Lectures 3 - 4 - 5
Vector space, basis and dimension

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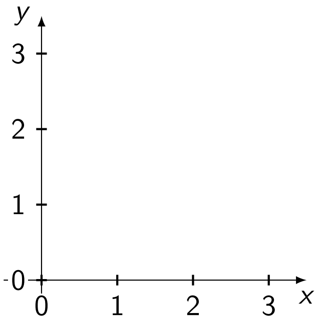
Vector addition in $\mathbb{R}^2$



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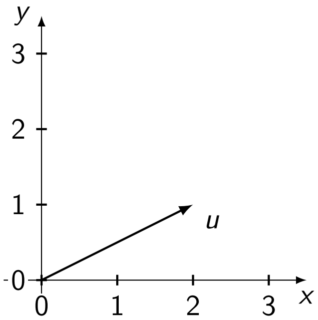

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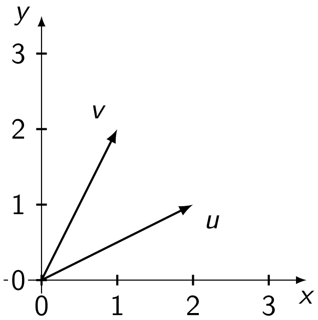

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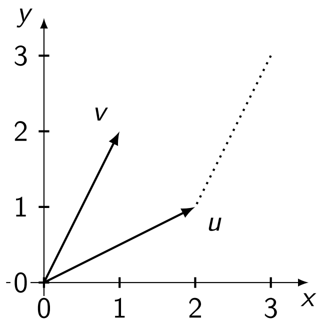

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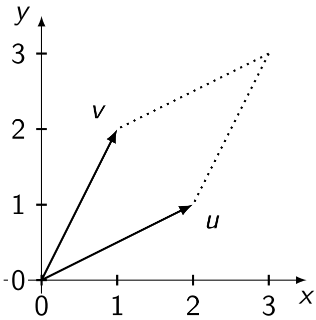

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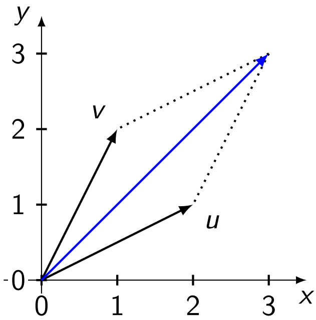

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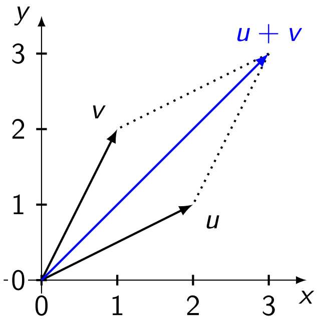

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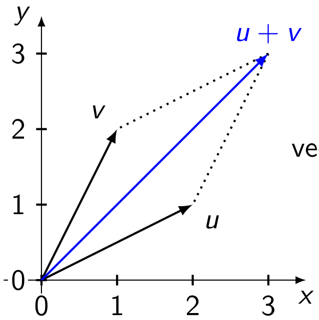

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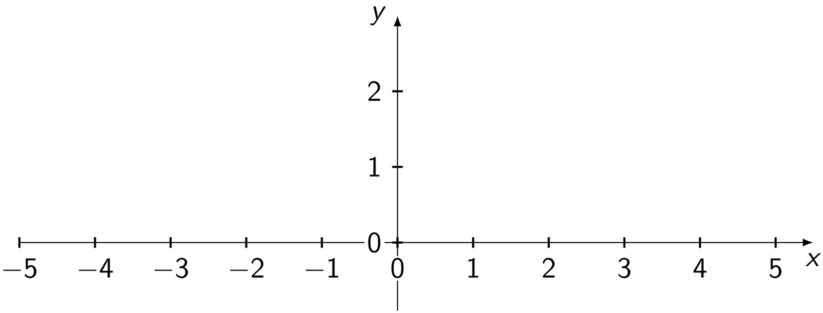

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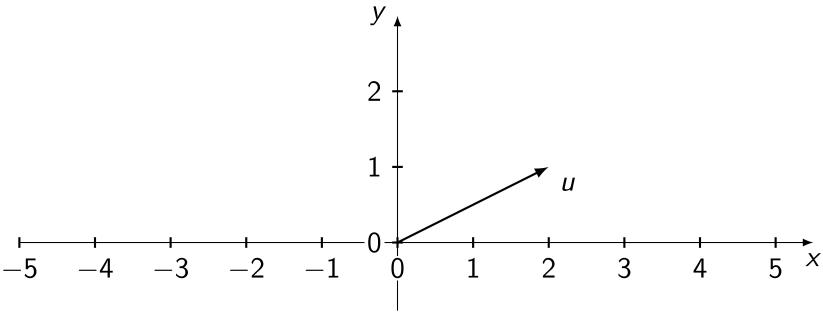

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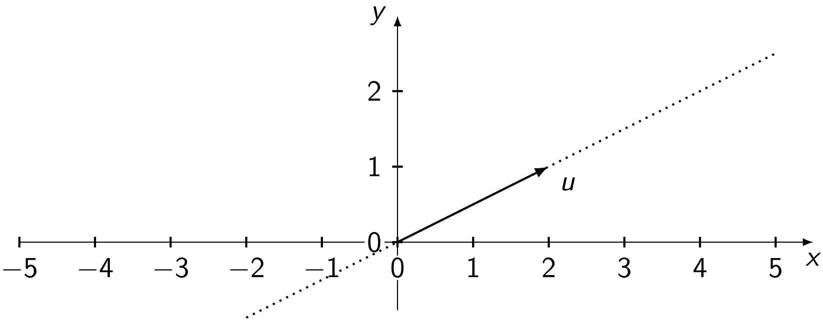

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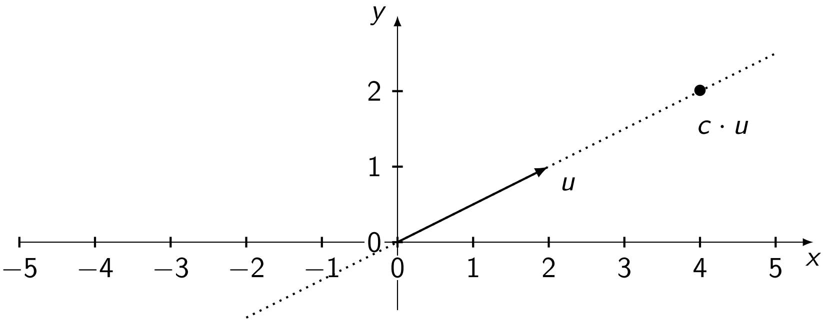

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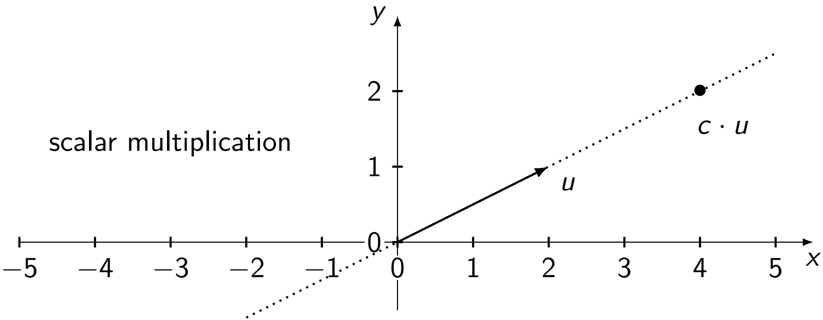

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4. From now, we work over the field $\mathbb{R}$.

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Component wise addition and component wise scalar multiplication.

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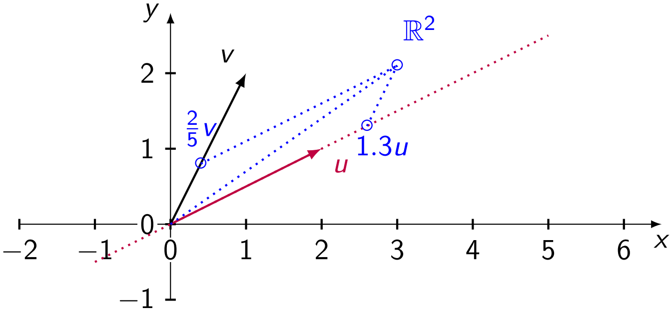

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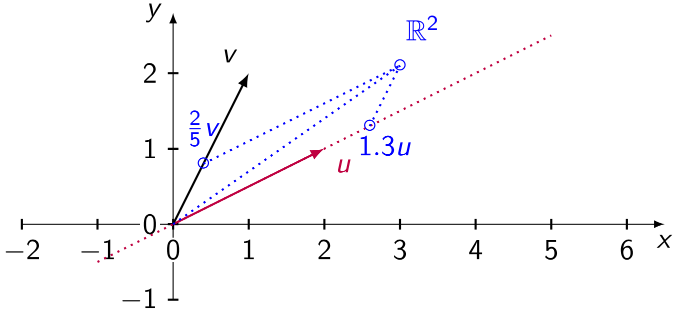

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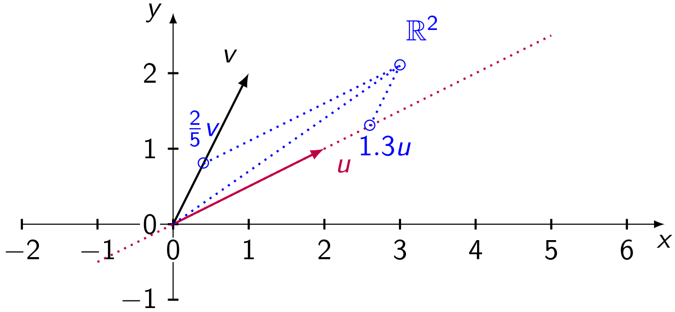

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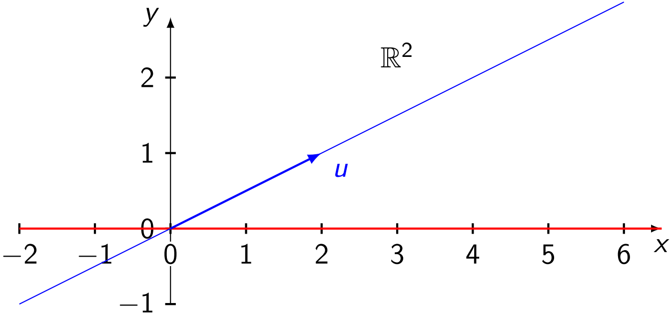

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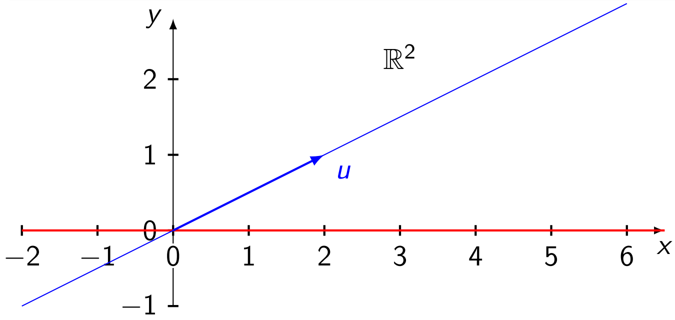

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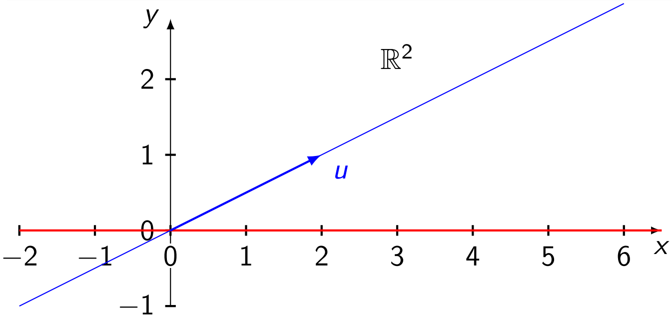

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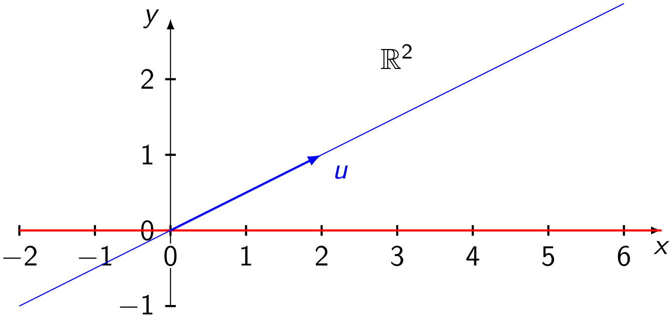

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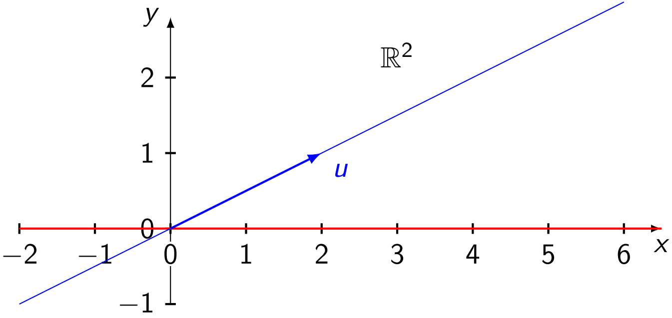

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Note that many properties of W will be inherited from V.

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Let S be a set of vectors in a vector space V. The **subspace spanned** by (or generated by) S is defined to be the smallest subspace of V containing S. We denote this subspace by $\mathrm{Span}(S)$. We write that S spans the subspace $\mathrm{Span}(S)$.

Theorem

Let S be a set of vectors in a vector space V. The following subspaces are equal.

1. *The intersection of all subspaces of V containing S.*
2. *The set of all linear combinations of vectors in S, i.e.*

$$\{c_1 v_1 + \cdots + c_r v_r : c_i \in \mathbb{R},\ v_i \in S\}.$$

3. *The subspace spanned by S (i.e. the smallest subspace of V containing S).*

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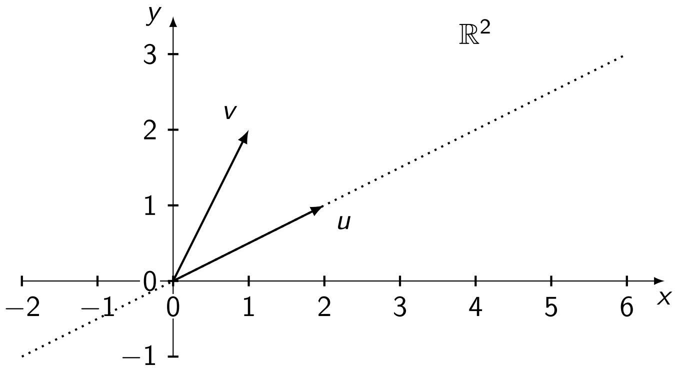

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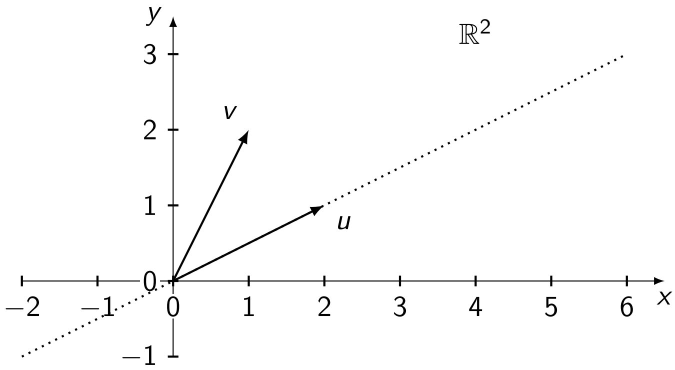

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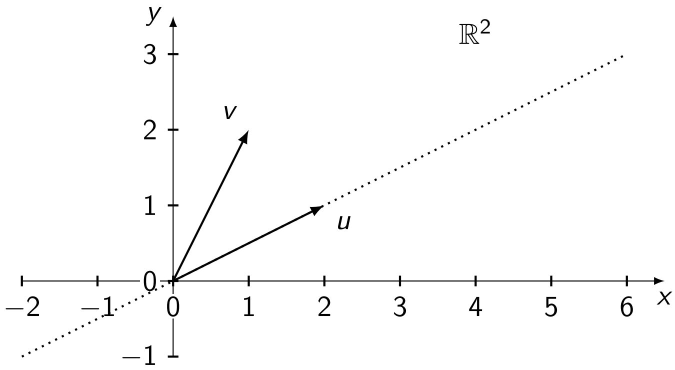

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If $S = \{v_1, \ldots, v_n\}$, we say that $v_1, \ldots, v_n$ are linearly dependent (or independent) instead of saying that S is so.

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Let $c_1 v_1 + \cdots + c_r v_r + cv = 0$ for some vectors $v_1, \ldots, v_r \in S$ and scalars $c_1, \ldots, c_r, c$.

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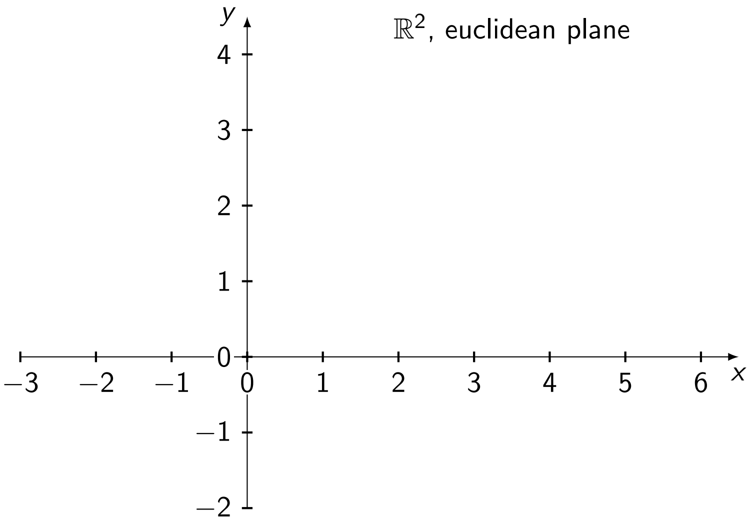
Therefore $c = 0$, and hence $c_1v_1 + \cdots + c_rv_r = 0$. Since S is linearly independent, it follows that $c_i = 0$ for all $1 \leq i \leq r$. $\square$

Vectors in $\mathbb{R}^2$ plane



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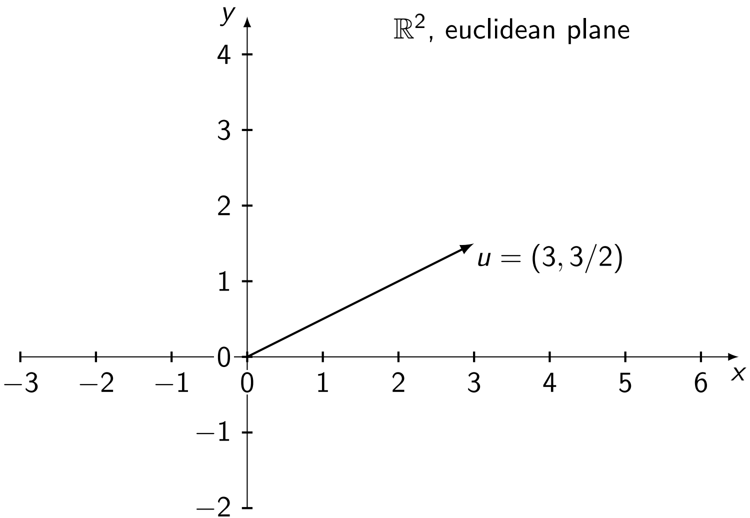

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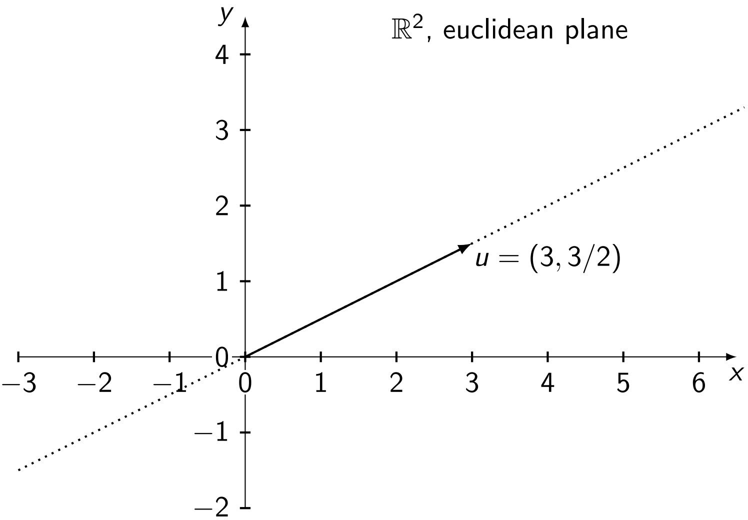

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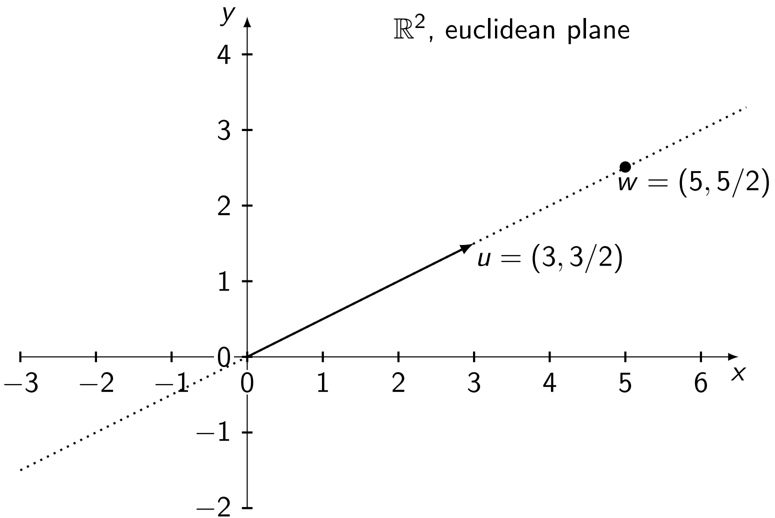

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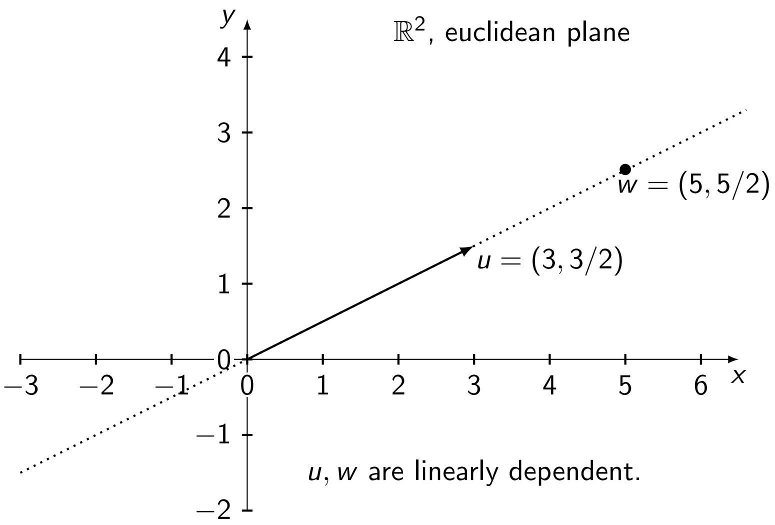

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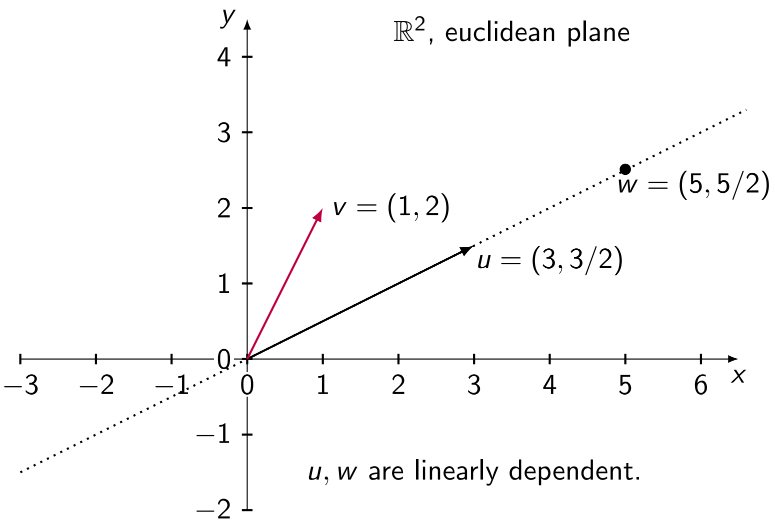

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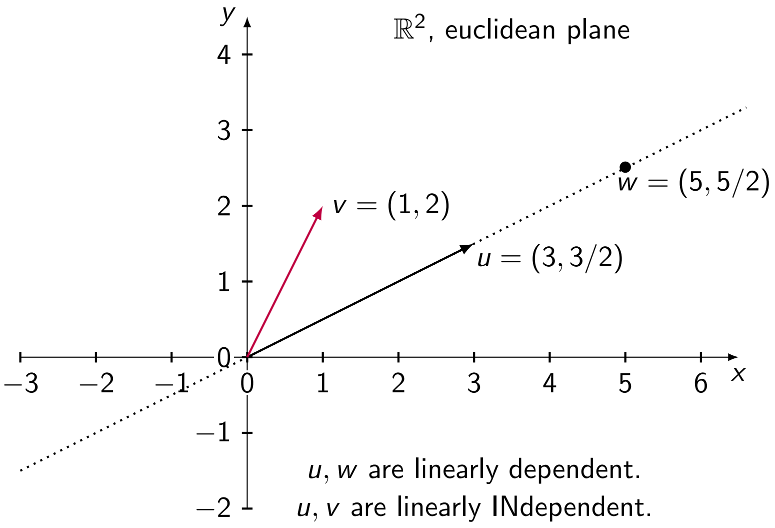

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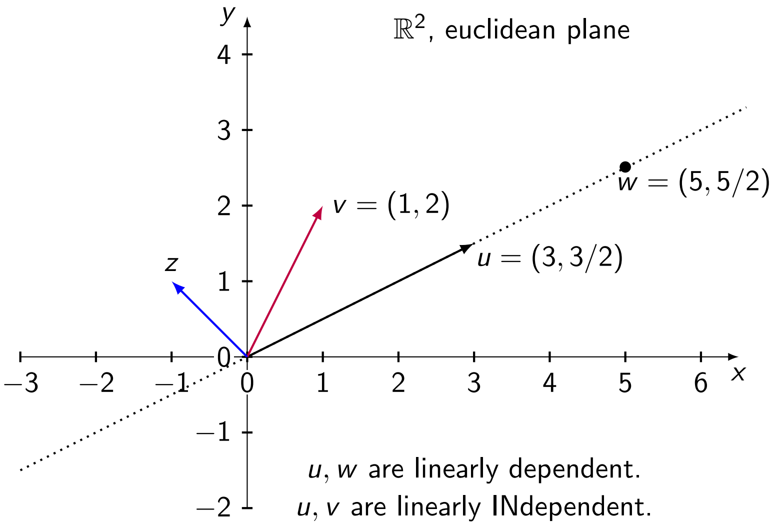

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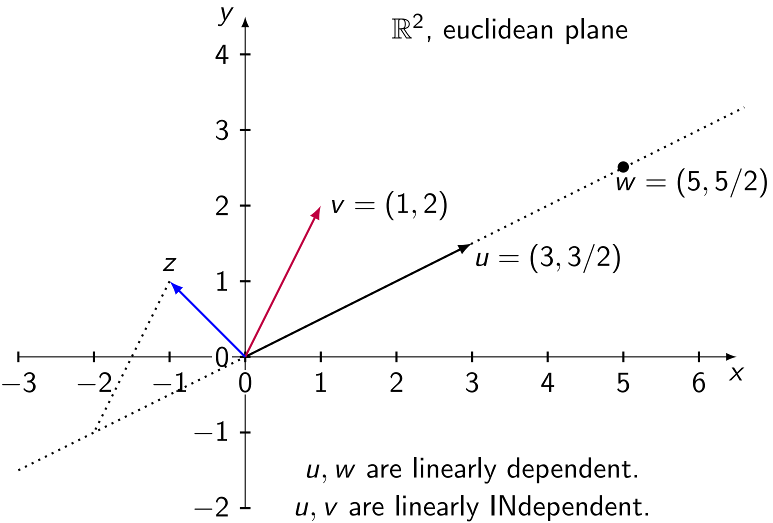

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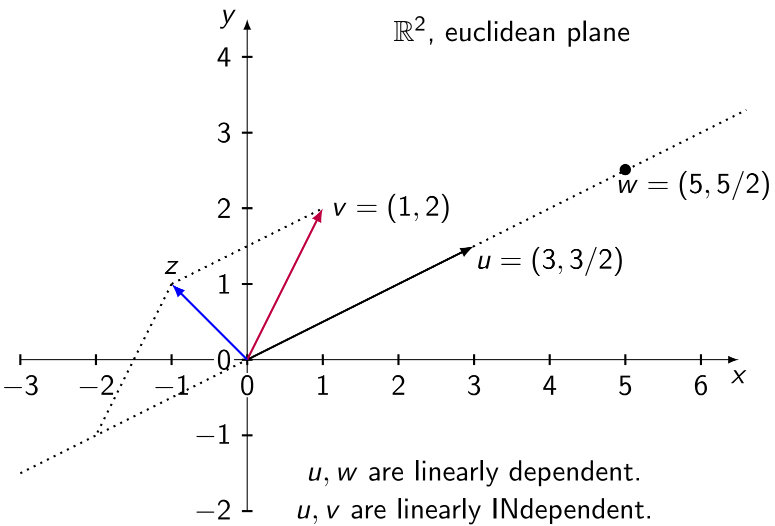

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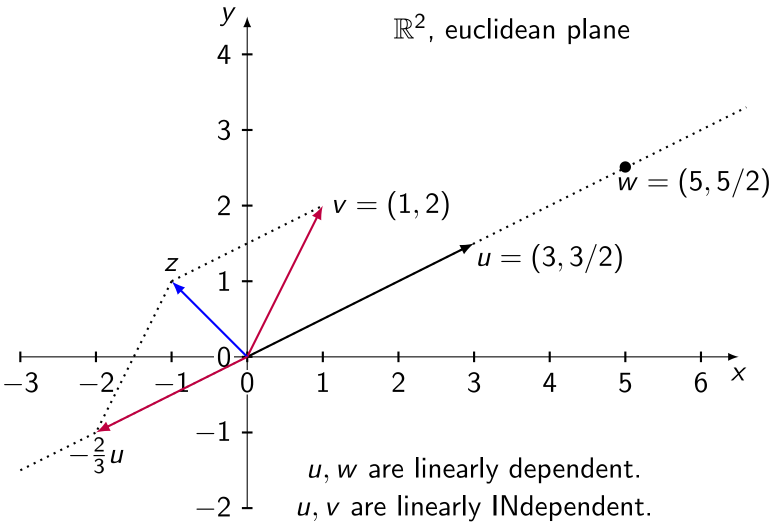

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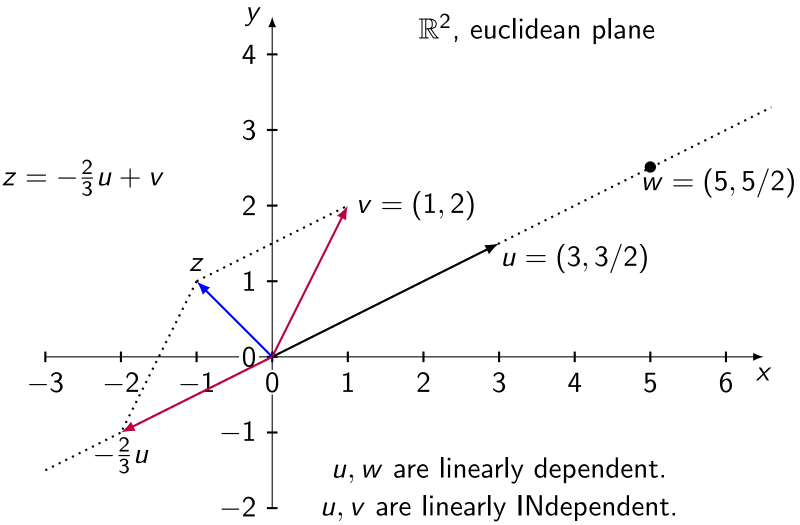

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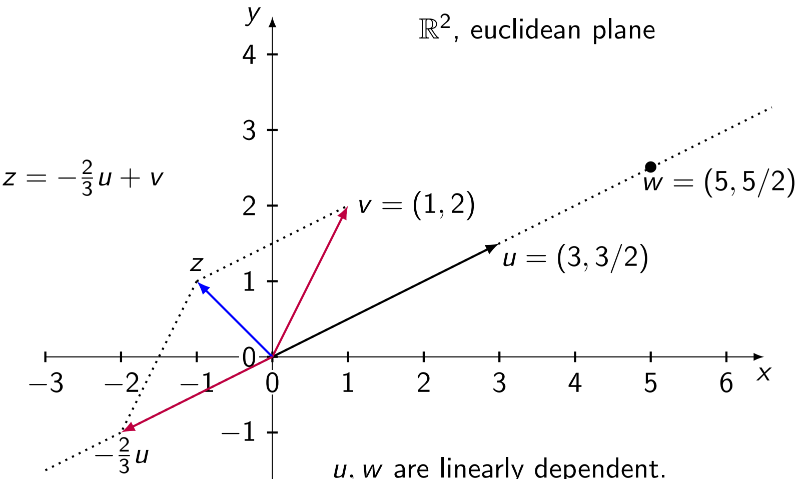

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The set $\left\{ u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ forms a basis of $\mathbb{R}^2$.

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2. Note that any vector $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be written as a linear combination $x_1 e_1 + \cdots + x_n e_n$. So S spans $\mathbb{R}^n$.
3. Is S linearly independent? Answer: Yes. Why? (Because $x_1 e_1 + \cdots + x_n e_n = 0 \implies$ every $x_i = 0$.)
4. Therefore S is a basis of $\mathbb{R}^n$.
5. This particular basis is called the **standard basis** of $\mathbb{R}^n$.
6. Thus $\dim(\mathbb{R}^n) = n$.

Basis of $\mathbb{R}[x]$

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forms a basis. Thus $\dim(\mathbb{R}[x]) = \infty$.

Theorems related to the dimension of a vector space

- In the rest of the slides, it is proved that if V is a finite dimensional vector space, then any two bases of V have the same number of elements.

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- Interested students can read it on their own

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$$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \text{ for some } c_1, \ldots, c_n \in \mathbb{R}. \tag{1}$$

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Linearly independent set has less vectors than spanning set

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Suppose $V = \mathrm{Span}\{v_1, v_2, \dots, v_n\}$, *and* $\{u_1, u_2, \dots, u_m\}$ *is a linearly independent subset of* V. *Then* $m \leq n$.

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Therefore $\{u_1, u_2, \dots, u_{n+1}\}$ is linearly dependent,

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Consequences of the fact that a linearly independent set has less or equal number of vectors than a spanning set

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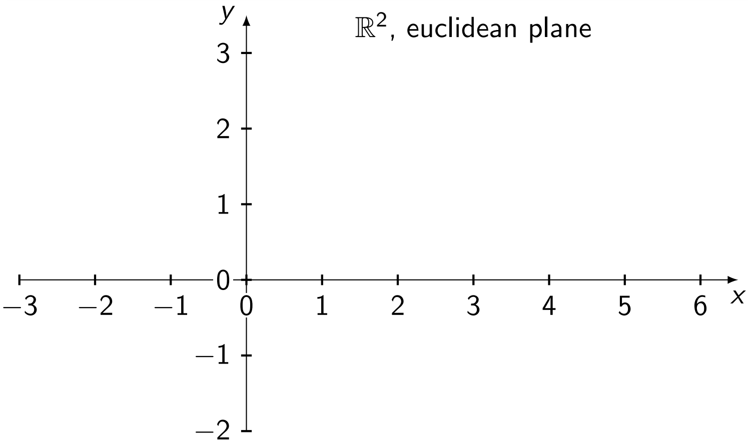

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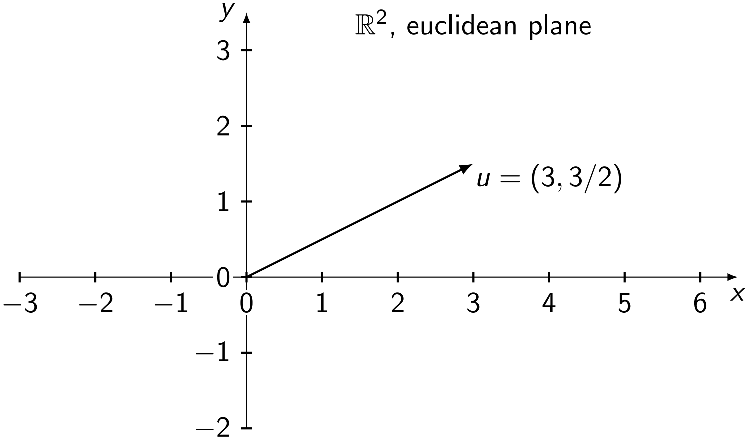

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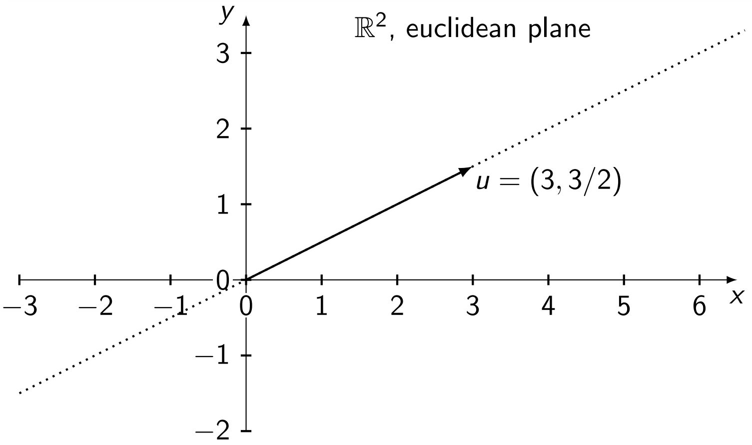

A linearly independent set can be extended to a basis

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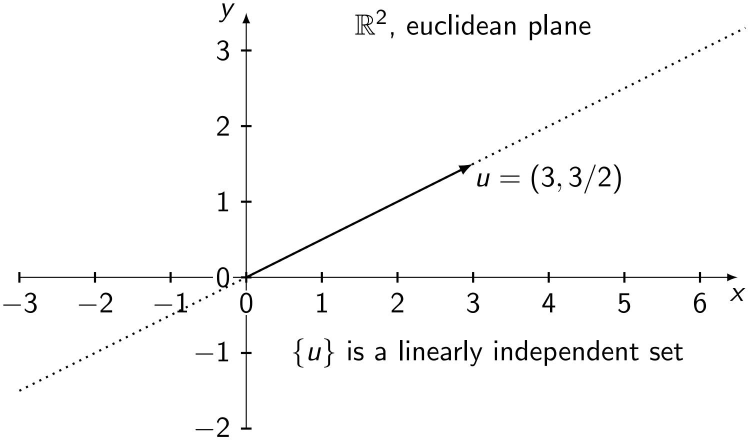

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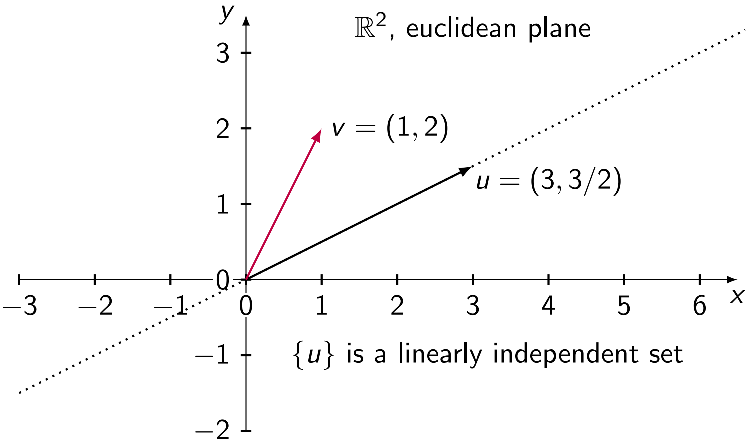

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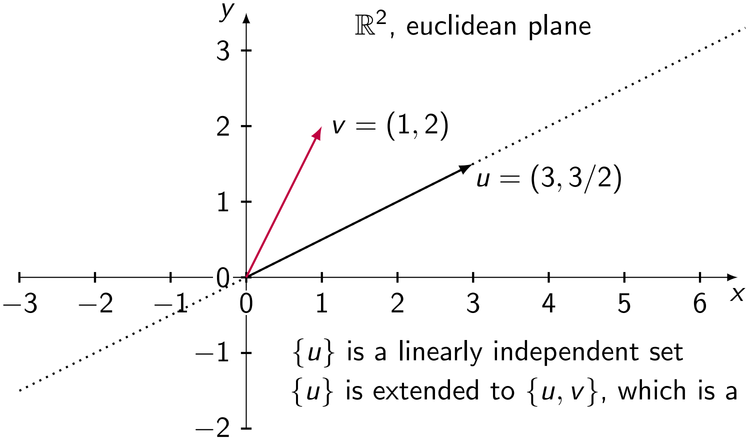

$\{u\}$ is a linearly independent set

$\{u\}$ is extended to $\{u, v\}$, which is a basis of $\mathbb{R}^2$

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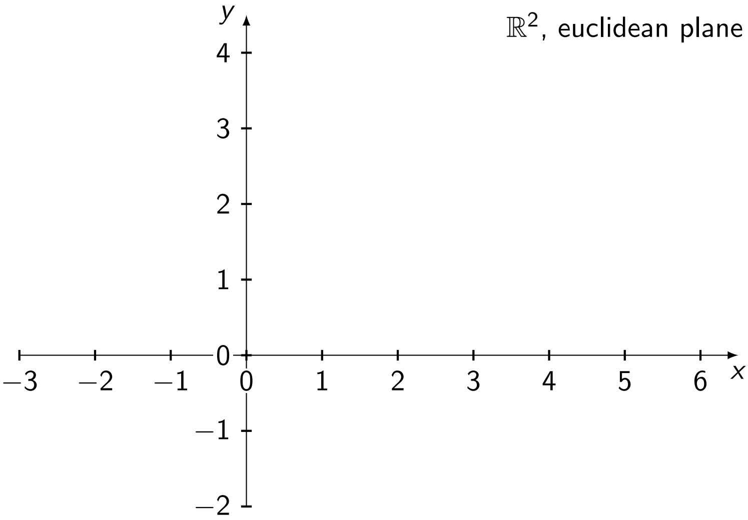
Therefore $\dim(W) < \dim(V)$.

Example: Proper subspaces of $\mathbb{R}^2$



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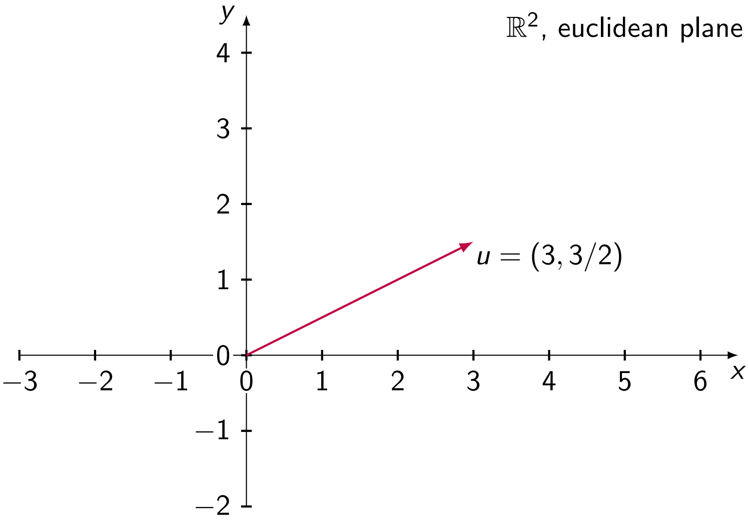

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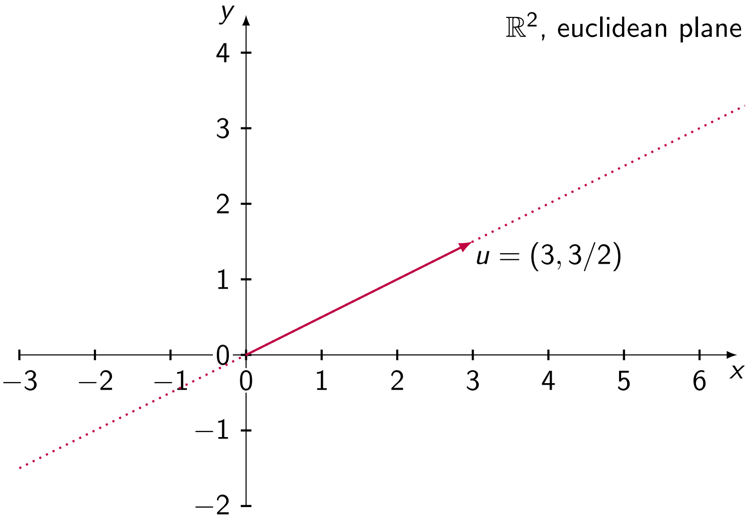

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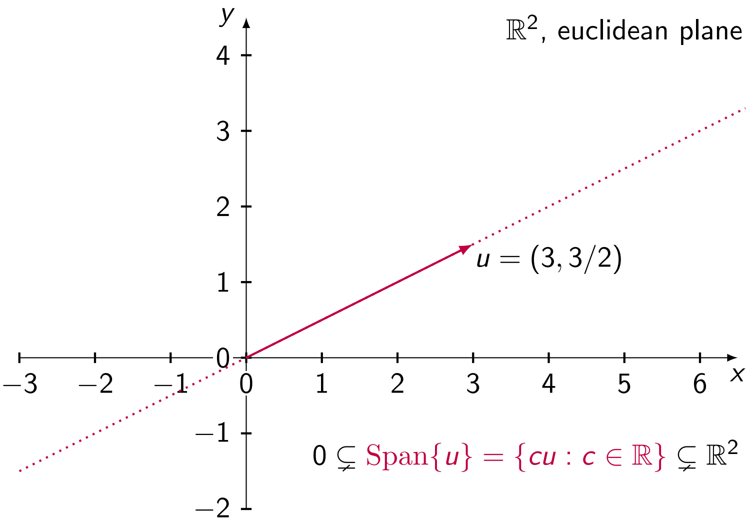

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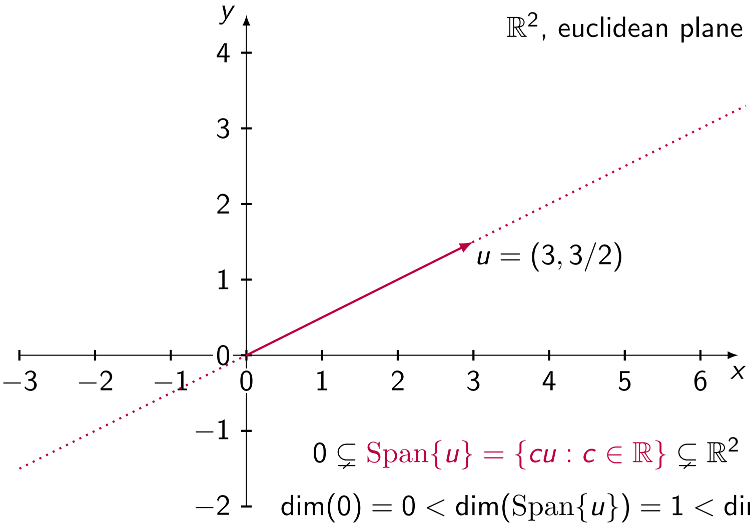

$$0 \subsetneq \mathrm{Span}\{u\} = \{cu : c \in \mathbb{R}\} \subsetneq \mathbb{R}^2$$

$$\dim(0) = 0 < \dim(\mathrm{Span}\{u\}) = 1 < \dim(\mathbb{R}^2) = 2$$

Sum of two subspaces

Definition

Let W_1 and W_2 be two subspaces of V.

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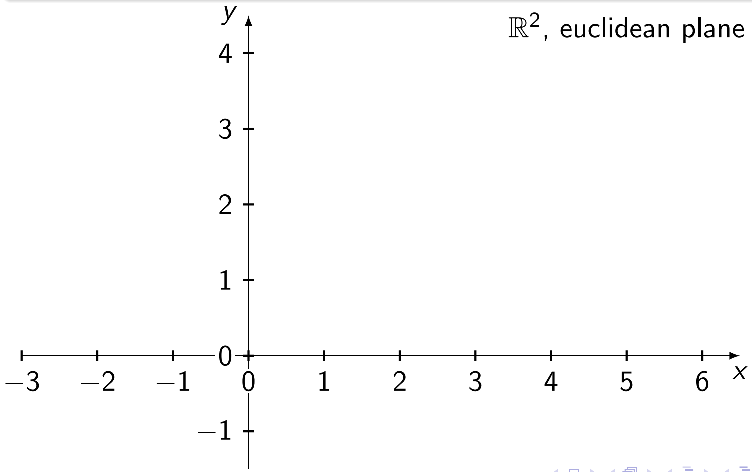

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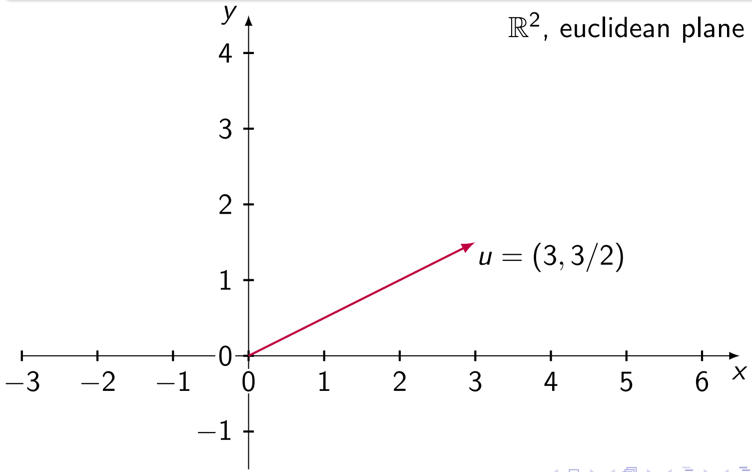

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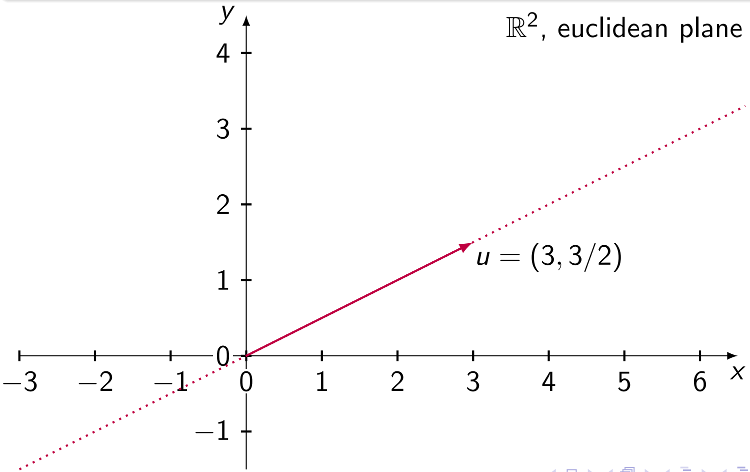

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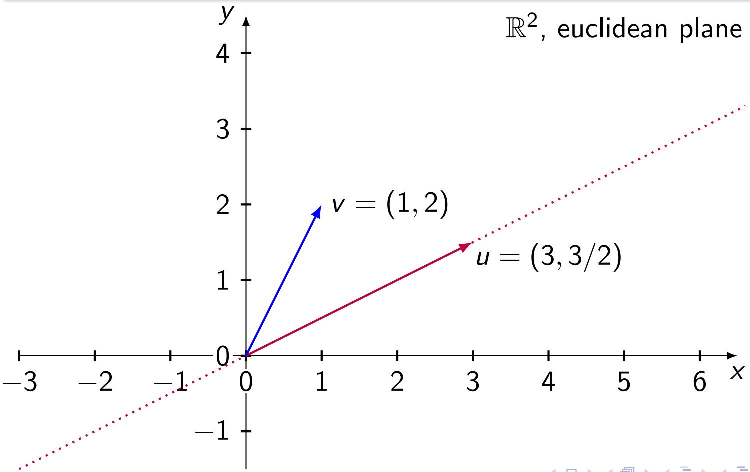

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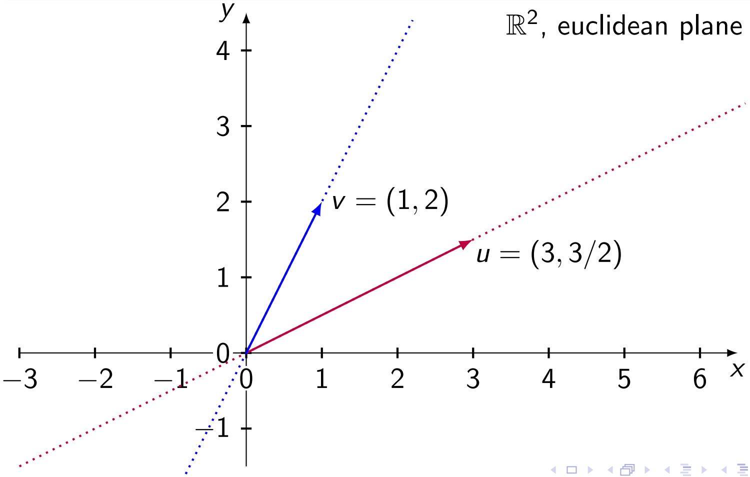

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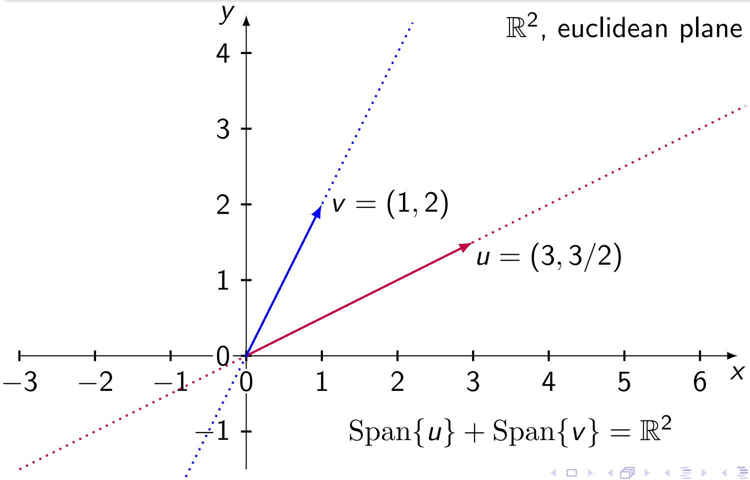

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Show that $\{u_1, \ldots, u_r, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is a basis of $W_1 + W_2$.

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between the set of all vectors in V and the set of all n-tuples in $\mathbb{R}^n$, and this correspondence respects both the vector additions and scalar multiplications.
3. This also shows that any vector space V of dimension n over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$.

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5. It follows from the above equalities that
$$[v]_{\mathcal{B}'} = P[v]_{\mathcal{B}} \quad \text{for every vector } v \in V.$$

Thank You!