Problems Set 1
Matrices, Linear equations and solvability

Before trying to solve each exercise, first you should be familiar with the terminologies, definitions and basic theory on that. You may read the lecture notes or lecture slides.

1. Solve (if solution exists) the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{lllll} u & +v & +w & +z & =6 \\ u & & +w & +z & =4 \\ u & & +w & & =2 \end{array}$$

   What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

   Remarks. The set of solutions of a non-homogeneous system does not form a subspace.

2. Find two points on the line of intersection of the three planes $t = 0$, $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

   Hint. It is just finding two solutions of the system: $t = 0$, $z = 0$ and $x + y + z + t = 1$.

3. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

   is *singular* (i.e., it does not have solutions at all). What value should replace the last zero on the right side to allow the system to have solutions, and what are the solutions over $\mathbb{R}$?

4. Under what condition on x_1, x_2 and x_3 do the points $(0, x_1), (1, x_2)$ and $(2, x_3)$ lie on a straight line?

5. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

6. Are the following systems equivalent:

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \end{aligned}$$

   and

$$\begin{aligned} 3x + y &= 0 \\ x + y &= 0 \end{aligned}$$

   If so, then express each equation in each system as a linear combination of the equations in the other system.

7. Set $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find all the solutions of $AX = 2X$, i.e., all X such that $AX = 2X$, where $2X$ is just componenntwise scalar multiplication.

8. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

9. Consider the system of equations $AX = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $\mathbb{R}$. Prove the following statements.

   (i) A is a zero matrix (i.e., all entries are zero) if and only if every pair (x_1, x_2) is a solution of $AX = 0$.

   (ii) $\det(A) \neq 0$, i.e., $ad - bc \neq 0$ if and only if the system has only the trivial solution.

   (iii) $\det(A) = 0$, i.e., $ad - bc = 0$ but A is a non-zero matrix (i.e., some entries are non-zero) if and only if there is $(y_1, y_2) \neq (0, 0)$ in $\mathbb{R}^2$ such that every solution of the system is given by $c(y_1, y_2)$ for some scalar c.

10. Prove that if two homogeneous systems each of two linear equations in two unknowns have the same solutions, then they are equivalent.

    Hint. You may use Q.9.

11. For the system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

    what is the triangular system after forward elimination, and what is the solution (by back substitution)? Also solve it by computing the equivalent system whose coefficient matrix is in row reduced echelon form. Verify whether both the solutions are same.

    Hint. You may follow the steps described as in the solution of Q.1.

12. Describe explicitly all 2×2 row reduced echelon matrices.

    Hint. Consider three cases that the number of non-zero rows of the matrix can be $0, 1$ or 2. When it is 1, then we will have two subcases. Think about the pivot positions.

13. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix over $\mathbb{R}$. Suppose that A is row reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

14. Find the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ using the elementary row operations.

15. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$, where I_3 is the 3×3 identity matrix. Deduce A^{-1}.

    Hint. Apply elementary row operations on $(A \,|\, I_3)$ to get A^{-1}, and keep track of the row operations to get the corresponding $E_1, E_2, \ldots, E_k$.