

# Binomial Heaps

# Outline for this Week

- ***Binomial Heaps (Today)***
  - A simple, flexible, and versatile priority queue.
- ***Lazy Binomial Heaps (Today)***
  - A powerful building block for designing advanced data structures.
- ***Fibonacci Heaps (Thursday)***
  - A heavyweight and theoretically excellent priority queue.

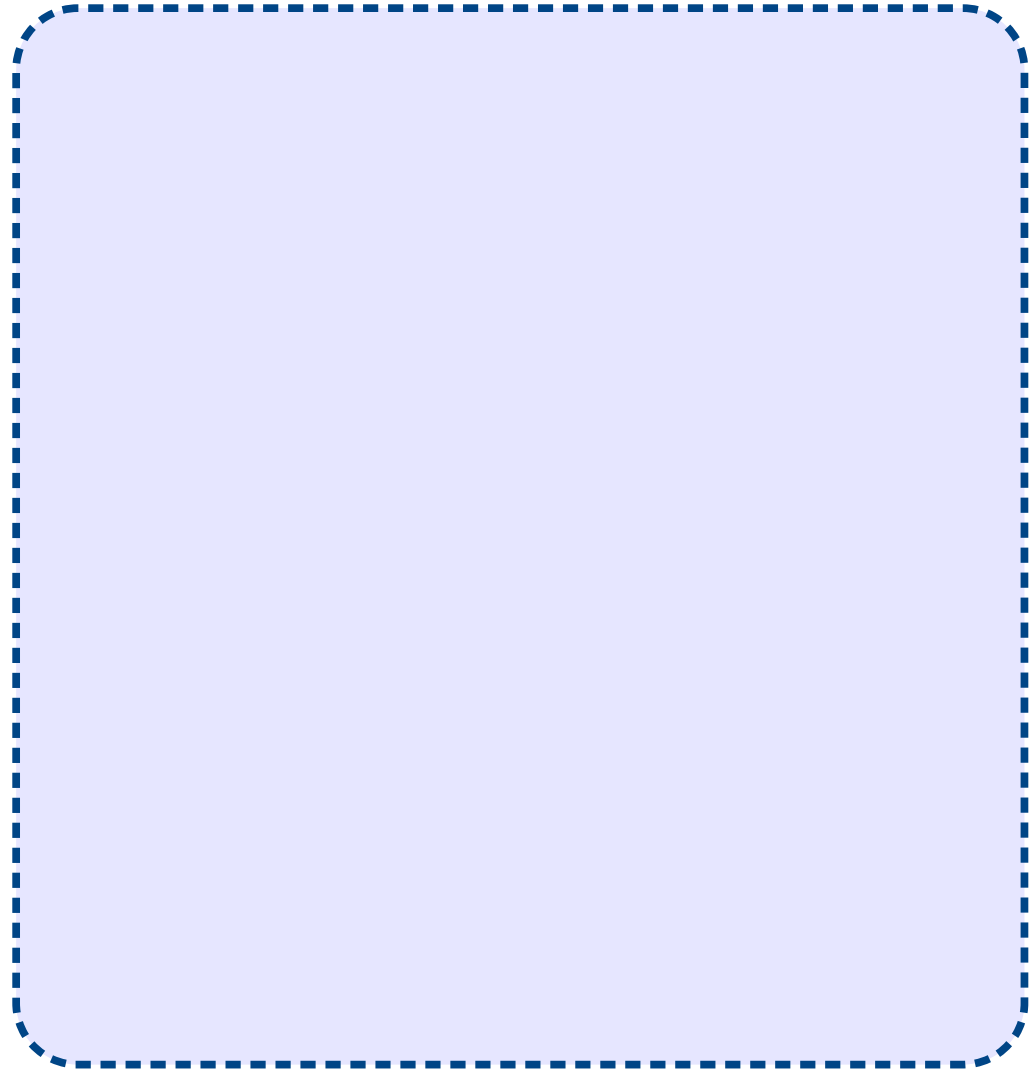
## ***Review:*** Priority Queues

# Priority Queues

- A **priority queue** is a data structure that supports these operations:
  - $pq.\text{enqueue}(v, k)$ , which enqueues element  $v$  with key  $k$ ;
  - $pq.\text{find-min}()$ , which returns the element with the least key; and
  - $pq.\text{extract-min}()$ , which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.

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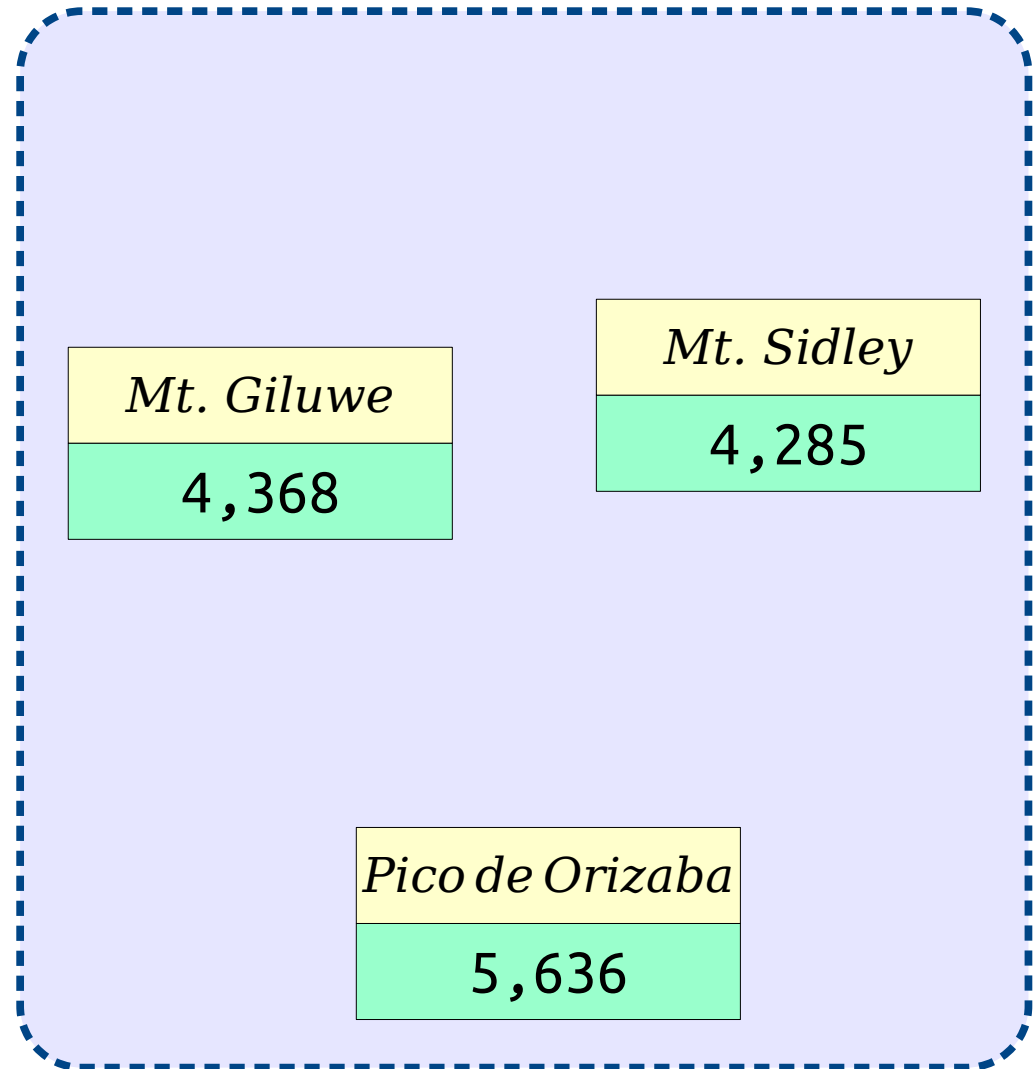
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*Pico de Orizaba*

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# Priority Queues

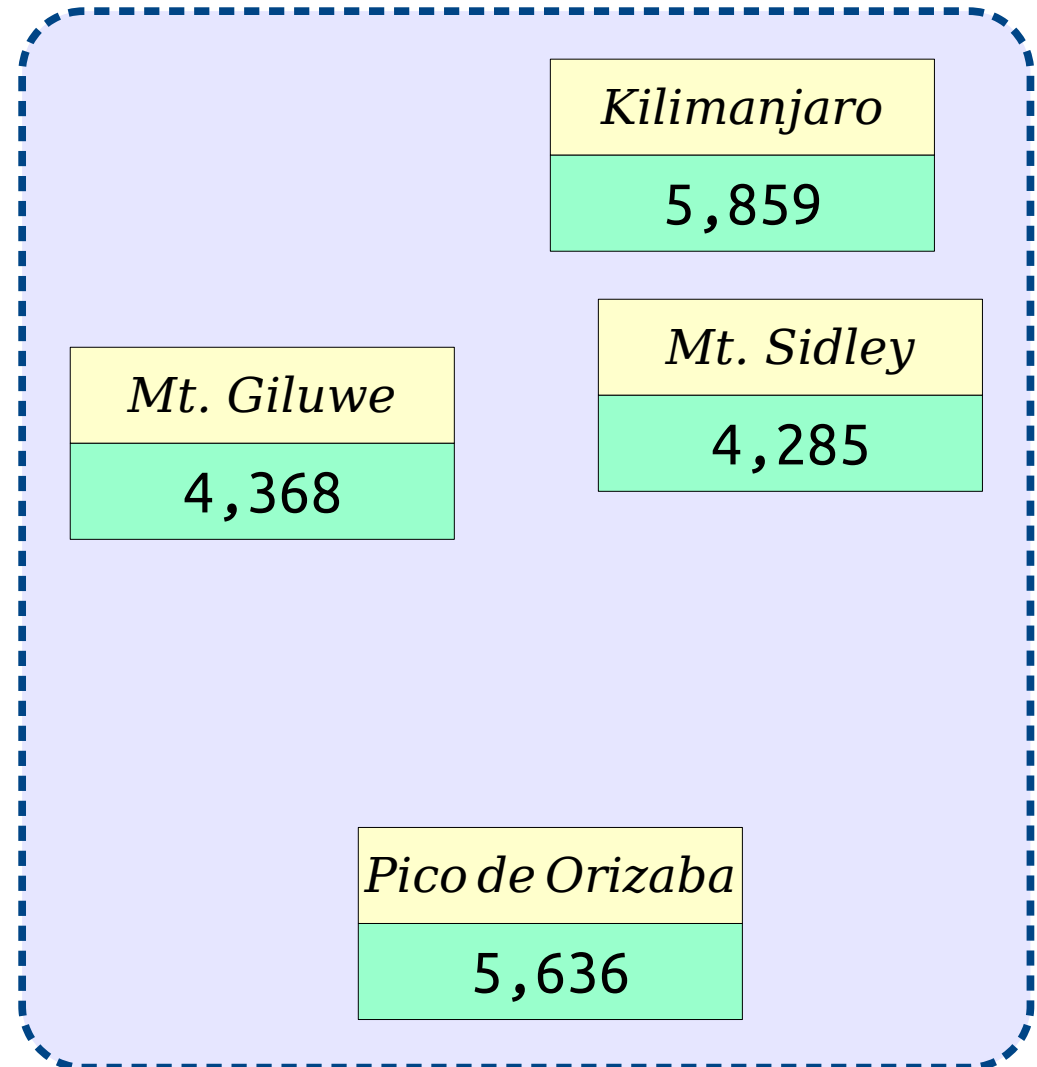
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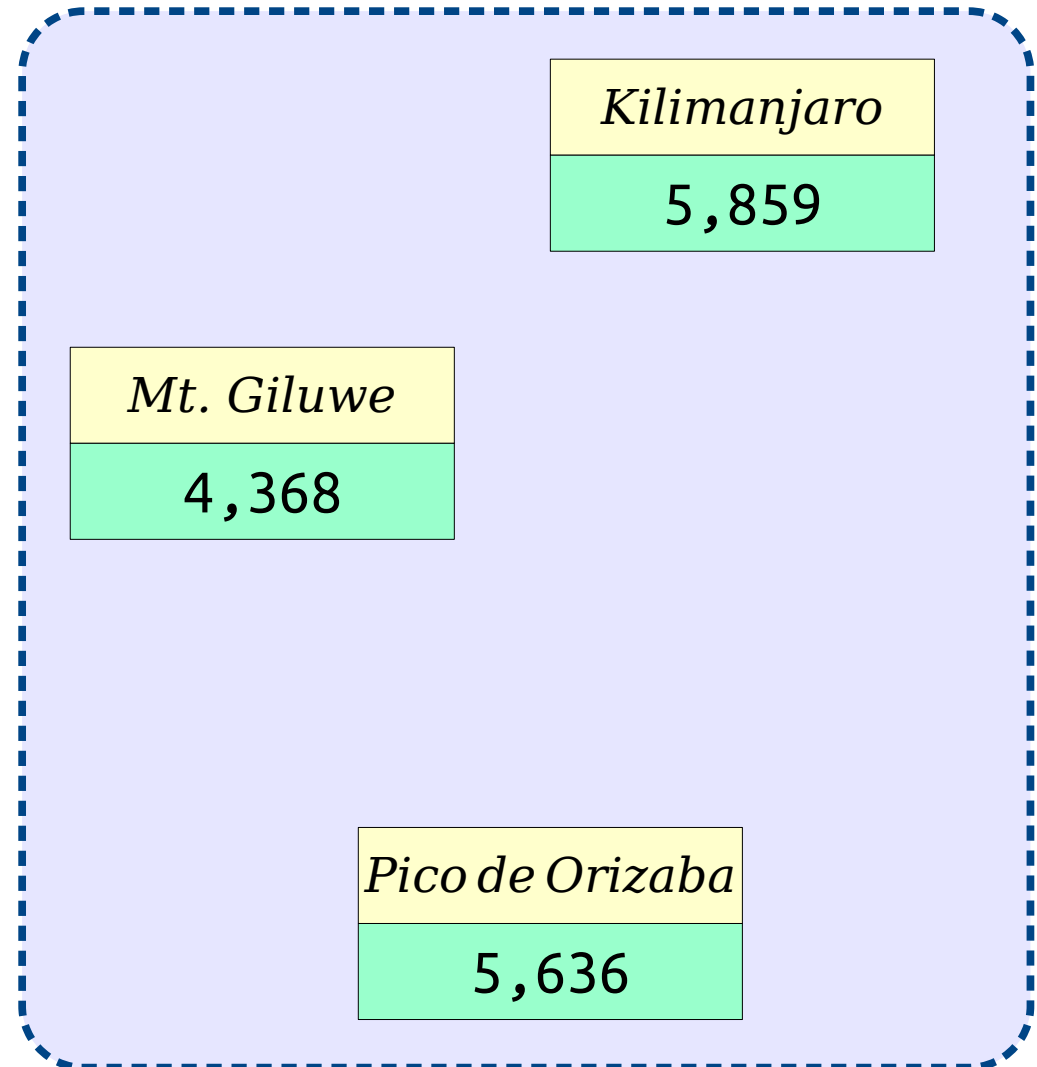
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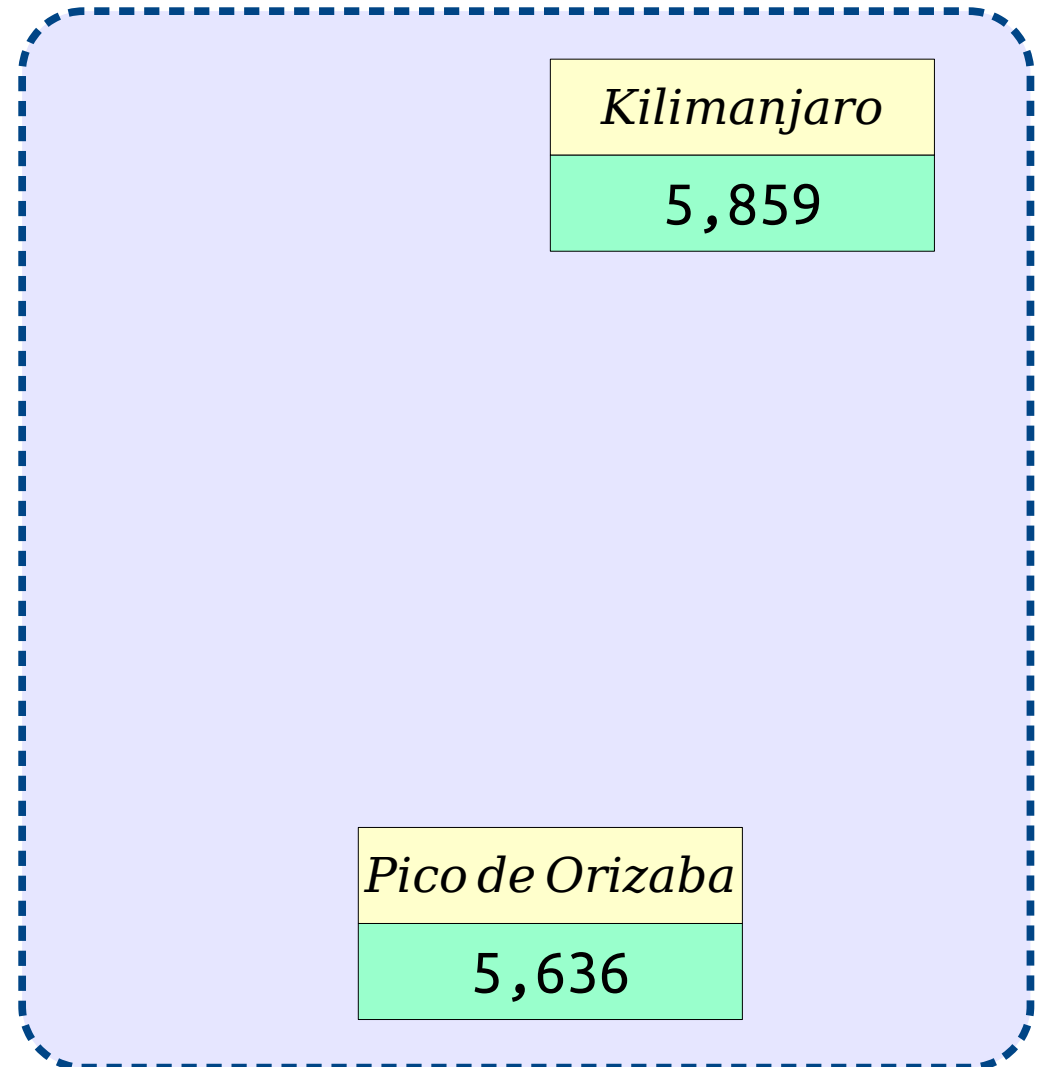
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# Binary Heaps

- Priority queues are frequently implemented as **binary heaps**.
  - **enqueue** and **extract-min** run in time  $O(\log n)$ ; **find-min** runs in time  $O(1)$ .
- These heaps are surprisingly fast in practice. It's tough to beat their performance!
  - $d$ -ary heaps can outperform binary heaps for a well-tuned value of  $d$ , and otherwise only the **sequence heap** is known to specifically outperform this family.
  - (Is this information incorrect as of 2020? Let me know and I'll update it.)
- In that case, why do we need other heaps?

# Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - **meld**( $pq_1, pq_2$ ): Destroy  $pq_1$  and  $pq_2$  and combine their elements into a single priority queue. (*MSTs via Cheriton-Tarjan*)
  - $pq$ .**decrease-key**( $v, k'$ ): Given a pointer to element  $v$  already in the queue, lower its key to have new value  $k'$ . (*Shortest paths via Dijkstra, global min-cut via Stoer-Wagner*)
  - $pq$ .**add-to-all**( $\Delta k$ ): Add  $\Delta k$  to the keys of each element in the priority queue, typically used with **meld**. (*Optimum branchings via Chu-Edmonds-Liu*)
- In lecture, we'll cover binomial heaps to efficiently support **meld** and Fibonacci heaps to efficiently support **meld** and **decrease-key**.
- Assuming the TAs sign off on it, you'll design a priority queue supporting **meld** and **add-to-all** on the next problem set.

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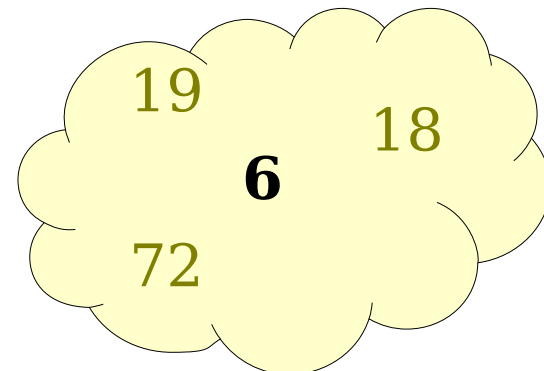
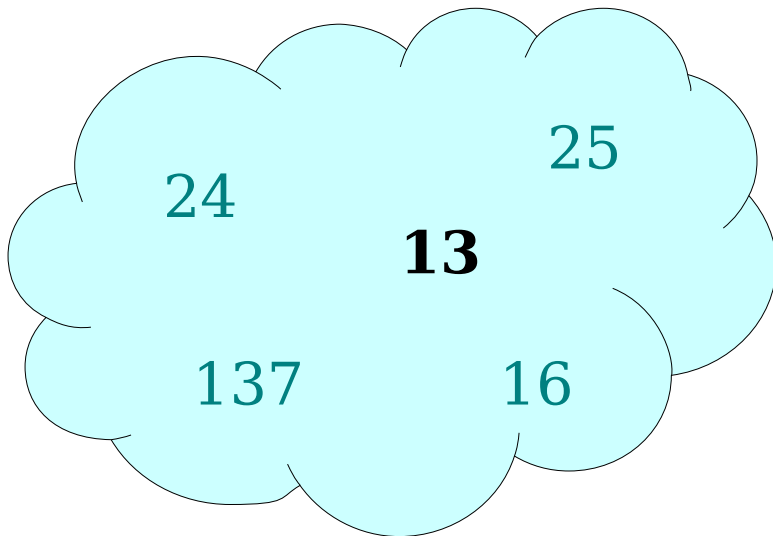
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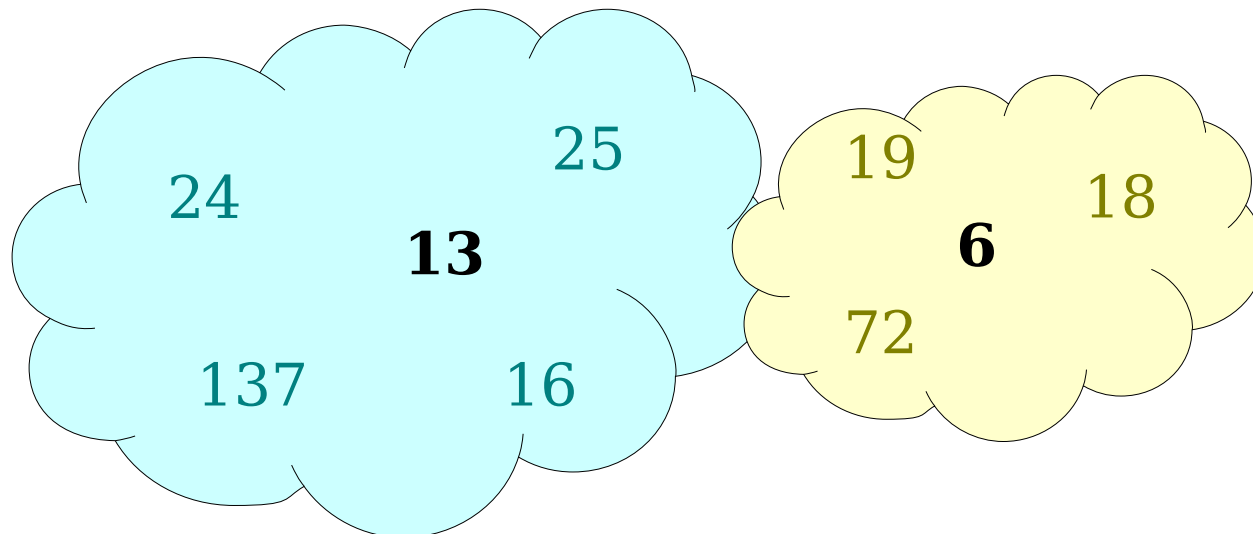
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- ***meld***( $pq_1$ ,  $pq_2$ ) destructively modifies  $pq_1$  and  $pq_2$  and produces a new priority queue containing all elements of  $pq_1$  and  $pq_2$ .





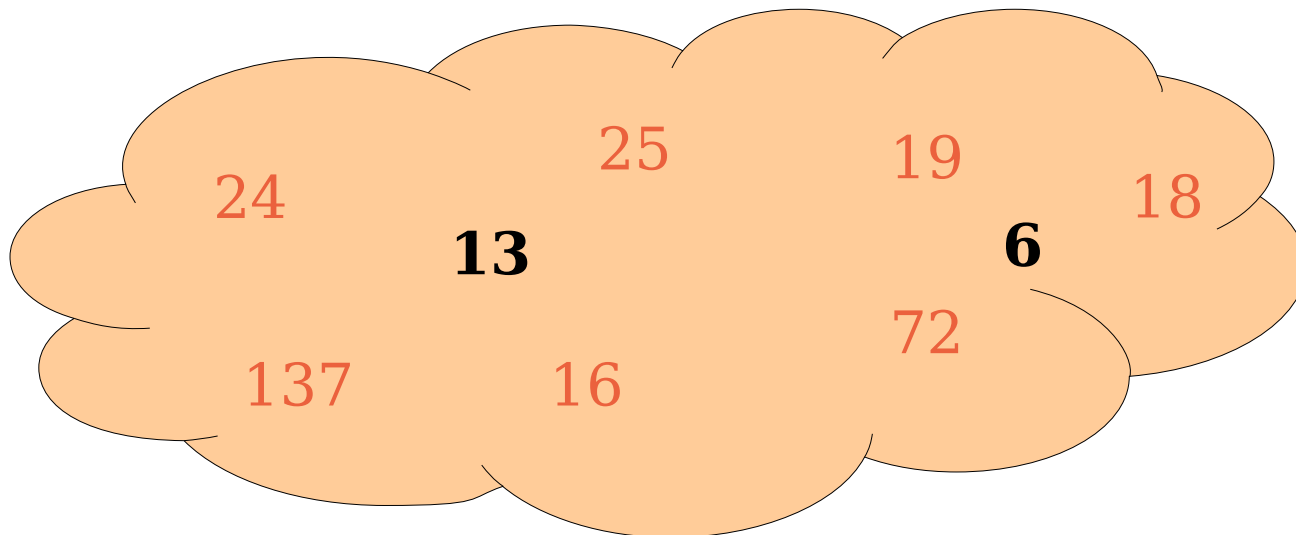
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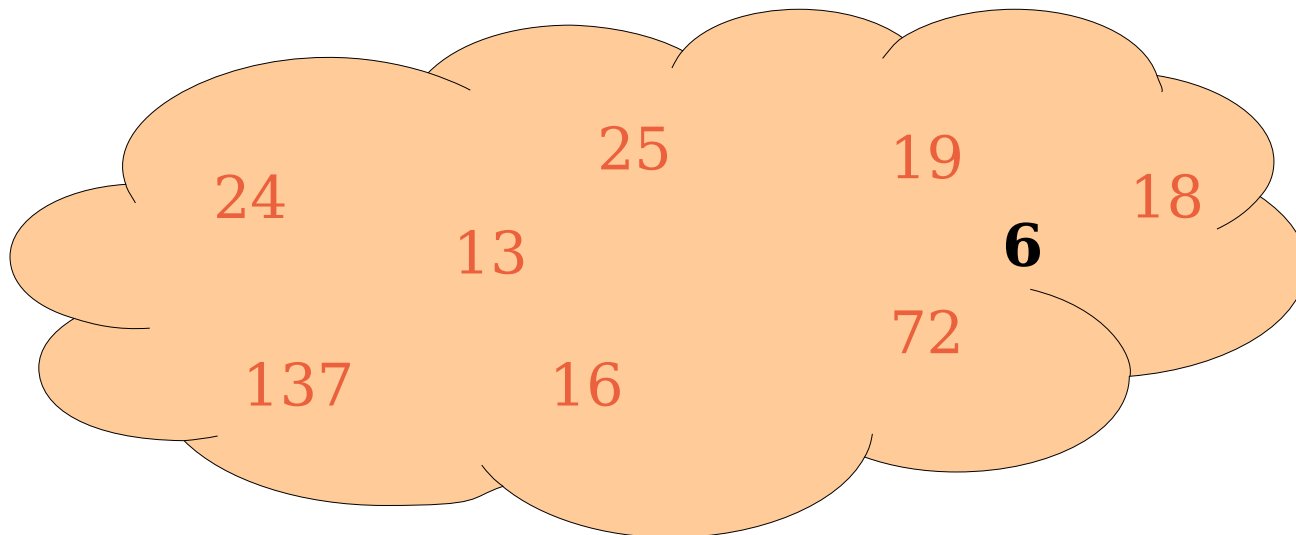
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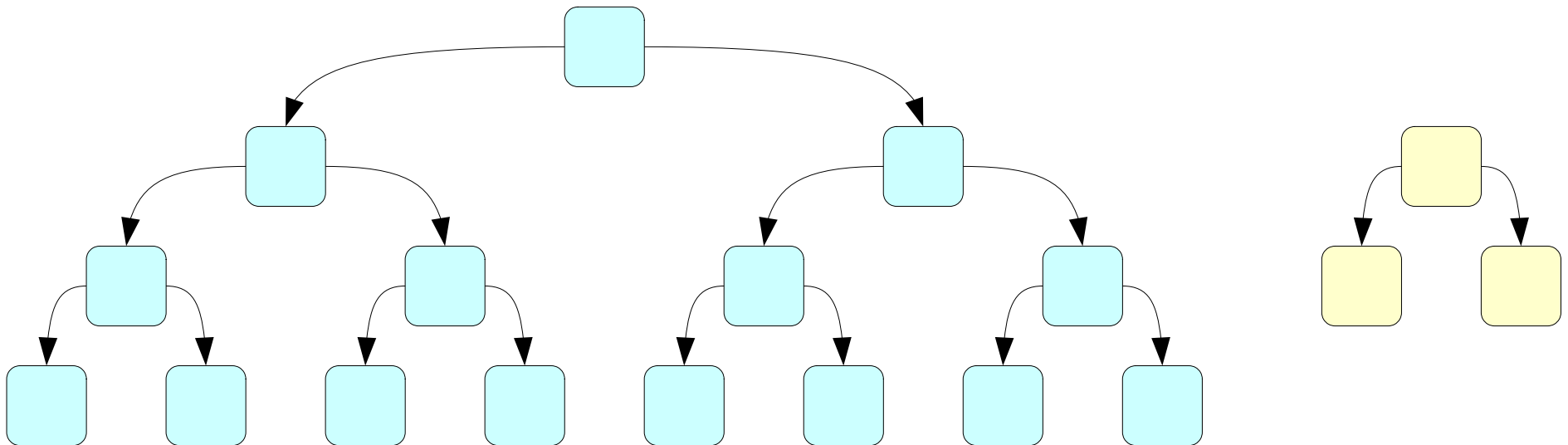
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# Efficiently Meldable Queues

- Standard binary heaps do not efficiently support ***meld***.
- ***Intuition***: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.



# Adding Binary Numbers

- Given the binary representations of two numbers  $n$  and  $m$ , we can add those numbers in time  $O(\log m + \log n)$ .

***Intuition:***

Writing out  $n$  in any “reasonable” base requires  $\Theta(\log n)$  digits.

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	1	0	1	1	0
+		1	1	1	1
<hr/>					

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A diagram illustrating the addition of two binary numbers. The numbers are aligned vertically, with a plus sign to the left. The top number is 10110 and the bottom number is 01111. A horizontal line is drawn under the bottom number. The sum is shown below the line as 10101. A carry of 1 is shown above the third column from the right. The fourth column from the right (the third column from the left) is highlighted in yellow, showing a 1 in both the top and bottom numbers, and a 1 in the sum below the line.

			1		
	1	0	1	1	0
+		1	1	1	1
<hr/>					
				0	1

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	1	0	1	1	0
+		1	1	1	1
<hr/>					
			0	1	

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A diagram illustrating the addition of two binary numbers, 1010 and 0101, resulting in 1101. The numbers are aligned by their least significant bits. The addition is performed column by column from right to left. The first column (rightmost) has 0 + 1 = 1. The second column has 1 + 0 = 1. The third column has 1 + 1 = 10, where 0 is written below the line and a carry of 1 is written above. The fourth column has 0 + 1 plus the carry of 1, resulting in 10, where 0 is written below the line and a carry of 1 is written above. The fifth column (leftmost) has 1 plus the carry of 1, resulting in 10, where 0 is written below the line and a final carry of 1 is written above. The final result is 1101.

		1	1		
	1	0	1	1	0
+		1	1	1	1
<hr/>					
			1	0	1

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A diagram illustrating the addition of two binary numbers, 10110 and 01111, resulting in 10011. The numbers are aligned by their least significant bits. The addition is performed column by column from right to left. The first column (rightmost) has 0 + 1 = 1. The second column has 1 + 1 = 0 with a carry of 1. The third column has 1 + 1 + 1 (carry) = 1 with a carry of 1. The fourth column has 0 + 1 + 1 (carry) = 0 with a carry of 1. The fifth column (leftmost) has 1 + 0 + 1 (carry) = 0 with a carry of 1. The final carry of 1 is placed to the left of the result. The result is 10011. The cells containing 0 and 1 in the second row are highlighted in yellow.

		1	1		
	1	0	1	1	0
+		1	1	1	1
<hr/>					
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A diagram illustrating the addition of two binary numbers, 10110 and 01111, resulting in 10101. The numbers are aligned by their least significant bits. A plus sign (+) is placed to the left of the first number. Above the first number, the carry bits from the previous steps are shown as 1, 1, and 1. The second number is aligned under the first. The result is shown below a horizontal line. The second column from the left (the second bit from the right) is highlighted in yellow, showing a 0 in the first row and a 1 in the second row, with a carry-out of 1 to the third column.

	1	1	1		
	1	0	1	1	0
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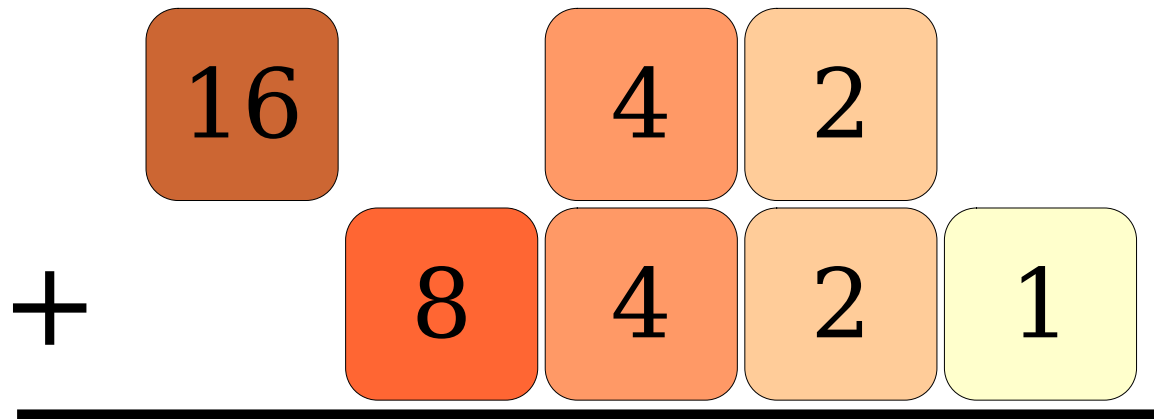
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- Represent  $n$  and  $m$  as a collection of “packets” whose sizes are powers of two.
- Adding together  $n$  and  $m$  can then be thought of as combining the packets together, eliminating duplicates

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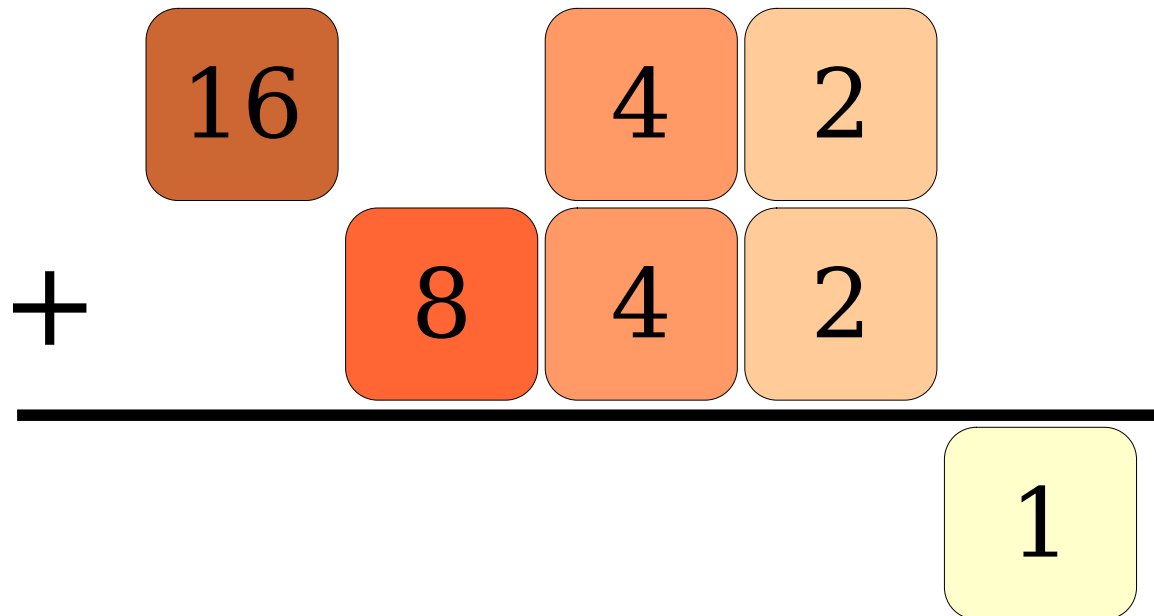
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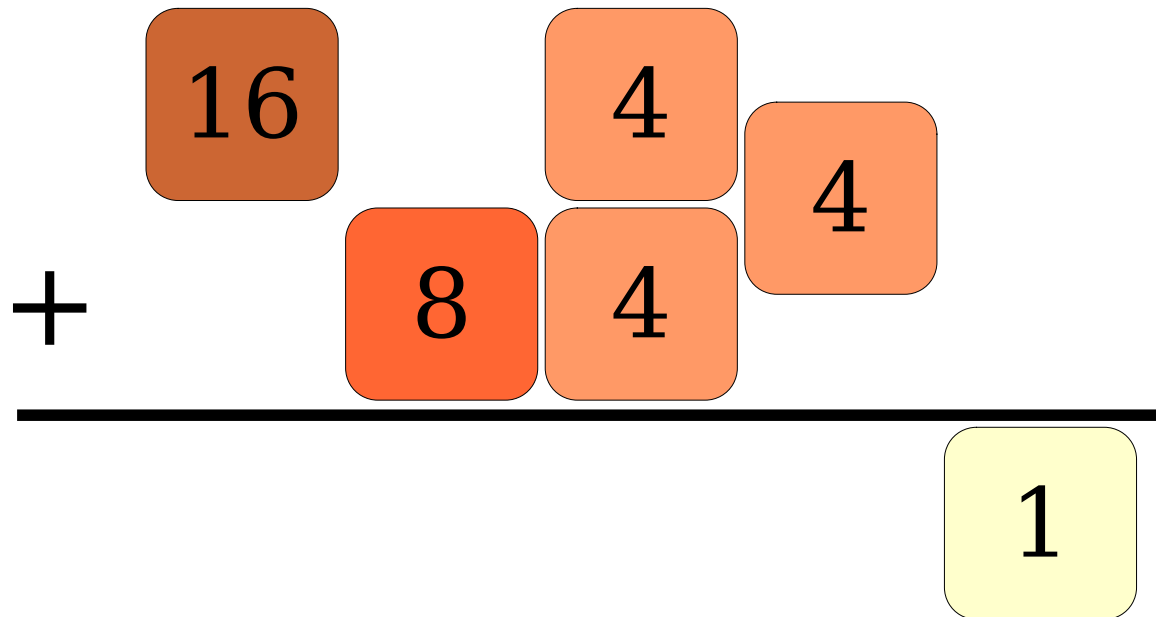
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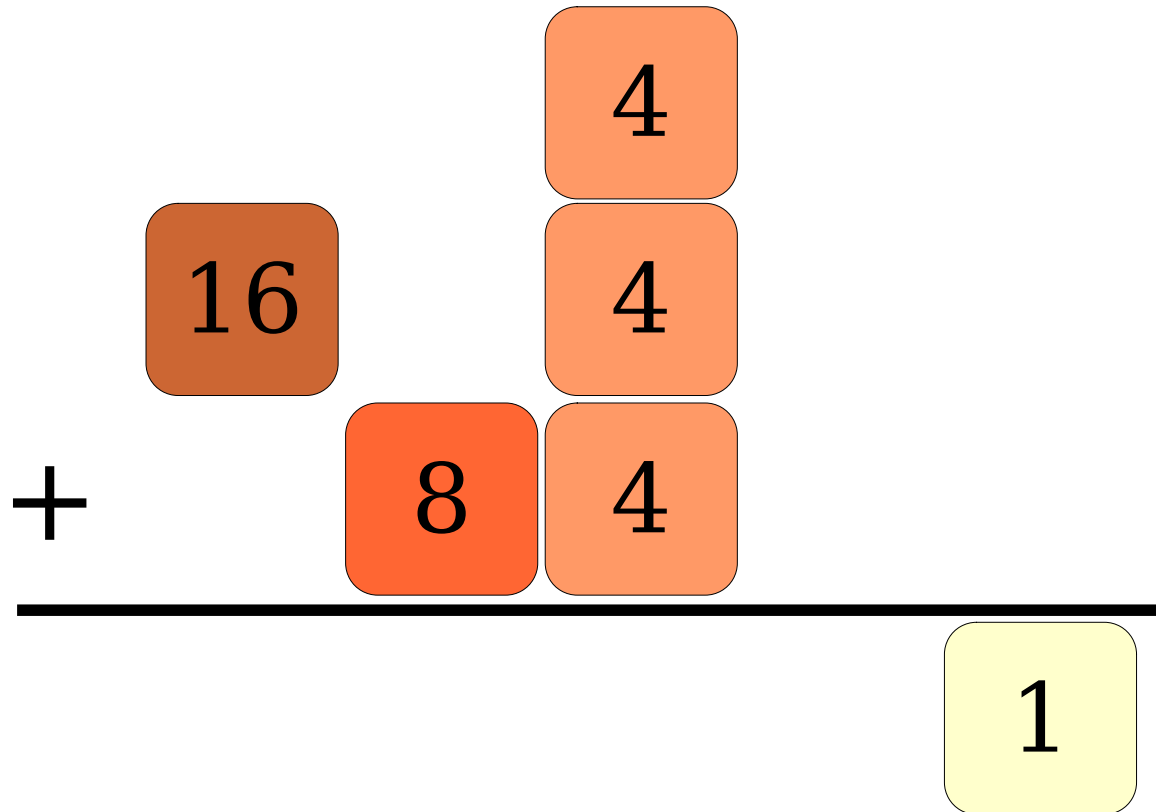
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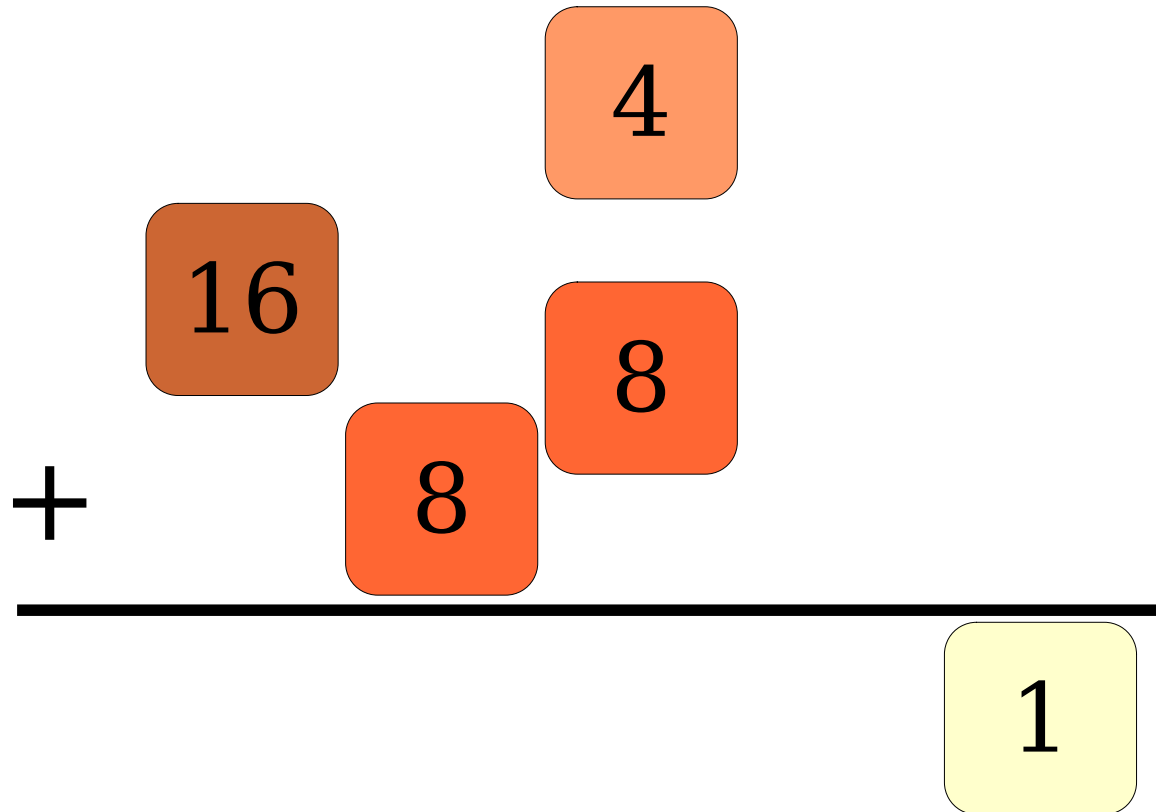
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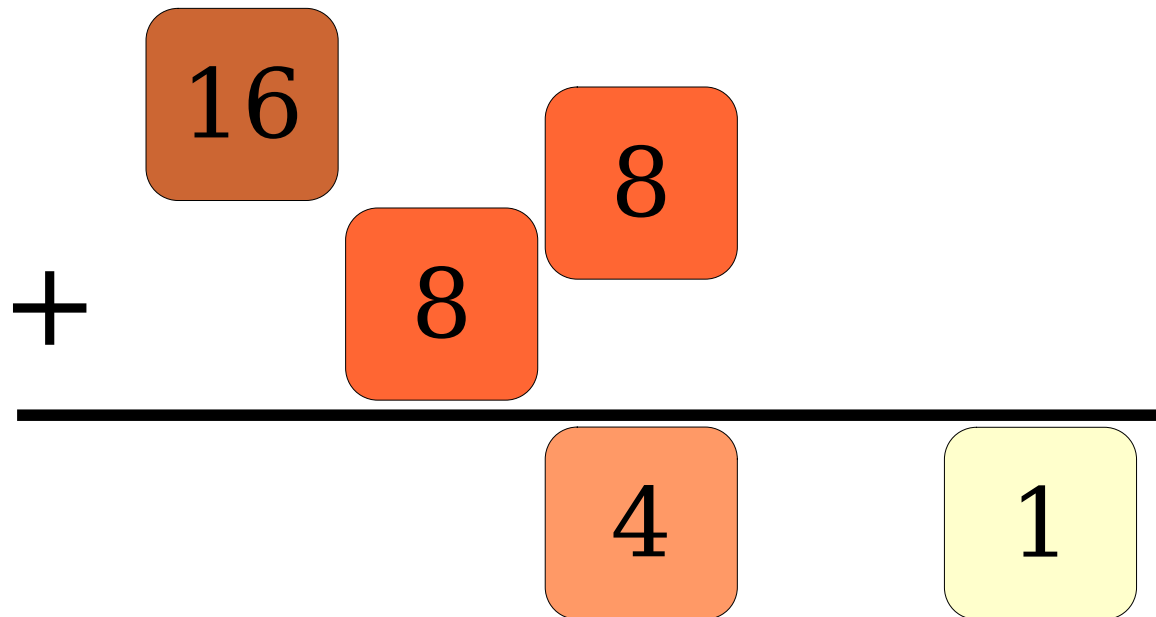
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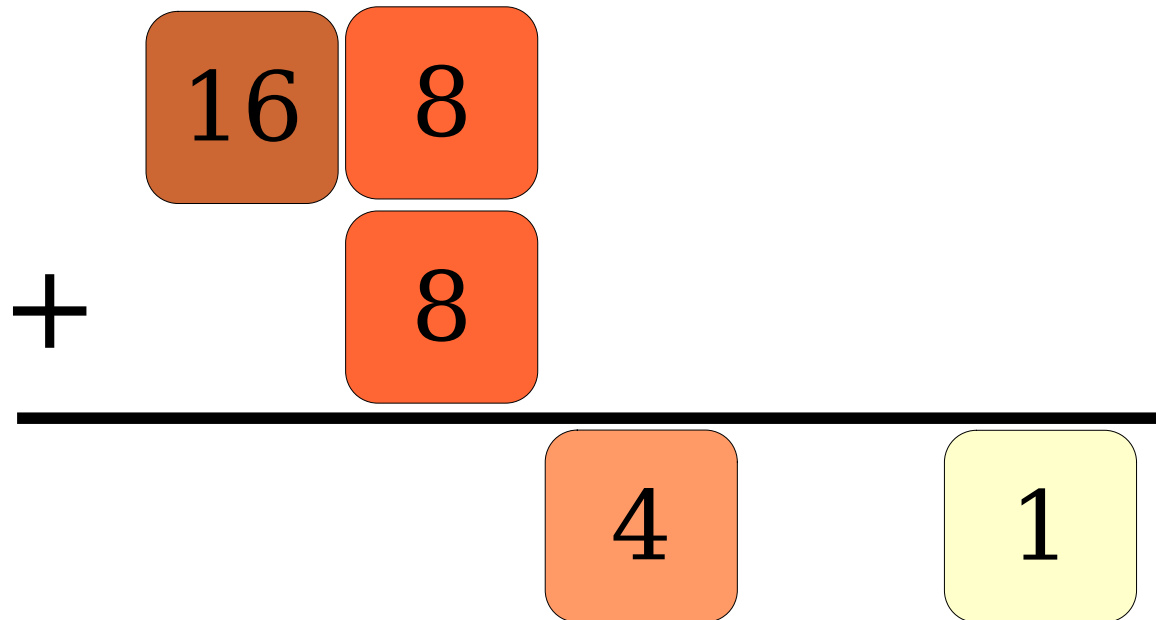
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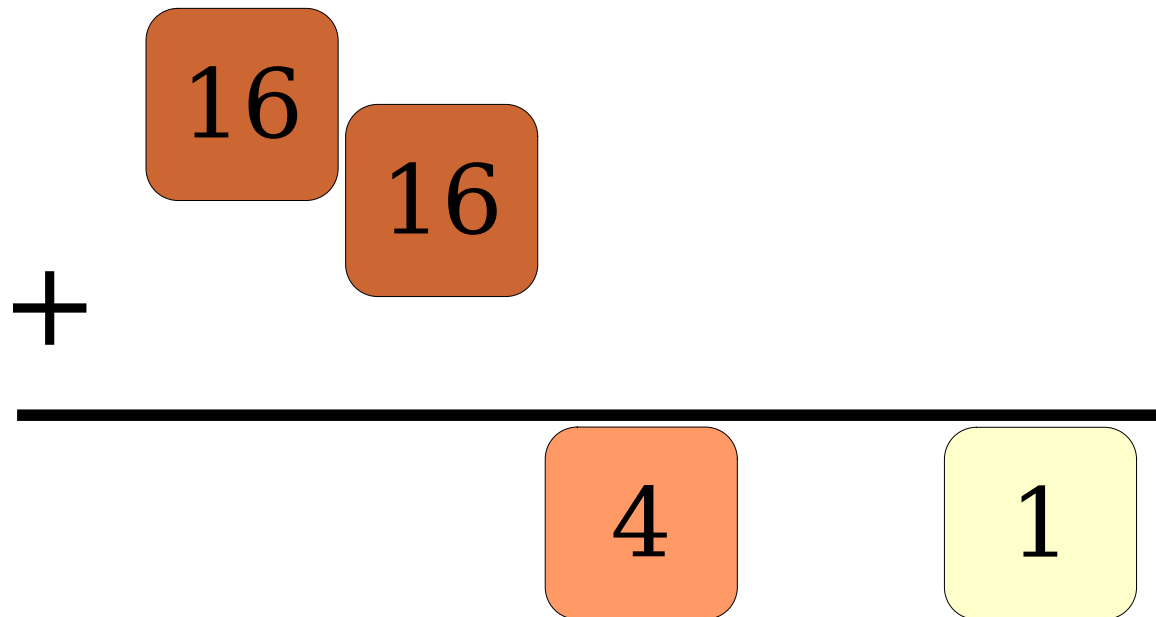
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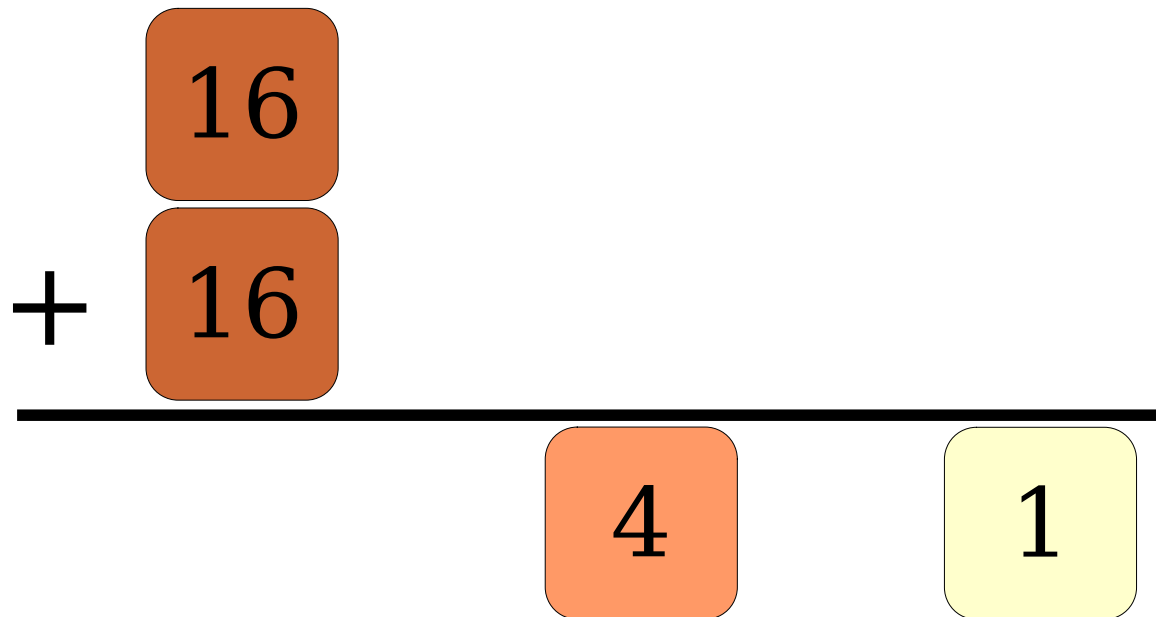
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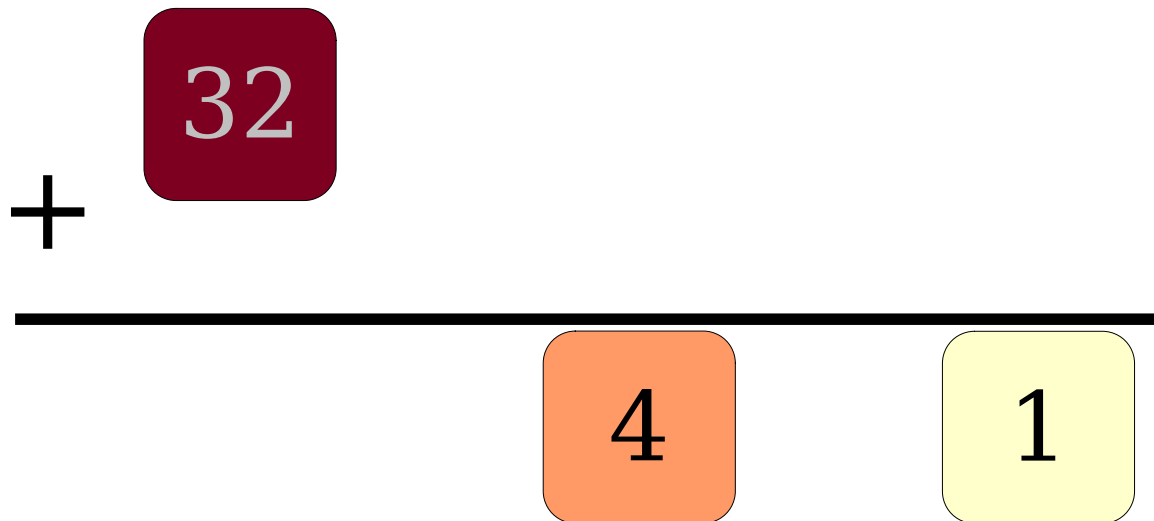
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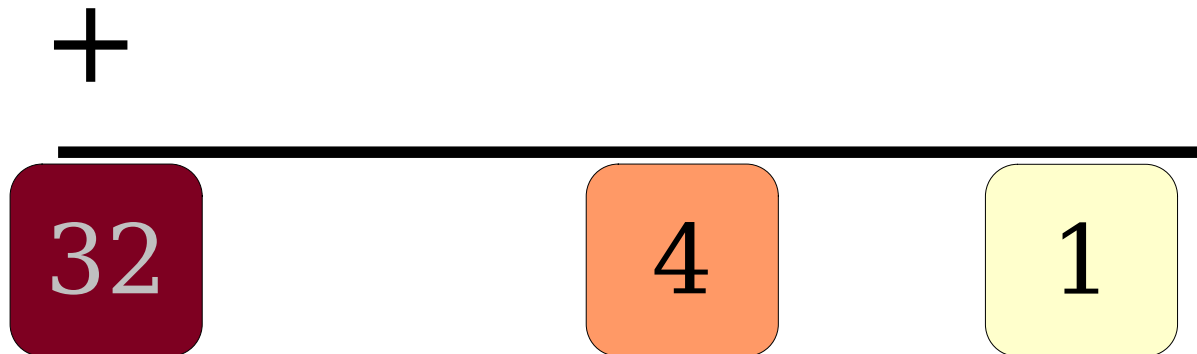
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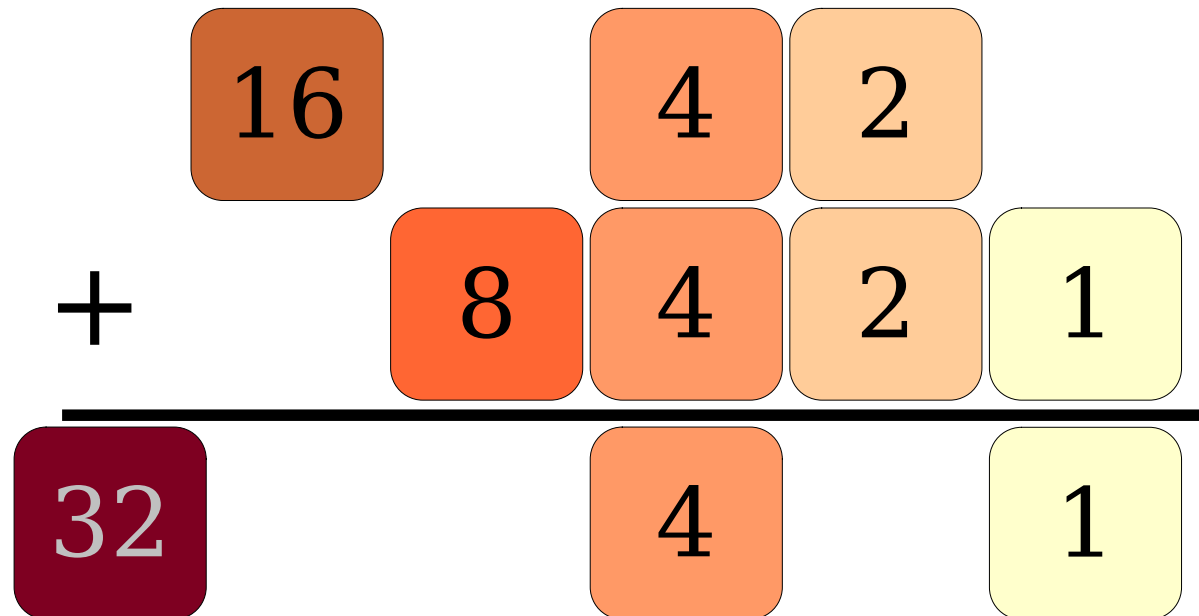
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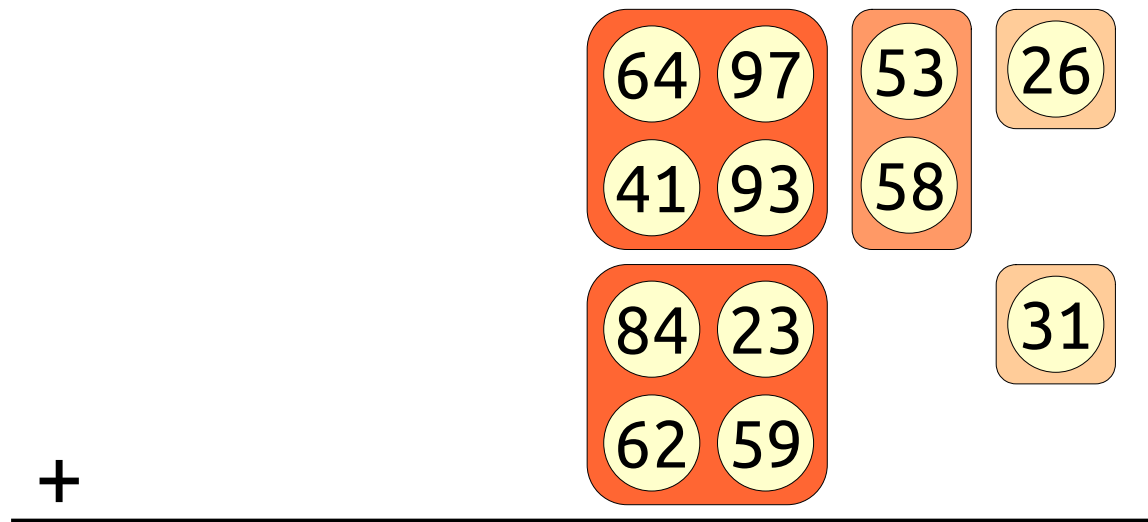
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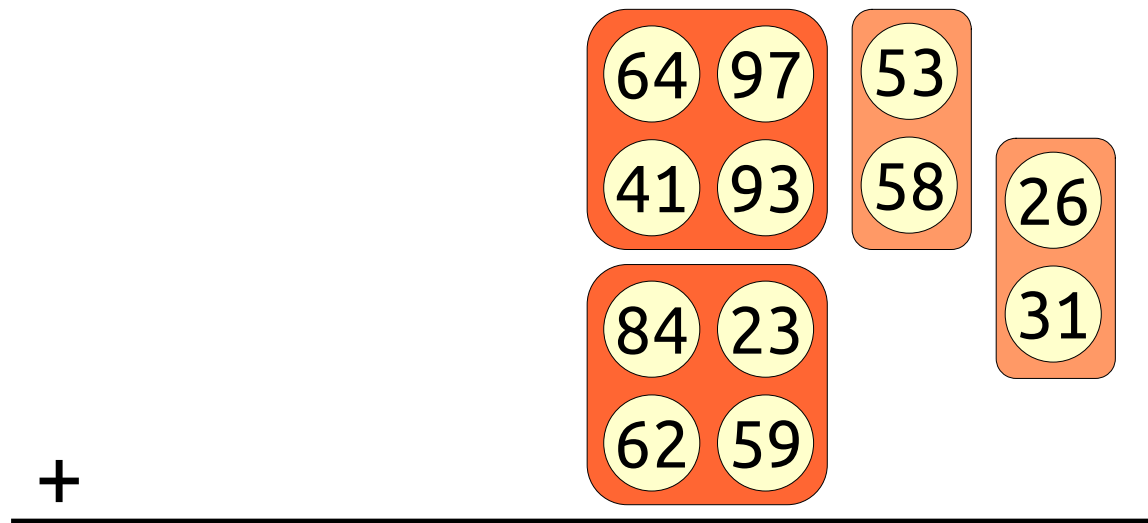
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- **Idea:** Store elements in “packets” whose sizes are powers of two and **meld** by “adding” groups of packets.



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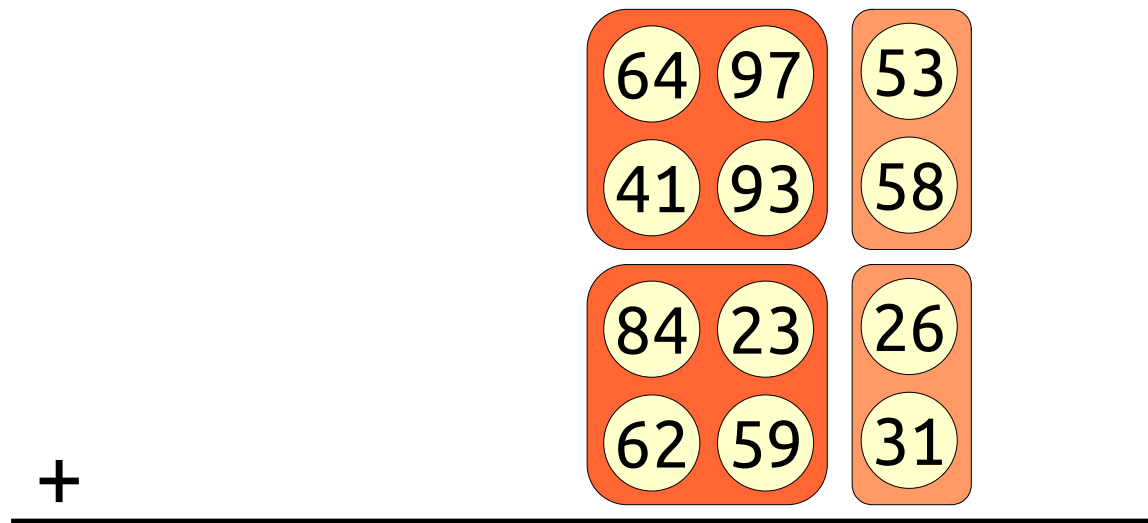
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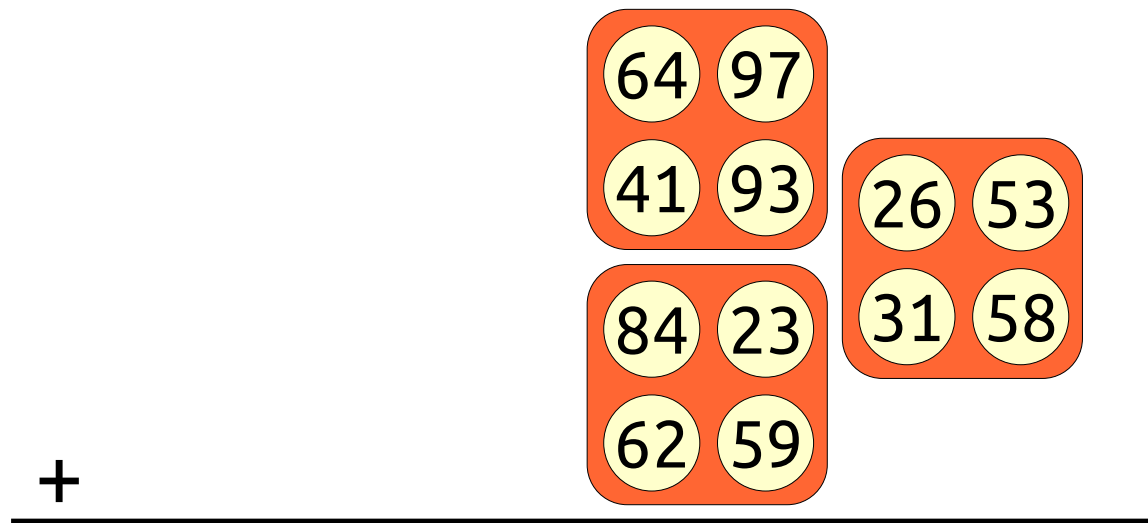
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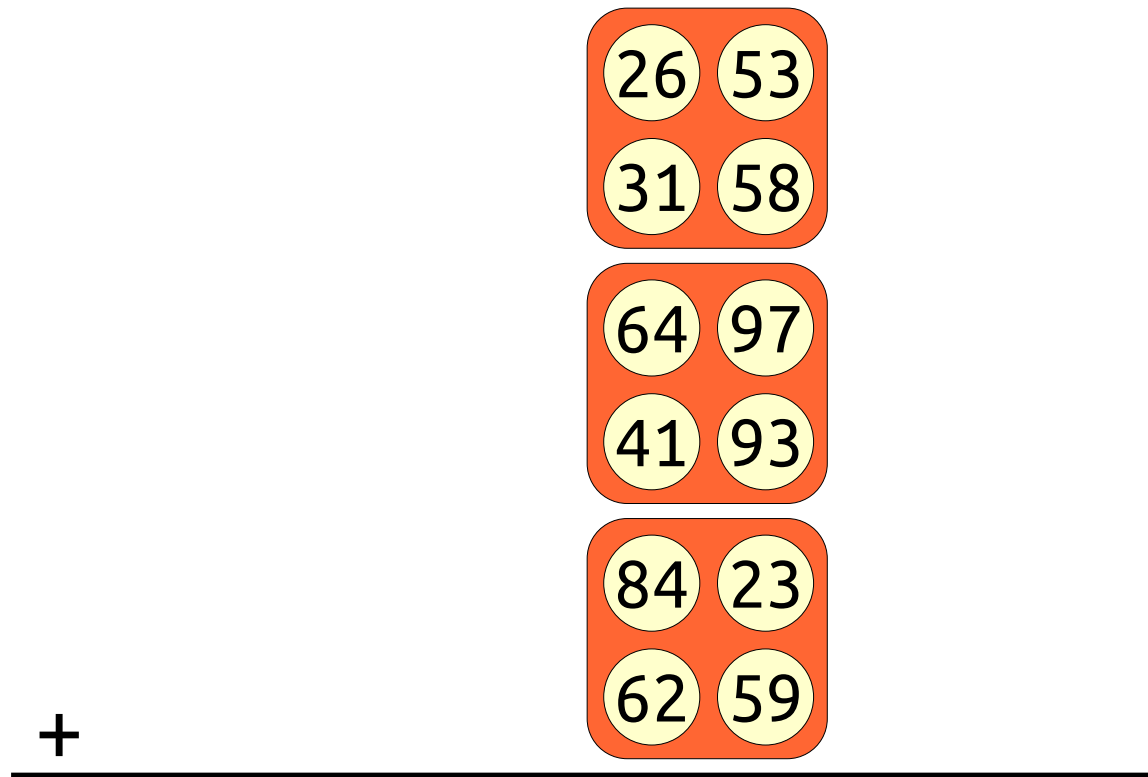
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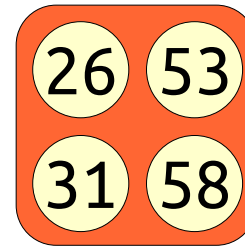
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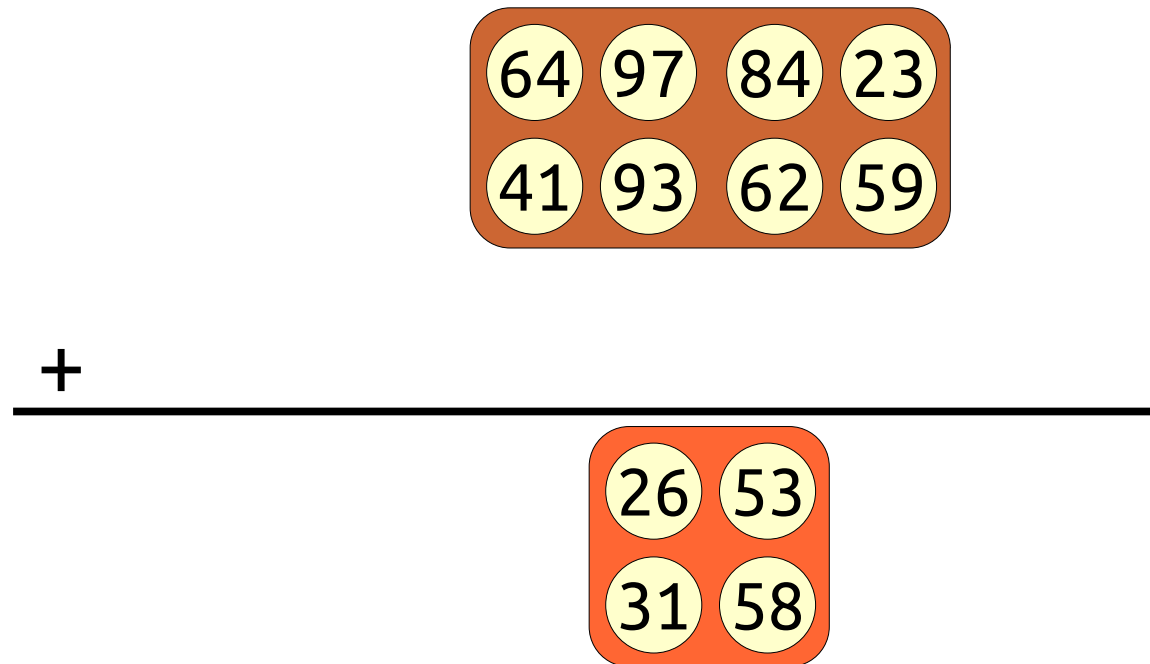


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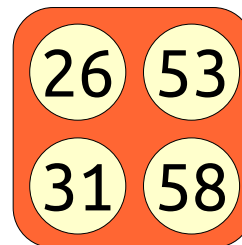


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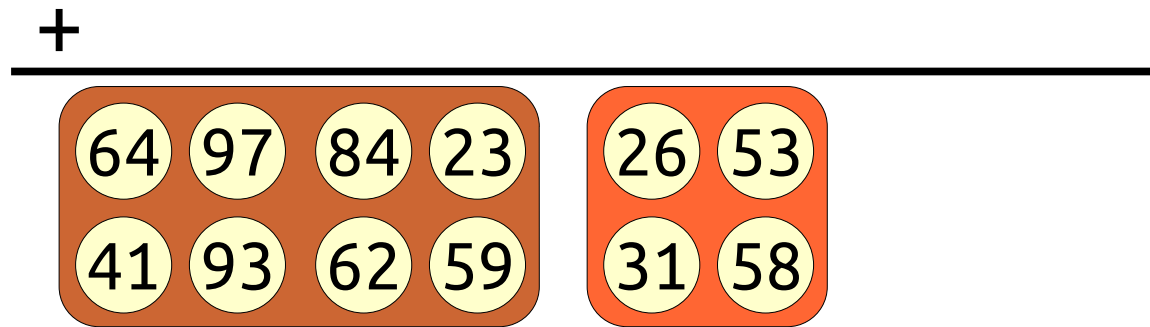


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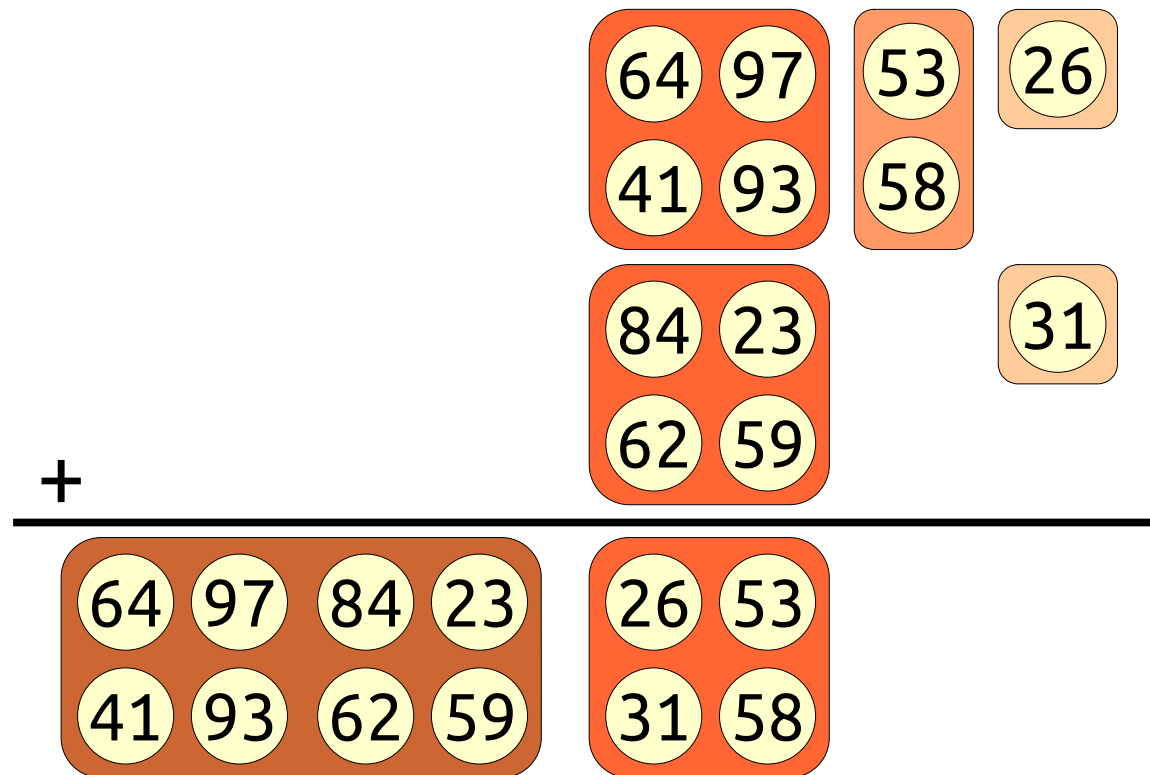
# Building a Priority Queue

- **Idea:** Store elements in “packets” whose sizes are powers of two and **meld** by “adding” groups of packets.



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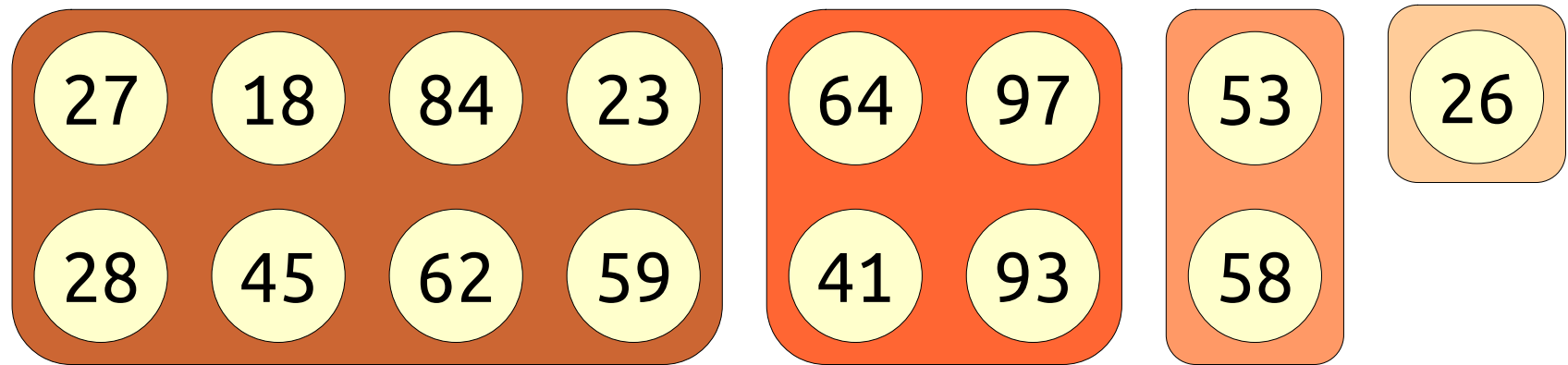


# Building a Priority Queue

- What properties must our packets have?

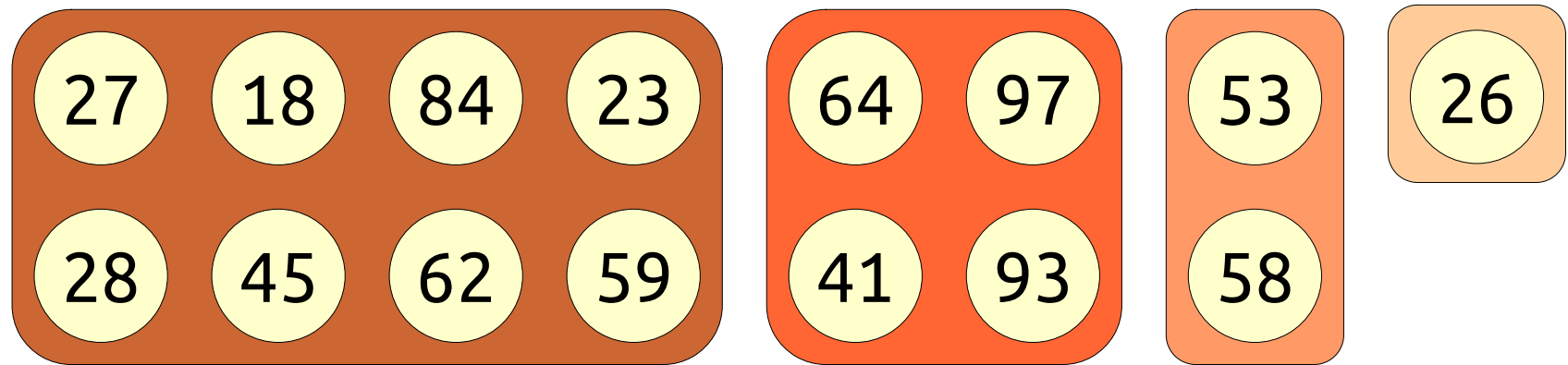
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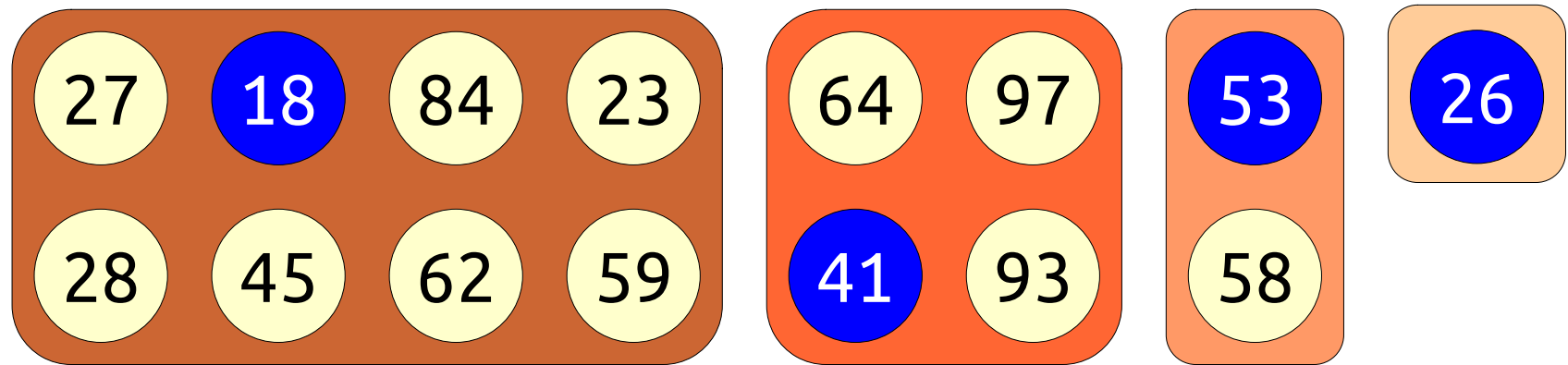
# Building a Priority Queue

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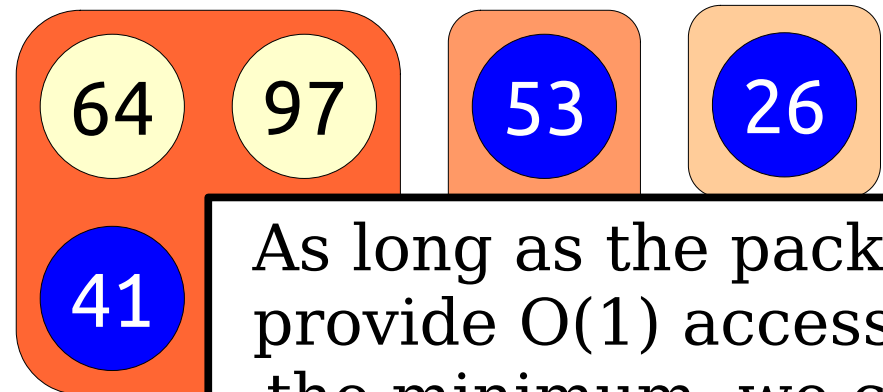
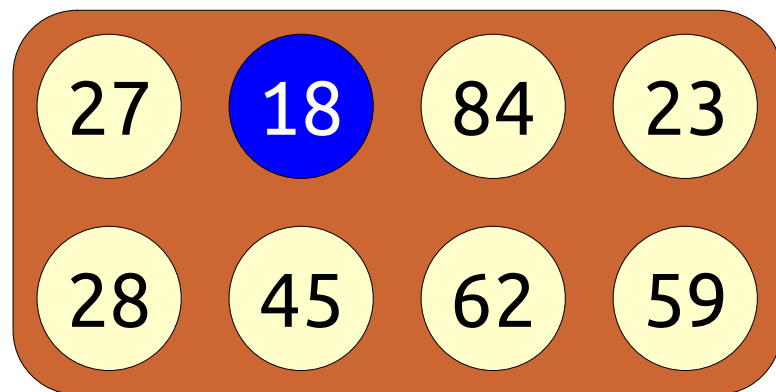
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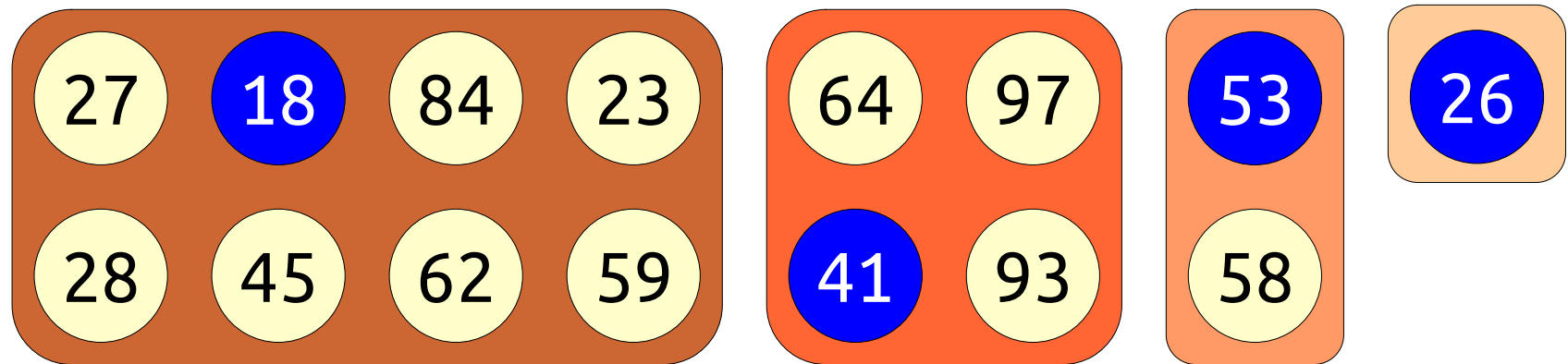
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As long as the packets provide  $O(1)$  access to the minimum, we can execute *find-min* in time  $O(\log n)$ .

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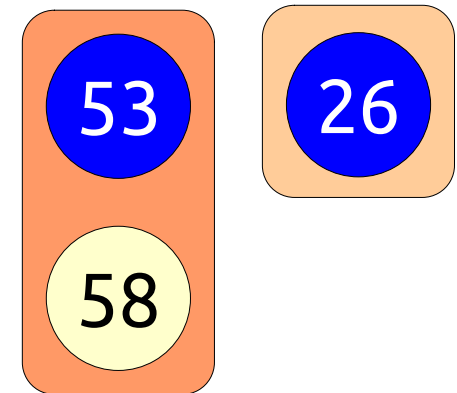
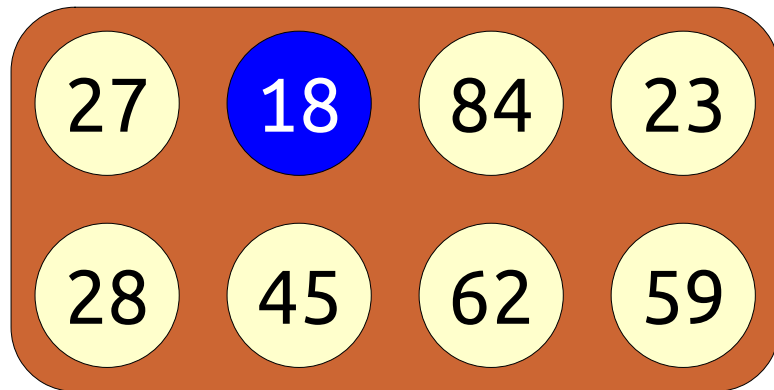


# Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.

# Inserting into the Queue

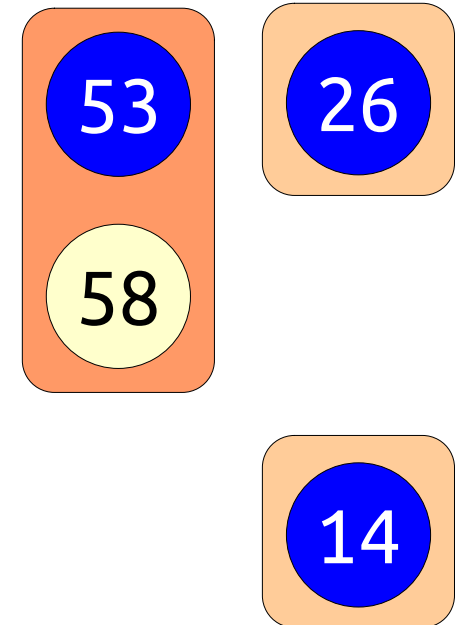
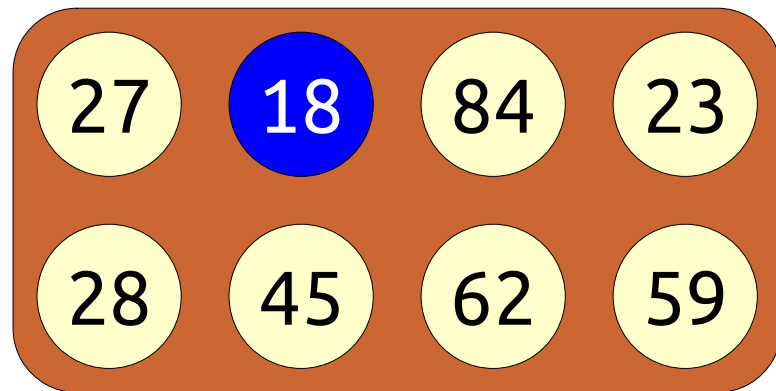
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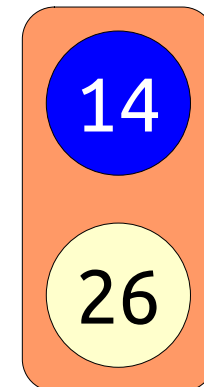
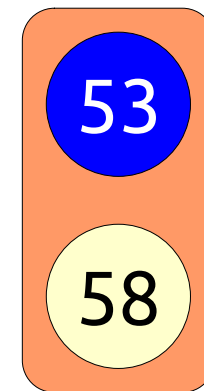
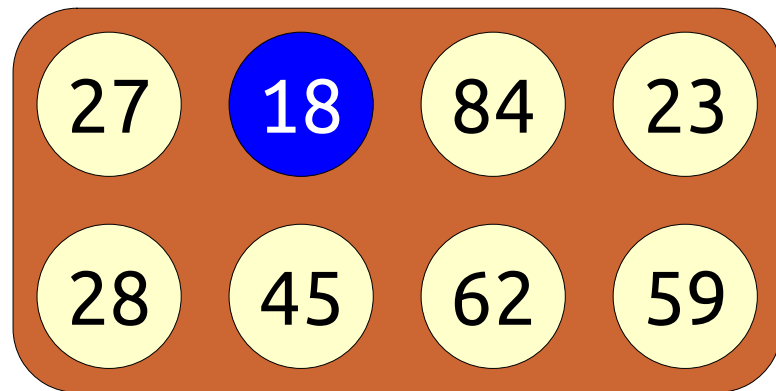
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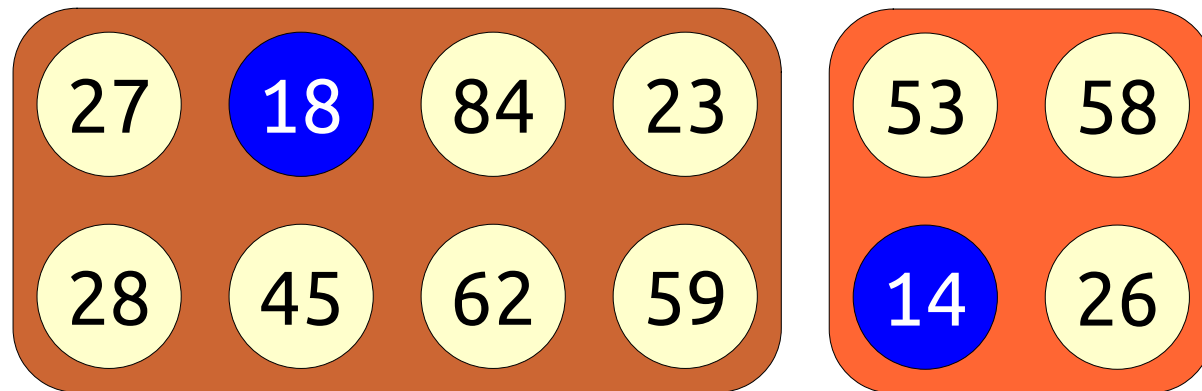
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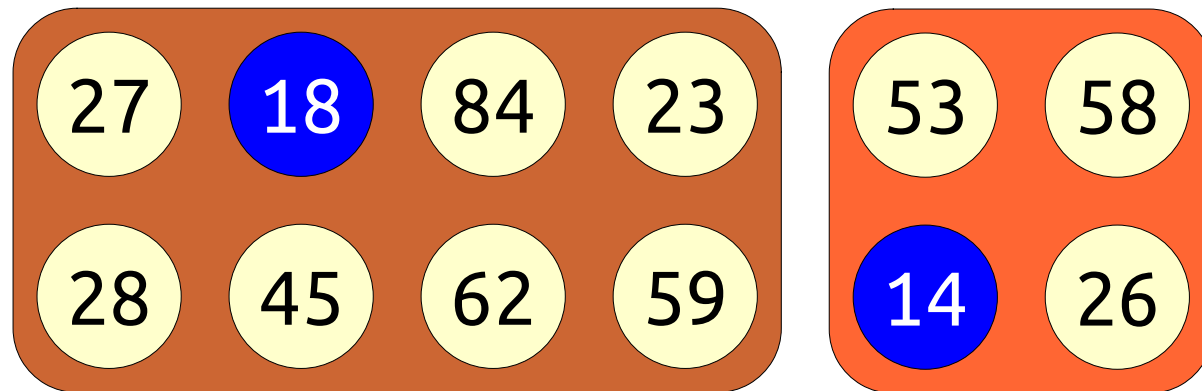
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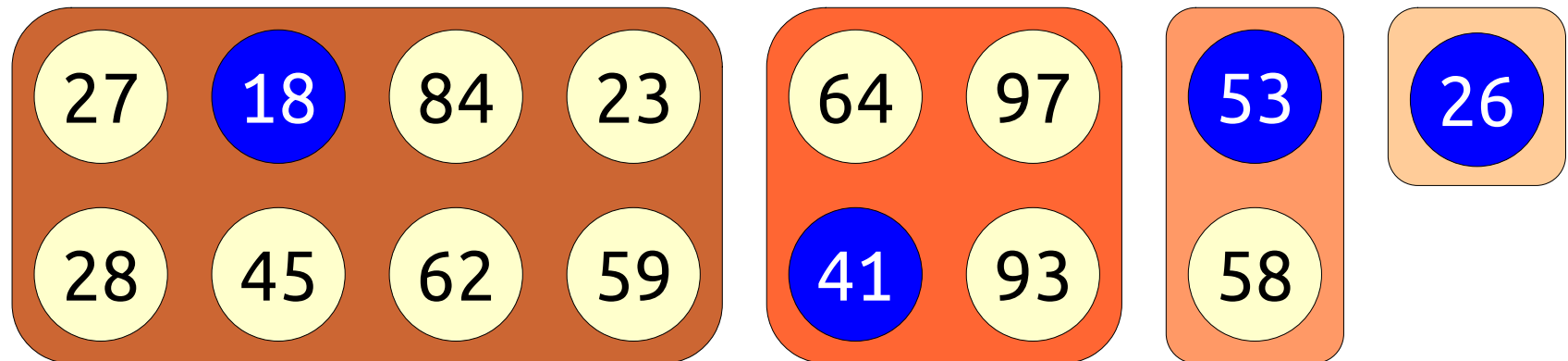
Time required:  
 $O(\log n)$  fuses.

# Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.

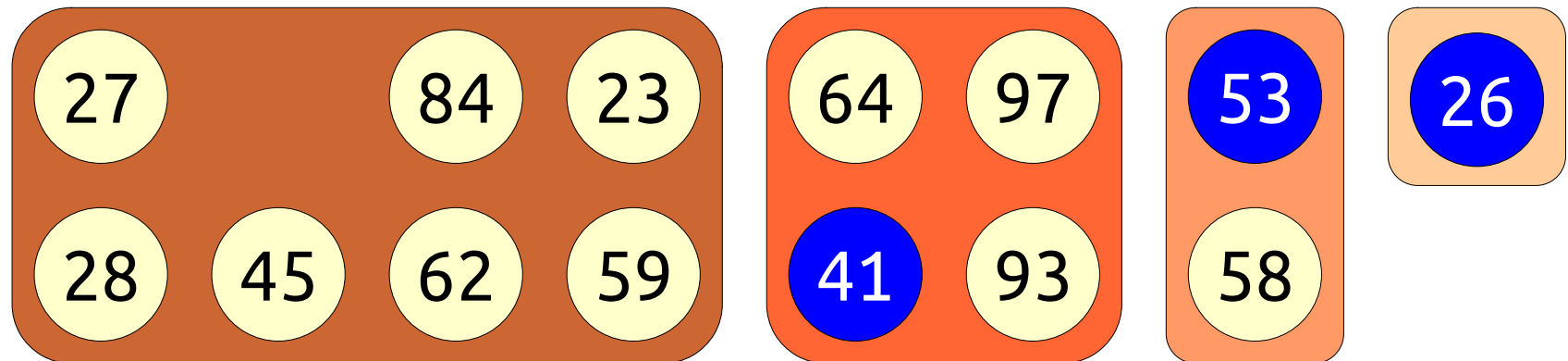
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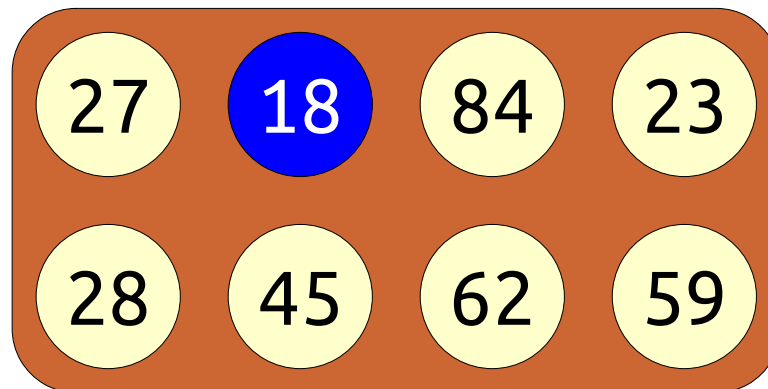
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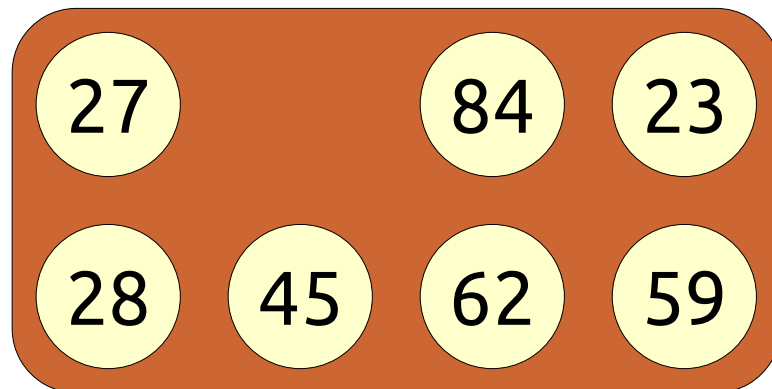
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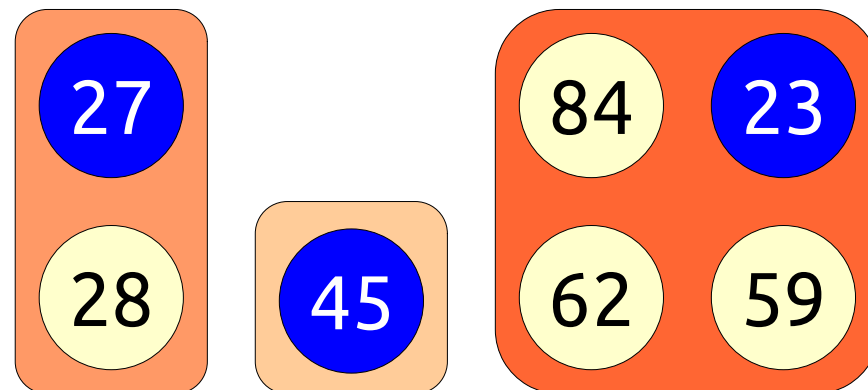
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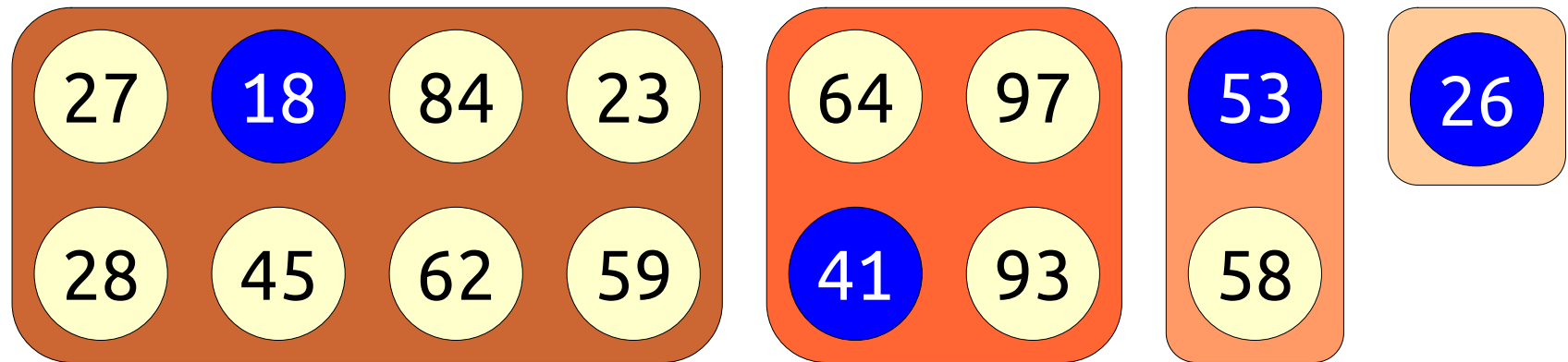
# Deleting the Minimum

- If we have a packet with  $2^k$  elements in it and remove a single element, we are left with  $2^k - 1$  remaining elements.
- **Fun fact:**  $2^k - 1 = 2^0 + 2^1 + 2^2 + \dots + 2^{k-1}$ .
- **Idea:** “Fracture” the packet into  $k$  smaller packets, then add them back in.



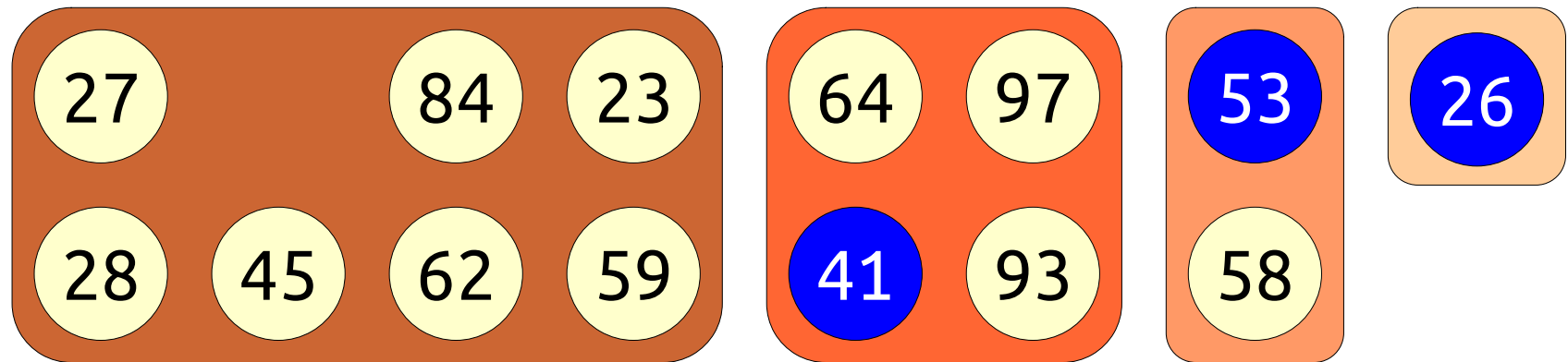
# Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.



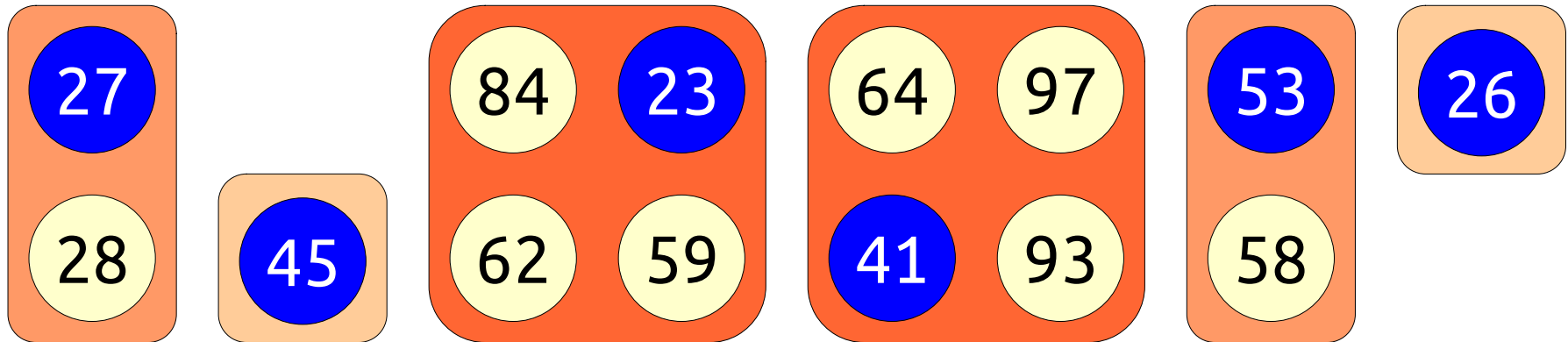
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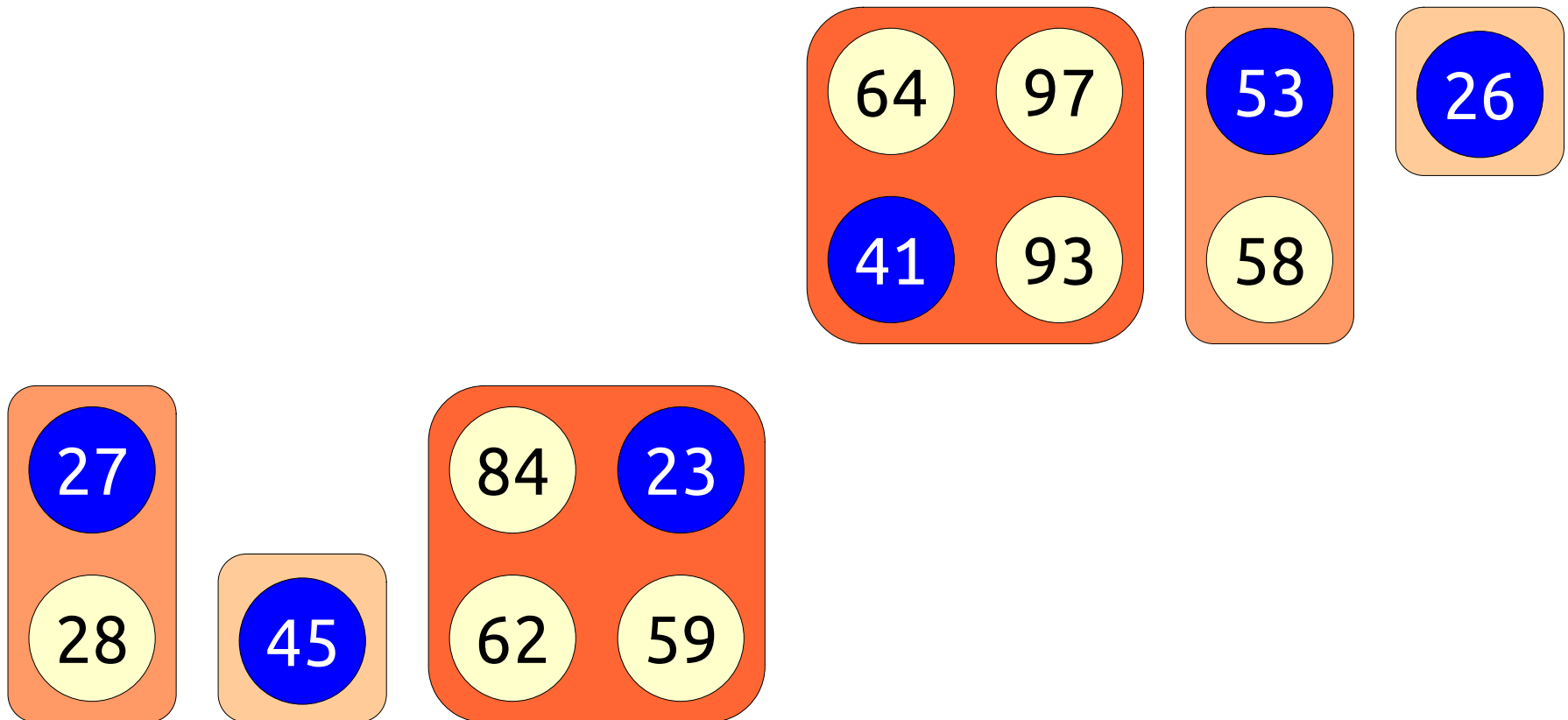
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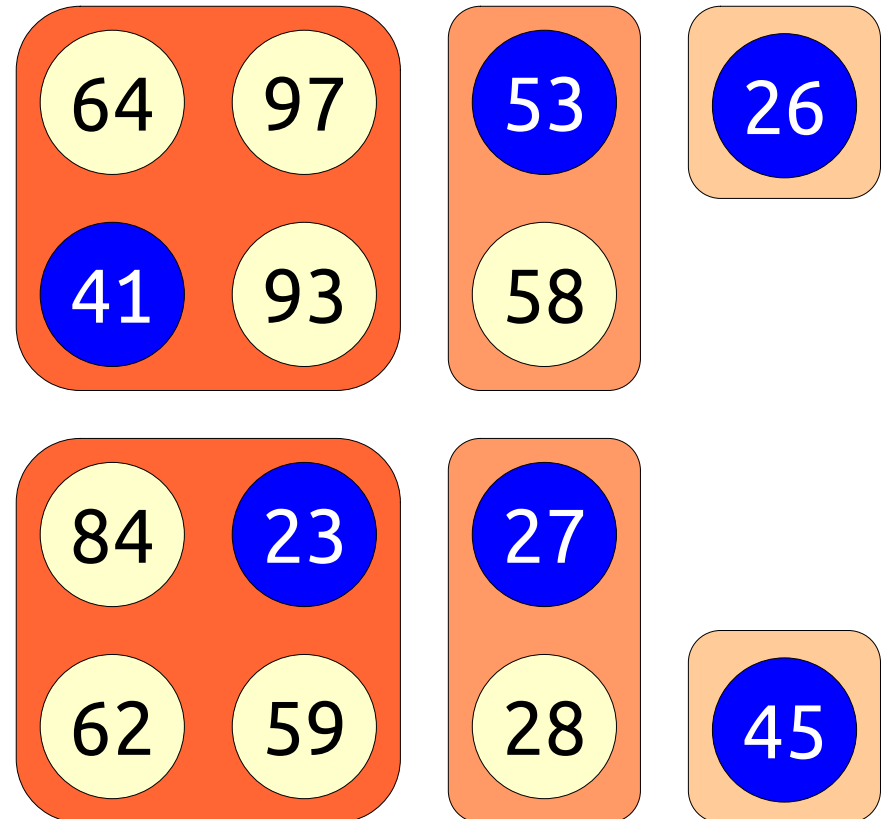
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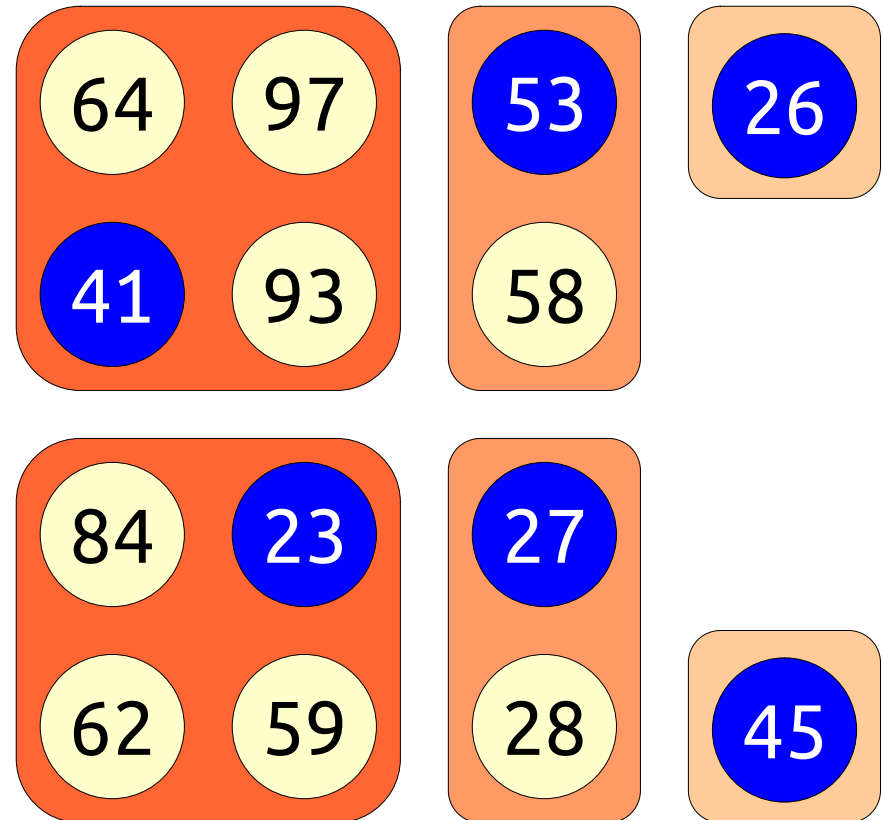
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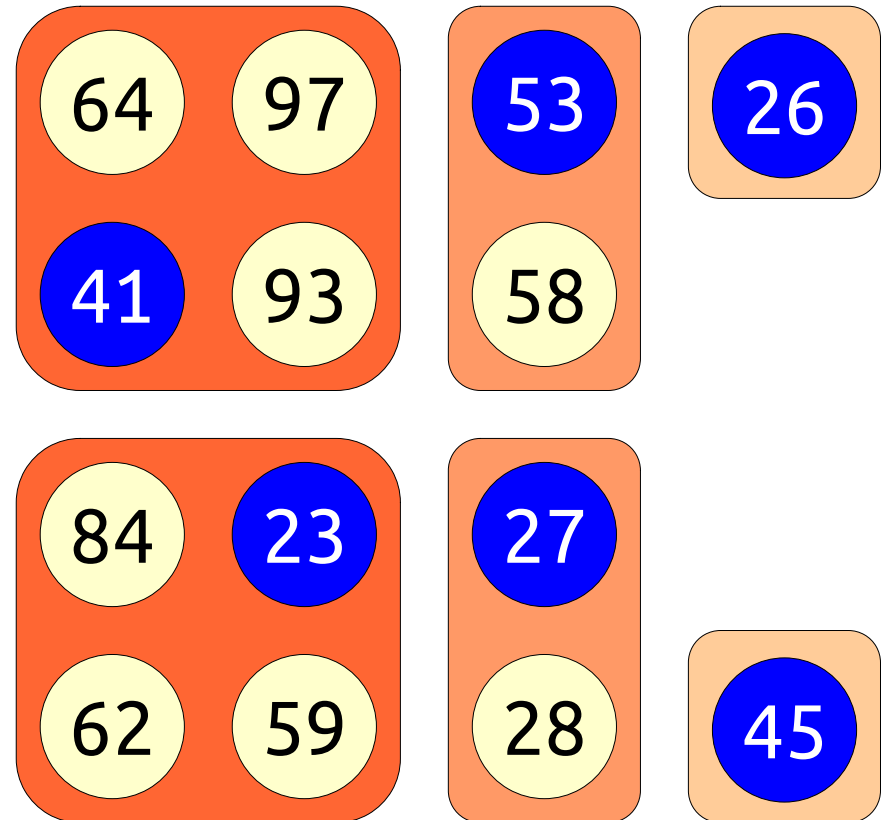




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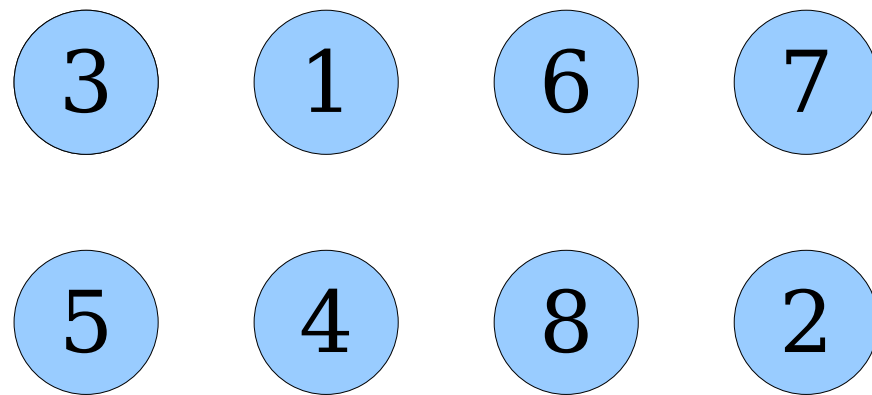
- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
- Runtime is  $O(\log n)$  fuses in *meld*, plus fracture cost.

+



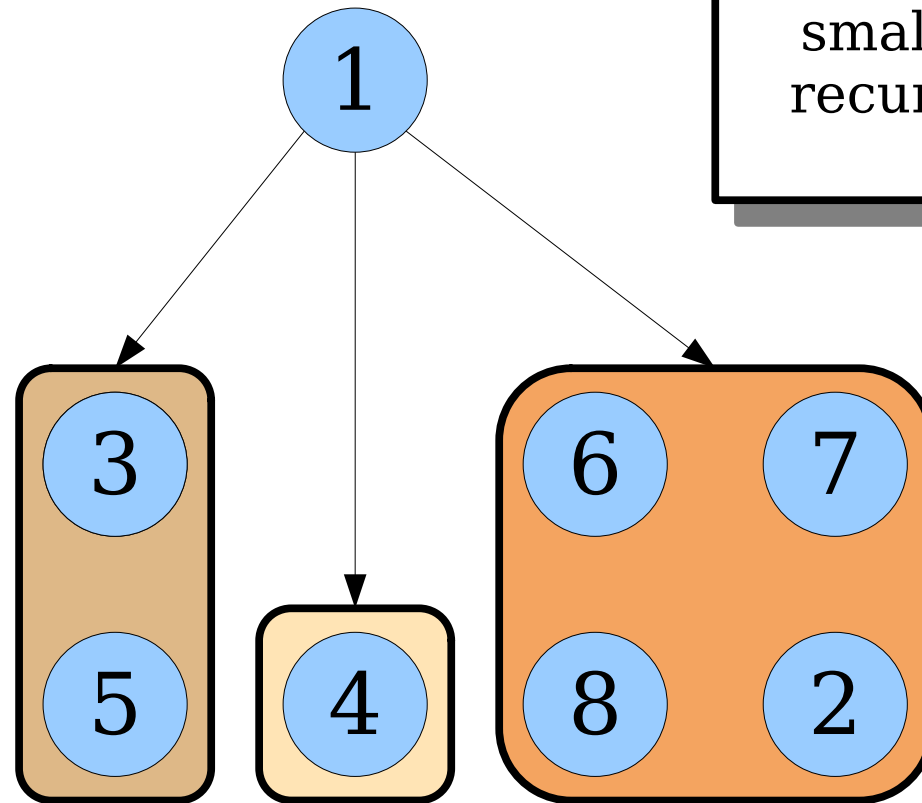
# Building a Priority Queue

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  - Can efficiently “fracture” a packet of  $2^k$  nodes into packets of  $2^0, 2^1, 2^2, 2^3, \dots, 2^{k-1}$  nodes.
- **Question:** How can we represent our packets to support the above operations efficiently?

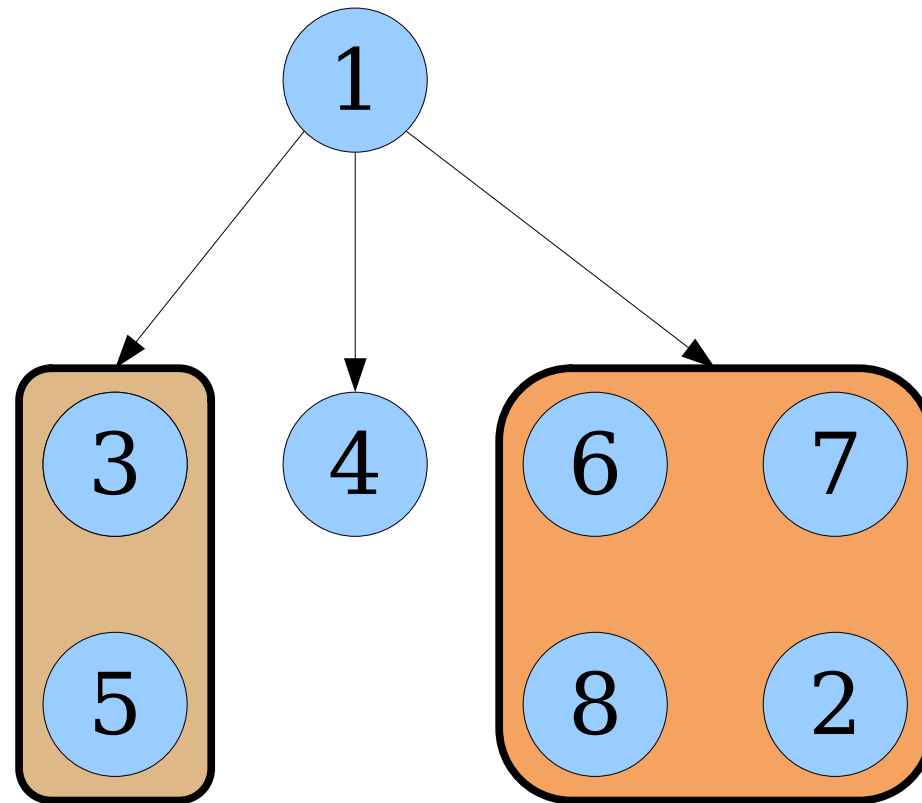


*Thanks to former CS166er  
Anna Zeng for this explanation!*

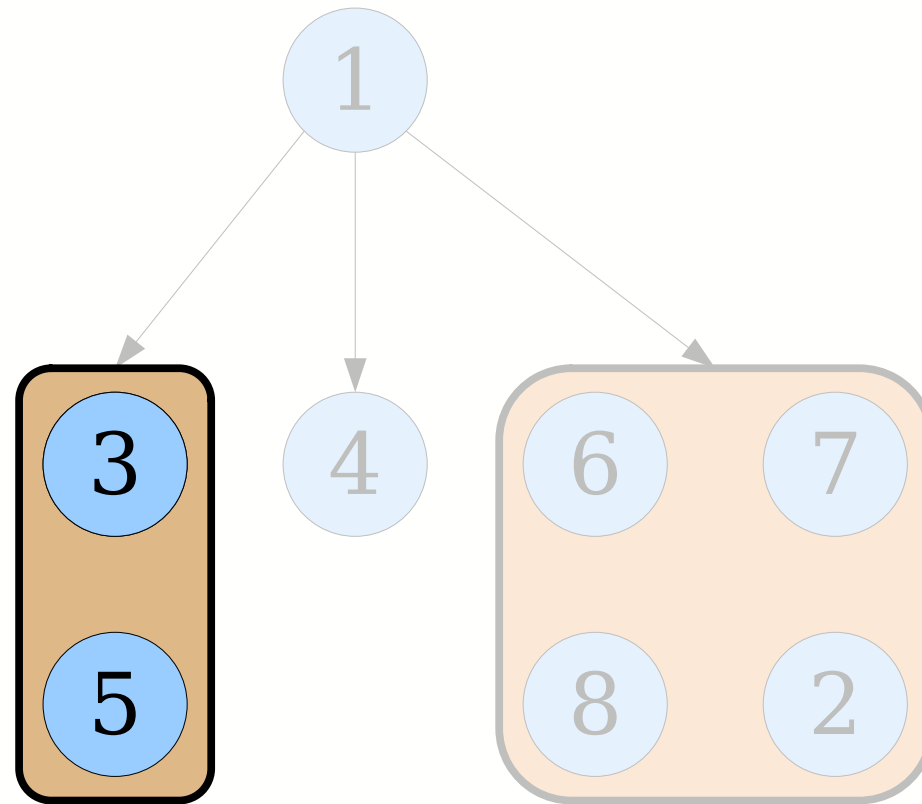
**Idea:** Have the minimum value as the root of a tree whose children are the smaller packets. Then recursively expand out the packets.



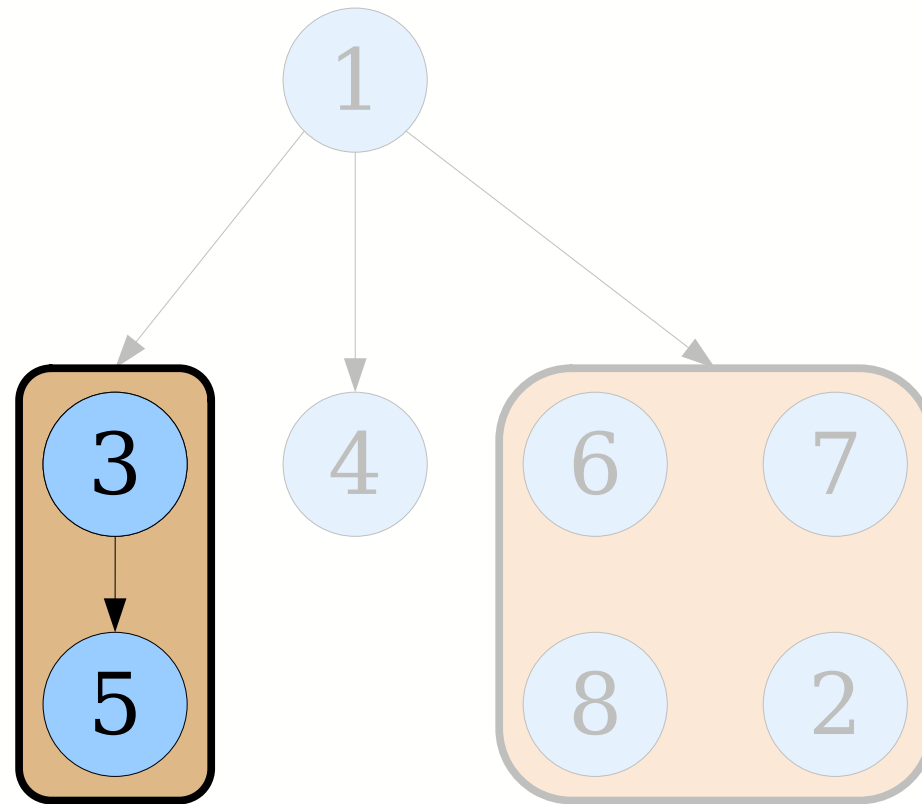
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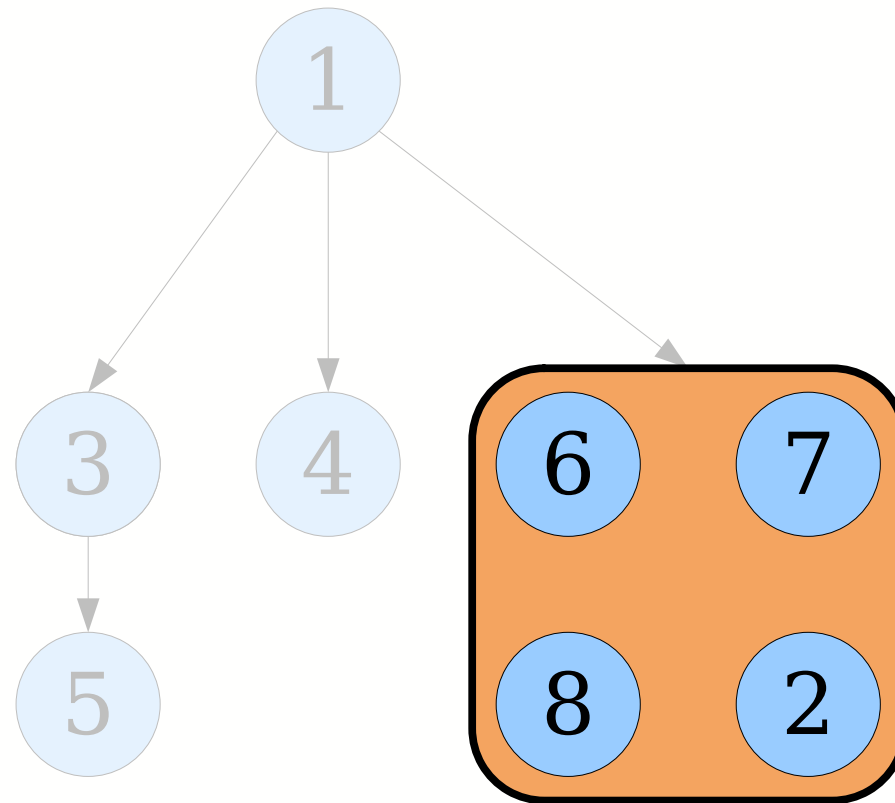
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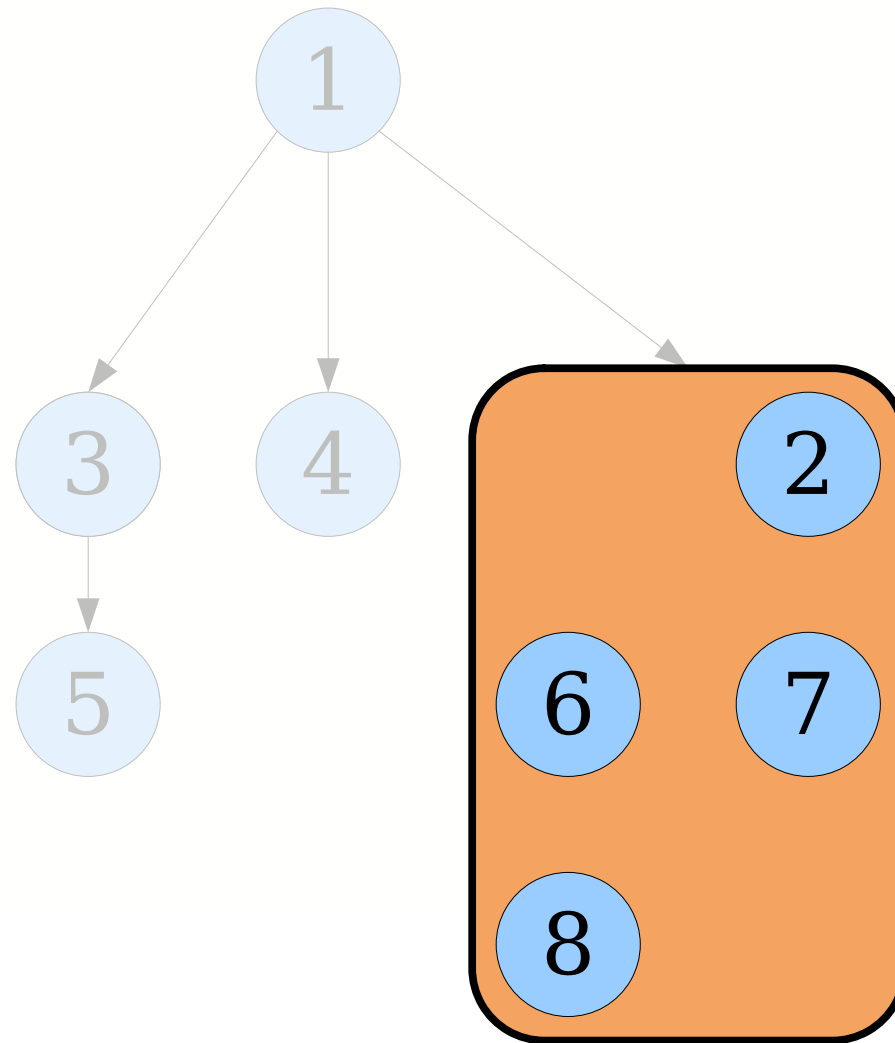


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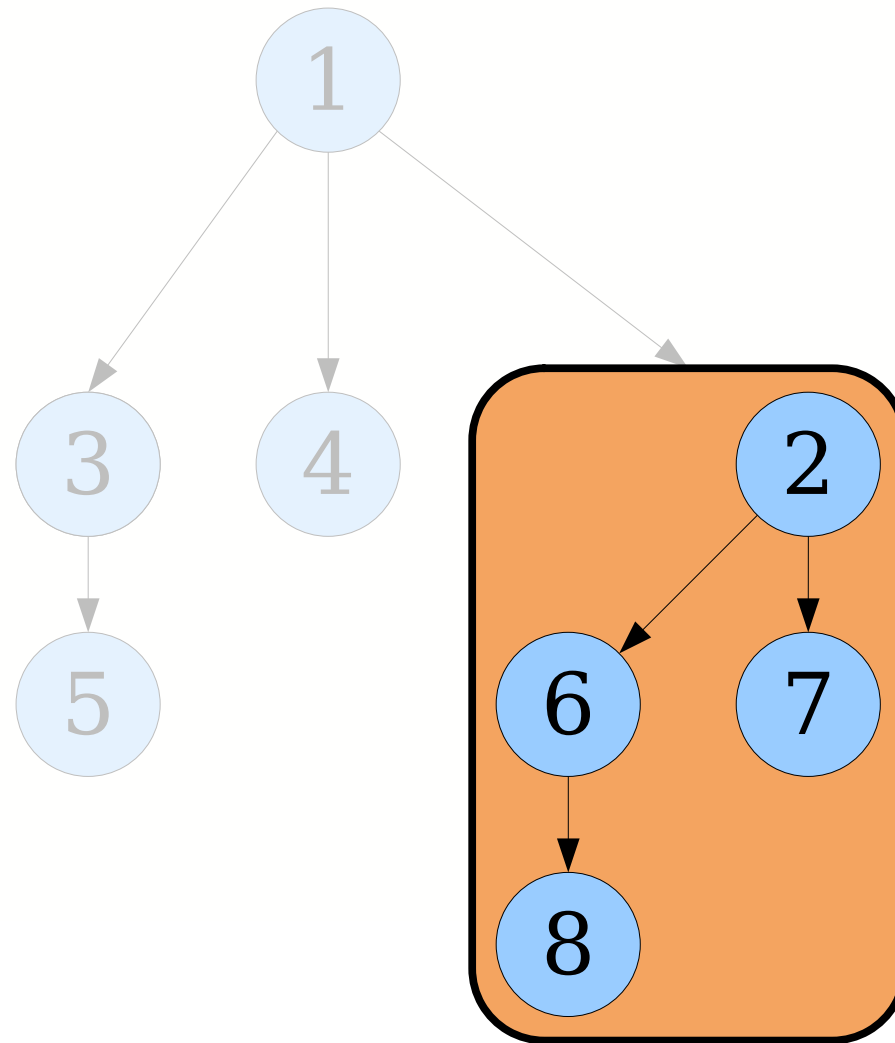


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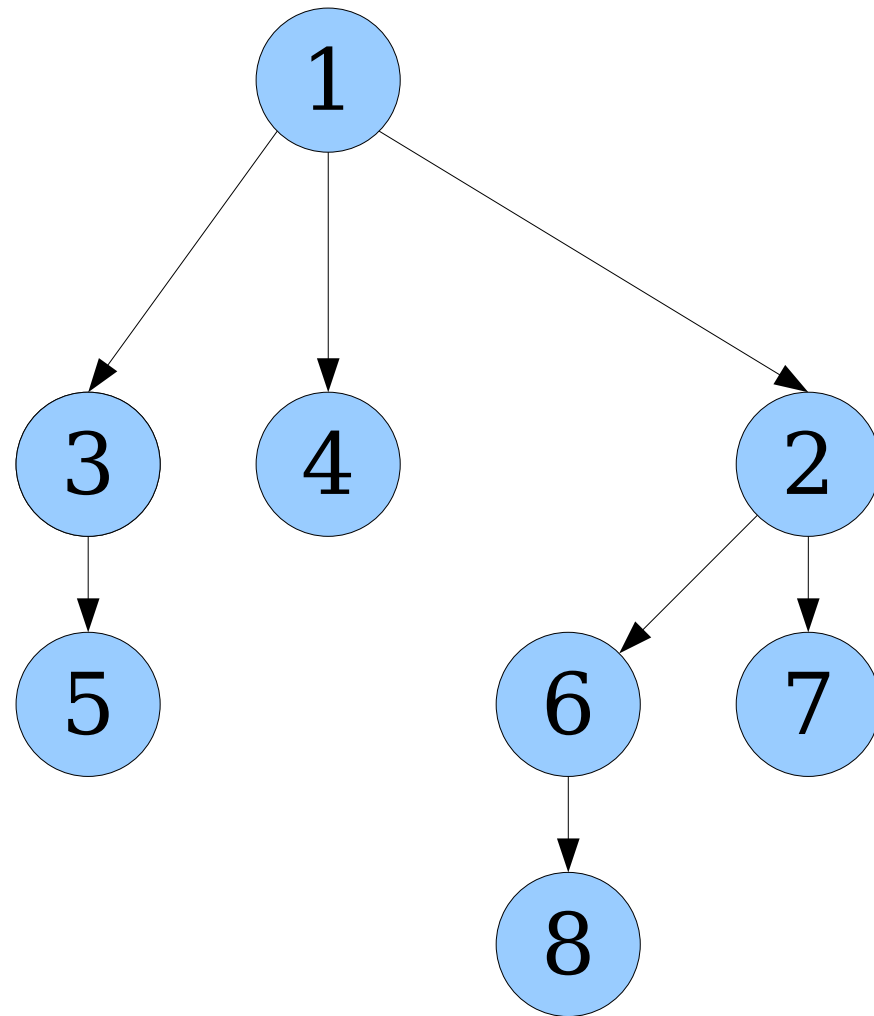




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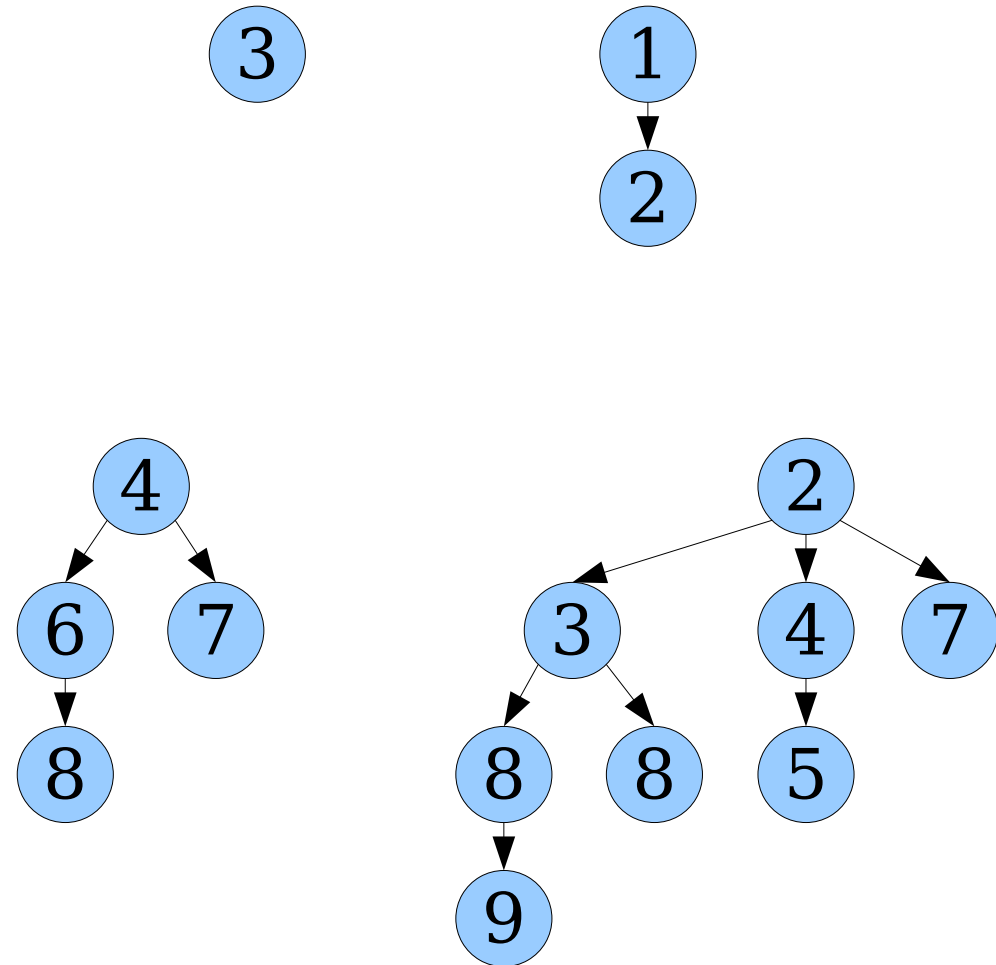
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# Binomial Trees

- A **binomial tree of order  $k$**  is a tree structure with  $2^k$  nodes.
- We can **mechanically** describe binomial trees as follows:
  - Place the minimum of the  $2^k$  keys at the root.
  - Regroup the remaining elements into  $k$  groups of sizes  $2^0, 2^1, 2^2, \dots$ , and  $2^{k-1}$ .
  - Recursively build binomial trees from each group.
  - Make those new trees children of the root.
- Can we operationally describe binomial trees?

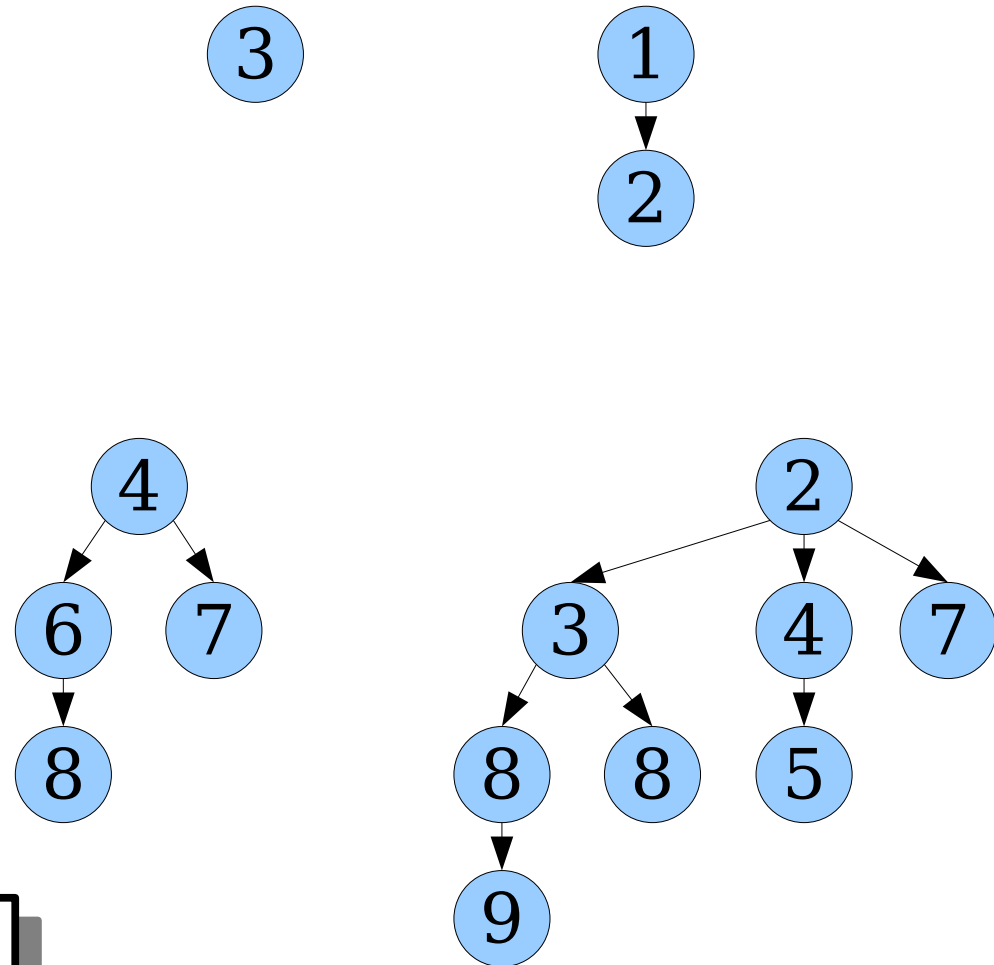


# Binomial Trees

- Here's an **operational definition** of a binomial tree of order  $k$ :

*A binomial tree of order  $k$  is a tree obeying the min-heap property consisting of a root node whose children are binomial trees of order  $0, 1, 2, \dots, k - 1$ .*

Why are these called binomial trees?  
Look across the layers of these trees  
and see if you notice anything!

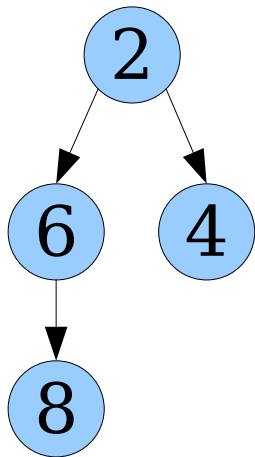


# Binomial Trees

- What properties must our packets have?
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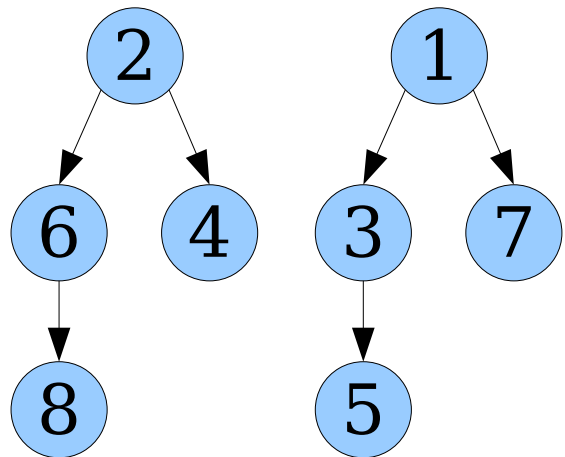
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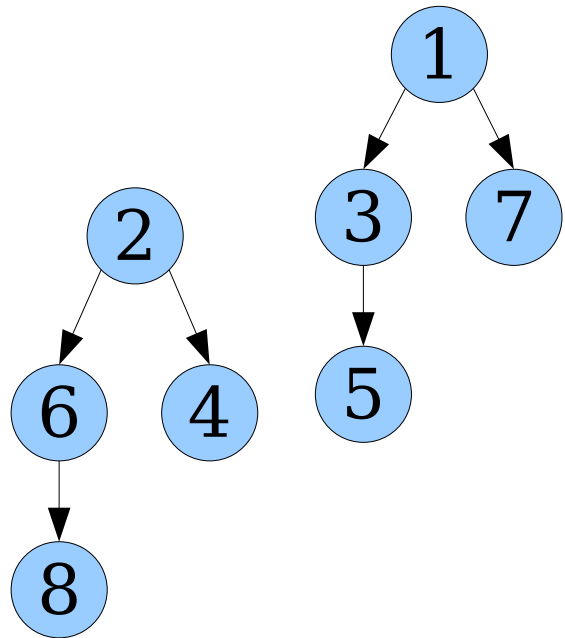
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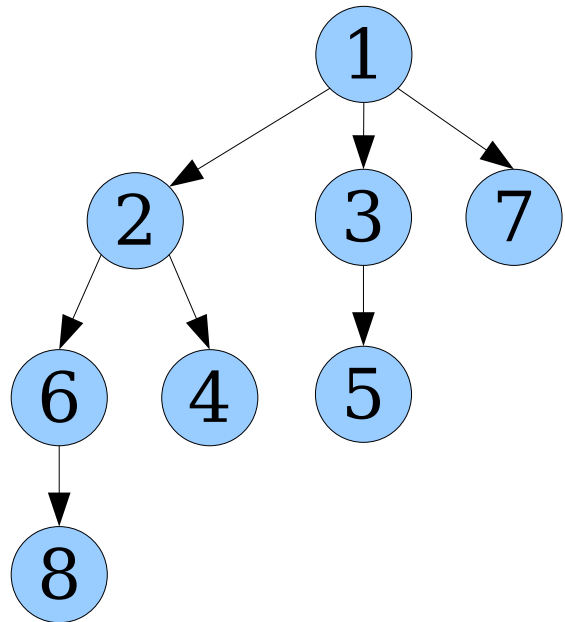
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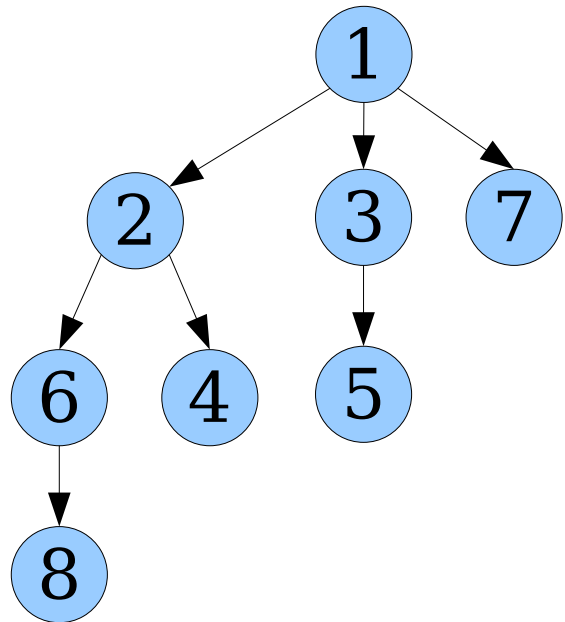
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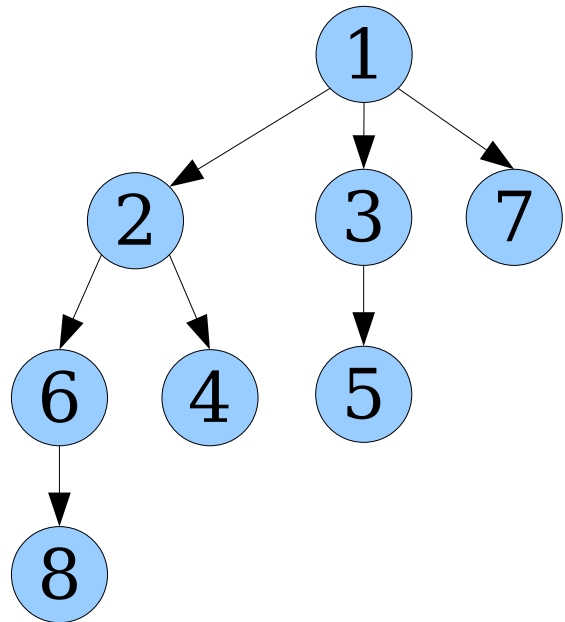
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Make the binomial tree with the larger root the first child of the tree with the smaller root.

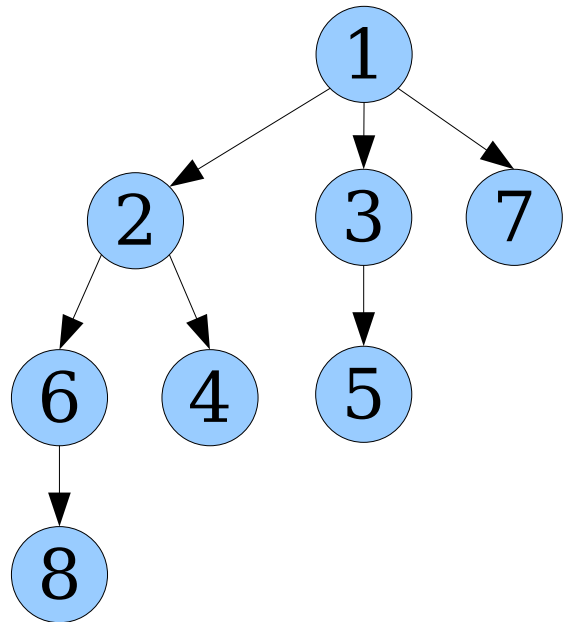
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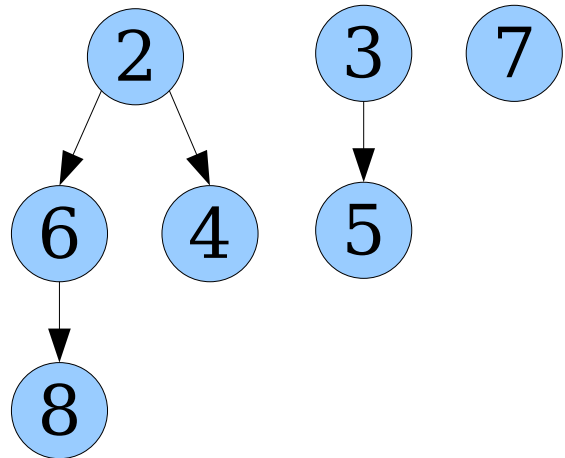
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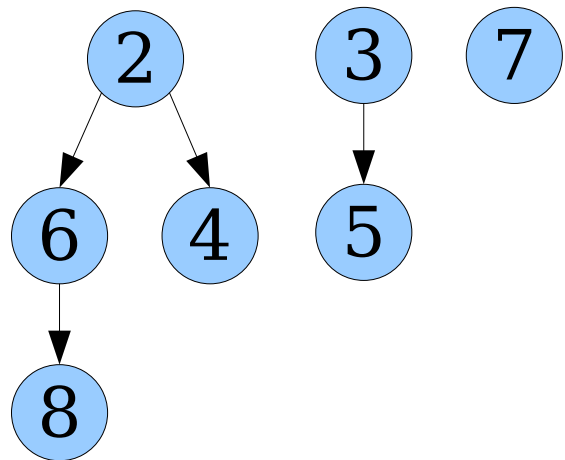
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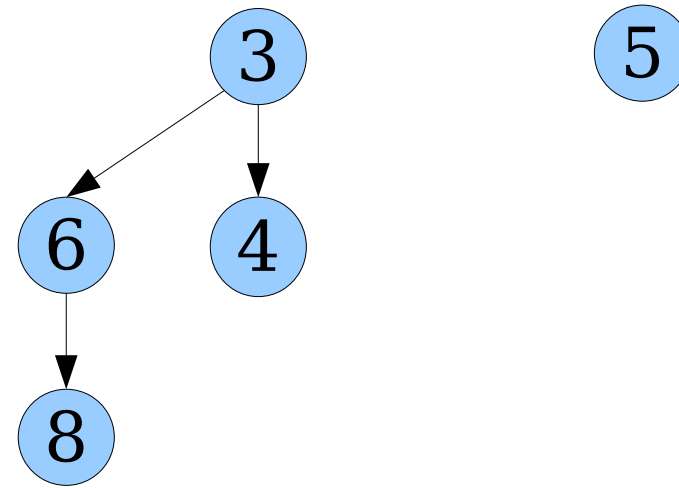
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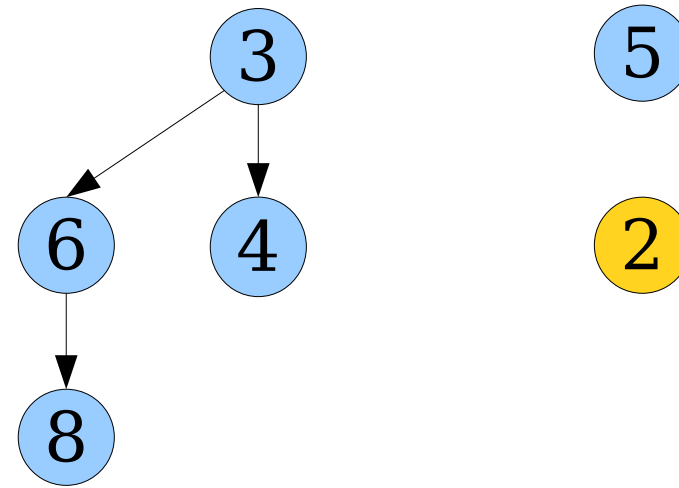


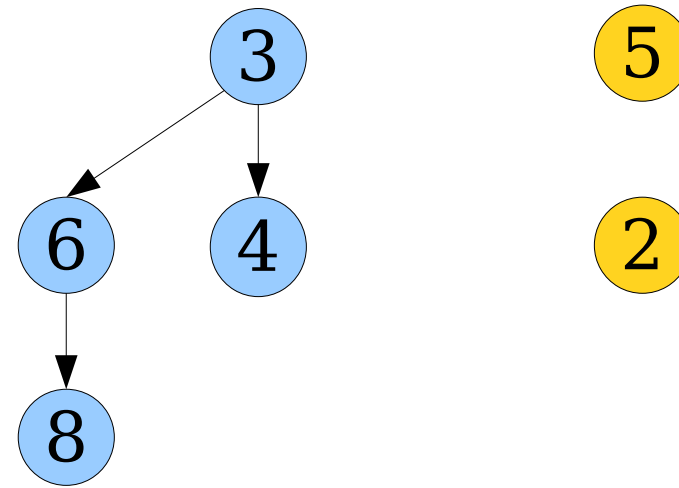
# The Binomial Heap

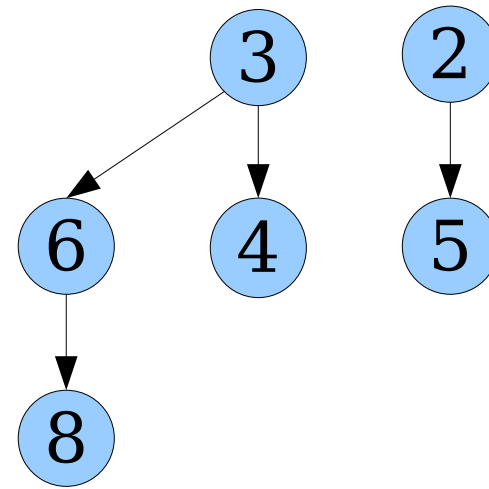
- A **binomial heap** is a collection of binomial trees stored in ascending order of size.
- Operations defined as follows:
  - **meld**( $pq_1, pq_2$ ): Use addition to combine all the trees.
    - Fuses  $O(\log n + \log m)$  trees. Cost:  $O(\log n + \log m)$ . Here, assume one binomial heap has  $n$  nodes, the other  $m$ .
  - **enqueue**( $v, k$ ): Meld  $pq$  and a singleton heap of  $(v, k)$ .
    - Total time:  $O(\log n)$ .
  - **find-min**(): Find the minimum of all tree roots.
    - Total time:  $O(\log n)$ .
  - **extract-min**(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time:  $O(\log n)$ .

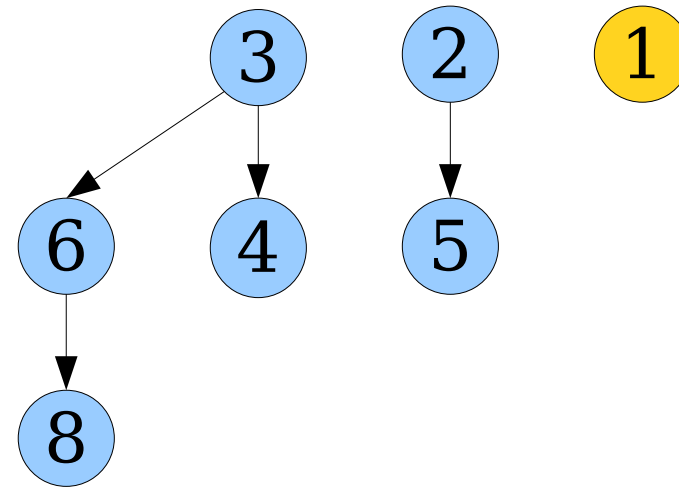


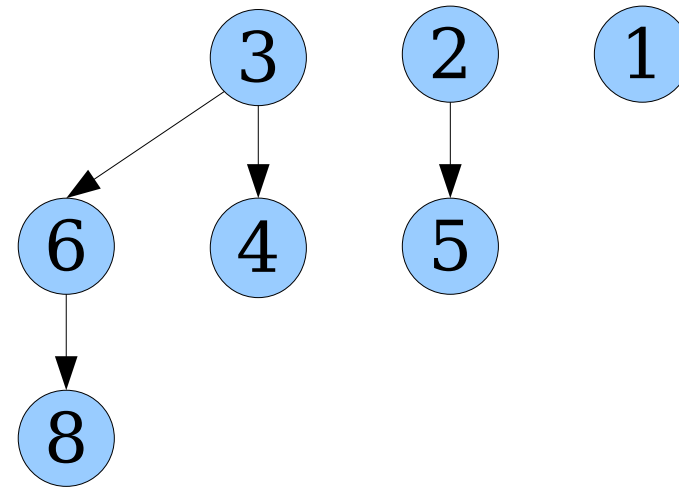


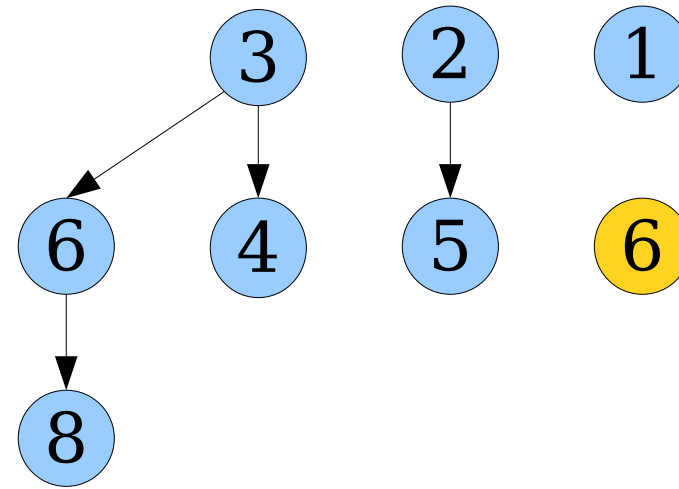


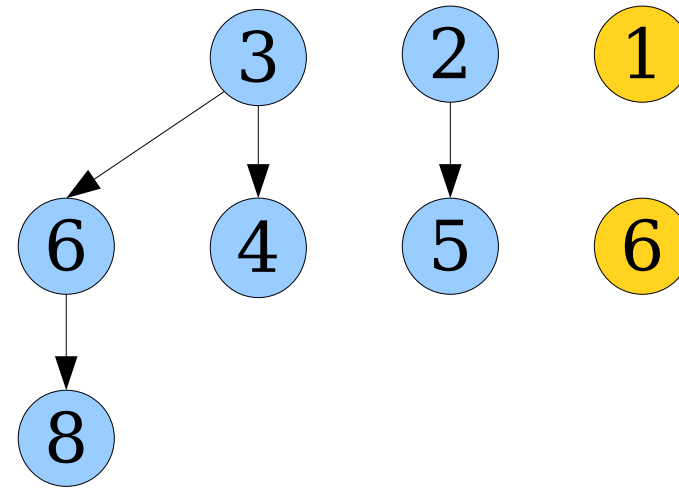




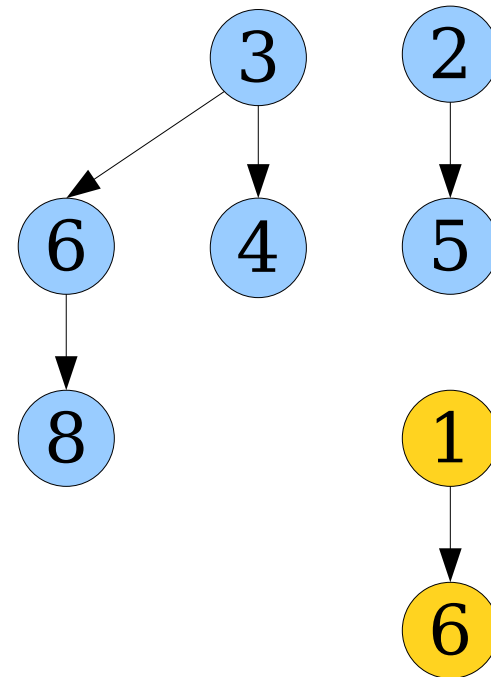


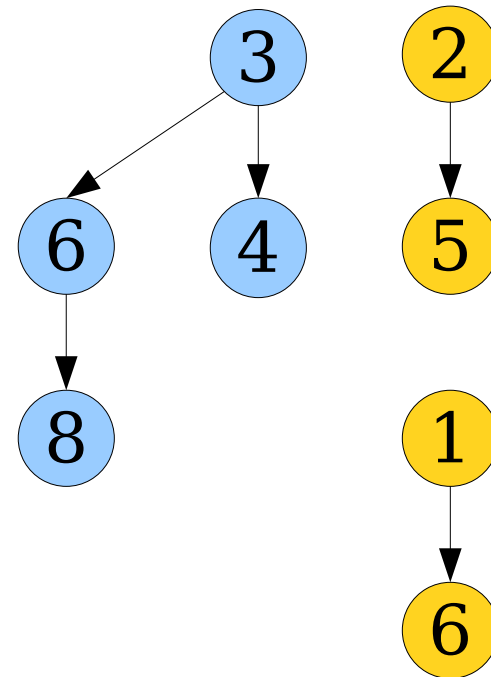


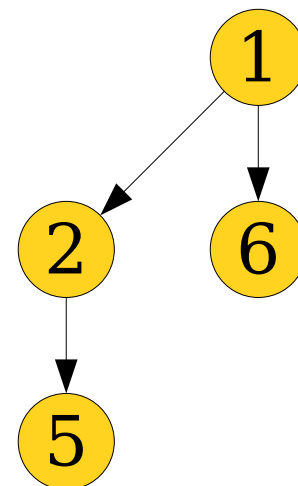
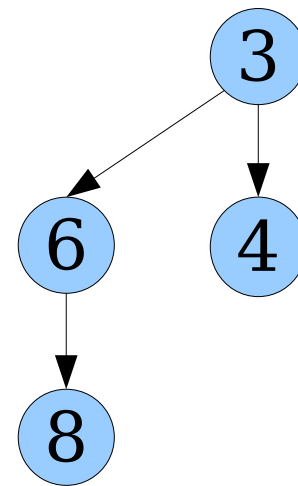


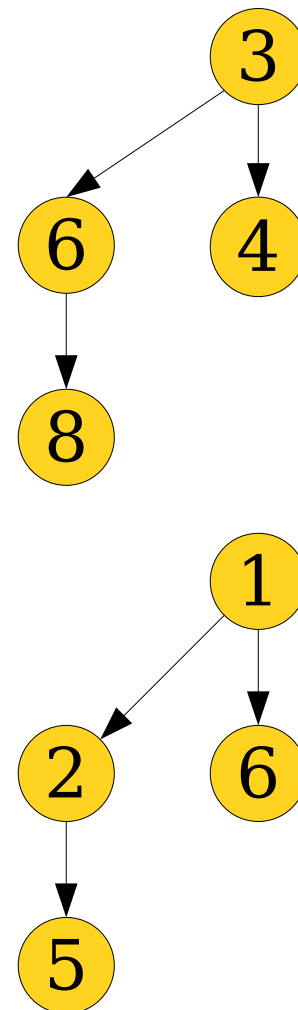


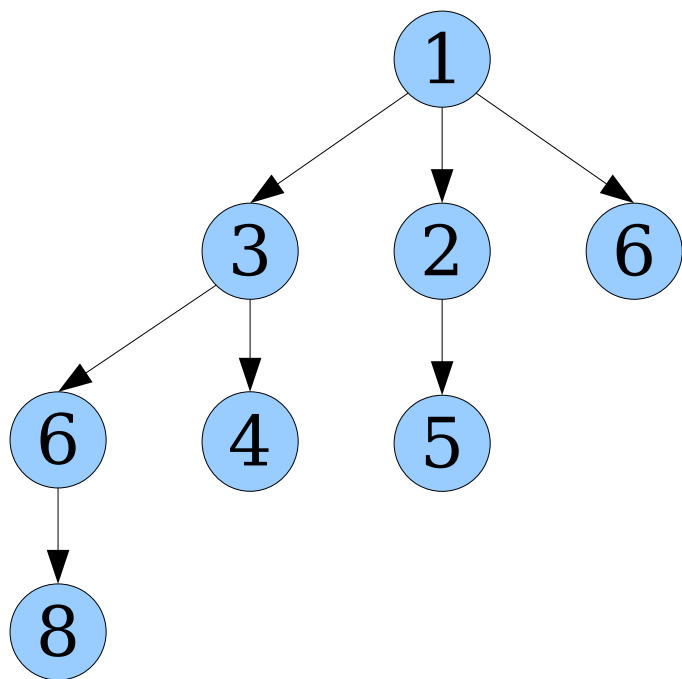


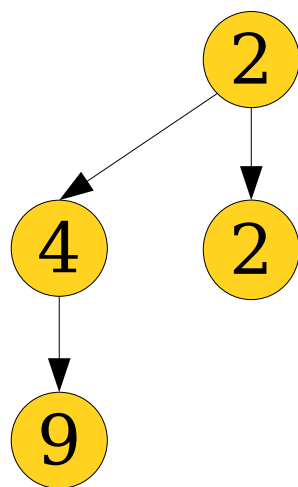
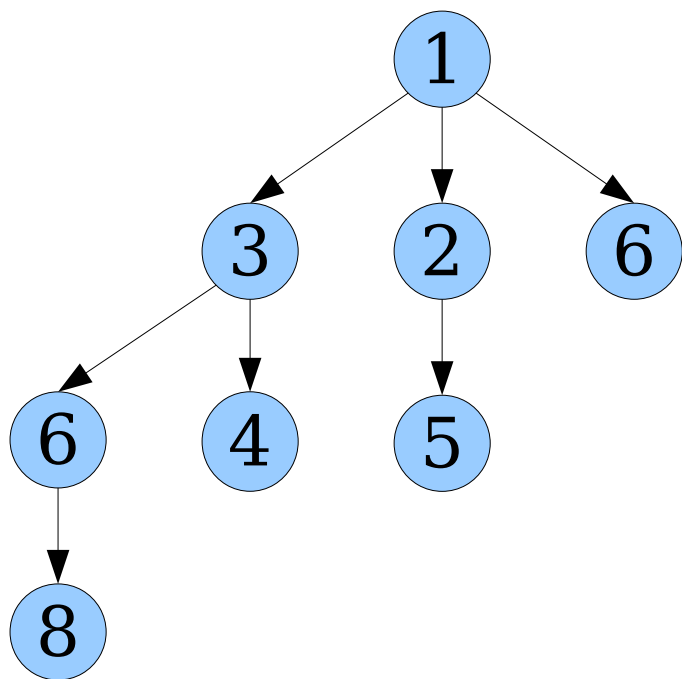


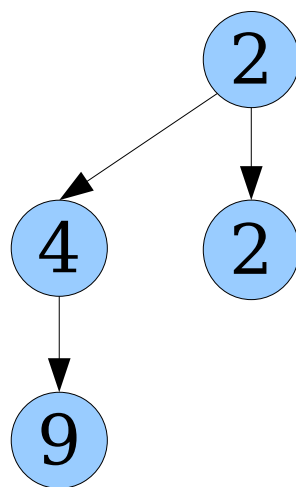
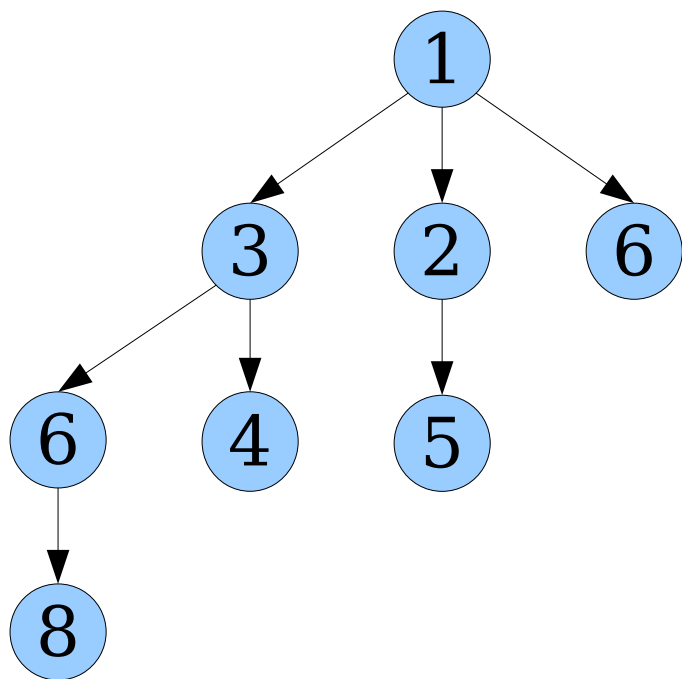


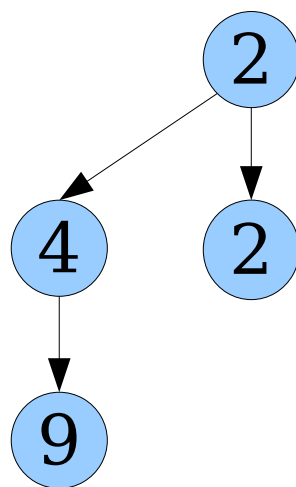
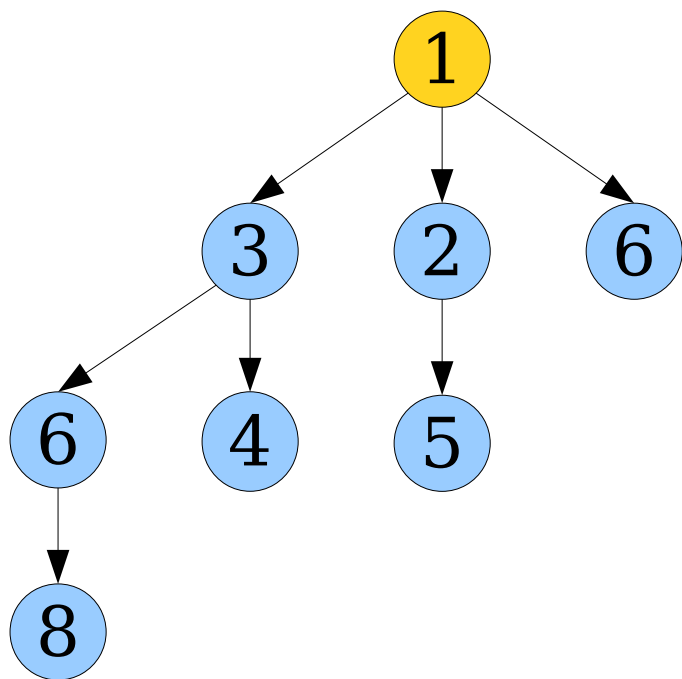




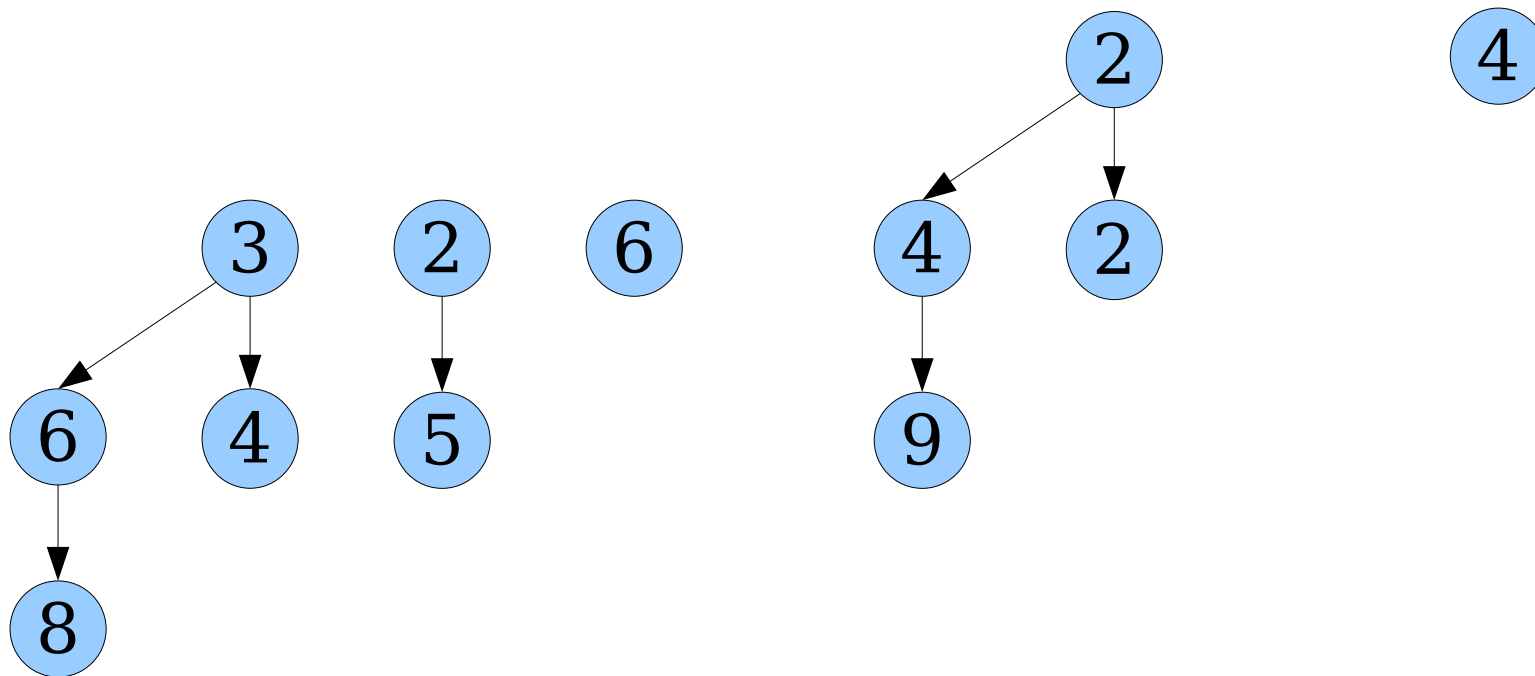


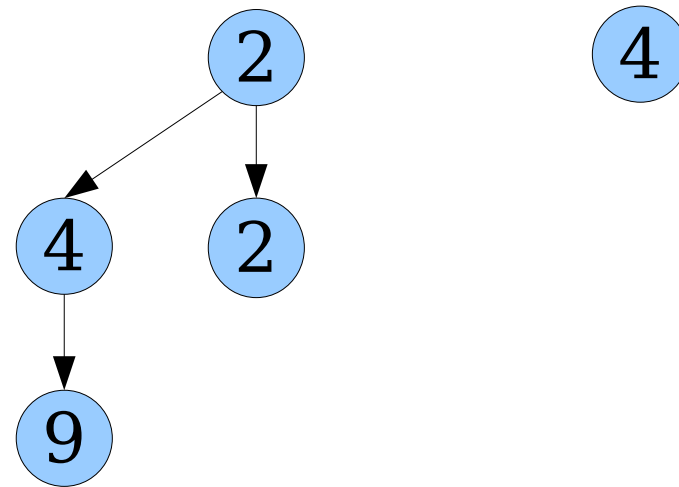
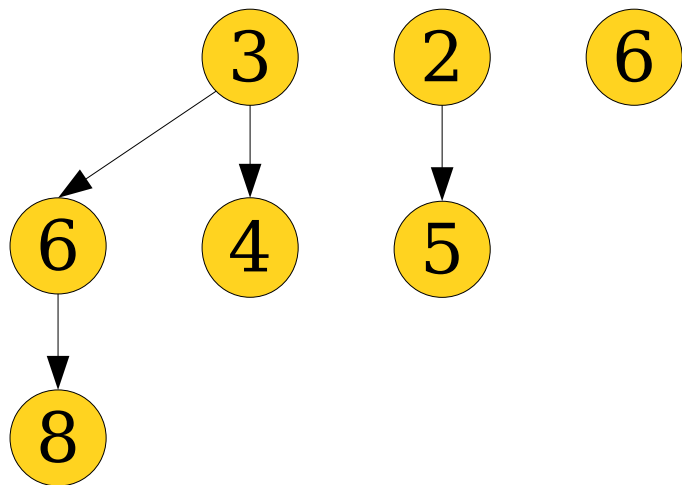


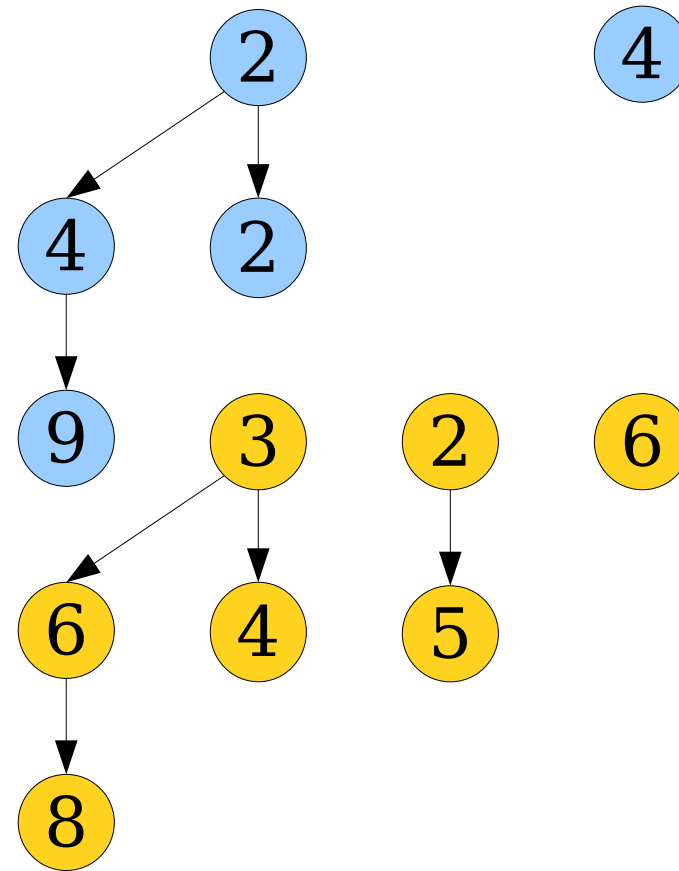


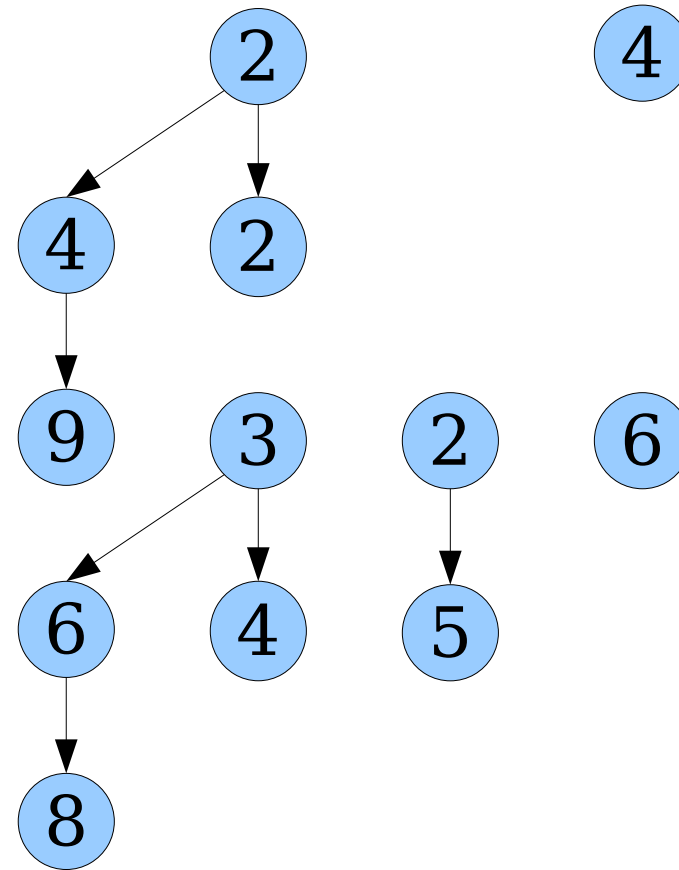


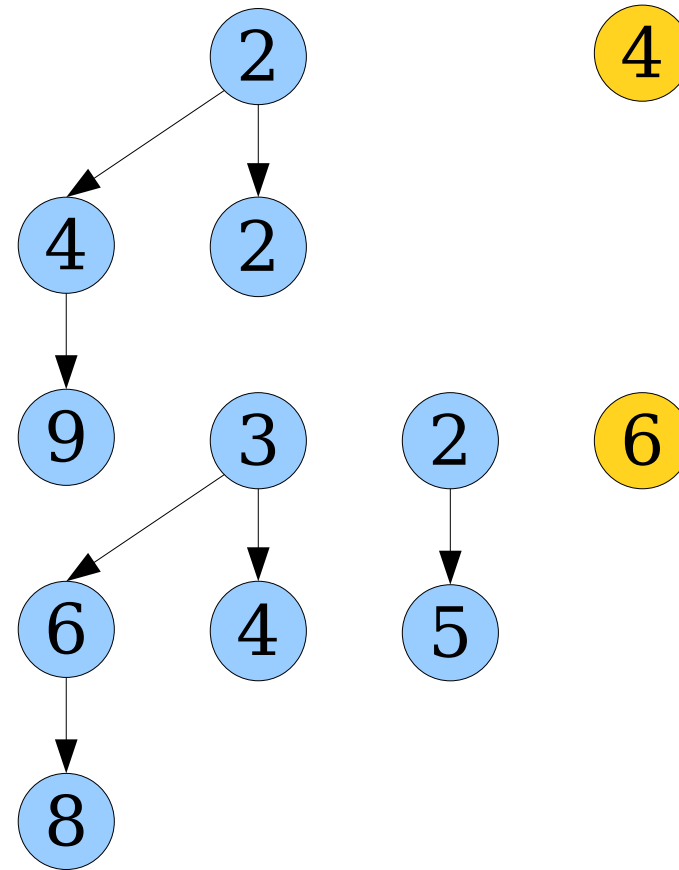


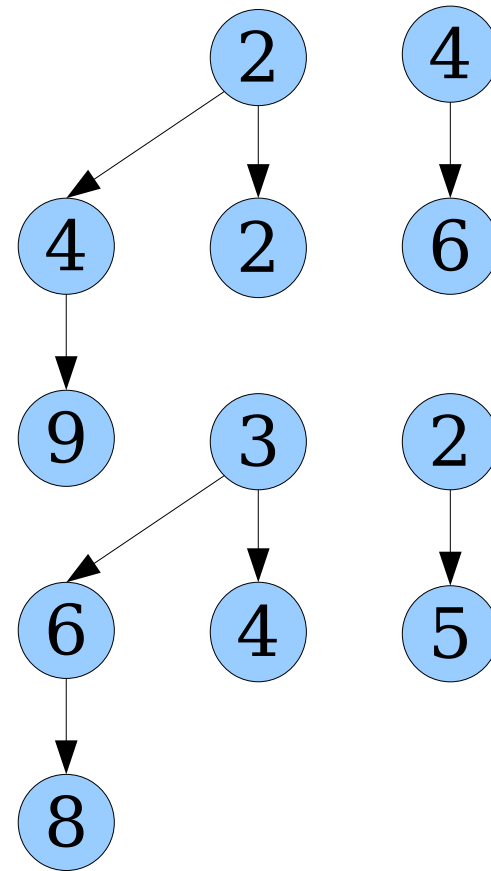


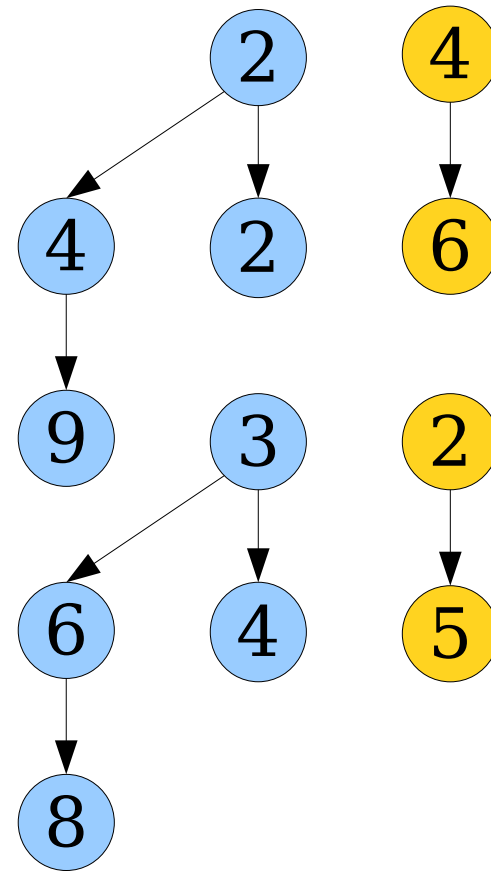


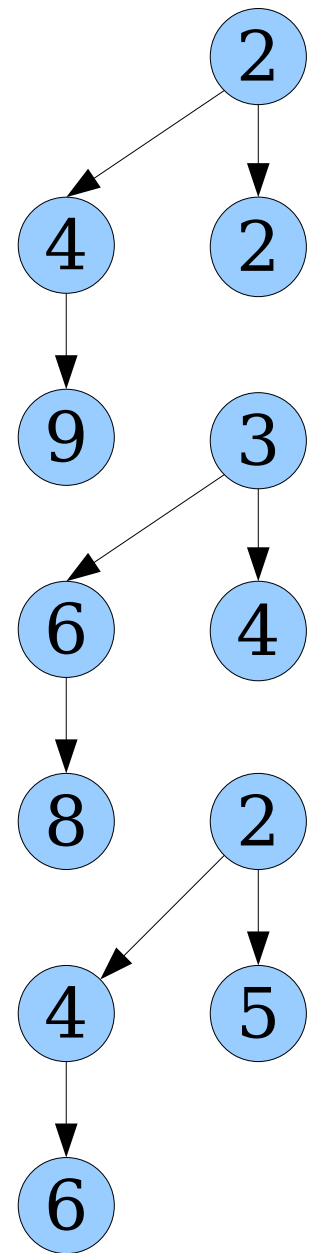




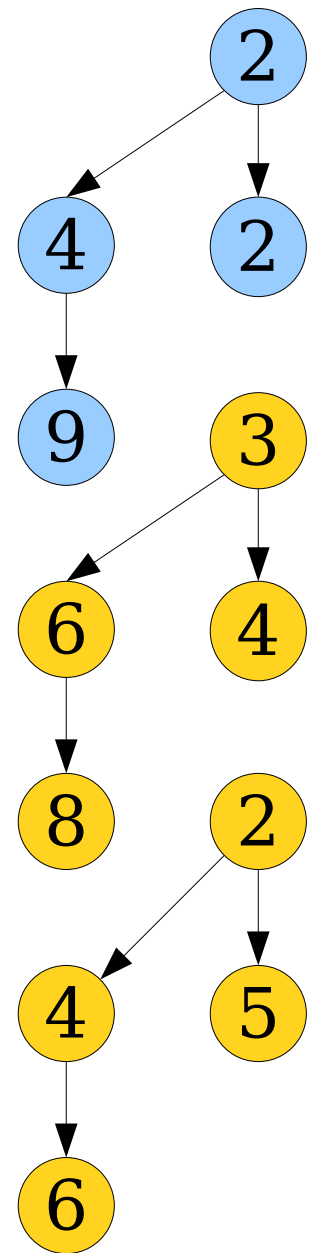


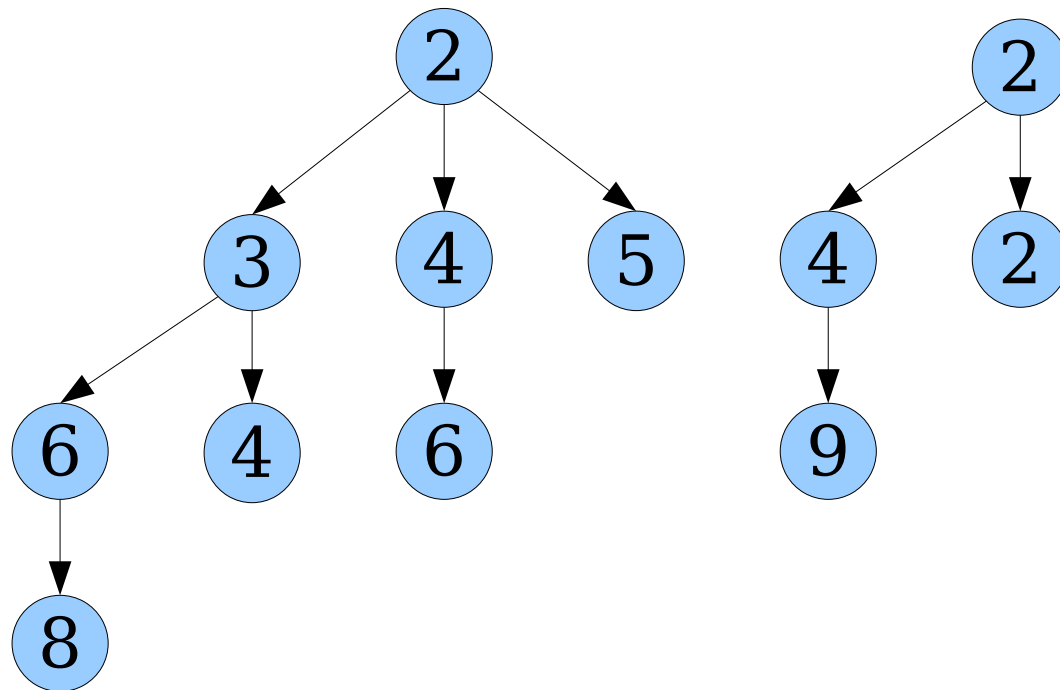












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Draw what happens if we *enqueue* the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.

1

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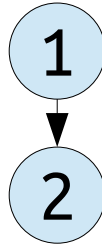
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1

2

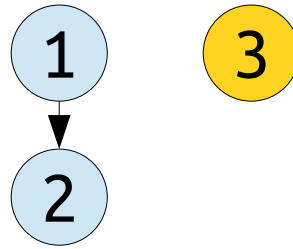
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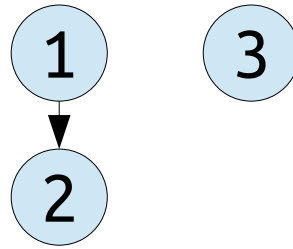
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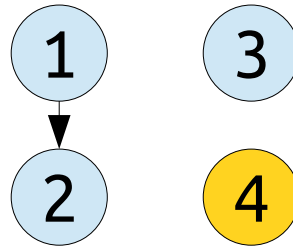
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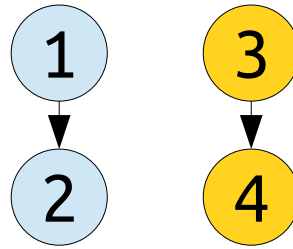
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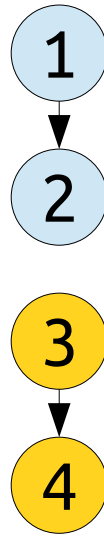
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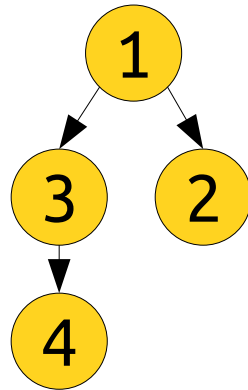
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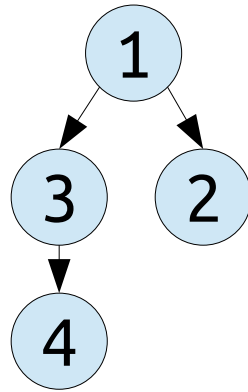
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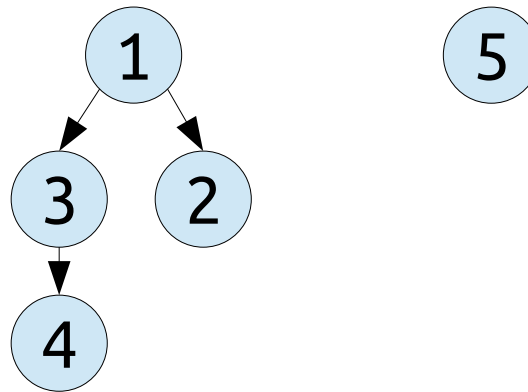
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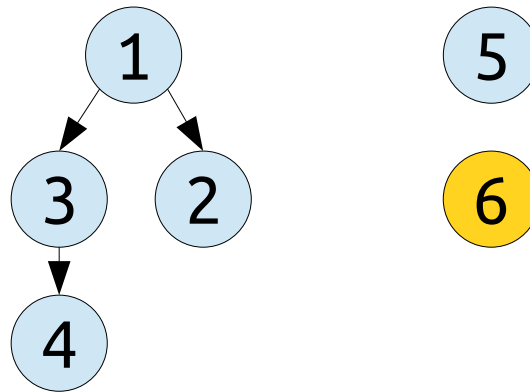
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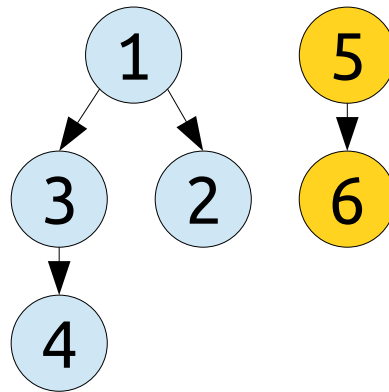
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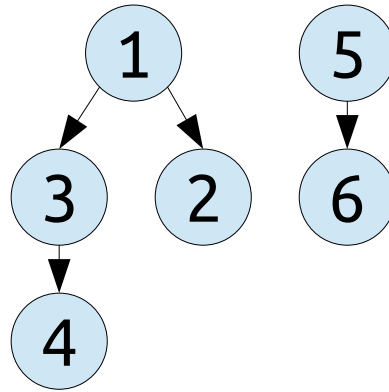
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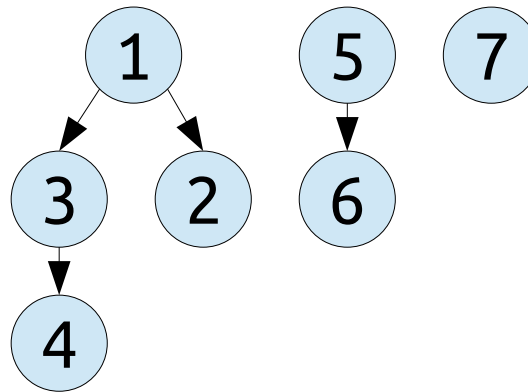
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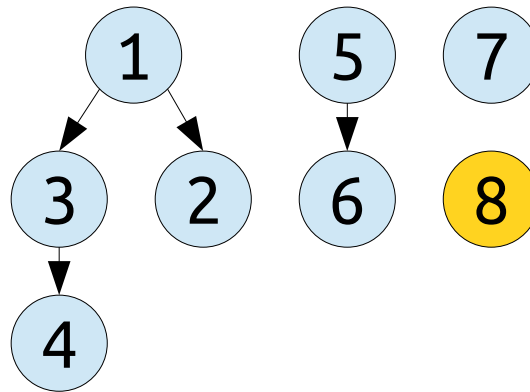
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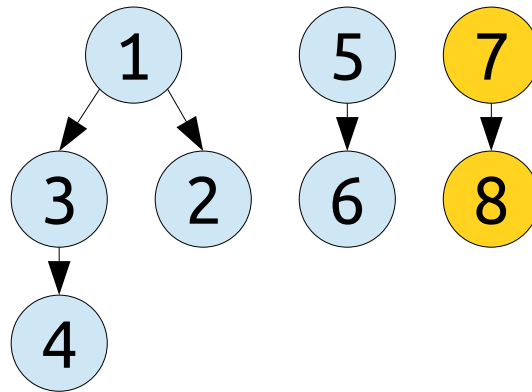
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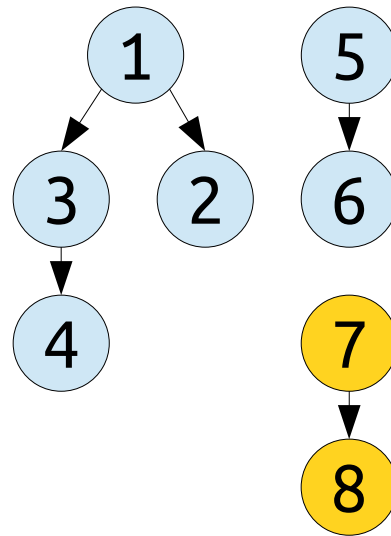
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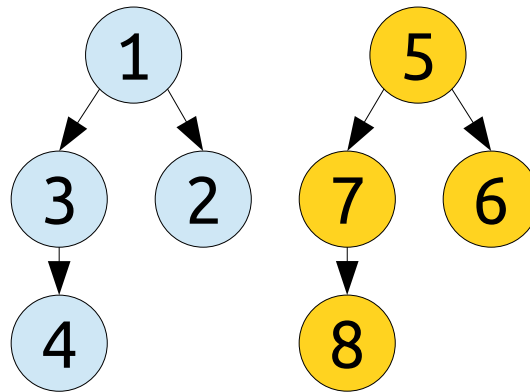
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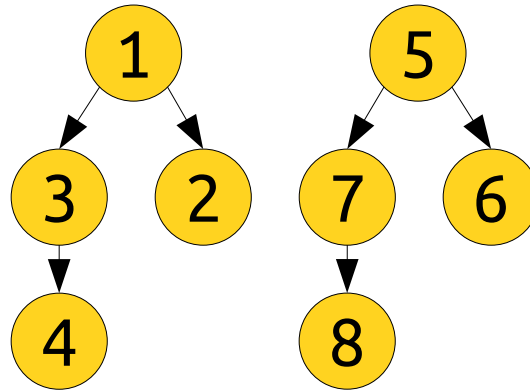
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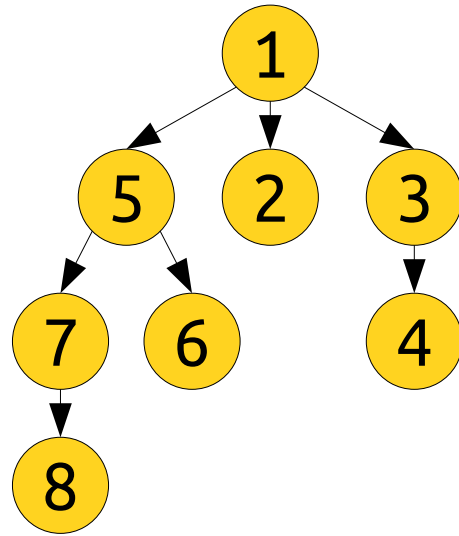
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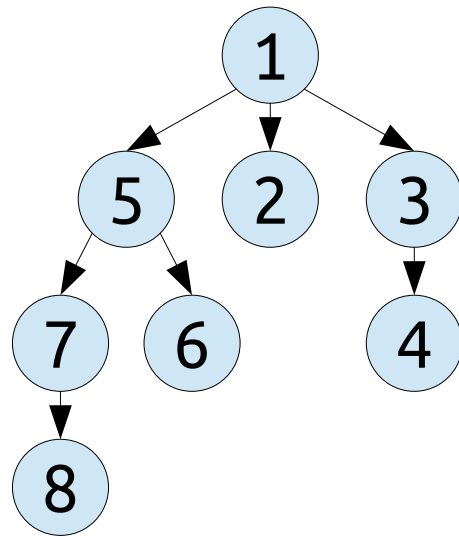
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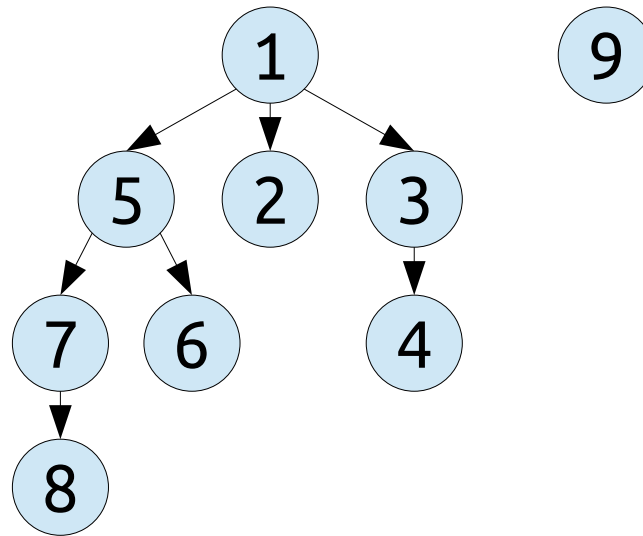
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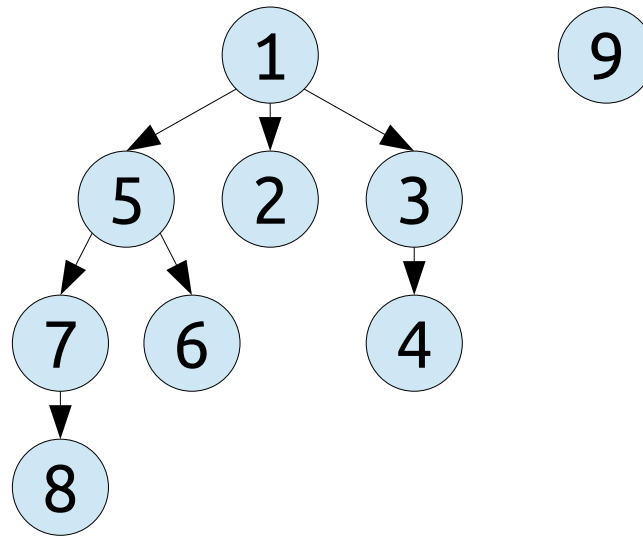
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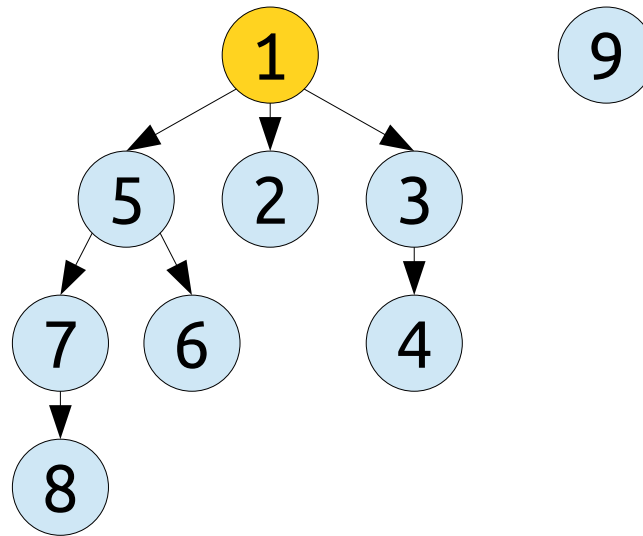
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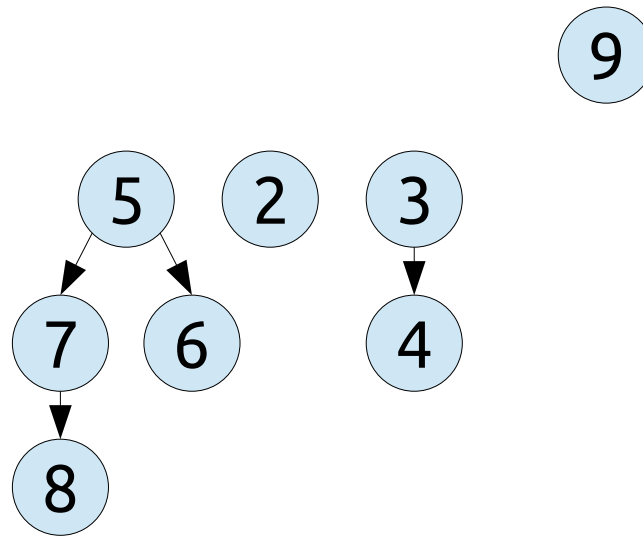
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Draw what happens after performing an ***extract-min*** in this binomial heap.



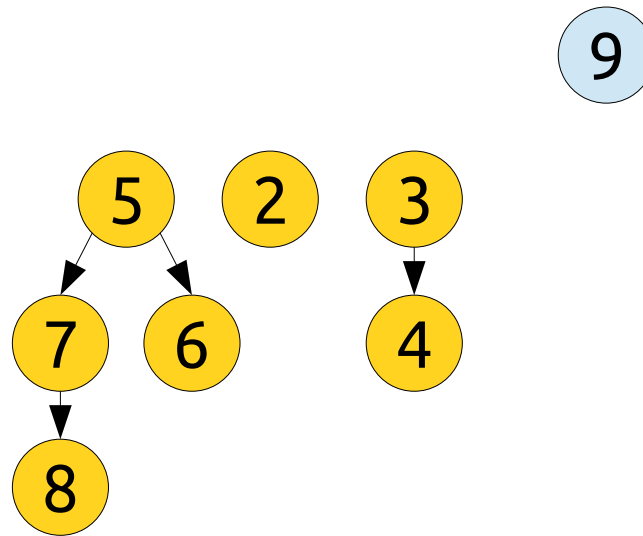
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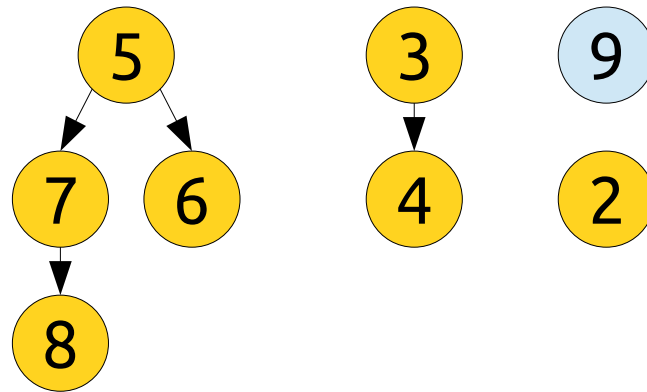
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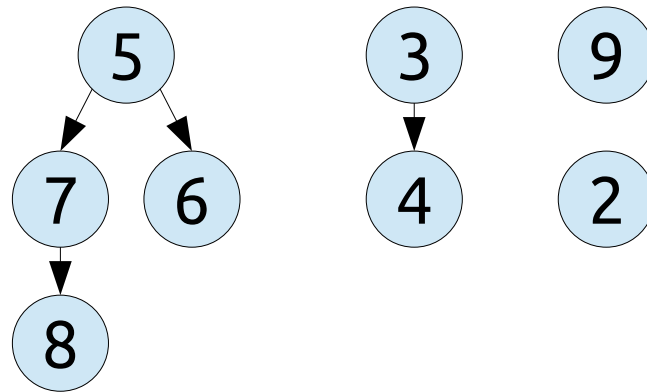
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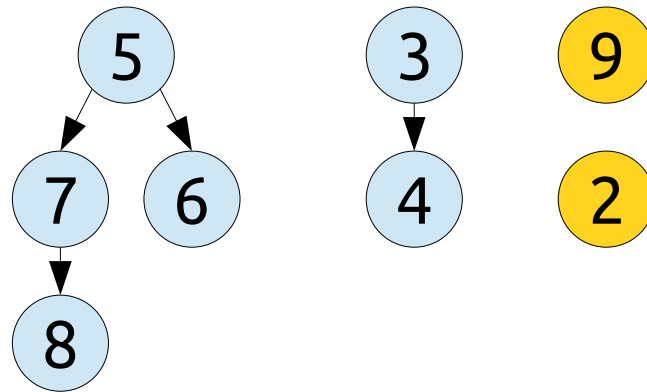
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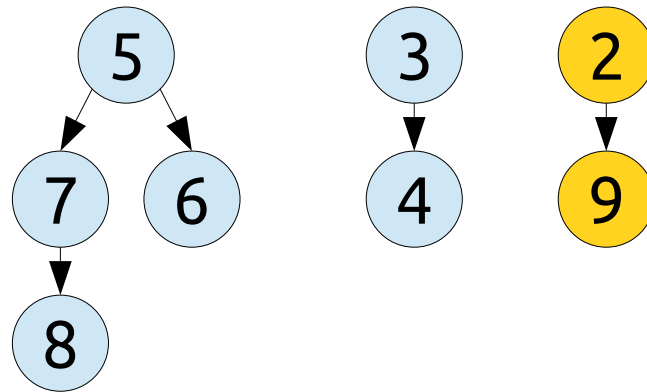
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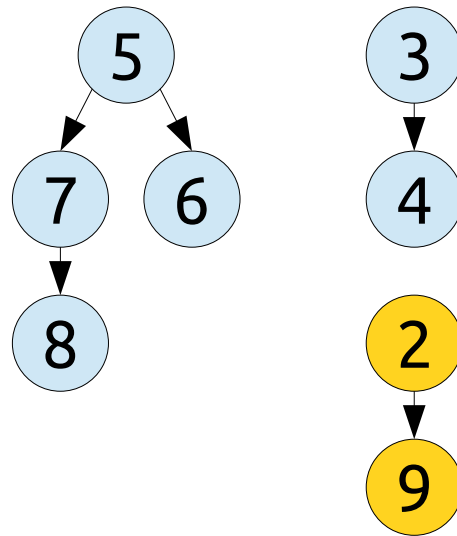
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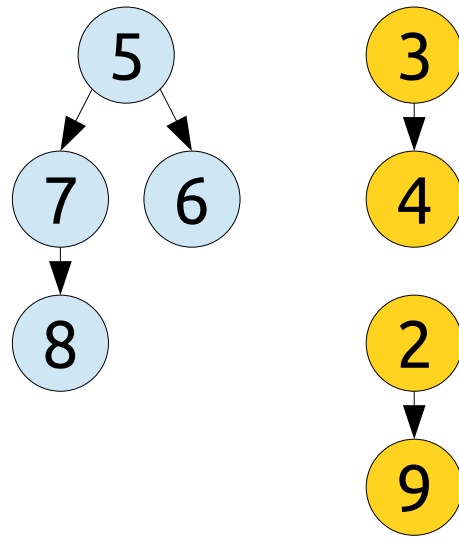
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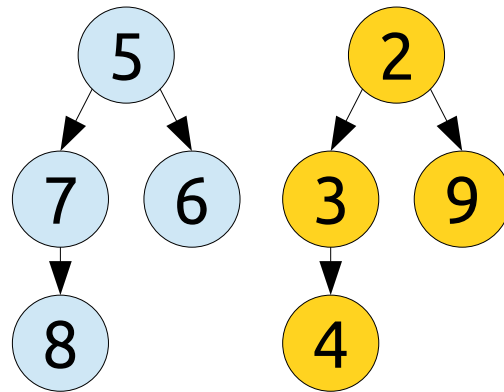
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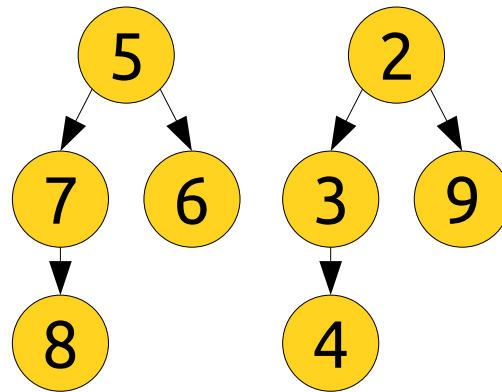
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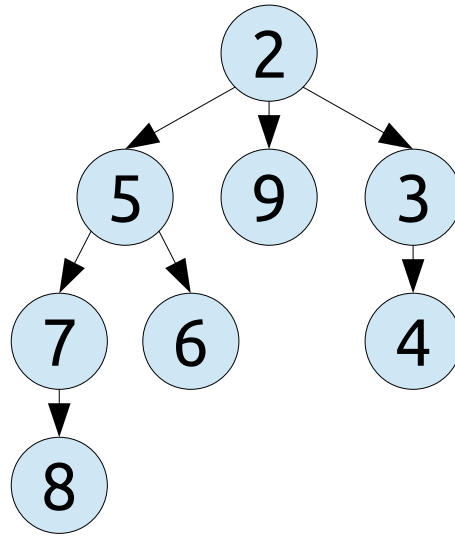
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Draw what happens after performing an ***extract-min*** in this binomial heap.



---

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---

Draw what happens after performing an ***extract-min*** in this binomial heap.

# Where We Stand

- Here's the current scorecard for the binomial heap.
- This is a fast, elegant, and clever data structure.
- **Question:** Can we do better?

## Binomial Heap

- **enqueue**:  $O(\log n)$
- **find-min**:  $O(\log n)$
- **extract-min**:  $O(\log n)$
- **meld**:  $O(\log m + \log n)$ .



# Where We Stand

- **Theorem:** No comparison-based priority queue structure can have **enqueue** and **extract-min** each take time  $o(\log n)$ .
- **Proof:** Suppose these operations each take time  $o(\log n)$ . Then we could sort  $n$  elements by perform  $n$  **enqueues** and then  $n$  **extract-mins** in time  $o(n \log n)$ . This is impossible with comparison-based algorithms. ■

## Binomial Heap

- **enqueue**:  $O(\log n)$
- **find-min**:  $O(\log n)$
- **extract-min**:  $O(\log n)$
- **meld**:  $O(\log m + \log n)$ .

# Where We Stand

- We can't make both **enqueue** and **extract-min** run in time  $O(\log n)$ .
- However, we could conceivably make one of them faster.
- **Question:** Which one should we prioritize?
- Probably **enqueue**, since we aren't guaranteed to have to remove all added items.
- **Goal:** Make **enqueue** take time  $O(1)$ .

## Binomial Heap

- **enqueue**:  $O(\log n)$
- **find-min**:  $O(\log n)$
- **extract-min**:  $O(\log n)$
- **meld**:  $O(\log m + \log n)$ .

# Where We Stand

- The *enqueue* operation is implemented in terms of *meld*.
- If we want *enqueue* to run in time  $O(1)$ , we'll need *meld* to take time  $O(1)$ .
- How could we accomplish this?

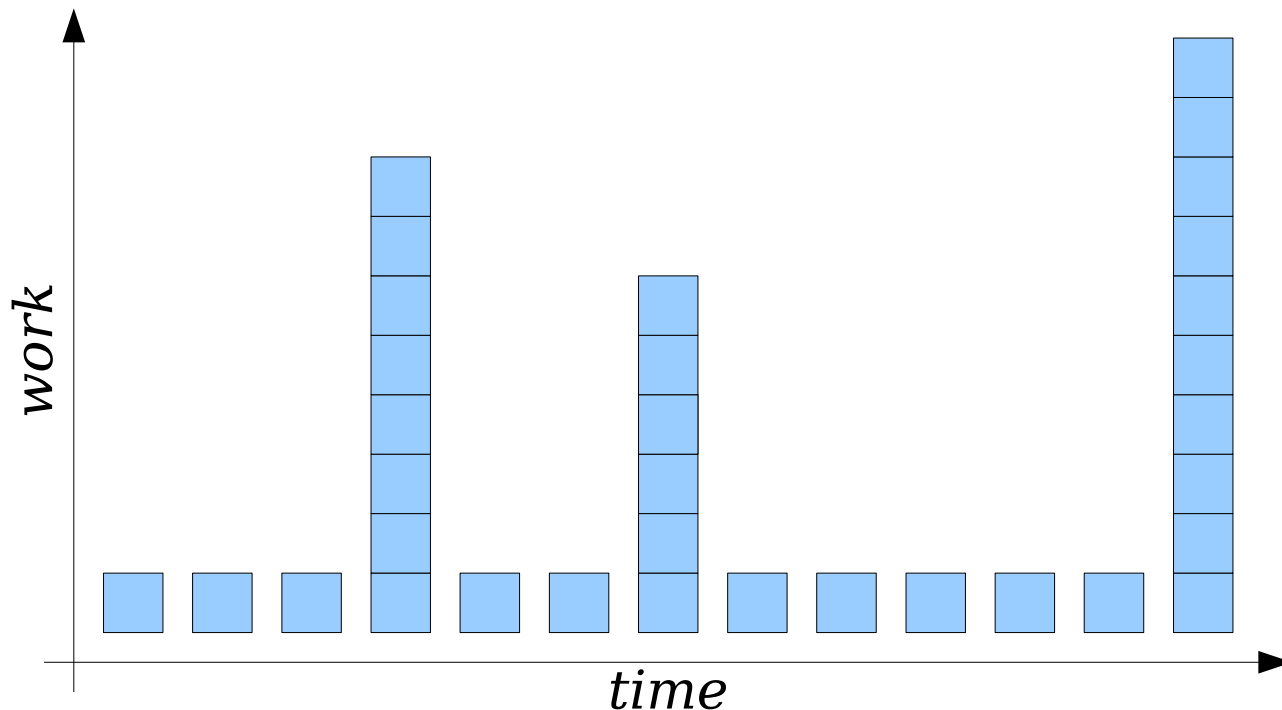
## Binomial Heap

- *enqueue*:  $O(\log n)$
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# Thinking With Amortization

# Refresher: Amortization

- In an amortized efficient data structure, some operations can take much longer than others, provided that previous operations didn't take too long to finish.
- Think dishwashers: you may have to do a big cleanup at some point, but that's because you did basically no work to wash all the dishes you placed in the dishwasher.

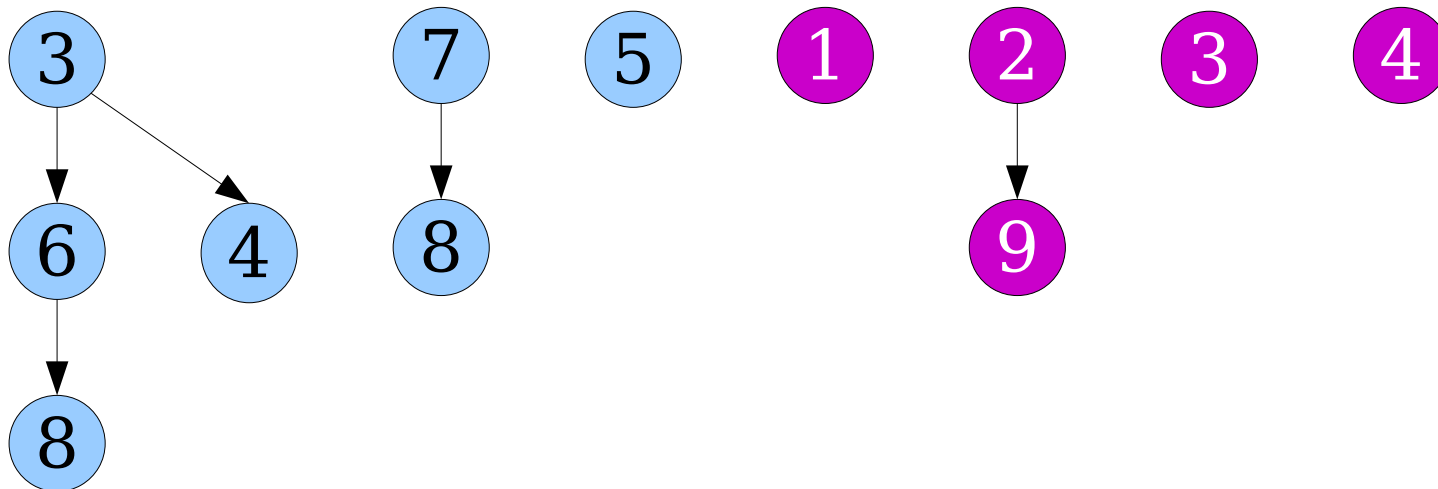


# Lazy Melding

- Consider the following lazy *melding* approach:

*To meld together two binomial heaps, just combine the two sets of trees together.*

- Intuition:** Why do any work to organize keys if we're not going to do an *extract-min*? We'll worry about cleanup then.

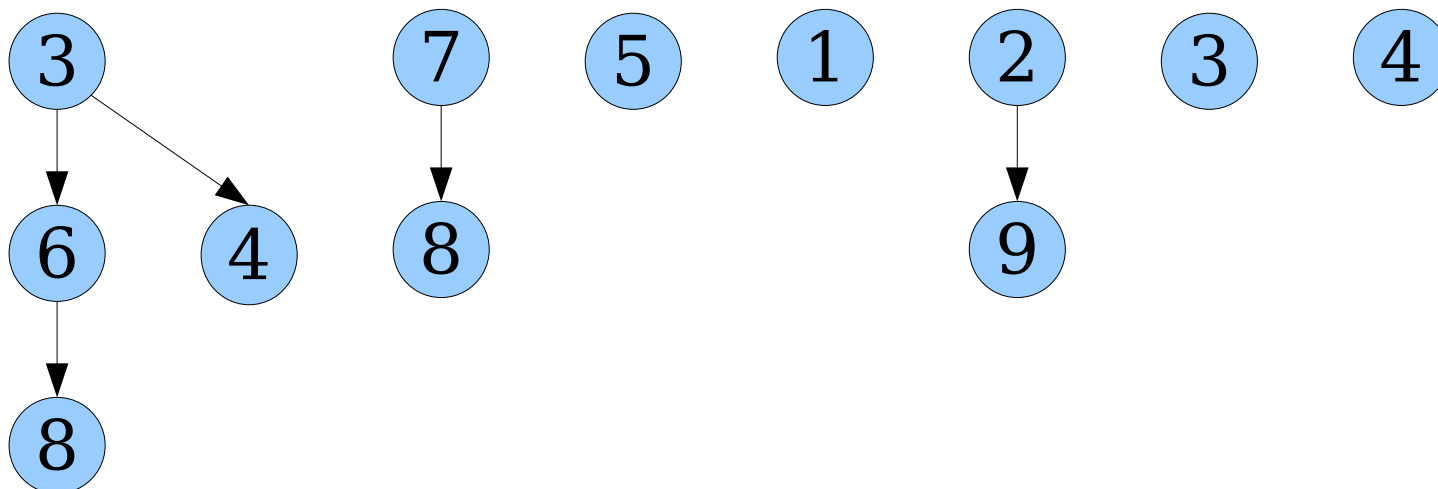


# Lazy Melding

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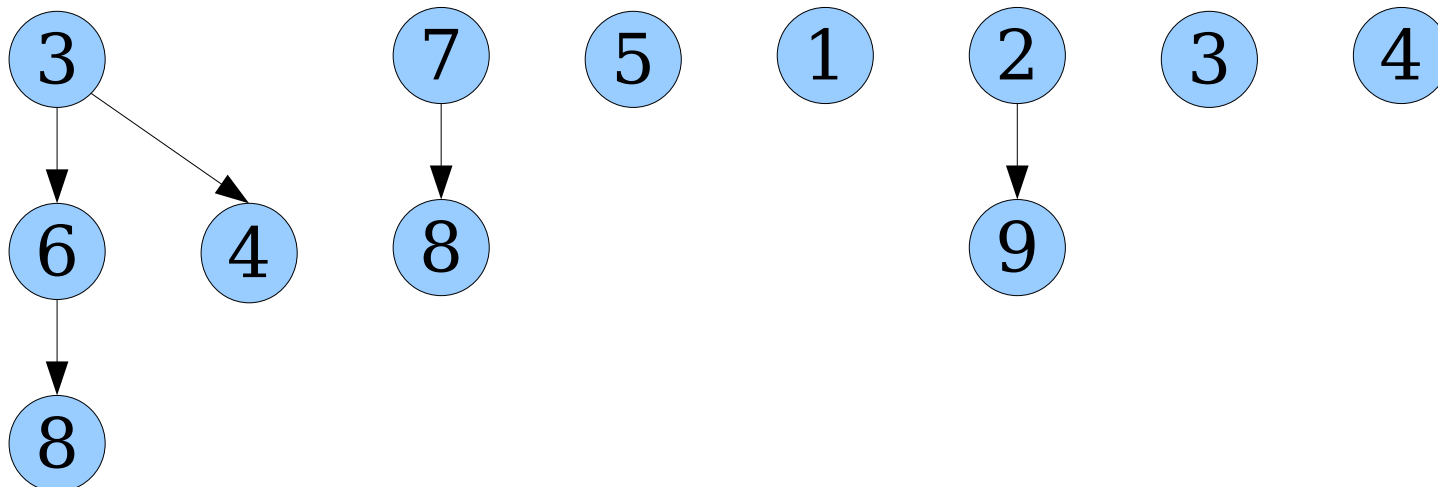
*To meld together two binomial heaps, just combine the two sets of trees together.*

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# Lazy Merging

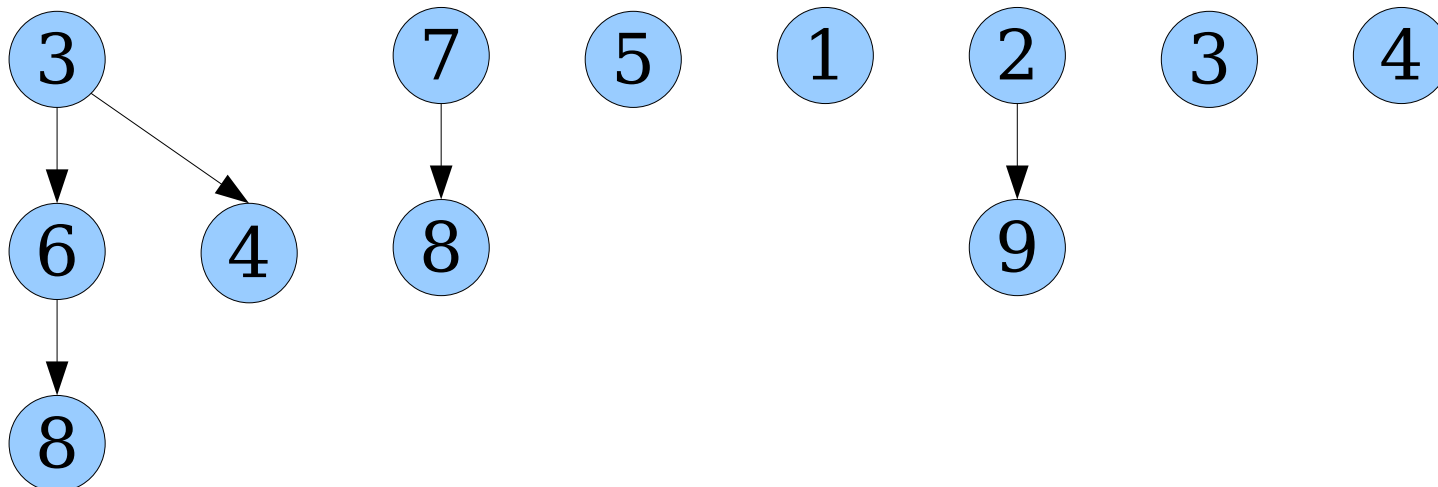
- If we store our list of trees as circularly, doubly-linked lists, we can concatenate tree lists in time  $O(1)$ .
  - Cost of a *meld*:  **$O(1)$** .
  - Cost of an *enqueue*:  **$O(1)$** .
- If it sounds too good to be true, it probably is.





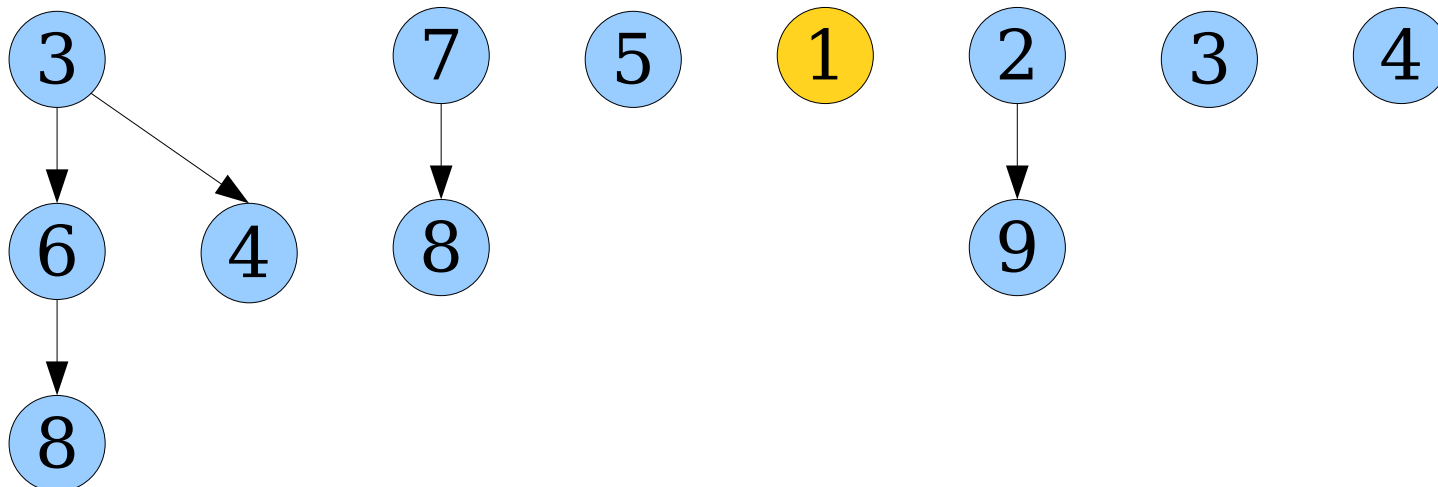
# Lazy Melding

- Imagine that we implement *extract-min* the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



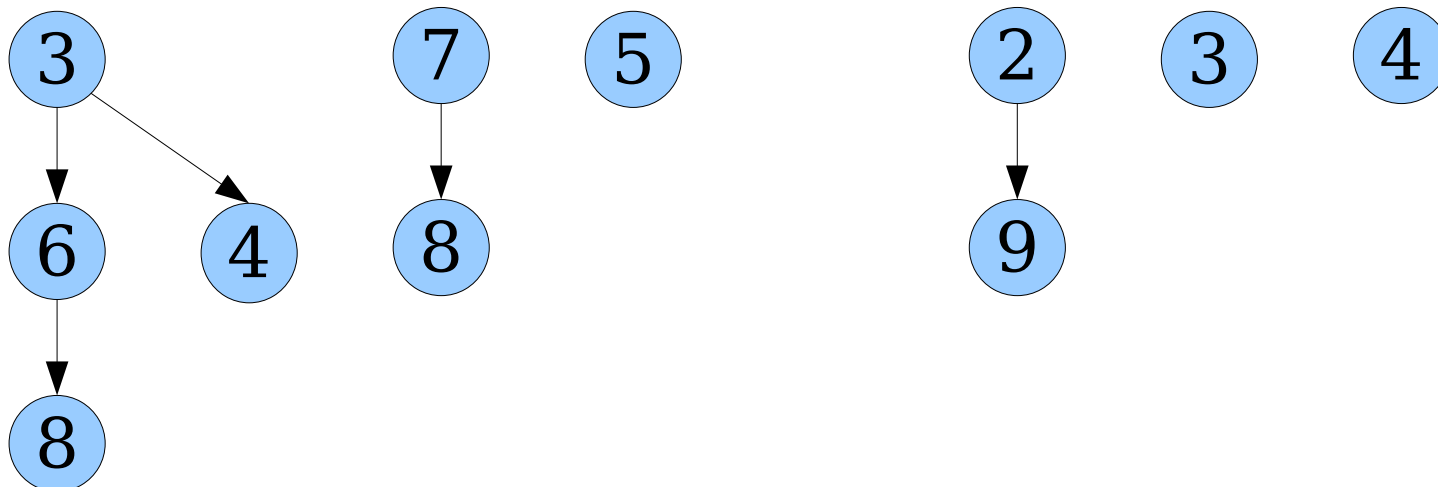
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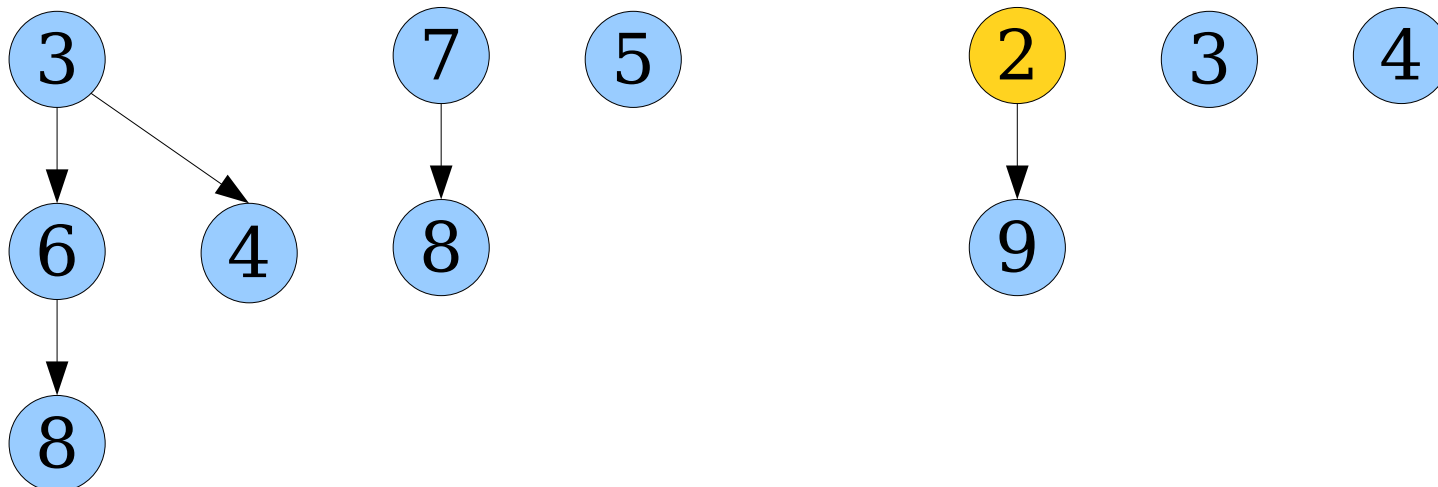
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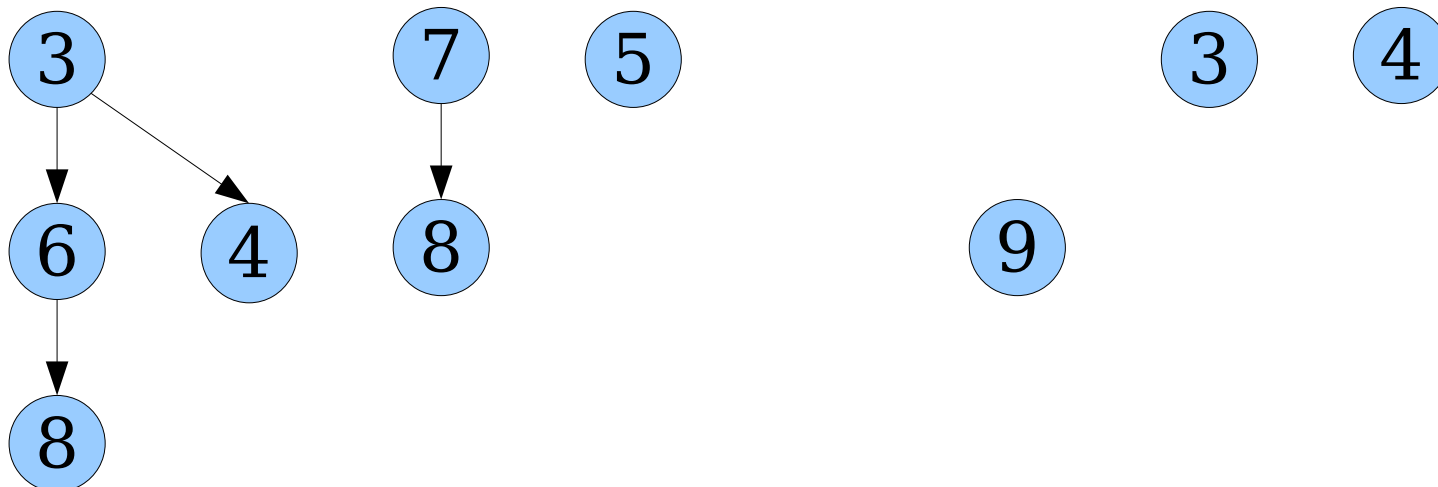
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  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



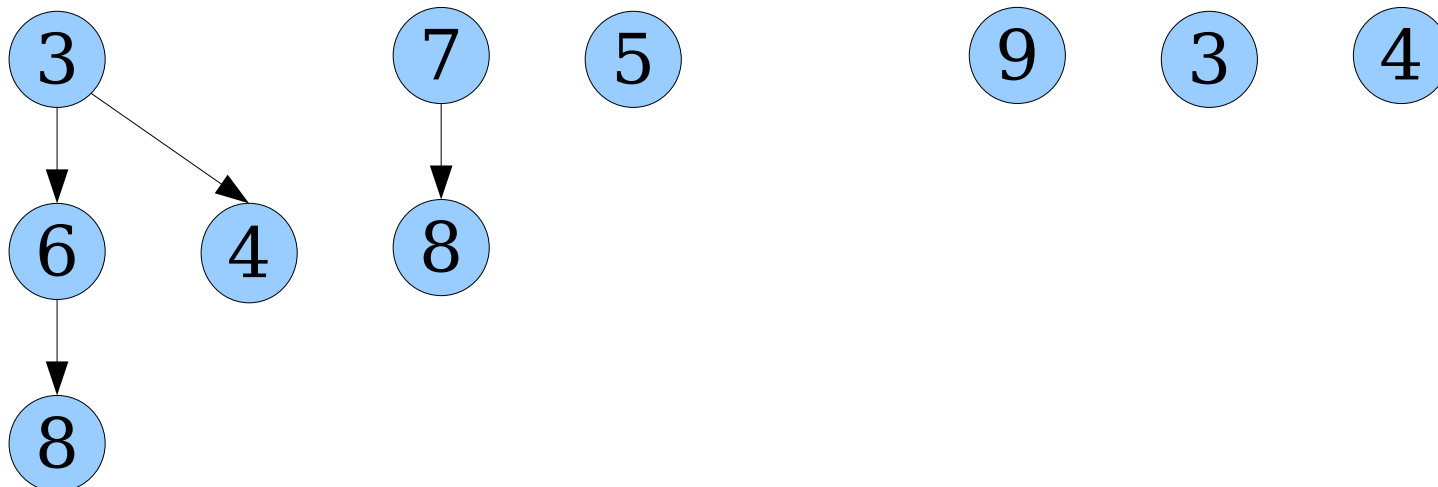
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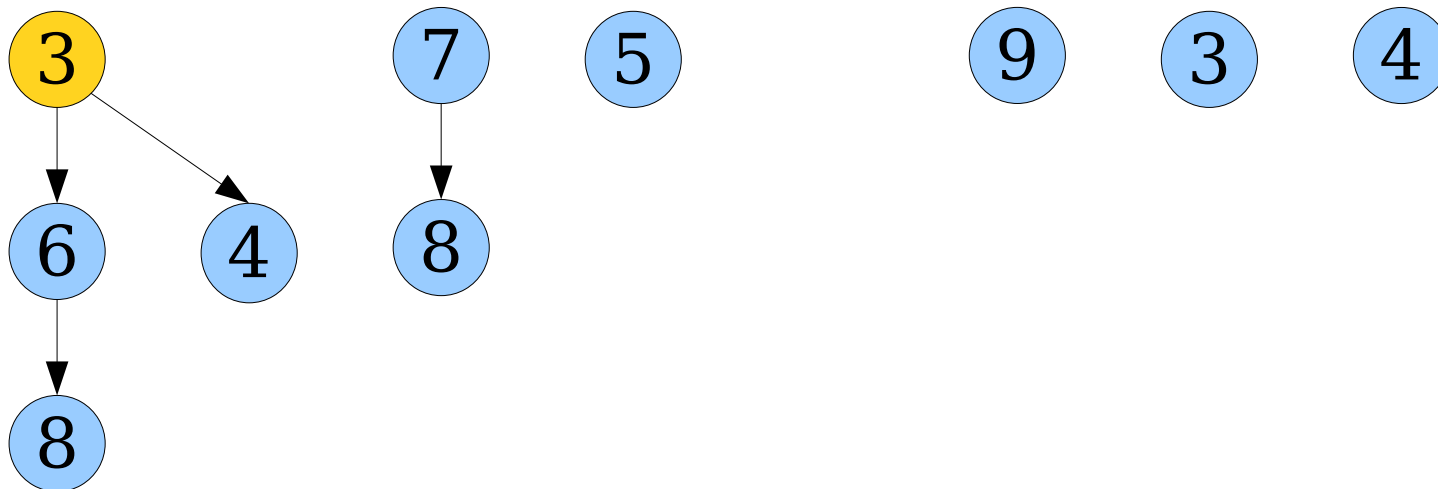
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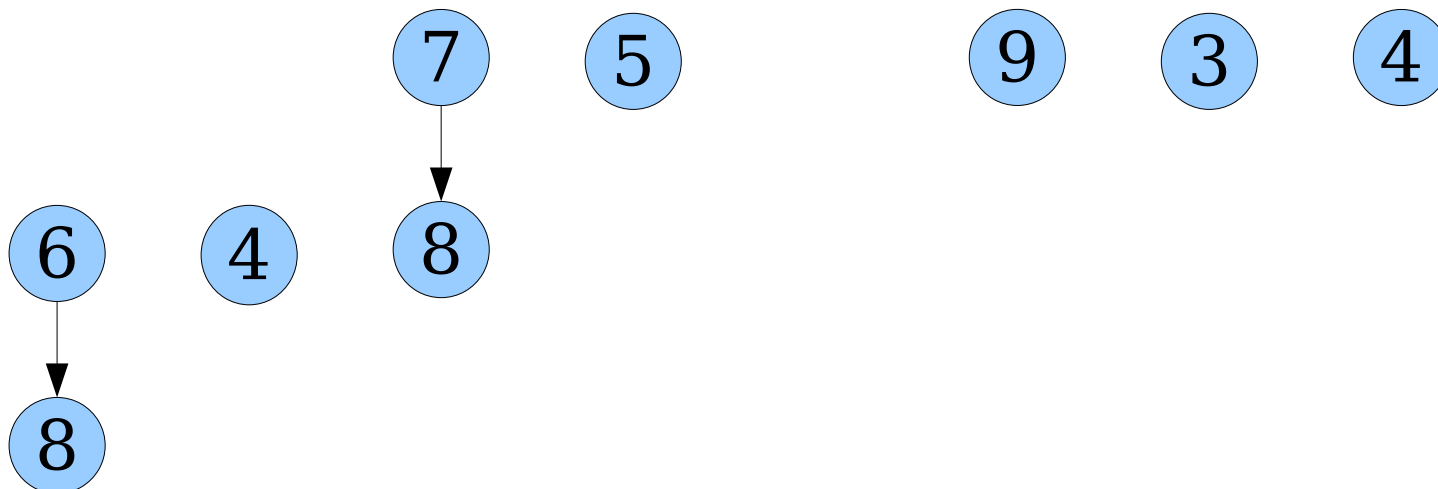
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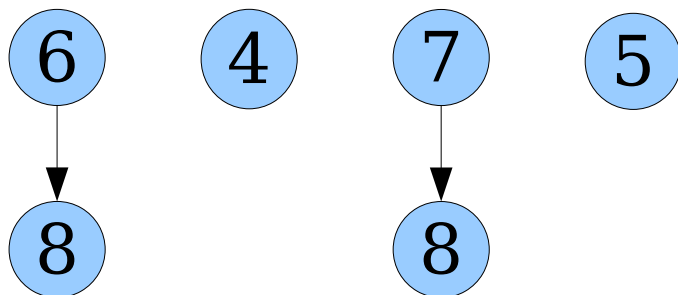
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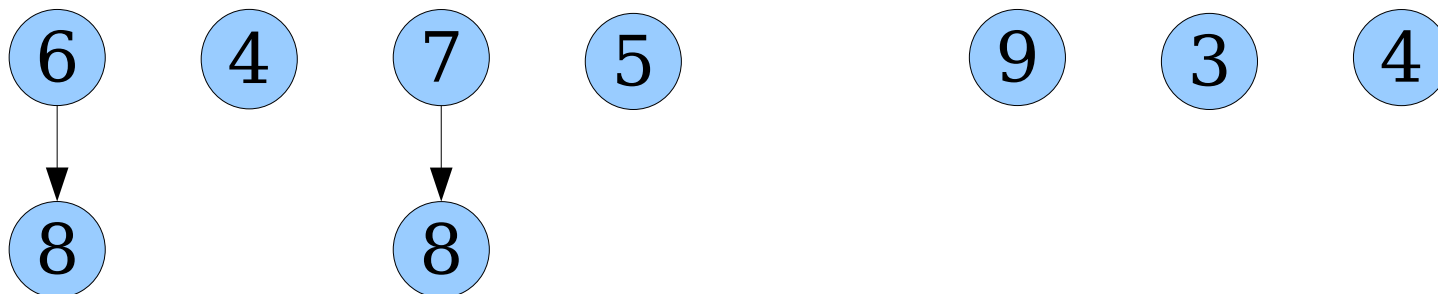
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Each pass of finding the minimum value takes time  $\Theta(n)$  in the worst case.  
We've lost our nice runtime guarantees!

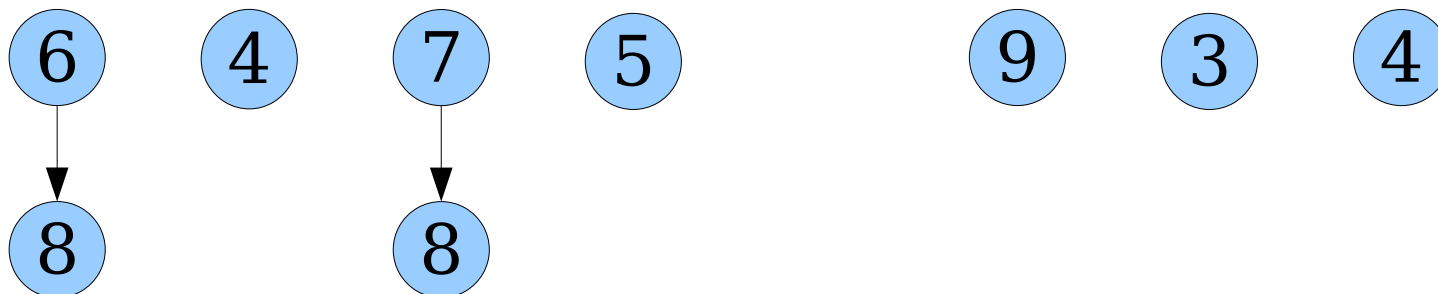
# Washing the Dishes

- Every *meld* (and *enqueue*) creates some “dirty dishes” (small trees) that we need to clean up later.
- If we never clean them up, then our *extract-min* will be too slow to be usable.
- **Idea:** Change *extract-min* to “wash the dishes” and make things look nice and pretty again.
- **Question:** What does “wash the dishes” mean here?



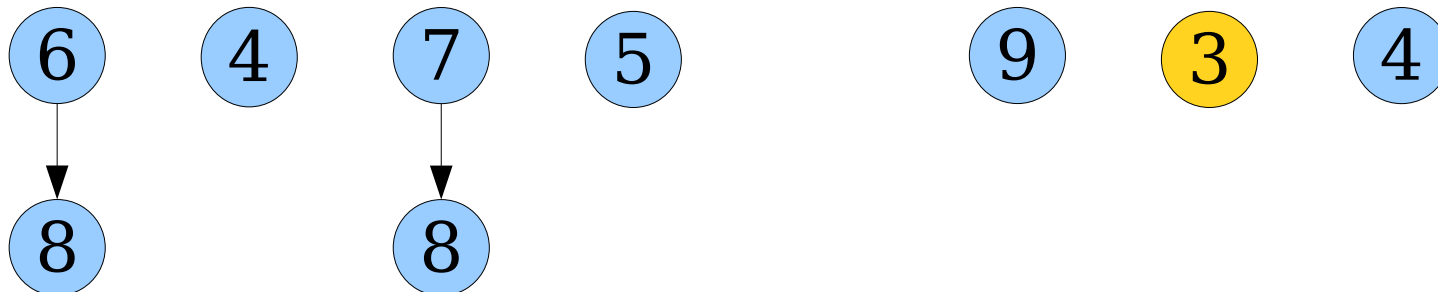
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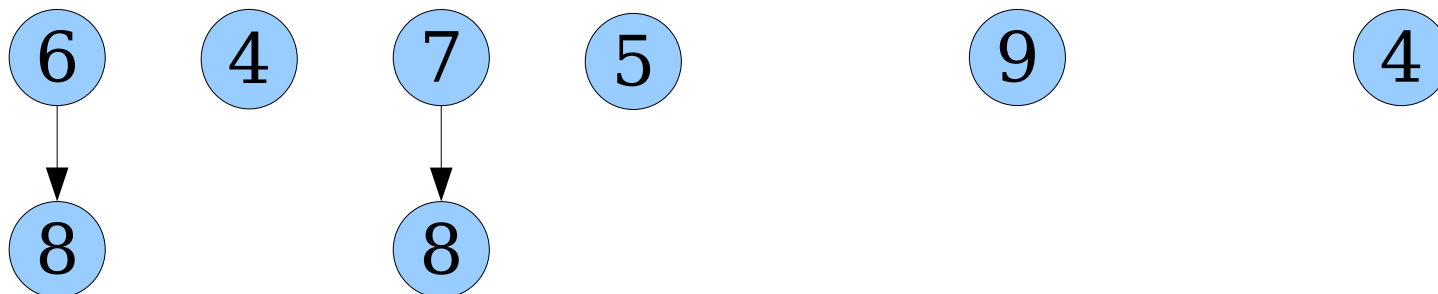
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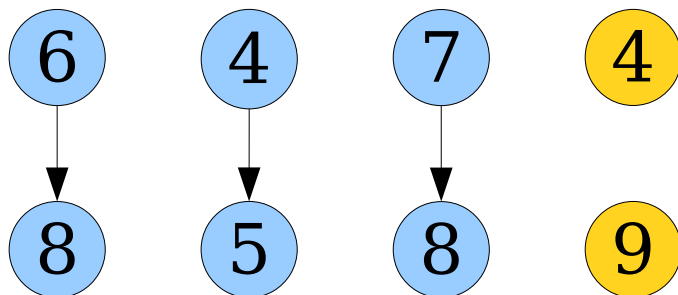
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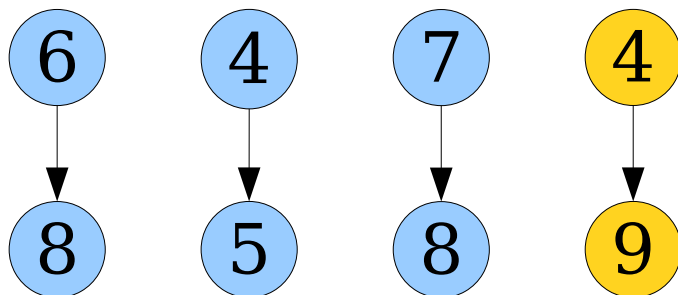
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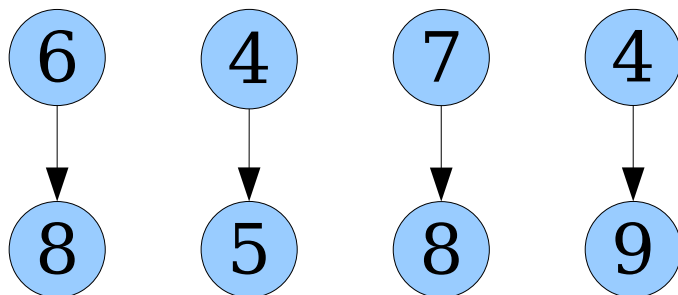
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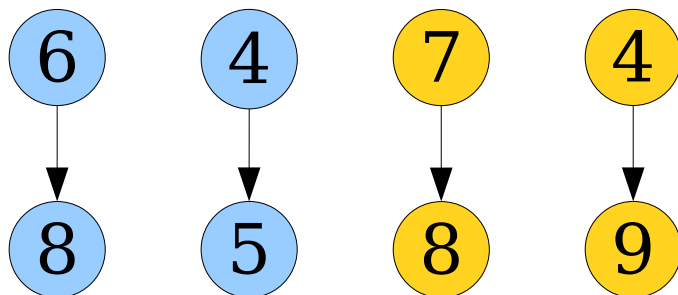
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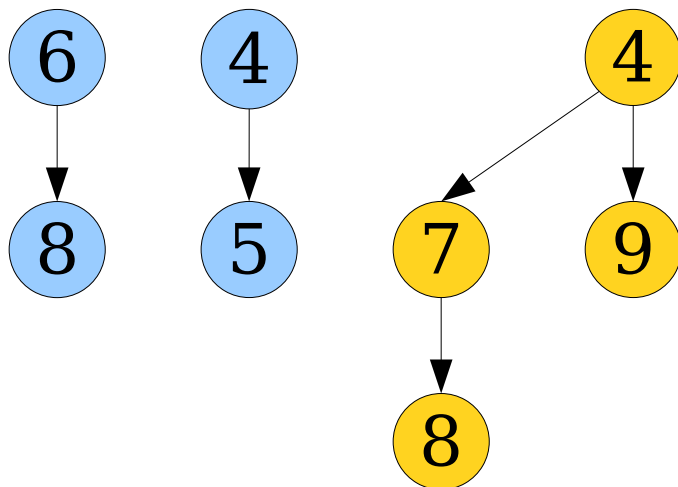
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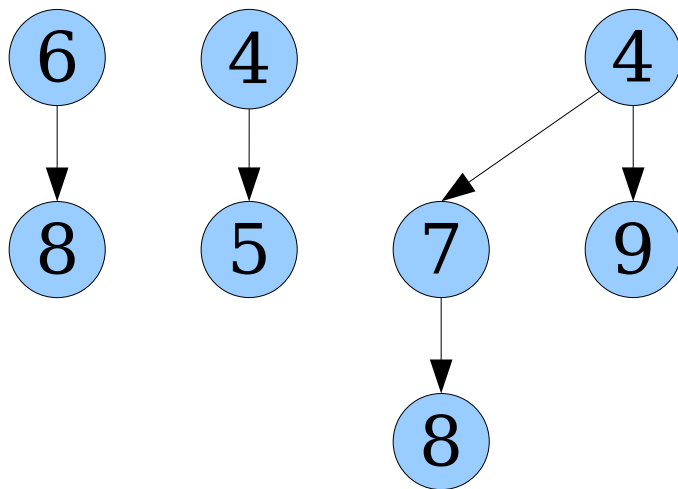
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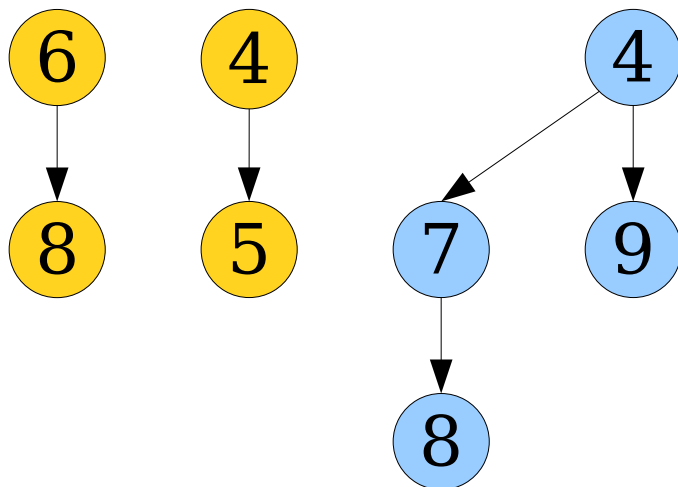
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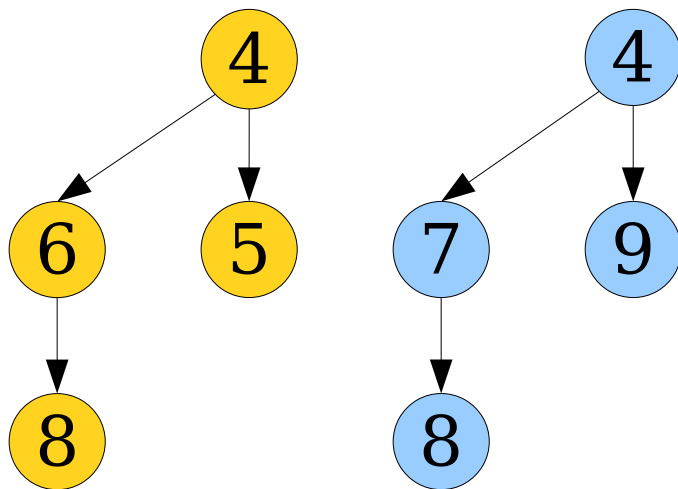
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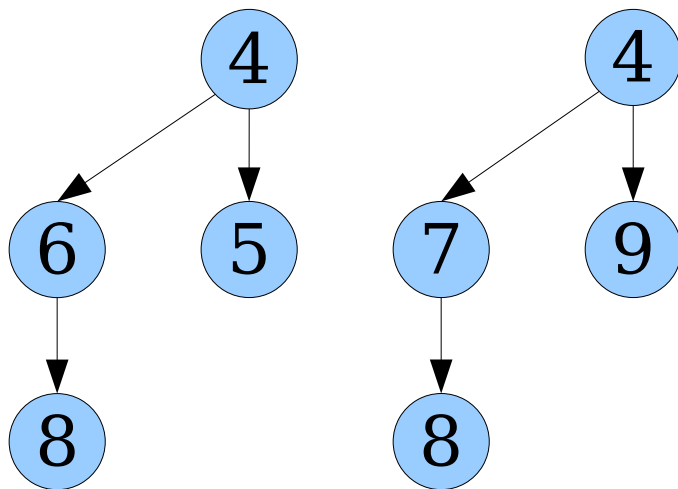
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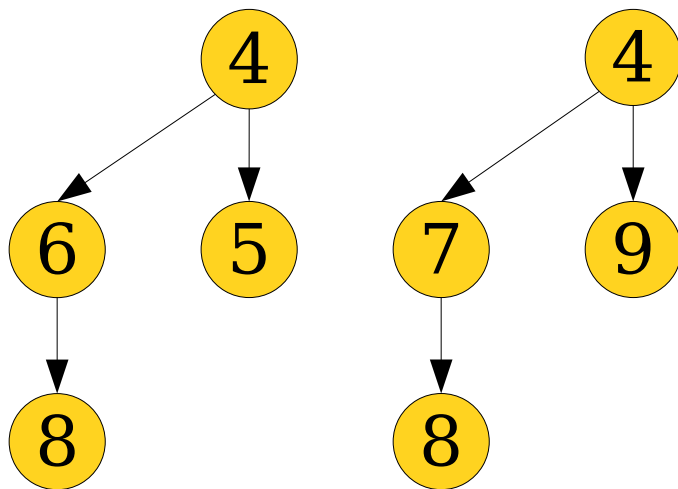
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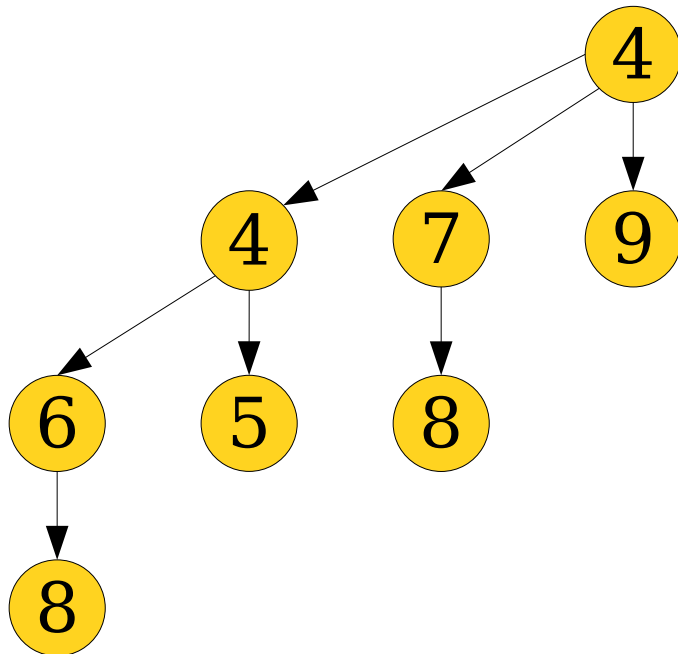
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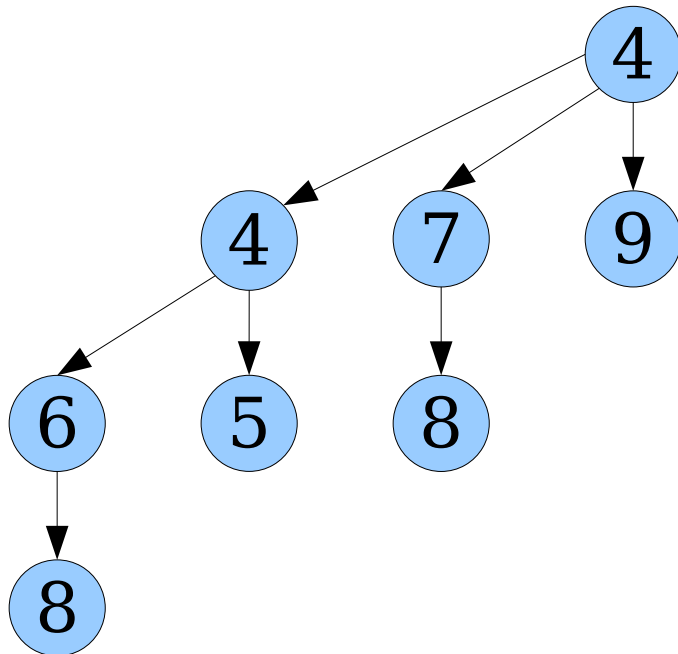
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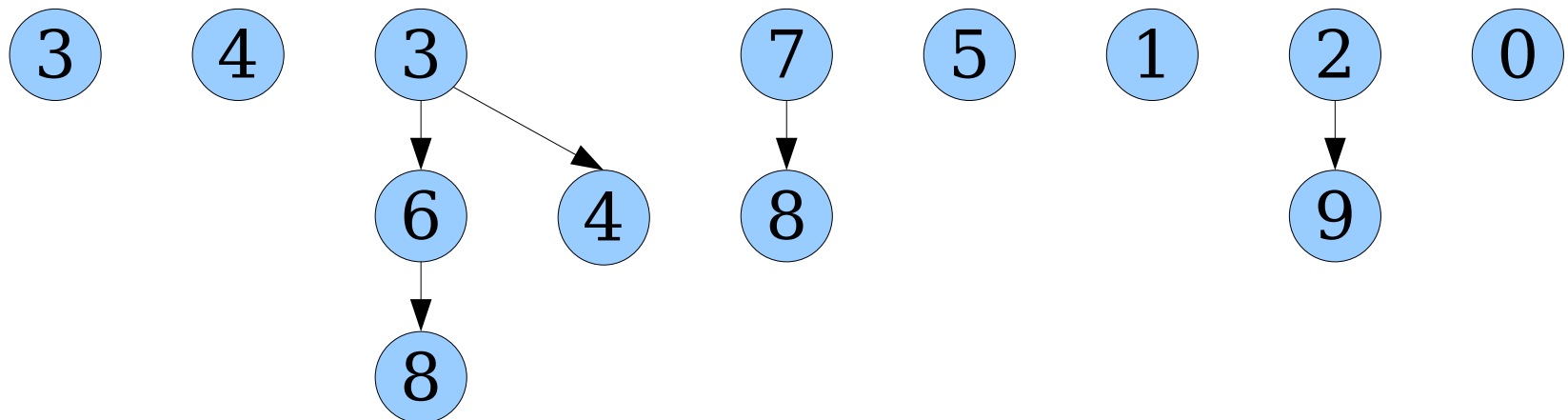
At this point, the mess is cleaned up, and we're left with what we would have had if we had been cleaning up as we go.

# Where We're Going

- A *lazy binomial heap* is a binomial heap, modified as follows:
  - The *meld* operation is lazy. It just combines the two groups of trees together.
  - After doing an *extract-min*, we do a *coalesce* to combine together trees until there's at most one tree of each order.
- Intuitively, we'd expect this to amortize away nicely, since the “mess” left by *meld* gets cleaned up later on by a future *extract-min*.
- Questions left to answer:
  - How do we efficiently implement the *coalesce* operation?
  - How efficient is this approach, in an amortized sense?

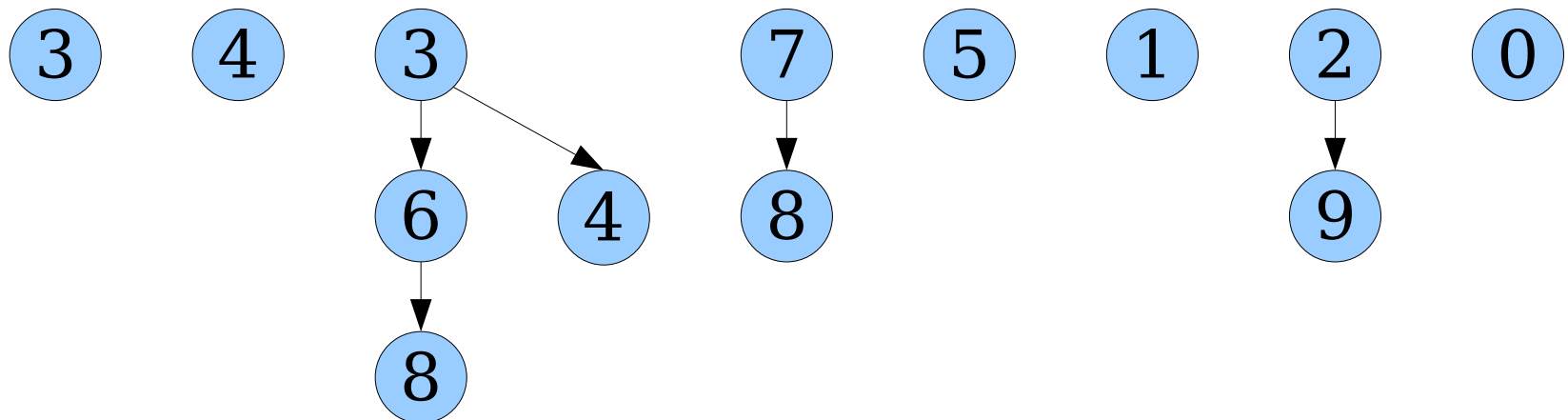
# Coalescing Trees

- The *coalesce* step repeatedly combines trees together until there's at most one tree of each order.
- How do we implement this so that it runs quickly?



# Coalescing Trees

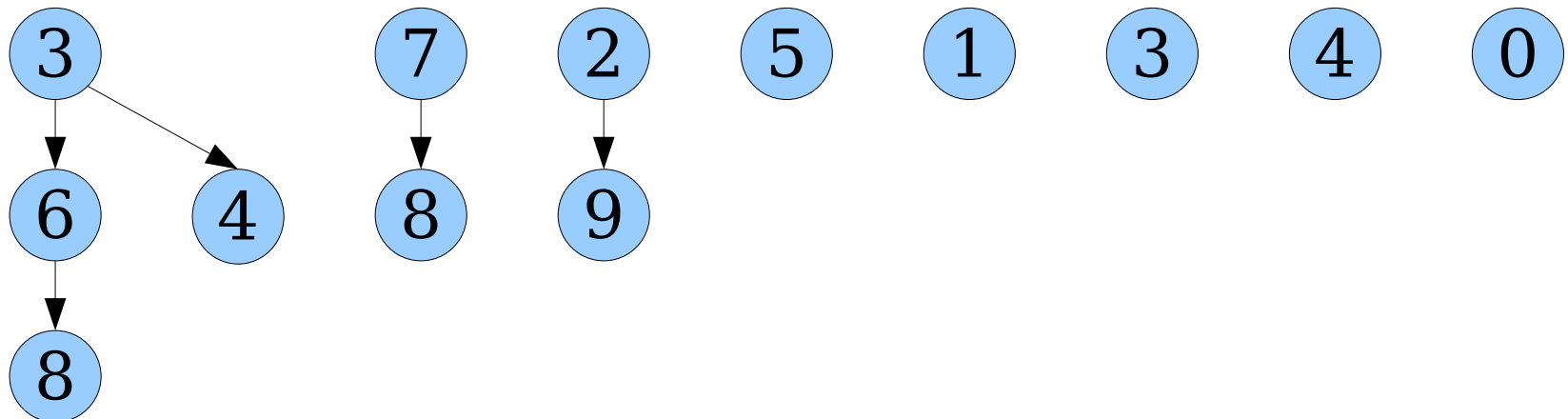
- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.





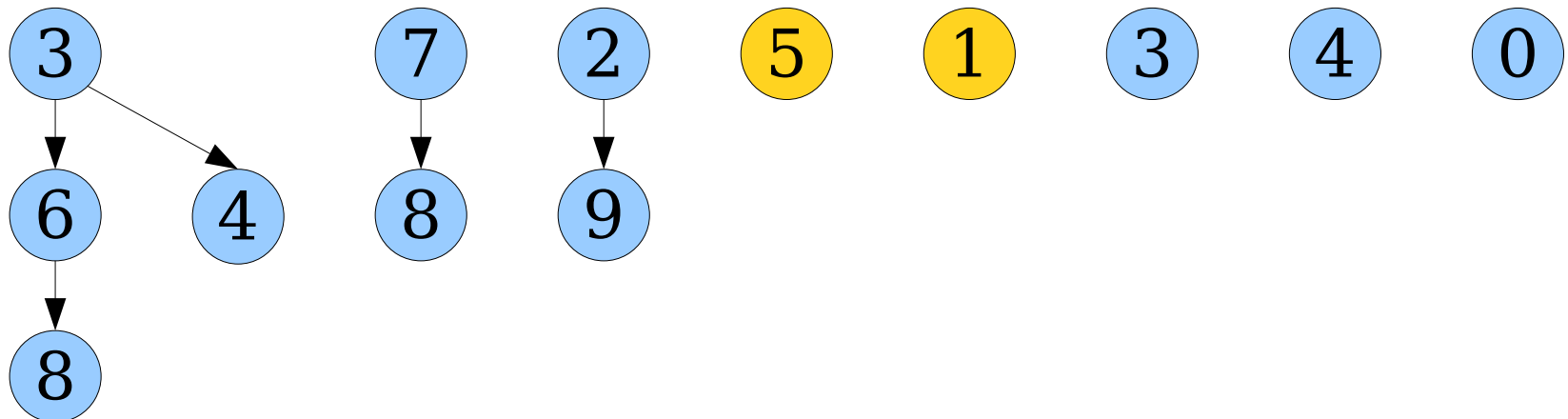
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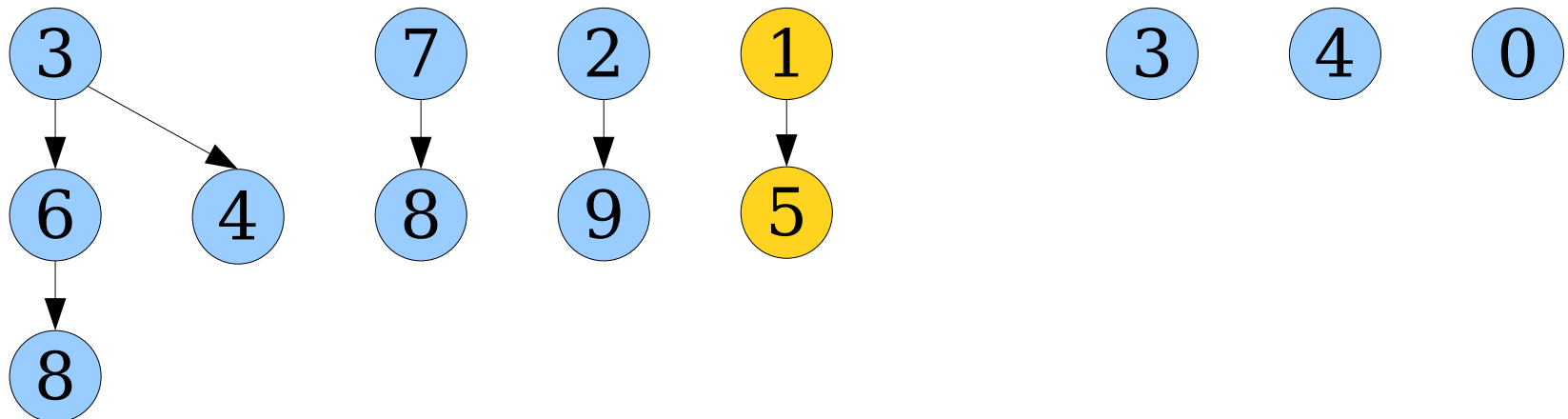
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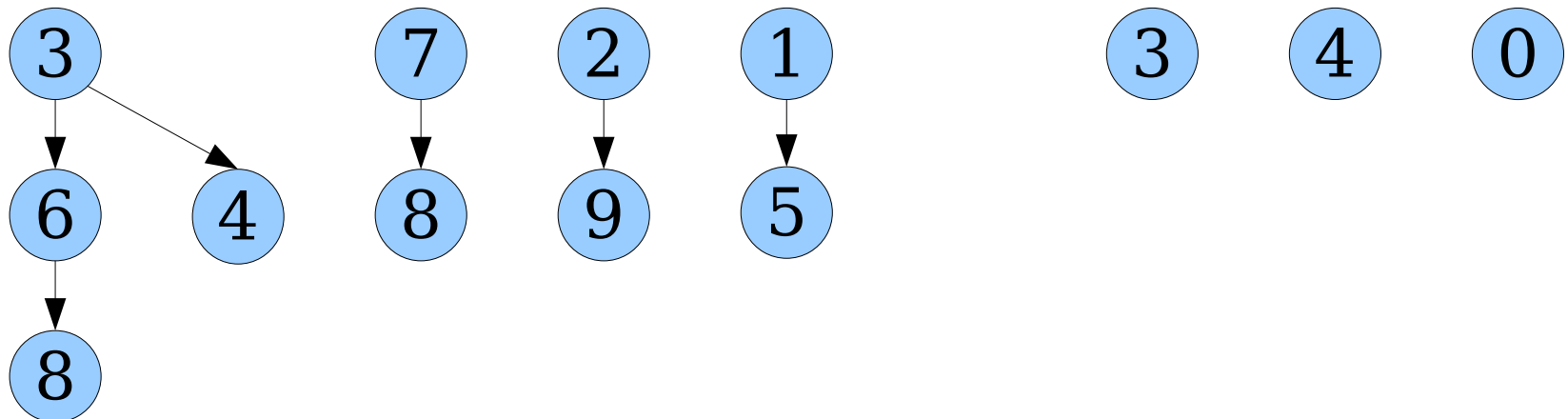
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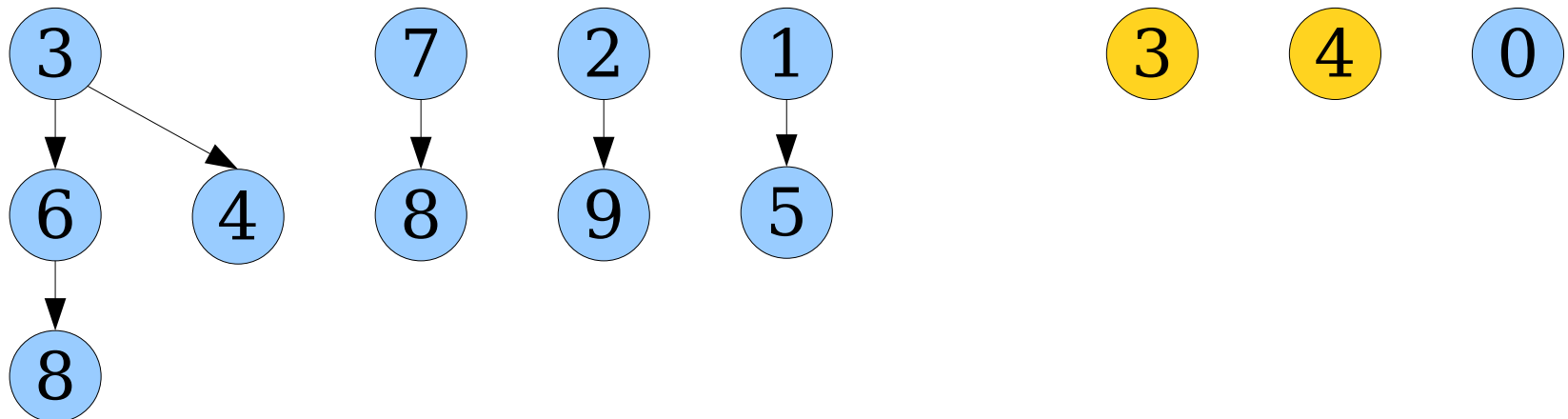
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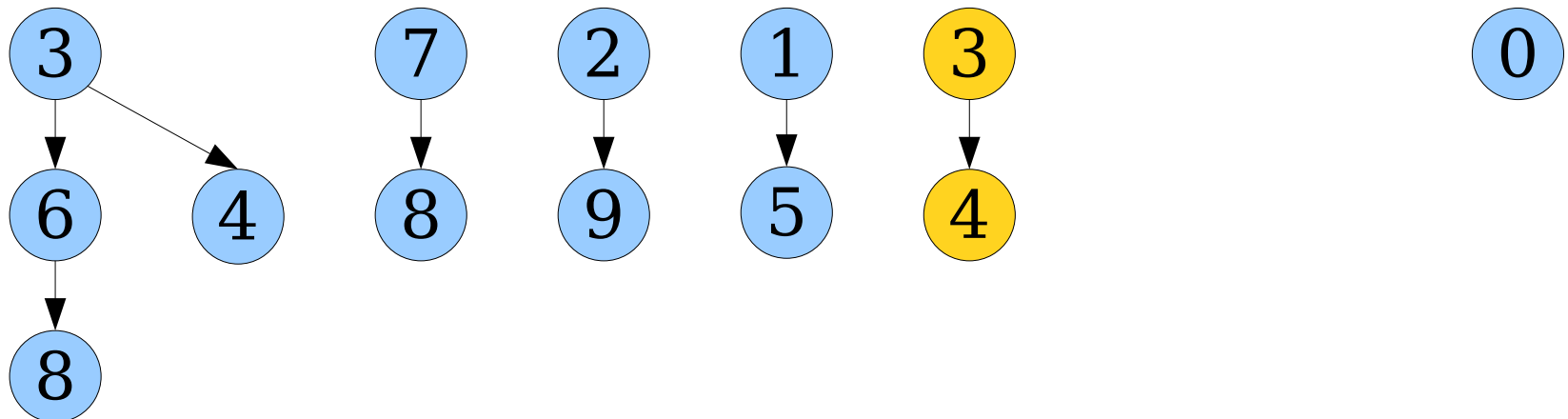
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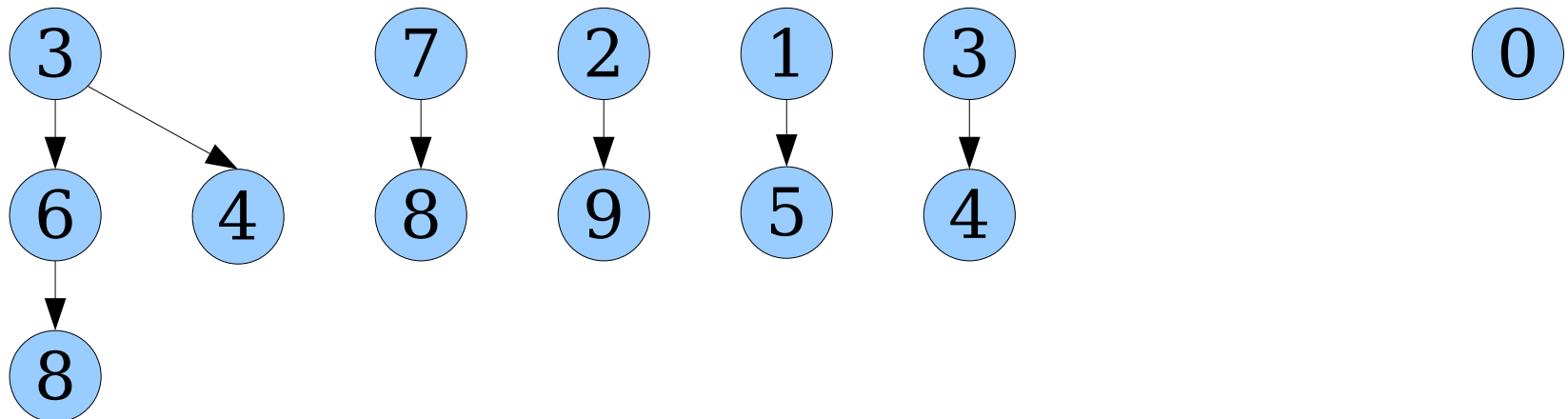
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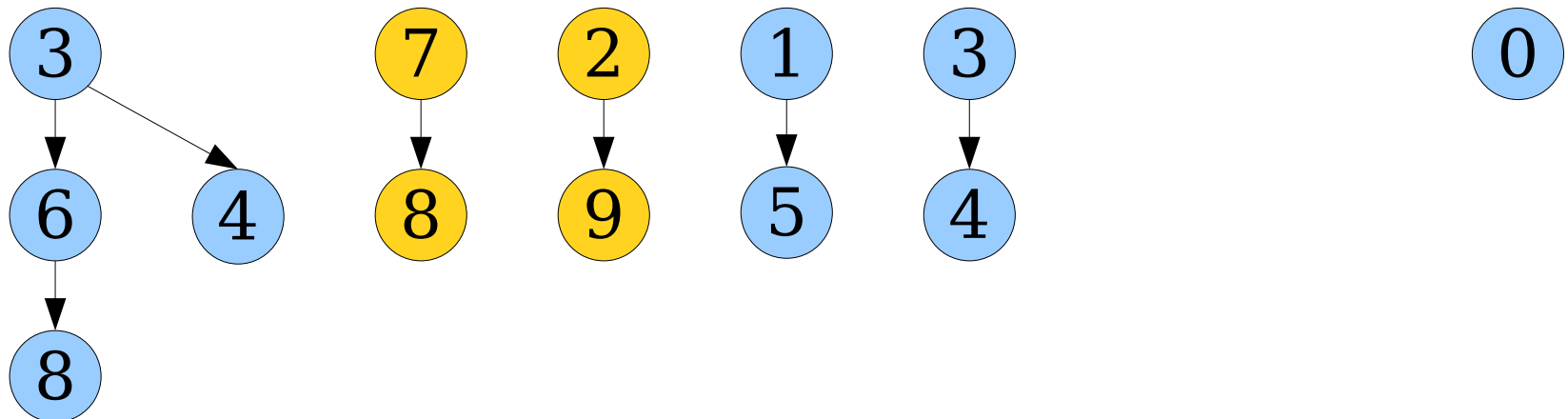
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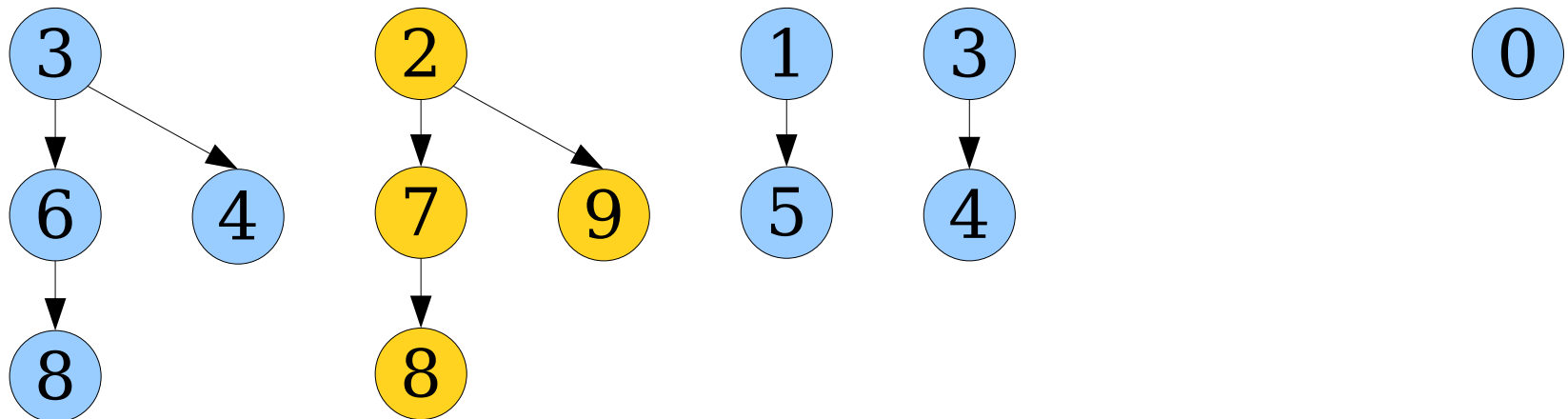
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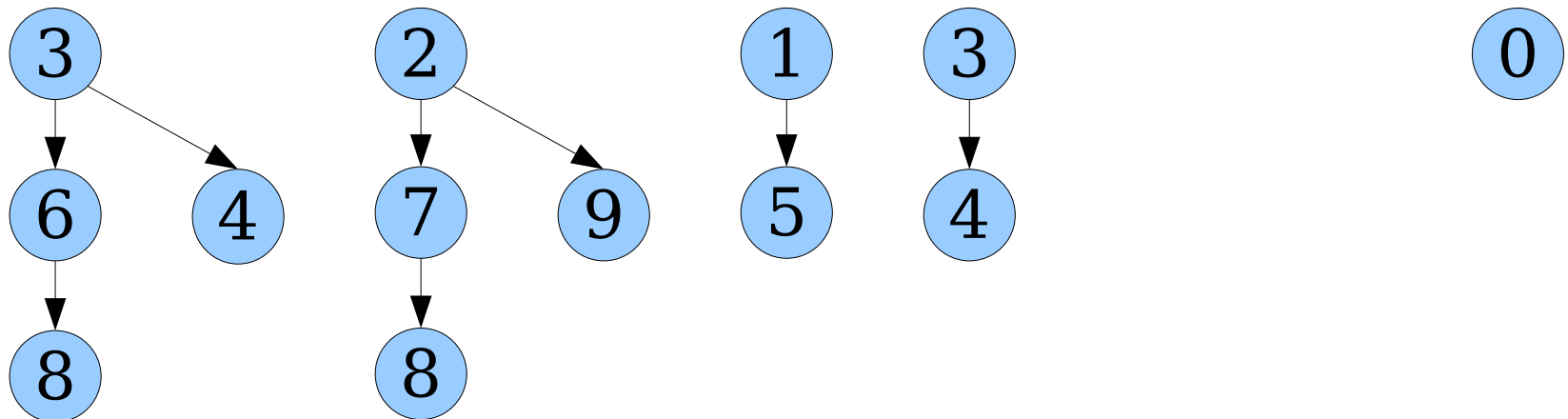
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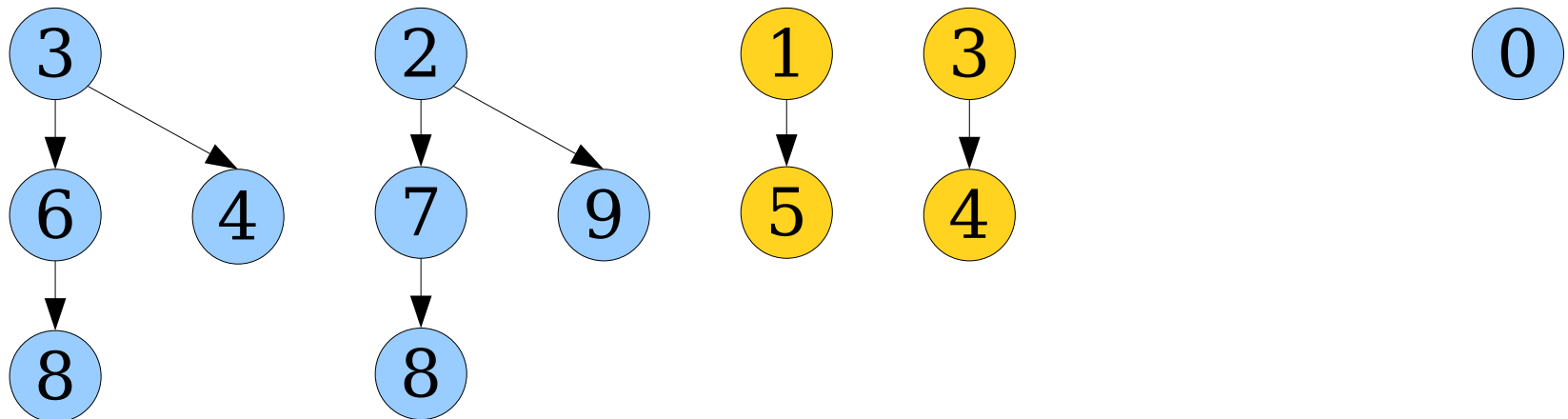
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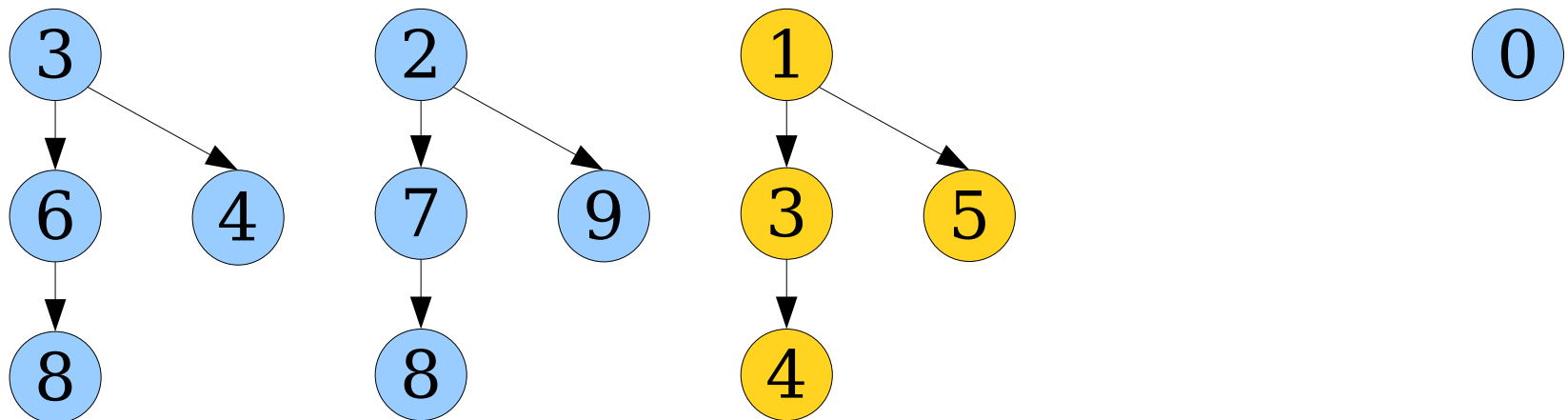
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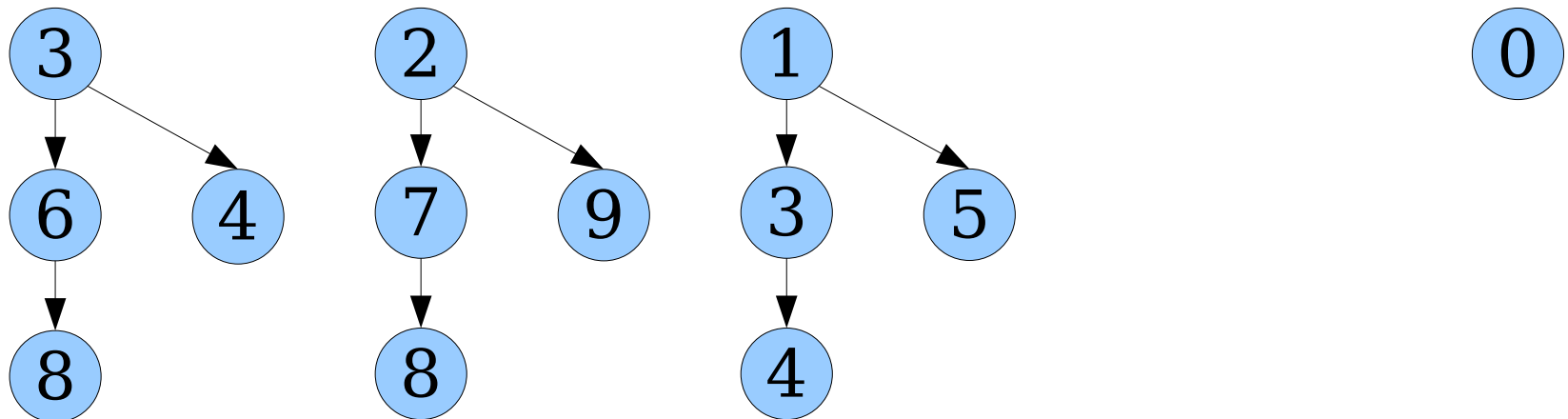
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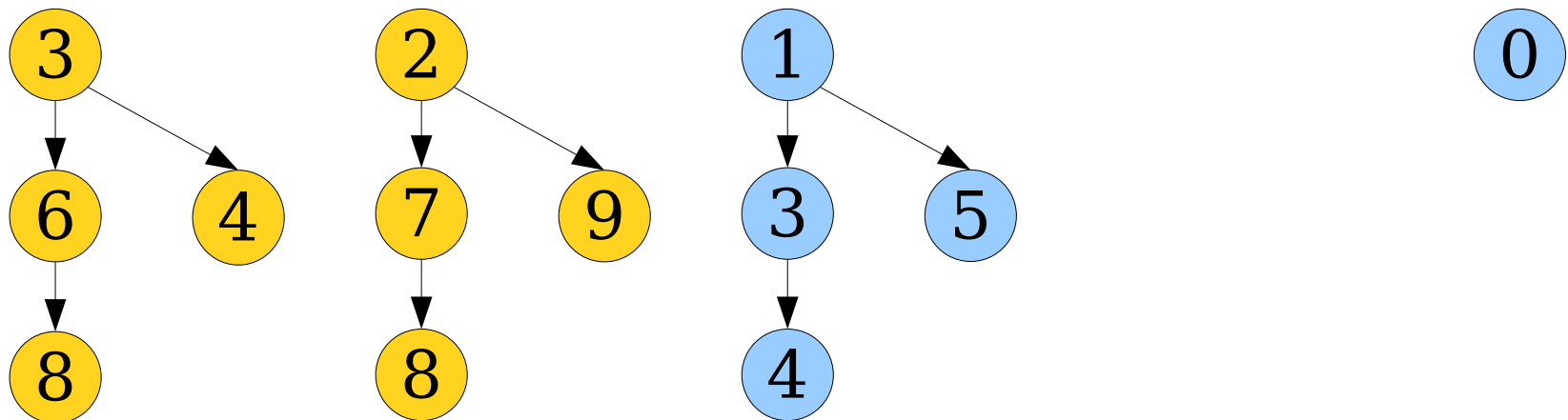
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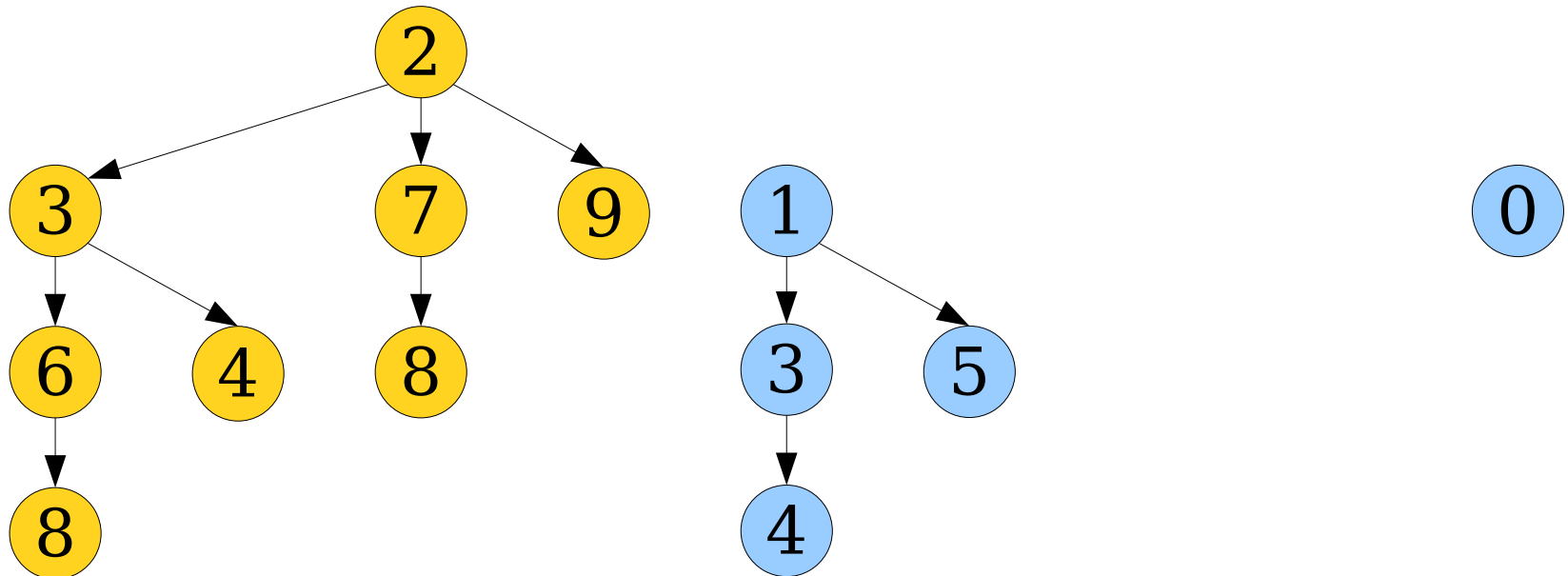
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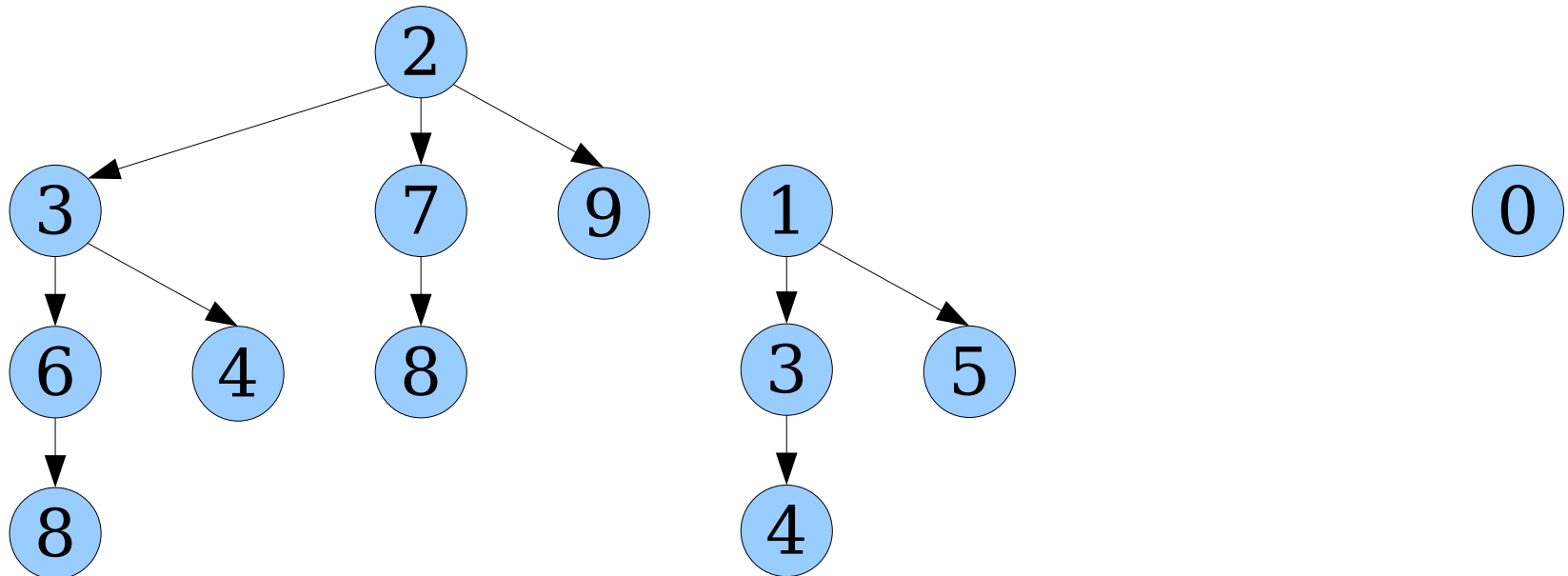
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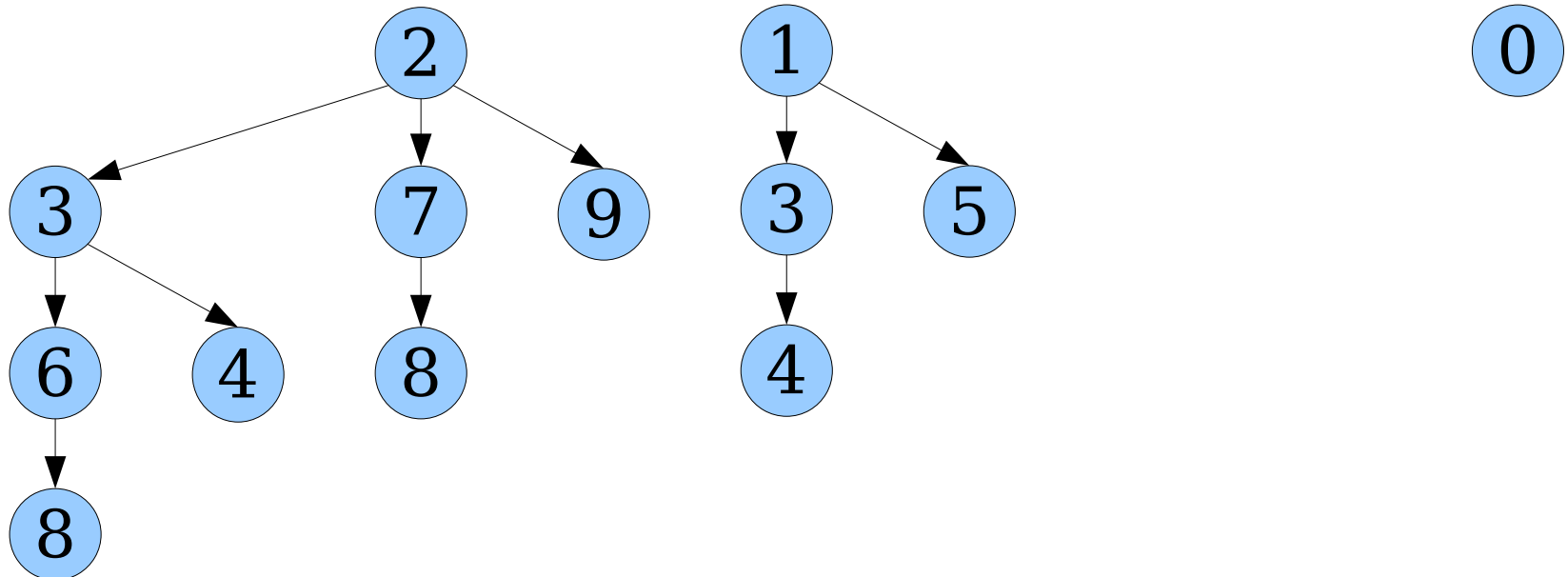
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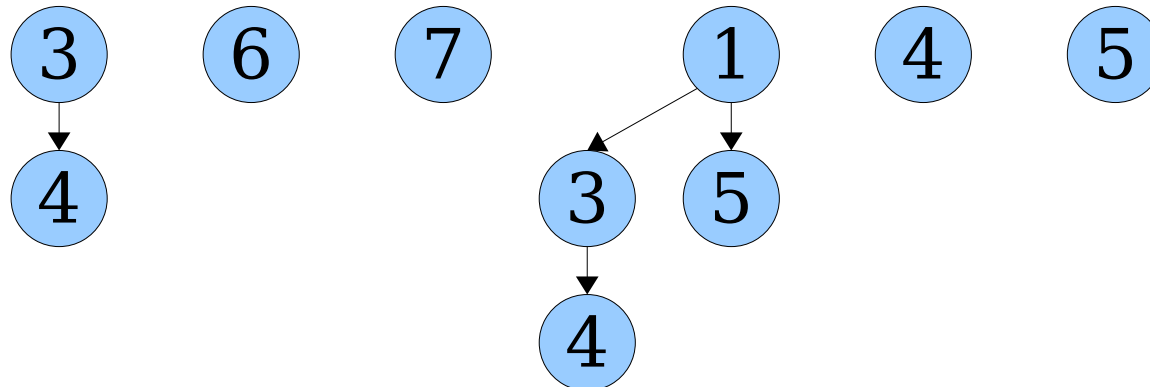


# Coalescing Trees

- ***Observation:*** This would be a *lot* easier to do if all the trees were sorted by size.
- We can sort our group of  $t$  trees by size in time  $O(t \log t)$  using a standard sorting algorithm.
- ***Better idea:*** All the sizes are small integers. Use counting sort!

# Coalescing Trees

- Here is a fast implementation of *coalesce*:
  - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, ...,  $\lceil \log_2 (n + 1) \rceil$ .
  - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



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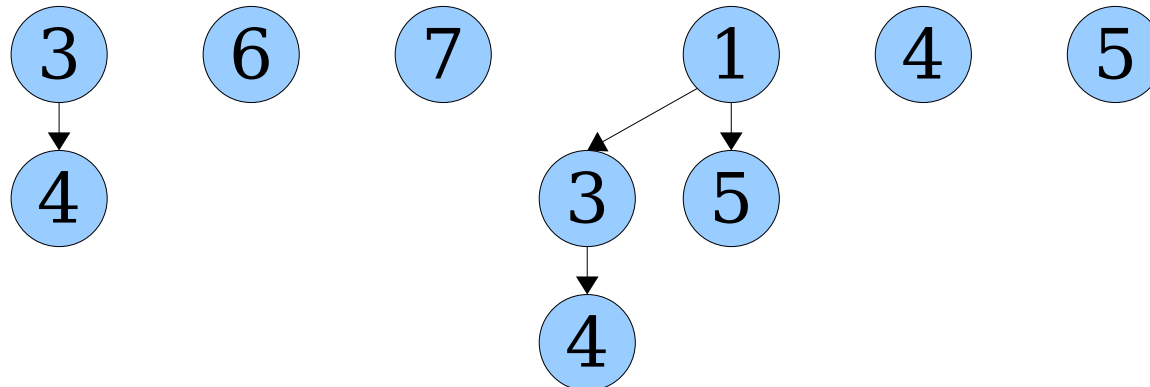
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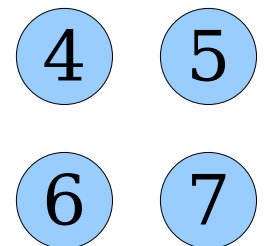
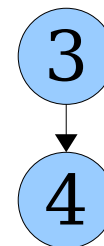
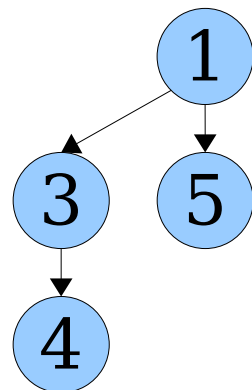
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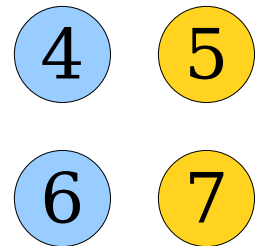
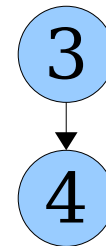
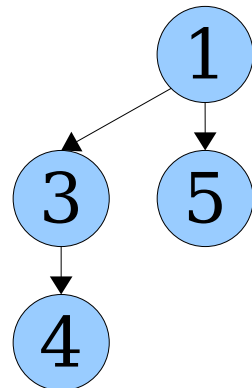
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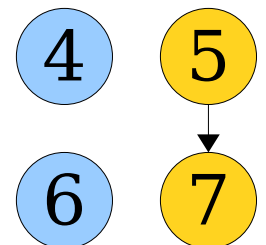
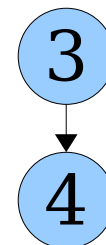
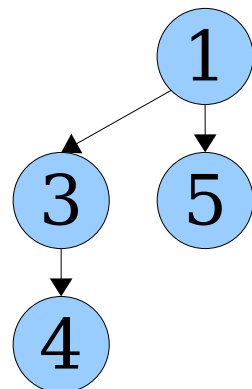
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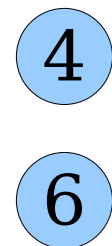
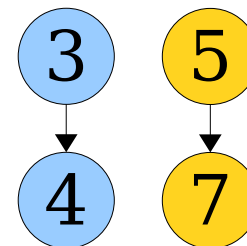
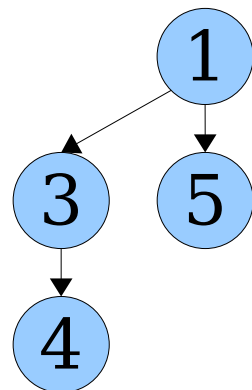
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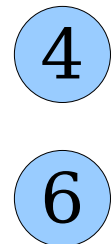
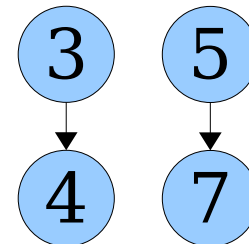
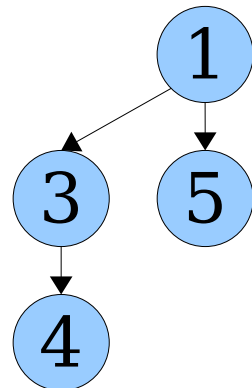
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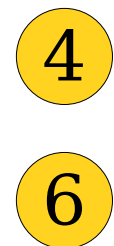
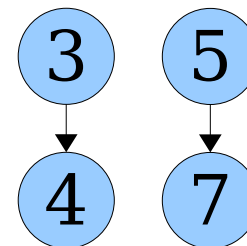
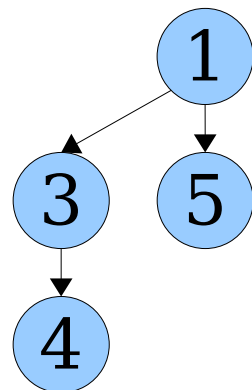
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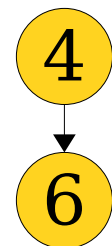
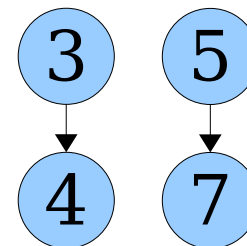
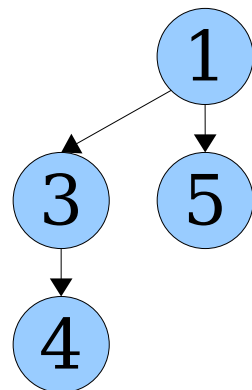
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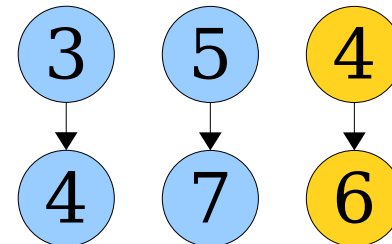
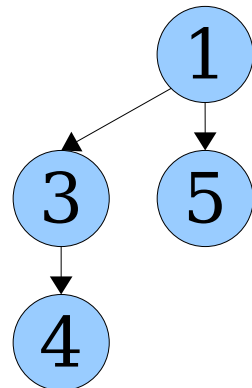
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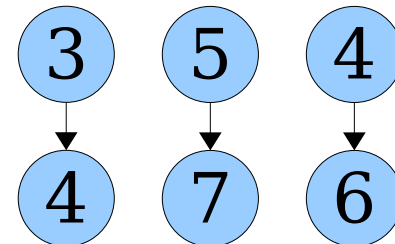
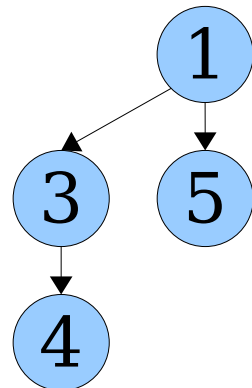
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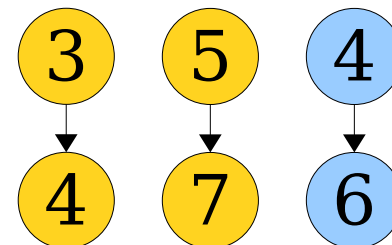
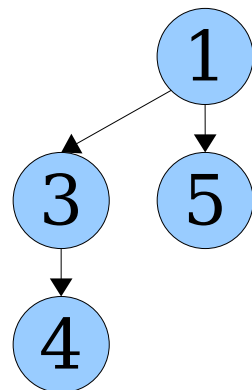
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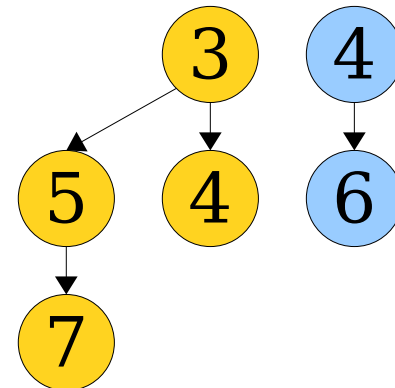
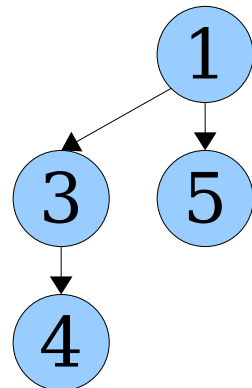
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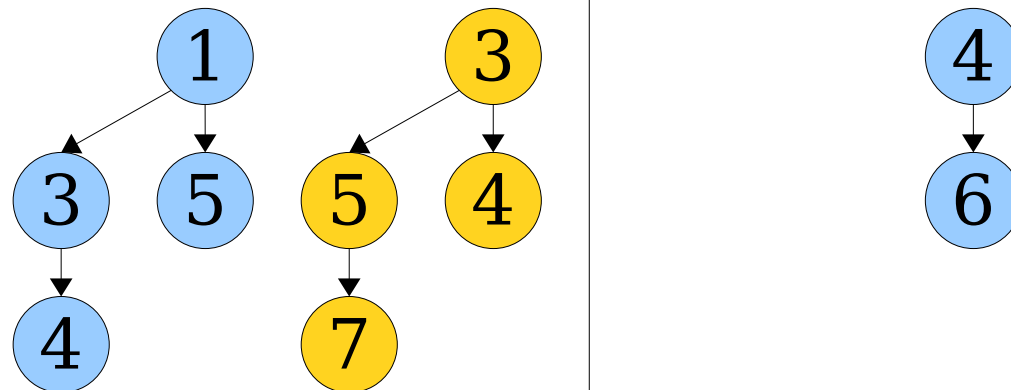
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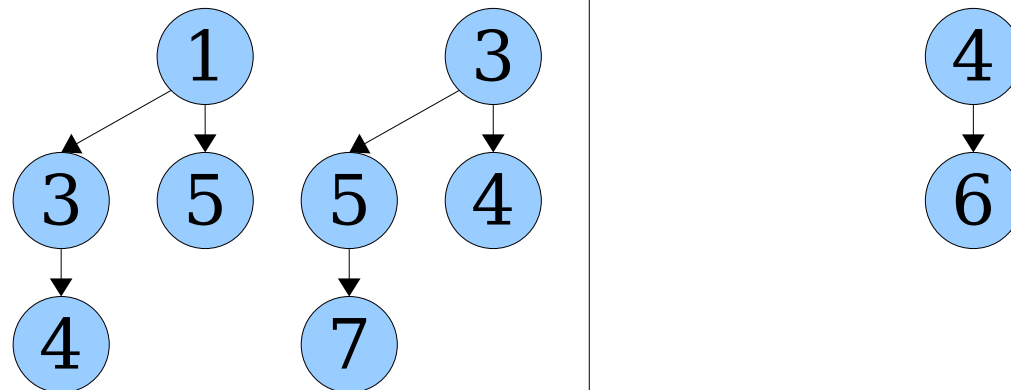
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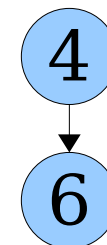
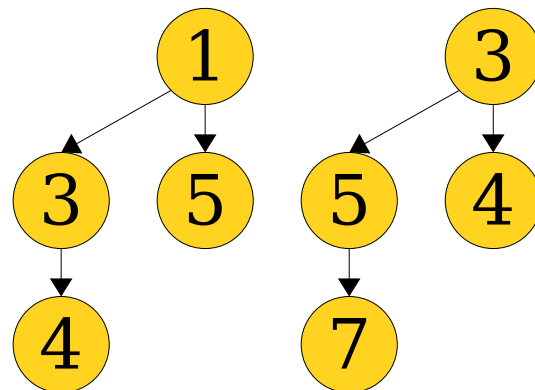
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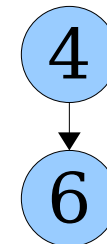
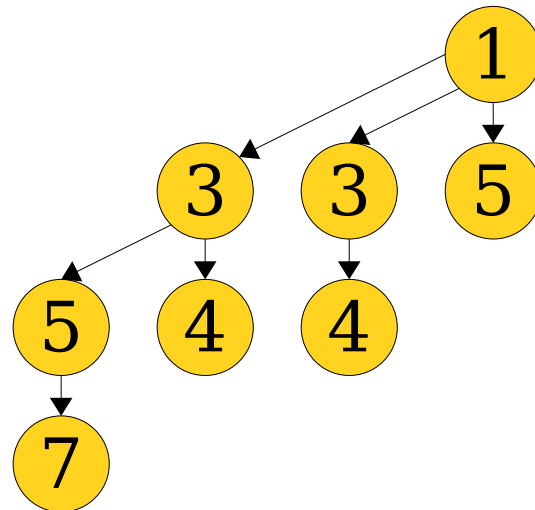
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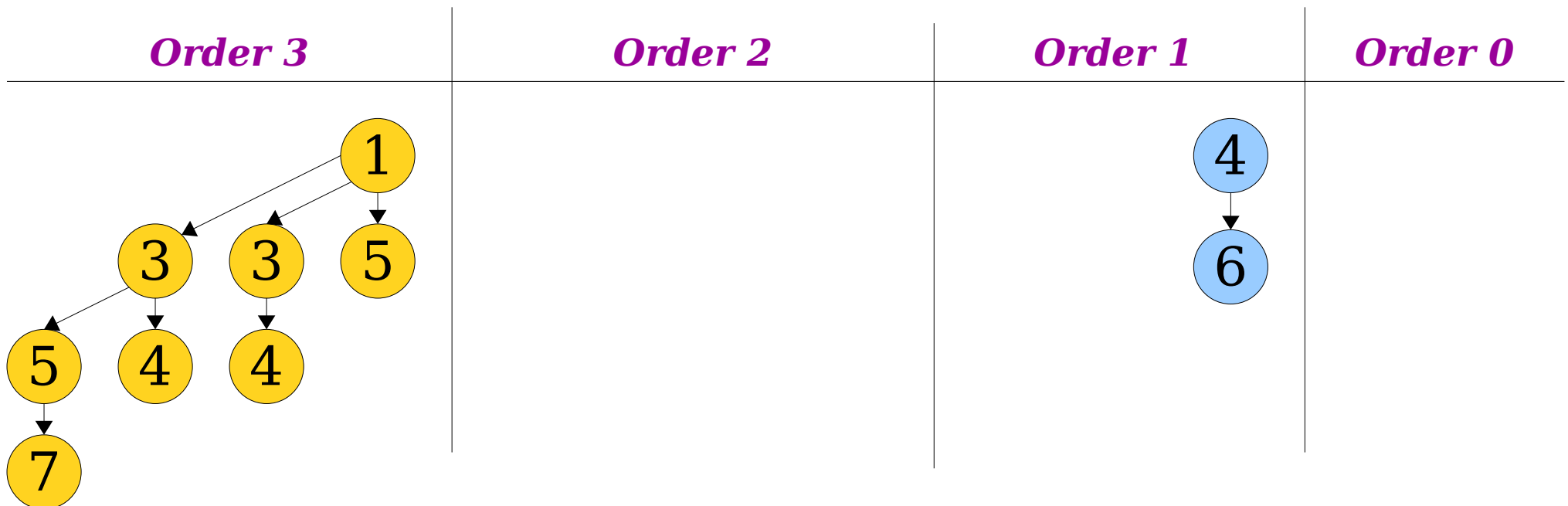
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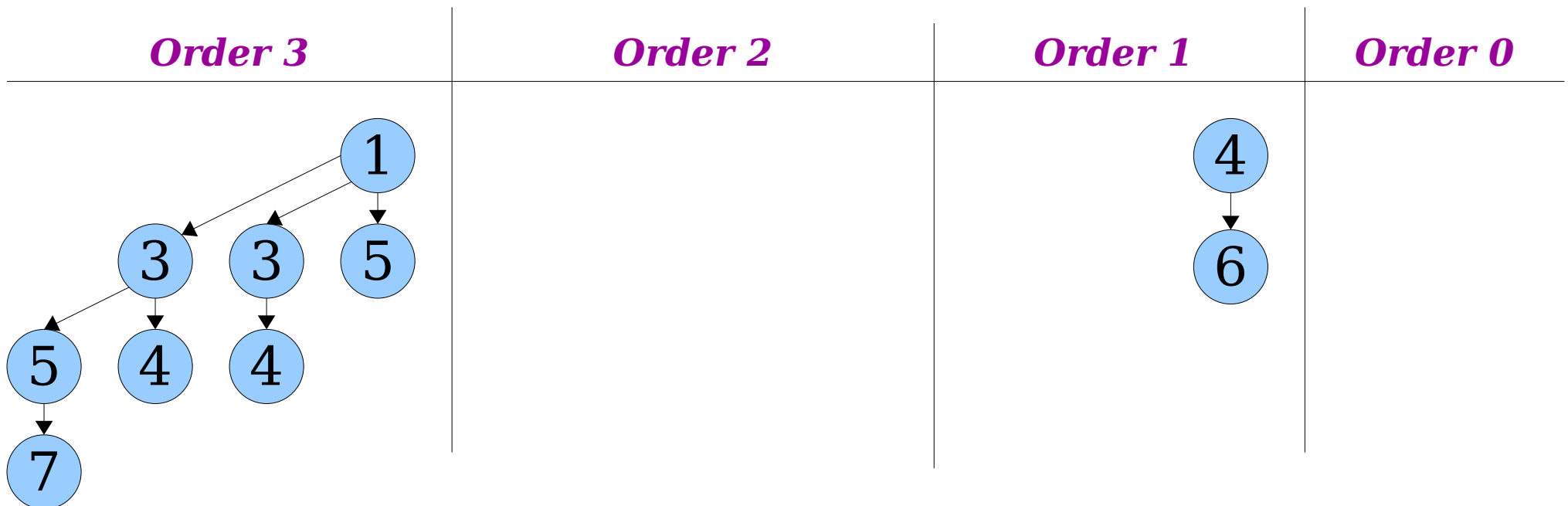
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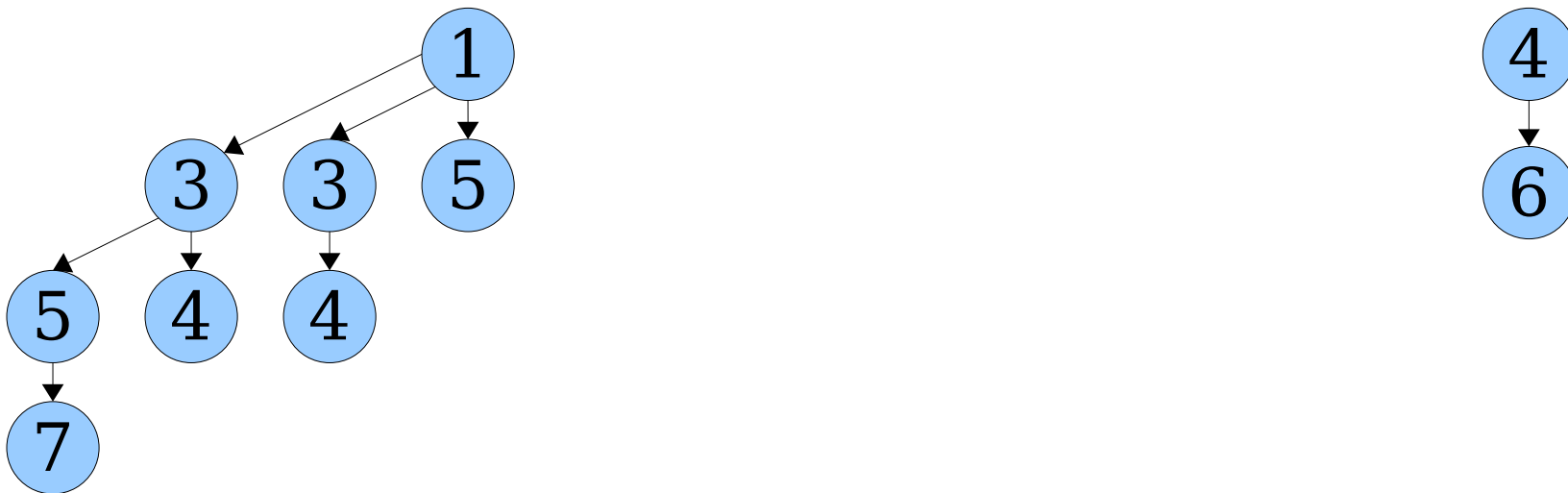
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# Analyzing Coalesce

- **Claim:** Coalescing a group of  $t$  trees takes time  $O(t + \log n)$ .
  - Time to create the array of buckets:  $O(\log n)$ .
  - Time to distribute trees into buckets:  $O(t)$ .
  - Time to fuse trees:  $O(t)$ 
    - Number of fuses is  $O(t)$ , since each fuse decreases the number of trees by one.
    - Cost per fuse is  $O(1)$ .
- Total work done:  **$O(t + \log n)$** .
- In the worst case, this is  $O(n)$ .

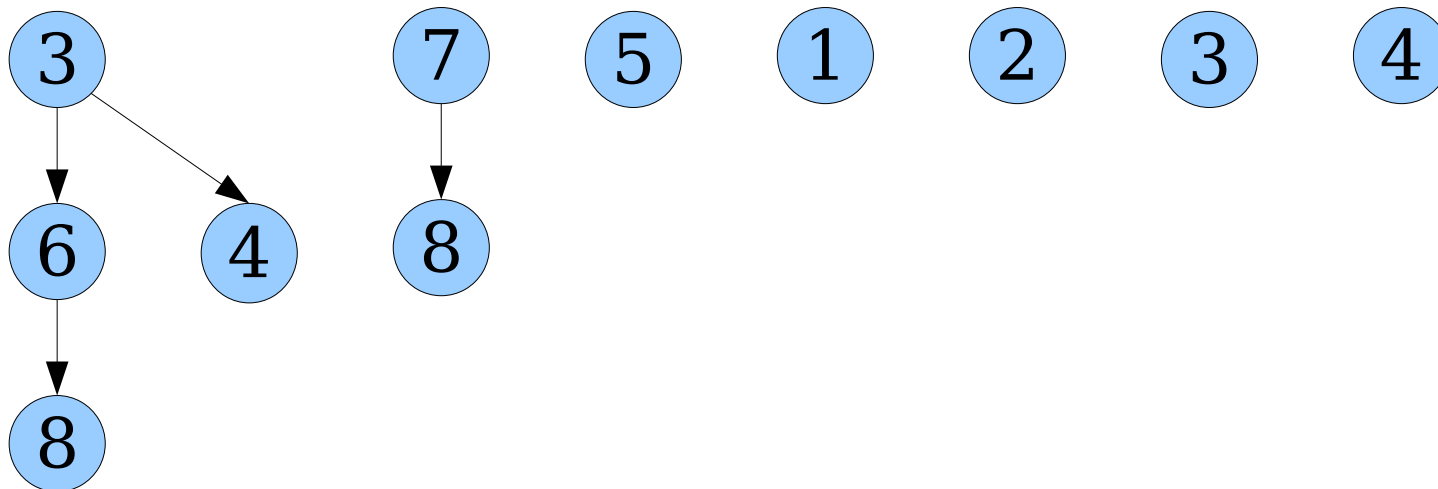
# The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - **enqueue**:  $O(1)$
  - **meld**:  $O(1)$
  - **find-min**:  $O(1)$
  - **extract-min**:  $O(n)$ .
- But these are *worst-case* time bounds. Intuitively, things should nicely amortize away.
  - The number of trees grows slowly (one per **enqueue**).
  - The number of trees drops quickly (at most one tree per order) after an **extract-min**).



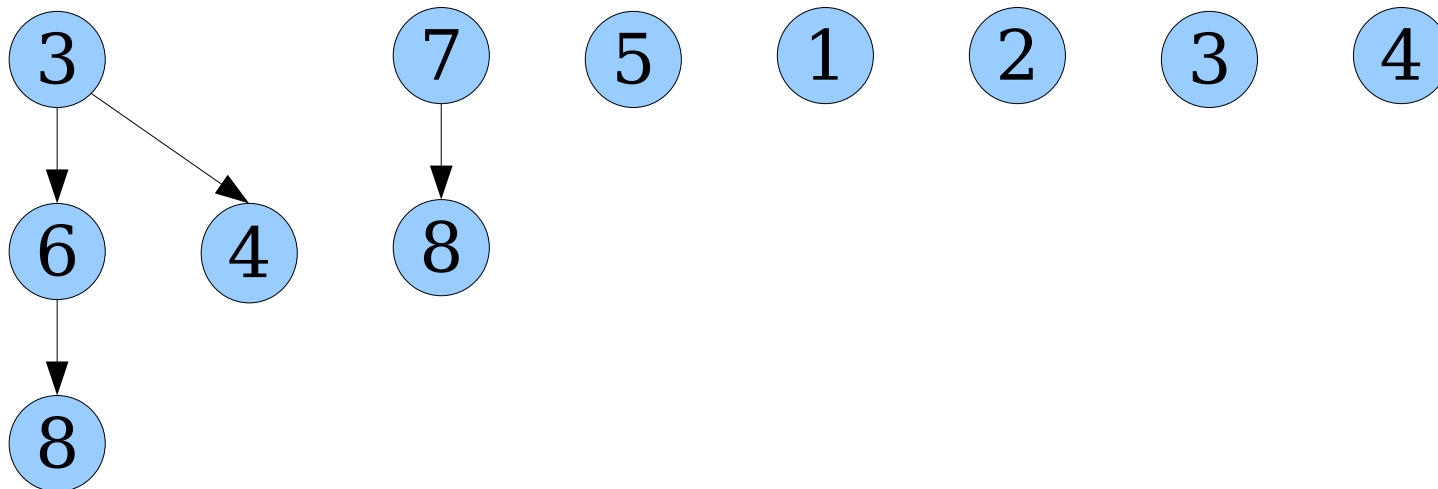
# An Amortized Analysis

- We've seen two methods for performing amortized analysis, the *banker's method* and the *potential method*.
- In each case, the idea is to clearly mark what “messes” we need to clean up.
- In our case, each tree is a “mess,” since our future *coalesce* operation has to clean it up.



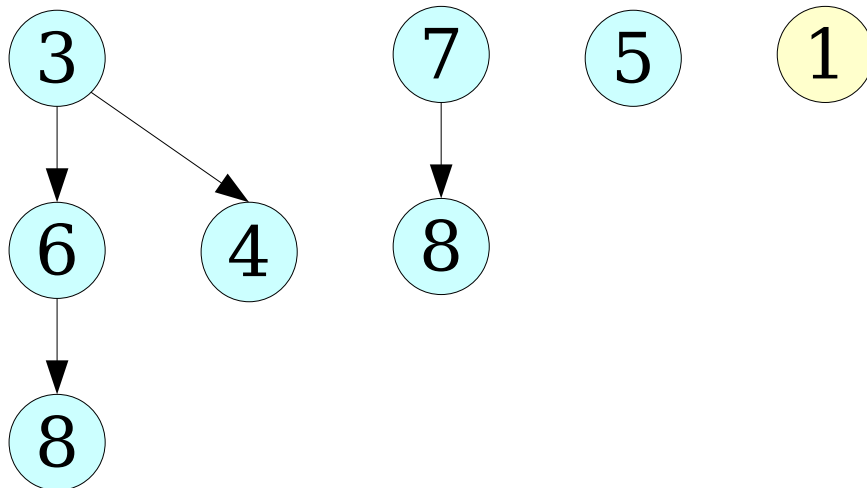
# An Amortized Analysis

- We'll use the **potential method** and set  $\Phi$  to be the number of trees in the lazy binomial heap.
- **Recall:** The amortized cost of each operation is the actual wall-clock time, plus  $O(1) \cdot \Delta\Phi$ .
- To perform the analysis, let's work out how much time each operation takes, plus how it changes the potential.



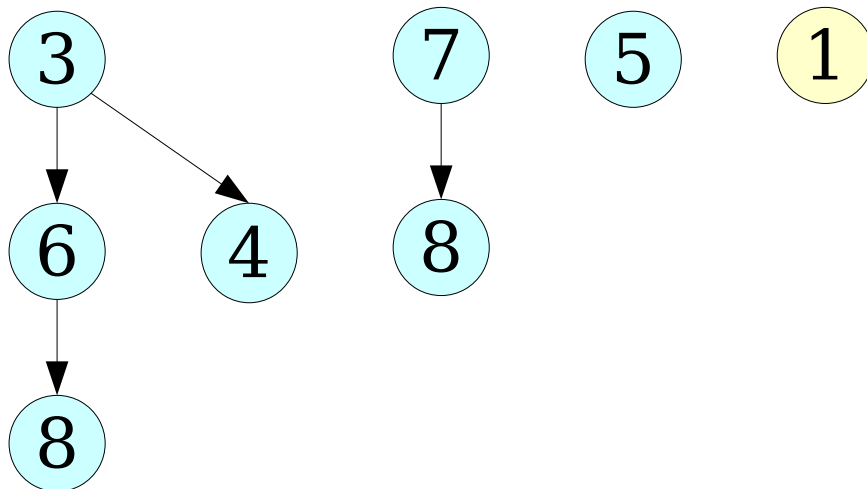
# Analyzing an Insertion

- To *enqueue* a key, we add a new binomial tree to the forest.



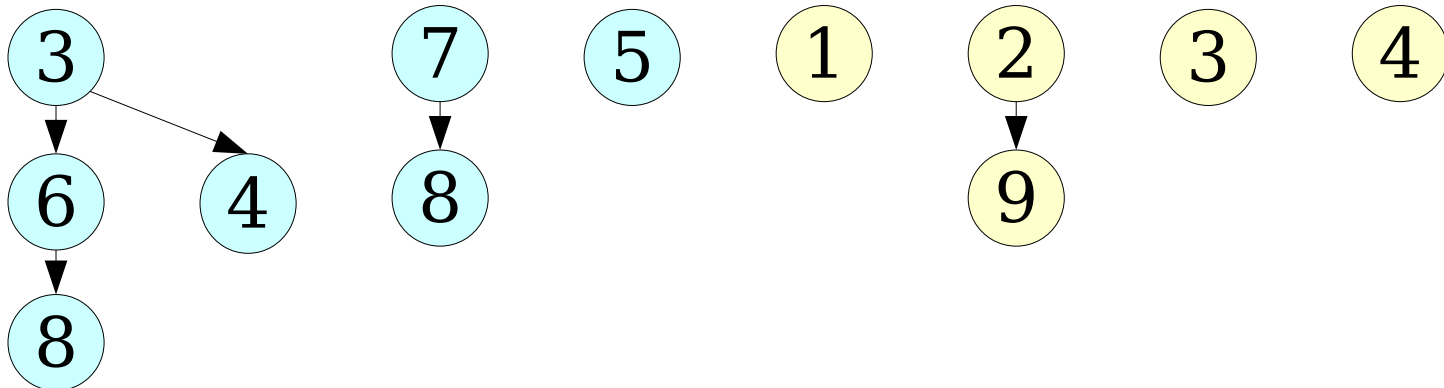
# Analyzing an Insertion

- To *enqueue* a key, we add a new binomial tree to the forest.
- Actual time:  $O(1)$ .  $\Delta\Phi$ :  $+1$
- Amortized cost:  **$O(1)$** .



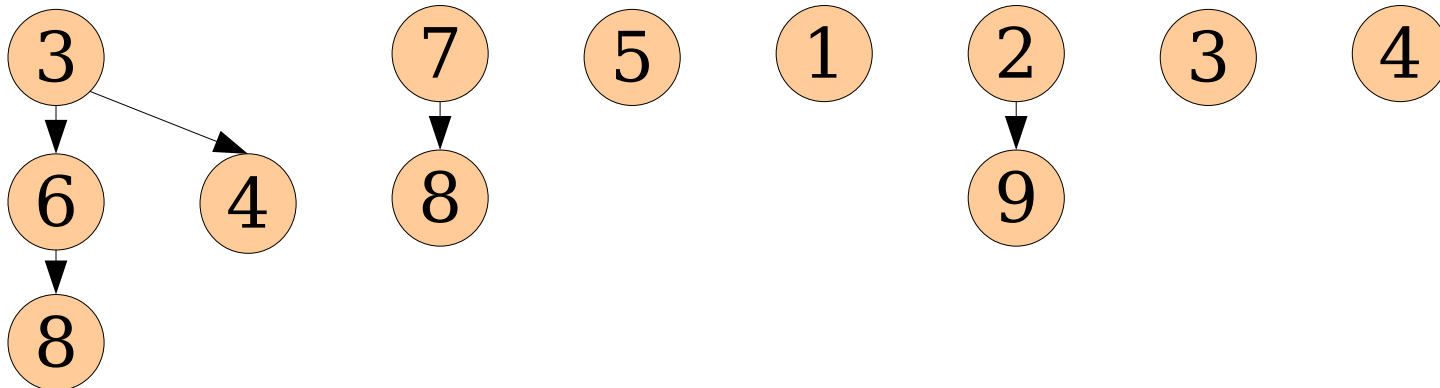
# Analyzing a Meld

- Suppose that we ***meld*** two lazy binomial heaps  $B_1$  and  $B_2$ . Actual cost:  $O(1)$ .



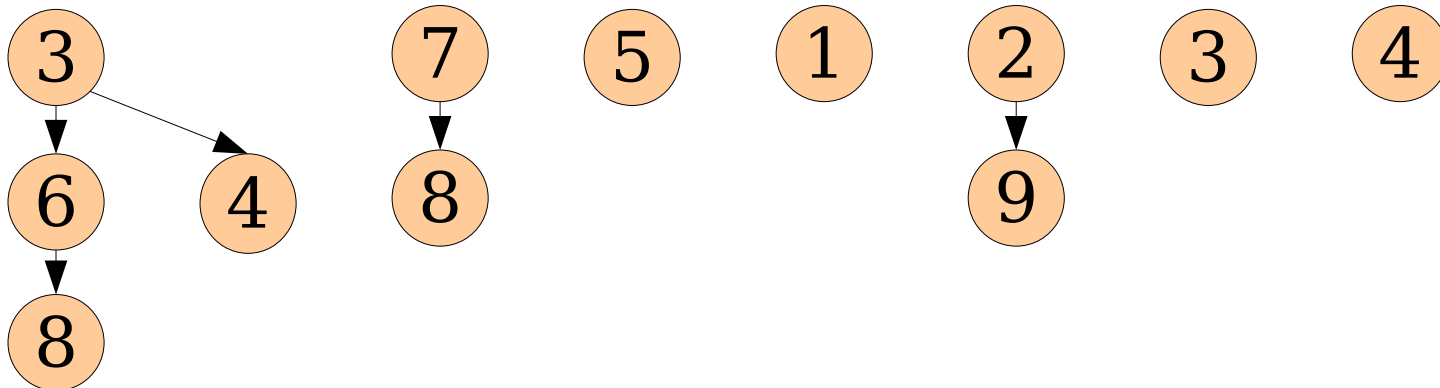
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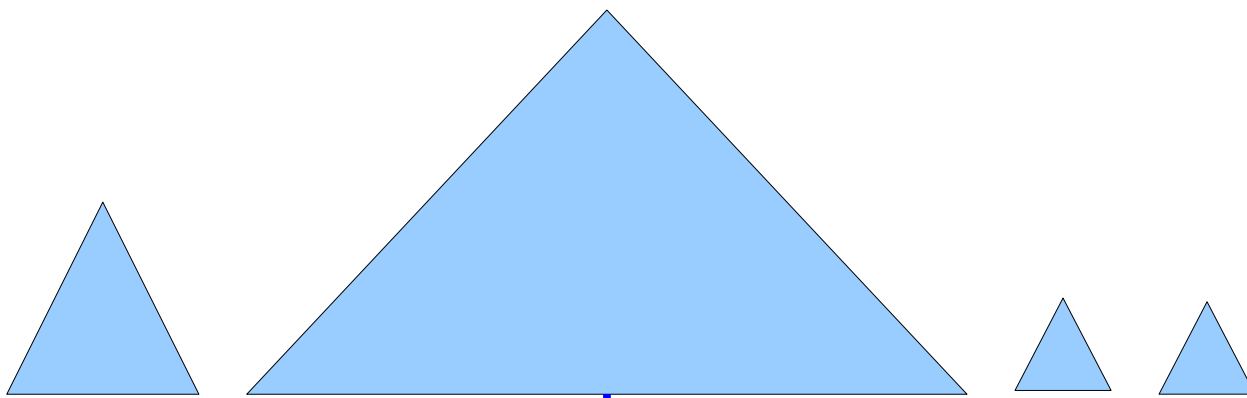
# Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps  $B_1$  and  $B_2$ . Actual cost:  $O(1)$ .
- We have the same number of trees before and after we do this, so  $\Delta\Phi = 0$ .
- Amortized cost:  **$O(1)$** .



Analyzing *extract-min*

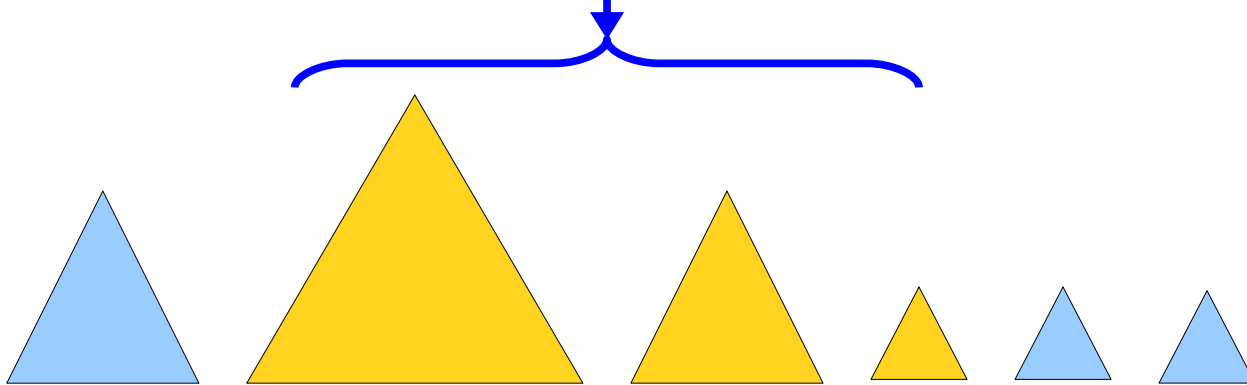




*Find tree with minimum key.*

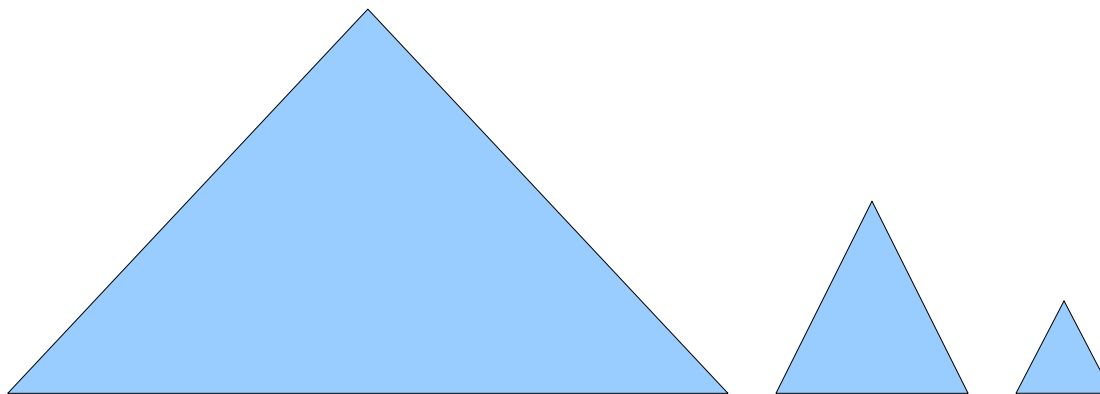
Work:  $O(t)$

$$\Phi = t$$



*Remove min.  
Add children to  
list of trees.*

Work:  $O(\log n)$



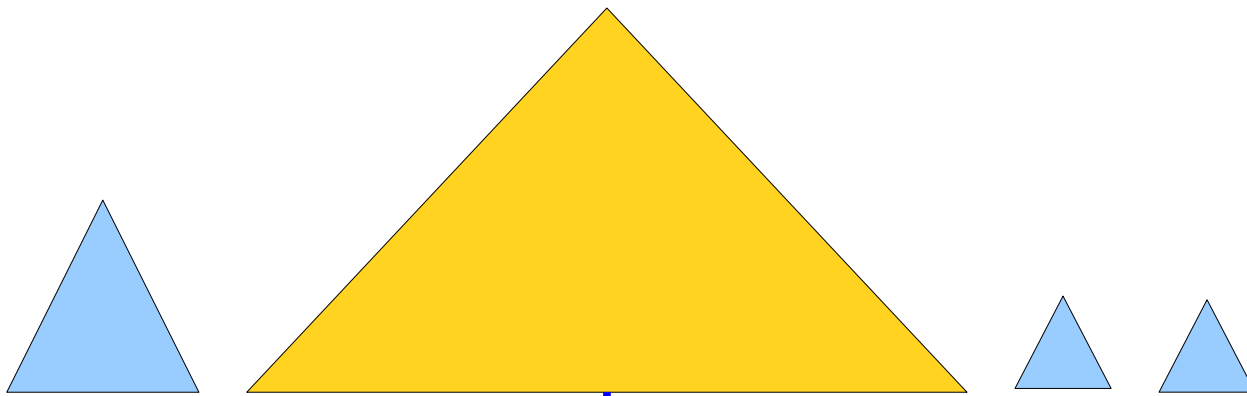
*Run the coalesce  
algorithm.*

Work:  $O(t + \log n)$

$$\Phi = O(\log n)$$

Work:  $O(t + \log n)$

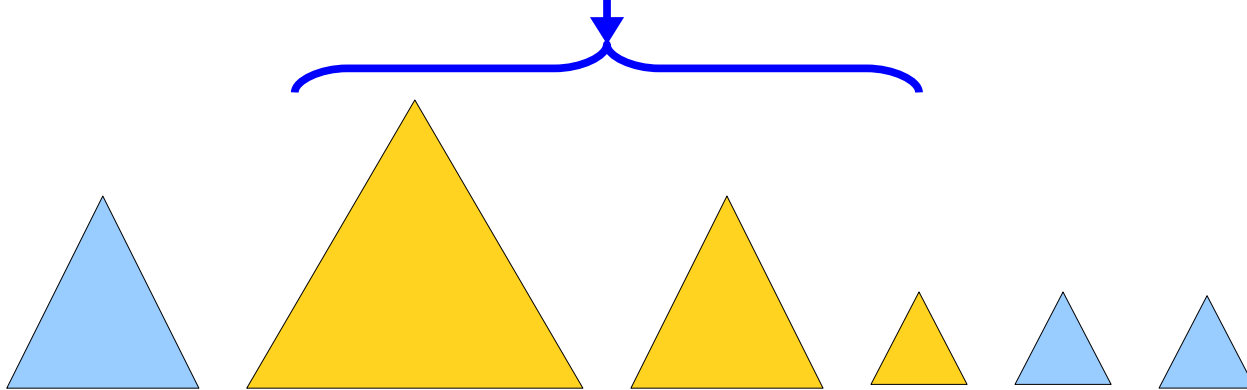
$\Delta\Phi: O(-t + \log n)$



*Find tree with minimum key.*

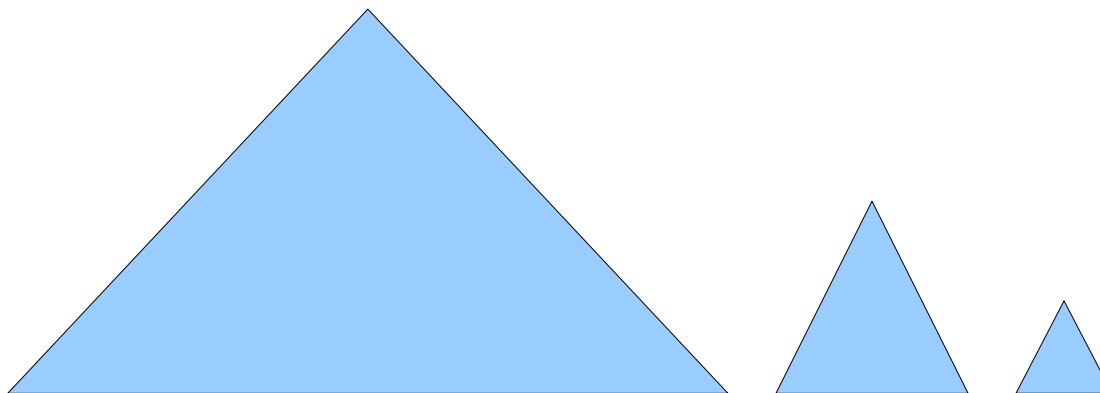
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*Run the coalesce  
algorithm.*

Work:  $O(t + \log n)$

$$\Phi = O(\log n)$$

Amortized cost:  **$O(\log n)$** .

# Analyzing Extract-Min

- Suppose we perform an **extract-min** on a binomial heap with  $t$  trees in it.
- Initially, we expose the children of the minimum element. This increases the number of trees to  $t + O(\log n)$ .
- The runtime for coalescing these trees is  $O(t + \log n)$ .
- When we're done merging, there will be  $O(\log n)$  trees remaining, so  $\Delta\Phi = -t + O(\log n)$ .
- Amortized cost is

$$\begin{aligned} & O(t + \log n) + O(1) \cdot (-t + O(\log n)) \\ &= O(t) - O(1) \cdot t + O(1) \cdot O(\log n) \\ &= O(\log n). \end{aligned}$$

# The Final Scorecard

- Here's the final scorecard for our lazy binomial heap.
- These are *great* runtimes! We can't improve upon this except by making **extract-min** worst-case efficient.
  - This is possible! Check out **bootstrapped skew binomial heaps** or **strict Fibonacci heaps** for details!

## Lazy Binomial Heap

- **Insert**:  $O(1)$
- **Find-Min**:  $O(1)$
- **Extract-Min**:  $O(\log n)^*$
- **Meld**:  $O(1)$

\* *amortized*

# Major Ideas from Today

- Isometries are a *great* way to design data structures.
  - Here, binomial heaps come from binary arithmetic.
- Designing for amortized efficiency is about building up messes slowly and rapidly cleaning them up.
  - Each individual *enqueue* isn't too bad, and a single *extract-min* fixes all the prior problems.

# Next Time

- ***The Need for decrease-key***
  - A powerful and versatile operation on priority queues.
- ***Fibonacci Heaps***
  - A variation on lazy binomial heaps with efficient decrease-key.
- ***Implementing Fibonacci Heaps***
  - ... is harder than it looks!