Differential Equations (MA 1150)

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Lecture 5

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Section 1

Bernoulli Equation

Transforming Non-Linear into Separable ODE

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If y_1 is a non-zero solution of $y' + p(x)y = 0$, then putting $y = u(x)y_1$ in ODE, we get

$$\begin{aligned} & u'y_1 + uy_1' + puy_1 = fu^r y_1^r \\ \implies & u'y_1 = fu^r y_1^r \\ \implies & \frac{u'}{u^r} = f(x)y_1(x)^{r-1} \end{aligned}$$

On integrating, we get

$$\frac{u^{-r+1}}{-r+1} = \int f(x)(y_1(x))^{r-1}\, dx \ + c.$$

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On integrating, we get

$$\begin{aligned}
& \frac{-1}{u} = -(1+x)e^{-x} + c \\
\Longrightarrow \quad & u = \frac{1}{(1+x)e^{-x} - c}
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Thus

$$y = u(x)e^{-x} = \frac{1}{(1+x) - ce^{x}}$$

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Since $u = \dfrac{y}{y_1}$ and $y_1 = x^2$, we get

$$y^7 = 7x^{14}\left[-\frac{1}{12}x^{-12} + c\right].$$

Section 2

Orthogonal Trajectories

Curves in $\mathbb{R}^n$

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Orthogonal Trajectories Two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be orthogonal at a point of intersection (x_0, y_0) if they have perpendicular tangents at (x_0, y_0).

A curve is said to be an orthogonal trajectory of a given family of curves if it is orthogonal to every curve in the family.

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Differentiating $y = cx^2$ w.r.t. x gives $y' = 2cx$. From this, we get $c = \dfrac{y'}{2x}$.

Now eliminate the parameter using the original equation for the family of curves and finally obtain

$$y = \frac{xy'}{2}.$$

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Section 3

Existence and uniqueness of a solution

Subsection 1

Linear ODE

Existence and Uniqueness Theorem for Linear Nonhomogeneous First Order Equations

Theorem: Suppose $p(x)$ and $f(x)$ are continuous functions on an open interval (a, b), and let y_1 be any nontrivial solution of the complementary equation

$$y' + p(x)y = 0$$

on (a, b). Then:

(i) The general solution of the non-homogeneous equation

$$y' + p(x)y = f(x) \tag{1}$$

on interval (a, b) is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right). \tag{2}$$

(ii) The initial value problem $y' + p(x)y = f(x), \quad y(x_0) = y_0$ has the following unique solution on (a, b),

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Proof of (i). We shall show that if y is a solution of (1) on (a, b) then y is of the form (2) for some constant c.

Suppose $y = uy_1$ is a solution of (1) on (a, b) for some u.

We know that y_1 has no zeros on (a, b) (why?), so the function $u = y/y_1$ is defined on (a, b). Moreover, since

$$y' = -py + f \quad \text{and} \quad y_1' = -py_1,$$

$$\begin{aligned} u' &= \frac{y_1 y' - y_1' y}{y_1^2} \\ &= \frac{y_1(-py + f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$, we get

$$u = \left(c + \int f(x)/y_1(x)\, dx \right).$$

Proof of (ii). We have seen in the proof of (i) that $\int f(x)/y_1(x)\,dx$ in (2) is an arbitrary antiderivative of f/y_1.

Now we may choose the antiderivative that equals zero when $x = x_0$, and so

$$y(x_0) = y_1(x_0)\left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt\right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = \dfrac{y_0}{y_1(x_0)}$.

Therefore unique solution on (a, b) is

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Problems:

For the following IVP:

(a) $y' = \sin x \quad y(x_0) = y_0$.

(b) $y' = x\sin(1/x) \quad y(x_0) = y_0$ for all $x > 0$.

(c) $y' = x + y \quad y(x_0) = y_0$.

Using Existence and Uniqueness theorem, Justify whether the following IVP has

- at least one solution on some interval containing x_0,
- has a unique solution on some interval containing x_0.

Subsection 2

Non-linear ODE

Existence and Uniqueness of solutions of non-linear ODE

Consider a non-linear ODE: $\quad y' = f(x, y)$.

Theorem Let $D = (a, b) \times (c, d)$ be an open rectangle containing the point (x_0, y_0) and consider the IVP

$$y' = f(x, y), \text{ where } y(x_0) = y_0.$$

(a) (Existence) Assume $f(x, y)$ is continuous on D. Then IVP has at least one solution on some interval $(a_1, b_1) \subset (a, b)$ containing x_0.

(b) (Uniqueness) If both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution on some interval $(a', b') \subset (a, b)$ containing x_0.

Proof. We will not discuss the proof of this.

(1) Note the theorem says that for non-linear ODE, the solution and the interval where the solution exists, depends on the choice of our initial condition.

(2) The solution of a non-linear ODE obtained using a particular method may not be a general solution.

Example 1 For non-linear ODE $y' = 2xy^2$, our solution $y = -\frac{1}{x^2+C}$ does not give the solution $y \equiv 0$ for any value of C.

Example 2 The circle $x^2 + y^2 = C$ is an implicit solution of $yy' = x$.

For $C = -1$, it does not give any solution to ODE, since the curve $x^2 + y^2 = -1$ is empty.

The above example shows that unlike linear ODE's, not every value of C will give an actual solution.

Example

Consider the IVP

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \text{ where } y(x_0) = y_0.$$

If $f(x,y) = \frac{x^2-y^2}{1+x^2+y^2}$, then

$$\frac{\partial f}{\partial y} = ??? = \frac{-2y(1+x^2)}{(1+x^2+y^2)^2}.$$

Since $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous for all $(x,y) \in \mathbb{R}^2$, by existence and uniqueness theorem, for any $(x_0, y_0) \in \mathbb{R}^2$, IVP has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \text{ where } y(x_0) = y_0. \qquad (3)$$

Let $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$, then

$$\frac{\partial f}{\partial y} = ??? = \frac{-4x^2 y}{(x^2+y^2)^2}.$$

Notice that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$.

Assume that $(x_0, y_0) \neq (0, 0)$.

There is an open rectangle R containing (x_0, y_0) but not containing $(0, 0)$.

$f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.

By existence and uniqueness theorem, (3) has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{x+y}{x-y}, \text{ where } y(x_0) = y_0. \tag{4}$$

Let $f(x, y) = \frac{x+y}{x-y}$, then

$$\frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}.$$

Note that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all $(x, y) \in \mathbb{R}^2$ except on the line $y = x$.

Assume that $x_0 \neq y_0$.

There is an open rectangle R containing (x_0, y_0) that does not intersect with the line $y = x$.

$f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.

By existence and uniqueness theorem, (4) has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{10}{3}xy^{2/5}, \text{ where } y(x_0) = y_0. \tag{5}$$

Here

$$f(x,y) = \frac{10}{3}xy^{2/5}, \text{ and } \frac{\partial f}{\partial y} = \frac{4}{3}xy^{-3/5}.$$

- Since $f(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$, IVP (5) has atleast one solution for all $(x_0, y_0) \in \mathbb{R}^2$.
- If $y \neq 0$, then $f(x,y)$ and $\frac{\partial f}{\partial y}$ both are continuous for all $(x,y) \in \mathbb{R}^2$.
- If $y \neq 0$, there is an open rectangle R containing (x_0, y_0) s.t. $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.
- Hence IVP (5) has a unique solution on some open interval containing x_0.

Example

Consider the IVP

$$y' = \frac{10}{3}xy^{2/5}, \text{ where } y(0) = 0. \tag{6}$$

Here $\frac{\partial f}{\partial y} = \frac{4}{3}xy^{-3/5}$ is not continuous if $y = 0$.

- IVP (6) may have more than one solution on every open interval containing x_0.
- If $y \equiv 0$ is one solution of IVP (6).

Let us find a nonzero solution of ODE (6).

$$\frac{y'}{y^{2/5}} = \frac{10}{3}x \implies \frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + C).$$

This implies that

$$y(x) = (x^2 + C)^{5/3}.$$

Example continued

Note that $y(x) = (x^2 + C)^{5/3}$ is defined for all $(x, y) \in \mathbb{R}^2$ and

$$y' = \frac{5}{3}(x^2 + C)^{2/3}(2x) = \frac{10}{3}xy^{3/5}, \text{ for all}(x, y) \in \mathbb{R}^2.$$

Thus $y(x)$ is a solution on $\mathbb{R}$ for all C.

$$y(0) = 0 \implies C = 0.$$

Thus the IVP

$$y' = \frac{10}{3}xy^{2/5}, \text{ where } y(0) = 0. \tag{7}$$

has two solutions, $y_1 \equiv 0$ and $y_2(x) = x^{\frac{10}{3}}$.

Example

Consider the IVP

$$y' = \frac{10}{3} x y^{2/5}, \text{ where } y(0) = -1. \tag{8}$$

Here $f(x, y)$ and $\frac{\partial f}{\partial y} = \frac{4}{3} x y^{-3/5}$ are continuous in an open rectangle containing $(0, 1)$. Hence the IVP has a unique solution on some open interval containing $x_0 = 0$.

Let us find the unique solution and its interval of validity.

Let $y \neq 0$ be the solution of $y' = \frac{10}{3} x y^{2/5}$. Then $y(x) = (x^2 + C)^{5/3}$ and $y(0) = -1$ implies that $C = -1$.

- Thus $y(x) = (x^2 - 1)^{5/3}$ is a solution of (8) on $(-\infty, \infty)$ (Existence part).
- If $y_0 \neq 0$, then by IVP, $y' = \frac{10}{3} x y^{2/5}$, where $y(x_0) = y_0$. has a unique solution on some open interval around x_0.
- Let us check that $y(x)$ is the unique solution to the IVP on the interval $(1, 1)$.

Example

- Suppose there is another solution $w(x)$ to the IVP. Then by the existence and uniqueness theorem, $y(x) = w(x)$ for all x in a neighbourhood (a, b) around 0. Choose a and b so that this interval is the largest.
- If $< a$, then since both $y(x)$ and $w(x)$ are continuous, we get $y(a) = w(a) = A$.
- Now we apply the existence and uniqueness theorem to the IVP with the condition $y(a) = A$. This will show that $y(x) = w(x)$ in a neighbourhood around a, which contradicts the assumption that (a, b) was the largest interval on which y and w agreed.

If we take any interval (a, b) with $a << 1 < b$ then, we can define another solution

$$y_1(x) = \begin{cases} (x^2 - 1)^{5/3}, & \text{if } -1 \leq x \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Section 4

Picard's Iteration

Picard's Iteration Method

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$$y' = f(x, y), \quad y(x_0) = y_0.$$

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A solution to the IVP ODE is equivalent to a solution to the integral equation and vice versa.

Taylor series solution to the IVP ODE.

Define

$$y_1(x) = y_0,$$

$$y_2(x) = y_1(x) + y'(x_0)(x - x_0) = y_0 + f(x_0, y_0)(x - x_0).$$

Similarly,

$$y_3(x) = y_2(x) + \frac{y''(x_0)}{2}(x - x_0)^2.$$

The $(n+1)$th iterate is

$$y_{n+1}(x) = y_n(x) + \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Picard's Iteration Method

Define

$$\phi_1(x) = y_0.$$

$$\phi_2(x) = \phi_1(x) + \int_{x_0}^{x} f(s, \phi_1(s))\, ds = y_0 + \int_{x_0}^{x} f(s, y_0)\, ds$$

Next,

$$\phi_3(x) = y_0 + \int_{x_0}^{x} f(s, \phi_2(s))\, ds.$$

Similarly, we define the $(n+1)$th iterate as below

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, \phi_n(s))\, ds.$$

Note. Each ϕ_n satisfies the initial condition $\phi_n(x_0) = y_0$.

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Suppose for some n, $\phi_{n+1} = \phi_n$. Then

$$\phi_{n+1}(t) = \phi_n(t) = y_0 + \int_0^t f(s, \phi_n(s))\, ds$$

and this implies

$$\frac{d}{dt}(\phi_n(t)) = f(t, \phi_n(t))$$

is a solution of the given IVP.

In general, the sequence $\{\phi_n\}$ may not terminate. In fact, all the ϕ_n may not even be defined outside a small region in the domain.

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However, it is possible to show that, if $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ is continuous in some open rectangle (hence continuous and bounded in a smaller closed rectangle), the sequence converges to a function

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

which will be the unique solution to the given IVP.

Example. Solve the IVP:

$$y' = xy; \quad y(-1) = 1.$$

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$$\phi_3(x) = 1 + \int_{-1}^{x} s\phi_2(s)\, ds = 1 + \int_{-1}^{x} s\left(\frac{1}{2} + \frac{s^2}{2}\right)\, ds = \frac{5}{8} + \frac{x^2}{4} + \frac{x^4}{8}$$

Similarly compute higher $\phi_i(x)$ for $i \geq 4$.

Picard's Iteration Method - Example

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Picard's Iteration Method - Example

Example (continued...) We claim

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \cdots + \frac{t^{2(n-1)}}{(n-1)}.$$

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$$= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \cdots + \frac{t^{2n}}{n!}.$$

Hence $\phi_{n+1}(t)$ is the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Picard's Iteration Method - Example

What about convergence!!!

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Applying the ratio test, we get

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{k \to \infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Uniqueness (Not in course)

Let's quickly see how to get uniqueness. Suppose ϕ and ψ are solutions of

$$y' = f(x, y), \quad y(0) = 0.$$

Thus, both these satisfy the integral equation as well. Then,

$$\phi(t) - \psi(t) = \int_0^t \left(f(s, \phi(s)) - f(s, \psi(s)) \right) \, ds.$$

Thus

$$|\phi(t) - \psi(t)| \leq \int_0^t |f(s, \phi(s)) - f(s, \psi(s)| \, ds. \tag{9}$$

Since f and $\dfrac{\partial f}{\partial y}$ both are continuous on some smaller rectangle $\mathcal{R}$, there is a constant M, such that

$$|f(s, \phi(s)) - f(s, \psi(s)| \leq M \left| \phi(s) - \psi(s) \right|. \tag{10}$$

Uniqueness (Not in course)

Let

$$W(t) = \int_0^t |\phi(s) - \psi(s)| \ ds.$$

Clearly, $W(0) = 0$, $W(t) \geq 0$. Also, $W' = |\phi(t) - \psi(t)|$.

Now using (9) and (10), we get

$$W'(t) - MW(t) \leq 0.$$

Thus,

$$\left[e^{-Mt} W(t)\right] \leq 0.$$

Integrate from 0 to t and use $W(0) = 0$ to conclude $W(t) \leq 0$.

Thus

$$W(t) \equiv 0,$$

and so $W'(t) \equiv 0$. Thus $\phi(t) \equiv \psi(t)$.