Sequences

Terminology of Sequences

An infinite sequence {a_n} is an ordered list of numbers of the form

$$a_1 + a_2 + a_3 + a_4 \ldots . . a_n$$

The subscript n is called the index variable of the sequence. Each number an is a term of the sequence. Sometimes sequences are defined by explicit formulas, in which case an = f(n) for some function f(n) defined over the positive integers. In other cases, sequences are defined by using a recurrence relation. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Limit of a Sequence

An important skill is to be able to derive the limit of a sequence or if it even has one. A sequence with a limit converges. A limit without diverges.

> Given a sequence {an}, if the terms an become arbitrarily close to a finite number L as n becomes sufficiently large, we say {an} is a convergent sequence and L is the limit of the sequence. In this case, we write
>
> $$\lim n \to \infty a_n = L$$
>
> If a sequence a_n is not convergeny we say it is divergent.

> A sequence {an} converges to a real number L if for all ε > 0, there exists an integer N such that |an −L| < ε if n ≥ N. The number L is the limit of the sequence and we write
>
> $$\lim n \to \infty a_n = L.$$
>
> or
>
> $$a_n \to L$$
>
> In this case, we say the sequence {an} is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist

It should be noted that as the convergence of a sequences is based on the behaviour of {a_n} as n approaches infinity, if terms are placed before a_0 the sequence will still have the same convergence as {a_n}

> Consider a sequence {an} such that an = f(n) for all n ≥ 1. If there exists a real number L such that
>
> $$\lim x \to \infty f(x) = L$$
>
> then {an} converges and
>
> $$\lim n \to \infty a_n = L$$

We can detrimine the limit of a geometric sequence of form:

$$r^n$$

using the following rules

$$r^n \to 0 : 0 < r < 1$$

$$r^n \to 1 : r = 1$$

$$r^n \to \infty : r > 1$$

We can also use algebraic limit laws to detrimin limits of sequences

> Given sequences {an} and{bn} and any real number c, if there exist constants A and B such that
>
> $$\lim n \to \infty a_n = A$$
>
> and
>
> $$\lim n \to \infty b_n = B$$
>
> then
>
> $$\lim n \to \infty c = c$$
>
> $$\lim n \to \infty c a_n = c \lim n \to \infty a^n = cA$$
>
> $$\lim n \to \infty (a_n \pm b_n) = \lim n \to \infty a_n \pm \lim n \to \infty b_n = A \pm B$$
>
> $$\lim n \to \infty (a_n * b_n) = \lim n \to \infty a_n * \lim n \to \infty b_n = A * B$$
>
> $$\lim n \to \infty (\frac{a_n}{b_n}) = \frac{\lim n \to \infty a_n}{\lim n \to \infty b_n} = \frac{A}{B}$$

> Consider a sequence {an} and suppose there exists a real number L such that the sequence {an} converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for n ≥ N, and the sequence f(an) converges to f(L)

Another important technique for finding limits is the squeeze theorem

> Consider sequences {an}, {bn}and {cn}. Suppose there exists an integer N such that
>
> $$a_n \leq b_n \leq c_n$$
>
> for all n ≥ N. If there exists a real number L such that
>
> $$\lim n \to \infty a_n = L = \lim n \to \infty c_n$$
>
> Then {bn} converges at L

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded

> A sequence {an} is bounded above if there exists a real number M such that
>
> $$a_n <= M$$
>
> for all positive integers n
>
> A sequence {an} is bounded below if there exists a real number M such that
>
> $$M <= a_n$$
>
> for all positive integers n
>
> A sequence {an} is a bounded sequence if it is bounded above and bounded below.
> If a sequence is not bounded, it is an unbounded sequence.

> If a seuqence {a_n} converges then it is bounded

a sequence being bounded is not a sufficient condition for a sequence to converge

> A sequence {an} is increasing for all n ≥ n0 if
>
> $$a_n <= a_{n+1} : n >= n_0$$
>
> A sequence {an} is decreasing for all n ≥ n0 if
>
> $$a_n >= a_{n+1} : n >= n_0$$
>
> A sequence {an} is a monotone sequence for all n ≥ n0 if it is increasing for all n ≥ n0 or decreasing for all n ≥ n0.

> If {an} is a bounded sequence and there exists a positive integer n0 such that {an} is monotone for all n ≥ n0, then {an} converges.

Infinite Series

Sums and Series

> An infinite series is an expression of the form
>
> $$\sum_{n=1}^{\infty} a_n = a_1 + a_2....$$
>
> for each positive number k the sum
>
> $$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2....a_k$$
>
> is called the kth partial sum of the infinite series. The partial sums form a sequence {S_k}. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the convergence of a series to S, we call S the sum of the series, and we write

> $$\sum_{n=1}^{\infty} a_n = S$$
>
> If the sequence of partial sums diverges, we have the divergence of a series

Algebraic Properties of Convergent Series

> let the following sieres be convergent
>
> $$\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n$$
>
> then
>
> $$\sum_{n=1}^{k}(a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n$$
>
> $$\sum_{n=1}^{k}(a_n - b_n) = \sum_{n=1}^{k} a_n - \sum_{n=1}^{k} b_n$$
>
> $$\sum_{n=1}^{k} c(a_n) = c\sum_{n=1}^{k} a_n$$

The Harmonic Series

A useful series to know about is the harmonic series. The harmonic series is defined as:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums {S_k} approach infinity, but do so very slowly

Geometric Series

A geometric series is any series that we can write in the form

$$a + ar + ar^2 + ar^3 + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

> A geometric series is a series of the form
>
> $$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ...$$
>
> if |r|, the series converges and
>
> $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} : |r| < 1$$
>
> if |r|>= 1 the series diverges

Telescoping Series

A telescoping series is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

The Divergence and Integral Tests

Divergence Test

For a series

$$\sum_{n=1}^{\infty} a_n$$

an to converge, the nth term an must satisfy an → 0 as n → ∞

$$\lim k \to \infty a_k = \lim k \to \infty (Sk - Sk - 1) = \lim k \to \infty S_k - \lim k \to \infty S_{k-1} = S - S = 0$$

> if the simit of a_n as n approches infinity, then {a_n} diverges. The converse is not true. A seeries with limit 0 is not necessarily convergent

Integral Test

This test, called the integral test, compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive

> Suppose
>
> $$\sum_{n=1}^{\infty} a_n$$
>
> is a series with positive terms an. Suppose there exist s a function f and a positive integer N such
>
> 1. f is continuous
> 2. f is decreasing
> 3. f(n) = a_n for all integers n>=N then
>
> $$\sum_{n=1}^{\infty} a_n and \int_N^{\infty} f(x)dx$$
>
> Both converge or diverge

That is to say so long asf :

$$\lim b \to \infty \int_N^b f(x)dx \neq \pm\infty$$

the series converges and if:

$$\lim b \to \infty \int_N^b f(x)dx = \pm\infty$$

the series diverges

The P-series

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

if p > 1 the series converges. If p ≤1 the series diverges

Estimating the Value of a Series

suppose

$$\sum_{n=1}^{\infty} a_n$$

is a convergent series with positive terms. Suppose there exists a function f satisfying the following

1. f is continuous
2. f is decreasing
3. f(n) = a_n for all integers n>= 1 then

Let SN be the Nth partial sum of

$$\sum_{n=1}^{\infty} a_n$$

For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_N^{\infty} f(x)dx$$

In other words the remainder

$$R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$$

Satisfies the following estimate

$$\int_{N+1}^{\infty} f(x)dx < R_N \int_N^{\infty} f(x)dx$$

Comparison Tests

Comparison Test

Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the comparison test.

> 1. Suppose there exists an integer N such that 0 ≤ an ≤ bn for all n ≥ N. If
>
> $$\sum_{n=1}^{\infty} b_n$$
>
> converges then
>
> $$\sum_{n=1}^{\infty} a_n$$
>
> converges also
>
> 2. suppose there exists an integer N such that an ≥ bn ≥ 0 for all n ≥ N. If
>
> $$\sum_{n=1}^{\infty} b_n$$
>
> diverges then
>
> $$\sum_{n=1}^{\infty} a_n$$
>
> diverges also

Limit Comparison Test

> Let an, bn ≥ 0 for all n ≥ 1.
>
> 1. If
>
> $$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$$
>
> then
>
> $$\sum_{n=1}^{\infty} a_n \& \sum_{n=1}^{\infty} b_n$$
>
> Both converge or diverge
>
> 2. If
>
> $$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$
>
> and
>
> $$\sum_{n=1}^{\infty} b_n$$
>
> converges, then

$$\sum_{n=1}^{\infty} a_n$$

converges

3. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

and

$$\sum_{n=1}^{\infty} b_n$$

diverges, then

$$\sum_{n=1}^{\infty} a_n$$

diverges

Alternating Series

The Alternating Series Test

A series whose terms alternate between positive and negative values is an alternating series

> Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form
>
> $$\sum_{n=1}^{\infty} -1^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

> An alternating series of the form
>
> $$\sum_{n=1}^{\infty} -1^{n+1} b_n or \sum_{n=1}^{\infty} -1^{n+} b_n$$
>
> Converges if
>
> 1. 0 ≤ b_(n + 1) ≤ bn for all n ≥ 1 and
> 2. nlim→ ∞bn = 0
>
>   This is known as the alternating series test

Remainder of an Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty} -1^{n+1} b_n or \sum_{n=1}^{\infty} -1^{n+} b_n$$

hat satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and SN denote the Nth partial sum. For any integer N ≥ 1, the remainder RN = S − SN satisfies

|R_N| ≤ b_(N + 1)

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the Nth partial sum SN is in magnitude at most the size of the next term bN + 1

Absolute and Conditional Convergence

> a series experiences absolute convergence if the absolute values of the series also converges. If a series converges but the absolute value of the series diverges it converges condionally

> If
>
> $$\sum_{n=1}^{\infty} |a_n|$$
>
> converges, then
>
> $$\sum_{n=1}^{\infty} a_n$$
>
> converges.

Ratio and Root Tests

Ratio Test

> Let
>
> $$\sum_{n=1}^{\infty} a_n$$
>
> be a series with nonzero terms. Let
>
> $$p = \lim_{n \to \infty} |\frac{a_{n+1}}{a_n}|$$
>
> 1. if 0 ≤ ρ < 1, then the series converges completely
> 2. If p> 1 or p = ∞, then the series diverges
> 3. if p = 1 the test does not provide any information

Root Test

> Let
>
> $$\sum_{n=1}^{\infty} a_n$$
>
> be a series with nonzero terms. Let
>
> $$p = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$
>
> 1. if 0 ≤ ρ < 1, then the series converges completely
> 2. If p> 1 or p = ∞, then the series diverges
> 3. if p = 1 the test does not provide any information

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n \to \infty} a_n$.	If $\lim_{n \to \infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n \to \infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r \rvert < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\lvert r \rvert \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n \to \infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Series or Test	Conclusions	Comments
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\| \frac{a_{n+1}}{a_n} \right\|$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1 \text{ or } \rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \to \infty} \sqrt[n]{\|a_n\|}$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\|a_n\| = b_n^n$.
	If $\rho > 1 \text{ or } \rho = \infty$, the series diverges.	