Introduction

RAMESH K. JALLU

LEC-01

DT. 12/01//22

Optimization problem

- An ***optimization problem*** is the problem of finding the ***best*** solution from all feasible solutions
- A ***discrete optimization problem*** seeks to determine the best possible solution from a finite set of feasible solutions
- For example, travelling salesman problem, MST, maximum independent set problem, minimum vertex cover problem, maximum matching problem, max network flow problem, knapsack problem, etc.

Combinatorial problems

- Combinatorial problems are ***computational problems*** involving arrangements of elements/objects from a finite set and selections from a finite set
  - For example, the MST problem, finding the inversions in a given array, the travelling salesman problem, etc.
- Combinatorial analysis is the mathematical study of the arrangement, grouping, ordering, or selection of discrete objects, usually finite in number

Combinatorial problems (Cont…)

- Traditionally, combinatorial problems concern with questions of existence or of enumeration
  - Does a particular type of arrangement exists? Or, how many such arrangements?
- Combinatorial optimization searches for an optimum object in a finite collection of objects
- Combinatorial optimization seeks to improve an algorithm by using mathematical methods either to reduce the size of the set of possible solutions or to make the search itself faster

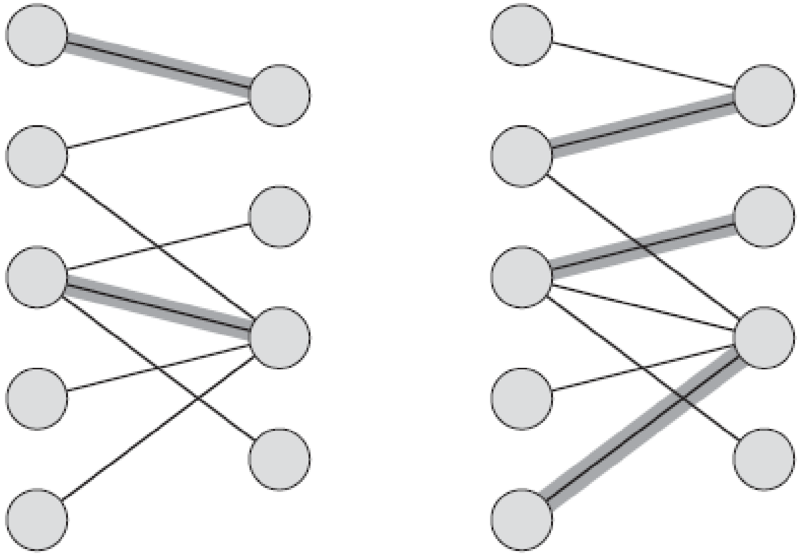
Why study optimization?

- Logistics
- Scheduling
- Machine learning
- Networking
- Neural networks
- ...

Matching algorithms

- Given a simple undirected graph $G = (V, E)$, a matching is a subset of edges $M \subseteq E$ such that such that for all vertices $v \in V$, at most one edge of M is incident on v
- We say that a vertex $v \in V$ is matched by the matching M if some edge in M is incident on v; otherwise, v is unmatched
- A ***maximum matching*** is a matching of maximum cardinality
  - That is, a matching M such that for any matching M', we have $|M| \geq |M'|$

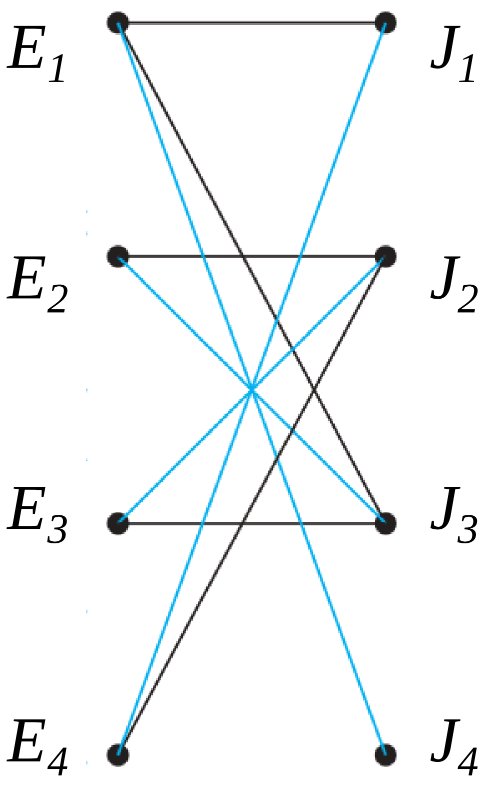
Example

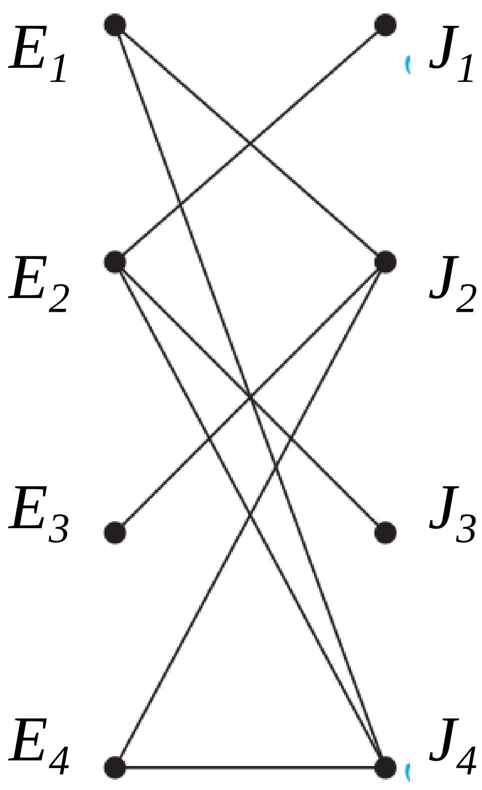


Bipartite graphs and matchings

- Bipartite graphs can be used to model many types of applications that involve matching the elements of one set to elements of another
- **Job assignments:** Suppose that there are m employees in a group and n different jobs that need to be done, where $m \geq n$
- Each employee is trained to do one or more of these n jobs
- We must also assign an employee to each job so that every job has an employee assigned to it and no employee is assigned more than one job

Example





Cont…

- Finding an assignment of jobs to employees can be thought of as finding a matching in the graph model
- If every vertex in G is matched, then we call such a matching ***perfect matching***
- A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1,V_2) is a ***complete matching*** from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$

Marriages based on matrimony

- Suppose that there are m men and n women: each person has a list of members of the opposite gender acceptable as a spouse
- We can construct a bipartite graph, an edge between a man and a woman if they find each other acceptable as a spouse
- A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair
- A ***maximum matching*** is a largest possible set of husband-wife pairs, and a complete matching of V_1 is a set of husband-wife pairs where every man is paired, but possibly not all women

Necessary and sufficient condition

PHILIP HALL (1904–1982)

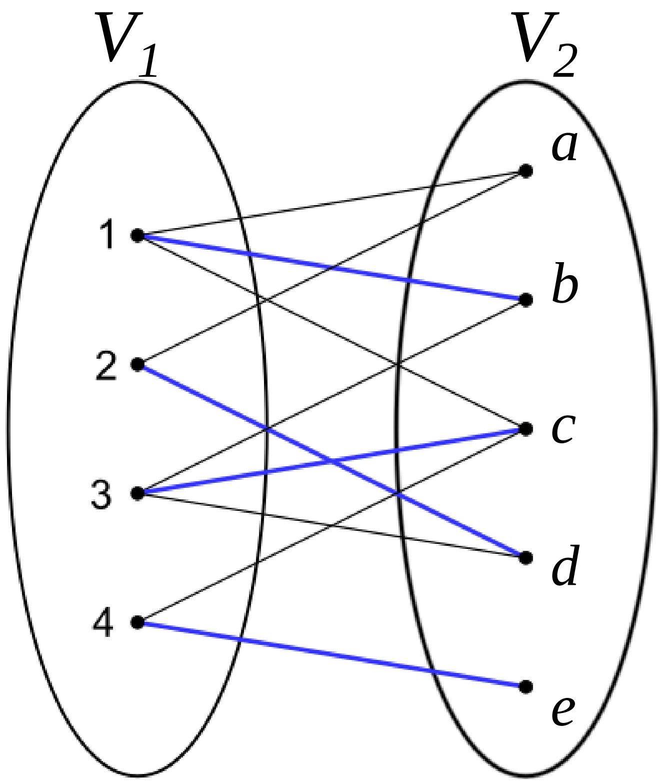
- If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

So, $N(A) = \bigcup_{v \in A} N(v)$.

- ***Hall's marriage theorem***: The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1

Proof

- (Only if part) Suppose that there is a complete matching M from V_1 to V_2
- Let A be any subset of V_1
- For every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2
- This implies, $|N(A)| \geq |A|$



Proof (Cont…)

- (If part) Suppose $|N(A)| \geq |A|$ for all $A \subseteq V_1$
- We need to prove that there is a complete matching M from V_1 to V_2
- Proof by strong mathematical induction on $|V_1|$
- ***Basis step***: If $|V_1| = 1$, then V_1 contains a single vertex, and it is true
- ***Inductive hypothesis***: Let k be a positive integer. If $|V_1| = j \leq k$, then there is a complete matching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met

Proof (Cont...)

- Now suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$
- ***Case*** *(i)*: Suppose that for all integers j with $1 \leq j \leq k$, the vertices in every subset A of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2
- That is for every proper subset A, $|N(A)| \geq |A| + 1$
- Select a vertex $v \in W_1$ and an element $w \in N(v)$

Proof (Cont...)

- We delete v and w and all edges incident to them from H
- This produces a bipartite graph H' with bipartition $(W_1 - \{v\}, W_2 - \{w\})$
- Because $|W_1 - \{v\}| = k$, the inductive hypothesis tells us there is a complete matching from $W_1 - \{v\}$ to $W_2 - \{w\}$
- Adding the edge from v to w to this complete matching produces a complete matching from W_1 to W_2
- The theorem is true in this case

Proof (Cont...)

- *Case (ii)*: Suppose that for some j with $1 \leq j \leq k$, there is a subset X of j vertices in W_1 such that there are exactly j neighbors of these vertices in W_2
- That is, there is X such that $|N(X)| = j$
- *Let* $N(X) = Y \subseteq W_2$
- Observe that there is a complete matching from X *to* Y
- Remove these $2j$ vertices from W_1 and W_2 and all incident edges to produce a bipartite graph K with bipartition $(W_1 - X, W_2 - Y)$

Proof (Cont...)

- We will show that the graph K satisfies the condition $|N(A)| \geq |A|$ for all subsets A of $W_1 - X$
- On the contrary, suppose it is not
- Then there is a subset A of $W_1 - X$ such that $|N(A)| < |A| \leq k+1-j$
- Now, observe that $|N(A)| < |A \cup X|$ and which is a contradiction to our assumption
- Hence, by the inductive hypothesis, the graph K has a complete matching
- Combining this complete matching with the complete matching from X to Y, we obtain a complete matching from W_1 to W_2

- Thank you!