Solutions of Problems Set 2
Vector Spaces, Basis and Dimension

Throughout, V is a vector space over $\mathbb{R}$, the set of real numbers. Note that in most of the cases, it is convenient to write a vector in $\mathbb{R}^n$ as a column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. But sometimes we also write it as an n-tuple $(x_1, \ldots, x_n)$.

1. Check whether $V = \mathbb{R}^2$ with each of the following operations is a vector space over $\mathbb{R}$.

(i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix}$ and $c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ y \end{pmatrix}$.

(ii) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ 0 \end{pmatrix}$ and $c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$.

(iii) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix}$ and $c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix}$.

Hint. Verify all 10 properties in the definition of a vector space. Show that in each case these operations do not give a vector space structure.

2. We call 'the' additive identity element of V as the zero vector. By definition, it is a vector $0 \in V$ such that $v + 0 = v$ for every $v \in V$. Its existence is there in the definition of 'vector space'. But, before saying it 'the' additive identity, can you prove its uniqueness?

Solution. Suppose θ_1 and θ_2 are two additive identity elements. By commutativity, $\theta_1+\theta_2 = \theta_2+\theta_1$. Since θ_1 is an additive identity, $\theta_2 = \theta_1 + \theta_2$. On the other hand, since θ_2 is an additive identity, $\theta_1 = \theta_1 + \theta_2$. Thus $\theta_1 = \theta_2$.

3. Let $0 \in V$ be the zero vector. Let $c \in \mathbb{R}$. Show that $c \cdot 0 = 0$.

Solution. For the zero vector, we have $0 = 0 + 0$. So $c \cdot 0 = c \cdot (0 + 0) = c \cdot 0 + c \cdot 0$. Hence conclude that $c \cdot 0 = 0$. (Observe that for a vector $v \in V$, if $v = v + v$, then $v = 0$.)

4. Let $v \in V$. Show that $0 \cdot v = 0$, where 0 in the right side is the zero vector, and 0 in the left side is the zero element of $\mathbb{R}$.

Solution. For the zero element in $\mathbb{R}$, $0 = 0 + 0$. So $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v$. Hence it follows that $0 \cdot v = 0$.

5. Let W be a subspace of V. Show that (the zero vector) $0 \in W$.

Hint. Use Q.4 and the fact that W is non-empty.

6. Which of the following sets of vectors $X = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geqslant 3)$?

(i) all X such that $x_1 \geqslant 0$;

(ii) all X such that $x_1 + 2x_2 = 3x_3$;

(iii) all X such that $x_1 = x_2^2$;

(iv) all X such that $x_1x_2 = 0$;

(v) all X such that x_1 is rational.

Hint. To verify whether a subset of $\mathbb{R}^n$ is a subspace, you need to verify whether that subset is non-empty, and closed under vector addition and scalar multiplication.

7. Prove that all the subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

8. Prove that a subspace of $\mathbb{R}^2$ is either 0, or $\mathbb{R}^2$, or a subspace consisting of all scalar multiples of some fixed non-zero vector in $\mathbb{R}^2$ (which is intuitively a straight line through the origin).

Hint. What are the possibilities of the dimension of a subspace of $\mathbb{R}^2$?

9. (i) Let W_1 and W_2 be subspaces of V such that the set-theoretic union $W_1 \cup W_2$ is also a subspace of V. Prove that one of the subspaces W_1 and W_2 is contained in the other.
(ii) Can you give examples of two subspaces U_1 and U_2 of $\mathbb{R}^2$ such that $U_1 \cup U_2$ is not a subspace.

Solution. (i) You may prove the statement by way of contradiction. If possible, assume $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. So there are $w_1 \in W_1 \smallsetminus W_2$ and $w_2 \in W_2 \smallsetminus W_1$. Since $w_1, w_2 \in W_1 \cup W_2$, and $W_1 \cup W_2$ is a subspace, the sum $w_1 + w_2 \in W_1 \cup W_2$. Hence either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. But $w_1 + w_2 \in W_1$ implies that $w_2 \in W_1$, which is a contradiction. Similarly, if $w_1 + w_2 \in W_2$, then $w_1 \in W_2$, which is again a contradiction. Thus we get a contradiction to the assumption that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Therefore either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(ii) What about the union of two distinct lines passing through the origin in $\mathbb{R}^2$?

10. Let W_1 and W_2 be subspaces of V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = 0$. Prove that for every vector $v \in V$, there are unique vectors $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$.

In this case, we write $V = W_1 \oplus W_2$, and call this as **direct sum** of W_1 and W_2.

Hint. Recall the definition of the sum of two (or more) subspaces.

11. (i) Let A be an $m \times n$ matrix. Suppose B is obtained from A by applying an elementary row operation. Prove that row space(A) = row space(B).

(ii) Deduce from (i) that if any two $m \times n$ matrices A and B are row equivalent, then row space(A) = row space(B).

(iii) Let $B = \begin{bmatrix} R_1 \\ \vdots \\ R_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ be an $m \times n$ row reduced echelon matrix with the non-zero rows $R_1, \ldots, R_r \in \mathbb{R}^n$ and the last $(m - r)$ zero rows. Prove that $\{R_1, \ldots, R_r\}$ is a basis of the row space of B.

(iv) Let A be an $m \times n$ matrix. Let A be reduced to a row reduced echelon matrix B. Then deduce from (ii) and (iii) that the non-zero rows of B gives a basis of the row space of A. Hence the row rank of A is same as the number of non-zero rows of B.

Hint. For (iii), the pivot positions will play a crucial role to show the linear independence.

12. Let A be an $m \times n$ matrix. By applying elementary row operations, how can you find a basis of the column space of A?

Solution. Note that the column space of A is same as the row space of A^t (transpose of A). Then apply elementary row operations on A^t to get a row reduced echelon matrix, say B. Then, by Q.11(iv), the non-zero rows of B gives a basis of the row space of A^t, which is same as the column space of A.

13. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 6 \\ 1 & -2 & 1 & 2 \\ 0 & 5 & -1 & 2 \end{pmatrix}$. Find a basis of the row space of A. Deduce the row rank of A. Find a basis of the column space of A as well. Deduce the column rank of A. Verify whether row rank of A is same as column rank of A. Furthermore, find the null space of A. Deduce the nullity of A. Verify the Rank-Nullity Theorem for the linear map $A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by A. (This part of the exercise should belong to the next section.) What is the range space of this map? Find a basis of this range space, and deduce the rank of $A : \mathbb{R}^4 \to \mathbb{R}^3$.

Hint. Do not forget Q.11(iv) and Q.12 to obtain the row and column spaces of A. Moreover, observe that the range space of the map $A : \mathbb{R}^4 \to \mathbb{R}^3$ is same as the column space of A.

Remarks. Given an $m \times n$ matrix A. One obtains four fundamental spaces: row space, column space, null space and range space of A. Note that row space and null space are subspaces of $\mathbb{R}^n$; while column space and range space are subspaces of $\mathbb{R}^m$. Moreover, the column space and the range space of A are same.

14. Consider some column vectors $v_1, \ldots, v_n \in \mathbb{R}^m$. By applying elementary row operations, how can you find a basis of the subspace $\operatorname{Span}(\{v_1, \ldots, v_n\})$ of $\mathbb{R}^m$?

Solution. Set $A := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, an $m \times n$ matrix with the columns $v_1, \ldots, v_n \in \mathbb{R}^m$. Then the subspace $\operatorname{Span}(\{v_1, \ldots, v_n\})$ is same as the column space of A. Now follow the solution of Q.12.

15. Let $A = \begin{pmatrix} 3 & -1 & 8 & 4 \\ 2 & 1 & 7 & 1 \\ 1 & -3 & 0 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. For which $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{R}^3$ does the system $AX = Y$ have a solution? Describe the answer in terms of subspaces of $\mathbb{R}^3$. Use the following approaches, and verify whether you get the same answer.

Two approaches: **(1st).** Apply elementary row eliminations on $(A \,|\, Y)$, conclude when the system $AX = Y$ has solutions. **(2nd).** Note that for every $X \in \mathbb{R}^4$, AX is nothing but a linear combination of the four column vectors of A:

$$AX = x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}.$$

So Y should belong into the column space of A. Furthermore, you may try to find a basis of the column space of A. To do that follow Q.12.

16. Let $S = \{v_1, \ldots, v_r\}$ be a collection of r vectors of a vector space V. Then show that S is linearly independent if and only if $\dim(\operatorname{Span}(S)) = r$.

Hint. See Corollary 2.26 in the lecture notes.

17. Check whether the following vectors in $\mathbb{R}^4$ are linearly independent:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

Two approaches: **(1st).** Consider a relation $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$. It yields a homogeneous system of linear equations:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that this system has a non-trivial solution if and only if $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. So you just need to check whether the system has only the trivial solution or not. For that, you may apply elementary row operations on the coefficient matrix.

(2nd). Set a matrix A whose rows are the vectors v_1, v_2, v_3, v_4. By Q.16, $\{v_1, v_2, v_3, v_4\}$ is linearly independent if and only if $\dim(\operatorname{Span}(\{v_1, v_2, v_3, v_4\})) = 4$. Since $\operatorname{Span}(\{v_1, v_2, v_3, v_4\})$ is nothing but the row space of A, we just have to compute row rank of A. So follow Q.11(iv).

18. Let V be the vector space of all $m \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that $\dim(V) = mn$.

Hint. Consider $\{A^{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, where A^{ij} is the $m \times n$ matrix with (i,j) entry 1 and all other entries 0. Is it a basis of V?

19. Let V be the vector space of all $n \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that the following are subspaces of V.

(i) The subset of V consisting of all symmetric matrices.

(ii) The subset of V consisting of all skew-symmetric (or anti-symmetric) matrices.

(iii) The subset of V consisting of all upper triangular matrices (i.e., $A_{ij} = 0$ for all $i > j$).

What is the dimension of each of these subspaces?

Show that the following are not subspaces of V.

(iv) The subset of V consisting of all invertible matrices.

(v) The subset of V consisting of all non-invertible matrices.

(vi) The subset of V consisting of all matrices A such that $A^2 = A$.

Hint. (i), (ii) and (iii). To find the dimension of each subspace, construct a basis of that subspace. E.g., for (i), consider $\{D^{ii}, A^{ij} : 1 \leqslant i \leqslant n, i < j \leqslant n\}$, where D^{ii} is the $n \times n$ matrix with (i,i) entry 1 and all other entries 0, and A^{ij} is the $n \times n$ matrix with (i,j) and (j,i) entries 1 and all other entries 0. Show that it is a basis of the subspace of all symmetric matrices, hence the dimension is $n + (n-1) + \cdots + 2 + 1 = n(n+1)/2$. Similarly, show that the dimension of the subspaces described in (ii) and (iii) are $n(n-1)/2$ and $n(n+1)/2$.

(iv) Does it contain the zero vector?

(v) Show that it is not closed under addition.

(vi) Consider $A = B = I_n$. Note that $A^2 = A$ and $B^2 = B$, but $(A+B)^2 \neq (A+B)$. So it is not closed under addition.

20. Let A and B be two matrices of order $l \times m$ and $m \times n$ respectively. Prove the following.
(i) Column space$(AB) \subseteq$ Column space(A).
(ii) Row space$(AB) \subseteq$ Row space(B).
(iii) $\text{rank}(AB) \leqslant \min\{\text{rank}(A), \text{rank}(B)\}$

Hint. (i) If $B = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}$, then $AB = \begin{bmatrix} AV_1 & AV_2 & \cdots & AV_n \end{bmatrix}$. Observe that each $AV_i \in$ Column space(A).

(ii) If $A = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_l \end{bmatrix}$, then $AB = \begin{bmatrix} U_1B \\ U_2B \\ \vdots \\ U_lB \end{bmatrix}$. Observe that each $U_iB \in$ Row space(B).

(iii) Conclude the inequality from (i) and (ii).