1.5 SOLUTION SET OF LINEAR SYSTEMS

* Def: Let A be some $m \times n$ matrix
the Linear System is called "Homogeneous" if it can be written
$A\vec{x} = \vec{0}$

* 2 types of Solutions to Homogeneous Systems:

① Trivial System: $A\vec{x} = \vec{0}$ always has a trivial solution $\rightarrow$
$\vec{x} = \vec{0}$

② Nontrivial solution: $A\vec{x} = \vec{0}$ has a non trivial solution when a free-variable(s) exists.

Strategy:

① Check for free-variables:
* Row reduce $[A \vdots \vec{b}]$ to echelon form $\Rightarrow$ IF a row of zeros exists, $A\vec{x} = \vec{0}$ has a free-variable(s) & a non trivial solution $\exists$

② Find a parametric description of the General Solution:
(vector)

- Continue row-reducing to RREF
- Write the Basic Variables in terms of any free variables
* Write solution set as a vector

③ Describe the Solution Set using the column picture of $\vec{x}$

Ex: Find a parametric-vector description of the solution set for the following

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \\ -x_1 + x_2 = 0 \end{cases}$$

Ans:

* Row Reduce $[A \vdots \vec{0}]$ to RREF:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\frac{R_1 + R_3}{nR_3}]{\frac{-2R_1 + R_2}{nR_2}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow{\frac{R_2 + R_3}{nR_3}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* Echelon Form

A free variable $\exists$ $\therefore$ $A\vec{x} = \vec{0}$ has non trivial solution ✓

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{-2R_2 + R_1}{nR_1}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} \bullet\ x_1 = -x_3 \\ \bullet\ x_2 = -x_3 \\ \bullet\ x_3 \text{ is free} \end{cases}$$

* RREF

$\Rightarrow$ Parametric vector for the General Solution $\vec{x} \in \mathbb{R}^3$

$$\bullet\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{ST} \quad x_3 \in \mathbb{R}$$

$\vec{x}$ is the set of all scalar-multi of $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (Line)

Homogenous Eq: $A\vec{x} = \vec{0}$ ST

* $\vec{A} \rightarrow m \times n$ matrix
* $\vec{x} \in \mathbb{R}^m$

- Trivial Solution: $\vec{x} = \vec{0}$
- Non-trivial solutions: One or more free variables $\exists$

Def: A Linear System is called a "Nonhomogeneous Eq" IF it can be defined $A\vec{x} = \vec{b}$,
* $A \rightarrow m \times n$ matrix
* $\vec{x}, \vec{b} \in \mathbb{R}^m$

Note: Some strategy for finding non trivial solution as $A\vec{x} = \vec{0}$

* Solution set of $A\vec{x} = \vec{b}$: $\vec{x} = \begin{pmatrix}\text{constant} \\ \text{vector}\end{pmatrix} + \begin{pmatrix}\text{Solution set} \\ \text{of } A\vec{x} = \vec{0}\end{pmatrix}$

Example: Find a parametric vector description for the following system:

$x_1 - 2x_2 + 3x_3 = 4$

Ans: * already in Echelon Form $\Rightarrow$ x_2 & x_3 are free variables

$\Rightarrow$ Nontrivial Solution $\exists$

$$\begin{cases} x_1 = 2x_2 - 3x_3 + 4 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases} \Rightarrow \begin{cases} x_1 = 4 + 2x_2 - 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

$$\rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + 2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$\therefore$ General Sol:

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \boxed{x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}}$$

collection of all linear combinations of $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ Span$\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$ (plane)

$$x_1 - 2x_2 + 3x_3 = 0 \xrightarrow[\text{const. vector}]{\text{remove}} \vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad x_2\ x_3 \in \mathbb{R}$$

* Nonhomogeneous Equation: $A\vec{x} = \vec{b}$ $\Rightarrow$ A non trivial solution $\exists$ if a free variable exist. $\Rightarrow$ A trivial solution $\exists$ if $[\vec{A} : \vec{b}]$ has one unique solution.

Favorite

Example: Considerder the following linear system

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -6x_1 - 6x_2 - 12x_3 = -24 \\ \quad\quad -5x_2 - 5x_3 = 15 \end{cases}$$

(a) Write the General Solution to $A\vec{x} = \vec{b}$ in parametric-vector Form

(b) Without calculations, use your answer form (a) to find the General Solution to the corresponding homogeneous equation

(c) Use a geometric interpretation to describe both solution set (i.e. the relationship between them)

(a) Row reduce $[A \vdots \vec{b}]$ to Echelon form to check for free-variables

$$\left[\begin{array}{cccc} 2 & 2 & 4 & 8 \\ -6 & -6 & -12 & -24 \\ 0 & -5 & -5 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free Variables $\exists$ Echelon Form

$\Rightarrow \therefore X_3$ is free variabe

$$\begin{array}{c} -R_2 \\ +R_1 \\ \hline nR_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x_1 = 7 - x_3 \\ x_2 = -3 - x_3 \\ x_3 \text{ is free} \end{cases}$$

* Write the solution in parametric vector form

$$\vec{x} = \underbrace{\cancel{\begin{bmatrix} 7 - x_3 \\ -3 - x_3 \\ x_3 \end{bmatrix}}}_{\text{(skipable)}} = \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad x_3 \in \mathbb{R}$$

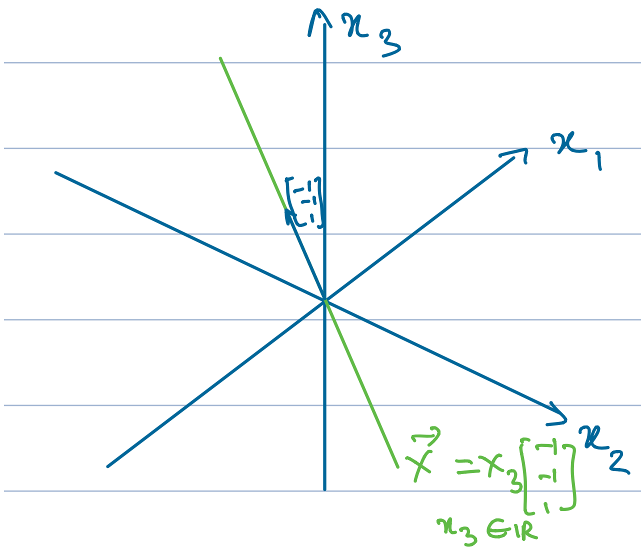
(b) General Solution for $A\vec{x} = \vec{0}$

Recall $A\vec{x} = \vec{b} \longrightarrow \vec{x} = \begin{bmatrix} \text{Constant} \\ \text{vector} \end{bmatrix} + \begin{bmatrix} \text{solution set} \\ \text{of } A\vec{x} = \vec{0} \end{bmatrix} \quad \therefore \boxed{\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}}$

(c) Geometric Interpretation:

$A\vec{x}=\vec{0}$: $\vec{x}$ is the set of ALL scalar multiples of $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ in $\mathbb{R}^3$

IOW: line in $\mathbb{R}^3$



$\boxed{A\vec{x}=\vec{b}: \vec{x} \text{ is a line passing through } (7,-3,0) \text{ \& parallel to the line } \begin{bmatrix}-1\\-1\\1\end{bmatrix}}$

$\boxed{1.5.1}$ $A\vec{x}=\vec{0}$

$$\Rightarrow \left[\begin{array}{ccc|c} 8 & -4 & 20 & 0 \\ -8 & -4 & -14 & 0 \\ 16 & 8 & 28 & 0 \end{array}\right] \xrightarrow[\frac{1}{4}R_3]{\frac{1}{4}R_1,\ -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & -1 & 5 & 0 \\ 4 & 2 & 7 & 0 \\ 4 & 2 & 7 & 0 \end{array}\right] \xrightarrow{\frac{R_2 - R_3}{nR_3}} \left[\begin{array}{ccc|c} 2 & -1 & 5 & 0 \\ 4 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_3$ is a free variable $\Rightarrow \therefore$ $\boxed{\text{Nontrivial}}$ solutions $\exists$

$\boxed{1.5.5}$ $A\vec{x}=\vec{0}$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -5 & 15 & 0 \end{array}\right] \xrightarrow[\frac{R_3}{-5}]{\frac{R_1}{2},\ \frac{R_2}{-4}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \xrightarrow{\frac{R_1 - R_2}{nR_2}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \end{array}\right]$$

Since x_3 is a free variable $\Rightarrow \therefore$ $\boxed{\text{Non Trivial}}$ solutions $\exists$

$x_2 - 3x_3 = 0 \Rightarrow x_2 = 3x_3$

$x_1 + x_2 + 2x_3 \Rightarrow x_1 = -x_2 - 2x_3$

$\Rightarrow x_1 = -3x_3 - 2x_3 = -5x_3$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-5x_3\\3x_3\\1\end{bmatrix} = \boxed{x_3\begin{bmatrix}-5\\3\\1\end{bmatrix}}$$

$\boxed{1.5.6}$ $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 18 & 0 \\ 2 & 1 & 18 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1+R_3 = nR_3]{2R_1 - R_2 = nR_2} \left[\begin{array}{ccc|c} 1 & 2 & 18 & 0 \\ 0 & 3 & 18 & 0 \\ 0 & 3 & 18 & 0 \end{array}\right] \xrightarrow[R_2 - R_3 = nR_3]{\frac{R_2}{3}, \frac{R_3}{3}} \left[\begin{array}{ccc|c} 1 & 2 & 18 & 0 \\ 0 & 1 & 6 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \end{array}\right]$$

R_3 is a free variable.

$\Rightarrow \therefore$ Nontrivial solution $\exists$

$\Rightarrow x_2 + 6x_3 = 0 \Rightarrow x_2 = -6x_3$

$\Rightarrow x_1 = -2x_2 - 18x_3 = 12x_3 - 18x_3 = -6x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6x_3 \\ -6x_3 \\ 1x_3 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} -6 \\ -6 \\ 1 \end{bmatrix}}$$

$\boxed{1.5.11}$

$A\vec{x} = \vec{0} \Rightarrow$

$$\begin{array}{c} \left[\begin{array}{cccccc|c} \underline{1} & 5 & -1 & 4 & 0 & -4 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \end{array}$$

Since columns of x_1, x_3, and x_4 have pivot

$\Rightarrow x_1, x_3, x_4$ are Basic Variables

$\Rightarrow x_2, x_5, x_6$ are Free Variables

- $x_4 = 9x_6$
- $x_3 = -9x_6$
- $x_1 = -5x_2 + x_3 - 4x_4 + 4x_6 \Rightarrow$

$\Rightarrow x_1 = -5x_2 - 9x_6 - 36x_6 + 4x_6$

$\Rightarrow x_1 = -5x_2 - 41x_6$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -41 \\ 0 \\ -9 \\ 9 \\ 0 \\ 1 \end{bmatrix}}$$

$\boxed{1.5.17}$

- $x_1 + 2x_2 - 3x_3 = 0$
- $x_1 + 2x_2 - 3x_3 = -7 \Rightarrow x_1 = -2x_2 + 3x_3 - 7$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}$$

$$x = \begin{bmatrix} -2x_2 + 3x_3 - 7 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}$$

$\boxed{1.5.15}$ The first system:

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -5 & 15 & 15 \end{array}\right] \xrightarrow[R_3/-5]{R_1/2 \quad R_2/-4} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 1 & -3 & -3 \end{array}\right] \xrightarrow{R_1-R_2 = nR_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_3$ is a free variable $\Rightarrow x_2 = -3 + 3x_3$

$$x_1 = -x_2 - 2x_3 + 4 = 3 - 3x_3 - 2x_3 + 4 = -5x_3 + 7$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 + 7 \\ 3x_3 - 3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}}$$

$\boxed{1.5.16}$ The first system $A\vec{x} = \vec{b}$

$$\left[\begin{array}{ccc|c} 1 & -3 & 7 & 5 \\ 2 & 1 & 7 & 3 \\ -1 & -4 & 0 & 2 \end{array}\right] \xrightarrow[2R_1 - R_2 = nR_2]{R_1 + R_3 = nR_3} \left[\begin{array}{ccc|c} 1 & -3 & 7 & 5 \\ 0 & -7 & 7 & 7 \\ 0 & -7 & 7 & 7 \end{array}\right] \xrightarrow[R_2 - R_3 = nR_3]{\frac{R_2}{-7} \quad \frac{R_3}{-7}} \left[\begin{array}{ccc|c} 1 & -3 & 7 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* R_3 is a Free Variable

- $x_2 = -1 + x_3$
- $x_1 = 5 + 3x_2 - 7x_3 = 5 - 3 + 3x_3 - 7x_3 = 2 - 4x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}}$$

$\boxed{1.5.22}$ $q - p = \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

$$\Rightarrow x = \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix} t$$