Network Flows (Cont...)

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Recap

- The Ford-Fulkerson method repeatedly augments the flow along augmenting paths until it has found a maximum flow
- How do we know that when the algorithm terminates, we have actually found a maximum flow?
- The max-flow min-cut theorem, which we shall prove shortly, tells us that a flow is maximum if and only if its residual network contains no augmenting path
- To prove this theorem, we must first explore the notion of a cut of a flow network

Cuts

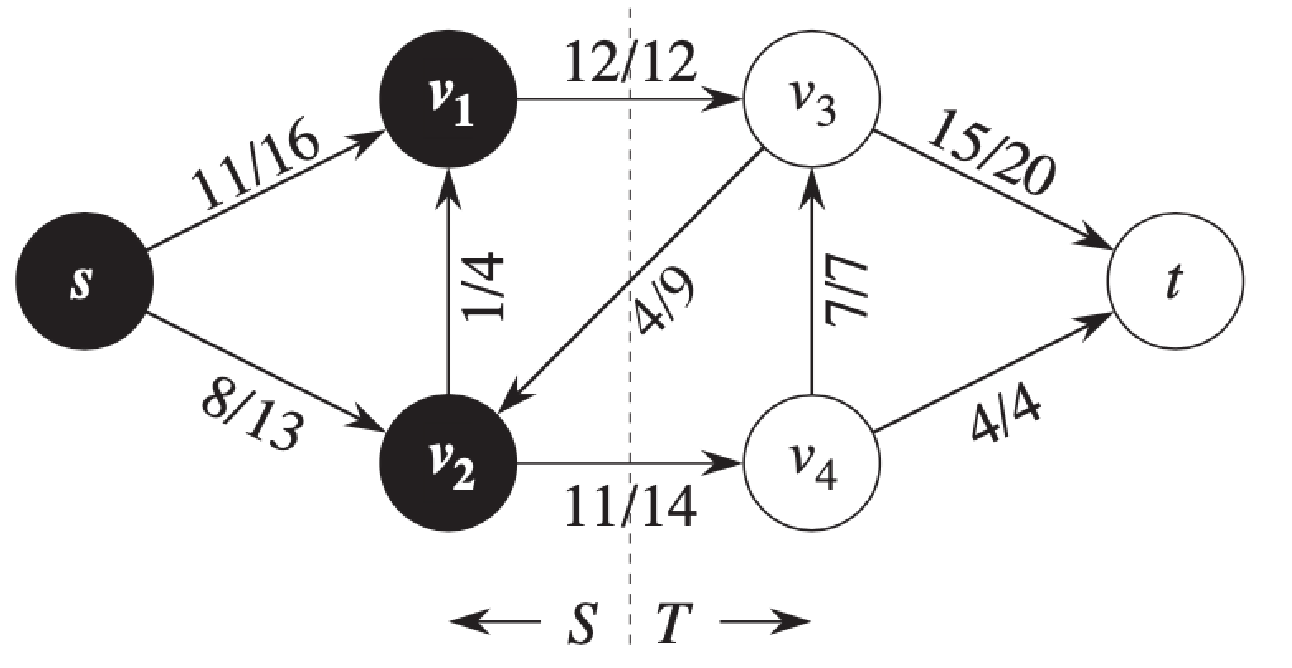
- We have already seen an upper bound on f is the sum of all the capacities of the edges leaving the source s
- Sometimes this bound is useful, but sometimes it is very weak
- We now use the notion of a *cut* to develop a much more general means of placing upper bounds on the maximum-flow value
- Suppose we divide the nodes in the given network G into two sets S and T, so that $s \in S$ and $t \in T$
- If f is a flow, then the ***net flow*** $f(S, T)$ across the cut (S, T) is defined to be

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

Cont ...

- Any flow that goes from *s* to *t* must cross from *S* into *T* at some point, and thereby use up some of the edge capacity from *S* to *T*
- The ***capacity*** of the cut (*S*, *T*) is $c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$
- This suggests that each such "cut" of the graph puts a bound on the maximum possible flow value
- A ***minimum cut*** of a network is a cut whose capacity is minimum over all cuts of the network
- Maximum-flow value equals the minimum capacity of any such division, called the *minimum cut*

Example



- $S = \{s, v_1, v_2\}$ and $T = \{v_3, v_4, t\}$ is a cut
- What is the net flow across the cut?
- What is the capacity of the cut?
- For a given flow f, the net flow across any (S, T) cut is the same, and it equals $|f|$, the value of the flow

Lemma

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- ***Proof***: We can rewrite the flow-conservation condition for any node $u \in V - \{s, t\}$ as $\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$.
- This implies, $\sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right)$ is zero
- WKT $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$

Proof (Cont …)

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right) .$$

Expanding the right-hand summation and regrouping terms yields

$$\begin{aligned}
|f| &= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \sum_{v \in V} f(u,v) - \sum_{u \in S-\{s\}} \sum_{v \in V} f(v,u) \\
&= \sum_{v \in V} \left(f(s,v) + \sum_{u \in S-\{s\}} f(u,v) \right) - \sum_{v \in V} \left(f(v,s) + \sum_{u \in S-\{s\}} f(v,u) \right) \\
&= \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u) .
\end{aligned}$$

Proof (Cont …)

Because $V = S \cup T$ and $S \cap T = \emptyset$, we can split each summation over V into summations over S and T to obtain

$$\begin{aligned} |f| &= \sum_{v \in S} \sum_{u \in S} f(u,v) + \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\ &= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\ &\qquad + \left(\sum_{v \in S} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) \right) . \end{aligned}$$

The two summations within the parentheses are actually the same, since for all vertices $x, y \in V$, the term $f(x,y)$ appears once in each summation. Hence, these summations cancel, and we have

$$\begin{aligned} |f| &= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) \\ &= f(S,T) . \end{aligned}$$

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Corollary

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G
- ***Proof***: Let (S, T) be any cut of G and let f be any flow

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \\
&\le \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T) .
\end{aligned}
$$

Max-flow min-cut theorem

- The corollary yields the immediate consequence that the value of a maximum flow in a network is bounded from above by the capacity of a minimum cut of the network
- **Theorem**: If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:
  1. f is a maximum flow in G
  2. The residual network/graph G_f contains no augmenting paths
  3. $|f| = c(S, T)$ for some cut (S, T) of G
- ***Proof***: $(1 \Rightarrow 2)$ If G_f contains an augmenting path, then we can increase the flow, contradicting the fact that f is maximum flow

Proof (Cont …)

- $(2 \Rightarrow 3)$ Suppose there is no path in G_f from s to t
- Let $S = \{v \in V \mid \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$ and $T = V - S$
- (S, T) is a cut such that $s \in S$ and $t \in T$
- Consider a pair of vertices $u \in S$ and $v \in T$
- If $(u, v) \in E$, then we must have $f(u, v) = c(u, v)$
  - If not, then $(u, v) \in E_f$ as a forward edge and hence v is reachable from u, so $v \in S$
- If $(v, u) \in E$, then we must have $f(v, u) = 0$
  - If not, then $(u, v) \in E_f$ as a back edge and hence v is reachable from u, so $v \in S$

Proof (Cont …)

- If u and v are not adjacent in G, then $f(u, v) = f(v, u) = 0$
- We thus have

$$\begin{aligned} f(S,T) &= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\ &= \sum_{u\in S}\sum_{v\in T} c(u,v) - \sum_{v\in T}\sum_{u\in S} 0 \\ &= c(S,T)\,. \end{aligned}$$

- By previous observation, $|f| = f(S, T) = c(S, T)$
- $(3 \Rightarrow 1)$ By previous corollary $|f| \leq c(S, T)$ for all cuts (S, T)
- By our assumption $f(S, T) = c(S, T)$, f must be a maximum flow

- Thank you!