* Recall : $A\vec{x} = \vec{0}$ has a non trivial solution when a free variable $\exists$

Def: An indexed set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ is said to be:

(i) Linearly Independent: if $A\vec{x} = \vec{0}$ (matrix Eq) has only the trivial solution ($\vec{x} = \vec{0}$)

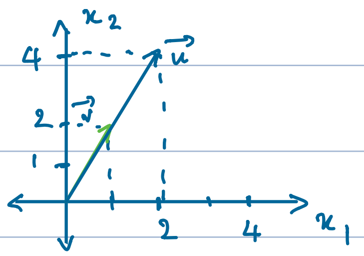
$\Rightarrow x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{0}$ has only the trivial solution (No Free variable)

(ii) Linearly Dependent: IF the $A\vec{x} = \vec{0}$ has a nontrivial solution $\nearrow$ A free variable $\exists$

Iow: A redundant vector $\exists$!

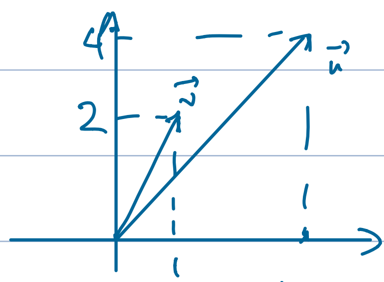
A linear combination of some of the vectors produces another vector in the set!

* Geometric Interpretation: Consider the set of vectors $\left\{ \underset{\vec{v}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{\vec{u}}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} \right\}$



* Since $\vec{u}$ & $\vec{v}$ are scalar-multiples $\vec{u} = 2\vec{v}$, these vectors are Linearly Dependent

Ex: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$



Since $\vec{u} \neq c\vec{v}$, the vectors are linearly Independent

Note: Sets of vectors with 3^+ vector $\Rightarrow$ let $A = [\vec{v}_1 \; \vec{v}_2 \dots \vec{v}_p]$

$\Rightarrow$ we can the row-reduce $[A \,\vdots\, \vec{0}]$ to echelon form to check for free variables

① A free variable $\exists$: $A\vec{x} = \vec{0}$ has a nontrivial sol & $\{\vec{v}_1, \dots \vec{v}_p\}$ is Linearly Dep.

② No free variables: $\{\vec{v}_1, \dots \vec{v}_p\}$ is Linearly Indep.

Example: Determine if the vectors are linearly Independent : Ans : Let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ and then row-reduce $[A \vdots \vec{0}]$ to Echelon Form to check for free variables

$$\underset{\vec{v}_1}{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} \quad \underset{\vec{v}_2}{\begin{bmatrix} 9 \\ 2 \\ -4 \end{bmatrix}} \quad \underset{\vec{v}_3}{\begin{bmatrix} 8 \\ 8 \\ -24 \end{bmatrix}} \qquad \vec{A} = \begin{bmatrix} 4 & 9 & 8 \\ 0 & 2 & 8 \\ 0 & -4 & -24 \end{bmatrix} \begin{bmatrix} 4 & 9 & 8 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \Rightarrow$$

$$\sim \begin{bmatrix} \underline{4} & 9 & 8 \\ 0 & \underline{1} & 4 \\ 0 & 0 & \underline{-8} \end{bmatrix}$$

Echelon Form

Since a pivot $\exists$ in each row $A\vec{x} = \vec{0}$ has one, unique Solution $\rightarrow$ trivial solution $\Rightarrow \vec{x} = \vec{0}$

$\therefore$ The vectors $\{\vec{v}_1\ \vec{v}_2\ \vec{v}_3\}$ are linearly Independent

Quiz 3: covers 1.5
Exam 1: chapter 1

Recall:

① An indexed set of vectors is <u>linearly Independent</u> if the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{0}$ has only the trivial solution $(\vec{x} = \vec{0})$

② An indexed set of vectors $\{\vec{v}_1, \dots \vec{v}_p\}$ is <u>linearly Dependent</u> if $\exists$ scalars / weights $c_1, c_2, \dots, c_p$ (NOT all zero) ST:

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0} \Rightarrow$ has a nontrivial solution

$A\vec{x} = \vec{0}$ must has at least one free-variable

$\Rightarrow$ vectors depend on each other ; A redundant vector $\exists$

Example: Find the value(s) of h ST the following set of vectors is Linear Dependant $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ h \end{bmatrix}$

to find "h" Row Reduce $[A \vdots \vec{0}]$ to Echelon Form

$$\begin{array}{c} \begin{bmatrix} 2 & 4 & -3 \\ -2 & -6 & 3 \\ 4 & 7 & h \end{bmatrix} \xrightarrow[2R_1 - R_3 = nR_3]{R_1 + R_2 = nR_2} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \begin{bmatrix} 2 & 4 & -3 \\ 0 & -2 & 0 \\ 0 & -1 & 6+h \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -3 \\ 0 & 1 & 0 \\ 0 & -1 & 6+h \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 2 & 4 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 6+h \end{bmatrix}$$

* Since we want $A\vec{x} = \vec{0}$ to have a <u>trivial Solution</u>, we want: $6 + h = 0 \Rightarrow$ $\boxed{\therefore \text{Vectors are Linearly Dependent if } h = -6}$

<u>Example:</u> Determine if the columns of matrix A are Linearly Dependent

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & -5 \\ -4 & -3 & 0 \\ 2 & 1 & -10 \end{bmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array}$$

* Check if $A\vec{x} = \vec{0}$ has a nontrivial solution... Does a Free Variable $\exists$?
* <u>To check:</u> Row - reduce to Echelon form

  ① If a F.V. $\exists$ $\Rightarrow$ columns are linearly Dependent

  ② If NO F.V. $\Rightarrow$ Columns are Linearly Independent.

$4R_1 + R_3 =$

$-2R_1 + R_4$

$$\begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & -5 \\ 0 & 1 & -20 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & 0 \\ 0 & 1 & -5 \\ 0 & 1 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

* Echelon Form

Since NO Free Variables exist $A\vec{x} = \vec{0}$ has only the trivial Solution

$\rightarrow \therefore$ By Def the columns of A are Linearly Independent $\Rightarrow$ $\boxed{\text{NOT Linear. Dep.}}$

* <u>Theorem:</u> IF the number of vectors in a set is greater than the number of entries per vector, then the set of vectors is Linearly Dep.

<u>Proof:</u> Let A be an $m \times n$ matrix

By definition, we know A has m - Equations (Rows), n: Unknowns (Columns)

- Suppose that $n > m$: IF $\begin{pmatrix}\# \text{ of} \\ \text{unknowns}\end{pmatrix} > \begin{pmatrix}\# \text{ of} \\ \text{Equations}\end{pmatrix}$, then a free variable exists.

If a free variable $\exists$, then $A\vec{x} = \vec{0}$ has a non trivial solution

If $A\vec{x} = \vec{0}$ has a non trivial solution, then the column-vectors of A are Linearly Dependent

Ex: Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$

Linearly Dep. / Indep. ?

Ans:

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

* At most 2 pivots, but 3 variables
* Since we only have 2 eq., we can only solve for 2 of the "unknowns" $\to$ The 3^{rd} must be Free.

$\boxed{\therefore \text{ Since } 3 \text{ unknowns} > 2 \text{ eq} \Rightarrow \text{Linearly Dependent.}}$

REVIEW: A system has free variable $\Rightarrow$ Linearly Dependent, NonTrivial Solutions
does NOT have free variable $\Rightarrow$ Linearly Independent, Trivial Solutions

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$\boxed{1.7.1}$ $\begin{bmatrix} 5 & 9 & 6 \\ 0 & 3 & 12 \\ 0 & -12 & -24 \end{bmatrix} \underset{-\frac{1}{12}R_3}{\overset{\frac{1}{3}R_2}{\sim}} \begin{bmatrix} 5 & 9 & 6 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\frac{R_1}{5}]{R_2 - R_3 = nR_3} \begin{bmatrix} 1 & 9/5 & 6/5 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \to$ 3 pivots in 3 rows of 3 vars.

$\Rightarrow$ The system does NOT have free vars. $\Rightarrow$ Trivial Solutions $\exists$

Linearly Independent

$\boxed{1.7.8}$ $\begin{bmatrix} 1 & -5 & 3 & 2 \\ -5 & 25 & -15 & 2 \end{bmatrix} \xrightarrow{5R_1 + R_2 = nR_2} \begin{bmatrix} 1 & -5 & 3 & 2 \\ 0 & 0 & 0 & 12 \end{bmatrix} \Rightarrow$ x_2, x_3 are free vars

x_1, x_4 are basic vars

$\Rightarrow$ Linearly Dependent

$\boxed{1.7.9}$

$$\begin{array}{ccc} \begin{bmatrix} 1 & -3 & 4 \\ -4 & 12 & 7 \\ 2 & -6 & h \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array}$$

(a) $\vec{v}_3$ spans $\{\vec{v}_1, \vec{v}_2\}$ if $\exists$ a solution $A\vec{x} = \vec{v}_3 \Leftrightarrow [\vec{v}_1 \ \vec{v}_2 \ \vdots \ \vec{v}_3]$ has nontrivial sol.

$$\xrightarrow[\substack{2R_1 - R_3 \\ = nR_3}]{\substack{4R_1 + R_2 \\ = nR_2}} \begin{bmatrix} 1 & -3 & 4 \\ \boxed{0 \quad 0 \quad 23} \\ 0 & 0 & 8-h \end{bmatrix} \leftarrow \text{contradiction!}$$

$\Rightarrow$ no values of h

(b) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent $\Rightarrow A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 0 & 23 & 0 \\ 0 & 0 & 8-h & 0 \end{array}\right] \xrightarrow[\frac{R_3}{}]{\frac{R_2}{R_3}} \left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 8-h & 0 \end{array}\right] \xrightarrow[\substack{-R_3 \\ = nR_3}]{(8-h)R_2} \left[\begin{array}{ccc|c} \underline{1} & -3 & 4 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 is a free var

$\rightarrow$ nontrivial

$\Rightarrow$ Linearly Dep.

$\forall$ values of h

$\boxed{1.7.11}$

$$\begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 4 & 7 & h \end{bmatrix} \xrightarrow[-\frac{1}{2}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -1 \\ 4 & 7 & h \end{bmatrix} \xrightarrow[\substack{4R_1 - R_3 \\ = nR_3}]{\substack{R_1 - R_2 \\ = nR_2}} \begin{bmatrix} \underline{1} & 2 & -4 \\ 0 & \underline{+1} & 0 \\ 0 & 1 & \underbrace{-4-h}_{=0} \end{bmatrix} \rightarrow [0 \ 0 \ h+4]$$

x_1, x_2 are basic vars

To let the vectors be dependent $\Rightarrow x_3$ will be a free variable.

$\rightarrow -4-h = 0 \Rightarrow h = -4$

$\boxed{1.7.14}$

$$\begin{bmatrix} 1 & -4 & 4 \\ -3 & 13 & 1 \\ -6 & 8 & h \end{bmatrix} \xrightarrow[\substack{6R_1 + R_3 \\ = nR_3}]{\substack{3R_1 + R_2 \\ = nR_2}} \begin{bmatrix} 1 & -4 & 4 \\ 0 & 1 & 13 \\ 0 & -16 & 24+h \end{bmatrix} \xrightarrow{\frac{R_3}{-16}} \begin{bmatrix} \underline{1} & -4 & 4 \\ 0 & \underline{1} & 13 \\ 0 & 1 & \frac{24+h}{-16} \end{bmatrix} \xrightarrow{\substack{R_2 - R_3 \\ = nR_3}} \begin{bmatrix} \underline{1} & -4 & 4 \\ 0 & \underline{1} & 13 \\ 0 & 0 & \underbrace{13 + \frac{24+h}{16}}_{=0} \end{bmatrix}$$

$\Rightarrow$ To let the vectors be dependent $\Rightarrow x_3$ must be a free var.

$$\Rightarrow 13 + \frac{24+h}{16} = 0 \Rightarrow h = -13 \times 16 - 24 = \boxed{-232}$$

$\boxed{1.7.15}$

$$\begin{bmatrix} 6 & 3 & 1 & -1 \\ 1 & 7 & 4 & 9 \end{bmatrix} \xrightarrow{R_1 - 6R_2} \begin{bmatrix} 6 & 3 & 1 & -1 \\ 0 & -39 & -23 & -55 \end{bmatrix}$$

2 pivots, 4 variables, 2 entries ⇒ free variables ∃

⇒ always has nontrivial solutions

⇒ Linearly Dependent

$\boxed{1.7.19}$

$$\begin{bmatrix} 10 & -2 \\ 20 & -4 \\ -5 & -1 \end{bmatrix} \xrightarrow[\frac{R_3}{-5}]{\frac{R_1}{+2},\ \frac{R_2}{-4}} \begin{bmatrix} 5 & -1 \\ 5 & -1 \\ -5 & -1 \end{bmatrix} \xrightarrow[R_1+R_3 = nR_3]{R_1-R_2 = nR_2} \begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 5 & -1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

Two pivots, system does NOT have free variable.

⇒ linearly independent

$\boxed{1.7.31}$

$$\begin{bmatrix} 2 & 2 & 4 \\ -4 & 2 & -2 \\ -4 & -1 & -5 \\ 5 & 0 & 5 \end{bmatrix} \xrightarrow[-R_3,\ \frac{R_4}{5}]{\frac{R_1}{2},\ \frac{R_2}{-2}} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 4 & 1 & 5 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[2R_1-R_4 = nR_4]{R_1-R_2 = nR_2,\ 4R_1-R_3 = nR_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow[3R_2-R_4 = nR_4]{3R_2-R_3 = nR_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

⇒ x_3 is a free variable ⇒ Linearly dependent

$$\left.\begin{aligned} x_2 &= -x_3 \\ x_1 &= -x_2 - 2x_3 = +x_3 - 2x_3 = -x_3 \end{aligned}\right\} \Rightarrow x = \{x_1\ x_2\ x_3\} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \leftarrow \text{Vector equation}}$$

$\boxed{1.7.6}$

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 1 & -4 \\ 2 & 1 & -8 \end{bmatrix} \xrightarrow[R_1+2R_4 = nR_4]{R_1+4R_3 = nR_3} \begin{bmatrix} -4 & -3 & 0 \\ 0 & +1 & -4 \\ 0 & 1 & -16 \\ 0 & -1 & -16 \end{bmatrix} \xrightarrow[R_2+R_4 = nR_4]{R_2-R_3 = nR_3} \left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & -20 & 0 \end{array}\right] \to \begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$