

## Theoretical Computer Science Cheat Sheet

Definitions		Series
$f(n) = O(g(n))$	iff $\exists$ positive $c, n_0$ such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$ .	$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$ .
$f(n) = \Omega(g(n))$	iff $\exists$ positive $c, n_0$ such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$ .	In general: $\sum_{i=1}^n i^m = \frac{1}{m+1} \left[ (n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$ $\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ .	Geometric series: $\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad c < 1,$ $\sum_{i=0}^n i c^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} i c^i = \frac{c}{(1-c)^2}, \quad c < 1.$
$\lim_{n \rightarrow \infty} a_n = a$	iff $\forall \epsilon \in \mathbb{R}, \exists n_0$ such that $ a_n - a  < \epsilon, \forall n \geq n_0$ .	Harmonic series: $H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n i H_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}.$ $\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left( H_{n+1} - \frac{1}{m+1} \right).$
$\sup S$	least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$ .	
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$ .	
$\liminf_{n \rightarrow \infty} a_n$	$\liminf_{n \rightarrow \infty} \{a_i \mid i \geq n, i \in \mathbb{N}\}$ .	
$\limsup_{n \rightarrow \infty} a_n$	$\limsup_{n \rightarrow \infty} \{a_i \mid i \geq n, i \in \mathbb{N}\}$ .	
$\binom{n}{k}$	Combinations: Size $k$ subsets of a size $n$ set.	
$[n]_k$	Stirling numbers (1st kind): Arrangements of an $n$ element set into $k$ cycles.	1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$
$\{n\}_k$	Stirling numbers (2nd kind): Partitions of an $n$ element set into $k$ non-empty sets.	4. $\binom{n}{k} = \frac{n(n-1)}{k(k-1)}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$
$\langle n \rangle_k$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with $k$ ascents.	6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n},$
$\langle\langle n \rangle\rangle_k$	2nd order Eulerian numbers.	8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$
$C_n$	Catalan Numbers: Binary trees with $n+1$ vertices.	10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \{n\}_{1,1} = \{n\}_{n,n} = 1,$
14. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$	15. $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1},$	12. $\{n\}_{2,2} = 2^{n-1} - 1, \quad 13. \{n\}_{k,k} = k \{n-1\}_{k,k} + \{n-1\}_{k-1,k},$
18. $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$	19. $\{n\}_{n-1,1} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2},$	16. $\begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad 17. \begin{bmatrix} n \\ k \end{bmatrix} \geq \begin{bmatrix} n \\ k \end{bmatrix},$
22. $\langle n \rangle_{0,1} = \langle n \rangle_{n-1,1} = 1,$	23. $\langle n \rangle_{k,1} = \langle n \rangle_{n-1-k,1},$	20. $\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = n!, \quad 21. C_n = \frac{1}{n+1} \binom{2n}{n},$
25. $\langle 0 \rangle_{k,1} = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$	26. $\langle n \rangle_{1,1} = 2^n - n - 1,$	24. $\langle n \rangle_{k,1} = (k+1) \langle n-1 \rangle_{k,1} + (n-k) \langle n-1 \rangle_{k-1,1},$
28. $x^n = \sum_{k=0}^n \langle n \rangle_{k,1} \binom{x+k}{n},$	29. $\langle n \rangle_{m,1} = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$	27. $\langle n \rangle_{2,1} = 3^n - (n+1)2^n + \binom{n+1}{2},$
31. $\langle n \rangle_{m,1} = \sum_{k=0}^n \{n\}_{k,1} \binom{n-k}{m} (-1)^{n-k-m} k!,$	32. $\langle\langle n \rangle\rangle_{0,1} = 1,$	30. $m! \{n\}_{m,1} = \sum_{k=0}^n \langle n \rangle_{k,1} \binom{k}{n-m},$
34. $\langle\langle n \rangle\rangle_{k,1} = (k+1) \langle\langle n-1 \rangle\rangle_{k,1} + (2n-1-k) \langle\langle n-1 \rangle\rangle_{k-1,1},$		33. $\langle\langle n \rangle\rangle_{n,1} = 0 \text{ for } n \neq 0,$
36. $\{x\}_{x-n,1} = \sum_{k=0}^n \langle\langle n \rangle\rangle_{k,1} \binom{x+n-1-k}{2n},$	37. $\{n+1\}_{m+1,1} = \sum_k \binom{n}{k} \{k\}_{m,1} = \sum_{k=0}^n \binom{k}{m} (m+1)^{n-k},$	35. $\sum_{k=0}^n \langle\langle n \rangle\rangle_{k,1} = \frac{(2n)^n}{2^n},$

## Theoretical Computer Science Cheat Sheet

Identities Cont.	Trees
38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \sum_{k=0}^n \begin{bmatrix} k \\ m \end{bmatrix} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix}$ ,	39. $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{x+k}{2n}$ ,
40. $\begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k+1 \\ m+1 \end{Bmatrix} (-1)^{n-k}$ ,	41. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k}$ ,
42. $\begin{Bmatrix} m+n+1 \\ m \end{Bmatrix} = \sum_{k=0}^m k \begin{Bmatrix} n+k \\ k \end{Bmatrix}$ ,	43. $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^m k(n+k) \binom{n+k}{k}$ ,
44. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \binom{k}{m} (-1)^{m-k}$ ,	45. $(n-m)! \begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{Bmatrix} \binom{k}{m} (-1)^{m-k}$ , for $n \geq m$ ,
46. $\begin{Bmatrix} n \\ n-m \end{Bmatrix} = \sum_k \begin{bmatrix} m-n \\ m+k \end{bmatrix} \binom{m+n}{n+k} \binom{m+k}{k}$ ,	47. $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \begin{bmatrix} m-n \\ m+k \end{bmatrix} \binom{m+n}{n+k} \binom{m+k}{k}$ ,
48. $\begin{Bmatrix} n \\ \ell+m \end{Bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{Bmatrix} k \\ \ell \end{Bmatrix} \binom{n-k}{m} \binom{n}{k}$ ,	49. $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \binom{n-k}{m} \binom{n}{k}$ .

## Recurrences

### Master method:

$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If  $\exists \epsilon > 0$  such that  $f(n) = O(n^{\log_b a - \epsilon})$  then

$$T(n) = \Theta(n^{\log_b a}).$$

If  $f(n) = \Theta(n^{\log_b a})$  then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If  $\exists \epsilon > 0$  such that  $f(n) = \Omega(n^{\log_b a + \epsilon})$ , and  $\exists c < 1$  such that  $af(n/b) \leq cf(n)$  for large  $n$ , then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that  $T_i$  is always a power of two.

Let  $t_i = \log_2 T_i$ . Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let  $u_i = t_i/2^i$ . Dividing both sides of the previous equation by  $2^{i+1}$  we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = 12,$$

which is simply  $u_i = i/2$ . So we find

that  $T_i$  has the closed form  $T_i = 2^{i2^{i-1}}$ .

Summing factors (example): Consider the following recurrence

$$T_i = 3T_{n/2} + n, \quad T_1 = n.$$

Rewrite so that all terms involving  $T$  are on the left side

$$T_i - 3T_{n/2} = n.$$

Now expand the recurrence, and choose a factor which makes the left side “telescope”

$$\begin{aligned} 1(T(n) - 3T(n/2)) &= n \\ 3(T(n/2) - 3T(n/4)) &= n/2 \\ &\vdots & \vdots & \vdots \\ 3^{\log_2 n - 1}(T(2) - 3T(1)) &= 2 \\ 3^{\log_2 n}(T(1) - 0) &= 1 \end{aligned}$$

Summing the left side we get  $T(n)$ . Summing the right side we get

$$\sum_{i=0}^{\log_2 n} \frac{n}{2^i} 3^i.$$

Let  $c = \frac{3}{2}$  and  $m = \log_2 n$ . Then we have

$$\begin{aligned} n \sum_{i=0}^m c^i &= n \left( \frac{c^{m+1} - 1}{c - 1} \right) \\ &= 2n(c \cdot c^{\log_2 n} - 1) \\ &= 2n(c \cdot c^{k \log_2 n} - 1) \\ &= 2n^{k+1} - 2n \approx 2n^{1.58496} - 2n, \end{aligned}$$

where  $k = (\log_2 \frac{3}{2})^{-1}$ . Full history recurrences can often be changed to limited history ones (example): Consider the following recurrence

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^i T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so  $T_{i+1} = 2T_i = 2^{i+1}$ .

Generating functions:

- Multiply both sides of the equation by  $x^i$ .
- Sum both sides over all  $i$  for which the equation is valid.
- Choose a generating function  $G(x)$ . Usually  $G(x) = \sum_{i=0}^{\infty} g_i x^i$ .
- Rewrite the equation in terms of the generating function  $G(x)$ .
- Solve for  $G(x)$ .
- The coefficient of  $x^i$  in  $G(x)$  is  $g_i$ .

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$$

We choose  $G(x) = \sum_{i \geq 0} x^i$ . Rewrite in terms of  $G(x)$ :

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for  $G(x)$ :

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x \left( \frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x \left( 2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\ &= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}. \end{aligned}$$

So  $g_i = 2^i - 1$ .

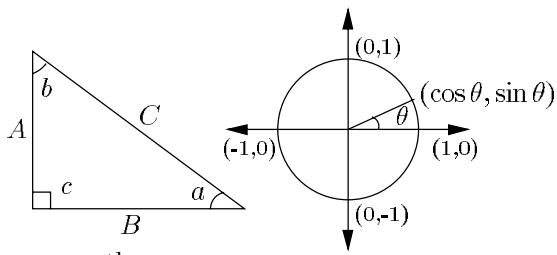
# Theoretical Computer Science Cheat Sheet

$$\pi \approx 3.14159, \quad e \approx 2.71828, \quad \gamma \approx 0.57721, \quad \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$$

$i$	$2^i$	$p_i$	General	Probability
1	2	2	Bernoulli Numbers ( $B_i = 0$ , odd $i \neq 1$ ): $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$ $B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$	Continuous distributions: If $\Pr[a < X < b] = \int_a^b p(x) dx,$
2	4	3	Change of base, quadratic formula: $\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	then $p$ is the probability density function of $X$ . If $\Pr[X < a] = P(a),$
3	8	5	Euler's number $e$ : $e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$ $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$	then $P$ is the distribution function of $X$ . If $P$ and $p$ both exist then $P(a) = \int_{-\infty}^a p(x) dx.$
4	16	7	$(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}.$	Expectation: If $X$ is discrete $E[g(X)] = \sum_x g(x) \Pr[X = x].$
5	32	11	$(1 + \frac{1}{n})^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$	If $X$ continuous then $E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$
6	64	13	Harmonic numbers: $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$	Variance, standard deviation: $\text{VAR}[X] = E[X^2] - E[X]^2,$ $\sigma = \sqrt{\text{VAR}[X]}.$
7	128	17	$\ln n < H_n < \ln n + 1,$	Basics: $\Pr[X \vee Y] = \Pr[X] + \Pr[Y] - \Pr[X \wedge Y]$
8	256	19	$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$	$\Pr[X \wedge Y] = \Pr[X] \cdot \Pr[Y],$ iff $X$ and $Y$ are independent.
9	512	23	Factorial, Stirling's approximation: $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$	$\Pr[X Y] = \frac{\Pr[X \wedge Y]}{\Pr[B]}$
10	1,024	29	$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$	$E[X \cdot Y] = E[X] \cdot E[Y],$ iff $X$ and $Y$ are independent.
11	2,048	31	Ackermann's function and inverse: $a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$	$E[X + Y] = E[X] + E[Y],$ $E[cX] = c E[X].$
12	4,096	37	$\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$	Bayes' theorem: $\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$
13	8,192	41		Inclusion-exclusion: $\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] + \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$
14	16,384	43		Moment inequalities: $\Pr[ X  \geq \lambda E[X]] \leq \frac{1}{\lambda},$ $\Pr[ X - E[X]  \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$
15	32,768	47		Geometric distribution: $\Pr[X = k] = p^{k-1} q, \quad q = 1 - p,$ $E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$
16	65,536	53		
17	131,072	59		
18	262,144	61		
19	524,288	67		
20	1,048,576	71		
21	2,097,152	73		
22	4,194,304	79		
23	8,388,608	83		
24	16,777,216	89		
25	33,554,432	97		
26	67,108,864	101		
27	134,217,728	103		
28	268,435,456	107	Binomial distribution: $\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$	
29	536,870,912	109	$E[X] = \sum_{k=1}^n k = 1k \binom{n}{k} p^k q^{n-k} = np.$	
30	1,073,741,824	113	Poisson distribution: $\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$	
31	2,147,483,648	127	Normal (Gaussian) distribution: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$	
32	4,294,967,296	131	The "coupon collector": We are given a random coupon each day, and there are $n$ different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we collect all $n$ types is $nH_n.$	
Pascal's Triangle				
	1			
	1 1			
	1 2 1			
	1 3 3 1			
	1 4 6 4 1			
	1 5 10 10 5 1			
	1 6 15 20 15 6 1			
	1 7 21 35 35 21 7 1			
	1 8 28 56 70 56 28 8 1			
	1 9 36 84 126 126 84 36 9 1			
1	10 45 120 210 252 210 120 45 10 1			

# Theoretical Computer Science Cheat Sheet

## Trigonometry



Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$

$$\csc a = C/A, \quad \sec a = C/B,$$

$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$$

$$\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$$

$$\sin x = \cos(\frac{\pi}{2} - x), \quad \sin x = \sin(\pi - x),$$

$$\cos x = -\cos(\pi - x), \quad \tan x = \cot(\frac{\pi}{2} - x),$$

$$\cot x = -\cot(\pi - x), \quad \csc x = \cot \frac{x}{2} - \cot x,$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$

$$\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$$

$$\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$

$$\sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y,$$

$$\cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$$

©1994 by Steve Seiden

sseiden@ics.uci.edu

<http://www.ics.uci.edu/~sseiden>

## Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$$

Determinants:  $\det A = 0$  iff  $A$  is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$$

$2 \times 2$  and  $3 \times 3$  determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - fha - ibd.$$

Permanents:

$$\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$$

## Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \csch x = \frac{1}{\sinh x},$$

$$\sech x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \sech^2 x = 1,$$

$$\coth^2 x - \csch^2 x = 1, \quad \sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$$

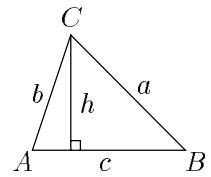
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$

$$2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$$

$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	$\infty$

... in mathematics you don't understand things, you just get used to them.  
- J. von Neumann

## More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Area:

$$A = \frac{1}{2}hc,$$

$$= \frac{1}{2}ab \sin C,$$

$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$

$$s = \frac{1}{2}(a+b+c),$$

$$s_a = s - a,$$

$$s_b = s - b,$$

$$s_c = s - c.$$

More identities:

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

$$\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{1 - \cos x}{\sin x},$$

$$= \frac{\sin x}{1 + \cos x},$$

$$\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$

$$= \frac{1 + \cos x}{\sin x},$$

$$= \frac{\sin x}{1 - \cos x},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$\sinh x = \frac{\sinh ix}{i},$$

$$\cos x = \cosh ix,$$

$$\tan x = \frac{\tanh ix}{i}.$$

## Theoretical Computer Science Cheat Sheet

Number Theory	Graph Theory								
<p>The Chinese remainder theorem: There exists a number <math>C</math> such that:</p> $C \equiv r_1 \pmod{m_1}$ $\vdots \vdots \vdots$ $C \equiv r_n \pmod{m_n}$ <p>if <math>m_i</math> and <math>m_j</math> are relatively prime for <math>i \neq j</math>.</p> <p>Euler's function: <math>\phi(x)</math> is the number of positive integers less than <math>x</math> relatively prime to <math>x</math>. If <math>\prod_{i=1}^n p_i^{e_i}</math> is the prime factorization of <math>x</math> then</p> $\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$ <p>Euler's theorem: If <math>a</math> and <math>b</math> are relatively prime then</p> $1 \equiv a^{\phi(b)} \pmod{b}.$ <p>Fermat's theorem:</p> $1 \equiv a^{p-1} \pmod{p}.$ <p>The Euclidean algorithm: if <math>a &gt; b</math> are integers then</p> $\gcd(a, b) = \gcd(a \bmod b, b).$ <p>If <math>\prod_{i=1}^n p_i^{e_i}</math> is the prime factorization of <math>x</math> then</p> $S(x) = \sum_{d x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$ <p>Perfect Numbers: <math>x</math> is an even perfect number iff <math>x = 2^{n-1}(2^n - 1)</math> and <math>2^n - 1</math> is prime.</p> <p>Wilson's theorem: <math>n</math> is a prime iff</p> $(n-1)! \equiv -1 \pmod{n}.$ <p>Möbius inversion:</p> $\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$ <p>If</p> $G(a) = \sum_{d a} F(d),$ <p>then</p> $F(a) = \sum_{d a} \mu(d) G\left(\frac{a}{d}\right).$ <p>Prime numbers:</p> $p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$ $\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$	<p><b>Definitions:</b></p> <ul style="list-style-type: none"> <li><b>Loop</b>: An edge connecting a vertex to itself.</li> <li><b>Directed</b>: Each edge has a direction.</li> <li><b>Simple</b>: Graph with no loops or multi-edges.</li> <li><b>Walk</b>: A sequence <math>v_0 e_1 v_1 \dots e_\ell v_\ell</math>.</li> <li><b>Trail</b>: A walk with distinct edges.</li> <li><b>Path</b>: A trail with distinct vertices.</li> <li><b>Connected</b>: A graph where there exists a path between any two vertices.</li> <li><b>Component</b>: A maximal connected subgraph.</li> <li><b>Tree</b>: A connected acyclic graph.</li> <li><b>Free tree</b>: A tree with no root.</li> <li><b>DAG</b>: Directed acyclic graph.</li> <li><b>Eulerian</b>: Graph with a trail visiting each edge exactly once.</li> <li><b>Hamiltonian</b>: Graph with a path visiting each vertex exactly once.</li> <li><b>Cut</b>: A set of edges whose removal increases the number of components.</li> <li><b>Cut-set</b>: A minimal cut.</li> <li><b>Cut edge</b>: A size 1 cut.</li> <li><b>k-Connected</b>: A graph connected with the removal of any <math>k-1</math> vertices.</li> <li><b>k-Tough</b>: <math>\forall S \subseteq V, S \neq \emptyset</math> we have <math>k \cdot c(G-S) \leq  S </math>.</li> <li><b>k-Regular</b>: A graph where all vertices have degree <math>k</math>.</li> <li><b>k-Factor</b>: A <math>k</math>-regular spanning subgraph.</li> <li><b>Matching</b>: A set of edges, no two of which are adjacent.</li> <li><b>Clique</b>: A set of vertices, all of which are adjacent.</li> <li><b>Ind. set</b>: A set of vertices, none of which are adjacent.</li> <li><b>Vertex cover</b>: A set of vertices which cover all edges.</li> <li><b>Planar graph</b>: A graph which can be embedded in the plane.</li> <li><b>Plane graph</b>: An embedding of a planar graph.</li> </ul> <p><b>Notation:</b></p> <ul style="list-style-type: none"> <li><math>E(G)</math>: Edge set</li> <li><math>V(G)</math>: Vertex set</li> <li><math>c(G)</math>: Number of components</li> <li><math>G[S]</math>: Induced subgraph</li> <li><math>\deg(v)</math>: Degree of <math>v</math></li> <li><math>\Delta(G)</math>: Maximum degree</li> <li><math>\delta(G)</math>: Minimum degree</li> <li><math>\chi(G)</math>: Chromatic number</li> <li><math>\chi_E(G)</math>: Edge chromatic number</li> <li><math>G^c</math>: Complement graph</li> <li><math>K_n</math>: Complete graph</li> <li><math>K_{n_1, n_2}</math>: Complete bipartite graph</li> <li><math>r(k, \ell)</math>: Ramsey number</li> </ul> <p><b>Geometry</b></p> <p>Projective coordinates: triples <math>(x, y, z)</math>, not all <math>x, y</math> and <math>z</math> zero.</p> $(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$ <table border="0"> <tr> <td>Cartesian</td> <td>Projective</td> </tr> <tr> <td><math>(x, y)</math></td> <td><math>(x, y, 1)</math></td> </tr> <tr> <td><math>y = mx + b</math></td> <td><math>(m, -1, b)</math></td> </tr> <tr> <td><math>x = c</math></td> <td><math>(1, 0, -c)</math></td> </tr> </table> <p>Distance formula, <math>L_p</math> and <math>L_\infty</math> metric:</p> $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$ $[ x_1 - x_0 ^p +  y_1 - y_0 ^p]^{1/p},$ $\lim_{p \rightarrow \infty} [ x_1 - x_0 ^p +  y_1 - y_0 ^p]^{1/p}.$ <p>Area of triangle <math>(x_0, y_0), (x_1, y_1)</math> and <math>(x_2, y_2)</math>:</p> $\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$ <p>Angle formed by three points:</p> $\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$ <p>Line through two points <math>(x_0, y_0)</math> and <math>(x_1, y_1)</math>:</p> $\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$ <p>Area of circle, volume of sphere:</p> $A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$ <p>If I have seen farther than others, it is because I have stood on the shoulders of giants. – Isaac Newton</p>	Cartesian	Projective	$(x, y)$	$(x, y, 1)$	$y = mx + b$	$(m, -1, b)$	$x = c$	$(1, 0, -c)$
Cartesian	Projective								
$(x, y)$	$(x, y, 1)$								
$y = mx + b$	$(m, -1, b)$								
$x = c$	$(1, 0, -c)$								

## Theoretical Computer Science Cheat Sheet

$\pi$

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \dots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \dots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\dots}}}}$$

Gregory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \dots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left( 1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \dots \right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots$$

### Partial Fractions

Let  $N(x)$  and  $D(x)$  be polynomial functions of  $x$ . We can break down  $N(x)/D(x)$  using partial fraction expansion. First, if the degree of  $N$  is greater than or equal to the degree of  $D$ , divide  $N$  by  $D$ , obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of  $N'$  is less than that of  $D$ . Second, factor  $D(x)$ . Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[ \frac{N(x)}{D(x)} \right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!} \left[ \frac{d^k}{dx^k} \left( \frac{N(x)}{D(x)} \right) \right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.  
— George Bernard Shaw

### Calculus

Derivatives:

1.  $\frac{d(cu)}{dx} = c \frac{du}{dx},$
2.  $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx},$
3.  $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$
4.  $\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx},$
5.  $\frac{d(u/v)}{dx} = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2},$
6.  $\frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$
7.  $\frac{d(c^u)}{dx} = (\ln c)c^u \frac{du}{dx},$
9.  $\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx},$
11.  $\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx},$
13.  $\frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx},$
15.  $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx},$
17.  $\frac{d(\arctan u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx},$
19.  $\frac{d(\text{arcsec } u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx},$
21.  $\frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx},$
23.  $\frac{d(\tanh u)}{dx} = \text{sech}^2 u \frac{du}{dx},$
25.  $\frac{d(\text{sech } u)}{dx} = -\text{sech } u \tanh u \frac{du}{dx},$
27.  $\frac{d(\text{arcsinh } u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx},$
29.  $\frac{d(\text{arctanh } u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx},$
31.  $\frac{d(\text{arcsech } u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$

Integrals:

1.  $\int cu \, dx = c \int u \, dx,$
2.  $\int (u+v) \, dx = \int u \, dx + \int v \, dx,$
3.  $\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1,$
4.  $\int \frac{1}{x} \, dx = \ln x,$
5.  $\int e^x \, dx = e^x,$
6.  $\int \frac{dx}{1+x^2} = \arctan x,$
8.  $\int \sin x \, dx = -\cos x,$
10.  $\int \tan x \, dx = -\ln |\cos x|,$
12.  $\int \sec x \, dx = \ln |\sec x + \tan x|,$
14.  $\int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$
7.  $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$
9.  $\int \cos x \, dx = \sin x,$
11.  $\int \cot x \, dx = \ln |\cos x|,$
13.  $\int \csc x \, dx = \ln |\csc x + \cot x|,$

## Theoretical Computer Science Cheat Sheet

### Calculus Cont.

15.  $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16.  $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17.  $\int \sin^2(ax) dx = \frac{1}{2a} (ax - \sin(ax) \cos(ax)),$
18.  $\int \cos^2(ax) dx = \frac{1}{2a} (ax + \sin(ax) \cos(ax)),$
19.  $\int \sec^2 x dx = \tan x,$
20.  $\int \csc^2 x dx = -\cot x,$
21.  $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22.  $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23.  $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24.  $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25.  $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26.  $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27.  $\int \sinh x dx = \cosh x,$
28.  $\int \cosh x dx = \sinh x,$
29.  $\int \tanh x dx = \ln |\cosh x|,$
30.  $\int \coth x dx = \ln |\sinh x|,$
31.  $\int \operatorname{sech} x dx = \arctan \sinh x,$
32.  $\int \operatorname{csch} x dx = \ln |\tanh \frac{x}{2}|,$
33.  $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2}x,$
34.  $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2}x,$
35.  $\int \operatorname{sech}^2 x dx = \tanh x,$
36.  $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37.  $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38.  $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39.  $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left( x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40.  $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41.  $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42.  $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44.  $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45.  $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46.  $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47.  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48.  $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49.  $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$
50.  $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51.  $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52.  $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53.  $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54.  $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56.  $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57.  $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58.  $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59.  $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60.  $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61.  $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

## Theoretical Computer Science Cheat Sheet

### Calculus Cont.

62.  $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \arccos \frac{a}{|x|}, \quad a > 0,$       63.  $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x},$

64.  $\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2},$       65.  $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3},$

66.  $\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right|, & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases}$

67.  $\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a} \sqrt{ax^2 + bx + c} \right|, & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases}$

68.  $\int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ax - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$

69.  $\int \frac{x dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$

70.  $\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{-1}{\sqrt{c}} \ln \left| \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right|, & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases}$

71.  $\int x^3 \sqrt{x^2 + a^2} dx = (\frac{1}{3}x^2 - \frac{2}{15}a^2)(x^2 + a^2)^{3/2},$

72.  $\int x^n \sin(ax) dx = -\frac{1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx,$

73.  $\int x^n \cos(ax) dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx,$

74.  $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx,$

75.  $\int x^n \ln(ax) dx = x^{n+1} \left( \frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right),$

76.  $\int x^n (\ln ax)^m dx = \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.$

### Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$

$$\mathrm{E} f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$$

$$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$$\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$$

$$\Delta(uv) = u\Delta v + \mathrm{E} v \Delta u,$$

$$\Delta(x^n) = nx^{n-1},$$

$$\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$$

$$\Delta(c^x) = (c-1)c^x, \quad \Delta(\binom{x}{m}) = \binom{x}{m-1}.$$

Sums:

$$\sum cu \delta x = c \sum u \delta x,$$

$$\sum (u+v) \delta x = \sum u \delta x + \sum v \delta x,$$

$$\sum u \Delta v \delta x = uv - \sum \mathrm{E} v \Delta u \delta x,$$

$$\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$$

$$\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$$

Falling Factorial Powers:

$$x^n = x(x-1)\cdots(x-m+1), \quad n > 0,$$

$$x^0 = 1,$$

$$x^n = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n < 0,$$

$$x^{n+m} = x^m (x-m)^n.$$

Rising Factorial Powers:

$$x^n = x(x+1)\cdots(x+m-1), \quad n > 0,$$

$$x^0 = 1,$$

$$x^n = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n < 0,$$

$$x^{n+m} = x^m (x+m)^n.$$

Conversion:

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (x-m+1)^{\overline{n}}$$

$$= 1/(x+1)^{\overline{-n}},$$

$$x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = (x+m-1)^{\underline{n}}$$

$$= 1/(x-1)^{\underline{-n}},$$

$$x^n = \sum_{k=1}^n \binom{n}{k} x^k = \sum_{k=1}^n \binom{n}{k} (-1)^{n-k} x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \binom{n}{k} (-1)^{n-k} x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \binom{n}{k} x^k.$$

$x^1 = \quad \quad \quad x^{\underline{1}} = \quad \quad \quad = \quad \quad \quad x^{\overline{1}}$	$x^2 = \quad \quad \quad x^{\underline{2}} + x^{\underline{1}} = \quad \quad \quad = \quad \quad \quad x^{\overline{2}} - x^{\overline{1}}$
$x^3 = \quad \quad \quad x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} = \quad \quad \quad = \quad \quad \quad x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$	
$x^4 = \quad \quad \quad x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} = \quad \quad \quad = \quad \quad \quad x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$	
$x^5 = \quad \quad \quad x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}} = \quad \quad \quad = \quad \quad \quad x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$	
$x^{\overline{1}} = \quad \quad \quad x^1 = \quad \quad \quad x^{\underline{1}} = \quad \quad \quad = \quad \quad \quad x^1$	
$x^{\overline{2}} = \quad \quad \quad x^2 + x^1 = \quad \quad \quad x^{\underline{2}} = \quad \quad \quad = \quad \quad \quad x^2 - x^1$	
$x^{\overline{3}} = \quad \quad \quad x^3 + 3x^2 + 2x^1 = \quad \quad \quad x^{\underline{3}} = \quad \quad \quad = \quad \quad \quad x^3 - 3x^2 + 2x^1$	
$x^{\overline{4}} = \quad \quad \quad x^4 + 6x^3 + 11x^2 + 6x^1 = \quad \quad \quad x^{\underline{4}} = \quad \quad \quad = \quad \quad \quad x^4 - 6x^3 + 11x^2 - 6x^1$	
$x^{\overline{5}} = \quad \quad \quad x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1 = \quad \quad \quad x^{\underline{5}} = \quad \quad \quad = \quad \quad \quad x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$	

## Theoretical Computer Science Cheat Sheet

### Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x}$$

$$= 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx}$$

$$= 1 + cx + c^2x^2 + c^3x^3 + \dots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n}$$

$$= 1 + x^n + x^{2n} + x^{3n} + \dots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2}$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \dots = \sum_{i=0}^{\infty} ix^i,$$

$$x^k \frac{d^n}{dx^n} \left( \frac{1}{1-x} \right)$$

$$= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x)$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \dots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$$

$$\ln \frac{1}{1-x}$$

$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}}$$

$$= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1}$$

$$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x})$$

$$= 1 + x + 2x^2 + 5x^3 + \dots = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}$$

$$= 1 + x + 2x^2 + 6x^3 + \dots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} \left( \frac{1-\sqrt{1-4x}}{2x} \right)^n$$

$$= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x} \ln \frac{1}{1-x}$$

$$= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2} \left( \ln \frac{1}{1-x} \right)^2$$

$$= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$$

$$\frac{x}{1-x-x^2}$$

$$= x + x^2 + 2x^3 + 3x^4 + \dots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$$

$$= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=0}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i-k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$x A'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If  $b_i = \sum_{j=0}^i a_j$  then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left( \sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers; all the rest is the work of man.

– Leopold Kronecker

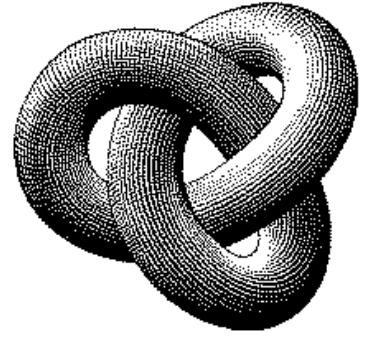
## Theoretical Computer Science Cheat Sheet

### Series

Expansions:

$$\begin{aligned}
 \frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} &= \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i, \\
 x^{\bar{n}} &= \sum_{i=0}^{\infty} \binom{n}{i} x^i, \\
 \left(\ln \frac{1}{1-x}\right)^n &= \sum_{i=0}^{\infty} \binom{i}{n} \frac{n! x^i}{i!}, \\
 \tan x &= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!}, \\
 \frac{1}{\zeta(x)} &= \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x}, \\
 \zeta(x) &= \prod_p \frac{1}{1-p^{-x}}, \\
 \zeta^2(x) &= \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d|n} 1, \\
 \zeta(x)\zeta(x-1) &= \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d|n} d, \\
 \zeta(2n) &= \frac{2^{2n-1}|B_{2n}|}{(2n)!} \pi^{2n}, \quad n \in \mathbb{N}, \\
 \frac{x}{\sin x} &= \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i-2)B_{2i}x^{2i}}{(2i)!}, \\
 \left(\frac{1-\sqrt{1-4x}}{2x}\right)^n &= \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i, \\
 e^x \sin x &= \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i, \\
 \sqrt{\frac{1-\sqrt{1-x}}{x}} &= \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2}(2i)!(2i+1)!} x^i, \\
 \left(\frac{\arcsin x}{x}\right)^2 &= \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.
 \end{aligned}$$

### Escher's Knot



### Stieltjes Integration

If  $G$  is continuous in the interval  $[a, b]$  and  $F$  is nondecreasing then

$$\int_a^b G(x) dF(x)$$

exists. If  $a \leq b \leq c$  then

$$\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$$

If the integrals involved exist

$$\begin{aligned}
 \int_a^b (G(x) + H(x)) dF(x) &= \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x), \\
 \int_a^b G(x) d(F(x) + H(x)) &= \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x), \\
 \int_a^b c \cdot G(x) dF(x) &= \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x), \\
 \int_a^b G(x) dF(x) &= G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).
 \end{aligned}$$

If the integrals involved exist, and  $F$  possesses a derivative  $F'$  at every point in  $[a, b]$  then

$$\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.$$

### Crammer's Rule

If we have equations:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n$$

Let  $A = (a_{i,j})$  and  $B$  be the column matrix  $(b_i)$ . Then there is a unique solution iff  $\det A \neq 0$ . Let  $A_i$  be  $A$  with column  $i$  replaced by  $B$ . Then

$$x_i = \frac{\det A_i}{\det A}.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.  
– William Blake (The Marriage of Heaven and Hell)

0	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	2	63
95	80	22	67	38	71	49	56	13	4
59	96	81	33	7	48	72	60	24	15
73	69	90	82	44	17	58	1	35	26
68	74	9	91	83	55	27	12	46	30
37	8	75	19	92	84	66	23	50	41
14	25	36	40	51	62	3	77	88	99
21	32	43	54	65	6	10	89	97	78
42	53	64	5	16	20	31	98	79	87

The Fibonacci number system:  
Every integer  $n$  has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where  $k_i \geq k_{i+1} + 2$  for all  $i$ ,  $1 \leq i < m$  and  $k_m \geq 2$ .

### Fibonacci Numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

Definitions:

$$F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$$

$$F_{-i} = (-1)^{i-1} F_i,$$

$$F_i = \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i),$$

Cassini's identity: for  $i > 0$ :

$$F_{i+1} F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$$

$$F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$