Section 1.6: Applications of Linear Systems:

Note: Here we explore applications of linear systems coming from economics, chemistry, & network flow.

* Purpose: To demonstrate how real-world problems involving linear systems may have only one solution, no solution, &/or many solutions :)

Application #1

Homogeneous Systems in Economics; Equilibrium Prices

Here we explore Leontief's simple "**exchange model**".

- $\exists$ a nation's economy is divided into many sectors
  - *Ex: Industries (manufacturing, communication, etc.)
- For each sector, $\exists$ we know we know its total output (per yr.) -AND- how this output is divided/exchanged amongst other sectors in the economy.
- Price of that output: Total dollar value of a sector's output.
- Conclusion: $\exists$ 'Equilibrium Prices' that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

Example (How to Find Equilibrium Prices): ①

\$ an economy consists of the Coal, Electric (power) & Steel sectors, and the output of each sector is distributed among the various sectors as shown in the table below (Note: The entries in a column represent the fractional parts of a sector's total output).

→ Denote the prices (\$) of the total annual outputs of the Coal, Electric, & Steel sectors by P_C, P_E, & P_S, respectively. IF possible, find the equilibrium prices that make each sector's income match its expenditures.

Distribution of Output Form

Coal:	Electric:	Steel:	Purchased By:
0.0	0.4	0.6	Coal
0.6	0.1	0.2	Electric
0.4	0.5	0.2	Steel

Note: Since all outputs are taken into account, the decimal fractions in each column must sum to 1.

Answer:

* A sector looks down the column to: See where its output goes

* A sector looks across the row to: See what it needs as inputs

⇒ The sum of the rows entries, multiplied by their respective outputs, provides the total expenses per sector.

Example (How to Find Equilibrium Prices) continued...

***Step 1: Define the Total Expenses for each sector:**

Note: Since the values of total outputs are p_c, p_E, p_S

- Total Expenses of Coal: $0.4p_E + 0.6p_S$
- Total Expenses of Electric: $0.6p_c + 0.1p_E + 0.2p_S$
- Total Expenses of Steel: $0.4p_c + 0.5p_E + 0.2p_S$

*To find the Equilibrium, set "Total Income" = "Total Expenses"

$$\begin{cases} p_c = 0.4p_E + 0.6p_S \\ p_E = 0.6p_c + 0.1p_E + 0.2p_S \\ p_S = 0.4p_c + 0.5p_E + 0.2p_S \end{cases}$$

***Step 2: Define the System of Equations**

To define the System, we bring all the unknowns (p_c, p_E, p_S) to the LHS:

$$\begin{cases} p_c = 0.4p_E + 0.6p_S \\ p_E = 0.6p_c + 0.1p_E + 0.2p_S \\ p_S = 0.4p_c + 0.5p_E + 0.2p_S \end{cases} \sim \begin{cases} p_c - 0.4p_E - 0.6p_S = 0 \\ -0.6p_c + 0.9p_E - 0.2p_S = 0 \\ -0.4p_c - 0.5p_E + 0.8p_S = 0 \end{cases}$$

*Note (Optional): Since we are Not allowed calculators in this course, I am going to multiply the entire system by "10" to remove the decimals → Provides exact solution (verses approx.)

Example (How to Find Equilibrium Price) Continued...

$$\sim \begin{cases} 10p_c - 4p_z - 6p_s = 0 \\ -6p_c + 9p_z - 2p_s = 0 \\ -4p_c - 5p_z + 8p_s = 0 \end{cases} \overset{\frac{1}{2}R_1}{\sim} \begin{cases} 5p_c - 2p_z - 3p_s = 0 \\ -6p_c + 9p_z - 2p_s = 0 \\ -4p_c - 5p_z + 8p_s = 0 \end{cases}$$

*Step 3: Solve the System:

Convert the system to its equivalent augmented matrix & row reduce as usual :

$$\Rightarrow [A \mid 0] = \left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ -6 & 9 & -2 & 0 \\ -4 & -5 & 8 & 0 \end{array}\right]$$

- $\dfrac{R_3 + R_1}{\text{new } R_1} \rightarrow \left[\begin{array}{ccc|c} 1 & -7 & 5 & 0 \\ -6 & 9 & -2 & 0 \\ -4 & -5 & 8 & 0 \end{array}\right]$

- $\dfrac{6R_1 + R_2}{\text{new } R_2} \rightarrow \begin{bmatrix} 1 & -7 & 5 & 0 \\ 0 & -33 & 28 & 0 \\ -4 & -5 & 8 & 0 \end{bmatrix}$

Example (How to Find Equilibrium Price) Continued...

- $\begin{array}{r} 4R_1 \\ + \; R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -7 & 5 & 0 \\ 0 & -33 & 28 & 0 \\ 0 & -33 & 28 & 0 \end{array}\right]$

- $\begin{array}{r} -R_2 \\ + \; R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -7 & 5 & 0 \\ 0 & -33 & 28 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{-\frac{1}{33}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -7 & 5 & 0 \\ 0 & 1 & \frac{-28}{33} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ *

*Note: x_3 is a 'free variable' → Nontrivial Solution ∃ .

- $\begin{array}{r} 7R_2 \\ + \; R_1 \\ \hline \text{new } R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -31/33 & 0 \\ 0 & 1 & -28/33 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} p_c = \frac{31}{33} p_s \\ p_E = \frac{28}{33} p_s \\ p_s \text{ is free} \end{cases}$

$$* \; 7\left(\frac{-28}{33}\right) + 5 = \frac{-196}{33} + 5 = \frac{-196+165}{33}$$

*Step 4: Write the General Solution for Equilibrium price in vector form:

$$\boxed{\vec{p} = \begin{bmatrix} p_c \\ p_E \\ p_s \end{bmatrix} = \begin{bmatrix} \frac{31}{33} p_s \\ \frac{28}{33} p_s \\ p_s \end{bmatrix} = p_s \begin{bmatrix} 31/33 \\ 28/33 \\ 1 \end{bmatrix}}$$

*Note: Any non-negative choice for p_s results in a choice of equilibrium prices :)

Answer.

Example: $ an economy has only 2 sectors: Goods & Services. Each year, Goods sells 75% of its outputs to Services & keeps the rest; Services sells 69% of its outputs to Goods & retains the rest. Find equilibrium prices for the annual outputs of the Goods & Services sectors that make each sector's income match its expenditures.

Answer:

*Create a table for the Distribution of Output:

Recall: A sector looks...

i) Down the Column: To see where the output goes
Sum of each column = 1

ii) Down the Row: To see what it needs as input
Sum of the entries, multiplied by their respective outputs, = total expenses (per sector)

Goods:	Services:	Purchased By:
0.25	0.69	Goods
0.75	0.31	Services

*Let p_G = price of total annual output of "Goods" (total income)

p_S = price of total annual output of "Services" (total income)

Example Continued...

***Define the Total Income (i.e. output) = Total Expenses:**

- Good's Expenses: $0.25p_G + 0.69p_S$
- Service's Expenses: $0.75p_G + 0.31p_S$

$$\Rightarrow \boxed{\therefore \begin{cases} p_G = 0.25p_G + 0.69p_S \\ p_S = 0.75p_G + 0.31p_S \end{cases}}$$

***Find the General Solution for the Good & Services System**

$$\begin{cases} 0.75p_G - 0.69p_S = 0 \\ -0.75p_G + 0.69p_S = 0 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 0.75 & -0.69 & 0 \\ -0.75 & 0.69 & 0 \end{array}\right]$$

- $\frac{R_1 + R_2}{\text{new } R_2} \rightarrow \left[\begin{array}{cc|c} 0.75 & -0.69 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{\frac{R_1}{0.75}}{\sim} \left[\begin{array}{cc|c} 1 & -0.92 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Answ

$$\Rightarrow \begin{cases} p_G = 0.92\,p_S \\ p_S \text{ is free} \end{cases}$$

$\therefore$ General Sol. for Equil. Price:

$$\vec{p} = \begin{bmatrix} p_G \\ p_S \end{bmatrix} = p_S \begin{bmatrix} 0.92 \\ 1 \end{bmatrix}$$

Notes: *If $p_S = \$1000$, then: $p_G = 0.92(1000) = \$920$

Problem: $ that an economy consists of Coal, Electric, & Steel sectors. Denote the prices ($) of the total annual outputs of the coal, electric, & steel sectors by P_C, P_E, & P_S respectively. $ the general solution to find the equilibrium prices that make each sector's income match its expenditures is:

$$\begin{cases} P_C = 0.94 P_S \\ P_E = 0.9 P_S \\ P_S \text{ is free} \end{cases}$$

(a)
*One Set of Equilibrium Prices is $P_C = \$94$, $P_E = \$90$, & $P_S = \$100$. Find another set.

(b)
*$ the same economy used Japanese yen instead of dollars to measure the value of the various sector's output. Would this change the problem in any way?

Answer:

- Part (a): Since P_S is free → Choose any $\mathbb{Z}^+$:)

$$\boxed{\text{IF } P_S = \$300: \begin{cases} \cdot P_C = 0.94(300) = \$282 \\ \cdot P_E = 0.9(300) = \$270 \end{cases}}$$

Not an exclusive ans.

- Part (b): Will need to convert dollars to yen

∴ Multiply the equilibrium prices in general solution by a scalar. ⇒ IOW: Prices change, but ratio is the same.

Application #2

Balancing Chemical Equations

A chemical equation describes the quantities of substances consumed and produced by different chemical reactions.

Note: Since atoms can neither be destroyed nor created in a chemical reaction, chemical equations must be "balanced" ∴

To "Balance" a Chemical Equation:

- We need the total # of atoms on the LHS to match the total # of atoms on the RHS

→ IOW: Find $\vec{x} = \{x_1, x_2, \ldots, x_n\} \in \mathbb{Z}^+$ ST the entries (atoms) are whole #s (positive integers).

A Systematic Method For Balancing Chemical Equations:

- Set-up a vector equation that describes the # of atoms of each type present in a reaction

→ IOW: $A\vec{x} = [a_1 \ a_2 \cdots a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$

Example (Balancing Chemical Equations):

When propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) & water (H_2O), according to an equation of the form:

$$(x_1)C_3H_8 + (x_2)O_2 \longrightarrow (x_3)CO_2 + (x_4)H_2O$$

Balance this equation.

$\Rightarrow$ find whole #s $x_1, \ldots, x_4$ such that the total # of carbon (C), hydrogen (H), & oxygen (O) atoms on the LHS match the RHS.

Answer:

*Step 1: Construct a vector in $\mathbb{R}^3$ for each reactant & product that lists the # of atoms per molecule:

- Propane, C_3H_8: $\begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$ ← * Carbon ← * Hydrogen ← * Oxygen } same order for remaining vectors :)

- Oxygen, O_2: $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

- Water, H_2O: $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

- Carbon Dioxide, CO_2: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Example (Balancing Chemical Equations): Continued...

*Step 2: Set-Up a vector equation:

- Chemical Equation:

$$(x_1)\, C_3H_8 + (x_2)\, O_2 \longrightarrow (x_3)\, CO_2 + (x_4)\, H_2O$$

- Vector Equation:

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} - x_3\begin{bmatrix}1\\0\\2\end{bmatrix} - x_4\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\-2\end{bmatrix} + x_4\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

*Step 3: Solve the augmented matrix using Row Reduction:

$$[A \mid 0] = \left[\begin{array}{cccc|c}3 & 0 & -1 & 0 & 0\\ 8 & 0 & 0 & -2 & 0\\ 0 & 2 & -2 & -1 & 0\end{array}\right] \underset{\sim}{\tfrac{1}{2}R_2} \left[\begin{array}{cccc|c}3 & 0 & -1 & 0 & 0\\ 4 & 0 & 0 & -1 & 0\\ 0 & 2 & -2 & -1 & 0\end{array}\right]$$

$$\underset{\sim}{\tfrac{1}{3}R_1 \ \& \ \tfrac{1}{2}R_3} \left[\begin{array}{cccc|c}1 & 0 & -1/3 & 0 & 0\\ 4 & 0 & 0 & -1 & 0\\ 0 & 1 & -1 & -1/2 & 0\end{array}\right]$$

Example (Balancing Chemical Equations) Continued...

- $-4R_1 + R_2$ = new R_2 $\longrightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \end{array}\right] \overset{\frac{3}{4}R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \end{array}\right]$$

- Interchanging R_2 & R_3 $\longrightarrow$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array}\right]$$

* x_4 is free

→ Nontrivial Solution $\exists$:)

- $R_3 + R_2$ = new R_2 $\longrightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array}\right]$$

- $\frac{1}{3}R_3 + R_1$ = new R_1 $\longrightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} x_1 = \frac{1}{4}x_4 \\ x_2 = \frac{5}{4}x_4 \\ x_3 = \frac{3}{4}x_4 \\ x_4 \text{ is free} \end{cases}$$

∴ The General Sol. for the Balanced Eq:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1/4 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$$

Note: Since $\vec{x} \in \mathbb{Z}^+$, the smallest coefficients are produced when

$x_4 = 4$:)

BUT: This is NOT an exclusive solution.

Balance the Following chemical equation. Assume that the coefficient on H_2S is '3'.

$$B_2S_3 + H_2O \longrightarrow H_3BO_3 + H_2S$$

"Boron Sulfide reacts violently with water to form Boric Acid & Hydrogen Sulfide gas"... stinky!

Answer:

*Let $\vec{x}$ be a vector whose entries are the amount (whole ideal number) of each compound needed to balance the given chemical eq:

$$\rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 3 \end{bmatrix} \begin{array}{l} \rightarrow \text{Boron Sulfide} \\ \rightarrow \text{Water} \\ \rightarrow \text{Boric Acid} \\ \rightarrow \text{Hydrogen Sulfide Gas} \end{array}$$

So,

$$(x_1)B_2S_3 + (x_2)H_2O \longrightarrow (x_3)H_3BO_3 + (3)H_2S$$

*Set-up a vector that lists the number of atoms per molecule for each reactant/product in the eq: $\begin{bmatrix} B \\ S \\ H \\ O \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} : B_2S_3, \quad \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} : H_2O, \quad \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} : H_3BO_3, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} : H_2S$$

So,

$$(x_1)\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} = (x_3)\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} + (3)\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

*To Balance the Chemical Eq., set-up a vector equation & then row-reduce the equivalent augmented matrix to produce a general solution:

$$x_1\begin{bmatrix}2\\3\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\\1\end{bmatrix} = x_3\begin{bmatrix}1\\0\\3\\3\end{bmatrix} + \begin{bmatrix}0\\3\\6\\0\end{bmatrix}$$

$$x_1\begin{bmatrix}2\\3\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\\1\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\-3\\-3\end{bmatrix} = \begin{bmatrix}0\\3\\6\\0\end{bmatrix} \iff \underbrace{\begin{bmatrix}2&0&-1\\3&0&0\\0&2&-3\\0&1&-3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}0\\3\\6\\0\end{bmatrix}}_{\vec{b}}$$

$$*\left[A \mid \vec{b}\right] = \left[\begin{array}{ccc|c}2&0&-1&0\\3&0&0&3\\0&2&-3&6\\0&1&-3&0\end{array}\right] \overset{\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c}2&0&-1&0\\1&0&0&1\\0&2&-3&6\\0&1&-3&0\end{array}\right] \overset{\text{*interchanging } R_1, R_2, \,\&\, R_4}{\sim} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&-3&0\\0&2&-3&6\\2&0&-1&0\end{array}\right]$$

$$*\ \begin{array}{r}-2R_1\\+\ R_4\\\hline \text{new } R_4\end{array} \longrightarrow \left[\begin{array}{ccc|c}1&0&0&1\\0&1&-3&0\\0&2&-3&6\\0&0&-1&-2\end{array}\right] \overset{-R_4}{\sim} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&-3&0\\0&2&-3&6\\0&0&1&2\end{array}\right]$$

$$*\ \begin{array}{r}-2R_2\\+\ R_3\\\hline \text{new } R_3\end{array} \longrightarrow \left[\begin{array}{ccc|c}1&0&0&1\\0&1&-3&0\\0&0&3&6\\0&0&1&2\end{array}\right] \overset{\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&-3&0\\0&0&1&2\\0&0&1&2\end{array}\right]$$

$$*\ \begin{array}{r}-R_3\\+\ R_4\\\hline \text{new } R_4\end{array} \longrightarrow \left[\begin{array}{ccc|c}1&0&0&1\\0&1&-3&0\\0&0&1&2\\0&0&0&0\end{array}\right]$$

Ex Continued...

* $3R_3 + R_2$ = new R_2 →

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note: While we have found a unique/trivial solution, this b/c we assumed $x_4 = 3$:)

* R_4 shows us that "x_4" is free! (the amount of H_2S defines the general solution; IOW x_4 determines how much of the other compounds we need :))

$$\Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 6 \\ x_3 = 2 \\ x_4 \text{ is free} \end{cases}$$ (specifically "$x_4 = 3$" in this example)

∴ The "Particular" Solution Set of the Nonhomogeneous Equation $A\vec{x} = \vec{b}$ is defined by the vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 3 \end{bmatrix} \begin{array}{l} \to \text{Boron Sulfide} \\ \to \text{Water} \\ \to \text{Boric Acid} \\ \to \text{Hydrogen Sulfide} \end{array}$$

*(i.e The # of each compound needed to balance eq.)

∴ The Balanced Eq. (Assuming "3" Hydrogen Sulfide):

$$\Rightarrow (1)B_2S_3 + (6)H_2O \longrightarrow 2H_3BO_3 + 3H_2S$$

Example: Balance the following equation:

$$H_3O + CaCO_3 \rightarrow H_2O + Ca + CO_2$$

Answer:

* Set-up a vector in $\mathbb{R}^4$ for each reactant & product that lists the # of atoms per molecule:

$$H_3O: \begin{array}{c} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} H \\ O \\ Ca \\ C \end{array}, \quad CaCO_3: \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad H_2O: \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad Ca: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad CO_2: \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

* Set-up a vector eq:

$$(x_1)\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (x_2)\begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} = (x_3)\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (x_4)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (x_5)\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

* Solve the augmented matrix using row reduction:

$$[A \mid 0] = \left[\begin{array}{ccccc|c} 3 & 0 & -2 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right] \underset{R_3 \ :)}{\overset{\text{interchange } R_1, R_2 \ \&}{\sim}} \left[\begin{array}{ccccc|c} 1 & 3 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$$

Example Continued...

- $\dfrac{-3R_1 + R_3}{\text{new } R_3} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 3 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -9 & 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$

- $\dfrac{9R_2 + R_3}{\text{new } R_3} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 3 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -9 & 6 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$

- $\dfrac{-3R_2 + R_1}{\text{new } R_1} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -9 & 6 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right]$

- $\dfrac{-R_2 + R_4}{\text{new } R_4} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -9 & 6 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$

- $\dfrac{R_3 + R_1}{\text{new } R_1} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -6 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -9 & 6 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$

Ex. Continued...

$$\xrightarrow[\text{new } R_1]{6R_4 + R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -9 & 6 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{new } R_2]{R_4 + R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -9 & 6 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{new } R_3]{9R_4 + R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = 2x_5 \\ x_2 = x_5 \\ x_3 = 3x_5 \\ x_4 = x_5 \\ x_5 \text{ is free} \end{cases}$$

$\therefore$ General Sol:

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}$$

Answer

Application #3

Network Flow

Systems of Linear Equations arise when we study the Flow of some quantity through a network.

- A **network** consists of a set of points called 'junctions' or 'nodes', w/ lines/arcs called 'branches' connecting some or all of the junctions.

  * The direction of Flow in each branch is indicated
  * The Flow amount/rate is either shown or denoted by a variable.

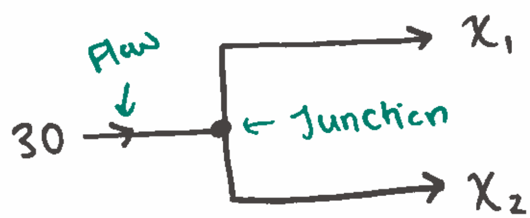
- The Basic Assumption of Network Flow is that:

  i) Total Flow into Network = Total Flow out of Network

  -AND-

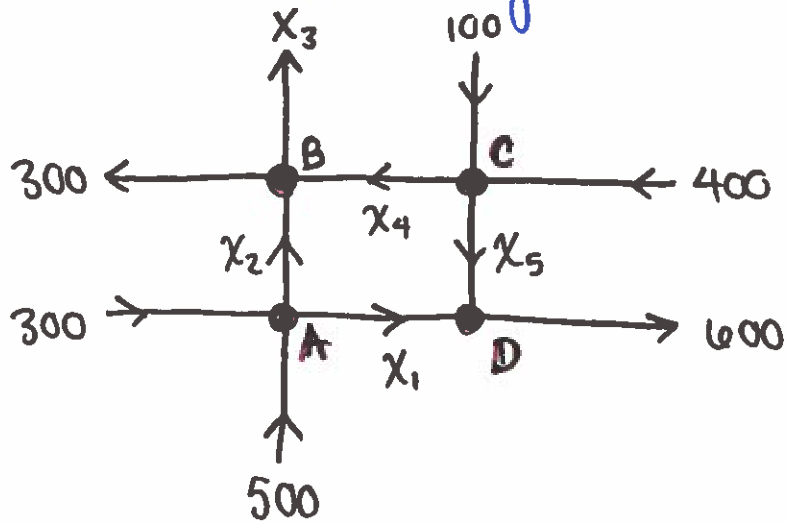
  ii) Total Flow into Junction = Total Flow out of Junction

- **Illustration:**



* 30 units Flows into a junction through 1 branch.
* x_1 & x_2 denote Flow out of junction through 2 branches.
* since "Flow" is conserved ⟹ $30 = x_1 + x_2$

Example (Network Flow): The network in the figure below shows the traffic flow (in vehicles per hour) over several one-way streets in a city during a typical early afternoon. Determine the general flow pattern for the network.



Answer:

*Step 1: Write an equation that describes the 'Flow' @ each junction:

Note:
- LHS of Eq = Flow into the junction.
- RHS of Eq = Flow out of the junction.

- Junction A: $300 + 500 = x_1 + x_2$
- Junction B: $x_2 + x_4 = x_3 + 300$
- Junction C: $100 + 400 = x_4 + x_5$
- Junction D: $x_1 + x_5 = 600$

*Step 2: Find the General Solution of the 'Flow' System:

- Bring all unknowns to LHS
- Use row reduction to solve the augmented matrix

Example (Network Flow): Continued...

$$\begin{cases} x_1 + x_2 = 800 \\ x_2 - x_3 + x_4 = 300 \\ x_4 + x_5 = 500 \\ x_1 + x_5 = 600 \end{cases} \iff \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{array}\right]$$

• $-R_1$
$+ R_4$
new R_2

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & -1 & 0 & 0 & 1 & -200 \end{array}\right] \underset{\sim}{\text{interchange } R_2, R_3, \& R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & -1 & 0 & 0 & 1 & -200 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

• R_2
$+ R_1$
new R_1

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & -1 & 0 & 0 & 1 & -200 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

R_2
$+ R_3$
new R_3

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & -1 & 0 & 0 & 1 & -200 \\ 0 & 0 & -1 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right] \underset{-R_3 \rightarrow}{\overset{-R_2 \rightarrow}{\sim}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & -1 & -1 & -100 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

R_4
$+ R_3$
new R_3

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{array}\right] \iff \begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \text{ is free} \end{cases}$$

Example (Network Flow) Continued...

∴ General Solution For the 'Flow' Pattern of the Network:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 600 - x_5 \\ 200 + x_5 \\ 400 \\ 500 - x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 600 \\ 200 \\ 400 \\ 500 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Notes: A negative flow in a network branch corresponds to flow in the direction opposite to that shown in the model

* For this specific example

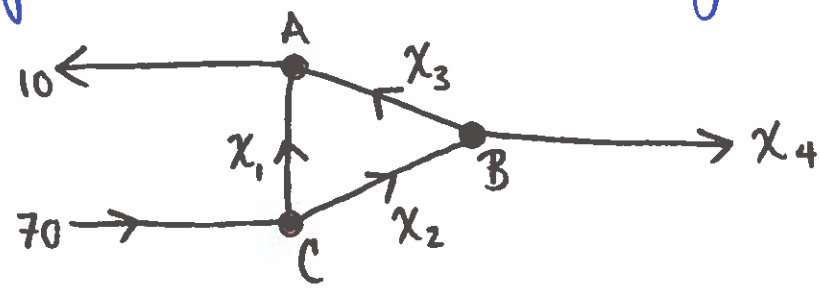
- Since the streets are all one-way ⇒ NO variable can be negative!

⇒ Since Eq(1) would produce ⊖ values, we can use Eq(4) to find restrictions/limitations on variables ∵

$500 - x_5 \geqslant 0 \rightarrow \boxed{500 \geqslant x_5}$ ✓

* Caution: If we use Eq(1), then $x_5 \leq 600$...
This is O.K. for Eq(1), but if $x_5 \in (500, 600]$ then $x_4 < 0$ ∴

Example: Find the general flow pattern for network shown in the figure below. Assuming that the flows are all non-negative, what is the largest possible value for x_3?



Answer:

* Write an equation that describes the flow @ each node:

Note: Here we will let

i) LHS = the flow into the node

ii) RHS = the flow out of the node.

- Node A: $x_1 + x_3 = 10$
- Node B: $x_2 = x_3 + x_4$
- Node C: $70 = x_1 + x_2$

$$\Longrightarrow \begin{cases} x_1 + x_3 = 10 \\ x_2 - x_3 - x_4 = 0 \\ x_1 + x_2 = 70 \end{cases}$$

* Find a General Solution for this Network Flow:

Note: Solve the equivalent augmented matrix using row reductio

$$[A \,\vdots\, \vec{b}] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 70 \end{array}\right]$$

Example Continued...

- $\begin{array}{r} -R_1 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 60 \end{array}\right]$

- $\begin{array}{r} -R_2 \\ +R_3 \\ \hline \text{new } R_3 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 60 \end{array}\right]$

- $\begin{array}{r} R_3 \\ +R_2 \\ \hline \text{new } R_2 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 10 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \end{array}\right]$

* Note: x_3 is free; Nontrivial sol. $\exists$

* Therefore: The General Solution is:

$$\begin{cases} x_1 = 10 - x_3 \\ x_2 = x_3 + 60 \\ x_3 \text{ is free} \\ x_4 = 60 \end{cases} \iff \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 - x^3 \\ 60 + x_3 \\ x_3 \\ 60 \end{bmatrix}$$

* Since $\vec{x} \geq 0$:

$10 - x_3 \geq 0 \rightarrow 10 \geq x_3$

$$\boxed{\vec{x} = \begin{bmatrix} 10 \\ 60 \\ 0 \\ 60 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$$

$\boxed{\therefore \text{Largest possible value for } x_3 \text{ is } 10.}$