Problems Set 4 : Eigenvalues and Eigenvectors

Before trying to solve each exercise, first you should be familiar with the terminologies, definitions and basic theory on that. You may read the lecture notes or lecture slides.

Throughout, the base field $\mathbb{F}$ is either the field of real numbers, or the field of complex numbers. All matrices here are over $\mathbb{F}$. Let A be an $n \times n$ matrix, and $f(x) = a_r x^r + \cdots + a_1 x + a_0$ be a polynomial over $\mathbb{F}$. Then we write $f(A) = a_r A^r + \cdots + a_1 A + a_0 I_n$, which is an $n \times n$ matrix.

1. Let $T : V \to V$ be a linear map.
(i) How many eigenvalues can be there for T corresponding to a fixed eigenvector v?
(ii) How many eigenvectors can be there for T associated with a fixed eigenvalue λ?

Remark. This is to understand the definitions of eigenvalues and eigenvectors.

2. Let A be an $n \times n$ matrix.
(i) Prove that $\det(A) = 0$ if and only if 0 is an eigenvalue of A.
(ii) If 0 is an eigenvalue of A, describe the eigenspace of A corresponding to 0. In this case, what is the geometric multiplicity $\mathrm{GM}_A(0)$ in terms of the invariants of A.

Hint. Consider the system of linear equations $AX = 0$.

3. Let A be an $n \times n$ matrix such that $A^2 - A = 0$ (zero matrix), or equivalently $A^2 = A$.
(i) What are the possibilities of eigenvalues of A. Provide examples for all such cases.
(ii) Find the eigenspace of A corresponding to each possible eigenvalue.

Hint. You may start with the definition of eigenvalues and eigenvectors. The possibilities of eigenvalues of A are 0 and 1.

4. Let A be an $n \times n$ matrix with THE eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Prove that

$$\operatorname{trace}(A) = \lambda_1 + \cdots + \lambda_n \quad \text{and} \quad \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Hint. We have proved that the eigenvalues can be obtained as the roots of the characteristic polynomial, i.e., $\det(xI_n - A) = (x - \lambda_1) \cdots (x - \lambda_n)$. Now compare the constant terms and the coefficients of x^{n-1} from both sides.

5. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Verify the statements stated in Q.4. Verify whether A is diagonalizable by answering all three equivalent questions:

(i) Is there an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. If yes, then what is that P, and what is that diagonal matrix $P^{-1}AP$?

(ii) For every eigenvalue λ of A, find $\mathrm{GM}_A(\lambda)$ and $\mathrm{AG}_A(\lambda)$. Also verify $\mathrm{GM}_A(\lambda) = \mathrm{AG}_A(\lambda)$.

(iii) Find a polynomial $p(x) \in \mathbb{C}[x]$ having distinct roots such that $p(A) = 0$ (zero matrix).

Hint. First find all the eigenvalues λ by computing the characteristic polynomial and its roots. Then in order to get the eigenspace corresponding to each λ, solve the homogeneous system $(A - \lambda I_2)X = 0$. We just need to check whether there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A; see the theorem proved in Lecture 8.

6. Let λ be an eigenvalue of an $n \times n$ matrix A with a corresponding eigenvector v.

(i) Show that λ is an eigenvalue of A^t (the transpose of A).

(ii) Show that v is an eigenvector of $B = A + cI_n$, where c is a fixed scalar. What is the corresponding eigenvalue of B?

(iii) Let r be a positive integer. Show that λ^r is an eigenvalue of A^r with the corresponding eigenvector v. Conclude that for every polynomial $f(x) \in \mathbb{F}[x]$, $f(\lambda)$ is an eigenvalue of $f(A)$ with the corresponding eigenvector v.

(iv) Let P be an $n \times n$ invertible matrix. Show that λ is an eigenvalue of $P^{-1}AP$ with a corresponding eigenvector $P^{-1}v$. Conclude from this statement that A and $P^{-1}AP$ have the same set of eigenvalues. Moreover, there is a one to one correspondence between the eigenvectors of A and that of $P^{-1}AP$ corresponding to every fixed eigenvalue λ.

(v) Suppose that A is invertible. Then, by Q.2, $\lambda \neq 0$. Show that v is also an eigenvector of A^{-1} with respect to the eigenvalue $1/\lambda$.

Hint. (i). It can be concluded from $\det(A - \lambda I_n) = \det(A^t - \lambda I_n)$.

(ii) and (iii). Verify directly by using the definition of eigenvalues and eigenvectors.

(iv). The 1st part can be verified directly. Using the 1st part, one also obtains that if λ is an eigenvalue of $P^{-1}AP$, then λ is an eigenvalue of $(P^{-1})^{-1}(P^{-1}AP)(P^{-1}) = A$, which concludes the 2nd part. By one to one correspondence, we mean bijective maps from both sides. Let E_λ and E'_λ be the eigenspaces of A and $P^{-1}AP$ corresponding to λ respectively. Define the maps $\varphi : E_\lambda \to E'_\lambda$ and $\psi : E'_\lambda \to E_\lambda$ by $\varphi(v) = P^{-1}v$ and $\psi(u) = Pu$ respectively. Clearly φ and ψ are inverse maps of each other.

(v). Apply A^{-1} on $(A - \lambda I_n)v = 0$.

7. Let A be an $n \times n$ matrix with only one eigenvalue $\lambda \in \mathbb{F}$ (in other words, $\det(xI_n - A) = (x - \lambda)^n$). Let E_λ be the eigenspace of A corresponding to λ. (Note that E_λ is a subspace of $\mathbb{F}^n$.) Show that $\dim(E_\lambda) = n$ if and only if A is diagonalizable.

8. Prove that the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable.

9. Let A be an $n \times n$ matrix. Show that if A is diagonalizable, then A^r is also diagonalizable, where r is a positive integer.

Hint. Note $P^{-1}A^rP = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$, multiplication of r many copies.

10. Let A be a nilpotent square matrix, i.e., $A^r = 0$ for some $r \geqslant 1$. Prove that if A is diagonalizable, then A is a zero matrix.

11. Let A be a 2×2 matrix. Suppose A has two eigenvalues λ_1 and λ_2 in $\mathbb{F}$ such that $\lambda_1 \neq \lambda_2$. Prove that A is diagonalizable.

Hint/Fact: (*This part is optional.*) More generally, by induction on n, one can prove the following: Let A be an $n \times n$ matrix, and $n \geqslant 2$. Let $v_1, v_2, \ldots, v_n$ be some eigenvectors correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Show that $v_1, v_2, \ldots, v_n$ are linearly independent. Conclude that a matrix having all distinct eigenvalues is diagonalizable.

12. Minimal polynomial. (*This exercise is optional.*) Let A be an $n \times n$ matrix over $\mathbb{F}$. Recall that a monic (i.e., with the leading coefficient 1) polynomial $p(x) \in \mathbb{R}[x]$ is said to be a minimal polynomial of A if $p(A) = 0$ (zero matrix) and p has minimal possible degree. Since a minimal polynomial is monic, it is a non-zero polynomial. Prove the following statements:

(i) *Existance.* The matrix A has a minimal polynomial $p(x)$.

(ii) *Uniqueness.* Let $f(x) \in \mathbb{R}[x]$ be such that $f(A) = 0$. Then a minimal polynomial $p(x)$ divides $f(x)$. Conclude that A has a unique minimal polynomial.

Solution. (i) By Cayley-Hamilton Theorem, there is a monic polynomial $h(x) \in \mathbb{F}[x]$ such that $h(A) = 0$. Set $\mathcal{B} := \{g(x) \in \mathbb{F}[x] : g(x) \text{ is monic and } g(A) = 0\}$. By Cayley-Hamilton Theorem,

$\mathcal{B}$ is non-empty. So, by Well-Ordering property of the set of natural numbers, there is an element $p(x)$ in $\mathcal{B}$ of minimal possible degree. Then $p(x)$ is a minimal polynomial of A.

(ii) By division algorithm, there are $q(x)$ (quotient) and $r(x)$ (remainder) in $\mathbb{F}[x]$ such that

$$f(x) = p(x)q(x) + r(x), \quad \text{where } r(x) = 0 \text{ or } \deg(r(x)) < \deg(p(x)).$$

Hence it follows from $f(A) = 0$ and $p(A) = 0$ that $r(A) = 0$. Since $p(x)$ has minimal possible degree and $\deg(r(x)) < \deg(p(x))$, we have $r(x) = 0$. Therefore $p(x)$ divides $f(x)$. For the last part, if possible, suppose $p(x)$ and $p'(x)$ are two minimal polynomials of A. Then, by the 1st part, $p(x)$ divides $p'(x)$. For the same reason, $p'(x)$ also divides $p(x)$. Hence $p(x) = p'(x)$.

13. **Diagonalizability via minimal polynomial** Let A be an $n \times n$ diagonalizable matrix over $\mathbb{F}$. Prove that the minimal polynomial of A has distinct roots in $\mathbb{F}$. (**Fact:** The converse is also true, i.e., a matrix having the minimal polynomial with distinct roots is diagonalizable. The proof is hard.)

    Hint. There is P such that $P^{-1}AP$ is a diagonal matrix. It is proved in Lecture 8 that for a polynomial $f(x)$, we have $f(A) = 0$ if and only if $f(P^{-1}AP) = 0$. Conclude that A and $P^{-1}AP$ have the same minimal polynomial. So, without loss of generality, we may assume that A is a diagonal matrix. Then pick all the distinct diagonal entries, say $d_1, \ldots, d_r$. Set $g(x) := (x - d_1) \cdots (x - d_r)$. Show that $g(A) = 0$. Hence conclude the statement by Q.12(ii).

14. **An application of Cayley-Hamilton Theorem:** Let A be an $n \times n$ matrix. Suppose the characteristic polynomial of A is $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{F}[x]$. Show that A is invertible if and only if $a_0 \neq 0$. Prove that when A is invertible, then

$$A^{-1} = \frac{-1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots a_2A + a_1I_n).$$

    Hint. Note that $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = f(x) = \det(xI_n - A)$, which yields that $a_0 = f(0) = (-1)^n \det(A)$. To compute A^{-1}, use the Cayley-Hamilton Theorem.