Problems Set 3 : Linear Transformations

Before trying to solve each exercise, first you should be familiar with the terminologies, definitions and basic theory on that. You may read the lecture notes or lecture slides.

Throughout, U, V and W are vector spaces over $\mathbb{R}$, the set of real numbers.

1. Let $T : V \to W$ be a linear transformation. What is $T(0)$, where 0 is the zero vector in V?

Hint. Note that $T(v) = T(v + 0) = T(v) + T(0)$ for any $v \in V$. Conclude that $T(0) = 0$, the zero vector in W.

2. Which of the following maps are linear? Justify your answer.

(i) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x + 2$ for every $x \in \mathbb{R}^1$.
(ii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = ax$ for every $x \in \mathbb{R}^1$, where $a \in \mathbb{R}$ is a constant.
(iii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x^2$ for every $x \in \mathbb{R}^1$.
(iv) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = \sin(x)$ for every $x \in \mathbb{R}^1$.
(v) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = e^x$ for every $x \in \mathbb{R}^1$.
(vi) $T : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $T(x_1, x_2) = x_1x_2$ for every $(x_1, x_2) \in \mathbb{R}^2$.
(vii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.
(viii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.
(ix) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.
(x) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, 1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

Hint. Verify $T(cu + dv) = cT(u) + dT(v)$ for all scalars $c, d \in \mathbb{R}$, and vectors u, v in the domain of T. If this is not true, then find particular c, d, u and v for which the above equality fails.

3. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$, $u_3 = (1, -1)$ and $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2, 3$?

Hint. A linear map should respect every linear combination.

4. Composition of linear maps: Let $T : U \to V$ and $S : V \to W$ be linear maps. The composition $S \circ T : U \to W$ is defined by $(S \circ T)(u) := S(T(u))$ every $u \in U$. Show that the map $S \circ T : U \to W$ is linear.

5. Matrix multiplication and composition of linear maps: Let A, B be matrices of order $l \times m$ and $m \times n$ respectively. Consider the corresponding linear maps $T_A : \mathbb{R}^m \to \mathbb{R}^l$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ given by A and B respectively. Prove that the matrix representation of the composition $T_A \circ T_B : \mathbb{R}^n \to \mathbb{R}^l$ is AB, or equivalently, prove that $T_A \circ T_B = T_{AB}$.

6. Application of composition of maps: Show that the matrix multiplication is associative.

Hint. Let A, B, C be matrices of order $k \times l$, $l \times m$ and $m \times n$ respectively. To show that $(AB)C = A(BC)$, consider T_A, T_B and T_C. Next use Q.5 and the fact that the composition of maps is associative.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Is it true that if we know $T(v)$ for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.

Hint. See what we have proved in Lecture 6. Try to analyze the statement when $n = 2$.

8. Define a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$, where $a_{ij} \in \mathbb{R}$ are constants. Is T linear? If yes, then write its matrix representation.

Hint. See the theorem concerning matrix representation of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ proved in Lecture 6.

9. Deduce from Q.8 that the map $S : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$S(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, x_2 + x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ is linear. Compute the range space and null space of S. Deduce the rank and nullity of S. Verify the Rank-Nullity Theorem. Conclude from the rank (resp. from the nullity), whether S is an isomorphism.

Hint. Write the matrix representation (say, A) of the linear map S. Observe that the null space of S is same as that of A. Moreover, the range space of S is same as the column space of A. Now one may follow the solution of Q.**??**. Recall the equivalent conditions for a linear operator to be an isomorphism (shown in Lecture 7).

Left/right inverse of an $n \times n$ matrix A. We know that if A has a left-inverse B (i.e., $BA = I_n$) and a right-inverse C (i.e., $AC = I_n$), then the two inverses are equal: $B = B(AC) = (BA)C = C$. If this is the case, we say that A is invertible. From the row rank and the column rank of A, we can actually decide when A has a left/right inverse; see Q.10 and Q.11.

10. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has full column rank, i.e., column rank of A is n.
(ii) The system $AX = b$ has at least one solution X for every $b \in \mathbb{R}^n$.
(iii) The rank of the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is n.
(iv) A has a right-inverse C, i.e., $AC = I_n$.

Hint. (i) $\Leftrightarrow$ (ii). Use that AX is nothing but a linear combination of the columns of A.

(ii) $\Leftrightarrow$ (iii). This is just an observation.

(ii) $\Leftrightarrow$ (iv). Prove the two implications one by one.

11. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has full row rank, i.e., row rank of A is n.
(ii) A has a left-inverse B, i.e., $BA = I_n$.

Hint. (i) $\Leftrightarrow$ (ii). Note that the row space of A is same as the column space of A^t (the transpose of A). So you may use the equivalence of (i) and (iv) in Q.10 for A^t.

12. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has a left-inverse.
(ii) A has a right-inverse.
(iii) A is invertible.

Hint. You may use Q.10, Q.11 and the fact that row rank(A) = column rank(A).

13. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$ and $v_1 = (1, 1)$, $v_2 = (0, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2$? If yes, then write the matrix representation of T.

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Let u, v be two non-zero vectors such that $T(u) = 0$ and $T(v) = 0$. What are the possibilities of nullity of T? What about rank of T?