Student: Phong Vo **Date:** 04/29/20	**Instructor:** Erica Yankowskas **Course:** Linear Algebra I (Spring 2020)	**Assignment:** Section 4.4 Homework

1. Find the vector **x** determined by the given coordinate vector $[\mathbf{x}]_B$ and the given basis B.

$$B = \left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \end{bmatrix} \right\}, \ [\mathbf{x}]_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

(Simplify your answers.)

2. Find the vector **x** determined by the given coordinate vector $[\mathbf{x}]_B$ and the given basis B.

$$B = \left\{ \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} \right\}, \ [\mathbf{x}]_B = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -15 \\ 14 \\ 3 \end{bmatrix}$$

(Simplify your answers.)

3. Find the vector **x** determined by the given coordinate vector $[\mathbf{x}]_B$ and the given basis B.

$$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -5 \end{bmatrix} \right\}, \ [\mathbf{x}]_B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -25 \\ 9 \\ 32 \end{bmatrix}$$

(Simplify your answers.)

4. Find the coordinate vector $[\mathbf{x}]_B$ of **x** relative to the given basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$\mathbf{b}_1 = \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(Simplify your answers.)

5. Find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to the given basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 4 \\ -3 \\ -16 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 7 \\ -6 \\ 0 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

(Simplify your answers.)

6. Find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to the given basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -4 \\ 2 \\ -10 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

(Simplify your answers.)

7. Find the change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$.

$$B = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right\}$$

$$P_B = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$$

8. Use an inverse matrix to find $[\mathbf{x}]_B$ for the given $\mathbf{x}$ and B.

$$B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} -13 \\ -9 \end{bmatrix}$$

9. The set $B = \{1 + t^2, 2t - t^2, 1 + t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = -5 + 8t - 10t^2$ relative to B.

$$[\mathbf{p}]_B = \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}$$

(Simplify your answers.)

10. Mark each statement true or false. Justify each answer. Unless otherwise stated, B is a basis for a vector space V.

a. If B is the standard basis for $\mathbb{R}^n$, then the B-coordinate vector of an $\mathbf{x}$ in $\mathbb{R}^n$ is $\mathbf{x}$ itself.

b. The correspondence $[\mathbf{x}]_B \mapsto \mathbf{x}$ is called the coordinate mapping.

c. In some cases, a plane in $\mathbb{R}^3$ can be isomorphic to $\mathbb{R}^2$.

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a. If B is the standard basis for $\mathbb{R}^n$, then the B-coordinate vector of an $\mathbf{x}$ in $\mathbb{R}^n$ is $\mathbf{x}$ itself. Choose the correct answer below.

- **A.** The statement is true. The standard basis consists of the columns of the $n \times n$ identity matrix. So $[\mathbf{x}]_B = \mathbf{e}_1 + \cdot \cdot \cdot +$
- **B.** The statement is true. The standard basis consists of the columns of the $n \times n$ identity matrix. So $[\mathbf{x}]_B = x_1\mathbf{e}_1 + \cdot \cdot$
- **C.** The statement is false. If B is the standard basis for $\mathbb{R}^n$, then the B-coordinate vector is the inverse of $\mathbf{x}$.
- **D.** The statement is false. If B is the standard basis for $\mathbb{R}^n$, then the B-coordinate vector does not exist.

b. The correspondence $[\mathbf{x}]_B \mapsto \mathbf{x}$ is called the coordinate mapping. Choose the correct answer below.

- **A.** The statement is false. By the definition, the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_B$ is called the coordinate mapping.
- **B.** The statement is true because B is linearly dependent.
- **C.** The statement is true. By the definition, the correspondence $[\mathbf{x}]_B \mapsto \mathbf{x}$ is called the coordinate mapping.
- **D.** The statement is false because B is linearly dependent.

c. In some cases, a plane in $\mathbb{R}^3$ can be isomorphic to $\mathbb{R}^2$. Choose the correct answer below.

- **A.** The statement is false. A plane cannot be isomorphic to a line.
- **B.** The statement is false. A space cannot be isomorphic to a plane.
- **C.** The statement is true. A plane in $\mathbb{R}^3$ that passes through the origin is isomorphic to $\mathbb{R}^2$.
- **D.** The statement is true. A plane in $\mathbb{R}^3$ that passes through any point except the origin is isomorphic to $\mathbb{R}^2$.

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11. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} -10 \\ 28 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Write $\begin{bmatrix} -10 \\ 28 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ when the coefficient of $\mathbf{v}_3$ is 0.

$$\begin{bmatrix} -10 \\ 28 \end{bmatrix} = (2)\mathbf{v}_1 + (-4)\mathbf{v}_2$$

Write $\begin{bmatrix} -10 \\ 28 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ when the coefficient of $\mathbf{v}_3$ is 1.

$$\begin{bmatrix} -10 \\ 28 \end{bmatrix} = (7)\mathbf{v}_1 + (-5)\mathbf{v}_2 + \mathbf{v}_3$$

12. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the B-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Which of the following statements are true? Select all that apply.

- [x] **A.** By the Unique Representation Theorem, for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\mathbf{x}$
- [x] **B.** By the definition of a basis, $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are in V.
- [ ] **C.** By the definition of a basis, $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly dependent.
- [ ] **D.** By the definition of an isomorphism, V is isomorphic to $\mathbb{R}^{n+1}$.

Since $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are in V and since for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$, what is true of each $\mathbf{b}_k$ for $k = 1, \ldots, n$?

- ( ) **A.** $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_{k-1}\mathbf{b}_{k-1} + c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n$ for some unique set of scalars $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots,$
- ( ) **B.** $\mathbf{b}_k = \mathbf{b}_1 + \cdots + \mathbf{b}_k$
- (x) **C.** $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ for some unique set of scalars $c_1, \ldots, c_n$

Rewrite the expression for $\mathbf{b}_k$ given that the scalars $c_1, \ldots, c_n$ are unique by the Unique Representation Theorem. Choose the correct answer below.

- ( ) **A.** $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = 1 \cdot \mathbf{b}_1 + \cdots + 1 \cdot \mathbf{b}_k + \cdots + 1 \cdot \mathbf{b}_n$
- ( ) **B.** $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = 1 \cdot \mathbf{b}_1 + \cdots + 0 \cdot \mathbf{b}_k + \cdots + 1 \cdot \mathbf{b}_n$
- (x) **C.** $\mathbf{b}_k = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = 0 \cdot \mathbf{b}_1 + \cdots + 1 \cdot \mathbf{b}_k + \cdots + 0 \cdot \mathbf{b}_n$

Thus, the coordinate vector $[\mathbf{b}_k]_B$ of $\mathbf{b}_k$ is $\mathbf{e}_k$, or the kth column of the $n \times n$ identity matrix.

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13. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is a linearly dependent spanning set for a vector space V. Show that each $\mathbf{w}$ in V can be expressed in more than one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_5$. [Hint: Let $\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_5\mathbf{v}_5$ be an arbitrary vector in V. Use the linear dependence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ to produce another representation of $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_5$.]

Let $\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_5\mathbf{v}_5$ be an arbitrary vector in V. Since the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is linearly dependent, there exist scalars $c_1, \ldots, c_5$, not all zero, such that $\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_5\mathbf{v}_5$.

Add $\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_5\mathbf{v}_5$ and $\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_5\mathbf{v}_5$. Choose the correct answer below.

- **A.** $\mathbf{w} + \mathbf{0} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_5 + c_5)\mathbf{v}_5$ ✓
- **B.** $\mathbf{w} + \mathbf{0} = (k_5 + c_1)\mathbf{v}_1 + \cdots + (k_1 + c_5)\mathbf{v}_5$
- **C.** $\mathbf{w} + \mathbf{0} = (k_5 + c_5)\mathbf{v}_1 + \cdots + (k_1 + c_1)\mathbf{v}_5$
- **D.** $\mathbf{w} + \mathbf{0} = (k_1 + c_5)\mathbf{v}_1 + \cdots + (k_5 + c_1)\mathbf{v}_5$

What conclusion can be drawn from the statements above?

- **A.** None of the weights in $\mathbf{w} + \mathbf{0} = \mathbf{w} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_5 + c_5)\mathbf{v}_5$ are equal to the corresponding weights in $\mathbf{w}$
- **B.** All of the weights in $\mathbf{w} + \mathbf{0} = \mathbf{w} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_5 + c_5)\mathbf{v}_5$ are equal to the corresponding weights in $\mathbf{w} = k$
- **C.** At least one of the weights in $\mathbf{w} + \mathbf{0} = \mathbf{w} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_5 + c_5)\mathbf{v}_5$ differs from the corresponding weight ✓

Thus, each $\mathbf{w}$ in V can be expressed in more than one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

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14. If B is the standard basis of the space $\mathbb{P}_3$ of polynomials, then let $B = \{1, t, t^2, t^3\}$. Use coordinate vectors to test the linear independence of the set of polynomials below. Explain your work.

$1 - 8t^2 - t^3,\ t + 3t^3,\ 1 + t - 8t^2$

Write the coordinate vector for the polynomial $1 - 8t^2 - t^3$.

(________, ________, ________, ________)

Write the coordinate vector for the polynomial $t + 3t^3$.

(________, ________, ________, ________)

Write the coordinate vector for the polynomial $1 + t - 8t^2$.

(________, ________, ________, ________)

To test the linear independence of the set of polynomials, row reduce the matrix which is formed by making each coordinate vector a column of the matrix. If possible, write the matrix in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -8 & 0 & -8 \\ -1 & 3 & 0 \end{bmatrix} \sim \underline{\qquad\qquad}$$

Are the polynomials linearly independent?

- **A.** Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent
- **B.** Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are not linearly indepen
- **C.** Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly in
- **D.** Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are not linear

15. Let B be the standard basis of the space $\mathbb{P}_2$ of polynomials.Use coordinate vectors to test whether the following set of polynomials span $\mathbb{P}_2$. Justify your conclusion.

$1 - 3t + 2t^2,\ -4 + 9t - 2t^2,\ -1 + 4t^2,\ +3t - 6t^2$

Does the set of polynomials span $\mathbb{P}_2$?

- **A.** No; since the matrix whose columns are the B-coordinate vectors of each polynomial does not have a pivot positic
- **B.** No; since the matrix whose columns are the B-coordinate vectors of each polynomial does not have a pivot positic
- **C.** Yes; since the matrix whose columns are the B-coordinate vectors of each polynomial has a pivot position in each
- **D.** Yes; since the matrix whose columns are the B-coordinate vectors of each polynomial has a pivot position in each