MA122: Final Examination

Duration: 2 hour Date: 10/02/2020 Time: 10 - 12 noon Total marks: 50

Name: ______________________ **Roll no.** ______________________ .

1. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. Fill in the following table. [3 x 3 + 1 x 2 = 11]

A basis of the row space of A	$\{(\mathbf{2}, \mathbf{1}, \mathbf{1}), (\mathbf{0}, \mathbf{3}, -\mathbf{1})\}$. Be careful as the answer is not unique. For example, any two rows of A form a basis of the row space.
A basis of the column space of A	Any two columns of A form a basis of the column space
A basis of the null space of A	$\left\{ \begin{pmatrix} -2/3 \\ 1/3 \\ 1 \end{pmatrix} \right\}$
rank of A	2
nullity of A	1

2. Answer any five of the following. Write True or False with at most 5 lines justification or counterexample (in case the statement is false) on the space below. [5 x 3 = 15]

(a) If u and v are eigenvectors of a matrix A, then $u + v$ is also an eigenvector of A.

(b) A minimal (with containment relation) generating set of a vector space V is a basis of V.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. If $\{u, v\}$ is a linearly independent subset of $\mathbb{R}^2$, then $\{T(u), T(v)\}$ is also linearly independent.

(d) The set $C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable function}\}$ is a subspace of the vector space of all functions from $\mathbb{R} \to \mathbb{R}$ under usual addition and scalar multiplication.

(e) Union of two subspaces of a vector space V is again a subspace.

(f) Let A be an $n \times n$ matrix such that the system $AX = 0$ has only the trivial solution. Then A is invertible.

(g) Any two elementary matrices of same order are always row equivalent.

(h) Let A and B be two square matrices of same order. If A and B are diagonalizable, then AB is also diagonalizable.

Answer with justifications:

$\boxed{(a)}$ False. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the matrix corresponding to the linear map 'reflection with respect to the x-axis'. Then $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are two eigenvectors of A, but $u + v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector of A.

$\boxed{(b)}$ True. Since a minimal generating set S of V is a generating set, we need to show that S is linearly independent. It is true, otherwise there is $v \in S$ which can be written as linear combination of vectors in $S \setminus \{v\}$. Thus $S \setminus \{v\}$ will be a generating set of V, contradicting the minimality of S.

$\boxed{(c)}$ False. Consider the zero map, i.e., $T(X) = 0$ (zero vector in $\mathbb{R}^2$) for all $X \in \mathbb{R}^2$. Then $\{T(u), T(v)\} = \{0\}$ cannot be linearly independent.

$\boxed{(d)}$ True. The set $C^1(\mathbb{R})$ is nonempty as it contains the zero function. Let $f, g \in C^1(\mathbb{R})$. Then $f + g \in C^1(\mathbb{R})$ because the sum of two differentiable functions is differentiable. Moreover $cf \in C^1(\mathbb{R})$ where $c \in \mathbb{R}$.

$\boxed{(e)}$ False. Let $V = \mathbb{R}^2$. Consider U_1 and U_2 as the x-axis and y-axis respectively. Then both U_1 and U_2 are subspaces of V, but $U_1 \cup U_2$ is not a subspace of V because $U_1 \cup U_2$ is not closed under addition. For example $(1,0), (0,1) \in U_1 \cup U_2$, but $(1,0) + (0,1) = (1,1) \notin U_1 \cup U_2$.

$\boxed{(f)}$ True. Applying elementary row operations, convert A to a row reduced echelon matrix B. Then $BX = 0$ has only the trivial solution. So there is no free variable. Hence $B = I_n$, the identity matrix. Moreover $E_r \cdots E_2 E_1 A = B = I_n$, i.e., $A^{-1} = E_r \cdots E_2 E_1$, where E_i are elementary matrices.

$\boxed{(g)}$ True. Any elementary matrix is row equivalent to identity matrix. Since the row equivalence of matrices is a transitive relation, any two elementary matrices (of same order) are always row equivalent.

$\boxed{(h)}$ False. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Each of these has distinct eigenvalues, hence they are diagonalizable. But the product $AB = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$ is not diagonalizable as $\mathrm{GM}_A(2) = 1 < 2 = \mathrm{AM}_A(2)$.

3. Write only the answers to the following questions: [2 x 1.5 = 3]

(a) The dimension of the vector space of all 2×2 diagonal matrices with usual operations? _ 2 _

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Suppose $T(v) \neq 0$ for some vector $v \in \mathbb{R}^2$, and there is a non-zero vector $u \in \mathbb{R}^2$ such that $T(u) = 0$. What is the rank of T? __ 1 __

4. Consider the functions 'det' and 'tr' from the collection of all $n \times n$ matrices $\mathbb{R}^{n \times n}$ to $\mathbb{R}$ given by determinant and trace values respectively. Are they linear maps? Write Yes/No. [1+1 = 2]

$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ __ **No** __ , $\mathrm{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$ __ **Yes** __

5. Write the matrix representation of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ 2x_1 + x_2 \\ -x_1 + 3x_2 \end{pmatrix}$.

. [3]

Answer (only): $\begin{pmatrix} -1 & 1 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$

6. Find the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ using the elementary row operations. [4]

Solution:

Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Apply elementary row operation on $(A|I_3) = \begin{bmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ to convert it to $(I_3|A^{-1})$. Hence write A^{-1}.

7. Consider $u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$. Find a basis of $\mathbb{R}^3$ containing $\{u, v\}$. [1+2 = 3]

Extended basis:

Note that the answer is not unique. You can adjoin any vector from $\mathbb{R}^3 \setminus \text{Span}\{u, v\}$. For example $\left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$ containing $\{u, v\}$.

Justification:

You have to verify the above given set is actually a basis of $\mathbb{R}^3$. For that you need to check two things spanning and linearly independency.

You can verify the above directly. Or you can argue in other ways also.

For example, since $e_2 \notin \text{Span}\{u, v\}$, the set $\{u, v, e_2\}$ is linearly independent. Now $\{u, v, e_2\}$ should span $\mathbb{R}^3$ otherwise there is one vector $w \in \mathbb{R}^3 \setminus \text{Span}\{u, v, e_2\}$, and after adjoining the vector w we get that $\{u, v, e_2, w\}$ is a linearly independent set in $\mathbb{R}^3$, contradicting that $\dim(\mathbb{R}^3) = 3$.

8. Let V be the collection of all $n \times n$ matrices over $\mathbb{R}$. Let $A \in V$. Without using the Caylay-Hamilton Theorem, prove that $\{I_n, A, A^2, A^3, \ldots\}$ is linearly dependent over $\mathbb{R}$. Conclude that A satisfies a polynomial in x over $\mathbb{R}$ with leading coefficient 1. [3]

Solution:

Let $W = \operatorname{Span}\{I_n, A, A^2, A^3, \ldots\}$. Then W is a subspace of V, and V has dimension n^2. Therefore $\dim(W) \leqslant n^2$. Since $\{I_n, A, A^2, A^3, \ldots, A^{n^2}\}$ has $n^2 + 1$ vectors, these vectors should be linearly dependent. Therefore there are $a_0, a_1, a_2, \ldots, a_{n^2} \in \mathbb{R}$, not all zero, such that

$$a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_{n^2} A^{n^2} = 0 \text{ (zero matrix of order } n \times n). \tag{1}$$

Since $\{I_n, A, A^2, A^3, \ldots, A^{n^2}\}$ is linearly dependent, the set $\{I_n, A, A^2, A^3, \ldots\}$ is also so.

Then, by multiplying (1) with suitable non-zero scalar, one concludes that A satisfies a polynomial in x over $\mathbb{R}$ with leading coefficient 1.

9. Let $A := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Then: [1 + 3 + 2 = 6]

(i) Find the characteristic polynomial of A.

(ii) Is it possible to find two linearly independent eigenvectors of A.

(iii) Is A diagonalizable?

Proof:

(i) The characteristic polynomial of A is $\det(xI_2 - A) = \det \begin{bmatrix} x-1 & 0 \\ -2 & x-1 \end{bmatrix} = (x-1)^2$.

(ii) A has only one eigenvalue which is 1. To obtain the eigenvectors of A, we solve the system $(A - I_2)X = 0$, i.e., $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. After applying elementary row operations, the system is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence $x_1 = 0$ and x_2 is a free variable. Thus the solution set is $\left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} : x_2 \in \mathbb{R} \right\}$ which has dimension 1. So A cannot have two linearly independent eigenvectors.

(iii) Since A cannot have two linearly independent eigenvectors, A is not diagonalizable.