Problems Set 3

Linear Transformations

Throughout, U, V and W are vector spaces over $\mathbb{R}$, the set of real numbers.

1. Let $T : V \to W$ be a linear transformation. What is $T(0)$, where 0 is the zero vector in V?

Hint. $T(0) = T(0+0) = T(0) + T(0)$. Conclude that $T(0) = 0$, the zero vector in W.

2. Which of the following maps are linear? Justify your answer.

(i) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x + 2$ for every $x \in \mathbb{R}^1$.

(ii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = ax$ for every $x \in \mathbb{R}^1$, where $a \in \mathbb{R}$ is a constant.

(iii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x^2$ for every $x \in \mathbb{R}^1$.

(iv) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = \sin(x)$ for every $x \in \mathbb{R}^1$.

(v) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = e^x$ for every $x \in \mathbb{R}^1$.

(vi) $T : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $T(x_1, x_2) = x_1 x_2$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(vii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(viii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(ix) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(x) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, 1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

Solution. Verify $T(cu + dv) = cT(u) + dT(v)$ for all scalars $c, d \in \mathbb{R}$, and vectors u, v in the domain of T. If this is not true, then find particular c, d, u and v for which the above equality fails. Examples: (i) Since $T(0) = 2 \neq 0$, the map is not linear. (ii) Since $T(cu + dv) = a(cu + dv) = c(au) + d(av) = cT(u) + dT(v)$, the map is linear. (iii) Since $T(1+1) = T(2) = 4 \neq 2 = T(1) + T(1)$, the map is not linear. Similarly prove that the maps in (iv), (v), (vi) and (x) are not linear, while the maps in (vii), (viii) and (ix) are linear.

3. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$, $u_3 = (1, -1)$ and $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2, 3$?

Solution. The answer is 'no'. If possible, suppose there is a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2, 3$. Then T should respect every linear combination. Note that $u_3 = u_2 - u_1$. After applying T on it, we should get $v_3 = v_2 - v_1$, which is not true, a contradiction.

4. Composition of linear maps: Let $T : U \to V$ and $S : V \to W$ be linear maps. The composition $S \circ T : U \to W$ is defined by $(S \circ T)(u) := S(T(u))$ every $u \in U$. Show that the map $S \circ T : U \to W$ is linear.

Solution. $(S \circ T)(c_1 u_1 + c_2 u_2) = S(T(c_1 u_1 + c_2 u_2)) = S(c_1 T(u_1) + c_2 T(u_2)) = c_1 (S \circ T)(u_1) + c_2 (S \circ T)(u_2)$ for all scalars c_i and vectors $u_i \in U$.

5. Matrix multiplication and composition of linear maps: Let A, B be matrices of order $l \times m$ and $m \times n$ respectively. Consider the corresponding linear maps $T_A : \mathbb{R}^m \to \mathbb{R}^l$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ given by A and B respectively. Prove that the matrix representation of the composition $T_A \circ T_B : \mathbb{R}^n \to \mathbb{R}^l$ is AB, or equivalently, prove that $T_A \circ T_B = T_{AB}$.

Solution. For every $X \in \mathbb{R}^n$, since $(T_A \circ T_B)(X) = T_A(T_B(X)) = T_A(BX) = ABX = T_{AB}(X)$, the map $(T_A \circ T_B)$ is represented by the matrix AB.

6. **Application of composition of maps:** Show that the matrix multiplication is associative.

Hint. Let A, B, C be matrices of order $k \times l$, $l \times m$ and $m \times n$ respectively. To show that $(AB)C = A(BC)$, consider T_A, T_B and T_C. Next use Q.5 and the fact that the composition of maps is associative.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Is it true that if we know $T(v)$ for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.

Hint. See what we have proved in Lecture 6. Try to analyze the statement when $n = 2$.

8. Define a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$, where $a_{ij} \in \mathbb{R}$ are constants. Is T linear? If yes, then write its matrix representation.

Hint. See the theorem concerning matrix representation of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ proved in Lecture 6.

9. Deduce from Q.8 that the map $S : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$S(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, x_2 + x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ is linear. Compute the range space and null space of S. Deduce the rank and nullity of S. Verify the Rank-Nullity Theorem. Conclude from the rank (resp. from the nullity), whether S is an isomorphism.

Hint. Write the matrix representation (say, A) of the linear map S. Observe that the null space of S is same as the solution space of the system $AX = 0$. Moreover, the range space of S is same as the column space of A. Recall the equivalent conditions for a linear operator to be an isomorphism (shown in Lecture 7).

Left/right inverse of an $n \times n$ matrix A. An $n \times n$ matrix B (resp., C) is called a left (resp., right) inverse of A if $BA = I_n$ (resp., $AC = I_n$).

If A has a left-inverse B and a right-inverse C, then the two inverses are equal: $B = B(AC) = (BA)C = C$. If this is the case, we say that A is invertible.

From the row rank and the column rank of A, we can actually decide when A has a left/right inverse; see Q.10 and Q.11.

10. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has full column rank, i.e., column rank of A is n.

(ii) The system $AX = b$ has at least one solution X for every $b \in \mathbb{R}^n$.

(iii) The rank of the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ (defined by $T_A(X) = AX$) is n.

(iv) A has a right-inverse C, i.e., $AC = I_n$.

Solution. (i) $\Leftrightarrow$ (ii). A has full column rank $\Leftrightarrow$ column space of A is $\mathbb{R}^n$ $\Leftrightarrow$ every vector $b \in \mathbb{R}^n$ can be written as a linear combination of the columns of A $\Leftrightarrow$ the system $AX = b$ has at least one solution X for every $b \in \mathbb{R}^n$ (because for some $X \in \mathbb{R}^n$, AX is nothing but a linear combination of the columns of A).

(ii) $\Leftrightarrow$ (iii): Note that (ii) is equivalent to that every $b \in \mathbb{R}^n$ has a preimage $X \in \mathbb{R}^n$ via the map $T_A : \mathbb{R}^n \to \mathbb{R}^n$. Thus (ii) $\Leftrightarrow$ Image$(T_A) = \mathbb{R}^n$ $\Leftrightarrow$ rank$(T_A) = n$.

(ii) $\Rightarrow$ (iv). Let $\{e_i : 1 \leqslant i \leqslant n\}$ be the standard basis of $\mathbb{R}^n$. By (ii), for every e_i, there is $v_i \in \mathbb{R}^n$ such that $Av_i = e_i$. Set $C = [v_1\ v_2\ \cdots\ v_n]$, an $n \times n$ matrix with v_i as the ith column. It follows that $AC = I_n$.

(iv) $\Rightarrow$ (ii). Let $b \in \mathbb{R}^n$. Since $AC = I_n$, we have $A(Cb) = (AC)b = I_n b = b$, i.e., Cb is a solution of the system $AX = b$.

11. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has full row rank, i.e., row rank of A is n.

(ii) A has a left-inverse B, i.e., $BA = I_n$.

Hint. (i) $\Leftrightarrow$ (ii). Note that the row space of A is same as the column space of A^t (the transpose of A). So you may use the equivalence of (i) and (iv) in Q.10 for A^t.

12. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has a left-inverse.

(ii) A has a right-inverse.

(iii) A is invertible.

Hint. You may use Q.10, Q.11 and the fact that row rank(A) = column rank(A).

13. Let $u_1 = (1,2)$, $u_2 = (2,1)$ and $v_1 = (1,1)$, $v_2 = (0,1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1,2$? If yes, then write the matrix representation of T.

Solution. Two approaches: **(1st)** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix representation. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These yield the following system of equations:

$$\begin{matrix} a + 2b = 1 \\ 2a + b = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} c + 2d = 1 \\ 2c + d = 1 \end{matrix}$$

After solving these systems, one obtains $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$.

(2nd) Since $u_1 = (1,2)$, $u_2 = (2,1)$ are linearly independent, they form a basis of $\mathbb{R}^2$. Hence there is a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1,2$. The matrix representation of $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $[T(e_1)\ T(e_2)]$, which is a 2×2 matrix with the columns $T(e_1)$ and $T(e_2)$. Write both e_1 and e_2 as linear combinations of u_1 and u_2, to get the vectors $T(e_1)$ and $T(e_2)$. Let $e_1 = x_1 u_1 + x_2 u_2$ and $e_2 = y_1 u_1 + y_2 u_2$. Then

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is invertible with the inverse $\frac{-1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, both the systems have unique solutions given by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$.

Hence $T(e_1) = T(x_1u_1 + x_2u_2) = x_1T(u_1) + x_2T(u_2) = x_1v_1 + x_2v_2 = \frac{-1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$.

Similarly $T(e_2) = \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{-1}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$. So the matrix representation of T is $\begin{pmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$.

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Let u, v be two non-zero vectors such that $T(u) = 0$ and $T(v) = 0$. What are the possibilities of nullity of T? What about rank of T?

Solution. Since $\text{NullSpace}(T) \subseteq \mathbb{R}^2$, $\text{nullity}(T) \leqslant 2$. But two non-zero vectors are there in $\text{NullSpace}(T)$. Note that u, v may not be linearly independent. In any case, $\text{NullSpace}(T) \neq 0$. Hence $\text{nullity}(T) \geqslant 1$. Thus the possibilities of nullity of T are 1 or 2. Therefore, by the Rank-Nullity Theorem, the possibilities of rank of T are 1 or 0.