COMP.4040 HW2 Solution

1. **Solution**(credit from Douglas Salvati):

(1) n^{-2}

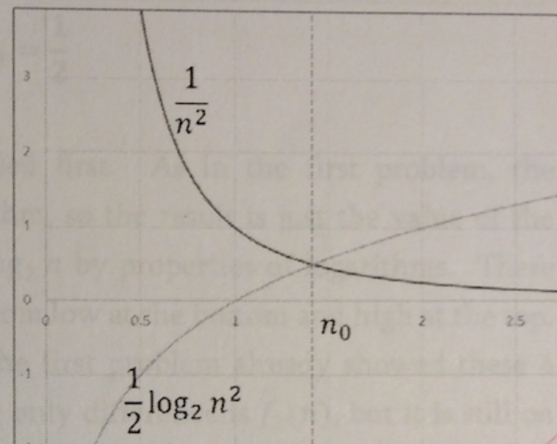
(2) $\log_2\left(2^{(\log_2 n^2)}\right)$

(3) $(\log_2 n)^2$

(4) n^2

The first pair is (1) and (2). Suppose $c = \frac{1}{2}$. Applying the laws of logarithms and exponents,

$$n^{-2} \le c \log_2\left(2^{(\log_2 n^2)}\right)$$
$$\frac{1}{n^2} \le c \log_2 n^2$$
$$\frac{1}{n_0^2} = 2c \log_2 n_0$$
$$\frac{1}{n_0^2} = \log_2 n_0$$

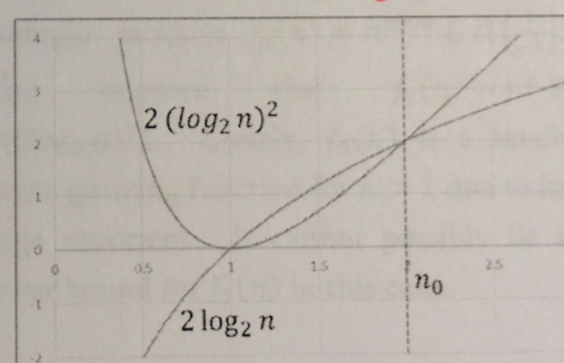


$\sqrt{2}$ is the solution to this equation since $\sqrt{2}^{-2} = \frac{1}{2} = \log_2 \sqrt{2}$.

$$\boldsymbol{c = \frac{1}{2}, n_0 = \sqrt{2}}$$

The second pair is (2) and (3). Suppose $c = 2$. Dividing each side by $\log_2 n$,

$$2 \log_2 n \le c\,(\log_2 n)^2$$
$$2 \le 2(\log_2 n)$$
$$1 = \log_2 n_0$$
$$2^1 = 2^{\log_2 n_0} = n_0$$



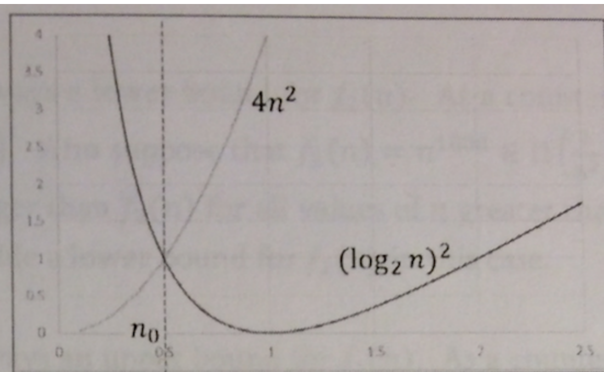
2 is the solution to this equation. Note that by dividing by $\log_2 n$ another solution was lost, in which $\log_2 n_0 = 0 \Rightarrow n_0 = 1$. However, this isn't the solution we were looking for, as shown in the graph.

$$\boldsymbol{c = 2, n_0 = 2}$$

The final pair is (3) and (4). Suppose $c = 4$.

$$(\log_2 n)^2 \leq cn^2$$
$$(\log_2 n_0)^2 = 4n_0^2$$

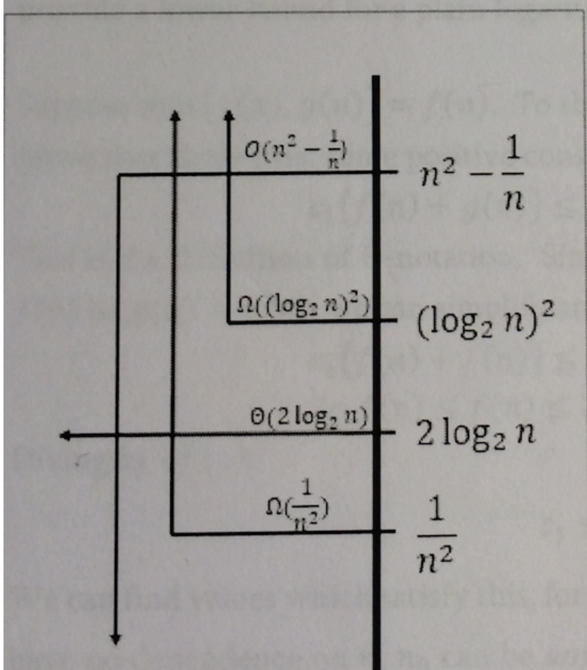
$\frac{1}{2}$ is the solution to this equation since $(\log_2 \left(\frac{1}{2}\right))^2 = 1 = 4\left(\frac{1}{2}\right)^2$.



$$c = 4, n_0 = \frac{1}{2}$$

2. Solution(credit from Douglas Salvati):

The function $\log_2(2^{\log_2 n^2})$ can be simplified first. As in the first problem, the exponential has the same base as the logarithm, so the result is just the value of the exponent, $\log_2 n^2$. This is equivalent to $2\log_2 n$ by properties of logarithms. These functions are ranked by asymptotic growth from low at the bottom and high at the top. For a proof of this diagram being correct, the first problem already showed these 4 functions and proved they are in order. The only difference is $f_2(n)$, but it is still on the order of n^2. Here is the arrow diagram for all four functions:



(a) FALSE. The claim is that $f_3(n)$ is always a lower bound for $f_1(n)$. As a counter-example, suppose $f_3(n) = n^{1000} \in \Omega\left(\frac{1}{n^2}\right)$. Also suppose that $f_1(n) = n^2 \in \Omega((\log_2 n)^2)$. Clearly, $f_3(n)$ is a much larger growing function for $n > 1$ due to its large exponent. It cannot possibly be a lower bound for $f_1(n)$ in this case.

(b) TRUE. There is no constant to multiply by a logarithm to bring it into the logarithm-squared domain. Therefore, the diagram shows that all possible values of $f_1(n) \in \Omega((\log_2 n)^2)$ are lower-bounded by the horizontal arrow representing $f_4(n) \in \Theta(2\log_2 n)$. In other words, any function $f_4(n)$ serves as an upper bound for $f_1(n)$. That is, $f_1(n) \in O(f_4(n))$.

(c) FALSE. The claim is that $f_3(n)$ is always a lower bound for $f_2(n)$. As a counter-example, suppose $f_2(n) = 1 \in O\left(n^2 - \frac{1}{n}\right)$. Also suppose that $f_3(n) = n^{1000} \in \Omega\left(\frac{1}{n^2}\right)$. The function $f_3(n)$ clearly gets much larger than $f_2(n)$ for all values of n greater than 1. Therefore, $f_3(n)$ cannot possibly provide a lower bound for $f_2(n)$ in this case.

(d) FALSE. The claim is that $f_2(n)$ is always an upper bound for $f_1(n)$. As a counter-example, suppose $f_1(n) = n^{1000} \in \Omega((\log_2 n)^2)$. Also suppose that $f_2(n) = 1 \in O\left(n^2 - \frac{1}{n}\right)$. The function $f_2(n)$ clearly gets much smaller than $f_1(n)$ once $n > 1$. Therefore, $f_2(n)$ cannot possibly be an upper bound on $f_1(n)$ in this case.

(e) FALSE. As a counter-example, take $f_4(n)$ to be $\log_2 n \in \Theta(2\log_2 n)$. If this function is to be bounded by $\Theta((\log_2 n)^3)$, that means that $c_1(\log_2 n)^3 \leq f_4(n) \leq c_2(\log_2 n)^3$ for some c_1 and c_2. However, there is no constant at all which can be multiplied by the cube of a logarithm to produce a regular logarithm. After all, to obtain $\log_2 n$ from $(\log_2 n)^3$, you must divide by $(\log_2 n)^2$, which is not a constant. In other words, the cube operation makes the logarithm very large, so it cannot possibly provide a lower bound for a plain logarithm.

3. **Solution**(credit from Denzel Pierre):

For $\max(f(n), g(n)) = \theta(f(n) + g(n))$ (Tight bound):

$\max(f(n), g(n)) = O(f(n) + g(n))$ for all $n > n_0$
$\max(f(n), g(n)) = \Omega(f(n) + g(n))$ for all $n > n_0$
c are some constants.

Because they are nonnegative functions, $f(n) \geq 0$ and $g(n) \geq 0$

$f(n) + g(n) \geq \max(f(n), g(n))$ for all $n \geq n_0$. Therefore,
$\max(f(n), g(n)) \leq c * (f(n) + g(n))$ for all $n > n_0$ when $c = 1$.
Confirmed $\mathbf{max(f(n), g(n)) = O(f(n) + g(n))}$ for all $n > n_0$

Because $\max(f(n), g(n)) \geq f(n)$, and $\max(f(n), g(n)) > g(n)$, $2 * \max(f(n), g(n)) > f(n) + g(n)$.

Therefore, $\max(f(n), g(n)) \geq \frac{1}{2}(f(n) + g(n))$ for all $n > n_0$ when $c = \frac{1}{2}$
Confirmed $\mathbf{max(f(n), g(n)) = \Omega(f(n) + g(n))}$ for all $n > n_0$

Confirmed $\boldsymbol{\max(f(n), g(n)) = \theta(f(n) + g(n))}$

4. **Solution**(credit from Denzel Pierre):

For $f_1(n) = n * \lg n$ and $f_2(n) = 256n$,

$f_2(n) = O(f_1(n))$ when $c = 1, n_0 = 2^{256}$. $f_2(n)$ is faster than $f_1(n)$ after 2^{256}, as that is where the who functions intersect. $f_1(n)$ is faster than $f_2(n)$ from 0 ... 2^{256}. Since 2^{256} is an unrealistically large number ($> 1.15 * 10^{77}$), $f_1(n)$ is the recommended choice.

5. Solution(credit from Denzel Pierre):

5. .

Mystery(n)

1 if n is an even number	c_1	1
2 for $i = \mathbf{1}$ to n	c_2	$n+1$
3 for $j = n$ downto $n/\mathbf{2}$	c_3	$\sum_{i=1}^{n}\sum_{j=\frac{n}{2}}^{n+1} 1$
4 print "even number"	c_4	$\sum_{i=1}^{n}\sum_{j=\frac{n}{2}}^{n} 1$
5 else	c_5	1
6 for $k = \mathbf{1}$ to $n/\mathbf{4}$	c_6	$\frac{n}{4}+1$
7 for $m = \mathbf{1}$ to n	c_7	$\sum_{k=1}^{n/4}\sum_{m=1}^{n+1} 1$
8 print "odd number"	c_8	$\sum_{k=1}^{n/4}\sum_{m=1}^{n} 1$

If n is an even number:

$$T(n) = c_1 + c_2(n+1) + c_3\left(\sum_{i=1}^{n}\sum_{j=\frac{n}{2}}^{n+1} 1\right) + c_4\left(\sum_{i=1}^{n}\sum_{j=\frac{n}{2}}^{n} 1\right)$$

$$T(n) = c_1 + c_2(n+1) + c_3\sum_{i=1}^{n}\left(n+2-\frac{n}{2}\right) + c_4\left(\sum_{i=1}^{n}\frac{n}{2}+1\right)$$

$$T(n) = c_1 + c_2(n+1) + c_3 n\left(\frac{n}{2}+2\right) c_4 n\left(\frac{n}{2}+1\right)$$

$$T(n) = (c_1 + c_2 + c_4) + (c_2 + 2c_3 + c_4)n + \left(\frac{c_3}{2}+\frac{c_4}{2}\right)n^2$$

$$T(n) = an^2 + bn + c \in \theta(n^2)$$

If n is an odd number:

$$T(n) = c_5 + c_6\left(\frac{n}{4}+1\right) + c_7\left(\sum_{k=1}^{\frac{n}{4}}\sum_{m=1}^{n+1} 1\right) + c_8\left(\sum_{k=1}^{\frac{n}{4}}\sum_{m=1}^{n} 1\right)$$

$$T(n) = c_5 + c_6\left(\frac{n}{4}+1\right) + c_7\sum_{k=1}^{\frac{n}{4}} n+1 + c_8\sum_{k=1}^{\frac{n}{4}} n$$

$$T(n) = c_5 + c_6\left(\frac{n}{4}+1\right) + c_7\left(\frac{n}{4}(n+1)\right) + c_8\left(\frac{n^2}{4}\right)$$

$$T(n) = \left(\frac{c_7}{4}+\frac{c_8}{4}\right)n^2 + \left(\frac{c_6+c_7}{4}\right)n + (c_5+c_6)$$

$T(n) = an^2 + bn + c \in \theta(n^2)$

Whether n is even or odd, at worst case, $T(n) \in \theta(n^2)$