NP Completeness

Jie Wang

University of Massachusetts Lowell
Department of Computer Science

Complexity Upper Bound

- The study of algorithms typically follows two directions:
  1. Seek more efficient algorithms. That is, try to reduce the complexity upper bound for solving a computational problem.
  2. Establish the fact that there are no better algorithms for solving a given problem. That is, provide a nontrivial complexity lower bound for solving a computational problem.
- We have seen a wide range of algorithm design principles and techniques in the first direction, which form a toolbox for tackling algorithmic problems.

Complexity Lower Bound

- We now deal with the second direction.
- It is customary to view polynomial-time algorithms as efficient.
- In practice, most problems that can be solved in polynomial time can indeed be solved in time that is a low-degree polynomial of the input size.
- How do we know if a problem **cannot** be solved by a polynomial-time algorithm?

Turing Machines

Unlike in the first direction we use random-access machines as our computation model for its programmability, we will use Turing machines for the study of complexity lower bound.

We use time-bounded (deterministic and nondeterministic) Turing machines to study computational complexity of decision problems. In this model, a Turing machine has two different halting states, called the *accepting* state and the *rejecting* state, respectively. We say that a Turing machine M is bounded by time T if for any input x, M on x always halts in $T(|x|)$ moves (steps). If M is deterministic and M on input x halts at the accepting state, then we say that x is accepted by M. If M is nondeterministic and there exists at least one computation path of M on input x that halts on the accepting state, then we say that x is accepted by M. We are particularly interested in polynomial-time bounds for the following reasons.

1. All problems in practice that are solvable efficiently can be solved by deterministic Turing machines in polynomial time.

2. Polynomials are closed under addition, multiplication, and composition.

P and NP

Let

$$\begin{aligned} P &= \{D \mid D \text{ is solvable by a DTM in polynomial time}\} \\ NP &= \{D \mid D \text{ is solvable by an NTM in polynomial time}\}. \end{aligned}$$

Problems in P are considered tractable. Clearly, P is included in NP. Many computational problems, while being in NP, are not known to be in P. It is the central open problem in computer science whether all problems in NP are tractable; namely, whether $P = NP$. While there is no known mathematical proof in either direction at this point, it is commonly believed that $P \neq NP$.

Polynomial-Time Reductions

We use the notion of NP-completeness to measure the hardest problems in NP. (Remark: NP-completeness is a worst-case measure.) First, we define the notion of polynomial-time reductions.

Definition 1 Let A and B be two decision problems. If there is a polynomial-time computable reduction f such that $A \leq_m B$ via f, namely, $\forall x : x \in A$ iff $f(x) \in B$, written as $A \leq_m^p B$, then we say that A is *polynomial-time reducible* to B.

The following proposition is straightforward (its proof is left to the reader).

Proposition 2 *1. Polynomial-time reductions are closed for P. Namely, if $A \leq_m^p B$ and $B \in P$, then $A \in P$.*

2. Polynomial-time reductions are transitive. Namely, if $A \leq_m^p B$ and $B \leq_m^p C$, then $A \leq_m^p C$.

NP Completeness

Proposition 2(1) indicates that, in the context of polynomial-time solvability, if $A \leq_m^p B$, then A is not harder than B. So the hardest problem in NP is one that can be reduced to from any problem in NP. We call such a problem an NP-complete problem.

Definition 3 A problem D is NP-complete if (1) $D \in NP$ and (2) $\forall A \in NP : A \leq_m^p D$.

Thousands of problems have been shown NP-complete. In the present note we will look at four classic NP-complete problems; they are bounded halting (BH), bounded tiling (BT), Boolean satisfiability (SAT), and 3-Satisfiability (3SAT).

Bounded Halting (BH)

Recall that the halting problem defined below is Turing-acceptable but not decidable.

$$H = \{\langle M, x\rangle \mid M \text{ is a TM and } M \text{ on } x \text{ halts}\}.$$

We can further show that every r.e. language is reducible to H. Let A be an r.e. set. Then there is a TM M_A such that M_A accepts A. Construct a reduction f by $f(x) = \langle M_A, x\rangle$. Clearly, f is a recursive function. We have $\forall x : x \in A$ iff M_A halts on x iff $f(x) \in H$. This result indicates that H is complete for all r.e. sets; in other words, H is the hardest problem in r.e. sets.

Bounded Halting Problem (BH)

Instance. A (nondeterministic) Turing machine M, a binary string x, and a unitary notion 0^n representing positive integer n.

Question. Does M accept x in n steps?

BH Is NP-Complete

For convenience we use BH to denote the positive instances of Bounded Halting; namely,

$$BH = \{\langle M, x, 0^n\rangle \mid M \text{ accepts } x \text{ in } n \text{ steps}\}.$$

Theorem 4 *BH is NP-complete.*

Proof. It is straightforward to show that $BH \in NP$, for $|\langle M, x, 0^n\rangle| = \Theta(|\langle M\rangle| + |x| + n)$ (the detailed proof is left to the reader). We will now polynomially reduce every problem in NP to BH. Let $A \in NP$. Then there exists a nondeterministic TM M_A such that M_A accepts A in p time, where p is a fixed polynomial. Construct a reduction f by

$$f(x) = \langle M_A, x, 0^{p(|x|)}\rangle.$$

Then $\forall x : x \in A$ iff M_A accepts x in $p(|x|)$ time iff $\langle M_A, x, 0^{p(|x|)}\rangle \in BH$ iff $f(x) \in BH$. Since M_A is a fixed TM and p is a polynomial, producing $\langle M_A, x, 0^{p(|x|)}\rangle$ from x can be carried out in time polynomial of $|x|$. Hence, f is polynomial-time computable. Thus, $A \leq_m^p BH$. This completes the proof. ∎

Bounded Tiling (BT)

A tile is a square with a symbol on each side. We assume that titles may not be rotated or turned over, and that there is an unlimited supply of each tile. By tiling of an $n \times n$ square it means to select and arrange n^2 tiles to cover the square so that the symbols on the common sides of adjacent tiles are the same. The size of a tile is the length of the binary representation of the four symbols at each edge in a fixed order.

BOUNDED TILING PROBLEM (BT)

Instance. A finite set of tiles T, a unitary notation 0^n representing positive integer n, and a sequence $S = s_1 s_2 \dots s_k$ $(k \leq n)$ of tiles that match each other, *i.e.* the symbol on the right side of s_i is the same as that on the left side of s_{i+1}. (The size of the instance is n plus the summation of the sizes of all tiles in T plus the summation of the sizes of all members in S.)

Question. Can S be extended to tile an $n \times n$ square using tiles from T?

BT Is NP-Complete

Denote by BT the positive instances of bounded tiling, namely,

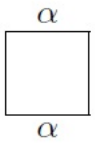
$$BT = \{\langle T, S, 0^n\rangle \mid S \text{ can be extended to tile an } n \times n \text{ square using tiles from } T\}.$$

Theorem 5 *Bounded tiling is NP-complete.*

Proof. That $BT \in NP$ is straightforward (the reader can easily fill in the details). We will now polynomially reduce every problem in NP to BT. Let $A \in NP$. Then there is a (nondeterministic) Turing machine $M_A = (Q, \Sigma, \delta, q_0, \{h_a, h_r\})$ that accepts A in polynomial time. Without loss of generality, we assume that $\Sigma = \{0, 1, B\}$, and all computation paths of M_A on input x have length strictly less than $p(|x|)$, where p is a fixed polynomial. Here Q is the finite set of states of M_A, δ the transition function, q_0 the initial state, h_a the accepting state, h_r the rejecting state, and B the blank symbol. We construct the following set $T(M_A)$ of legal tiles:

Proof Continued

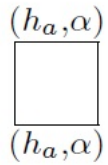
for $\alpha \in \{0, 1, B\}$:



for $\alpha, \beta, \gamma \in \{0, 1, B\}, q \in Q - \{h_a, h_r\}, p \in Q$, $\delta(q, \alpha) = (p, \beta, R)$:

for $\alpha, \beta, \gamma \in \{0, 1, B\}, q \in Q - \{h_a, h_r\}, p \in Q$, $\delta(q, \alpha) = (p, \beta, L)$:

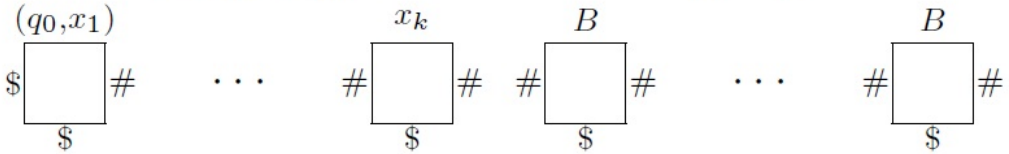
for $\alpha \in \{0, 1, B\}$:



and for $\alpha \in \{0, 1, B\}$:

Proof Continued

The desired reduction is defined by $f(x) = \langle T(M_A), S, 0^{p(|x|)} \rangle$, where if x is written as $x_1 x_2 \ldots x_k$ for $x_i \in \{0, 1\}$, S is the following sequence of tiles with $p(|x|) - k$ blanks:



From the construction, it is easy to see that f is polynomial-time computable. If S extends to a set of legal tiles which tile a $p(|x|) \times p(|x|)$ square, then S will have to occupy the bottom row of the square. In this case, the symbols at the top of the bottom row represents the initial configuration of M_A on input x. Likewise, the symbols at the top of the ith row from the bottom will represent the configuration of a position reachable by M_A within i steps. Since M_A will reach the accepting or rejecting state in less than $p(|x|)$ steps, and the tiling can only duplicate the accepting state, S can extend to a tiling only if M_A accepts x. Similarly, if M_A accepts x, then S can be extended to a tiling of a $p(|x|) \times p(|x|)$ square. This completes the proof. ∎

Boolean Satisfiability (SAT)

We consider Boolean formulas in conjunctive normal form. Namely, each formula F of n variables is given as a set of clauses $\{C_1, C_2, \ldots, C_m\}$, where each C_i is a set of literals $\{l_{i,1}, \ldots, l_{i,j_i}\}$. A literal is a variable or negation of a variable. Then $F = \prod_{i=1}^{m} \sum_{k=1}^{j_i} l_{i,j_k}$, where $\prod$ denotes conjunction, and $\sum$ denotes disjunction.

Boolean Satisfiability Problem (SAT)

Instance. A boolean formula ψ in conjunctive normal form.

Question. Is ψ satisfiable?

For convenience, denote by SAT the positive instances of Boolean satisfiability problem; namely,

$$SAT = \{\psi \mid \psi \text{ is satisfiable}\}.$$

Cook's Theorem

Theorem 6 (Cook's Theorem) *SAT is NP-complete.*

Proof. We will polynomially reduce bounded tiling to SAT. Then by Proposition 2(2), we know that every problem in NP is polynomially reducible to SAT.

Let $\langle T, S, 0^n \rangle$ be an instance of bounded tiling. Label the different tiles occurred in S as $t_1, \ldots, t_\ell$, and the tiles in T but not in S as $t_{\ell+1}, \ldots, t_{|T|}$. Let H_T denote the set of tile pairs $[t_i, t_j]$ such that t_i can be placed horizontally left to t_j, and V_T the set of tile pairs $[t_i, t_j]$ such that t_i can be placed vertically below t_j. Create $N = n^2|T|$ variables $v_{i,j,k}$, where $1 \leq i, j \leq n$, and $1 \leq k \leq |T|$. We will later want $v_{i,j,k}$ to be 1 if the (i, j)th cell of the square is covered by the kth tile. We construct a formula $\psi = \psi_1\psi_2\psi_3$ as follows.

Proof Continued

1. Let $S = s_1 \cdots s_m$. Assume that $s_j = t_{k_j}$ for $j = 1, \ldots, m$, where $k_j \in \{1, \ldots, \ell\}$. Let

$$\psi_1 = \prod_{j=1}^{m} v_{1,j,k_j}. \tag{1}$$

Then $\psi_1 = 1$ if and only if the jth cell in the bottom row is covered by t_{k_j} for $1 \leq j \leq m$.

2. For all k, k' with $[t_k, t_{k'}] \notin H_T$, where $1 \leq k, k' \leq |T|$, let

$$\psi_{2,1} = \prod_{i=1}^{n} \prod_{j=1}^{n-1} \prod_{[t_k, t_{k'}] \notin H_T} (\neg v_{i,j,k} + \neg v_{i,j+1,k'}). \tag{2}$$

For all k, k' with $[t_k, t_{k'}] \notin V_T$, where $1 \leq k, k' \leq |T|$, let

$$\psi_{2,2} = \prod_{i=1}^{n-1} \prod_{j=1}^{n} \prod_{[t_k, t_{k'}] \notin V_T} (\neg v_{i,j,k} + \neg v_{i+1,j,k'}). \tag{3}$$

Let $\psi_2 = \psi_{2,1}\psi_{2,2}$. Then $\psi_2 = 1$ if and only if the adjacency conditions of tiling are satisfied.

Proof Continued

3. Finally, let $\psi_3 = \psi_{3,1}\psi_{3,2}$, where

$$\psi_{3,1} = \prod_{i=1}^{n}\prod_{j=1}^{n}\sum_{k=1}^{|T|} v_{i,j,k} \tag{4}$$

and

$$\psi_{3,2} = \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k\neq k'} (\neg v_{i,j,k} + \neg v_{i,j,k'}) \tag{5}$$

Then $\psi_{3,1} = 1$ if and only if each cell is covered by at least one tile, and $\psi_{3,2} = 1$ if and only if each cell is covered by at most one tile. Thus, $\psi_3 = 1$ if and only if each cell is covered by exactly one tile.

Proof Continued

We note that $|\psi| = \Theta(n^2|T|^2)$. Let $f(\langle T, S, 0^n\rangle) = \psi$, then it is easy to see that f is polynomial-time computable. If $\langle T, S, 0^n\rangle$ is a positive instance of bounded tiling, then for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$, there must be a $1 \leq k \leq |T|$ such that the square at location (i, j) is tiled by t_k. Set $v_{i,j,k} = 1$. We conclude that every clause in (4) and (5) is satisfied. Clearly, every unit clause in (1) is satisfied. Now for each pair (k, k') with $(t_k, t_{k'}) \notin H_T$, and for all (i, j) with $1 \leq i, j \leq n$, at least one of $v_{i,j,k}$ and $v_{i,j,k'}$ has not been set to 1. Set that variable to 0. Thus, every clause in (2) is satisfied. Similarly, every clause in (3) is also satisfied. Conversely, we can show that if ψ is satisfiable then $\langle T, S, 0^n\rangle$ is a positive instance of bounded tiling. This completes the proof. ∎

3SAT

We consider CNF Boolean formula F of n variables $\{C_1, C_2, \ldots, C_m\}$ in which each clause consists of exactly three literals. We call such a formula a 3CNF formula.

3-Satisfiability Problem (3SAT)

Instance. A 3CNF formula ψ.

Question. Is ψ satisfiable?

Denote by 3SAT the positive instances of 3-Satisfiability problem; namely,

$$3SAT = \{\psi \mid \psi \text{ is a 3CNF and is satisfiable}\}.$$

3SAT Is NP-Complete

Theorem 7. 3SAT is NP-complete.

We consider CNF Boolean formula F of n variables $\{C_1, C_2, \ldots, C_m\}$ in which each clause consists of exactly three literals. We call such a formula a 3CNF formula.

3-Satisfiability Problem (3SAT)

Instance. A 3CNF formula ψ.

Question. Is ψ satisfiable?

Denote by 3SAT the positive instances of 3-Satisfiability problem; namely,

$$3SAT = \{\psi \mid \psi \text{ is a 3CNF and is satisfiable}\}.$$

Proof

Proof. It suffices to reduce a CNF formula to a 3CNF formula. Let F be a CNF formula. We introduce three new variables y_1, y_2, y_3 and seven new clauses:

$$\{y_1, y_2, y_3\}, \{y_1, y_2, \neg y_3\}, \{y_1, \neg y_2, y_3\}, \{\neg y_1, y_2, y_3\}, \{y_1, \neg y_2, \neg y_3\}, \{\neg y_1, \neg y_2, y_3\}, \{\neg y_1, y_2, \neg y_3\}.$$

Namely, these seven new clauses are all possible 3-literal (nontrivial) clauses over variables y_1, y_2, y_3 except $\{\neg y_1, \neg y_2, \neg y_3\}$. It is easy to see that the only truth-value assignment to satisfy all these seven clauses is $y_1 = y_2 = y_3 = 1$.
Let C be a clause of F. Then $C = \{z_1, z_2, \ldots, z_k\}$ for some $k \geq 1$, where z_i is a literal. We will replace C by 3-clause(s) as follows.
Case 1: $k = 1$. Replace C by $\{z_1, \neg y_1, \neg y_2\}$.
Case 2: $k = 2$. Replace C by $\{z_1, z_2, \neg y_1\}$.
Case 3: $k = 3$. Do nothing.
Case 4: $k = 4$. Introduce one new variable $u_1 = y_C$, and replace C by the following 3-clauses:

$$\{z_1, z_2, u_1\}, \{z_3, z_4, \neg u_1\}, \{\neg z_3, u_1, \neg y_1\}, \{\neg z_4, u_1, \neg y_1\}.$$

Note that the last three 3-clauses imply that $z_3 + z_4$ is equivalent to u_1, and so $z_1 + z_2 + u_1$ is equivalent to $z_1 + z_2 + z_3 + z_4$.
Case 5: $k > 4$. Inductively apply Case 4 until only clauses of 3 literals left. For example, when $k = 5$, first replace C with the following clauses:

Proof Continued

Then the last three clauses of three literals imply that $z_4 + z_5$ is equivalent to u_1. Thus, $z_1 + z_2 + z_3 + z_4 + z_5$ is equivalent to $z_1 + z_2 + z_3 + u_1$. Now replace the first clause of four literals $\{z_1, z_2, z_3, u_1\}$ by the following 3-clauses, where u_2 is a new variable:

$$\{z_1, z_2, u_2\}, \{z_3, u_1, \neg u_2\}, \{\neg z_3, u_2, \neg y_1\}, \{\neg u_1, u_2, \neg y_1\}.$$

As shown before that the last three clauses imply that u_2 is equivalent to $z_3 + u_1$, and so $z_1 + z_2 + z_3 + u_1$ is equivalent to $z_1 + z_2 + u_2$.
Let G be the product of all new clauses. It is straightforward to see that F is satisfiable if and only if G is satisfiable, and the above transformation from C to a set of 3-clauses can be carried out in time polynomial in k. Thus, $SAT \leq_m^p 3SAT$. This completes the proof. ∎