1. Links

Definition 1.1: Let $\mathcal{K}$ be a simplicial complex and $\sigma \in \mathcal{K}$ be a simplex. The **link** of $Lk(\sigma)$ consists of all simplices $\tau \in \mathcal{K}$ such that $\sigma \cap \tau = \emptyset$ and $\sigma \cup \tau$ is a simplex in $\mathcal{K}$.

Theorem 1.1: $Lk(\sigma)$ is a subcomplex of $\mathcal{K}$.

Proof: We need to show that $Lk(\sigma)$ is downward closed, so let $\tau \in Lk(\sigma)$ and $\tau' \subseteq \tau$. Then $\tau' \cap \sigma \subseteq \tau \cap \sigma = \emptyset$, so $\tau' \cap \sigma = \emptyset$. Similarly, $\tau' \cup \sigma \subseteq \tau \cup \sigma$, so by downward closure of $\mathcal{K}$, $\tau' \cup \sigma$ is also a simplex in $\mathcal{K}$. Thus, $\tau' \in Lk(\sigma)$, so $Lk(\sigma)$ is downward closed. This makes $Lk(\sigma)$ a subcomplex of $\mathcal{K}$. $\square$