Pattern matching for (213, 231)-avoiding permutations

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Abstract. Given permutations $\sigma \in S_k$ and $\pi \in S_n$, the *pattern containment* problem is to decide whether π contains σ as an order-isomorphic subsequence. We give a linear-time algorithm in case both π and σ avoid the two size-3 permutations 213 and 231. For the special where only σ avoids 213 and 231, we present a $O(kn^2)$ time algorithm. Finally We extend our research to bivincular pattern that avoid 213 and 231 and present a $O(kn^5)$ time algorithm.

1 Introduction

A permutation π is said to contain another permutation σ, in symbols $\sigma \preceq \pi$, if there exists a subsequence of entries of π that has the same relative order as σ, and in this case σ is said to be a *pattern* of π. Otherwise, π is said to *avoid* the permutation σ. For example a permutation contains the pattern 123 (resp. 321) if it has an increasing (resp. decreasing) subsequence of length 3. Here, note that members need not actually to be consecutive, merely increasing (resp. descreasing). As another example, 6152347 contains 213 but not 231. During the last decade, the study of the pattern containment on permutations has become a very active area of research [15] and a whole annual conference (Permutation Patterns) is now devoted to this topic.

We consider here the so-called *pattern containment* problem (also sometimes referred to as the *pattern involvement problem*): Given two permutations σ and π, this problem is to decide whether $\sigma \preceq \pi$ (the problem is ascribed to Wilf in [6]). The permutation containment problem is known to be **NP**-hard [6]. It is, however, polynomial-time solvable by brute-force enumeration if σ has bounded size. Improvements to this algorithm were presented in [2] and [1], the latter describing a $O(|\pi|^{0.47|\sigma|+o(|\sigma|)})$ time algorithm. Bruner and Lackner [8] gave a fixed-parameter algorithm solving the pattern containment problem with an exponential worst-case runtime of $O(1.79^{\mathrm{run}(\pi)})$, where $\mathrm{run}(\pi)$ denotes the number of alternating runs of π. (This is an improvement upon the $O(2^{|\pi|})$

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runtime required by brute-force search without imposing restrictions on σ and π.) A recent major step was taken by Marx and Guillemot [12]. They showed that the permutation containment problem is fixed-parameter tractable (FPT) for parameter $|\sigma|$.

A few particular cases of the pattern containment problem have been attacked successfully. The case of increasing patterns is solvable in $O(|\pi| \log \log |\sigma|)$ time in the RAM model [9], improving the previous 30-year bound of $O(|\pi| \log |\sigma|)$. (The algorithm also improves on the previous $O(|\pi| \log \log |\pi|)$ bound.) Furthermore, the patterns 132, 213, 231, 312 can all be handled in linear-time by stack sorting algorithms. Any pattern of length 4 can be detected in $O(|\pi| \log |\pi|)$ time [2]. Algorithmic issues for 321-avoiding patterns containment has been investigated in [13]. The pattern containment problem is also solvable in polynomial-time for separable patterns [14, 6] (see also [7] for LCS-like issues of separable permutations). Separable permutations are those permutations that contain neither 2413 nor 3142, and they are enumerated by the Schröder numbers (sequence A006318 in OEIS). To the best of our knowledge, separable permutations first arose in the work of Avis and Newborn [3], who showed that they are precisely the permutations which can be sorted by an arbitrary number of pop-stacks in series, where a pop-stack is a restricted form of stack in which any pop operation pops all items at once. (Notice that the separable permutations include as a special case the stack-sortable permutations, which avoid the pattern 231.)

There exist many generalisation of patterns that are worth considering in the context of algorithmic issues in pattern involvement (see [15] for an up-to-date survey). *Vincular patterns*, also called *generalized patterns*, resemble (classical) patterns with the additional constraint that some of the letters in an occurrence must be consecutive. Of particular importance in our context, Bruner and Lackner [8] proved that deciding whether a vincular pattern σ of length k occurs in a longer permutation π is $W[1]$-complete for parameter k; for an up to date survey of the $W[1]$ class and related material, see [10]. *Bivincular patterns* generalize classical patterns even further than vincular patterns. Indeed, in bivincular patterns, not only positions but also values of elements involved in a matching may be forced to be adjacent

We focus in this paper on pattern matching issues for $(213, 231)$-avoiding permutations (*i.e.*, those permutations that avoid both 213 and 231). The number of n-permutations that avoid both 213 and 231 is $t_0 = 1$ for $n = 0$ and $t_n = 2^{n-1}$ for $n \geq 1$ [17]. On an individual basis, the permutations that do not contain the permutation pattern 231 are exactly the *stack-sortable permutations* and they are counted by the Catalan numbers [16]. A stack-sortable permutation is a permutation whose elements may be sorted by an algorithm whose internal storage is limited to a single stack data structure. As for 213, it is well-known that if $\pi = \pi_1 \pi \ldots \pi_n$ avoids 132, then its complement $\pi' = (n + 1 - \pi_1)(n + 1 - \pi_2) \ldots (n + 1 - \pi_n)$ avoids 312, and the reverse of π' avoids 213. From a combinatorial point of view, Bóna [5] showed the rather surprising fact that the cumulative number of occurrences of the classical patterns 231 and 213 are the same on the set of permutations avoiding 132, beside the pattern based statistics 231 and 213 do not have the same distribution on this set. *Almost avoidance* has also been considered; a permutation π almost avoids a set X of permutations if there is a way to remove a single element of π to get a permutation that avoids all elements in P and $L_n(P)$ denotes the set of permutations of length n that almost avoid P. It is shown in [11] that $L_n(213, 231) = L_n(132, 231) = L_n(132, 312) = L_n(213, 312)$.

This paper is organized as follows. In Section 2 the needed definition are presented. Section 3 is devoted to presenting an online linear-time algorithm in case both the pattern and the target permutation are $(213, 231)$-avoiding, whereas Section 4 focuses on the case where the pattern only is $(213, 231)$-avoiding. In Section 5 we give a polynomial-time algorithm for $(213, 231)$-avoiding bivincular patterns. In Section 6 we consider the problem of finding the longest $(213, 231)$-avoiding pattern in a permutation. In Section **??** we give a polynomial-time algorithm for computing the longest $(213, 231)$-avoiding pattern common to two permutations.

2 Definitions

A *permutation* of length n is a one-to-one function from an n-element set to itself. We write permutations as words $\pi = \pi_1\pi_2 \dots \pi_n$, whose letters are distinct and uasually consist of the integers $12 \dots n$, and we let $\pi[i]$ stands for π_i. For the sake of convenience, we let $\pi[i : j]$ stand for $\pi_i\pi_{i+1} \dots \pi_j$, $\pi[: j]$ stand for $\pi[1 : j]$ and $\pi[i :]$ stand for $\pi[i : n]$. As usual, we let S_n denote the set of all permutations of length n.

A permutation π is said to *contain* the permutation σ if there exists a subsequence of (not necessarily consecutive) entries of π that has the same relative order as σ, and in this case σ is said to be a *pattern* of π, written $\sigma \preceq \pi$. Otherwise, π is said to *avoid* the permutation σ. For example, the permutation $\pi = 391867452$ contains the pattern $\sigma = 51342$, as can be seen in the highlighted subsequence of $\pi =$ 3**91**8**674**52 (or $\pi =$ 3**91**8**67**4**5**2 or $\pi =$ 3**91**8**67**45**2**). Each subsequence 91674, 91675, 91672 in π is called a *copy*, (or *instance* or *occurrence*) of σ. Since the permutation $\pi = 391867452$ contains no increasing subsequence of length four, π avoids 1234.

Suppose P is a set of permutations. We let $\mathrm{Av}_n(P)$ denote the set of all n-permutations avoiding each permutation in P. For the sake of convenience (and as it is customary [15]), we omit P's braces thus having e.g. $\mathrm{Av}_3(213, 231)$ instead of $\mathrm{Av}_3(\{213, 231\})$. If $\pi \in \mathrm{Av}_n(P)$, we also say that π is *P-avoiding*.

An *ascent* of a permutation $\pi \in S_n$ is any position $1 \leq i < n$ where the following value is bigger than the current one. That is, if $\pi = \pi_1\pi_2 \dots \pi_n$, then i is an ascent if $\pi_i < \pi_{i+1}$. For example, the permutation 3452167 has ascents (at positions) $1, 2, 5$ and 6. Similarly, a *descent* is a position $1 \leq i < n$ with $\pi_i > \pi_{i+1}$, so every $1 \leq i < n$ is either an ascent or is a descent of π. To clarify the exposition, we let a and d denote an ascend and a descend, respectively. The *stripe* s_π of a permutation $\pi \in S_n$ is the word $\mathrm{s}_\pi[1]\,\mathrm{s}_\pi[2] \dots \mathrm{s}_\pi[n-1] \in \{a, d\}^{n-1}$ defined by $\mathrm{s}_\pi[i] = a$ if i an ascent in π and $\mathrm{s}_\pi[i] = d$ if i a descent in π. For example the stripe of the permutation $\pi = 981234765$ is $\mathrm{s}_\pi = ddaaaadd$. The stripe s_σ is said to be a *pattern* of the stripe s_π (or s_σ *occurs* of the stripe s_π) if s_σ occurs in s_π as a subsequence.

A *bivincular pattern* (abbreviated BVP) σ of length k is a permutation in S_k written in two-line notation (that is the top row is $12\,\ldots\,k$ and the bottom row is a permutation $\sigma_1\sigma_2\,\ldots\,\sigma_k$). We have the following conditions on the top and bottom rows of σ:

– If the bottom line of σ contains $\underline{\sigma_i\sigma_{i+1}\,\ldots\,\sigma_j}$ then the letters corresponding to $\sigma_i\sigma_{i+1}\,\ldots\,\sigma_j$ in an occurrence of σ in a permutation must be adjacent, whereas there is no adjacency condition for non-underlined consecutive letters. Moreover if the bottom row of σ begins with $\llcorner\sigma_1$ then any occurrence of σ in a permutation π must begin with the leftmost letter of π, and if the bottom row of σ begins with $\sigma_k\lrcorner$ then any occurrence of σ in a permutation π must end with the rightmost letter of π.

– If the top line of σ contains $\overline{i\,i+1\,\ldots\,j}$ then the letters corresponding to $\sigma_i, \sigma_{i+1}, \ldots, \sigma_j$ in an occurrence of σ in a permutation must be adjacent in values, whereas there is no value adjacency restriction for non-overlined letters. Moreover, if the top row of σ begins with $\ulcorner 1$ then any occurrence of σ is a permutation π must begin with the smallest letter of π, and if top row of σ ends with $k\urcorner$ then any occurrence of σ is a permutation π must end with the largest letter of π.

For example, let $\sigma = \genfrac{}{}{0pt}{}{\overline{123}}{\llcorner 213\lrcorner}$. If $\pi_i\pi_j\pi_\ell$ is an occurrence of σ in $\pi \in S_n$, then $\pi_i\pi_j\pi_k = (x+1)x(x+2)$ for some $1 \leq x \leq n-2$, $i = 1$ and $\ell = n$. For example 316524 contains one occurrence of σ (the subsequence 324), whereas 42351 avoids it. It is straightforward to see that an n-permutation π avoids $\genfrac{}{}{0pt}{}{\overline{123}}{\llcorner 213\lrcorner}$ unless $n \geq 3$, $\pi_1 = x > 1$ and $\pi_n = x+1$ (in which case there is one occurrence of the pattern). The best general reference is [15].

3 Both π and σ are $(213, 231)$-avoiding

This section is devoted to presenting a fast algorithm for deciding if $\sigma \preceq \pi$ in case both π and σ are $(213, 231)$-avoiding. We begin with an easy but crucial structure lemma.

Lemma 1 (Folklore). *The first element of any* $(213, 231)$*-avoiding permutation must be either* 1 *or* n*.*

Proof. Any other initial element would serve as a '2' in either a 231 or 213 with 1 and n as the '1' and '3' respectively. $\square$

Notice that Lemma 1 gives as an immediate consequence that the number of $(213, 231)$-avoiding n-permutations is 2^{n-1} for $n \geq 1$ [17]. The above structural lemma gains in interest in the form of the following corollaries.

Corollary 1. *Let* $\pi \in S_n$*. Then* $\pi \in \mathrm{Av}_n(213, 231)$ *if and only if for* $1 \leq i \leq n$*,* $\pi[i]$ *is the minimal or the maximal element of* $\pi[i:]$*.*

Corollary 2. *Let $\pi \in\in \mathrm{Av}_n(213, 231)$ and $1 \leq i < n$. Then, (1) if $\pi[i]$ is an ascent element then $\pi[i]$ is the minimal element of $\pi[i:]$ and (2) if if $\pi[i]$ is a descent element then $\pi[i]$ is the maximal element of $\pi[i:]$*

The following lemma is central to all our algorithm.

Lemma 2. *Let π and σ be two $(213, 231)$-avoiding permutations, and t be a subsequence of π of length $|\sigma|$ such that $s_t = s_\sigma$. Then, t is an occurrence of σ in π if and only if s_t occurs as a subsequence in s_π.*

Proof. The forward direction is obvious. We prove the backward direction by induction on the size of the motif σ. The base case is a motif of size 2. Suppose first that $\sigma = 12$ and hence $s_\sigma = a$. Let $t = \pi_{i_1}\pi_{i_2}$, $i_1 < i_2$, be a subsequence of π such that s_t occurs as a subsequence in s_π. Since $s_\sigma = s_t = a$, this reduces to saying that $\pi_{i_1} < \pi_{i_2}$, and hence t is an occurrence of $\sigma = 12$ in π. A similar argument shows that the lemma holds true for $\sigma = 21$. Now, assume that the lemma is true for all motifs up to size $k \geq 2$. Let $\sigma \in \mathrm{Av}_{k+1}(231, 213)$ and let $t = \pi_{i_1}\pi_{i_2} \ldots \pi_{i_{k+1}}$, $i_1 < i_2 < \ldots < i_{k+1}$, be a subsequence of π of length $k+1$ such that $s_t = s_\sigma$. Suppose that s_t occurs in s_π as a subsequence. Then, $s_t[2:] = s_\sigma[2:]$ occurs as a subsequence in $s_\pi[2:]$. But $\sigma[2:] \in \mathrm{Av}_k(231, 213)$ and hence, by the inductive hypothesis, it follows that $t[2:]$ is an occurrence of $\sigma[2:]$ in $\pi[2:]$. Moreover, if $s_\sigma[1] = a$ (resp. $s_\sigma[1] = d$) then $\sigma[1]$ is the minimal (resp. maximal) element of σ (Lemma 1), and hence, since $s_t = s_\sigma$ occurs as a subsequence in s_π, $t[1]$ is the minimal (resp. maximal) element of t. Therefore, t is an occurrence of σ in π. $\square$

We are now ready to solve the pattern containment problem in case both π and σ are $(213, 231)$-avoiding.

Proposition 1. *Let π and σ be two $(213, 231)$-avoiding permutations. One can decide whether π contains σ in linear-time.*

Proof. According to Lemma 2 the problem reduces to deciding whether s_σ occurs as a subsequence in s_π. A straightforward greedy approach solves this issue in linear-time. $\square$

Observe that, according to Corollary 1, the effective construction of the two stripes is not required for the above proposition, and hence we have an online algorithm.

4 σ only is $(231, 213)$-avoiding

This section focuses on the pattern containment problem in case only the pattern σ avoids 231 and 213. We need to consider a specific decomposition F of σ into factors. For $\sigma \in S_k$, the *factorization* $\sigma = \mathrm{F}_\sigma(p)\,\mathrm{F}_\sigma(p-1)\ldots\mathrm{F}_\sigma(1)$ is defined as follows: (i) $\sigma[k]$ is the last letter of $\mathrm{F}_\sigma(1)$, (ii) for every $1 \leq i < k$, if $s_\sigma[i] = s_\sigma[i+1]$ then $\sigma[i]\sigma[i+1]$ is part of the same factor. For every factor $\mathrm{F}_\sigma(j)$

of the decomposition, we let $\mathrm{L}_\sigma(j)$ stand for the leftmost letter of $\mathrm{F}_\sigma(j)$, and $\mathrm{LI}_\sigma(j)$ stand for the index in σ of the leftmost letter of $\mathrm{F}_\sigma(j)$. For example, for $\sigma = 981237654$, we have $s_\sigma = ddaaaddd$ and hence $\sigma = \mathrm{F}_\sigma(3)\,\mathrm{F}_\sigma(2)\,\mathrm{F}_\sigma(1)$ with $\mathrm{F}_\sigma(1) = 7654$, $\mathrm{F}_\sigma(2) = 123$ and $\mathrm{F}_\sigma(3) = 98$. Furthermore, $\mathrm{L}_\sigma(1) = 7$, $\mathrm{L}_\sigma(2) = 1$, $\mathrm{L}_\sigma(3) = 9$, $\mathrm{LI}_\sigma(1) = 6$, $\mathrm{LI}_\sigma(2) = 3$ and $\mathrm{LI}_\sigma(3) = 1$.

Remark 1. A Factor is either an increasing or a decreasing subsequence.

Lemma 3. *Given a permutation* $\sigma \in \mathrm{Av}(213, 231)$ *and a suffix of its decomposition* $\mathrm{F}_\sigma(i)\,\mathrm{F}_\sigma(i-1)\ldots\mathrm{F}_\sigma(1)$*, if* $\mathrm{LI}_\sigma(i)$ *is an ascent (resp. descent) element then the maximal (resp. minimal) element of* $\mathrm{F}_\sigma(i)\,\mathrm{F}_\sigma(i-1)\ldots\mathrm{F}_\sigma(1)$ *is* $\mathrm{L}_\sigma(i-1)$*.*

Proof. Consider the rightmost element of $\mathrm{F}_\sigma(i)$ and assume this element is an ascent. Then, it follows that it is smaller than the following element which is $\mathrm{L}_\sigma(i-1)$. Now, since $\mathrm{F}_\sigma(i)$ is an increasing subsequence, every element of $\mathrm{F}_\sigma(i)$ is smaller than $\mathrm{L}_\sigma(i-1)$. According to Lemma1, $\mathrm{L}_\sigma(i-1)$ is the maximal (minimal) element of $\mathrm{F}_\sigma(i-1)\ldots\mathrm{F}_\sigma(1)$, and hence $\mathrm{L}_\sigma(i-1)$ is the maximal element of $\mathrm{F}_\sigma(i)\,\mathrm{F}_\sigma(i-1)\ldots\mathrm{F}_\sigma(1)$.

A similar argument shows that the lemma holds true if the rightmost element of $\mathrm{F}_\sigma(i)$ is a descent. □

we now define the set $\mathrm{S}^\pi_\sigma(i,j)$ as follows: given a subsequence s of $\pi[j\ :]$, $s \in \mathrm{S}^\pi_\sigma(i,j)$ if and only if s is an occurrence of $\sigma[\mathrm{LI}_\sigma(i)\ :]$ in $\pi[j\ :]$ and the element$\mathrm{L}_\sigma(i)$ occurs in $\pi[j]$.

Lemma 4. *Let* σ *be a permutation with* $\mathrm{F}_\sigma(i)$ *as an ascent (respectively descent) factor,* s *a subsequence such that,* $s \in \mathrm{S}^\pi_\sigma(i,j)$ *and* $s[\mathrm{L}_\sigma(i-1)]$ *is minimal (resp. maximal). For all subsequence* $s' \in \mathrm{S}^\pi_\sigma(i,j)$ *and for all subsequence* t *of* π*, such that* $t = t's'$*, if* t *is an occurrence of* $\sigma[\mathrm{LI}_\sigma(i+1)\ :]$ *then the subsequence* $t's$ *is an occurrence of* $\sigma[\mathrm{LI}_\sigma(i+1)\ :]$*.*

Proof. By definition s is an occurrence of $\sigma[\mathrm{LI}_\sigma(i)\ :]$. To prove that ts is an occurrence of $\sigma[\mathrm{LI}_\sigma(i+1)\ :]$ we prove that every element of t is larger than every element of s. If ts' is an occurrence of $\sigma[\mathrm{LI}_\sigma(i+1)\ :]$ so every element of t is larger than every element of s'. Moreover the maximal element of s is smaller than the maximal element of s' so every element of s is smaller than every element of s' thus every element of s is smaller than every element of t. We use a similar argument if $\mathrm{F}_\sigma(i)$ a descent factor. □

Corollary 3. *Let* σ *be a permutation with* $\mathrm{F}_\sigma(i)$ *as an ascent (respectively descent) factor, and let* $s \in \mathrm{S}^\pi_\sigma(i,j)$*, such that* $s[\mathrm{L}_\sigma(i-1)]$ *is minimal (resp. maximal). Those following statements are equivalent :*

- *There exists an occurrence of* σ *in* π *with element* $\mathrm{L}_\sigma(i)$ *occurs in* $\pi[j]$*.*
- *There exists a subsequence* t *of* $\pi[1 : j-1]$ *such that* ts *is an occurrence of* σ *in* π *with element* $\mathrm{L}_\sigma(i)$ *occurs in* $\pi[j]$*.*

Proposition 2. *Let* $\sigma \in \mathrm{Av}_k(213, 231)$ *and* $\pi \in S_n$*. One can decide in* $O(kn^2)$ *time and* $O(kn^2)$ *space if* π *contains* σ*.*

Proof. Consider the following problem :

$$\mathrm{LM}^{\pi}_{\sigma}(i,j) = \begin{cases} \text{The maximal value of a subsequence} & \text{If } \mathrm{L}_\sigma(i) \text{ is} \\ \text{ which is an occurence of } \sigma[\mathrm{LI}_\sigma(i) :] \text{ in } \pi[j :] & \text{ an ascent element} \\ \text{ and that minimize the maximal value} & \\ \text{ and } \mathrm{L}_\sigma(i) \text{ occurs in } \pi[j] & \\ & \\ \text{The minimal value of a subsequence} & \text{If } \mathrm{L}_\sigma(i) \text{ is} \\ \text{ which is an occurence of } \sigma[\mathrm{LI}_\sigma(i) :] \text{ in } \pi[j :] & \text{ a descent element} \\ \text{ and that maximize the minimal value} & \\ \text{ and } \mathrm{L}_\sigma(i) \text{ occurs in } \pi[j] & \end{cases}$$

This problem can be solve by induction as follows:
BASE :

$$\mathrm{LM}^{\pi}_{\sigma}(1,j) = \begin{cases} \min_{j<j'}\{\infty\} \cup \{\pi[j'] |\ j' \text{ such that} & \text{If } \mathrm{L}_\sigma(i) \text{ is} \\ \ |\,\mathrm{F}_\sigma(1)| \leq LIS(j,j',\pi[j'])\} & \text{ an ascent element} \\ & \\ \max_{j<j'}\{0\} \cup \{\pi[j'] |\ j' \text{ such that} & \text{If } \mathrm{L}_\sigma(i) \text{ is} \\ \ |\,\mathrm{F}_\sigma(1)| \leq LDS(j,j',\pi[j'])\} & \text{ a descent element} \end{cases}$$

STEP :

$$\mathrm{LM}^{\pi}_{\sigma}(i,j) = \begin{cases} \min\{\infty\} \cup \mathrm{AF}^{\pi}_{\sigma}(i,j) & \text{If } \mathrm{L}_\sigma(i) \text{ is an ascent element} \\ \max\{0\} \cup \mathrm{DF}^{\pi}_{\sigma}(i,j) & \text{If } \mathrm{L}_\sigma(i) \text{ is a descent element} \end{cases}$$

where $\mathrm{AF}^{\pi}_{\sigma}(i,j)$ and $\mathrm{DF}^{\pi}_{\sigma}(i,j)$ are the sets of elements such that if $\pi[k] \in \mathrm{AF}^{\pi}_{\sigma}(i,j)$ or if $\pi[k] \in \mathrm{DF}^{\pi}_{\sigma}(i,j)$ then there exists an occurrence of $\sigma[\mathrm{LI}_\sigma(i) :]$ in $\pi[j :]$ and the element $\mathrm{L}_\sigma(i-1)$ occurs in element $\pi[k]$. Given that one has the set $\mathrm{S}^{\pi}_{\sigma}(i-1,k)$, to decide if $\pi[k] \in \mathrm{AF}^{\pi}_{\sigma}(i,j)$, one needs to find an occurrence t of $\mathrm{F}_\sigma(i)$ in $\pi[j : k-1]$ such there exists $s \in \mathrm{S}^{\pi}_{\sigma}(i-1,k)$ and ts is an occurrence of $\sigma[\mathrm{LI}_\sigma(i) :]$. Formally we define $\mathrm{AF}^{\pi}_{\sigma}(i,j)$ and $\mathrm{DF}^{\pi}_{\sigma}(i,j)$ as follows:

$$\begin{aligned} \mathrm{AF}^{\pi}_{\sigma}(i,j) = \{\pi[j'] \mid j < j' \text{ and } \mathrm{LM}^{\pi}_{\sigma}(i-1,j') \neq 0 \text{ and} \\ |\,\mathrm{F}_\sigma(i)| \leq LIS_\pi(j, j'-1, \mathrm{L}_\sigma(i-1))\} \\ \mathrm{DF}^{\pi}_{\sigma}(i,j) = \{\pi[j'] \mid j < j' \text{ and } \mathrm{LM}^{\pi}_{\sigma}(i-1,j') \neq \infty \text{ and} \\ |\,\mathrm{F}_\sigma(i)| \leq LDS_\pi(j, j'-1, \mathrm{L}_\sigma(i-1))\} \end{aligned}$$

where $LIS_\pi(i,j,k)$ (resp. $LDS_\pi(i,j,k)$) is the longest increasing (resp. decreasing) sequence in π starting at i and ending at j, with every element of this sequence smaller (resp. bigger) than k. LIS_π and LDS_π can be computed in $O(|\pi|^2 \log(\log(|\pi|)))$ (see [4]).

For the base case, one is looking for an occurrence for the first factor. If the factor is an ascent (respectively descent) factor, then one has to find an

increasing (resp. decreasing) subsequence in the text of same size or longest that the size of the first factor. To find the "safest" solution one must assure that the right most element of the sequence is minimal (resp. maximal).
For the induction, it follows the same idea except that one must assure that every element of the subsequence is larger (resp. smaller) than the maximal (resp. minimal) element of the rest of the occurrence which is given by the inductive problem of the previous factor.

There exists a occurrence of σ in π if and only if there exists a $LM(n,i) \neq 0$ and $LM(n,i) \neq \infty$ for $1 \leq i \leq |\pi|$, with n the number of factor in σ. Moreover the basic case can be computed in $O(n^2)$ and the induction on $O(kn^2)$ time and space. □

5 (213, 231)-avoiding bivincular patterns

This section is devoted to presenting a polynomial-time algorithm for deciding whether a $(213, 231)$-avoiding bivincular pattern occurs in a permutation. We start by presenting a property of bivincular motifs avoiding both 213 and 231.

Lemma 5. *If $\overline{\sigma[i]\sigma[j]}$ (thus $\sigma[j] = \sigma[i]+1$) and if $\sigma[i]$ is an ascent (resp. decent) element and $\sigma[i]+1$ is an ascent (resp. decent) element then $i < j$ ($j > i$) and for every k, $i < k < j$ ($j > k > i$), $\pi[k]$ is a descent (resp. ascent) element.*

Proof. Suppose that there exists k, $i < k < j$, such that $\sigma[k]$ is ascent. Ascent elements are increasing so $\sigma[i] < \sigma[k] < \sigma[j]$ which is in contradiction that with $\sigma[j] = \sigma[i] + 1$. We use a similar argument if $\sigma[i]$ is a descent element □

Proposition 3. *Let σ be a bivincular motif avoiding 213 and 231 of length k and $\pi \in S_n$. One can decide in $O(kn^5)$ time and $O(kn^5)$ space if σ occurs in π.*

Proof. We consider the following problem : Given a bivincular motif σ^+ with $\sigma \in \mathrm{Av}_{n_\sigma}(231, 213)$, and a text $\pi \in \mathrm{Av}_{n_\pi}(231, 213)$.

$$\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(i,j) = \begin{cases} true & \text{If } \sigma^+[i:] \text{ occurs in } \pi[j:] \\ & \text{with every element is in } [\mathrm{lb}, \mathrm{ub}] \\ & \text{and element } \sigma[i] \text{ occurs in } \pi[j] \\ false & otherwise \end{cases}$$

$\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(i,j)$ is closed under induction. It can be solved by means by the following relations:

BASE :

$$
\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(n_\sigma, j) = \begin{cases} \mathit{true} & \text{if } \pi[j] \in [\mathrm{lb}, \mathrm{ub}] \\ & \text{and if } \sigma[n_\sigma]_{\lrcorner} \text{ then } j = n_\pi \\ & \text{and if } \sigma[n_\sigma]^{\urcorner} \text{ then } \pi[j] = \mathrm{ub} = n_\sigma \\ & \text{and if } {}^{\ulcorner}\sigma[n_\sigma] \text{ then } \pi[j] = \mathrm{lb} = 1 \\ & \text{and if } \overline{(\sigma[n_\sigma]-1)\sigma[n_\sigma]} \text{ then } \pi[j] = \mathrm{lb} \\ & \text{and if } \overline{(\sigma[n_\sigma]\sigma[n_\sigma]-1)} \text{ then } \pi[j] = \mathrm{ub} \\ \mathit{false} & \text{otherwise} \end{cases}
$$

The base case finds an occurrence for the right most element of the motif. If the last element does not have any restriction on position and on value, then $\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(n_\sigma, j)$ is true if and only if $\sigma[n_\sigma]$ can occur in $\pi[j]$. This condition is checked by assuring that $\pi[j] \in [\mathrm{lb}, \mathrm{ub}]$.

STEP :

We consider 3 cases for the problem $\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(i, j)$:

– If $\pi[j] \notin [\mathrm{lb}, \mathrm{ub}]$ then :

$$
\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(i, j) = \mathit{false}
$$

this is straightforward with the definition. If $\pi[j] \notin [\mathrm{lb}, \mathrm{ub}]$ then it can not be part of an occurrence of $\sigma[i:]$ in $\pi[i:]$ with every element is in $[\mathrm{lb}, \mathrm{ub}]$.

– If $\pi[j] \in [\mathrm{lb}, \mathrm{ub}]$ and $\sigma[i]$ is an ascent element then :

$$
\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(i, j) = \begin{cases} \bigcup_{j<k} \mathrm{PM}_{\sigma^+}^{\pi,\pi[j]+1,\mathrm{ub}}(i+1, k) & \text{if } \sigma[i] \text{ is not underlined} \\ & \text{ and } \sigma[i] \text{ is not overlined} \\ \bigcup_{j<k} \mathrm{PM}_{\sigma^+}^{\pi,\pi[j]+1,\mathrm{ub}}(i+1, k) & \text{if } \sigma[i] \text{ is not underlined} \\ & \text{ and } \overline{(\sigma[i]-1)\sigma[i]} \text{ or } {}^{\ulcorner}\sigma[i] \\ & \text{ and } \pi[j] = \mathrm{lb} \\ \mathrm{PM}_{\sigma^+}^{\pi,\pi[j]+1,\mathrm{ub}}(i+1, j+1) & \text{if } \underline{\sigma[i]\sigma[i+1]} \\ & \text{ and } \overline{(\sigma[i]-1)\sigma[i]} \text{ or } {}^{\ulcorner}\sigma[i] \\ & \text{ and } \pi[j] = \mathrm{lb} \\ \mathrm{PM}_{\sigma^+}^{\pi,\pi[j]+1,\mathrm{ub}}(i+1, j+1) & \text{if } \underline{\sigma[i]\sigma[i+1]} \\ & \text{ and } \sigma[i] \text{ is not overlined} \\ \mathit{false} & \text{otherwise} \end{cases}
$$

Remark that $\sigma[i]$ can occur in $\pi[j]$ because $\pi[j] \in [\mathrm{lb}, \mathrm{ub}]$. Thus if $\sigma[i+1:]$ occurs in $\pi[j+1:]$ with every element in $[\sigma[i]+1, \mathrm{ub}]$ then $\sigma[i:]$ occurs in $\pi[j:]$ which correspond to know if there exists k, $j < k$ such that $\mathrm{PM}_{\sigma^+}^{\pi,\pi[j]+1,\mathrm{ub}}(i+1, k)$ is true.

– If $\pi[j] \in [\mathrm{lb}, \mathrm{ub}]$ and $\sigma[i]$ is a descent element then :

$$
\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\mathrm{ub}}(i,j) = \begin{cases}
\bigcup_{j<k} \mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\pi[j]-1}(i+1,k) & \text{if } \sigma[i] \text{ is not underlined} \\
 & \quad\text{and } \sigma[i] \text{ is not overlined} \\
\bigcup_{j<k} \mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\pi[j]-1}(i+1,k) & \text{if } \sigma[i] \text{ is not underlined} \\
 & \quad\text{and } \overline{\sigma[i](\sigma[i]+1)} \text{ or } \sigma[i]^{\urcorner} \\
 & \quad\text{and } \pi[j] = \mathrm{ub} \\
\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\pi[j]-1}(i+1,j+1) & \text{if } \underline{\sigma[i]\sigma[i+1]} \\
 & \quad\text{and } \overline{\sigma[i](\sigma[i]+1)} \text{ or } \sigma[i]^{\urcorner} \\
 & \quad\text{and } \pi[j] = \mathrm{ub} \\
\mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\pi[j]-1}(i+1,j+1) & \text{if } \underline{\sigma[i]\sigma[i+1]} \\
 & \quad\text{and } \sigma[i] \text{ is not overlined} \\
\mathit{false} & \text{otherwise}
\end{cases}
$$

We can show symmetrically as the last case that $\sigma[i:]$ occurs in $\pi[j:]$ with $\sigma[i]$ occurs in $\pi[j]$. starting at element j if and only if $\bigcup_{j<k} \mathrm{PM}_{\sigma^+}^{\pi,\mathrm{lb},\pi[j]-1}(i+1,k)$.

Clearly if $\bigcup_{0<j} \mathrm{PM}_{\sigma^+}^{\pi,1,n_\pi}(1,j)$ is true then σ occurs in π. We now discuss about the position and value constraints.

Position Constraint. There are 3 types of position constraints

- If $\llcorner\sigma[1]$ then the left most element of σ must occur in the left most element of π ($\sigma[1]$ occurs in $\pi[1]$ on an occurrence of σ in π). This constraint is assured by demanding that the occurrence starts at the left most element of π : if $\mathrm{PM}_{\sigma^+}^{\pi,1,n_\pi}(1,1)$ is true.
- If $\sigma[n_\sigma]\lrcorner$ then the right most element σ must occurs in the right most element of π ($\sigma[n_\sigma]$ occurs in $\pi[n_\pi]$ on a occurrence of σ in π). This constraint is assured in the base case.
- If $\underline{\sigma[i]\sigma[i+1]}$ then the index of the occurrence of $\sigma[i]$ and $\sigma[i+1]$ must be consecutive. In other word if $\sigma[i]$ occurs in $\pi[j]$ then $\sigma[i+1]$ must occur in $\pi[j+1]$. We assure this restriction in the induction by demanding that occurrences of $\sigma[i+1:]$ start at index $j+1$.

Value Constraint. There are 3 types of position constraints

- If $^{\ulcorner}\sigma[i]$ (and thus $\sigma[i]=1$) then the minimal value of σ must occur in the minimal value of π.
  - If $\sigma[i]$ is an ascent element then $\sigma[i]$ is the left most ascent element (if not $\sigma[i]$ would not be the minimal element, because ascent elements are increasing). Remark that the problem starts with the lower bound as 1, and inductive problems before the index i are called with descent elements which do not modified the lower bound, so for every problem with index i has for lower bound 1. $\mathrm{PM}_{\sigma^+}^{\pi,1,*}(i,*)$ is either an inductive case or a base case, in either way $\mathrm{PM}_{\sigma^+}^{\pi,1,*}(i,*)$ is true if and only if the occurrence of $\sigma[i]$ is equal to lower bound which is 1.

- If $\sigma[i]$ is a descent element then $i = n_\sigma$ ($\sigma[i]$ is the right most element). Which is solved in the case case.

– If $\sigma[i]^{\urcorner}$ (and thus $\sigma[i] = n_\sigma$) then the maximal value of σ must occur in the maximal value of π.
  - If $\sigma[i]$ is a descent element then $\sigma[i]$ is the left most ascent element (if not $\sigma[i]$ would not be the maximal element, because descent elements are decreasing). Remark that the problem starts with the upper bound as n_π, and any inductive problems before the index i are called with ascent elements which do not modified the upper bound, every problem with index i has for upper bound n_π. $\mathrm{PM}^{\pi,*,n_\sigma}_{\sigma^+}(i,*)$ is either an inductive case or a base case, in either way $\mathrm{PM}^{\pi,*,n_\sigma}_{\sigma^+}(i,*)$ is true if and only if the occurrence of $\sigma[i]$ is equal to upper bound which is n_σ.
  - If $\sigma[i]$ is an ascent element then $\sigma[i]$ then $i = n_\sigma$ ($\sigma[i]$ is the right most element) which is solved in the base case.

– If $\overline{\sigma[i]\sigma[i']}$, with $\sigma[i'] = \sigma[i] + 1$ then if $\sigma[i]$ occurs in $\pi[j]$ then $\sigma[i']$ must occur in $\pi[j] + 1$.
  - The case $\sigma[i]$ is a descent element, $\sigma[i']$ is an ascent element and $i < i'$ (remark that this case is equivalent to the case $\sigma[i]$ is an ascent element, $\sigma[i']$ is a descent element and $i' < i$) is not possible. Indeed $\sigma[i]$ is the maximal element of $\sigma[i:]$ thus $\sigma[i] > \sigma[i']$ which is in contradiction with $\sigma[i'] = \sigma[i] + 1$.
  - If $\sigma[i]$ is an ascent element, $\sigma[i']$ is a descent element and $i < i'$ (remark that this case is equivalent to the case $\sigma[i]$ is a descent element, $\sigma[i']$ is an ascent element and $i' < i$) , then $\sigma[i]$ is the right most ascent element and $\sigma[i']$ is the right most element (or $\sigma[i'] \neq \sigma[i] + 1$). Thus inductive problems between $i + 1$ and i' do not modified the lower bound. If $\sigma[i]$ occurs in $\pi[j]$ then the inductive problems after $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i,*)$ have for lower bound $\pi[i] + 1$. The problem $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i',*)$ is a base case and is true if and only if the occurrence of $\sigma[i']$ is equal the lower bound which is equal to $\pi[i] + 1$ so the occurrence of $\sigma[i]$ and $\sigma[i']$ are consecutive in value.
  - If $\sigma[i]$ is a descent element and $\sigma[i']$ is a descent element then $i' < i$ and there is no descent element between $\sigma[i]$ and $\sigma[i']$ (lemma 5), Thus inductive problems between $i'+1$ and i do not modified the upper bound. Remark that if $\sigma[i']$ occurs in $\pi[j]$ then the inductive problems after $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i',*)$ have for upper bound $\pi[j] - 1$. $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i,*)$ is either an inductive case or a base case, in either way $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i,*)$ is true if and only if the occurrence of $\sigma[i]$ is equal to upper bound which is equal to $\sigma[i'] - 1$ so the occurrence of $\sigma[i]$ and $\sigma[i']$ are consecutive in value.
  - If $\sigma[i]$ is an ascent element and $\sigma[i']$ is an ascent element then $i < i'$ and there is no ascent element between $\sigma[i]$ and $\sigma[i']$ (lemma 5), Thus inductive problems between $i+1$ and i' do not modified the lower bound. Remark that if $\sigma[i]$ occurs in $\pi[j]$ then
    the inductive problems after $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i,*)$ have for lower bound $\pi[j] + 1$. $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i',*)$ is either an inductive case or a base case, in either way $\mathrm{PM}^{\pi,*,*}_{\sigma^+}(i',*)$ is true if and only if the occurrence of $\sigma[i']$ is equal to

lower bound which is equal to $\sigma[i'] + 1$ so the occurrence of $\sigma[i]$ and $\sigma[i']$ are consecutive in value.

$\square$

6 Computing the longest $(213, 231)$-avoiding subsequence

This section is devoted to finding longest $(213, 231)$-avoiding subsequences in permutations: Given a set of permutations, find a longest $(213, 231)$-avoiding that occurs as a pattern in each input permutation.

We start with the easiest case where we are given just one input permutation. We need the set of descent elements and the set of ascent elements. $P(\pi) = \{i | \mathrm{s}_\pi[i] = a\} \cup \{n\}$ and $M(\pi) = \{i | \mathrm{s}_\pi[i] = d\} \cup \{n\}$.

Proposition 4. *If s is a longest $(213, 231)$-avoiding subsequence with last element at index f in π then $P(\pi)$ is a longest increasing subsequence with last element at index f and $M(\pi)$ is a longest decreasing subsequence with last element at index f.*

Proof. Let s be a longest subsequence avoiding (213,231) with last element at index f in π, suppose that $P(\pi)$ is not a longest increasing subsequence with last element at index f. Let s_m be a longest increasing subsequence with last element f. Thus $|s_m| > |P(\pi)|$, clearly the sequence $s_m \cup M(\pi)$ is $(213, 231)$-avoiding and is longer than s, this is a contradiction. The same ideacan be used to show that $M(\pi)$ is the longest decreasing subsequence. $\square$

Proposition 5. *Let π be a permutation. Computing the longest $(213, 231)$-pattern that occurs in π can be done in $O(|\pi| \log(\log(|\pi|)))$ time and in $O(n)$ space.*

Proof. The proposition 4 lead to an algorithm where one has to compute longest increasing and decreasing subsequence ending at every index. Then finding the maximal sum of longest increasing and decreasing subsequence ending at the same index. Computing the longest increasing subsequence and the longest decreasing subsequence can be done in $O(|\pi| \log(\log(|\pi|)))$ time and $O(n)$ space (see [4]), then finding the maximal can be done in linear time. $\square$

We now turn to considering the case where the input is composed of two permutations.

Proposition 6. *Given two permutations π_1 and π_2, the longest common subsequence avoiding (231,213) can be compute in $O(|\pi_1|^3 |\pi_2|^3)$ time and space.*

Proof. Consider the following problem, that compute the longest stripe common to π_1 and π_2. Given two permutations π_1 and π_2.

$$\mathrm{LCS}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}(i_1, i_2)$$

$= \max \{|s| \mid s$ occurs in $\pi_1[i_1 :]$ with every element is in $[\mathrm{lb}_1, \mathrm{ub}_1]$ and s occurs in $\pi_2[i_2 :]$ with every element is in $[\mathrm{lb}_2, \mathrm{ub}_2]$ $\}$

We show that this family of problems are closed under induction.

BASE :

$$\mathrm{LCS}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}(|\pi_1|, |\pi_2|) = \begin{cases} 1 & \text{if } \mathrm{lb}_1 \le \pi_1[j] \le \mathrm{ub}_1 \\ & \quad \text{and } \mathrm{lb}_2 \le \pi_2[j] \le \mathrm{ub}_2 \\ 0 & \textit{otherwise} \end{cases}$$

STEP :

$$\mathrm{LCS}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}(i_1, i_2) = max \begin{cases} \mathrm{LCS}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}(i_1, i_2 + 1) \\ \\ \mathrm{LCS}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}(i_1 + 1, i_2) \\ \\ \mathrm{M}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}(i_1, i_2) \end{cases}$$

with

$$\mathrm{M}^{\pi_2,\mathrm{lb}_2,\mathrm{ub}_2}_{\pi_1,\mathrm{lb}_1,\mathrm{ub}_1}(i_1, i_2) = \begin{cases} 1 + \mathrm{LCS}^{\pi_2,\pi_2[i_2]+1,\mathrm{ub}_2}_{\pi_1,\pi_1[i_1]+1,\mathrm{ub}_1}(i_1, i_2 + 1) & \pi_1[i_1] < \mathrm{lb}_1 \\ & \text{and } \pi_2[i_2] < \mathrm{lb}_2 \\ \\ 1 + \mathrm{LCS}^{\pi_2,\mathrm{lb}_2,\pi_2[i_2]-1}_{\pi_1,\mathrm{lb}_1,\pi_1[i_1]-1}(i_1 + 1, i_2) & \pi_1[i_1] > \mathrm{ub}_1 \\ & \text{and } \pi_2[i_2] > \mathrm{ub}_2 \\ \\ 0 & \text{otherwise} \end{cases}$$

For every pair i, j we either ignore the element of π_1, ot we ignore the element of π_2, or we match as the same step (if possible). Those relations lead to a $O(|\pi_1|^3|\pi_2|^3)$ time and $O(|\pi_1|^3|\pi_2|^3)$ space algorithm. $\square$

References

1. S. Ahal and Y. Rabinovich. On complexity of the subpattern problem. *SIAM Journal on Discrete Mathematics*, 22(2):629–649, 2008.
2. M.H. Albert, R.E.L. Aldred, M.D. Atkinson, and D.A. Holton. Algorithms for pattern involvement in permutations. In *Proc. International Symposium on Algorithms and Computation (ISAAC)*, volume 2223 of *Lecture Notes in Computer Science*, pages 355–366, 2001.
3. D. Avis and M. Newborn. On pop-stacks in series. *Utilitas Mathematica*, 19:129140, 1981.
4. Sergei Bespamyatnikh and Michael Segal. Enumerating longest increasing subsequences and patience sorting, 2000.
5. M. Bóna. Surprising symmetries in objects counted by catalan numbers. *The Electronic Journal of Combinatorics*, 19(1):P62, 2012.

6. P. Bose, J.F.Buss, and A. Lubiw. Pattern matching for permutations. *Information Processing Letters*, 65(5):277–283, 1998.
7. M. Bouvel, D. Rossin, and S. Vialette. Longest common separable pattern between permutations. In B. Ma and K. Zhang, editors, *Proc. Symposium on Combinatorial Pattern Matching (CPM'07), London, Ontario, Canada*, volume 4580 of *Lecture Notes in Computer Science*, pages 316–327, 2007.
8. M.-L. Bruner and M.Lackner. A fast algorithm for permutation pattern matching based on alternating runs. In F.V. Fomin and P. Kaski, editors, *13th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT), Helsinki, Finland*, pages 261–270. Springer, 2012.
9. M. Crochemore and E. Porat. Fast computation of a longest increasing subsequence and application. *Information and Compututation*, 208(9):1054–1059, 2010.
10. R.G. Downey and M. Fellows. *Fundamentals of Parameterized Complexity.* Addison Wesley Longman Publishing Co., Inc., Redwood City, CA, USA, 2013.
11. W. Griffiths, R. Smith, and D.Warren. Almost avoiding pairs of permutations. *Pure Mathematics and Applications*, 22(2):129–139, 2011.
12. S. Guillemot and D. Marx. Finding small patterns in permutations in linear time. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), Portland, Oregon, USA*, pages 82–101. SIAM, 2014.
13. S. Guillemot and S. Vialette. Pattern matching for 321-avoiding permutations. In Y. Dong, D.-Z. Du, and O. Ibarra, editors, *Proc. 20-th International Symposium on Algorithms and Computation (ISAAC), Hawaii, USA*, volume 5878 of *LNCS*, page 10641073. Springer, 2009.
14. L. Ibarra. Finding pattern matchings for permutations. *Information Processing Letters*, 61(6):293–295, 1997.
15. S. Kitaev. *Patterns in Permutations and Words.* Springer-Verlag, 2013.
16. Donald E. Knuth. *The Art of Computer Programming, Volume 1 (3rd Ed.): Fundamental Algorithms.* Addison Wesley Longman Publishing Co., Inc., Redwood City, CA, USA, 1997.
17. R. Simion and F.W.Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6(4):383406, 1985.