# Calculus III 2D Integration Problem Requiring Change of Variables and Jacobian

Integrate $f(x,y)=x^2+y^2$ over the region, $R$ , bounded by $-\frac{1}{4}x \le y \le \frac{2}{3}x$ and $\sqrt{y^2+2} \le x \le \sqrt{y^2+7}$ .

$\iint\limits_R (x^2+y^2)dA = ??$ **ANSWER USING HYPERBOLIC SUBSTITUTION:** Let $x = h\cosh(\alpha)$ and $y = h\sinh(\alpha)$ .

The integrand becomes $x^2+y^2 = h^2(\cosh^2(\alpha)+\sinh^2(\alpha)) = h^2(1+2\sinh^2(\alpha))$ .

The first boundary constraints become: $-\frac{1}{4} \le \frac{y}{x} \le \frac{2}{3}$ or $-\frac{1}{4} \le \tanh(\alpha) \le \frac{2}{3}$ or $-\tanh^{-1}(\frac{1}{4}) \le \alpha \le \tanh^{-1}(\frac{2}{3})$ .

The second set of boundary constraints become:

$x^2-y^2=2$ or $h^2=2$ or $h=\sqrt{2}$ and $x^2-y^2=7$ or $h^2=7$ or $h=\sqrt{7}$ . Therefore, $\sqrt{2} \le h \le \sqrt{7}$ .

For our definition of $x$ and $y$ , the Jacobian is: $J = \left| \begin{pmatrix} \cosh(\alpha) & h\sinh(\alpha) \\ \sinh(\alpha) & h\cosh(\alpha) \end{pmatrix} \right| = h(\cosh^2(\alpha)-\sinh^2(\alpha)) = h$ .

Therefore:

$$\iint\limits_R (x^2+y^2)dA = \int_{-\tanh^{-1}(1/4)}^{\tanh^{-1}(2/3)} \int_{\sqrt{2}}^{\sqrt{7}} h^2(1+2\sinh^2(\alpha))h\,dh\,d\alpha \text{ where } h\,dh\,d\alpha = dA$$

$$= \int_{-\tanh^{-1}(1/4)}^{\tanh^{-1}(2/3)} (1+2\sinh^2(\alpha))d\alpha \int_{\sqrt{2}}^{\sqrt{7}} h^3 dh$$

$$= \frac{1}{4}(49-4)\int_{-\tanh^{-1}(1/4)}^{\tanh^{-1}(2/3)} (1+2\sinh^2(\alpha))d\alpha$$

$$= \frac{45}{4}\int_{-\tanh^{-1}(1/4)}^{\tanh^{-1}(2/3)} \cosh(2\alpha)d\alpha = \frac{45}{8}(\sinh(2\tanh^{-1}(2/3)) - \sinh(-2\tanh^{-1}(1/4))$$

$$= \frac{45}{8}(\sinh(2\tanh^{-1}(2/3)) + \sinh(2\tanh^{-1}(1/4))$$

$$= \frac{45}{8}(2\sinh(\tanh^{-1}(2/3))\cosh(\tanh^{-1}(2/3)) + 2\sinh(\tanh^{-1}(1/4)\cosh(\tanh^{-1}(1/4)))$$

$$= \frac{45}{4}(\frac{2\times 3}{3^2-2^2} + \frac{1\times 4}{4^2-1^2}) = \frac{45}{4} * \frac{22}{15} = \frac{33}{2}$$ {See note below}

Note: $\sinh(2\Theta) = 2\sinh(\Theta)\cosh(\Theta)$ and $\sinh(\tanh^{-1}(a/b)) = \frac{a}{\sqrt{b^2-a^2}}$ and $\cosh(\tanh^{-1}(a/b)) = \frac{b}{\sqrt{b^2-a^2}}$

So, $\sinh(2\tanh^{-1}(a/b)) = 2\sinh(\tanh^{-1}(a/b))\cosh(\tanh^{-1}(a/b)) = \frac{2ab}{b^2-a^2}$

**ANSWER USING POLAR COORDINATES:** Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$ .

The integrand becomes $x^2 + y^2 = r^2$ .

The first boundary constraints become: $-\frac{1}{4} \le \frac{y}{x} \le \frac{2}{3}$ or $-\frac{1}{4} \le \tan(\theta) \le \frac{2}{3}$ or $-\tan^{-1}(\frac{1}{4}) \le \theta \le \tan^{-1}(\frac{2}{3})$.

The second set of boundary constraints become:

$$x^2 - y^2 = 2 \text{ or } r^2(\cos^2(\theta) - \sin^2(\theta)) = 2 \text{ or } r^2(\cos(2\theta)) = 2 \text{ and therefore } r = \sqrt{2}\big/\sqrt{\cos(2\theta)}$$

Similarly, for the outer limit of $r$ , we get $r = \sqrt{7}\big/\sqrt{\cos(2\theta)}$ .

The polar coordinate integral thus becomes: $\int_{-\tan^{-1}(1/4)}^{\tan^{-1}(2/3)} \int_{\sqrt{2}/\sqrt{\cos(2\theta)}}^{\sqrt{7}/\sqrt{\cos(2\theta)}} r^2 r\,dr\,d\theta$

$$\int_{-\tan^{-1}(1/4)}^{\tan^{-1}(2/3)} \int_{\sqrt{2}/\sqrt{\cos(2\theta)}}^{\sqrt{7}/\sqrt{\cos(2\theta)}} r^2 r\,dr\,d\theta = \int_{-\tan^{-1}(1/4)}^{\tan^{-1}(2/3)} d\theta \left[\frac{1}{4}r^4\right]_{\sqrt{2}/\sqrt{\cos(2\theta)}}^{\sqrt{7}/\sqrt{\cos(2\theta)}} = \frac{45}{4}\int_{-\tan^{-1}(1/4)}^{\tan^{-1}(2/3)} \sec^2(2\theta)\,d\theta$$

$$= \frac{45}{8}\left[\tan(2\theta)\right]_{-\tan^{-1}(1/4)}^{\tan^{-1}(2/3)} = \frac{45}{8}\left[\tan(2\tan^{-1}(2/3)) + \tan(2\tan^{-1}(1/4))\right]$$

Using the double angle formula for tangent, $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$ , yields

$$= \frac{45}{8}\left[\frac{2\times 2/3}{1 - 4/9} + \frac{2\times 1/4}{1 - 1/16}\right] = \frac{33}{2}$$

Note: Why didn't we use the Jacobian for this substitution? We did! The replacement of $dA = dx\,dy = r\,dr\,d\theta$ has the factor of $r$ that comes from the Jacobian for the polar substitution of variables.

For $x = r\cos(\theta)$ and $y = r\sin(\theta)$ the Jacobian is:

$$J = \left|\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}\right| = r\left(\cos^2(\theta) + \sin^2(\theta)\right) = r$$ which is exactly the extra $r$ that is required.