Assignment 2 (ver 1)
(CS5061 Topics in Computing)

Deadline: 21st Oct.

1. Let X be a random variable taking values in $[0,1]$ such that $\mathbb{E}[X] \geq 1/4$

   Show that $\Pr[X \geq 1/8] \geq 1/7$.

   (Hint: Markov's inequality after some preparation)

2. For any $d \in \mathbb{N}$ and $w \in \mathbb{R}^{d+1}$, the degree d polynomial threshold function $h_w^d : \mathbb{R} \to \{0,1\}$ is defined as:

   $$h_w^d(x) \triangleq \operatorname{sign}\left(\sum_{i=0}^{d} w_i x^i\right)$$

   Denote the set of all polynomial threshold functions of degree d by $\mathcal{P}_d$. i.e., $\mathcal{P}_d = \{h_w^d \mid w \in \mathbb{R}^{d+1}\}$. Further, define $\mathcal{P} = \cup_d \mathcal{P}_d = \{h_w^d \mid w \in \mathbb{R}^{d+1}, d \in \mathbb{N}\}$ as the set of all polynomial threshold functions.

   i. Show that for every fixed $d \in \mathbb{N}$, the class $\mathcal{P}_d$ has finite VC-dim by calculating it.

   ii. Show that $\mathcal{P}$ does not have finite VC-dim.

3. Let the domain be $\{-1,1\}^n$. For any subset $S \subseteq [n]$, we define the parity function $\chi_S : \{-1,1\}^n \to \{-1,1\}$ as follows:

   $$\chi_S(x) = \prod_{i \in S} x_i$$

   Find the VC-dim of the class $\mathcal{H}_{\text{parity}} = \{h_S \mid S \subseteq [n]\}$.

4. Let Rec^d be the set of all axis parallel rectangles in $\mathbb{R}^d$. More formally, let $\vec{a} = (a_1^1, a_1^2, \ldots, a_d^1, a_d^2) \in \mathbb{R}^{2d}$ be such that $a_i^1 \leq a_i^2$ for all i. Then the axis parallel rectangle $h_{\vec{a}} : \mathbb{R}^d \to \{0,1\}$ is the function:

   $$h_{\vec{a}}(x) = \begin{cases} 1 \text{ if } a_i^1 \leq x_i \leq a_i^2 \ \forall i \in [d] \\ 0 \text{ otherwise} \end{cases}$$

   We can now define $\mathsf{Rec}^d = \{h_{\vec{a}} \mid a \in \mathbb{R}^{2d}\}$. Show that VC-dim$(\mathsf{Rec}^d) = 2d$.
   (Hint: Solve $d = 2$ first)

5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypothesis classes over the same domain, with VC-dim d_1 and d_2 respectively. Establish an upper bound of the VC-dim m of the class $\mathcal{H}_1 \cup \mathcal{H}_2$ in terms of $\hat{d} = \max\{d_1, d_2\}$.
   (Hint: Sauer Lemma. Then use inequality A.2 from the text)