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Final Exam

Q1

a) $\dim[Nul(T)] = 2$

$\Rightarrow \dim[rank(T)] = n - \dim[Nul(T)]$

$= 4 - 2 = \boxed{2}$

$\therefore \dim[rank(T)] = 2$

b) $\dim[rank(T)] = 3$

$\Rightarrow \dim[Nul(T)] = n - \dim[rank(T)]$

$= 4 - 3 = \boxed{1}$

$\therefore \dim[Nul(T)] = 1$

Q3

$p_1(t) = 1 \rightarrow [\vec{p_1}]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$p_2(t) = -2 + 4t^2 \longrightarrow [p_2]_\beta = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$p_3(t) = 2t \longrightarrow [p_3]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$p_4(t) = -12t + 8t^3 \longrightarrow [p_4]_\beta = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

$$[P_\beta \mid \vec{0}] = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -12 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\xrightarrow[\frac{R_4}{8} = nR_4]{\frac{R_2}{2} = nR_2,\ \frac{R_3}{4} = nR_3} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the matrix has 4 pivots, the columns of the matrix are linearly Independent $\Rightarrow \therefore$ Polynomials are <u>NOT</u> linearly Dependent

$\boxed{Q4}$ $B = \{t-1,\ t+1,\ t^2-1\}$ $\quad C = \{1,\ t+1,\ t^2+t\}$

$$\Rightarrow B = \left\{ \underset{b_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{b_3}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

a)

$$\underset{C \leftarrow B}{P} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - R_2 = nR_1]{R_2 - R_3 = nR_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_3 = nR_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{\Rightarrow \therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

b)

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 5 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_1 = nR_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -5 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_1+R_2=nR_2}\left[\begin{array}{ccc|c}1 & -1 & 1 & -5\\ 0 & 2 & -1 & 4\\ 0 & 0 & 1 & 1\end{array}\right]\xrightarrow{2R_1+R_2=nR_1}\left[\begin{array}{ccc|c}2 & 0 & 1 & -6\\ 0 & 2 & -1 & 4\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\xrightarrow[\substack{R_2+R_3\\=nR_2}]{\substack{R_1-R_3\\=nR_1}}\left[\begin{array}{ccc|c}2 & 0 & 0 & -7\\ 0 & 2 & 0 & 5\\ 0 & 0 & 1 & 1\end{array}\right]\xrightarrow[\frac{1}{2}R_2=nR_2]{\frac{1}{2}R_1=nR_1}\left[\begin{array}{ccc|c}1 & 0 & 0 & -\frac{7}{2}\\ 0 & 1 & 0 & \frac{5}{2}\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\boxed{\therefore\ [\vec{p}]_B = \begin{bmatrix}-7/2\\ 5/2\\ 1\end{bmatrix}}$$

$\boxed{Q5}$ $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - x_3 + x_4,\ 2x_1 + x_2 + 4x_3 + x_4,\ 3x_1 + x_2 + 9x_3)$

$$\Rightarrow T = \begin{bmatrix}1 & 1 & -1 & 1\\ 2 & 1 & 4 & 1\\ 3 & 1 & 9 & 0\end{bmatrix}\xrightarrow[\substack{3R_1-R_3\\=nR_3}]{\substack{2R_1-R_2\\=nR_2}}\begin{bmatrix}1 & 1 & -1 & 1\\ 0 & 1 & -6 & 1\\ 0 & 2 & -12 & 3\end{bmatrix}$$

$$\xrightarrow[R_3-2R_2=nR_3]{R_1-R_2=nR_1}\begin{bmatrix}\underline{1} & 0 & 5 & 0\\ 0 & \underline{1} & -6 & 1\\ 0 & 0 & 0 & \underline{1}\end{bmatrix}$$

a) Basis for the column space of T

$$\boxed{= \beta_c = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}}$$

b) Basis for the Row space of T

$$= B_R = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c) $[T \,\vdots\, \vec{0}\,]$

$$T \to \begin{bmatrix} \underline{1} & 0 & 5 & 0 \\ 0 & \underline{1} & -6 & 1 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix} \Rightarrow \begin{cases} x_1 = -5x_3 \\ x_2 = 6x_3 - x_4 = 6x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$$

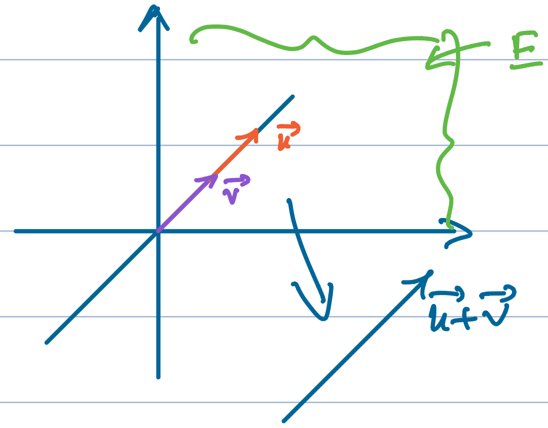
$$\Rightarrow T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \boxed{\text{Basis for Nul}\,T = \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix}}$$

d) $\text{rank}(T) = \boxed{3}$

$$\dim \text{Nul}(T) = n - \text{rank}(T) = 4 - 3 = \boxed{1}$$

$\boxed{Q_2}$

a) $E = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x \right\} =$



Since $\vec{u} = c\vec{v}$

$\Rightarrow \vec{u} + \vec{v} = c\vec{v} + \vec{v} = (c+1)\vec{v} \in E$

$\boxed{\Rightarrow \therefore E \text{ is a vector space}}$

b) $R = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; y \geq 0 \right\}$

Example: $\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in R$

Assume $c = -2$

$\Rightarrow c\vec{u} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \notin R$ (b/c $y < 0$)

$\boxed{\Rightarrow \therefore R \text{ is } \underline{\underline{\text{NOT}}} \text{ a vector space.}}$

$Y = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x + 1 \right\}$

Example: $\vec{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ Let $c = 2 \Rightarrow \vec{v} = 2\vec{u} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \end{bmatrix} \notin Y : y = x + 1$

$\boxed{\Rightarrow \therefore Y \text{ is NOT a vector space}}$