CMPSC 623 Problem Set 6. Solution.
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Out: November 9, 2006
Due: November 16, 2006, before class.

Problem 1. Page 398, Exercise 16.4-1. Show it is closed under inclusion, and then show it satisfies exchange property.

Closed under inclusion, because any subset of a set in L_k is a subset of S and its size cannot be larger than k, i.e., it is still in L_k.

(S, L_k) also satisfies exchange property, because take any two sets i, i' in L_k and $|i| < |i'|$, it must be true that $i + e \subseteq L_k$ for any $e \in i' - i$.

Problem 2. Read the description of activity-selection problem on page 371. Define a valid subset S' of S to be a set in which all activities are mutually compatible. For example, $S' = \{a_3, a_9, a_{11}\}$ is a valid subset. Define I as the collection of all those valid subsets of S. Is (S, I) a subset system that is closed under inclusion? Is it a matroid? Justify your answer.

Yes, (S, I) a subset system that is closed under inclusion because any subset of a set in I will only include compatible activities.

No, it is not a matriod. For example, consider valid subsets $S_1 = \{a_3, a_9, a_{11}\}$ and $S_2 = \{a_1, a_4, a_8, a_{11}\}$. $|S_2| > |S_1|$. Take $a_1 \in S_2 - S_1$, and we find that $a_1 + S_1$ is not a valid subset, thus, exchange property cannot be satisfied.

Problem 3. Page 398, Exercise 16.4-4. Show it is closed under inclusion, and then show it satisfies exchange property.

First we note that S is a finite nonempty set and $I \subseteq 2^S$. Consider $i \in I$ and suppose that $j \subset i$. As $j \cap S_n \subset i \cap S_n$ for $n = 1, 2, ..., k$, and $|j \cap S_n| \leq |i \cap S_n| \leq 1$ for all $n = 1, 2, ..., k$. So I is closed under inclusion. It follows that (S, I) is a subset system.

For a valid set i, define $i_n = i \cap S_n$, for $n = 1, 2, ..., k$. For each i_n, $|i_n| \leq 1$; $i_n \subset i$, so $i_n \in I$, but $i_n \subset S_n$, so $|i_n \cap S_j| = 0$ for $j \neq n$. Thus, for each valid set i there are indices $n_1, n_2, ..., n_r$ such that $i = \cup_{j=1}^{r} i_{n_j}$, with $i_{n_j} \neq \emptyset$. These i_{n_k}s are disjoint and $|i_{n_k}| = 1; 0 < |i_{n_k}| \leq 1$. It follows that $r = |i|$.

Now suppose that $i, j \in I$ and $|i| < |j|$. Then $i = \cup_{k=1}^{q} i_{n_k}$ and $j = \cup_{k=1}^{r} j_{m_k}$, with $q < r$. So there is an m_l such that $i_{m_l} = \emptyset$ and $j_{m_l} \subset j \cap i^c$. Thus, $i \cup j_{m_l} \in I$. The exchange property follows. j_{m_l} contains only one element.