Nondeterministic Polynomial Time

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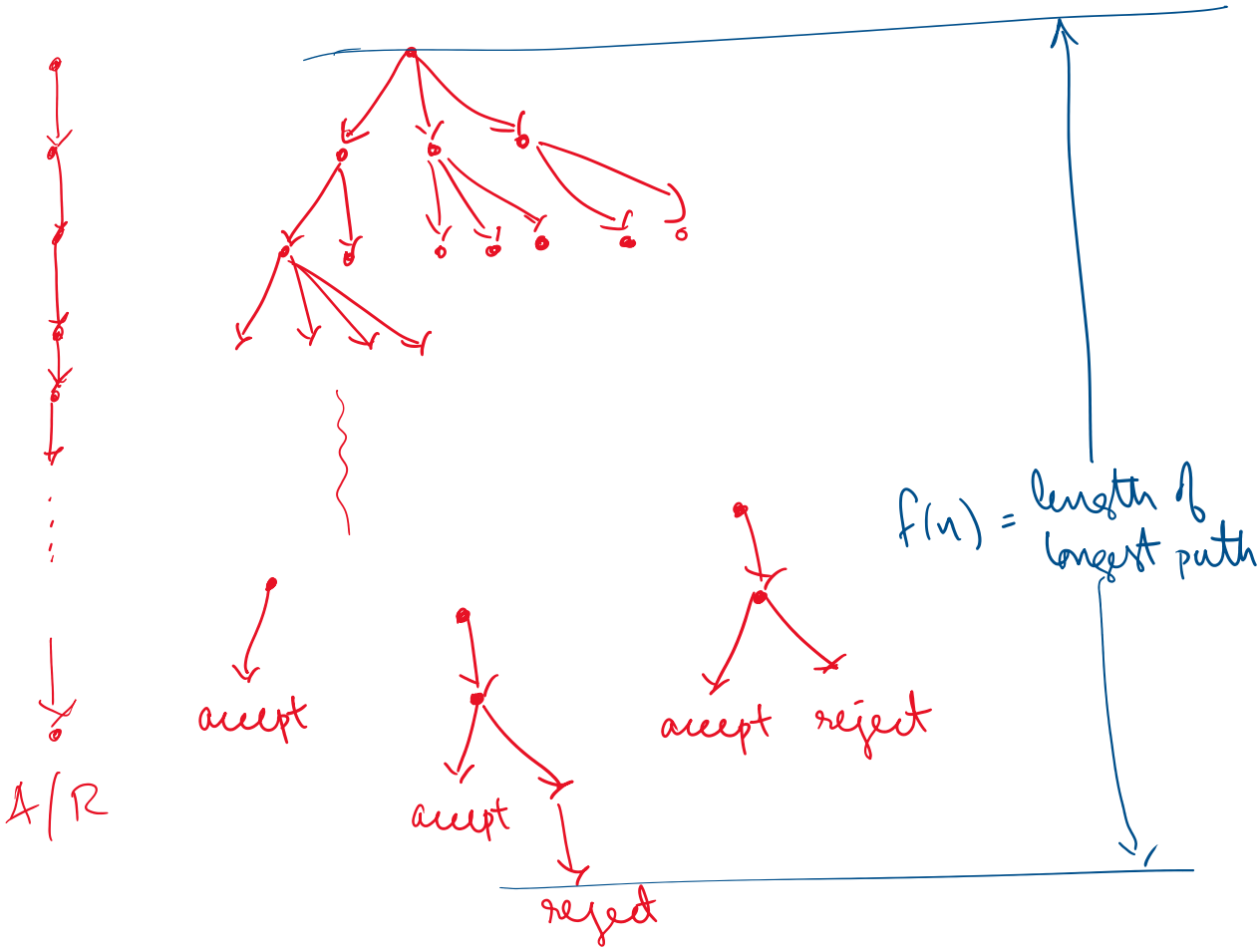
Def 7.9: Let N be a non-det TM that is a decider. The running time of N is $f: \mathbb{N} \to \mathbb{N}$ where $f(n)$ is the maximum no. of steps that N uses on any branch of its computation on an input of length n.



Def 7.21: $NTIME(t(n)) = \{L \mid L$ is decided by an NTM in $O(t(n))$ time$\}$

Def 7.22: $NP = \bigcup_{k=1}^{\infty} NTIME(n^k)$.

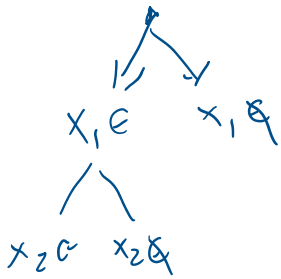
[illegible] resource model.

Def 1.[illegible]

$k=1$

This is equivalent to the guess and verify model.

$$\text{SUBSET-SUM} = \{ \langle S, t \rangle \mid S = \{x_1, x_2 \ldots x_k\},\ \exists T \subseteq \{1, 2, \ldots k\} \text{ s.t } \sum_{i \in T} x_i = t \}$$



Example: SUBSET-SUM $\in$ NP.

On input $\langle S, t \rangle$:

1. Non deterministically select / reject each of $x_1, x_2, x_3 \ldots x_k$.
2. Add all the selected x_i and verify if they add up to t.
3. If sum $= t$, then accept. Else, reject.

3-COLORABLE $= \{ \langle G \rangle \mid G$ is 3-colorable $\}$.

On input $\langle G \rangle$:

1. Go through the vertices $1, 2, \ldots n$, non-deterministically assigning colors R, G, B.
2. Go through each edge of G and check whether it is properly colored.

whether it is properly colored.

3. Accept if coloring is valid. Else reject.

Notice that a DTM can also be considered as an NTM.

So $\quad DTIME(t(n)) \subseteq NTIME(t(n))$

$$DTIME(n^k) \subseteq NTIME(n^k)$$

$$\underline{P \subseteq NP}$$

<u>P vs. NP question</u>: Is the above containment strict?

Is <u>$P = NP$</u> or Is <u>$P \subsetneq NP$</u>?

This is a big open question.

<u>SAT</u>: Short for Satisfiability

There is a Boolean formula with

Variables x_i, Negated variables $\bar{x}_i$

Literals

$$\boxed{x_1 \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge \bar{x}_3}$$

$x_1 \quad \bar{x}_1$

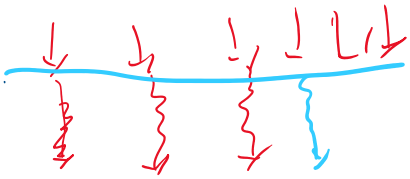
$x_1 \bar{x}_2 \vee \bar{x}_1 x_2$

AND $\wedge$ OR $\vee$

SAT = $\{ \langle \phi \rangle \mid \phi$ is a Boolean formula that is satisfiable $\}$

There is an assignment of True | False to the Boolean variables such that the whole formula evaluates to True.

CNF formula: Conjunctive Normal Form



$(x_1 \vee \bar{x}_2 \vee x_3 \vee x_4) \wedge (x_2 \vee x_3 \vee \bar{x}_5) \wedge (\bar{x}_3 \vee \bar{x}_6)$

$C_1 \wedge C_2 \wedge C_3 \wedge \ldots C_m$

AND's of OR's.

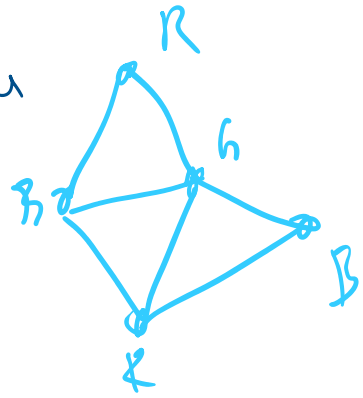
AND's of clauses. Each clause is an OR of literals.

CNF SAT = $\{ \langle \phi \rangle \mid \phi$ is a CNF formula that is satisfiable $\}$

3-CNF SAT or 3-SAT = $\{ \langle \phi \rangle \mid \phi$ is a CNF formula where each clause has exactly 3-literals, ϕ is satisfiable $\}$

SAT, CNFSAT, 3-SAT are all in NP.

We saw that SAT $\in$ NP, 3-COLORABLE $\in$ NP and SUBSET-SUM $\in$ NP. In each of these three, we "guess" a solution and then verify it. This makes one ask the question: Can we connect every nondeterministic machine in this format?



A <u>verifier</u> for a language L is a decider (algorithm) DTM V such that

$$L = \{ x \mid \exists y,\ V \text{ accepts } \langle x, y \rangle \}.$$

For x to be in L, there should exist y (y is called proof / certificate / witness) such that we can use y to verify that x is in the language.

Note that the existence of y is a lesser requirement than independently being able to test if $x \in L$.

Theorem 7.20: NP is the class of languages that have polynomial time verifiers.

$L \in NP \iff \exists$ a polynomial time verifier algorithm V such that

$$x \in L \iff \exists y,\ |y| = \text{poly}(|x|),\ V(x,y) = 1$$

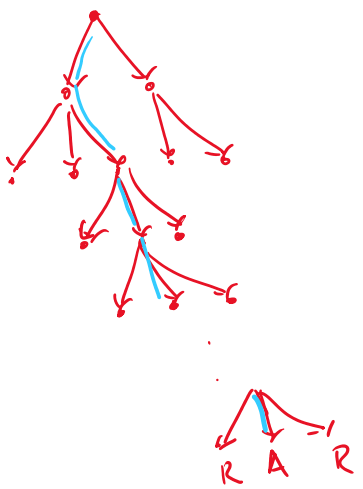
Contrast this with P. $x \notin L \Rightarrow$

$L \in P \iff \exists$ a polynomial time decider alg. A such that $x \in L \iff A(x) = 1$.

Proof: Two directions.

(1) NP $\Rightarrow$ Poly time verifiers.

Suppose $L \in NP$. Then there is a NTM N that runs in poly time, say n^k time, that decides L.



If $x \in L$, there is an accepting computation path. This path can be verified. The proof/certificate can be the encoding of this path

Verifier machine : V on input x, y.

1. Simulates N on input x. When N has to make a choice, V is guided by y.

2. Accept iff N accepts.

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(2) Poly time verifier $\Rightarrow$ NP.

Suppose there is a poly time verifier V. Let us say V runs in time n^k.

NTM machine N : (1) Guess a string y of length n^k.

(2) Run V on $\langle x, y \rangle$.

(3) Accept if and only if V accepts.

What is a proof / certificate ?

SUBSET-SUM : The subset itself

CNF-SAT : The satisfying assignment

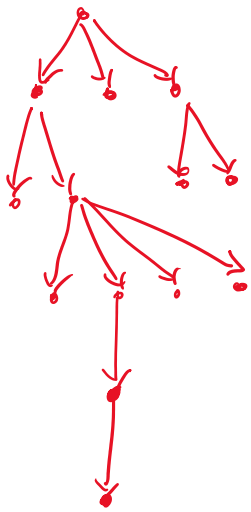
3-COLORING : The valid 3-coloring.

Theorem 7.11 : Let $t(n)$ be such that $t(n) \geq n$.

Every NTM which runs in $t(n)$ time has an equivalent

Every NTM which runs in $t(n)$ time has an equivalent DTM that runs in $2^{O(t(n))}$ time.

Proof: Similar to the proof that NTM is equivalent in computation to DTM. We simulate all computation paths and check if any one leads to an accept.



Suppose b is the largest number of children of any configuration. Note that b is a constant that depends only on the NTM and not the input to the NTM.

Then we have $\leq b^{t(n)}$ possible computation paths.

Total time needed for the simulation of NTM using DTM is $\leq b^{t(n)} * t(n)$

$$= 2^{\log b * t(n)} * 2^{\log t(n)}$$

$$= 2^{[t(n) \log b + \log t(n)]} = 2^{O(t(n))}$$

Note: Any NTM is equivalent to an NTM that makes ≤ 2 choices at every step, not $\leq b$.

makes ≤ 2 choices at every step, not ≤ 6.
This can be accomplished with only a constant blow up in time.