APPLICATIONS OF INTEGRATION

Areas between Curves

Area of a Region between Two Curves

To find the area between two curves we fin the points of intersection or the points on the x axis if given. and take these as the bounds for the integrals. We then take the integral of the lower curve from the integral of the upper curve. If the curves swap positions this section must be taken as a separate calculation.

Regions Defined with Respect to y

It can sometimes be easier to find the area between curves by using points on the y axis for bounds. To do this we must rearrange the function for x and then take the integrals between the two curves.

Determining Volumes by Slicing

Volume and the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section
3. Integrate the area formula over the appropriate interval to get the volume

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a solid of revolution. A slicing method is one way of calculating these

The Disk Method

When we use the slicing method with solids of revolution, it is often called the disk method because, for solids of revolution, the slices used to over approximate the volume of the solid are disks

> Let f(x) be continuous and nonnegative. Define R as the region bounded above by the graph of f(x), below by the x-axis, on the left by the line x = a, and on the right by the line x = b. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by
>
> $$V = \int_a^b \pi |f(x)|^2 dx$$

> Let g(y) be continuous and nonnegative. Define Q as the region bounded on the right by the graph of g(y), on the left by the y-axis, below by the line y = c, and above by the line y = d. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by
>
> $$V = \int_c^d \pi |f(y)|^2 dx$$

The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happenis when an axis of revolution other than the x-axis or y-axis is selected.

> Suppose f(x) and g(x) are continuous, nonnegative functions such that f(x) ≥ g(x) over (a,b). Let R denote the region bounded above by the graph of f(x), below by the graph of g(x), on the left by the line x = a, and on the right by the line x = b. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by
>
> $$V = \int_a^b \pi[f(x)^2 - g(x)^2]dx$$

> Suppose u(y) and v(y) are continuous, nonnegative functions such that v(y) ≤ u(y) for y ∈ [[c, d]]. Let Q denote the region bounded on the right by the graph of u(y), on the left by the graph of v(y),
> below by the line y = c, and above by the line y = d. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by
>
> $$V = \int_c^d \pi[u(y)^2 - v(y)^2]dy$$

Volumes of Revolution: Cylindrical Shells

The Method of Cylindrical Shells

> Let f(x) be continuous and nonnegative. Define R as the region bounded above by the graph of f(x), below by the x-axis, on the left by the line x = a, and on the right by the line x = b. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by
>
> $$V = \int_a^b (2\pi x f(x))dx$$

> Let g(y) be continuous and nonnegative. Define Q as the region bounded on the right by the graph of g(y), on the left by the y-axis, below by the line y = c, and above by the line y = d. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by
>
> $$V = \int_a^b (2\pi x g(y))dx$$

Arc Length of a Curve and Surface Area

We can think of arc length as the distance you would travel if you were walking along the path of the curve. We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y.

Arc Length of the Curve y = f(x)

> Let f(x) be a smooth function over the interval [a,b]. Then the arc length of the portion of the graph of f(x) from the point (a,f(a)) to the point (b,f(b)) is given by
>
> $$length = \int_a^b \sqrt{1 + |f'(x)|^2} dx$$

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>
> $$length = \int_a^b \sqrt{1 + |f'(y)|^2} dx$$

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. Surface area is the total area of the outer layer of an object.

> Let f(x) be a nonnegative smooth function over the interval [a,b]. Then, the surface area of the surface of revolution formed by revolving the graph of f(x) around the x-axis is given by
>
> $$area = \int_a^b 2\pi f(x) \sqrt{1 + |f'(y)|^2} dx$$
>
> Similarly, let g(y) be a nonnegative smooth function over the interval
> [c,d] Then, the surface area of the surface of revolution formed by revolving the graph of g(y) around the y-axis is given by
>
> $$area = \int_c^d 2\pi f(x) \sqrt{1 + |f'(y)|^2} dx$$

Moments and Centers of Mass

Center of Mass and Moments

> Let m1, m2 ,..., mn be point masses placed on a number line at points x1, x2 ,..., xn, respectively, and let
>
> $$m = \sum_{i=1}^{n} m_1$$
>
> denote the total mass of the system. Then, the moment of the system with respect to the origin is given by
>
> $$m = \sum_{i=1}^{n} m_i x_i$$
>
> and the centre of mass of the system is given by
>
> $$\bar{x} = \frac{M}{m}$$

Let m1, m2 ,..., mn be point masses located in the xy-plane at points (x1, y1), (x2, y2),..., (xn, yn), respectively,
and let

$$m = \sum_{i=1}^{n} m_1$$

denote the total mass of the system. Then the moments Mx and My of the system with respect
to the x- and y-axes, respectively, are given by

$$M_y = \sum_{i=1}^{n} m_i x_i$$

$$M_x = \sum_{i=1}^{n} m_i y_i$$

Also, the coordinates of the center of mass (x̄, ȳ) of the system are

$$\bar{x} = \frac{M}{m}$$

$$\bar{y} = \frac{M}{m}$$

Center of Mass of Thin Plates

If a region R is symmetric about a line l, then the centroid of R lies on l

Let R denote a region bounded above by the graph of a continuous function f(x), below by the x-axis, and on the left and right by the lines x = a and x = b, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements

1. The mass of the lamina is

$$m = p \int_a^b f(x) dx$$

2. The moments Mx and My of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = p \int_a^b \frac{|f(x)|^2}{2} dx$$

$$M_y = p \int_a^b x f(x) dx$$

3. The coordinates of the centre of mass (x̄,ȳ) are

$$\bar{x} = \frac{M_y}{m}$$

$$\bar{y} = \frac{M_x}{m}$$

Let R denote a region bounded above by the graph of a continuous function f(x), below by the graph of the continuous function g(x), and on the left and right by the lines x = a and x = b, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements

1. The mass of the lamina is

$$m = p\int_a^b [f(x) - g(x)]dx$$

2. The moments Mx and My of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = p\int_a^b \frac{1}{2}[f(x)^2 - g(x)^2]dx$$

$$M_y = p\int_a^b x[f(x) - g(x)]dx$$

3. The coordinates of the center of mass (x̄,ȳ) are:

$$\bar{x} = \frac{M_y}{m}$$

$$\bar{y} = \frac{M_x}{m}$$

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Theorem of Pappus

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R.

Exponential Growth and Decay

Exponential Growth Model

Systems that exhibit exponential growth increase according to the mathematical model

$$y = y_0 e^{kt}$$

If a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double. It is given by

$$t = \frac{ln2}{k}$$

Systems that exhibit exponential decay behave according to the mode

$$y = y_0 e^{-kt}$$

where y0 represents the initial state of the system and k > 0 is a constant, called the decay constant.

If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by

$$half - life = \frac{ln2}{k}$$