Pattern matching for $(231, 213)$-avoiding permutations

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Abstract. bla bla. . .

1 Introduction

A permutation π is said to contain another permutation σ, in symbols $\sigma \preceq \pi$, if there exists a subsequence of entries of π that has the same relative order as σ, and in this case σ is said to be a *pattern* of π. Otherwise, π is said to avoid the permutation σ. For example a permutation contains the pattern 123 (resp. 321) if it has an increasing (resp. decreasing) subsequence of length 3. Here, note that members need not actually to be consecutive, merely increasing (resp. descreasing). During the last decade, the study of the pattern containment on permutations has become a very active area of research.

We consider here the so-called *pattern containment* problem (also sometimes referred to as the pattern involvement problem): Given two permutations σ and π, this problem is to decide whether $\sigma \preceq \pi$ (the problem is ascribed to Wilf in [**?**]). The permutation containment problem is **NP**-hard [**?**]. It is, however, polynomial time solvable by brute-force enumeration if σ has bounded size. Improvements to this algorithm were presented in [**?**] and [**?**], the latter describing a $O(|\pi|^{0.47k+o(|\sigma|)})$ time algorithm. Bruner and Lackner [**?**] gave a fixed-parameter algorithm solving the pattern containment problem with an exponential worst-case runtime of $O(1.79^{\mathrm{run}(\pi)})$, where $\mathrm{run}(\pi)$ denotes the number of alternating runs of π. (This is an improvement upon the $O(2^{|\pi|})$ runtime required by brute-force search without imposing restrictions on σ and π.)

A few particular cases of the pattern containment problem have been attacked successfully. The case of increasing patterns is solvable in $O(|\pi| \log \log |\sigma|)$ time in the RAM model [**?**], improving the previous 30-year bound of $O(|\pi| \log |\sigma|)$. (The algorithm also improves on the previous $O(|\pi| \log \log |\pi|)$ bound.) Furthermore, the patterns 132, 213, 231, 312 can all be handled in linear time by stack sorting algorithms. Any pattern of length 4 can be detected in $O(|\pi| \log |\pi|)$ time [**?**]. Algorithmic issues for 321-avoiding patterns containment has been investigated in [**?**].

Of particular interest in our context, the pattern containment problem is also solvable in polynomial time for separable patterns. Separable permutations are those permutations that contain neither 2413 nor 3142, and they are enumerated by the Schrder numbers (sequence A006318 in OEIS). Separable permutations has gained a lot of interest in the pattern containment context during the last years. Indeed, the pattern containment problem is solvable $O(kn^4)$ time and

$O(kn^3)$ space for separable patterns [?] (see also [?]), where k is the length of the pattern and n is the length of the target permutation. Notice that there are numerous characterizations of separable permutations. To mention just a few examples, they are the permutations whose permutation graphs are cographs (*i.e.* P_4-free graphs); equivalently, a separable permutation is a permutation that can be obtained from the trivial permutation 1 by *direct sums* and *skew sums*. To the best of our knowledge, separable permutations first arose in the work of Avis and Newborn [?], who showed that they are precisely the permutations which can be sorted by an arbitrary number of pop-stacks in series, where a pop-stack is a restricted form of stack in which any pop operation pops all items at once. (Notice that the separable permutations include as a special case the stack-sortable permutations, which avoid the pattern 231.) Separable permutations have been widely studied in the last decade, both from a combinatorial [?] and an algorithmic [?,?,?,?] point of view.

There exist many generalisation of patterns that are worth considering in the context of algorithmic issues in pattern involvement (see [?] for an up-to-date survey). Vincular patterns, also called generalized patterns, resemble (classical) patterns, with the constraint that some of the letters in an occurrence must be consecutive. Of particular importance in our context, Bruner and Lackner [?] proved that deciding whether a vincular pattern σ of length k occurs in a permutation $\pi \in \mathfrak{S}_n$ is $W[1]$-complete for parameter k. bivincular patterns generalize classical patterns even further than vincular patterns. Indeed, in bivincular patterns, not only positions but also values of elements involved in a matching may be forced to be adjacent

2 Definition

This section presents the decomposition used and some notation.

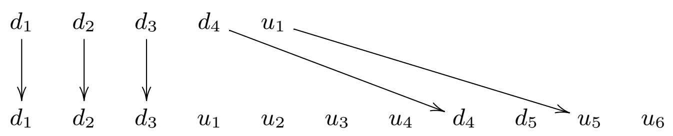
We denote by $\mathrm{Av}_n(213, 231)$ the class of permutation of size n which avoid the motif 213 and 231. $\mathcal{S}_n$ the permutation of size n. $[n]$ the set $\{1, 2, ..., n\}$. For a permutation p, let $p[i : j]$, be the permutation from element i to j, $p[: j]$ be the prefix ending at index j and $p[i :]$ be the suffix starting at index i.

Given two permutations σ and π, we say that σ is a motif/match of π, if and only if there exists and increasing function $\sigma : [|\sigma|] \to [|\sigma|]$ such that : $\forall i$ and $\forall j$ $i, j \in |\sigma|$ if $\sigma(i) < \sigma(j)$ then $\pi(\sigma(i)) < \pi(\sigma(j))$.

A **Stripe** of a permutation is a mapping into **up step** (noted u) or a **down step** (noted d). An element i of a permutation p is an up step if and only if i is not the last element and $p(i) < p(i + 1)$. An element i of a permutation p is a down step if and only if i is not the last element and $p(i) > p(i + 1)$.
Formally a Stripe is a function $\mathcal{S} : [n - 1] \to \{d, u\}$ define as :
$$\mathcal{S}(p, i) = \begin{cases} u & \text{if } 1 \leq i < n \text{ and } p(i + 1) - p(i) > 0 \\ d & \text{if } 1 \leq i < n \text{ and } p(i + 1) - p(i) < 0 \end{cases}$$

By extension we define $\mathcal{S} : \mathcal{S}_n \to \{d, u\}^n$ such as : $\mathcal{S}(p) = \mathcal{S}(p, 1)\mathcal{S}(p, 2)...\mathcal{S}(p, n-1)$. We call $\mathcal{S}(p)$ the stripe of p. For example the stripe of the permutation $(12, 11, 10, 1, 2, 3, 4, 9, 8, 5, 6, 7)$, is $d, d, d, u, u, u, u, d, d, u, u$.

Formally we say that the stripe of $\sigma \in \mathcal{S}_m$ is a motif of a stripe of $\pi \in \mathcal{S}_n$ if and only if there exist an increasing function $\phi : [|\sigma|] \to [|\sigma|]$ such that :
$\forall i \in [|\sigma|]$, $\mathcal{S}(\sigma, i) = \mathcal{S}(\pi, \phi(i))$
If a stripe R_2 is a motif of a stripe R_1, it means that there exists a subsequence of R_1 such as this sequence is equal to R_2. For example, the stripe $(d_1, d_2, d_3, d_4, u_1)$ is a motif of the stripe
$(d_1, d_2, d_3, u_1, u_2, u_3, u_4, d_4, d_5, u_5, u_6)$, because we can match
$(d_1, d_2, d_3, d_4, u_1)$ to $(d_1, d_2, d_3, d_4, u_5, u_6)$. But the stripe (d, u, d, d, d) is not a



motif of the stripe
$(d_1, d_2, d_3, u_1, u_2, u_3, u_4, d_4, d_5, u_5, u_6, u_7)$.

A **bivincular motif** generalize a motif. A bivincular motif allows to force element to be adjacent in position or/and in value. A bivincular motif is a motif with two set (σ, X, Y), the first set contains the first index of two consecutive indexes and the second set contains the first indexes of the two consecutive elements. Formally, Let (σ, X, Y) be a bivincular motif and π a permutation. if $i \in X$, and i is matched with j in π then $i + 1$ must be matched with $j + 1$, with the special case of 0 and $|\sigma|$: if $0 \in X$ then the first element of σ must be matched with the first element of π, and if $|\sigma| \in X$ then the last element of σ must be matched with the last element of π. if $i \in Y$, and i is matched with j in π then $i + 1$ must be matched with l such as $\pi(l) = \pi(j) + 1$, with the special case of 0 and $|\sigma|$: if $0 \in Y$ then the minimal element of σ must be matched with the minimal element of π, and if $|\sigma| \in Y$ then the maximal element of σ must be matched with the maximal element of π.

3 Pattern Matching Problem With Text and Motif Avoiding (231,213)

In this section we focus on the problem of pattern matching if both the motif and the text avoid (231,213). We first exhibit some properties of a stripe of a permutation avoiding (231,213).

Lemma 1 If a permutation is in $\mathrm{Av}_n()$ then the first element is either the minimal or the maximal element.

Proof Suppose that we have $p \in \mathcal{S}_n$, and $p(1) \neq 1$ and $p(1) \neq n$ then we have either $p(1)...n...1$ or $p(1)...1...n$, either the motif 231 or 213, which is not possible by hypothesis. $\square$

Corollary 2 A permutation $p \in \mathcal{S}_n$ is in $\mathrm{Av}_n(213, 231)$ if and only if for $1 \leq i \leq n$, $p[i :]$ start either with the maximal element or the minimal element of $p[i :]$.

Corollary 3 If $p \in \mathrm{Av}_n(213, 231)$ its stripe can be define such as,

$$\mathcal{S}(p, i) = \begin{cases} u & \text{if } p(i) \text{ is the minimal element of } p[i :] \\ d & \text{if } p(i) \text{ is the maximal element of } p[i :] \end{cases}$$

Lemma 4 Let $p \in \mathrm{Av}_n(213, 231)$, $\forall i, j \in [n]$, $i < j$, if $\mathcal{S}(p, i) = u$ and $\mathcal{S}(p, j) = d$ then $p(i) < p(j)$.

In other word, for every index i and j of an (231,213) avoiding permutation p, if i is a up step and if j is a down step then $p(i) < p(j)$.

Proof By induction on the size of the permutation : the proposition is true for the permutations $(1, 2)$ and $(2, 1)$. Suppose the proposition true for $\mathrm{Av}_n(213, 231)$. Let $p \in \mathrm{Av}_n(213, 231)$, we show that the proposition holds for p. Let p' be the permutation $p[2 :]$. $p' \in \mathrm{Av}_n(213, 231)$, so every element of p' verify our lemma (by induction). We still need to compare the first element with the rest. But the first element is the maximal (minimal) element of all the permutation (by Lemma 1). Moreover if the first element is the maximal (minimal) it will mapped to an down step (up step). So the proposition is also true for the first element. $\square$

Lemma 5 Let $p \in \mathrm{Av}_n(213, 231)$, $\forall i, j \in [n]$, $i < j$, if $\mathcal{S}(p, i) = u$ and $\mathcal{S}(p, j) = u$ then $p(i) < p(j)$.

In other word, for every index i and j of an (231,213) avoiding permutation p, if i is a up step and j is a up step and $i < j$ then $p(i) < p(j)$.

Lemma 6 Let $p \in \mathrm{Av}_n(213, 231)$, $\forall i, j \in [n]$, $i < j$, if $\mathcal{S}(p, i) = d$ and $\mathcal{S}(p, j) = d$ then $p(i) > p(j)$.

In other word, for every index i and j of an (231,213) avoiding permutation p, if i is a down step and j is a down step and $i < j$ then $p(i) > p(j)$.

Proposition 7 Let $\pi, \sigma \in \mathrm{Av}_n(213, 231)$. σ is a motif of π if and only if the stripe of σ is a motif of the stripe of π.

Proof
$\Rightarrow$ Suppose that σ is a motif of π by subsequence s. By definition s normalized is equal to σ so they have the same stripe.
$\Leftarrow$ Let π be a permutation avoiding $(213, 231)$, R_1 be its stripe, σ be a permutation avoiding $(213, 231)$, and R_2 its stripe. Let ϕ be the function that match R_2 to R_1. We define φ as, $\forall x \in [|\sigma| - 1]$, $\varphi(x) = \phi(x)$, and $\varphi(|\sigma|) = \phi(|\sigma| - 1) + 1$ Suppose that $\exists x, y \in [|\sigma|]$, $\sigma(x) < \sigma(y)$ and $\pi(\varphi(x)) > \pi(\varphi(y))$, without lost of generality we can suppose that $x < y$, Let see all the case possible that can make this hypothesis possible :

- if $\sigma(x) < \sigma(y)$, $\mathcal{S}(\sigma, x) = u$ and $\mathcal{S}(\sigma, y) = u$ (if x and y are increasing in σ), then $\mathcal{S}(\pi, \varphi(x)) = u$ and $\mathcal{S}(\pi, \varphi(y)) = u$, (the match are also increasing). As $x < y$ and as φ is an increasing function, $\varphi(x) < \varphi(y)$. And thus, by **Lemma 5**, $\pi(\varphi(x)) < \pi(\varphi(y))$. Which contradict our hypothesis, so this case is not possible.
- if $\sigma(x) < \sigma(y)$, $\mathcal{S}(\sigma, x) = u$ and $\mathcal{S}(\sigma, y) = d$ then $\mathcal{S}(\pi, \varphi(x)) = u$ and $\mathcal{S}(\pi, \varphi(y)) = d$, As $x < y$ and as φ is an increasing function, $\varphi(x) < \varphi(y)$. And thus, by **Lemma 4**, $\pi(\varphi(x)) < \pi(\varphi(y))$. Which contradict our hypothesis, so this case is not possible.
- if $\sigma(x) > \sigma(y)$, $\mathcal{S}(\sigma, x) = d$ and $\mathcal{S}(\sigma, y) = d$ then $\mathcal{S}(\pi, \varphi(x)) = d$ and $\mathcal{S}(\pi, \varphi(y)) = d$, As $x < y$ and as φ is an increasing function, $\varphi(y) < \varphi(x)$. And thus, by **Lemma 6**, $\pi(\varphi(y)) > \pi(\varphi(x))$. Which contradict our hypothesis, so this case is not possible.
- if $\sigma(x) > \sigma(y)$, $\mathcal{S}(\sigma, x) = d$ and $\mathcal{S}(\sigma, y) = u$ then $\mathcal{S}(\pi, \varphi(x)) = d$ and $\mathcal{S}(\pi, \varphi(y)) = u$, As $x < y$ and as φ is an increasing function, $\varphi(y) < \varphi(x)$. And thus, by **Lemma 4**, $\pi(\varphi(y)) > \pi(\varphi(x))$. Which contradict our hypothesis, so this case is not possible.$\Box$
- If $y = |\sigma|$ and $x = |\sigma| - 1$, Clearly if $\mathcal{S}(\sigma, x) = u$, then $\sigma(y) - \sigma(x) > 0$ and because we have a match in the ribbon $\pi(\varphi(y)) - \pi(\varphi(x)) > 0$. Same idea goes if $\mathcal{S}(\sigma, x) = d$. So this case is not possible.
- if $y = |\sigma|$ and $x < |\sigma| - 1$, if $\mathcal{S}(\sigma, x) = u$ ie $\sigma(x+1) - \sigma(x) > 0$. Then we must have either $\sigma(x) < \sigma(x+1) < \sigma(|\sigma|)$ or $\sigma(x) < \sigma(|\sigma|) < \sigma(x+1)$ to avoid 231 or 213 in the motif. Because we have a match in the stripe and what we prove above, $\pi(\varphi(x+1) - \pi(\varphi(x)) > 0$. But by hypothesis $\pi(\varphi(|\pi|)) < \pi(\varphi(x))$, so we have either $\sigma(|\sigma|) < \sigma(x) < \sigma(x+1)$ or $\sigma(|\sigma|) < \sigma(x) < \sigma(x+1)$ which both lead to a 231 or a 213 motif in the text which is not possible. The same idea goes if $\mathcal{S}(\sigma, x) = d$.

So our hypothesis is not realisable.So our assertion is true. $\Box$

Proposition 8 Let σ and π be two permutations avoiding (231,213), we can decide in linear time if σ appear in π.

Proof To solve the permutation pattern for a permutation in $\mathrm{Av}_n(213, 231)$, we find a find a match of the stripe of the pattern in the stripe of the text. This can be done linearly by a greedy algorithm, by matching a step whenever it is possible. And thank to the definition of the definition of a stripe in **Corollary**

3, we do not have to compute the stripe advance. Therefore we have an linear (online) algorithm.□

4 Pattern Matching With Motif Avoiding (231,213)

Proposition 9 Let σ, π be two permutation, with σ a permutation avoiding (231,213), we can decide in $O(|\sigma| * |\pi|^2)$ time and $O(|\sigma| * |\pi|^2)$ space if σ is a motif of π.

For the algorithm we introduce a decomposition, we decompose the motif into adjacent element of same step in the stripe (called such sequence segment), add the last element to the right most segment, and label them form right to left with increasing number. Let's call this decomposition S. For each segment, we call important the leftmost element denoted by IMP. For example, given the motif $(12, 11, 10, 1, 2, 3, 4, 9, 8, 5, 6, 7)$, $S(1) = (5, 6, 7)$, $IMP(1) = 5$, $S(2) = (9, 8)$, $IMP(2) = 9$, $S(3) = (1, 2, 3, 4)$, $IMP(3) = 1$, $S(4) = (12, 11, 10)$ and $IMP(4) = 12$.

Given a up step (down step) segment $S(s)$, if we want to match the suffix $s_n s_{s-1}...s_1$ starting with an up step (down step), and given that we know every match of $s_{s-1}...s_1$, the optimal way of matching $s_{s-1}...s_1$ is to choose the match that minimize (maximize) the maximal (minimal) element of the match and that allow a match of s_n. In other word if we want to match $\sigma[IMP(s) :]$ with $\pi[i :]$ and match $IMP(s)$ with i. The optimal way of matching $\sigma[IMP(s-1) :]$ is to choose the minimal (maximal) element $\pi(i')$ such that : (1) $\sigma[IMP(s-1) :]$ is matched with $\pi[i' :]$ and $IMP(s-1)$ is matched with e. (2) The segment $S(s)$ is matched with $\pi[i : i'-1]$ and $IMP(s)$ is matched with i. (3) Every elements of the match of (2), is inferior (superior) to the minimal (maximal) element of the match of (1). Indeed when we will match $s_{n+1}s_n s_{s-1}...s_1$ more element will be available, because every matched element of s_{n+1} must be superior (inferior) to every matched element of $s_{s-1}...s_1$.

Consider the following problem :
$LM(s, i) =$ the optimal match of $IMP(s-1)$ if there exists a match of $\sigma[IMP(s) :]$ with $\pi[i :]$ and $IMP(s)$ is matched with element i.

This problem can be solve by induction.

BASE :

$$LM(1, i) = \begin{cases} MIN_{i<i'}\{0\} \cup \{i' |\ i' \text{ such that } & \text{if } \mathcal{S}(\sigma, IMP(1)) = u \\ \ LIS(i, i', \pi(i')) \leq S(1)\} & \\ & \\ MAX_{i<i'}\{0\} \cup \{i' |\ i' \text{ such that } & \text{if } \mathcal{S}(\sigma, IMP(1)) = d \\ \ LDS(i, i', \pi(i')) \leq S(1)\} & \end{cases}$$

STEP :

$$LM(s,i) = \begin{cases} MIN\{0\} \cup I(s,i) & \text{if } \mathcal{S}(\sigma, IMP(s)) = u \\ MAX\{0\} \cup D(s,i) & \text{if } \mathcal{S}(\sigma, IMP(s)) = u \end{cases}$$

With $I(s,i)$ ($D(s,i)$) the set of element such that if $e \in I(s,i)$ then there exists a match of $\sigma[IMP(s-1):]$ in $\pi[e:]$ and there is a match of s in $\pi[i:e-1]$ with every element inferior (superior) to the match of $\sigma[IMP(s-1):]$ in $\pi[e:]$. Formally we define $I(s,i)$ and $D(s,i)$ such as:

$$I(s,i) = \{e | i < e \text{ and } LM(s-1,e) \neq 0 \\ \text{and } LIS(i, e-1, IMP(s-1)) \leq S(s)\}$$

$$D(s,i) = \{e | i < e \text{ and } LM(s-1,e) \neq 0 \\ \text{and } LDS(i, e-1, IMP(s-1)) \leq S(s)\}$$

With LIS(i,j,k) (LDS(i,j,k)) is the longest increasing (decreasing) sequence in π starting at i and ending at j, with every element of this sequence inferior (superior) to k. LIS and LDS can be computed in $O(|\pi|^2 * log(log(|\pi|)))$ (see [**?**]).

In the base case, we are looking for a match for the first segment. Each segment is either up step or down step. If the segment is an up step (down step) we have to find an increasing (decreasing) sequence in the text of same size of longest that the size of the segment, and to find the optimal solution we must assure that the last element of the sequence is minimal (maximal).
For the induction, it is the same idea except that we must assure that every element of the sequence is superior to the maximum element of the rest of the match or inferior to the minimal element of the rest of the match which is given by LM of the previous segment.

There exists a match of σ in π if and only if there exists a $LM(n,i)$ for $1 \leq i \leq |\pi|$, with n the number of segment in σ. Moreover the basic case can be computed in $O(|\pi|^2)$ and the induction on $O(|\sigma| * |\pi|^2)$ $\square$.

5 Pattern Matching With Bivincular Motif Avoiding (231,213)

Proposition 10 Let (σ, X, Y) be a bivincular motif avoiding (231,213), π a permutation, we can decide in $O(|\sigma| * |\pi|^3)$ time and $O(|\pi|^3)$ space if σ appear in π.

Proof We consider the following problem :

Given a bivincular motif (σ, X, Y) avoiding $(231, 213)$, a text π, i,j , $i < j$.

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(i,j) = \begin{cases} true & \text{If } \sigma[i:] \text{ is a motif of } \pi[j:] \\ & \text{with every element in} \\ & [min, max] \\ false & otherwise \end{cases}$$

$\mathbb{P}_{\sigma,\pi,min,max}$ is closed under induction. it can be solved by means of the following relations:

BASE :

if $|\sigma| \notin X$:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|,j) = \begin{cases} true & \text{if } min < \pi(j) < max \\ false & otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1,j) = true$$

if $|\sigma| \in X$:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|,|\pi|) = \begin{cases} true & \text{if } min < \pi(|\pi|) < max \\ false & otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1,|\pi|) = true$$

if $|\sigma| \notin X$ and $|\sigma| \in Y$ and $\sigma(|\sigma|)$ is the maximal element:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|,j) = \begin{cases} true & \text{if } min < \pi(j) < max \\ & \quad \pi(j) \text{ is the maximal element} \\ false & otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1,j) = true$$

if $|\sigma| \in X$ and $|\sigma| \in Y$ and $\sigma(|\sigma|)$ is the maximal element:

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|,|\pi|) = \begin{cases} true & \text{if } min < \pi(|\pi|) < max \\ & \quad \pi(|\pi|) \text{ is the maximal element} \\ false & otherwise \end{cases}$$

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(|\sigma|+1,|\pi|) = true$$

STEP :

$$\mathbb{P}_{(\sigma,X,Y),\pi,min,max}(i,j) =$$

$$
\begin{cases}
\bigcup_{j<k} \mathbb{P}_{(\sigma,X,Y),\pi,t(j),max}(i+1,k) & \text{if } \mathcal{S}(\sigma,i)=u \text{ and } i\notin X \\ & \text{and } i\notin Y \\ & \text{and } min<\pi(j)<max \\
\bigcup_{j<k} \mathbb{P}_{(\sigma,X,Y),\pi,min,t(j)}(i+1,k) & \text{if } \mathcal{S}(\sigma,i)=d \\ & \text{and } i\notin X \text{ and } i\notin Y \\ & \text{and } min<\pi(j)<max \\
\mathbb{P}_{(\sigma,X,Y),\pi,t(j),max}(i+1,j+1) & \text{if } \mathcal{S}(\sigma,i)=u \\ & \text{and } i\in X \text{ and } i\notin Y \\ & \text{and } min<\pi(j)<max \\
\mathbb{P}_{(\sigma,X,Y),\pi,min,t(j)}(i+1,j+1) & \text{if } \mathcal{S}(\sigma,i)=d \\ & \text{and } i\in X \text{ and } i\notin Y \\ & \text{and } min<\pi(j)<max \\
\mathbb{P}_{(\sigma,X,Y),\pi,t(j),max}(i+1,\pi^{-1}(\pi(j)+1)) & \text{if } \mathcal{S}(\sigma,i)=u \\ & \text{and } i\notin X \text{ and } i\in Y \\ & \text{and } min<\pi(j)<max \\ & \text{and } \pi^{-1}(\pi(j)+1)>j \\
\mathbb{P}_{(\sigma,X,Y),\pi,min,t(j)}(i+1,\pi^{-1}(\pi(j)-1)) & \text{if } \mathcal{S}(\sigma,i)=d \\ & \text{and } i\notin X \text{ and } i\in Y \\ & \text{and } min<\pi(j)<max \\ & \text{and } \pi^{-1}(\pi(j)-1)>j \\
\mathbb{P}_{(\sigma,X,Y),\pi,t(j),max}(i+1,j+1) & \text{if } \mathcal{S}(\sigma,i)=u \\ & \text{and } i\in X \text{ and } i\in Y \\ & \text{and } \pi(j)+1=\pi(j+1) \\ & \text{and } min<\pi(j)<max \\
\mathbb{P}_{(\sigma,X,Y),\pi,min,t(j)}(i+1,j+1) & \text{if } \mathcal{S}(\sigma,i)=d \\ & \text{and } i\in X \text{ and } i\in Y \\ & \text{and } \pi(j)-1=\pi(j+1) \\ & \text{and } min<\pi(j)<max \\
False & Otherwise
\end{cases}
$$

At each step (i,j), If $i \notin X$ and $i \notin Y$, we match the current element of σ ($\sigma(i)$) with the current element of π ($\pi(j)$), if possible. Then we match $\sigma[i+1:]$ with every suffixes π starting after j.

If $i \in X$ and $i \notin Y$, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[j+1:]$ with the condition that $\sigma(i+1)$ is matched to $\pi(j+1)$.

If $i \notin X$, $i \in Y$ and i is a down step, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[\pi^{-1}(\pi(j)-1):]$ with the condition that $\sigma(i+1)$ is matched to the element $\pi(j)-1$.

If $i \notin X$, $i \in Y$ and i is an up step, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[\pi^{-1}(\pi(j)+1):]$ with the condition that $\sigma(i+1)$ is matched to the element $\pi(j)+1$.

If $i \in X$, $i \in Y$ and i is a down step, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[j+1:]$ with the condition that $\sigma(i+1)$ is matched to $\pi(j+1)$, and that $\pi(j+1) = \sigma(i) - 1$.

If $i \in X$, $i \in Y$ and i is an up step, we match $\sigma(i)$ to $\pi(j)$, then we match $\sigma[i+1:]$ with $\pi[j+1:]$ with the condition that $\sigma(i+1)$ is matched to $\pi(j+1)$, and that $\pi(j+1) = \sigma(i) + 1$.

There are two cases to consider searching for a match :
If $|\sigma| \in X$ there exists a match if and only if
$\bigcup_{0<min<max<|\pi|} \mathbb{P}_{\sigma,\pi,min,t(j)}(0,0)$ is true.

If $|\sigma| \notin X$ there exists a match if and only if
$\bigcup_{0<k<|\pi|, 0<min<max<|\pi|} \mathbb{P}_{\sigma,\pi,min,t(j)}(0,k)$ is true. $\square$

6 Longest Subsequence Avoiding (231,213) for a Permutation

In this section we present an algorithm to solve the problem of the longest subsequence avoiding (231,213). To do so, we need the set of element that are matched to an up step and the set of element that are matched to a down step. For a permutation p of size n, we define $P(p) = \{i | \mathcal{S}(p,i) = u\} \cup \{n\}$ and $M(p) = \{i | \mathcal{S}(p,i) = d\} \cup \{n\}$.

Proposition 11 If s is a longest subsequence avoiding (213,231) with last element at index f in p then $P(p)$ is a longest increasing subsequence with last element at index f and $M(p)$ is a longest decreasing subsequence with last element at index f.

Proof Let's s is a longest subsequence avoiding (213,231) with last element at index f in p, suppose that $P(p)$ is not a longest increasing subsequence with last element at index f. Let's s_m be a longest increasing subsequence with last element f. Thus $|s_m| > P(p)$, clearly the sequence $s_m \cup M(p)$ is avoiding (213,231) and is longer than s witch is not possible. The case for $M(p)$ follows the same idea.$\square$

Proposition 12 Given a permutation p, finding the longest subsequence avoiding (231,213) can be done in $O(|\pi| * log(log(|\pi|)))$ time and in $O(n)$ space.

Proof The proposition 11 lead to algorithm where we have to compute longest increasing and decreasing subsequence ending at every index. Then finding the maximum sum of longest increasing and decreasing subsequence ending at the same index. Computing the longest increasing and decreasing can be done in $O(|\pi| * log(log(|\pi|)))$ time and $O(n)$ space (see [**?**]), then finding the maximum can be done in linear time.$\square$

7 Longest Subsequence Avoiding (231,213) Common for Two Permutation

In this section we present an algorithm to find the longest common subsequence avoiding (231,213) between two permutations.

Proposition 13 Given two permutation p_1 and p_2 The longest common subsequence avoiding (231,213) can be solve in $O(|p_1|^3 * |p_2|^3)$.

Proof Consider the following problem, that compute the longest stripe common to p_1 and p_2. Given two permutation p_1 and p_2.

$S_{p_1,p_2}(min_1, max_1, min_2, min_2, i_1, i_2) = \max\{\ |m| \mid$ s is a pattern occurring in $p_1[i_1 :]$ by the subsequence s_1 and $min(s_1) = min_1$ and $max(s_1) = max_1$ and s is occurring in $p_2[i_2 :]$ by the subsequence s_2 and $min(s_2) = min_2$ and $max(s_2) = max_2\ \}$

We show that this family of problems are closed under induction.

BASE :

$$S_{p_1,p_2,min_1,max_1,min_2,max_2}(|p_1|, |p_2|) = \begin{cases} 1 \text{ if } min_1 < p_1(j) < max_1 \\ \quad \text{and } min_2 < p_2(j) < max_2 \\ 0\ otherwise \end{cases}$$

STEP :

$$S_{p_1,p_2,min_1,max_1,min_2,max_2}(i_1, i_2) = \\ max \begin{cases} S_{p_1,p_2,min_1,max_1,min_2,max_2}(i_1, i_2 + 1) \\ S_{p_1,p_2,min_1,max_1,min_2,max_2 2}(i_1 + 1, i_2) \\ S'_{p_1,p_2,min_1,max_1,min_2,max_2}(i_1, i_2) \end{cases}$$

with $S'_{p_1,p_2,min_1,max_1,min_2,max_2}(i_1,i_2) =$

$$\begin{cases} 1 + S_{p_1,p_2,p_1(i_1),max_1,p_2(i_2)}(max_2, i_1+1, i_2+1) & p_1(i_1) < min_1 \\ & \text{and } p_2(i_2) < min_2 \\ \\ 1 + S_{p_1,p_2,min_1,p_1(i_1),min_2,p_2(i_2)}(i_1+1, i_2+1) & p_1(i_1) > max_1 \\ & \text{and } p_2(i_2) > max_2 \\ \\ 0 & \text{otherwise} \end{cases}$$

For every pair i, j we either ignore the element of p_1, either ignore the element of p_2, either we match as the same step (if possible). Those relation lead to a $O(|p_1|^3 * |p_2|^3)$ time and $O(|p_1|^3 * |p_2|^3)$ space algorithm.□

8 Conclusion