Linear Approximations and Differentials

Linear Approximation of a Function at a Point

we can find a rough approximation of f(b) where b approaches a, where f(x) is a differentiable function , by using the tangent line at (a,f(a)

Differentials

We can also find the rate at which an output changes relative to a change in input at a certain point (the differential) by rearranging

$$f'(x) = \frac{dy}{dx}$$

$$dy = f'(x)dx$$

for example: Find the differential at f(3) and dx = 0.1 for the function:

$$f(x) = x^2 + 2x$$

$$f'(x) = 2x + 2$$

$$dy = (2x + 2) * 0.1$$

$$dy = (2(3) + 2) * 0.1$$

$$dy = 0.8$$

We can connect this concept to linear approximations as it shows that:

$$\Delta y = f(a + dx) - f(a)$$

$$f(a + dx) \approx L(a + dx)$$

$$f(a + dx) - f(a) \approx L(a + dx) - L(a) = f'(a)dx$$

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Maxima and Minima

Absolute Extrema

> Let f be a function defined over an interval I and let c ∈ I. We say f has an absolute maximum on I at c if f(c) ≥ f(x) for all x ∈ I. We say f has an absolute minimum on I at c if f(c) ≤ f(x) for all x ∈ I. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c.

For example the function f(x) = x^2 + 1 over the interval {−∞, ∞} has no absolute maximum as the limit of f(x) as x → ±∞ is ∞. However is has an absolute minimum value of 1 when x = 0.

If f is a continuous function over the closed, **bounded** interval [a, b], then there is a point in [a, b] at which f has an absolute maximum over [a, b] and there is a point in [a, b] at which f has an absolute minimum over [a, b].

Local Extrema and Critical Points

A local extrema is a value which is a high/low point relative to surrounding values but is not definitely the highest/lowest value of f(x) over interval {−∞, ∞}

if f has an absolute extremum at c and f is defined over an interval containing c, then f(c) is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum

> Let c be an interior point in the domain of f . We say that c is a critical point of f if f '(c) = 0 or f '(c) is undefined

If f has a local extremum at c and f is differentiable at c, then f '(c) = 0. (Ferrmats theorem)

Locating Absolute Extrema

> Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

We can find the location of the absolute extrema by following these steps

1. Evaluate f at the endpoints x = a and x = b.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2), the largest of these values is the absolute maximum of f . The
smallest of these values is the absolute minimum of f

The Mean Value Theorem

Rolle’s Theorem

Rolle’s theorem states that if the outputs of a differentiable function f are equal at the endpoints of an interval,
then there must be an interior point c where f '(c) = 0

> Let f be a continuous function over the closed interval [a, b] and differentiable over the open interval (a, b) such that f(a) = f(b). There then exists at least one c ∈ (a, b) such that f '(c) = 0.

An important point about Rolle’s theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function f(x) = |x| − 1 is continuous over [−1, 1] and f(−1) = 0 = f(1), but f '(c) /= 0 for any c ∈ (−1, 1)

The Mean Value Theorem and Its Meaning

The Mean Value Theorem generalizes Rolle’s theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle’s theorem. The Mean Value Theorem states that if f is continuous over the
closed interval [a, b] and differentiable over the open interval (a, b), then there exists a point c ∈ (a, b) such that the tangent line to the graph of f at c is parallel to the secant line connecting (a,f(a)) and (b,f(b)).

> Let f be continuous over the closed interval [a, b] and differentiable over the open interval (a, b). Then, there
> exists at least one point c ∈ (a, b) such that

> $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Corollaries of the Mean Value Theorem

1. > Let f be differentiable over an interval I. If f '(x) = 0 for all x ∈ I, then f(x) = constant for all x ∈ I
2. > If f and g are differentiable over an interval I and f '(x) = g'(x) for all x ∈ I, then f(x) = g(x) + C for some constant C
3. > Let f be continuous over the closed interval [a, b] and differentiable over the open interval (a, b).
   >
   > 1. If f '(x) > 0 for all x ∈ (a, b), then f is an increasing function over [a, b]
   > 2. If f '(x) < 0 for all x ∈ (a, b), then f is a decreasing function over [a, b].

Derivatives and the Shape of a Graph

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I

> Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then f(c) satisfies one of the following descriptions:
>
> 1. If f ' changes sign from positive when x < c to negative when x > c, then f(c) is a local maximum of f .
> 2. ii. If f ' changes sign from negative when x < c to positive when x > c, then f(c) is a local minimum of f .
> 3. iii. If f ' has the same sign for x < c and x > c, then f(c) is neither a local maximum nor a local minimum off .

Concavity and Points of Inflection

If the graph curves, does it curve upward or curve downward? This notion is called the concavity of the function

> Let f be a function that is differentiable over an open interval I. If f ' is increasing over I, we say f is concave up over I. If f ' is decreasing over I, we say f is concave down over I.

We test for concavity using the following theorem

> Let f be a function that is twice differentiable over an interval I.
>
> 1. If f ''(x) > 0 for all x ∈ I, then f is concave up over I.
> 2. ii. If f ''(x) < 0 for all x ∈ I, then f is concave down over I.

If f''(x) = 0, it meant the concavity changes. This is called a point of inflection.

> If f is continuous at a and f changes concavity at a, the point (a, f(a)) is an inflection point of f .

The Second Derivative Test

Using the second derivative can sometimes be a simpler method than using the first derivative. If a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. The second derivative test can be used to determine whether a function has a local extremum at a critical point

> Suppose f '(c) = 0, f '' is continuous over an interval containing c.
>
> 1. If f ''(c) > 0, then f has a local minimum at c.
> 2. If f ''(c) < 0, then f has a local maximum at c.
> 3. If f ''(c) = 0, then the test is inconclusive

Limits at Infinity and Asymptotes

Limits at Infinity

Limits at Infinity and Horizontal Asymptotes

f(x) becomes arbitrarily close to L as long as x is sufficiently close to a because:

> If the values of f(x) becomes arbitrarily close to L for x < 0 as |x| becomes sufficiently large, we say that the function f has a limit at negative infinity and write
>
> $$\lim x \rightarrow^{-} \infty f(x) = L$$

This means that as x approaches infinity f(x) approaches the line y=L making it a horizontal asymptote

> If x lim→ ∞ f(x) = L or x →lim−∞ f(x) = L, we say the line y = L is a horizontal asymptote of f

Infinite Limits at Infinity

Sometimes f(x) will approach ±∞ as x approaches ±∞

> We say a function f has an infinite limit at infinity and write
>
> $$\lim x \rightarrow^{-} \infty f(x) = \infty$$
>
> if f(x) becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and
> write
>
> $$\lim x \rightarrow^{-} \infty f(x) = -\infty$$
>
> if f(x) < 0 and | f(x)| becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as
>
> $$x \rightarrow -\infty$$

End Behaviour

The behavior of a function as x → ±∞ is called the function's end behavior. At each of the function's ends, the function
could exhibit one of the following types of behaviour:

1. The function f(x) approaches a horizontal asymptote y = L.
2. The function f(x) → ∞ or f(x) → −∞.
3. The function does not approach a finite limit, nor does it approach ∞ or −∞. In this case, the function may have
   some oscillatory behaviour.

End Behavior for Polynomial Functions

for function:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} \ldots\ldots a_1 x + x^0$$

we can rearrange to

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1} x^{n-1}}{a_n x^n} + \frac{a_{n-2} x^{n-2}}{a_n x^n} \ldots\ldots \frac{a_1 x}{a_n x^n} + \frac{x^0}{a_n x^n}\right)$$

This means that as x approaches infintiy all terms expect for anxn will equate to 0. Meaning:

$$\lim x \to \infty f(x) = \lim x \to \infty a_n x^n$$

End Behavior for Algebraic Functions

To evaluate the limits at infinity for a rational function,we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as x → ±∞. For example, sinx oscillates between 1 and −1. The tangent function x has an infinite number of vertical asymptotes as x → ±∞; therefore, it does not approach a finite limit nor does it approach ±∞ as x → ±∞

Applied Optimization Problems

Solving Optimization Problems over a Closed, Bounded Interval

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables
   (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may
   involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to
   write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for
   critical points and evaluating a function at endpoints.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

Let’s now consider functions for which the domain is neither closed nor bounded. Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded For example, the function f(x) = x^2 + 4 over (−∞, ∞) has an absolute minimum of 4 at x = 0. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema.

L’Hôpital’s Rule

Applying L’Hôpital’s Rule

L’Hôpital’s Rule can be applied when a limit involving a quotient isn't definable. For example:

1. When lim f(x)/g(x) evaluates to 0/0
2. When lim f(x)/g(x) evaluates to ∞/∞

> Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If
>
> $$\lim x \to a \frac{f(x)}{g(x)} = \frac{0}{0}$$
>
> then
>
> $$\lim x \to a \frac{f(x)}{g(x)} = \lim x \to a \frac{f'(x)}{g'(x)}$$
>
> assuming the limit on the right exists or is ∞ or −∞. This result also holds if we are considering one-sided limits, or if a = ∞ and − ∞.

> Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If
>
> $$\lim x \to a \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$
>
> then
>
> $$\lim x \to a \frac{f(x)}{g(x)} = \lim x \to a \frac{f'(x)}{g'(x)}$$
>
> assuming the limit on the right exists or is ∞ or −∞. This result also holds if we are considering one-sided limits, or if a = ∞ and − ∞.

Growth Rates of Functions

suppose f and g are two functions that approach infinity as x → ∞. We say g grows more rapidly than f as x → ∞ if

$$\lim x \to a \frac{f(x)}{g(x)} = 0$$

$$\lim x \to a \frac{g(x)}{f(x)} = \infty$$

On the other hand, we say that f(x) and g(x) have the same growth rate if there exists a constant M /= 0 such that

$$\lim x \to a \frac{f(x)}{g(x)} = M$$

Antiderivatives

The Reverse of Differentiation

If we can find a function F derivative f , we call F an antiderivative of f

> A function F is an antiderivative of the function f if
>
> $$F\prime(x) = f(x)$$
>
> for all x in the domain of f .

> Let F be an antiderivative of f over an interval I. Then,
>
> 1. for each constant C, the function F(x) + C is also an antiderivative of f over I;
> 2. if G is an antiderivative of f over I, there is a constant C for which G(x) = F(x) + C over I.
>
>   In other words, the most general form of the antiderivative of f over I is F(x) + C.

Indefinite Integrals

If F is an antiderivative of f , we say that F(x) + C is the most general antiderivative of f and write

$$\int f(x)dx = F(x) + C.$$

∫ f(x)dx is called the indefinite integral of f

> Given a function f , the indefinite integral of f , denoted
>
> $$\int f(x)dx$$
>
> is the most general antiderivative of f . If F is an antiderivative of f , then
>
> $$\int f(x)dx = F(x) + C$$
>
> The expression f(x) is called the integrand and the variable x is the variable of integration

The collection of all functions of the form x^2 + C, where C is any real number, is known as the family of antiderivatives-

> For n /= −1,
>
> $$\int x^n = \frac{x^{n+1}}{n+1} + C$$