Lecture 12 February 03, 2022

Problems [Subgroups section].

$\overset{\sigma}{(12)} \circ \overset{\tau}{(13)}$

10. (a) $x \in G$ and $\overset{\text{ord}(x)}{|x|} = rs$, then $|x^r| = ?$

(b) $x \in G$ and $|x| = n$, then $|x^r| = ?$

$|x^r| =$ depend on n, r

Exercise

Solution.

Part (a). $|x| = rs \Rightarrow x^{rs} = 1.$

$$\Rightarrow (x^r)^s = 1$$

$\leq$

Can we conclude that $|x^r| = s$?

What we can conclude that $|x^r| \leq s$.

Assume that $|x^r| = \underset{}{\overset{\text{some integer}}{m}} < s$

$$\Rightarrow (x^r)^m = 1$$

$$\Rightarrow x^{rm} = 1 = x^{rs}$$

$$\Rightarrow x^{rm-rs} = 1 \quad / \text{or.} \quad x^{rs-rm} = 1$$

$rm < rs$

$\Rightarrow$

Given $|x| = rs$

$$\Rightarrow rs - rm = 0$$

$$\Rightarrow s = m$$

$$|x^r| = s$$

Part (b). Exercise.

Quiz.

1 (B).

$$\left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| \, a \neq 0 \right\}$$

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\frac{1}{4a}$$

11. $a, b \in G$, Prove that $\underset{\substack{\parallel \\ m}}{|ab|} = |ba|$

$|ab| = m$ (say)

$\Rightarrow (ab)^m = 1$ $\quad \underbrace{ab}\cdot\underbrace{ab}\cdot \cdots \underbrace{ab} = 1$

$\Rightarrow ab \cdot ab \cdot \cdots \cdot ab = 1$

$\Rightarrow a \cdot (ba)\cdot(ba) \cdots (ba)\cdot \underline{b} = 1$

$\Rightarrow b \cdot [\qquad]$

$[\qquad]\, b^{-1}$

$\Rightarrow (ba)^m = 1$

12, 13, 14 : Good exercises. (Try these).

16. G is cyclic and $|G| = 6$.

(i) How many of ~~etc~~ its elements generate G.

$G = \langle a \rangle$ $\quad$ ($\because$ G is cyclic)

$\parallel$

$\langle e, a, a^2, a^3, a^4, a^5 \rangle$ & $a^6 = e$

Clearly a generates G $\left(\because \quad G = \langle a \rangle \right)$

How about a^2 ? Is it true that $G = \langle a^2 \rangle$

Similarly " " a^3 ?

a^4 ?

a^5 ?

(See Recording, solved from Text book)

Observe that $|a^5| = 6$

$\Rightarrow$ a^5 generates G.

Infact $\varphi(6) = 2$ two generators precisely

(ii) Similar question for

G cyclic and $|G| = 5$, or
$|G| = 8$, or
$|G| = 10$.

(iii) In general $|G| = n$

Ans: $\varphi(n)$

17. $G = \{e, \underbrace{a, b, c, \ldots\ldots}_{\text{order } 2}\}$ any other elt

Claim. G is abelion group. [$xy = yx$ for all $x, y \in G$]

$a^2 = e$ or $(ab)^2 = e$

[$\Rightarrow$ $a \cdot a = e$ $\Rightarrow$ a is its own inverse.]

$$(ab)^{-1} = b^{-1}a^{-1}$$

$$\begin{array}{c} \parallel \\ ab = ba \end{array}$$

19. Determine the number of elements of order 2 in S_4

S_4 : permutation group on $\{1, 2, 3, 4\}$.

Example. $(12) \in S_4$

$|(12)| = 2$

$1 \to 2$, $2 \to 1$

$3 \to 3$

$4 \to 4$

All such pair of two elements while other elements are mapped to itself $\binom{4}{2} = 6$

Now, consider elements of the form

$$(12)(34) \in S_4$$

All such pair also are of order 2.

$$\left.\begin{array}{l} (12)(34) \\ (13)(24) \\ (14)(23) \end{array}\right\} \text{ 3 elements.}$$

20. G: Abelian Group.

(a) Given $|a| = m$ and $|b| = n$

then $|ab| = ?$ (Exercise)

(b) If G is non-abelian, and consider elements of finite order in G.

Construct example: product of elements of finite order need not be of finite order.

Problems [Isomorphisms]

2. $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$ $\qquad$ $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$

$$x \longmapsto e^{x}$$

5. $\varphi : G \longrightarrow G'$ isomorphism of group

Claim. $\varphi^{-1} : G' \longrightarrow G$ is also an isomorphism.

Prove that $\varphi^{-1}(g_1' g_2') = \varphi^{-1}(g_1')\, \varphi^{-1}(g_2')$

for all $g_1', g_2' \in G'$

write down the details here.

7. Prove that the matrices

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are conjugate in $GL_2(\mathbb{R})$

not conjugate in $SL_2(\mathbb{R})$.

Definition. Two elements a and a' of a group G are called *conjugate* if

$$a' = b\,a\,b^{-1} \quad \text{for some } b \in G$$

$$\Updownarrow$$

[Linear Algebra, think of equivalent notion]

$$a'b = ba$$

Question. $a, a' \in G$ are given. Does such a $b \in G$ exists?

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad a' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \overset{0}{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$G = GL_2(\mathbb{R})$

$G = SL_2(\mathbb{R})$

$$\begin{bmatrix} 0 & \alpha \\ \alpha & \beta \end{bmatrix}$$

$\det(\ ) = -\alpha^2 \neq 1$

$$\boxed{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$\Downarrow$$

$$x_{11} = 0, \quad x_{12} = x_{21}$$

$$\begin{bmatrix} \underline{x_{11}} & x_{12} \\ x_{11} + x_{21} & x_{12} + x_{22} \end{bmatrix} = \begin{bmatrix} \underline{x_{11}} & x_{11} + x_{12} \\ x_{21} & \underline{x_{21} + x_{22}} \end{bmatrix}$$

$0 = x_{11}$

$$b = \begin{bmatrix} 0 & \alpha \\ \alpha & \beta \end{bmatrix}$$

$\det(b) = -\alpha^2$

$x_{11} = 0$

$x_{12} = x_{21} = \alpha$ (say)

12. G group, $\varphi: G \longrightarrow G$

$x \longmapsto x^{-1}$

(a) Prove that φ is bijective

(b) φ is an automorphism $\Longleftrightarrow$ G is abelian

Easy

Exercise.

13. (a) Let G be a group of order 4. Prove that every element of G has order 1, 2 or 4.

(b) $G = \{e, a, b, c\}$

$|a| \in \{1, 2, 4\}$

$a = e$

then G is cyclic

$|a| \in \{1,2\}$ ~~Late~~ G could be Klein four group

Why $|a| = 3$ is not possible ?

If order of a is 3, then

$$G = \{ \underbrace{1, a, a^2}_{\langle a \rangle}, b \}$$

Draw multiplication table and find why it will not work.

14. Determine the group of automorphism of the following group $(Aut(G), \circ)$ Group

(a) $(\mathbb{Z}, +)$ (b) Cyclic group of order 10 (c) S_3

Part (a): $Aut(\mathbb{Z}, +)$.

$$\varphi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}, +) \quad \text{isomorphism}$$

1_G

Let $n \in \mathbb{Z}$. $n \longmapsto n\,\underline{\underline{\varphi(1)}}$

$\left.\begin{matrix} 1 \\ -1 \end{matrix}\right\}$ isomorphisms

$0 \; \}$ 1-1 fail

If $n > 0$, $n = 1 + 1 + \cdots + 1$ (n times)

$$\varphi(n) = \varphi(1 + 1 + \cdots + 1)$$
$$= \varphi(1) + \varphi(1) + \cdots + \varphi(1)$$

[Use $\varphi(a_1 a_2 \cdots a_k) = \varphi(a_1) \cdots \varphi(a_k)$]

$$\Rightarrow \quad \varphi(n) = n\,\varphi(1)$$

If $n < 0$, then $-n > 0$

then $\varphi(-n) = \varphi((-1) + (-1) + \cdots + (-1))$

$$= n \cdot \varphi(-1) = -\varphi(n)$$

Also $\boxed{\varphi(0) = 0}$

$\qquad \parallel$

$\varphi(1 + (-1)) = 0 \quad \Rightarrow \quad \boxed{\varphi(-1) = -\varphi(1)}$

$\qquad \parallel$

$\boxed{\varphi(1) + \varphi(-1) = 0}$

Thus $\varphi(-n) = n \cdot \varphi(-1) = -n\,\varphi(1) = -\varphi(n)$

$$\boxed{\varphi(-n) = -\varphi(n)}$$

$(\mathbb{Z}/10\mathbb{Z}, +)$

generators are ~~$\{1, 3, 5, 7, 9\}$~~

$\{1, 3, 7, 9\}$ $\qquad\qquad \varphi(10)$

generators $\qquad\qquad$ generators

$1_G \longrightarrow \varphi_1(1_G) = 1$

$1_G \longmapsto \varphi_2(1_G) = 3$

$1_G \longmapsto \varphi_3(1_G) = 7$

$1_G \longmapsto \varphi_4(1_G) = 9$

Need justification why they are isomorphism

$Aut(\mathbb{Z}/10\mathbb{Z}, +) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$