Lecture 8 (Introduction to Group Theory)

January 27, 2022

Symmetric Group

Set S_n := set of all bijections from

$= \{ f : \{1, \dots n\} \longrightarrow \{1, 2, \dots, n\}$ s.t.

f is a bijection $\}$

$(S_n, \circ)$ composition as binary operation

GROUP.

Examples.

$\{1,2\} \to \{1,2\}$

S_2 : $\{1, 2\}$

i : $1 \longrightarrow 1$

$2 \longrightarrow 2$

τ : $1 \longrightarrow 2$

$2 \longrightarrow 1$

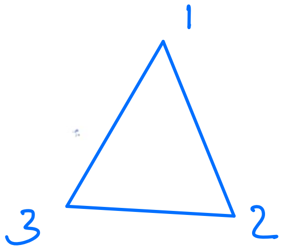
Note that $\tau^2 = i$

$\tau \circ \tau$

S_3 : Group of permutations of $\{1, 2, 3\}$.

123
132
213
231
312
321

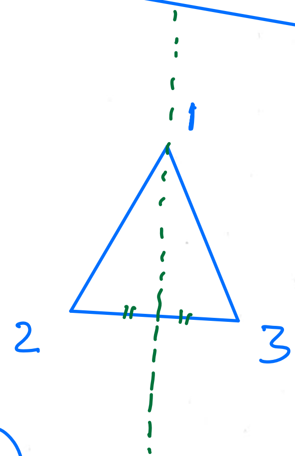
1 2 3



$$g_1 : \{1,2,3\} \longrightarrow \{1,2,3\}$$
$$i \longmapsto i$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1 3 2

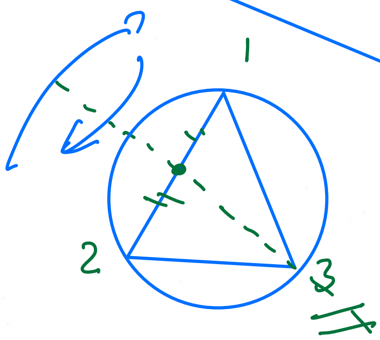


(fixed)

$$g_2 : \{1,2,3\} \longrightarrow \{1,2,3\}$$
$$1 \longmapsto 1$$
$$2 \longmapsto 3$$
$$3 \longmapsto 2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

2 1 3



fixed

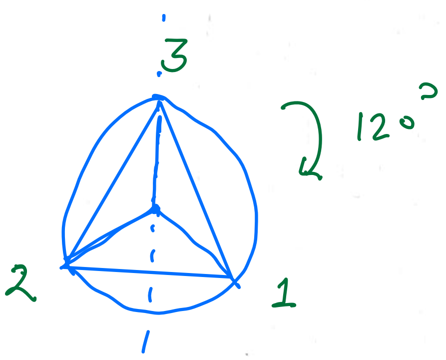
$$g_3 : \{1,2,3\} \longrightarrow \{1,2,3\}$$
$$1 \longmapsto 2$$
$$2 \longmapsto 1$$
$$3 \longmapsto 3$$

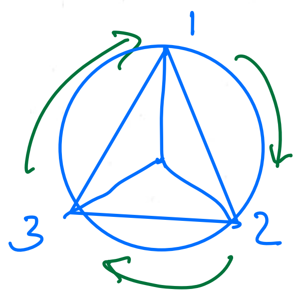
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 3 1

$$g_4 : \{1,2,3\} \longrightarrow \{1,2,3\}$$
$$1 \longmapsto 2$$
$$2 \longmapsto 3$$
$$3 \longmapsto 1$$

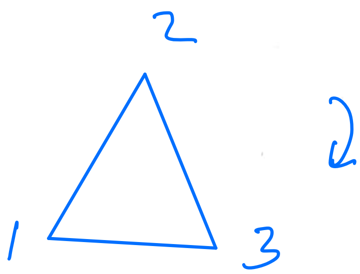
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$





312

$g_5 : \{1,2,3\} \longrightarrow \{1,2,3\}$



$1 \longmapsto 3$

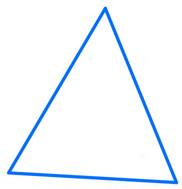
$2 \longmapsto 1$

$3 \longmapsto 2$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

321

$g_6 : \{1,2,3\} \longrightarrow \{1,2,3\}$



$1 \longmapsto 3$

$2 \longmapsto 2$

$3 \longmapsto 1$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\{1,2,3\}$ as bijection

$$\begin{array}{cccccccc} & & 123 & 132 & 213 & 231 & 312 & 321 \\ & & \| & \| & \| & \| & \| & \| \\ S_3 = & \{ & g_1, & g_2, & g_3, & g_4, & g_5, & g_6 \} \\ & & \| & \| & \| & \| & \| & \| \\ & & I & RF & F & R & R^2 & R^2F \end{array}$$

R : rotation by 120°

F : flip

$R^3 = I$

I, R, R^2

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad RF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad R^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R^2F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

S_4 := Group of permutations of $\{1,2,3,4\}$ with $*$ as composition.

Symmetries of a square:

$\{1,2,3,4\}$

4! permutations

6

1 2 3 4

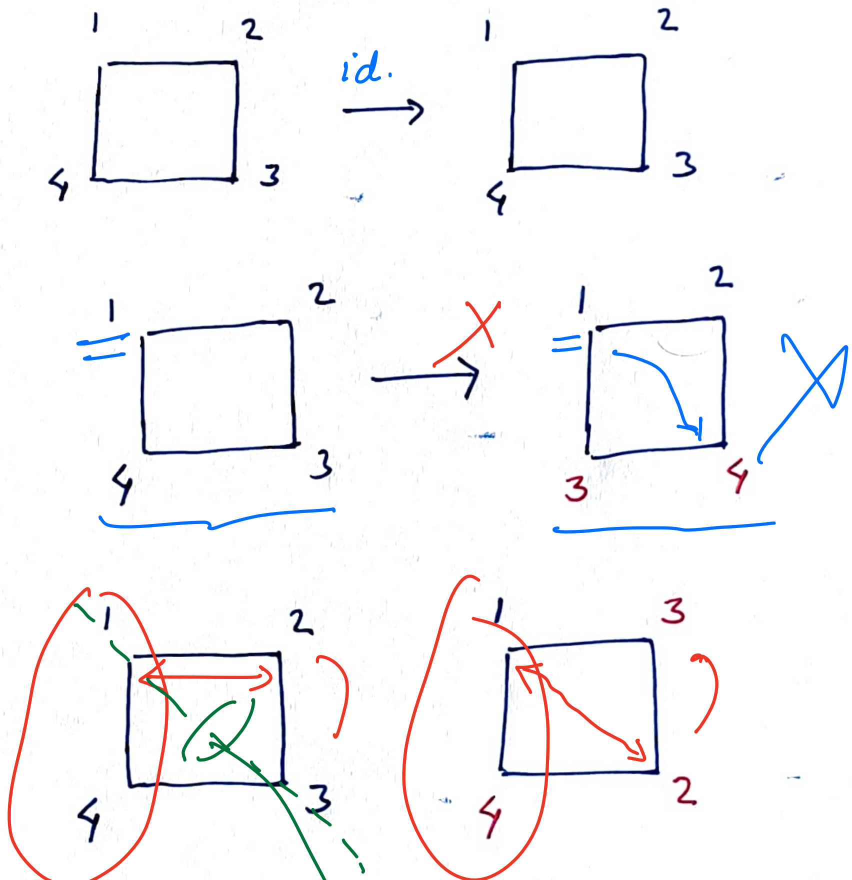
1 2 4 3

1 3 2 4

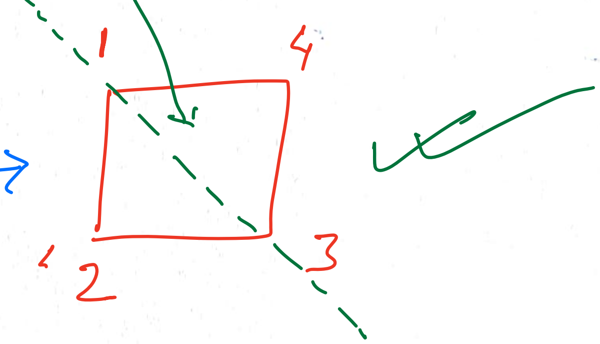
1 3 4 2

1 4 2 3

1 4 3 2



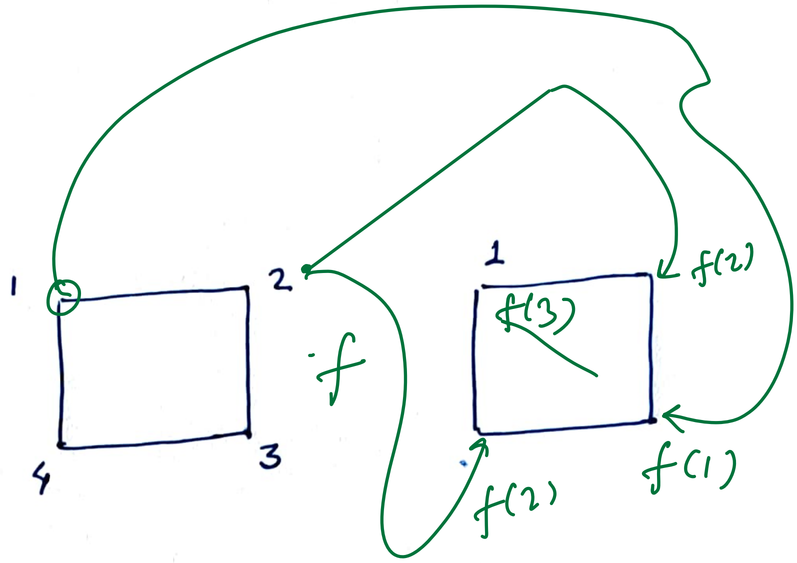
Similarly work out other cases:



$\{1, 2, 3, 4\}$

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2

2 choices



We have only two choices for 2.

$1 \longmapsto 1$

$2 \longmapsto$ either 2 or 4.

$\left.\begin{array}{l} 3 \longmapsto \\ 4 \longmapsto \end{array}\right\}$ gets fixed by the previus choice

Similarly

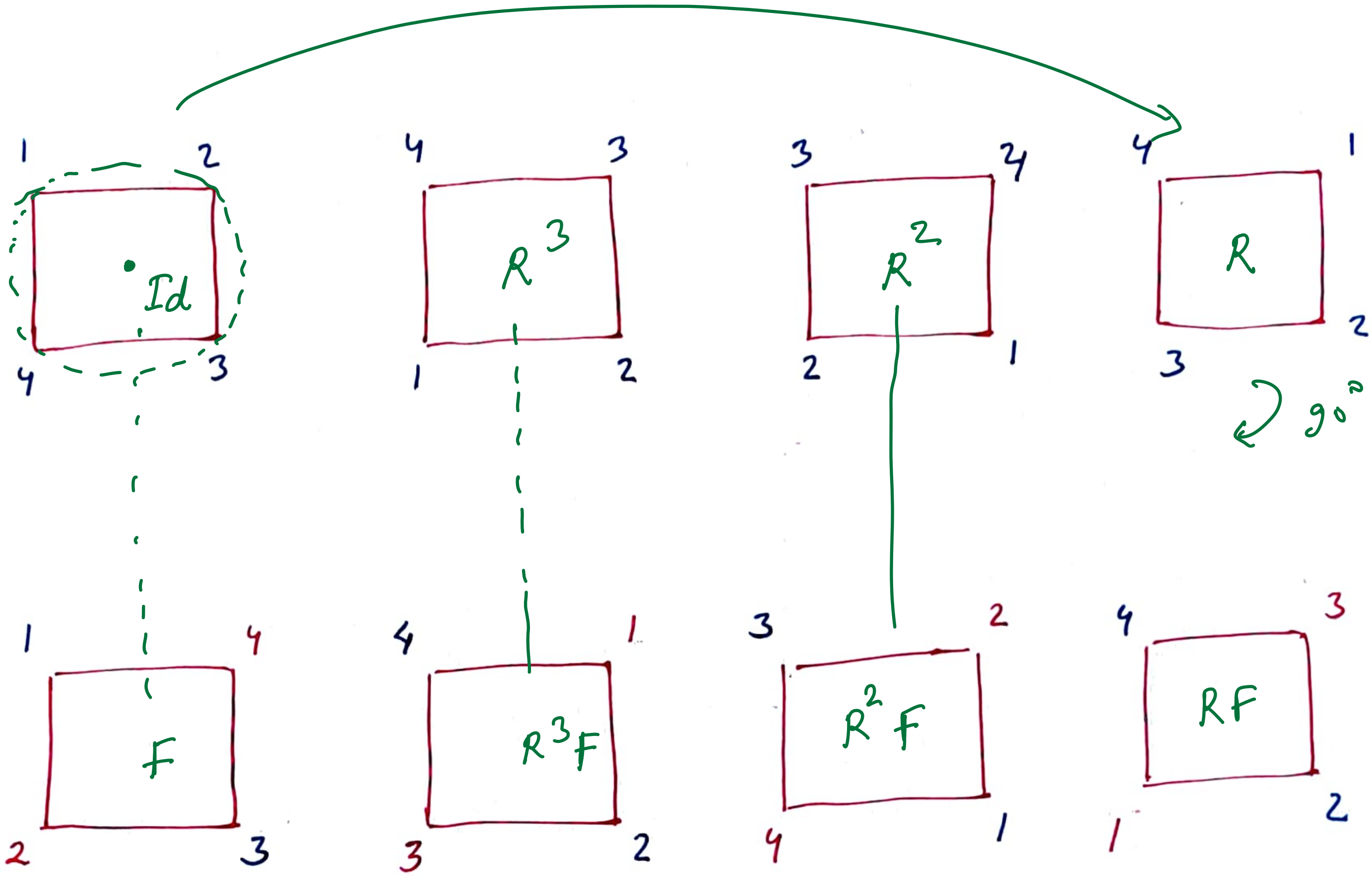
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1

3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1

4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

$$\boxed{\mathrm{Aut}(C_4) = 8}$$

Cycle on 4 vertices.



$S_4 =$

Symmetries of squares are

$$\{ \underbrace{Id, R, R^2, R^3, F, RF, R^2F, R^3F}_{\parallel} \}$$

$$G = \langle \underbrace{R, F}_{\text{two elements generating } G} \text{ such that } \boxed{R^4 = Id, \; F^2 = Id} \text{ and } RF = FR^{-1} \rangle$$

$\boxed{\text{Dihedral Group of order 8.}}$

D_{2n} : Dihedral Group

Set of symmetries of regular n-gon.

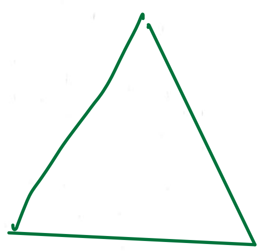
$\downarrow$

Fix $n \geq 3$.

rigid motions (distance-preserving transformations)

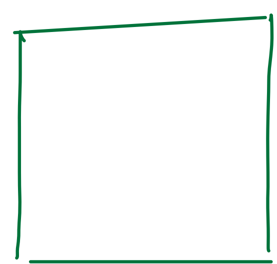
taking a regular n-gon back to itself, with the operation being composition.

D_6 =



3-regular gon

D_8



4-regular gon

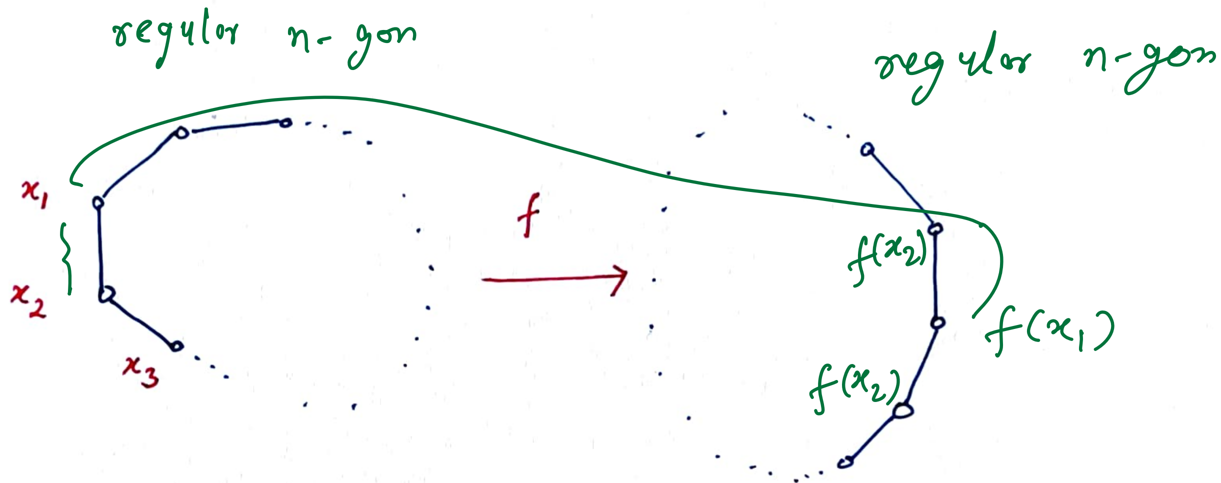
~~D_8~~ THEOREM. $|D_{2n}| = 2n$

order of Dihedral group is $2n$.

Proof.

Proof.

Step 1. We shall show that $|D_{2n}| \leqslant 2n$.



$$f : \{x_1, x_2, \ldots, x_n\} \longrightarrow \{x_1, x_2, \ldots, x_n\}$$

$x_1 \longmapsto f(x_1)$ — n choices

$x_2 \longmapsto f(x_2)$ — 2 choices

$\Big\} \longmapsto$ All other's get fixed.

" Think of distance-preserving graph automorphism of n-gon "

$$|D_{2n}| \leqslant 2n.$$

Step 2. We will show that $|D_{2n}| = 2n$.

(a) Rotations.

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$
$$i \longmapsto i+1 \pmod n$$
$$i = 1, 2, \ldots, n$$

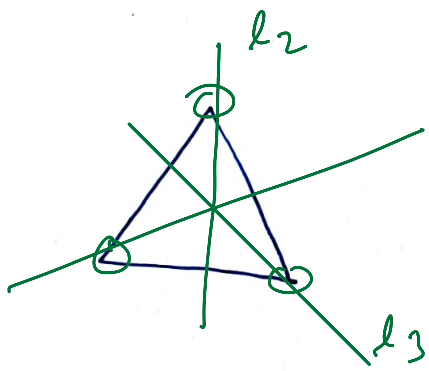
Geometrically rotation by $\frac{2k\pi}{n}$; $k = 0, 1, \ldots, n-1$

(b). Reflections :

Case (i). n is odd

Reflection across the line connecting each vertex to the mid-point of the opposite side.

For example,

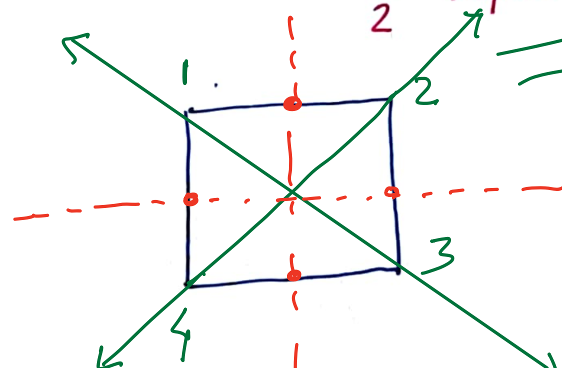


n reflections, one for each vertex.

Case (ii). n is even

There is a reflection across the line connecting each pair of opposite vertices

$\frac{n}{2}$ reflections



There is a reflection across the line connecting mid-point of opposite sides

$\frac{n}{2}$ reflections

Total : $\frac{n}{2} + \frac{n}{2} = n$ reflections

Note. Each of these are different reflections because each one fixes different vertices.

$$|D_{2n}| = 2n.$$

Discussion. $|D_{2n}| = \underline{\underline{2n}}$ if all of them are different

$$\left\{\begin{array}{l} n\text{-rotations are } \{ \overbrace{Id, R, R^2, \ldots, R^{n-1}}^{\text{distinct}} \} \\ n\text{-reflections are } \{ \cancel{F, FR, F} \quad F, RF, R^2F, \ldots, R^{n-1}F \} \end{array}\right.$$

distinct

Question: Are all these listed above different?

Clearly $\{ I, R, R^2, \ldots, R^{n-1} \}$ are all different.

Assume that $\boxed{R^k F = R^l}$ for some k and l.

$$\Downarrow$$

$$\boxed{F = R^{(l-k) \bmod n}}$$

Contradiction as $\underline{F \text{ is not a rotation.}}$