① Let $B = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$ & $C = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$ be 2 bases of $\mathbb{R}^3$

(a) Find the Change of Coordinates Matrix from B to C.

<u>Answer</u>

*To Find $\underset{C \leftarrow B}{P}$: Row-reduce $\left[\vec{c}_1\ \vec{c}_2\ \vec{c}_3 \,\vdots\, \vec{b}_1\ \vec{b}_2\ \vec{b}_3\right]$ to RREF

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \underset{\text{N.}R_3}{\overset{-R_2 + R_3}{\sim}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \underset{\text{N.}R_2}{\overset{-R_3 + R_2}{\sim}}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Longrightarrow \boxed{\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

<u>Ans</u>

(b) Find the Change of Coordinates Matrix from C to B

<u>Answer:</u>

*To Find $\underset{B \leftarrow C}{P}$: Row-reduce $\left[\vec{b}_1\ \vec{b}_2\ \vec{b}_3 \,\vdots\, \vec{c}_1\ \vec{c}_2\ \vec{c}_3\right]$ to RREF

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-R_1 + R_3 \\ \text{N.}R_3}}{\overset{\substack{-R_1 + R_2 \\ \text{N.}R_2}}{\sim}} \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{array}\right] \overset{-R_2 \leftrightarrow -R_3}{\sim}$$

$$\underbrace{\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]}_{\text{Same augmented matrix as (a)}} \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\therefore \underset{B \leftarrow C}{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

Same augmented matrix as (a) :)

<u>Ans</u>

(c) Let $\vec{x} \in \mathbb{R}^3$ ST $[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\vec{x}$ & $[\vec{x}]_c$

Ans.

Given: $\vec{x} \in \mathbb{R}^3$ ST $[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\Rightarrow \vec{x} = 1\vec{b}_1 + 2\vec{b}_2 + 3\vec{b}_3 = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

***Find $\vec{x}$**

$$\vec{x} = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3 \\ 1+2+0 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} \qquad \boxed{\therefore \vec{x} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}}$$ Ans.

***Find $[\vec{x}]_c$:** (*Not an exclusive solution ∵ See next pg.)

$$\Rightarrow \vec{x} = [P_c][\vec{x}]_c = [\vec{c}_1 \; \vec{c}_2 \; \vec{c}_3][\vec{x}]_c$$

$\Rightarrow$ Row-reduce $[\vec{c}_1 \; \vec{c}_2 \; \vec{c}_3 \,\vdots\, \vec{x}]$ to Find $[\vec{x}]_c$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\text{N.}R_1}{\overset{-R_2+R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\text{N.}R_2}{\overset{-R_3+R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{\therefore [\vec{x}]_c = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$ Ans

*Alternative Solution: A quicker method :)

* Find $[\vec{x}]_C$:

* Recall (By Def): $[\vec{x}]_C = \underset{C \leftarrow B}{P} [\vec{x}]_B$

$$[\vec{x}]_C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+0+3 \\ 0+2+0 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\therefore [\vec{x}]_C = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

Ans