Differential Equations (MA 1150)

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Lecture 8

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Section 1

Higher Order Linear Differential Equations

Definition An nth order differential equation is said to be linear if it can be written in the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x).$$

Higher Order Linear Differential Equations

More generally, nth order linear differential equations can be written as

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Suppose $P_0(x) \neq 0$ for all x in some open interval (a, b), then we can divide by $P_0(x)$ in (1) on (a, b), and obtain the equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x).$$

where $p_i(x) = \dfrac{P_i(x)}{P_0(x)}$ and $f(x) = \dfrac{F(x)}{P_0(x)}$.

Convention The left side of nth order linear differential equation

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We say that the equation $Ly = F$ is *normal* on (a, b) if P_0, P_1, $\ldots$, P_n and F are continuous on (a, b) and P_0 has no zeros on (a, b).

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Theorem Suppose $Ly = F$ is normal on (a, b) and let $x_0 \in (a, b)$. Then the initial value problem

$$Ly = F, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \ \ldots, \ y^{(n-1)}(x_0) = k_{n-1}$$

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You may compare this with the 1st order and 2nd order linear ODE.

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Theorem If $Ly = 0$ is normal on (a, b), then a set $\{y_1, y_2, \ldots, y_n\}$ of n solutions of $Ly = 0$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b).

Subsection 1

The Wronskian and Abel's Formula

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Abel's formula

$$W(x) = W(x_0) e^{-\int_{x_0}^{x} p(t)\, dt}, \quad a < x < b.$$

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To check linear independence of $\{y_1, y_2, \ldots, y_n\}$ on (a, b), where $y_1, y_2, \ldots, y_n$ are solutions of nth order equation $Ly = 0$. (Assume that $Ly = 0$ is normal).

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Suppose that there are constants c_1, c_2,..., c_n such that

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Differentiating $n - 1$ times leads to the $n \times n$ system of equations

$$\begin{array}{rcl} c_1 y_1(x) + c_2 y_2(x) + & \cdots & + c_n y_n(x) = 0 \\ c_1 y_1'(x) + c_2 y_2'(x) + & \cdots & + c_n y_n'(x) = 0 \\ & \vdots & \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + & \cdots & + c_n y_n^{(n-1)}(x) = 0 \end{array} \tag{4}$$

Wronskian of $\{y_1, y_2, \ldots, y_n\}$.

For a fixed x, let

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

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If $W(x) \neq 0$ for some x in (a, b) then the system (4) has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

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Thus using previous theorem we conclude that

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is the general solution of $Ly = 0$ on (a, b).

Wronskian of $\{y_1, y_2, \ldots, y_n\}$.

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Theorem Suppose the homogeneous linear nth order equation

$$P_0(x)y^{(n)} + P_1(x)y^{n-1} + \cdots + P_n(x)y = 0 \tag{5}$$

is normal on (a, b), let y_1, y_2, $\ldots$, y_n be solutions of (5) on (a, b), and let x_0 be in (a, b).

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Subsection 2

Particular solution

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Theorem Suppose $Ly = F$ is normal on (a, b). Let y_p be a particular solution of $Ly = F$ on (a, b), and let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of the complementary equation $Ly = 0$ on (a, b).

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is a particular solution on (a, b) of

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Compare these with homogeneous linear 2nd order ODE.

Subsection 3

Higher order constant coefficient equations

If $a_0, a_1, \ldots, a_n$ are constants and $a_0 \neq 0$, then

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Similar to 2nd order ODE: we call

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the *characteristic polynomial* of (7).

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If k is a positive integer, let D^k stand for the k-th derivative operator; that is

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We call $q(D)$ a *polynomial* operator.

Recall $p(m) = a_0 m^n + a_1 m^{n-1} + \cdots + a_n$ is the *characteristic polynomial*, we have

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If m is a constant then

$$\begin{aligned} p(D)e^{mx} &= \left(a_0 D^n e^{mx} + a_1 D^{n-1} e^{mx} + \cdots + a_n e^{mx}\right) \\ &= (a_0 m^n + a_1 m^{n-1} + \cdots + a_n) e^{mx}; \end{aligned}$$

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Therefore solution to IVP is

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Therefore $y_3 = e^x$ is solution of (13).

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Hence $\{\cos x, \sin x, e^x\}$ is linearly independent and

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Therefore y is a solution if it is a solution of any of the three equations

$$(D - 2)y = 0, \quad (D + 2)y = 0, \quad (D^2 + 4)y = 0.$$

Hence, $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is a set of solutions.

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$$y_1 = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

is the general solution of $y^{(4)} - 16y = 0$.

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Therefore the general solution of (15) is

$$y = e^{-x}(c_1 + c_2 x + c_3 x^2).$$

Higher order constant coefficient equations

Theorem: If $\omega \neq 0$ and m is a positive integer, then

$$\begin{aligned}\{ & e^{\lambda x}\cos\omega x,\ xe^{\lambda x}\cos\omega x,\quad \dots,\ x^{m-1}e^{\lambda x}\cos\omega x,\\ & e^{\lambda x}\sin\omega x,\ xe^{\lambda x}\sin\omega x,\quad \dots,\ x^{m-1}e^{\lambda x}\sin\omega x\}\end{aligned}$$

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Example: Find the general solution of

$$(D^2+4D+13)^3 y = 0. \tag{16}$$

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$$y = (c_1+c_2x+c_3x^2)e^{-2x}\cos 3x + (c_4+c_5x+c_6x^2)e^{-2x}\sin 3x.$$

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Higher order constant coefficient equations

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