Section 3.2: Properties of Determinants:

Note: The secret of determinants lies in how they change when row operations are performed.

⇒ We observed these changes in the previous section while exploring how elementary row operations effect det(A)

*Theorem 3 (Row Operations):

Let A be a square matrix.

(i) Combining:

If a multiple of one row A is added to another row to produce matrix B, then: $\det(B) = \det(A)$

(ii) Interchanging:

If two rows of A are interchanged to produce B, then: $\det(B) = -\det(A)$

(iii) Scaling:

If one row of A is multiplied by some scalar "K" to produce B, then: $\det(B) = K\det(A)$

* **Note:** The 3 properties of this theorem help us to find the determinant of a matrix more efficiently & effectively!

Example: State which property of determinants is illustrated in the equation below:

$$\begin{bmatrix} -3 & 3 & -5 \\ 6 & -4 & 2 \\ -9 & 5 & -3 \end{bmatrix} = -\begin{bmatrix} 6 & -4 & 2 \\ -3 & 3 & -5 \\ -9 & 5 & -3 \end{bmatrix}$$

Answer:

* **Given:**

• $A = \begin{bmatrix} -3 & 3 & -5 \\ 6 & -4 & 2 \\ -9 & 5 & -3 \end{bmatrix}$

• $B = -\begin{bmatrix} 6 & -4 & 2 \\ -3 & 3 & -5 \\ -9 & 5 & -3 \end{bmatrix}$

* **Describe the Row-Operation:**

Matrix B is attained by interchanging R_1 & R_2 of Matrix A.

$\therefore$ The Property of Determinants used is Interchanging

$\Rightarrow$ $\boxed{\text{If two rows of } A \text{ are interchanged to produce } B\text{, then: } \det(B) = -\det(A)}$

Answer

Example: State which property of determinants is illustrated below:

$$\begin{bmatrix} -9 & -1 & -1 \\ -27 & -6 & -7 \\ 9 & -8 & -9 \end{bmatrix} = \begin{bmatrix} -9 & -1 & -1 \\ 0 & -3 & -4 \\ 9 & -8 & -9 \end{bmatrix}$$

Answer:

* Given:

• $A = \begin{bmatrix} -9 & -1 & -1 \\ -27 & -6 & -7 \\ 9 & -8 & -9 \end{bmatrix}$ & $B = \begin{bmatrix} -9 & -1 & -1 \\ 0 & -3 & -4 \\ 9 & -8 & -9 \end{bmatrix}$

* Describe the Elementary Row Operation:

$-3R_1 + R_2$ (of A) = New R_2 (of B)

⇒ The Property of Determinants used: "Combining"

$$\boxed{\text{If a multiple of one Row of A is added to another Row to produce B, then: } \det(B) = \det(A)}$$

Answer

Example: Find the Determinant below, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3. \implies B = \begin{bmatrix} a & b & c \\ d & e & f \\ 8g & 8h & 8i \end{bmatrix}$$

Answer:

* **Given:**

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ & $\det(A) = 3$

* **Describe the Row-Operation on A to produce B:**

- transformation: $8R_3 \mapsto \text{New } R_3$
- elementary matrix: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$\Rightarrow$ Since R_3 of A is multiplied by a scalar "8", then: $\det(B) = 8\det(A)$

$$\boxed{\therefore \det(B) = 8(3) = 24}$$

Answer

Example: If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ & $\det(A) = 4$,

Find the determinant of the matrix: $B = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$

Answer:

* **Describe the Row-Operation on A to produce B:**

• Transformation: $R_3 \leftrightarrow R_2$ (Rows 2 & 3 are interchanged)

• Elementary Matrix: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow$ Since R_2 & R_3 of A are interchanged to produce B, then: $\det(B) = -\det(A)$

$$\boxed{\therefore \det(B) = -4}$$

Ans.

Example: If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ & $\det(A) = 7$, then find the determinant of matrix: $B = \begin{bmatrix} a & b & c \\ 6d+g & 6e+h & 6f+i \\ g & h & i \end{bmatrix}$

Answer:

Recall: Elementary Matrices can represent one and only one row-operation at a time :)

* **Rewrite Matrix B to identify the Row-Operation:**

$$B = \begin{bmatrix} a & b & c \\ 6d+g & 6e+h & 6f+i \\ g & h & i \end{bmatrix} \sim 6 \begin{bmatrix} a & b & c \\ d+\frac{g}{6} & e+\frac{h}{6} & f+\frac{i}{6} \\ g & h & i \end{bmatrix}$$

⇒ By factoring the common "6" out of R_2 of B ↑, we can now identify the row operation performed on A.

CAUTION: We ignore the 6 below to identify the Row-Operation on A, but do not forget it in your final calculations.

* **Describe the Row-Operation on A to produce B:**

· Transformation: $6\left[R_2 + \frac{1}{6}R_3\right] \longmapsto \overset{\text{NEW}}{6R_2}$

· Elementary Matrix: $E = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

⇒ Since "$R_2 + \frac{1}{6}R_3$" of A is multiplied by a scalar "6" to produce R_2 of B, then: $\det(B) = 6\det(A)$

$\boxed{\therefore \det(B) = 6(7) = 42}$ Ans.

*Recall: (Thm #2 ; Section 3.1)

If Matrix A is triangular, then $\det(A)$ is equal to the product of the entries along the main diagonal

Example (Finding the determinant w/ row operations):

Compute the $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Answer:

Note: Here we use both Thm's 2 & 3 to find the $\det(A)$ w/o Cofactor Expansion ∴

*First Row-Reduce $[A \,\vdots\, 0]$ to echelon form, being mindful of how each row operation w/ affect the $\det(A)$:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ -2 & 8 & -9 & 0 \\ -1 & 7 & 0 & 0 \end{array}\right] \underset{\text{(i)}}{\overset{\frac{2R_1 + R_2}{N.R_2}}{\sim}} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 0 & -5 & 0 \\ -1 & 7 & 0 & 0 \end{array}\right] \underset{\text{(i)}}{\overset{\frac{R_1 + R_2}{N.R_2}}{\sim}}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \underset{\text{(ii)*}}{\overset{\text{interchange } R_2 \,\&\, R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

*Echelon Form ∴

*Note: While (i) has NO effect on $\det(A)$, (ii) changes the sign

$\det(B) = -\det(A)$

*Compute the $\det(A)$:

By theorem #2 & theorem #3 (prop. 2), we know

$$\det(A) = -(1)(3)(-5) = 15$$

$$\boxed{\therefore \det(A) = 15}$$

Ans ✓

Note: Another efficient application of theorem #3 is to Factor a common scalar out of a single row, leaving the other rows untouched (Prop. #3: $\det(B) = K\det(A)$)

Example (Using Row Operations to Find the Determinant):

Compute $\det(A)$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

Answer:

* Here are (at least) 2 options to get started here:

i) Interchange R_1 & R_4

ii) Factor common scalar, $K=2$, out of R_1

$\Rightarrow$ Lets explore the later ∵ (...why? We need "1" in the 1st pivot.)

* Row-Reduce "A" to echelon form, being mindful of how each row operation will affect det (A):

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \underset{\text{(iii)*}}{\overset{\text{*Factor 2 out of } R_1}{\sim}} 2\begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \underset{\text{(i)}}{\overset{\frac{-3R_1 + R_2}{N.R_2}}{\sim}} 2\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

$$\underset{\text{(i)}}{\overset{\frac{3R_1 + R_3}{N.R_3}}{\sim}} 2\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 1 & -4 & 0 & 6 \end{bmatrix} \underset{\text{(i)}}{\overset{\frac{-R_1 + R_4}{N.R_4}}{\sim}} 2\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix} \underset{\text{(i)}}{\overset{\frac{4R_2 + R_3}{N.R_3}}{\sim}} 2\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Example Continued...

$(\frac{1}{2})R_3 + R_4$
N. R_4

$\sim$ (i) $2\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Echelon Form :)

*Note: We are ready to compute the det(A). Keep in mind:

- Prop (i): det(B) = det(A)
- Prop.(iii): det(B) = K det(A) (ST K=2 here)

*Compute the Determinant:

By theorem[2] (3.1) & theorem[3] (prop. 3), we know that:

$$\det(A) = 2[(1)(3)(-6)(1)] = 2[-18] = -36$$

$$\boxed{\therefore \det(A) = -36}$$

Answer.

Row Operations & the Determinant

\$ a square matrix A has been reduced to echelon form "U" by row-replacement & row-interchanges. If there are "r" interchanges, then by Thm #3 (ii):

$$\det(A) = (-1)^r \det(U)$$

Since U is in Echelon Form, it is a triangular matrix, & so we compute $\det(U)$ by taking the product of the entries along the main diagonal:

$$\det(U) = (u_{11})(u_{22})\cdots(u_{nn})$$

① **If A is invertible, then:**

- The entries u_{ii} are all pivots (b/c $A \sim I_n$ & u_{ii} NOT scaled to 1)
- $\det(U) \neq 0$

Ex: $\det(U) \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

· $n = 4$ pivot positions

② **If A is NOT invertible, then:**

- At least u_{nn} is zero (i.e. NOT a pivot position)
- $\det(U) = 0$

Ex: $\det(U) = 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

· $n = 3 \neq 4$ pivots.

· **Conclusions:**

$$\det(A) = \begin{cases} \bullet\ (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots of } U \end{pmatrix}, & \text{when } A \text{ is invertible.} \\ \bullet\ 0, & \text{when } A \text{ is NOT invertible.} \end{cases}$$

Note: Although the echelon form U described on the previous page is NOT unique (b/c not rref) & the pivot are NOT unique $\Longrightarrow$ The Product of the pivots (i.e. The det(A)) is unique! (* w/ the except of a possible $\ominus$:))

*Theorem[4]:

A square matrix is invertible IFF $\det(A) \neq 0$.

We can now add this as the 13th logically equivalent statement to the "Invertible Matrix Theorem" (sect. 2.3)

*Fun/Useful Corollaries:

① $\operatorname{Det}(A) = 0$ When the Columns of A are Linearly Dependent.

② $\operatorname{Det}(A) = 0$ when the ROWS of A are Linearly Dependent.

*Recall:
- The Rows of A = The Columns of A^T
- Linearly Dependent Columns of $A^T \Rightarrow A^T$ is singular (invertible)
- If A^T is invertible, then A is invertible :)

Note: (Helpful Hint): Remember that it is easy to identify Linear Dependence by simply comparing the columns or the rows ("Same" $\Longrightarrow$ Dependence) &/or a row is zero :)

Example (Linear Dependence & the Determinant):

Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

Answer:

*Start by row-reducing "A" to echelon form, being mindful of how each row-operation affects the $\det(A)$:

$$A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix} \underset{\substack{+\,2R_1 \\ R_3 \\ \hline N.\ R_3}}{\sim} \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

*STOP: $R_2 = R_3$

$\boxed{\therefore \text{ Since } R_2 = R_3 \text{ the rows of } A \text{ are linearly dependent and thus } \det(A) = 0.}$

Answer.

Example: Use determinants to find out if the matrix is invertible:

$$A = \begin{bmatrix} 5 & 0 & -1 \\ 2 & -6 & -4 \\ 0 & 5 & 3 \end{bmatrix}$$

Answer:

* Row-reduce A to echelon form, being mindful of how each row operation affects the determinant:

(iii) $R_2 \mapsto 2R_2$ (ii) $R_1 \leftrightarrow R_2$ $\frac{-5R_1 + R_2}{N.R_2}$

$$\overset{\text{(iii)}}{\sim} 2\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix} \overset{\text{(ii)}}{\sim} -2\begin{bmatrix} 1 & -3 & -2 \\ 5 & 0 & -1 \\ 0 & 5 & 3 \end{bmatrix} \overset{\text{(i)}}{\sim} -2\begin{bmatrix} 1 & -3 & -2 \\ 0 & 15 & 9 \\ 0 & 5 & 3 \end{bmatrix}$$

(iii) $R_2 \mapsto 3R_2$

$$\sim -6\begin{bmatrix} 1 & -3 & -2 \\ 0 & 5 & 3 \\ 0 & 5 & 3 \end{bmatrix}$$

* $R_2 = R_3$

$$\boxed{\therefore \det(A) = 0 \implies \text{So matrix } A \text{ is } \underline{\text{NOT}} \text{ invertible}}$$

Answer

Example: Use determinants to find out if the matrix is invertible:

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 4 & -2 & -3 & 4 \\ -1 & 2 & 8 & 5 \end{bmatrix}$$

Answer:

* Row-Reduce Matrix A to Echelon Form, being mindful of how each row operation affects the determinant:

$$\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 4 & -2 & -3 & 4 \\ -1 & 2 & 8 & 5 \end{bmatrix} \underset{(i)}{\overset{\substack{-4R_1 \\ +R_3 \\ \hline N.R_3}}{\sim}} \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 2 & 9 & 4 \\ -1 & 2 & 8 & 5 \end{bmatrix} \underset{(i)}{\overset{\substack{R_1 \\ +R_4 \\ \hline N.R_4}}{\sim}} \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 2 & 9 & 4 \\ 0 & 1 & 5 & 5 \end{bmatrix}$$

$$\underset{(i)}{\overset{\substack{-2R_2 \\ +R_3 \\ \hline N.R_3}}{\sim}} \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 5 & 5 \end{bmatrix} \underset{(i)}{\overset{\substack{-R_2 \\ +R_4 \\ \hline N.R_4}}{\sim}} \underbrace{\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{*Echelon Form :) *}}$$

$\therefore$ $\boxed{\det(A) = -1 \neq 0 \text{ \& so } A \text{ is invertible}}$

Answer.

Example: Use determinants to decide if the set of vectors is Linearly Independent: $\begin{bmatrix}3\\5\\-6\\4\end{bmatrix}, \begin{bmatrix}2\\-6\\0\\8\end{bmatrix}, \begin{bmatrix}-2\\-1\\3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\3\end{bmatrix}$

Answer:

* **Given:**

$$A = \begin{bmatrix}3 & 2 & -2 & 0\\5 & -6 & -1 & 0\\-6 & 0 & 3 & 0\\4 & 8 & 0 & 3\end{bmatrix}$$

* **Note:** Since Column 4 has only ONE nonzero entry, lets start w/ a Cofactor expansion down Col. # 4 :)

* **Compute the det(A) by applying a Cofactor Expansion down Column 4:**

$$\det(A) = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

$$\det(A) = 0 + 0 + 0 + 3(-1)^8 \det\begin{bmatrix}3 & 2 & -2\\5 & -6 & -1\\-6 & 0 & 3\end{bmatrix}$$

$$= 3\det\begin{bmatrix}3 & 2 & -2\\5 & -6 & -1\\-6 & 0 & 3\end{bmatrix}$$

* **Note:** Lets now apply the Recursive Def. (since matrix is 3×3)

$$= 3\left(3\begin{bmatrix}-6 & -1\\0 & 3\end{bmatrix} - 2\begin{bmatrix}5 & -1\\-6 & 3\end{bmatrix} + (-2)\begin{bmatrix}5 & -6\\-6 & 0\end{bmatrix}\right)$$

$$= 3\left[3(-18-0) - 2(15-6) - 2(0-36)\right]$$

$$= 3\left[3(-18) - 2(9) - 2(-36)\right] = 3\left[-54-18+72\right] = 3(0) = 0$$

$$\boxed{\therefore \det(A) = 0 \text{ \& so the Columns of } A \text{ are a Linearly Dependent set.}}$$

Example (Row-Operations & the Determinant):

Compute det(A), where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$

Answer:

* Note: Since Column 1 only has 2 nonzero entries, a cofactor expansion down Column 1 is a good place to start

First row-reduce A to remove the last entry in Column 1 & then apply a cofactor expansion down Col. 1:

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix} \underset{(i)}{\overset{\frac{R_2 + R_4}{N.\,R_4}}{\sim}} \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

So,

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$$

$$= 0 + (-1)^3 2 \det \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{bmatrix} + 0 + 0$$

$$= -2 \det \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{bmatrix}$$

* Note: From here there are several ways to proceed in computing the det(A): ① Apply Recursive Def. ② Cofactor Expansion ③ Row-reduce to attain a triangular matrix, if

Example Continued...

* Lets apply row-operations to further simplify the matrix:

$$-2\det\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{bmatrix} \underset{\text{(i)}}{\overset{\substack{-3R_1 \\ +R_2 \\ \hline N.R_2}}{\sim}} -2\det\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{bmatrix} \underset{\text{(ii)}}{\overset{\substack{\text{Interchange} \\ R_2 \,\&\, R_3}}{\sim}}$$

$$\Rightarrow \overset{+}{-}(-2)\det\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

* Echelon form ⇒ A triangular Matrix

$$\therefore \boxed{\det(A) = 2\left[(1)(-3)(5)\right] = 2(-15) = -30}$$

Example: Combine the methods of row-reduction & cofactor expansion to compute the determinant:

$$A = \begin{bmatrix} -1 & 3 & 8 & 0 \\ 4 & 2 & 4 & 0 \\ 6 & 6 & 8 & 6 \\ 5 & 3 & 5 & 3 \end{bmatrix}$$

Answer:

* **Note:** Keep in mind that ∃ infinitely many ways to find the det(A). The following is only one of many correct ways to approach this problem :)

* Since Column 4 has only two nonzero entries, lets try a cofactor expansion down Column 4!

① Use R_3 as a pivot to eliminate the 4^{th} entry in R_4:

$$A = \begin{bmatrix} -1 & 3 & 8 & 0 \\ 4 & 2 & 4 & 0 \\ 6 & 6 & 8 & 6 \\ 5 & 3 & 5 & 3 \end{bmatrix} \underset{\text{N. } R_4}{\overset{\left(-\frac{1}{2}\right)R_3 + R_4}{\sim}} \begin{bmatrix} -1 & 3 & 8 & 0 \\ 4 & 2 & 4 & 0 \\ 6 & 6 & 8 & 6 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

② Compute the det(A) using a Cofactor Expansion Down Col. 4:

$$\det(A) = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

so,

$$\det(A) = 0 + 0 + 6(-1)^7 \det \begin{bmatrix} -1 & 3 & 8 \\ 4 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} + 0$$

Example Continued...

$$= -6 \det \begin{bmatrix} -1 & 3 & 8 \\ 4 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}$$

③ Compute the det(A) by the Recursive DeF:

$$\det(A) = -6\left(-1 \det \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} - 3 \det \begin{bmatrix} 4 & 4 \\ 2 & 1 \end{bmatrix} + 8 \det \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix} \right)$$

$$= -6\left[-\underset{2}{(2-0)} - 3\underset{-4}{(4-8)} + 8\underset{-4}{(0-4)} \right]$$

$$= -6\left[-2 + 12 - 32 \right]$$

$$= -6\left[-22 \right]$$

$$= 132$$

$$\boxed{\therefore \det(A) = 132}$$

Ans.

Column Operations

Note: We can perform operations on the columns of a matrix in a way that is similar to the row operations we have considered :)

Theorem 5:

If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

IOW: Each statement in theorem 3 is true when the word "row" is replaced everywhere by "column".

To Verify:

- Apply each property in thm. 3 to the transpose of A, A^T

  * A Row-Operation on A^T $\Leftrightarrow$ A Column-Operation on A *

Note: Column Operations are useful for theoretical purposes & some hand-calculations, but we will continue to only use row-operations in numerical computations.

*Theorem # 6 (The Multiplicative Property):

If A & B are $n \times n$ matrices, then:

$$\det(AB) = \det(A)\det(B).$$

WARNING: This ↑ does NOT hold true for the sum of matrices. In General, $\det(A+B) \neq \det(A) + \det(B)$.

Example: Verify Th^m #6 for the following matrices:

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

Answer:

① Find the product AB & then compute its determinant:

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 24+1 & 18+2 \\ 12+2 & 9+4 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

So, $\det(AB) = (25)(13) - (14)(20) = 325 - 280 = \boxed{45}$ ✓

② Find $\det(A)$ & $\det(B)$, and take their product:

- $\det(A) = (6)(2) - (1)(3) = 12 - 3 = 9$
- $\det(B) = (4)(2) - (3)(1) = 8 - 3 = 5$

So, $\det(A) \cdot \det(B) = (9)(5) = \boxed{45}$ ✓

$$\boxed{\therefore \det(AB) = 45 = \det(A)\det(B)}$$ Answer.

Example: Compute $[\det(B)]^3$ where: $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

Answer:

Recall (Thm 6): If A & B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

*First, let's compute the $\det(B)$, to verify $\exists$:

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix} \underset{\text{(ii)}}{\overset{*R_1 \leftrightarrow R_2}{\sim}} - \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \underset{\text{(i)}}{\overset{\frac{-2R_1 + R_2}{N.\,R_2}}{\sim}} - \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -7 \\ 1 & 2 & 1 \end{bmatrix} \underset{\text{(i)}}{\overset{\frac{-R_1 + R_3}{N.\,R_3}}{\sim}}$$

$$\underbrace{- \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -7 \\ 0 & 0 & -3 \end{bmatrix}}_{\text{Echelon Form :)}} \Rightarrow \boxed{\therefore \det(B) = -(1)(-4)(-3) = -12}$$

*Since $\det(B) \neq 0$, B is invertible.

*Applying Theorem # 6:

Since B is a 3×3, invertible matrix, then:

$$\det(B^3) = \det(B)\det(B)\det(B) = (-12)(-12)(-12)$$

$$\boxed{\therefore \det(B^3) = -1728}$$

Answer

Example // Property: Show that if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Proof:

\$ A is an $n \times n$, invertible matrix.

* Then, by definition:
    - $\det(A) \neq 0$
    - $AA^{-1} = I = A^{-1}A$

Goal: Show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

* Since A is invertible, then by prop. of inverses:
    - A^{-1} is an $n \times n$, invertible matrix
    - $\det(A^{-1}) \neq 0$

* Since A & A^{-1} are $n \times n$ matrices, then by theorem 6:

$$\begin{aligned} \det(A)\det(A^{-1}) &= \det(AA^{-1}) \\ &= \det(I_n) \\ &= 1 \end{aligned}$$

* Dividing both sides by "$\det(A)$":

$$\det(A)\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$\boxed{X}$

Example: Find a formula for $\det(rA)$ when A is an $n \times n$ matrix & r is any scalar.

Answer:

* **Recall**: (Theorem 3, property 3)
If one row of A is multiplied by some scalar "K" to produce B, then: $\det(B) = K\det(A)$

* Since A has n-rows, if we factor the scalar "r" out of each of the n-rows, then:

$$\boxed{\det(rA) = r^n \det(A)}$$

A Linearity Property of the Determinant Function

For an $n \times n$ matrix A, we can consider the $\det(A)$ as a function of the n Column-Vectors in A.

*Note: Here we show that if all columns except one are held fixed, then $\det(A)$ is a Linear Function of that one variable (vector).

Let A be an $n \times n$ matrix ST:

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n] \quad , \text{ where } \{\vec{a}_1, \ldots \vec{a}_n\} = \text{Columns of } A.$$

- $ that all columns are held fixed except the j^{th} column, which is allowed to vary:

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_{j-1} \quad \vec{x} \quad \vec{a}_{j+1} \ \cdots \ \vec{a}_n]$$

- Define a Linear Transformation, $T: \mathbb{R}^n \rightarrow \mathbb{R}$ ST:

$$T(\vec{x}) = \det(A) = \det[\vec{a}_1 \cdots \vec{a}_{j-1} \ \vec{x} \ \vec{a}_{j+1} \cdots \vec{a}_n]$$

- Then, by definition of a Linear Transformation:

(i) $T(c\vec{x}) = cT(\vec{x}) \quad , \forall$ scalars c & $\forall \vec{x} \in \mathbb{R}^n$.

(ii) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad , \forall \vec{u} \ \& \ \vec{v} \in \mathbb{R}^n$.