1.3 & 1.4

Recall: solve this problem

$$\begin{cases} x_1 + 2x_2 + 4x_3 = -8 \\ 2x_1 + 4x_2 + 5x_3 = -10 \\ 4x_1 + 5x_2 + 4x_3 = -11 \end{cases} \Rightarrow \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = -2 \end{cases}$$

* Rewrite the Linear System as → Vector Equation → Matrix Eq

$$\begin{bmatrix} x_1 \\ 2x_1 \\ 4x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 4x_2 \\ 5x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 5x_3 \\ 4x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ -11 \end{bmatrix} \longrightarrow \underbrace{x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ -11 \end{bmatrix}}_{\text{(* Vector Eq)}}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \\ 4 & 5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -8 \\ -10 \\ -11 \end{bmatrix}}_{\vec{b}}$$

* the Matrix Eq

where
- A: coefficient Matrix
- $\vec{x}$: the solution vector (vector of unknowns)
- $\vec{b}$: constant vector (RHS)

* Check substitute

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$ into the matrix eq to verify $A\vec{x} \overset{?}{=} b$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} -8 \\ -10 \\ -11 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ -11 \end{bmatrix}$$

Sections 1.3 & 1.4 (Vector Eq and Matrix Equation)

Example in $\mathbb{R}^2$ (2 Equations / 2 Unknowns)

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 = 3 \end{cases}$$

Breakdown the system:

* Coefficient Matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

* Augmented Matrix $\left[\begin{array}{cc:c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]$

* Solution Vector

  $\vec{x} \in \mathbb{R}^2$ ST $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

* 3 Equivalent Ways to write the system

- Linear Eq: $\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 = 3 \end{cases}$

- Vector Equation: $x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

  $x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$

- Matrix Equation:

$$\underset{A}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

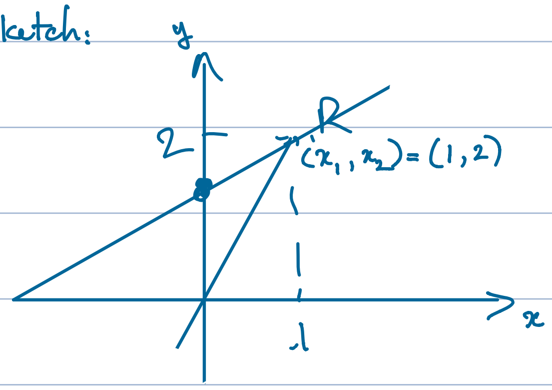
$$[\vec{a}_1 \ \vec{a}_2]\, \vec{x} = \vec{b}$$

* <u>Consider the "Row Picture"</u>: Look accross a row of the system, sketching one eq. @ a time

$R_1:\ 2x_1 - x_2 = 0 \longleftrightarrow 2x - y = 0 \longrightarrow y = 2x$

$R_2:\ -x_1 + 2x_2 = 3 \longleftrightarrow -x + 2y = 3 \longrightarrow y = \frac{1}{2}x + \frac{3}{2}$

* Sketch:

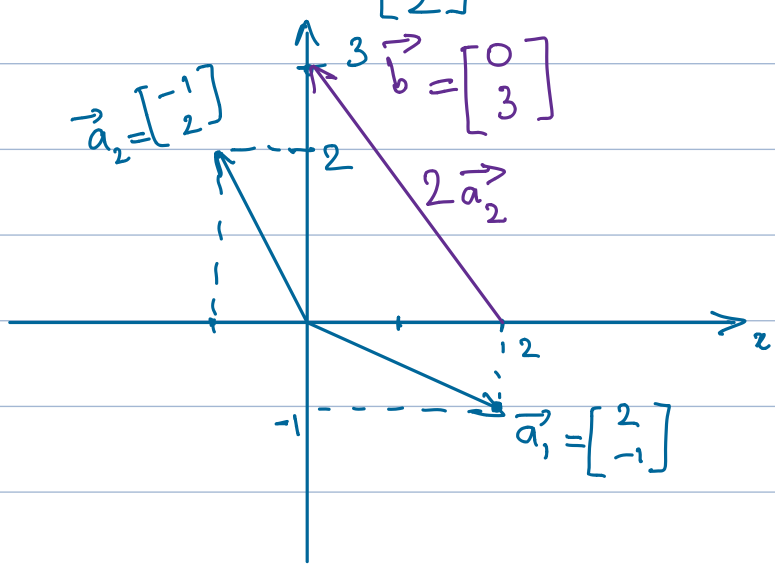


* Consider the "Column Picture": Asks us: Find the correct combination of the columns of matrix A ST they are equal to $\vec{b}$

• Vector Eq:

$$\underbrace{x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\text{Linear Combination of}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

* Hint: We already know that the correct combo. is "1" of $\vec{a}_1$ and "2" $\vec{a}_2$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Example in $\mathbb{R}^3$ (3 eq , 3 unknowns) $\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ -3x_2 + 4x_3 = 0 \end{cases}$ * Linear eq

* Vector Equation:

$$x_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}}_{\vec{a}_2} + x_3 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}_{\vec{a}_3} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}_{\vec{b}}$$

* Matrix Eq :

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}_{\vec{b}}$$

where $\vec{A} = [\vec{a}_1 \ \ \vec{a}_2 \ \ \vec{a}_3]$

* Augmented Matrix

$$[A \,\vdots\, \vec{b}] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -3 & 4 & 4 \end{array}\right]$$

* To find the general solution ⇒ Row reduce A to RREF

* The Row Picture ($\mathbb{R}^3$ ⇒ Equations are Planes)
* Consider the Column Picture

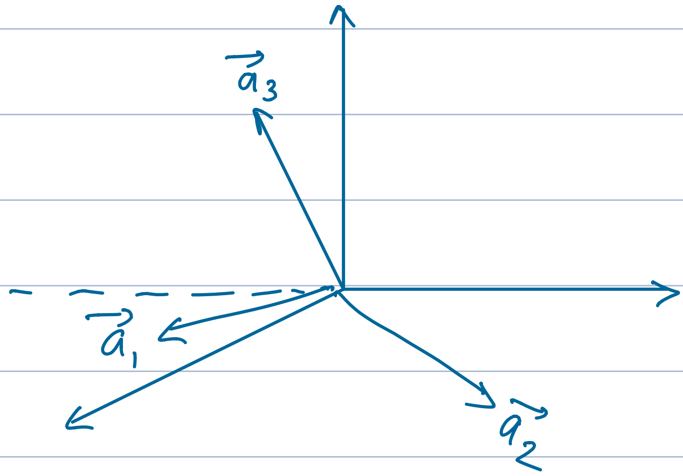
$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

* Find the correct combo. of the columns of A to produce $\vec{b}$

$$\Rightarrow 0\begin{bmatrix}2\\-1\\0\end{bmatrix} + 0\begin{bmatrix}-1\\2\\3\end{bmatrix} + 1\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

* Graphically



Ex Extension: Same A (coeff. matrix), but change $\vec{b} = \begin{bmatrix}1\\1\\-3\end{bmatrix}$

$$\Rightarrow \vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

* $\vec{a}_1 + \vec{a}_2 = \vec{b}$

* Conclusion Questions: Does this ALWAYS WORK?

$\forall \vec{b} \in \mathbb{R}^3$, can we find a Linear Combo. of col. of A to produce $\vec{b}$?

* Follow up questions to Ex in $\mathbb{R}^3$

Q: For any $\vec{b} \in \mathbb{R}^3$, can we always find a linear combination of the column vectors of A to produce $\vec{b}$?

<u>Alternative way:</u> Do the column vectors of A span $\mathbb{R}^3$

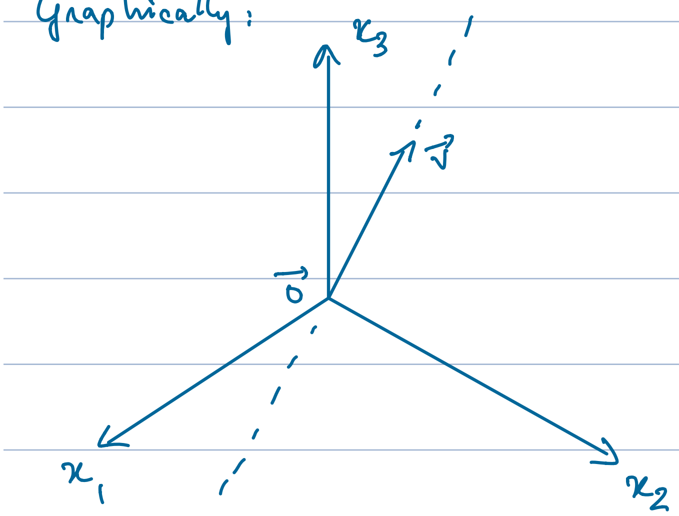
Span: (of a set of vectors)

$\rightarrow$ the collection of all linear combinations of the vectors

* <u>Geometric Interpretation in $\mathbb{R}^3$:</u> Let $\vec{u}, \vec{v} \in \mathbb{R}^3$

① span $\{\vec{v}\}$ The collection of all scalar multiples of $\vec{v}$

Graphically:

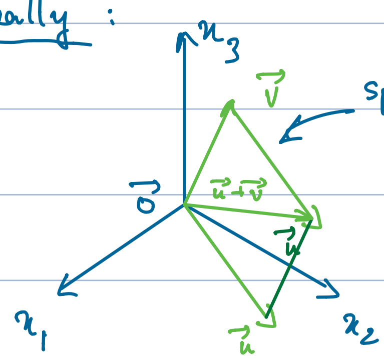


* span $\{\vec{v}\}$ is a line passing through $\vec{v}$ ~~&~~ $\vec{0}$

② span $\{\vec{v}, \vec{u}\}$ the collection of all linear combinations of $\vec{u}$ & $\vec{v}$

$(c_1 \vec{v} + c_2 \vec{u}$ ST $c_1, c_2 \in \mathbb{R})$

<u>Graphically</u>:



span $\{\vec{v}, \vec{u}\}$ is a plane.

<u>Strategy:</u> Determine if $\vec{b}$ spans a set of vectors

① Create a matrix A ST the vectors are the columns of A

② Row-reduce the augmented matrix $[\vec{A} \,\vdots\, \vec{b}]$ to Echelon Form

i) IF we attain E.F.: System is consistent $\Rightarrow \vec{b}$ can be written as a linear combination of the vectors

$\therefore \vec{b}$ spans the vectors.

ii) otherwise, $\vec{b}$ does NOT span the vectors

* **Example:** Consider $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$

Midterm Summer '19

(a) Determine if $\vec{b}$ spans $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (Can I row-reduce $[A \vdots \vec{b}]$ to Echelon form ST $A = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3]$]

(b) If it does, write $\vec{b}$ as a linear combination of the vectors.

(a):

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ -1 & -1 & -1 & -4 \\ 0 & -1 & -3 & -7 \end{array}\right] \xrightarrow[nR_2]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & -3 & 2 & 1 \\ 0 & -1 & -3 & 7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[nR_3]{3R_1 + R_3} \left[\begin{array}{ccc|c} \underline{\underline{1}} & -2 & 3 & 5 \\ 0 & \underline{\underline{1}} & 3 & 7 \\ 0 & 0 & \underline{\underline{11}} & 22 \end{array}\right]$$

Echelon form

* $\boxed{\text{System is consistent}} \Rightarrow \vec{b}$ can be written as a Linear Combo of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

(b): Continue Row-Reduce to R.R.E.F.:

$$\frac{1}{11}R_3 \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[nR_1]{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{c} -9R_3 + R_1 \\ \overline{nR_1} \\ -3R_3 + R_2 \\ \overline{nR_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} \bullet\, x_1 = 1 \\ \bullet\, x_2 = 1 \\ \bullet\, x_3 = 2 \end{array} \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

* Linear Combination: $\vec{b} \boxed{= 1\vec{v}_1 + 1\vec{v}_2 + 2\vec{v}_3}$ ← The column Picture

* Checking your Solution ($\vec{x}$) 2 methods

① Column-by-column → Saw this at the start of 1.3//Lone

② The Row-Column Rule (aka The Dot Product) Multiply each row of A by $\vec{x}$

* Check solution For previous ex: $A\vec{x} \stackrel{?}{=} \vec{b} \Rightarrow$

$$\Rightarrow \underset{A}{\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} \stackrel{?}{=} \underset{\vec{b}}{\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \checkmark$$

* Equivalence Thm: Let A be an $m \times n$ matrix. The following 4 statements are logically equivalent (either all TRUE or all FALSE)

① The columns of A span $\mathbb{R}^m$

$\Updownarrow$

② Each $\vec{b} \in \mathbb{R}^m$ can be written as a linear Combination of the col. of A

③ $\forall\, \vec{b} \in \mathbb{R}^m$, the matrix equation has solution

$\Updownarrow$

④ A pivot position exists in every row

* Example: Consider

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \;\&\; \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) Solve the problem

(b) Is it possible to solve $A\vec{x} = \vec{b}$ for ANY $\vec{b} \in \mathbb{R}^3$

(c) Describe the set of all $\vec{b}$ ST $A\vec{x} = \vec{b}$ does have a solution

Ans: (a) Row-reduce $[A \,\vdots\, \vec{b}] = [A \,\vdots\, \vec{0}] = [A]$ to R.R.E.F,

Only works b/c $\vec{b} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \xrightarrow[\substack{-R_1 + R_3 \\ nR_3}]{\substack{-R_1 + R_2 \\ nR_2}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow[\substack{R_2 + R_3 \\ nR_3}]{\substack{-2R_2 + R_1 \\ nR_1}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

* RREF

$$\boxed{\Rightarrow \begin{cases} \bullet\ x_1 = -3x_3 \\ \bullet\ x_2 = x_3 \\ \bullet\ x_3 \text{ is free variable}, x_3 \in \mathbb{R} \end{cases}}$$

(b) Does this work $\forall\, \vec{b} \in \mathbb{R}^3$? $\Rightarrow$ NO! RREF only has a pivot in R_1 & R_2

By the Equivalence Thm: $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ if a pivot position $\exists$ in each row.

(c) Find $\vec{b}$ ST $A\vec{x} = \vec{b}$ has a solution: Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be same arbitary vector

Row-reduce $[A \,\vdots\, \vec{b}]$ to RREF

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 1 & 3 & 0 & b_2 \\ 1 & 1 & 2 & b_3 \end{array}\right] \begin{array}{c} -R_1+R_2 \\ \longrightarrow \\ -R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & -1 & 1 & b_3 - b_1 \end{array}\right] \xrightarrow[\substack{R_2 + R_3 \\ nR_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right]$$

$\therefore$ $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^3$ when $b_3 + b_2 - 2b_1 = 0$

1.3.11

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ -4 & 1 & -6 & -3 \\ 0 & 4 & 40 & 36 \end{array}\right] \xrightarrow[=nR_2]{4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 10 & 9 \\ 0 & 4 & 40 & 36 \end{array}\right] \xrightarrow{\frac{R_3}{4}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 10 & 9 \\ 0 & 1 & 10 & 9 \end{array}\right]$$

$$\xrightarrow[nR_3]{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 10 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

where the last column is $\vec{b}$.

consistent $\longrightarrow$ Vector $\vec{b}$ is a linear combination of a_1, a_2 & a_3

1.3.12 $A\vec{x} = \vec{b}$

$$\left[\begin{array}{ccc|c} 1 & -5 & -4 & 13 \\ 0 & 3 & 9 & -5 \\ 1 & -5 & 1 & 8 \end{array}\right] \xrightarrow[\frac{R_1-R_3}{nR_3}]{\frac{R_2}{3}=nR_2} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 13 \\ 0 & 1 & 3 & -5/3 \\ 0 & 0 & -5 & 5 \end{array}\right] \xrightarrow[=nR_3]{\frac{R_3}{-5}} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 13 \\ 0 & 1 & 3 & -5/3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

A has pivot at each row $\rightarrow$ the system is consistent $\rightarrow$ $\vec{b}$ is a combonation of the linear System.

1.3.13

$$A = \left[\begin{array}{ccc|c} 1 & -5 & 3 & 4 \\ 0 & 4 & 7 & -7 \\ -3 & 15 & -9 & -4 \end{array}\right] \xrightarrow[\frac{R_2}{4}=nR_2]{\frac{3R_1+R_3}{nR_3}} \left[\begin{array}{ccc|c} 1 & -5 & 3 & 4 \\ 0 & 1 & 7/4 & -7/4 \\ 0 & 0 & 0 & 8! \end{array}\right]$$

no solution

$\rightarrow$ The system is inconsistent

$\rightarrow$ b is a NOT ✓ linear combo. of vectors formed from A

1.3.17 $[A \vdots b]$

$$\rightarrow \begin{bmatrix} 1 & -6 & 5 \\ 3 & -11 & -13 \\ -1 & 2 & h \end{bmatrix} \xrightarrow[\frac{R_1+R_3}{nR_3}]{\frac{3R_1-R_2}{nR_2}} \begin{bmatrix} 1 & -6 & 5 \\ 0 & -7 & 28 \\ 0 & -4 & 5+h \end{bmatrix} \xrightarrow[\frac{R_3}{-4}]{R_2/-7} \begin{bmatrix} 1 & -6 & 5 \\ 0 & 1 & -4 \\ 0 & 1 & \frac{5+h}{-4} \end{bmatrix} \xrightarrow{\frac{R_2-R_3}{nR_3}}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -6 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & -4+\frac{5+h}{4} \end{array}\right] \longrightarrow$$

Need to find h ST equation 3 is still TRUE

so, $-4 + \dfrac{5+h}{4} = 0 \Rightarrow 5+h = 16$

$\Rightarrow \boxed{h = 11}$

$\boxed{1.3.25}$ a) Asking $\vec{b}$ in $\{a_1, a_2, a_3\}$ is to check whether $c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3} = \vec{b}$

$$A = \left[\begin{array}{ccc|c} 1 & 0 & -7 & 4 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 3 & -4 \end{array}\right] \text{ is consistent? } \xrightarrow[nR_3]{3R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 4 \\ 0 & 2 & -3 & 1 \\ 0 & 4 & -18 & 8 \end{array}\right] \xrightarrow{\frac{R_3}{2}} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 4 \\ 0 & 2 & -3 & 1 \\ 0 & 2 & -9 & 4 \end{array}\right]$$

$$\xrightarrow[nR_3]{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 4 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 6 & -3 \end{array}\right]$$

A has pivot at 3 lines

→ System is consistent

→ $\vec{b}$ is in $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$

$\boxed{1.3.31}$ a) Balance point $= \frac{1}{3}\left(v_{1x} + v_{2x} + v_{3x}, v_{1y} + v_{2y} + v_{3y}\right) = \left(\frac{14}{3}, \frac{8}{3}\right)$

b) Since 6 grams are added to the plate → the total weight is now 9g

$\boxed{1.4.9}$

$$\begin{cases} 9x_1 + x_2 - 3x_3 = 9 \\ 9x_2 + 4x_3 = 0 \end{cases}$$

* Vector equation form:

$$\begin{bmatrix} 9 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 9 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 4 \end{bmatrix} x_3 = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

* Matrix Equation form:

$$\begin{bmatrix} 9 & 1 & -3 \\ 0 & 9 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$\boxed{1.4.10}$

$$\begin{cases} 2x_1 - x_2 = 5 \\ 8x_1 + 3x_2 = 4 \\ 5x_1 - x_2 = 1 \end{cases}$$

* Vector Equation form:

$$\begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} x_2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

* Matrix Equation form:

$$\begin{bmatrix} 2 & -1 \\ 8 & 3 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

$\boxed{1.4.11}$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -8 \\ 1 & 5 & 2 & 10 \\ 4 & 2 & 4 & -26 \end{array}\right] \xrightarrow[4R_1 - R_3 = nR_3]{R_1 - R_2 = nR_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & -8 \\ 0 & -2 & -4 & -18 \\ 0 & 10 & -12 & -6 \end{array}\right] \xrightarrow{\frac{R_2}{-2} = nR_2}$$

$$\begin{bmatrix} 1 & 3 & -2 & -8 \\ 0 & 1 & 2 & 9 \\ 0 & 10 & -12 & -6 \end{bmatrix} \xrightarrow{10R_2 - R_3 = nR_3} \left[\begin{array}{ccc|c} 1 & 3 & -2 & -8 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 32 & 96 \end{array}\right] \xrightarrow{\frac{R_3}{32} = nR_3} \begin{bmatrix} 1 & 3 & -2 & -8 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow[R_2 - 2R_3 = nR_2]{} \begin{bmatrix} 1 & 3 & -2 & -8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 - 3R_2 = nR_2} \begin{bmatrix} 1 & 0 & -2 & -17 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[nR_3]{R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 11 \\ 3 \\ 13 \end{bmatrix}$$

$\boxed{1.4.12}$

$$\left[\begin{array}{ccc|c} 1 & 6 & -7 & -14 \\ -4 & -2 & 6 & 56 \\ 2 & 3 & 3 & -4 \end{array}\right] \xrightarrow[2R_1 - R_3 = nR_3]{4R_1 + R_2 = nR_2} \begin{bmatrix} 1 & 6 & -7 & -14 \\ 0 & 22 & -22 & 0 \\ 0 & 9 & -17 & -24 \end{bmatrix} \xrightarrow{\frac{R_2}{22}} \begin{bmatrix} 1 & 6 & -7 & -14 \\ 0 & 1 & -1 & 0 \\ 0 & 9 & -17 & -24 \end{bmatrix}$$

$$\xrightarrow{9R_2 - R_3 = nR_3} \begin{bmatrix} 1 & 6 & -7 & -14 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 24 \end{bmatrix} \xrightarrow{R_3/3} \begin{bmatrix} 1 & 6 & -7 & -14 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 + R_3 = nR_2}$$

$$\begin{bmatrix} 1 & 6 & -7 & -14 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 - 6R_2 + 7R_3 = nR_1} \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 3 \end{bmatrix}$$

$\boxed{1.4.22}$ $\begin{bmatrix} 0 & 0 & 5 \\ 0 & -3 & -2 \\ -5 & 10 & -15 \end{bmatrix} = \begin{bmatrix} -5 & 10 & -15 \\ 0 & -3 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[R_2/-3,\ R_3/5]{R_1/-5} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$ Matrix A has a pivot in each line So, $\vec{v}$ spans $\mathbb{R}^3$

$\boxed{1.4.26}$

$\begin{bmatrix} -4 & 5 \\ 3 & -3 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\begin{matrix} -4 & 5 \end{matrix}$

$\begin{bmatrix} 3 & -3 \\ -3 & -6 \end{bmatrix}$ $\begin{matrix} 3 & -3 & 30 \\ -3 & -6 & 15 \end{matrix}$ $\begin{matrix} 1 & -1 & 10 \\ +1 & 2 & 5 \end{matrix}$ $\rightarrow$ $\begin{matrix} 1 & -10 & 10 \\ 0 & -3 & 5 \end{matrix}$

$\begin{cases} 3x_1 - 3x_2 = 30 \\ -3x_1 - 6x_2 = 15 \end{cases} \longrightarrow$ $-9x_2 = 45 \longrightarrow x_2 = 5$

$\rightarrow x_1 = \dfrac{10 + x_2}{1} = 15$