MACHINE INTELLIGENCE

UNIT - 4

Dimensionality Reduction

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Q: Comment on the usefulness of the following feature vectors

Set 1

Feature 1	5	17	13	29	72	7
Feature 2	0.9	1.1	1.0	0.95	1.05	0.91

Retain feature 1

Set 2

Feature 1	5	17	13	29	72	7
Feature 2	1.9	35	65	3.8	11.8	45.8

Retain both

Set 3

Feature 1	5	20	30	10	45	71
Feature 2	5.5	19	28	9.5	44	70.5

Transform by 45°

$$\theta = \pi/4 \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 30 \\ 28 \end{bmatrix} = \begin{bmatrix} 41 \\ -1.4 \end{bmatrix}$$

Feature 1'	7.4	27.5	41	13.6	63	100
Feature 2'	0.35	0.7	-1.4	0.35	0.70	0.35

Retain Feature 1'

Assumptions

1. Features with high variance are useful
2. Only one feature from set of highly correlated features is useful

COVARIANCE MATRIX

$$\hat{\Sigma} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})(x_i - \overline{x})^T$$

Covariance Matrix with Mean Centered Data

- Unbiased estimate

$$C_X = \frac{1}{n-1} X X^T$$

Principle Component Analysis

Feature 1':	7.4	27.5	41	13.6	63	100	Mean = 42
Feature 2':	0.35	0.7	1.4	0.35	0.70	0.35	Mean = 0.64

Mean-Centered Data:

Feature 1':	-34.6	-14.5	-1.0	-28.4	21	58
Feature 2':	-0.29	0.06	0.76	-0.29	0.06	-0.29

Covariance Matrix

$$\begin{bmatrix} 1204 & 0.216 \\ 0.216 & 0.167 \end{bmatrix}$$

- Desirable covariance matrix:
  off diagonal entries ≈ 0 & sorted by mag

$$\begin{bmatrix} 1204 & 0.216 \\ 0.216 & 0.167 \end{bmatrix}$$ good cov matrix

- Change coordinate axis to make cov matrix better

- Transform x (original feature space) such that new covariance matrix is desirable

$$Y = P X$$

new feature space

transformation matrix

- Rows of P are called principle components

$$C_Y = \frac{1}{n-1} Y Y^T$$

$$= \frac{1}{n-1} (PX)(PX)^T$$

$$= \frac{1}{n-1} (PX) X^T P^T$$

$$= \frac{1}{n-1} P (X X^T) P^T$$

→ original cov matrix

let $XX^T = E D E^T$ (diagonalisation of square symmetric matrix)

eigenvectors of XX^T

$$C_Y = \frac{1}{n-1} P E D E^T P^T$$

let us take $P = E^T$

$$C_Y = \frac{1}{n-1} (E^T E) D (E^T E)$$

identity matrix (orthonormal)

$$C_Y = \frac{1}{n-1} D$$

→ diagonal cov matrix

Q: Reduce to 1 feature

x	y
4	11
8	4
13	5
7	14

Mean-centric

$\bar{x} = 8$

$\bar{y} = 8.5$

x'	y'
-4	2.5
0	-4.5
5	-3.5
-1	5.5

$$\Sigma = \frac{1}{4-1}\begin{bmatrix} -4 & 0 & 5 & -1 \\ 2.5 & -4.5 & -3.5 & 5.5 \end{bmatrix}\begin{bmatrix} -4 & 2.5 \\ 0 & -4.5 \\ 5 & -3.5 \\ -1 & 5.5 \end{bmatrix}$$

$$\Sigma = \frac{1}{3}\begin{bmatrix} 42 & -33 \\ -33 & 69 \end{bmatrix} = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$|\Sigma - \lambda I| = 0$$

$$\begin{vmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{vmatrix} = (\lambda-23)(\lambda-14) - 121$$

$$= \lambda^2 - 37\lambda + 201 = 0$$

$$\lambda_1 = \frac{37+\sqrt{565}}{2} \qquad\qquad \lambda_2 = \frac{37-\sqrt{565}}{2}$$

$$= 30.39 \qquad\qquad = 6.62$$

Eigenvectors

$$(\Sigma - \lambda I)\, v = 0$$

$$\begin{bmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(14-\lambda)v_1 - 11 v_2 = 0 \longrightarrow (1)$$

$$-11 v_1 + (23-\lambda) v_2 = 0 \longrightarrow (2)$$

(1) $(14-\lambda)v_1 = 11 v_2$

$$\frac{v_1}{v_2} = \frac{11}{14-\lambda} = 1$$

$$v_1 = 11 \qquad\qquad v_2 = 14-\lambda$$

(a) for $\lambda = \frac{37 + \sqrt{565}}{2}$

$v_1 = 11 \qquad v_2 = -\left(\frac{9 + \sqrt{565}}{2}\right)$

$= -16.38$

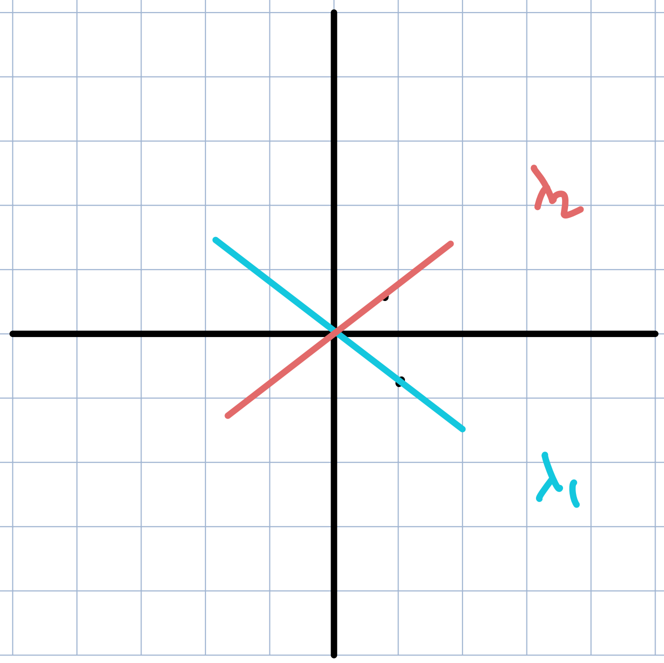
normalised : $\|v\| = 19.73$

$$v_1 = \begin{bmatrix} 0.557 \\ -0.830 \end{bmatrix}$$

New data

$$\begin{bmatrix} 0.557 & -0.830 \end{bmatrix} \begin{bmatrix} -4 & 0 & 5 & -1 \\ 2.5 & -4.5 & -3.5 & 5.5 \end{bmatrix}$$

$$= \begin{bmatrix} -4.303 & 3.735 & 5.69 & -5.122 \end{bmatrix}$$



(b) for $\lambda = \frac{37 - \sqrt{565}}{2}$

$v_1 = 11 \qquad v_2 = 14 - \lambda$

$= \frac{-9 + \sqrt{565}}{2}$

$= 7.39$

normalised : $\|v\| = 13.25$

$$V_2 = \begin{bmatrix} 0.83 \\ 0.53 \end{bmatrix}$$

New data

$$\begin{bmatrix} 0.83 & 0.53 \end{bmatrix} \begin{bmatrix} -4 & 0 & 5 & -1 \\ 2.5 & -4.5 & -3.5 & 5.5 \end{bmatrix}$$

$$= \begin{bmatrix} -1.93 & -2.5 & 2.2 & 2.24 \end{bmatrix}$$

Non-parametric method

Q: PCA

x	y	$x-\bar{x}$	$y-\bar{y}$
2.5	2.4	0.69	0.49
0.5	0.7	-1.31	-1.21
2.2	2.9	0.39	0.99
1.9	2.2	0.09	0.29
3.1	3.0	1.29	1.09
2.3	2.7	0.49	0.79
2	1.6	0.19	-0.31
1	1.1	-0.81	-0.81
1.5	1.6	-0.31	-0.31
1.1	0.9	-0.71	-1.01

$\bar{x} = 1.81 \qquad \bar{y} = 1.91$

$$\Sigma = \begin{bmatrix} 0.616 & 0.615 \\ 0.615 & 0.716 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 1.28303$ $\qquad v_1 = (0.922, 1)$

$\lambda_2 = 0.04897$ $\qquad v_2 = (-1.085, 1)$

Normalising

$$\|v_1\| = 1.3602$$

$$v_1 = \begin{bmatrix} 0.678 \\ 0.735 \end{bmatrix}$$

$$\|v_2\| = 1.4755$$

$$v_2 = \begin{bmatrix} -0.7353 \\ 0.6777 \end{bmatrix}$$

Singular Value Decomposition

- If $A_{n\times n}$ is a square matrix,

$$(A_{n\times n} - \lambda I) V_{n\times 1} = 0$$

or $$A_{n\times n} V_{n\times 1} = \lambda V_{n\times 1}$$

eigen value

eigen vector

- For rectangular matrix $A_{m\times n}$

$$A_{m\times n} V_{n\times 1} = \text{some scalar } U_{m\times 1}$$

  - we will need to find two vectors & a scalar

- For $A_{m\times n}$, $(A^T A)_{n\times n}$ and $(AA^T)_{m\times m}$ are square symmetric matrices

- Decompose using Eigenvalue decomposition
  - Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be eigenvalues and (sorted in desc)

    $v_1, v_2, \ldots, v_r$ be the corresponding eigenvectors

$$(A^TA)\,v_i = \lambda_i v_i$$

Premultiply with A (associative)

$$(AA^T)\,A v_i = \lambda_i v_i$$

Divide by $\|Av_i\|$

$$(AA^T)\,u_i = \lambda u_i$$

where $u_i = \dfrac{Av_i}{\|Av_i\|}$

To Summarise

- A^TA is $n \times n$

$$(A^TA)\,v_i = \lambda_i v_i$$

- AA^T is $m \times m$

$$(AA^T)\,u_i = \lambda_i u_i$$

$$u_i = \frac{Av_i}{\|Av_i\|}$$

- $\|Av_i\|^2 = \lambda_i$

$$(A^T A) v_i = \lambda_i v_i$$

$$v_i^T (A^T A) v_i = v_i^T \lambda_i v_i$$

$$(A v_i)^T (A v_i) = \lambda_i v_i^T v_i$$

$$\| A v_i \|^2 = \lambda_i$$

$$\| A v_i \| = \sqrt{\lambda_i} = \sigma_i$$

- $u_i = \dfrac{A v_i}{\| A v_i \|} = \dfrac{A v_i}{\sigma}$

- $A v_i = u_i \sigma$ → singular value of $A = \sqrt{\lambda_i}$

  v_i: $n \times 1$ unit vector

  u_i: $m \times 1$ unit vector

- $AV = U \Sigma$

$$A = U \Sigma V^T$$

Optionally:

- Extend $\{U_1, U_2, \dots, U_r\}$ by adding $(m-r)$ zero vectors of size $m \times 1$ to get an $m \times m$ matrix U

- Extend Vs similarly by adding $(n-r)$ zero vectors of size $n \times 1$ to get an $n \times n$ matrix V

Not Done in Practice - reduced SVD

SVD and PCA

- Let $Y = \frac{1}{\sqrt{n-1}} X^T$ (mean centered)

$$Y^T Y = C_X$$

- Columns of SVD of $Y^T Y$ are V
- SVD of X^T as $U \Sigma V^T$
- Columns of V are principle components of X
- No need to explicitly calc cov $X^T X$

Q: $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ (not sym). Do SVD.

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} = Y$$

Find eigenvalues & eigenvectors

$$\begin{vmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{vmatrix} = 0$$

$$(\lambda-25)^2 - 49 = 0$$

$$\lambda^2 - 50\lambda + 576 = 0$$

$$\lambda_1 = 32 \qquad \lambda_2 = 18$$

(i) $(A - \lambda_1 I)\, v_1 = 0$

$$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 7x_2 = 0$$

$$\frac{x_1}{x_2} = 1 \Rightarrow x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = \frac{1}{\sqrt{2}}$$

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(ii) $(A - \lambda_2 I) V_2 = 0$

$$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$7x_1 + 7x_2 = 0 \Rightarrow -7x_1 = 7x_2$$

$$\frac{x_1}{x_2} = -1 \Rightarrow V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Find Σ

$$\Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \quad V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

$$u_1 = \frac{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{\sqrt{32}} = \frac{\begin{bmatrix} 4\sqrt{2} \\ 0 \end{bmatrix}}{4\sqrt{2}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{\sqrt{18}} = \frac{\begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix}}{3\sqrt{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\uparrow \quad \uparrow$

attributes

Q: $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ (tall)

$$A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

(i) eigenvalues

$$\begin{vmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{vmatrix} = 0$$

$$(\lambda-9)^2-81=0$$

$$\lambda^2-18\lambda=0$$

$$\lambda_1=18 \qquad \lambda_2=0$$

$$\Sigma=\begin{bmatrix}\sqrt{18} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

(ii) eigenvectors (v)

$$\begin{bmatrix}9-18 & -9\\ -9 & 9-18\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}9 & -9\\ -9 & 9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$9x_1-9x_2=0$$

$$x_1=x_2=1$$

normalised

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$v_2 : \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(iii) Solve for U

$$U_1 = \frac{A V_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{\sqrt{18}}$$

$$U_1 = \frac{\begin{bmatrix} -\sqrt{2} \\ 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}}{3\sqrt{2}} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

Find v_2 & v_3 from orthogonality

$$\begin{bmatrix} -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

row space & null space

y & z are independent

$$-\frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z = 0$$

$$x = 2y - 2z$$

$$\therefore \quad NS = \begin{bmatrix} 2y-2z \\ y \\ z \end{bmatrix} = \left\{ y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\therefore \quad v_2' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad v_3' = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Gram-Schmidt process:

$$q_1 = \frac{(2\ \ 1\ \ 0)}{\sqrt{5}} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - q_1^T \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} q_1$$

$$B = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + 4/\sqrt{5} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{\begin{bmatrix} -2/5 \\ 4/5 \\ 5/5 \end{bmatrix}}{\frac{3\sqrt{5}}{5}} = \frac{\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}}{3\sqrt{5}} = \begin{bmatrix} -2/(3\sqrt{5}) \\ 4/(3\sqrt{5}) \\ 5/(3\sqrt{5}) \end{bmatrix}$$

$$q_2 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$