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# Exam 1 Review Sheet / Discrete Structures II / Spring 2018

**Exam 1 is on Friday, February 16 and will cover Chapter 6.** No notes, calculator, book, etc.

*Extra office hours:* Thursday, February 15 from noon to 3 pm in 428L Olney.

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## Review Problems

1. A group of eight people consists of four husband and wife couples.

   (a) In how many different ways can the eight people be seated in a row if $2 \times (4^2 \times 3^2 \times 2^2 \times 1^2) = 1152$

      i. no two men are seated next to each other, and no two women are seated next to each other?

      ii. each husband sits next to his wife? $8 \times 1 \times 6 \times 1 \times 4 \times 1 \times 2 \times 1 = 384$

   (b) In how many different ways can a group of four people be selected

      i. if there are no restrictions? $C(8,4) = 70$

      ii. if one member of a couple is selected, the person's partner must also be selected? 4 couples, $C(4,2) = 6$

      iii. if the group contains either the husband or the wife from a couple, but not both? $2^4 = 16$ (Pick H or W from group

      iv. if exactly one couple is included in the group?

2. A password for a certain website consists of 5 digits chosen from $\{0, 1, \ldots, 9\}$. How many passwords

   (a) are there in total?

   (b) contain the digit 1?

   (c) consist of five different digits?

   (d) consist of five different digits and contain 145 as a consecutive substring?

   (e) have digits that are in strictly increasing order? That is, strings that look like *abcde*, where $a < b < c < d < e$.
   Every password satisfying condition corresponds uniquely to a subset of 5 elements from: $\{0, 1, \ldots, 9\}$. Count $C(10,5)$

   (f) have digits that are alternating? (For example, 34343 or 51515.)

   (g) contain a string of four or more consecutive 5's?

   (h) consist of three 2's and two 4's? (For example, 34433.)

   (i) contain exactly three 2's?

3. There are four different types of cookies available from a bakery (chocolate chip, oatmeal, peanut butter, and lemon drop). Assume that cookies of the same type are considered to be identical.

   (a) How many ways are there to pick select 6 cookies if ...

      i. there's no restrictions?

      ii. at least two chocolate chip cookies and one lemon drop cookie are selected? 2 choco + 1 lemon + 3 add'l cookies
      Count # of 3-combinations of 4 cookies ($r = 3$, $n = 4$): $C(6,3) = 20$

      iii. the selection doesn't include at least one of the types of cookies?
      = Total # of ways to pick 6 cookies − # of ways to pick so that at least 1 cookie of each type is selected = $84 - C(5,2) =$ 1[illegible]
      ($r = 2$, $n = 4$)

      iv. at most three chocolate chip cookies are picked?

   (b) Bob buys three chocolate chip cookies, two oatmeal cookies, and five peanut butter cookies. He eats exactly one cookie per day for ten days. In how many different orders can he consume the cookies? Count # of strings of length 10 with exactly 3 C's, 2 O's, 5 P's: $10!/(3!\,2!\,5!)$

   (c) How many ways can 12 chocolate chip cookies be distributed among five people if every person gets at least one cookie?

   (d) How many ways can 5 oatmeal cookies and 6 peanut butter cookies be distributed among four people?

   (e) Alice has one of each of the four types of cookies. In how many ways can she distribute the cookies among four friends if ...

      i. each person gets one cookie? C O P L: $4 \times 4 \times 4 \times 4 = 256$

      ii. there's no restrictions? (In particular, this means that someone could get no cookies or more than one cookie.)

3/b/Alt. method

Procedure: $C(10,3) \times C(7,2)$

+ Pick 3 out of 10 days for choco chip: $C(10,3)$
+ Pick 2 out of 7 days for oatmeal: $C(7,2)$
+ The rest of days: Peanut

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SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS. This exam is worth 100 points.

1. (6 points) Complete the following statement of the generalized pigeonhole principle:
Let $m, p$ be positive integers. If $p$ objects are placed into $m$ boxes, then ...there is at least one box which contains at least $\lceil \frac{p}{m} \rceil$ objects.

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Excellent

2. (14 points) One of the following statements below is true, and the other is false. Use the generalized pigeonhole principle to explain why the true statement is true. For the statement that's false, give a specific counterexample demonstrating why it's false.

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Statements:

(a) A sock drawer contains 10 red socks, 10 blue socks, and 10 green socks. If eleven socks are taken out of the drawer at random, then there will be least three socks which are the same color.

(b) Suppose that a class of 20 students consists of 10 freshmen, 6 juniors, and 4 sophomores. If 11 students are selected at random, then there must be at least three students who are freshmen.

a/ The statement is true

- Because based on the generalized pigeohole principle, 11 socks are placed into 3 boxes (3 different colors: red, blue and green), then there is at least one box which contains at least $\lceil \frac{11}{3} \rceil = \lceil 3\frac{2}{3} \rceil = 4$ socks with the same color

It does not contradict the statement that there is at least 3 socks with the same color

b/ The statement is false

If the 10 first students which are selected are 6 juniors and 4 sophomores, then the last student will be the freshman

Therefore, 11 students which are selected include 6 juniors, 4 sophomores and 1 freshman

It contradicts the statement there must be at least 3 students who are freshmen.

$\boxed{20 \text{ pts}}$

3. (6 points) Consider the 5-combinations of the set $S = \{1, 2, \ldots, 9\}$. What's the next larger 5-combination after $\{1, 3, 5, 8, 9\}$ in lexicographic order?

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$n = 9$

$r = 5$

$n - r + i = 9 - 5 + 1 = 4 + i$

$\{1, 3, \boxed{5}, 8, 9\}$ ✓

$\{1, 3, 6, 7, \boxed{8}\}$ ①

$\{1, 3, 6, \boxed{7}, 9\}$ ②

$\{1, 3, \boxed{6}, 8, 9\}$ ③

$\{1, \boxed{3}, 7, 8, 9\}$ ④

$\{1, 4, 5, 6, 7\}$ ⑤

4. (6 points) Consider permutations of the set $S = \{1, 2, \ldots, 7\}$. What's the next larger permutation in lexicographic order after 3125764?

6

$3\,1\,2\,\underset{j}{\boxed{5}}\,7\,\underset{k}{\boxed{6}}\,4$

$3\,1\,2\,6\,7\,5\,4$

$3\,1\,2\,6\,4\,5\,7$

The next larger permutation : 3126457 ✓

5. (6 points) Suppose that a quiz has 8 true-false questions. How many different ways can a student answer the questions if exactly three questions are answered false and the rest are answered true?

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The number of ways :

$$\frac{8!}{3!\,5!} = \frac{8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \boxed{56} \text{ (ways)}$$ ✓

$\boxed{18\text{pt}}$

6. (20 points) Consider strings of length 6 containing only letters from the set $\{a, b, c\}$. Note that there are $3^6 = 729$ such strings. How many of these strings ...

(a) ... contain at least one $c$?

6/6 $U$ = { numbers of strings of length 6 }

$A$ = { numbers of strings of length 6 contain at least one c }

$\bar{A}$ = { numbers of strings of length 6 contain no c }

$$|A| = |U| - |\bar{A}|$$

$$= 3^6 - 2^6 = 729 - 64 = \boxed{665}$$

(b) ... contain an equal number of $a$'s, $b$'s, and $c$'s?

6/6 This means there are 2 a's, 2 b's and 2c's

There are: $\frac{6!}{2!\,2!\,2!} = \frac{6 \times 5 \times 4 \times 3 \times \cancel{2} \times 1}{\cancel{2} \times 1 \times \cancel{2} \times 1 \times \cancel{2} \times 1} = \boxed{90}$ (ways)

8/8 (c) ... contain exactly one $b$?

There are 6 ways to pick a position for b

There are $2^5 = 32$ ways to pick for the rest 5 positions.

Product rule: $6 \times 32 = \boxed{192}$ (ways) to choose the strings of length 6 containing exactly one b

20pt

14

7. (14 points) How many ways are there to distribute six different toys to three children ...

(a) ... if each child gets exactly two toys?

$$\frac{6!}{2!\,2!\,2!} = \frac{6\times5\times\cancel{4}\times3\times\cancel{2}\times1}{\cancel{2}\times1\times2\times1\times2\times1} = \boxed{90} \text{ (ways)}$$

(b) ... if there are no restrictions?

Each different toy has 3 selections :

$$3^6 = \boxed{729} \text{ (ways)}$$

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8. (8 points) Suppose that a department contains 7 men and 5 women. How many ways are there to form a committee with six members if there must be an equal number of men and women?

6 members and there must be an equal number of men and women, that means there are 3 women and 3 men :

The total number of ways will be :

$$C(7,3) \times C(5,3) = \frac{7\times\cancel{6}\times5}{\cancel{3}\times\cancel{2}\times1} \times \frac{5\times\overset{2}{\cancel{4}}\times\cancel{3}}{\cancel{3}\times\cancel{2}\times1}$$

$$= 7\times5\times5\times2$$

$$= \boxed{350} \text{ (ways)}$$

22pt

9. (6 points) Bob buys five different types of bagels. He plans to eat a bagel a day for five days. In how many different orders can he consume the bagels?

6

$5! = 5 \times 4 \times 3 \times 2 \times 1 = \boxed{120}$ (ways)

10. (14 points) A bagel shop has five types of bagels, including onion and poppy seed bagels. How many ways are there to select eight bagels if ...

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(a) ... at least one bagel of each type is selected?

There are 5 bagels of each type which are selected, so need to pick 3 more bagels

Count the number of 3-combinations of 5 objects

$n = 5$

$r = 3$

$C(n+r-1, r) = C(7,3) = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \boxed{35}$ (ways)

(b) ... exactly three onion bagels are selected?

Count the number of 5-combinations of 4 objects

$n = 4$

$r = 5$

$C(n+r-1, n-1) = C(8,3) = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = \boxed{56}$ (ways)

20 pts

x2

EXTRA CREDIT (5 POINTS): Consider the bagel shop from the last question. How many ways are there to pick eight bagels if at most two onion bagels or at most two poppy seed bagels are selected? (Hint: Count the complement.)

0 onion bagel : $n = 4$, $r = 8$ $\quad C(11,3) = \frac{11 \times \cancel{10}^{5} \times \cancel{9}^{3}}{\cancel{3} \times \cancel{2} \times 1} = 55 \times 3 = 165$

1 onion bagel : $n = 4$, $r = 7$ $\quad C(10,3) = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{3} \times \cancel{2} \times 1} = 120$

2 onion bagels : $n = 4$, $r = 6$ $\quad C(9,3) = \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7}{\cancel{3} \times \cancel{2} \times 1} = 84$

Same with poppy seed bagels : $(165 + 120 + 84) \times 2$

$= (285 + 84) \times 2$

$= 369 \times 2$

$= \boxed{738}$ (ways)

The cases have overlap, e.g. you can have 1 onion and 2 poppy bagels.

Use inclusion-exclusion

($n = 5, r = 8$) $\quad$ ($n = 5, r = 2$)

[total # ways] − [# ways so that at least 3 onions & at least 3 poppy]

$= C(12,4) - C(6,4)$

$= 495 - 15$

$= \boxed{480}$

Dvengnhr

# Exam 2 Review - Discrete Structures II - Spring 2018

**Exam 2 is on Monday, March 26** in class and will cover Sections 7.1 through 7.3 plus the principle of inclusion-exclusion. No notes, textbook, calculator, or other electronic devices allowed! We will go over any questions you have from this sheet in class on Friday, March 23. # 1 - 4 on HW from "notes on inclusion - exclusion"

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## Review Problems:

1. Review the assigned homework problems from section 7.1 through 7.3, section 6.5, and the additional notes for the principle of inclusion-exclusion.

2. Be able to use the methodology of Section 7.1 to compute probability (i.e. use the $p(E) = |E|/|S|$ formula).

3. Be able to state the principle of inclusion-exclusion for any number of sets.

4. Let $A_1, A_2, A_3, A_4$ be four sets satisfying the following conditions:

   - $|A_1 \cup A_2 \cup A_3 \cup A_4| = 100$
   - $|A_j \cap A_k| = 3k$ for all $k > j$.
   - $|A_i \cap A_j \cap A_k| = 3$ for all $k > j > i$.
   - $A_1 \cap A_2 \cap A_3 \cap A_4 = \varnothing$

   Suppose that the cardinalities of the sets $A_1, A_2, A_3, A_4$ are all equal. Use the principle of inclusion-exclusion to find $|A_1|$.

5. A coin is flipped seven times. What's the probability that

   (a) exactly three tails occurs? $C(7,3)/2^7 = 35/128$

   (b) at most three heads occurs? $\frac{C(7,0)+C(7,1)+C(7,2)+C(7,3)}{2^7} = 64/128 = 1/2$

   (c) at most five heads occurs? $b + \frac{C(7,4)+C(7,5)}{128}$

   (d) three heads occurs given that the first flip was heads? $\frac{C(6,2)}{2^6} = 15/64$

6. Same question as the previous question, but assume the coin is biased so that probability of heads is 1/3. (You can leave your answers as a sum/difference/product/quotient of numbers.) $b(n,k,p) = C(n,k)\,p^k q^{n-k}$

7. In a certain lottery, a subset of 4 different numbers is picked from the set $\{1, 2, \ldots, 10\}$. A person wins \$100 if they pick all four numbers correctly, \$25 if they pick three out of four numbers correctly.

   (a) What's the probability of winning \$100? $1/C(10,4) = 1/210$

   (b) What's the probability of winning \$25? $\frac{C(4,3)\times 3!}{C(10,4)} = 4/35$

   (c) What's the probability that a person picks a subset that does not contain any of the winning numbers?

8. An urn contains 3 red marbles, 2 yellow marbles, and 5 green marbles. Suppose you are blindfolded, and you pick three marbles from the urn, one at a time, without replacement. Let $E_1$ be the event that the first marble selected is red, and let $E_2$ be the event that the second marble selected is green.

   $= 5/9$ $= 3/10$ $= 1/2$ $= p(E_2|E_1)p(E_1) = 1/6$

   (a) Compute the following probabilities: $p(E_2|E_1)$, $p(E_1)$, $p(E_2)$, $p(E_1 \cap E_2)$, and $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

   (b) What's the probability that all three marbles are green? $= 5/10 \times 4/9 \times 3/8 = 1/12$

   (c) What's the probability that all three marbles are different colors? $= 3/10 \times 2/9 \times 5/8 \times 3! = 1/4$

9. Suppose the same urn from the previous problem is used. Four marbles are picked at random from the urn one at a time, with replacement (i.e. each marble chosen is put back into the urn before the next marble is selected at random). Let $E_1$ be the event that the first marble selected is red, and let $E_2$ be the event that the second marble selected is green.

   (a) Compute the following probabilities: $p(E_2|E_1)$, $p(E_1)$, $p(E_2)$, $p(E_1 \cap E_2)$, and $p(E_1 \cup E_2)$.

   (b) What's the probability all of the marbles are green?

   (c) What's the probability that exactly two of the four choices are yellow?

10. Suppose that $E, F$ are events in a sample space which satisfy $p(E \cup F) = 0.5$, $p(E) = 0.3$, and $p(F) = 0.4$. Compute $p(E \cap F)$, $p(E\,|\,F)$, and $p(E \cap \overline{F})$.

11. Suppose a die is loaded so that the probability of rolling an even number is twice as likely as rolling an odd number. Also, the probabilities of rolling the even numbers are all equal, and the probabilities of rolling the odd numbers are all equal. (The sides of the die are labeled with the numbers 1 through 6, as usual.) $p(2) = p(4) = p(6) = 2/9$

   (a) Find the probability distribution for the die. $p(2) = 2p(1)$ $\quad 3p(1) + 6p(1) = 1 \Rightarrow p(1) = 1/9 = p(3) = p(5)$

   (b) A pair of the loaded dice is thrown. What is the probability that

      i. the numbers 2 and 3 are rolled on the first and second die, respectively? $p(2) \times p(3) = 2/9 \times 1/9 = 2/81$

      ii. the sum of the dice is 10? $p(4) \times p(6) + p(5)\, p(5) + p(6)\, p(4) = 1/9$

      ✓ iii. one of the dice is even and the other is odd?

      iv. the first die is less than 5? $p(1) + p(2) + p(3) + p(4) = 2/3$

      v. the sum is 9 given that the first die is less than 5? $\dfrac{p(\text{sum is } 9 \text{ \& first die} < 5)}{p(\text{IV})} = \dfrac{p(3)\,p(6) + p(4)\,p(5)}{p(\text{IV})} = 2/27$

12. Suppose that 2% of people who do not use opium test positive for opium, and 5% of opium users test negative for opium. Furthermore, suppose that 1% of people actually use opium. $p(\overline{N}|\overline{D}) = 0.02$ $\quad p(D) = 0.01$
$p(N|D) = 0.05$

   Suppose that a person is selected at random. Let $N$ be the event that the person tests negative, and let $D$ be the event that the person uses opium. Compute the following probabilities. (You can leave your answers as a sum, different, product, and/or quotient of numbers.) $1 - p(N|D) \leftarrow p(\overline{N}|D)$

   (a) $p(\overline{D}) = 0.99$ (b) $p(D \cap N) = p(N|D) \times p(D)$ (c) $p(\overline{N} \cap D) = p(\overline{N}|D) \times p(D)$ (d) $p(D|N)$ (e) $p(N|D) = \dfrac{p(N \cap D)}{p(D)}$ (f) $p(\overline{N}|D) = \dfrac{p(\overline{N} \cap D)}{p(D)}$

   (g) $p(N)$ (h) the probability the person uses opium, given that it's known they tested positive. $p(D|\overline{N})$

   (i) the probability that the person uses opium *and* tested positive. $p(D \cap \overline{N}) = c$ ✓

   (*Hint:* I suggest using the tree diagram for Bayes' Theorem. Also, do the parts in the order that makes sense, not necessarily in the given order!)

   (j) the probability that the person tested negative or uses opium. $p(N \cup D) = p(N) + p(D) - p(N \cap D)$

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Answers:

4. 37

5. (a) 35/128
   (b) 1/2
   (c) 15/16
   (d) 15/64

6. (a) $35 \cdot \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^4$

   (b) $\left(\frac{2}{3}\right)^7 + 7 \cdot \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right) + 21 \cdot \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^2 + 35 \cdot \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3$

   (c) $1 - 7 \cdot \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right) - \left(\frac{1}{3}\right)^7$

   (d) $15 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$

7. (a) 1/210
   (b) 4/35
   (c) 1/14

8. (a) $p(E_2|E_1) = 5/9, \quad p(E_1) = 3/10, \quad p(E_2) = 1/2,$
   $p(E_1 \cap E_2) = 1/6, \quad p(E_1 \cup E_2) = 19/30$
   (b) 1/12
   (c) 1/4

9. (a) $p(E_2|E_1) = 1/2, \quad p(E_1) = 3/10, \quad p(E_2) = 1/2,$
   $p(E_1 \cap E_2) = 3/20, \quad p(E_1 \cup E_2) = 13/20$
   (b) 1/16
   (c) 96/625

10. $p(E \cap F) = 0.2, \quad p(E \mid F) = 0.2/0.4 = 0.5,$
    $p(E \cap \overline{F}) = 0.3 - 0.2 = 0.1$

11. (a) $p(1) = p(3) = p(5) = 1/9,$
    $p(2) = p(4) = p(6) = 2/9$
    (b) i. 2/81
       ii. 1/9
       iii. 4/9
       iv. 2/3
       v. 2/27

12. (a) 0.99
    (b) 0.0005
    (c) 0.0095
    (d) $\dfrac{(0.05)(0.01)}{(0.05)(0.01) + (0.98)(0.99)}$ $\quad p(D|N) = \dfrac{p(N|D)\,p(D)}{\underbrace{p(N|D)\,p(D) + p(N|\overline{D})}_{p(N)}}$
    (e) 0.05
    (f) 0.95
    (g) $(0.05)(0.01) + (0.98)(0.99)$ $\quad p(N) = p(N|D)\,p(D) + p(N|\overline{D})\,p$
    (h) $\dfrac{(0.95)(0.01)}{(0.95)(0.01) + (0.02)(0.99)}$ $\quad p(D|\overline{N}) = \dfrac{p(\overline{N}|D)\,p(D)}{p(\overline{N}|D)\,p(D) + p(\overline{N}|\overline{D})\,p}$
    (i) 0.0095
    (j) $p(N \cup D) =$
    $(0.05)(0.01) + (0.98)(0.99) + 0.01 - 0.0005$
    (using inclusion-exclusion with parts (b) and (g))

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Show all work and circle your final answers! No calculator, notes, etc.! This exam is worth 100 points.

**Unless otherwise indicated, give exact numerical answers expressed as a simplified fraction or decimal.**

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Excellent

1. (7 points) Let $A, B, C$ be any finite sets. Use the principle of inclusion-exclusion to complete the right hand side of the following formula:

$|A \cup B \cup C| = \ldots |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

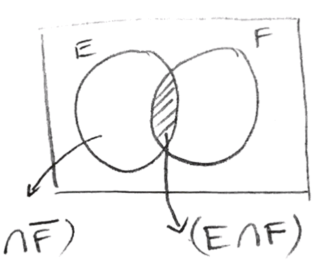
2. (10 points) Let $E, F$ be events in a sample space. Suppose that $p(E) = 0.4$, $p(E\,|\,F) = 0.2$, and $p(E \cap F) = 0.1$. Compute the following probabilities:

(a) $p(F)$ (*Hint:* Conditional probability formula.)

$$p(F) = \frac{p(E \cap F)}{p(E|F)} = \frac{0.1}{0.2} = \boxed{0.5}$$

(b) $p(E \cap \overline{F})$ (*Hint:* Sketch the Venn diagram for $E \cap \overline{F}$ and $E \cap F$.)

$$p(E \cap \overline{F}) = p(E) - p(E \cap F)$$
$$= 0.4 - 0.1$$
$$= \boxed{0.3}$$




17 pt

3. (10 points) A class contains four freshmen, six sophomores, and two juniors. Suppose that five students will be selected to win identical prizes. How many ways are there to do this if exactly two of the prize winners are freshmen, or exactly two of the prize winners are juniors? (*Hint:* This is not a probability question!)

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$A = \{$ two of the prize winners are freshmen $\}$

$B = \{$ two of the prize winners are juniors $\}$

$$|A| = C(4,2) \times C(8,3) = \frac{4\times 3}{2\times 1} \times \frac{8\times 7\times 6}{3\times 2\times 1} = 336$$

$$|B| = C(2,2) \times C(10,3) = \frac{2\times 1}{2\times 1} \times \frac{10\times 9\times 8}{3\times 2\times 1} = 120$$

$$|A \cap B| = C(4,2) \times C(2,2) \times C(6,1)$$

$$= \frac{4\times 3}{2\times 1} \times \frac{2\times 1}{2\times 1} \times 6$$

$$= 36$$

Total number of ways for picking exactly two of the prize winners are freshmen or exactly two of the prize winners are juniors:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 336 + 120 - 36$$

$$= 456 - 36$$

$$= \boxed{420}$$

4 6 2

4. (20 points) A class contains four freshmen, six sophomores, and two juniors. Three different students will be selected at random to win three different prizes ($100, $50, and $20).

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(a) Let $E_1$ be the event a freshman wins the $100 prize, and let $E_2$ be the event that a freshman wins the $50 prize. Compute $p(E_1)$, $p(E_2)$, $p(E_1 \mid E_2)$, and $p(E_1 \cap E_2)$.

$$p(E_1) = \frac{4}{12} = \frac{1}{3}$$

$$p(E_2) = \frac{4}{12} = \frac{1}{3}$$

$$p(E_1 | E_2) = \frac{3}{11}$$

$$p(E_1 \cap E_2) = p(E_1 | E_2)\, p(E_2) = \frac{3}{11} \times \frac{1}{3} = \frac{1}{11}$$

(b) What's the probability that a freshman wins the $50 prize, and a sophomore and a junior win the other two prizes?

$$p(E) = \frac{4}{12} \times \frac{6}{11} \times \frac{2}{10} \times 2!$$

$$= \frac{1}{\cancel{3}} \times \frac{\overset{2}{\cancel{6}}}{11} \times \frac{1}{5} \times 2 \times 1$$

$$= \boxed{\frac{4}{55}}$$

5. (23 points) For a certain unfair die, the probability of rolling each outcome is given below:

$$p(1) = p(2) = p(3) = 0.2$$
$$p(4) = 0.3$$
$$p(5) = p(6) = x$$

22

(a) Determine $x$.

$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$

$0.2 + 0.2 + 0.2 + 0.3 + x + x = 1$

$0.9 + 2x = 1$

$2x = 0.1$

$x = \frac{0.1}{2}$

$\boxed{x = 0.05}$

(b) The unfair die is rolled once. What's the probability of getting a number less than 4?

$p(1) + p(2) + p(3) = 0.2 + 0.2 + 0.2$

$= \boxed{0.6}$

(c) The unfair die is rolled three times. What's the probability that the number 2 appears on the first two rolls, and the number 4 appears on the last roll?

$p(2) \times p(2) \times p(4) = 0.2 \times 0.2 \times 0.3$

$= 0.04 \times 0.3$

$= \boxed{0.012}$

(d) The unfair die is rolled four times. What's the probability that you get four different numbers, none of which are 5 or 6?

$p(1) \times p(2) \times p(3) \times p(4) \times 4!$

$= 0.2 \times 0.2 \times 0.2 \times 0.3 \times 4 \times 3 \times 2 \times 1$

$= 0.0024 \times 24$

$= \boxed{0.00576}$ X 0.0576

**For this question only, you can leave your answer as a sum, difference, product, and/or quotient of numbers.**

---

24

6. (24 points; 4-6 points per part)

Suppose that 2% of patients tested in a clinic are infected with a certain disease. Furthermore, suppose that when a test for this disease is given, 3% of the patients infected with the disease test negative, and 5% of the patients not infected with the disease test positive.

A patient is selected at random from the clinic. Let $D$ be the event that the patient has the disease, and let $N$ be the event that the patient tests negative.

Compute the following probabilities. (You can do the parts in any order, but clearly label your answers!)

(a) $p(\overline{D})$

(b) $p(D \mid N)$

(c) $p(N \mid D)$

(d) the probability that the random patient tests negative.

(e) the probability that the random patient tests positive and has the disease.

Given: $p(D) = 0.02$

$p(N|D) = 0.03 \Rightarrow p(\overline{N}|D) = 1 - p(N|D) = 1 - 0.03 = 0.97$

$p(\overline{N}|\overline{D}) = 0.05 \Rightarrow p(N|\overline{D}) = 1 - p(\overline{N}|\overline{D}) = 1 - 0.05 = 0.95$

a/ $p(\overline{D}) = 1 - p(D) = 1 - 0.02 = 0.98$

b/ $p(D|N) = \dfrac{p(N|D) \times p(D)}{p(N|D) \times p(D) + p(N|\overline{D})\, p(\overline{D})} = \dfrac{0.03 \times 0.02}{0.03 \times 0.02 + 0.95 \times 0.98}$

c/ $p(N|D) = 0.03$

d/ $p(N) = p(N|D) \times p(D) + p(N|\overline{D})\, p(\overline{D})$

$= 0.03 \times 0.02 + 0.95 \times 0.98$

e/ $p(\overline{N} \cap D) = p(\overline{N}|D) \times p(D)$

$= 0.97 \times 0.02$

EXTRA CREDIT (5 POINTS)

Alice has 10 identical cookies which she wants to distribute to four of her friends (Bob, Carl, Dave, and Elsie). How many different ways can she distribute the cookies if Bob gets less than three cookies or Dave gets less than four cookies? (*Hint:* Count the complement.)

X5

A = { Bob get less than 3 cookies or Dave gets less than 4 cookies}

S = { total number of ways }

$\bar{A}$ = { Bob get more than 3 cookies and Dave gets more than 4 cookies }

$$|A| = |S| - |\bar{A}| = C(13,3) - C(6,3)$$

$$= \frac{13 \times \cancel{12}^{2} \times 11}{\cancel{3} \times 2 \times 1} - \frac{\cancel{6} \times 5 \times 4}{\cancel{3} \times \cancel{2} \times 1}$$

$$= 13 \times 22 - 20$$

$$= \boxed{266}$$

# Exam 3 Review / Discrete Structures II / Spring 2018

Exam 3 is on Friday, April 20, and will cover material from sections 2.4, 8.1, 8.2, 10.1, and 10.2. No notes, calculator, books, etc!

Extra office hours: Thursday, April 19 from noon to 3 pm, or by appointment.

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## Review the assigned homework problems: (See the course website!)

## Some additional review problems

1. For the following problem, tiles can be of the following types:
   - $1 \times 1$ tiles that are red, blue, or green.
   - $1 \times 2$ tiles that are green or blue (which can be rotated to be vertical or horizontal)

   For each sequence $a_n$ below, find a linear recurrence relation for $a_n$. Also, compute $a_1$, $a_2$, $a_3$, and $a_4$.

   (a) $a_n$ is the number of tilings of a $1 \times n$ rectangle.

   (b) $a_n$ is the number of tilings of a $1 \times n$ rectangle where only blue or green tiles are used.

   (c) $a_n$ is the number of tilings of a $1 \times n$ rectangle so that only $1 \times 1$ tiles are used, and red tiles occur consecutively.

   ✓ (d) $a_n$ is the number of tilings of a $1 \times n$ rectangle using only red and green tiles and so that red tiles do not occur consecutively.

   (e) $a_n$ is the number of tilings of a $2 \times n$ rectangle using only blue or green $1 \times 2$ tiles which can be placed horizontally or vertically.

   (f) $a_n$ is the number of tilings of a $2 \times n$ rectangle using only blue or green $1 \times 2$ tiles, where only the green tiles can be placed horizontally. (*Note:* For the $2 \times n$ rectangle, the vertical dimension is 2 and the horizontal dimension is $n$.)

2. A *ternary* string is a string of numbers from the set $\{0, 1, 2\}$. For each sequence $a_n$ below, find a linear recurrence relation for $a_n$. Also, compute $a_1$, $a_2$, $a_3$, $a_4$, and $a_5$. : no case $*000$ : no $3^{n-3}$

   (a) $a_n$ is the number of ternary strings that [don't] contain 000 consecutively. $*1$, $*2$, $*10$, $*20$, $*100$, $*200$, $*000$

   (b) $a_n$ is the number of ternary strings that don't [contain 00] contain 00 consecutively. $1*$, $2*$, $01*$, $02*$, $00*$ : ... $+ 3^{n-}$

3. What's the characteristic equation of the recurrence relation $a_n = 3a_{n-2} + 4a_{n-3} - 7a_{n-6}$?

   $n = 6$

4. In each part, prove that the given sequence is a solution to the given recurrence relation using the method of section $\boxed{2.4}$ Then find the general solution for the recurrence relation.

   1/ Prove seq. is a solution

   (a) Sequence: $a_n = 4^{n+2}$
   Recurrence: $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$

   $4^{n+2} \stackrel{?}{=} -5(4^{n+1}) - 6 \cdot 4^n + 42 \cdot 4^n = 4^n(-5 \cdot 4 - 6 + 42) = 4^n \cdot 16 = 4^{n+2}$ ✓

   (b) Sequence: $a_n = 3n + 5$
   Recurrence: $a_n = 2a_{n-1} - 3n + 1$

   2/ Find gen. soln to: $a_n = \underbrace{-5a_{n-1} - 6a_{n-2}}_{} + \underbrace{42 \cdot 4^n}_{F(n)}$ $(\star)$

   - Gen. soln. for homogenous: → Char. equ. $x^2 + 5x + 6 = 0$ ($x = -3$, $x = -2$)
     $a_n = \alpha(-3)^n + \beta(-2)^n$
   - Gen. soln. for $(\star)$: $a_n = \alpha(-3)^n + \beta(-2)^n + 4^{n+2}$

5. Solve the following recurrence relations:

   (a) $a_n = 9a_{n-1}$, $a_0 = 18$

   (b) $a_n = 9a_{n-2}$, $a_0 = 2$, $a_1 = -3$

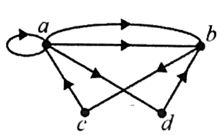
   (c) $a_n = a_{n-1} + 6a_{n-2}$, $a_0 = 9$, $a_1 = 7$

   (d) $a_n = -4a_{n-1} - 4a_{n-2}$, $a_0 = 5$, $a_1 = -6$

6. Suppose that the characteristic equation of a linear homogenous recurrence relation with constant coefficients for $\{a_n\}$ can be factored as $(x^2 - 9)^2(x^2 + x - 12)^3 = 0$. Find the general solution for the recurrence.

7. An undirected graph has 10 vertices. Suppose that six of the vertices have degree 5 and the remaining vertices have degree 3. Find the number of edges using the Handshaking Theorem.

8. Let $G$ be the directed graph below. Find $\deg^+(a)$, $\deg^-(a)$, $\deg^+(b)$, and $\deg^-(b)$.




$\deg^+(v)$ : out - degree
$\deg^-(v)$ : in - degree
Loop : $+1$ for both

remove arrows

9. Let $H$ be the underlying undirected graph of $G$, where $G$ is the directed graph from the previous problem.

   (a) Compute $\deg(a)$ and $\deg(b)$. Compare your answers with part (a) of the previous problem.

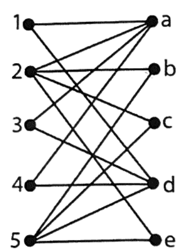
   (b) Draw the subgraph of $H$ induced by the vertices $\{a, b, d\}$.

10. Consider the bipartite graph $G$ given below.

   (a) Find a complete matching, if possible. If not, find a subset $A$ of the vertex set that violates Hall's Marriage Theorem. Show why your answer works.

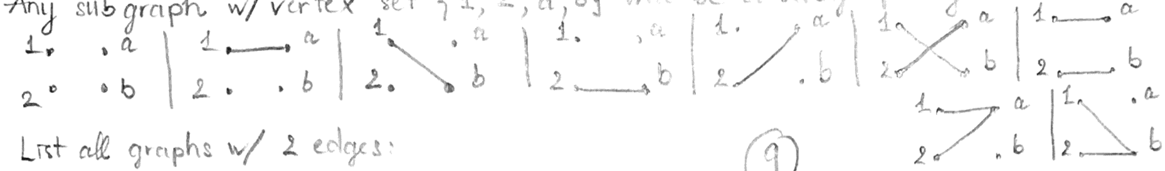
   (b) Find all subgraphs of $G$ with vertex set $\{1, 2, a, b\}$.

   (c) Do part (a) with the new graph $G'$ obtained by adding the edge $\{3, e\}$ to $G$.




b/

Any subgraph w/ vertex set $\{1, 2, a, b\}$ will be a subgraph of this



List all graphs w/ 2 edges:

9

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Answers:

1. (a) $a_n = 3a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 3$, $a_2 = 11$, $a_3 = 39$, $a_4 = 139$

   (b) $a_n = 2a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 6$, $a_3 = 16$, $a_4 = 44$

   (c) $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$, $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 5$, $a_4 = 21$

   (d) $a_n = a_{n-1} + 2a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 9$, $a_4 = 19$

   (e) $a_n = 2a_{n-1} + 4a_{n-2}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 8$, $a_3 = 24$, $a_4 = 80$

   (f) $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 5$, $a_3 = 12$, $a_4 = 29$

2. (a) $a_n = 2a_{n-1} + 2a_{n-2} + 2a_{n-3} + 3^{n-3}$, $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 5$, $a_5 = 21$

   (b) $a_n = 2a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 3$, $a_2 = 8$, $a_3 = 22$, $a_4 = 60$, $a_5 = 164$

3. $x^6 - 3x^4 - 4x^3 + 7 = 0$ (*Note:* You can use $r$ instead of $x$ if you prefer the book's notation.)

4. (a) $a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$, where $\alpha, \beta$ are any real numbers

   (b) $a_n = 3n + 5 + \alpha \cdot 2^n$, where $\alpha$ is any real number.

---

5. (a) $a_n = 2 \cdot 9^{n+1}$

   (b) $a_n = \frac{1}{2} \cdot 3^n + \frac{3}{2} \cdot (-3)^n$

   (c) $a_n = 5 \cdot 3^n + 4(-2)^n$ or $a_n = 5 \cdot 3^n + (-2)^{n+2}$

   (d) $a_n = 5(-2)^n + n(-2)^{n+1}$

---

6. $a_n = (\alpha_0 + \alpha_1 n)(-3)^n + (\beta_0 + \beta_1 n + \beta_2 n^2 + \beta_3 n^3 + \beta_4 n^4)(3^n) + (\gamma_0 + \gamma_1 n + \gamma_2 n^2)(-4)^n$, where $\alpha_i, \beta_i, \gamma_i$ are any real numbers. $(x+3)^2(x-3)^2(x+4)^3(x-3)^3 = 0 \Leftrightarrow (x+3)^2(x-3)^5(x+4)^3 = 0 \Rightarrow$ Roots: $-3$ w/ multi. $2$; $3$ — $5$; $-4$ — $3$

7. 21 edges

8. $\deg^+(a) = 4$, $\deg^-(a) = 2$, $\deg^+(b) = 1$, $\deg^-(b) = 3$

9. (a) $\deg(a) = 6$ and $\deg(b) = 4$

10. (a) No complete matching; $A = \{1, 3, 4\}$. $N(A) = \{a, d\}$ $|N(A)| < |A|$

   (b) Draw the 8 subgraphs!

   (Question: Note that the number of edges on the subgraph induced by $\{1, 2, a, b\}$ is 3. How can you count the number of subgraphs only using this fact?)

   (c) Complete matching: $\{\{1, a\}, \{2, b\}, \{3, e\}, \{4, d\}, \{5, c\}\}$

**Exam 3 / Discrete Structures II / Spring 2018** Name: Dangnhi Ngo

Show all work and circle your final answers! No calculator, notes, etc.! This exam is worth 100 points.

94

Good

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1. (18 points) Solve the following recurrence relation with initial condition:

$$a_n = 7a_{n-1} + 18a_{n-2}$$
$$a_0 = 2, \quad a_1 = 3$$

$x^2 - 7x - 18 = 0$

$(x+2)(x-9) = 0$

$x = -2,\ x = 9$

$a_n = \alpha_1 (-2)^n + \alpha_2 9^n$

$a_0 = 2 \Rightarrow \alpha_1 \cdot (-2)^0 + \alpha_2 \cdot 9^0 = 2 \Leftrightarrow \alpha_1 + \alpha_2 = 2 \Leftrightarrow 2\alpha_1 + 2\alpha_2 = 4$ (1)

$a_1 = 3 \Rightarrow \alpha_1 \cdot (-2)^1 + \alpha_2 \cdot 9^1 = 3 \Leftrightarrow -2\alpha_1 + 9\alpha_2 = 3$ (2)

(1) + (2), we have: $11\alpha_2 = 7 \Rightarrow \alpha_2 = 7/11$

$\alpha_1 = 2 - \alpha_2 = 2 - 7/11 = 15/11$

$$a_n = \frac{15}{11}(-2)^n + \frac{7}{11} \cdot 9^n$$

$$a_n = \frac{15 \cdot (-2)^n + 7 \cdot 9^n}{11}$$

2. (8 points) Suppose that the characteristic equation for a linear homogeneous recurrence relation with constant coefficients can be factored as

7

$$(x^2 + 5x - 14)(x - 2)^3 = 0$$

Find the general solution for the recurrence.

$(x^2 + 5x - 14)(x-2)^3 = 0$

$(x+7)(x-2)(x-2)^3 = 0$

Roots: $x = 2$ with multiplicity 4

$x = -7$ with multiplicity 1

$a_n = (\alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^4) \cdot 2^n + \beta (-7)^n$

3

25 pt

18

3. (18 points) Consider the following recurrence:

$$a_n = 4a_{n-1} - 4a_{n-2} + 3n - 10$$

(a) Prove that the sequence $a_n = 3n + 2$ is a solution for the recurrence. Show at least a couple of algebraic steps!

(b) Find the general solution for the recurrence.

a/ $3n + 2 \overset{?}{=} 4 \cdot [3(n-1)+2] - 4 \cdot [3(n-2)+2] + 3n - 10$

$3n + 2 \overset{?}{=} 4 \cdot (3n-1) - 4(3n-4) + 3n - 10$

$3n + 2 \overset{?}{=} \cancel{12n} - 4 - \cancel{12n} + 16 + 3n - 10$

$3n + 2 \overset{?}{=} 3n + 2$ ✓

Therefore, the sequence $a_n = 3n+2$ is a solution for the recurrence

b/ $x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$ with multiplicity 2

$a_n = (\alpha_0 + \alpha_1 n) \cdot 2^n$

$a_n = 3n + 2$

The general solution for the recurrence:

$a_n = (\alpha_0 + \alpha_1 n) 2^n + 3n + 2$

4. (20 points) Let $a_n$ be the number of tilings of a $1 \times n$ rectangle using $1 \times 1$ red tiles, $1 \times 1$ green tiles, and $1 \times 2$ blue tiles so that two consecutive tiles $1 \times 1$ green tiles are not used.

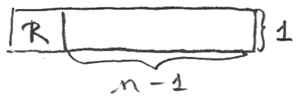
20

(a) Set up a recurrence relation for $a_n$. Be sure to show how you're setting it up by describing what each term in your recurrence represents.

(b) Find $a_0$, $a_1$, $a_2$, $a_3$.

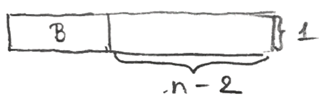
a/ Let $a_n$ be the number of tilings of a $1 \times n$ rectangle

Case 1 :




There are $a_{n-1}$ ways

Case 2 :

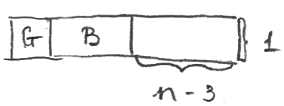



There are $a_{n-2}$ ways

Case 3 : a/


There are $a_{n-2}$ ways

b/




There are $a_{n-3}$ ways

Therefore : $a_n = a_{n-1} + 2a_{n-2} + a_{n-3}$

b/ Initial values :

$a_0 = 1$

$a_1 = 2$

$a_2 = 4$

$a_3 = 9$

6

5. (6 points)Suppose that a graph has 15 vertices and 35 edges. Also, suppose that 10 of the vertices have degree 3 and the remaining vertices all have degree $d$. Use the Handshaking Theorem to determine $d$.

Handshaking Theorem : Sum of all number of vertices = 2 × Number of edges

$$10 \times 3 + (15-10) \times d = 2 \times 35$$

$$30 + 5d = 70$$

$$5d = 40$$

$$d = \frac{40}{5} = 8$$

$$\boxed{d = 8}$$

5

6. (10 points) One of the 5-tuples below represents the degree sequence of a graph, and the other does not:

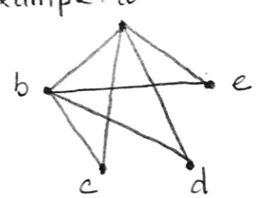
✓ • (4, 4, 2, 2, 2)

✗ • (3, 3, 3, 2, 2)

For the 5-tuple that is a degree sequence, draw a simple graph with the given degree sequence. For other 5-tuple, explain why the graph can't represent the degree sequence of any graph.

– The 5-tuples (4, 4, 2, 2, 2) represents the degree sequence of a graph:

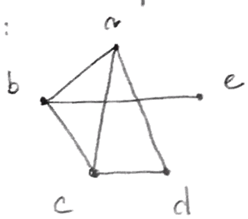
For example:




deg a = 4
deg b = 4
deg c = 2
deg d = 2
deg e = 2

– The 5-tuples (3, 3, 3, 2, 2) does not represent the degree sequence of a graph.

For example:



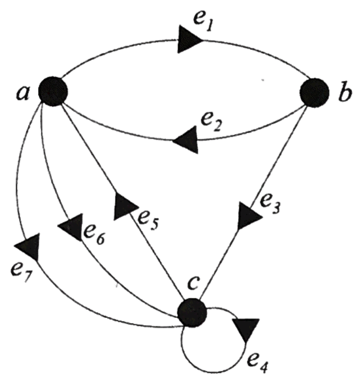

deg a = 3
deg b = 3
deg c = 3
deg d = 2
deg e = 1 ✗ (not correct)

What if there's another way to draw the graph to get the degree sequence? Explain why this can't happen!

11pt

7. (6 points) Consider the directed graph below. Find the in-degrees $\deg^-(a)$ and $\deg^-(c)$ and the out-degree $\deg^+(a)$.

6




$\deg^-(a) = 2$ ✓

$\deg^-(c) = 4$ ✓

$\deg^+(a) = 3$ ✓

20pt

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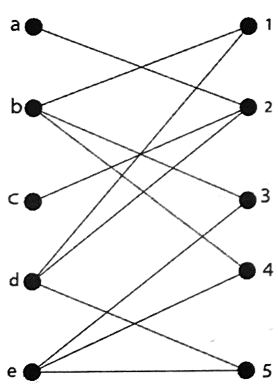
4 8. (4 points) Draw the graph of $K_5$.

10 9. (10 points)

Consider the bipartite graph below with bipartition $(V_1, V_2)$, where $V_1 = \{a, b, c, d, e\}$ and $V_2 = \{1, 2, 3, 4, 5\}$.




| | | | |
|---|---|---|---|
| a | 2 | | |
| b | 1 | 3 | 4 |
| c | 2 | | |
| d | 1 | 2 | 5 |
| e | 3 | 4 | 5 |

(a) The graph does not have a complete matching. Find a subset $A$ of $V_1$ that violates the condition in Hall's Marriage Theorem. Show that your answer works.

$A = \{a, c\}$ ✓

$N(A) = \{2\}$ ✓

$|N(A)| = 1 < 2 = |A|$

It violates the Hall's Marriage Theorem. $(|N(A)| \geq |A|)$

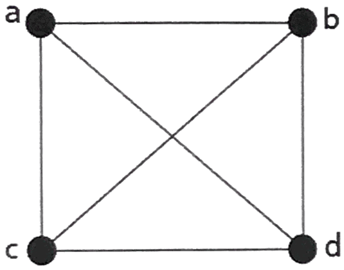
Therefore, the graph does not have a complete matching.

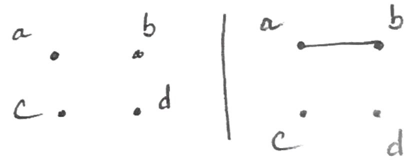
(b) Add one edge to the graph so that the resulting graph has a complete matching. Show what this complete matching is.

Add one edge $\{a, 3\}$

Complete matching : $\{\{a, 3\}, \{b, 1\}, \{c, 2\}, \{d, 5\}, \{e, 4\}\}$ ✓

Extra Credit (5 points): Let $G$ be the graph below. Compute the number of different subgraphs of $G$ with at least one vertex. (Note: Graphs are considered different if they have different vertex sets and/or edge sets.)




# subgraphs:

$1 + 3 \times 4 = 13$

# Exam 4 Review / Discrete Structures II / Spring 2018

Exam 4 is on Friday, May 11 from 8 to 10 am and will cover material from sections 10.3, 10.4, 10.5, 10.7, and 10.8. The exam will be in the usual classroom (Olsen 412). No notes, calculator, books, etc!

Final office hours: Noon to 3 pm on Wednesday, May 9, and Thursday, May 10; or by appointment.

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## Review the assigned homework problems:

- Section 10.3: #11-23 (odd), 29, 35-41 (odd)
- Section 10.4: #1-3 (all), 5, 11, 19, 25
- Section 10.5: #1-7 (odd)
- Section 10.7: #3-7 (odd), 13, 21-25 (odd)
  (Use Kuratowski's Theorem on #5, #21.)
- Section 10.8: #1-11 (odd), 17, 19

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## Some additional review problems

1. Let $G$ be the graph given below.

   $5 = \chi(K_5) \leq \chi(G)$

   (a) Find a subgraph of $G$ which is isomorphic to $K_5$.

   (b) Find the chromatic number $\chi(G)$. Prove your answer works. $K_5$ is a subgraph of $G$

   (c) Is $G$ planar? If so, give a planar representation. If not, explain why not using the appropriate theorem.

   No. It contains a subgraph homeomorphic to $K_5$

2. Let $G'$ be the graph given below.

   (a) Write down the adjacency matrix of $G'$ using the vertex order $a, b, c$.

   | | a | b | c |
   |---|---|---|---|
   | a | 0 | 1 | 1 |
   | b | 0 | 0 | 1 |
   | c | 1 | 0 | 1 |

   (b) Using the appropriate matrix computations, find the number of paths from $a$ to $c$ of length 3. Then list all such paths. $A^2 A$

   (c) Is $G'$ strongly connected? Is it weakly connected? Explain your answers. Both

   (d) Suppose that $H'$ is the directed graph obtained from $G'$ by removing the edge from $c$ to $a$. Is $H'$ strongly connected? weakly connected?

3. Let $G''$ be the graph given below.

   G" contains subgraph (age,bhf) which is isomorphic to K3,3. So it is not planar. $K_{3,3}$

   (a) Is $G''$ planar? If it is, draw a planar representation. If not, use Kuratowski's Theorem to explain why not.

   (b) Explain why $G''$ does not have an Euler path.

   (c) Add an edge to $G''$ so that the result is a simple graph $H''$ that does have an Euler path, and state what this Euler path is. Also, explain why $H''$ does not have an Euler circuit using the appropriate theorem.

   has exactly 2 vertices of odd degree

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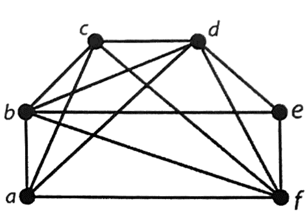
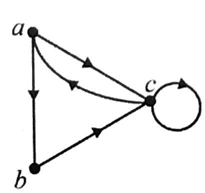
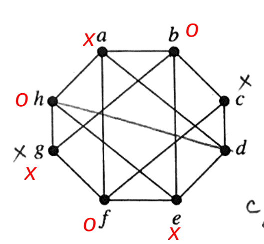


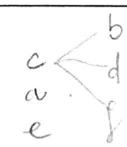

Graph $G$




Graph $G'$




Graph $G''$




cfadcbahdehgfebg

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Theorem 1 (Kuratowski' s Theorem). Let G be a graph. Then G is nonplanar if and only if G contains a subgraph is homeomorphic to either K3,3 or K5.

Eulerian Path is a path in graph that visits every edge exactly once.
Eulerian Circuit is an Eulerian Path which starts and ends on the same vertex.

4. Let $G_1$ be simple (undirected) graph, and suppose that $G_2$ is a subgraph of $G_1$ with at least one vertex. Answer the following questions. Explain why your answers work! (Each part is independent of the other parts.)

   (a) Suppose that $G_1$ can be colored using 4 colors. What are the possible values for the chromatic number of $G_1$?

   (b) If the chromatic number of $G_1$ is 4, is it possible that the chromatic number of $G_2$ is 5?

   (c) If $G_1$ is planar, is it possible that $G_2$ is not planar?

   $G_2$ is not planar $\rightarrow$ $G_1$ must be not planar

---

Answers:

1. (a) subgraph induced by the vertices $a, b, c, d, f$
   (b) 5
   (c) No. (For example, you can use Kuratowski's Theorem.)

2. (a) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
   (b) 3 paths (given in terms of vertices):
   - $a, b, c, c$
   - $a, c, c, c$
   - $a, c, a, c$

   (c) both strongly and weakly connected
   (d) not strongly connected but still weakly connected

3. (a) Not planar. (Find a subgraph that's homeomorphic to $K_{3,3}$.)
   (b) Use Theorems 1 and 2 from section 10.5.
   (c) You can add the edge $\{d, h\}$. One Euler path would be $cfadcbahdehgfebg$.

4. (a) 1, 2, 3, 4
   (b) no
   (c) no

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# Exam 4 Topics Outline

**Section 10.3**

- Represent graphs in terms of adjacency matrices.
- Given an adjacency matrix, draw the corresponding graph.
- If two graphs are isomorphic, be able to find a graph isomorphism. If the graphs are not isomorphic, explain why not.

**Section 10.4**

- For undirected graphs: Definition of connected.
- For directed graphs: Understand and be able to apply the definitions for strongly connected, weakly connected.
- Use Theorem 2 to count the number of paths of length $r$ for a fixed pair of vertices. Be able to list the paths.

**Section 10.5**

- Apply Theorems 1 and 2 to determine if an undirected graph has an Euler circuit or Euler path.
- Given a graph, find an Euler path or circuit if there is one.

**Section 10.7**

- If a graph is planar, draw its planar representation.
- Be able to state and apply Kuratowski's Theorem. If a graph is not planar, be able to find a subgraph homeomorphic to $K_{3,3}$ or $K_5$.
- Know Euler's formula, and be able to apply it in conjunction with the Handshaking Theorem from section 10.2.

**Section 10.8**

- Find the dual graph for a map. Be able to relate a coloring of the dual graph to a coloring of the map.
- Compute the chromatic number of a graph. If $\chi(G) = k$ for a graph $G$, you should be able to exhibit a coloring of $G$ and also show why fewer than $k$ colors is insufficient. Facts you can use:
  - If $H$ is a subgraph of $G$, then $\chi(H) \leq \chi(G)$.
  - $\chi(K_n) = n$
  - $\chi(C_n) = 2$ if $n$ is even and $\chi(C_n) = 3$ if $n$ is odd.
- Model scheduling problems using graphs.

# Revision to the syllabus

Your grade will be computed using two methods given below:

Method 1: Each of the four exams is worth 20%, and quizzes are worth 20%. (This is the same as the original syllabus.)

Method 2: Quizzes are worth 20%, the first three exams are worth 22% each, and the last exam is worth 14%.

Your final grade will be the maximum of the two grades, and the grade scale on the syllabus will be used to compute your letter grade.

Note that two of your lowest quiz scores will be dropped.