Lecture 11 – Introduction to Group Theory

PROBLEMS DISCUSSION SESSION

February 01, 2022

Quiz : Google form at 6 pm "Today".

Deadline 11:59 pm

~~5 hr~~ 6 hr duration

Section 2. Subgroups.

1. Cyclic group generated by $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\left\langle \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}}_{= a} \right\rangle = \langle a, a^2, a^3, \dots \rangle$$

2. $a, b \in G$ with $|a| = 5$ and $a^3 b = b a^3$.

Prove that $ab = ba$.

3. (a) Is $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$ a subgroup?

(b) $$\begin{array}{c} (H,\cdot) \\ H = \{1, -1\} \subset \end{array} \begin{array}{c} (\mathbb{R}-\{0\}, \cdot) \\ \shortparallel \\ G \end{array}$$

Is H a subgroup of G ?

Notations (Artin Textbook).

$\mathbb{Z}^+ := (\mathbb{Z}, +)$, $\mathbb{R}^+ := (\mathbb{R}, +)$, $\mathbb{C}^+ = (\mathbb{C}, +)$

$\mathbb{R}^\times := (\mathbb{R}-\{0\}, \cdot)$, $\mathbb{C}^\times := (\mathbb{C}-\{0\}, \cdot)$

(c) $(\mathbb{N}, +) \subset (\mathbb{Z}, +)$

(d) $(\mathbb{R}_{>0}, \cdot) \subset (\mathbb{R}-\{0\}, \cdot)$

3 (e).

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \;\middle|\; a \neq 0,\ a \in \mathbb{R} \right\} \subseteq \underset{\underset{G}{\shortparallel}}{GL_2(\mathbb{R})}$$

Is H a subgroup of G ?

4. $\underset{\substack{\uparrow \\ \text{non-empty set}}}{\phi \neq H} \subseteq G \leftarrow$ Group , H is a subgroup of G

if for all $x, y \in H$,

$xy^{-1} \in H$.

5. $U_n = \{ z \in \mathbb{C} \text{ s.t. } z^n = 1 \}$.

Is U_n a cyclic ~~group~~ subgroup of $(\mathbb{C} - \{0\}, \cdot)$ of order n ?

Q. 6. Klein 4-group is the smallest group which is not cyclic.

Multiplication Table.

Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group with $g_1 = e$.
(identity element)

The multiplication table or group table of G is the $n \times n$ matrix whose (i,j)th entry is the group element $g_i g_j$.

	$g_1 = e$	g_2	g_3	$\cdot\ g_j\ \cdot$	g_n
$g_1 = e$	e	$g_1 * g_2$	$g_1 * g_3$	$\cdot$	$\cdot$
g_2		$g_2 * g_2$	$g_2 * g_3$	$\cdots$	$g_2 * g_n$
g_3		$g_3 * g_2$	$g_3 * g_3$	$\cdots$	$g_3 * g_n$
$\vdots$		$\vdots$			
g_i				$g_i * g_j$	
$\vdots$					
g_n		$g_n * g_2$	$\cdots$	$\cdots$	$g_n * g_n$

$$\boxed{g_i * g_j = g_j * g_i} \quad \forall\, i,j$$

Also know as Cayley's Table ~1850

Remark.

G is Abelian $\Leftrightarrow$ Table is Symmetric.

Case Study:

$|G| = 1$

	e
e	e

i.e. $G = \{e\}$

$|G| = 2$, assume $G = \{\overset{1}{\overset{\shortparallel}{e}}, a\}$ $a \neq 1$

	$\overset{1}{\overset{\shortparallel}{e}}$	a
$=e$	$e=1$	a
$=a$	a	$a * a = a^2,$

$\uparrow$
?

$$\boxed{a^2 = 1 \quad \text{or} \quad a^2 = a}$$

Note that $a * a = a^2$ but G has only two elements $\{1, a\}$. Then the possibilities are $a^2 = a$

$\Rightarrow \boxed{a = 1}$

[Use cancellation law]

But $a \neq 1$ in $G = \{1, a\}$

If $a^2 \neq a$, then a^2 must be 1.

$\Rightarrow \quad a * a = 1 = a * a$

$\Rightarrow \quad a^{-1} = a$

	1	a
1	1	a
a	a	1

Here:

$o(1) = 1$

$o(a) = 2$

$|G| = 3$ $\qquad G = \{1, a, b\}$

$$\begin{array}{c|ccc} & 1 & a & b \\ \hline 1 & 1 & a & b \\ a & a & a^2 & ab \\ b & b & ba & b^2 \end{array}$$

$a^2 \in \{1, a, b\}$

$a^2 \in G$
$ab \in G$
$ba \in G$
$b^2 \in G$

Note that $\boxed{a^2 \neq a \quad \text{and} \quad b^2 \neq b.}$

Is $ab = a$? If Yes, then $b = 1$, but $b \neq 1$

Is $ba = b$? If Yes, then $a = 1$, but $a \neq 1$.

$ab \in \{1, \cancel{a}, \cancel{b}\}$, hence $\boxed{ab = 1.}$

Similarly, $ba \neq a$ & $ba \neq b$. $\Rightarrow$ $\boxed{ba = 1}$

$$\begin{array}{c|ccc} & 1 & a & b \\ \hline 1 & 1 & a & b \\ a & a & a^2 = b & 1 \\ b & b & 1 & b^2 = a \end{array}$$

$a^2 \in \{1, \cancel{a}, b\}$

Can a^2 be equal to 1 ?

Suppose $a^2 = 1$

$\Rightarrow a^2 b = b$

$\Rightarrow a \cdot \underbrace{ab}_{\text{\textpi}} = b$

$\Rightarrow a \cdot 1 = b$

$\Rightarrow \boxed{a = b}$ but $a \neq b$.

$a^2 \neq 1$, $a^2 \neq a$

$a^2 \in \{1, a, b\}$

$\Rightarrow a^2 = b$

$a^2 \in \{\overset{\times}{1}, \overset{\times}{a}, b\}$, Hence $a^2 = b$

Similarly,

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	b^2

$b^2 \neq 1$ (similar to above)

$b^2 \neq b$

& $b^2 \in \{\overset{\times}{1}, a, \overset{\times}{b}\}$

$\Rightarrow b^2 = a$

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

Group of order 3 is Abelian $\Leftarrow$

[Symmetric Table]

	1	-1	i	-i	j	-j	k	-k

Quaternion Group H.

$$H = \{ \pm 1, \pm i, \pm j, \pm k \}$$

where $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$$H = \langle i, j \text{ such that } i^4 = 1,\ i^2 = j^2,\ ji = i^3 j \rangle$$

This H is a subgroup of $GL_2(\mathbb{C})$.

Klein four group V.

$$V = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{= e}, \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{= a}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{= b}, \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{= c} \right\}$$

with

$$\left.\begin{array}{l} a^2 = e \\ b^2 = e \\ c^2 = e \end{array}\right\} \quad \begin{array}{l} \langle e, a \rangle = \langle a \rangle \neq V \\ \langle e, b \rangle = \langle b \rangle \neq V \\ \langle c \rangle \neq V \end{array}$$

Smallest group which is not cyclic.

Question. What are non-trivial proper subgroups of the klein four group?

whole group.

$$\begin{tikzcd}[column sep=small]
 & V = \{e, a, b, c\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{e, a\} \arrow[dr, dash] & \{e, b\} \arrow[d, dash] & \{e, c\} \arrow[dl, dash] \\
 & \{e\} &
\end{tikzcd}$$

subgroups of order 2

trivial subgro

Three proper subgroups each of order 2.

Question. Is klein four group abelion?

$$(x * y = y * x \text{ for all } x, y \in V)$$

$$(\mathbb{Z}/4\mathbb{Z}, +) = \{[0], [1], [2], [3]\} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$$

	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$			
$\bar{3}$	$\bar{3}$			

$$\begin{tikzcd}
\mathbb{Z}/4\mathbb{Z} \arrow[d, dash] \\
\{\bar{0}, \bar{2}\} \arrow[d, dash] \\
\{\bar{0}\}
\end{tikzcd}$$

only one subgroup of order 2

Exercise. Multiplication Table for group of order 4 and 5.

Multiplication table for S_6.

Re-visit to Symmetric group.

Let A be a non-empty set.

Define $S_A = \{ f : A \to A$ such that f is a bijection$\}$

Lemma. $(S_A, \circ)$ is a Group.

Binary operation as composition of functions.

$$\circ : S_A \times S_A \longrightarrow S_A$$
$$(f, g) \longmapsto f \circ g \in S_A$$

Definition. $(S_A, \circ)$ is called "Symmetric Group" or "Permutation Group" on the set A.

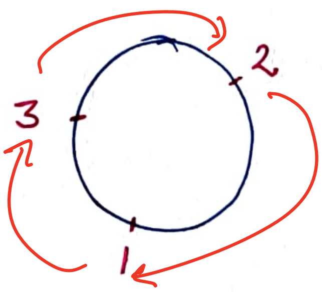
Special Case: $A = \{1, \cdots, n\}$, we denote this by S_n.

Cycle. A cycle is a string of integers which represents the elements of S_n which cyclically permutes these integers and fixes all other integers.

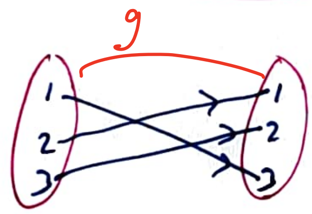
Example.

$(213) \in S_3$

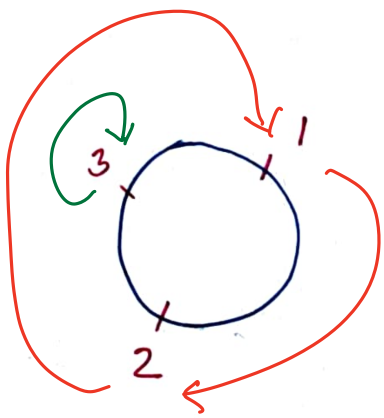
Notation: $(\;\;)$ for cycle, $\underline{\underline{213}} \in S_3$

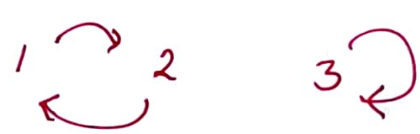


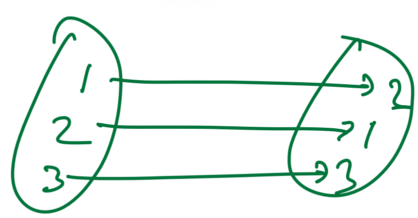
$2 \rightsquigarrow 1 \rightsquigarrow 3$



$\underbrace{(12)}_{\text{Cycle}}\underbrace{(3)}_{\text{Cycle}}$







Consider a cycle $(a_1 a_2 \cdots a_\ell)$ in S_n.

$$a_1 \rightsquigarrow a_2 \rightsquigarrow a_3 \rightsquigarrow \cdots \rightsquigarrow a_{\ell-1} \rightsquigarrow a_\ell$$

Convention. Greek words σ, τ etc. are often used in litrature for elements in S_n.

$\sigma \in S_n$, then

$$\sigma = (a_1 a_2 \cdots a_{\ell_1})(a_{\ell_1+1} \cdots a_{\ell_2}) \cdots (a_{\ell_{k-1}+1} \cdots a_{\ell_k})$$

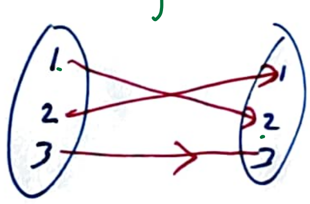
Cycle decomposition of permutation σ.

$$a_1 \rightsquigarrow a_2 \rightsquigarrow \cdots \rightsquigarrow a_{\ell_1} \qquad a_{\ell_1+1} \rightsquigarrow a_{\ell_1+2} \rightsquigarrow \cdots \rightsquigarrow a_{\ell_2} \quad \cdots \quad a_{\ell_{k-1}+1} \rightsquigarrow \cdots \rightsquigarrow a_{\ell_k}$$

$\sigma = (12)(3) \in S_3$

$1 \rightleftarrows 2 \qquad 3$

f



$\sigma = (1357)(2468) \in S_8$ etc..

Note.

1. The length of a cycle is the number of integers which appear in it. In particular, a cycle of length m is called an m-cycle.

$$\sigma = \underbrace{(1357)}_{4\text{-cycle}}\underbrace{(2468)}_{4\text{-cycle}}\underbrace{(9,10)}_{2\text{ cycle}} \in S_{10}$$

2. Two cycles are disjoint if they have no members in common.

Group S_3 [6 bijections]

3 of them comes as a result of rotation by 120° (equilateral Δ)

123 ~ 1 (identity)

231 ~ (123)

312 ~ (132)

3 of them come as a reflection, one for each rotation

(12)(3) ~ 213

(13)(2) ~ 321

(23)(1) ~ 132

Writing elements of S_3

$f_1 : 123 \to 123$	$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$e = 1 = (1)(2)(3)$ or $(\,)$
$f_2 : 123 \to 132$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$(23) \in S_3$
$f_3 : 123 \to 213$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$(12) \in S_3$
$f_4 : 123 \to 231$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$(123) \in S_3$
$f_5 : 123 \to 312$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}_{2\times 3}$	$(132) \in S_3$
$f_6 : 123 \to 321$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$(13) \in S_3$

multiplication

Verify $(12)\circ(13) \neq (13)\circ(12)$.

In general, S_n $(n \geq 3)$ is a non-abelian group.

$ab \neq ba$ for some a & b.

Multiplication Table for S_3

	( )	(12)	(23)	(13)	(123)	(132)
( )	( )	(12)	(23)	(13)	(123)	(132)
(12)	(12)					
(23)	(23)					
(13)	(13)					
(123)	(123)					
(132)	(132)					

work out details

7. Easy (a) $(a\mathbb{Z} + b\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$

80

(b) $\langle a, b + 7a \rangle = a\mathbb{Z} + b\mathbb{Z}$

over $\mathbb{Z}$

8. Draw a multiplication table for the quaternion group H.

Exercise.

9. H is a subgroup of G generated by two elements a, b of a group G.

Prove that if $ab = ba$, then H is an abelian group.

<u>Solution</u>. [H is Abelian means for any $h_1, h_2 \in H$ we need to show $h_1 h_2 = h_2 h_1$]

Here $|G|$ is not given.

Let $h \in H$, then

$$h \left\{ \begin{array}{l} a \\ b \end{array} \right.$$

h could be any of the following form

$$\begin{cases} h = a \\ h = b \\ h = \prod a & \text{(finite product of } a \text{ with itself)} \\ h = \prod b & \text{(finite product of } b \text{ with itself} \\ h = & \text{mix product of } a \text{ and } b \end{cases}$$

Let $h = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_n} b^{\beta_n}$

where $\alpha_1, \ldots, \alpha_n$ are +ve integers
$\beta_1, \ldots, \beta_n$ are +ve integers

Similarly $K \in H$, then

$$K = a^{\gamma_1} b^{\delta_1} a^{\gamma_2} b^{\delta_2} \cdots a^{\gamma_n} b^{\delta_n}$$

$\boxed{h \cdot k = k \cdot h}$

where $\gamma_1, \ldots, \gamma_n$ and $\delta_1, \ldots, \delta_n$ are

$ab = ba$

positive integers.

Claim. $hK = Kh$.

1^{st} consider a special case:

$$h = a^{\alpha_1} b^{\beta_1} \quad \text{and} \quad k = a^{\gamma_1} b^{\delta_1}$$

then

$$h \cdot k = \underbrace{a \cdots a}_{\alpha_1 \text{ times}} \underbrace{b \cdots b}_{\beta_1 \text{ times}} \underbrace{a \cdots a}_{\gamma_1 \text{ times}} \underbrace{b \cdots b}_{\delta_1 \text{ times}}$$

Use $ab = ba$

and rewrite it as.

$$h \cdot k = a^{\alpha_1} a^{\gamma_1} b^{\beta_1} b^{\delta_1}$$

Again re-write the expression as

$$= a^{\gamma_1} b^{\delta_1} a^{\alpha_1} b^{\beta_1}$$

$$= k \cdot h.$$

This approach works in general, and conclude

that $\quad h \cdot k = k \cdot h \quad \forall \; h, k \in H$