

4.6.2

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & 5 \\ 3 & 9 & -9 & 3 & -2 \\ -3 & -9 & 6 & 0 & 15 \\ -3 & -9 & 6 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & -4 & 2 & 5 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $A \sim B$ has 3 pivot positions

$$\Rightarrow \text{rank}(A) = \dim[\text{col}(A)] = \boxed{3}$$

$$\dim \text{Nul } A = n - \text{rank } A = 5 - 3 = \boxed{2}$$

• Basis for $\text{col } A$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 15 \\ 0 \end{bmatrix} \right\}$$

• Basis for $\text{Row } A$ is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \right\}$$

• Basis for Nul A: $[B: \vec{0}]$

$$\begin{bmatrix} 1 & 3 & -4 & 2 & 5 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{4R_2 + R_1 \\ = nR_1 \\ R_3 / -2}} \begin{bmatrix} 1 & 3 & 0 & -2 & -15 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} R_1 + 15R_3 \\ = nR_1 \\ R_2 + 5R_3 \\ = nR_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -3x_2 + 2x_4 \\ x_3 = x_4 \\ x_5 = 0 \\ x_2, x_4: \text{free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{x}_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \vec{x}_4 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ Basis for Nul A}$$

4.6.4:

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & -3 & -1 \\ 1 & 2 & -3 & 0 & -4 & 1 \\ 1 & -1 & 0 & 0 & 3 & 7 \\ 1 & 3 & -3 & 1 & -2 & 2 \\ 1 & -2 & 1 & 0 & -3 & -11 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & -2 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

• • $\text{rank}(A) = \dim[\text{Col}(A)] = \boxed{5}$

$$\therefore \dim[\text{Nul}(A)] = n - \text{rank}(A) = 6 - 5 = \boxed{1}$$

$$\therefore \text{Basis for Col } A \text{ is: } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 7 \\ 2 \end{bmatrix} \right\}$$

$$\therefore \text{Basis for Row } A \text{ is } \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\therefore \text{Basis for Nul } A : [B; \vec{0}]$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l}
 R_4 - 3R_5 = nR_4 \\
 R_3 - 3nR_4 = nR_3 \\
 \longrightarrow \\
 R_2 + nR_3 = nR_2 \\
 R_1 - nR_2 = nR_1
 \end{array}
 \left[\begin{array}{cccccc}
 1 & 0 & -2 & -1 & -2 & -2 \\
 0 & 1 & 0 & 1 & -1 & 1 \\
 0 & 0 & 1 & 1 & 0 & -1 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right]$$

$$\begin{array}{l}
 R_3 + R_5 = nR_3 \\
 R_2 + R_4 - R_5 = nR_2 \\
 \longrightarrow \\
 R_1 + 2R_4 + 2R_5 = nR_1
 \end{array}
 \left[\begin{array}{cccccc}
 1 & 0 & -2 & -1 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right]$$

$$\begin{array}{l}
 R_1 + 2R_3 = nR_1 \\
 \longrightarrow
 \end{array}
 \left[\begin{array}{cccccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right]
 \begin{array}{l}
 x_1 = -x_4 \\
 x_2 = -x_4 \\
 x_3 = -x_4 \\
 x_4 : \text{free} \\
 x_5 = 0 \\
 x_6 = 0
 \end{array}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{4.6.6} \quad []_{6 \times 4}, \text{ rank} = 3, \Rightarrow \dim \text{Col}(A) = \text{rank} = 3$$

$$\dim \text{Null } A = 4 - 3 = \boxed{1}$$

$$\dim \text{Row } A = 6 - 3 = \boxed{3}$$

$$\text{rank}(A^T) = \dim \text{Row } A = \boxed{3}$$

$$\boxed{4.6.11} \quad A: []_{6 \times 9}, \dim \text{Null}(A) = 8 \Rightarrow \text{rank} = n - \dim \text{Null}(A) \\ \Rightarrow \text{rank} = 9 - 8 = 1$$

$$\dim \text{Row } A = 9 - 8 = \boxed{1}$$

$$= n - \dim \text{Null}$$