Differential Equations (MA 1150)

Sukumar

Lecture 7

May 05, 2020

Section 1

Homogeneous second order linear ODE

Connection:

$$\text{ODE} \longleftrightarrow \text{Recurrence relation}$$

Subsection 1

Power series solution

If $a, b, c \in \mathbb{R}$ with $a \neq 0$, then

$$ay'' + by' + cy = 0 \tag{1}$$

is called a constant coefficient 2nd order homogeneous ODE.

Recall that solution to ODE (7) will have linear combination of the following form

If $a, b, c \in \mathbb{R}$ with $a \neq 0$, then

$$ay'' + by' + cy = 0 \tag{1}$$

is called a constant coefficient 2nd order homogeneous ODE.

Recall that solution to ODE (7) will have linear combination of the following form

- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$.

Discussion

Recall

- Power series expansion of e^x around 0.

Discussion

Recall

- Power series expansion of e^x around 0.
- Power series expansion of $\sin x$ around 0.

Discussion

Recall

- Power series expansion of e^x around 0.
- Power series expansion of $\sin x$ around 0.
- Power series expansion of $\cos x$ around 0.

Discussion

Recall

- Power series expansion of e^x around 0.
- Power series expansion of $\sin x$ around 0.
- Power series expansion of $\cos x$ around 0.

Consider the following Taylor series:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty, \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty, \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty,
\end{aligned}
$$

Taylor Series If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about x_0 is defined by

Taylor Series If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about x_0 is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor Series If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about x_0 is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Power Series A power series for f at a point $x = x_0$ is a function

Taylor Series If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about x_0 is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Power Series A power series for f at a point $x = x_0$ is a function

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

Taylor Series If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about x_0 is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Power Series A power series for f at a point $x = x_0$ is a function

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

(We are assuming positive radius of convergence on open interval!!)

Theorem A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is, Theorem A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad (2)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \quad (3)$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}. \quad (4)$$

Moreover, all of these series have the same radius of convergence R.

Example Let $f(x) = \sin x$. Then

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Example Let $f(x) = \sin x$. Then

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Hence ,

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

Example Let $f(x) = \sin x$. Then

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Hence ,

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series for $\cos x$.

Discussion

Recall Constant coefficient 2nd order homogeneous ODE:

$$ay'' + by' + cy = 0 \tag{5}$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$.

Recall Constant coefficient 2nd order homogeneous ODE:

$$ay'' + by' + cy = 0 \tag{5}$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$.

Recall The solution to ODE (7) will have linear combination of the following form

- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$.

Discussion

Recall Constant coefficient 2nd order homogeneous ODE:

$$ay'' + by' + cy = 0 \tag{5}$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$.

Recall The solution to ODE (7) will have linear combination of the following form

- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$.

Lets slightly change the condition Consider 2nd order homogeneous ODE:

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{6}$$

where $a(x), b(x), c(x)$ are some polynomials and $a(x) \neq 0$.

Lets slightly change the condition Consider 2nd order homogeneous ODE:

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{7}$$

where $a(x), b(x), c(x)$ are some polynomials and $a(x) \neq 0$.

Lets slightly change the condition Consider 2nd order homogeneous ODE:

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{7}$$

where $a(x), b(x), c(x)$ are some polynomials and $a(x) \neq 0$.

We may not have solutions y_1 and y_2 in the standard form:

- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$.

Lets slightly change the condition Consider 2nd order homogeneous ODE:

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{7}$$

where $a(x), b(x), c(x)$ are some polynomials and $a(x) \neq 0$.

We may not have solutions y_1 and y_2 in the standard form:

- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$.

Note that y_1 and y_2 may be some power series (depends upon the initial conditions).

Example

Example

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Derive an expression for

$$(2 - x)y'' + 2y$$

as a power series in x on I.

Example

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Derive an expression for

$$(2-x)y'' + 2y$$

as a power series in x on I.

Note that

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore $(2-x)y'' + 2y = 2y'' - xy' + 2y =$

$$= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n.$$

Example

Shift indices in the first two so that all three series will start with $n = 0$; thus, we get

$$(2-x)y'' + 2y = \sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n]x^n.$$

Example

Shift indices in the first two so that all three series will start with $n = 0$; thus, we get

$$(2-x)y'' + 2y = \sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n]x^n.$$

Observation: We see that y satisfies ODE

$$(2-x)y'' + 2y = 0$$

on open interval I if the following recurrence relation holds

$$2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n = 0, \quad n = 0, 1, 2, \ldots.$$

Example

Shift indices in the first two so that all three series will start with $n = 0$; thus, we get

$$(2-x)y'' + 2y = \sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n]x^n.$$

Observation: We see that y satisfies ODE

$$(2-x)y'' + 2y = 0$$

on open interval I if the following recurrence relation holds

$$2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n = 0, \quad n = 0, 1, 2, \ldots.$$

Conclusion: The power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of an ODE gives rise to recurrence relations among the coefficients a_n's.

Subsection 2

Cauchy-Euler Equations

Cauchy-Euler Equations The equation of the following form

$$x^2y'' + axy' + by = 0 \quad x > 0 \tag{8}$$

where $a, b \in \mathbb{R}$, is called a Cauchy-Euler equation.

Cauchy-Euler Equations The equation of the following form

$$x^2y'' + axy' + by = 0 \quad x > 0 \tag{8}$$

where $a, b \in \mathbb{R}$, is called a Cauchy-Euler equation.

This equation can be transformed into one with constant coefficients by change of variables($x = e^t$) on the interval $(0, \infty)$. Consider the function of t given by

$$h(t) = y(e^t) = y(x)$$

Cauchy-Euler Equations The equation of the following form

$$x^2y'' + axy' + by = 0 \quad x > 0 \tag{8}$$

where $a, b \in \mathbb{R}$, is called a Cauchy-Euler equation.

This equation can be transformed into one with constant coefficients by change of variables$(x = e^t)$ on the interval $(0, \infty)$. Consider the function of t given by

$$h(t) = y(e^t) = y(x)$$

Then

$$\frac{dh}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t.$$

Therefore

$$\frac{dy}{dx} = \frac{dh}{dt}e^{-t}.$$

Therefore

$$\frac{dh}{dt} = \frac{dy}{dx}e^t,$$

Cauchy-Euler Equations

Therefore

$$\frac{dh}{dt} = \frac{dy}{dx}e^{t},$$

and

$$\frac{d^2h}{dt^2} = \frac{d^2y}{dx^2}e^{2t} + \frac{dh}{dt}$$

Therefore

$$\frac{dh}{dt} = \frac{dy}{dx}e^t,$$

and

$$\frac{d^2h}{dt^2} = \frac{d^2y}{dx^2}e^{2t} + \frac{dh}{dt}$$

Substituting these, we get a second order ODE with constant coefficients

$$h''(t) + (a-1)h'(t) + bh(t) = 0$$

Therefore

$$\frac{dh}{dt} = \frac{dy}{dx} e^t,$$

and

$$\frac{d^2 h}{dt^2} = \frac{d^2 y}{dx^2} e^{2t} + \frac{dh}{dt}$$

Substituting these, we get a second order ODE with constant coefficients

$$h''(t) + (a-1)h'(t) + bh(t) = 0$$

If $h(t)$ is a solution to this equation, then $y(x) = h(\ln x)$ is a solution to the Cauchy-Euler equation.

Therefore

$$\frac{dh}{dt} = \frac{dy}{dx}e^t,$$

and

$$\frac{d^2h}{dt^2} = \frac{d^2y}{dx^2}e^{2t} + \frac{dh}{dt}$$

Substituting these, we get a second order ODE with constant coefficients

$$h''(t) + (a-1)h'(t) + bh(t) = 0$$

If $h(t)$ is a solution to this equation, then $y(x) = h(\ln x)$ is a solution to the Cauchy-Euler equation.

If $h_1(t)$ and $h_2(t)$ are solutions to the constant coefficient ODE, then the general solution is given by

$$y = c_1 h_1(\ln x) + c_2 h_2(\ln x).$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 5y = 0$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 5y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 5h(t) = 0$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 5y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 5h(t) = 0$$

The characteristic equation is

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 5y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 5h(t) = 0$$

The characteristic equation is

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$$

Therefore, $h_1(t) = e^{-t}$ and $h_2(t) = e^{-5t}$. Hence $h_1(\ln x) = \dfrac{1}{x}$ and $h_2(\ln x) = \dfrac{1}{x^5}$.

Example Solve the following ODE:

$$x^2y'' + 7xy' + 5y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 5h(t) = 0$$

The characteristic equation is

$$p(m) = m^2 + 6m + 5 = (m+1)(m+5)$$

Therefore, $h_1(t) = e^{-t}$ and $h_2(t) = e^{-5t}$. Hence $h_1(\ln x) = \dfrac{1}{x}$ and $h_2(\ln x) = \dfrac{1}{x^5}$. Hence the general solution is

$$y = c_1\frac{1}{x} + c_2\frac{1}{x^5}.$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 9y = 0$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 9y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 9h(t) = 0$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 9y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 9h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (7-1)m + 9 = (m+3)^2.$$

Example Solve the following ODE:

$$x^2y'' + 7xy' + 9y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 9h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (7-1)m + 9 = (m+3)^2.$$

Therefore, $h_1(t) = e^{-3t}$ and $h_2(t) = te^{-3t}$.

Example Solve the following ODE:

$$x^2y'' + 7xy' + 9y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 9h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (7-1)m + 9 = (m+3)^2.$$

Therefore, $h_1(t) = e^{-3t}$ and $h_2(t) = te^{-3t}$. Hence $h_1(\ln x) = \dfrac{1}{x^3}$ and $h_2(\ln x) = \ln x \dfrac{1}{x^3}$.

Example Solve the following ODE:

$$x^2y'' + 7xy' + 9y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (7-1)h'(t) + 9h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (7-1)m + 9 = (m+3)^2.$$

Therefore, $h_1(t) = e^{-3t}$ and $h_2(t) = te^{-3t}$. Hence $h_1(\ln x) = \frac{1}{x^3}$ and $h_2(\ln x) = \ln x \frac{1}{x^3}$. Hence the general solution is

$$y = c_1 \frac{1}{x^3} + c_2 \ln x \frac{1}{x^3}.$$

Example Solve the following ODE:

$$x^2y'' + 5xy' + 13y = 0$$

Example Solve the following ODE:

$$x^2y'' + 5xy' + 13y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (5-1)h'(t) + 13h(t) = 0$$

Example Solve the following ODE:

$$x^2y'' + 5xy' + 13y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (5-1)h'(t) + 13h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9.$$

Example Solve the following ODE:

$$x^2y'' + 5xy' + 13y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (5-1)h'(t) + 13h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9.$$

Therefore, $h_1(t) = e^{-2t}\cos 3t$ and $h_2(t) = e^{-2t}\sin 3t$.

Example Solve the following ODE:

$$x^2y'' + 5xy' + 13y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (5-1)h'(t) + 13h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9.$$

Therefore, $h_1(t) = e^{-2t}\cos 3t$ and $h_2(t) = e^{-2t}\sin 3t$.
Hence $h_1(\ln x) = \dfrac{1}{x^2}\cos(3\ln x)$ and $h_2(\ln x) = \dfrac{1}{x^2}\sin(3\ln x)$.

Example Solve the following ODE:

$$x^2y'' + 5xy' + 13y = 0$$

Putting $t = \ln x$, we get

$$h''(t) + (5-1)h'(t) + 13h(t) = 0$$

The characteristic equation of associated constant coefficient ODE is

$$p(m) = m^2 + (5-1)m + 13 = (m+2)^2 + 9.$$

Therefore, $h_1(t) = e^{-2t}\cos 3t$ and $h_2(t) = e^{-2t}\sin 3t$.
Hence $h_1(\ln x) = \dfrac{1}{x^2}\cos(3\ln x)$ and $h_2(\ln x) = \dfrac{1}{x^2}\sin(3\ln x)$. Hence the general solution is

$$y = \frac{1}{x^2}\left[c_1\cos(3\ln x) + c_2\sin(3\ln x)\right]$$

Theorem Consider the Cauchy-Euler Equation

$$x^2y'' + axy' + by = 0 \quad x > 0 \tag{9}$$

where $a, b \in \mathbb{R}$. Putting $t = \ln x$, ODE becomes

$$h''(t) + (a-1)h'(t) + bh(t) = 0 \tag{10}$$

Let $p(m) = m^2 + (a-1)m + b = 0$ be the characteristic equation.

Theorem Consider the Cauchy-Euler Equation

$$x^2 y'' + axy' + by = 0 \quad x > 0 \tag{9}$$

where $a, b \in \mathbb{R}$. Putting $t = \ln x$, ODE becomes

$$h''(t) + (a-1)h'(t) + bh(t) = 0 \tag{10}$$

Let $p(m) = m^2 + (a-1)m + b = 0$ be the characteristic equation.

(a) If $m_1 \neq m_2 \in \mathbb{R}$, then the general solution of (11) is

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

Theorem Consider the Cauchy-Euler Equation

$$x^2y'' + axy' + by = 0 \quad x > 0 \tag{9}$$

where $a, b \in \mathbb{R}$. Putting $t = \ln x$, ODE becomes

$$h''(t) + (a-1)h'(t) + bh(t) = 0 \tag{10}$$

Let $p(m) = m^2 + (a-1)m + b = 0$ be the characteristic equation.

(a) If $m_1 \neq m_2 \in \mathbb{R}$, then the general solution of (11) is

$$y = c_1x^{m_1} + c_2x^{m_2}.$$

(b) If $m_1 = m_2 \in \mathbb{R}$, then the general solution of (11) is

$$y = c_1x^m + c_2x^m \ln x.$$

Theorem Consider the Cauchy-Euler Equation

$$x^2y'' + axy' + by = 0 \quad x > 0 \tag{9}$$

where $a, b \in \mathbb{R}$. Putting $t = \ln x$, ODE becomes

$$h''(t) + (a-1)h'(t) + bh(t) = 0 \tag{10}$$

Let $p(m) = m^2 + (a-1)m + b = 0$ be the characteristic equation.

(a) If $m_1 \neq m_2 \in \mathbb{R}$, then the general solution of (11) is

$$y = c_1x^{m_1} + c_2x^{m_2}.$$

(b) If $m_1 = m_2 \in \mathbb{R}$, then the general solution of (11) is

$$y = c_1x^m + c_2x^m \ln x.$$

(c) If $m = \lambda \pm i\omega$ and then the general solution of (11) is

$$y = x^\lambda \left(c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)\right).$$

Solve the following Cauchy-Euler Equation

1. $x^2y'' - 3xy' + 4y = 0$.
2. $x^2y'' - 3xy' + 5y = 0$.
3. $x^2y'' - xy' - 3y = 0$.

Something to think!!! Assume that we have Cauchy-Euler Equation of the form

$$x^2 y'' + axy' + by = 0 \quad x < 0 \tag{11}$$

where $a, b \in \mathbb{R}$.

Something to think!!! Assume that we have Cauchy-Euler Equation of the form

$$x^2y'' + axy' + by = 0 \quad x < 0 \tag{11}$$

where $a, b \in \mathbb{R}$.

Shall one substitute $x = -e^t$ and proceed as did before!!!

Section 2

Non-homogeneous second order linear ODE

Non-homogeneous second order linear ODE

Consider the non-homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{12}$$

where $f(x) \neq 0$ identically.

Non-homogeneous second order linear ODE

Consider the non-homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{12}$$

where $f(x) \neq 0$ identically.

Theorem: Suppose $p(x)$, $q(x)$ and $f(x)$ are continuous functions on an open interval (a, b), let x_0 be any point in (a, b). Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = w_0, \quad y'(x_0) = w_1$$

has a unique solution on (a, b).

Consider the non-homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{12}$$

where $f(x) \neq 0$ identically.

Theorem: Suppose $p(x)$, $q(x)$ and $f(x)$ are continuous functions on an open interval (a, b), let x_0 be any point in (a, b). Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = w_0, \quad y'(x_0) = w_1$$

has a unique solution on (a, b).

We know how to solve the associated homogeneous equation, initial value problem

$$y'' + p(x)y' + q(x)y = 0 \quad y(x_0) = w_0, \quad y'(x_0) = w_1$$

Set-up: The homogeneous IVP

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) is called complementary equation.

Non-homogeneous second order linear ODE

Set-up: The homogeneous IVP

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) is called complementary equation.

Particular solution: Any solution of non-homogeneous IVP

$$y'' + p(x)y' + q(x)y = f(x)$$

on (a, b) is called particular solution.

Non-homogeneous second order linear ODE

Set-up: The homogeneous IVP

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) is called complementary equation.

Particular solution: Any solution of non-homogeneous IVP

$$y'' + p(x)y' + q(x)y = f(x)$$

on (a, b) is called particular solution.

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7.$$

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{13}$$

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{13}$$

The characteristic equation is

$$p(m) = m^2 + 1 = 0.$$

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{13}$$

The characteristic equation is

$$p(m) = m^2 + 1 = 0.$$

Thus $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation.

Particular solution of non-homogeneous second order linear ODE

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{13}$$

The characteristic equation is

$$p(m) = m^2 + 1 = 0.$$

Thus $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation.

The general solution of (13) is

$$y = y_p + \cos x + 7 \sin x. \tag{14}$$

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{13}$$

The characteristic equation is

$$p(m) = m^2 + 1 = 0.$$

Thus $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation.

The general solution of (13) is

$$y = y_p + \cos x + 7 \sin x. \tag{14}$$

Any guess for y_p.

Particular solution of non-homogeneous second order linear ODE

Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{13}$$

The characteristic equation is

$$p(m) = m^2 + 1 = 0.$$

Thus $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation.

The general solution of (13) is

$$y = y_p + \cos x + 7 \sin x. \tag{14}$$

Any guess for y_p. take $y_p = 1$.

Particular solution

Example: Solve IVP

$$y'' + 6y' + 5y = x^2 + 1, \text{ where } y(0) = 3, y'(0) = 1. \tag{15}$$

Example: Solve IVP

$$y'' + 6y' + 5y = x^2 + 1, \text{ where } y(0) = 3, y'(0) = 1. \tag{15}$$

Thus the solution to IVP is

$$y = c_1 e^{-x} + c_2 e^{-5x} + y_p.$$

Example: Solve IVP

$$y'' + 6y' + 5y = x^2 + 1, \text{ where } y(0) = 3, y'(0) = 1. \tag{15}$$

Thus the solution to IVP is

$$y = c_1 e^{-x} + c_2 e^{-5x} + y_p.$$

Find y_p:

Theorem: Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous function on (a, b). Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{16}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Solution of 2nd order non-homogeneous linear ODE

Theorem: Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous function on (a, b). Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{16}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (16) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are arbitrary constants.

Solution of 2nd order non-homogeneous linear ODE

Theorem: Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous function on (a, b). Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{16}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (16) on (a, b) if and only if

$$y = y_p + c_1y_1 + c_2y_2,$$

where c_1 and c_2 are arbitrary constants.

Proof. Exercise (Think about it!)

Example

Example: Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{17}$$

Example: Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{17}$$

Answer: The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $m^2 - 2m + 1 = (m-1)^2$.

Example: Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{17}$$

Answer: The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $m^2 - 2m + 1 = (m-1)^2$.

Therefore $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation.

Example: Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{17}$$

Answer: The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $m^2 - 2m + 1 = (m-1)^2$.

Therefore $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation.

Notice that y is a solution of (17) if and only if

$$y = y_p + c_1 e^x + c_2 xe^x,$$

where c_1 and c_2 are arbitrary constants.

Example: Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{17}$$

Answer: The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $m^2 - 2m + 1 = (m-1)^2$.

Therefore $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation.

Notice that y is a solution of (17) if and only if

$$y = y_p + c_1 e^x + c_2 x e^x,$$

where c_1 and c_2 are arbitrary constants. Need to find a particular solution:

Example (continued...)

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Example (continued...)

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Notice that y is a solution if and only if

$$y = y_p + c_1 e^x + c_2 x e^x.$$

How to find a particular solution?:

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Notice that y is a solution if and only if

$$y = y_p + c_1 e^x + c_2 x e^x.$$

How to find a particular solution?:

- Can we choose y_p to be of the form $A + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$? (Yes/No and why)

Example (continued...)

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Notice that y is a solution if and only if

$$y = y_p + c_1 e^x + c_2 x e^x.$$

How to find a particular solution?:

- Can we choose y_p to be of the form $A + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$? (Yes/No and why)

Example (continued...)

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Notice that y is a solution if and only if

$$y = y_p + c_1 e^x + c_2 x e^x.$$

How to find a particular solution?:

- Can we choose y_p to be of the form $A + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$? (Yes/No and why)
- From (18), the better candidate for y_p seems to be of the form $A + A_1 x + A_2 x^2$.

Example (continued...)

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Notice that y is a solution if and only if

$$y = y_p + c_1 e^x + c_2 x e^x.$$

How to find a particular solution?:

- Can we choose y_p to be of the form $A + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$? (Yes/No and why)
- From (18), the better candidate for y_p seems to be of the form $A + A_1 x + A_2 x^2$.
- Set $y_p = A + A_1 x + A_2 x^2$ and substitute in (18). Find constants A, A_1, A_2.

Example (continued...)

$$y'' - 2y' + y = -3 - x + x^2. \tag{18}$$

Notice that y is a solution if and only if

$$y = y_p + c_1 e^x + c_2 x e^x.$$

How to find a particular solution?:

- Can we choose y_p to be of the form $A + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4$? (Yes/No and why)
- From (18), the better candidate for y_p seems to be of the form $A + A_1 x + A_2 x^2$.
- Set $y_p = A + A_1 x + A_2 x^2$ and substitute in (18). Find constants A, A_1, A_2.
- (Verify) $y_p = 1 + 3x + x^2$.

Part (a) General solution of given ODE is

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2 x).$$

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{19}$$

Example (continued...)

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{19}$$

Recall that general solution of this ODE is

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2 x).$$

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{19}$$

Recall that general solution of this ODE is

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2 x).$$

Using the initial conditions, $y(0) = -2, \quad y'(0) = 1$, solution of IVP is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Recall: Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{20}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Recall: solution of 2nd order non-homogeneous linear ODE

Recall: Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{20}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (20) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

Recall: Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{20}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (20) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

Observation

Recall: Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{20}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (20) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

Observation How to pick y_p.

Recall: Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{20}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (20) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

Observation How to pick y_p. It is decided by $f(x)$

Recall: Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{20}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Then y is a solution of (20) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

Observation How to pick y_p. It is decided by $f(x)$ (or may be more than that!!).

Theorem: Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b).

The Principle of Superposition

Theorem: Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Observation

- Split the non-homogeneous equation into simpler parts,

The Principle of Superposition

Theorem: Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Observation

- Split the non-homogeneous equation into simpler parts,
- Find a particular solution for each part, and

The Principle of Superposition

Theorem: Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Observation

- Split the non-homogeneous equation into simpler parts,
- Find a particular solution for each part, and
- Combine their solutions to obtain a particular solution of the original ODE.

Questions

Question:

Question:

- In the first order non-linear ODE $y' + p(x)y = f(x)$, we have used $y = u(x)y_1$, where y_1 is a solution of complementary equation $y' + p(x)y = 0$.

Question:

- In the first order non-linear ODE $y' + p(x)y = f(x)$, we have used $y = u(x)y_1$, where y_1 is a solution of complementary equation $y' + p(x)y = 0$.
- We found y_p in some different way to get the complete solution in 2nd order non-homogeneous linear ODE.

Question:

- In the first order non-linear ODE $y' + p(x)y = f(x)$, we have used $y = u(x)y_1$, where y_1 is a solution of complementary equation $y' + p(x)y = 0$.
- We found y_p in some different way to get the complete solution in 2nd order non-homogeneous linear ODE.
- Then for getting solutions, what is the difference between 1st and 2nd order ODE ? Especially, in the case of non-homogeneous linear ODE.

Question:

- In the first order non-linear ODE $y' + p(x)y = f(x)$, we have used $y = u(x)y_1$, where y_1 is a solution of complementary equation $y' + p(x)y = 0$.
- We found y_p in some different way to get the complete solution in 2nd order non-homogeneous linear ODE.
- Then for getting solutions, what is the difference between 1st and 2nd order ODE ? Especially, in the case of non-homogeneous linear ODE.

Subsection 1

The method of undetermined coefficients

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.
Let's try:

- Derivative of an exponential function e^{mx} is again an exponential function $*e^{mx}$ up to a constant $*$. Same applies for higher derivatives too.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.
Let's try:

- Derivative of an exponential function e^{mx} is again an exponential function $*e^{mx}$ up to a constant $*$. Same applies for higher derivatives too.
- If we substitute some solution y in (21), L.H.S and R.H.S should agree.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.
Let's try:

- Derivative of an exponential function e^{mx} is again an exponential function $*e^{mx}$ up to a constant $*$. Same applies for higher derivatives too.
- If we substitute some solution y in (21), L.H.S and R.H.S should agree.
- Possible candidate for $y_p = Ae^{2x}$ for some constant A.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.
Let's try:

- Derivative of an exponential function e^{mx} is again an exponential function $*e^{mx}$ up to a constant $*$. Same applies for higher derivatives too.
- If we substitute some solution y in (21), L.H.S and R.H.S should agree.
- Possible candidate for $y_p = Ae^{2x}$ for some constant A.

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

hence $A = 2$.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.
Let's try:

- Derivative of an exponential function e^{mx} is again an exponential function $*e^{mx}$ up to a constant $*$. Same applies for higher derivatives too.
- If we substitute some solution y in (21), L.H.S and R.H.S should agree.
- Possible candidate for $y_p = Ae^{2x}$ for some constant A.

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

hence $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution.

General solution:

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{21}$$

Then find the general solution.
Let's try:

- Derivative of an exponential function e^{mx} is again an exponential function $*e^{mx}$ up to a constant $*$. Same applies for higher derivatives too.
- If we substitute some solution y in (21), L.H.S and R.H.S should agree.
- Possible candidate for $y_p = Ae^{2x}$ for some constant A.

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

hence $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution.

General solution:

$$y = 2e^{2x} + c_1 e^{3x} + c_2 e^{4x}.$$

Question: Can we always substitute $y_p = Ae^{mx}$ for some constant A?

Example: Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{22}$$

Then find the general solution.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{22}$$

Then find the general solution.
Let's try:

- Let $y_p = Ae^{4x}$ as a possible candidate for particular solution.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{22}$$

Then find the general solution.
Let's try:

- Let $y_p = Ae^{4x}$ as a possible candidate for particular solution.
- Something is wrong! Notice that L.H.S $= 0$, whereas R.H.S $\neq 0$.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{22}$$

Then find the general solution.

Let's try:

- Let $y_p = Ae^{4x}$ as a possible candidate for particular solution.
- Something is wrong! Notice that L.H.S $= 0$, whereas R.H.S $\neq 0$.
- We know why it is wrong. Because e^{4x} is a solution of the complementary equation.

Example: Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{22}$$

Then find the general solution.
Let's try:

- Let $y_p = Ae^{4x}$ as a possible candidate for particular solution.
- Something is wrong! Notice that L.H.S $= 0$, whereas R.H.S $\neq 0$.
- We know why it is wrong. Because e^{4x} is a solution of the complementary equation.

Let's try: $y_p = u(x)e^{4x}$. Substituting y_p, y_p' and y_p'' into (22) and canceling the common factor e^{4x} gives

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

that is,

$$u'' + u' = 5.$$

Example (continued...)

Now if

$$u'' + u' = 5,$$

then we may take $u(x) = 5x$ as a possible candidate.

Example (continued...)

Now if

$$u'' + u' = 5,$$

then we may take $u(x) = 5x$ as a possible candidate.

Therefore, $y_p = 5xe^{4x}$ is a particular solution.

Example (continued...)

Now if

$$u'' + u' = 5,$$

then we may take $u(x) = 5x$ as a possible candidate.

Therefore, $y_p = 5xe^{4x}$ is a particular solution.

Characteristic polynomial is $m^2 - 7m + 12 = 0$, that is, $(m-3)(m-4) = 0$.

Therefore

$$y = 5xe^{4x} + c_1 e^{3x} + c_2 e^{4x}$$

is the general solution.

Example: Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{23}$$

Then find the general solution.

Example: Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{23}$$

Then find the general solution.

Let's try: Note that characteristic polynomial is $(m-4)^2 = 0$.

- Can we have $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$ as a possible candidate for particular solution?

Example: Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{23}$$

Then find the general solution.
Let's try: Note that characteristic polynomial is $(m-4)^2 = 0$.

- Can we have $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$ as a possible candidate for particular solution? (No!)

Example: Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{23}$$

Then find the general solution.
Let's try: Note that characteristic polynomial is $(m-4)^2 = 0$.

- Can we have $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$ as a possible candidate for particular solution? (No!)

Let's try: $y_p = u(x)e^{4x}$. Substituting y_p, y_p' and y_p'' into (23) and canceling the common factor e^{4x} gives

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2 \implies u'' = 2.$$

Therefore $u_p = x^2 e^{4x}$ is a particular solution of (23).

Example: Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{23}$$

Then find the general solution.
Let's try: Note that characteristic polynomial is $(m-4)^2 = 0$.

- Can we have $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$ as a possible candidate for particular solution? (No!)

Let's try: $y_p = u(x)e^{4x}$. Substituting y_p, y_p' and y_p'' into (23) and canceling the common factor e^{4x} gives

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2 \implies u'' = 2.$$

Therefore $u_p = x^2 e^{4x}$ is a particular solution of (23).

Therefore the general solution to given ODE is

$$y = e^{4x}(x^2 + c_1 + c_2 x).$$

The method of undetermined coefficients

The form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

The form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

- If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{24}$$

  then $y_p = Ae^{\alpha x}$, where A is a constant.

The method of undetermined coefficients

The form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

- If $e^{\alpha x}$ isn't a solution of the complementary equation

  $$ay'' + by' + cy = 0, \tag{24}$$

  then $y_p = Ae^{\alpha x}$, where A is a constant.
- If $e^{\alpha x}$ is a solution of (24) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant.

The method of undetermined coefficients

The form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

- If $e^{\alpha x}$ isn't a solution of the complementary equation

  $$ay'' + by' + cy = 0, \tag{24}$$

  then $y_p = Ae^{\alpha x}$, where A is a constant.
- If $e^{\alpha x}$ is a solution of (24) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant.
- If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (24), then $y_p = Ax^2e^{\alpha x}$, where A is a constant.

The method of undetermined coefficients

The form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

- If $e^{\alpha x}$ isn't a solution of the complementary equation

  $$ay'' + by' + cy = 0, \tag{24}$$

  then $y_p = Ae^{\alpha x}$, where A is a constant.
- If $e^{\alpha x}$ is a solution of (24) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant.
- If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (24), then $y_p = Ax^2e^{\alpha x}$, where A is a constant.

One can substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A,

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{25}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{25}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

We want to find a particular solution of (25).

The method of undetermined coefficients

We want to find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

We want to find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos \omega x + Q(x)\sin \omega x\right)$$

Case study: Assume that $\lambda = 0$.

We want to find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

Case study: Assume that $\lambda = 0$. Then we have

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

We want to find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

Case study: Assume that $\lambda = 0$. Then we have

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

If

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

The method of undetermined coefficients

We want to find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

Case study: Assume that $\lambda = 0$. Then we have

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

If

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where F and G are some other polynomials. By comparing find the co-efficients so that $F(x) = P(x)$ and $G(x) = Q(x)$.

The method of undetermined coefficients

Theorem: Suppose $\omega > 0$ and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

The method of undetermined coefficients

Theorem: Suppose $\omega > 0$ and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{26}$$

where

The method of undetermined coefficients

Theorem: Suppose $\omega > 0$ and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{26}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

Theorem: Suppose $\omega > 0$ and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{26}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation.

Theorem: Suppose $\omega > 0$ and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{26}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x$$

are of the form (26),

The method of undetermined coefficients

Theorem: Suppose $\omega > 0$ and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{26}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x$$

are of the form (26),

$$A(x) = A_0 x + A_1 x^2 + \cdots + A_k x^{k+1} \quad \text{and} \quad B(x) = B_0 x + B_1 x^2 + \cdots + B_k x^{k+1}.$$

Example: Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{27}$$

Example: Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{27}$$

Set $y_p = A\cos 2x + B\sin 2x.$.

Example: Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{27}$$

Set $y_p = A\cos 2x + B\sin 2x$.. Now

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) \\ &\quad +(A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{27}$$

Set $y_p = A\cos 2x + B\sin 2x$.. Now

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) \\ &\quad +(A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

Solve for A and B.

Example: Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{27}$$

Set $y_p = A\cos 2x + B\sin 2x$.. Now

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) \\ &\quad +(A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

Solve for A and B. Get $A = 1$, $B = -2$ and hence

$$y_p = \cos 2x - 2\sin 2x$$

is a particular solution

Example: Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{28}$$

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{28}$$

Set $y_p = x(A\cos 2x + B\sin 2x)$..

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{28}$$

Set $y_p = x(A\cos 2x + B\sin 2x)$.. Then

$$\begin{aligned} y_p' &= A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x) \\ y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p\ , \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{28}$$

Set $y_p = x(A\cos 2x + B\sin 2x)$.. Then

$$\begin{aligned} y_p' &= A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x) \\ y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p \ , \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore $A = -3$ and $B = 2$ and hence

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{29}$$

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{29}$$

Set $y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x$.

The method of undetermined coefficients - Example

Example: Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{29}$$

Set $y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x$. Then

$$y_p' = (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x$$

and

Example: Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{29}$$

Set $y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x$. Then

$$y_p' = (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x$$

and

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x.$$

Example: Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{29}$$

Set $y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x$. Then

$$y_p' = (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x$$

and

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x.$$

Therefore

$$\begin{aligned} y_p'' + 3y_p' + 2y_p \quad = \quad & [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x \\ & + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \end{aligned} \tag{30}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ with the corresponding coefficients in (29)

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + \ \ B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + \ \ B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

The method of undetermined coefficients - Example

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ with the corresponding coefficients in (29)

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + \ \ B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + \ \ B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

We get constants, $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ with the corresponding coefficients in (29)

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + \ B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + \ B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

We get constants, $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$

Therefore

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{31}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{31}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

To find a particular solution of (31),when $\lambda \neq 0$.

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{31}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

To find a particular solution of (31),when $\lambda \neq 0$.

Substituting $y = ue^{\lambda x}$ into (31).

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{31}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

To find a particular solution of (31),when $\lambda \neq 0$.

Substituting $y = ue^{\lambda x}$ into (31).

The new equation after substitution will be constant coefficient equation for u of the $P(x)\cos\omega x + Q(x)\sin\omega x$.

Consider the 2nd order constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{31}$$

where λ and ω are real numbers, $\omega \neq 0$, and $P(x)$ and $Q(x)$ are polynomials.

To find a particular solution of (31),when $\lambda \neq 0$.

Substituting $y = ue^{\lambda x}$ into (31).

The new equation after substitution will be constant coefficient equation for u of the $P(x)\cos\omega x + Q(x)\sin\omega x$.

We can find a particular solution u_p of this equation by earlier approach.

Example: Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{32}$$

Example: Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{32}$$

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

Example: Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{32}$$

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

Need to solve

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{33}$$

Notice that $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

The method of undetermined coefficients

A particular solution of

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x.$$

is of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{34}$$

The method of undetermined coefficients

A particular solution of

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x.$$

is of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{34}$$

Hence

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

A particular solution of

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x.$$

is of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{34}$$

Hence

$$\begin{aligned} u_p'' - 7u_p' + 12u_p \quad &= \quad [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

u_p is a solution if

A particular solution of

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x.$$

is of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{34}$$

Hence

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

u_p is a solution if

$$\begin{array}{rcr} 3A_1 - 21B_1 &=& 0 \\ 21A_1 + 3B_1 &=& 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &=& 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &=& -34. \end{array} \tag{35}$$

Solving the system of equations, we get $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$.

Solving the system of equations, we get $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$.

Therefore

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of (33).

Solving the system of equations, we get $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$.

Therefore

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of (33).

Hence

$$y_p = e^{-2x}\left[(1 + 7x)\cos 3x - (2 - x)\sin 3x\right]$$

is a particular solution of our original ODE.