Approximating Areas

Sigma (Summation) Notation

Using shapes of known area to approximate the area of an irregular region bounded by curves often requires adding up long strings of numbers. To make it easier to write down these lengthy sums we will use sigma notation (Summation notation). In sigma notation the follow two expressiosnare equivalent.

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20$$

$$\sum_{i=1}^{20} i$$

The properties associated with the summation process are given in the following rule:

> Let a1, a2 ,..., an and b1, b2 ,..., bn represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with 1 ≤ m ≤ n
>
> $$\sum_{i=1}^{n} c = nc$$
>
> $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$
>
> $$\sum_{i=1}^{n}(a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$
>
> $$\sum_{i=1}^{n}(a_i-b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$
>
> $$\sum_{i=1}^{n}(a_i) = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for sums and powers of integers, and we use them in the next set of examples.

> 1. The sum of n integers is given by
>
> $$\sum_{i=1}^{n} i = \frac{n(n=1)}{2}$$
>
> 2. The sum of consecutive integers squared is given by
>
> $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
>
> 3. The sum of consecutive integers cubed is given by
>
> $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Approximating Area

We find the area between a curve by dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area we begin by dividing the interval [a,b] into n subintervals of equal width

$$\Delta x = \frac{a-b}{n}$$

> A set of points P = {xi} for i = 0, 1, 2,..., n with a = x0 < x1 < x2 < ⋯ < xn = b, which divides the interval [a,b] into subintervals of the form [x0, x1], [x1, x2],..., [x_n − 1, x_n] is called a partition of [a, b]. If the subintervals all have the same width, the set of points forms a regular partition of the interval
> [a, b].

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two
methods: the left-endpoint approximation and the right-endpoint approximation.

Left-endpoint approximation

> On each subinterval [xi − 1, xi] (for i = 1, 2, 3,..., n), construct a rectangle with width Δx and height equal to f(xi − 1), which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is f(xi − 1)Δx. Adding the areas of all these rectangles, we get an approximate value for A (Figure 1.3). We use the notation Ln to denote that this is a left-endpoint approximation of A using n subintervals
>
> $$A \approx Ln = f(x_0)\mathit{\Delta} x + f(x_1)\mathit{\Delta} x + \cdots + f(x_n - 1)\mathit{\Delta} x$$
> $$= \sum_{i=1}^{n} f(xi-1)\mathit{\Delta} x$$

Right-Endpoint Approximation

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval

> Construct a rectangle on each subinterval [xi − 1, xi], only this time the height of the rectangle is determined by the function value f(xi) at the right endpoint of the subinterval. Then, the area of each rectangle is f(xi)Δx and the approximation for A is given by
>
> $$A \approx Rn = f(x_1)\mathit{\Delta} x + f(x_2)\mathit{\Delta} x + \cdots + f(x_n)\mathit{\Delta} x$$
> $$= \sum_{i=1}^{n} f(x_i)\mathit{\Delta} x.$$
>
> The notation Rn indicates this is a right-endpoint approximation for A

if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely

Forming Riemann Sums

We could evaluate the function at any point c_i in the subinterval [x_i − 1, x_i], and use f *(x_i) as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

A sum of this form is called a Riemann sum

> Let f(x) be defined on a closed interval [a, b] and let P be a regular partition of [a, b]. Let Δx be the width of each subinterval [xi − 1, xi] and for each i, let x*i be any point in [xi − 1, xi]. A Riemann sum is defined for f(x) as
>
> $$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Riemann sums give better approximations for larger values of n. This means that the accuracy of the area improves as n approaches infinity

> Let f(x) be a continuous, nonnegative function on an interval [a, b], and let
>
> $$\sum_{i=1}^{n} f(x_i^*)\Delta x$$
>
> be a Riemann sum for f(x). Then, the area under the curve y = f(x) on [a, b] is given by
>
> $$A = \lim n \to \infty \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The Definite Integral

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that f(x) be continuous and nonnegative, and define the definite integral as follows

> If f(x) is a function defined on an interval [a,b], the definite integral of f from a to b is given by
>
> $$\int_b^a f(x)dx = \lim n \to \infty \sum_{i=1}^{n} f(x_i^*)\Delta x$$
>
> provided the limit exists. If this limit exists, the function f(x) is said to be integrable on [a, b] or is an integrable function.

> If f(x) is continuous on [a, b], then f is integrable on [a, b].

Evaluating Definite Integrals

We define the integrals by substituting the end values into the Reiman sum and taking the difference.

Area and the Definite Integral

Net Signed Area

The net signed area is the area above the x-axis take away the area under the x-axis

Properties of the Definite Integral

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_b^a (f(x) - g(x))dx = \int_b^a f(x) - \int_b^a g(x)$$

$$\int_b^a (f(x) + g(x))dx = \int_b^a f(x) + \int_b^a g(x)$$

$$\int_b^a cf(x)dx = c\int_b^a f(x)dx$$

$$\int_b^a (f(x) - g(x))dx = \int_b^a f(x) - \int_b^a g(x)$$

Comparison Properties of Integrals

> i. If f(x) ≥ 0 for a ≤ x ≤ b, then
>
> $$\int_b^a f(x)dx \geq 0$$
>
> If f(x) ≥ g(x) for a ≤ x ≤ b, then
>
> $$\int_b^a f(x)dx \geq \int_b^a g(x)dx.$$
>
> If m and M are constants such that m ≤ f(x) ≤ M for a ≤ x ≤ b, then
>
> $$m(b-a) \leq \int_b^a f(x)dx \leq M(b-a)$$

Average Value of a Function

Let f(x) be continuous over the interval [a, b]. Then, the average value of the function f(x) (or f_ave) on [a, b] is given by

$$f_{ave} = \frac{1}{b-a}\int_b^a f(x)dx.$$

The Fundamental Theorem of Calculus

The Mean Value Theorem for Integrals.

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at the same point in that interval

> If f(x) is continuous over an interval [a, b], then there is at least one point c ∈ [a, b] such that

$$f(c) = \frac{1}{b-a}\int_b^a f(x)dx.$$

This formula can also be stated as

$$\int_b^a f(x)dx = f(c)(b-a)$$

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

This establishes the relationship between differentiation and integration.

> If f(x) is continuous over an interval [a, b] and the function F(x) is defined by
>
> $$F(x) = \int_a^x f(t)dt$$
>
> then F' (x) = f(x) over [a,b]

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

If f is continuous over the interval [a, b] and F(x) is any antiderivative of f(x), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integration Formulas and the Net Change Theorem

The Net Change Theorem

The net change theorem considers the integral of a rate of change. It says that when a quantity changes, the new value
equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways.
The second is more familiar; it is simply the definite integral.

$$F(b) = F(a) + \int_a^b F'(x)dx$$

Integrating Even and Odd Functions

Integrals of even functions, when the limits of integration are from −a to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly [−a, a], evaluate to zero because the areas above and below the x-axis are equal

> For continuous even functions such that f(−x) = f(x),
>
> $$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$
>
> For continuous odd functions such that f(−x) = −f(x)
>
> $$\int_{-a}^a f(x)dx = 0$$

Substitution

> Let u = g(x), where g' (x) is continuous over an interval, let f(x) be continuous over the corresponding range of
> g, and let F(x) be an antiderivative of f(x). Then
>
> $$\int f(|g(x)|)g\prime(x)dx = \int f(u)du$$
>
> $$= F(u) + C$$
>
> $$= F|g(x)| + C$$

The steps taken to find an integral through substitution

> 1. Look carefully at the integrand and select an expression g(x) within the integrand to set equal to u. Let's select
> g(x). such that g' (x) is also part of the integrand.
> 2. Substitute u = g(x) and du = g' (x)dx. into the integral.
> 3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to
> go back and select a different expression to use as u.
> 4. Evaluate the integral in terms of u.
> 5. Write the result in terms of x and the expression g(x).

Substitution for Definite Integrals

When using substitution on definite integrals we need convert the bounds to be in terms of the substitution

Let u = g(x) and let g' be continuous over an interval (a,b), and let f be continuous over the range of u = g(x). Then,

$$\int_a^b f(g(x))g'(x) = \int_{g(a)}^{g(b)} f(x)dx$$