Week 3 - Derivatives

Defining the Derivative

Tangent Lines

We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points (a, f(a)) and (x,f(x)). The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{(f(x) - f(a))}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with *a* + *h*, where h is a value close to 0.

$$m_{sec} = \frac{f(a+h) - f(a)}{h}$$

The closer the value of a to x the more accurate the slope at point x . Because of this fact we can find the accurate slope at x by finding the limit of f(x) as x approaches a. Let f(x) be a function defined in an open interval containing a. The tangent line to f(x) at a is the line passing through the point (a,f(a)) having slope (provided a limit exists).

$$m_{tan} = \lim x \to a \frac{(f(x) - f(a))}{x - a}$$

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function is known as the derivative.

Let f(x) be a function defined in an open interval containing a. The derivative of the function f(x) at a, denoted by f '(a), is defined by:

$$f'(a) = \frac{f(a+h) - f(a)}{h}$$

Velocities and Rates of Change

Recall that if s(t) is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval [a, t] if t > a or [t, a] if t < a is given by the difference quotient

$$v_{avg} = \frac{s(t) - s(a)}{t - a}$$

As the values of t approach a, the values of v_avg approach the value we call the instantaneous velocity at a. That is, instantaneous velocity at a, denoted v(a), is given by:

$$v(a) = s'(a) = \lim t \to a \frac{s(t) - s(a)}{t - a}$$

The Derivative as a Function

Derivative Functions

Let f be a function. The derivative function, denoted by f ', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim h \to 0 \frac{f(x+h) - f(h)}{h}$$

A function is said to be differentiable on S if it is differentiable at every point in an open set S, and a differentiable function is one in which f '(x) exists on its domain.

Derivatives and Continuity

Let f(x) be a function and a be in its domain. If f(x) is differentiable at a, then f(x) is continuous at a. However, if a function is continuous, however even if a function is continuous it may still fail to be differentiable, for example f(x) = |x| and f(x) = This is because f(x) = |x| turns at a sharp point due to the limit having different values on each side of 0. And f(x) =$\sqrt[3]{x}$ has a vertical slope at 0.

Higher-Order Derivatives

We can find the derivative of a derivative. These are know as higher order derivatives.

Differentiation Rules

The Basic Rules

Let c be a constant. If f(x) = c, then

$$f'(c) = 0$$

Let n be a positive integer. If f(x) = x^n, then

$$f'(x) = nx^{n-1}$$

The derivative of the sum/difference of a function f and a function g is the same as the sum/difference of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$$

Product Rule

Let f(x) and g(x) be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}f(x) * g(x) + \frac{d}{dx}g(x) * f(x)$$

The Quotient Rule

Let f(x) and g(x) be differentiable functions. Then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{d}{dx}f(x) * g(x) - \frac{d}{dx}g(x) * f(x)}{g(x)^2}$$

Derivatives of Trigonometric Functions

Derivatives of the Sine and Cosine Functions

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Derivatives of Other Trigonometric Functions

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Higher-Order Derivatives

The higher-order derivatives of sinx and cos x follow a repeating pattern. By following the pattern, we can find any higher-order derivative of sinx and cos x.

The Chain Rule

Deriving the Chain Rule

We can use the chain rule to simplify differentiate composite functions

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at g(x), the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

if given by

$$h'(x) = f'(g(x))g'(x)$$

The Chain and Power Rules Combined

We can also combine the chain rule and power rule to differentiates functions . For all values of x for which the derivative is defined, if

$$h(x) = (g(x))^n$$

Then

$$h'(x) = n(g(x))^{n-1}g'(x)$$

Composites of Three or More Functions

For functions with 3 or more functions we can use the chain rule twice. This looks like:

$$k'(x) = h'(f(g(x)))f'(g(x))g\prime(x)$$

Derivatives of Inverse Functions

The Derivative of an Inverse Function

If f(x) is both invertible and differentiable, the inverse of f(x) is also differentiable. If we recall that recalling that

$$x = f(f^{-1}(x)))$$

Then by differntiating this equation we get

$$1 = f'(f^{-1}(x))(f^{-1}(x)')$$

Solving for x we get

$$f^{-1}(x)' = \frac{1}{f'(f^{-1}(x))}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-(x)^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-(x)^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+(x)^2}$$

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+(x)^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{(x)^2-1}}$$

$$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{(x)^2-1}}$$

Implicit Differentiation

Implicit Differentiation

While we normal deal with equation where y is the subjects. We may occasionally need to diferenciate y in terms of x. In these cases we simply differenciate the term and multiple by dy/dx and then factor dy/dx out of these terms. For example

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Finding Tangent Lines Implicitly

We can use the technique to find the slope of tangent lines. Take the equation:

$$y^3 + x^3 + 3xy = 0$$

Do find the slope at any point in this line we must differentiate it:

$$y^3 + x^3 + 3xy = 0$$

$$3y^2\frac{dy}{dx} + 3x^2 + 3y + \frac{dy}{dx}3x = 0$$

$$\frac{dy}{dx}(3y^2 + 3x) + 3x^2 + 3y = 0$$

$$\frac{dy}{dx}(3y^2 + 3x) = -(3x^2 + 3y)$$

$$\frac{dy}{dx} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)}$$

Now we must pick a point we want to find the tangent as. for this example we will use (3/2,3/2)

$$\frac{dy}{dx} = -\frac{(3(\frac{3}{2})^2 + 3(\frac{3}{2}))}{(3(\frac{3}{2})^2 + 3(\frac{3}{2}))}$$

$$\frac{dy}{dx} = -1$$

Then using this value in the slope point equation we get:

$$y - b = \frac{dy}{dx}(x - a)$$

$$y - \frac{3}{2} = -1(x - \frac{3}{2})$$

$$y - \frac{3}{2} = -x + \frac{3}{2}$$

$$y = -x + 3$$

Derivatives of Exponential and Logarithmic Functions

Derivative of the Exponential Function

Let E(x) = e^x be the natural exponential function. Then

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x)$$

Derivative of the Logarithmic Function

If x > 0 and y = ln(x), then

$$h'(x) = \frac{1}{g(x)}g'(x)$$

However if we have a logarithm that is of base b we must use the following formula

$$h'(x) = \frac{g'(x)}{g(x)\ln b}$$