Pattern matching for (231, 213)-avoiding permutations

Both Emerite Neou[1,3] Romeo Rizzi[2] and Stéphane Vialette[3]

[1] Department of Computer Science, Universit degli Studi di Trento, Italy
[2] Department of Computer Science, Universit degli Studi di Verona, Italy
romeo.rizzi@univr.it
[3] Université' Paris-Est, LIGM (UMR 8049), CNRS, UPEM, ESIEE Paris, ENPC, F-77454, Marne-la-Valle, France
vialette@univ-mY.fr

Abstract. bla bla...

1 Introduction

A permutation π is said to contain another permutation σ, in symbols $\sigma \preceq \pi$, if there exists a subsequence of entries of π that has the same relative order as σ, and in this case σ is said to be a *pattern* of π. Otherwise, π is said to avoid the permutation σ. For example a permutation contains the pattern 123 (resp. 321) if it has an increasing (resp. decreasing) subsequence of length 3. Here, note that members need not actually to be consecutive, merely increasing (resp. descreasing). During the last decade, the study of the pattern containment on permutations has become a very active area of research.

We consider here the so-called *pattern containment* problem (also sometimes referred to as the *pattern involvement problem*): Given two permutations σ and π, this problem is to decide whether $\sigma \preceq \pi$ (the problem is ascribed to Wilf in [5]). The permutation containment problem is **NP**-hard [5]. It is, however, polynomial-time solvable by brute-force enumeration if σ has bounded size. Improvements to this algorithm were presented in [2] and [1], the latter describing a $O(|\pi|^{0.47k+o(|\sigma|)})$ time algorithm. Bruner and Lackner [6] gave a fixed-parameter algorithm solving the pattern containment problem with an exponential worst-case runtime of $O(1.79^{\mathrm{run}(\pi)})$, where $\mathrm{run}(\pi)$ denotes the number of alternating runs of π. (This is an improvement upon the $O(2^{|\pi|})$ runtime required by brute-force search without imposing restrictions on σ and π.) Of particular importance, it was recently proved that the permutation containment problem is fiexed-parameter tractable for parameter $|\sigma|$ [8].

A few particular cases of the pattern containment problem have been attacked successfully. The case of increasing patterns is solvable in $O(|\pi| \log\log|\sigma|)$ time in the RAM model [7], improving the previous 30-year bound of $O(|\pi| \log|\sigma|)$. (The algorithm also improves on the previous $O(|\pi| \log\log|\pi|)$ bound.) Furthermore, the patterns 132, 213, 231, 312 can all be handled in linear time by stack sorting algorithms. Any pattern of length 4 can be detected in $O(|\pi| \log|\pi|)$ time [2].

Algorithmic issues for 321-avoiding patterns containment has been investigated in [9]. The pattern containment problem is also solvable in polynomial-time for separable patterns [10] (see also [5]). Separable permutations are those permutations that contain neither 2413 nor 3142, and they are enumerated by the Schrder numbers (sequence A006318 in OEIS). To the best of our knowledge, separable permutations first arose in the work of Avis and Newborn [3], who showed that they are precisely the permutations which can be sorted by an arbitrary number of pop-stacks in series, where a pop-stack is a restricted form of stack in which any pop operation pops all items at once. (Notice that the separable permutations include as a special case the stack-sortable permutations, which avoid the pattern 231.)

There exist many generalisation of patterns that are worth considering in the context of algorithmic issues in pattern involvement (see [11] for an up-to-date survey). *Vincular patterns*, also called *generalized patterns*, resemble (classical) patterns, with the constraint that some of the letters in an occurrence must be consecutive. Of particular importance in our context, Bruner and Lackner [6] proved that deciding whether a vincular pattern σ of length k occurs in a permutation $\pi \in \mathfrak{S}_n$ is $W[1]$-complete for parameter k. *Bivincular patterns* generalize classical patterns even further than vincular patterns. Indeed, in bivincular patterns, not only positions but also values of elements involved in a matching may be forced to be adjacent

This paper is organized as follows.

2 Definition

A *permutation* of length n is a one-to-one function from an n-element set to itself. We write permutations as words $\sigma = \sigma_1\,\sigma_2\,\ldots\,\sigma_n$, whose letters are distinct and uasually consist of the integers $1\,2\,\ldots\,n$. For the sake of convenience, we let $\pi[i:j]$ stand for $\pi_i\,\pi_{i+1}\,\ldots\,\pi_j$, $\pi[:j]$ stand for $\pi[1:j]$ and $\pi[i:]$ stand for $\pi[i:n]$. We let S_n denote the set of all permutations of length n.

A permutation π is said to *contain* the permutation σ if there exists a subsequence of (not necessarily consecutive) entries of π that has the same relative order as σ, and in this case σ is said to be a *pattern* of π, written $\sigma \preceq \pi$. Otherwise, π is said to *avoid* the permutation σ. For example, the permutation $\pi = 3\,9\,1\,8\,6\,7\,4\,5\,2$ contains the pattern $\sigma = 51342$, as can be seen in the highlighted subsequence of $\pi = \mathbf{3\,9\,1}\,8\,\mathbf{6\,7\,4}\,5\,2$ (or $\pi = \mathbf{3\,9\,1}\,8\,\mathbf{6\,7}\,4\,\mathbf{5}\,2$ or $\pi = \mathbf{3\,9\,1}\,8\,\mathbf{6\,7}\,4\,5\,\mathbf{2}$). Each subsequence $(9\,1\,6\,7\,4,\ 9\,1\,6\,7\,5,\ 9\,1\,6\,7\,2)$ is called a *copy*, *instance*, or *occurrence* of σ. Since the permutation $\pi = 3\,9\,1\,8\,6\,7\,4\,5\,2$ contains no increasing subsequence of length four, π avoids $1\,2\,3\,4$.

Suppose P is a set of permutations. We let $\mathrm{Av}_n(p)$ denote the set of all n-permutations avoiding each permutation in P. For the sake of convenience, we omit P's braces thus having e.g. $\mathrm{Av}_3(132, 4312)$ instead of $\mathrm{Av}_3(\{132, 4312\})$. A basic example is if $\pi = \pi_1\,\pi_2\,\ldots\,\pi_n$ avoids $3\,2\,1$, i.e., has no decreasing subsequence of length 3, then its reverse, $\pi' = \pi_n\,\pi_{n-1}\,\ldots\,\pi_1$ avoids $1\,2\,3$, i.e., has no increasing subsequence of length 3.

An *ascent* of a permutation $\pi \in S_n$ is any position $1 \leq i < n$ where the following value is bigger than the current one. That is, if $\pi = \pi_1 \pi_2 \ldots \pi_n$, then i is an ascent if $\pi_i < \pi_{i+1}$. For example, the permutation 3 4 5 2 1 6 7 has ascents (at positions) $1, 2, 5, 6$. Similarly, a *descent* is a position $1 \leq i < n$ with $\pi_i > \pi_{i+1}$, so every i with $1 \leq i < n$ either is an ascent or is a descent of π. Let a and d denote an ascend and a descend, respectively. The *stripe* s_π of a permutation $\pi \in S_n$ is the word $\mathrm{s}_\pi(1)\,\mathrm{s}_\pi(2)\ldots\mathrm{s}_\pi(n-1) \in \{a,d\}^{n-1}$ defined by $\mathrm{s}_\pi(i) = a$ if i an ascent in π and $\mathrm{s}_\pi(i) = d$ if i a descent in π. For example the stripe of the permutation $\pi = 12\,11\,10\,1\,2\,3, 4\,9\,8\,5\,6\,7$ is $\mathrm{s}_\pi = dddaaaaddaa$. The stripe $s\,\mathrm{s}_\sigma$ is a *pattern* of the strip s_π (or $s\,\mathrm{s}_\sigma$ *occurs* of the strip s_π) if s_σ occurs in s_π as a subsequence.

A *bivincular pattern* (abbreviated BVP) σ of length k is a permutation in S_k written in two-line notation (that is the top row is $1\,2\,\ldots\,k$ and the bottom row is a permutation $\sigma_1\,\sigma_2\,\ldots\,\sigma_k$). We have the following conditions on the top and bottom rows of σ:

- If the bottom line of σ contains $\underline{\sigma_i\,\sigma_{i+1}\,\ldots\,\sigma_j}$ then the letters corresponding to $\sigma_i\,\sigma_{i+1}\,\ldots\,\sigma_j$ in an occurrence of σ in a permutaton must be adjacent, whereas there is no adjacency condition for non-underlined consecutive letters. Moreover if the bottom row of σ begins with $\llcorner\sigma_1$ then any occurrence of σ in a permutation π must begin with the leftmost letter of π, and if the bottom row of σ begins with $\sigma_{k\lrcorner}$ then any occurrence of σ in a permutation π must end with the rightmost letter of π.
- If the top line of σ contains $\overline{i\,i+1\,\ldots\,j}$ then the letters corresponding to $\sigma_i, \sigma_{i+1}, \ldots, \sigma_j$ in an occurrence of σ in a permutation must be adjacent in values, whereas there is no value adjacency restriction for non-overlined letters. Moreover, if the top row of σ begins with $^{u}lcorner1$ then any occurrence of σ is a permutation π must begin with the smallest letter of π, and if top row of σ ends with $k^{\urcorner}$ then any occurrence of σ is a permutation π must end with the largest letter of π.

3 Pattern Matching Problem With Text and Motif Avoiding (231,213)

In this section we focus on the problem of pattern matching if both the motif and the text avoid (231,213). We first exhibit some properties of a stripe of a permutation avoiding (231,213).

Lemma 1. *If a permutation is in* $\mathrm{Av}_n(213, 231)$ *then the first element is either the minimal or the maximal element.*

Proof Suppose that we have $\pi \in \mathcal{S}_n$, and $\pi(1) \neq 1$ and $\pi(1) \neq n$ then we have either $\pi(1)...n...1$ or $\pi(1)...1...n$, either the motif 231 or 213, which is not possible by hypothesis. $\square$

Corollary 1. *A permutation $\pi \in \mathcal{S}_n$ is in $\mathrm{Av}_n(213, 231)$ if and only if for $1 \leq i \leq n$, $\pi[i :]$ start either with the maximal element or the minimal element of $\pi[i :]$.*

Corollary 2. *If $\pi \in \mathrm{Av}_n(213, 231)$ its stripe can be define such as,*
$$\mathrm{s}_\pi(i) = \begin{cases} a & \text{if } \pi(i) \text{ is the minimal element of } \pi[i :] \\ d & \text{if } \pi(i) \text{ is the maximal element of } \pi[i :] \end{cases}$$

Lemma 2. *Let $\pi \in \mathrm{Av}_n(213, 231)$, $\forall i, j \in [n]$, $i < j$, if $\mathrm{s}_\pi(i) = a$ and $\mathrm{s}_\pi(j) = d$ then $\pi(i) < \pi(j)$.*

In other word, for every index i and j of an (231,213) avoiding permutation π, if i is an ascent and if j is a descent then $\pi(i) < \pi(j)$.

Proof By induction on the size of the permutation : the proposition is true for the permutations $(1, 2)$ and $(2, 1)$. Suppose the proposition true for $\mathrm{Av}_n(213, 231)$. Let $\pi \in \mathrm{Av}_n(213, 231)$, we show that the proposition holds for π. Let π' be the permutation $\pi[2 :]$. $\pi' \in \mathrm{Av}_n(213, 231)$, so every element of p' verify our lemma (by induction). We still need to compare the first element with the rest. But the first element is the maximal (minimal) element of all the permutation (by Lemma 1). Moreover if the first element is the maximal (minimal) it will mapped to an descent (ascent). So the proposition is also true for the first element. □

Lemma 3. *Let $\pi \in \mathrm{Av}_n(213, 231)$, $\forall i, j \in [n]$, $i < j$, if $\mathrm{s}_\pi(i) = a$ and $\mathrm{s}_\pi(j) = a$ then $\pi(i) < \pi(j)$.*

In other word, for every index i and j of an (231,213) avoiding permutation π, if i is a ascent and j is a ascent and $i < j$ then $p(i) < p(j)$.

Lemma 4. *Let $\pi \in \mathrm{Av}_n(213, 231)$, $\forall i, j \in [n]$, $i < j$, if $\mathrm{s}_\pi(i) = d$ and $\mathrm{s}_\pi(j) = d$ then $\pi(i) > \pi(j)$.*

In other word, for every index i and j of an (231,213) avoiding permutation π, if i is a descent and j is a descent and $i < j$ then $\pi(i) > \pi(j)$.

Proposition 1. *Let $\pi, \sigma \in \mathrm{Av}_n(213, 231)$. σ is a motif of π if and only if the stripe of σ is a motif of the stripe of π.*

Proof

$\Rightarrow$ Suppose that σ is a motif of π by subsequence s. By definition s normalized is equal to σ so they have the same stripe.

$\Leftarrow$ Let π be a permutation avoiding $(213, 231)$, S_1 be its stripe, σ be a permutation avoiding $(213, 231)$, and S_2 its stripe. Let ϕ be the function that match S_2 to S_1. We define φ as, $\forall x \in [|\sigma| - 1]$, $\varphi(x) = \phi(x)$, and $\varphi(|\sigma|) = \phi(|\sigma| - 1) + 1$ Suppose that $\exists x, y \in [|\sigma|]$, $\sigma(x) < \sigma(y)$ and $\pi(\varphi(x)) > \pi(\varphi(y))$, without lost of generality we can suppose that $x < y$, Let see all the case possible that can make this hypothesis possible :

- if $\sigma(x) < \sigma(y)$, $\mathrm{s}_\sigma(x) = a$ and $\mathrm{s}_\sigma(y) = a$ (if x and y are increasing in σ), then $\mathrm{s}_\pi(\varphi(x)) = a$ and $\mathrm{s}_\pi(\varphi(y)) = a$, (the match are also increasing). As $x < y$ and as φ is an increasing function, $\varphi(x) < \varphi(y)$. And thus, by **Lemma 3**, $\pi(\varphi(x)) < \pi(\varphi(y))$. Which contradict our hypothesis, so this case is not possible.
- if $\sigma(x) < \sigma(y)$, $\mathrm{s}_\sigma(x) = a$ and $\mathrm{s}_\sigma(y) = d$ then $\mathrm{s}_\pi(\varphi(x)) = a$ and $\mathrm{s}_\pi(\varphi(y)) = d$, As $x < y$ and as φ is an increasing function, $\varphi(x) < \varphi(y)$. And thus, by **Lemma 2**, $\pi(\varphi(x)) < \pi(\varphi(y))$. Which contradict our hypothesis, so this case is not possible.
- if $\sigma(x) > \sigma(y)$, $\mathrm{s}_\sigma(x) = d$ and $\mathrm{s}_\sigma(y) = d$ then $\mathrm{s}_\pi(\varphi(x)) = d$ and $\mathrm{s}_\pi(\varphi(y)) = d$, As $x < y$ and as φ is an increasing function, $\varphi(y) < \varphi(x)$. And thus, by **Lemma 4**, $\pi(\varphi(y)) > \pi(\varphi(x))$. Which contradict our hypothesis, so this case is not possible.
- if $\sigma(x) > \sigma(y)$, $\mathrm{s}_\sigma(x) = d$ and $\mathrm{s}_\sigma(y) = a$ then $\mathrm{s}_\pi(\varphi(x)) = d$ and $\mathrm{s}_\pi(\varphi(y)) = a$, As $x < y$ and as φ is an increasing function, $\varphi(y) < \varphi(x)$. And thus, by **Lemma 2**, $\pi(\varphi(y)) > \pi(\varphi(x))$. Which contradict our hypothesis, so this case is not possible.□
- If $y = |\sigma|$ and $x = |\sigma| - 1$, Clearly if $\mathrm{s}_\sigma(x) = a$, then $\sigma(y) - \sigma(x) > 0$ and because we have a match in the ribbon $\pi(\varphi(y)) - \pi(\varphi(x)) > 0$. Same idea goes if $\mathrm{s}_\sigma(x) = d$. So this case is not possible.
- if $y = |\sigma|$ and $x < |\sigma| - 1$, if $\mathrm{s}_\sigma(x) = a$ ie $\sigma(x+1) - \sigma(x) > 0$. Then we must have either $\sigma(x) < \sigma(x+1) < \sigma(|\sigma|)$ or $\sigma(x) < \sigma(|\sigma|) < \sigma(x+1)$ to avoid 231 or 213 in the motif. Because we have a match in the stripe and what we prove above, $\pi(\varphi(x+1) - \pi(\varphi(x)) > 0$. But by hypothesis $\pi(\varphi(|\pi|)) < \pi(\varphi(x))$, so we have either $\sigma(|\sigma|) < \sigma(x) < \sigma(x+1)$ or $\sigma(|\sigma|) < \sigma(x) < \sigma(x+1)$ which both lead to a 231 or a 213 motif in the text which is not possible. The same idea goes if $\mathrm{s}_\sigma(x) = d$.

So our hypothesis is not realisable.So our assertion is true. □

Proposition 1 Let σ and π be two permutations avoiding (231,213), we can decide in linear time if σ appear in π.

Proof To solve the permutation pattern for a permutation in $\mathrm{Av}_n(213, 231)$, we find a find a match of the stripe of the pattern in the stripe of the text. This can be done linearly by a greedy algorithm, by matching a step whenever it is possible. And thank to the definition of the definition of a stripe in **Corollary 1**, we do not have to compute the stripe advance. Therefore we have an linear (online) algorithm.□

4 Pattern Matching With Motif Avoiding (231,213)

Proposition 2. *Let σ, π be two permutation, with σ a permutation avoiding (231,213), we can decide in $O(|\sigma| * |\pi|^2)$ time and $O(|\sigma| * |\pi|^2)$ space if σ is a motif of π.*

For the algorithm we introduce a decomposition, we decompose the motif into adjacent element of same step in the stripe (called such sequence segment), add the last element to the right most segment, and label them form right to left with increasing number. Let's call this decomposition S. For each segment, we call important the leftmost element denoted by IMP. For example, given the motif $(12, 11, 10, 1, 2, 3, 4, 9, 8, 5, 6, 7)$, $S(1) = (5, 6, 7)$, $IMP(1) = 5$, $S(2) = (9, 8)$, $IMP(2) = 9$, $S(3) = (1, 2, 3, 4)$, $IMP(3) = 1$, $S(4) = (12, 11, 10)$ and $IMP(4) = 12$.

Given a ascent (descent) segment $S(s)$, if we want to match the suffix $s_n s_{s-1} ... s_1$ starting with an ascent (descent), and given that we know every match of $s_{s-1} ... s_1$, the optimal way of matching $s_{s-1} ... s_1$ is to choose the match that minimize (maximize) the maximal (minimal) element of the match and that allow a match of s_n. In other word if we want to match $\sigma[IMP(s) :]$ with $\pi[i :]$ and match $IMP(s)$ with i. The optimal way of matching $\sigma[IMP(s-1) :]$ is to choose the minimal (maximal) element $\pi(i')$ such that : (1) $\sigma[IMP(s-1) :]$ is matched with $\pi[i' :]$ and $IMP(s-1)$ is matched with e. (2) The segment $S(s)$ is matched with $\pi[i : i'-1]$ and $IMP(s)$ is matched with i. (3) Every elements of the match of (2), is inferior (superior) to the minimal (maximal) element of the match of (1). Indeed when we will match $s_{n+1} s_n s_{s-1} ... s_1$ more element will be available, because every matched element of s_{n+1} must be superior (inferior) to every matched element of $s_{s-1} ... s_1$.

Consider the following problem :
$LM(s, i) =$ the optimal match of $IMP(s-1)$ if there exists a match of $\sigma[IMP(s) :]$ with $\pi[i :]$ and $IMP(s)$ is matched with element i.

This problem can be solve by induction.

BASE :

$$LM(1, i) = \begin{cases} MIN_{i<i'}\{0\} \cup \{i' |\ i' \text{ such that } & \text{if } s_\sigma(IMP(1)) = a \\ \ LIS(i, i', \pi(i')) \leq S(1)\} & \\ & \\ MAX_{i<i'}\{0\} \cup \{i' |\ i' \text{ such that } & \text{if } s_\sigma(IMP(1)) = d \\ \ LDS(i, i', \pi(i')) \leq S(1)\} & \end{cases}$$

STEP :

$$LM(s, i) = \begin{cases} MIN\{0\} \cup I(s, i) & \text{if } s_\sigma(IMP(s)) = a \\ MAX\{0\} \cup D(s, i) & \text{if } s_\sigma(IMP(s)) = a \end{cases}$$

With $I(s, i)$ $(D(s, i))$ the set of element such that if $e \in I(s, i)$ then there exists a match of $\sigma[IMP(s-1) :]$ in $\pi[e :]$ and there is a match of s in $\pi[i : e-1]$ with every element inferior (superior) to the match of $\sigma[IMP(s-1) :]$ in $\pi[e :]$. Formally we define $I(s, i)$ and $D(s, i)$ such as:

$$I(s,i) = \{i' | i < i' \text{ and } LM(s-1,i') \neq 0$$
$$\text{and } LIS(i, i'-1, IMP(s-1)) \leq S(s)\}$$

$$D(s,i) = \{i' | i < i' \text{ and } LM(s-1,i') \neq 0$$
$$\text{and } LDS(i, i'-1, IMP(s-1)) \leq S(s)\}$$

With LIS(i,j,k) (LDS(i,j,k)) is the longest increasing (decreasing) sequence in π starting at i and ending at j, with every element of this sequence inferior (superior) to k. LIS and LDS can be computed in $O(|\pi|^2 * log(log(|\pi|)))$ (see [4]).

In the base case, we are looking for a match for the first segment. Each segment is either ascent or descent. If the segment is an ascent (descent) we have to find an increasing (decreasing) sequence in the text of same size of longest that the size of the segment, and to find the optimal solution we must assure that the last element of the sequence is minimal (maximal).
For the induction, it is the same idea except that we must assure that every element of the sequence is superior to the maximum element of the rest of the match or inferior to the minimal element of the rest of the match which is given by LM of the previous segment.

There exists a match of σ in π if and only if there exists a $LM(n,i)$ for $1 \leq i \leq |\pi|$, with n the number of segment in σ. Moreover the basic case can be computed in $O(|\pi|^2)$ and the induction on $O(|\sigma| * |\pi|^2)$ $\square$.

5 Pattern Matching With Bivincular Motif Avoiding (231,213)

Proposition 3. *Let (σ, X, Y) be a bivincular motif avoiding (231,213), π a permutation, we can decide in $O(|\sigma| * |\pi|^3)$ time and $O(|\pi|^3)$ space if σ appear in π.*

Proof We consider the following problem :

Given a bivincular motif (σ, X, Y) avoiding $(231, 213)$, a text π, i,j , $i < j$.

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(i,j) = \begin{cases} true & \text{If } \sigma[i:] \text{ is a motif of } \pi[j:] \\ & \text{with every element in} \\ & [min, max] \\ false & otherwise \end{cases}$$

$\mathbb{P}_{\sigma,\pi,min,max}$ is closed under induction. it can be solved by means of the following relations:

BASE :

if $|\sigma| \notin X$:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|, j) = \begin{cases} true & \text{if } min < \pi(j) < max \\ false\ otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1, j) = true$$

if $|\sigma| \in X$:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|, |\pi|) = \begin{cases} true & \text{if } min < \pi(|\pi|) < max \\ false\ otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1, |\pi|) = true$$

if $|\sigma| \notin X$ and $|\sigma| \in Y$ and $\sigma(|\sigma|)$ is the maximal element:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|, j) = \begin{cases} true & \text{if } min < \pi(j) < max \\ & \quad \pi(j) \text{ is the maximal element} \\ false\ otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1, j) = true$$

if $|\sigma| \in X$ and $|\sigma| \in Y$ and $\sigma(|\sigma|)$ is the maximal element:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|, |\pi|) = \begin{cases} true & \text{if } min < \pi(|\pi|) < max \\ & \quad \pi(|\pi|) \text{ is the maximal element} \\ false\ otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1, |\pi|) = true$$

STEP :

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(i, j) =$$

$$
\begin{cases}
\bigcup_{j<k} \mathbb{P}_{(\sigma,X,Y),\pi,\pi(j),max}(i+1,k) & \text{if } \mathrm{s}_\sigma(i)=a \text{ and } i\notin X \\ & \text{and } i\notin Y \\ & \text{and } min<\pi(j)<max \\[1em]
\bigcup_{j<k} \mathbb{P}_{(\sigma,X,Y),\pi,min,\pi(j)}(i+1,k) & \text{if } \mathrm{s}_\sigma(i)=d \\ & \text{and } i\notin X \text{ and } i\notin Y \\ & \text{and } min<\pi(j)<max \\[1em]
\mathbb{P}_{(\sigma,X,Y),\pi,t(j),max}(i+1,j+1) & \text{if } \mathrm{s}_\sigma(i)=a \\ & \text{and } i\in X \text{ and } i\notin Y \\ & \text{and } min<\pi(j)<max \\[1em]
\mathbb{P}_{(\sigma,X,Y),\pi,min,t(j)}(i+1,j+1) & \text{if } \mathrm{s}_\sigma(i)=d \\ & \text{and } i\in X \text{ and } i\notin Y \\ & \text{and } min<\pi(j)<max \\[1em]
\mathbb{P}_{(\sigma,X,Y),\pi,\pi(j),max}(i+1,\pi^{-1}(\pi(j)+1)) & \text{if } \mathrm{s}_\sigma(i)=a \\ & \text{and } i\notin X \text{ and } i\in Y \\ & \text{and } min<\pi(j)<max \\ & \text{and } \pi^{-1}(\pi(j)+1)>j \\[1em]
\mathbb{P}_{(\sigma,X,Y),\pi,min,\pi(j)}(i+1,\pi^{-1}(\pi(j)-1)) & \text{if } \mathrm{s}_\sigma(i)=d \\ & \text{and } i\notin X \text{ and } i\in Y \\ & \text{and } min<\pi(j)<max \\ & \text{and } \pi^{-1}(\pi(j)-1)>j \\[1em]
\mathbb{P}_{(\sigma,X,Y),\pi,\pi(j)),max}(i+1,j+1) & \text{if } \mathrm{s}_\sigma(i)=a \\ & \text{and } i\in X \text{ and } i\in Y \\ & \text{and } \pi(j)+1=\pi(j+1) \\ & \text{and } min<\pi(j)<max \\[1em]
\mathbb{P}_{(\sigma,X,Y),\pi,min,\pi(j)}(i+1,j+1) & \text{if } \mathrm{s}_\sigma(i)=d \\ & \text{and } i\in X \text{ and } i\in Y \\ & \text{and } \pi(j)-1=\pi(j+1) \\ & \text{and } min<\pi(j)<max \\[1em]
False & Otherwise
\end{cases}
$$

At each step (i,j), If $i \notin X$ and $i \notin Y$, we match the current element of σ ($\sigma(i)$) with the current element of π ($\pi(j)$), if possible. Then we match $\sigma[i+1:]$ with every suffixes π starting after j.

If $i \in X$ and $i \notin Y$, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[j+1:]$ with the condition that $\sigma(i+1)$ is matched to $\pi(j+1)$.

If $i \notin X$, $i \in Y$ and i is a descent, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[\pi^{-1}(\pi(j)-1):]$ with the condition that $\sigma(i+1)$ is matched to the element $\pi(j)-1$.

If $i \notin X$, $i \in Y$ and i is an ascent, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[\pi^{-1}(\pi(j)+1):]$ with the condition that $\sigma(i+1)$ is matched to the element $\pi(j)+1$.

If $i \in X$, $i \in Y$ and i is a descent, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[j+1:]$ with the condition that $\sigma(i+1)$ is matched to $\pi(j+1)$, and that $\pi(j+1) = \sigma(i) - 1$.

If $i \in X$, $i \in Y$ and i is an ascent, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[j+1:]$ with the condition that $\sigma(i+1)$ is matched to $\pi(j+1)$, and that $\pi(j+1) = \sigma(i) + 1$.

There are two cases to consider searching for a match :
If $|\sigma| \in X$ there exists a match if and only if
$\bigcup_{0<min<max<|\pi|} \mathbb{P}_{\sigma,\pi,min,max}(0,0)$ is true.

If $|\sigma| \notin X$ there exists a match if and only if
$\bigcup_{0<k<|\pi|, 0<min<max<|\pi|} \mathbb{P}_{\sigma,\pi,min,max}(0,k)$ is true. $\square$

6 Longest Subsequence Avoiding (231,213) for a Permutation

In this section we present an algorithm to solve the problem of the longest subsequence avoiding (231,213). To do so, we need the set of element that are matched to an ascent and the set of element that are matched to a descent. For a permutation π of size n, we define $P(\pi) = \{i|\,\mathrm{s}_\pi(i) = a\} \cup \{n\}$ and $M(\pi) = \{i|\,\mathrm{s}_\pi(i) = d\} \cup \{n\}$.

Proposition 4. *If s is a longest subsequence avoiding (213,231) with last element at index f in π then $P(\pi)$ is a longest increasing subsequence with last element at index f and $M(\pi)$ is a longest decreasing subsequence with last element at index f.*

Proof Let's s is a longest subsequence avoiding (213,231) with last element at index f in π, suppose that $P(\pi)$ is not a longest increasing subsequence with last element at index f. Let's s_m be a longest increasing subsequence with last element f. Thus $|s_m| > P(\pi)$, clearly the sequence $s_m \cup M(\pi)$ is avoiding (213,231) and is longer than s witch is not possible. The case for $M(\pi)$ follows the same idea.$\square$

Proposition 5. *Given a permutation π, finding the longest subsequence avoiding (231,213) can be done in $O(|\pi| * log(log(|\pi|)))$ time and in $O(n)$ space.*

Proof The proposition 4 lead to algorithm where we have to compute longest increasing and decreasing subsequence ending at every index. Then finding the maximum sum of longest increasing and decreasing subsequence ending at the same index. Computing the longest increasing and decreasing can be done in $O(|\pi| * log(log(|\pi|)))$ time and $O(n)$ space (see [4]), then finding the maximum can be done in linear time.□

7 Longest Subsequence Avoiding (231,213) Common for Two Permutation

In this section we present an algorithm to find the longest common subsequence avoiding (231,213) between two permutations.

Proposition 6. *Given two permutation π_1 and π_2 The longest common subsequence avoiding (231,213) can be solve in $O(|\pi_1|^3 * |\pi_2|^3)$.*

Proof Consider the following problem, that compute the longest stripe common to π_1 and π_2. Given two permutation π_1 and π_2.

$S_{\pi_1,\pi_2}(min_1, max_1, min_2, min_2, i_1, i_2) = \max\{\ |m| \mid$ s is a pattern occurring in $\pi_1[i_1 :]$ by the subsequence s_1 and $min(s_1) = min_1$ and $max(s_1) = max_1$ and s is occurring in $\pi_2[i_2 :]$ by the subsequence s_2 and $min(s_2) = min_2$ and $max(s_2) = max_2\ \}$

We show that this family of problems are closed under induction.

BASE :

$$S_{\pi_1,\pi_2,min_1,max_1,min_2,max_2}(|\pi_1|, |\pi_2|) = \begin{cases} 1 \text{ if } min_1 < \pi_1(j) < max_1 \\ \quad \text{and } min_2 < \pi_2(j) < max_2 \\ 0 \ otherwise \end{cases}$$

STEP :

$$S_{\pi_1,\pi_2,min_1,max_1,min_2,max_2}(i_1, i_2) = \\ max \begin{cases} S_{\pi_1,\pi_2,min_1,max_1,min_2,max_2}(i_1, i_2 + 1) \\ S_{\pi_1,\pi_2,min_1,max_1,min_2,max_2 2}(i_1 + 1, i_2) \\ S'_{\pi_1,\pi_2,min_1,max_1,min_2,max_2}(i_1, i_2) \end{cases}$$

with $S'_{\pi_1,\pi_2,min_1,max_1,min_2,max_2}(i_1,i_2) =$

$$\begin{cases} 1 + S_{\pi_1,\pi_2,\pi_1(i_1),max_1,\pi_2(i_2)}(max_2, i_1+1, i_2+1) & \pi_1(i_1) < min_1 \\ & \text{and } \pi_2(i_2) < min_2 \\ \\ 1 + S_{\pi_1,\pi_2,min_1,\pi_1(i_1),min_2,\pi_2(i_2)}(i_1+1, i_2+1) & \pi_1(i_1) > max_1 \\ & \text{and } \pi_2(i_2) > max_2 \\ \\ 0 & \text{otherwise} \end{cases}$$

For every pair i, j we either ignore the element of π_1, either ignore the element of π_2, either we match as the same step (if possible). Those relation lead to a $O(|\pi_1|^3 * |\pi_2|^3)$ time and $O(|\pi_1|^3 * |\pi_2|^3)$ space algorithm.□

8 Conclusion

References

1. S. Ahal and Y. Rabinovich. On Complexity of the Subpattern Problem. 22(2):629–649, 2008.
2. M.H. Albert, R.E.L. Aldred, M.D. Atkinson, and D.A. Holton. Algorithms for pattern involvement in permutations. In *Proc. International Symposium on Algorithms and Computation (ISAAC)*, volume 2223 of *Lecture Notes in Computer Science*, pages 355–366, 2001.
3. D. Avis and M. Newborn. On pop-stacks in series. *Utilitas Mathematica*, 19:129140, 1981.
4. Sergei Bespamyatnikh and Michael Segal. Enumerating longest increasing subsequences and patience sorting, 2000.
5. P. Bose, J.F.Buss, and A. Lubiw. Pattern matching for permutations. *Information Processing Letters*, 65(5):277–283, 1998.
6. M.-L. Bruner and M.Lackner. A fast algorithm for permutation pattern matching based on alternating runs. In F.V. Fomin and P. Kaski, editors, *13th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT), Helsinki, Finland*, pages 261–270. Springer, 2012.
7. M. Crochemore and E. Porat. Fast computation of a longest increasing subsequence and application. *Information and Compututation*, 208(9):1054–1059, 2010.
8. S. Guillemot and D. Marx. Finding small patterns in permutations in linear time. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), Portland, Oregon, USA*, pages 82–101. SIAM, 2014.
9. S. Guillemot and S. Vialette. Pattern matching for 321-avoiding permutations. In Y. Dong, D.-Z. Du, and O. Ibarra, editors, *Proc. 20-th International Symposium on Algorithms and Computation (ISAAC), Hawaii, USA*, volume 5878 of *LNCS*, page 10641073. Springer, 2009.
10. L. Ibarra. Finding pattern matchings for permutations. *Information Processing Letters*, 61(6):293–295, 1997.
11. S. Kitaev. *Patterns in Permutations and Words*. Springer-Verlag, 2013.