Math Background

- Induction Proof
- Mathematical notation
- Limits, Series, and Combinatorics

Mathematical Induction

The principle of mathematical induction.

Consider an integer a known as the *basis*. If

1. $P(a)$ holds and
2. $P(n)$ must hold whenever $P(n-1)$ holds, for each integer $n > a$.

Therefore, a typical proof consists of two steps

- basis
- induction step

A more general principle of mathematical induction.

Consider any property P of the integers, and two integers a and b such that $a \leq b$

1. $P(n)$ holds for $a \leq n < b$
2. for any integer $n \geq b$, the fact $P(n)$ holds follows from the assumption that $P(m)$ holds for all m such that $a \leq m < n$.

Therefore, a typical proof consists of two steps

- basis
- induction step

Constructive Induction

Example: Fibonacci sequence

$f_0 = 0; f_1 = 1$ and

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

Easy to prove if we know

$$f_n = \frac{1}{\sqrt{5}}[\phi^n - ((-\phi)^{-n})], \phi = \frac{1+\sqrt{5}}{2}$$

How about we don't know f_n.

Conjecture: $\exists x > 1, N_0 \in \mathcal{N}$, for all $n > N_0$, $f_n \geq x^n$.

Constructive Induction

Let $g(n)$ be the number of times that the marked instruction is executed. Show that there exist positive constants a and b such that $af_n \leq g(n) \leq bf_n$ for any sufficiently large n.

```
double fibRecursive(int n)
{
  double ret;

    if (n<2)
      ret = (double)n;
    else
      ret =  fibRecursive(n-1)
        + fibRecursive(n-2); // ***
    return ret;
}
```

Mathematical Notation

- Propositional calculus
- Set theory
- Integers, reals, and intervals
- Functions and relations
- Quantifiers
- Sums and products
- Logarithm equations

propositional calculus

- Boolean variable can be either *true* or *false*
- Conjunction, $p \bigwedge q$
- Disjunction, $p \bigvee q$
- Negation, $\neg p$
- Implication, $p \Longrightarrow q$
- Equivalence, $p \Longleftrightarrow q$

Set theory

- A *set* is an unordered collection of distinct elements.
- finite, infinite, empty set (ϕ)
- Cardinality of X, $|X|$.
- $x \in X$, $x \notin X$
- $X \subseteq Y$, $X \subset Y$
- $X \supseteq Y$, $X \supset Y$

Integers, reals, and intervals

- $\mathcal{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
- $\mathcal{N} = \{0, 1, 2, ...\}$
- $\mathcal{N}^{+} = \{1, 2, ...\}$
- $\mathcal{R}$ for real numbers and $\mathcal{R}^{+}$ for positive real numbers
- An open interval $(a, b) = \{x \in \mathcal{R} | a < x < b\}$.
- An close interval $[a, b] = \{x \in \mathcal{R} | a \leq x \leq b\}$.
- An semi-open interval $(a, b] = \{x \in \mathcal{R} | a < x \leq b\}$. Similarly, $[a, b)$.
- An integer interval $[i..j] = \{n \in \mathcal{Z} | i \leq n \leq j\}$. $|[i..j]| = j - i + 1$.

Functions and relations

- Any subset ρ of Cartesian product XxY is a relation.
- A relation f between X and Y is a function if for each $x \in X$, there exists one and only one $y \in Y$ such that $(x, y) \in f$. It is denoted as $f : X \to Y$.
  1. *domain*
  2. *image*
  3. *range*
- A function $f : X \to Y$ is *injective* if there do not exist two distinct $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$
- *surjective, bijective*

Quantifiers

- $\forall n \in \mathcal{N} \quad [\sum_{i=1}^{n} i = \frac{n(n+1)}{2}]$
- $\exists n \in \mathcal{N}^{+} \quad [\sum_{i=1}^{n} i = n^2]$
- Definition of "exist infinite", "finite exceptions"
- Duality principle

Sums and Products

- Sum, $\sum_{i=1}^{n} f(n)$
- Conditional sum, $\sum_{i=1,P(i)}^{n} f(n)$
- Conditional product, $\Pi_{i=1,P(i)}^{n} f(n)$

Logarithm equations

- $log_a(xy) = log_a x + log_a y$
- $log_a x^y = y log_a x$
- $log_a x = \frac{log_b x}{log_b a}$
- $x^{log_b y} = y^{log_b x}$

Limits

- Definitions of $lim_{n \to \infty} f(n) = a$ and $lim_{n \to \infty} f(n) = \infty$
- Properties
  If $lim_{n \to \infty} f(n) = a$ and $lim_{n \to \infty} f(n) = b$ then
  $lim_{n \to \infty} f(n)\ op\ g(n) = a\ op\ b$.

Series

- Arithmetic series: $a, a+d, a+2d, ...$
  $s_n = \sum_{i=0}^{n-1} a + i * d = an + n(n-1)d/2.$
- Geometric series: $a, ar, ar^2, ...,$
  $s_n = \sum_{i=0}^{n-1} ar^i = a(1-r^n)/(1-r).$
- The infinite geometric series: $a + ar + ar^2 + ...$ is convergent and has the sum $a/(1-r)$ if and only of $-1 < r < 1.$
- Harmonic series. Let $H_n = \sum_{i=1}^{n} 1/n.$
  $log(n+1) < H_n \leq 1 + logn.$

Combinatorics

- A *permutation* of n objects is an ordered arrangement of the objects. $n!$.
- A *combination* of r objects from n objects is a selection of r objects without regard to order. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
- $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x + ... + \binom{n}{n-1}x^{n-1} + x^n$.