Unit 8: Differential equations

Basics of Differential Equations

General Differential Equations

A differential equation is an equation involving an unknown function y = f(x) and one or more of its derivatives. A solution to a differential equation is a function y = f(x) that satisfies the differential equation when f and its derivatives are substituted into the equation.

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero

The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.

General and Particular Solutions

In the equation:

$$y' = 2x$$

The differential equation is of the form

$$y = x^2 + C$$

As c can take any value here the equation is a **general solution** In this example, we are free to choose any solution we wish; for example, y = x2 − 3 is a member of the family of solutions to this differential equation. This is called a **particular solution** to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Initial-Value Problems

A differential equation together with one or more initial values is called an initial-value problem. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation

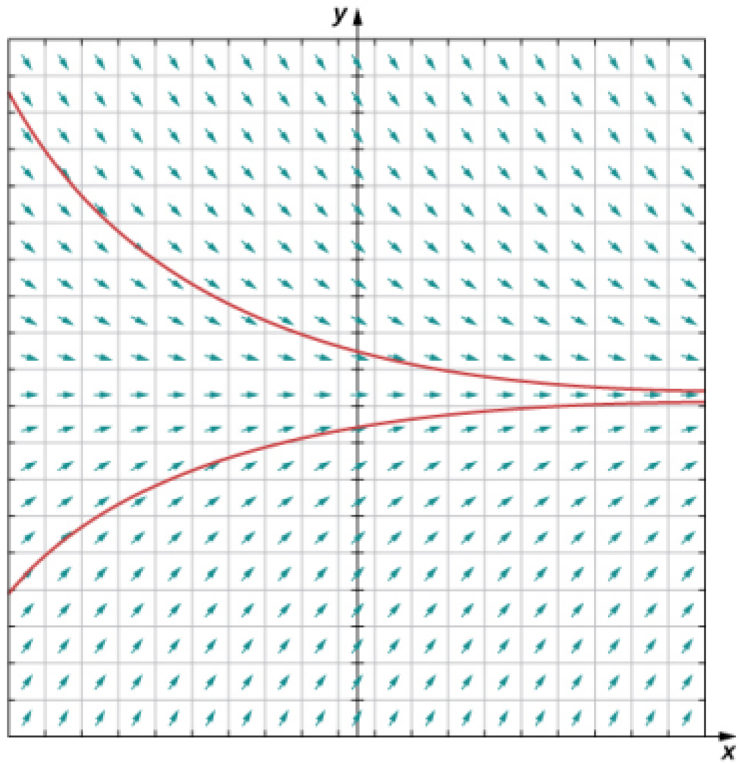
Direction Fields and Numerical Methods

Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form.

$$y' = f(x, y)$$

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point



Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution.

An equilibrium solution is any solution to the differential equation of the form y = c, where c is a constant.

To determine the equilibrium solutions to the differential equation y' = f(x, y), set the right-hand side equal to zero. An equilibrium solution of the differential equation is any function of the form y = k such that f(x, k) = 0 for all values of x in the domain of f

> Consider the differential equation y' = f(x, y), and assume that all solutions to this differential equation are defined for x ≥ x0. Let y = k be an equilibrium solution to the differential equation.
>
> 1. y = k is an asymptotically stable solution to the differential equation if there exists ε > 0 such that for any value c ∈ (k − ε, k + ε) the solution to the initial-value problem
>
> $$y' = f(x, y), y(x_0) = c$$
>
> approaches k as x approaches infinity.
>
> 2. y = k is an asymptotically unstable solution to the differential equation if there exists ε > 0 such that for any value c ∈ (k − ε, k + ε) the solution to the initial-value problem
>
> $$y' = f(x, y), y(x_0) = c$$
>
> 3. y = k is an asymptotically semi-stable solution to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

Euler’s Method

Consider the initial-value problem

$$y' = f(x, y), y(x_0) = y_o$$

To approximate a solution to this problem using Euler’s method, define

$$x_n = x_0 + nh$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

Here h > 0 represents the step size and n is an integer, starting with 1. The number of steps taken is counted by the variable n

Separable Equations

Separation of Variables

> A separable differential equation is any equation that can be written in the form
>
> $$y' = f(x)g(y).$$
>
> for xeample
>
> $$\frac{dy}{dx} = (x^2 - 4)(3y + 2)dx$$
>
> $$\frac{1}{3y+2}dy = (x^2 - 4)$$

The Logistic Equation

Population Growth and Carrying Capacity

We can use separating variables to find the growth/decay rates such as for half lives, or bacterial populations,

First-order Linear Equations

A first-order differential equation is linear if it can be written in the form

$$a(x)y' + b(x)y = c(x)$$

where a(x), b(x), and c(x) are arbitrary functions of x.

Standard Form

The standard for of a first order linear equation is:

$$y' + p(x)y = q(x)$$

Integrating Factors

The integrating factor is used to solve First-order Linear Equations. it is calculated as such

$$\mu(x) = e^{\int p(x)}$$

We then multiple both sides of the equation by μ(x), integrate both sides, and divide by μ(x) once more.

for example

$$xy\prime + 3y = 4x^2 - 3x$$

rearrange into the standard for

$$y' + \frac{3}{x}y = 4x - 3$$

calculate the integrating factor

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3lnx} = x^3$$

Multiplying both sides of the differential equation by μ(x) gives us

$$\frac{d}{dx}(x^3y) = 4x^4 - 3x^3$$

Integrate both sides of the equation

$$y = \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}$$