Pblm $\rightarrow AX = b$

① Fixed point iterative scheme

$$X^{k+1} = MX^k + C$$

where $M = (I - A)$, $C = b$

② Jacobi-iterative Method

$$X^{k+1} = MX^k + C$$

Here $M = -D^{-1}(L+U)$, $b = D^{-1}b$

③ Gauss-Seidel Method $X^{k+1} = MX^k + C$

Here $M = (D-L)^{-1}U$

$C = (D-L)^{-1}b$

$\|M\| < 1$

$\Downarrow$

$A = [a_{ij}]_{n \times n}$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

Suppose d is the eigen-value of the matrix M. and V is the eigenvector corresponding to d

$$M \cdot V = d \cdot V$$

$\|d v\| \leq |d| \cdot \|v\|$

$\Rightarrow$

$$\|Mv\| = \|dv\|$$

$$\|Mv\| \leq \|M\| \cdot \|v\|$$

$$\Rightarrow \|dv\| \leq \|M\| \cdot \|v\|$$

$[\|Tx\| \leq \|T\| \cdot \|x\|$

$$\Rightarrow |\lambda| \cdot \|v\| \le \|M\| \cdot \|v\|$$

$$\Rightarrow |\lambda| \le \|M\|$$

$$\boxed{\max |\lambda| \le \|M\|}$$

$$\Downarrow$$

$$\rho(M) \le \|M\| < 1$$

$$\Rightarrow \rho(M) < 1$$

Spectral Radius and rate of convergence.

$$M = [m_{ij}]_{N \times N}$$

$$\Downarrow$$

M has N linearly independent eigen-vectors

$\bar{X}$ $\Downarrow$ exact solution

$k+1^{th}$ $\quad X^{k+1} - X =$

$$X^{k+1} = M X^{k} + c$$

$$\Rightarrow X = MX + c$$

X^k

$X = \begin{bmatrix} \\ \\ \end{bmatrix}_{N \times 1}$

$$X^{k+1} - X = M(X^k - X) \Leftarrow$$

$$\Downarrow$$

$$V^{k+1} = X^{k+1} - X$$

$$V^k = X^k - X$$

X^0

$X^0 - X = V^0$

$$V^{k+1} = M \cdot V^k \Leftarrow$$

R^N (N-)

$$V^0 = \sum_{i=1}^{N} \alpha_i \vec{e_i}$$, where $\vec{e_i}$ is an eigenvector of M.

$\alpha_i \in R|F$

$$V^1 = M \cdot V^0$$
$$= M \cdot \left(\sum_{i=1}^{N} \alpha_i \vec{e_i} \right)$$
$$= \sum_{i=1}^{N} \alpha_i M \vec{e_i}$$
$$= \sum_{i=1}^{N} \underbrace{\alpha_i d_i}_{\beta} \vec{e_i}$$

d_i

$(M \vec{e_i} = d_i \vec{e_i}$ $\forall\, i = 1, 2 \ldots N)$

$$V^2 = \sum_{i=1}^{N} \alpha_i d_i^2 \vec{e_i}$$

$$\vdots$$

$$V^K = \sum_{i=1}^{N} \alpha_i d_i^K \vec{e_i}$$

$\Rightarrow \frac{|d_i|}{|d_1|} < 1$

d_1 is the maximum eigenvalue of M

then

$$V^K = \alpha_1 d_1^K e_1 + \sum_{i=2}^{N} \alpha_i d_i^K \vec{e_i} \Rightarrow |d_1| \geq |d_i|$$

$i = 2, 3 \ldots N$

$$v^k = \lambda_1^k \left[\alpha_1 e_1 + \sum_{i=2}^{N} \alpha_i \left(\frac{d_i}{d_1} \right)^k \vec{e}_i \right]$$

$$\left| \frac{d_i}{d_1} \right| < 1$$

$$\Rightarrow k = 10$$

$$d_1 < \frac{1}{2}$$

$$0.000000\,01$$

$$(0.5)^3$$

$$= 0.25$$

$$= 0.00$$

$$v^k \approx \alpha_1 d_1^k e_1$$

$$x^k - x \approx \alpha_1 d_1^k e_1$$

$$\Rightarrow \quad \rho(M)$$

$$\Rightarrow \quad \rho(M) < 1$$

$$x^k - x \approx \alpha_1 \rho(M)^k e_1 \, (d_1)^k$$

Gerschgorin thm:

$$\vec{e} \rightarrow d \text{ (eigenvalue)}$$

$$\Rightarrow \quad M \cdot \vec{e} = d \cdot \vec{e} \qquad d \in \mathbb{R}$$

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2N} \\ m_{i1} & m_{i2} & \cdots & m_{iN} \\ m_{N1} & m_{N2} & \cdots & m_{NN} \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_i \\ \vdots \\ e_N \end{bmatrix} = d \begin{bmatrix} d \cdot e_1 \\ d \cdot e_2 \\ \vdots \\ d \cdot e_i \\ \vdots \\ d e_N \end{bmatrix}$$

$$\Rightarrow m_{i1} e_1 + m_{i2} e_2 + m_{i3} e_3 \cdots + m_{iN} e_N = d e_i$$

$$\Rightarrow \sum_{j=1}^{N} m_{ij} e_j = d e_i \qquad j = i$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} m_{ij} e_j + m_{ii} e_i = d e_i$$

$$\Rightarrow \underline{(m_{ii} - d)} e_i = -\sum_{\substack{j=1 \\ i \neq j}}^{N} m_{ij} e_j$$

$$\|\underline{(m_{ii} - d)} e_i\| = \left\| -\sum_{\substack{j=1 \\ i \neq j}}^{N} m_{ij} e_j \right\|$$

$$|m_{ii} - d| \cdot \|e_i\| = \left\| \sum_{\substack{j=1 \\ i \neq j}}^{N} m_{ij} e_j \right\|$$

$$|(m_{ii} - d)| \cdot \|e_i\| \leq \sum_{\substack{j=1 \\ i \neq j}}^{N} |m_{ij}| \cdot |e_j| \qquad (1)$$

$i = l$

let $|e_l| \geq |e_j| \ \forall j$

$$\Rightarrow |(m_{ll} - d)| \cdot |e_l| \leq \sum_{j=1}^{N} |(m_{ij})| \cdot |e_l|$$

$$\Rightarrow \underline{|(m_{ll} - d)|} \leq \sum_{\substack{j=1 \\ j \neq l}}^{N} (m_{lj}) \qquad (0)$$

$\Rightarrow$ each eigenvalues lies inside a circle with centre m_{ll} and radius $\sum_{j=1}^{N} |a_{lj}|$ with $l \neq j$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \qquad A = [a_{ij}]_{N \times N} \quad N = 4$$

$$\Rightarrow |a_{ll} - d| \leq \sum_{\substack{j=1 \\ j \neq l}}^{N} |a_{lj}|$$

$l = 1$

$$① \quad |a_{11} - d| \leq \sum_{j=1}^{N} |a_{1j}| = |a_{12}| + |a_{13}| + |a_{14}| = 1 + 0 + 0$$

$$\Rightarrow |a_{11} - d| < 1 \quad -①$$

$$\Rightarrow |2 - d| < 1$$

$l = 2$ $\quad |a_{22} - d| \leq \sum_{\substack{j=1 \\ j \neq 2}}^{4} |a_{2j}| = |a_{21}| + |a_{23}| + |a_{24}|$

$$|2 - d| < 1 + 1 + 0 \Rightarrow |2 - d| < 2 \quad -②$$

$l = 3$ $\quad |2 - d| \leq 2 \quad -③$

$l = 4$ $\quad |2 - d| < 1 \quad -④$

$|2-\alpha| < 1$

$\Rightarrow |2-\alpha| < 2$

$0 < \alpha < 4$

$AX = b$

$3.618, \quad 2.618, \quad 1.382$

0.382

SOR($\quad x^{k+1} \Rightarrow x^{k+1} - x^{k}$

X – exact solution / true solution of $AX = b$

x^k is the approximate solution at k^{th} iteration.

$b - A\vec{x}^k = 0.00005$

The quantity $\vec{r}^k = \vec{b} - A x^k$ is called a residual and it is a measure of the accuracy of x^k

G. Seidel

$$X^k = \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_i^k \\ x_N^k \end{bmatrix}$$

$$X^{k+1} = M x^k + b$$

$$M = (D-L)^{-1} U$$

$$AX = b$$

$$(D-L-U)X = b$$

$$(D-L)X = UX + b$$

$$\Rightarrow DX = LX + (UX + b)$$

$$(x_i)^{k+1} = \frac{1}{a_{ii}}\left[b - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{N} a_{ij} x_j^{k}\right]$$

$$(x_i)^{k} = \frac{1}{a_{ii}}\left[b - \sum_{j=1}^{i-1} a_{ij} x_j^{k} - \sum_{j=i+1}^{N} a_{ij} x_j^{k-1}\right]$$

$$\Downarrow$$

$$(x_i)^{k} \qquad r^k = b - Ax^k$$

$$\vec{r}^k = b \qquad \left[Dx^k + Lx^{k+1} + Ux^k = b \right]$$

then

$$\vec{r}^k = b - Dx^k + Lx^{k+1} + Ux^k$$

$$\downarrow$$

$$\underline{x} - x^k \qquad D(\underline{x^{k+1}} - x^k) = \omega(\vec{b} - Dx^k + Lx^{k+1} + Ux^k)$$