Solutions of Problems Set 1

Matrices, Linear equations and solvability

I have written the solutions or hints for most of the exercises. But if you want to learn the subject, it is better to try on your own before seeing the solution. Please verify each and everything, as there might be mistakes.

1. Solve (if solution exists) the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{lllll} u & +v & +w & +z & =6 \\ u & & +w & +z & =4 \\ u & & +w & & =2 \end{array}$$

What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

Solution. Apply elementary row operations on the augmented matrix to obtain:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ (upper triangular matrix, i.e., } A_{ij} = 0 \text{ for } i > j\text{).}$$

The corresponding triangular system is

$$\begin{aligned} u + v + w + z &= 6 \\ v \qquad\quad &= 2 \\ z &= 2 \end{aligned}$$

The back substitution yields $z = 2$, $v = 2$ and $u + w = 2$. The set of solutions are given by

$$\begin{pmatrix} 2-w \\ 2 \\ w \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where } w \text{ varies in } \mathbb{R}.$$

If the fourth plane $u = -1$ is included, then it can be observed that the original system is equivalent to the system $u + v + w + z = 6$, $v = 2$, $z = 2$ and $u = -1$. In this case, the system will have only one solution $u = -1$, $v = 2$, $z = 2$ and $w = 3$.

For the last part, you may include a fourth equation as $z = a$ for some scalar $a \neq 2$ in $\mathbb{R}$.

Remarks. The set of solutions of a non-homogeneous system does not form a subspace.

To solve a system, one may also reduce the augmented matrix to its row reduced echelon form. For instance, in the first part of Q.1, we may reduce the system as follows;

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ (row reduced echelon form).}$$

Now solve the corresponding system. Note that u, v, z are pivot variables, and w is a free variable (that means you can assign any value to w, and then u, v, z are uniquely determined by that value of w).

2. Find two points on the line of intersection of the three planes $t = 0$, $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

Hint. It is just finding two solutions of the system: $t = 0$, $z = 0$ and $x + y + z + t = 1$.

3. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is *singular* (i.e., it does not have solutions at all). What value should replace the last zero on the right side to allow the system to have solutions, and what are the solutions over $\mathbb{R}$?

Solution. Let us write b in place of the last zero on the right side. Then reduce the system:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & b \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & b+1 \end{bmatrix},$$

which yields a triangular system, whose last equation is $0 = b + 1$. So the original system (i.e., when $b = 0$) does not have solutions. Moreover, the last zero on the right side should be replaced by -1 to allow the system to have solutions. When $b = -1$, then the system can be reduced to its row reduced echelon form as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & b \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So u and v are pivot variables, and w is a free variable. The solutions of the system corresponding to the last augmented matrix (i.e., the system $u - w = 3$ and $v + 2w = -1$) are given by

$$\begin{pmatrix} w+3 \\ -2w-1 \\ w \end{pmatrix} = w \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \quad \text{where } w \text{ varies in } \mathbb{R}.$$

4. Under what condition on x_1, x_2 and x_3 do the points $(0, x_1), (1, x_2)$ and $(2, x_3)$ lie on a straight line?

Solution. **1st approach**. The equation of the straight line passing through $(0, x_1)$ and $(1, x_2)$ is given by

$$\frac{y - x_1}{x - 0} = \frac{x_2 - x_1}{1 - 0}, \quad \text{i.e., } y - x_1 = x(x_2 - x_1).$$

It passes through $(2, x_3)$ if and only if $x_3 - x_1 = 2(x_2 - x_1)$, which is the desired condition.

2nd approach. The equation of a straight line in an euclidean plane is given by $ax + by = c$ for some scalars $a, b, c \in \mathbb{R}$. If all the three points lie in this line, then we have

$$\begin{aligned} bx_1 &= c \\ a + \ bx_2 &= c \\ 2a + \ bx_3 &= c \end{aligned} \quad \textit{i.e.,} \quad \begin{aligned} x_1 b - c &= 0 \\ a + \ x_2 b - c &= 0 \\ 2a + \ x_3 b - c &= 0 \end{aligned} \quad \textit{i.e.,} \quad \begin{pmatrix} 0 & x_1 & -1 \\ 1 & x_2 & -1 \\ 2 & x_3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{matrix} 0 \\ 0. \\ 0 \end{matrix}$$

The desired condition is equivalent to that the above system has a non-trivial solution, which is equivalent to that the determinant of the coefficient matrix is zero, i.e., $(x_1 - x_3) + 2(x_2 - x_1) = 0$.

Remark. The 2nd approach can be treated as an application of system of linear equations.

5. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$ax + 2y = 0$$
$$2x + ay = 0$$

Solution. The determinant of the coefficient matrix is $a^2 - 4$. We have only these two possibilities: Case 1. $a^2 - 4 \neq 0$. In this case, the system has only the trivial **Solution.** Case 2. $a^2 - 4 = 0$. In this case, the system can be reduced as follows:

$$\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix} \overset{R2 \to R2 - \frac{a}{2}R1}{\Longrightarrow} \begin{pmatrix} a & 2 \\ 0 & 0 \end{pmatrix}.$$

Therefore the original system is equivalent to the system of one equation $ax + 2y = 0$, which is nothing but a line. So $a = \pm 2$ are the desired values of a.

6. Are the following systems equivalent:

$$x - y = 0$$
$$2x + y = 0$$

and

$$3x + y = 0$$
$$x + y = 0$$

If so, then express each equation in each system as a linear combination of the equations in the other system.

Hint. (1st part). You may use the fact that for two systems $AX = 0$ and $BX = 0$, if A and B are row equivalent to the same row reduced echelon form C, then by transitivity, A and B are row equivalent, and hence $AX = 0$ and $BX = 0$ are equivalent.

(2nd part). Observe that writing $x - y = 0$ as a linear combination of $3x + y = 0$ and $x + y = 0$ is equivalent to write $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So you just have to find c and d such that $c\begin{pmatrix} 3 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, i.e., $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The solution of this system is $c = 1$ and $d = -2$. Therefore $\{x - y = 0\} = \{3x + y = 0\} + (-2)\{x + y = 0\}$ in the obvious sense. Similarly, you can find the other linear combinations.

7. Set $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find all the solutions of $AX = 2X$, i.e., all X such that $AX = 2X$, where $2X$ is just componenntwise scalar multiplication.

Hint. $AX = 2X$ can be written as $(A - 2I_3)X = 0$, where I_3 is the 3×3 identity matrix. Now it can be solved by applying elementary row operations on the coefficient matrix $A - 2I_3$.

8. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Hint. Try with a matrix consisting of two rows $\begin{bmatrix} R1 \\ R2 \end{bmatrix}$.

9. Consider the system of equations $AX = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $\mathbb{R}$. Prove the following statements.

(i) A is a zero matrix (i.e., all entries are zero) if and only if every pair (x_1, x_2) is a solution of $AX = 0$.

(ii) $\det(A) \neq 0$, i.e., $ad - bc \neq 0$ if and only if the system has only the trivial solution.

(iii) $\det(A) = 0$, i.e., $ad - bc = 0$ but A is a non-zero matrix (i.e., some entries are non-zero) if and only if there is $(y_1, y_2) \neq (0, 0)$ in $\mathbb{R}^2$ such that every solution of the system is given by $c(y_1, y_2)$ for some scalar c.

Hint. (i) Observe that $AX = 0$ for every $X \in \mathbb{R}^2$ if and only if $x_1 \begin{pmatrix} a \\ c \end{pmatrix} + x_2 \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all $x_1, x_2 \in \mathbb{R}$ if and only if $A = 0$ (to show that you may take $(x_1, x_2) = (1, 0)$ or $(0, 1)$).

(ii) and (iii). We already have proved (ii) in the class. Or directly, you can try to reduce the coefficient matrix to its row reduced echelon form. Observe that $ad - bc \neq 0$ if and only if A is reduced to the identity matrix if and only if the system has only the trivial **Solution.** For (iii), when A is a non-zero matrix, then $ad - bc = 0$ if and only if A is row equivalent to $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ if and only if the system $AX = 0$ is equivalent to $ax + by = 0$; see the solution of Q.5. The rest is left as an exercise. Now you may try to understand the statement in (iii) geometrically.

10. Prove that if two homogeneous systems each of two linear equations in two unknowns have the same solutions, then they are equivalent.

Hint. You may use Q.9.

11. For the system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution (by back substitution)? Also solve it by computing the equivalent system whose coefficient matrix is in row reduced echelon form. Verify whether both the solutions are same.

Hint. You may follow the steps described as in the solution of Q.1.

12. Describe explicitly all 2×2 row reduced echelon matrices.

Hint. Consider three cases that the number of non-zero rows of the matrix can be $0, 1$ or 2. When it is 1, then we will have two subcases. Think about the pivot positions.

13. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix over $\mathbb{R}$. Suppose that A is row reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

14. Find the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ using the elementary row operations.

15. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$, where I_3 is the 3×3 identity matrix. Deduce A^{-1}.

Hint. Apply elementary row operations on $(A \,|\, I_3)$ to get A^{-1}, and keep track of the row operations to get the corresponding $E_1, E_2, \ldots, E_k$.