Differential Equations (MA 1150)

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Lecture 3

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Linear ODE's

Definition An ODE of order n is called linear if it can be written as

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x) \text{ where } a_0(x) \neq 0.$$

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- Linear $Ay = b$
- Homogeneous $Ay = 0$ – Find all solutions
- Non-homogenous $Ay = b$ – Find one solution

Recall: Linear first order non-homogeneous differential equations

Consider the most general form of linear first order differential equation. By definition, this looks like

$$p(x)y' + q(x)y = h(x), \text{ where } p(x) \neq 0.$$

Recall: Linear first order non-homogeneous differential equations

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Dividing this by $p(x)$, let us rewrite this in the standard form

$$y' + a(x)y = f(x).$$

Assume that $a(x)$ and $f(x)$ are defined on the interval $I = (x_0 - \epsilon, x_0 + \epsilon)$.

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$$y' + a(x)y = f(x).$$

Assume that $a(x)$ and $f(x)$ are defined on the interval $I = (x_0 - \epsilon, x_0 + \epsilon)$. The homogeneous equation / complementary equation of it is

$$y' + a(x)y = 0.$$

Solving first order linear ODE

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1. Complementary function (CF) - All solutions of homogeneous.
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2. Particular integral (PI) - One solution of non-homogeneous.
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Then the general solution of $y' + a(x)y = f(x)$ is

$$y = CF + PI$$

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Recall you have done the same in linear algebra in solving $Ay = b$. For example solving $u + v = 3$ for (u, v)

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$$\begin{aligned}
y' + a(x)y &= 0 \\
\frac{dy}{dx} &= -a(x)y \\
\frac{dy}{y} &= -a(x)dx, \text{ by separating variables} \\
\ln y &= -\int a(x)dx + c, \text{ on integration} \\
y &= Ce^{-\int a(x)dx}, \text{ on integration and renaming } e^c \text{ as } C
\end{aligned}$$

Hence the CF is $y_h(x) = Ce^{-\int a(x)dx}$

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Hence the CF is $y_h(x) = Ce^{-\int a(x)dx}$
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If the co-efficient function is a constant (say α) then $y_h(x) = Ce^{-\alpha x}$.

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$$u'y_h + uy_h' + auy_h = f(x).$$

Since $y_h' + a(x)y_h = 0$, we get

$$u'y_h = f(x).$$

$$u(x) = \int_{x_0}^{x} \frac{f(s)}{y_h(s)} ds + c$$

Hence a particular integral is

$$y_p(x) = y_h(s)\left(\int_{x_0}^{x} \frac{f(s)}{y_h(s)} ds + c\right).$$

The general solution of non-homogeneous equation is

$$\begin{aligned} y(x) &= CF + PI \\ &= y_h(x) + y_p(x) \\ y_(x) &= \alpha y_h(x) + y_h(s)\left(\int_{x_0}^{x} \frac{f(s)}{y_h(s)} ds\right) \end{aligned}$$

Example

(1) Solve the ODE $y' - 2xy = 1.$

Note that $y' - 2xy = 0$ has a solution $y_1(x) = e^{x^2}$.

The solution of ODE is $y = uy_1$, where

$$u' y_1 = 1 \implies u(x) = \int_0^x e^{s^2} ds + C$$

and this implies that

$$y(x) = e^{x^2} \left(\int_0^x e^{s^2} ds + C \right).$$

(2) Solve the ODE $y' - 2xy = 1,$ where $y(0) = y_0.$ Write the solution of ODE as

$$y(x) = e^{x^2} \left(\int_0^x e^{s^2} ds + C \right).$$

Then $y(0) = y_0$ gives $C = y_0$.

ODE of order n

Definition. An ordinary differential equation of order n is an equation

$$F(x, y, y', y^{(2)}, \dots, y^{(n)}) = 0,$$

where F is a function of $(n+2)$-variables.

Definition. Suppose we are given an ODE of order n

Let $x_0 \in \mathbb{R}$ and suppose there is a function $y(x)$ which is defined in a small neighborhood $(x_0 - \epsilon, x_0 + \epsilon)$, and which is n times differentiable in this interval.

If $F(x, y, y', y^{(2)}, \dots, y^{(n)}) = 0$, then we say that the function $y(x)$ is a solution to the ODE in the interval $(x_0 - \epsilon, x_0 + \epsilon)$.

(a) In view of the above definition, we can ask what is the largest open interval around x_0 on which the given ODE has a solution

(b) If $y(x)$ is a solution to the ODE around x_0, then the largest interval around x_0 in which $y(x)$ is defined is called the interval of validity for the solution y.

Example

(1) The function

$$y = \frac{x^2}{3} + \frac{1}{x}$$

satisfies

$$xy' + y = x^2$$

on $(-\infty, 0) \cup (0, \infty)$.

(2) For IVP

$$xy' + y = x^2, \text{ where } y(1) = \frac{4}{3}$$

the interval of validity of $y(x)$ is $(0, \infty)$.

(3) For IVP

$$xy' + y = x^2, \text{ where } y(-1) = -\frac{4}{3}$$

the interval of validity of $y(x)$ is $(-\infty, 0)$.

Recall from Calculus

Definition of continuous function!

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Definition and geometrical meaning of partial derivatives!

Theorem. If $G = G(x, y)$ has continuous partial derivatives G_x and G_y, then

$$G(x, y) = c \qquad (c\text{=constant}), \tag{1}$$

is an implicit solution of the differential equation

$$G_x(x, y)\, dx + G_y(x, y)\, dy = 0. \tag{2}$$

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Thus (1) is an implicit solution of (2).

Exact ODE

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is said to be exact on open rectangle $\mathcal{R}$ if

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$$\frac{\partial G}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial G}{\partial y} = N(x,y) \tag{4}$$

for all (x,y) in $\mathcal{R}$.

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Question 1. Given an equation (3), how can we determine whether it's exact?

Question 2. If (3) is exact, how do we find a function G satisfying (4)?

When is an ODE exact ?

Theorem.(The Exactness Condition) Consider the ODE

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Assume that functions M, N, $\dfrac{\partial M}{\partial y}$, $\dfrac{\partial N}{\partial x}$ are continuous on an open rectangle $\mathcal{R} := \{(x,y) \in \mathbb{R}^2 \ : \ (x,y) \in (a,b) \times (c,d)\}$.

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Then (5) is an exact ODE on open rectangle $\mathcal{R}$ if and only if M and N satisfies the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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In other words, there exists a function $G = G(x, y)$ such that

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Which of the following ODE's are exact?

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- $(2x+3)+(2y-2)\dfrac{dy}{dx}=0$.
- $\dfrac{dy}{dx}=\dfrac{ax+by}{bx+cy}$.
- $\left(\dfrac{y}{x}+6x\right)+(\ln x-2)\,\dfrac{dy}{dx}=0$, where $x,y>0$.
- $(3x^2y+2xy+y^3)+(x^2+y^2)\dfrac{dy}{dx}=0$.

Solve $(2x+3)+(2y-2)\dfrac{dy}{dx}=0$.

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The ODE is exact, so we need to find $G(x,y)$ such that

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Exact Equations

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Integrating first equation w.r.t. x gives

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Therefore, an implicit solution to ODE is

$$G(x,y)=x^2+3x+y^2-2y+c.$$

Solve

$$(3x^2 + 6xy^2) + (6x^2y + 4y^3)\frac{dy}{dx} = 0.$$

Exact Equations

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Here $M(x, y) = 3x^2 + 6xy^2$ and $N(x, y) = 6x^2y + 4y^3$.

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Here $M(x, y) = 3x^2 + 6xy^2$ and $N(x, y) = 6x^2y + 4y^3$.

Notice that $\dfrac{\partial M}{\partial y} = 12xy \equiv 12xy = \dfrac{\partial N}{\partial x}$.

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The solution to ODE is

$$G(x, y) = x^3 + 3x^2y^2 + y^4 + c.$$

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Check that given ODE is exact. We need to find $G(x, y)$ such that

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Differentiate w.r.t. y to obtain

$$\frac{\partial G}{\partial y} = \ln |x| + \frac{dh}{dy} = \ln x - 2 \implies h(y) = -2y + c.$$

Solve $\left(\frac{y}{x} + 6x\right) + (\ln x - 2)\frac{dy}{dx} = 0$, where $x, y > 0$.

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Therefore, an implicit solution to ODE is

$$G(x, y) = y \ln|x| + 3x^2 - 2y + c.$$

Method of integrating factor

Example. Solve

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Therefore note that ODE is not exact.

Question. Can we multiply the ODE by a function $\mu(x, y)$ so that it becomes exact.

Integrating factors

Definition. We say that a function $\mu(x, y)$ is an integrating factor of ODE

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Question. What if the ODE was already exact?

The Exactness Theorem says that (7) is exact on an open rectangle $\mathcal{R}$ if μM, μN, $\frac{\partial}{\partial y}(\mu M)$, and $\frac{\partial}{\partial x}(\mu N)$ are continuous

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$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)\frac{dy}{dx} = 0. \tag{7}$$

is exact.

Question. What if the ODE was already exact?

The Exactness Theorem says that (7) is exact on an open rectangle $\mathcal{R}$ if μM, μN, $\frac{\partial}{\partial y}(\mu M)$, and $\frac{\partial}{\partial x}(\mu N)$ are continuous and

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

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Integrating factors

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It's useful to rewrite the last equation as
$$\mu(M_y - N_x) = \mu_x N - \mu_y M, \tag{8}$$

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Case study We will divide the discussion in three cases.

Case 1 If we assume that $\mu = \mu(x)$ is independent of y, then

Finding the integrating factors

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$$\begin{aligned} \mu(M_y - N_x) &= \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M \\ \implies \mu(M_y - N_x) &= \frac{\partial \mu}{\partial x} N \\ \implies \frac{1}{\mu}\frac{\partial \mu}{\partial x} &= \frac{M_y - N_x}{N} := p(x) \end{aligned}$$

If $\dfrac{M_y - N_x}{N}$ is a function of x only, say $p(x)$, then

$$\mu(x) = e^{\int p(x)\, dx}$$

is an integrating factor of ODE $\quad M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$ on $\mathcal{R}$.

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Integrating factors

Theorem Let M, N, M_y, and N_x be continuous on an open rectangle $\mathcal{R}$. Consider the ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{9}$$

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Question Is an Integrating factor unique (up to a constant) for a given ODE?

Example

Example Solve

$$\cos x \cos y \; dx + (\sin x \cos y - \sin x \sin y + y) \; dy = 0.$$

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This implies that the integrating factor is $\mu = e^y$.

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is exact. So there exists $G(x, y)$ such that

$$\frac{\partial G}{\partial x} = e^y \cos x \cos y, \text{ and } \frac{\partial G}{\partial y} = e^y(\sin x \cos y - \sin x \sin y + y)$$

Example (continued ...)

Since $\dfrac{\partial G}{\partial x} = e^y \cos x \cos y$, integrating w.r.t. x, we get

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Therefore $\dfrac{dh}{dy} = ye^y$, and hence $h(y) = ye^y - e^y + c$.

Thus

$$G(x, y) = e^y(\sin x \cos y + y - 1) + c$$

is an implicit solution of ODE.

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$$(3x^2y^3 - y^2 + y)dx + (-xy + 2x)dy = 0$$

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Here $M(x,y) = 3x^2y^3 - y^2 + y$ and $N(x,y) = -xy + 2x$. Note that

$$M_y - N_x = 9x^2y^2 - 2y + 1 - 2 = 9x^2y^2 - y - 1.$$

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Observe that

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Can we write

$$M_y - N_x = p(x)N - q(y)M$$

for some $p(x)$ and $q(y)$?

Example (continued ...)

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$$M_y - N_x = p(x)N - q(y).$$

The integrating factor is then given by

$$\mu(x, y) = e^{\int \frac{-2}{x} dx} e^{\int \frac{-3}{y} dy} = \frac{1}{x^2 y^3}.$$

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We get an exact ODE

$$\frac{1}{x^2y^3}\left[(3x^2y^3 - y^2 + y)dx + (-xy + 2x)dy\right] = 0.$$