Differential Equations (MA 1150)

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Lecture 6

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Second order linear ODE

Fundamental solutions : Linearly independent solution

Second order ODE with Constant coefficients

Distinct real roots

Repeated real roots

Complex conjugate roots

Wronskian

Non-homogeneous second order linear ODE

Section 1

Second order linear ODE

Linear Second Order ODE

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Let us try to understand how to solve homogeneous linear second order ODE:

$$y'' + p(x)y' + q(x)y = 0.$$

Theorem: Suppose $p(x)$ and $q(x)$ are continuous function on an open interval (a, b), let x_0 be any point in (a, b). Then the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = y_0,\ y'(x_0) = y_1$$

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Step 2: Verify $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of $y'' - y = 0$.

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Step 3: Verify that if c_1 and c_2 are arbitrary constants, $y = c_1 e^x + c_2 e^{-x}$ is a solution of (2) on $(-\infty, \infty)$.

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$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

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Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of (2) on $(-\infty, \infty)$.

Since $y = c_1 e^x + c_2 e^{-x}$ is a solution of ODE $y'' - y = 0$ on $(-\infty, \infty)$.

Setting $y(0) = 1$ and $y'(0) = 3$, we get

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Therefore $y = 2e^x - e^{-x}$ is the unique solution on $(-\infty, \infty)$.

Subsection 1

Fundamental solutions : Linearly independent solution

Theorem: Consider the homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

If y_1 and y_2 are solutions of (3) on (a, b), then any linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{4}$$

is also a solution of (3) on (a, b).

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Definition: If every solution of (3) on (a, b) can be written as a linear combination of y_1 and y_2 as in (4), we say that $\{y_1, y_2\}$ is a fundamental set of solutions of (3) on (a, b).

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Moreover: We say that (4) is general solution of (3) on (a, b).

Theorem: Suppose p and q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5}$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

Revision: Linearly independent

$V = C^\infty(a, b)$: Vector space of real valued differentiable functions of any order.

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By definition if $c_1 f_1 + c_2 f_2 = 0$ for all $x \in (a, b)$, then c_1 and c_2 must be zero.

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Why are we discussing this Linear Algebra problem in Differential Equations course?

Section 2

Second order ODE with Constant coefficients

Second Order Linear Homogeneous ODE with constant coeff.

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Let us look for a solution of the type e^{mx}, where m is a constant. Then,

$$\begin{aligned} am^2 e^{mx} + bme^{mx} + ce^{mx} &= 0, \\ e^{mx}\left(am^2 + bm + c\right) &= 0. \end{aligned}$$

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$$\begin{aligned} am^2e^{mx} + bme^{mx} + ce^{mx} &= 0, \\ e^{mx}\left(am^2 + bm + c\right) &= 0. \end{aligned}$$

Since $e^{mx} \neq 0$ for all $x \in \mathbb{R}$ and for any constant m, we get that

$$am^2 + bm + c = 0.$$

The quadratic polynomial

$$p(m) = am^2 + bm + c = 0.$$

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The roots of the characteristic equation are given by $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

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Case 2: When $b^2 - 4ac = 0$, the characteristic equation $p(m) = 0$ has repeated real roots.

Case 3: When $b^2 - 4ac < 0$, the characteristic equation $p(m) = 0$ has two distinct complex roots which are conjugates.

Subsection 1

Distinct real roots

Distinct real roots case

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$$y'' + 6y' + 5y = 0 \tag{7}$$

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Note that any linear combination of the above two is also a solution

$$y = c_1 e^{-x} + c_2 e^{-5x}.$$

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If we solve for c_1 and c_2 using the initial conditions, we get $c_1 = 4$ and $c_2 = -1$.

Thus the solution to IVP is

$$y = 4e^{-x} - e^{-5x}.$$

Subsection 2

Repeated real roots

A repeated real root case

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$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ &= u''e^{-3x} = 0. \end{aligned}$$

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This shows that $u(x) = c_1x + c_2$, where c_1 and c_2 are constants. Therefore

$$y = e^{-3x}(c_1 + c_2x) \tag{10}$$

is a solution of (9).

A repeated real root case

Example: Solve the initial value problem

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Recall

$$y = e^{-3x}(c_1 + c_2 x) \tag{12}$$

is a solution of (27). Using initial conditions, get $c_1 = 3$ and $c_2 = 10$.

Thus solution of IVP is

$$y = e^{-3x}(3 + 10x). \tag{13}$$

Subsection 3

Complex conjugate roots

Two distinct complex conjugate roots case

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It is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (14).

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In fact that is true. But they are complex valued solutions and we want real solutions.

Two distinct complex conjugate roots case

Let us write

$$y_1 = e^{(-2+3i)x} = e^{-2x}e^{3ix} \quad \text{and} \quad y_2 = e^{(-2-3i)x} = e^{-2x}e^{-3ix}.$$

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Assume that linear combination of y_1 and y_2 can be written as $y = ue^{-2x}$, where u depends upon x. Let us find u.

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Since $y = ue^{-2x}$ then

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x}.$$

Now from ODE, we get

$$\begin{aligned} y'' + 4y' + 13y &= e^{-2x}\left[(u'' - 4u' + 4u) + 4(u' - 2u) + 13u\right] \\ &= e^{-2x}\left[u'' - (4-4)u' + (4-8+13)u\right] = e^{-2x}(u'' + 9u). \end{aligned}$$

Therefore $y = ue^{-2x}$ is a solution of (14) if and only if

$$u'' + 9u = 0.$$

Two distinct complex conjugate roots case

The general solution of the equation $u'' + 9u = 0$ is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Check that $\cos 3x$ and $\sin 3x$ are solutions of ODE $u'' + 9u = 0$, and hence their linear combinations.

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \tag{15}$$

is a solution of (14).

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Using initial conditions, get $c_1 = 2$ and $c_2 = \frac{1}{3}$ and hence the solution is

$$y = e^{-2x}(2\cos 3x + \frac{1}{3}\sin 3x).$$

Theorem: Let $p(m) = am^2 + bm + c$ be the characteristic polynomial of

$$ay'' + by' + cy = 0. \tag{17}$$

Then:

(a) If $p(m) = 0$ has distinct real roots m_1 and m_2, then the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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(c) If $p(m) = 0$ has complex conjugate roots $m_1 = \lambda + i\omega$ and $m_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution is

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

How to prove linearly independent!!!

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- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$.

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Let $p(m) = am^2 + bm + c$ be the characteristic polynomial of

$$ay'' + by' + cy = 0. \tag{18}$$

Then $p(m) = 0$ can have at most two distinct roots.

Observe that solution to ODE (18) will have linear combination of the following form

- $e^{m_1 x}$ and $e^{m_2 x}$;
- e^{mx} and xe^{mx}, or
- $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$.

Something to think about!!!

Suppose $p(m) = a_0 m^\ell + a_1 m^{\ell-1} + \cdots + a_{m-1} m + a_m = 0$ be the characteristic polynomial of

$$a_0 y^{(\ell)} + a_1 y^{(\ell-1)} + \cdots + a_{m-1} y' + a_m y = 0 \tag{19}$$

Recall: Second Order Linear ODE

Theorem: Suppose $p(x)$ and $q(x)$ are continuous function on an open interval (a, b), let x_0 be any point in (a, b). Then the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = w_0,\ y'(x_0) = w_1 \tag{20}$$

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$$y(x) = c_1 y_1 + c_2 y_2.$$

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The system of equation

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= w_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= w_1 \end{aligned} \tag{21}$$

has a solution (c_1, c_2) for every choice of (w_0, w_1).

Second Order Linear ODE

Theorem: Suppose $p(x)$ and $q(x)$ are continuous on (a, b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{22}$$

on (a, b), and define

$$W = y_1 y_2' - y_1' y_2. \tag{23}$$

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$$W(x) = W(x_0) e^{-\int_{x_0}^{x} p(t)\, dt}, \quad a < x < b. \tag{24}$$

Therefore either W has no zeros in (a, b) or $W \equiv 0$ on (a, b).

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Proof Given $W = y_1 y_2' - y_1' y_2$. On differentiating (23), we get

$$W' = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2.$$

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Also notice that

$$y_1'' = -py_1' - qy_1 \quad \text{and} \quad y_2'' = -py_2' - qy_2.$$

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Proof (continued...) Simplify the expression of W' and get

$$\begin{aligned} W' &= -y_1(py_2' + qy_2) + y_2(py_1' + qy_1) \\ &= -p(y_1y_2' - y_2y_1') - q(y_1y_2 - y_2y_1) \\ &= -p(y_1y_2' - y_2y_1') = -pW. \end{aligned}$$

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- W decides linearly independence of y_1 and y_2 (important!).

Subsection 4

Wronskian

Definition: The Wronskian of y_1 and y_2 is

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Example: Solve IVP

$$y'' + 6y' + 5y = 0, \text{ where } y(0) = 3, y'(0) = 1. \tag{26}$$

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Since $p(t) = 6$ and $W(0) = -4$, we get $W(x) = -4e^{-6x}$.

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Recall that the solution to ODE is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

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Section 3

Non-homogeneous second order linear ODE

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$$y'' + p(x)y' + q(x)y = f(x), \tag{29}$$

where $f(x) \neq 0$ identically.

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Theorem: Suppose $p(x)$, $q(x)$ and $f(x)$ are continuous functions on an open interval (a, b), let x_0 be any point in (a, b). Then the initial value problem

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We know how to solve the associated homogeneous equation, initial value problem

$$y'' + p(x)y' + q(x)y = 0 \quad y(x_0) = w_0, \quad y'(x_0) = w_1$$

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Set-up: The homogeneous IVP

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Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7.$$

Particular solution of non-homogeneous second order linear ODE

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Thus $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation.

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The general solution of (30) is

$$y = y_p + \cos x + 7 \sin x. \tag{31}$$

Any guess for y_p.

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Any guess for y_p. take $y_p = 1$.