1.9 THE MATRIX OF A TRANSFORMATION

Thm: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation. Then $\exists$ a unique matrix A ST $T(\vec{x}) = A\vec{x}$ ST $A \to m \times n$ matrix

$$A = [T(\vec{e}_1) \;\; T(\vec{e}_2) \; \cdots \; T(\vec{e}_n)] \to \text{Standard Matrix of } T$$

Proof: (2 parts) $\to$ ① Show A $\exists$

② Show A is unique.

* Show Matrix A $\exists$: Let I_n be the $m \times n$ Identity matrix $I_n = [\vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n]$

- Consider $\vec{x} \in \mathbb{R}^n$: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

- Since T is a Linear Transformation:

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n)$$

$$= [T(\vec{e}_1) \;\; T(\vec{e}_2) \cdots T(\vec{e}_n)]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A\vec{x} \longrightarrow \therefore A \; \exists \Rightarrow A = [T(\vec{e}_1) \;\; T(\vec{e}_2) \cdots T(\vec{e}_n)]$$

* Standard Matrix of T

* Show that Matrix A is unique:

Let A be the Standard Matrix of T

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a Linear Transformation ST $T(\vec{x}) = B\vec{x}$, where B is $m \times n$ matrix ✓

* Since A is Standard Matrix of T: $A = [T(\vec{e}_1) \cdots T(\vec{e}_j) \cdots T(\vec{e}_n)]$

* Since $T(\vec{x}) = B\vec{x}$, then by matrix-vector multiplication.

$$T(\vec{e}_j) = B\vec{e}_j = [\vec{b}_1 \;\; \vec{b}_2 \cdots \vec{b}_j \cdots \vec{b}_n]\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{b}_j$$

* In order for both statements to be true: $A = [\vec{b}_1 \cdots \vec{b}_n] = B$ * so A is unique

Ex: Define the Linear Transformation $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ as: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + x_2 - x_3 + x_4 \\ 2x_1 + x_2 + 4x_3 + x_4 \\ 3x_1 + x_2 + 9x_3 \end{bmatrix}$

(a) Find the Standard Matrix of T

(b) Find $\vec{x}$ (if one exists) ST $T(\vec{x}) = \vec{b}$ where $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Ans: Recall $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ ST

$$T(\vec{x}) = A\vec{x} = [T(\vec{e}_1) \cdots T(\vec{e}_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(a) * Rewrite the Mapping:

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{x} \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Standard matrix of T: $\boxed{A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix}}$

(b) Find $\vec{x}$ ST: $A\vec{x} = \vec{b} \iff [A \,\vdots\, \vec{b}\,]$ * Row-reduce the augmented matrix to RREF

$$[A \,\vdots\, \vec{b}\,] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 1 & 4 & 1 & 2 \\ 3 & 1 & 9 & 0 & -1 \end{array}\right] \underset{\substack{-3R_1 + R_3 \\ = nR_3}}{\overset{\substack{-2R_1 + R_2 \\ = nR_2}}{\sim}} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 6 & -1 & 0 \\ 0 & -2 & 12 & -3 & -4 \end{array}\right] \overset{\substack{R_2 + R_1 \\ nR_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -6 & 1 & 0 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -6 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & -6 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \Longrightarrow$$

- $x_1 = 1 - 5x_3$
- $x_2 = -4 + 6x_3$
- x_3 is free
- $x_4 = 4$

* General Solution as a Parametric Vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix} \qquad x_3 \in \mathbb{R}$$

*

* Particular Solution:

$$\begin{bmatrix} 1 \\ -4 \\ 0 \\ 4 \end{bmatrix}$$ Let $x_3 = 0$

* Ex: Define T: $T(\vec{x}) = \vec{x} \cdot \vec{x}$

Determine if the transformation is linear
① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
② $T(c\vec{u}) = cT(\vec{u})$

Check ①

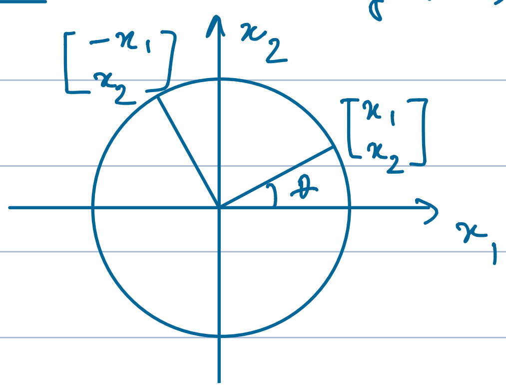
$$T(\vec{u} + \vec{v}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v}\,\vec{v}$$

$$= T(\vec{u}) + 2\vec{u}\,\vec{v} + T(\vec{v}) \neq T(\vec{u}) + T(\vec{v})$$

$\therefore$ Prop 1 fails $\Rightarrow$ T is not a linear Transformation

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation that rotates each point in $\mathbb{R}^2$ about the origin, through a ⊕ angle θ, with ccw rotation from a positive angle. Find the standard Matrix of T.

Ans.: * Geometrically ($\mathbb{R}^2$)

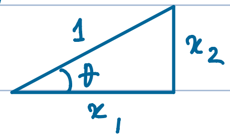


Mapping:

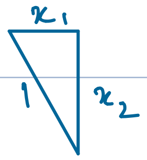
$$\vec{e}_1 \longrightarrow T(\vec{e}_1) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ?$$

$$\vec{e}_2 \longrightarrow T(\vec{e}_2) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

* Note: Need to use Right Triangle Trig to define components in terms of θ "SOH / CAH / TOA"

① 

$\therefore T(\vec{e}_1) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

② 

$\therefore T(\vec{e}_2) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos\theta \end{bmatrix}$

* Standard Matrix of T:

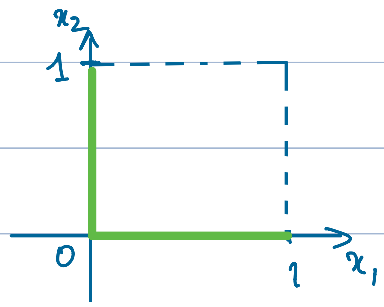
Rotation Matrix

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

* Geometric Transformations of the Unit Square:

* The Unit Square: $I_2 = [\vec{e}_1 \quad \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

① Projections: ① Onto x_1- axis:

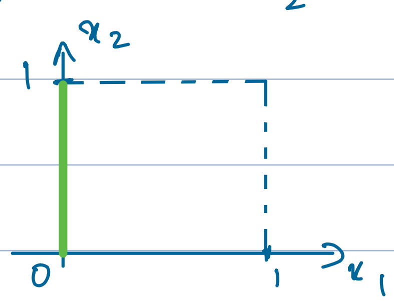


Mapping:

- $\vec{e}_1 \to T(\vec{e}_1) = \vec{e}_1$
- $\vec{e}_2 \to T(\vec{e}_2) = \vec{e}_2$

Standard Matrix of A: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

② Onto the x_2-axis:



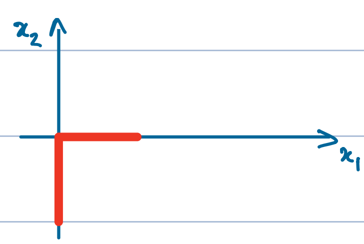
Mapping:

$\left.\begin{array}{l} \vec{e}_1 \to T(\vec{e}_1) = \vec{0} \\ \vec{e}_2 \to T(\vec{e}_2) = \vec{e}_2 \end{array}\right\}$ $\therefore$ Standard Matrix of T:

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

* Reflections:

① Accross the x_1-axis:

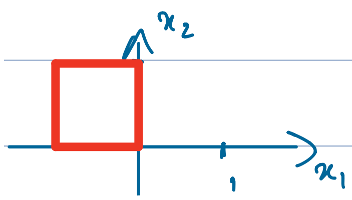


$\vec{e}_1 \to T(\vec{e}_1) = \vec{e}_1$

$\vec{e}_2 \to T(\vec{e}_2) = -\vec{e}_2$

$\therefore$ Standard Matrix of T: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

② Accross the x_2-axis:



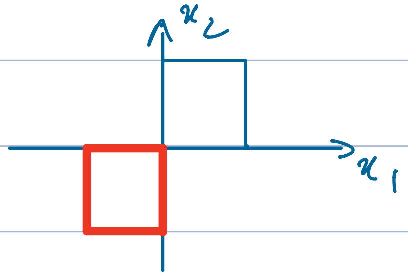
$\vec{e}_1 \longrightarrow T(\vec{e}_1) = -\vec{e}_1$

$\vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{e}_2$ $\Rightarrow$

Standard matrix of T:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

③ Accross the origin:



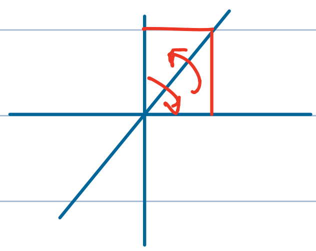
Mapping:

$\vec{e}_1 \longrightarrow T(\vec{e}_1) = -\vec{e}_1$

$\vec{e}_2 \longrightarrow T(\vec{e}_2) = -\vec{e}_2$

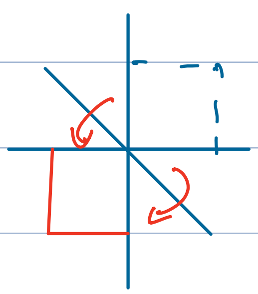
$$\Rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

④ Accross the line $x_2 = x_1$



$$\left.\begin{array}{l} \vec{e}_1 \longrightarrow T(\vec{e}_1) = \vec{e}_2 \\ \vec{e}_2 \longrightarrow T(\vec{e}_2) = \vec{e}_1 \end{array}\right\} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

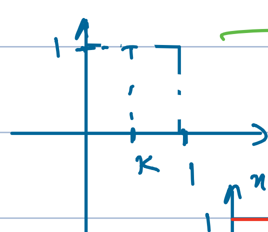
⑤ Accross the line $x_2 = -x_1$



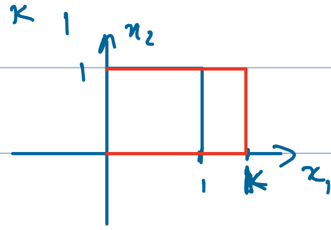
$$\left.\begin{array}{l} \vec{e}_1 \longrightarrow T(\vec{e}_1) = -\vec{e}_2 \\ \vec{e}_2 \longrightarrow T(\vec{e}_2) = -\vec{e}_1 \end{array}\right\} A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

* Expansions/Contractions: Let k be any scalar, $k \in \mathbb{R}$

- Horizontal $(0 < k < 1)$

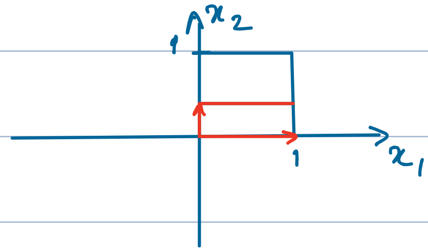


$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



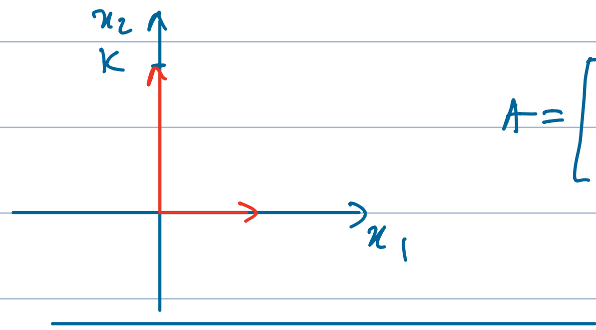
- Horizontal $(k > 1)$

* Vertical $(0<k<1)$



$$A=\begin{bmatrix}1 & 0\\ 0 & k\end{bmatrix}$$

* Vertical $(k>1)$



$$A=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

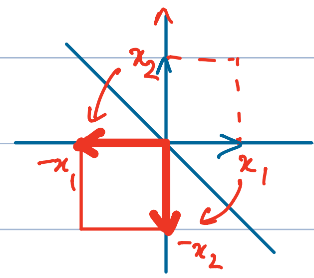
1.9.9

$$e_1:\ T(e_1)=T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = e_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$e_2:\ T(e_2) = 15e_1+e_2 = \begin{bmatrix}15\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}15\\1\end{bmatrix}$$

$$T = \begin{bmatrix}T(e_1) & T(e_2)\end{bmatrix} = \begin{bmatrix}1 & 15\\ 0 & 1\end{bmatrix}$$

Reflection matrix through the line $x_2 = -x_1$



$$A=\underbrace{\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}}_{x_2=-x_1}\begin{bmatrix}1 & 15\\ 0 & 1\end{bmatrix}=\boxed{\begin{bmatrix}0 & -1\\ -1 & -15\end{bmatrix}}$$

1.9.21 Solve $A\vec{x}=\vec{b}$

$$\left[\begin{array}{cc|c}1 & 1 & 1\\ 4 & 2 & 14\end{array}\right]\xrightarrow[=nR_2]{4R_1-R_2}\left[\begin{array}{cc|c}1 & 1 & 1\\ 0 & 2 & -10\end{array}\right]\xrightarrow{\frac{R_2}{2}}\left[\begin{array}{cc|c}1 & 1 & 1\\ 0 & 1 & -5\end{array}\right]\xrightarrow[=nR_1]{R_1-R_2}\boxed{\left[\begin{array}{cc|c}1 & 0 & 6\\ 0 & 1 & -5\end{array}\right]=\vec{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}}$$