Differential Equations (MA 1150)

D. Sukumar

Lecture 1

April 7, 2020

Overview

Section 1

Course information

Information about this course

- Instructor: D Sukumar
- Office: Academic Block A, 506 What is the use?
- Email: suku@math.iith.ac.in

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- Reference Textbooks
  - Elementary Differential Equations by William Trench, available at ramanujan.math.trinity.edu/wtrench/texts/index.shtml.

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Marks Distribution. Check the Course plan file uploaded in Google Classroom

Course content

Reference textbook: Elementary Differential Equations by William Trench

Syllabus. Ordinary Differential equations: First order linear equations, Bernoulli's equations, Exact equations and integrating factor, second order and higher order linear differential equations with constant coefficients.

- Lectures $1-4$: Ordinary Differential equations: First order linear equations, Bernoulli's equations, Exact equations and integrating factor.
- Lectures $5-8$: Second order and higher order linear differential equations with constant coefficients.
- Revision in Lecture 9.

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- Modeling and solving

Section 2

Introduction

Subsection 1

ODE and its order

Ordinary Differential Equation

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- Ordinary differential equation by ODE.
- Derivative of y by y', $\dfrac{dy}{dx}$ or $y^{(1)}$.

The order of an ODE is the highest order of derivative of y occurring in the ODE.

Examples.

- $y' = x^3y^4 + y$ is a 1st order ODE.
- $y'' + x^5y' + y = \cos x$ is a 2nd order ODE.
- $y^{(4)} + xy^{(1)}y^{(2)} + 2xy = \sin x$ is a 4th order ODE.

Open Intervals

Given any $a, b \in \mathbb{R}$, we define the open interval from a to b to be the set

$$(a, b) = \{x \in \mathbb{R} \ : \ a < x < b\},$$

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We use the symbols ∞ and $-\infty$ to define, for any $a \in \mathbb{R}$, the following semi-infinite (open) intervals:

$$(-\infty, b) = \{x \in \mathbb{R} \ : \ x < b\} \text{ and } (a, \infty) = \{x \in \mathbb{R} \ : \ a < x\}.$$

The set $\mathbb{R}$ can also be thought of as the doubly infinite (open) interval $(-\infty, \infty)$.

Subsection 2

General and particular solution

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Applications. Modeling. Initial Value Problems

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Examples Radioactive substance decay, Growth of bacteria, Falling stone from certain height, Pendulum movement, Population growth etc. (For more details, refer to textbook).

Subsection 3

IVP

Initial Value Problem (IVP)

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Solution to IVP. A function $y = y(x)$ defined on some open interval (a, b) containing x_0 is a solution of the IVP if y satisfies the ODE on open interval (a, b) and $y(x_0) = y_0$.

Subsection 4

Geometry

Geogebra Webpage

Section 3

Solving methods

Subsection 1

Separation of variables

Separation of Variables to solve ODE: Example

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We can solve this equation by separating the variables.

$$\begin{aligned} &\frac{1}{y}dy = -2x\ dx \\ \implies &\int \frac{1}{y}dy = \int -2x\ dx, \\ \implies &\ln|y| = -x^2 + c, \\ \implies &y = c_1 e^{-x^2}. \end{aligned}$$

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Assume that ODE is separable.

Let H_1 and H_2 be anti derivatives of M and N respectively. This means ${H_1}'(x) = M(x)$ and ${H_2}'(y) = N(y)$. Then our ODE is

$${H_1}'(x) + {H_2}'(y)\frac{dy}{dx} = 0.$$

Separation of Variables: General Method (continued...)

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In general, the separable variables method only gives us the implicit solution to the given ODE.

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- Any non linear ODE of the form $y' = q\left(\frac{y}{x}\right)$ can be converted into a separable ODE by substituting $y = vx$.

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$$v'x + v = v^2 + v - 1.$$

Re-write it as

$$\begin{aligned} \frac{v'}{v^2 - 1} &= \frac{1}{x} \\ \frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right) v' &= \frac{1}{x}. \end{aligned}$$

Example

Example Solve $x^2y' = y^2 + xy - x^2$.

Re-write it as

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1.$$

Substituting $y = vx$, we get

$$v'x + v = v^2 + v - 1.$$

Re-write it as

$$\frac{v'}{v^2 - 1} = \frac{1}{x}$$
$$\frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right)v' = \frac{1}{x}.$$

Integrating, we get

$$\frac{1}{2}\left(\ln|v-1| - \ln|v+1|\right) = \ln|x| + c_1.$$

Example (Continued ...)

$$\frac{v-1}{v+1} = cx^2 \implies v = \frac{1+cx^2}{1-cx^2}.$$

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Both $y = x$ and $y = -x$ are also solutions, but only $y = x$ can be obtained by choosing a particular value of $c = 0$.

Converting to Separable Equation

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$$y' = \frac{2x + y + 1}{x + 2y - 4}$$

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Use substitution $X = x + 2$, and $Y = y - 3$.

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$$y' + ay = 0, \ a \in \mathbb{R}.$$

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Question. Is this y an explicit solution of ODE $y' + ay = 0$?