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List of Abbreviations

SITPN	**S**ynchronously executed **I**nterpreted **T**ime **P**etri **N**et with priorities
VHDL	**V**ery high speed integrated circuit **H**ardware **D**escription **L**anguage
CIS	**C**omponent **I**nstantiation **S**tatement
PCI	**P**lace **C**omponent **I**nstance
TCI	**T**ransition **C**omponent **I**nstance
GPL	**G**eneric **P**rogramming **L**anguage
HDL	**H**ardware **D**escription **L**anguage
LRM	**L**anguage **R**eference **M**anual
DSL	**D**omain **S**pecific **L**anguage
MDE	**M**odel-**D**riven **E**ngineering

For/Dedicated to/To my...

Chapter 1

Preliminary Notions

In this chapter, we introduce the mathematical formalisms and notations used throughout this thesis to express and formalize our ideas. In Section 1.3, we provide the basics to understand the Coq proof assistant, which is the system we use to write our programs and to mechanize our proofs. This chapter is inspired by the book *The Formal Semantics of Programming Languages: an Introduction* [8], the courses of the University of Cambridge on the semantics of programming languages[1], the documentation of the Coq proof assistant[2], and the book *Certified Programming with Dependent Types* [2].

1.1 Mathematical notations

1.1.1 Intuitionistic first order logic

The intuitionistic first-order logic [5] constitutes our framework for the expression and the interpretation of logical formulas. The language to express logical formulas is the same between classical and intuitionistic first-order logic. A logical formula is either:

- $\bot$ (bottom), the always *false* formula, or $\top$ (top), the always *true* formula
- a predicate (i.e. an atomic formula). A predicate P possibly takes n parameters as inputs. We write $P(x_0, \ldots, x_n)$ to denote an n-ary predicate P applied to the inputs $x_0, \ldots, x_n$.
- the composition of two subformulas with one of the following binary operators: the conjunction $\wedge$, the disjunction $\vee$, the implication $\Rightarrow$, the double implication $\Leftrightarrow$
- the composition of one subformula with the negation operator $\neg$
- a subformula prefixed by the universal quantifier $\forall$ or the existantial quantifier $\exists$. For instance, the formula $\forall x.P(x)$ denotes the atomic formula $P(x)$ where the parameter x is a universally quantified variable of the formula. As a shorthand notation, we write $\forall x, y, z. \ldots$ to denote $\forall x, \forall y, \forall z. \ldots$. The same stands for the existantial quantifier $\exists$. Variables that are introduced in a logical formula by one of the previous quantifiers are called the *bound* variables of the formula. Variables that appear in a logical formula without being introduced by

[1] https://www.cl.cam.ac.uk/teaching/2021/Semantics/
[2] https://coq.inria.fr/distrib/current/refman/index.html

a quantifier are called *free* variables. For instance, in the logical formula $\forall x.\ P(x) \wedge Q(y)$, x is a bound variable and y is a free variable of the formula.

The difference between the classical first-order logic and the intuitionistic one relies in the absence of the *law of the excluded middle* in the latter logic. In classical logic, the *law of the excluded middle* considers that a formula can always be either interpreted as true or false, i.e. a formula is always *decidable* regardless the valuation of its inputs. In intuitionistic logic, such an assumption is discarded. Thus, a formula of the intuitionistic logic can be *undecidable*, i.e. given a valuation of its input, one can not always state that the formula is true or false. The consequence is that one as to exhibit a *explicitly-built* proof to show that a given formula is true. One can not rely on the law of excluded middle to perform a proof by contradiction. As so, the intuitionistic logic is said to be *constructive* in the sense that one as to *construct* a concrete proof object to show that a formula is true. The intuitionistic first-order logic is built in the Coq proof assistant. Thus, a logic formula expressed with the Coq proof assistant is an intuitionistic formula by default. For this reason, we choose the intuitionistic first-order logic as our mathematical framework for the expression of logic formulas.

In what follows, we often give the definition of a predicate by relating its semantics to the semantics of an underlying formula. For instance, we define the predicate $IsEven(n)$, for all $n \in \mathbb{N}$, as follows: $IsEven(n) \equiv \exists k \in \mathbb{N}\ s.t.\ 2k = n$.

1.1.2 Set theory

In this thesis, we use set theory as the base formalism for all our mathematical definitions and proofs. In set theory, a set represents a group of elements called the members of the set. For every set X, we write $x \in X$ to denote that the element x is a member of set X. We note $X \subseteq Y$ the fact that a set X is a subset of set Y such that for all x if $x \in X$ then $x \in Y$. From here, there are multiple ways to define a set.

Extensional definition

A set is defined by *extension* with the enumeration of all its members. For instance, $\{1, 0, -1\}$, $\{a, b, c\}$ or $\{p_0, \ldots, p_n\}$ are all sets defined by extension.

Intensional definition

The intensional definition of a set specifies a given property verified by all the members of the set. For instance, here is the intensional definition of the set of even numbers: $\{n \in \mathbb{N} \mid \exists k \in \mathbb{N}\ s.t.\ n = 2k\}$ or $\{n \in \mathbb{N} \mid IsEven(n)\}$.

Set operators

Set theory also defines some operators to compose sets together. Given two sets A and B, the following sets are formed:

- $A \cup B$ denotes the set formed by the union of the members of A and the members of B, i.e. $A \cup B = \{x \mid x \in A \vee x \in B\}$.

- $A \cap B$ denotes the set formed by the intersection of set A and B, i.e. $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- $A \setminus B$ denotes the set formed by the elements of set A that are not elements of set B (the difference between set A and B), i.e. $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$.
- $A \times B$ denotes the cartesian product between the elements of set A and set B, i.e. the set of all ordered pairs defined by $\{(x, y) \mid x \in A \wedge y \in B\}$. We generalize the definition to build the set of n-tuples $A_0 \times A_1 \times \cdots \times A_n$ defined by $\{(x_0, (x_1, \ldots (\ldots, x_n))) \mid x_0 \in A_0, x_1 \in A_1, \ldots, x_n \in A_n\}$. It is sometimes useful to give a name to the elements of a tuple without refering to their index. In such a case, a tuple is called a record where each element, called a field, has been given an explicit name. This formalism is useful to represent rather complex data structures. For instance, say that we want to represent the set of humans by a triplet composed of the size, weight, and eye color of a given human. We can define this set as the set of triplet $\mathbb{R} \times \mathbb{R} \times \{green, blue, brown\}$. If we want to give a concrete name to the elements of the triplet, we can equivalently define such a triplet as a record, written $<size, weight, eye>$, where $size \in \mathbb{R}$, $weight \in \mathbb{R}$ and $eye \in \{green, blue, brown\}$.
- $A \sqcup B$ denotes the set formed by the disjoint union of set A and set B. The disjoint union is obtained by adjoining an index i to the elements of A and an index j to the elements of B such that $i \neq j$. Then, the two sets of couples are joined together to build the disjoint union of A and B. For instance, consider that $A = \{a, b, c\}$ and $B = \{a, b\}$. To obtain the disjoint union $A \sqcup B$, we create the two sets $A_i = \{(i, a), (i, b), (i, c)\}$ and $B_i = \{(j, a), (j, b)\}$, and then join the sets together s.t. $A \sqcup B = \{(i, a), (i, b), (i, c), (j, a), (j, b)\}$. When the two sets A and B are disjoint, i.e. $A \cap B = \emptyset$, then $A \sqcup B$ is isomorphic to $A \cup B$. To stress the fact that we are building a set from the union of two disjoint sets, we prefer to use the disjoint union operator. For instance, we write $\mathbb{N} \sqcup \{\infty\}$, instead of $\mathbb{N} \cup \{\infty\}$, to denote the set of values ranging from the set of natural numbers with the addition of the infinite value ∞.
- $\mathcal{P}(A)$ denotes the powerset of A defined by all the possible subsets formed with the elements of set A, i.e. $\mathcal{P}(A) = \{X \mid X \subseteq Y\}$.

Relations and functions

A binary relation R between two sets X and Y is a subset of the set of pairs $X \times Y$, i.e. $R \subseteq X \times Y$, or an element of the powerset $\mathcal{P}(X \times Y)$, i.e. $R \in \mathcal{P}(X \times Y)$. We write $R(x, y)$ to denote $(x, y) \in R$. We generalize the definition to n-ary relations. A n-ary relation between sets $X_0, \ldots, X_n$ is a subset of the set of n-tuples $X_0 \times \cdots \times X_n$, i.e. $R \subseteq X_0 \times \cdots \times X_n$, or an element of the powerset $\mathcal{P}(X_0 \times \cdots \times X_n)$, i.e. $R \in \mathcal{P}(X_0 \times \cdots \times X_n)$. We write $R(x_0, \ldots, x_n)$ to denote $(x_0, \ldots, x_n) \in R$.

A partial function f from set X to set Y is a binary relation from X to Y verifying that $\forall x \in X, y, y' \in Y,\ (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'$, i.e. x appears at most once as the first element of a pair in f. We note $f \in X \nrightarrow Y$ to denote a partial function from X to Y. The set of the first elements of the pairs defined in f is called the *domain* of f. We write it $\texttt{dom}(f) = \{x \mid \exists y \ s.t. (x, y) \in f\}$. When there is no ambiguity, and given an $x \in X$ and

$f \in X \nrightarrow Y$, we write $x \in f$ as a shorthand to $x \in \texttt{dom}(f)$.

A total function a, or application, from X to Y is a partial function verifying that all the elements of X appear as the first element of a pair in a, i.e. for all $x \in X$, there exists $y \in Y$ such that $(x, y) \in a$. In other words, the domain of an application from X to Y is equal to the set X. We note $a \in X \to Y$ to denote an application from X to Y.

1.1.3 Rule-based definition of sets

While describing the operational semantics of a subset of the VHDL language in Chapter **??**, we define some of the sets (e.g, simulation relations) with rule instances, also called inference rules or judgments. A rule instance, defining the members of a set A, can take the following forms:

$$\frac{}{C} \text{ or } \frac{P_1, \ldots, P_n}{C}$$

The left form of rule instance is called an axiom; it states that $C \in A$. The rule instance $\frac{P_1, \ldots, P_n}{C}$ states that $C \in A$ if $P_1, \ldots, P_n$ hold. Also, we say that C is derivable, if $P_1, \ldots, P_n$ are derivable. $P_1, \ldots, P_n$ are the premises of the rule, and C is the conclusion of the rule. The premises $P_1, \ldots, P_n$ hold if there exists a *finite* derivation tree for each one of them. A *finite* derivation tree is obtained by applying the rule instances defining the set until all branches of the derivation reach axioms. For instance, the two following rules, named rules Ev0 and Ev2, define the *IsEven* unary relation that states that a given natural number is even:

$$\frac{}{IsEven(0)}\ \text{Ev0} \qquad \frac{IsEven(n-2)}{IsEven(n)}\ \text{Ev2}$$

The rule Ev0 states as an axiom that 0 is an even number; the rule Ev2 states that for all natural number n, n is an even number if one can derive that fact that $n - 2$ is an even number. Thus, we can derive from the previous rules that 4 is an even number by building the following derivation tree:

$$\dfrac{\dfrac{\dfrac{}{IsEven(0)}\ \text{Ev0}}{IsEven(2)}\ \text{Ev2}}{IsEven(4)}\ \text{Ev2}$$

Starting from $IsEven(4)$, we can apply the Ev2 rule to derive $IsEven(2)$. Then, another application of the Ev2 rule leads to $IsEven(0)$, and we can close the derivation branch by applying the Ev0 rule. Here, the only branch of the tree has reached an axiom, and thus the derivation tree is finite. To further illustrate the use of rule instances in the definition of a set, let us consider the following minimal language of arithmetic expressions expressed in the Backus-Naur form:

$$e \quad ::= \quad n \mid id \mid e_0 + e_1$$

Here, n ranges over the set of natural numbers $\mathbb{N}$; id ranges over the set `string` of non-empty strings (i.e. it is the set of identifiers). To evaluate the arithmetic expressions, we need a state $s \in \texttt{string} \rightharpoonup \mathbb{N}$ that maps each variable identifier to a natural number value. We assume that only a certain set of declared identifiers can appear in an arithmetic expression. Thus, the state is a partial function from the set of non-empty strings to the set of natural numbers. We define the evaluation relation for the arithmetic expressions with the three following rules:

$$\frac{}{s \vdash n \to n}\ \textsc{Nat} \qquad \frac{}{s \vdash id \to s(id)}\ \textsc{Var}\ id \in \texttt{dom}(s) \qquad \frac{s \vdash e_0 \to n \qquad s \vdash e_1 \to m}{s \vdash e_0 + e_1 \to n + m}\ \textsc{Add}$$

Here, the evaluation relation is a subset of the set of triplets $(\texttt{string} \rightharpoonup \mathbb{N}) \times e \times \mathbb{N}$. In the rule instances defining the evaluation relation, the $\vdash$ symbol (pronounced *thesis*) stresses that the left part implies the right part, or it is involved in the evaluation of the right part. For instance, the second rule can be read: in the context of state s, id evaluates to $s(id)$ if $id \in \texttt{dom}(s)$. When the *context* is not involved in the evaluation of the syntactic constructs at the right side of the $\vdash$ symbol, we permit ourselves to remove the context and the $\vdash$ from the rule instances. For example, we can define to the NAT rule by

$$\frac{}{n \to n}\ \textsc{Nat}$$

as the state s in not involved in the evaluation of expressions that are natural numbers.

Note that in the VAR rule, there appears an extra statement, at the right of the judgment line, called a *side condition*. This is an extra condition that must hold with all the premises of the rule instance, but which does not generate a derivation tree of its own.

Finally, here is an example of a derivation tree for the evaluation of the expression $x + (y + 1)$ in the context of state $\{(x, 1), (y, 2)\}$:

$$\dfrac{\dfrac{}{\{(x,1),(y,2)\} \vdash x \to 1}\ \textsc{Var}\ x \in \{x,y\} \qquad \dfrac{\dfrac{}{\{(x,1),(y,2)\} \vdash y \to 2}\ \textsc{Var}\ y \in \{x,y\} \qquad \dfrac{}{1 \to 1}\ \textsc{Nat}}{\{(x,1),(y,2)\} \vdash y + 1 \to 3}\ \textsc{Add}}{\{(x,1),(y,2)\} \vdash x + (y + 1) \to 4}\ \textsc{Add}$$

Here again, all the branches of the derivation tree have reached axioms, and thus $\{(x, 1), (y, 2)\} \vdash x + (y + 1) \to 4$ is a member of the evaluation relation of arithmetic expressions. It states that the expression $x + (y + 1)$ evaluates to 4 in the state $\{(x, 1), (y, 2)\}$.

1.2 Induction principles

In the proceedings of proofs laid out in this thesis, we often perform proofs by induction. Here are some reminders on some induction principles to help the reader understand the proofs of Chapter **??** and Appendix **??**.

1.2.1 Mathematical induction

The principle of mathematical induction is tied to the set of natural numbers. It states that, in order to prove that a property P holds for all natural numbers, it is sufficient to prove that:

- P holds for 0
- if P holds for a given n then it holds at $n+1$

Mathematical induction describes a way to deduce that a property holds for the set of natural numbers, first by stating that the property holds for zero, i.e. the minimal element of the set, then by stating that if the property holds for a given number then it holds for its successor. Thus, knowing that $P(0)$ holds, we can deduce that $P(1)$ holds, $P(2)$ holds, $P(3)$ holds, etc.

1.2.2 Structural induction

Sometimes, reasoning by induction necesitates to follow the structure of a given set, i.e. the formation rules of a given set. For instance, if we want to prove that a given property P holds for the set of arithmetic expressions given as an example in Section 1.1.3, we must prove that:

- P holds for all natural number n
- P holds for all identifiers id
- if P holds for all sub-expressions e_0 and e_1, then P holds for $e_0 + e_1$

A proof that leverages structural induction follows the structure of the elements we are reasoning upon. Many times in this thesis, we are using structural induction to prove that a sum expression verifies a certain property. Thus, the structural induction follows the recursive definition of the sum term, which is, for any set A, function $f \in A \to \mathbb{N}$ and $X \subseteq A$ and :

$$\sum_{x \in X} f(x) = \begin{cases} 0 \texttt{ if } X = \emptyset \\ f(x) + \sum\limits_{x' \in X'} f(x') \texttt{ if } X = \{x\} \cup X' \end{cases}$$

In the second computation branch, it is left implicit that set X' is strict subset of X such that $x \notin X'$ or $X' = X \setminus \{x\}$. Given a set A and a function $f \in A \to \mathbb{N}$, to prove that for all $X \subseteq A$, the property $P(X, \sum\limits_{x \in X} f(x))$ holds, we must show that:

- $\forall X \subseteq A,\ X = \emptyset \Rightarrow P(\emptyset, 0)$

- $\forall X \subseteq A, x \in X, X' \subset X,\ X = \{x\} \cup X' \Rightarrow P(X', \sum_{x' \in X'} f(x'))$
  $\Rightarrow P(\{x\} \cup X', f(x) + \sum_{x' \in X'} f(x'))$

The induction follows the structure of the function. In this specific case, structural induction is often refered to as *functional* induction.

1.2.3 Rule induction

A specific kind of induction, called *rule* induction, applies when wanting to prove properties over sets that are defined by inference rules (or rule instances). For instance, let us take the evaluation relation for arithmetic expressions used in Section 1.1.3 to illustrate the principle of rule induction. To prove that a property P holds for the evaluation relation of arithmetic expressions, which is a subset of triplets $(\texttt{string} \to \mathbb{N}) \times e \times \mathbb{N}$, we must prove that:

- $P(s, n, n)$ for all s, n
- if $id \in \texttt{dom}(s)$ then $P(s, id, s(id))$ for all s, id
- if $s \vdash e_0 \to n$ and $P(s, e_0, n)$, and $s \vdash e_1 \to m$ and $P(s, e_1, m)$ then $P(s, e_0 + e_1, n + m)$ for all s, e_0, e_1, n, m

Rule induction states that in order to prove a property over a set defined by rule instances, the property must hold in any construction case of the considered set. The idea is that if the property is preserved from the premises of rules to the conclusions then the property holds for all the elements of the set.

Let us give an application of rule induction to prove a property over the evaluation relation of arithmetic expressions. First, we define, through the three following rules, the relation $\in_r$ stating that a given identiifer id is referenced in an arithmetic expression e, written $id \in_r e$:

$$\frac{}{id \in_r id}\ \textsc{InRId} \qquad \frac{id \in_r e_0}{id \in_r e_0 + e_1}\ \textsc{InRAddL} \qquad \frac{id \in_r e_1}{id \in_r e_0 + e_1}\ \textsc{InRAddR}$$

Then, the property of Proposition states that an arithmetic expression that contains references to identifiers that are not part of the current state's domain can not be evaluated.

Proposition 1. *Let $id \in \texttt{string}$. For all state s, arithmetic expression e, and natural number n,*

$$id \notin \texttt{dom}(s) \wedge id \in_r e \Rightarrow \neg s \vdash e \to n$$

Proof. Let us define the property P as follows:

$$P(s, e, n) \equiv id \notin \texttt{dom}(s) \wedge id \in_r e \Rightarrow \neg s \vdash e \to n$$

Then, let us use rule induction to prove $P(s, e, n)$.

First, we must prove $P(s, n, n)$. Assuming $id \in_r n$, there is a contradiction as no rule instance defining the relation $\in_r$ includes the case where the considered expression is a natural number.

Then, we must prove $P(s, id', s(id'))$, assuming that $id' \in \texttt{dom}(s)$. We know that $id \in_r id'$, and thus $id = id'$. Then, there is a contradiction between $id \in \texttt{dom}(s)$ and $id \notin \texttt{dom}(s)$.

Finally, we must prove $P(s, e_0 + e_1, n + m)$, assuming that $s \vdash e_0 \to n$ and $P(s, e_0, n)$, and $s \vdash e_1 \to m$ and $P(s, e_1, m)$. We know that $id \in_r e_0 + e_1$; this hypothesis has either be constructed by applying Rule InRAddL or Rule InRAddR. If Rule InRAddL has been applied, then we know $id \in_r e_0$; thus, from $P(s, e_0, n)$, we can deduce $\neg s \vdash e_0 \to n$, which contradicts $s \vdash e_0 \to n$. We can perform the proof in a similar way if Rule InRAddR has been applied. □

1.3 The Coq proof assistant

In this section, we present the Coq proof assistant [7]. The Coq proof assistant is the framework we use to encode the different semantics and programs involved in the HILECOP model-to-text transformation, and also to formally verify our proof of semantic preservation. Here, we give an overview of the different concepts underlying the Coq proof assistant. The aim is to give to the reader the tools to understand the different listings presenting Coq code in the following chapters. For a thorough presentation of the Coq proof assistant, the reader can refer to the reference manual[3], or to [2, 6, 1].

1.3.1 The Calculus of Inductive Constructions (CIC)

At the heart of the Coq proof assistant, there is a kernel that implements the Calculus of Inductive Constructions (CIC) [3]. The CIC is a typed lambda-calculus extended with the possibility to define inductive types. Thus, the CIC permits to define programs and types in a similar way; both are *terms* of the language. A program is a term with a certain type, and a type is a term with a certain type. The type of a type is called a *sort*. We can mention two basic sorts built in the Coq proof assistant: the `Prop` sort which is the type of logical formulas, and the `Set` sort which is the type of *small* sets.

The Coq proof assistant permits to express logic formulas and to interactively build proofs of these formulas by using a high-level tactic language. The sequence of tactics that builds a proof for a given formula is called a proof script. The execution of a proof script builds a proof term. As agreed with the CIC, a logic formula can be seen as a *type* and a proof for this formula is an *inhabitant* of the type denoted by the logic formula. Thus, when building a proof term by executing a proof script, the Coq kernel checks that the proof term is of the type of the logic formula by applying typing rules[4]. For instance, let us take two logical propositions `A` and `B`. In Coq, we can declare these propositions as elements of the `Prop` type in the Coq top-level loop:

```
Coq < Variables A B : Prop.
```

[3] https://coq.inria.fr/distrib/current/refman/

[4] https://coq.inria.fr/distrib/current/refman/language/cic.html

The `Variables` keyword adds the propositions `A` and `B` to the global environment accessed by the Coq kernel. Now, say that we want to prove the *modus ponens* expressed with the propositions `A` and `B`, namely that $\texttt{A} \Rightarrow (\texttt{A} \Rightarrow \texttt{B}) \Rightarrow \texttt{B}$. In Coq, we can express it as follows:

```
Coq < Theorem modus_ponens : A → (A → B) → A.
```

Here, we declare the modus ponens theorem as an element of type $\texttt{A} \to (\texttt{A} \to \texttt{B}) \to \texttt{A}$. Here, the arrow represents the functional arrow; in fact, $\texttt{A} \to \texttt{B}$ is a notation for the product type $\Pi x : \texttt{A}.\texttt{B}$ where x is not referenced in `B`. According to the Curry-Howard correspondence [4], there is an equivalence between a proof term and a program. Thus, a proof term of the logical implication $\texttt{A} \Rightarrow \texttt{B}$ is equivalent to a program, or a function, of type $\texttt{A} \to \texttt{B}$, i.e. a program that takes an element of type `A` and yields an element of type `B`. Thus, the type $\texttt{A} \to (\texttt{A} \to \texttt{B}) \to \texttt{A}$ is a valid encoding of the formula $\texttt{A} \Rightarrow (\texttt{A} \Rightarrow \texttt{B}) \Rightarrow \texttt{B}$. The `Theorem` keyword triggers the interactive proof mode through which the user will build a proof term for the corresponding formula. A simple proof term for the modus ponens theorem is a function that takes an element x of type `A` and a function f of type $\texttt{A} \to \texttt{B}$ as inputs, and yields an element of type `B` by applying the function f to parameter x, i.e. $(f\,x)$. The function takes the form of the following term of the typed lambda-calculus:

$$\lambda(x : \texttt{A}).\lambda(f : \texttt{A} \to \texttt{B}).(f\ x)$$

While passing this *lambda-term* as a proof term of the modus ponens theorem, the Coq kernel checks the well-typedness of the term by building the following derivation tree, which is a simplified version of the full derivation tree according to the typing rules of the CIC:

$$\dfrac{\dfrac{\dfrac{}{\texttt{A B} : \texttt{Prop}[x : \texttt{A}, f : \texttt{A} \to \texttt{B}] \vdash f\ :\ \texttt{A} \to \texttt{B}}\ \textsc{Var} \qquad \dfrac{}{\texttt{A B} : \texttt{Prop}[x : \texttt{A}, f : \texttt{A} \to \texttt{B}] \vdash x\ :\ \texttt{A}}\ \textsc{Var}}{\texttt{A B} : \texttt{Prop}[x : \texttt{A}, f : \texttt{A} \to \texttt{B}] \vdash (f\ x)\ :\ \texttt{B}}\ \textsc{App}}{\texttt{A B} : \texttt{Prop}[\texttt{x} : \texttt{A}] \vdash \lambda(f : \texttt{A} \to \texttt{B}).(f\ x)\ :\ (\texttt{A} \to \texttt{B}) \to \texttt{B}}\ \textsc{Lam}}{\texttt{A B} : \texttt{Prop}[] \vdash \lambda(x : \texttt{A}).\lambda(f : \texttt{A} \to \texttt{B}).(f\ x)\ :\ \texttt{A} \to (\texttt{A} \to \texttt{B}) \to \texttt{B}}\ \textsc{Lam}$$

In the above derivation tree, the global and the local environment are represented at the left of the thesis symbol. The local environment is represented by square brackets. The global environment is represented at the left of the local environment. At the root of the derivation tree, the global environment contains our two previously declared logical propositions `A` and `B`, whereas the the local environment is empty. The application of the LAM rule adds new entry to the local environment; the APP triggers the type-checking of the left and the right part of an application; the VAR rule checks that a term is well-typed if it is referenced as an element of the given type in the global or the local environment.

As said before, the `Theorem` keyword triggers the interactive proof mode. The interactive proof mode will accompany the user to an incremental building of a proof term for the current goal, i.e. the current logic formula we want to prove. Then, to prove the modus ponens theorem, the following interface is first presented to the user:

```
Coq < Theorem modus_ponens : A → (A → B) → B.
```

```
1 subgoal

============================
A → (A → B) → B
```

The term under the horizontal bar represents the current goal to proof, i.e. the current formula for which we are building a proof term. Above the horizontal bar are referenced the variables constituting the local environment. At the beginning of the proof, the local environment is empty – and so is the local environemnt at the root of the derivation tree presented above. To build a proof term in interactive mode, the user will then invoke commands called *tactics*. Each tactic invocation corresponds to the invocation of a typing rule of the CIC performed by the Coq kernel. To build a proof term for the modus ponens theorem, the first thing to do is to invoke the LAM rule; this is done by appealing to the `intros` tactic.

```
Coq < intros x.
1 subgoal

x : A
============================
(A → B) → B
```

Here, the user passes to the system the name of the variable that will be introduce in the local environment by the LAM rule, i.e the variable `x`. Then, we repeat the operation, applying the LAM rule a second time to introduce an element of type `A → B` in the environment.

```
Coq < intros f.
1 subgoal

x : A
f : A → B
============================
B
```

Then, based on the local environment, we can build an object of type `B` by applying `f` to the input `x`. We can do it by appealing to the `apply` tactic. The `apply` tactic invokes the APP rule.

```
Coq < apply (f x).
No more subgoals.

Coq < Qed.
```

After the invocation of the `apply` tactic, the proof term for the modus ponens theorem is completely built, thus, the Coq top-level loop displays the message that all goals are completed. Then, we can close the interactive proof mode and store the proof of the modus ponens theorem in the global environment by using the `Qed` keyword.

1.3.2 Inductive types

One of the major strength of the CIC, and therefore of the Coq proof assistant, is the possibility to enrich the global type system with the definition of inductive types. For instance, here is the definition of the type of natural numbers, named `nat`:

```
Inductive nat : Set :=
| O : nat
| S : nat → nat.
```

The `nat` type is of the `Set` sort (remember that the type of a type is called a sort). The `nat` type is defined through two constructors, represented by the pipe-separated entries. The `O` constructor states that zero is a natural number. The `S` constructor takes a natural number as input and yields the succesor to this natural number. This corresponds to the structural definition of natural numbers in Peano's arithmetic. Thus, in this setting, the number 2 is represented by `(S (S O))`, the number 3 by `(S (S (S O))`, etc. The result of the evaluation of an inductive type, declared through the `Inductive` keyword, is the addition of this type and each of its constructors to the global environment accesible by the Coq kernel. Also, a corresponding induction principle is generated at the evaluation of an inductive type definition. For instance, the `nat_ind` induction principle is generated at the evaluation of the `nat` type. The `nat_ind` induction principle is equivalent to the so-called mathematical induction presented in Section 1.2.1. It is a proof term of the logical formula denoting the mathematical induction, i.e.:

```
forall P : nat → Prop, P 0 → (forall n : nat, P n → P (S n)) → forall n : nat, P n
```

Leveraging the definition of inductive types, the syntactic constructs of programming languages are also easily implemented. Here is the implementation of the syntax of arithmetic expressions presented in the previous section:

```
Inductive e : Set :=
| enat : nat → e
| eid : string → e
| eadd e → e → e.
```

Each constructor corresponds to a construction case in the definition of arithmetic expressions in the Backus-Naur form. The Coq system also generates the induction principle following the structure of arithmetic expressions (thus, a structural induction principle). The induction principle is a proof term of the following logical formula:

```
forall P : e → Prop,
    (forall n : nat, P (enat n)) →
    (forall id : string, P (eid id)) →
    (forall e0 : e, P e0 → forall e1 : e, P e1 → P (eadd e0 e1)) →
    forall e : e, P e
```

The evaluation relation for arithmetic expressions is defined in a similar way:

```
Inductive evale (s : string → option nat) : e → nat → Prop :=
| evalnat : forall n : nat, evale s (enat n) n
| evalid : forall (id : string) (n : nat),
           s id = Some n →
```

```
            evale s (eid id) n
| evaladd : forall (e0 e1 : e) (n m : nat),
             evale s e0 n →
             evale s e1 m →
             evale s (eadd e0 e1) (n + m).
```

In the above listing, the state that yields the value of identifiers present in an arithmetic expression is a named parameter of the `evale` relation, i.e. `s`. Parameters which are not varying from one construction case to another can be passed as named parameters while defining an inductive type. The state `s` takes a string identifier as input and yields an `option` to a natural number. As so, the `option` type permits to represent partial functions. The identifiers that belong to the domain of state `s` will be: associated with `Some` natural number, whereas the unreferenced identifiers will be associated with the `None` value of the `option` type. The `Some` and the `None` constructors are the two constructors of the `option` type which is defined in Coq as follows:

```
Inductive option (A : Type) : Type :=
| Some : A → option A
| None : option A.
```

The `option` type is parameterized by a type `A` that will set the type of elements passed to the `Some` constructor.

Maybe present record types?

1.3.3 Functional programming

As told in the presentation of the CIC, the Coq proof assistant permits to write functional programs, including the definition of recursive functions. The definition of a recursive function is performed with `Fixpoint` keyword. Here is an example of recursive function defined in Coq. The `pow` function takes two natural numbers `a` and `n` as inputs and yields `a` to the power `n`.

```
Fixpoint pow (a n : nat) {struct n} : nat :=
  match n with
  | O ⇒ 1
  | S m ⇒ a * pow a m
  end.
```

In the body of the `pow` function, the `match` construct performs pattern-matching over the structure of the input `n`. The input `n` is an element of the `nat` type, and thus it could either have been built with the `O` constructor or as the succesor of another element of the `nat` type, i.e. with the `S` constructor. The `match` construct enumerates all the possible construction cases for the given input. Each construction case leads to a pipe-separated entry; for each entry, the structure of the input appears at the left of the arrow, and the result returned appears at the right the arrow. In the above example, `1` is returned if `n` equals `O`, and the result of the multiplication of `a` with the recursive call `pow a m` is returned if `n` is the succesor of a certain `m`. In that case, we have $m = n - 1$, and then the recursive call `pow a m` can be read as `pow a` $(n - 1)$.

When declaring a recursive function, the user must specify which parameter is structurally decrementing through the recursive call. This is performed through {`struct` *id*} annotation,

where *id* denotes one parameter of the declared function. This information permits to the Coq kernel to generate the fixpoint equation for the function, thus proving that the function is always terminating.

1.3.4 Dependent types

In the listings that the reader will find in the following chapters, and also in the code repository associated with this thesis, some data structures are *dependently-typed* structures. Thus, we introduce here the notion of dependent type and how it is expressible with the Coq proof assistant.

A type is said to be dependent when its expression depends on one or more elements of other types. Give an example of dependent type, let us take the definition of polymorphic lists that carry their own length. In Coq, these lists are defined as follows:

```
Inductive listn (A : Type) : nat → Set :=
| niln : list A 0
| consn : forall n : nat, A → listn A n → listn A (S n).
```

The `listn` takes the type `A` of its elements as its first parameter, then its second parameter is an element of the `nat` type which represent the actual length of the list. Note that the first parameter, i.e. the `A` parameter, of the `listn` type alone is not sufficient to qualify `listn` as a dependent type. The `A` parameter is the expression of the polymorphism of the elements of the list involved in generic programming. Polymorphism relates to the fact that the `A` type is general enough to accept multiple types as the type of the list's elements. The `niln` constructor of the `listn`, i.e. the constructor of the empty list, is the type of lists of length 0. The `consn` constructor permits to add a new element at the head of an existing list to build a new list. Thus, the type of the resulting list is the type of lists of length $n + 1$, where n is the length of the tail list.

To further illustrate the use of dependent types, let us say that we want to write a function that takes two natural numbers n and m as inputs, and yields $n - m$ only if $n \geq m$. Thus, the function takes two parameters n and m, and a third parameter which is the proof that $n \geq m$. This third parameter *depends* on the two previous parameters, and thus the function is said to be a dependently-typed function. In Coq, it would be written as follows:

```
Definition my_sub (n m : nat) (pf : m <= n) : nat := n − m.
```

Even though, in its definition body, the `my_sub` function simply appeals to the Coq built-in substraction function, passing a proof that m is less than or equal to n adds a constraint to the computation of the substraction. One can see how dependent types can help check that the parameters of programs meet some properties at definition time. Constraining the type of parameters during the definition of programs reduces the proof efforts afterwards, but adds programming complexities at the moment of the definition. Thus, there is a trade-off between using dependent types to constraint the structures and programs at the moment of their definition, or letting the structures and programs as constraint loose as possible at the cost of having to prove much more properties afterwards.

To conclude the subject of dependent types, we often use *sigma* types to types of elements that meets a given property. A sigma type expresses the dependence between a parameter and

a proof of a given property that possibly depends on another parameter. As so, sigma types are useful to express intentional sets (cf. Section 1.1.2). In the Coq standard library, the definition of the sigma type is as follows:

```
Inductive sig (A:Type) (P:A → Prop) : Type := exist : forall x:A, P x → sig P.
```

The `sig` type only constructor takes an element `x` of type `A` along with a proof that `x` meets a certain property `P`. For instance, if we want to define in the type of natural numbers that are strictly greater than zero, we can do it as follows:

```
Definition natstar := sig nat (fun n : nat ⇒ n > 0).
```

The property passed as the second argument of the `sig` type is expressed by a lambda abstraction (denoted by the `fun` keyword) that takes a parameter `n` of type `nat` and returns a proof that `n` is strictly greater than zero. The Coq standard library defines a notation to write sigma types as intensional sets. Thus, we can write the `natstar` type as follows:

```
Definition natstar2 := { x : nat | x > 0}.
```

We can leverage sigma types to rewrite the `my_sub` function presented above. In the following version, the type of the m parameter carries the proof that m is less than or equal to n:

```
Definition my_sub2 (n : nat) (m : { x : nat | x <= n }) : nat := n − (proj1_sig m).
```

Here, we can no longer directly substract n with m as the type of m is no longer `nat` but `{ x : nat x <= n } |`. We have to extract the first part of the m parameter with the help of the `proj1_sig` function. The first part of an element of the `{ x : nat x <= n } |` type corresponds to the natural number `x` verifying the following property `x <= n`, and the second part corresponds to the proof that `x` verifies the property.

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