MA1140: Lecture 8
Eigenvalues and Eigenvectors

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Indian Institute of Technology Hyderabad

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- An eigenvalue can be positive, negative or zero.

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Example 1: eigenvalues and eigenvectors of stretching

Let $c \in \mathbb{R}$. Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix} \text{ for every } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

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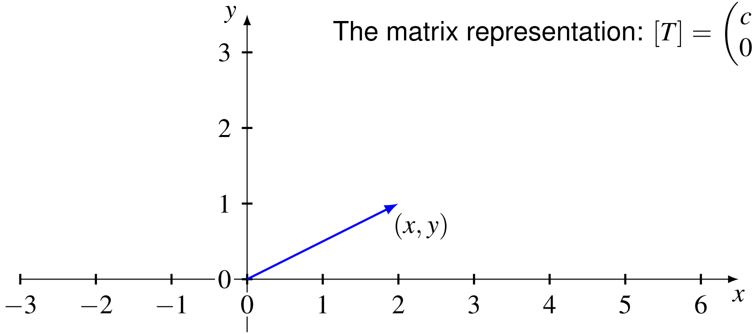

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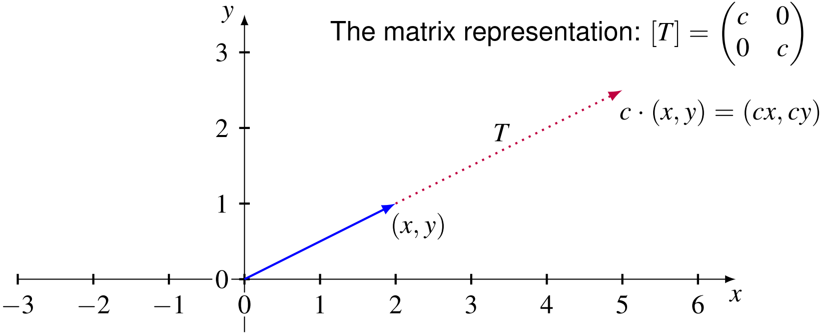

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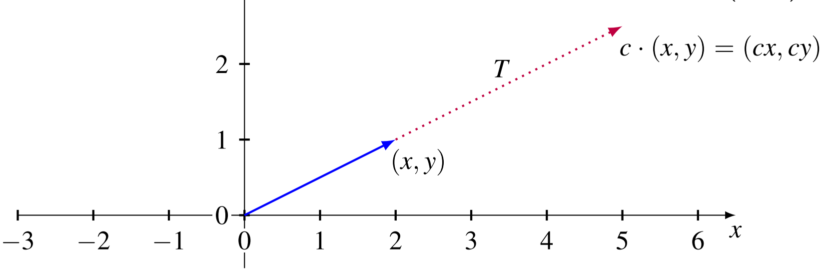

Every $v\,(\neq 0) \in \mathbb{R}^2$ is an eigenvector of T with the eigenvalue c.

Example 2: eigenvalues and eigenvectors of reflection

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

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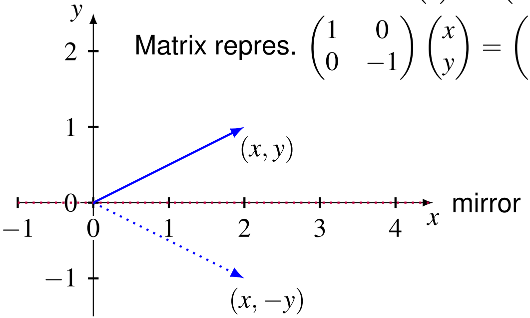

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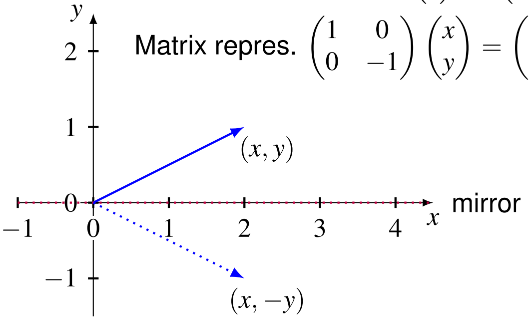

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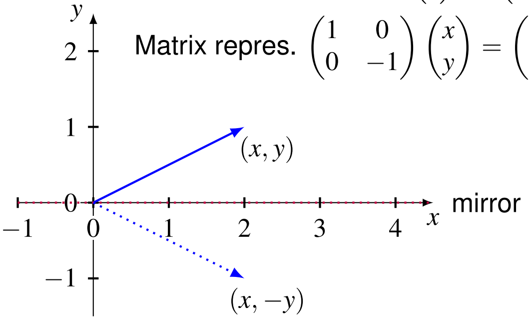

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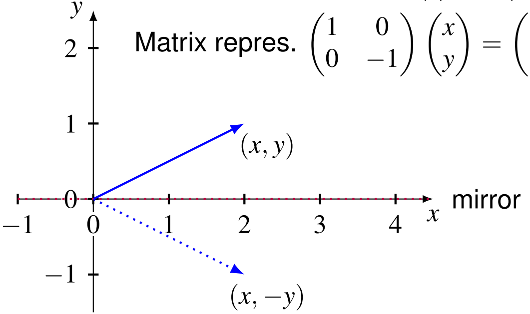

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These are ALL the eigenvectors of T. (Verify it!)

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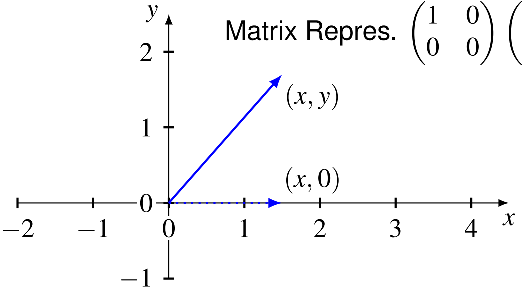

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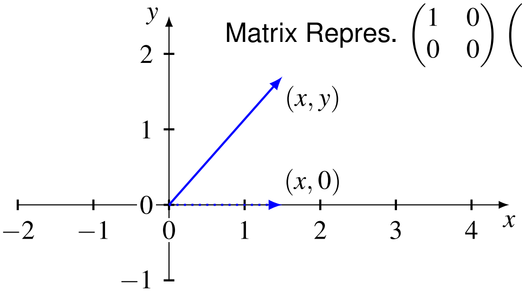

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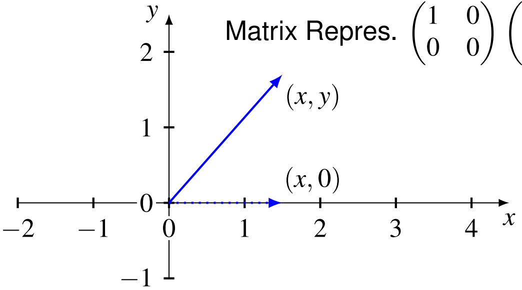

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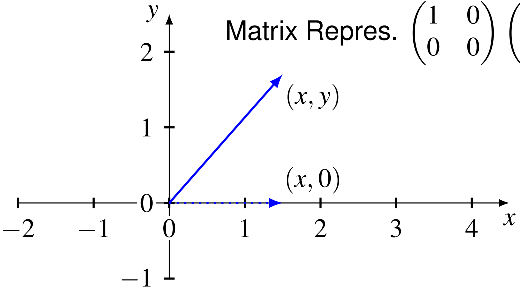

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Example 4: A may not have eigenvalues and eigenvectors over a particular field

- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{R}$.

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- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{R}$.
- Does A have eigenvalues and eigenvectors over $\mathbb{R}$?

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- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{R}$.
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Since $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0$.

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Since $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0$. Hence $\lambda^2 + 1 = 0$. But no such λ exists in $\mathbb{R}$.

Example 4: A may not have eigenvalues and eigenvectors over a particular field

- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{R}$.
- Does A have eigenvalues and eigenvectors over $\mathbb{R}$?
- If yes, then there are $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$ and $\lambda \in \mathbb{R}$ such that

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- So A does not have eigenvalues and eigenvectors over $\mathbb{R}$.

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- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{C}$, the set of complex numbers.
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- **Conclusion:** The matrix A has eigenvalues and eigenvectors over $\mathbb{C}$, but not over $\mathbb{R}$.

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Some inequalities on algebraic/geometric multiplicities

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Proof.

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   Since $\mathrm{GM}_A(\lambda) = \dim\left(\mathrm{Null}(A - \lambda I_n)\right)$,

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Let A be an $n \times n$ matrix over $\mathbb{C}$. For every eigenvalue λ of A, we have

1. $1 \leqslant \mathrm{AM}_A(\lambda) \leqslant n$ and $1 \leqslant \mathrm{GM}_A(\lambda) \leqslant n$.
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Proof.

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$\square$

Example: Characteristic polynomial and eigenvalues

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- The algebraic multiplicities of both λ_1 and λ_2 are 1.

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- Recall that in order to solve a linear system, you may apply elementary row operations to make it into a system with row reduced echelon coefficient matrix.

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The set of eigenvectors helps us to test whether a matrix is diagonalizable or not.

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Hence left multiply by P from the left side.

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(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): We will skip it. $\square$

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If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,then $p_A(x) = x^2 - 5x - 2$. The Cayley-Hamilton Theorem says that $A^2 - 5A - 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

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Similar matrices have same characteristic polynomial

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Let A and B be similar, i.e., $B = P^{-1}AP$ for some invertible matrix P. Then $\det(xI_n - A) = \det(xI_n - B)$.

Proof. $\det(xI_n - B) = \det(xP^{-1}I_nP - P^{-1}AP) = \det(P^{-1}(xI_n - A)P)$
$= (1/\det(P))\det(xI_n - A)\det(P) = \det(xI_n - A)$. □

Theorem

Let A and B be similar, i.e., $B = P^{-1}AP$. For a polynomial $f(x) \in \mathbb{R}[x]$, $f(A) = 0$ if and only if $f(B) = 0$ (zero matrix).

Proof. Note: $P^{-1}A^rP = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) = B^r$,
and $P^{-1}(c_1D_1 + c_2D_2)P = c_1(P^{-1}D_1P) + c_2(P^{-1}D_2P)$.
Verify that $P^{-1}f(A)P = f(B)$ and $Pf(B)P^{-1} = f(A)$.
Hence the theorem follows. □

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Remark

It can be observed from the proof that if A is diagonalizable, then A satisfies a polynomial having distinct roots.

Applications of Eigenvalues and Eigenvectors

Some real life applications of the use of eigenvalues and eigenvectors in science, engineering and computer science can be found here:
`https://www.intmath.com/matrices-determinants/8-applications-eigenvalues-eigenvectors.php`

Thank You!