Section 4.7: Change of Basis

Note: In some cases, a problem may be initially described in terms of some Basis $\mathcal{B}$, BUT the problem's solution is aided by changing $\mathcal{B}$ to a new Basis "$\mathcal{C}$".

* In this section: We explore how $[\vec{x}]_{\mathcal{C}}$ & $[\vec{x}]_{\mathcal{B}}$ are related $\forall\ \vec{x} \in V$:)

Intro Example (2 coordinate systems for the same vector space):

Consider two bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ & $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ for some Vector Space V, ST : $\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$ & $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$.

\$ that $\vec{x} = 3\vec{b}_1 + \vec{b}_2$; which implies: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Find the coordinates of $\vec{x}$, relative to Basis $\mathcal{C}$ (i.e. $[\vec{x}]_{\mathcal{C}} = ?$).

Answer:

*Apply the Coordinate Mapping determined by $\mathcal{C}$ to $\vec{x}$:

Recall: A coordinate mapping is a Linear Transformation (4.4)

$$\vec{x} = 3\vec{b}_1 + \vec{b}_2 \longrightarrow [\vec{x}]_{\mathcal{C}} = [3\vec{b}_1 + \vec{b}_2]_{\mathcal{C}}$$

$$= 3[\vec{b}_1]_{\mathcal{C}} + [\vec{b}_2]_{\mathcal{C}} \quad \text{*Vector-Eq.}$$

$$= \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{*Matrix Eq}$$

*Find the column-vectors of the Change-of-Coord. Matrix:

- $\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2 = \underbrace{[\vec{c}_1\ \vec{c}_2]}_{P_{\mathcal{C}}} \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{[\vec{b}_1]_{\mathcal{C}}} \longrightarrow \boxed{[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}}$

Intro Ex Continued...

- $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2 = \underbrace{\begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix}}_{P_C} \underbrace{\begin{bmatrix} -6 \\ 1 \end{bmatrix}}_{[\vec{b}_2]_C} \Rightarrow \boxed{[\vec{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}}$

- **Then:**

$$[\vec{x}]_C = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 - 6 \\ 3 + 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\boxed{\therefore [\vec{x}]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}}$$

Answer

*Theorem[15]:

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ & $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be bases of a vector space V. Then, $\exists$ a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ ST :

$$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the Basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \ [\vec{b}_2]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right]$$

*Additional Notes on Theorem:

(i) The matrix, $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, is called the "Change of Coordinates Matrix from $\mathcal{B}$ to $\mathcal{C}$".

(ii) Multiplication by "$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$" converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.

*** To Remember how to construct this matrix:**

- Think of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$ as a Linear Combo. of the columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$. The matrix-vector product is a $\mathcal{C}$-coord. vector, so the columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ should be $\mathcal{C}$-coord. vectors too :)

Example: Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ & $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for a vector space V, and $ that $\vec{b}_1 = -6\vec{c}_1 + 5\vec{c}_2$ & $\vec{b}_2 = -3\vec{c}_1 + 2\vec{c}_2$.

(a) Find the Change-of-Coordinates Matrix from B to C.

(b) Find $[\vec{x}]_C$ for $\vec{x} = 3\vec{b}_1 - 7\vec{b}_2$ using part (a).

Answer:

*Part (a): Find the Change-of-Coordinates Matrix from B to C:

Note: Here we want to find $\underset{C \leftarrow B}{P} = \left[[\vec{b}_1]_C \; [\vec{b}_2]_C \right] = ?$

*Use Basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ to find the column-vectors:

- $\vec{b}_1 = -6\vec{c}_1 + 5\vec{c}_2 = [\vec{c}_1 \; \vec{c}_2]\begin{bmatrix} -6 \\ 5 \end{bmatrix} \Rightarrow \boxed{\therefore [\vec{b}_1]_C = \begin{bmatrix} -6 \\ 5 \end{bmatrix}}$

- $\vec{b}_2 = -3\vec{c}_1 + 2\vec{c}_2 = [\vec{c}_1 \; \vec{c}_2]\begin{bmatrix} -3 \\ 2 \end{bmatrix} \Rightarrow \boxed{\therefore [\vec{b}_2]_C = \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$

$$\boxed{\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} -6 & -3 \\ 5 & 2 \end{bmatrix}}$$ Answer.

*Part (b): Find $[\vec{x}]_C$ for $\vec{x} = 3\vec{b}_1 - 7\vec{b}_2$:

① Apply the coordinating mapping determined by C to $\vec{x}$:

$\vec{x} = 3\vec{b}_1 - 7\vec{b}_2 \Rightarrow [\vec{x}]_C = [3\vec{b}_1 - 7\vec{b}_2]_C$

*Since $\vec{x} \mapsto [\vec{x}]_C$ is a Linear Transformation

$= 3[\vec{b}_1]_C - 7[\vec{b}_2]_C$

⇘

Example Continued...

$$[\vec{x}]_C = 3[\vec{b}_1]_C - 7[\vec{b}_2]_C = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

② Substitute " $\underset{C \leftarrow B}{P}$ " found in part (a) into the matrix eq. & then compute the product:

$$[\vec{x}]_C = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$$= 3\begin{bmatrix} -6 \\ 5 \end{bmatrix} + (-7)\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -18 \\ 15 \end{bmatrix} + \begin{bmatrix} 21 \\ -14 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\boxed{\therefore [\vec{x}]_C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

Ans✓

Example: Let $A = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ & $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be bases for some vector space V, and $ that $\vec{a}_1 = 6\vec{b}_1 - \vec{b}_2$, $\vec{a}_2 = -\vec{b}_1 + 5\vec{b}_2 + \vec{b}_3$, & $\vec{a}_3 = \vec{b}_2 - 4\vec{b}_3$.

(a) Find the change-of-coordinates matrix from A to B.

(b) Find $[\vec{x}]_B$ for $\vec{x} = 2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3$.

Answer:

*Part (a): Find $\underset{B \leftarrow A}{P} = \left[[\vec{a}_1]_B \; [\vec{a}_2]_B \; [\vec{a}_3]_B \right] = ?$

Note: Here we rewrite the vectors of Basis A to find the column-vectors of the change-of-coord. matrix ∴

- $\vec{a}_1 = 6\vec{b}_1 - \vec{b}_2 = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3]\begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} \Rightarrow [\vec{a}_1]_B = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$

- $\vec{a}_2 = -\vec{b}_1 + 5\vec{b}_2 + \vec{b}_3 = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3]\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \Rightarrow [\vec{a}_2]_B = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$

- $\vec{a}_3 = \vec{b}_2 - 4\vec{b}_3 = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3]\begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \Rightarrow [\vec{a}_3]_B = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$

$$\boxed{\therefore \underset{B \leftarrow A}{P} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & -4 \end{bmatrix}}$$

Ans.

Example Continued...

***Part (b): Find $[\vec{x}]_\beta$ for $\vec{x} = 2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3$**

Note: Appy the coordinate mapping (a Linear Transformation :)) determined by $\mathcal{B}$ to $\vec{x}$.

$$[\vec{x}]_\beta = \left[2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3\right]_\beta = 2[\vec{a}_1]_\beta + 3[\vec{a}_2]_\beta + [\vec{a}_3]_\beta$$

$$= \underbrace{\begin{bmatrix} [\vec{a}_1]_\beta & [\vec{a}_2]_\beta & [\vec{a}_3]_\beta \end{bmatrix}}_{} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

* $\underset{\beta \leftarrow A}{P}$ found in (a) :

Substitute Change-of-Coord. Matrix, $\underset{\beta \leftarrow A}{P}$, in & compute:

$$[\vec{x}]_\beta = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 - 3 + 0 \\ -2 + 15 + 1 \\ 0 + 3 - 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \\ -1 \end{bmatrix}$$

$$\boxed{\therefore\ [\vec{x}]_\beta = \begin{bmatrix} 9 \\ 14 \\ -1 \end{bmatrix}}$$

Ans.

Example: Let $A = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ & $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be bases for some vector space V and $ $\vec{a}_1 = 6\vec{b}_1 - \vec{b}_2$, $\vec{a}_2 = -\vec{b}_1 + 2\vec{b}_2 + \vec{b}_3$, $\vec{a}_3 = \vec{b}_2 - 5\vec{b}_3$.

(a) Find the Change-of-Coordinates Matrix from A to B.

(b) Find $[\vec{x}]_B$ for $\vec{x} = 2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3$

Answer:

*Part (a): Find $\underset{B \leftarrow A}{P} = \begin{bmatrix} [\vec{a}_1]_B & [\vec{a}_2]_B & [\vec{a}_3]_B \end{bmatrix} = ?$

Note: Rewrite the vectors of Basis A to find the column vectors of matrix $\underset{B \leftarrow A}{P}$.

- $\vec{a}_1 = 6\vec{b}_1 - \vec{b}_2 = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \therefore [\vec{a}_1]_B = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$

- $\vec{a}_2 = -\vec{b}_1 + 2\vec{b}_2 + \vec{b}_3 = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \therefore [\vec{a}_2]_B = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

- $\vec{a}_3 = \vec{b}_2 - 5\vec{b}_3 = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} \Rightarrow \therefore [\vec{a}_3] = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

$$\boxed{\therefore \underset{B \leftarrow A}{P} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -5 \end{bmatrix}}$$ Answ

Example Continued...

***Part (b): Find $[\vec{x}]_\beta$ for $\vec{x} = 2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3$:**

Note: Apply the coordinate mapping determined by B to $\vec{x}$.

$$[\vec{x}]_\beta = [2\vec{a}_1 + 3\vec{a}_2 + \vec{a}_3]_\beta = 2[\vec{a}_1]_\beta + 3[\vec{a}_2]_\beta + [\vec{a}_3]_\beta$$

$$= \underbrace{\begin{bmatrix} [\vec{a}_1]_\beta & [\vec{a}_2]_\beta & [\vec{a}_3]_\beta \end{bmatrix}}_{\underset{B\leftarrow A}{P}} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

:)

*Substitute "$\underset{B\leftarrow A}{P}$" into the expression & compute:

$$[\vec{x}]_\beta = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

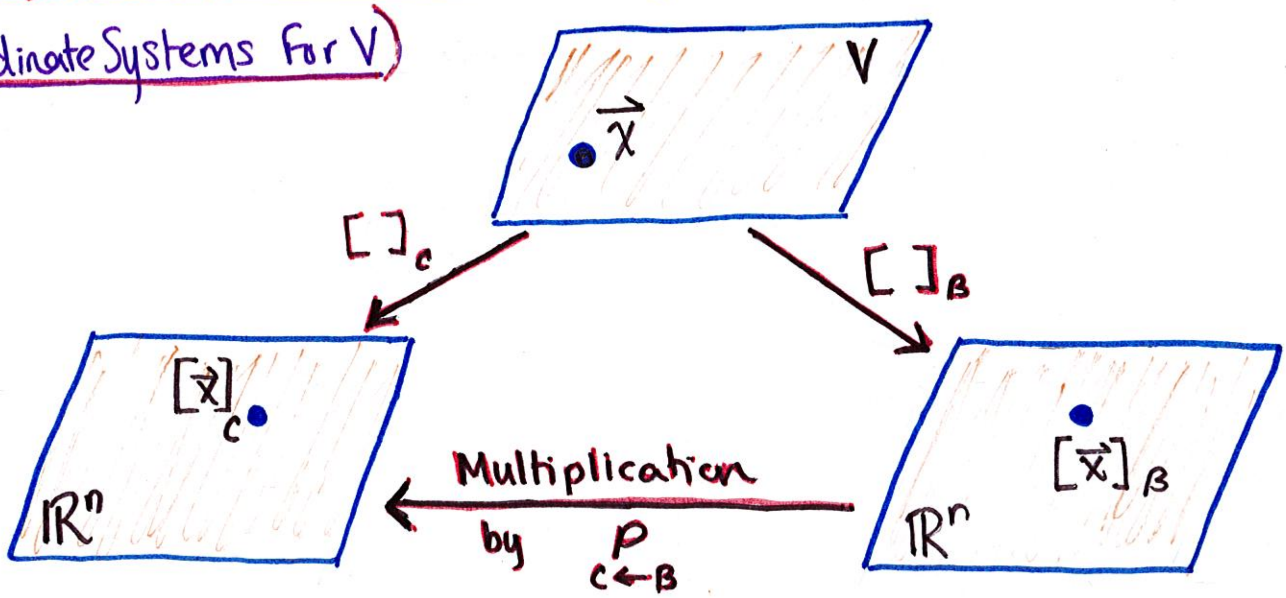
$$= \begin{bmatrix} 12 - 3 + 0 \\ -2 + 6 + 1 \\ 0 + 3 - 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -2 \end{bmatrix}$$

$$\boxed{\therefore [\vec{x}]_\beta = \begin{bmatrix} 9 \\ 5 \\ -2 \end{bmatrix}}$$

Ansr

* Illustration of the Change of Coordinates Equation *

(2 Coordinate Systems for V)



- The columns of $\underset{C \leftarrow B}{P}$ are Linearly Independent b/c they are the coordinate vectors of a Linearly Independent set B (by Def. of a Basis.)

- By the Invertible Matrix Th^m:
Since $\underset{C \leftarrow B}{P}$ is square, it must be invertible.

- LH-Multiplying the Change-of-Coordinates Eq. by "$\left(\underset{C \leftarrow B}{P}\right)^{-1}$":

$$\left(\underset{C \leftarrow B}{P}\right)^{-1}[\vec{x}]_C = \cancel{\left(\underset{C \leftarrow B}{P}\right)^{-1}}^{\,I}\ \underset{C \leftarrow B}{P}\,[\vec{x}]_B$$

$$\left(\underset{C \leftarrow B}{P}\right)^{-1}[\vec{x}]_C = [\vec{x}]_B$$

- Therefore, "$\left(\underset{C \leftarrow B}{P}\right)^{-1}$" is a matrix that converts C-coord. into B-coord:

$$\left(\underset{C \leftarrow B}{P}\right)^{-1} = \underset{B \leftarrow C}{P}$$

Change of Basis in $\mathbb{R}^n$

- If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ & $\mathcal{E}$ is the Standard Basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ in $\mathbb{R}^n$, then:

$$[\vec{b}_1]_{\mathcal{E}} = \vec{b}_1 \ , \ [\vec{b}_2]_{\mathcal{E}} = \vec{b}_2 \ , \dots , \ [\vec{b}_n]_{\mathcal{E}} = \vec{b}_n$$

- In this case, $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ is the same Change-of-Coordinates matrix we were first introduced to:

$$\Rightarrow P_{\mathcal{B}} = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_n]$$

*Note:

To change coordinates btw 2 nonstandard bases in $\mathbb{R}^n$, we apply theorem 15 ∴

$\Rightarrow$ The Th^m shows us that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

Example² (Change of Basis in $\mathbb{R}^n$):

Let $\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$,

& consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Answer:

Note: The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\vec{b}_1$ & $\vec{b}_2$ $\Longrightarrow$ Goal: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}} \right] = ?$

- Let $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ ST $x_1, x_2, y_1, y_2 \in \mathbb{R}$ (scalars)

- Then, by Definition:

$$\begin{cases} * \; \vec{b}_1 = x_1\vec{c}_1 + x_2\vec{c}_2 = [\vec{c}_1 \; \vec{c}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ * \; \vec{b}_2 = y_1\vec{c}_1 + y_2\vec{c}_2 = [\vec{c}_1 \; \vec{c}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{cases}$$

ST $P_{\mathcal{C}} = [\vec{c}_1 \; \vec{c}_2]$ (Coeff. Matrix)

- To solve both systems simultaneously, we augment the coefficient matrix w/ $\vec{b}_1$ & $\vec{b}_2$:

$$[\vec{c}_1 \; \vec{c}_2 \mid \vec{b}_1 \; \vec{b}_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

$\Downarrow$

Example² (Change of Basis in $\mathbb{R}^n$) Continued...

* **Row-reduce the augmented matrix to rref:**

$$\left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \overset{\substack{4R_1 \\ +R_2 \\ \hline N.R_2}}{\sim} \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{array}\right] \overset{\frac{1}{7}R_2}{\sim}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{array}\right] \overset{\substack{-3R_2 \\ +R_1 \\ \hline N.R_1}}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$$\Longrightarrow \begin{cases} \bullet\ [\vec{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \\ \bullet\ [\vec{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{cases}$$

* **Therefore, the Change-of-Coordinate Matrix from $\mathcal{B}$ to $\mathcal{C}$:**

$$\therefore \boxed{\underset{C \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2] \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}}$$ Ans.

* **Note:** An analogous procedure (to this example) works for finding the change-of-coord. matrix btw any 2 bases in $\mathbb{R}^n$

$$\Longrightarrow \left[\begin{array}{cc|cc} \vec{c}_1 & \vec{c}_2 & \vec{b}_1 & \vec{b}_n \end{array}\right] \sim \left[\begin{array}{c|c} I & \underset{C \leftarrow \mathcal{B}}{P} \end{array}\right]$$

Example[3] (Change of Basis in $\mathbb{R}^n$):

Let $\vec{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ & $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$.

(a) Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

(b) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Answer:

* Part (a): Find the change-of-coord. matrix from $\mathcal{C}$ to $\mathcal{B}$

Note: Here we want to find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[[\vec{c}_1]_{\mathcal{B}} \ [\vec{c}_2]_{\mathcal{B}} \right] = ?$

- Augment the coefficient matrix w/ $\vec{c}_1$ & $\vec{c}_2$, then rows-reduce to rref:

$$[\ \vec{b}_1\ \vec{b}_2 \ \vdots\ \vec{c}_1\ \vec{c}_2\ ] = \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \underset{N.R_2}{\overset{3R_1 + R_2}{\sim}}$$

$$\left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right] \overset{-\frac{1}{2}R_2}{\sim} \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & 1 & 6 & 4 \end{array}\right] \underset{N.R_1}{\overset{2R_2 + R_1}{\sim}}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right] \Longrightarrow \quad * \ [\vec{c}_1]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad * \ [\vec{c}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\therefore \boxed{\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[[\vec{c}_1]_{\mathcal{B}} \ [\vec{c}_2]_{\mathcal{B}} \right] = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}}$$ Ans ✓

Example[3] (Change of Basis in $\mathbb{R}^n$) continued...

*** Part (b): Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$:**

Note: Here we want to find $\underset{C \leftarrow B}{P} = \left[[\vec{b}_1]_C \;\; [\vec{b}_2]_C \right]$

• Since we already know the change-of-coord. matrix from C to B, we can apply the following property:

$$\underset{C \leftarrow B}{P} = \left(\underset{B \leftarrow C}{P}\right)^{-1} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}^{-1} = \ldots ?$$

*Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$

• $\det(A) = ad - bc = 20 - 18 = 2 \neq 0$ ✓

• $A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{2} \begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$

$$\therefore \boxed{\underset{C \leftarrow B}{P} = \left[[\vec{b}_1]_C \;\; [\vec{b}_2]_C \right] = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}}$$ Ans.

Example: Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ & $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases in $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ -AND- the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$:

$$\vec{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \vec{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \vec{c}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Answer:

Recall:

$$* \left[\vec{c}_1 \ \vec{c}_2 \;\vdots\; \vec{b}_1 \ \vec{b}_2\right] \sim \left[I \;\vdots\; \underset{C \leftarrow B}{P} \right]$$

-AND-

$$* \left(\underset{C \leftarrow B}{P}\right)^{-1} = \underset{B \leftarrow C}{P}$$

*Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, $\underset{C \leftarrow B}{P}$:

Note: Augment the coefficient matrix $P_C = [\vec{c}_1 \ \vec{c}_2]$ with $\vec{b}_1$ & $\vec{b}_2$, and then row-reduced to rref.

$$\left[\vec{c}_1 \ \vec{c}_2 \;\vdots\; \vec{b}_1 \ \vec{b}_2\right] = \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 4 & 3 & 3 & -2 \end{array}\right] \begin{array}{c} -4R_1 \\ +R_2 \\ \hline N.R_2 \end{array} \sim$$

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & -1 & 7 & -6 \end{array}\right] \begin{array}{c} R_2 \\ +R_1 \\ \hline N.R_1 \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & -5 \\ 0 & -1 & 7 & -6 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & -5 \\ 0 & 1 & -7 & 6 \end{array}\right]$$

$$\Rightarrow \begin{cases} \cdot \ [\vec{b}_1]_C = \begin{bmatrix} 6 \\ -7 \end{bmatrix} \\ \cdot \ [\vec{b}_2]_C = \begin{bmatrix} -5 \\ 6 \end{bmatrix} \end{cases}$$

$$\boxed{\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}}$$

Ansv

Example Continued...

* Find the Change-of-Coordinates matrix from C to B, $\underset{B\leftarrow C}{P}$:

Note: Since $\underset{B\leftarrow C}{P} = \left(\underset{C\leftarrow B}{P}\right)^{-1}$, we can simply find the inverse of the Change-of-Coordinates Matrix from B to C:

Let $\underset{C\leftarrow B}{P} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- Find the Inverse: $\left(\underset{C\leftarrow B}{P}\right)^{-1} = \dfrac{1}{\det\left(\underset{C\leftarrow B}{P}\right)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(i) $\det\left(\underset{C\leftarrow B}{P}\right) = ad - bc = 36 - 35 = 1 \neq 0$ ✓

(ii) $\left(\underset{C\leftarrow B}{P}\right)^{-1} = \dfrac{1}{1}\begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$

$$\therefore \boxed{\underset{B\leftarrow C}{P} = \left(\underset{C\leftarrow B}{P}\right)^{-1} = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}}$$ Answer

*An Alternative Description For $\underset{C \leftarrow B}{P}$ *

- **Recall that $\forall\ \vec{x} \in \mathbb{R}^n$:**

$$P_B[\vec{x}]_B = \vec{x}\ ,\quad P_C[\vec{x}]_C = \vec{x}\ ,\quad \&\quad [\vec{x}]_C = P_C^{-1}\vec{x}$$

- **Thus, this implies:**

$$[\vec{x}]_C = P_C^{-1}\vec{x} = P_C^{-1}P_B[\vec{x}]_B$$

- **So, in $\mathbb{R}^n$, the Change-of-Coordinates matrix $\underset{C \leftarrow B}{P}$ may be computed as:** $P_C^{-1}P_B$

<u>Note</u>: For matrices larger than 2×2, an algorithm similar to "Example[3] (Change of Basis in $\mathbb{R}^n$)" is <u>faster</u> than computing P_C^{-1} & then $P_C^{-1}P_B$:) (* See section 2.2)

Example: In $\mathbb{P}_2$, find the change-of-coordinates matrix from the Basis $\mathcal{B} = \{1-2t+t^2, 3-5t+4t^2, 2-2t+5t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$.

Then find the $\mathcal{B}$-coordinate vector for $2-4t+3t^2$.

Answer:

(a) First, find the Change-of-Coord. Matrix from $\mathcal{B}$ to $\mathcal{C}$:

$$\rightarrow \boxed{\underset{C \leftarrow B}{P} = \left[[\vec{b}_1]_C \; [\vec{b}_2]_C \; [\vec{b}_3]_C \right] = ?}$$

Note: Rewrite the polynomial-vectors of Basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ to find the column-vectors of matrix $\underset{C \leftarrow B}{P}$:

- $\vec{b}_1 = 1-2t+t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \longrightarrow [\vec{b}_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

- $\vec{b}_2 = 3-5t+4t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} \longrightarrow [\vec{b}_2]_C = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$

- $\vec{b}_3 = 2-2t+5t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \longrightarrow [\vec{b}_3]_C = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$

$$\boxed{\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -2 \\ 1 & 4 & 5 \end{bmatrix}}$$

Ans.

Example Continued...

(b) Next, find the β-coordinate vector for $2-4t+3t^2$:

- Let $\vec{p} = 2-4t+3t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \longrightarrow \boxed{[\vec{p}]_C = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}}$

Note: We want to find $[\vec{p}]_\beta = ?$ ST it satisfies the equation: $\boxed{\left(\underset{C \leftarrow B}{P}\right)[\vec{p}]_\beta = [\vec{p}]_C}$

So,

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -2 \\ 1 & 4 & 5 \end{bmatrix}[\vec{p}]_\beta = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ -2 & -5 & -2 & -4 \\ 1 & 4 & 5 & 3 \end{array}\right]$$

- Row-reduce the augmented matrix to rref:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ -2 & -5 & -2 & -4 \\ 1 & 4 & 5 & 3 \end{array}\right] \underset{\frac{-R_1+R_3}{N.R_3}}{\overset{\frac{2R_1+R_2}{N.R_2}}{\sim}} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] \underset{\frac{-R_2+R_3}{N.R_3}}{\overset{\frac{-3R_2+R_1}{N.R_1}}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \boxed{\therefore [\vec{p}]_\beta = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}}$$

Answer.

Example: In $\mathbb{P}_2$, find the change of coordinates matrix from the Basis $\mathcal{B} = \{1-7t^2, -2+t+15t^2, 1+6t\}$ to the standard Basis $\mathcal{C} = \{1, t, t^2\}$. Then write "$t^2$" as a Linear Combination of the Polynomials in $\mathcal{B}$.

Answer:

① In $\mathbb{P}_2$, find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \ [\vec{b}_2]_{\mathcal{C}} \ [\vec{b}_3]_{\mathcal{C}} \right] = ?$

Note: Rewrite the polynomial-vectors of basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ to find the column-vectors of matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ ü

- $\vec{b}_1 = 1-7t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix} \longrightarrow [\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$

- $\vec{b}_2 = -2+t+15t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 15 \end{bmatrix} \longrightarrow [\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 1 \\ 15 \end{bmatrix}$

- $\vec{b}_3 = 1+6t = \begin{bmatrix} 1 & t & t^2 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \longrightarrow [\vec{b}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$

$$\boxed{\therefore \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 6 \\ -7 & 15 & 0 \end{bmatrix}}$$ Ans

↘

Example Continued...

② Write t^2 as a Linear Combination of the polynomials in $\mathcal{B}$:

- Let $\vec{p} = t^2 = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \boxed{[\vec{p}]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

Note: Use $[\vec{p}]_C$ to find the coordinate vector $[\vec{p}]_\mathcal{B} = ?$ ST it satisfies: $\boxed{\left(\underset{C\leftarrow\mathcal{B}}{P}\right)[\vec{p}]_\mathcal{B} = [\vec{p}]_C}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 6 \\ -7 & 15 & 0 \end{bmatrix} [\vec{p}]_\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ -7 & 15 & 0 & 1 \end{array}\right]$$

- Row-reduce the augmented matrix to rref:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ -7 & 15 & 0 & 1 \end{array}\right] \underset{N.R_3}{\overset{7R_1 + R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 1 & 7 & 1 \end{array}\right] \overset{\substack{-R_2 + R_3 \\ N.R_3}}{\underset{\substack{2R_2 + R_1 \\ N.R_1}}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 13 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -13 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \boxed{\therefore [\vec{p}]_\mathcal{B} = \begin{bmatrix} -13 \\ -6 \\ 1 \end{bmatrix}}$$

- Therefore:

$$\boxed{t^2 = (-13)(1-7t^2) + (-6)(-2+t+15t^2) + (1)(1+6t)}$$

Answer