1. Unit 7 Techniques of Integration

Integration by Parts

The Integration-by-Parts Formula

Let u = f(x) and v = g(x) be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int uv' = uv - \int vu'$$

We choose which function to be u and which to be v' based on this acronym:

LIATE (Logarithmic Functions, Inverse Trigonometric Functions, Algebraic Functions, Trigonometric Functions, and Exponential Functions.) the function with the highest president in the acronym will be u

Integration by Parts for Definite Integrals

Let u = f(x) and v = g(x) be functions with continuous derivatives on [a, b]. Then

$$\int_a^b uv' = uv|_a^b - \int_a^b vu'$$

Trigonometric Integrals

Integrating Products and Powers of sinx and cosx

A key idea behind the strategy used to integrate combinations of products and powers of sinx and cos x involves rewriting these expressions as sums and differences of integrals of the form

$$\int \sin^j x \cos x$$

or

$$\int \cos^j x \sin x$$

After rewriting these integrals, we evaluate them using u-substitution.

> To integrate products involving sin(ax), sin(bx), cos(ax), and cos(bx), use the substitutions
>
> $$sin(ax)sin(bx) = \frac{1}{2}cos((a-b)x) - \frac{1}{2}cos((a+b)x)$$
>
> $$sin(ax)cos(bx) = \frac{1}{2}sin((a-b)x) + \frac{1}{2}cos((a+b)x)$$
>
> $$cos(ax)cos(bx) = \frac{1}{2}cos((a-b)x) + \frac{1}{2}cos((a+b)x)$$

Reduction Formulas

$$\int sec^n xdx = \frac{1}{n-1} sec^{n-2} xtanx + \frac{n-2}{n-1} \int sec^{n-2} xdx$$

$$\int tan^n xdx = \frac{1}{n-1} tan^{n-1} x - \int tan^{n-2} xdx$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of tanx.

Trigonometric Substitution

Integrals Involving

$$\sqrt{a^2 - x^2}$$

we solve these integrals using the following steps:

1. It is a good idea to make sure the integral cannot be evaluated easily in another way
2. Make the substitution x = asinθ and dx = acosθdθ. Note: This substitution yields
3. Simplify the expression
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 3.4 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship θ = sin−1 (x/a)
   .

Problem-Solving Strategy: Integrating Expressions Involving

$$\sqrt{a^2 + x^2}$$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method
2. Substitute x = atanθ and dx = asec2 θdθ.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 3.7 to rewrite the result in terms of x. You may also need to use
   some trigonometric identities and the relationship θ = tan−1 (x/a)

Integrals Involving

$$\sqrt{x^2 - a^2}$$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute x = asecθ and dx = asecθtanθdθ.
3. Simplify the expression
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure 3.9 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship θ = sec−1(x/a)

Partial Fractions

Nonrepeated Linear Factors

Say we have a fraction in the form:

$$\frac{p(x)}{q(x)}$$

if q(x) an be factories into non repeating terms

$$q(x) = (a_1x + b_1)(a_2x + b_2)(a_3x + b_3)(a_nx + b_n)$$

The fraction can be represented

$$\frac{p(x)}{q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \dots \frac{A_n}{(a_nx + b_n)}$$

we can then rearange this:

$$p(x) = A_1(a_2x + b_2)(a_nx + b_n) + A_2(a_1x + b_1)(a_nx + b_n) + A_n(a_1x + b_1)(a_2x + b_2)$$

Now we can pick values to substitute in to isolate values A_n, and placed back into the orginal equation to find the partial fractions.

Repeated Linear Factors

if q(x) an be factories into non repeating terms

$$q(x) = (a_1x + b_1)^2(a_2x + b_2)$$

this fraction can be represented at

$$\frac{p(x)}{q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_1x + b_1)^2} + \frac{A_3}{(a_2x + b_2)}$$

we can then rearange this:

$$p(x) = A_1(a_2x + b_2)(a_1x + b_1) + A_2(a_2x + b_2) + A_3(a_1x + b_1)^2$$

Now we can pick values to substitute in to isolate values A_n, and placed back into the orginal equation to find the partial fractions.

Improper Integrals

Integrating over an Infinite Interval

> 1. Let f(x) be continuous over an interval of the form [a, +∞). Then
>
> $$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^I f(x)dx$$
>
> provided this limit exists
>
> 2. Let f(x) be continuous over an interval of the form (−∞, b]. Then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_{I}^{b} f(x)dx$$

provided this limit exists.
In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let f(x) be continuous over (−∞, +∞). Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

Provided the functions both wither converge if either diverge than the unbounded integral is divergent

Let f(x) and g(x) be continuous over [a, +∞). Assume that 0 ≤ f(x) ≤ g(x) for x ≥ a.

If:

$$\int_{a}^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_{a}^{t} f(x)dx = +\infty$$

then

$$\int_{a}^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_{a}^{t} g(x)dx = +\infty$$

and If

$$\int_{a}^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_{a}^{t} g(x)dx = L$$

where L is a real number, then

$$\int_{a}^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_{a}^{t} f(x)dx = M$$

for some real number M ≤ L