Lectures 1 and 2
System of Linear Equations

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January 03, 2020

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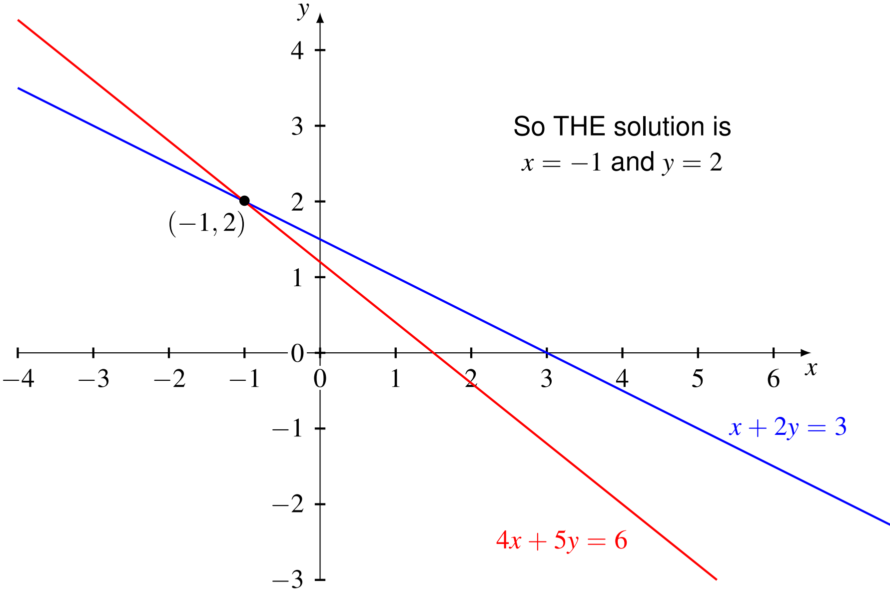
What does it mean geometrically?



So THE solution is
$x = -1$ and $y = 2$

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- Eliminating x from the 2nd equation, we obtain a **triangulated** system:

$$\begin{aligned} x + 2y &= 3 \qquad \text{(equation1)} \\ -3y &= -6 \qquad \text{(equation 2) - 4(equation 1)}. \end{aligned} \tag{2}$$

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- In this case, the solution is $y = 2,\ x = -1$.

Another method to solve the system: Cramer's Rule

- The system can be written as

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \quad \text{or} \quad \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

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$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 6 \cdot 2}{1 \cdot 5 - 4 \cdot 2} = \frac{3}{-3} = -1$$

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- We understand the Gaussian Elimination method by examples.

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- This is absurd. So the system does not have solutions.

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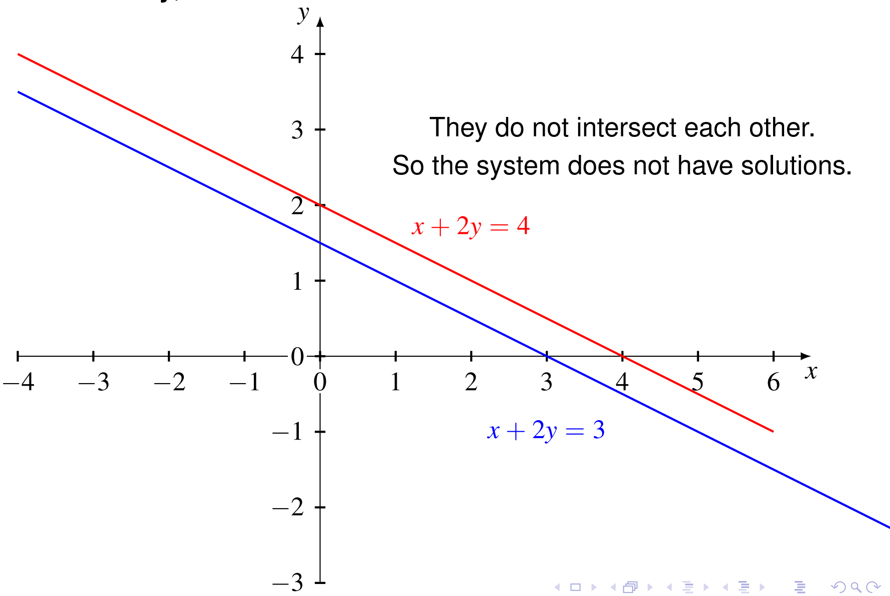
Geometrically,



They do not intersect each other.

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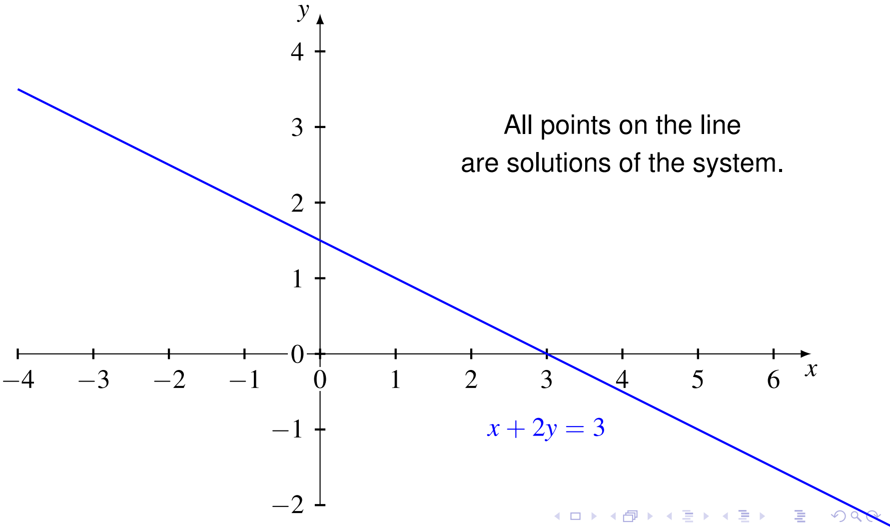

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- We will discuss two such applications:
  1. Polynomial Curve Fitting.
  2. Networks and Kirchhoff's Laws for electricity.

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$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

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  whose graph passes through these n points. To solve for the coefficients of $p(x)$, substitute each of the points into the polynomial function and obtain linear equations in variables $a_0, a_1, \dots, a_{n-1}$.

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- Then we can compute the (approximate) traffic $p(x')$ for the road at every time x'.

Kirchhoff’s circuit laws: Physics concept

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- Kirchhoff's laws are fundamental in circuit theory. They quantify how current flows through a circuit and how voltage varies around a loop in a circuit.
- Kirchhoff's current law (1st Law) states that current flowing into a node (or a junction) must be equal to current flowing out of it.

An example to understand Gaussian Elimination

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- What was the aim?

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- What was the aim? To change the system so that the coefficient of u in the 1st equation becomes non-zero.

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- We call the coefficient 4 as the **first pivot**.

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Triangular system and back-substitution

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Gaussian Elimination process in short

Original System

$\Downarrow$ Forward Elimination

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Solution

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In the Gaussian Elimination process, the 3 types of row operations are called **elementary row operations.**

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The forward elimination steps

- The forward elimination steps can be described as follows.

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{R3 \rightarrow R3+(1/2)R1}$$

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- In this case, where we have a full set of 3 pivots, there is only one solution.

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- Consider a system of m linear equations in n variables $x_1, \ldots, x_n$.

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- Here $A_{ij}, b_i \in \mathbb{R}$, and $x_1, \ldots, x_n$ are unknown. We try to find the values of $x_1, \ldots, x_n$ in $\mathbb{R}$ satisfied by the system.
- Any n tuple $(x_1, \ldots, x_n)$ of elements of $\mathbb{R}$ which satisfies the system (i.e., which satisfies every equation of the system) is called a **solution** of the system.
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Equivalent systems of linear equations

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- Since the elementary row operations are invertible with inverses of same types, the 1st system can also be obtained from the 2nd system by applying elementary row operations.
- In this case, we call that the **two systems are equivalent**.

Equivalent systems have same solutions set

- Consider two **equivalent systems:**

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

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- Then they have the same set of solutions.

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- For a non-homogeneous system, we apply **elementary row operations** on the augmented matrix $(A \mid b)$.

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- i.e., when the system is $Ax = 0$, then it is enough to consider the coefficient matrix A.
- So, in this case, we apply elementary row operations on A.

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1. Interchange of two rows of A, say rth and sth rows.

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(1) the 1st non-zero entry in each non-zero row of A is equal to 1;
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(i) $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

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Here (ii) and (iii) are not row reduced matrices.

Row reduced echelon matrix

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The matrix in (ii) is row reduced, but NOT row reduced echelon.

Every matrix is row equivalent to a row reduced echelon matrix

Theorem

Every $m \times n$ matrix over $\mathbb{R}$ is **row equivalent** *to a row reduced echelon matrix.*

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$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{Row\ operations} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

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Exercise. Let A be an $n \times n$ row reduced echelon matrix over $\mathbb{R}$.

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Exercise. Let A be an $n \times n$ row reduced echelon matrix over $\mathbb{R}$. Show that A is invertible if and only if A is the identity matrix.

Example: A matrix $\rightarrow$ Row reduced echelon matrix

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

Example: A matrix $\rightarrow$ Row reduced echelon matrix

$$\begin{bmatrix} 0 & 1 & 1 & \color{purple}{5} \\ 4 & -6 & 0 & \color{purple}{-2} \\ -2 & 7 & 2 & \color{purple}{9} \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2}$$

Example: A matrix $\to$ Row reduced echelon matrix

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$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{R3 \to R3 + 2 \cdot R1}$$

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Example: A matrix $\rightarrow$ Row reduced echelon matrix

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$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -12 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -12 \end{bmatrix} \xRightarrow{R3 \to (-1/2)R3}$$

Example: A matrix $\rightarrow$ Row reduced echelon matrix

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(Triangular system with pivot entries 1)

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$$\overset{R2 \to R2 - R3}{\Longrightarrow}$$

Example: A matrix $\to$ Row reduced echelon matrix

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$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \overset{R3 \to R3+2\cdot R1}{\Longrightarrow} \begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 2 & 8 \end{bmatrix} \overset{R3 \to R3-4\cdot R2}{\Longrightarrow}$$

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$$\xrightarrow{R2 \to R2 - R3} \begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{R1 \to R1 + (3/2)R2}$$

Example: A matrix $\rightarrow$ Row reduced echelon matrix

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{R1 \to (1/4)R1}$$

$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{R3 \to R3 + 2\cdot R1} \begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 2 & 8 \end{bmatrix} \xrightarrow{R3 \to R3 - 4\cdot R2}$$

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So it is just combination of forward and backward eliminations.

Solution of a system corr. to a row reduced echelon

Example

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{\textit{Row operations}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

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$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \quad \overset{Row\ operations}{\Longrightarrow} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

Considering the corresponding system, we have the solution $u = -2$, $v = -1$ and $w = 6$.

Solution of a system corr. to a row reduced echelon

Consider the homogeneous system corr. to the coefficient matrix

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The values of x_1, x_3 and x_5 can be chosen freely.

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- Consider $Ax = 0$, where A is a row reduced echelon matrix.

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- We may assign any values to $u_1, \ldots, u_{n-r}$. Then $x_{k_1}, \ldots, x_{k_r}$ are determined uniquely by those assigned values.

Solution to a homogeneous system (when $m < n$)

Theorem

Let A be an $m \times n$ matrix over $\mathbb{R}$ with $m < n$.

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$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = c_1, \ \ldots, \ x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = c_r$$

$$0 = c_{r+1}$$

$$\cdots$$

$$0 = c_m$$

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- The systems $Ax = b$ and $Bx = c$ are equivalent, and hence they have the same solutions.
- Let $1, \ldots, r$ be the non-zero rows of B, and the leading non-zero entry of row i occurs in column k_i.
- The variables $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are the pivot variables.
- Let $u_1, \ldots, u_{n-r}$ denote the remaining $n - r$ (free) variables.
- Then the system $Bx = c$ can be written as

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- Assign any value to x_3, and compute x_1, x_2.

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These are all the 2×2 elementary matrices.

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- So applying an *elementary row operation* on a matrix is same as left multiplying by the corresponding *elementary matrix*.

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Corollary

Let A and B be two $m \times n$ matrices. Then
A and B are equivalent

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$B = PA$, where P is a product of some $m \times m$ elementary matrices.

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Proof.

Let E be an elementary matrix corresponding to the elementary row operation e,

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$$EE' = e(E') = e(e'(I)) = I \text{ and}$$
$$E'E = e'(E) = e'(e(I)) = I.$$

$\square$

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$$A^{-1} = (E_k \cdots E_2 E_1) = E_k \cdots E_2 E_1 (I).$$

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So $A^{-1} = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{pmatrix}$.

Thank You!