MATH 3220: Discrete Structures II - Spring 2018 Syllabus

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Office Hours: MWF 10:45-11:45 am, or by appointment

Prerequisite: MATH 3210 (Discrete Structures I)

Course Website: http://faculty.uml.edu/thao_tran/discrete2

Textbook: *Discrete Mathematics and Its Applications* by Kenneth Rosen, 7th edition

Coverage: We will cover parts of Chapters 6-8, 10, and 11. Topics include counting techniques, recurrence relations, discrete probability, graphs, and trees.

Course Format:

1. **Quizzes** will be given approximately once per week. Quizzes are worth 20% of your final grade. The lowest two quiz grades will be dropped. No makeups will be given unless you have a valid, documented reason.

2. **Homework** and reading assignments will be announced on the course website each Friday. In general, the homework problems will not be collected. (Occasionally, assignments may be collected and counted as part of your quiz / exam grade.) However, questions related to the homework problems will appear on the quizzes and exams.

3. **Exams**: There will be four exams. Three of the exams are given in class, and the last exam will be at the time your final exam is scheduled. Each exam is worth 20% of your final grade. No makeups will be given unless you have a valid, documented reason for your absence.

4. **Attendance** will not be taken, but you are responsible for everything said in class.

Grade scale:

Average	[0, 59]	[60, 63]	[64, 66]	[67, 69]	[70, 73]	[74, 76]	[77, 79]	[80, 83]	[84, 86]	[87, 89]	[90, 100]
Grade	F	D	D+	C-	C	C+	B-	B	B+	A-	A

Academic integrity policy:

Academic dishonesty will not be tolerated in this class. The penalty for the first offense is a zero on the assignment. The penalty for the second offense is a grade of F for the course. More information on the university's policy on academic integrity can be found here:

https://www.uml.edu/Catalog/Undergraduate/Policies/Academic-Policies/Academic-Integrity.aspx

Quiz 1 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

25/25

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS. This quiz is worth 25 points.

15

1. (15 points) Consider strings of length 4 consisting of letters from the set $\{a, b, c\}$. How many such strings ...

Very good!

(a) ... do not contain the letter a?

$\square\square\square\square$

$2^4 = \boxed{16}$

(b) ... contain exactly three b's (e.g. $bbcb$)?

bbb_ bb_b b_bb _bbb

Case 1 : bbb_ : $C(2,1) = 2$

Case 2 : bb_b : $C(2,1) = 2$

Case 3 : b_bb : $C(2,1) = 2$

Case 4 : _bbb : $C(2,1) = 2$

Sum: $2 + 2 + 2 + 2 = \boxed{8}$ (strings)

(c) ... contain the letter b at least once?

U = { numbers of strings of length 4 }

A = { numbers of strings contain the letter b at least once }

$\bar{A}$ = { numbers of strings contain no letter b }

$|U| = 3^4 = 81$

$|\bar{A}| = 2^4 = 16$

$|A| = |U| - |\bar{A}| = 81 - 16 = \boxed{65}$

2. (10 points) There are four women and two men in a group. How many ways are there to arrange the ~~five~~ 6 people in a row if ...

10

(a) ... the two men are in the leftmost positions?

Case 1 : $M_1 M_2$ _ _ _ _

$4! = 4 \times 3 \times 2 \times 1 = 24$

Case 2 : $M_2 M_1$ _ _ _ _

$4! = 24$

Sum: $24 + 24 = \boxed{48}$

(b) ... the two men must be seated next to each other?

$$\left.\begin{array}{l} M_1 M_2 \_ \_ \_ \_ \\ \_ M_1 M_2 \_ \_ \_ \\ \_ \_ M_1 M_2 \_ \_ \\ \_ \_ \_ M_1 M_2 \_ \\ \_ \_ \_ \_ M_1 M_2 \end{array}\right\}$$ 5 ways to arrange M_2 to the right of M_1

Replace the position of M_1 by M_2 and M_2 by M_1, there are $5 \times 2 = 10$ (ways) to arrange position for two men

Number of ways to arrange for 3^{rd} seat : 4

——— // ——— 4^{th} ——— : 3

——— // ——— 5^{th} ——— : 2

——— // ——— 6^{th} ——— : 1

Product rule : $10 \times 4 \times 3 \times 2 \times 1 = \boxed{240}$

Quiz 2 / Discrete Structures II / Spring 2018 Name: Dongzhi Ngo

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! This quiz is worth 25 points.

25/25

Very good

10

1. (10 points)

(a) Complete the following statement of the generalized pigeonhole principle:

Let m, r be two positive integers. If r objects are placed into m boxes, then ...

there is at least one box which contains at least $\lceil \frac{r}{m} \rceil$ objects.

(b) Use the generalized pigeonhole principle to explain why the following statement is true:

Statement: If there are 20 students in a class, and every student is a freshman, sophomore, or junior, then there are at least seven freshmen, at least seven sophomores, or at least seven juniors in the class.

Place 20 students into 3 boxes (freshman, sophomore, junior),

there is at least 1 box contains at least

$$\left\lceil \frac{20}{3} \right\rceil = \left\lceil 6\tfrac{2}{3} \right\rceil = 7 \text{ (students)}$$

Therefore, it is true with the statement that there are at least 7 freshmen, at least 7 sophomores, or at least 7 juniors in the class

2. (4 points) How many strings of length 8 consist of exactly 3 a's and 5 b's (e.g. $abbaabbb$)?

4

$$C(8,3) = \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1} = \boxed{56} \text{ (strings)}$$

3. (5 points) 12 people (including Abby and Ben) are competing for 1st, 2nd, 3rd, and 4th prize. How many ways are there to award the prizes if Abby and Ben both win a prize?

5

$$I(4,2) \times I(10,2)$$

$$= 4 \times 3 \times 10 \times 9 = \boxed{1080} \text{ (ways)}$$

Pick prize for A : 4 ways

B : 3 ways

1st remaining prize : 10 ways

2nd : 9 ways

4. (6 points) A class contains 5 freshmen and 7 sophomores. How many groups of four students from the class contain an equal number of freshmen and sophomores?

6

A group of 4 students contain an equal number of freshmen and sophomores, it means there are 2 freshmen and 2 sophomores:

$$C(5,2) \times C(7,2) = \frac{5 \times \cancel{4}^{2}}{\cancel{2} \times 1} \times \frac{7 \times \cancel{6}^{3}}{\cancel{2} \times 1} = 10 \times 21 = \boxed{210} \text{ (ways)}$$

15pt

Quiz 3 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! This quiz is worth 25 points. (Each question is worth 5 points.)

~~22/25~~ 25/25

5

1. Find the number of solutions for the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 7$ in nonnegative integers.

The number of 7-combinations of 5 objects which allows repetition:

$n = 5$

$r = 7$

$C(n+r-1, n-1) = C(11, 4) = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 11 \times 30 = \boxed{330}$

Good job

Check: $x_1 + x_2 + .. + x_5 = 7$

7 stars

4 bars

$\star|\star\star|\star|\star\star\star|\star$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Count: the # of strings of length 11 with exactly 7 stars & 4 bars $\frac{11!}{7!\,4!} = C(11, 4)$

5

2. A bagel shop sells seven different types of bagels. How many different ways can you select 10 bagels if you are planning to buy at least one bagel of each type?

Buying at least one bagel of each type means that you have 7 bagels in each type and find the 3-combinations of 7 objects which repetition is allowed:

$n = 7$

$r = 3$

$C(n+r-1, n-1) = C(9, 6) = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= 3 \times 4 \times 7$

$= \boxed{84}$

5

3. How many strings of length 9 consist of exactly three x's, four y's, and two z's? (For example, $yxxzyyxyz$.)

$\frac{9!}{3!\;4!\;2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1}$

$= 9 \times 4 \times 7 \times 5$

$= \boxed{1260}$

2

4. How many ways are there to assign four different tasks to three employees if each employee can be given more than one task to do?

Each employee can be assigned 4 different tasks:

$4 \times 4 \times 4 = \boxed{64}$

Tasks : 1, 2, 3, 4

Employees : A, B, C

One possibility : $1 \to A$, $2 \to C$, $3 \to A$, $4 \to B$

or $1, 2, 3 \to B$, $4 \to A$

Product Rule : Task $\to$ $\underset{1}{\square}\ \underset{2}{\square}\ \underset{3}{\square}\ \underset{4}{\square}$

$3 \times 3 \times 3 \times 3 = 81$

Note that each task will be assigned to exactly 1 employee.

$3^4 = 81$

5

5. Find the next two larger permutations in lexicographic order after the permutation 5143762.

Find a_j, a_k : $5\,1\,4\,\overset{j}{\boxed{3}}\,7\,\overset{k}{\boxed{6}}\,2$

Replace a_j & a_k : $5\,1\,4\,6,\,7\,3\,2$

Reorder : 5146 237

$5\,1\,4\,6\,2\,\overset{j}{\boxed{3}}\,\overset{k}{\boxed{7}}$

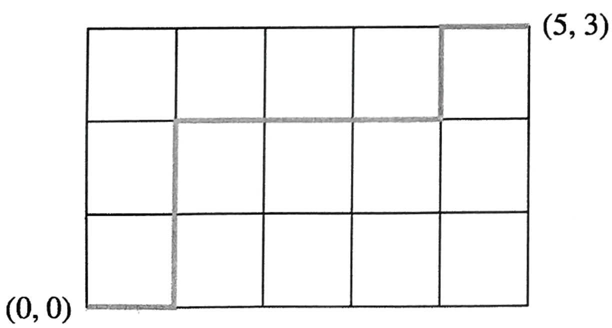
5146273

5146273

Next 2 larger permutations : 5146237, 5146273

Extra Credit (3 points): Consider paths in the xy plane from the point $(0, 0)$ to the point $(5, 3)$ so that each path is made up of a series of steps, where each step must be a move one unit to the right or a move one unit upward. (See below for an example.) How many such paths are there? Explain *why* your answer works. (No credit without explanation.)

See prob. # 33 in §6.4



Quiz 4 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! This quiz is worth 25 points.

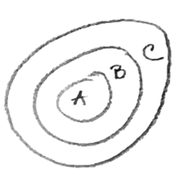
28/25 Very good

5 1. (5 points) Let A, B, C be any sets. State the formula for $|A \cup B \cup C|$ given by the principle of inclusion-exclusion. (No partial credit; check your answer!)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

11 2. (11 points) Let A, B, C be sets so that $|A| = 30$, $|B| = 40$, and $|C| = 50$. Answer each question below with explanation. (Each parts are independent of each other.)

(a) How many elements does $A \cup B \cup C$ have if $A \subseteq B$ and $B \subseteq C$?



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 30 + 40 + 50 - 30 - 30 - 40 + 30$$
$$= 50$$

(b) Suppose that the cardinality of the intersection between any two different sets on the list A, B, C is 15, and the intersection of all three sets contains exactly 5 elements. How many elements does $A \cup B \cup C$ have?

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 30 + 40 + 50 - 15 - 15 - 15 + 5$$
$$= 80$$

9

3. (9 points) In a class of 30 students, there are 14 math majors, 10 sophomores, and 6 students who are both math majors and sophomores.

(a) How many students are math majors or sophomores?

$M = \{\#$ students are Math major$\}$

$S = \{\#$ student are sophomores$\}$

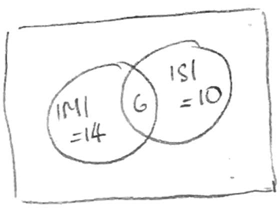
$M \cup S = \{\#$ students are math majors or sophomores$\}$

$M \cap S = \{\#$ students are math majors and sophomores$\}$

$$|M \cup S| = |M| + |S| - |M \cap S|$$
$$= 14 + 10 - 6 = 18$$

(b) How many students are neither math majors nor sophomores?

$A = \{\#$ students are neither Math majors nor sophomores$\}$



$$|A| = |\text{Total students}| - |M \cup S|$$
$$= 30 - 18$$
$$= 12$$

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x3

EXTRA CREDIT (3 POINTS): Let A_1, A_2, A_3, A_4 be sets. Suppose that $|A_k| = 8$ for all k, $|A_i \cap A_j| = 4$ for all $i \neq j$, $|A_i \cap A_j \cap A_k| = 2$ for all distinct i, j, k, and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$. Compute $|A_1 \cup A_2 \cup A_3 \cup A_4|$ using the principle of inclusion-exclusion.

$$|A_1 \cup A_2 \cup A_3 \cup A_4|$$
$$= |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 - A_3| - |A_2 \cap A_4| - |A_3 \cap A_4|$$
$$|A_1 + A_3 + A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$
$$= 8 \times 4 - 4 \times 6 + 2 \times 4 - 1$$
$$= 32 - 24 + 8 - 1$$
$$= 15$$

Quiz 5 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

25/25

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! This quiz is worth 25 points.

Note: For all probability questions, you must use $p(E) = |E|/|S|$. Also, leave your answers as simplified fractions.

Very good

15

1. A club contains 4 freshmen, 5 sophomores, and 3 juniors.

   (a) (12 points) How many groups consisting of five club members contain exactly two freshmen or exactly three sophomores? Use the inclusion-exclusion method given in class.

A_1 = { group of five contains exactly 2 freshmen }

A_2 = { group of five contains exactly 3 sophomores }

$A_1 \cap A_2$ = { group of five contains exactly 2 freshmen and 3 sophomores }

$A_1 \cup A_2$ = { group of five contains exactly 2 freshmen or 3 sophomores }

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$= C(4,2) \times C(8,3) + C(5,3) \times C(7,2) - C(4,2) \times C(5,3)$$

$$= \frac{4 \times 3}{2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} + \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times \frac{7 \times 6}{2 \times 1} - \frac{4 \times 3}{2 \times 1} \times \frac{5 \times 4 \times 3}{3 \times 2 \times 1}$$

$$= 336 + 210 - 60$$

$$= \boxed{486}$$

   (b) (3 points) A person is selected at random from the club. What's the probability that the person is a freshman?

$$p(E) = \frac{|E|}{|S|} = \frac{4}{4+5+3} = \frac{4}{12} = \boxed{\frac{1}{3}}$$

Problem 1 continues ...

(c) (5 points) Three people from the club are selected at random to serve as president, vice-president, and treasurer. What's the probability that the selection includes a freshman, a sophomore, and a junior?

$$p(E) = \frac{|E|}{|S|} = \frac{C(4,1) \times C(5,1) \times C(3,1) \times 3!}{C(12,3)}$$

$$= \frac{4 \times 5 \times 3}{12 \times 11 \times 10} \times (3 \times 2 \times 1)$$

$$= \boxed{\frac{3}{11}}$$

S : the set of all ordered triples (president, VP, treasurer)

$|S| = 12 \times 11 \times 10 = 1320$

E : ordered triple includes a freshman, a sophomore and a junior

$|E| = \underbrace{4 \times 5 \times 3}_{F,S,J} \times \underbrace{3!}_{\text{assigning}} = 360$

$p(E) = \frac{360}{1320} = \frac{3}{11}$

2. (5 points) A coin is flipped six times. What's the probability that heads occurs at most twice?

Case 1 : Head occurs 0 time : $p(E) = \frac{|E|}{|S|} = \frac{1}{2^6} = \frac{1}{64}$

Case 2 : Head occurs 1 time : $p(E) = \frac{|E|}{|S|} = \frac{6}{2^6} = \frac{6}{64}$ $C(6,1)$

Case 3 : Head occurs 2 times ; $p(E) = \frac{|E|}{|S|} = \frac{15}{2^6} = \frac{15}{64}$: $C(6,2)$

Sum : $\frac{1}{64} + \frac{6}{64} + \frac{15}{64} = \frac{22}{64} = \boxed{\frac{11}{32}}$

S : set of all sequences of 6 coin flips

$|S| = 2^6 = 64$

E : head occurs at most twice

$|E| = C(6,2) + C(6,1) + C(6,0) = 15 + 6 + 1 = 22$

Ans : $p(E) = \frac{22}{64} = \frac{11}{32}$

10

Quiz 5 Makeup / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

25/25

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! This quiz is worth 25 points.

Instructions: For all questions:

- Describe in words what the sample space S is.
- Describe in words the event E for which you want to compute the probability.
- Use these descriptions to compute $|E|$ and $|S|$, then use these numbers to compute $p(E)$. Leave your answer as a simplified fraction.

8

1. (8 points) A coin is flipped seven times. What's the probability that heads occurs at least five times?

sequences of coin flips when....

$S = \{$coin is flipped 7 times$\}$

$E = \{$heads occurs at least 5 times$\}$

elements of S where...

$$p(E) = \frac{|E|}{|S|} = \frac{C(7,5) + C(7,6) + C(7,7)}{2^7}$$

$$= \frac{\frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} + \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2 \times 1} + 1}{128}$$

$$= \frac{21 + 7 + 1}{128}$$

$$= \boxed{\frac{29}{128}}$$

17

2. (17 points) A club contains 3 women and 5 men.

(a) A group of four people is selected at random from the club. What's the probability that the selected group contains an equal number of women and men?

S = { group of 4 people is selected }

E = { group of 4 people contains an equal number of women and men }

$$p(E) = \frac{|E|}{|S|} = \frac{C(3,2) \times C(5,2) \times 4!}{8 \times 7 \times 6 \times 5}$$

$$= \frac{\frac{3 \times 2}{2 \times 1} \times \frac{5 \times 4}{2 \times 1} \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5}$$

$$= \boxed{\frac{3}{7}}$$

all possible ways to select 3 club members.

(b) Three club members are selected at random for three different prizes (\$100, \$50, and \$25). What's the probability that exactly two of the winners are men?

S = { 3 club members are selected }

E = { exactly 2 of winners are men }

$$p(E) = \frac{|E|}{|S|} = \frac{C(5,2) \times C(3,1) \times 3!}{8 \times 7 \times 6}$$

$$= \frac{\frac{5 \times 4}{2 \times 1} \times 3 \times 3 \times 2 \times 1}{8 \times 7 \times 6}$$

$$= \boxed{\frac{15}{28}}$$

Quiz 6 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

20/25

22/25

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! Leave each answer as a decimal or simplified fraction. This quiz is worth 25 points.

5

1. (5 points) Suppose that E, F are events in a sample space satisfying $p(E) = 0.3$, $p(F) = 0.5$, and $p(E \mid F) = 0.2$. Compute $p(E \cap F)$ and $p(F \mid E)$.

$$p(E \cap F) = p(E \mid F)\, p(F) = 0.2 \times 0.5 = \boxed{0.1}$$ ✓

$$p(F \mid E) = \frac{p(E \cap F)}{p(E)} = \frac{0.1}{0.3} = \boxed{\frac{1}{3}}$$ ✓

4

2. (9 points) There are nine cards, each labeled with one of the numbers 1 through 9. Four cards are drawn from the deck, one at a time, without replacement. What's the probability that ...

(a) ... the first two cards drawn are both even numbers?

2/4

$$p(E) = \frac{C(4,2) \times 2! \times C(7,2) \times 2!}{9 \times 8 \times 7 \times 6} = \frac{\frac{4 \times 3}{2 \times 1} \times 2 \times 1 \times \frac{7 \times 6}{2 \times 1}}{9 \times 8 \times 7 \times 6}$$

$$= \frac{2}{24} = \boxed{\frac{1}{12}} \quad \frac{1}{6}$$

E_1 : 1^{st} card is even

E_2 : 2^{nd} card is even

$$p(E_1 \cap E_2) = p(E_1)\, p(E_2 \mid E_1) = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$$

The order of the numbers matters! Need to mulitply by $4! = 4 \cdot 3 \cdot 2 \cdot 1$ to get all possible orderings of the 1 even number and 3 odd numbers.

2/5

(b) ... the selection contains exactly one even number?

$$p(E) = \frac{C(4,1) \times C(5,3)}{9 \times 8 \times 7 \times 6} = \frac{4 \times \frac{5 \times 4 \times 3}{3 \times 2 \times 1}}{9 \times 8 \times 7 \times 6}$$

$$= \frac{10}{756} = \boxed{\frac{5}{378}}$$

4 cases
- $p(\text{even}, \text{odd}, \text{odd}, \text{odd}) = \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{5}{63}$
- $p(O, E, O, O) = 5/63$
- $p(O, O, E, O) = 5/63$
- $p(O, O, O, E) = 5/63$

Sum: $4 \times \frac{5}{63} = \frac{20}{63}$

3. (11 points) For a certain unfair die, the probability of each possible outcome is given below:

$$p(1) = p(2) = p(4) = 0.1$$

$$p(3) = p(6) = 0.2$$

$$p(5) = 0.3$$

A pair of the unfair dice are tossed. What's the probability that

(a) ...the sum is equal to 10?

$$p(4) \times p(6) + p(5) \times p(5) + p(6) \times p(4) = 0.1 \times 0.2 + 0.3 \times 0.3 + 0.2 \times 0.1$$

$$= 0.02 + 0.09 + 0.02$$

$$= \boxed{0.13}$$

(b) ... the sum is equal to 10 given that the first die was a 6?

$E = \{$ the sum is equal to 10 $\}$

$F = \{$ the first die was a 6 $\}$

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{p(6) \times p(4)}{p(6) \times p(1) + p(6) \times p(2) + p(6) \times p(3) + p(6) \times p(4) + p(6) \times p(5) + p(6) \times p(6)}$$

$p(4 \text{ on } 2^{nd} \text{ die}) = 0.1$

$$= \frac{0.2 \times 0.1}{0.2 \times 0.1 + 0.2 \times 0.1 + 0.2 \times 0.2 + 0.2 \times 0.1 + 0.2 \times 0.3 + 0.2 \times 0.2}$$

$$= \frac{0.02}{0.2} = \boxed{\frac{1}{10}}$$

Extra Credit (3 points): Consider the setup in problem 2. What's the probability that the cards are drawn in increasing order (e.g. the cards are 1, 3, 4, and 7, in that order)?

Quiz 7 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! No calculator, book, notes, etc. allowed! This quiz is worth 25 points.

25/25

Excellent

19

1. (19 points) Consider the recurrence relation $a_n = a_{n-1} + 6a_{n-2} - 6n + 13$.

One of the sequences below is a solution for the recurrence relation, and the other sequence is not:

1st sequence: $a_n = 2^n + 1$

2nd sequence: $a_n = 3^n + n$

For the sequence that is a solution, prove it using the method given in class. (Show at least a few algebraic steps!)

For the sequence that is not a solution, show why not.

a/ 1st sequence: $a_n = 2^n + 1$

$a_0 = 2^0 + 1 = 2$

$a_1 = 2^1 + 1 = 3$

$a_2 = 2^2 + 1 = 5$

Check with Recurrence relation: $a_n = a_{n-1} + 6a_{n-2} - 6n + 13$

$a_2 \stackrel{?}{=} a_1 + 6a_0 - 6(2) + 13$

$5 \stackrel{?}{=} 3 + 6 \times 2 - 12 + 13$

$5 \stackrel{?}{=} 16$ ✗

Therefore, the sequence ($a_n = 2^n + 1$) is not the solution for the recurrence relation

b/ 2nd sequence: $a_n = 3^n + n$

Prove with the recurrence relation:

$a_n = a_{n-1} + 6a_{n-2} - 6n + 13$

$3^n + n \stackrel{?}{=} 3^{n-1} + n - 1 + 6(3^{n-2} + n - 2) - 6n + 13$

$3^n + n \stackrel{?}{=} 3^{n-1} + n - \cancel{1} + 6.3^{n-2} + \cancel{6n} - \cancel{12} - \cancel{6n} + \cancel{13}$

$3^n + n \stackrel{?}{=} 3^{n-1} + 2.3.3^{n-2} + n$

$3^n + n \stackrel{?}{=} 3^{n-1} + 2.3^{n-1} + n$

$3^n + n \stackrel{?}{=} 3.3^{n-1} + n$

$3^n + n \stackrel{?}{=} 3^n + n$

Therefore, the sequence ($a_n = 3^n + n$) is a solution for the recurrence relation

6

2. (6 points) Consider the following recurrence relation with initial conditions:

$$a_n = 3a_{n-1} - 2na_{n-2}$$
$$a_0 = 2,\ a_1 = 5$$

Compute a_2, a_3, a_4.

* $a_2 = 3a_1 - 2.2.a_0$

$a_2 = 3.5 - 4.2$

$a_2 = 15 - 8$

$\boxed{a_2 = 7}$

* $a_3 = 3a_2 - 2.3.a_1$

$a_3 = 3.7 - 6.5$

$a_3 = 21 - 30$

$\boxed{a_3 = -9}$

* $a_4 = 3a_3 - 2.4.a_2$

$a_4 = 3.(-9) - 8.7$

$a_4 = -27 - 56$

$\boxed{a_4 = -83}$

Quiz 8 / Discrete Structures II / Spring 2018 Name: Danghi Ngo

22/25

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! This quiz is worth 25 points.

Instructions for both problems:
For each sequence $\{a_n\}$, set up a recurrence relation for a_n. Be sure to show how you're setting it up by describing what each term in your recurrence represents. Also, compute a_n for $n = 0, 1, 2, 3$.

Nice job!

11/11 1. (11 points) Let a_n be the number of ways to tile a $1 \times n$ grid of squares, where the tiles you're allowed to use are 1×1 red tiles, 1×2 blue tiles, and 1×2 green tiles.

11/14 2. (14 points) Let a_n be the number of strings of length n using letters from the set $\{a, b\}$ so that at least two consecutive a's occur in the string (e.g. *baaabab* and *aabaabb*).

EXTRA CREDIT (3 POINTS): Use the method given in Section 8.2 to find a closed formula for the sequence a_n given in problem 1.

2/ a_n is the number of strings of length n with at least 2 consecutive a's

The string start with b*: there is a_{n-1} ways to insert the rest with 2 consecutive a's

The string start with ab* there is a_{n-2} ways

The string start with aa*: there is 2^{n-2} ways

Recurrence relation: $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ (for $n \geq 2$)

Initial values: $a_0 = 0$

$a_1 = 1$ ✗ $\underline{\underline{a_1 = 0}}$ (A string of length 1 can't contain consecutive a's)

$a_2 = a_1 + a_0 + 2^0 = 1 + 0 + 1 = 2$ ✗ $a_2 = 1$

$a_3 = a_2 + a_1 + 2^1 = 2 + 1 + 2 = 5$ ✗ $a_3 = 3$

1/ a_n is the number of ways to tile a $1 \times n$ grid of squares:

1×1 red tiles: there is a_{n-1} ways

1×2 blue tiles: there is a_{n-2} ways

1×2 green tiles: there is a_{n-2} ways

$a_n = a_{n-1} + a_{n-2} + a_{n-2}$

$a_n = a_{n-1} + 2a_{n-2}$ ($n \geq 2$)

Initial values: $a_0 = 1$

$a_1 = 1$

$a_2 = a_1 + 2a_0 = 1 + 2.1 = 3$

$a_3 = a_2 + 2a_1 = 3 + 2.1 = 5$

Quiz 9 / Discrete Structures II / Spring 2018 Name: Dangnhi Ngo

SHOW ALL WORK AND CIRCLE YOUR FINAL ANSWERS! No calculator, book, notes, etc. allowed! This quiz is worth 25 points.

26/25

Very good

20

1. (6 points; 14 points)

   Solve the following recurrence relations with initial conditions:

   (a) $a_n = 4a_{n-1}$
   $a_0 = 2$

$$a_n = \alpha 4^n$$

$$a_0 = 2 \Rightarrow \alpha . 4^0 = 2$$

$$\Leftrightarrow \alpha = 2$$

$$a_n = 2 . 4^n = 2^{2n+1}$$

   (b) $a_n = 3a_{n-1} + 10a_{n-2}$
   $a_0 = 1, \quad a_1 = 3$

$$x^2 - 3x - 10 = 0$$

$$(x-5)(x+2) = 0$$

$$x = 5, \ x = -2$$

$$a_n = \alpha_1 . 5^n + \alpha_2 (-2)^n$$

$$a_0 = 1 \Rightarrow \alpha_1 . 5^0 + \alpha_2 (-2)^0 = 1 \Rightarrow \alpha_1 + \alpha_2 = 1 \Rightarrow 2\alpha_1 + 2\alpha_2 = 2 \ (1)$$

$$a_1 = 3 \Rightarrow \alpha_1 . 5^1 + \alpha_2 (-2)^1 = 3 \Rightarrow 5\alpha_1 - 2\alpha_2 = 3 \quad (2)$$

(1) + (2), we have: $7\alpha_1 = 5$

$$\Rightarrow \alpha_1 = \frac{5}{7}$$

$$\Rightarrow \alpha_2 = 1 - \alpha_1 = 1 - \frac{5}{7} = \frac{2}{7}$$

$$a_n = \frac{5}{7} \cdot 5^n + \frac{2}{7}(-2)^n$$

$$= \frac{5^{n+1} - (-2).(-2)^n}{7}$$

$$= \frac{5^{n+1} - (-2)^{n+1}}{7}$$

5

2. (5 points) Suppose that the roots of a linear homogenous recurrence relation with constant coefficients for $\{a_n\}$ are given below:

- The root 7 has multiplicity 3.
- The root 2 has multiplicity 1.

Find the general solution for the recurrence.

$$a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3)7^n + \beta \cdot 2^n$$

X

EXTRA CREDIT (4 POINTS): Find the general solution of the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2} + 3^n$. (*Hint:* The recurrence has a particular solution of the form $a_n = c \cdot 3^n$. Determine what c must be by plugging $a_n = c \cdot 3^n$ into the recurrence relation. Also, you can use problem 1(b).)

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot (-2)^n$$

$$a_n = c \cdot 3^n$$

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot (-2)^n + 3^n \quad X$$

need to put a particular soln. here ...

$$3 \cdot (c\, 3^{n-1}) + 10(c \cdot 3^{n-2}) + 3^n = c \cdot 3^n$$

$$c \cdot 3^n + 10c\, 3^{n-2} + 3^n = c\, 3^n$$

$$3^n\left(c + \frac{10}{9}c + 1\right) = c \cdot 3^n$$

$$\frac{19}{9}c + 1 = c$$

$$19c + 9 = 9c$$

$$10c = -9$$

$$c = -9/10$$