

4.4.1

$$B = \left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \end{bmatrix} \right\} \quad [x]_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{matrix} \sim c_1 \\ \sim c_2 \end{matrix}$$

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 = 4 \begin{bmatrix} -3 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

4.4.3

$$B = \left\{ \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} \right\} \quad [x]_B = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \begin{matrix} \sim c_1 \\ \sim c_2 \\ \sim c_3 \end{matrix}$$

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$$

$$= 2 \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -15 \\ 14 \\ 3 \end{bmatrix}$$

4.4.3

$$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -5 \end{bmatrix} \right\} \quad [x]_B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \begin{matrix} \sim c_1 \\ \sim c_2 \\ \sim c_3 \end{matrix}$$

$$\Rightarrow \vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 = 3 \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ -2 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -6 - 4 - 15 \\ 3 - 0 + 6 \\ 15 + 2 + 15 \end{bmatrix} = \begin{bmatrix} -25 \\ 9 \\ 32 \end{bmatrix}$$

4.4.5

$$\vec{b}_1 = \begin{bmatrix} -4 \\ -5 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{cases} -4c_1 - 2c_2 = 4 \\ -5c_1 - 1c_2 = 8 \end{cases} \Rightarrow \begin{bmatrix} -4 & -2 & | & 4 \\ -5 & -1 & | & 8 \end{bmatrix}$$

A

$$\det(A) = 4 - 10 = -6$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} -1 & 2 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}_{2 \times 2} \begin{bmatrix} 4 \\ 8 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} \frac{2}{3} - \frac{8}{3} \\ -\frac{10}{3} + \frac{16}{3} \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

4.4.7

$$b_1 = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} \quad b_2 = \begin{bmatrix} 4 \\ -3 \\ -16 \end{bmatrix} \quad b_3 = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} \quad x = \begin{bmatrix} 7 \\ -6 \\ 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \end{bmatrix} \Rightarrow c_1 b_1 + c_2 b_2 + c_3 b_3 = x$$
$$\Leftrightarrow \begin{bmatrix} 1 & 4 & 2 & | & 7 \\ -1 & -3 & -2 & | & -6 \\ -4 & -16 & 6 & | & 0 \end{bmatrix}$$

$$\begin{aligned}
 & R_1 + R_2 = nR_2 \rightarrow \begin{bmatrix} 1 & 4 & 2 & | & 7 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 14 & | & 28 \end{bmatrix} \xrightarrow[R_1 - 4R_2 = nR_1]{R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 2 & | & 3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \\
 & 4R_1 + R_3 = nR_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \xrightarrow[R_1 - 2R_3 = nR_1]{R_1 - 2R_3} \Rightarrow [x]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}
 \end{aligned}$$

$$\boxed{4.4.7} \quad \vec{x}_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 = \vec{x}$$

$$\begin{aligned}
 (\Leftrightarrow) & \begin{bmatrix} 1 & 2 & 1 & | & -4 \\ 1 & -2 & -1 & | & 2 \\ 2 & 6 & 2 & | & -10 \end{bmatrix} \xrightarrow[R_3 - 2R_1 = nR_3]{R_1 - R_2 = nR_2} \begin{bmatrix} 1 & 2 & 1 & | & -4 \\ 0 & 4 & 2 & | & -6 \\ 0 & 2 & 0 & | & -2 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & | & -4 \\ 0 & 2 & 0 & | & -2 \\ 0 & 4 & 2 & | & -6 \end{bmatrix} \\
 & \xrightarrow[R_3 - 2R_2 = nR_3]{R_1 - R_3 = nR_1} \begin{bmatrix} 1 & 0 & 1 & | & -2 \\ 0 & 2 & 0 & | & -2 \\ 0 & 0 & 2 & | & -2 \end{bmatrix} \xrightarrow{R_1 - R_3 = nR_1} \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 2 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \\
 & \Rightarrow [x]_B = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}
 \end{aligned}$$

$$\boxed{4.4.11} \quad B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \end{bmatrix} \right\} \quad x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}}_{P_B} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \vec{x}_B = \vec{x}$$

$$P_B = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} \Rightarrow \det(P_B) = 21 - 20 = 1 \neq 0$$

$$P_B^{-1} = \frac{1}{\det P_B} \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$$

$$[x]_B = P_B^{-1} \vec{x} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -21 + 8 \\ -15 + 6 \end{bmatrix} = \begin{bmatrix} -13 \\ -9 \end{bmatrix}$$

4.4.13 $B = \{b_1, b_2, b_3\}$, find $[p]_B = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ s.t.

$$a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3 = -5 + 8t - 10t^2$$

$$\Leftrightarrow a_1(1+t^2) + a_2(2t-t^2) + a_3(1+t+t^2) = -5 + 8t - 10t^2$$

$$\Rightarrow \begin{cases} a_1 + a_3 = -5 \\ 2a_2 + a_3 = 8 \\ a_1 - a_2 + a_3 = -10 \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 2 & 1 & 8 \\ 1 & -1 & 1 & -10 \end{array} \right]$$

$$\xrightarrow{R_1 - R_3 = nR_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & 5 \end{array} \right] \xrightarrow{R_2 - 2R_3 = nR_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\xrightarrow{R_1 - R_3 = nR_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right] \Rightarrow \bullet \bullet [p]_B = \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}$$

4.4.17

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ -8 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -10 \\ 28 \end{bmatrix}$$

①

$$\left[\begin{array}{cc|c} 1 & 3 & -10 \\ -2 & -8 & 28 \end{array} \right] \xrightarrow[= nR_2]{2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 0 & -2 & 8 \end{array} \right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & -10 \\ 0 & 1 & -4 \end{array} \right]$$

$$\xrightarrow[= nR_2]{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -4 \end{array} \right] \Rightarrow \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

②

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -8 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ 28 \end{bmatrix}$$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -8 \end{bmatrix} = \begin{bmatrix} -8 \\ 26 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 3 & -8 \\ -2 & -8 & 26 \end{array} \right] \xrightarrow[= nR_2]{2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 3 & -8 \\ 0 & -2 & 10 \end{array} \right] \xrightarrow{R_2 / -2} \left[\begin{array}{cc|c} 1 & 3 & -8 \\ 0 & 1 & -5 \end{array} \right]$$

$$\xrightarrow[= nR_1]{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -5 \end{array} \right] \Rightarrow \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

4.4.28

$$(b_1, b_2, b_3, b_4) = ?$$