Power Series

Power Series and Functions

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x_2 + \ldots\ldots a$$

is a power series centered at x = 0. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \ldots\ldots$$

s a power series centered at x = a.

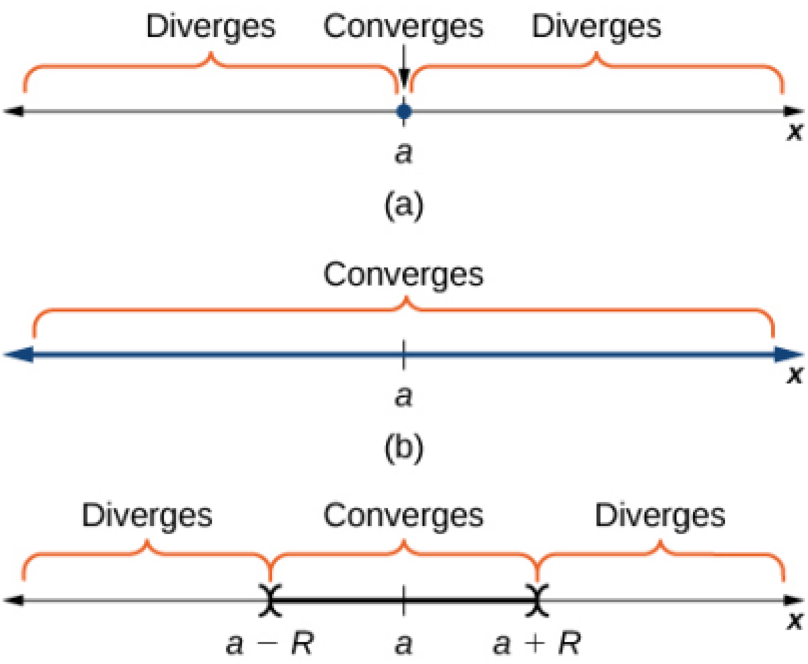
Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series entered at x = a, the value of the series at x = a is given by c0.

> The following series meets exactly one of the bellow properites
>
> $$\sum_{n=0}^{\infty} c_n (x-a)^n$$
>
> 1. The series converges at x = a and diverges for all x /= a
> 2. The series converges for all real numbers x
> 3. There exists a real number R > 0 such that the series converges if |x − a| < R and diverges if |x − a| > R. At the values x where |x − a| = R, the series may converge or diverge

If a series falls into case iii. of Convergence of a Power Series, then the series converges for all x such that |x − a| < R for some R > 0,and diverges for all x such that |x − a| > R. The set of values for which the series converges is the Interval of converges. Since the series diverges for all values x where |x − a| > R, the length of the interval is 2R, and therefore, the radius of the interval is R

> Consider the power series
>
> $$\sum_{n=0}^{\infty} c_n (x-a)^n$$
>
> The set of real numbers x where the series converges is the interval of convergence. If there exists a real number R > 0 such that the series converges for |x − a| < R and diverges for |x − a| > R, then R is the radius of convergence. If the series converges only at x = a, we say the radius of convergence is R = 0. If the series converges for all real numbers x, we say the radius of convergence is R = ∞



Representing Functions as Power Series

// TODO

Properties of Power Series

Combining power series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence.

> Suppose that the two power series converge to the functions f and g, respectively, on a common interval I.
>
> 1. The sum of the power series converge to f+g on I
> 2. A power series multiplies by constant b will converge at the function b*f
> 3. For any integer m ≥ 0 and any real number b, the series
>
> $$\sum_{n=0}^{\infty} c_n (bx^m)^n$$
>
> converges to f(bxm) for all x such that bx m is in I.

Multiplication of Power Series

We can multiply power series just as we would any other polynomial so long as the converge to the same interval

Differentiating and Integrating Power Series

power series can me differentiated and integrated on a perm by term basis so long as f(x) are differentiable on I

> let

> $$\sum_{n=0}^{\infty} c_n(x-a)^n$$
>
> and
>
> $$\sum_{n=0}^{\infty} d_n(x-a)^n$$
>
> be two convergent power series such that
>
> $$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} d_n(x-a)^n$$
>
> for all x in an open interval containing a. Then cn = dn for all n ≥ 0.

Taylor and Maclaurin Series

Overview of Taylor/Maclaurin Series

if f has derivatives of all orders at x = a, then the Taylor series for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{n!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

if the talyor series is centred at 0 it is called the Maclaurin series for f

> If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth Taylor polynomial

If f has n derivatives at x = a, then the nth Taylor polynomial for f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{n!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f

Taylor's Theorem with Remainder

Recall that the nth Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials {pn} converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f . To answer this question, we define the remainder Rn (x) as

$$R_n(x) = f(x) = p_n(x)$$

> Let f be a function that can be differentiated n+1 times on an interval I containing the real number a. Lets p_n be the nth Taylor polynomial of f at a ad let
>
> $$R_n(x) = f(x) - p_n(x)$$

> be the nth remainder then for each x in the interval I there exists a real number c between a and x such that
>
> $$R_n(x) = \frac{f^{n+1}c}{(n+1)!}(x-a)^{n+1}$$
>
> if there exist a real number M such that
>
> $$|f^{(n+1)}(x)| \leq M$$
>
> for all x $\in$ I then
>
> $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$
>
> for all x in I

Representing Functions with Taylor and Maclaurin Series

> Suppose that f has derivatives of all orders on an interval I containing a. the taylor series
>
> $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
>
> converges to f(x) for all x in I if and only if
>
> $$\lim_{n\to\infty} R_n(x) = 0$$
>
> for all x in I

Working with Taylor Series

The Binomial Series

The binomial series is the Maclaurin series ofr f(x) =(1+x)^r. It converges to f for |x| < 1 and we write

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

$$= 1 + rx + \frac{r(r-1)x^2}{2!} + \frac{r(r-1).\dots.(r-n+1)x^n}{n!}$$

Common Functions Expressed as Taylor Series

Function	Maclaurin Series	Interval of Convergence
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x \leq 1$
$f(x) = \tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 < x \leq 1$
$f(x) = (1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y$$

Recall that this is a first-order separable equation and its solution is y = Cex.

Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

> Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x_2 + \ldots\ldots a$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 \ldots$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x_2 + \ldots = c_1 + 2c_2 x + \ldots\ldots$$

Using Uniqueness of Power Series on the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$c_0 = c_1$$

$$c_1 = 2c_2$$

$$c_2 = 3c_3$$

$$c_3 = 4c_4$$

Using the initial condition y(0) = 3 combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x_2 + \ldots\ldots a$$

we find that c0 = 3. We are now ready to solve for the rest of the coefficients. Using the fact that c0 = 3, we have

$$c_1 = c_0 = 3 = \frac{3}{1!}$$
$$c_2 = \frac{c_1}{2} = \frac{3}{2} = \frac{3}{3!}$$
$$c_3 = \frac{c_2}{3} = \frac{3}{3 * 2} = \frac{3}{3!}$$
$$c_4 = \frac{c_4}{4} = \frac{3}{4 * 3 * 2} = \frac{3}{4!}$$

Therefore,

$$y = 3 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x so:

$$y = 3e^x$$

Nonelementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions