Counting Using the Principle of Inclusion-Exclusion

Let U be a universal set, and let A be the set of elements of U which satisfies at least one property on the list of properties $P_1, P_2, \ldots, P_n$. Suppose we're interested in counting how many elements the set A has.

For $i = 1, \ldots, n$, let A_i be the set of elements in U that satisfy property P_i. Then

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

Roughly speaking, the principle of inclusion-exclusion implies that $|A|$ is equal to a certain sum / difference of cardinalities $|A_{i_1} \cap \ldots \cap A_{i_k}|$.

Each intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ is the set of elements of U which satisfy all of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$. If we can figure out how to use this description to calculate the cardinality $|A_{i_1} \cap \ldots \cap A_{i_k}|$, then we can compute $|A|$ using the principle of inclusion-exclusion.

Counting problems with two sets A_1 and A_2:

Using the notation above, A is the set of elements of U that satisfy property P_1 or property P_2. The law of inclusion-exclusion for two sets states that $|A| = |A_1| + |A_2| - |A_1 \cap A_2|$

This implies the following: The number of elements in U satisfying property P_1 or P_2 equals ...
[# elements in U satisfying property P_1] plus
[# elements in U satisfying property P_2] minus
[# elements in U satisfying both properties]

Example 1: Find the number of solutions in nonnegative integers for the equation

$$x_1 + x_2 + x_3 + x_4 = 10 \tag{1}$$

if $x_1 \geq 2$ or $x_2 \geq 3$.

Solution: Let U be the set of solutions (x_1, x_2, x_3, x_4) in nonnegative integers for equation (1). Also, let A_1 be the set of elements (x_1, x_2, x_3, x_4) in U so that $x_1 \geq 2$, and let A_2 be the set of elements (x_1, x_2, x_3, x_4) in U so that $x_2 \geq 3$. It follows that $A = A_1 \cup A_2$ is the set of elements in U so that $x_1 \geq 2$ or $x_2 \geq 3$.

Next, we compute $|A|$ using the principle of inclusion-exclusion. For $|A_1|$, note that finding all nonnegative integer solutions for (1) satisfying $x_1 \geq 2$ is equivalent to finding all nonnegative integer solutions (x_1', x_2, x_3, x_4) to

$$x_1' + x_2 + x_3 + x_4 = 8$$

where $x_1 = x_1' - 2$. Therefore, $|A_1| = C(8 + 4 - 1, 4 - 1) = C(11, 3) = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$.

By similar reasoning, $|A_2| = C(7 + 4 - 1, 4 - 1) = C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$.

Finally, to compute $|A_1 \cap A_2|$, note that finding all nonnegative integer solutions for (1) satisfying $x_1 \geq 2$ and $x_2 \geq 3$ is equivalent to finding all nonnegative integer solutions (x_1', x_2', x_3, x_4) to

$$x_1' + x_2 + x_3 + x_4 = 5$$

where $x_1 = x_1' - 2$ and $x_2 = x_2' - 3$. Therefore, $|A_1 \cap A_2| = C(5 + 4 - 1, 4 - 1) = C(8, 3) = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$.

Using the principle of inclusion-exclusion, we get $|A| = 165 + 120 - 56 = \boxed{229}$

Example 2: How many ways are there to distribute eight identical cookies to three different children (Alice, Bob, and Candice) if Alice gets at least two cookies or Bob gets at least three cookies?

Solution: Let U be the set of all ways of distributing the cookies to the children with no restrictions. Also, let A_1 be the set of all ways to distribute the cookies so that Alice gets at least two, and let A_2 be the set of of all ways to distribute the cookies so that Bob gets at least three. We want to compute the cardinality of $A = A_1 \cup A_2$ using the principle of inclusion-exclusion.

To compute $|A_1|$, note that if we are distributing the cookies so that Alice gets at least two, then that means we can distribute the cookies by automatically giving Alice two cookies, then taking the remaining 6 cookies and distributing to the three children in any way. (Note that Alice may get additional cookies beyond the two she initially received.) There are $C(6+3-1, 3-1) = C(8,2) = \frac{8 \cdot 7}{2 \cdot 1} = 28$ ways to do this.

Similar, we can show that $|A_2| = C(5+3-1, 3-1) = C(7,2) = \frac{7 \cdot 6}{2 \cdot 1} = 21$.

Finally, to compute $|A_1 \cap A_2|$, note that if we are distributing the cookies so that Alice gets at least two and Bob gets at least three, then we can automatically give Alice two cookies and Bob three cookies, which means there are three cookies left to distribute among the three children in any way. There are $C(3+3-1, 3-1) = C(5,2) = 10$ ways to do this.

Therefore, $|A| = 28 + 21 - 10 = \boxed{39}$

Note: We could reframe the problem in Example 2 as follows:
Let x_1, x_2, x_3 be the number of cookies that Alice, Bob, and Candice get, respectively. How many solutions in nonnegative integers does the equation $x_1 + x_2 + x_3 = 8$ have if $x_1 \geq 2$ and $x_2 \geq 3$?

Example 3: How many ways are there to distribute eight identical cookies to three different children (Alice, Bob, and Candice) if Alice gets at most one cookie and Bob gets at most two cookies?

Solution: Let U be the same set as in the solution to the previous example. Then $|U| = C(8+3-1, 3-1) = \frac{10 \cdot 9}{2 \cdot 1} = 45$.

Note that the following statements are equivalent by DeMorgan's Law for logic:

- It's not true that Alice gets at most one cookie and Bob gets at most two cookies.
- Alice gets at least two cookies or Bob gets at least three cookies.

Using "counting the complement", we want to compute $|U|$ minus the number of ways of distributing the cookies so that Alice gets at least two cookies or Bob gets at least three cookies. Using the previous example, this is equal to $45 - 39 = \boxed{6}$

Note: It's pretty easy to list out all the possibilities in this case – try it!

Counting problems with three or more sets:

Naturally, things are more complicated in this situation, but we can still use the same idea outlined above. Let's see how this works by example.

Example 4: How many ways can 8 different tasks be assigned to three different people so at least one employee is assigned no tasks?

Note: There are $3^8 = 6561$ ways to assign the tasks with no restrictions. (Use the product rule: There's three ways to assign an employee to task 1, three ways to assign an employee to task 2, etc.)

Solution: For $i = 1, 2, 3$, let A_i be the set of all ways to assign the tasks so that the ith employee is assigned no tasks. Therefore, $A_1 \cup A_2 \cup A_3$ is the set of all ways to assign the tasks so that at least one employee is assigned no tasks.

We want to compute $|A_1 \cup A_2 \cup A_3|$ using the principle of inclusion-exclusion.

For $|A_1|$, note that if no tasks are assigned to employee 1, then the tasks must all be assigned to the other two employees. There are $2^8 = 256$ ways to do this. By similar reasoning, $|A_2| = |A_3| = 256$.

For $|A_1 \cap A_2|$, note that $A_1 \cap A_2$ is the set of all ways of assigning the tasks so that employees 1 and 2 both get no tasks. This means employee 3 gets all the tasks, so $|A_1 \cap A_2| = 1$. Likewise, $|A_1 \cap A_3| = 1$ and $|A_2 \cap A_3| = 1$.

For $|A_1 \cap A_2 \cap A_3|$, note that $A_1 \cap A_2 \cap 3$ is the set of all ways of assigning the tasks so that employees 1,2, and 3 all get no tasks. This can't happen, so $|A_1 \cap A_2 \cap A_3| = 0$.

Putting it all together using the principle of inclusion-exclusion, we get

$$|A| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 256 + 256 + 256 - 1 - 1 - 1 + 0 = \boxed{765}$$

Example 5: How many ways can 8 different tasks be assigned to three different people if every employee has at least one task?

Solution: Use "counting the complement":

- The total number of ways to assign the tasks is $3^8 = 6561$.
- The number of ways to assign tasks where it's *not* true each employee has at least one task is 765 (by the previous example).

Therefore, the answer is $6561 - 765 = \boxed{5796}$

Next, let's consider a different type of problem: Let U be a universal set, and let B be the set of elements of U which satisfies all of the properties $P_1, P_2, \ldots, P_n$. Suppose we're interested in counting how many elements the set B has. Using "counting the complement", we can do this by computing $|U|$ minus the number of elements of U which don't satisfy at least one of the properties $P_1, P_2, \ldots,$ or P_n. Equivalently, we can compute $|U| - |\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n|$. The cardinality of $\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n$ can be computed using the principle of inclusion-exclusion.

For $n = 2$, we get

$$|B| = |U| - |\overline{A_1}| - |\overline{A_2}| + |\overline{A}_1 \cap \overline{A}_2|,$$

where $|\overline{A}_1|$ is the number of elements in U that don't satisfy property P_1, $|\overline{A}_2|$ is the number of elements in U that don't satisfy property P_2, and $|\overline{A}_1 \cap \overline{A}_2|$ is the number of elements that don't satisfy both properties P_1 and P_2.

Example 6: A bagel shop sells five different types of bagels (poppy, onion, everything, raisin, and cheddar). How many ways can ten bagels be selected with at most two onion bagels and at most two poppy seed bagels?

Solution: Let U be the set of ways to pick ten bagels with no restrictions, let A_1 be the set of ways to pick ten bagels at most two onion bagels, and let A_2 be the set of ways to pick ten bagels with at most two poppy seed bagels. We want to compute the cardinality of $B = A_1 \cap A_2$.

Note that $\overline{A}_1$ is the set of ways to pick ten bagels with at least three onion bagels, and $\overline{A}_2$ is the number of ways to pick ten bagels with at least three poppy seed bagels.

You can verify that:

$|U| = C(10 + 5 - 1, 5 - 1) = C(14, 4) = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2 \cdot 1} = 1001$

$|\overline{A}_1| = C(7 + 5 - 1, 5 - 1) = C(11, 4) = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} = 330, \quad |\overline{A}_2| = C(7 + 5 - 1, 5 - 1) = C(11, 4) = 330$

$|\overline{A}_1 \cap \overline{A}_2| = C(4 + 5 - 1, 5 - 1) = C(8, 4) = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70.$

So $|A_1 \cap A_2| = 1001 - 330 - 330 + 70 = \boxed{411}$

Homework problems:

1. A password consists of a string of seven letters taken from the set $\{a, b, c, d\}$. How many passwords ...
   (a) ... contain exactly three a's or exactly four b's?
   (b) ... start with a string of four different letters or end with a string of four different letters? (e.g. *bacdaab* starts with a string of four different letters)
2. Suppose that there are five freshmen, three sophomores, and four juniors in a class.
   (a) Suppose that 1st, 2nd, and 3rd prizes will be awarded to three of the students (no ties). How many ways are there to award these prizes if a freshmen wins first or a junior wins second?
   (b) Suppose that five students will selected to win identical prizes. How many ways are there to do this if exactly two freshmen are winners or exactly two juniors are winners?
3. A fruit stand sells four different types of fruit (apples, bananas, pears, and oranges). How many different ways are there to select nine pieces of fruit if ...
   (a) ... at least two apples or at least four bananas are picked?
   (b) ... exactly three apples or exactly two pears are picked?

   (Assume that fruit pieces are identical if they're of the same type – only the numbers of each type of fruit matters.)
4. Alice has a dozen identical kiwis she wants to distribute among five friends who include Bob and Candice. How many ways can she do this if ...
   (a) ... Bob gets at least three or Candice gets at least four?
   (b) ... Bob gets at most three and Candice gets at most four?
5. How many ways are there to distribute five different toys among three children if ...
   (a) ... at least one child gets no toys?
   (b) ... each child gets at least one toy?

Answers:

1. (a) $2835 + 945 - 35 = \boxed{3745}$
   (b) $1536 + 1536 - 144 = \boxed{2928}$
2. (a) $550 + 440 - 200 = \boxed{790}$
   (b) $350 + 336 - 180 = \boxed{506}$
3. (a) $120 + 56 - 20 = \boxed{156}$
   (b) $28 + 36 - 5 = \boxed{59}$
4. (a) $715 + 495 - 126 = \boxed{1084}$
   (b) $1820 - 495 - 330 + 35 = \boxed{1030}$
5. (a) $32 + 32 + 32 - 1 - 1 - 1 + 0 = \boxed{93}$
   (b) $243 - 93 = \boxed{150}$