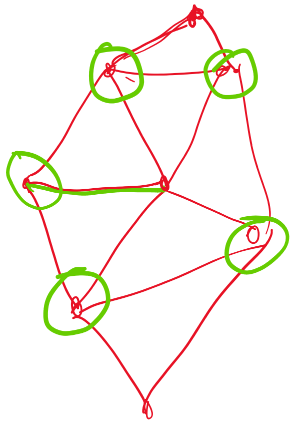
Other NP Complete Problems

18 April 2021 23:09

We've seen SAT, CNF-SAT, 3-SAT are NP-complete. Let us see some other languages.

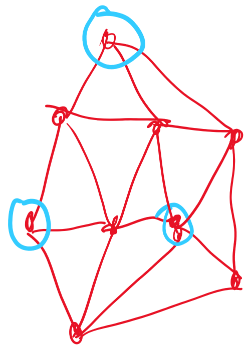
CLIQUE: (1) CLIQUE $\in$ NP

$$\underbrace{(2)\ 3\text{-SAT} \le_p \text{CLIQUE}}_{\forall A \in NP,\ A \le_p \text{CLIQUE}}$$



Vertex Cover: Given a graph $G = (V, E)$, a subset $U \subseteq V$ is a vertex cover if $\forall e \in E$, $\exists u \in U$ that is incident on e.

VERTEX-COVER = { ⟨G, k⟩ | G is an undirected graph that has a vertex cover of size k }



VERTEX-COVER is NP-complete.

(1) VERTEX-COVER $\in$ NP. Easy

(2) 3-SAT $\le_p$ VERTEX-COVER.

(Check the book)

Easier reduction: CLIQUE $\le_p$ IND-SET $\le_p$ VERTEX-COVER.

Hamiltonian Path : Is there a path from s to t in graph G, which goes through each vertex exactly once?

HAM-PATH = $\{ \langle G, s, t \rangle \mid G$ is a directed graph, G has a Ham path from s to t $\}$

HAM-PATH $\in$ NP. Easy.

* Guess $n-2$ vertices : $v_1, v_2, \dots v_{n-2}$.
* Check $s v_1 v_2 \dots v_{n-2} t$ is a path and that $s, v_1, v_2 \dots v_{n-2}, t$ are each distinct.

3-SAT $\leq_p$ HAM-PATH.

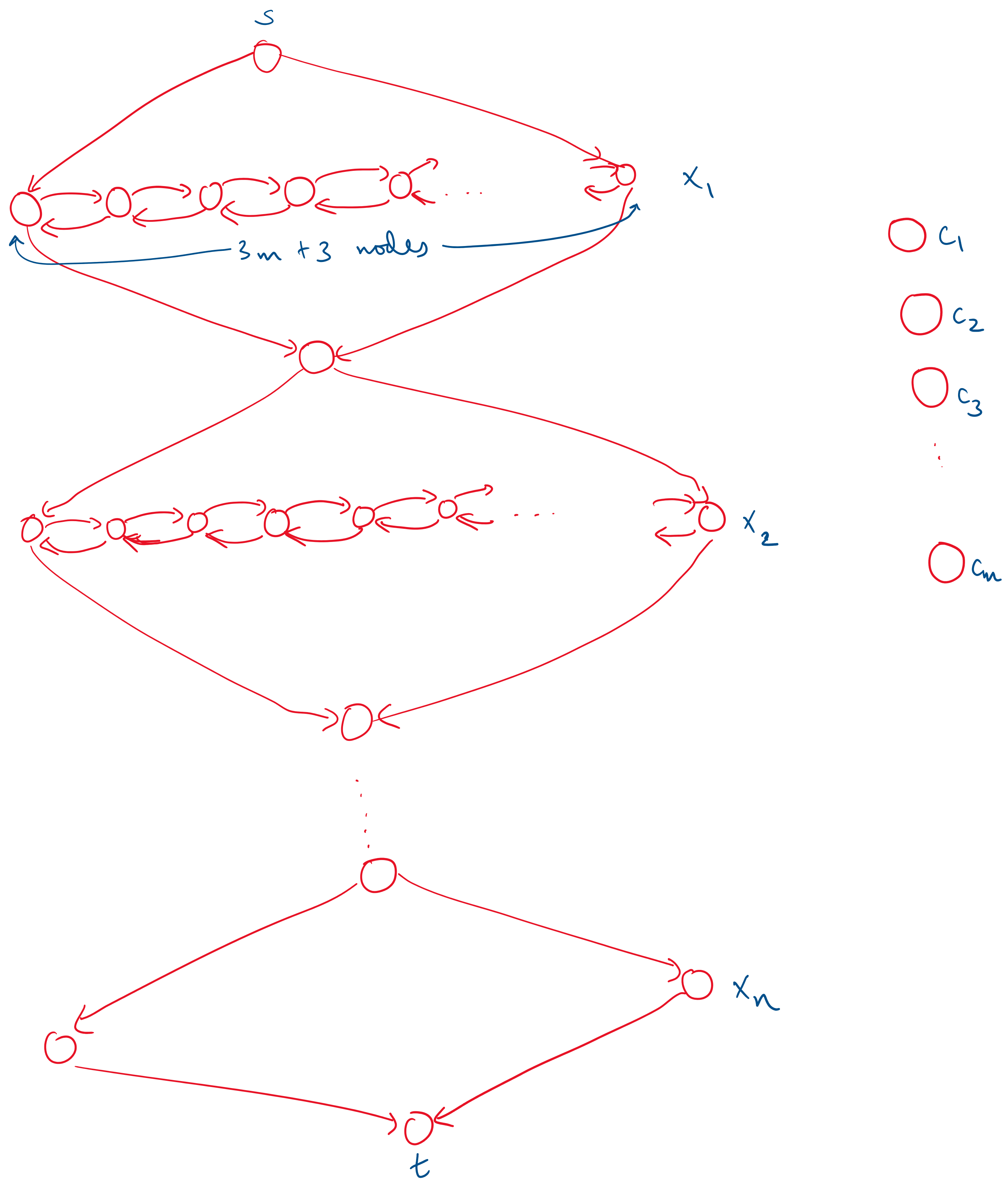
Given ϕ, construct $\langle G, s, t \rangle$ such that

$$\langle \phi \rangle \in \text{3-SAT} \iff \langle G, s, t \rangle \in \text{HAM-PATH}.$$

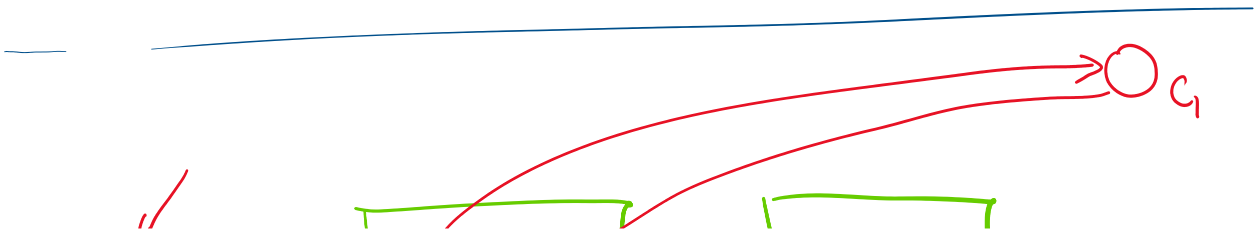
Let $\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \dots \wedge (a_m \vee b_m \vee c_m)$

m clauses, n variables.

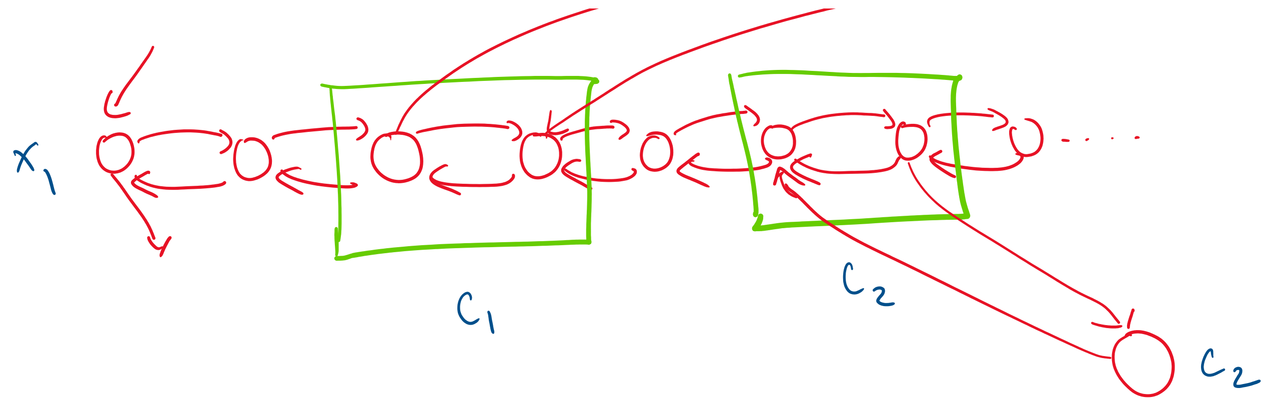
Each a_i, b_i, c_i is x_j or $\bar{x}_j$ for some j.



s
x_1
3m + 3 nodes
x_2
x_n
t
c_1
c_2
c_3
c_m



c_1

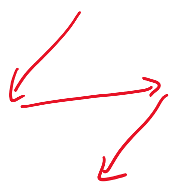


x_1 appears in C_1

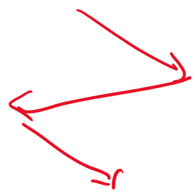
$\bar{x}_1$ appears in C_2

How can we go from s to t? How many ways are there? At each x_i "diamond", we can go zigzag or zag zig

This will cover all the nodes except the clauses. To cover the clause nodes, we need to take detours.



: ZIGZAG. We can cover clauses in which x_i is present.



: ZAG-ZIG We can cover clauses in which $\bar{x}_i$ is present.

Let $\phi \in SAT$. That is, $\exists$ a satisfying assignment.

Let $\phi \in SAT$. That is, $\exists$ a satisfying assignment. $\forall$ Clauses C_j, $\exists$ a literal x_i or $\overline{x_i}$ which is true.

For each literal x_i, zigzag if x_i is true in the satisfying assignment. Zagzig if x_i is false. Each clause can be covered by a detour, from one of the true variables in it.

So G has an s-t Ham. path.

Suppose G has an s-t Ham path.
There is a path from s to t that goes through all the vertices.

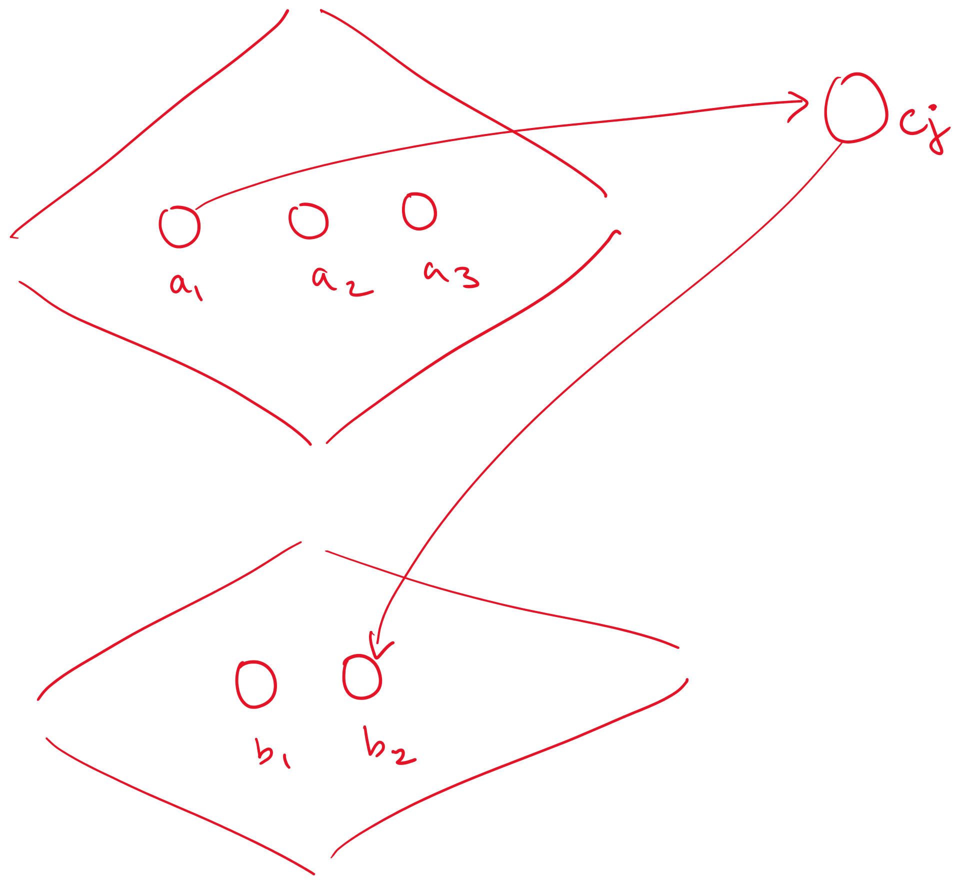
If this path is normal (goes through all the diamonds in order from top to bottom), set each variable to True / False accordingly.
Each clause is covered and has to have at least one true literal.

How can a Ham path be not normal?

How can a Ham path be not normal?

This can only happen through C_j nodes.

Go into C_j from x_1, but go out to x_{10}, say.



Either a_2 or a_3 is separator node.

- If a_2 is separator, then only way to enter a_2 is through a_1 (covered already) or a_3. If we enter a_2 through a_3, then no exit possible.

- If a_3 is separator, then only way

- If a_3 is separator, then only way to enter a_2 will be through a_1 (covered), c_j (covered) or a_3. If entering via a_3, then no exit possible.

So the path has to be normal. So we can can assign each x_i to T/F as per whether the path zig-zags or zag-zigs.

ϕ will be satisfiable.

Q1: Show UHAMPATH is NP-complete.

Q2: Show HAMCYCLE is NP-complete.