Analysis of Algorithms 91.404, Fall, 2012
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Homework #2

1. (25 points) **Rank the following three functions by order of asymptotic growth.**
Ans:
Let:

$$g_1(n) = \left(\frac{1}{2}\right)^{n^3}$$

$$g_2(n) = 3^{4\log_3 n} = n^4$$

$$g_3(n) = 5\lg n + n^2 \lg\lg n$$

1) $g_1(n) \in O(g_2(n))$:
Since $g_1(n) < 1$ and $g_2(n) \geq 1$ for $n \geq 2$, we have $g_1(n) \leq c_0 g_2(n) \quad \forall n \geq n_0$ where $c = 1$ and $n_0 = 2$.

2) $g_3(n) \in O(g_2(n))$:
Let $G(n) = g_3(n) - g_2(n) = n^4 - 5\lg n - n^2 \lg\lg n$, we assume $n \geq 3$ then

$$G(n) = n^2\left(n^2 - \frac{5}{n^2}\lg n - \lg\lg n\right)$$

$$\geq n^2(n^2 - \lg n - \lg n)\left(\forall n \geq 3, \frac{5}{n^2} \leq 1 \, and \, \lg\lg n \leq \lg n\right)$$

$$= n^2(n^2 - 2\lg n)$$

$$\geq n^2(n^2 - 2n)(\forall n \geq 3, \lg n \leq n)$$

$$\geq 0$$

So, we have $g_3(n) \leq c_0 \, g_2(n) \; \forall n \geq n_0, where\ c = 1\ and\ n_0 = 3$

2. (25 points) i) $f_1(n) \in \Omega\left(\left(\frac{1}{2}\right)^n\right)$ **ii)** $f_2(n) \in \Theta(n^2 \lg n)$ **iii)** $f_3(n) \in O(\lg^3 n)$

a) If statements i) – iii) are true, can we conclude that $f_3(n) \in O(f_2(n))$ **?**

b) If statements i) – iii) are true, can we conclude that $f_2(n) \in \Omega(f_1(n))$ **?**

Ans:

Since i) – iii) are true, we can have following statements:

(1) There exist positive constants c_1 and n_1 such that $0 \leq c_1\left(\frac{1}{2}\right)^n \leq f_1(n), \forall n \geq n_1$.

(2) There exist positive constants a, b and n_2 such that

$0 \le an^2 lgn \le f_2(n) \le bn^2 lgn, \forall n \ge n_2$.

(3) There exist positive constant c_3 and n_3 such that $0 \le f_3(n) \le c_3(lg^3 n), \forall n \ge n_3$.

a) True. Let $c_{23} = \frac{c_3}{a}$ and $n_{23} = max(1, n_2, n_3)$, from (2) and (3), and we already know that $lg^3 n \le n^2 lgn \forall n \ge 1$, we have:

$$f_2(n) \le c_3 lg^2 n \le c_3 n^2 lgn = \frac{c_3}{a} an^2 lgn \le c_{23} f_2(n), \quad \forall n \ge n_{23}.$$

So we can conclude that $f_3(n) \in O(f_2(n))$.

b) False. Because i) only shows that the lower bound of $f_1(n)$, but we do not know its exact upper bound, we can not say that $f_2(n)$ would be the upper bound of $f_1(n)$ even if $n^2 lgn$ is the upper bound of $(1/2)^n$. It is possible that $f_1(n)$ is the upper bound of $f_2(n)$, for example, $f_1(n) = n^4$ and $f_2(n) = n^2 lgn$. That satisfies statements i) and ii), but it is obvious that $f_1(n) \in \Omega(f_2(n))$.

So we cannot conclude that $f_2(n) \in \Omega(f_1(n))$.

3. (25 points) **True or False.**

Ans: a) True; b) False; c) False; d) True; e) True

For b), use the limit rule and we get 0, which means $n lg^2 n \in O(n^{1.05})$.

For d) and e), the cost function T(n) could be: $T(n) = c_1(log_2 n + 1) + c_2 log_2 n$.

d) is true since $T(n) \le c_1(log_2 n + 1) + c_2 log_2 n = O(n)$, and e) is true since $T(n) \ge c_1 log_2 n + c_2(log_2 n - 1) = \Omega(n)$.

4. (25 points) **Pseudocode Analysis: find the tight upper-and-lower bounds on the asymptotic worst-case running time *f(n)*.**

Ans:

Mystery(n)	**Cost**	**Times**
1. c ← 1	c_1	1
1. for i ← 1 to n	c_2	n+1
2. do for j ← i to n	c_3	$\sum_{i=1}^{n} \sum_{j=i}^{n+1} 1$
3. do for k ← n down to n/2	c_4	$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=n/2}^{n+1} 1$
4. do c ← c + 1	c_5	$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=n/2}^{n+1} 1$
5. print c	$\mathbf{c_6}$	**1**

The procedure Mystery(n) is a 3-level loop, and the worst-case running time is:

$$f(n) = c_1 + c_2(n+1) + c_3 \sum_{1}^{n} \sum_{i}^{n+1} 1 + c_4 \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=n/2}^{n+1} 1 + c_5 \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=n/2}^{n+1} 1 + c_6$$

Simplify this,

$$f(n) = c_1 + c_2(n+1) + c_3 \sum_{1}^{n} \sum_{i}^{n+1} 1 + c_4 \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=n/2}^{n+1} 1 + c_5 \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=n/2}^{n+1} 1 + c_6$$

=

$$f(n) = (c_1 + c_2 + c_6) + (c_2 + c_3)n + c_3 \sum_{1}^{n} \sum_{i}^{n+1} 1 + c_4 \sum_{i=1}^{n} \sum_{j=i}^{n} (n - \frac{n}{2} + 1) + c_5 \sum_{i=1}^{n} \sum_{j=i}^{n} (n - \frac{n}{2}) + c_6$$

$= (c_1 + c_2 + c_6) + (c_2 + c_3)n + \frac{n(n+1)}{2}(c_3 + \ c_4 + \frac{n}{2}(c_4 + c_5))$

$= an^3 + bn^2 + cn + d \in \Theta(n^3)$

That is $f(n) \in \Theta(g(n))$ which $g(n)=n^3$.