① Use the coordinate vectors to determine whether the given polynomials are Linearly Dependent in $\mathbb{P}_2$. Let $\mathcal{B}$ be the Standard Basis of the space $\mathbb{P}_2$ of polynomials, that is $\mathcal{B} = \{1, t, t^2\}$.

$$1+2t \quad , \quad 3+6t^2 \quad , \quad 1+3t+4t^2$$

* Find the Coordinate Vectors of the polynomials, relative to $\mathcal{B}$:

- $p_1(t) = 1+2t \longrightarrow [\vec{p}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
- $p_2(t) = 3+6t^2 \longrightarrow [\vec{p}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$
- $p_3(t) = 1+3t+4t^2 \longrightarrow [\vec{p}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

* Check if coord. vectors are Linearly Dependent:

- Row-reduce Augmented matrix, whose columns are the coord. vectors, to Echelon Form:

$$[P_{\mathcal{B}} \,\vdots\, \vec{0}] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 3 \\ 0 & 6 & 4 \end{bmatrix} \underset{\sim}{\frac{-2R_1 + R_2}{N.R_2}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -6 & 1 \\ 0 & 6 & 4 \end{bmatrix} \underset{\sim}{\frac{R_2 + R_3}{N.R_3}}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -6 & 1 \\ 0 & 0 & 5 \end{bmatrix} \Longrightarrow$$

* Echelon Form

Since matrix has $n=3$ pivots, the columns of matrix are Linearly Independent.

$\therefore$ Polynomials are NOT Linearly Dependent in $\mathbb{P}_2$.

Answer.

② For the given matrix A, Find the:

(a) Basis and Dimension of the Column Space of A.

(b) Basis and Dimension of the Null Space of A.

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 5 & -4 \\ 0 & 0 & 1 & -6 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*Part (a):

Note: Matrix A is already in Echelon Form! Pivot Columns: $\vec{a}_1, \vec{a}_3, \vec{a}_6$

$$\boxed{\begin{array}{l} \therefore \text{Basis for } Col(A): \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\} \\ \therefore \text{Dimension of } Col(A): \quad \dim[Col(A)] = 3 \end{array}}$$

Ans

*Part (b):

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 5 & -4 \\ 0 & 0 & 1 & -6 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{-3R_2+R_1}{\sim} \begin{bmatrix} 1 & -2 & 0 & 19 & -6 & 11 & -4 \\ 0 & 0 & 1 & -6 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{2R_3+R_2}{\overset{-11R_3+R_1}{\sim}}$$

$$\begin{bmatrix} 1 & -2 & 0 & 19 & -6 & 0 & -37 \\ 0 & 0 & 1 & -6 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = 2x_2 - 19x_4 + 6x_5 + 37x_7 \\ x_3 = 6x_4 - 2x_5 - 6x_7 \\ x_6 = -3x_7 \end{cases}$$

RREF

* x_2, x_4, x_5, x_7 are free!

General Solution to $A\vec{x}=\vec{0}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2x_2 - 19x_4 + 6x_5 + 37x_7 \\ x_2 + 0 + 0 + 0 \\ 0 + 6x_4 - 2x_5 - 6x_7 \\ 0 + x_4 + 0 + 0 \\ 0 + 0 + x_5 + 0 \\ 0 + 0 + 0 - 3x_7 \\ 0 + 0 + 0 \quad x_7 \end{bmatrix}$$

← you do not need to write this part, but just in case you need/want a reminder.

$$= x_2\begin{bmatrix}2\\1\\0\\0\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-19\\0\\6\\1\\0\\0\\0\end{bmatrix} + x_5\begin{bmatrix}6\\0\\-2\\0\\1\\0\\0\end{bmatrix} + x_6\begin{bmatrix}37\\0\\-6\\0\\0\\-3\\1\end{bmatrix} \quad \text{ST} \quad x_2, x_4, x_5, x_6 \in \mathbb{R}$$

$$\boxed{\begin{array}{l} \text{*}\underline{\text{Basis for Nul}(A)}\text{:} \quad \mathcal{B} = \left\{ \begin{bmatrix}2\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-19\\0\\6\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}6\\0\\-2\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}37\\0\\-6\\0\\0\\-3\\1\end{bmatrix} \right\} \\ \\ \text{*}\underline{\text{Dimension of Nul}(A)}\text{:} \quad \dim[\text{Nul}(A)] = 4 \end{array}}$$

Ans.