$x_0, x_1, x_2 \ldots x_n$ $\qquad x_1 - x_0 \neq x_2 - x_1$

$y = f(x)$

$x_1 = (x_0, x_n) - x_{n+1}$

Derivatives Using Newton's Divided Difference Formula

The Newton's divided difference formula (from Eq. (6.5.13) is

$$y = y_0 + \Delta_d y_0(x - x_0) + \Delta_d^2 y_0 (x - x_0)(x - x_1) + \Delta_d^3 y_0 (x - x_0)(x - x_1)(x - x_2) + \ldots\ldots + \Delta_d^{n-1} y_0 (x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1}) \qquad \ldots\ldots\ldots (8.5.1)$$

$$= y_0 + \Delta_d y_0(x - x_0) + \Delta_d^2 y_0 \{x^2 - (x_0 + x_1)x + x_0 x_1\}$$
$$+\Delta_d^3 y_0 \{x^3 - (x_0 + x_1 + x_2)x^2 + (x_0x_1 + x_0x_2 + x_1x_2)x - x_0x_1x_2\}$$
$$+ \Delta_d^4 y_0 \{x^4 - (x_0 + x_1 + x_2 + x_3)x^3 + (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)x^2 - (x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3)x + x_0x_1x_2x_3\} + \ldots\ldots$$

Differentiating both sides of Eq. (8.5.1) with respect to x, we get

$$\frac{dy}{dx} = \Delta_d y_0 + \Delta_d^2 y_0\{2x - (x_0 + x_1)\}$$
$$+\Delta_d^3 y_0 \{3x^2 - 2(x_0 + x_1 + x_2)x + (x_0x_1 + x_0x_2 + x_1x_2)\} +$$
$$\Delta_d^4 y_0 \{4x^3 - 3(x_0 + x_1 + x_2 + x_3)x^2 + 2(x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)x - (x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3)\} + \ldots\ldots \qquad \ldots\ldots\ldots (8.5.2)$$

$$\frac{d^2y}{dx^2} = 2\Delta_d^2 y_0 + \{6x - 2(x_0 + x_1 + x_2)\}\Delta_d^3 y_0 +$$
$$\{12x^2 - 6(x_0 + x_1 + x_2 + x_3)x + 2(x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)\}\Delta_d^4 y_0$$
$$+ \ldots\ldots \qquad \ldots\ldots\ldots (8.5.3)$$

$$\frac{d^3y}{dx^3} = 6\Delta_d^3 y_0 + \{24x - 6(x_0 + x_1 + x_2 + x_3)\}\Delta_d^4 y_0 + \ldots\ldots \qquad \ldots\ldots\ldots (8.5.4)$$

Example 8.7 Find the value of $f'(8)$ and $f''(9)$ from the following data, using an appropriate interpolation formula:

x:	4	5	7 (x_2)	10	11
$f(x)$:	48	100	294	900	1210

x	$f(x)$	Δd	$\Delta^2 d$	$\Delta^3 d y$	$\Delta^4 d y$
4	48	52	15	1	0
5	100	97	21	1	x
7	294	202	27	x	x
10	900	310	x	x	x
11	1210	x	x	x	

$$y = y_0 + \Delta y_0 (x - x_0) + \Delta^2 y_0 (x - x_0)(x - x_1) + \Delta^3 y_0 (x - x_0)(x - x_1)(x - x_2)$$

$$= 48 + 52 \cdot (x - 4) + 15 \cdot (x-4)(x-5) + 1(x-4)(x-5)(x-7)$$

$$y' = 0 + 52 + 15\left(x - 5 + (x-4)\right] + 1\left[(x-5)(x-7) + (x-4)(x-7) + (x-5)(x-4)\right]$$

$x = x_0,\ x = x_1$

$x = 8$

$$y'(8) = 52 + 15(2 \times 8 - 9) + 1\left((8-5)(8-7) + (8-4)(8-7) + (8-5)(8-4)\right]$$

$$= 176$$

Derivatives Using Lagrange's Interpolating Polynomial

The Lagrange's interpolating polynomial (from Eq. (6.5.2) is

$$y = f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \qquad \ldots\ldots(8.5.5)$$

Differentiating both sides of Eq. (8.5.5) with respect to x, we get

$$\Rightarrow \quad f'(x) = \frac{(x - x_1) + (x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0) + (x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0) + (x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$x = x_0$

$$\therefore\ f'(x_0) = \frac{(x_0 - x_1) + (x_0 - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x_0 - x_0) + (x_0 - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x_0 - x_0) + (x_0 - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$= \frac{(x_0 - x_1) + (x_0 - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x_0 - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x_0 - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \qquad \ldots\ldots(8.5.6)$$

Similarly,

$$f''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{2}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{2}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$x = x_0$

$$\therefore f''(x_0) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{2}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{2}{(x_2 - x_0)(x_2 - x_1)} y_2$$

..... (8.5.7)

Example 8.8 Find the value of $\cos 33^0$ (33°) using the following data by Lagrange's interpolation formula.

	x_0	x_1	x_2	x_3
x^0:	30	32	35	38
$y = \sin x^0$:	0.5	0.5299	0.5736	0.6157

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y' = f'(x) = \frac{(x-x_1)(x-x_2)+(x-x_1)(x-x_3)+(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$x = 33^\circ$

$$+ \frac{(x-x_0)(x-x_2)+(x-x_3)(x-x_0)+(x-x_3)(x-x_2)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$y'(33) = \cos(33^\circ)$

$x_i = x_0 + ih.$

MAXIMA AND MINIMA OF A TABULATED FUNCTION

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24} \times \Delta^4 y_0 \cdots \right]$$

For max. and minimum $\qquad u = \frac{x-x_0}{h}$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 \cdots = 0$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3	0.205	0.035	-0.016	0.000	0.001	-0.001
4	0.240	0.019	-0.016	0.001	0.000	×
5	0.259	0.003	-0.015	0.001	×	×
6	0.262	-0.012	-0.014	×	×	✓
7	0.250	-0.026	×	×	×	
8	0.224	×	×	×	✓	

$x_0 = 3, \quad h = 1$

$$y = y_0 + \frac{\Delta y_0}{\lfloor 1}u + \frac{\Delta^2 y_0}{\lfloor 2}u(u-1) + \cdots + \frac{\Delta^n y_0}{\lfloor n}u(u-1)\cdots(u-(n-1))$$

$$y = 0.205 + u(0.035) + \frac{(-0.016)}{\lfloor 2}u(u-1) + 0 + \frac{0.001}{\lfloor 4}u(u-1)(u-2)(u-3)$$

$$+ \frac{(-0.001)}{\lfloor 5}\left[u(u-1)(u-2)(u-3)(u-4)\right]$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0\right]$$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 0.035 - 0.008(2u-1) = 0$$

$$\Rightarrow \quad u = 2.6875$$

$$u = \frac{x-x_0}{h}$$

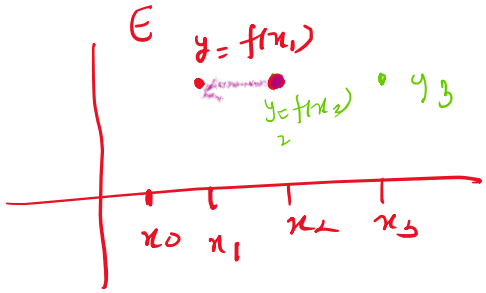
$$\Rightarrow \quad 2.6875 = \frac{x-3}{1} \Rightarrow x = 3+2.6875$$
$$= 5.6875$$

Remark: we have to take upto the term containing $\Delta^3 y$ only, otherwise $\frac{dy}{dx} = 0$ will be higher degree Eq^n which can not be solved easily.

Partial derivatives :→ (1) Shift operator E (displacement / Translation)

let $y_i = f(x_i)$ be a function

$x_i = x_0 + ih$, $i = 0, 1, 2 \ldots n$.



$$E f(x) = f(x+h)$$

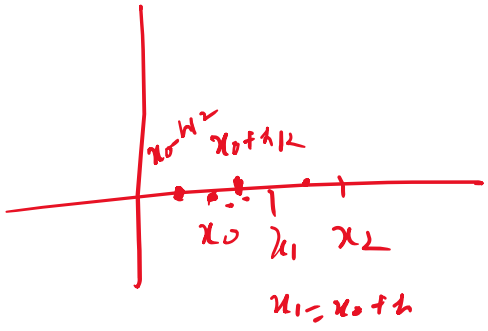
$$E[E f(x)] = E^2 f(x) = E[f(x+h)] = f(x+2h)$$

$$E^n f(x) = f(x+nh).$$

Inverse shift operator E^{-1},

$$E^{-1} f(x) = f(x-h)$$

$$E^{-1} f(x_i) = f(x_{i-1})$$



$$\boxed{E^{-n} f(x) = f(x-nh)}$$

Mean operator (Averaging operator μ)

$$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2) \right]$$

Differential operator (D

$$D f(x) = \frac{df(x)}{dx}.$$

$$D^2 f(x) = \frac{d}{dx}\left[\frac{df}{dx}\right] = \frac{d^2 f}{dx^2}$$

derivative.

$$D^n f(x) = f^{n}(x) = \frac{d^n f}{dx^n}$$

$$\Delta y(x) = y(x+h) - y(x) \qquad \Delta y(x_i) = y(x_{i+1}) - y(x_i)$$

$$\Delta y(x) = E y(x) - I y(x)$$

$$\Rightarrow \Delta y(x) = (E - I)\, y(x) \qquad E = \Delta + I$$

$$\boxed{\Delta = E - I}$$

$$\nabla y(x) = y(x) - y(x-h)$$
$$= I y(x) - E^{-1} y(x)$$

$$E^{-1} y(x) = y(x-h)$$

$$\nabla y(x) = (I - E^{-1})\, y(x) \qquad \forall x.$$

$$\Rightarrow \boxed{\nabla = I - E^{-1}}$$

$$D^0 = I$$

$$D f(x) = \frac{d}{dx} f(x)$$

$$f(x+h) = f(x) + \frac{h}{\lfloor 1} f'(x) + \frac{h^2}{\lfloor 2} f''(x) \cdots$$

$$= D^0 f(x) + \frac{h}{\lfloor 1} D f(x) + \frac{h^2}{\lfloor 2} D^2 f(x) \cdots$$

$$= \left[I + \frac{h}{\lfloor 1} D + \frac{h^2}{\lfloor 2} D^2 \cdots \right] f(x)$$

$x_1 - x_0 = h$

$x_i = x_0 + ih.$

$$e^{hD} \cdot f(x)$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{\lfloor 2} + \frac{\Delta^3}{\lfloor 3} \cdots \right]$$

$$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 \cdots \right]$$

$$\boxed{\delta f(x) = f(x + h/2) - f(x - h/2)}$$

$$\mu = \sqrt{1 + \frac{1}{4}\delta^2}$$

1

$$\delta = 2\sinh\left(\frac{hD}{2}\right)$$

(11)

$$D^2 = \frac{1}{h}\frac{\mu}{\sqrt{1+\frac{1}{4}\delta^2}} \cdot 2\sinh^{-1}(\delta/2)$$

$$\delta/2 = \sinh(hD/2)$$

$$\sinh^{-1}(\delta/2) = hD/2$$

$$D = \frac{2}{h}\sinh^{-1}(\delta/2)$$

$$= \frac{2}{h}\frac{\mu}{\mu} \cdot \sinh^{-1}(\delta/2)$$

$$D = \frac{1}{h} \cdot \frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}}\left[2\sinh^{-1}(\delta/2)\right]$$

$$= \frac{1}{h}\mu\left[1+\delta^2/4\right]^{-1/2}\left[\delta - \frac{1^2}{2^2 \cdot \lfloor 3}\delta^3 - \cdots\right]$$

$$= \frac{1}{h}\mu\left(1-\frac{1}{8}\delta^2+\frac{3}{8}\delta^4 - \cdots\right)\left(\delta - \frac{1^2}{2^2 \cdot \lfloor 3}\delta^3 - \cdots\right)$$

$$D^2 = \frac{1}{h}\mu\left(\delta - \frac{1^2}{\lfloor 3}\delta^3 + \frac{1^2 \cdot 2^2}{\lfloor 5}\delta^5 - \cdots\right)$$

$$Df(x_k) = f'(x)\Big|_{x=x_k} = \frac{1}{h}\left[\Delta f(x_k) - \frac{1}{2}\Delta^2 f_k + \cdots\right]$$

$$= \frac{1}{h}\mu\left[\delta f(x_k) - \frac{1^2}{\lfloor 3}\delta^3 f(x_k) + \frac{1^2\cdot 2^2}{\lfloor 5}\delta^5 f(x_k) - \cdots\right]$$

$$f'(x_k) = \frac{1}{h}\Delta f(x_k) = \frac{1}{h}\left[f(x_{k+1}) - f(x_k)\right]$$

$$f'(x_k) = \frac{1}{h}\mu\left[\delta f(x_k)\right] = \frac{1}{h}\mu\left[f(x_k + h/2) - f(x_k - h/2)\right]$$

$$= \frac{1}{h}\,\frac{f(x_{k+1}) - f(x_{k-1})}{2h}$$

$$f''(x_k) = D^2 f(x_k) = \frac{1}{h^2}\left[\delta - \frac{1^2}{2^2\cdot\lfloor 3}\delta^3 - \cdots\right]^2 f_k$$

$$= \frac{1}{h^2}\delta^2 f_k = \frac{1}{h^2}\left[E^{+1/2} - E^{-1/2}\right]^2 f_k$$

$$\frac{1}{h^2}\left(E - 2 + E^{-1}\right) f_k = \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2}$$

$$\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)} = \begin{cases} \dfrac{f_{i+1,j} - f_{i,j}}{h} & \Delta \\[2ex] \dfrac{f_{i,j} - f_{i-1,j}}{h} & \nabla \\[2ex] \dfrac{f_{i+1,j} - f_{i-1,j}}{2h} & \end{cases}$$

$$\left(\frac{\partial f}{\partial y}\right)_{(x_i, y_i)} = \begin{cases} \dfrac{f_{i,j+1} - f_{i,j}}{k} & \Delta \\ \dfrac{f_{i,j} - f_{i,j-1}}{k} & \nabla \\ \dfrac{f_{i,j+1} - f_{i,j-1}}{2k} & \delta \end{cases}$$

$$=$$

$$f(x+h) = e^{hD} \cdot f(x)$$

$$E f(x) = e^{hD} \cdot f(x) \quad \forall x$$

$$\boxed{E = e^{hD}}$$

$$\Delta + I = e^{hD}$$

$$hD = \log(\Delta + I)$$

$$hD = \Delta - \frac{\Delta^2}{(2)} + \frac{\Delta^3}{(3)} - \cdots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{(1)}\right]$$

$y = f(x, y)$

$(x_i, y_j) = f_{ij}$

$x_i = x_0 + ih$

$y_j = y_0 + jk$