CSCI-B561 Practice Problems on Assumed Knowledge Monday, January 13, 2020
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The following is a list of problems that covers topics included in the assumed knowledge for B561. The purpose of this not-graded homework is for you to review these topics and to practice your skills to solve the problems.

1. Let p, q, and r be propositional variables.
   (a) Use the *truth table method* to show that $p \wedge \neg(q \rightarrow r)$ is logically equivalent with $(p \wedge q) \vee (p \wedge \neg r)$.
   (b) Use *logical equivalences* to prove that $p \wedge \neg(q \rightarrow r)$ is logically equivalent with $(p \wedge q) \vee (p \wedge \neg r)$.

2. Let p, q, and r be propositional variables.
   (a) Use the truth table method to prove that $(p \wedge (p \rightarrow q) \rightarrow q$ is a tautology.
   (b) Use logical equivalences to prove that $(p \wedge (p \rightarrow q) \rightarrow q$ is a tautology.

3. Let $\mathbb{N}$ denote the set of natural numbers. Define the relation $\equiv_{\text{mod } 3}$ over $\mathbb{N}$ such that for each pair of natural numbers m and n

$$m \equiv_{\text{mod } 3} n \text{ if } m \text{ mod } 3 = n \text{ mod } 3.$$

   (a) Prove that $\equiv_{\text{mod } 3}$ is an *equivalence relation*.
   (b) Specify the *partition* of $\mathbb{N}$ induced by this equivalence relation.

4. Let A be a set and let 2^A denote the powerset of A. Define the relation $\subset$ on 2^A such that for each pair of sets X and Y in 2^A, $X \subset Y$ if X is a strict subset of Y.

   Prove that $\subset$ is an *order relation*.

5. Let A be a set and let f and g be *one-to-one correspondences* (i.e., *bijections*) from A to A. Prove that the *composition* of f and g, i.e., the function $g \circ f$, is a one-to-one correspondence from A to A.

   Recall that for an element $a \in A$, $g \circ f(a)$ is defined as the value $g(f(a))$.

6. Let A and B be two finite sets. Prove that the number of functions from A to be B is $|B|^{|A|}$. Here, $|A|$ and $|B|$ denote the *sizes*, i.e., *cardinalities*, of A and B, respectively.

7. Let $\mathbb{N}$ denote the set of natural numbers and let $2\mathbb{N}$ denote the set of even natural numbers. Prove that the *cardinalities* of $\mathbb{N}$ and $2\mathbb{N}$ are the same.

8. Let $P(x)$ and $Q(x)$ be predicates. In the following problem, $\Leftrightarrow$ denotes *logical equivalence* and $\Rightarrow$ denotes *logical implication*.
   (a) Show that

$$\exists x(P(x) \wedge \neg\exists y Q(x)) \Leftrightarrow \exists x(P(x) \wedge \neg(\exists y(P(x) \wedge P(y))))$$

   (b) Show that

$$\forall x(P(x) \rightarrow Q(x)) \Leftrightarrow \neg\exists x(P(x) \wedge \neg Q(x))$$

(c) Show that the existential quantifier $\exists$ distributes over the $\vee$ connective. I.e. show that

$$\exists x(P(x) \vee Q(x)) \Leftrightarrow \exists x P(x) \vee \exists x Q(x)$$

(d) Show that the $\exists$ quantifier does not distribute over the $\wedge$ connective. That is, give an example to show that

$$\exists x(P(x) \wedge Q(x)) \not\Leftarrow \exists x P(x) \wedge \exists x Q(x)$$

(e) Show that the $\forall$ quantifier does not distribute over the $\vee$ connective. That is, give an example to show that

$$\forall x(P(x) \vee Q(x)) \not\Rightarrow (\forall x P(x)) \vee (\forall x Q(x))$$

(f) Show that

$$\forall x P(x) \vee \forall x Q(x) \Rightarrow \forall x(P(x) \vee Q(x))$$

9. Consider the predicates $male(x)$, $female(x)$, and $parentof(x, y)$ in a domain of persons. The predicate $male(x)$ states of a person that he is male, the predicate $female(x)$ states of a person that she is female, the predicate $parentof(x, y)$ states that x is the parent of y. Write a sentence in Predicate Logic for the following natural languages sentences.
   (a) Each person has a mother and a father.
   (b) Each person has exactly one mother and has exactly one father.
   (c) There exists a person who has granddaughters.
   (d) There exists a person who has at least two granddaughters but who has no grandsons.
   (e) There exists a person who has exactly one daughter.
   (f) There exists a person who has a cousin. (Recall that a cousin is the son or daughter of an uncle or aunt.)
   (g) For each mother of a child, there exists another person who is the father of that child.

10. Consider the predicates $male(x)$, $female(x)$, and $parentof(x, y)$ in a domain of persons. The predicate $male(x)$ states of a person that he is male, the predicate $female(x)$ states of a person that she is female, the predicate $parentof(x, y)$ states that x is the parent of y. Write the following queries in the Predicate Logic.
   (a) Find parents who have both sons and daughters.
   (b) Find the person pairs (x, y) such that x and y have the same children.
   (c) Find the person pairs (x, y) such that x is a sister of y.
   (d) Find the persons who have daughters but who do not have sons.

(e) Find the grandmothers with at least 3 grandchildren.

11. In the theory of rational behavior, which has applications in economics, ethics and psychology, the notion of an individual preferring one object or state of affairs to another is of importance. We may say that an individual *weakly* prefers x to y if he does not strictly prefer y to x. We use the notion of weak preference for formal convenience, for if we use strict preferences, we also need a notion of *indifference*. The point of this question is to ask you to show that on the basis of two simple postulated properties of weak preference and the appropriate definition of strict preference and indifference in terms of weak preference, we may logically infer all the expected properties of strict preference and indifference. Let us use $Q(x, y)$ to denote that individuals weakly prefer x over y, $P(x, y)$ to denote that individuals strictly prefer x over y, and $I(x, y)$ to denote that individuals are indifferent about their weak preference of x versus y. Our two premises on the predicate $Q(x, y)$ just say that Q is transitive and that of any objects in the domain of objects under consideration, one is weakly preferred to the other. In Predicate Logic:

$$\forall x \forall y \forall z((Q(x, y) \wedge Q(y, z)) \rightarrow Q(x, z))$$

$$\forall x \forall y(Q(x, y) \vee Q(y, x))$$

As additional premises we introduce the two obvious definition for I and P.

$$\forall x \forall y(I(x, y) \leftrightarrow Q(x, y) \wedge Q(y, x))$$

$$\forall x \forall y(P(x, y) \leftrightarrow \neg Q(y, x))$$

Derive the following conclusions from these four premises:

(a) $\forall x I(x, x)$.

(b) $\forall x \forall y(I(x, y) \rightarrow I(y, x))$.

(c) $\forall x \forall y(P(x, y) \rightarrow \neg P(y, x))$.

(d) $\forall x \forall y(I(x, y) \rightarrow \neg(P(x, y) \wedge P(y, x)))$.

(e) $\forall x \forall y \forall z((I(x, y) \wedge P(y, z)) \rightarrow P(x, z))$.

12. Prove by *mathematical induction* that

$$\forall n \geq 0 : 11^n - 6 \text{ is divisible by } 5.$$

13. Consider the following recursively defined function f:

$$\begin{aligned} f(2) &= \tfrac{1}{2} \\ f(n) &= f(n-1)\left(1 - \tfrac{1}{n}\right) \quad \text{if } n \geq 3 \end{aligned}$$

Prove by mathematical induction that

$$\forall n \geq 2 : f(n) = \frac{1}{n}.$$

14. Prove by mathematical induction that for each $n \geq 3$, if $A = \{1, \ldots, n\}$ then the number of sets of the form $\{i, j, k\}$, where $1 \leq i < j < k \leq n$, is $\frac{n(n-1)(n-2)}{6}$.

    In this problem, you are allowed to use the fact that A has $\frac{n(n-1)}{2}$ subsets of the form $\{i, j\}$ with $1 \leq i < j \leq n$.

15. Prove by mathematical induction one can make any amount of postage (8 cents or more) using only 5-cent and 3-cent stamps.

16. Let a and b be two characters. Define the set of strings $\mathcal{S}$ over these two characters recursively as follows:

$$\begin{array}{ll} \lambda & \text{is in } \mathcal{S} \\ a & \text{is in } \mathcal{S} \\ b & \text{is in } \mathcal{S} \\ uv & \text{is in } \mathcal{S} \text{ if } u \text{ and } v \text{ are in } \mathcal{S} \end{array}$$

λ is called the *empty-string*: it is a string with no characters.

uv is called the *concatenation* of strings u and v. For example, if u is the string aba and v is the string $bbaa$, then uv is the string $ababbaa$.

We define several functions.

$$\begin{array}{lcl} l(\lambda) & = & 0 \\ l(a) & = & 1 \\ l(b) & = & 1 \\ l(uv) & = & l(u) + l(v) \end{array}$$

So $l(x)$ is the length of the string x.

$$\begin{array}{lcl} f_a(\lambda) & = & 0 \\ f_a(a) & = & 1 \\ f_a(b) & = & 0 \\ f_a(uv) & = & f_a(u) + f_a(v) \end{array}$$

So $f_a(x)$ is the number of a characters in the string x. For example $f_a(abbaa) = 3$.

$$\begin{array}{lcl} f_b(\lambda) & = & 0 \\ f_b(a) & = & 0 \\ f_b(b) & = & 1 \\ f_b(uv) & = & f_b(u) + f_b(v) \end{array}$$

So $f_b(x)$ is the number of b characters in the string x. For example $f_b(abbaa) = 2$.

$$\begin{array}{lcl} r(\lambda) & = & \lambda \\ r(a) & = & a \\ r(b) & = & b \\ r(uv) & = & r(v)r(u) \end{array}$$

So $r(x)$ is the reverse of the string x. For example, $r(abb) = bba$.

$$\begin{array}{lcl} s(\lambda) & = & \lambda \\ s(a) & = & b \\ s(b) & = & a \\ s(uv) & = & s(u)s(v) \end{array}$$

So $s(x)$ is the string x wherein a and b are interchanged. For example $s(abb) = baa$.

$$\begin{array}{lcl} d(\lambda) & = & \lambda \\ d(a) & = & aa \\ d(b) & = & bb \\ d(uv) & = & d(u)d(v) \end{array}$$

So $d(x)$ is the string x wherein each a is replaced by aa and each b is replaced by bb. For example $d(abb) = aabbbb$.

Prove the following properties about these functions by *structural induction*:

1. For each string x in $\mathcal{S}$, $f_a(x) = f_a(r(x))$.
2. For each string x in $\mathcal{S}$, $f_a(x) = f_b(r(s(x)))$.
3. For each string x in $\mathcal{S}$, $l(x) = f_b(s(x)) + f_b(x)$.
4. For each string x in $\mathcal{S}$, $l(d(x)) = 2l(s(x))$.
5. Let r be the binary relation $\{(a, b), (b, c), (b, a), (b, d), (a, a), (a, e), (e, f)\}$. Determine the *reflexive transitive closure* r^* of r.

17. Let $f(n)$ and $g(n)$ be functions from the natural numbers to the natural numbers.

    1. Formulate the definition of $f(n) \in O(g(n))$.
    2. Formulate the definition of $f(n) \notin O(g(n))$.
    3. Prove that $\frac{1}{3}n^2 + 5n + 7 \in O(n^3)$.
    4. Prove that $n \notin O(\log_2(n))$.

18. Let L be an ordered array of n natural numbers. Let a be a number. Show that the *binary search algorithm* can determine in $O(\log_2(n))$ that a occurs in L.

19. Consider the list L of numbers $(5, 1, 4, 3, 2, 7, 6, 8)$. Show how the *mergesort algorithm* can sort these numbers.

    Argue that time-complexity of mergesort algorithm to sort a list of n natural numbers is $O(n \log(n))$.

20. Let H be a *hash-table* consisting of 3 buckets. Consider the *hash function* h such that for a number n, $h(n) = n \bmod 3$. Show how the numbers 5, 1, 4, 3, 2, 7, 6, and 8 are hashed into H.

21. A graph is a structure $G = (V, E)$ where V denote the set of vertices of G and E the set of edges of G.

    Give an example of a graph with 5 vertices and 10 edges and show of the *depth-first search algorithm* visits each of 5 vertices in this graph. Then show how the *breadth-first search algorithm* visits these 5 vertices.