Solution

Name:

Linear Algebra I: Exam 2 (Summer 2019)

Show ALL work, as unjustified answers may receive no credit. Calculators are not allowed on any quiz or test paper. *Make sure to exhibit skills discussed in class.* Box all answers and simplify answers as much as possible.

Good Luck! ☺

1. ***The Inverse of a Matrix***

[8pts] Use the *Algorithm for Finding A^{-1}* to find the inverse of the given matrix, if it exists:

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 2 & 1 & 4 \\ -3 & -2 & 4 \end{bmatrix}$$

$$[A \,|\, I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ -3 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \overset{\frac{-2R_1 + R_2}{N.R_2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 12 & -2 & 1 & 0 \\ -3 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \overset{\frac{3R_1 + R_3}{N.R_3}}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 12 & -2 & 1 & 0 \\ 0 & -2 & -8 & 3 & 0 & 1 \end{array}\right] \overset{\frac{2R_2 + R_3}{N.R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 12 & -2 & 1 & 0 \\ 0 & 0 & 16 & -1 & 2 & 1 \end{array}\right] \overset{\frac{1}{16}R_3}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 12 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1/16 & 1/8 & 1/16 \end{array}\right] \overset{\frac{-12R_3 + R_2}{N.R_2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5/4 & -1/2 & -3/4 \\ 0 & 0 & 1 & -1/16 & 1/8 & 1/16 \end{array}\right] \overset{\frac{4R_3 + R_1}{N.R_1}}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & -5/4 & -1/2 & -3/4 \\ 0 & 0 & 1 & -1/16 & 1/8 & 1/16 \end{array}\right] \Longrightarrow \boxed{\therefore A^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ -5/4 & -1/2 & -3/4 \\ -1/16 & 1/8 & 1/16 \end{bmatrix}}$$

2. **_Characteristics of Invertible Matrices_**

Define a Linear Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ 5x_1 + 2x_2 \end{bmatrix}$.

(a) [3pts] Is T an invertible Linear Transformation? Explain.

(b) [2pts] If T is invertible, find the formula for T^{-1}.

(a) $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow \det(A) = 4 \overset{+}{\cancel{-}} (\cancel{-}15) = 19$

$\boxed{\therefore \text{Since } \det(A) = 19 \neq 0, \ T \text{ is invertible}}$

(b) $A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{19}\begin{bmatrix} 2 & 3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2/19 & 3/19 \\ -5/19 & 2/19 \end{bmatrix}$

$\boxed{\therefore T^{-1}(\vec{x}) = A^{-1}\vec{x} = \begin{bmatrix} 2/19\, x_1 + 3/19\, x_2 \\ -5/19\, x_1 + 2/19\, x_2 \end{bmatrix}}$

3. **_Matrix Factorizations_**

[5pts] Find the LU Factorization of the matrix A (with L unit lower triangular):

$$A = \begin{bmatrix} 6 & 9 \\ 18 & 25 \end{bmatrix}$$

* Find U: $[A \,\vdots\, \vec{0}] = \begin{bmatrix} 6 & 9 \\ 18 & 25 \end{bmatrix} \underset{N\ R_2}{\overset{-3R_1 + R_2}{\sim}} \boxed{\begin{bmatrix} 6 & 9 \\ 0 & -2 \end{bmatrix} = U}$

* Echelon Form

* Find L: $\begin{bmatrix} 6 \\ 18 \end{bmatrix}\begin{bmatrix} -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix}$ $\boxed{\therefore L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$

$\therefore$ LU Factorization: $\boxed{A = LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ 0 & -2 \end{bmatrix}}$

NOT an exclusive solution
(ans. is the same)

4. ***Introduction to Determinants***

[7pts] Compute the determinant by *Cofactor Expansion*. At each step, choose a row or column that involves the least amount of computation:

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

*Cofactor - Expansion Down Col. # 5:

$$\det(A) = 0 + 0 + 0 + 1(-1)^9 \begin{vmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{vmatrix} + 0$$

$$= - \begin{vmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{vmatrix}$$

*Cofactor Expansion Across Row 2:

$$\det(A) = -\left[0 + 0 + (-1)(-1)^5 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} \right] = - \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

*Cofactor Expansion Across Row 2:

$$\det(A) = -\left[0 + 0 + (-1)(-1)^5 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \right] = - \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -(1-1) = 0$$

$$\boxed{\therefore \det(A) = 0}$$

NOT an exclusive solution
(answ in the same)

5. **The Properties of Determinants**

[8pts] Find the determinant of the provided matrix. Specify whether the matrix has an inverse without trying to compute the inverse:

$$A = \begin{bmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & 3 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & 3 & 0 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{vmatrix} \underset{\sim}{\overset{R_1 + R_2}{\overline{N.R_2}}} \begin{vmatrix} 2 & -2 & -2 & -2 \\ 0 & 0 & 1 & -2 \\ -2 & -2 & 2 & 0 \\ 1 & -1 & -3 & -1 \end{vmatrix} \underset{\sim}{\overset{R_1 + R_3}{\overline{N.R_3}}}$$

$$\begin{vmatrix} 2 & -2 & -2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & -4 & 0 & -2 \\ 1 & -1 & -3 & -1 \end{vmatrix} \overset{\frac{1}{2}R_1}{\sim} 2\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & -4 & 0 & -2 \\ 1 & -1 & -3 & -1 \end{vmatrix} \underset{\sim}{\overset{-R_1 + R_4}{\overline{N.R_4}}} 2\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{\sim}$$

$$-2\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{vmatrix} \underset{\sim}{\overset{2R_3 + R_4}{\overline{N.R_4}}} -2\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

Echelon Form

$$\boxed{\therefore \det(A) = -2\left[(1)(-4)(1)(-4)\right] = -2(16) = -32}$$

$\Rightarrow$ $\boxed{\text{Since } \det(A) = -32 \neq 0, \text{ A is invertible.}}$

6. **Cramer's Rule, Volume, and Linear Transformations**

[pts] Find the inverse of the following matrix using the *Inverse Formula*:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$* \boxed{\det(A) = (1)(2)(3) = 6}$

*Find the Matrix of Cofactors, C:

$C_{11} = (-1)^2 \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} = 6$, $C_{12} = (-1)^3 \begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} = 0$, $C_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} = -6$, $C_{22} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = 3$, $C_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 0$, $C_{32} = (-1)^5 \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = -3$, $C_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$

$$\Rightarrow C = \begin{bmatrix} 6 & 0 & 0 \\ -6 & 3 & 0 \\ 0 & -3 & 2 \end{bmatrix} \quad \& \quad \boxed{\therefore \operatorname{adj}(A) = C^T = \begin{bmatrix} 6 & -6 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}}$$

*Find/Check adj(A) & det(A):

$$\operatorname{adj}(A)\, A = \begin{bmatrix} 6 & -6 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 6+0+0 & 12-12+0 & 18-18+0 \\ 0+0+0 & 0+6+0 & 0+9-9 \\ 0+0+0 & 0+0+0 & 0+0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 I_3 \Longrightarrow \boxed{\therefore \det(A) = 6}$$

* Find A^{-1}:

$$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{6} \begin{bmatrix} 6 & -6 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/3 \end{bmatrix}}$$

NOT an exclusive solution
(answ is the same)

7. **Cramer's Rule, Volume, and Linear Transformations**

[8pts] Find the volume of the box with one vertex at the origin and adjacent vertices $(1,0,-2),(1,2,4),(7,1,0)$.

$$*A = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 0 \end{bmatrix}$$

$$* \text{Volume} = |\det(A)| :$$

$$\det(A) = 1(-1)^2 \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} + 0 + (-2)(-1)^4 \begin{vmatrix} 1 & 7 \\ 2 & 1 \end{vmatrix}$$

$$= (0-4) - 2(1-14)$$

$$= -4 - 2(-13)$$

$$= -4 + 26$$

$$= 22$$

$$\therefore \text{Volume}: \boxed{22 \text{ cubic units}}$$

∴ Solution, $\vec{x}$:

$$\boxed{\vec{X} = \begin{bmatrix} -5/11 \\ 36/11 \\ 76/11 \end{bmatrix}}$$

Bonus Question: ***Cramer's Rule, Volume, and Linear Transformations***

[5 pts] Solve the linear system using *Cramer's Rule*:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ 3x_1 - 2x_2 + x_3 &= -1 \\ -5x_1 - 4x_2 + 2x_3 &= 3 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ -5 & -4 & 2 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}} \Longrightarrow \det(A) = 2\begin{vmatrix} -2 & 1 \\ -4 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ -5 & -4 \end{vmatrix}$$

$$= 2(\underbrace{\cancel{-4+4}}_{0}) - 3(\underbrace{6+5}_{11}) - (\underbrace{-12-10}_{-22}) = -33 + 22$$

$$\boxed{\therefore \det(A) = -11}$$

$$\bullet A_1(\vec{b}) = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ 3 & -4 & 2 \end{bmatrix} \Rightarrow \det[A_1(\vec{b})] = 2\begin{vmatrix} -2 & 1 \\ -4 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} - \begin{vmatrix} -1 & -2 \\ 3 & -4 \end{vmatrix}$$

$$= 0 - 3(\underbrace{-2-3}_{-5}) - (\underbrace{4+6}_{10}) = 15 - 10 = 5$$

$$\boxed{\therefore \det[A_1(\vec{b})] = 5}$$

$$\bullet A_2(\vec{b}) = \begin{bmatrix} 2 & 2 & -1 \\ 3 & -1 & 1 \\ -5 & 3 & 2 \end{bmatrix} \Rightarrow \det[A_2(\vec{b})] = 2\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} - \begin{vmatrix} 3 & -1 \\ -5 & 3 \end{vmatrix}$$

$$= 2(\underbrace{-2-3}_{-5}) - 2(\underbrace{6+5}_{11}) - (\underbrace{9-5}_{4}) = -10 - 22 - 4$$

$$\boxed{\therefore \det[A_2(\vec{b})] = -36}$$

$$\bullet A_3(\vec{b}) = \begin{bmatrix} 2 & 3 & 2 \\ 3 & -2 & -1 \\ -5 & -4 & 3 \end{bmatrix} \Rightarrow \det[A_3(\vec{b})] = 2\begin{vmatrix} -2 & -1 \\ -4 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ -5 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ -5 & -4 \end{vmatrix}$$

$$= 2(\underbrace{-6-4}_{-10}) - 3(\underbrace{9-5}_{4}) + 2(\underbrace{-12-10}_{-22}) = -20 - 12 - 44$$

$$\boxed{\therefore \det[A_3(\vec{b})] = -76}$$