Lectures 6 and 7
Linear Transformation and Rank-Nullity Theorem

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Linear Transformations

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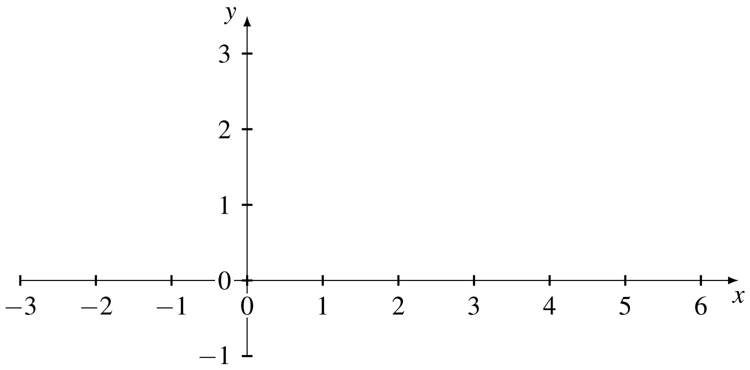

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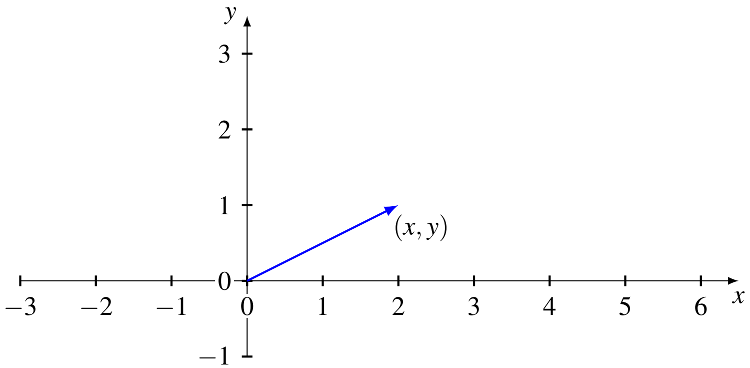

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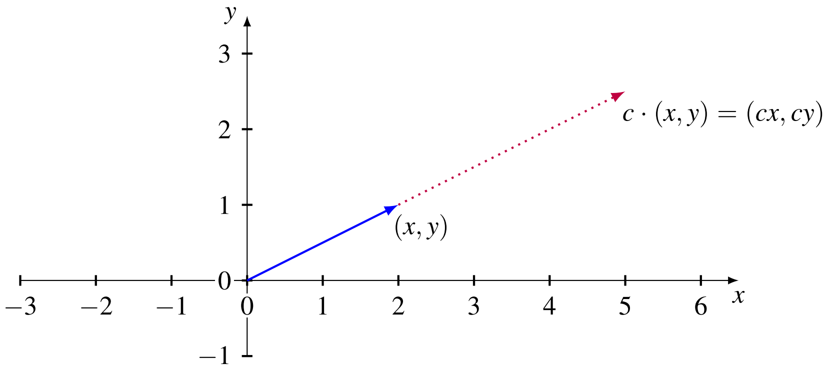

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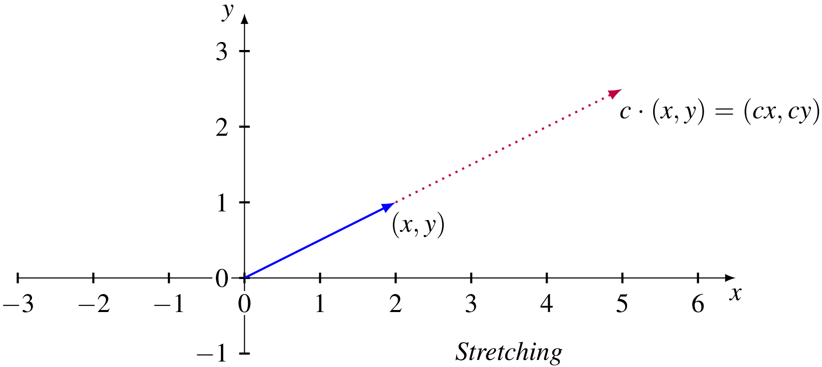

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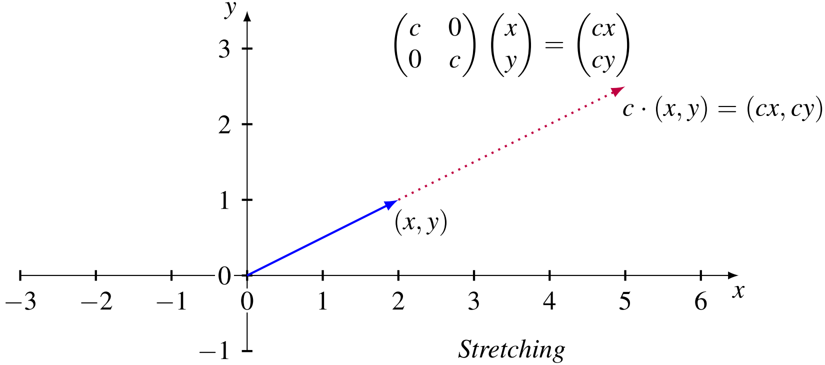
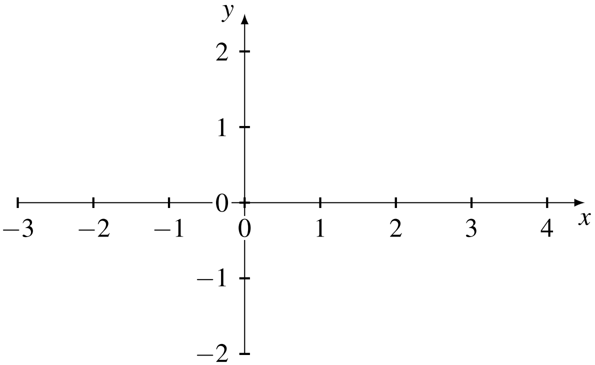


Stretching

Reflection with x-axis as mirror



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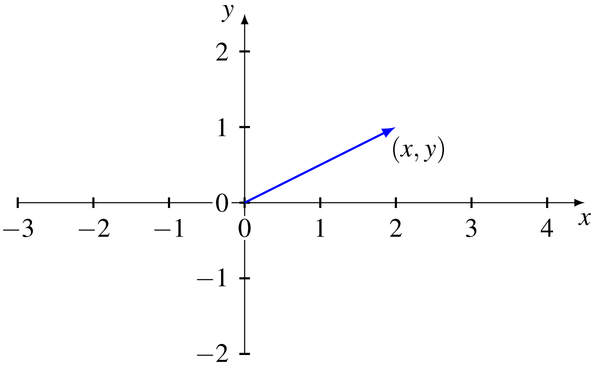

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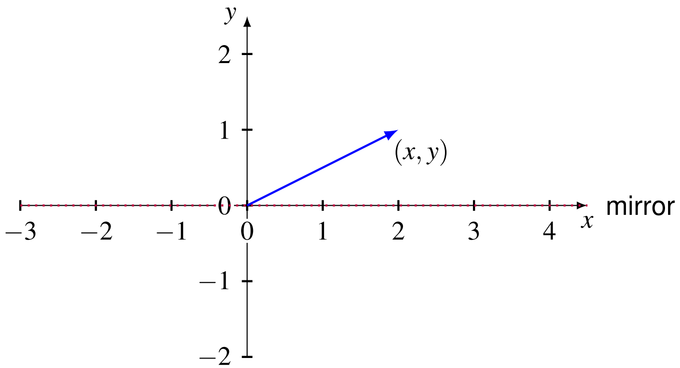

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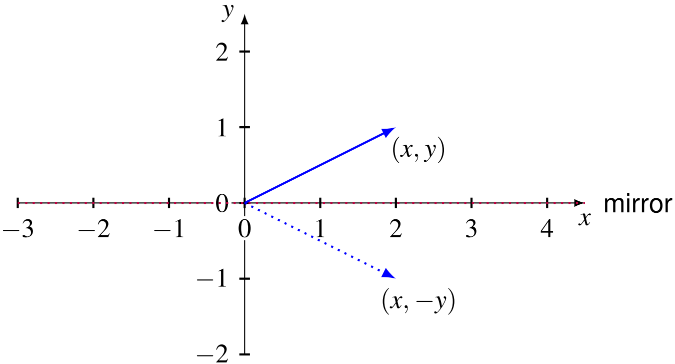

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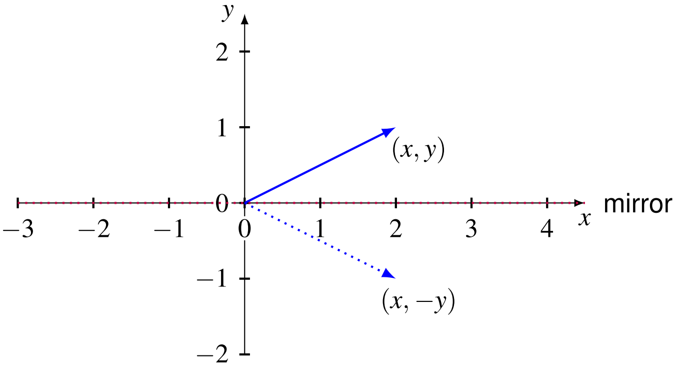

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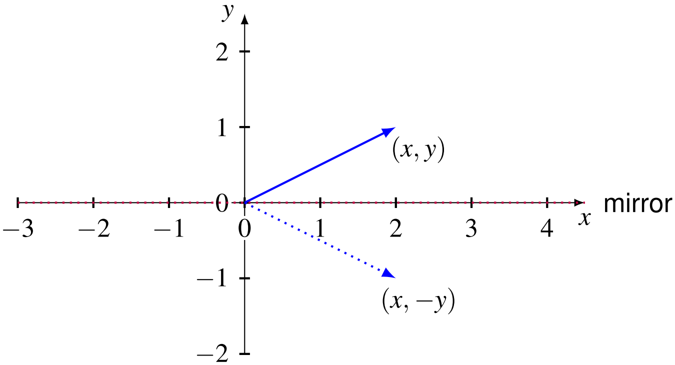

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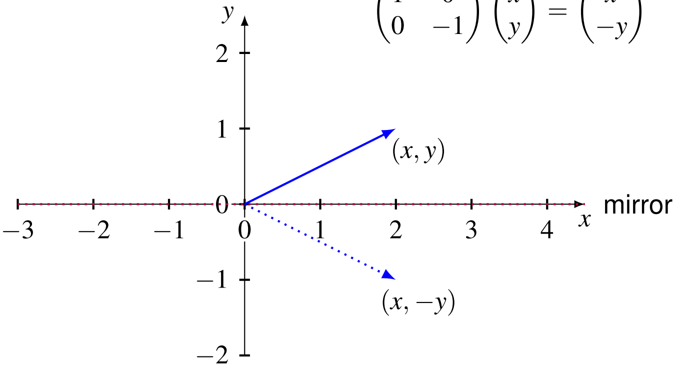
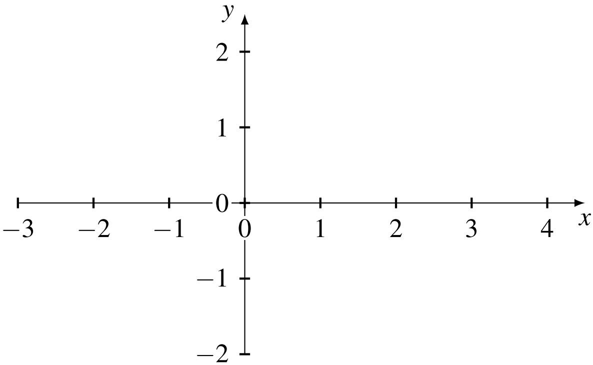


Projection on the x-axis



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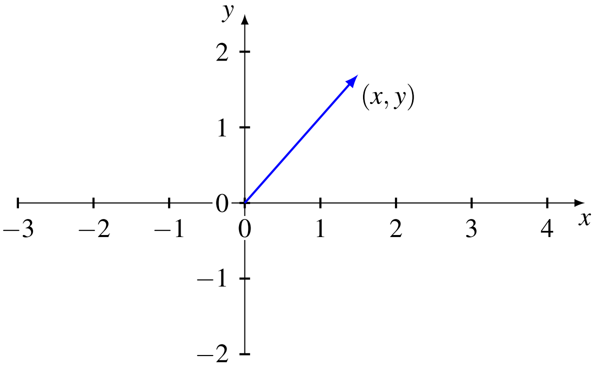

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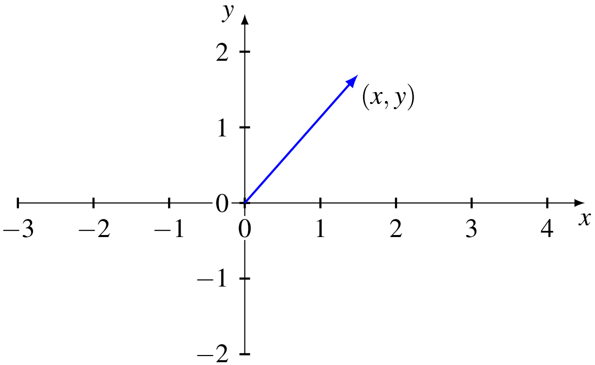

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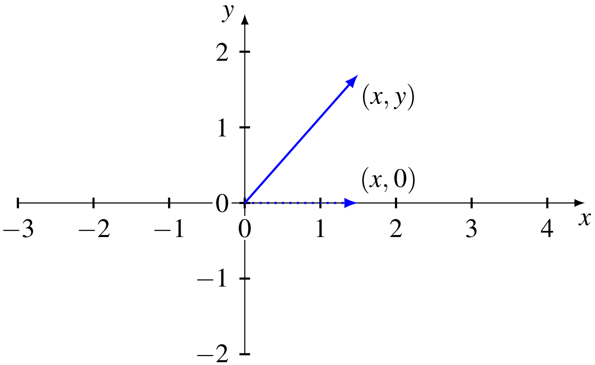

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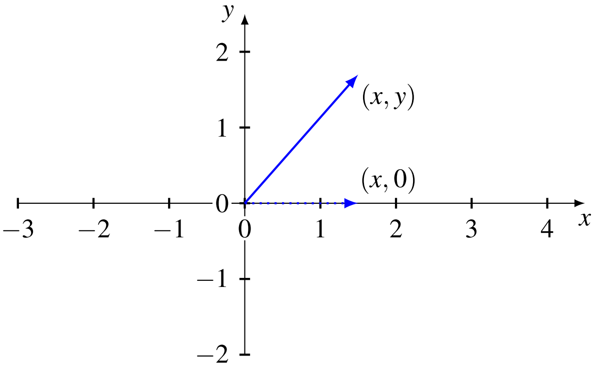

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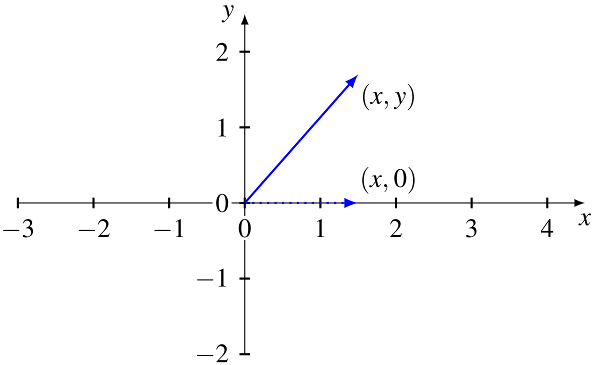

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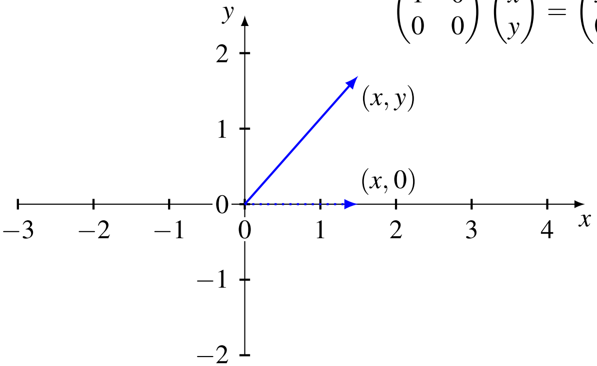

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$S : \mathbb{R}^2 \longrightarrow \mathbb{R}$

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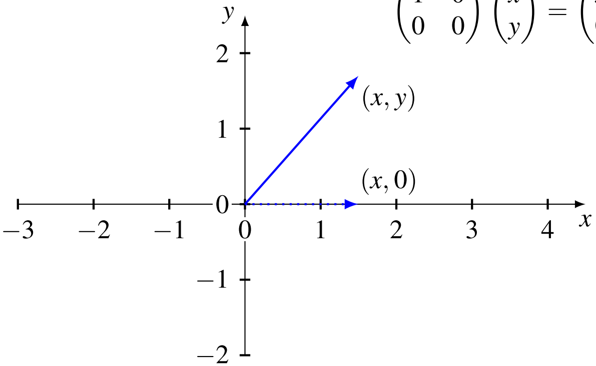

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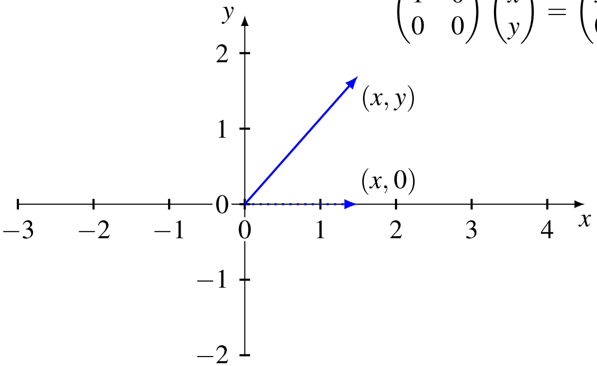

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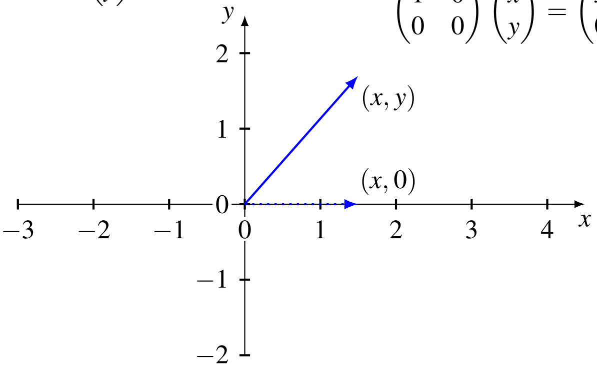

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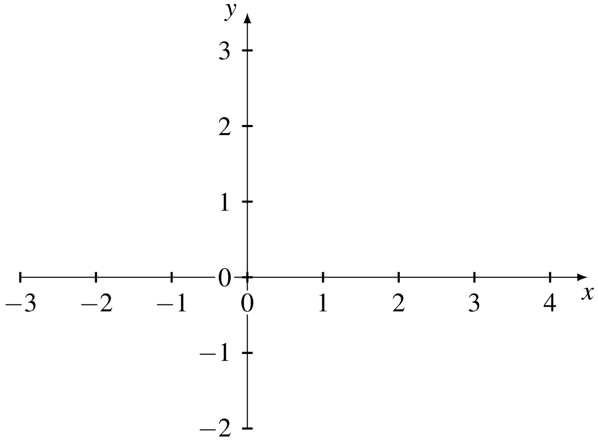

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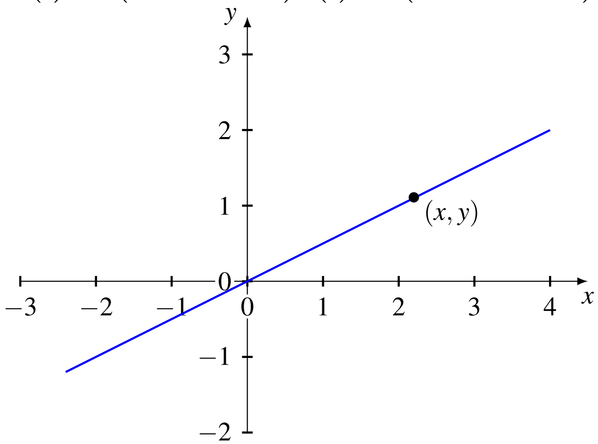

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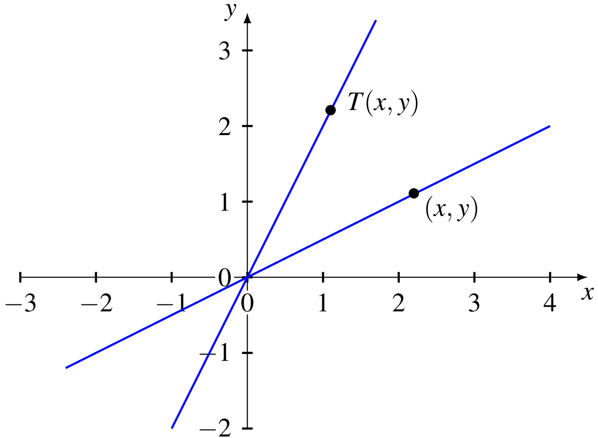

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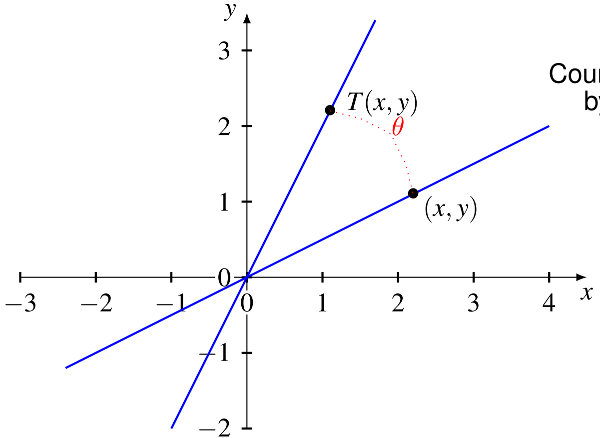

Counterclockwise rotation by an angle θ

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$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$$

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Starting with an $m \times n$ matrix A over $\mathbb{R}$, one can construct a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(X) := AX$ for all $X \in \mathbb{R}^n$.

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Remark. A linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined by its action on $\{e_1, \ldots, e_n\}$, i.e., by $T(e_i)$ for all $1 \leqslant i \leqslant n$.

Corollary

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By the last theorem, there exists A such that the following diagram is commutative.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow] & W \arrow[d, leftrightarrow] \\ \mathbb{R}^n \arrow[r, dashed, "A"] & \mathbb{R}^m \end{tikzcd}.$$

Matrix representation of a linear map $T : V \to W$

Theorem

Let $T : V \to W$ be a linear transformation. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{w_1, \ldots, w_m\}$ be two ordered bases of V and W respectively. Then there exists an $m \times n$ matrix A such that T can be represented by A, i.e., $A[v]_{\mathcal{B}} = [T(v)]_{\mathcal{B}'}$ for every $v \in V$. The ith column of A, which is same as Ae_i, will be obtained by $[T(v_i)]_{\mathcal{B}'}$.

Sketch of the Proof.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow] & W \arrow[d, leftrightarrow] \\ \mathbb{R}^n & \mathbb{R}^m \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, "T"] \arrow[d, leftrightarrow] & T(v) \arrow[d, leftrightarrow] \\ {[v]_{\mathcal{B}}} & {[T(v)]_{\mathcal{B}'}} \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow] & W \arrow[d, leftrightarrow] \\ \mathbb{R}^n \arrow[r, dashed, "???"] & \mathbb{R}^m \end{tikzcd}.$$

By the last theorem, there exists A such that the following diagram is commutative.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow] & W \arrow[d, leftrightarrow] \\ \mathbb{R}^n \arrow[r, dashed, "A"] & \mathbb{R}^m \end{tikzcd}.$$

Observe that the ith column of A is $[T(v_i)]_{\mathcal{B}'}$.

A linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by $T(e_i)$

Theorem

Consider the standard basis $\{e_1, \ldots, e_n\}$ *of* $\mathbb{R}^n$.

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Consider the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$.
Then any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined by $T(e_i)$ for all $1 \leqslant i \leqslant n$.

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Then any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ *is uniquely determined by* $T(e_i)$ *for all* $1 \leqslant i \leqslant n$.

Proof. Every vector $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ has a unique expression:

$$v = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

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Hence, by linearity, $T(v) = x_1 T(e_1) + \cdots + x_n T(e_n)$,

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Hence, by linearity, $T(v) = x_1 T(e_1) + \cdots + x_n T(e_n)$, which has a unique choice

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Hence, by linearity, $T(v) = x_1 T(e_1) + \cdots + x_n T(e_n)$, which has a unique choice once $T(e_i)$ is given for every i.

A linear map is uniquely determined by its action on a basis

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Let V be finite dimensional,

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Proof. Every vector $v \in V$

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Proof. Every vector $v \in V$ has a unique expression:
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Proof. Every vector $v \in V$ has a unique expression: $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, because if $v = d_1v_1 + d_2v_2 + \cdots + d_nv_n$ is another expression,

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Hence, by linearity, $T(v) = c_1T(v_1) + \cdots + c_nT(v_n)$, which has a unique choice, once $T(v_i)$ is given for every i.

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Let V be finite dimensional, and $\{v_1, \ldots, v_n\}$ be a basis of V. Let $\{w_1, \ldots, w_n\}$ be any collection of n vectors in W.

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Let V be finite dimensional, and $\{v_1, \ldots, v_n\}$ be a basis of V.
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Proof. Once we show the existence,

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Proof. Once we show the existence, uniqueness follows from the last theorem.

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Then there is EXACTLY one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $1 \leqslant i \leqslant n$.

Proof. Once we show the existence, uniqueness follows from the last theorem. We define a map as follows:

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Proof. Once we show the existence, uniqueness follows from the last theorem. We define a map as follows: Every vector $v \in V$ has a UNIQUE expression:

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Proof. Once we show the existence, uniqueness follows from the last theorem. We define a map as follows: Every vector $v \in V$ has a UNIQUE expression: $v = c_1 v_1 + \cdots + c_n v_n$ as before.

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Let V be finite dimensional, and $\{v_1, \ldots, v_n\}$ be a basis of V.
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Define $T(v)$

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Define $T(v) := c_1 w_1 + \cdots + c_n w_n$.

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Define $T(v) := c_1 w_1 + \cdots + c_n w_n$. Then

- $T : V \to W$ is a linear map because:

A linear map is determined by its action on a basis

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- $T : V \to W$ is a linear map because:
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Proof. Once we show the existence, uniqueness follows from the last theorem. We define a map as follows: Every vector $v \in V$ has a UNIQUE expression: $v = c_1v_1 + \cdots + c_nv_n$ as before.
Define $T(v) := c_1w_1 + \cdots + c_nw_n$. Then

- $T : V \to W$ is a linear map because:
- If $v = c_1v_1 + \cdots + c_nv_n$ and $u = d_1v_1 + \cdots + d_nv_n$, then $v + u = (c_1 + d_1)v_1 + \cdots + (c_n + d_n)v_n$.

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Let V be finite dimensional, and $\{v_1, \ldots, v_n\}$ be a basis of V. Let $\{w_1, \ldots, w_n\}$ be any collection of n vectors in W. Then there is EXACTLY one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $1 \leqslant i \leqslant n$.

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- $T : V \to W$ is a linear map because:
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- $T : V \to W$ is a linear map because:
- If $v = c_1v_1 + \cdots + c_nv_n$ and $u = d_1v_1 + \cdots + d_nv_n$, then $v + u = (c_1 + d_1)v_1 + \cdots + (c_n + d_n)v_n$. Hence $T(v + u)$ $= (c_1 + d_1)w_1 + \cdots + (c_n + d_n)w_n = T(v) + T(u)$.

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Theorem

Let V be finite dimensional, and $\{v_1, \dots, v_n\}$ be a basis of V. Let $\{w_1, \dots, w_n\}$ be any collection of n vectors in W. Then there is EXACTLY one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $1 \leqslant i \leqslant n$.

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Define $T(v) := c_1w_1 + \cdots + c_nw_n$. Then

- $T : V \to W$ is a linear map because:
- If $v = c_1v_1 + \cdots + c_nv_n$ and $u = d_1v_1 + \cdots + d_nv_n$, then $v + u = (c_1 + d_1)v_1 + \cdots + (c_n + d_n)v_n$. Hence $T(v + u)$ $= (c_1 + d_1)w_1 + \cdots + (c_n + d_n)w_n = T(v) + T(u)$.
- If $v = c_1v_1 + \cdots + c_nv_n$,

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- If $v = c_1 v_1 + \cdots + c_n v_n$ and $u = d_1 v_1 + \cdots + d_n v_n$, then $v + u = (c_1 + d_1)v_1 + \cdots + (c_n + d_n)v_n$. Hence $T(v + u)$ $= (c_1 + d_1)w_1 + \cdots + (c_n + d_n)w_n = T(v) + T(u)$.
- If $v = c_1 v_1 + \cdots + c_n v_n$, then $cv = (cc_1)v_1 + \cdots + (cc_n)v_n$.

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Theorem

Let V be finite dimensional, and $\{v_1, \dots, v_n\}$ be a basis of V. Let $\{w_1, \dots, w_n\}$ be any collection of n vectors in W. Then there is EXACTLY one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $1 \leqslant i \leqslant n$.

Proof. Once we show the existence, uniqueness follows from the last theorem. We define a map as follows: Every vector $v \in V$ has a UNIQUE expression: $v = c_1v_1 + \cdots + c_nv_n$ as before.
Define $T(v) := c_1w_1 + \cdots + c_nw_n$. Then

- $T : V \to W$ is a linear map because:
- If $v = c_1v_1 + \cdots + c_nv_n$ and $u = d_1v_1 + \cdots + d_nv_n$, then $v + u = (c_1 + d_1)v_1 + \cdots + (c_n + d_n)v_n$. Hence $T(v + u)$ $= (c_1 + d_1)w_1 + \cdots + (c_n + d_n)w_n = T(v) + T(u)$.
- If $v = c_1v_1 + \cdots + c_nv_n$, then $cv = (cc_1)v_1 + \cdots + (cc_n)v_n$. Hence $T(cv)$

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Proof. Once we show the existence, uniqueness follows from the last theorem. We define a map as follows: Every vector $v \in V$ has a UNIQUE expression: $v = c_1v_1 + \cdots + c_nv_n$ as before.
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- $T : V \to W$ is a linear map because:
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A linear map is determined by its action on a basis

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As a consequence of Rank-Nullity Theorem, we will prove that for an arbitrary matrix D, row rank(D) = column rank(D).

For every matrix, row rank = column rank (an application of the Rank-Nullity Theorem)

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For an $m \times n$ *matrix* A *over* $\mathbb{R}$*,* $\text{row rank}(A) = \text{column rank}(A)$*.*

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4. We will study some examples to observe this inequality. But I will leave it as an exercise to verify this inequality in the general situation.

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- We claim that $\text{nullity}(B)$ is the number of free variables, because ...

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$$\begin{aligned} x_1 \qquad + \frac{3}{5}x_3 + \frac{7}{5}x_4 &= 0 \\ x_2 - \frac{1}{5}x_3 + \frac{1}{5}x_4 &= 0 \end{aligned}$$

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Proof of the theorem contd...

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Set $v_1 := S(w_1)$ and $v_2 := S(w_2)$.

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Hence, since T is inverse of S (as a set map), it follows that
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Similarly, one can prove that $S(cw) = cS(w)$ for every scalar $c \in \mathbb{R}$ and every vector $w \in W$.

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Conditions for a linear transformation to be isomorphism

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Proof.

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Hence, by Rank-Nullity Theorem, $\mathrm{rank}(T) = \dim(V)$.

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So $\mathrm{Image}(T) = V$, i.e., T is surjective.
(5) $\Rightarrow$ (2): Since T is surjective, $\mathrm{rank}(T) = \dim(V)$, hence $\mathrm{nullity}(T) = 0$, i.e., $\mathrm{Ker}(T) = 0$. Then, by linearity, T is injective.

Conditions for a square matrix to be invertible

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Proof. Consider A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$.

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Proof. Consider A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then (2) is same as $\mathrm{Null}(A) = 0$.

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Proof. Consider A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then (2) is same as $\mathrm{Null}(A) = 0$. Moreover (3) is same as A is surjective.

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Proof. Consider A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then (2) is same as $\mathrm{Null}(A) = 0$. Moreover (3) is same as A is surjective. Thus, by the previous theorem, we have (1), (2) and (3) are equivalent.

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Since AX is nothing but a linear combination of column vectors of A,

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Since AX is nothing but a linear combination of column vectors of A, it follows that (3) and (4) are equivalent.

Conditions for a square matrix to be invertible contd...

Theorem

Let A be an $n \times n$ matrix over $\mathbb{R}$. The following are equivalent:

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Proof. We already proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

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Let A be an $n \times n$ matrix over $\mathbb{R}$. The following are equivalent:

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Proof. We already proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
(4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7):

Conditions for a square matrix to be invertible contd...

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Proof. We already proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
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(4) $\Leftrightarrow$ (6): It follows from the above equivalences "(4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7)" and the fact that $\text{column rank}(A) = \text{row rank}(A)$.

Thank You!