Matching algorithms (Cont...)

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LEC-03

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Recap

- Matching in graphs
  - Maximum matching
  - Perfect matching
  - Complete matching in bipartite graphs
- Necessary and sufficient condition
  - The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 ***if and only if*** $|N(A)| \geq |A|$ for all subsets A of V_1
  - **Corollary**: G has a perfect matching ***if and only if*** $|N(A)| \geq |A|$ for all subsets A of V_1

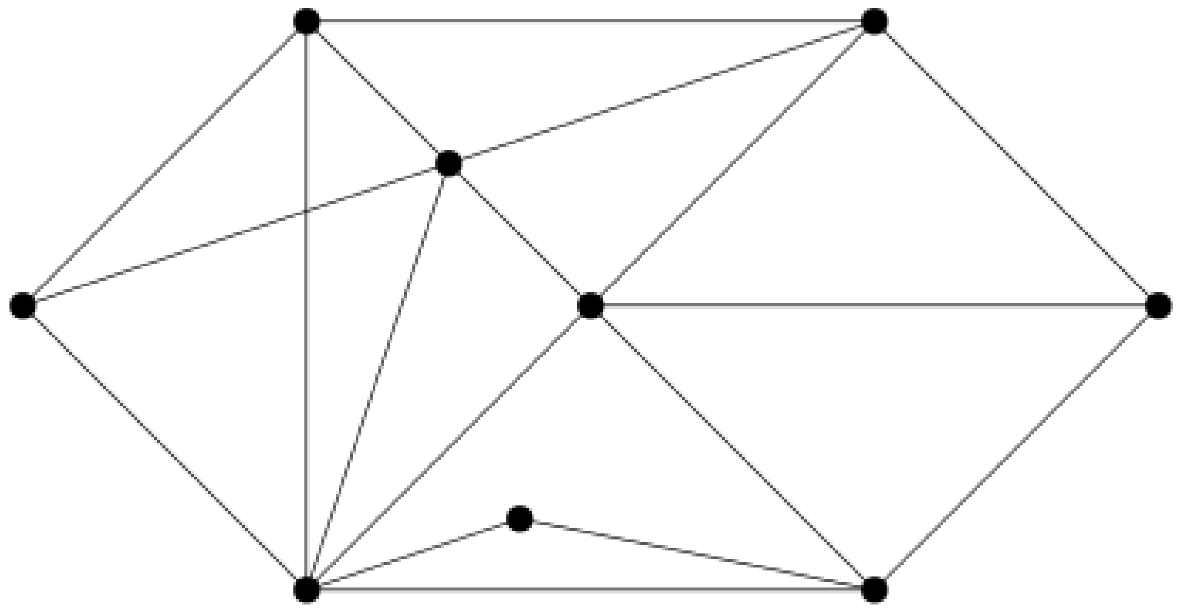
Questionnaire: True or False

1. Every bipartite graph has perfect matching
2. If a graph has a perfect matching then it is a bipartite graph
3. Every perfect matching a maximum matching
4. Every maximum matching is a perfect matching
5. If a bipartite graph has a perfect matching then it is a complete matching
6. What about the converse of 5?

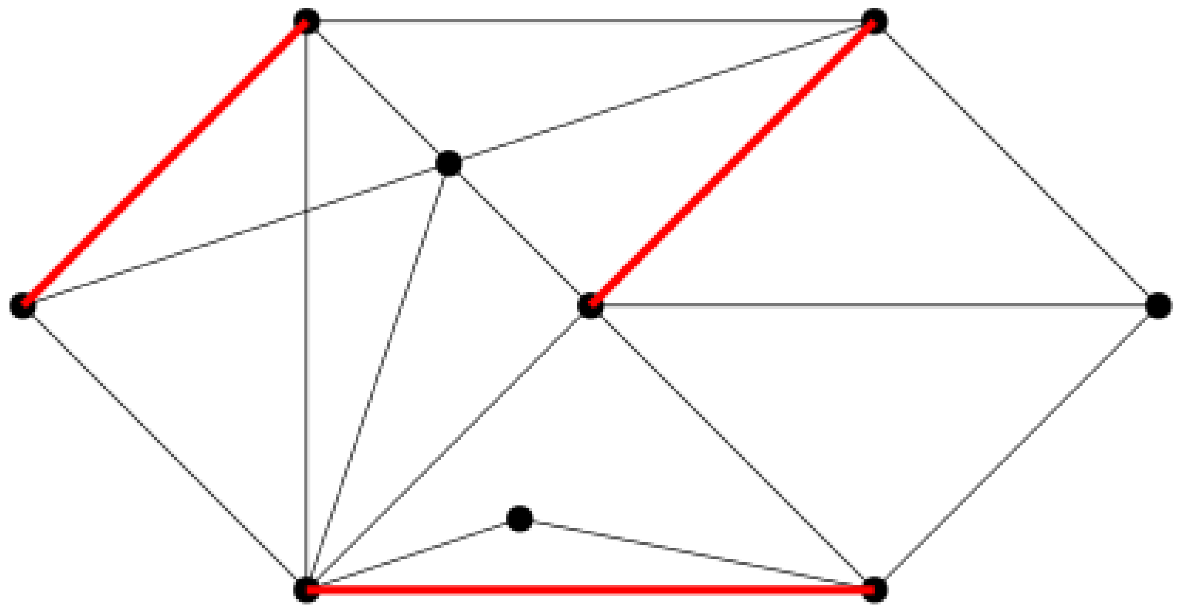
Augmenting path

- Let G be a graph (bipartite or not), and let M be some matching in G
- A path P is said to be an ***M-alternating path*** if its edges are alternately in and not in M
- That is, an alternating path with respect to a matching M is a path in which edges alternate between those in M and those not in M
- An ***M-augmenting path*** is an alternating path that starts and ends with an unmatched vertex (aka free vertex)

Example



Example



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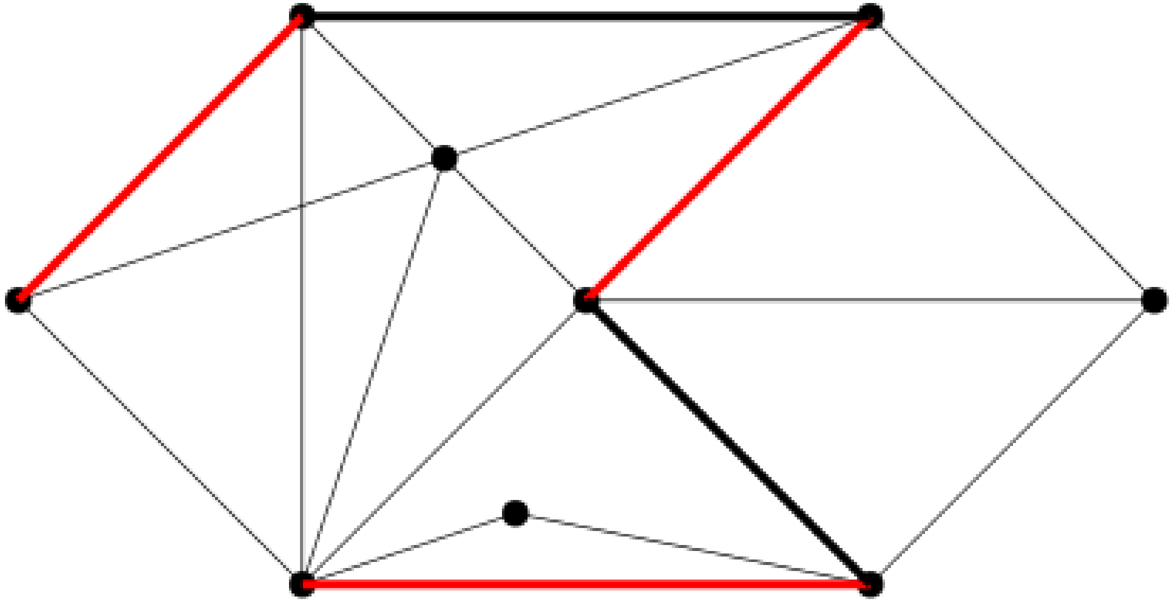

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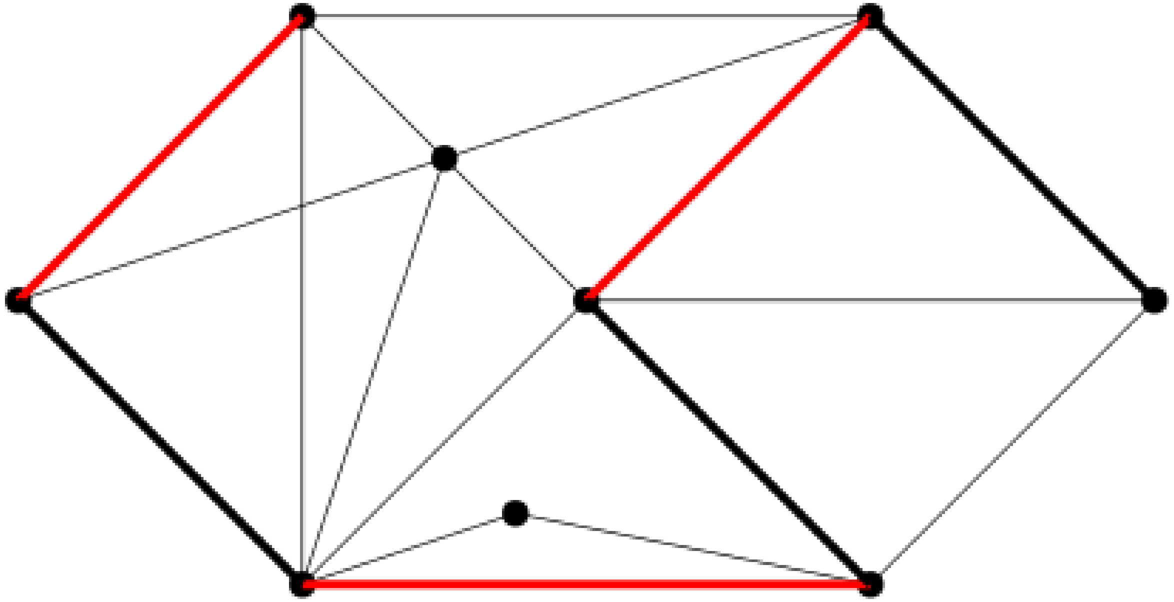

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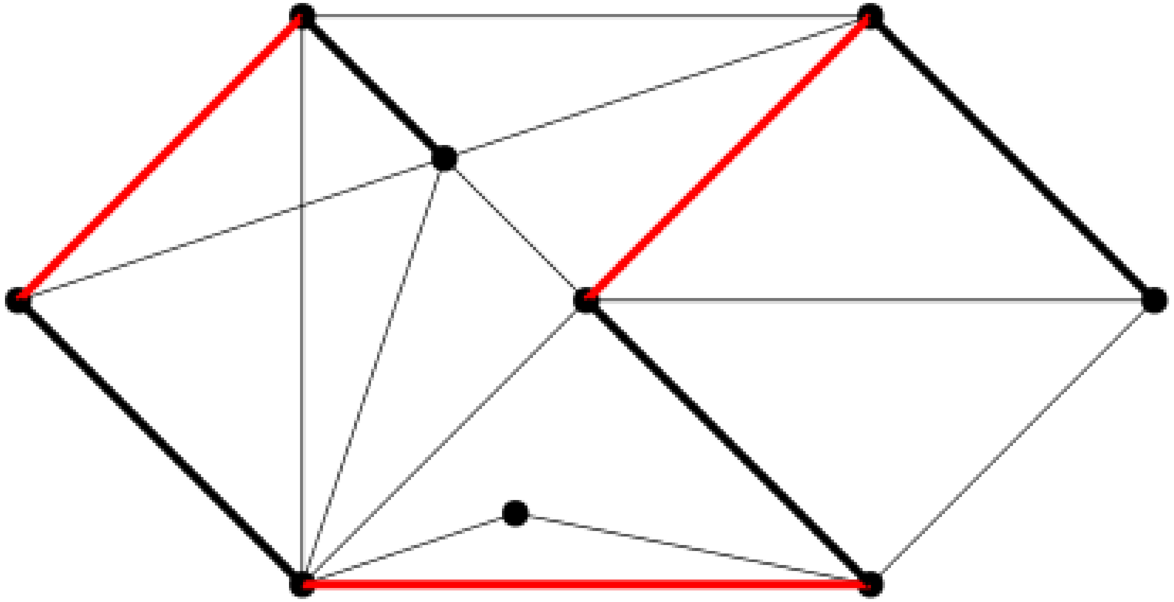

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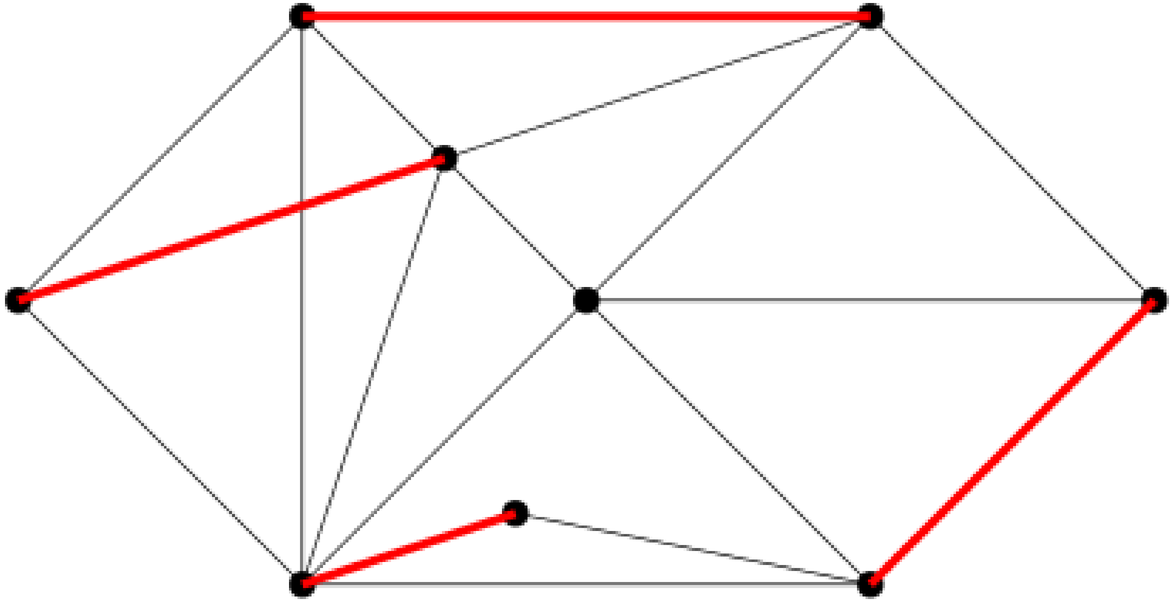

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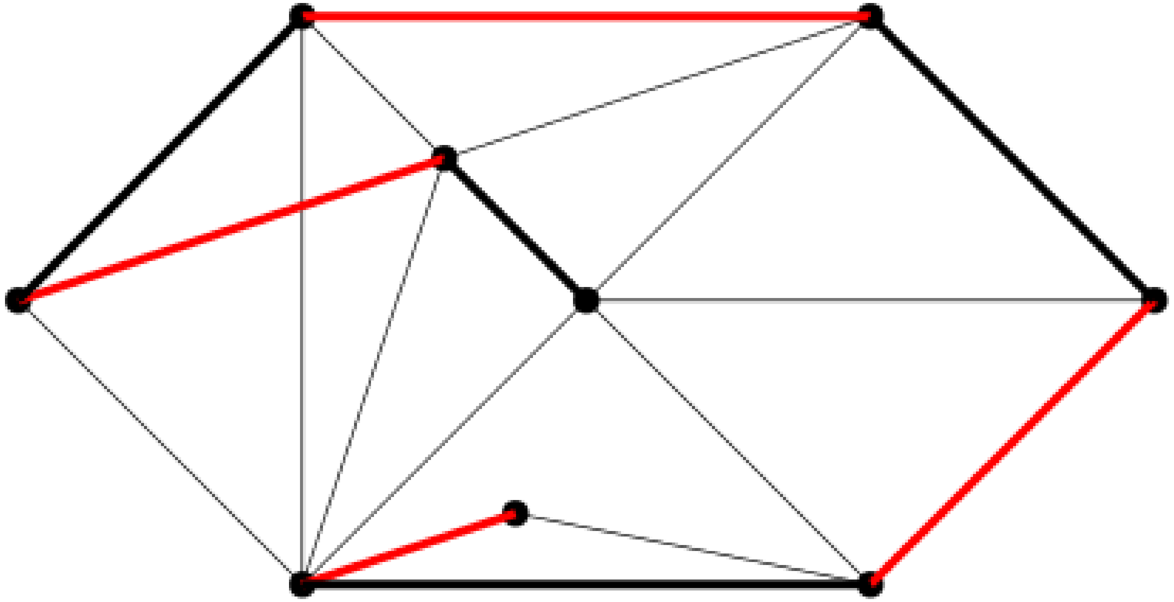

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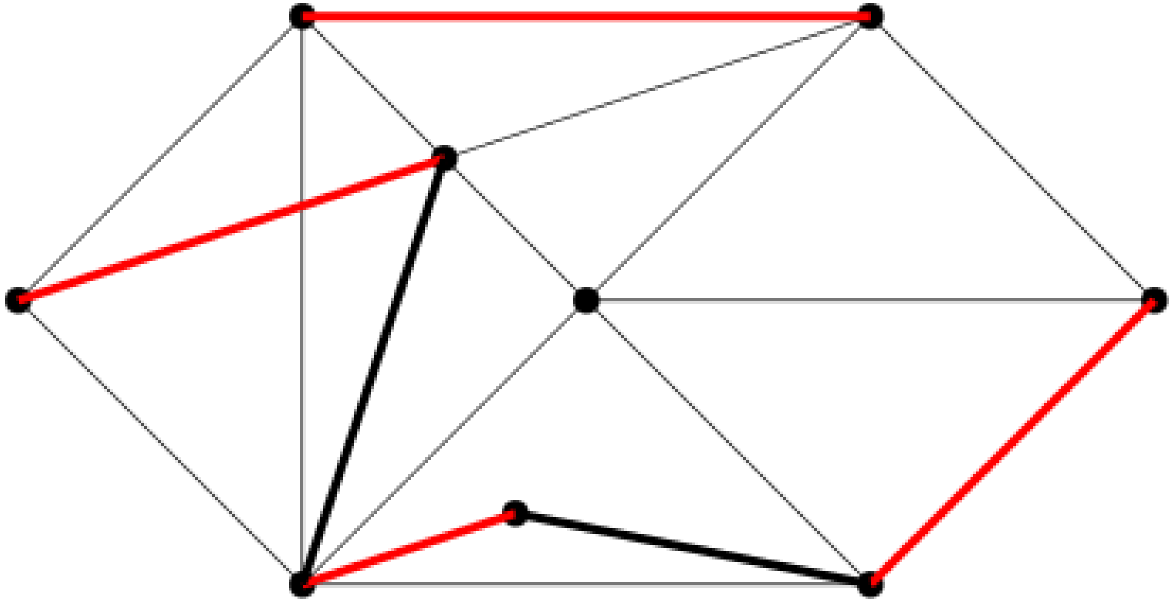

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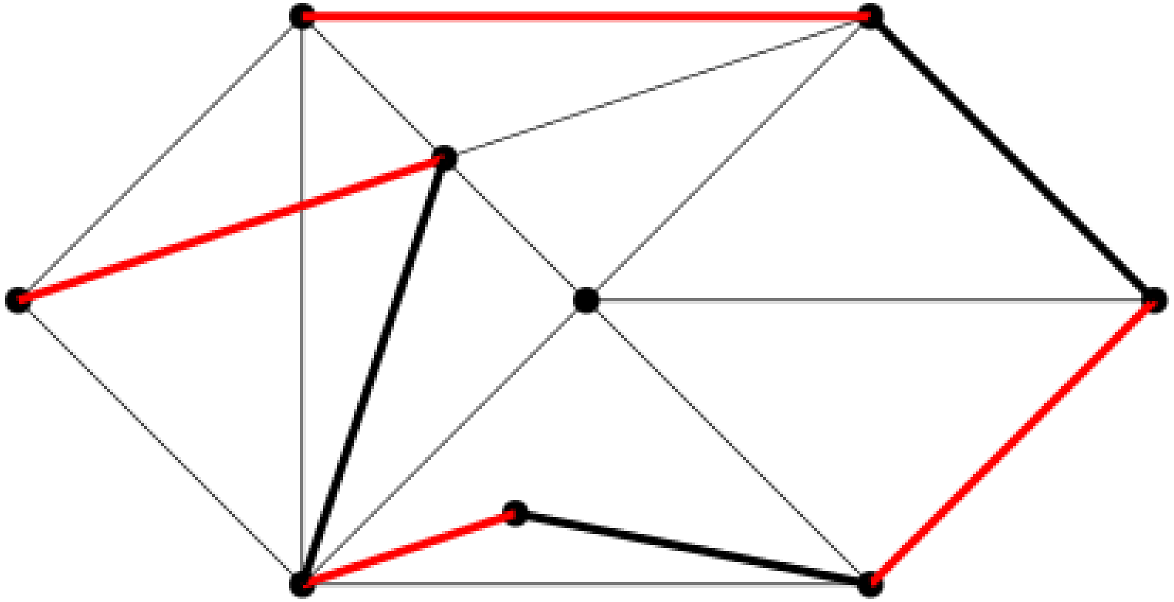

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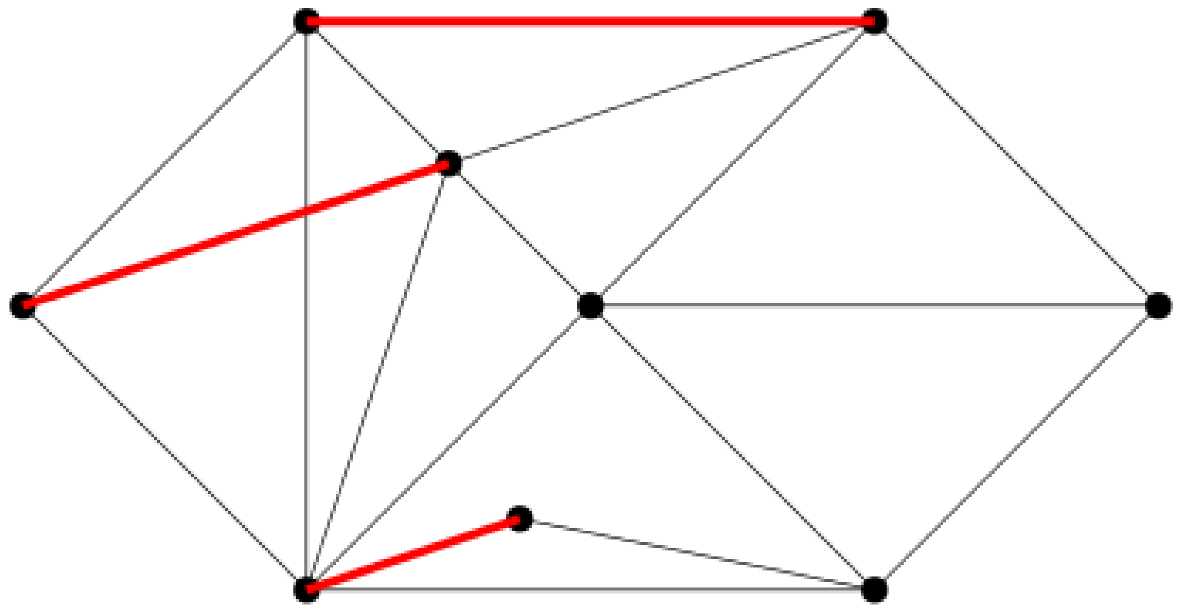

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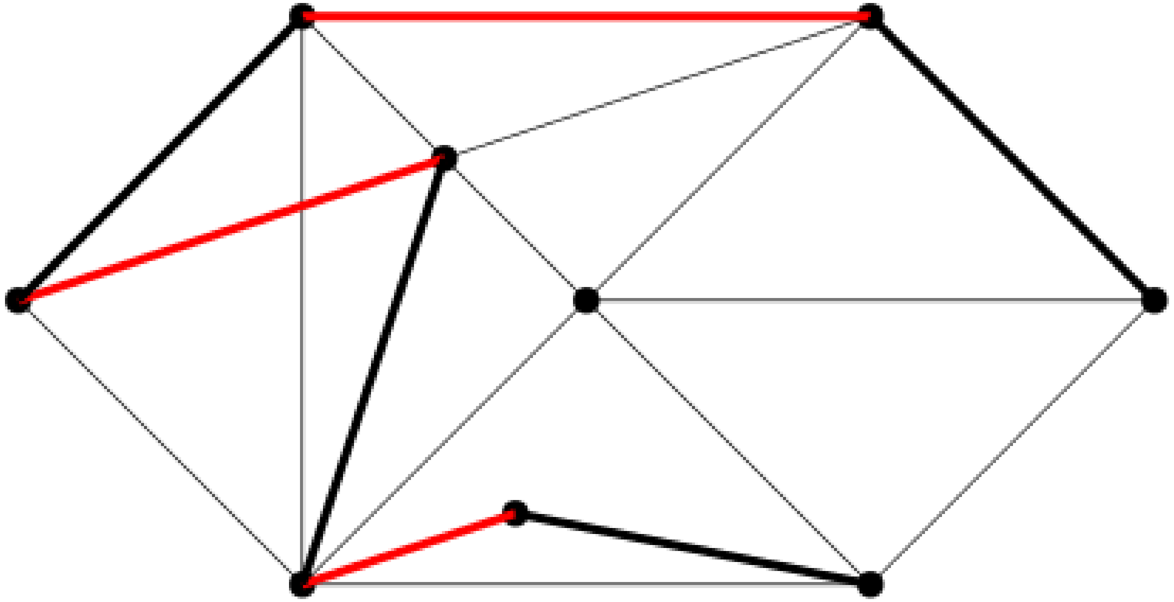

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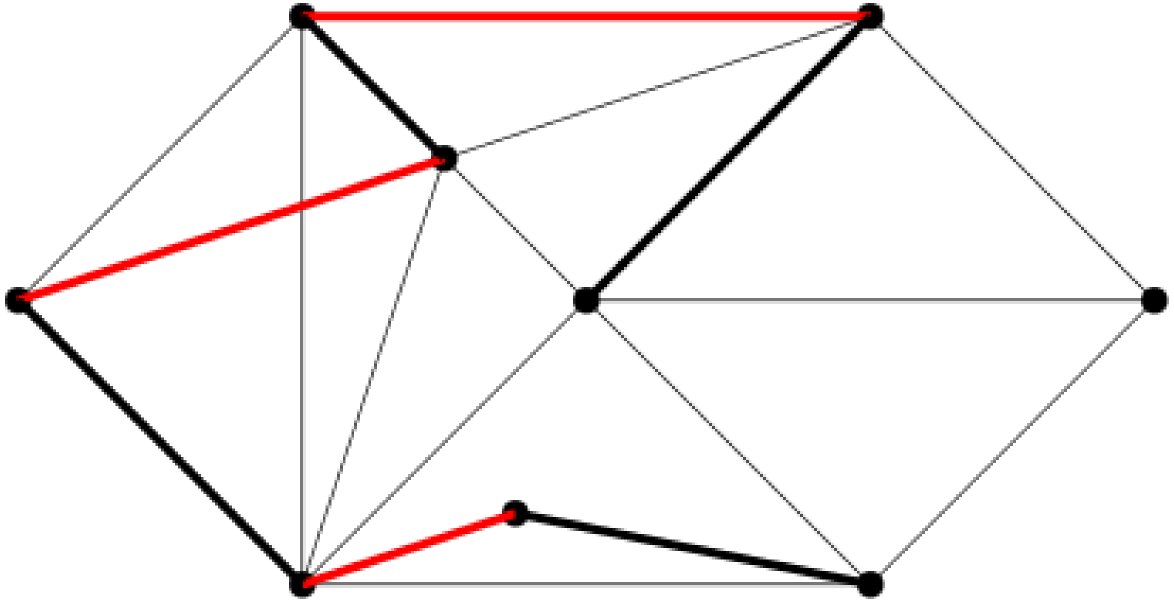

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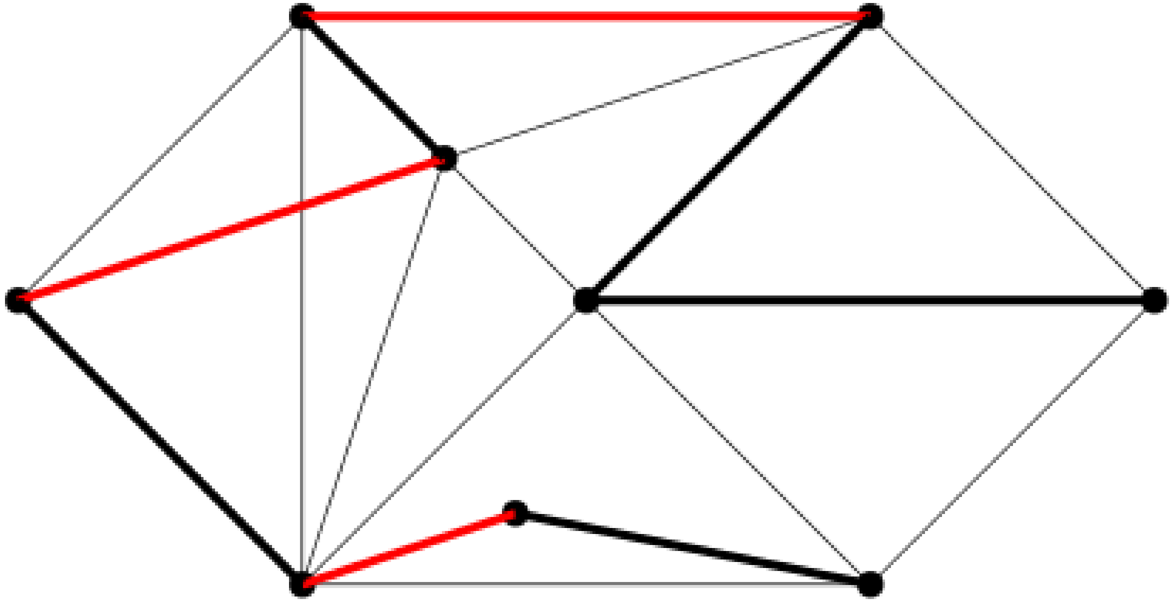
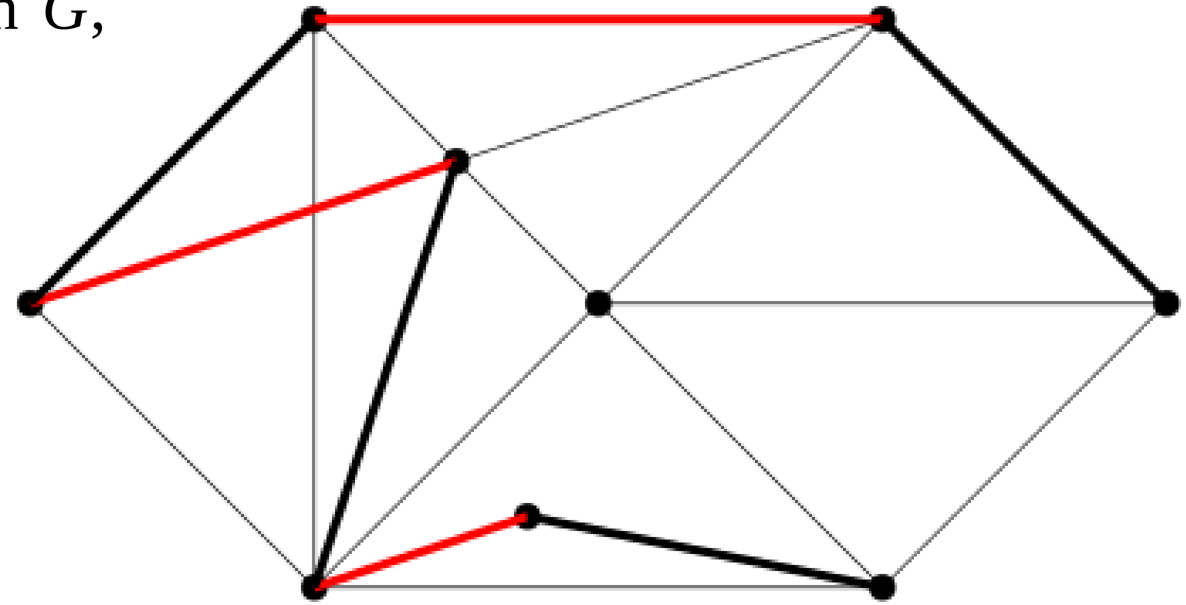


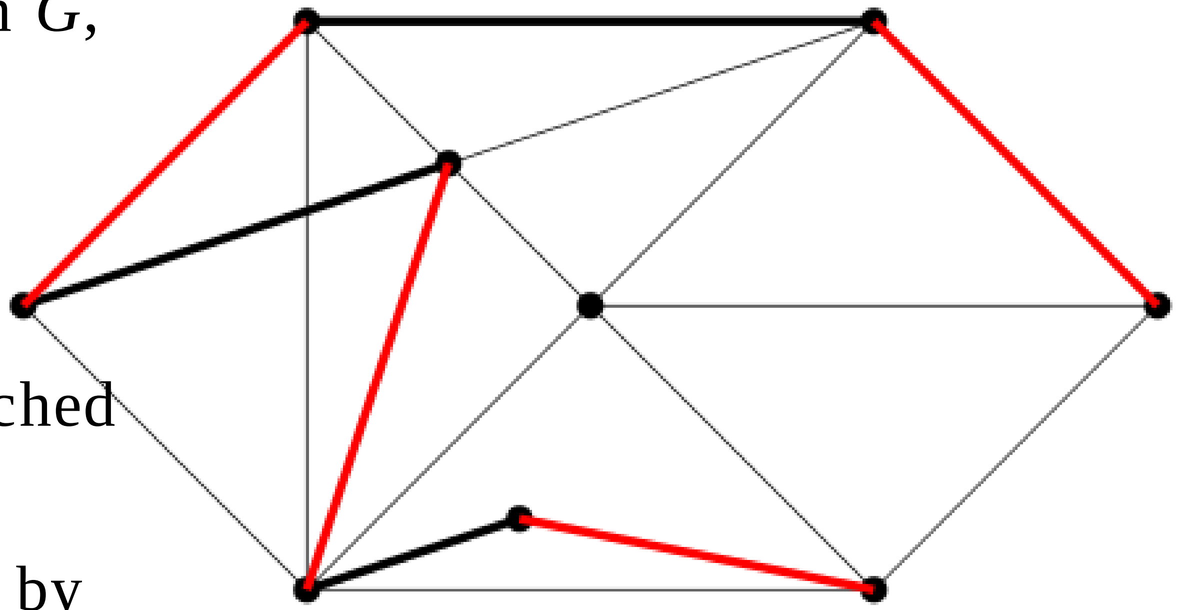
Observation

- If there is an M-augmenting path in G, then we can obtain a new matching M'. Also, $|M'| = |M| + 1$



Observation

- If there is an M-augmenting path in G, then we can obtain a new matching M'. Also, $|M'| = |M| + 1$
- ***Augmentation***: given an augmenting path, change its unmatched edges to matched and vice-versa, increasing the size of the matching by one



Augmenting path theorem

Claude Jacques Berge (1926-2002)

- A matching M is a maximum matching in graph G ***if and only*** if there are **no** M-augmenting paths in G
  - The theorem is also known as Berge's Optimality Criterion
- ($\Rightarrow$) Let M be a maximum matching
- It is trivial that there are no M-augmenting paths in G
- ($\Leftarrow$) Suppose there are no M-augmenting paths in G
- On the contrary, let us assume that M is not a maximum matching

Proof (Cont...)

- Suppose there is a matching M' with larger cardinality, I.e., $|M'| > |M|$
- Consider the symmetric difference of M and M' (i.e., only edges that are in either M or M' but not in both)
- $M' \bigoplus M = (M' \setminus M) \cup (M \setminus M')$
- Each vertex can be incident to at most two edges (one from M and one from M')
- Hence, the connected components in $M' \bigoplus M$ are alternating cycles or alternating paths

Proof (Cont...)

- On each such path or cycle, edges of M' and M alternate
- Each cycle contains the same number of edges in M' as in M
- As $|M'| > |M|$, there must be a path P for which both endpoints are incident to edges from M'
- P is an alternating path
- Which is a contradiction to our assumption
- Therefore, M is a maximum matching

Algorithm

THE MATCHING ALGORITHM

```
{
    1.  Start with any matching.
    2.  Find an augmenting path with respect to the current matching.
    3.  Augment the current matching.
    4.  Repeat the above two steps as long as possible.
}
```

How long does our algorithm take?

- **Correctness**: When the algorithm terminates we have a maximum matching
- In steps 2 and 3, we increase the size of the matching by 1
- How many times do steps 2 and 3 repeat?
- Questions:
  1. How to start with an initial matching in step 1?
  2. How to find out an augmenting path and how long does it take to find an augmenting path in step 2?
  3. What is the time spent in augmentation?

- Thank you!