2b + 5c = 0 \right\}$$

Answer:

*Part (a): Find the Basis of H:

Note: We can think of the vector-eq. as $A\vec{x} = \vec{0}$ here :)

$$a - 2b + 5c = 0 \Longleftrightarrow \boxed{\begin{cases} a = 2b - 5c \\ b \text{ is free} \\ c \text{ is free} \\ d \text{ is free} \end{cases}}$$

*General Solution to $A\vec{x} = \vec{0}$

*We can now decompose the General Sol. to a Linear Combination of column-vectors to find the Spanning Set of $A\vec{x} = \vec{0}$ (IOW: The Basis of Nul(A))

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2b - 5c \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{ST } b,c,d \in \mathbb{R}$$

(a)

$$\boxed{\therefore H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

*Basis for Nul(A) = H

Ans

Recall: These vectors are also automatically Linearly Ind. :)

Part (b) $\therefore$ Dimension of H: $\boxed{\dim(H) = 3}$

Ans

Example: Find the Dimension of the subspace H of $\mathbb{R}^2$ spanned by: $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

Answer:

* Given: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \end{bmatrix} \right\}$

Note: To find dim(H), we need to first find a Basis, $\mathcal{B}$, for H to identify the # of vectors in $\mathcal{B}$:)

* Check for Dependence-Relation(s) w/ set of vectors:

- $\begin{bmatrix} -1 \\ 3 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -3 \end{bmatrix} \Rightarrow \vec{v}_2 = (-1)\vec{v}_1$

- $\begin{bmatrix} -2 \\ 6 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -3 \end{bmatrix} \Rightarrow \vec{v}_3 = (-2)\vec{v}_1$

$\therefore$ By the Spanning Set Th^m: Since $\vec{v}_2 = (-1)\vec{v}_1$ & $\vec{v}_3 = (-2)\vec{v}_1$, $\{\vec{v}_1\} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ still spans H. ✓

$\Rightarrow$ Since the set contains only one vector, $\{\vec{v}_1\}$ is Linearly Independent ✓

Therefore: $\boxed{\text{A Basis for H is } \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\} \text{ \& } \dim(H) = 1}$

Ans.

*Subspaces of a Finite-Dimensional Space *

Note: The next theorem is a natural counterpart to the spanning set thm :)

*Theorem 11: Let H be a subspace of a finite-dimensional vector space V. Any Linearly Independent set in H can be expanded, if necessary, to a Basis for H. Also, H is finite-dimensional & $\dim(H) \le \dim(V)$

Note: When the dimension of a vector space or subspace is known, the search for a basis is simplified by the next theorem.
*It says that if a set has the right # of elements, then one has only to show either that:

i) The Set is Linearly Independent

-or-

ii) The Set spans the space.

*Theorem 12 (The Basis Theorem):

Let V be a p-dimensional vector space, $p \ge 1$.
Any Linearly Independent set of exactly p-elements in V is automatically a basis for V.
Any set of exactly p-elements that spans V is automatically a Basis for V.

Example: The 1^{st} 4 Hermite polynomials are 1, $2t$, $-2+4t^2$, & $-12t+8t^3$. (Those polynomials naturally arise in the study of certain Diff. Eq. in mathematical physics.)

Show that the first four Hermite Polynomials are a Basis for $\mathbb{P}_3$.

Answer:

Recall: The 'Standard Basis of $\mathbb{P}_3$' : $\{1, t, t^2, t^3\}$

*Set-up a matrix 'A' whose columns are the coordinate-vectors of the Hermite Polynomials, relative to $\{1, t, t^2, t^3\}$:

- $\vec{p}_1(t) = 1 \longrightarrow \vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $\vec{p}_3(t) = -2+4t^2 \Longrightarrow \vec{p}_3 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$
- $\vec{p}_2(t) = 2t \Longrightarrow \vec{p}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$
- $\vec{p}_4(t) = -12t+8t^3 \Longrightarrow \vec{p}_4 = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$

$$\therefore \boxed{A = [\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4] = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}}$$

$\searrow$

Example Continued...

$$A = \begin{bmatrix} \underline{1} & 0 & -2 & 0 \\ 0 & \underline{2} & 0 & -12 \\ 0 & 0 & \underline{4} & 0 \\ 0 & 0 & 0 & \underline{8} \end{bmatrix}$$

Notes:
- A is a 4×4 matrix
- A has $n = 4$ pivot rows/columns.

* Since Matrix A has 4-pivot columns:

$\Rightarrow$ NO Free-variables

$\Rightarrow$ So, the Columns of A are Linearly Independent.

Since Columns of $A = [\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4]$

$\therefore$ $\boxed{\text{The coordinate-vectors of the Hermite Polynomials are thus Linearly Independent in } \mathbb{P}_3.}$

* Since there are 4 Hermite Polynomials & the $\dim(\mathbb{P}_3) = 4$, then by the Basis Th^m:

$\therefore$ $\boxed{\text{The Hermite Polynomials form a Basis in } \mathbb{P}_3}$

$\Rightarrow$ Basis for $\mathbb{P}_3$: $\boxed{\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\} = \{1, 2t, -2+4t^2, -12t+8t^3\}}$

Example: Let $\mathcal{B}$ be the Basis of $\mathbb{P}_3$ consisting of the Hermite Polynomials: 1, $2t$, $-2+4t^2$, & $-12t+8t^3$. Let $\vec{p}(t) = 7-20t^2+8t^3$. Find the coordinate vector of $\vec{p}$ relative to $\mathcal{B}$, $[\vec{p}]_\beta$.

Answer:

Want: $[\vec{p}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = ?$

*Given:

- Basis of $\mathbb{P}_3$: $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\} = \{1, 2t, -2+4t^2, -12t+8t^3\}$

- Vector-Eq. for $\vec{p}(t) \in \mathbb{P}_3$: $\vec{p}(t) = 7-20t^2+8t^3$

*The Coordinates of $\vec{p}(t) = 7-20t^2+8t^3$, with respect to $\mathcal{B}$ satisfy the equation:

$$\vec{p}(t) = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 = 7-20t^2+8t^3$$

*Substituting $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ in:

$$\Rightarrow c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) = 7-20t^2+8t^3$$

*Group Like Terms:

$$c_1 + 2c_2t - 2c_3 + 4c_3t^2 - 12c_4t + 8c_4t^3 = 7 - 20t^2 + 8t^3$$

$$(c_1 - 2c_3) + (2c_2 - 12c_4)t + 4c_3t^2 + 8c_4t^3 = 7 - 20t^2 + 8t^3$$

Example Continued...

***Create a System of Eq. by equating the coefficients of the powers of t:**

- **Constants:** $c_1 - 2c_3 = 7$
- **Linear:** $2c_2 - 12c_4 = 0$
- **Quadratic:** $4c_3 = -20$
- **Cubic:** $8c_4 = 8$

$$\Rightarrow \boxed{\begin{cases} c_1 - 2c_3 = 7 \\ c_2 - 6c_4 = 0 \\ c_3 = -5 \\ c_4 = 1 \end{cases}}$$

***Solve the System:**

- Since $c_3 = -5 \longrightarrow$

$$c_1 - 2(-5) = 7$$
$$c_1 + 10 = 7$$
$$\boxed{c_1 = -3}$$

- Since $c_4 = 1 \longrightarrow$

$$c_2 - 6(1) = 0$$
$$c_2 - 6 = 0$$
$$\boxed{c_2 = 6}$$

***Therefore:**

$$\boxed{[\vec{p}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{matrix} -3 \\ 6 \\ -5 \\ 1 \end{matrix}}$$

Ans.

* The Dimensions of Nul(A) & Col(A) *

- **Dimension of Nul(A):**

The dimension of the Nul(A) is the number of free-variables in the homogeneous-equation, $A\vec{x} = \vec{0}$.

- **Dimension of Col(A):**

The dimension of the Col(A) is the number of pivot columns in A.

Note: Keeping the above summary in mind, we can find both the dimension of Nul(A) & Col(A) by row-reducing the augmented matrix [A] to Echelon Form :)

Example (Dimension of Nul(A) & Col(A)):

Find the Dimension of the Nul(A) & Col(A) for the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Answer:

*To find the Dimensions of the Nul(A) & Col(A), first row-reduce the augmented matrix [A] to Echelon Form:

$$[A] = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \overset{R_1 \Leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \overset{\frac{3R_1 + R_2}{N.R_2}}{\sim}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \underset{\frac{1}{5}R_2}{\overset{\frac{-2R_1 + R_3}{N.R_3}}{\sim}} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \overset{\frac{-R_2 + R_3}{N.R_3}}{\sim}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

* Echelon Form *

∴ Pivot Columns of A: $\{\vec{a}_1, \vec{a}_3\}$

∴ Free Variables: x_2, x_4, x_5

*Since ∃ 3-free variables: $\boxed{\therefore \dim[\mathrm{Nul}(A)] = 3}$ Ans.

*Since ∃ 2-pivot columns: $\boxed{\therefore \dim[\mathrm{Col}(A)] = 2}$ Ans.

Example: Find the dimensions of the Nul(A) & Col(A) for the matrix:

$$A = \begin{bmatrix} 1 & 6 & -8 & -6 & 1 \\ 0 & 0 & 0 & -8 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

*** Recall:**

- $\text{Dim}[\text{Nul}(A)]$ = # of free variables in $A\vec{x} = \vec{0}$
- $\text{Dim}[\text{Col}(A)]$ = # of pivot columns in A

*** Row-Reduce $[A \,\vdots\, \vec{0}]$ to Echelon Form:**

$$\left[\begin{array}{ccccc|c} \underline{1} & 6 & -8 & -6 & 1 & 0 \\ 0 & 0 & 0 & \underline{-8} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

* A is already in Echelon Form! Woohoo :)

*** Dimension of Nul(A):** x_2, x_3, x_5 are free-variables.

$\therefore$ Since A has 3 Free Variables $\Longrightarrow$ $\boxed{\therefore \dim[\text{Nul}(A)] = 3}$ Ans.

*** Dimension of Col(A):** $\vec{a}_1$ & a_4 are pivot columns.

$\therefore$ Since A has 2 Pivot Columns $\Longrightarrow$ $\boxed{\therefore \dim[\text{Col}(A)] = 2}$ Ans.

Example: Find the dimensions of the Nul(A) & Col(A)

For the matrix:

$$A = \begin{bmatrix} 1 & 4 & -6 & 5 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

Recall:

- $\dim[\text{Nul}(A)]$ = # of free-variables in $A\vec{x} = \vec{0}$
- $\dim[\text{Col}(A)]$ = # of pivot-columns in A

*Row-reduce $[A \mid \vec{0}]$ to Echelon Form:

$$\left[\begin{array}{ccccccc|c} 1 & 4 & -6 & 5 & -3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

*A is already in Echelon Form :)

∴ Dimension of Nul(A): Since A has 3 free-variables (i.e. $x_2, x_3, \& \ x_4$)

$$\Rightarrow \boxed{\dim[\text{Nul}(A)] = 3} \quad \text{Ans.}$$

∴ Dimension of Col(A): Since A has 4 pivot-columns (i.e. $\vec{a}_1, \vec{a}_5, \vec{a}_6, \vec{a}_7$)

$$\rightarrow \boxed{\dim[\text{Col}(A)] = 4} \quad \text{Ans.}$$

Example: Find the dimensions of the Nul(A) & Col(A) for the matrix:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

Recall:

- dim[Nul(A)] = # of free-variables in $A\vec{x} = \vec{0}$
- dim[Col(A)] = # of pivot-columns in A

* Row-reduce $[A \,|\, \vec{0}]$ to Echelon Form:

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

* A is already in Echelon Form :)

∴ Dimension of Nul(A): Since A has NO free-variables

$$\Longrightarrow \boxed{\dim[\text{Nul}(A)] = 0}$$

∴ Dimension of Col(A): Since A has a pivot in each column

$$\Longrightarrow \boxed{\dim[\text{Col}(A)] = 3}$$