CS 747, Autumn 2020: Week 5, Lecture 1

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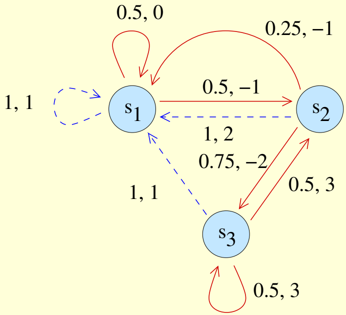
Autumn 2020

Summary of Previous Lecture

1. Definitions
   - MDP (S, A, T, R, γ)
   - Policy (π)
   - Value Function (V^{π})
2. MDP planning
3. Alternative formulations
4. Applications
5. Policy Evaluation



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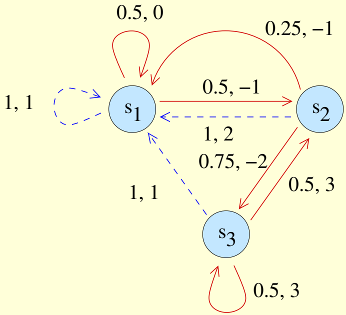

What is coming up this week?

Markov Decision Problems

1. Bellman optimality
   - Banach's fixed-point theorem
   - Bellman optimality operator

2. Value Iteration

3. Linear Programming formulation
   - Review of LP
   - MDP Planning as LP

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Complete, Normed Vector Spaces

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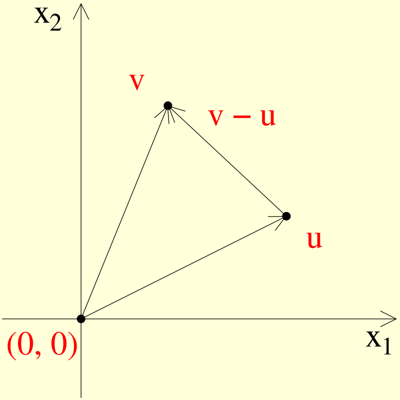

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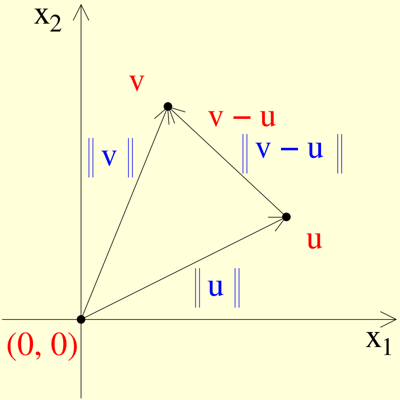

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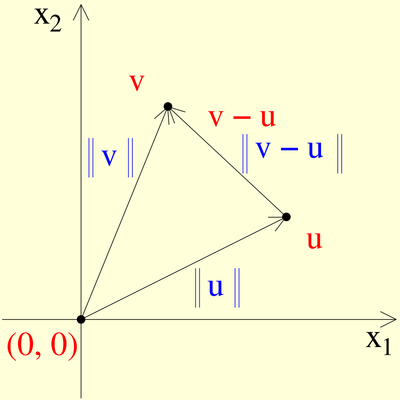
- A complete, normed vector space $(X, \|\cdot\|)$ is one in which every Cauchy sequence has a limit in X.



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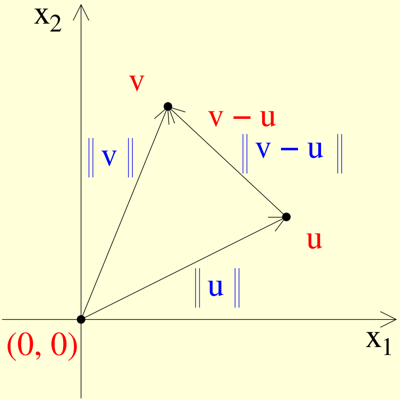

- A complete, normed vector space is called a Banach space.

Two Definitions

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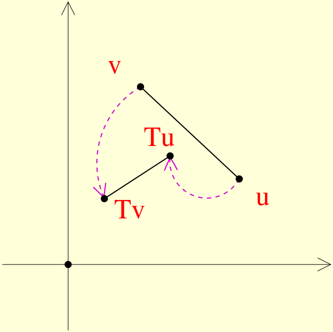
$$\|Tv - Tu\| \leq L\|v - u\|.$$



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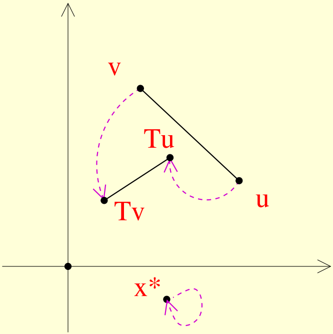

- **Fixed-point.** $x^\star \in X$ is called a fixed-point of T if $Tx^\star = x^\star$.

Banach's Fixed-point Theorem

(Adapted from Szepesvári, 2010 (see Appendix A.1).)

Let $(X, \|\cdot\|)$ be a Banach space, and let $T : X \to X$ be a contraction mapping with contraction factor $L \in [0, 1)$. Then:

1. T has a unique fixed point $x^\star \in X$.
2. For $x \in X, m \geq 0$: $\|T^m x - x^\star\| \leq L^m \|x - x^\star\|$.

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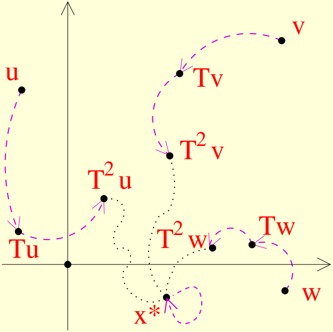

Bellman Optimality Operator

- The Bellman optimality operator $B^\star : (S \to \mathbb{R}) \to (S \to \mathbb{R})$ for an MDP (S, A, T, R, γ) is defined as follows.
  For $F : S \to \mathbb{R}$ and $s \in S$:

$$(B^\star(F))(s) \stackrel{\text{def}}{=} \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma F(s')\}.$$

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- Recall that the max norm $\|\cdot\|_\infty$ of $F = (f_1, f_2, \ldots, f_n) \in \mathbb{R}^n$ is

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Fact. $B^\star$ is a contraction mapping in the $(\mathbb{R}^n, \|\cdot\|_\infty)$ Banach space with contraction factor γ.

Proof that $B^\star$ is a Contraction Mapping

We use: $|\max_a f(a) - \max_a g(a)| \leq \max_a |f(a) - g(a)|$.

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$$\|B^\star(F) - B^\star(G)\|_\infty = \max_{s \in S} |(B^\star(F))(s) - (B^\star(G))(s)|$$

$$= \max_{s \in S} \left| \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma F(s')\} - \right.$$

$$\left. \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma G(s')\} \right|$$

$$\leq \gamma \max_{s \in S} \max_{a \in A} \left| \sum_{s' \in S} T(s, a, s')\{F(s') - G(s')\} \right|$$

$$\leq \gamma \max_{s \in S} \max_{a \in A} \sum_{s' \in S} T(s, a, s') |F(s') - G(s')|$$

$$\leq \gamma \max_{s \in S} \max_{a \in A} \sum_{s' \in S} T(s, a, s') \|F - G\|_\infty = \gamma \|F - G\|_\infty.$$

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- Denote the fixed point $V^{\star} : S \to \mathbb{R}$. Note that $B^{\star}(V^{\star}) = V^{\star}$. In other words, for $s \in S$:

$$\boxed{V^{*}(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s') \{R(s, a, s') + \gamma V^{*}(s')\}.}$$

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- **Fact**. $V^\star$ is the value function of every policy $\pi^\star : S \to A$ that satisfies, for all $s \in S$:

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- We shall prove next week that every such policy $\pi^\star$ is an optimal policy. Hence $V^\star$ is the optimal value function.

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$V_0 \leftarrow$ Arbitrary, element-wise bounded, n-length vector.
$t \leftarrow 0$.
Repeat:
  For $s \in S$**:**
    $V_{t+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} T(s, a, s') \left(R(s, a, s') + \gamma V_t(s')\right)$.
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Until $V_t \approx V_{t-1}$ (up to machine precision).

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- Popular; easy to implement; quick to converge in practice.

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  For $s \in S, a \in A$:
  $$Q^\star(s, a) = \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma V^\star(s')\}.$$
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  Solve Bellman equations for $\pi^\star$!

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- Today's solvers (commercial, as well as open source) can handle LPs with millions of variables.

Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.

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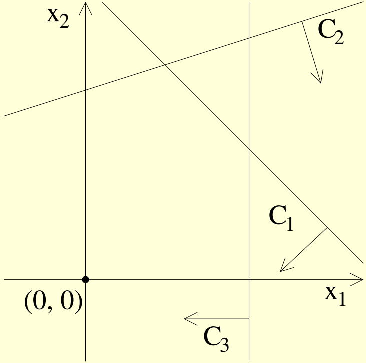

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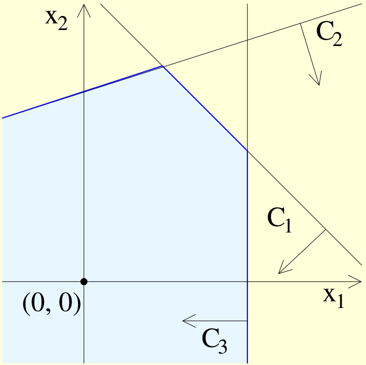

Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.
- **Step 2**: Identify points within the feasible set that maximise the objective. Usually a single point.

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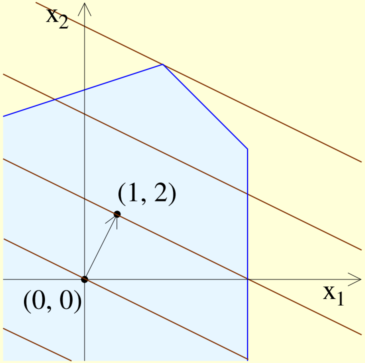

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Can we construct an objective function for which $V^\star$ is the sole optimiser?

Vector Comparison

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$$X \succeq Y \iff \forall s \in S : X(s) \geq Y(s),$$
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- Also note that if $\pi_1 \succeq \pi_2$ and $\pi_2 \succeq \pi_1$, then $V^{\pi_1} = V^{\pi_2}$.

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- **Fact.** For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$,
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$$\begin{aligned}
&(B^\star(X))(s) - (B^\star(Y))(s) \\
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&\qquad \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma Y(s')\} \\
&\geq \gamma \min_{a \in A} \sum_{s' \in S} T(s, a, s')\{X(s') - Y(s')\} \geq 0.
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Examining the Feasible Set of our LP

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- We "linearise" this result: for $V: S \to R$ in the feasible set.
$$\sum_{s \in S} V(s) \geq \sum_{s \in S} V^\star(s).$$

Linear Programming Formulation

Maximise $\left(-\sum_{s \in S} V(s)\right)$

subject to

$$V(s) \geq \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma V(s')\}, \forall s \in S, a \in A.$$

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- There is also a *dual* LP formulation with nk variables and n constraints. See Littman et al. (1995) if interested.

Markov Decision Problems

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   - Bellman optimality operator

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Next week: Policy Iteration, proof of optimality of $\pi^\star$.