Asymptotic Notations

Text

Chapters 3

Asymptotic Notation

- What does "the order of" mean
- Big O, Ω, and Θ notations
- Properties of asymptotic notation
- Limit rule

A notation for "the order of"

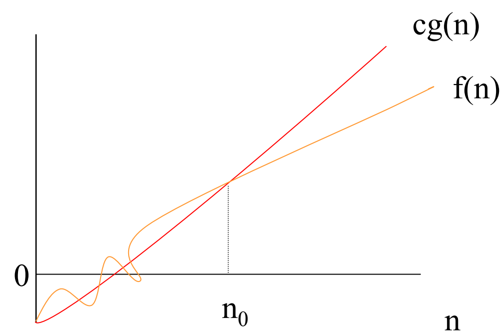
- We'd like to measure the efficiency of an algorithm
  - Determine mathematically the resources needed
- There is no such a computer which we can refer to as a standard to measure computing time
- We introduce "asymptotic" notation
  - An asymptotically superior algorithm is often preferable even on instances of moderate size

Definition of big O

$$O(g(n)) = \{f(n) \mid (\exists c \in R^+, n_0 \in N)(\forall n \geq n_0)[0 \leq f(n) \leq cg(n)]\}$$

- Typically used for *asymptotic upper bound*
- Attention
  - O(f(n)) is a **set** of functions
- Pitfall
  - Traditionally we say $n^2 = O(n^2)$ as used in our text book
  - It really means $n^2 \in O(n^2)$

A graphical view of asymptotic definition



Example

- Prove that following statements

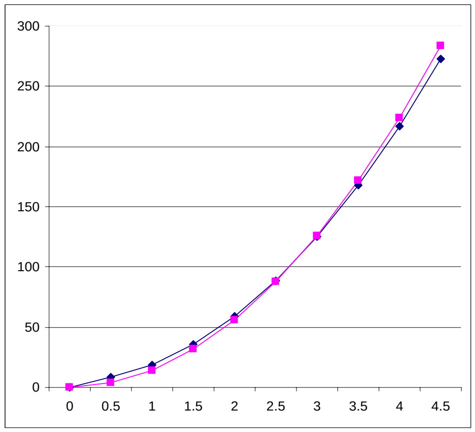
$$13n^2 + n + 5 \in O(n^2)$$

$$13n^2 + n + 5 \in O(n^2 \log n)$$

$$f(n) \in O(n) \rightarrow f^2(n) \in O(n^2)$$

$$O(n) \subset O(n^2)$$

$$13n^2 + n + 5 \in O(n^2) \qquad \forall n \geq 3, \quad 13n^2 + n + 5 \leq 14n^2$$



$14n^2$

$13n^2 + n + 5$

What are c and n_0?

Several notations

- Logarithm time O(log n)
- Linear time O(n)
- Quadratic time O(n^2)
- Cubic time O(n^3)
- Exponential time O(c^n), c>1

$$O(\lg n) \subset O(n^c) \subset O(n^c \lg n) \subset O(n^{c+\varepsilon} \lg n) \subset O(d^n) \qquad c, \varepsilon > 0, d > 1$$

- Order of growth

The Maximum rule

- Let $f, g : N \rightarrow R^{\geq 0}$,

  then $O(f(n) + g(n)) = O(\max(f(n), g(n)))$
- Proof
  - the key is max(f(n),g(n)) <= f(n)+g(n) <= 2*max(f(n),g(n))
- Examples
  - $O(12n^3-5n+n\log n+36)$
- The maximum rule let us ignore lower-order terms

Example

- True or false
  - ? $5 = O(\log n)$
  - ? $\log n = O(5)$
  - ? $n = O(n^{0.6}\log n)$
  - ? $n^{0.6}\log n = O(n)$
  - ? $n^8 = O((n^2-3n+5)^4)$

Definition of Ω

$$\Omega(g(n)) = \{ f(n) \mid (\exists c \in R^+, n_0 \in N)(\forall n \geq n_0)[f(n) \geq cg(n) \geq 0]\}$$

- Ω is typically used to describe *asymptotic lower bound*
  - For example, insertion sort take time in Ω(n)
- Ω for algorithm complexity
  - We use it to give the lower bounds on the intrinsic difficulty of solving problems
  - Example, any comparison-based sorting algorithm takes time Ω(*nlogn*)

Example of Ω(n^2)

- n^2
- n^2+n
- n^2-n
- $n^{2.0001}$
- $n^2\log n$
- 2^n

The Θ notation

Definition:

$$\Theta(g(n)) = \{f(n) \mid (\exists c_1, c_2 \in R^+, n_0 \in N)(\forall n \geq n_0)[0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)]\}$$

Equivalent to:

$$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$$

- Used to describe *asymptotically tight bound*
- Example: selection sort take time in $\Theta(n^2)$

The Limit Rule

- Let $f, g : N \to R^{\geq 0}$, then

1. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} \in R^+$ then $f(n) \in \Theta(g(n))$
2. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ then $f(n) \in O(g(n))$ but $f(n) \notin \Omega(g(n))$
3. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = +\infty$ then $f(n) \in \Omega(g(n))$ but $f(n) \notin O(g(n))$

Example

$$(n^c)' = cn^{c-1}$$

$$(\ln n)' = \frac{1}{n}$$ (ln n means $\log_e n$, the text use log)

When c>0

$$\lim_{n\to\infty}\frac{\ln n}{n^c} = \lim_{n\to\infty}\frac{(\ln n)'}{(n^c)'} = \lim_{n\to\infty}\frac{1/n}{cn^{c-1}} = \lim_{n\to\infty}\frac{1}{cn^c} = 0$$

$$\ln n \in O(n^c)$$ for any $c>0$

Semantics of big-O and Ω

- When we say an algorithm takes worst-case time $t(n) = O(f(n))$, then there exist a real constant c such that $c*f(n)$ is an upper bound for any instances of size of sufficiently large n
- When we say an algorithm takes worst-case time $t(n) = \Omega(f(n))$, then there exist a real constant d such that there exists at least one instance of size n whose execution time >= $d*f(n)$, for any sufficiently large n
- Example
  - Is it possible an algorithm takes worst-case time $O(n)$ and $\Omega(n\log n)$?

Practice Problems

```
anAlgorithm( int n)
{
  // if (x) is an elementary
  // operation
  if (x) {
    some work done
    by n^2 elementary
    operations;
  } else {
    some work done
    by n^3 elementary
    operations;
  }
}
```

- True or false
  - The algorithm takes time in $O(n^2)$
  - The algorithm takes time in $\Omega(n^2)$
  - The algorithm takes time in $O(n^3)$
  - The algorithm takes time in $\Omega(n^3)$
  - The algorithm takes time in $\Theta(n^3)$
  - The algorithm takes time in $\Theta(n^2)$
  - The algorithm takes worst case time in $O(n^3)$
  - The algorithm takes worst case time in $\Omega(n^3)$
  - The algorithm takes worst case time in $\Theta(n^3)$
  - The algorithm takes best case time in $\Omega(n^3)$

Definition of o and ω

- Definition

$$o(g(n)) = \{f(n) \mid (\forall c \in R^+, \exists n_0 \in N, \forall n \geq n_0)[0 \leq f(n) < cg(n)]\}$$

$$\omega(g(n)) = \{f(n) \mid (\forall c \in R^+, \exists n_0 \in N, \forall n \geq n_0)[f(n) > cg(n) \geq 0]\}$$

- Denote upper/lower bounds that are not asymptotically tight
- Example $1000n \in o(n^2); \quad 1000n^2 \notin o(n^2)$
  $1000n^2 \in \omega(n); \quad 1000n^2 \notin \omega(n^2)$
- Properties $f(n) \in o(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$$f(n) \in \omega(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Relational Properties

- Transitivity: *O, o, Ω, ω,Θ*
- Reflexivity: *O, Ω, Θ*
- Symmetry: $f(n) = \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- Transpose symmetry (Duality)

$$f(n) = O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$$
$$f(n) = o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$$

- Analogy

$$f(n) \in O(g(n)) \approx a \le b$$
$$f(n) \in \Omega(g(n)) \approx a \ge b$$
$$f(n) \in \Theta(g(n)) \approx a = b$$
$$f(n) \in o(g(n)) \approx a < b$$
$$f(n) \in \omega(g(n)) \approx a > b$$