CS 747, Autumn 2020: Week 6, Lecture 1

Shivaram Kalyanakrishnan

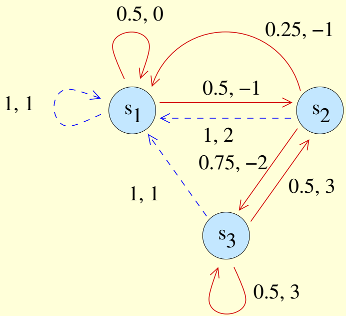
Department of Computer Science and Engineering
Indian Institute of Technology Bombay

Autumn 2020

MDPs: Weeks 4 and 5

1. MDP, policy, value function
2. MDP planning
3. Alternative formulations
4. Applications
5. Policy Evaluation

6. Banach's Fixed-point theorem
7. Bellman optimality operator
8. Value Iteration
9. Linear Programming



Markov Decision Problems

1. Policy Iteration
   - Policy evaluation (review)
   - Policy improvement
   - Algorithm and variants

2. Proof of Policy Improvement Theorem
   - Bellman operator
   - Proof

3. Computational complexity of MDP planning

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Policy Evaluation (Review)

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- V^π and Q^π computable in poly(n, k) arithmetic operations.

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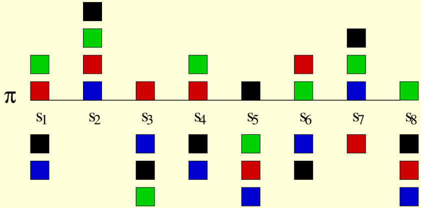

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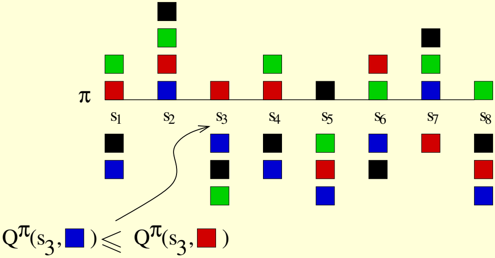

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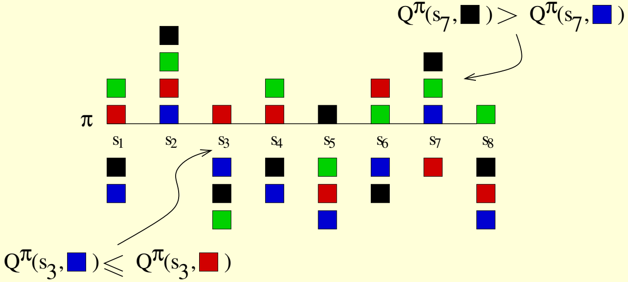

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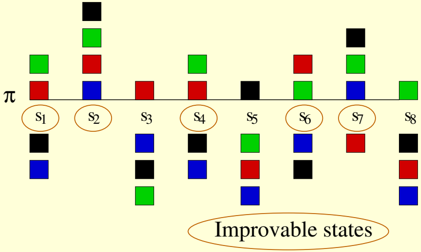

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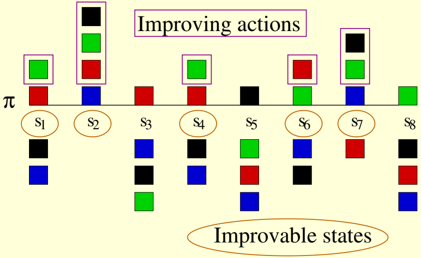

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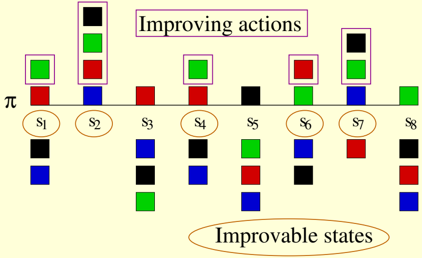

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Pick one or more improvable states, and in them,

Switch to an arbitrary improving action.

Let the resulting policy be π'.

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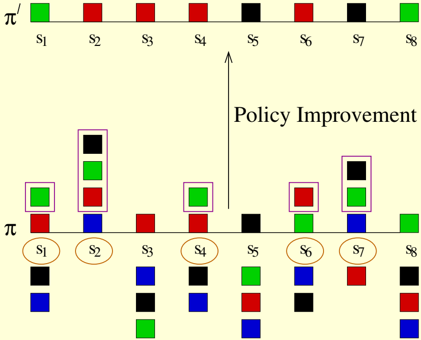

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(1) If $\mathbf{IS}(\pi) = \emptyset$, then π is optimal, else
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- Observe that $\mathbf{IS}(\pi^\star) = \emptyset \iff B^\star(V^{\pi^\star}) = V^{\pi^\star}$.
- In other words, $V^{\pi^\star}$ satisfies the Bellman optimality equations—which we know has a unique solution. It is a convention to denote $V^{\pi^\star} = V^\star$.

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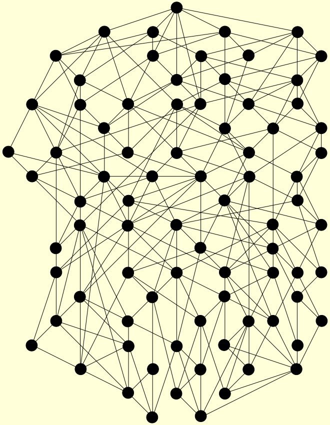

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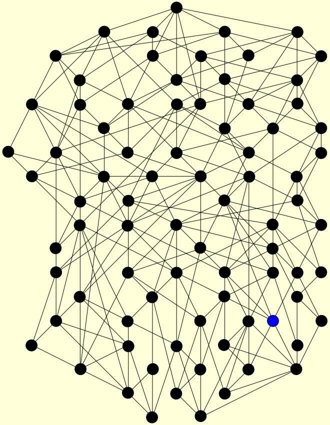

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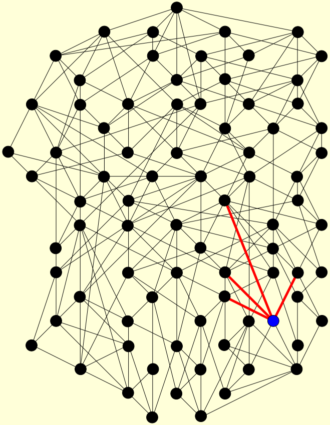

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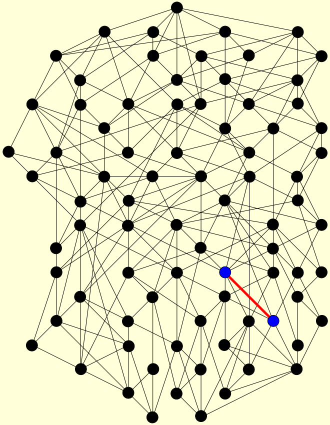

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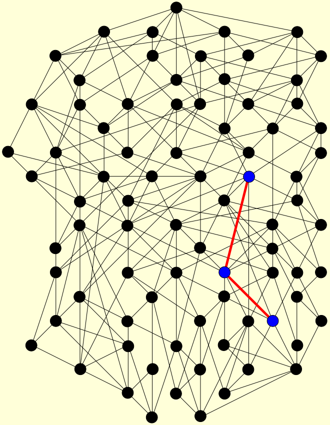

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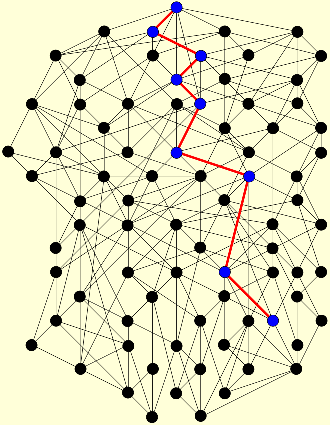

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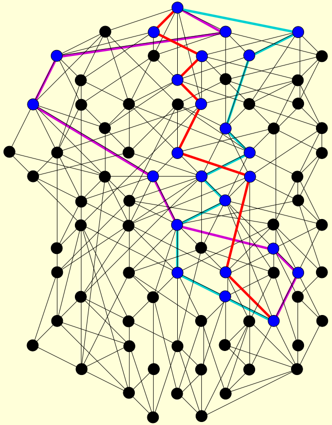
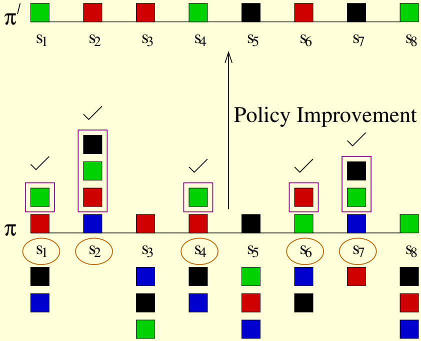


Path taken (and hence the number of iterations) in general depends on the switching strategy.

Howard's Policy Iteration

- Reference: Howard (1960).
- Greedy; switch all improvable states.



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- Reference: Mansour and Singh (1999).
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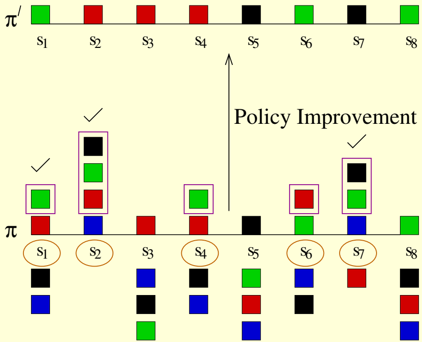

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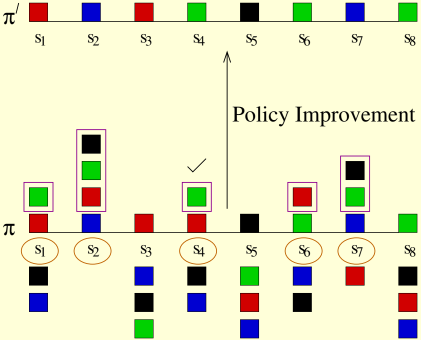

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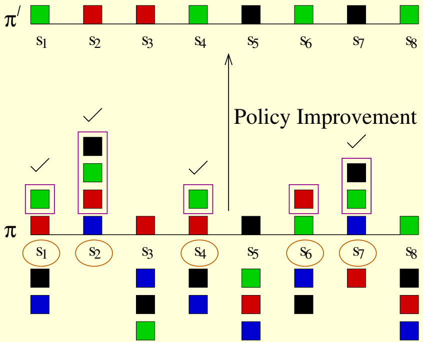
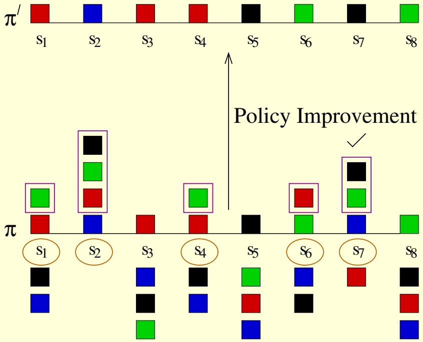


Simple Policy Iteration

- Reference: Melekopoglou and Condon (1994).
- Assume a fixed indexing of states.
- Switch the improvable state with the highest index.



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- Observe that for $\pi, \pi' \in \Pi, \forall s \in S$:

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&\implies \forall \pi' \in \Pi : V^{\pi} \succeq V^{\pi'}.
\end{aligned}
$$

$\mathbf{IS}(\pi) \neq \emptyset$ and policy improvement on π yields π'

$$
\begin{aligned}
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A More General Class of Policies

- In principle, an agent can follow a policy λ that maps every possible history $s^0, a^0, r^0, s^1, a^1, r^1, \ldots, s^t$ for $t \geq 0$ to a probability distribution over A.
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Could there exist $\lambda \in \Lambda \setminus \Pi$ such that $\neg(\pi^\star \succeq \lambda)$? No.

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- Optimal policies for the finite horizon reward setting are in general non-stationary (time-dependent).
- Optimal policies ("strategies") in many types of multi-player games are in general stochastic ("mixed").

Markov Decision Problems

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Running-time Bounds

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- **Value Iteration**: $V_0 \to V_1 \to \cdots \to V_\infty = V^{\pi^\star}$.
  Upper Bound: poly$(n, k, B, \frac{1}{1-\gamma})$. iterations [LKM95]; B is the number of bits used to represent the MDP.

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  $\text{poly}(n, k, B)$ [K80, K84].
  $\text{poly}(n, k) \cdot \exp(O(\sqrt{n \log(n)}))$ (Expected) [MSW96].

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- Strong bounds are those that depend solely on n and k.
- **Policy Iteration** naturally yields strong bounds (also enjoys good weak bounds [P94]). We review strong bounds.

PI: Switching Strategies and Bounds

Upper bounds on number of iterations

PI Variant	**Type**	$k = 2$	**General** k
Howard's (Greedy) PI [H60, MS99]	Deterministic	$O\left(\frac{2^n}{n}\right)$	$O\left(\frac{k^n}{n}\right)$
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Lower bounds on number of iterations

$\Omega(n)$ Howard's PI on n-state, 2-action MDPs [HZ10].

$\Omega(2^n)$ Simple PI on n-state, 2-action MDPs [MC94].

PI: Some Recent Results ($k = 2$)

(Authors with names underlined once took CS 747!)

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- Kalyanakrishnan, Mall, and Goyal (2016) devise the Batch-switching PI algorithm (deterministic), and show an upper bound of 1.6479^n iterations.

- Taraviya and Kalyanakrishnan (2019) use a similar approach to provide an upper bound of 1.6001^n iterations (in expectation) for a randomised PI variant.

PI: Some Recent Results ($k \geq 2$)

- Gupta and Kalyanakrishnan (2017) give a deterministic PI variant with an upper bound of $k^{0.7207n}$ iterations, improved by Taraviya and Kalyanakrishnan to $k^{0.7019}$ iterations.

- Kalyanakrishnan, Misra, and Gopalan (2016) show an upper bound of $(2 + \ln(k-1))^n$ iterations for a randomised PI variant.

- Taraviya and Kalyanakrishnan (2019) show an upper bound of $(\sqrt{k \log(k)})^n$ iterations for a randomised variant of Howard's PI.

- Ashutosh, Consul, Dedhia, Khirwadkar, Shah, and Kalyanakrishnan (2020) show a lower bound of $\sqrt{k}^n$ iterations for a particular deterministic variant of PI.

Open Problems

- Is the complexity of Howard's PI on 2-action MDPs upper-bounded by the Fibonacci sequence ($\approx 1.6181^n$)?
- Is Howard's PI the most efficient among deterministic PI algorithms (worst case over all MDPs)?
- Is there a super-linear lower bound on the number of iterations taken by Howard's PI on 2-action MDPs?
- Is Howard's PI strongly polynomial on deterministic MDPs?
- Is there a variant of PI that can visit all k^n policies in some n-state, k-action MDP—implying an $\Omega(k^n)$ lower bound?
- Is there a strongly polynomial algorithm for MDP planning?

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Summary of MDP Planning

- MDPs are an abstraction of sequential decision making.
- Many applications; many different formulations.
- Essential solution concept: optimal policy (known to exist).

- Three main families of planning algorithms: Value Iteration, Linear Programming, Policy Iteration.
- Have strengths and weaknesses in theory and in practice.
- Can combine (especially Value Iteration, Policy Iteration).

- We showed correctness of all three methods.
- Used Banach's fixed-point theorem, Bellman optimality operator, Bellman operator.

- What if T, R were not given, but have to be *learned* from interaction? Can we still learn to act optimally?
- Yes: that's the Reinforcement Learning problem. Next week!