Section 1.4: The Matrix Equation, $A\vec{x} = \vec{b}$:

Note: Here we view a linear combination of vectors as the product of a matrix & a vector ∵
(* simply rephrasing what we learned in 1.3)

*** Def:**

Let A be an $m \times n$ matrix, with columns

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$.

Let $\vec{x}$ be a vector in $\mathbb{R}^n$.

The product of matrix A & vector $\vec{x}$, $A\vec{x}$, is the linear combination of the columns of A using the corresponding entries in $\vec{x}$ as weights:

$$A\vec{x} = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Notes:

(i) $A\vec{x}$ is defined IFF the # of columns in A equals the # of entries in $\vec{x}$

(ii) Each entry in $A\vec{x}$ is the sum of products (or "Dot Product") using the corresponding row of A & entries of $\vec{x}$ ∵ more later!

Numerical Example: Find $A\vec{x}$ if:

$$A = \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \quad \& \quad \vec{x} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Answer:

* matrix A is a 2×3 matrix $\rightarrow$ 2 columns
* vector $\vec{x}$ is an $\mathbb{R}^2$ vector $\rightarrow$ 2 entries

✓

*Find the Product:

$$A\vec{x} = \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

*multiply column 1 of A w/ entry 1 of $\vec{x}$

*multiply colum 2 of A w/ entry 2 of $\vec{x}$

$$= 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix}$$

*take the sum of like terms

$$= \begin{bmatrix} 8-21 \\ 32+0 \\ -20+14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

$$\boxed{\therefore A\vec{x} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}}$$

Answer

Row-Vector Rule for Computing $A\vec{x}$

If the product $A\vec{x}$ is defined, then the i^{th} entry in $A\vec{x}$ is the sum of the products of corresponding entries from row "i" of A & from the vector $\vec{x}$

Note: This is sometimes called the 'Dot Product', like in multivariable calculus :)

Algebraic Example: Find $A\vec{x}$ if:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \& \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Answer:

Note: The first entry in the product $A\vec{x}$ is the sum of the products (i.e. the dot product :)) using the 1^{st} row of A & entries in $\vec{x}$:

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ \\ \\ \end{bmatrix}$$

*The other entries are found similarly:

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix}$$

$\boxtimes$

*Numerical Example (w/ a fun conclusion :)):

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

↓

Note: By definition, a matrix w/ "1s" along its diagonal & zeros elsewhere is called: The Identity Matrix, I.

Conclusion: $\forall \ \vec{x} \in \mathbb{R}^3, \ I\vec{x} = \vec{x}$.

(analogous, $\forall \ \vec{x} \in \mathbb{R}^n \Rightarrow I_n \vec{x} = \vec{x}$)

*Theorem:

Let A be an $m \times n$ matrix in $\mathbb{R}^n$.

Let $\vec{u}$ & $\vec{v}$ be vectors in $\mathbb{R}^n$.

Let $c \in \mathbb{R}$ be a scalar.

① $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

② $A(c\vec{u}) = c(A\vec{u})$

*Note: The proof for both prop. can be easily verified w/ vector arithmetic :)

Example: Compute the product using:

(a) the Def. where $A\vec{x}$ is the linear combination of the columns of A using the entries of $\vec{x}$ as weights.

(b) the row-vector rule for computing $A\vec{x}$

*If the product is undefined, explain why.

$$\begin{bmatrix} -3 & 5 \\ 3 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Answer:

Recall: The product $A\vec{x}$ $\exists$ IFF the # of columns of matrix A matches the # of entries in vector $\vec{x}$.

*The given matrix has 2 columns, but there are 3 entries in the given vector.

$\Rightarrow$ $\therefore$ The product is undefined.

Example: For $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^m$, write a linear combination $3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3$ as a matrix times a vector.

Answers:

* Recall: The product "$A\vec{x}$" is the linear combination of the columns of A using the corresponding entries in $\vec{x}$ as weights:

$$\Rightarrow A\vec{x} = [\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

* Given: $3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3$, ST $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^m$

* Rewrite the vector equation as a matrix eq:

$$3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3 = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$$

$$\boxed{\therefore A\vec{x} = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}}$$

Answer.

Important Conclusion

A system of linear equations may now be viewed in 3 different, but equivalent ways:

(i) As a matrix equation

(ii) As a vector equation

(iii) As a system of linear equations

*Theorem:

Let A be an $m \times n$ matrix, with columns $\vec{a}_1, \dots, \vec{a}_n$.

Let $\vec{b}$ be a vector in $\mathbb{R}^m$.

The following 3 forms have the SAME solution set:

① The matrix equation: $A\vec{x} = \vec{b}$

② The vector equation: $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$

③ The system of linear eq: (augmented matrix) $[\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n \mid \vec{b}]$

Note: The above 3 different, but equivalent forms are all solved the same way!

$\Rightarrow$ By row-reducing the augmented matrix :)

Example (of Equivalence): Consider the system of equations:

$$\begin{cases} 2x_1 - 3x_2 + 4x_3 = -5 \\ \qquad\quad 6x_2 - 7x_3 = 8 \end{cases}$$

*This system is equivalent to the following:

① **The Matrix Equation:**

$$\begin{bmatrix} 2 & -3 & 4 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\Updownarrow$$

② **The Vector Equation:**

$$x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$$

$$\Updownarrow$$

③ **The System of Linear Eq:** (whose augmented matrix)

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & -5 \\ 0 & 6 & -7 & 8 \end{array}\right]$$

Note: From this point forward, you are free to choose which ever form is your favorite!

⇒ Again, they are all solved the same exact way (row-reduce the augmented matrix to verify the system is consistent)

Example: Use the definition of $A\vec{x}$ to write the vector equation as a matrix equation:

$$x_1\begin{bmatrix}6\\-8\\-6\\4\end{bmatrix} + x_2\begin{bmatrix}-7\\-6\\-4\\-7\end{bmatrix} + x_3\begin{bmatrix}-3\\-1\\-4\\7\end{bmatrix} = \begin{bmatrix}1\\3\\7\\5\end{bmatrix}$$

Answer:

*Recall: $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n = \vec{b} \iff A\vec{x} = \vec{b}$

(vector equation) (matrix eq.)

$\forall\ \vec{x}, A, \vec{b} \in \mathbb{R}^n$

*Rewrite the given vector equation in matrix eq. Form:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = b \iff [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \vec{b}$$

$$\therefore [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \vec{b}$$

$$\boxed{\begin{bmatrix}6 & -7 & -3\\-8 & -6 & -1\\-6 & -4 & -4\\4 & -7 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\3\\7\\5\end{bmatrix}}$$ Ans.

Example: Write the system first as a vector equation & then as a matrix equation:

$$\begin{cases} 8x_1 + x_2 - 3x_3 = 6 \\ \quad\quad 9x_2 + 6x_3 = 0 \end{cases}$$

Answer:

Note: I am going to live dangerously :) & do it in reverse (matrix eq, then vector eq.)

⇒ Mostly b/c I find this conversion order more natural

* Matrix Equation, $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 8 & 1 & -3 \\ 0 & 9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

* Vector Equation, $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$:

$$x_1 \begin{bmatrix} 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Example: Write the following system first as a vector equation & then as a matrix equation:

$$\begin{cases} 5x_1 - x_2 = 2 \\ 6x_1 + 2x_2 = 4 \\ 9x_1 - x_2 = 1 \end{cases}$$

Answer:

* **Recall:** $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b} \iff [\vec{a}_1\ \vec{a}_2]\vec{x} = \vec{b}$

(Vector Eq.) (Matrix Eq.)

This time I will play be the rule :) ...

$\therefore$ Vector Equation Form:

$$\boxed{\begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}}$$

Answers

$\therefore$ Matrix Equation Form:

$$\boxed{\begin{bmatrix} 5 & -1 \\ 6 & 2 \\ 9 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}}$$

* Existence of Solutions *

The equation $A\vec{x} = \vec{b}$ has a solution IFF $\vec{b}$ is a linear combination of the columns of A.

Recall: (Section 1.3)

* A vector $\vec{b}$ is in the $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ if the vector eq $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{b}$ and/or linear system w/ the augmented matrix $[\vec{v}_1 \cdots \vec{v}_p \mid \vec{b}]$ has a solution.

* In particular, $\vec{b}$ can generated a linear combination of $\vec{v}_1, \dots \vec{v}_p$ IFF $\exists$ a solution to $[\vec{v}_1 \cdots \vec{v}_p \mid \vec{b}]$

Resulting Existence Questions:

(i) Is $A\vec{x} = \vec{b}$ consistent?

* If yes, the $A\vec{x} = \vec{b}$ has at least one solution & so $\vec{b}$ is a linear combination of the columns of A.

(ii) A harder existence problem is to determine whether the equation $A\vec{x} = \vec{b}$ is consistent $\forall$ possible $\vec{b}$. ... Lets take a look!

Example: Given matrix A & vector $\vec{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\vec{x} = \vec{b}$. Then solve the system & write the solution as a vector:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 2 \\ -3 & -4 & 3 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} 8 \\ 18 \\ -5 \end{bmatrix}$$

Answer:

Recall: $\forall$ $A, \vec{x}, \& \vec{b}$ in $\mathbb{R}^3$ the following are equivalent

$$A\vec{x} = \vec{b} \iff [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \mid \vec{b}]$$

* Write the augmented matrix:

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 8 \\ 2 & 4 & 2 & 18 \\ -3 & -4 & 3 & -5 \end{array}\right]$$

* Solve the System $\Rightarrow$ Row-Reduce the Augmented Matrix:

- $\dfrac{-2R_1 + R_2}{\text{new } R_2}$ $\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 8 \\ 0 & -2 & 6 & 2 \\ -3 & -4 & 3 & -5 \end{array}\right] \underset{(-\frac{1}{2})R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 8 \\ 0 & 1 & -3 & -1 \\ -3 & -4 & 3 & -5 \end{array}\right]$

Example (Continued...)

- $\begin{array}{l} 3R_1 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 8 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -3 & 19 \end{array}\right]$

- $\begin{array}{l} -5R_2 \\ + R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 8 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 12 & 24 \end{array}\right] \underset{(\frac{1}{12})R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 8 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

∴ solution ∃ :)

*Lets now find the row-reduced echelon form:

- $\begin{array}{l} -3R_2 \\ + R_1 \\ \hline \text{new } R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 7 & 11 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

- $\begin{array}{l} -7R_3 \\ + R_1 \\ \hline \text{new } R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

- $\begin{array}{l} 3R_3 \\ + R_2 \\ \hline \text{new } R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$

∴ Therefore:

$$\boxed{\vec{x} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}}$$

Answer

Example[2]: Given matrix A & vector $\vec{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\vec{x} = \vec{b}$. Then solve the system & write the solution as a vector:

$$A = \begin{bmatrix} 1 & 5 & -5 \\ -3 & -3 & 3 \\ 2 & 3 & 5 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} 1 \\ 9 \\ 19 \end{bmatrix}$$

Answer:

*Recall: $\forall\ A, \vec{x}, \vec{b} \in \mathbb{R}^3$, the following are equivalent:

$$A\vec{x} = \vec{b} \iff [\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \,\vdots\, \vec{b}]$$

*Write $A\vec{x} = \vec{b}$ as an augmented matrix:

$$[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \,\vdots\, \vec{b}] \iff \left[\begin{array}{ccc|c} 1 & 5 & -5 & 1 \\ -3 & -3 & 3 & 9 \\ 2 & 3 & 5 & 19 \end{array}\right]$$

*Solve the System → Row Reduce ↑ the echelon form:

- $\begin{array}{l} 3R_1 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & -5 & 1 \\ 0 & 12 & -12 & 12 \\ 2 & 3 & 5 & 19 \end{array}\right] \underset{(\frac{1}{12})R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 5 & -5 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & 5 & 19 \end{array}\right]$

- $\begin{array}{l} -2R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & -5 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 15 & 17 \end{array}\right]$

✓

Example[2] Continued...

- $\begin{array}{l} -5R_2 \\ +\ R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 15 & 17 \end{array}\right]$

- $\begin{array}{l} 7R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 8 & 24 \end{array}\right] \underset{\frac{1}{8}(R_3)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

  A Solution $\exists$:)

- $\begin{array}{l} R_3 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Therefore:

$$\boxed{\vec{x} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}}$$

Answer

Example (Existence of Solutions): Is $A\vec{x} = \vec{b}$ consistent for all possible b_1, b_2, b_3 if:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer:

*Note: $A\vec{x} = \vec{b}$ is consistent if at least one solution $\exists$.

*Rewrite $A\vec{x} = \vec{b}$ as its equivalent augmented matrix:

$$A\vec{x} = \vec{b} \iff [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \mid \vec{b}]$$

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right]$$

*Row-Reduce the augmented matrix:

- $\begin{array}{r} 4R_1 \\ +\ R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 2b_1 + \frac{b_2}{2} \\ -3 & -2 & -7 & b_3 \end{array}\right]$

$*\frac{R_2}{2} \uparrow$

Example (existence of solutions): Continued...

- $\begin{array}{l} 3R_1 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 2b_1 + \frac{b_2}{2} \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right]$

- $\begin{array}{l} -R_2 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 2b_1 + b_2/2 \\ 0 & 0 & 0 & 3b_1 + b_3 - 2b_1 - \frac{b_2}{2} \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 2b_1 + b_2/2 \\ 0 & 0 & 0 & b_1 - b_2/2 + b_3 \end{array}\right]$$

*

Caution: If $b_1 - \frac{b_2}{2} + b_3 \neq 0$, then a contradiction will be produced!

Therefore:

$\boxed{\text{* } A\vec{x} = \vec{b} \text{ is NOT consistent } \forall\ \vec{b} \text{ as some choices of } \vec{b} \text{ may/will produce a nonzero entry for the column.}}$

Note: $A\vec{x} = \vec{b}$ is consistent IFF the entries of $\vec{b}$ satisfy the equation $b_1 - \frac{b_2}{2} + b_3 = 0$:)

Existence #2

Example: For the given matrix A & vector $\vec{b}$, show that the equation $A\vec{x} = \vec{b}$ does not have a solution $\forall$ possible $\vec{b}$, & describe the set of all $\vec{b}$ for which $A\vec{x} = \vec{b}$ does have a solution:

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -4 & 4 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer:

* Write the product $A\vec{x} = \vec{b}$ in its equivalent augmented matrix form:

$$A\vec{x} = \vec{b} \iff \left[\vec{a}_1 \;\; \vec{a}_2 \;\; \vec{a}_3 \;\middle|\; \vec{b}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 4 & 0 & b_2 \\ 3 & -2 & 1 & b_3 \end{array}\right]$$

* Solve the System → Row-Reduce the Augmented Matrix:

$$\begin{array}{l} 4R_1 \\ + R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -4 & -4 & 4b_1 + b_2 \\ 3 & -2 & 1 & b_3 \end{array}\right] \underset{\left(-\frac{1}{4}\right)R_2}{\sim} \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 - \frac{b_2}{4} \\ 3 & -2 & 1 & b_3 \end{bmatrix}$$

Example Continued...

- $\begin{array}{r} -3R_1 \\ +\ R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 - \frac{b_2}{4} \\ 0 & 4 & 4 & -3b_1 + b_3 \end{array}\right]$

$$\underset{\left(\frac{1}{4}\right)R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 - \frac{b_2}{4} \\ 0 & 1 & 1 & -\frac{3b_1}{4} + \frac{b_3}{4} \end{array}\right]$$

- $\begin{array}{r} -R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 - \frac{b_2}{4} \\ 0 & 0 & 0 & -\frac{3b_1}{4} + \frac{b_3}{4} + b_1 + \frac{b_2}{4} \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -b_1 - \frac{b_2}{4} \\ 0 & 0 & 0 & \frac{b_1}{4} + \frac{b_2}{4} + \frac{b_3}{4} \end{array}\right]$$

Caution

Caution: If $\frac{b_1 + b_2 + b_3}{4} \neq 0$, then a contradiction is produced.

Therefore:

Answers

(i) $A\vec{x} = \vec{b}$ is NOT consistent $\forall\ \vec{b}$ as some choices of $\vec{b}$ will produce a nonzero entry for the column of $\vec{b}$

(ii) $A\vec{x} = \vec{b}$ is consistent IFF the entries of $\vec{b}$ satisfy the equation $\frac{b_1 + b_2 + b_3}{4} = 0 \leftrightarrow b_1 + b_2 + b_3 = 0$:)

*Theorem:

*This theorem is important
⇒ Continues to help throughout entire course.

Let A be an $m \times n$ matrix.

The following 4 statements are logically equivalent:

(IOW: They are all true -or- They are all false :))

① For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

$\Updownarrow$

② Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

$\Updownarrow$

③ The columns of A span $\mathbb{R}^m$

$\Updownarrow$

④ A has a pivot position in every row.

*Caution:

This theorem refers to a coefficient matrix, NOT an augmented matrix. If an augmented matrix $[A\ \vec{b}]$ has a pivot position in every row, then the equation $A\vec{x} = \vec{b}$ may or may not be consistent.

Example: For the following $\vec{v}$, does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$? Why or why not?

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 12 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 6 \\ -1 \\ -8 \end{bmatrix}$$

Answer:

* **Recall:** The columns of matrix A span $\mathbb{R}^m$ means that every vector $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

Furthermore, $\vec{b}$ is a linear combination of A if the equation $A\vec{x} = \vec{b}$ has a solution (i.e. A has a pivot in every row :))

⇒ The last theorem told us that the above 4 statements are logically equivalent (for a coeff. matrix); either ALL true or ALL false

* Write $\vec{v}$ as a matrix:

$$\vec{v} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 0 & 0 & 6 \\ 0 & -3 & -1 \\ -4 & 12 & -8 \end{bmatrix}$$

* Row-reduce the matrix to see if a solution $\exists$:

$$\begin{bmatrix} 0 & 0 & 6 \\ 0 & -3 & -1 \\ -4 & 12 & -8 \end{bmatrix} \sim \begin{bmatrix} -4 & 12 & -8 \\ 0 & -3 & -1 \\ 0 & 0 & 6 \end{bmatrix} \checkmark$$

Ans.

$\therefore$ Since a pivot $\exists$ in each row, a solution $\exists$ & $\vec{v}$ spans $\mathbb{R}^3$.

Example: Let $\vec{u} = \begin{bmatrix} 5 \\ -5 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}$, & $\vec{w} = \begin{bmatrix} -30 \\ 45 \\ -9 \end{bmatrix}$.

It can be shown that $-2\vec{u} - 5\vec{v} - \vec{w} = 0$.

Use this fact (& no row operations) to find x_1 & x_2 that satisfy the equation: $\begin{bmatrix} 5 & 4 \\ -5 & -7 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -30 \\ 45 \\ -9 \end{bmatrix}$

Answer:

Note: $\Rightarrow [\vec{u}\ \vec{v}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{w}$

* Given: $-2\vec{u} - 5\vec{v} - \vec{w} = 0 \Rightarrow -2\vec{u} - 5\vec{v} = \vec{w}$

* Rewrite the vector eq as a matrix eq:

$$-2\vec{u} - 5\vec{v} = \vec{w} \iff [\vec{u}\ \vec{v}] \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \vec{w}$$

$$\boxed{\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}}$$

Answer ✓

Example: Determine if the columns of the matrix span $\mathbb{R}^4$:

$$\begin{bmatrix} 10 & -8 & 1 & -17 & 9 \\ -8 & 5 & -5 & 8 & -6 \\ -6 & 12 & -5 & 13 & -9 \\ 3 & -2 & 8 & 3 & 14 \end{bmatrix}$$

Answer:

* Given: 4×5 matrix (A coefficient matrix)

- $m = 4$ rows
- $n = 5$ columns

Since the matrix is a 4×5 matrix, then A can have at most 4-pivots

$\therefore$ Since the given matrix can have a pivot in every row $(m = 4)$, then the columns of A span $\mathbb{R}^4$:)

Answer.

Notes to Self:

① The equation $A\vec{x} = \vec{b}$ is called: <u>Matrix Eq</u>

② A vector $\vec{b}$ is a linear combination of the columns of matrix A IFF $A\vec{x} = \vec{b}$ is consistent (2 statements are logically equivalent).

③ If the augmented matrix $[A \,\vdots\, \vec{b}]$ has a pivot in every row, the equation $A\vec{x} = \vec{b}$ may or may not be consistent.

* If pivot appears in the column for $\vec{b}$, then $A\vec{x} = \vec{b}$ is inconsistent.

④ 'Dot Product' Rule → The first entry in $A\vec{x}$ is the sum of the products of $\vec{x}$ & the first entry in each column of A.

⑤ If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, the eq. $A\vec{x} = \vec{b}$ is consistent $\forall\ \vec{b}$ in $\mathbb{R}^m$
(i.e. a solution $\exists\ \forall\ \vec{b}$ in $\mathbb{R}^m$)

⑥ If matrix A $(m \times n)$ & $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$ in $\mathbb{R}^m$ then A cannot have a pivot in every row (b/c $A\vec{x} = \vec{b}$ has no solution for some $\vec{b}$ in $\mathbb{R}^m$)

Notes to Self:

(7) The eq. $A\vec{x} = \vec{b}$ is consistent IFF $\vec{b}$ is a linear combination of the columns of A
(IOW: $\vec{b}$ is the set spanned by the columns of A)

(8) A linear combination of vectors can always be written in the For $A\vec{x}$ For a suitable matrix & vector (# of columns in A = # of entries in $\vec{x}$), where A is a matrix of the coeff. of the system of vectors

(9) If A is an $m \times n$ matrix w/ $m > n$, then A can have at most n-pivot positions, which is NOT enough to fill all m-rows.