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Course: Linear Algebra I (Spring 2020)

Assignment: Section 4.6 Homework

1. Assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & 5 \\ 3 & 9 & -9 & 3 & -2 \\ -3 & -9 & 6 & 0 & 15 \\ -3 & -9 & 6 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -4 & 2 & 5 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank A = 3

dim Nul A = 2

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 15 \\ 0 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

A basis for Row A is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

A basis for Nul A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

2. Assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & -3 & -1 \\ 1 & 2 & -3 & 0 & -4 & 1 \\ 1 & -1 & 0 & 0 & 3 & 7 \\ 1 & 3 & -3 & 1 & -2 & 2 \\ 1 & -2 & 1 & 0 & -3 & -11 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & -2 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

rank A = 5

dim Nul A = 1

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 3 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 7 \\ 2 \\ -11 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

A basis for Row A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

A basis for Nul A is $\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

3. If a 6×4 matrix A has rank 3, find dim Nul A, dim Row A, and rank A^T.

dim Nul A = 1

dim Row A = 3

rank A^T = 3

4. Suppose a 5×7 matrix A has five pivot columns. Is Col A = $\mathbb{R}^5$? Is Nul A = $\mathbb{R}^2$? Explain your answers.

Is Col A = $\mathbb{R}^5$?

- **A.** No. Since A has five pivot columns, dim Col A is 5. Thus, Col A is a five-dimensional subspace of $\mathbb{R}^5$, so Col A is not equal to $\mathbb{R}^5$.
- **B.** No, the column space of A is not $\mathbb{R}^5$. Since A has five pivot columns, dim Col A is 0. Thus, Col A is equal to **0**.
- **C.** No, Col A is not $\mathbb{R}^5$. Since A has five pivot columns, dim Col A is 2. Thus, Col A is equal to $\mathbb{R}^2$.
- **D.** Yes. Since A has five pivot columns, dim Col A is 5. Thus, Col A is a five-dimensional subspace of $\mathbb{R}^5$, so Col A is equal to $\mathbb{R}^5$.

Is Nul A = $\mathbb{R}^2$?

- **A.** No, Nul A is not equal to $\mathbb{R}^2$. It is true that dim Nul A is equal to 2, but Nul A is a subspace of $\mathbb{R}^7$.
- **B.** No, Nul A is not equal to $\mathbb{R}^2$. Since A has five pivot columns, dim Nul A is equal to 5. Thus, Nul A is equal to $\mathbb{R}^5$.
- **C.** No, Nul A is equal to $\mathbb{R}^2$. Since A has five pivot columns, dim Nul A is equal to 0. Thus, Nul A is equal to **0**.
- **D.** Yes, Nul A is equal to $\mathbb{R}^2$. Since A has five pivot columns, dim Nul A is equal to 2. Thus, Nul A is equal to $\mathbb{R}^2$.

5. If the null space of a 6×9 matrix A is 8-dimensional, what is the dimension of the row space of A?

dim Row A = 1

6. If A is a 9×7 matrix, what is the largest possible rank of A? If A is a 7×9 matrix, what is the largest possible rank of A? Explain your answers.

Select the correct choice below and fill in the answer box(es) to complete your choice.

- **A.** The rank of A is equal to the number of columns of A. Since there are 7 columns in a 9×7 matrix, the largest possible rank of a 9×7 matrix is ____. Since there are 9 columns in a 7×9 matrix, the largest possible rank of a 7×9 matrix is ____.
- **B.** The rank of A is equal to the number of pivot positions in A. Since there are only 7 columns in a 9×7 matrix, and there are only 7 rows in a 7×9 matrix, there can be at most 7 pivot positions for either matrix. Therefore, the largest possible rank of either matrix is 7.
- **C.** The rank of A is equal to the number of non-pivot columns in A. Since there are more rows than columns in a 9×7 matrix, the rank of a 9×7 matrix must be equal to ____. Since there are 7 rows in a 7×9 matrix, there are a maximum of 7 pivot positions in A. Thus, there are 2 non-pivot columns. Therefore, the largest possible rank of a 7×9 matrix is ____.

7. If A is an 8×6 matrix, what is the largest possible dimension of the row space of A? If A is a 6×8 matrix, what is the largest possible dimension of the row space of A? Explain.

Select the correct choice below and fill in the answer box(es) to complete your choice.

A. The dimension of the row space of A is equal to the number of pivot positions in A. Since there are only 6 columns in an 8×6 matrix, and there are only 6 rows in a 6×8 matrix, there can be at most 6 pivot positions for either matrix. Therefore, the largest possible dimension of the row space of either matrix is 6.

B. The dimension of the row space of A is equal to the number of rows of A, which is equal to the number of pivot positions in A. Since there are 8 rows in an 8×6 matrix, the largest possible dimension of the row space of an 8×6 matrix is _____. Since there are 6 rows in a 6×8 matrix, the largest possible dimension of the row space of a 6×8 matrix is _____.

C. The dimension of the row space of A is equal to the number of non-pivot columns in A. Since there are more rows than columns in an 8×6 matrix, the dimension of the row space of an 8×6 matrix must equal _____. Since there are 6 rows in a 6×8 matrix, there are a maximum of 6 pivot positions in A and 2 non-pivot columns. Therefore, the largest possible dimension of the row space of a 6×8 matrix is _____.

8. For parts a. through e., A is an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

a. If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

Is this statement true or false?

- **A.** The statement is false. The columns of an echelon form B of A span the column space of A.
- **B.** The statement is true. The columns of an echelon form B of A are always in the column space of A.
- **C.** The statement is false. The columns of an echelon form B of A are often not in the column space of A.
- **D.** The statement is true. The columns of an echelon form B of A span the column space of A.

b. Row operations preserve the linear dependence relations among the rows of A.

Is this statement true or false?

- **A.** The statement is false because the rows of a matrix B that is an echelon form of A are linearly dependent.
- **B.** The statement is false. Row operations may change the linear dependence relations among the rows of A.
- **C.** The statement is true because the nonzero rows of a matrix B that is an echelon form of A are linearly independent.
- **D.** The statement is true. Row operations may change the linear dependence relations among the columns of A, but not the rows.

c. The dimension of the null space of A is the number of columns of A that are not pivot columns.

Is this statement true or false?

- **A.** The statement is false. The dimension of Nul A equals the number of pivot columns.
- **B.** The statement is true. The dimension of Nul A equals the number of columns of A minus the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
- **C.** The statement is false. The dimension of the column space of A is the number of columns of A that are not pivot columns.
- **D.** The statement is true. The dimension of Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

d. The row space of A^T is the same as the column space of A.

Is this statement true or false?

- **A.** The statement is true because the number of pivot columns of A^T is the same as the number of pivot columns of $(A^T)^T = A$.
- **B.** The statement is true because the rows of A^T are the columns of $(A^T)^T = A$.
- **C.** The statement is false because the rows of A^T are also the rows of $(A^T)^T = A$.
- **D.** The statement is false because the number of free variables in the equation $A^T\mathbf{x} = \mathbf{0}$ is the same as the number of pivot columns of $(A^T)^T = A$.

e. If A and B are row equivalent, then their row spaces are the same.

Is this statement true or false?

- **A.** The statement is true. If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A and vice-versa.
- **B.** The statement is false. If B is obtained from A by row operations, the columns of B are linear combinations of the columns of A and vice-versa.

C. The statement is true. If B is obtained from A by row operations, the columns of B are linear combinations of the rows of A and vice-versa.

D. The statement is false. If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A and vice-versa.

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9. Consider an $m \times n$ matrix A. Which of the subspaces Row A, Col A, Nul A, Row A^T, Col A^T, and Nul A^T are in $\mathbb{R}^m$ and which are in $\mathbb{R}^n$? How many distinct subspaces are in this list?

Select each subspace that is in $\mathbb{R}^m$.

- [ ] A. Nul A
- [ ] B. Col A^T
- [x] C. Row A^T
- [x] D. Col A
- [ ] E. Row A
- [x] F. Nul A^T

Select each subspace that is in $\mathbb{R}^n$.

- [x] A. Col A^T
- [x] B. Row A
- [ ] C. Col A
- [ ] D. Nul A^T
- [x] E. Nul A
- [ ] F. Row A^T

There are 4 distinct subspaces in the given list.

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10. Let A be an $m \times n$ matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Choose the correct answer below.

A. The system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$, or dim Col A = m. The equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if dim Nul A = 0. By the Rank Theorem, dim Col A = rank A = m − dim Nul A. Thus, dim Col A = m if and only if dim Nul A = 0.

(selected) B. The system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$, or dim Col A = m. The equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if dim Nul A^T = 0. Since Col A = Row A^T, dim Col A = dim Row A^T = rank A^T = m − dim Nul A^T by the Rank Theorem. Thus, dim Col A = m if and only if dim Nul A^T = 0.

C. The system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$, or dim Row A = m. The equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if dim Nul A^T = 0. Since Row A = Col A^T, dim Row A = dim Col A^T = m − dim Nul A^T by the Rank Theorem. Thus, dim Row A = m if and only if dim Nul A^T = 0.