Matching algorithms (Cont…)

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Basic definitions (Recap)

- Given a graph $G = (V, E)$ and a matching M of G
  1. A ***stem*** with respect to M is an alternating path of even length from an unmatched vertex v (called the ***root*** of the stem)
     - Observation: The last edge on the stem belongs to M
  2. A ***blossom*** B with respect to M is a
     I. a cycle in G consisting of $2k + 1$ edges
     II. exactly k edges of the cycle belong to M, and
     III. one of the vertices v of the cycle (we call it the ***base***) is such that there exists a stem from an unmatched vertex w to v
        - Observation: The two edges of the blossom touching the base are not in M. Other than that, every second edge on the blossom belongs to M

Modifying the existing algorithm

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1. It can detect existing blossoms, and
2. It can go on finding augmenting paths even in the presence of blossoms

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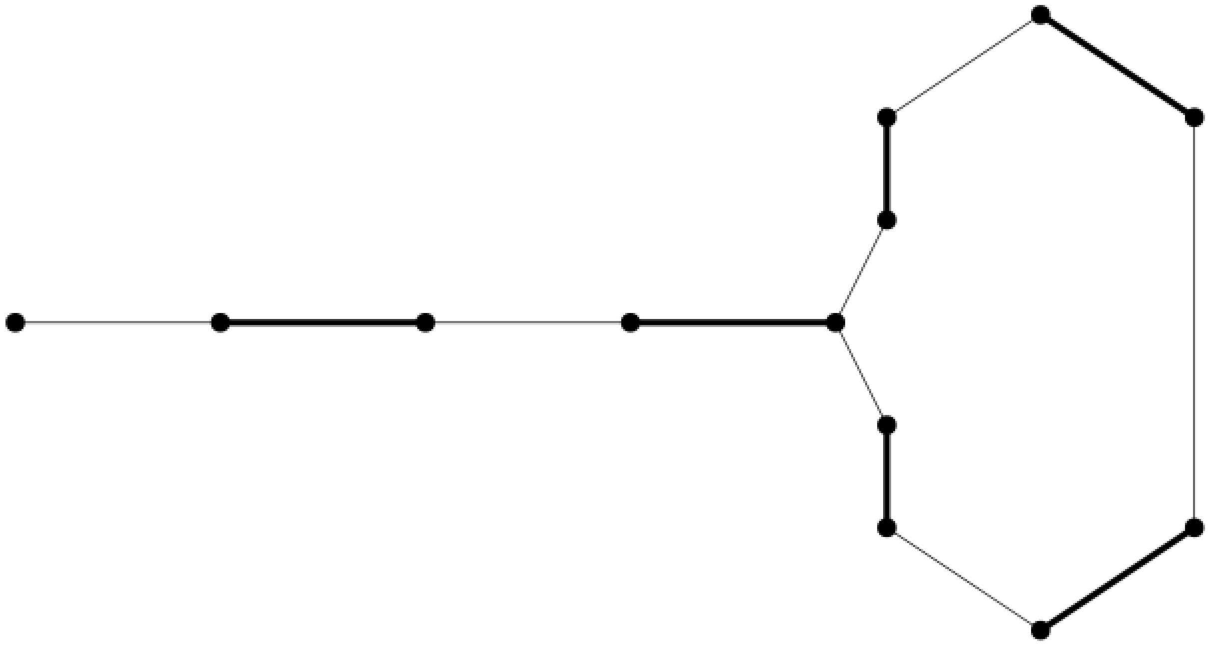
  2. It can go on finding augmenting paths even in the presence of blossoms

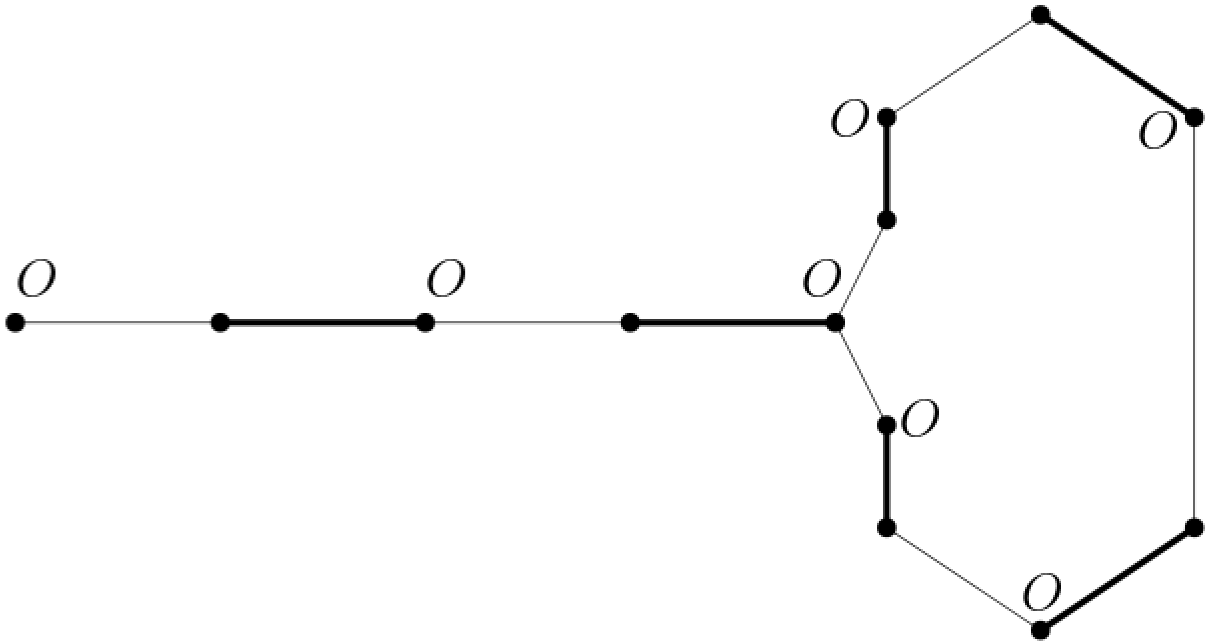
Detecting blossoms

- Perform alternating path search of the bipartite matching algorithm: start from an unmatched vertex
  1. Label vertices at even distances from the root as ***outer***
  2. Label vertices at odd distances from the root as ***inner***
- If two outer vertices found to be adjacent in the graph, we have a blossom

Example



Example

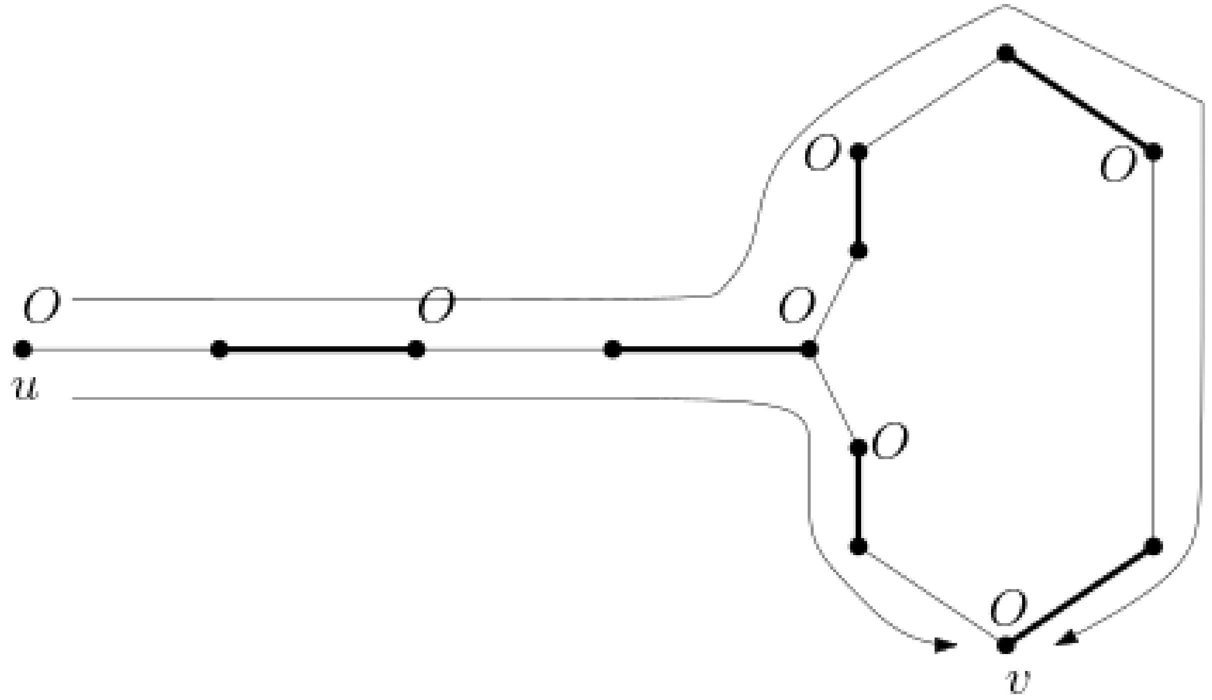


Algorithm idea

- To find an augmenting path in general graph, we will modify the procedure for bipartite graphs, so that it also detects blossoms
- If blossoms found, we shrink the blossom and restart on the new graph
- Any augmenting path found in new graph can easily be translated to an augmenting path in the original graph
- We will prove that, if the matching is maximum in the new graph, then it is also maximum in the original graph

Key observation 1

- Let M be a matching of G and B be a blossom. While searching for an augmenting path from an unmatched node u in G, if we discover B, then there is an alternating path from u to any node on B ending with a matched edge



Key observation 2

- Let M be a matching of G and B be a blossom. Consider the graph G' obtained by contracting B to a single vertex. Then the matching M' of G' induced by M is maximum in G' if and only if M is maximum in G
- *Proof* : If M' is not maximum in G', then G' contains an augmenting path P' wrt M'
- ***Case 1***: If P' doesn't have any vertex common in B, then P' is also an augmenting path in G. Contradiction to fact that M is maximum
- ***Case 2***: If P' intersects B, then some portion of the blossom is in interior of P' in G'
- Let P' enters B at a vertex v, and let u be the vertex where P' leaves B
- If $u = v$, then u is nothing but the base of B
- If $u \neq v$, then consider the augmented path in G' up to v and one of the alternating path from v to u in B and the rest of the path of P'
- The path P' is then an augmenting path in G. A contradiction

Proof (Cont⋯)

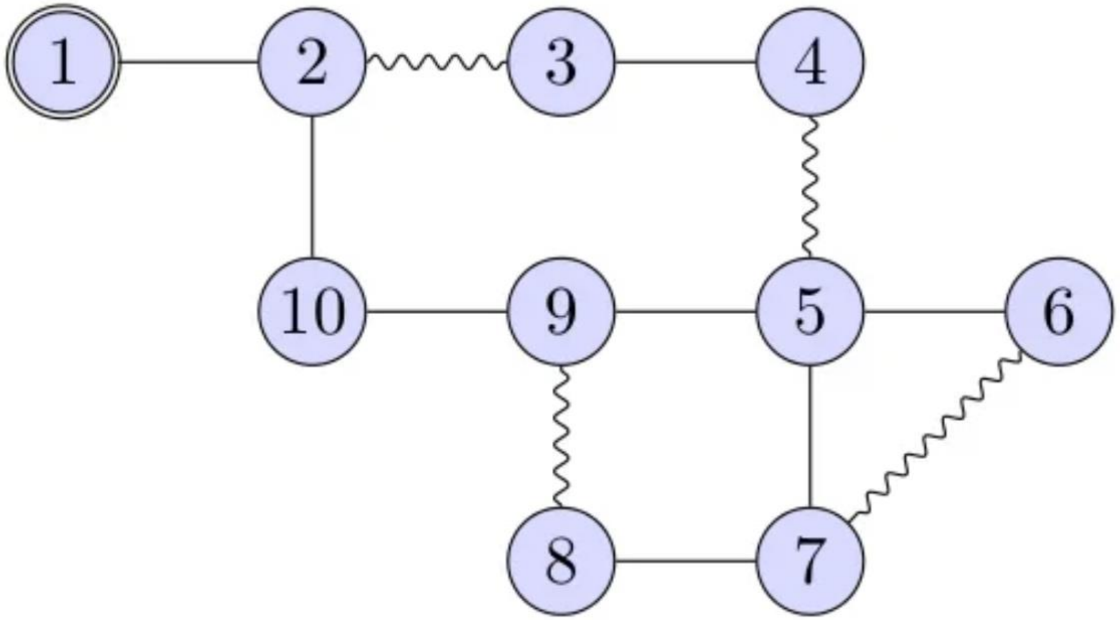
- Now assume that M is not a maximum matching in G
- We will show that M' is not a maximum matching in G'
- Let P be an augmenting path in G (Observe P is of odd length)
- P must intersect the blossom (otherwise P is also an augmenting path in G')
- P must pass through the stem of B
- If there are blossom's edges included in the augmenting path, contracting the blossom will eliminate an even number edges
- Thus, with an even number of alternating edges removed, the path in G remains an odd-length augmenting path in G'
- We arrived at a contradiction as P' is an augmenting path in G'

Edmond's algorithm

1. Start with an empty matching M
2. For all unmatched vertices $v \in V$
   a. Search for simple alternate path P from v
      - Shrink any blossom found
   b. If the path P ends at an unmatched vertex
      - Update M
   c. Else if no augmenting path found then
      - Ignore v for future searches
3. Return M

Example

$|M| = 4$



Example

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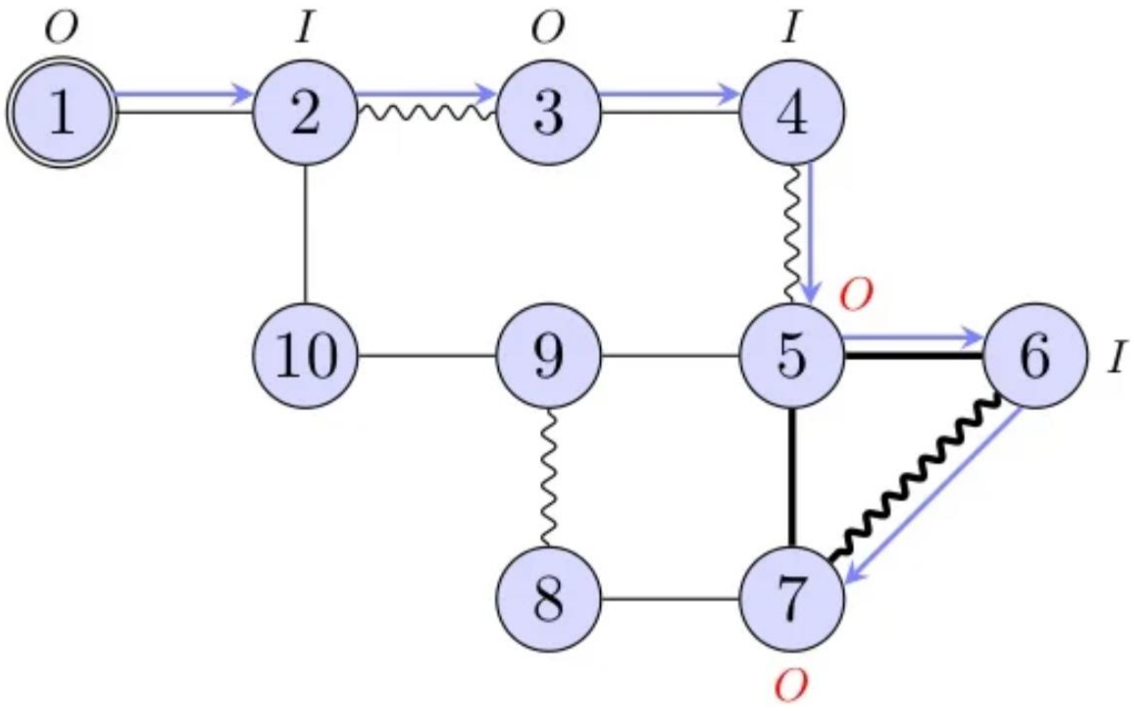

Example

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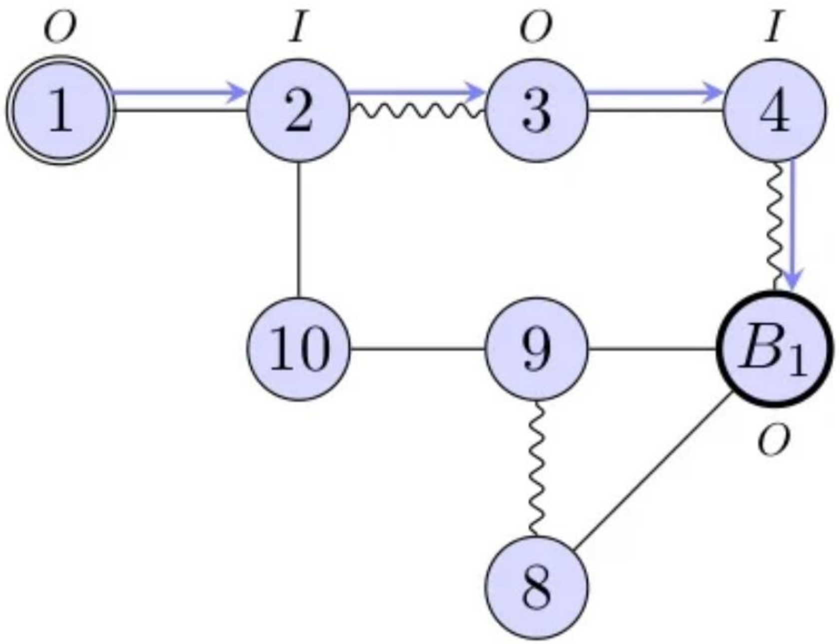
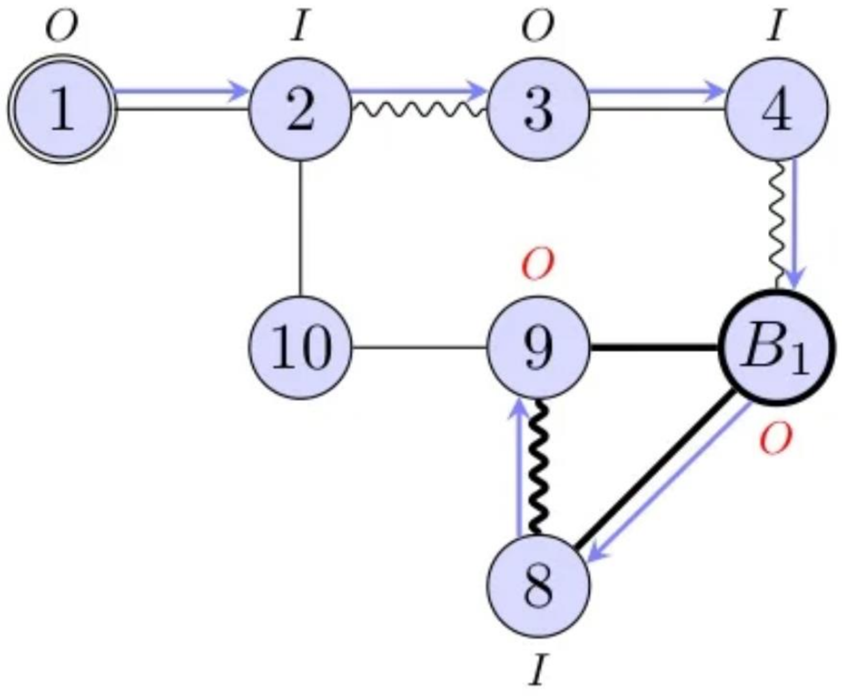


$B_1 = 5, 6, 7$

Example

$|M| = 4$



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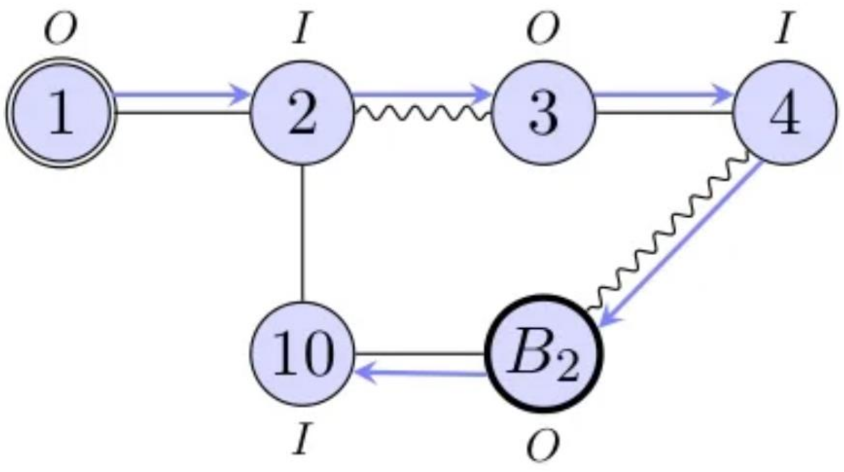

$B_1 = 5, 6, 7$
$B_2 = B_1, 8, 9 = 5, 6, 7, 8, 9$

Example

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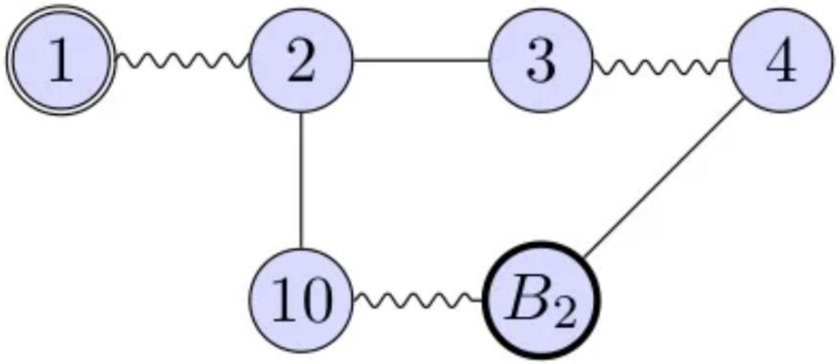

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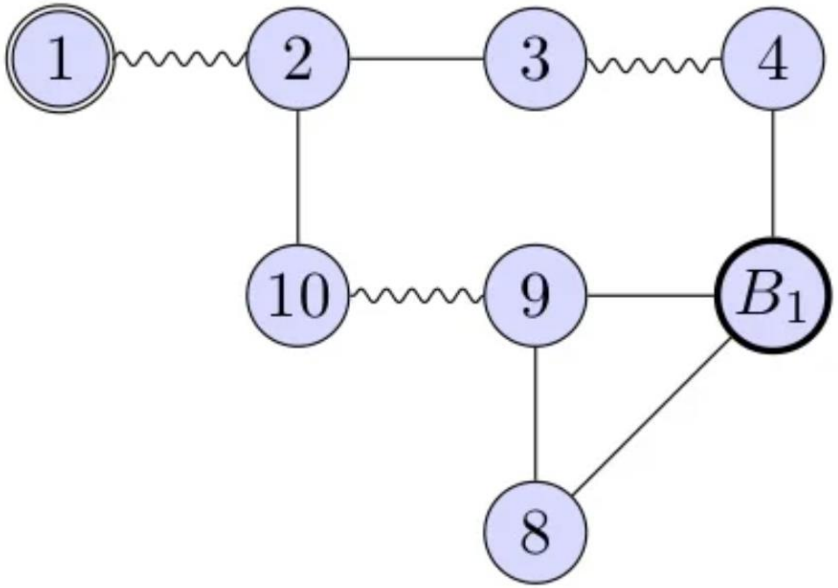
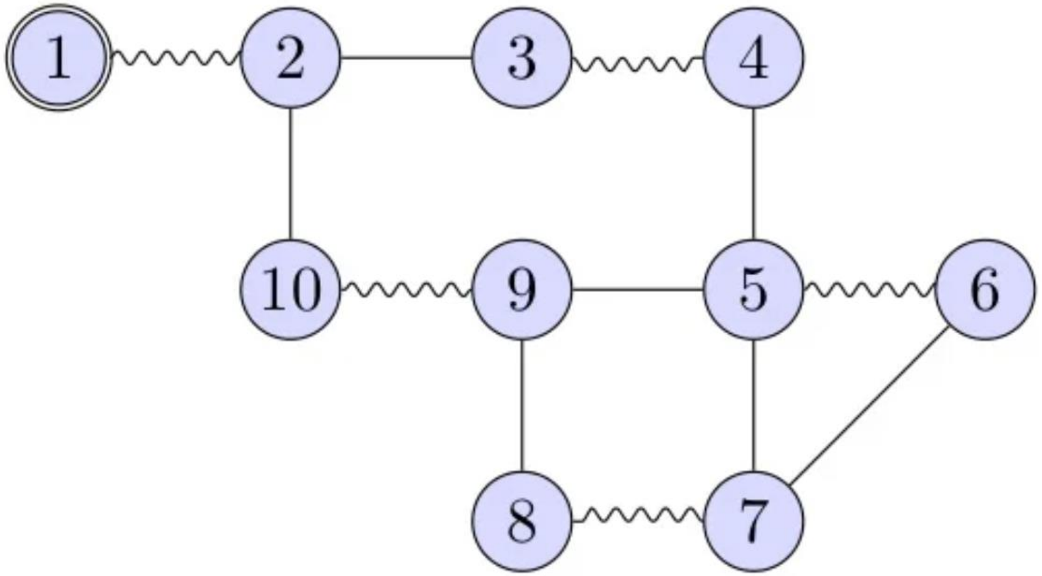


$B_1 = 5, 6, 7$
$B_2 = B_1, 8, 9 = 5, 6, 7, 8, 9$

Example

$|M| = 4$

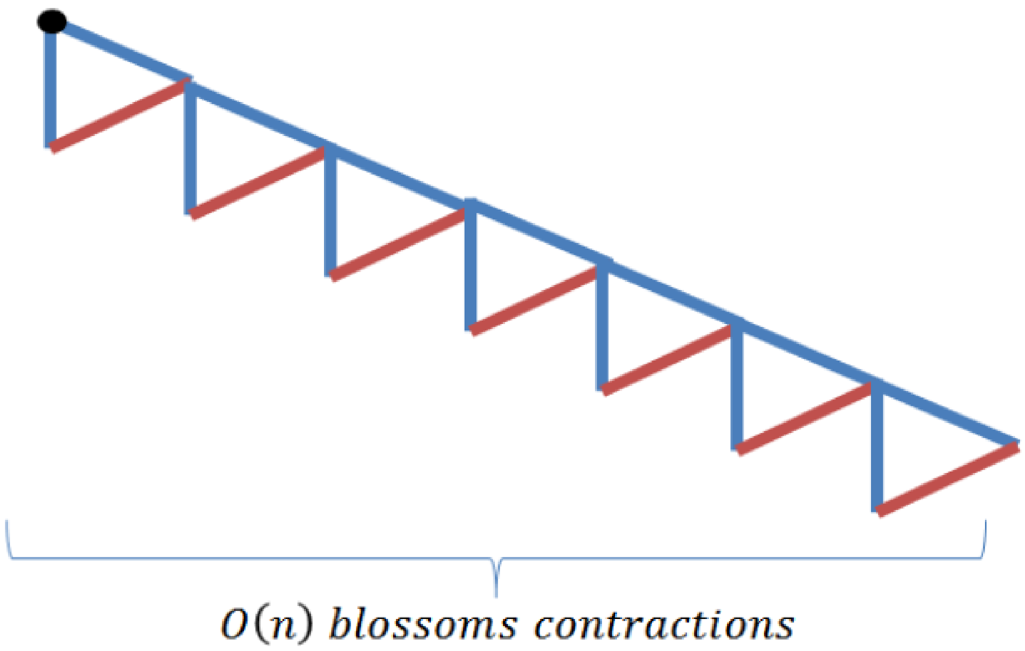
$|M| = 5$



$B_1 = 5, 6, 7$
$B_2 = B_1, 8, 9 = 5, 6, 7, 8, 9$

Lemma

- Each augmenting path iteration can have $O(n)$ Blossom contraction steps
- ***Proof***: One contraction of a blossom reduces the nodes by at least two
- Therefore, we can have at most $n/2$ blossom contractions
- As a result, each augmenting path iteration cannot have more than $O(n)$ blossom contraction steps



Complexity

- Contraction of a blossom takes $O(n + m)$ time
- An augmenting path can have at most $O(n)$ blossoms
- Total cost = No. of iterations•(time spent per iteration)

$= O(n)$•(time spent for case a + case b + case c)

$= O(n) \cdot (O(m) + (O(m) + O(m)) \cdot O(n))$

$= O(n^2 m)$

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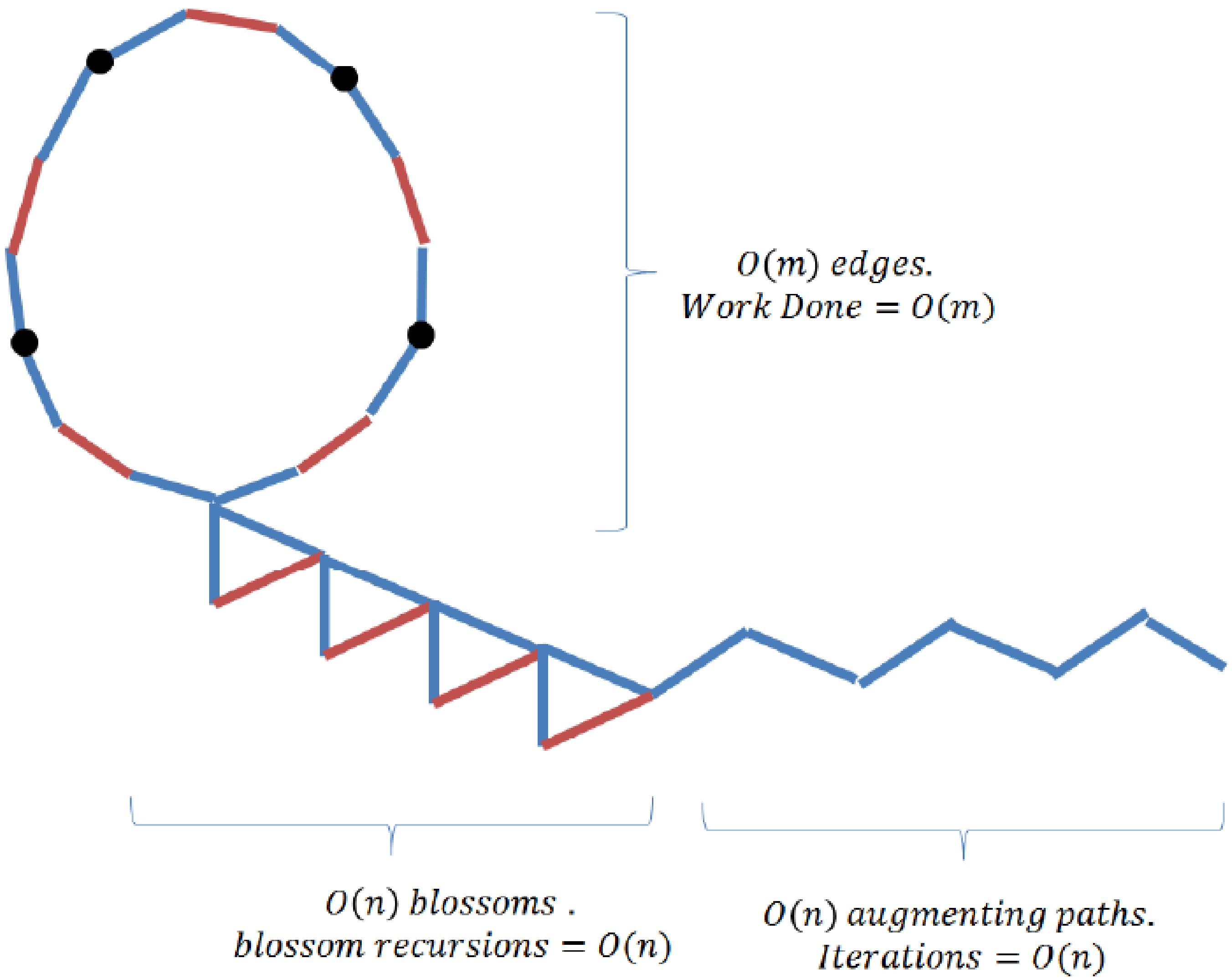
Tight analysis



- Thank you!