Section 1.8: Introduction to Linear Transformations:

Note: A matrix eq. $A\vec{x} = \vec{b}$ can arise in Linear Algebra in a way that is not directly connected w/ linear combination of vectors

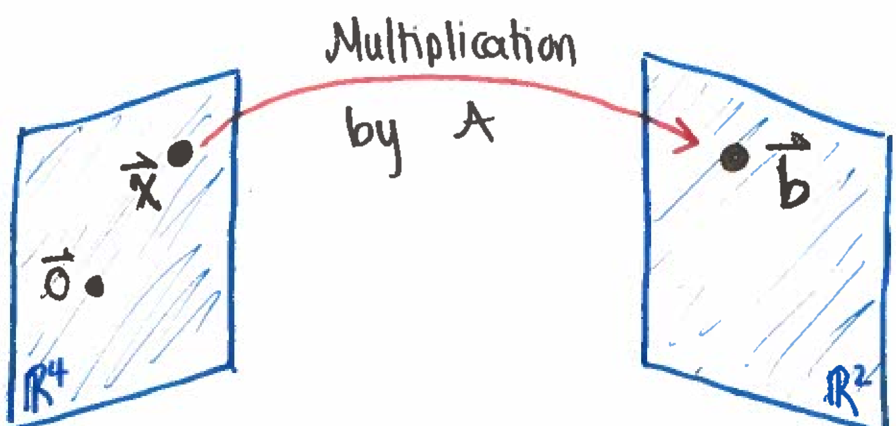
→ **This occurs when:**

We think of the matrix A as an object that 'acts' on a vector $\vec{x}$ by multiplication to produce a new vector called $A\vec{x}$.

Illustration:

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{\vec{b}}$$

Note: This equation says that multiplication by A transforms vector $\vec{x}$ into vector $\vec{b}$.



This new P.O.V.: Solving the eq. $A\vec{x} = \vec{b}$ amounts to finding all vectors $\vec{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\vec{b}$ in $\mathbb{R}^2$ under the action of multiplication by A.

Transforming Vectors via Matrix Multiplication

Note: The correspondence from $\vec{x}$ to $A\vec{x}$ is a function from one set of vectors to another.

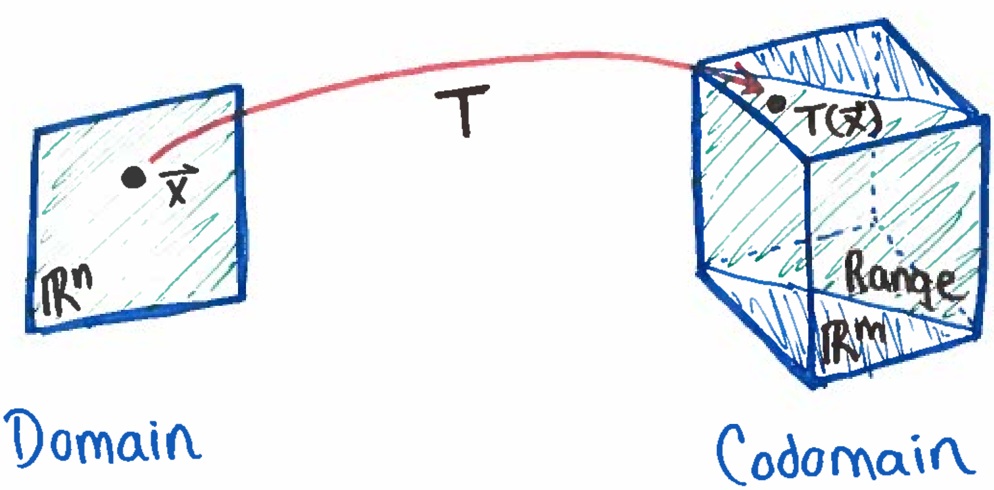
Definition:

A Transformation (or Function, or Mapping) 'T' from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\vec{x}$ in $\mathbb{R}^n$ to a vector $T(\vec{x})$ in $\mathbb{R}^m$

→ **Denoted:** $T : \mathbb{R}^n \to \mathbb{R}^m$

* The Set $\mathbb{R}^n$ is called: The **Domain** of T
* The Set $\mathbb{R}^m$ is called: The **Codomain** of T
* The vector $T(\vec{x})$ in $\mathbb{R}^m$ is called: The **Image** of $\vec{x}$ *Under the action of T*
* The set of all images $T(\vec{x})$ is called: The **Range** of T

Illustration of the Domain, Codomain, & Range of $T : \mathbb{R}^n \to \mathbb{R}^m$:



Example (Matrix Transformations):

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

and define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$

such that:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

Find the following:

(a) Find $T(\vec{u})$, the image of $\vec{u}$ under the transformation T.

(b) Find a $\vec{x}$ in $\mathbb{R}^2$ whose image under T is $\vec{b}$.

(c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?
* A uniqueness question! "Is $\vec{b}$ the image of a unique $\vec{x}$ in $\mathbb{R}^n$?"

(d) Determine if $\vec{c}$ is in the range of the transformation T.
* An existence question! "Does $\exists\ \vec{x}$ whose image is $\vec{c}$?"

Answer:

*Part (a): Find the image of $\vec{u}$ under the transformation T:

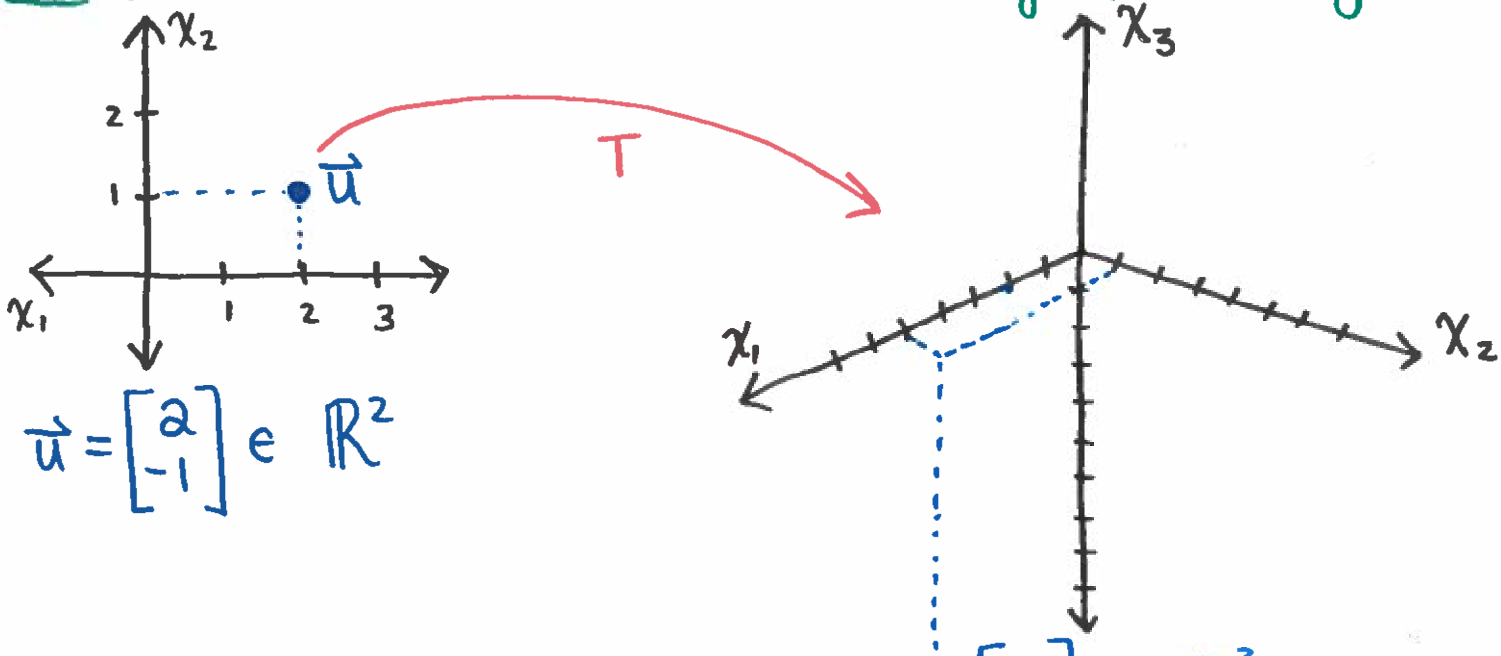
Note: Since the transformation is defined by $T(\vec{x}) = A\vec{x}$ to find $T(\vec{u})$ we compute: $T(\vec{u}) = A\vec{u}$:)

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2(1) - 1(-3) \\ 2(3) - 1(5) \\ 2(-1) - 1(7) \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}}$$

Ans.

Example (Matrix Transformation) Continued...

Note: Lets consider how $T(\vec{u})$ looks graphically :



$\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in \mathbb{R}^2$

$T(\vec{u}) = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix} \in \mathbb{R}^3$

*Part (b): Find a $\vec{x}$ in $\mathbb{R}^2$ whose image under T is $\vec{b}$:

Note: Here we are asked to solve $T(\vec{x}) = \vec{b}$ for $\vec{x}$.

$\Rightarrow$ IOW: Since $T(\vec{x}) = A\vec{x}$, solve $A\vec{x} = \vec{b}$:

- Convert $A\vec{x} = \vec{b}$ to the equivalent aug. matrix $[A \,\vdots\, \vec{b}]$:

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \iff \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]$$

*We want to verify that this system is consistent (i.e. $\exists$ @ least 1 solution)

- Row reduce the aug. matrix $[A \,\vdots\, \vec{b}]$ to find general solution: See next page :

Example (Matrix Transformation) Continued...

- $\begin{array}{r} -3R_1 \\ + \ R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ -1 & 7 & -5 \end{array}\right] \overset{\frac{1}{14}R_2}{\sim} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ -1 & 7 & -5 \end{array}\right]$

- $\begin{array}{r} R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 4 & -2 \end{array}\right] \overset{\frac{1}{4}R_3}{\sim} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 1 & -1/2 \end{array}\right]$

- $\begin{array}{r} -R_2 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$

- $\begin{array}{r} 3R_2 \\ + \ R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \checkmark \Longrightarrow \boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}}$

Answer. reads

$\boxed{\therefore \text{The image of } \vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \text{ under } T \text{ is } \vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}}$

Part (c): Is the more than one $\vec{x}$ whose image under T is $\vec{b}$?

Note: Any $\vec{x}$ whose image under T is $\vec{b}$ must satisfy the equation: $A\vec{x} = \vec{b} \rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$\boxed{\text{No.}}$ $\Rightarrow$ There is only one, unique $\vec{x}$ whose image under T is $\vec{b}$ (as shown in part (b) :))

Example (Matrix Transformation) Continued...

Part (d): Determine if $\vec{c}$ is in the range of the transformation T:

Note: $\vec{c}$ is in the Range of T if: $\vec{c}$ is the image of some $\vec{x}$ in $\mathbb{R}^2 \Rightarrow \vec{c} = T(\vec{x})$

* IOW: Since $T(\vec{x}) = A\vec{x}$, solve $A\vec{x} = \vec{c}$:)

*Convert $A\vec{x} = \vec{c}$ to the equivalent aug. matrix $[A \,|\, \vec{c}]$:

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \longleftrightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right]$$

*Want to verify/check that this system is consistent :)

*Row reduce the augmented matrix:

- $\dfrac{-3R_1 + R_2}{\text{new } R_2} \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ -1 & 7 & 5 \end{array}\right] \overset{\frac{1}{14}R_2}{\sim} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ -1 & 7 & 5 \end{array}\right]$

- $\dfrac{R_1 + R_3}{\text{new } R_3} \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 4 & 8 \end{array}\right] \overset{\frac{1}{4}R_3}{\sim} \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 1 & 2 \end{array}\right]$

- $\dfrac{-R_2 + R_3}{\text{new } R_3} \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 5/2 \end{array}\right]$ * $\rightarrow\leftarrow$

R_3 produces a contradiction $\Rightarrow \therefore$ The system is inconsistent (i.e. No solution $\exists$)

$\boxed{\therefore \text{ Since } A\vec{x} = \vec{c} \text{ produces an inconsistent system, } \vec{c} \text{ is NOT in the Range of the transformation } T.}$

Example: Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$. Find $T(\vec{u})$ & $T(\vec{v})$.

Let $A = \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 8 \\ 16 \\ -24 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} b \\ a \\ d \end{bmatrix}$

Answer:

* **Transformation:** $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$

* **Find $T(\vec{u}) = A\vec{u}$:** Reads: "The image of $\vec{u}$ under the transformation T"

$$T(\vec{u}) = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 8 \\ 16 \\ -24 \end{bmatrix} = \begin{bmatrix} 8(\frac{1}{8}) + 0 + 0 \\ 0 + 16(\frac{1}{8}) + 0 \\ 0 + 0 - 24(\frac{1}{8}) \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}}$$

Answer.

* **Find $T(\vec{v}) = A\vec{v}$:** Reads: "The image of $\vec{v}$ under the transformation T"

$$T(\vec{v}) = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} b \\ a \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{8}b + 0 + 0 \\ 0 + \frac{1}{8}a + 0 \\ 0 + 0 + \frac{1}{8}d \end{bmatrix} = \boxed{\begin{bmatrix} \frac{b}{8} \\ \frac{a}{8} \\ \frac{d}{8} \end{bmatrix}}$$

Answer.

Example: If T is defined by $T(\vec{x}) = A\vec{x}$, find a vector $\vec{x}$ whose image under T is $\vec{b}$, and determine whether $\vec{x}$ is unique. Let $A = \begin{bmatrix} 1 & -4 & 4 \\ 0 & 1 & -4 \\ 4 & -17 & 16 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} -4 \\ -10 \\ -2 \end{bmatrix}$

Answer:

*Recall: $A\vec{x} = \vec{b}$ is equivalent to the aug. matrix $[A \mid \vec{b}]$

*Transformation: $T(\vec{x}) = A\vec{x}$

*Find $\vec{x}$ whose image under T is $\vec{b}$ $\Longrightarrow$ Solve $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & -4 & 4 \\ 0 & 1 & -4 \\ 4 & -17 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -10 \\ -2 \end{bmatrix} \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & -4 & 4 & -4 \\ 0 & 1 & -4 & -10 \\ 4 & -17 & 16 & -2 \end{array}\right]$$

Now solve the aug. matrix to row-reduced echelon form.

- $\begin{array}{r} -4R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array}$ $\longrightarrow \left[\begin{array}{ccc|c} 1 & -4 & 4 & -4 \\ 0 & 1 & -4 & -10 \\ 0 & -1 & 0 & 14 \end{array}\right]$

- $\begin{array}{r} 4R_2 \\ + R_1 \\ \hline \text{new } R_1 \end{array}$ $\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -12 & -44 \\ 0 & 1 & -4 & -10 \\ 0 & -1 & 0 & 14 \end{array}\right]$

- $\begin{array}{r} R_2 \\ + R_1 \\ \hline \text{new } R_1 \end{array}$ $\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -12 & -44 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -4 & 4 \end{array}\right] \overset{\frac{-1}{4}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -12 & -44 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Example Continued...

- $\begin{array}{r} 4R_3 \\ + R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -12 & -44 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -1 \end{array}\right]$

- $\begin{array}{r} 12R_3 \\ + R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -56 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -1 \end{array}\right] \Longrightarrow \boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -56 \\ -14 \\ -1 \end{bmatrix}}$

Therefore: A vector $\vec{x}$ whose image under T is $\vec{b}$ is

$$\vec{x} = \begin{bmatrix} -56 \\ -14 \\ -1 \end{bmatrix}$$

*Since NO free variables $\exists$, this solution is unique.

Answer.

Example: If T is defined by $T(\vec{x}) = A\vec{x}$, find a vector $\vec{x}$ whose image under T is $\vec{b}$, and determine if $\vec{x}$ is unique. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 1 & -12 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}$

Answer:

* Transformation: $T(\vec{x}) = A\vec{x}$

* Find a vector $\vec{x}$ whose image under T is $\vec{b}$:

IOW: Find $\vec{x}$ ST $A\vec{x} = \vec{b}$:)

Recall: $A\vec{x} = \vec{b} \Leftrightarrow [A \,\vdots\, \vec{b}]$
Solve the aug. matrix for $\vec{x}$

$$\begin{bmatrix} 1 & -3 & -4 \\ -3 & 1 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ -3 & 1 & -12 & -11 \end{array}\right]$$

Note: $\exists$ more unknowns than equations $\Rightarrow$ linear dependent.

- $3R_1 + R_2$ = new R_2 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ 0 & -8 & -24 & -8 \end{array}\right] \overset{-\frac{1}{8}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right]$

- $3R_2 + R_1$ = new R_1 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 3 & 1 \end{array}\right] \Rightarrow \boxed{\vec{x} = \begin{cases} x_1 = 4 - 5x_3 \\ x_2 = 1 - 3x_3 \\ x_3 \text{ is free} \end{cases}}$

$\therefore$ General Solution for vector $\vec{x}$ whose image under T is $\vec{b}$:

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4-5x_3 \\ 1-3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}}$$

* Since x_3 is a free variable, solution is NOT unique.

Example: Let $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ & Let $A = \begin{bmatrix} 1 & -4 & 4 & -4 \\ 0 & 1 & -4 & 4 \\ 3 & -10 & 4 & -3 \end{bmatrix}$.

Is $\vec{b}$ in the range of the linear transformation $\vec{x} \mapsto A\vec{x}$? Explain why or why not.

Answer:

* Recall: The range of T is the set of all linear combinations of the columns of A

$\Rightarrow$ IOW: $\vec{b}$ is in the range of T if $\vec{b}$ is the image of some $\vec{x}$ $\Rightarrow$ Solve: $A\vec{x} = \vec{b}$

* Convert $A\vec{x} = \vec{b}$ to its equivalent aug. matrix form $[A \,\vdots\, \vec{b}]$ & row reduce:

* Note: If $[A \,\vdots\, \vec{b}]$ is a consistent system, then $\vec{b}$ is in the range of $\vec{x} \mapsto A\vec{x}$:)

$$\begin{bmatrix} 1 & -4 & 4 & -4 \\ 0 & 1 & -4 & 4 \\ 3 & -10 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \Leftrightarrow \underbrace{\left[\begin{array}{cccc|c} 1 & -4 & 4 & -4 & -1 \\ 0 & 1 & -4 & 4 & 1 \\ 3 & -10 & 4 & -3 & -1 \end{array}\right]}$$

* Note: # of unknowns > # of equations

$\Rightarrow$ Linear Dependent :)

Therefore: The columns of A are linearly dependent & $A\vec{x} = \vec{b}$ has a nontrivial solution

Answer.

$\Rightarrow$ $\boxed{[A \,\vdots\, \vec{b}] \text{ is a consistent system } \& \ \vec{b} \text{ is in the range of T.}}$

Geometric Representations of Matrix Transformations

Note: Looking at matrix transformations geometrically can help to reinforce the view of a matrix as something that transforms vectors into other vectors :)

Illustration (A projection transformation):

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\vec{x} \mapsto A\vec{x}$ projects points in $\mathbb{R}^3$ onto the x_1x_2-plane.

*Computation: $\vec{x} \mapsto A\vec{x}$

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 0x_2 + 0x_3 \\ 0x_1 + 1x_2 + 0x_3 \\ 0x_1 + 0x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in \mathbb{R}^2$$

$$\boxed{\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \longmapsto A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in \mathbb{R}^2}$$

*Geometrically:

A projection transformation

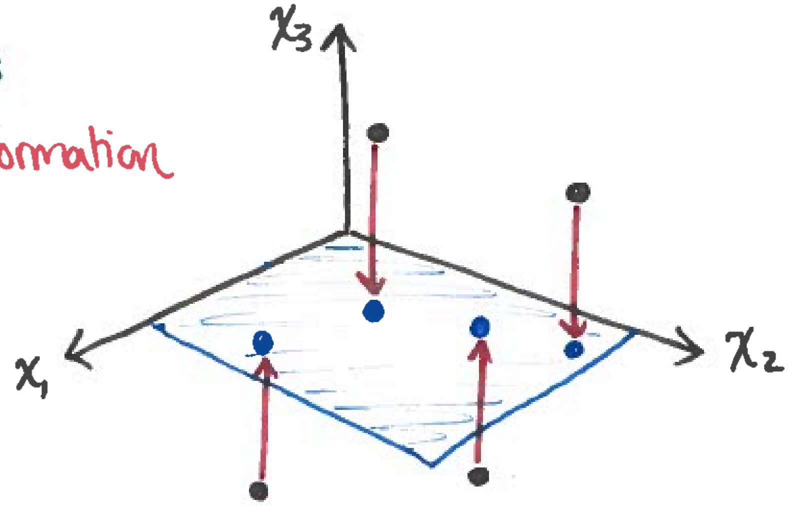


*Illustration (A Shear Transformation):

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ is called a <u>Shear Transformation</u>. If T acts on each point in the 2×2 square, then the set of images forms the shaded parallelogram.

*<u>Note</u>: The key idea is to show that T maps line segments to line segments, and then to check that the corners of the squares map to the vertices of the paralleogram.

Let $\vec{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find $T(\vec{u})$ & $T(\vec{v})$.

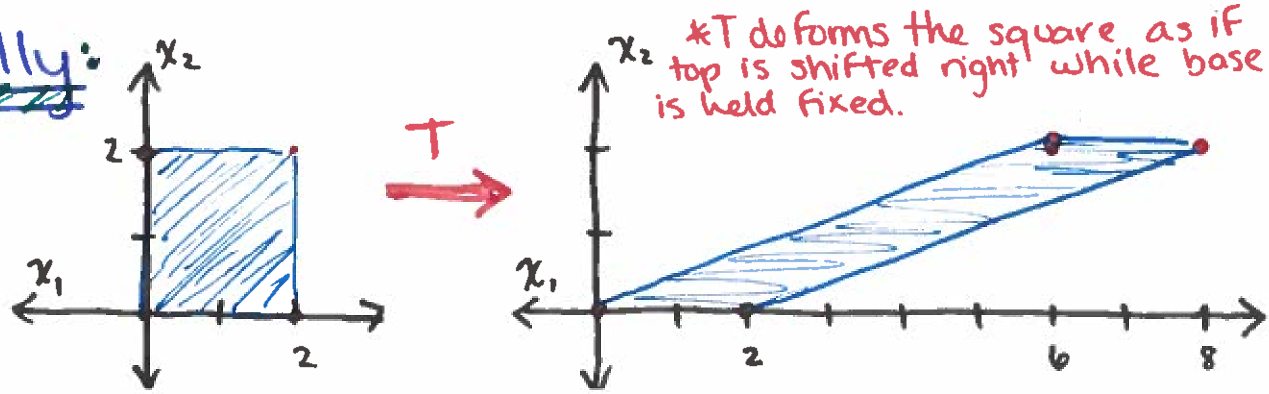
*Computation:

* $T(\vec{u}) = A\vec{u} \to$ The image of $\vec{u}$ under the transformation T
* $T(\vec{v}) = A\vec{v} \to$ The image of $\vec{v}$ under the transformation T

$$*T(\vec{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0(1) + 2(3) \\ 0(0) + 2(1) \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$*T(\vec{v}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(1) + 2(3) \\ 2(0) + 2(1) \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

*Geometrically:



Matrix Transformations

For each $\vec{x}$ in $\mathbb{R}^n$, $T(\vec{x})$ is computed as $A\vec{x}$, where A is an $m \times n$ matrix.

Note: We sometimes denote such a <u>matrix transformation</u> by: $\vec{x} \mapsto A\vec{x}$

Observations:

- The domain of T is $\mathbb{R}^n$ when A has n columns
- The codomain of T is $\mathbb{R}^m$ when each column of A has m entries.
- The range of T is the set of all linear combinations of the columns of A
  - → because each image of $T(\vec{x})$ is of the form $A\vec{x}$:)
  - → A vector $\vec{b}$ is in the range of T if $[A \,\vdots\, \vec{b}]$ is a <u>consistent system</u> :)

*General Observations/ Conclusions *

① A <u>linear transformation</u> is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that assigns to each vector in $\mathbb{R}^n$, a vector in $\mathbb{R}^m$.

② If A is a 3×5 matrix & T is a transformation defined by $T(\vec{x}) = A\vec{x}$, Then...

* # of columns in A = # of rows in $\vec{x}$ ⇒ Domain of T

∴ Since A has $n=5$ columns, the domain is $\mathbb{R}^5$

⇒ <u>In the product $A\vec{x}$</u>: If A is a $m \times n$ matrix, then $\vec{x} \in \mathbb{R}^n$:)

* Domain ⇒ # of columns

③ If A is an $m \times n$ matrix, then <u>the RANGE of the transformation $\vec{x} \mapsto A\vec{x}$</u> is:

The set of all linear combinations of the columns of A b/c each image of the transformation is of the form $A\vec{x}$

* $\mathbb{R}^n \to$ Domain
  $\mathbb{R}^m \to$ Codomain

Question: How many rows & columns must matrix A have in order to define a mapping from $\mathbb{R}^4$ into $\mathbb{R}^7$ by the rule $T(\vec{x}) = A\vec{x}$?

$\Rightarrow$ Let $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^7$ such that $T(\vec{x}) = A\vec{x}$

*Note: For the product $A\vec{x}$ to $\exists$

$\Rightarrow$ The # of columns in matrix A must match the # of rows in vector $\vec{x}$

- $\vec{x} \in \mathbb{R}^4$ ($\vec{x}$ is in the Domain of T) $\rightarrow$ 4 rows in $\vec{x}$

- $A \rightarrow$ 7×4 matrix $\begin{cases} \cdot\, 7 \text{ rows} \\ \cdot\, 4 \text{ columns} \end{cases}$

Check:

$$A\vec{x} = \begin{bmatrix} a_{11} & \cdot & \cdot & a_{14} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & & & \\ a_{71} & \cdot & \cdot & a_{74} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} + x_4 a_{14} \\ \vdots \\ \vdots \\ x_1 a_{71} + x_2 a_{72} + x_3 a_{73} + x_4 a_{74} \end{bmatrix}$$

$\uparrow$

Each column has 7 entries :)

Linear Transformations

The properties of the Matrix-Vector Product $A\vec{x}$, written in function notation, identify the most important class of transformations in Linear Algebra!

Definition:

Let $\vec{u}$ & $\vec{v}$ be vectors in $\mathbb{R}^n$. Let $c \in \mathbb{R}$ be a scalar.

A transformation (or mapping) T is **linear** if:

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ $\quad \forall\ \vec{u},\vec{v}$ in the domain of T

(ii) $T(c\vec{u}) = cT(\vec{u})$ $\quad \forall$ scalars c & $\forall\ \vec{u}$ in the domain of T

Every matrix transformation is a linear transformation.

*CAUTION: The reverse is NOT necessarily true → As we will see later in chapters 4 & 5 :)

Linear Transformations preserve the operations of vector addition & scalar multiplication, leading to the following:

Additional Properties:

If T is a Linear Transformation, then:

(i) $T(\vec{0}) = \vec{0}$

(ii) $T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p) = c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_pT(\vec{u}_p)$

$\forall$ vectors $\vec{u} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ & $\forall$ scalars $c = \{c_1, c_2, \ldots, c_p\}$

Example 1 (Linear Transformation):

Given a scalar r, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$.

T is called a contraction when $0 \le r \le 1$ and called a dilation when $r > 1$.

Let $r = 3$ and show that T is a linear transformation.

Answer:

* Given:
  - A transformation (or mapping): $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$, where r is a scalar.
  - A scalar: $r = 3 \Rightarrow \boxed{T(\vec{x}) = 3\vec{x}}$

Recall: If T is a linear transformation, then:

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p) = c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_pT(\vec{u}_p)$$

$\forall$ vectors $\vec{u}$ is the domain of T & $\forall$ scalars c.

Let $\vec{u}_1$ & $\vec{u}_2$ be vectors in $\mathbb{R}^2$ & Let c_1, c_2 be scalars

Since $T(\vec{x}) = 3\vec{x}$:

$$\begin{aligned} T(c_1\vec{u}_1 + c_2\vec{u}_2) &= 3(c_1\vec{u}_1 + c_2\vec{u}_2) \\ &= 3c_1\vec{u}_1 + 3c_2\vec{u}_2 \\ &= c_1(3\vec{u}_1) + c_2(3\vec{u}_2) \\ &= c_1T(\vec{u}_1) + c_2T(\vec{u}_2) \end{aligned}$$

$\therefore$ Since $T(c_1\vec{u}_1 + c_2\vec{u}_2) = c_1T(\vec{u}_1) + c_2T(\vec{u}_2)$, T is a Linear Transformation

Example 2 (Linear Transformation):

Define a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Find the images under T of $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and

$\vec{u} + \vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Answer:

Notes: i) $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

ii) $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

*Find $T(\vec{u})$:

$$T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix}}$$

*Find $T(\vec{v})$:

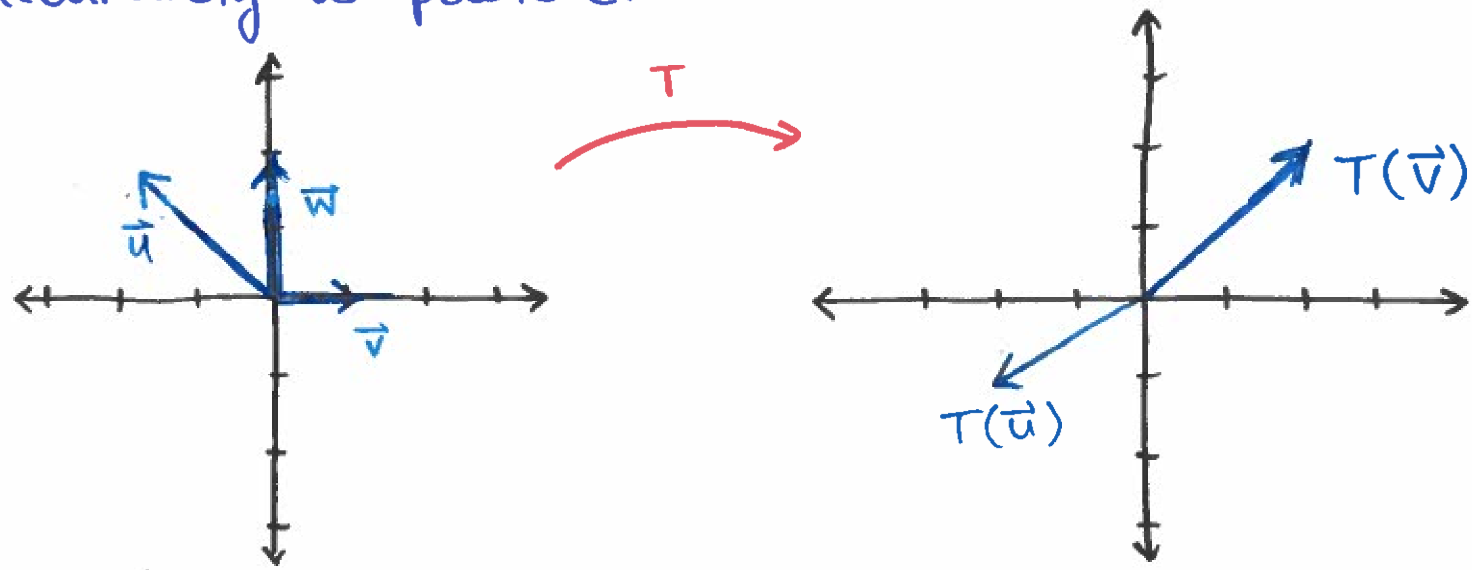
$$T(\vec{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$

*Find $T(\vec{u} + \vec{v})$:

Recall: $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

$$T(\vec{u} + \vec{v}) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ 6 \end{bmatrix}}$$

Example: The figure below shows vectors $\vec{u}$, $\vec{v}$, & $\vec{w}$, along with $T(\vec{u})$ & $T(\vec{v})$ under the action of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$. Draw the image of $T(\vec{w})$ as accurately as possible.

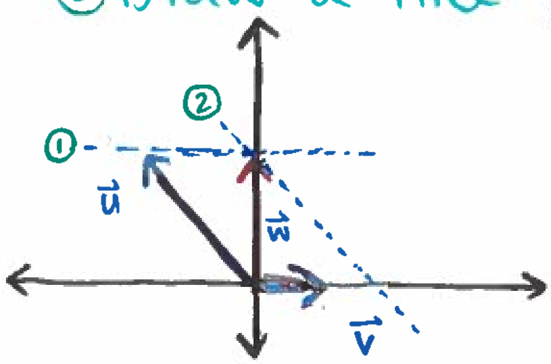


Answer:

Note: We can create a parallelogram using $\vec{u}$, $\vec{v}$, & $\vec{w}$.

① Draw a line parallel to $\vec{v}$, through $\vec{w}$

② Draw a line parallel to $\vec{u}$, through $\vec{w}$



Notes:

- One side length is $\vec{u}$
- Shorter side length $\approx 2\vec{v}$

$\Rightarrow \boxed{\vec{w} = \vec{u} + 2\vec{v}}$

*Using the Properties of a Linear Transformation:

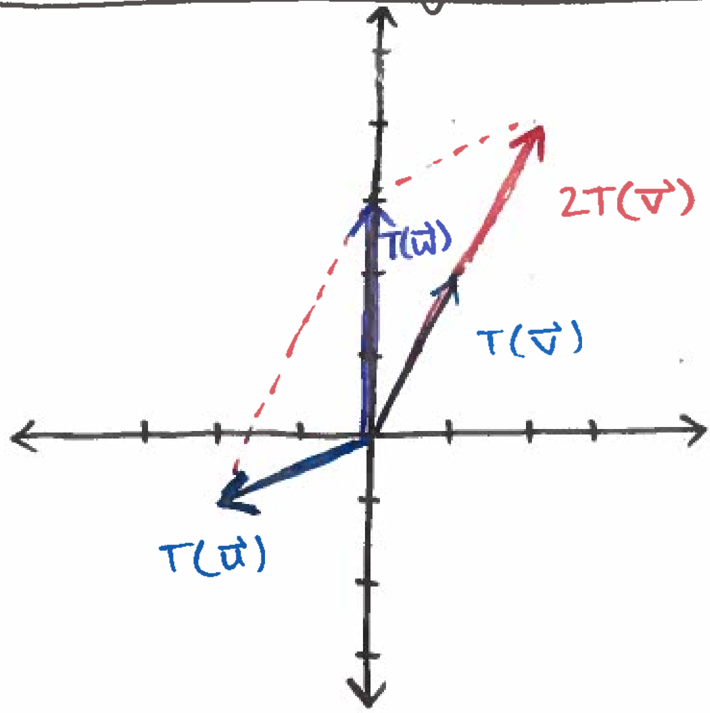
$$T(\vec{w}) = T(\vec{u} + 2\vec{v}) = T(\vec{u}) + T(2\vec{v}) = T(\vec{u}) + 2T(\vec{v})$$

$$\therefore \boxed{T(\vec{w}) = T(\vec{u}) + 2T(\vec{v})}$$

Sketch on the next page :)

Example Continued...

* **Draw the image of $T(\vec{w})$:**



Notes: To sketch $T(\vec{w})$, we again use a parallelogram ∵ Since $T(\vec{w}) = T(\vec{u}) + 2T(\vec{v})$

① Side Lengths are "$T(\vec{u})$" & "$2T(\vec{v})$"

② $T(\vec{w})$ is the diagonal of the parallelogram

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ and maps $\vec{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Use the fact that T is linear to find the images under T of $3\vec{u}$, $2\vec{v}$, & $3\vec{u} + 2\vec{v}$.

Answer:

* Linear Transformation: $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that

  - For $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $T(\vec{u}) = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

  - For $\vec{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $T(\vec{v}) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Recall: If T is a linear transformation, then:

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p) = c_1T(\vec{u}_1) + \cdots + c_pT(\vec{u}_p)$$

for vectors $\{\vec{u}_1, \ldots, \vec{u}_p\}$ & scalars $\{c_1, \ldots, c_p\}$

Since T is a linear transformation:

* $T(3\vec{u}) = 3T(\vec{u}) = 3\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 18 \\ 3 \end{bmatrix}}$ * The image of $3\vec{u}$ under T.

* $T(2\vec{v}) = 2T(\vec{v}) = 2\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ 8 \end{bmatrix}}$ * The image of $2\vec{v}$ under T.

* $T(3\vec{u} + 2\vec{v}) = T(3\vec{u}) + T(2\vec{v}) = \begin{bmatrix} 18 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \boxed{\begin{bmatrix} 16 \\ 11 \end{bmatrix}}$ * The image of $3\vec{u} + 2\vec{v}$ under T

Example: Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $\vec{y}_1 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ & $\vec{y}_2 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps e_1 into y_1 and e_2 into y_2. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Answer:

* Linear Transformation: $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that

  - For $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $T(\vec{e}_1) = \vec{y}_1 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

  - For $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $T(\vec{e}_2) = \vec{y}_2 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$

* Find the image of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$: Let $\vec{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

Goal: Find $T(\vec{x})$

i) Rewrite $\vec{x}$ in terms of $\vec{e}_1$ & $\vec{e}_2$:

$$\vec{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\vec{e}_1 - 3\vec{e}_2$$

$$\Rightarrow \boxed{\vec{x} = 5\vec{e}_1 - 3\vec{e}_2}$$

ii) Use prop. of Linear Transformations to find $T(\vec{x})$:

$$T(\vec{x}) = T(5\vec{e}_1 - 3\vec{e}_2) = 5T(\vec{e}_1) - 3T(\vec{e}_2) = 5\begin{bmatrix} 4 \\ 6 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 8 \end{bmatrix} = \cdots$$

Example Continued...

$$T(\vec{x}) = 5\begin{bmatrix}4\\6\end{bmatrix} - 3\begin{bmatrix}-1\\8\end{bmatrix} = \begin{bmatrix}20+3\\30-24\end{bmatrix} = \begin{bmatrix}23\\6\end{bmatrix}$$

$$\boxed{\therefore T(\vec{x}) = \begin{bmatrix}23\\6\end{bmatrix}}$$

*Find the image of $\begin{bmatrix}x_1\\x_2\end{bmatrix}$: Let $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

Note: Want to find $T(\vec{x})$ ∴

i) Rewrite $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ in terms of $\vec{e}_1$ & $\vec{e}_2$:

$$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2$$

$$\Rightarrow \boxed{\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2}$$

ii) Find $T(\vec{x}) \rightarrow$ Use properties of Linear Transformations:

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1\vec{y}_1 + x_2\vec{y}_2 = x_1\begin{bmatrix}4\\6\end{bmatrix} + x_2\begin{bmatrix}-1\\8\end{bmatrix}$$

$$= \begin{bmatrix}4x_1 - x_2\\6x_1 + 8x_2\end{bmatrix}$$

$$\boxed{\therefore T(\vec{x}) = \begin{bmatrix}4x_1 - x_2\\6x_1 + 8x_2\end{bmatrix}}$$

Example: Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$, & $\vec{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{x}$ into $x_1\vec{v}_1 + x_2\vec{v}_2$. Find a matrix A such that $T(\vec{x})$ is $A\vec{x}$ for each $\vec{x}$.

Answer:

* **Want to Find:** Matrix A ST : $T(\vec{x}) = A\vec{x}$, $\forall \vec{x}$

* **Given the Linear Transformation:**

$T: \mathbb{R}^2 \to \mathbb{R}^2$ ST : $T(\vec{x}) = x_1\vec{v}_1 + x_2\vec{v}_2$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= A\vec{x}$$

$$\therefore \boxed{\text{Matrix } A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 2 & 7 \end{bmatrix}}$$

Answer.

Example: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.

Answer:

* Want to show: The Set $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.

IOW (Goal) $\Rightarrow$ $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent if $\exists$ weights $\{c_1, c_2, c_3\}$, not all zero, ST: $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = \vec{0}$

* Given:

- Linear Transformation: $T: \mathbb{R}^n \to \mathbb{R}^m$
- Linearly Dependent Set: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \in \mathbb{R}^n$

By Definition (1.7), since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \in \mathbb{R}^n$ are linearly dependent, $\exists$ weights/scalars (not all zero) ST:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

By Properties of Linear Transformations (1.8), Since T is a linear transformation:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0})$$

$$c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = \vec{0} \quad \checkmark$$

$\boxed{\therefore \{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \text{ is linearly dependent.}}$