Section 1.5: Solution Sets of Linear Solutions

<u>Note</u>: Here we will give explicit & geometric descriptions of solution sets of linear systems.

Homogeneous Linear Systems

A system of linear equations is said to be 'Homogeneous' if it can be written in the form:

$A\vec{x} = \vec{0}$, where:
- * $A \rightarrow m \times n$ matrix
- * $\vec{0} \rightarrow$ zero vector in $\mathbb{R}^m$

*The Trivial Solution:

- $A\vec{x} = \vec{0}$ always has @ least 1 solution ; $\vec{x} = \vec{0}$ (i.e. The zero vector :))

*The Nontrivial Solution:

- We are more concerned w/ determining if $\exists$ a non-zero vector $\vec{x}$ ST $A\vec{x} = \vec{0}$
- As a result of the Existence & Uniqueness Th^m (Section 1.2)

→ The homogeneous eq. $A\vec{x} = \vec{0}$ has a nontrivial solution IFF the eq. has at least one free variable :)

*Strategy For Determining if a Homogeneous Eq. $A\vec{x}=\vec{0}$ (&/or Nonhomogeneous Eq. $A\vec{x}=\vec{b}$) has a Free Variable -AND- Describing Solution Sets:

Given a Linear System ⇒ Convert to Matrix Eq. Form.

① Check the System For the existence of Free Variable(s):

Row-reduce the equivalent augmented matrix to echelon Form:
(i) IF NO Free Variables: ⇒ Trivial Solution Only.
(ii) Free Variable(s) ∃ : ⇒ NonTrivial Solution ∃!
*Proceed to Step 2.

② Find a Description For the Solution Set:

(i) Row-reduce the aug. matrix to row-reduced echelon form.
(ii) Solve For the Basic Variable(s) in terms of the Free Variable(s).
(iii) Write the general solution of the Homogeneous Eq. $A\vec{x}=0$ as a vector ($\vec{x}$)

③ Use the Column Picture of $\vec{x}$ & its corresponding Geometric Interpretation to describe the solution set of the given linear system.

We rely heavily on our understanding of linear combinations of vectors here & how the vector(s) "spans" the m-dimensional space we are working in.

* Writing a Solution Set (of a Consistent System) in Parametric Vector Form:

⇒ Given $A\vec{x} = \vec{0}$ or $A\vec{x} = \vec{b}$

① Row reduce the augmented matrix, $[A \mid \vec{0}]$ -or- $[A \mid \vec{b}]$, to reduced echelon form.

② Express each Basic Variable in terms of any Free Variables that appear in the equation.

③ Write a typical solution $\vec{x}$ as a vector whose entries depend on the Free Variable(s), if any $\exists$.

④ Decompose $\vec{x}$ into a linear combination of vectors (w/ numerical entries) using the Free Variables as parameters

Example: Determine if the system has a nontrivial solution:

$$\begin{cases} 8x_1 - 8x_2 + 23x_3 = 0 \\ -8x_1 - 4x_2 - 14x_3 = 0 \\ 16x_1 + 8x_2 + 28x_3 = 0 \end{cases}$$

Answer:

*Write the system $A\vec{x} = \vec{0}$ as an augmented matrix:

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{ccc|c} 8 & -8 & 23 & 0 \\ -8 & -4 & -14 & 0 \\ 16 & 8 & 28 & 0 \end{array}\right]$$

*Row reduce the augmented matrix:

- $R_1 + R_2$ = new R_2

$$\longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & 23 & 0 \\ 0 & -12 & 9 & 0 \\ 16 & 8 & 28 & 0 \end{array}\right] \underset{-\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 8 & -8 & 23 & 0 \\ 0 & 4 & -3 & 0 \\ 16 & 8 & 28 & 0 \end{array}\right]$$

- $-2R_1 + R_3$ = new R_3

$$\longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & 23 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & 24 & -18 & 0 \end{array}\right] \underset{\frac{1}{6}R_3}{\sim} \left[\begin{array}{ccc|c} 8 & -8 & 23 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & 4 & -3 & 0 \end{array}\right]$$

- $-R_2 + R_3$ = new R_3

$$\longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & 23 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* x_3 is a free variable ✓

$\boxed{\therefore \text{Nontrivial Solutions } \exists}$

Ans.

Example[1] (Homogeneous Linear Systems):

Determine if the following homogeneous system has a nontrivial solution, and then describe the solution set:

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

Answer:

* First convert the linear system $A\vec{x} = \vec{0}$ to its aug. matrix form $[A \vdots \vec{0}]$ & then row reduce to echelon form:

$$[A \vdots \vec{0}] = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

• $\dfrac{R_1 + R_2}{\text{new } R_2} \rightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \overset{\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$ *

• $\dfrac{-2R_1 + R_3}{\text{new } R_3} \rightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right] \overset{-\frac{1}{9}R_3}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

* Note: You can leave the pivot as "3" here, but since we need ref...:)

Example[1] Continued...

• $-R_2 + R_3$ = new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ✓

* So x_3 = free variable & nontrivial sol. $\exists$

$\boxed{\therefore \text{ Since } x_3 = \text{free variable, then } A\vec{x} = \vec{0} \text{ has nontrivial solutions (IOW: one solution per choice of } x_3)}$

*Next, to describe the solution set, we row reduce the augmented matrix to reduced echelon form:

• $-5R_2 + R_1$ = new R_1 $\longrightarrow$ $\left[\begin{array}{ccc|c} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ✓

*Solve the Associated System for the Basic Variables to obtain the General Solution

$$x_1 - \frac{4}{3}x_3 = 0$$
$$x_2 = 0$$
$$0 = 0$$

$$\Longrightarrow \boxed{\begin{cases} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}}$$

Notes:

* x_1 & x_2 $\rightarrow$ "Basic Variables"
* x_3 $\rightarrow$ "Free Variable"

Example 1 Continued...

*Write the General Solution for $A\vec{x}=\vec{0}$ as a vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\underbrace{\vec{x} = x_3\vec{v}}_{\text{*parametric vector equation}}, \quad \text{where} \quad \vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}$$

General Solution Vector for Nontrivial Solutions (in parametric vector form)

Notes:

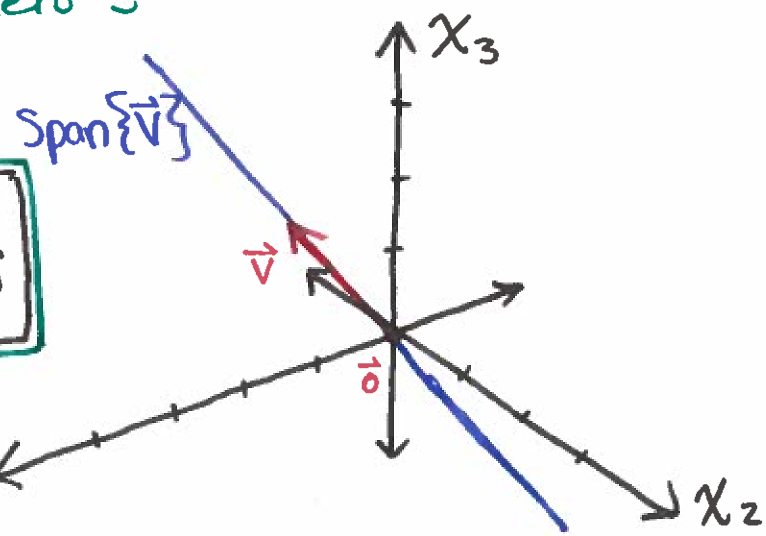
- x_3 is factored out of the general solution vector

$\Rightarrow$ IOW: Every solution of $A\vec{x}=\vec{0}$ (in this case) is a scalar multiple of $\vec{v}$.

- Nontrivial Solutions CAN have zero entries ST NOT ALL entries are zero :)

*Geometrically:

The Solution Set is the line through $\vec{0}$ in $\mathbb{R}^3$.



*Trivial Solution? Let $x_3 = 0$:) $\rightarrow \boxed{\vec{x} = 0\vec{v} = \vec{0}}$

Example: Write the solution set of the given homogeneous system in parametric vector form:

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -6x_1 - 6x_2 - 12x_3 = 0 \\ \quad\quad -7x_2 + 7x_3 = 0 \end{cases}$$

, where: the solution set is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Answer:

*Write the given system as an augmented matrix:

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -6 & -6 & -12 & 0 \\ 0 & -7 & 7 & 0 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{7}R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

*Row reduce $[A \,\vdots\, \vec{0}]$ to reduced-echelon form:

- $\begin{array}{r} -R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$ *Nontrivial Solutions ∃ :)

- Switch R_2 & R_3 $\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- $\begin{array}{r} -R_2 \\ +R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - x_3 = 0 \\ \quad\quad 0 = 0 \end{cases}$

Example Continued...

***Solving for the Basic Variables:**

$$\Rightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$$

***Write the General Solution as a Vector:**

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ **Parametric Vector Form:** $\boxed{\vec{x} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}}$

Answer.

Example: Write the solution set of the given homogeneous system in parametric vector form:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \\ -x_1 + x_2 = 0 \end{cases}, \quad \text{where: } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is the Solution Set}$$

Answer:

*Write the given system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$$

*Write $[A \mid \vec{0}]$ in reduced echelon form:

- $-2R_1 + R_2$ / new R_2 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \overset{-\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$

- $R_1 + R_3$ / new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

- $-R_2 + R_3$ / new R_3 $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

* x_3 is a free variable
→ Nontrivial Solution(s) $\exists$

Example Continued...

- $\begin{array}{r} -2R_2 \\ +\ R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$

So, $\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases} \Longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ General Solution in Parametric Vector Form:

$$\boxed{\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

Answer ✓

Example: Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form, where A is row equivalent to the given matrix:

$$A = \begin{bmatrix} 1 & -4 & -2 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

*Write the augmented matrix $[A \,\vdots\, \vec{0}]$:

$$\left[\begin{array}{cccccc|c} 1 & -4 & -2 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note: Since the given matrix is row-reduced, we observe:

- $x_1, x_3, x_4 \rightarrow$ Basic Variables
- $x_2, x_5, x_6 \rightarrow$ Free Variables

*Write $[A \,\vdots\, \vec{0}]$ in reduced-echelon form:

- $\begin{array}{r} 2R_2 \\ +\; R_1 \\ \hline \text{new } R_1 \end{array} \rightarrow \left[\begin{array}{cccccc|c} 1 & -4 & 0 & 2 & 0 & 18 & 0 \\ 0 & 0 & 1 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- $\begin{array}{r} -2R_3 \\ +\; R_1 \\ \hline \text{new } R_1 \end{array} \rightarrow \left[\begin{array}{cccccc|c} 1 & -4 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 - 4x_2 + 8x_6 = 0 \\ x_2 \text{ is free} \\ x_3 + 7x_6 = 0 \\ x_4 + 5x_6 = 0 \\ x_5 \text{ is free} \\ 0 = 0 \end{cases}$

Example Continued...

So, $\begin{cases} x_1 = 4x_2 - 8x_6 \\ x_2 \text{ is free} \\ x_3 = -7x_6 \\ x_4 = -5x_6 \\ x_5 \text{ is free} \\ x_6 \text{ is free} \end{cases}$

*Write the General Solution as a Vector, $\vec{x}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 8x_6 \\ x_2 \\ -7x_6 \\ -5x_6 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5 \\ 0 \end{bmatrix} + \begin{bmatrix} -8x_6 \\ 0 \\ -7x_6 \\ -5x_6 \\ 0 \\ x_6 \end{bmatrix}$$

$$\boxed{\therefore \vec{x} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -8 \\ 0 \\ -7 \\ -5 \\ 0 \\ 1 \end{bmatrix}}$$

*Parametric Vector Form of the Solution

Example² (Homogeneous Linear System):

Describe all solutions of the homogeneous system:

$$10x_1 - 3x_2 - 2x_3 = 0$$

Answer:

Note: Since the system consists of a single linear equation, no matrix needed :)

*Solve this "simple system" for the basic variable, x_1, to obtain the General Solution:

$$10x_1 - 3x_2 - 2x_3 = 0 \sim 10x_1 = 3x_2 + 2x_3$$

$$\xrightarrow{\frac{1}{10}Eq_1} \sim \boxed{\begin{cases} x_1 = \frac{3}{10}x_2 + \frac{1}{5}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}}$$

*Write the General Solution for $A\vec{x} = \vec{0}$ as a vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x_2 + \frac{1}{5}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{5}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix}$$

Example² Continued...

$$\vec{x} = \underbrace{x_2\vec{u} + x_3\vec{v}}_{\text{parametric vector equation}}, \quad \text{where: } \vec{u} = \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} \ \& \ \vec{v} = \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix}$$

General Solution Vector For Nontrivial Solutions
(in "Parametric Vector Form" :))

***Notes:**

- Every solution of the given system is a linear combination of the vectors $\vec{u}$ & $\vec{v}$

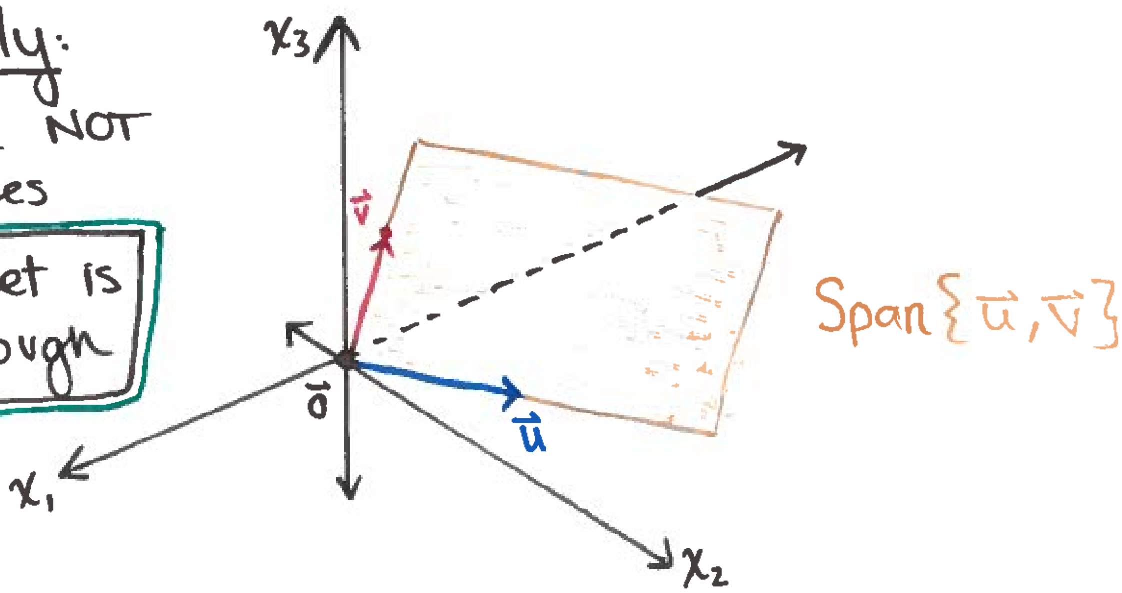
  $\Rightarrow$ IOW: The Solution Set is Span$\{\vec{u}, \vec{v}\}$

- $\vec{u}$ & $\vec{v}$ are NOT scalar multiples.

***Geometrically:**

Since $\vec{u}$ & $\vec{v}$ are NOT scalar multiples

$\Rightarrow$ The Solution Set is the plane through the origin.



* Generalizations About Homogeneous Equations *

- The Solution Set of a Homogeneous Equation $A\vec{x} = \vec{0}$ can always be expressed explicitly as:

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$$

For suitable vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$

- If the only solution is trivial (zero vector, $\vec{0}$) then the solution set is: $\text{Span}\{\vec{0}\}$

- If the eq. $A\vec{x} = \vec{0}$ has one free variable, then the geometric representation of the solution set is:

A Line through the origin (Ex 1)

- If the eq. $A\vec{x} = \vec{0}$ has two (or more) free variables, then the geometric representation of the solution set is (or similar to):

A plane through the origin (Ex. 2)

*Note: A similar geometric rep. ↑ can be used to visualize $\text{Span}\{\vec{u}, \vec{v}\}$ even if $\vec{u}$ & $\vec{v}$ are NOT solutions to $A\vec{x} = \vec{0}$
→ Just be mindful of what you are considering :)

Solutions of Nonhomogeneous Systems

A **nonhomogeneous linear system** is defined:

$A\vec{x} = \vec{b}$, where:
* $A \rightarrow m \times n$ matrix
* $\vec{b} \rightarrow$ a vector in $\mathbb{R}^m$ (nonzero vector)

When a nonhomogeneous linear system has MANY solutions, the General Solution can be written in Parametric Vector Form as:

"(One Vector)" + "(An arbitrary linear combo. of vectors)"

that satisfy the corresponding homogeneous system :)

Example: Describe & compare the solution sets of

$x_1 + 6x_2 - 7x_3 = 0$ and $x_1 + 6x_2 - 7x_3 = -4$.

Answer:

* **System 1:** $x_1 + 6x_2 - 7x_3 = 0$ $\quad (A\vec{x} = \vec{0})$

$$\Rightarrow \begin{cases} x_1 = -6x_2 + 7x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

Parametric Vector Form:

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6x_2 + 7x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}}$$

* **System 2:** $x_1 + 6x_2 - 7x_3 = -4$ $\quad (A\vec{x} = \vec{b})$

b/c 2 free variables :)

Note: The Solution Set of $A\vec{x} = \vec{b}$ is the plane through $\vec{p}$, parallel to the solution set of $A\vec{x} = \vec{0}$

IOW: The Solution Set of System 2 is the Sol. Set of System 1 translated by $\vec{p}$ ST $\vec{p} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$:)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 - 6x_2 + 7x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

Example (Solutions to Nonhomogeneous Systems):

Describe all solutions of $A\vec{x} = \vec{b}$, where:

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Answer:

*Write the linear system, $A\vec{x} = \vec{b}$, in its augmented matrix form, $[A \mid \vec{b}]$:

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right]$$

*To describe the solution set, use row operations to write $[A \mid \vec{b}]$ in its reduced echelon form:

- $\begin{array}{r} R_1 \\ + R_2 \\ \hline \text{new } R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 6 & 1 & -8 & -4 \end{array}\right] \overset{\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 6 & 1 & -8 & -4 \end{array}\right]$

- $\begin{array}{r} -2R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -9 & 0 & -18 \end{array}\right] \overset{-\frac{1}{9}R_3}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{array}\right]$

Example Continued...

- $\begin{array}{r} -5R_2 \\ +\ R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{array}\right]$

- $\begin{array}{r} -R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ * x_3 is a free variable $\Rightarrow$ Nontrivial Solutions $\exists$:)

$$\tfrac{1}{3}R_1 \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \\ 0 = 0 \end{cases}$$

Solving for the Basic Variables, we obtain:

$$\begin{cases} x_1 = \frac{4}{3}x_3 - 1 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases},$$

* x_1 & $x_2 \rightarrow$ The Basic Variables
* $x_3 \rightarrow$ The Free Variable

Write the General Solution of $A\vec{x} = \vec{b}$ as a Vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\,x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\,x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Example Continued...

$$\therefore \boxed{\vec{x} = \vec{p} + x_3\vec{v}, \text{ where: } \vec{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \text{ \& } \vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}$$

Answer:

*The Solution Set of $A\vec{x} = \vec{b}$ in Parametric Vector Form

Note: For a general parameter "t", we can also write

$\Rightarrow \vec{x} = \vec{p} + t\vec{v}$, where $t \in \mathbb{R}$

*Conclusion:

The solutions of $A\vec{x} = \vec{b}$ are obtained by adding $\vec{p}$ to the solutions of the homogeneous system $A\vec{x} = \vec{0}$.

*The vector $\vec{p}$ is just one particular solution of $A\vec{x} = \vec{b}$ (i.e. When $t = 0 \rightarrow \vec{x} = \vec{p} + 0\vec{v} = \vec{p}$:))

*Geometric Representation of $A\vec{x} = \vec{b}$ *

To describe the solution set of $A\vec{x} = \vec{b}$ geometrically, we consider the addition (of $\vec{p}$) as a translation.

1) Illustration: (Adding $\vec{p}$ to $\vec{v}$)

$ $\vec{v}$ & $\vec{p}$ are two vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$:)).

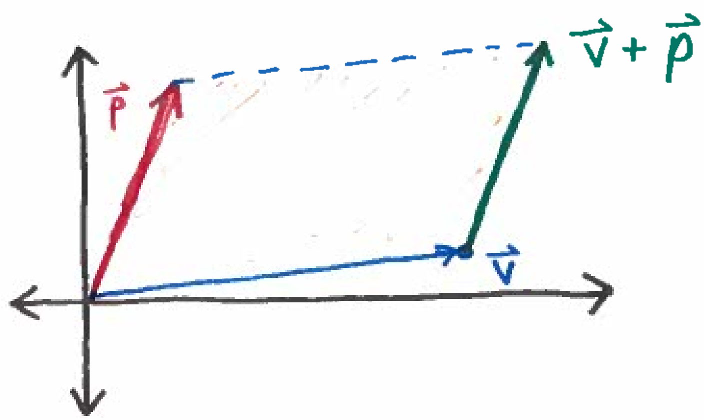
- The effect of "Adding $\vec{p}$ to $\vec{v}$" is:

→ To move $\vec{v}$ in a direction parallel to the line through $\vec{p}$ & $\vec{0}$

- Defined as:

→ $\vec{v}$ is traslated by $\vec{p}$ to "$\vec{v} + \vec{p}$"

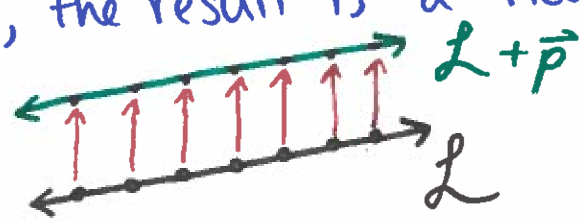
- Graphically:



2) Illustration: (Translated Line)

If each point on a line $\mathcal{L}$ (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is translated by $\vec{p}$, the result is a new line parallel to $\mathcal{L}$

- Graphically:



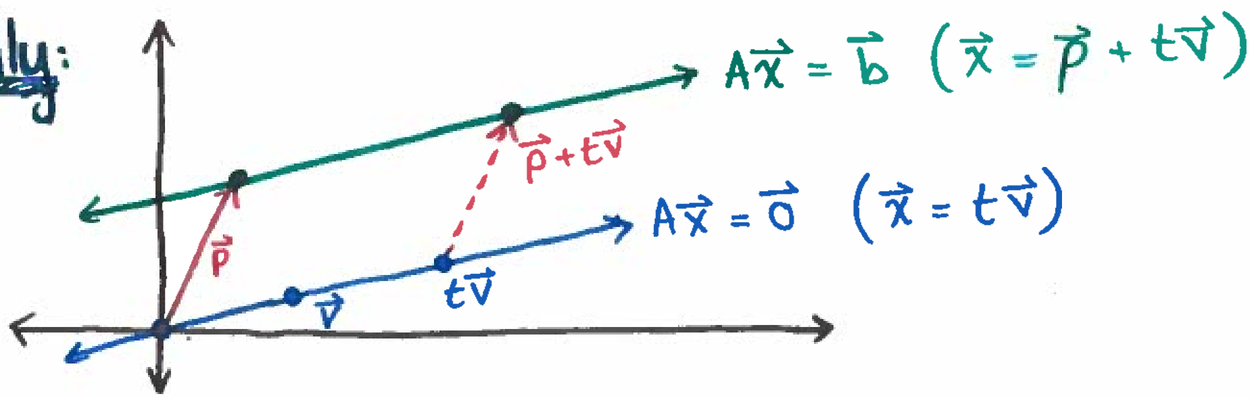
③ Illustration: Parallel Solution Sets of $A\vec{x}=\vec{b}$ & $A\vec{x}=\vec{0}$

* Suppose $\mathcal{L}$ is a line described by the solution set for $A\vec{x}=\vec{0} \rightarrow \boxed{\vec{x}=t\vec{v}}$
  - IOW: $\mathcal{L}$ passes through $\vec{v}$ & $\vec{0}$.

* Adding $\vec{p}$ to each point on $\mathcal{L}$ produces the translated line described by the solution set for $A\vec{x}=\vec{b} \rightarrow \boxed{\vec{x}=\vec{p}+t\vec{v}}$
  - Called: The Eq. of the Line passing through $\vec{p}$ & parallel to $\vec{v}$

* **Therefore:** The solution set of $A\vec{x}=\vec{b}$ is the line through $\vec{p}$, parallel to the solution set of $A\vec{x}=\vec{0}$

* **Graphically:**



The above geometric representation generalizes to any consistent equation $A\vec{x}=\vec{b}$ (at least 1 sol. $\exists$)

* Note: Solution set will be larger for 2^+ free variables ⁚)

Note: The previous illustration (# 3) can be generalized by the following theorem.

* Theorem:

$ the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution to the system.

Then, the <u>solution set of $A\vec{x} = \vec{b}$</u> is the set of all vectors of the form:

$\vec{w} = \vec{p} + \vec{v}_h$, <u>where:</u> $\vec{v}_h \to$ ANY solution of $A\vec{x} = \vec{0}$

$\Downarrow$

*<u>IOW</u>: Putting this into "easier to understand" terms ↴

"If $A\vec{x} = \vec{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\vec{x} = \vec{0}$, using any particular solution $\vec{p}$ of $A\vec{x} = \vec{b}$."

↰ (Like we saw in the 1st example for Nonhomogeneous Sys.)

<u>Caution:</u>

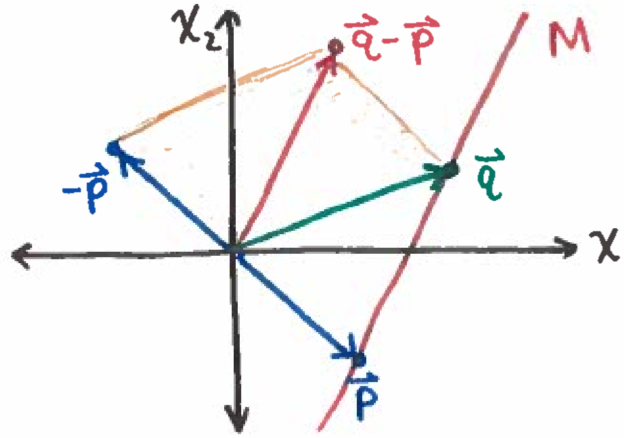
This theorem only applies to equations $A\vec{x} = \vec{b}$ that have at least one nonzero solution $\vec{p}$!

* <u>When $A\vec{x} = \vec{b}$ has NO solution, the solution set is empty.</u>

Example: Find a parametric equation of the line M through $\vec{p}$ and $\vec{q}$ for the given values.

Hint: M is parallel to the vector $\vec{q}-\vec{p}$ a shown in the figure below. *<u>Figure is NOT to scale</u>*

$$\vec{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}, \quad \vec{q} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$



Answer:

Recall: The solution set of $A\vec{x}=\vec{b}$ (Line M) is the line through $\vec{p}$, parallel to the line that is the solution set of $A\vec{x}=\vec{0}$ (i.e. Line passing through $\vec{0}$ & $(\vec{q}-\vec{p})$).

*Let L be the line described by the solution set for $A\vec{x}=\vec{0}$ $\Rightarrow$ $\boxed{\vec{x} = t\vec{v}}$, where:
- $t \to$ parameter
- $\vec{v} \to (\vec{q}-\vec{p})$

$$\bullet\ \vec{v} = \vec{q}-\vec{p} = \begin{bmatrix} 0-(-6) \\ -5-3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

$$\therefore \text{Line } L: \boxed{\vec{x} = t\begin{bmatrix} 6 \\ -8 \end{bmatrix}}$$

* Add $\vec{p}$ to each point on L to produce the translated Line "M" described by the Solution Set for $A\vec{x}=\vec{b}$.

$$\therefore \text{Line } M: \boxed{\vec{x} = \vec{p} + t\begin{bmatrix} 6 \\ -8 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 6 \\ -8 \end{bmatrix}}$$ Ans ✓

Question: Describe the solutions of the first system of equations below in parametric vector form. Provide a geometric comparison w/ the solution set of the second system of equations below:

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -6x_1 - 6x_2 - 12x_3 = -24 \\ \quad -5x_2 - 5x_3 = 15 \end{cases} \quad ; \quad \begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -6x_1 - 6x_2 - 12x_3 = 0 \\ -5x_2 - 5x_3 = 15 \end{cases}$$

Sol. Set 1 *Sol. Set 2*

Answer:

*Write the first system as an augmented matrix:

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -6 & -6 & -12 & -24 \\ 0 & -5 & -5 & 15 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{5}R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

*Find the reduced-echelon form:

- $\begin{array}{r} -R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -3 \end{array}\right] \xrightarrow{\text{swap } R_2 \,\&\, R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- $\begin{array}{r} -R_2 \\ +R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} x_1 + x_3 = 7 \\ x_2 + x_3 = -3 \\ 0 = 0 \end{cases}$

Question Continued...

*** Solving for the Basic Variables:**

$$\Rightarrow \begin{cases} x_1 = 7 - x_3 \\ x_2 = -3 - x_3 \\ x_3 \text{ is free} \end{cases}$$

***The General Solution as a Vector:**

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 - x_3 \\ -3 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

Solution of 1st System in Parametric Vector Form

Geometric Comparisen of Systems 1 & 2

- The first system of eq. ($A\vec{x} = \vec{b}$) has one free variable (line)
- The second system of eq. ($A\vec{x} = \vec{0}$).

*Recall: The solutions of $A\vec{x} = \vec{b}$ are obtained by adding $\vec{p}$ to the solutions of $A\vec{x} = \vec{0}$ (Parallel Lines)

∴ The solution set of the first system $A\vec{x} = \vec{b}$ is the line through $\vec{p}$, parallel to the line of the solution set of the second system $A\vec{x} = \vec{0}$; $\vec{p} = \begin{bmatrix} -7 \\ -3 \\ 0 \end{bmatrix}$

Example: Describe the solution set of the first system of equations below in parametric form. Provide a geometric comparison w/ the solution set of the second system of equations shown below:

$$\begin{cases} x_1 - x_2 + 3x_3 = 3 \\ 2x_1 + x_2 + 3x_3 = 3 \\ -x_1 - 2x_2 = 0 \end{cases} \quad \& \quad \begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \\ -x_1 - 2x_2 &= 0 \end{aligned}$$

Answer:

Note: · System 1 has the nonhomogeneous eq $\Rightarrow A\vec{x} = \vec{b}$

· System 2 has the homogeneous eq $\Rightarrow A\vec{x} = \vec{0}$

* Write the 1st system of Eq. $A\vec{x} = \vec{b}$ as an augmented matrix:

$$[A \,|\, \vec{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 2 & 1 & 3 & 3 \\ -1 & -2 & 0 & 0 \end{array}\right]$$

* Find the reduced-echelon form:

• $\begin{array}{l} -2R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 3 & -3 & -3 \\ -1 & -2 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ -1 & -2 & 0 & 0 \end{array}\right]$

• $\begin{array}{l} R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 3 & 3 \end{array}\right] \overset{-\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$

Example Continued...

- $\begin{array}{l} -R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ * x_3 is a free variable :)

- $\begin{array}{l} R_2 \\ +R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 + 2x_3 = 2 \\ x_2 - x_3 = -1 \\ 0 = 0 \end{cases}$

*Solve for the Basic Variables & then write the Gen. Solution as a vector:

$$\Rightarrow \begin{cases} x_1 = 2 - 2x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ is free} \end{cases}$$

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-2x_3 \\ -1+x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

Parametric Vector Form.

*Geometric Comparison:

Recall: The solutions of $A\vec{x} = \vec{b}$ are obtained by adding $\vec{p}$ to the solutions of $A\vec{x} = 0$; 1 free variable $\Rightarrow$ Lines

$\therefore$ The solution set of the 1st system $(A\vec{x} = \vec{b})$ is a Line passing through $\vec{p} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ & parallel to the Line that is the solution of the 2nd system $(A\vec{x} = \vec{0})$

*Note: 2nd system's solution: $\vec{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Notes to Self:

① $ A is a 3×3 matrix w/ 3 pivot positions

• Does the eq. $A\vec{x} = \vec{0}$ have a nontrivial solution?

No. $A\vec{x} = \vec{0}$ must have a free variable for nontrivial solutions to ∃, 3 pivots prevents this.

• Does the eq. $A\vec{x} = \vec{b}$ have @ least one solution ∀ b?

Yes. Matrix A has 3 rows & 3 pivots (1 pivot/row)
∴ Since A has a pivot in each row, ∀ $b \in \mathbb{R}^3$, the eq. $A\vec{x} = \vec{b}$ has a solution.

② $ A is a 2×5 matrix w/ 2 pivot positions

• Does the eq. $A\vec{x} = \vec{0}$ have a nontrivial solution?

Yes. Matrix A has 2 rows w/ 5 columns.
2 pivots ⇒ 2 Basic Variables & 3 Free Variables
∴ Nontrivial Sol. can ∃!

• Does the eq. $A\vec{x} = \vec{b}$ have @ least one sol. ∀ b?

Yes. Matrix A has 2 rows & 2 pivots (1 pivot/row)
⇒ ∴ Since A has a pivot in each row, ∀ $b \in \mathbb{R}^2$, the eq. $A\vec{x} = \vec{b}$ has a solution.