Solving for y gives us

$$y = \frac{2C_3e^{2t}}{C_3e^{2t} - 1} = \frac{2}{1 + Ce^{-2t}},$$

where C is arbitrary.

Note that from the general solution we can obtain the equilibrium solution $y = 2$ by setting $C = 0$. However, there is no value of C for which this general solution gives us the equilibrium solution $y = 0$. Thus, $y = 0$ is a singular solution for this equation. Combining these two solutions together means that all solutions of the equation $dy/dt = 2y - y^2$ can be written as

$$y = \frac{2}{1 + Ce^{-2t}} \quad \text{or} \quad y = 0,$$

where C is constant. □

EXERCISES 2.2

In Exercises 1–30, solve each equation.

1. $dy/dx = x/y^2$
2. $\frac{1}{2}t^{-1/2}\,dt + y^2\,dy = 0$
3. $dy/dx = \sqrt{y}/x^2$
4. $dy/dt = (1 + y^2)/y$
5. $(6 + 4t^3)\,dt + (5 + 9y^{-8})\,dy = 0$
6. $(6t^{-9} - 6t^{-3} + t^7)\,dt + (9 + s^{-2} - 4s^8)\,ds = 0$ (Solve for $s = s(t)$.)
7. $4\sinh 4y\,dy = 6\cosh 3x\,dx$
8. $dy/dt = (y + 1)/(t + 1)$
9. $dy/dt = (y + 2)/(2t + 1)$
10. $3t^{-2}\,dt = (y^{-1/2} + y^{1/2})\,dy$
11. $3\sin x\,dx - 4\cos y\,dy = 0$
12. $\cos y\,dy = 8\sin 8t\,dt$
13. $y' + ky = 0$ (k constant)
14. $(5x^5 - 4\cos x)\,dx + (2\cos 9t + 2\sin 7t)\,dt = 0$
15. $(\cosh 6t + 5\sinh 4t)\,dt + 20\sinh y\,dy = 0$
16. $dy/dt = e^{2y+10t}$
17. $dy/dt = e^{3y+2t}$
18. $\sin^2\theta\,d\theta = \cos^2\phi\,d\phi$
19. $(3\sin\theta - \sin 3\theta)\,d\theta = (\cos 4y - 4\cos y)\,dy$
20. $dx/dt = \sec^2 t/(\sec x \tan x)$
21. $(2 - 5/y^2)\,dy + 4\cos^2 x\,dx = 0$
22. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2 + 1}$
23. $\dfrac{dy}{dx} = \dfrac{x^2 + 1}{y^2}$
24. $\dfrac{dy}{dt} = \dfrac{1 + 2e^y}{e^y\, t\, \ln t}$
25. $x\sin\left(x^2\right)\,dx = \dfrac{\cos\sqrt{y}}{\sqrt{y}}\,dy$
26. $\dfrac{dy}{dx} = \dfrac{y + 1}{x + 2}$
27. $\dfrac{dy}{dx} = \dfrac{y^2 + 1}{x + 2}$
28. $\dfrac{dy}{dx} = \dfrac{x + 2}{y + 1}$
29. $\dfrac{\sqrt{\ln x}}{x}\,dx = \dfrac{e^{3/y}}{y^2}\,dy$
30. $dy/dt = 5^{-t}/y^2$
31. $dy/dt = t^2y^2 + y^2 - t^2 - 1$
32. $dy = \left(y^2 - 3y + 2\right)dx$
33. $4(x - 1)^2\,dy - 3(y + 3)^2\,dx = 0$
34. $dy/dt = \sin(t - y) + \sin(t + y)$
35. $dy/dt = y^3 + 1$
36. $dy/dt = y^3 - 1$
37. $dy/dt = y^3 + y$
38. $dy/dt = y^3 - y^2$
39. $dy/dt = y^3 - y$
40. $dy/dt = y^3 + y$

In Exercises 36–40, draw the phase line for the equation and classify the equilibrium solutions.

In Exercises 41–52, solve the initial value problem. Graph the solution on an appropriate interval.

41. $dy/dx = x^3$, $y(0) = 0$
42. $dy/dt = \cos t$, $y(\pi/2) = -1$
43. $dx = \cos y\,dy$, $x(0) = 2$
44. $\sin^2 y\,dy = dx$, $x(0) = 0$
45. $dy/dt = \sqrt{t}/y$, $y(0) = 2$
46. $d\phi/d\theta = \sqrt{\phi/\theta}$, $\phi(1) = 2$
47. $dy/dt = e^t/(y + 1)$, $y(0) = -2$

48. $dy/dy = e^{t-y}$, $y(0) = 0$
49. $dy/dx = y/\ln y$, $y(0) = e$
50. $dy/dt = t\,\sin\left(t^2\right)$, $y(\sqrt{\pi}) = 0$
51. $dy/dx = 1/(1+x^2)$, $y(0) = 1$
52. $\dfrac{dy}{d\theta} = \dfrac{\sin\theta}{\cos y + 1}$, $y(0) = 0$
53. $dy/dx = (y+3)/(3x+1)$, $y(0) = 1$
54. $dy/dx = e^{x-y}$, $y(0) = 1$
55. $dy/dx = e^{2x-y}$, $y(0) = 1$
56. $dy/dx = (3y+1)/(x+3)$, $y(0) = 1$
57. Solve each of the following initial value problems and graph the results.

(a) $\begin{cases} dy/dt = y\,\cos t \\ y(0) = 1 \end{cases}$

(b) $\begin{cases} dy/dt = y^2\,\cos t \\ y(0) = 1 \end{cases}$

(c) $\begin{cases} dy/dt = \sqrt{y}\,\cos t \\ y(0) = 1 \end{cases}$

58. Carefully show that the solution to the initial value problem $y' + f(t)\,y = 0$, $y(0) = 0$ is $y = y_0 e^{-\int f(t)\,dt}$.
59. Find an equation of the curve that passes through the point $(0,0)$ and has slope $dy/dx = -(y-2)/(x-2)$ on each point on the curve (x,y).
60. In some cases, substitutions can convert an equation to a form that we recognize and can solve.

For example, consider the first order equation

$$\frac{dy}{dx} = \frac{ax+ay+c}{bx+by+d}, \tag{2.1}$$

Note that if $ad - bc = 0$, $ad = bc$ and Eq. (2.1) reduces to $dy/dx = a/b$.

where a, $b \neq 0$, c, and d are constants. Let $Y = x + y$. Then, $dY/dx = 1 + dy/dx$ and substituting into Eq. (2.1) gives us

$$\frac{dY}{dx} - 1 = \frac{aY+c}{bY+d}$$

$$\frac{dY}{dx} = 1 + \frac{aY+c}{bY+d}$$

$$\frac{dY}{dx} = \frac{(a+b)Y+(c+d)}{bY+d},$$

which is separable. Separating, solving for Y and replacing Y with $x+y$ gives an implicit solution of the equation. Use this substitution to solve

(a) $\dfrac{dy}{dx} = \dfrac{x+y+3}{3x+3y+1}$

(b) $\dfrac{dy}{dx} = \dfrac{x-y+2}{2x-2y-1}$

61. A differential equation of the form $dy/dx = f(ax+by+k)$ is separable if $b = 0$. However, if $b \neq 0$ the substitution $u(x) = ax + by + k$ yields a separable equation. Use this substitution to solve
(a) $dy/dx = (x+y-4)^2$
(b) $dy/dx = (3y+1)^4$
62. Let $\omega > 0$ be constant. (a) Show that the system $\begin{cases} dx/dt = x(1-r) - \omega y \\ dy/dt = y(1-r) + \omega x \end{cases}$, $r = \sqrt{x^2+y^2}$ can be rewritten as the system $\begin{cases} dr/dt = r(1-r) \\ d\theta/dt = \omega \end{cases}$ by changing to polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$ (b) Show that the solution to $\begin{cases} dr/dt = r(1-r) \\ d\theta/dt = \omega \\ r(0) = r_0,\ \theta(0) = \theta_0 \end{cases}$ is $\begin{cases} r(t) = r_0 e^t/(1-r_0) + r_0 e^t) \\ \theta(t) = \omega t + \theta_0 \end{cases}$, and the solution to $\begin{cases} dx/dt = x(1-r) - \omega y \\ dy/dt = y(1-r) + \omega x \end{cases}$, $r = \sqrt{x^2+y^2}$ is $\begin{cases} x = r(t)\cos\theta(t) \\ y = r(t)\sin\theta(t) \end{cases}$. (c) How does the solution change for various initial conditions? (*Hint:* First determine how ω and θ_0 affect the solution. Then, determine how r_0 affects the solution. Try graphing the