CS 747, Autumn 2020: Week 3, Lecture 1

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Autumn 2020

Multi-armed Bandits

1. Concentration bounds
2. Analysis of UCB
3. Understanding Thompson Sampling
4. Other bandit problems

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- Then, for or any fixed $\epsilon > 0$, we have

$$\mathbb{P}\{\bar{x} \geq \mu + \epsilon\} \leq e^{-2u\epsilon^2}, \text{ and}$$
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- Note the bounds are trivial for large ϵ, since $\bar{x} \in [0, 1]$.

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  With probability at least $1 - \delta$, the empirical mean $\bar{x}$ exceeds the true mean μ by at most $\epsilon_0 =$ __________.
  We can write $\epsilon_0 = \sqrt{\frac{1}{2u} \ln(\frac{1}{\delta})}$; by Hoeffding's Inequality:

$$\mathbb{P}\{\bar{x} \geq \mu + \epsilon_0\} \leq e^{-2u(\epsilon_0)^2} \leq \delta.$$

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  Consider $Y = \frac{X-a}{b-a}$; for $1 \leq i \leq u$, $y_i = \frac{x_i - a}{b-a}$; $\bar{y} = \frac{1}{u}\sum_{i=1}^{u} y_i$.

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  Since Y is bounded in $[0, 1]$, we get

$$\mathbb{P}\{\bar{x} \geq \mu + \epsilon\} = \mathbb{P}\left\{\bar{y} \geq \frac{\mu - a}{b-a} + \frac{\epsilon}{b-a}\right\} \leq e^{-\frac{2u\epsilon^2}{(b-a)^2}}, \text{ and}$$

$$\mathbb{P}\{\bar{x} \leq \mu - \epsilon\} = \mathbb{P}\left\{\bar{y} \leq \frac{\mu - a}{b-a} - \frac{\epsilon}{b-a}\right\} \leq e^{-\frac{2u\epsilon^2}{(b-a)^2}}.$$

A "KL" Inequality

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- Then, for or any fixed $\epsilon \in [0, 1 - \mu]$, we have

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and for or any fixed $\epsilon \in [0, \mu]$, we have

$$\boxed{\mathbb{P}\{\bar{x} \leq \mu - \epsilon\} \leq e^{-uKL(\mu-\epsilon, \mu)},}$$

where for $p, q \in [0, 1]$, $KL(p, q) \stackrel{\text{def}}{=} p \ln(\frac{p}{q}) + (1 - p) \ln(\frac{1-p}{1-q})$.

Some Observations

- The KL inequality gives a tighter upper bound:
  For $p, q \in [0, 1]$,

$$KL(p, q) \geq 2(p - q)^2 \implies e^{-uKL(p,q)} \leq e^{-2u(p-q)^2}.$$

- Both bounds are instances of "Chernoff bounds", of which there are many more forms.

- Similar bounds can also be given when X has infinite support (such as a Gaussian), but might need additional assumptions.

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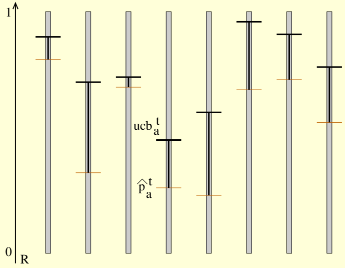
UCB (Auer *et al.*, 2002)

- Algorithm
  - Pull each arm once.
  - At time $t \in \{n, n+1, \dots\}$, for every arm a, $\text{ucb}_a^t \stackrel{\text{def}}{=} \hat{p}_a^t + \sqrt{\frac{2\ln(t)}{u_a^t}}$; pull $\text{argmax}_a \, \text{ucb}_a^t$.



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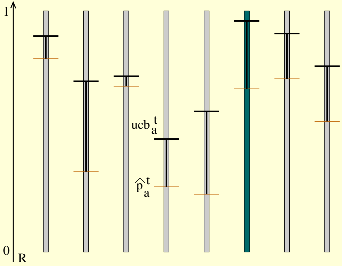

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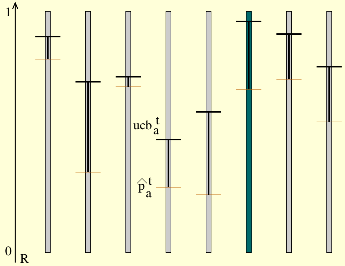

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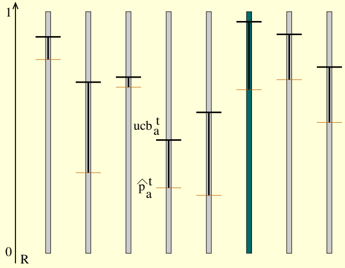

- Recall that $R_T = Tp^\star - \sum_{t=0}^{T-1} \mathbb{E}[r^t]$.
- We shall show that UCB achieves
  $R_T = O\left(\sum_{a:p_a \neq p^\star} \frac{1}{p^\star - p_a} \log(T)\right)$.

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- We define an instance-specific **constant** $\bar{u}_a^T \stackrel{\text{def}}{=} \left\lceil \frac{8}{(\Delta_a)^2} \ln(T) \right\rceil$ that will serve in our proof as a "sufficient" number of pulls of arm a for horizon T.

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&= \sum_{a \in A} \mathbb{E}[u_a^T] (p^\star - p_a)
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&= \sum_{a \in A} \mathbb{E}[u_a^T](p^\star - p_a) = \sum_{a: p_a \neq p^\star} \mathbb{E}[u_a^T] \Delta_a.
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$$= \sum_{a \in A} \mathbb{E}[u_a^T](p^\star - p_a) = \sum_{a:p_a \neq p^\star} \mathbb{E}[u_a^T]\Delta_a.$$

To show the regret bound, we shall show for each sub-optimal arm a that

$$\mathbb{E}[u_a^T] = O\left(\frac{1}{(\Delta_a)^2} \log(T) \right).$$

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\end{aligned}$$

We show A is upper-bounded by $\bar{u}_a^T$ and B is upper-bounded by a constant.

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$$\leq \sum_{m=0}^{\bar{u}_a^T - 1} 1$$

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$$A = \sum_{t=0}^{T-1} \mathbb{P}\{Z_a^t \text{ and } (u_a^t < \bar{u}_a^T)\}$$

$$= \sum_{t=0}^{T-1} \sum_{m=0}^{\bar{u}_a^T - 1} \mathbb{P}\{Z_a^t \text{ and } (u_a^t = m)\} = \sum_{m=0}^{\bar{u}_a^T - 1} \sum_{t=0}^{T-1} \mathbb{P}\{Z_a^t \text{ and } (u_a^t = m)\}$$

$$= \sum_{m=0}^{\bar{u}_a^T - 1} \mathbb{P}\{Z_a^0, (u_a^0 = m) \text{ or } Z_a^1, (u_a^1 = m) \text{ or } \dots \text{ or } \ Z_a^{T-1}, (u_a^{T-1} = m)\}$$

$$\leq \sum_{m=0}^{\bar{u}_a^T - 1} 1 = \bar{u}_a^T.$$

Step 3: Bounding A

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\begin{aligned}
A &= \sum_{t=0}^{T-1} \mathbb{P}\{Z_a^t \text{ and } (u_a^t < \bar{u}_a^T)\} \\
&= \sum_{t=0}^{T-1} \sum_{m=0}^{\bar{u}_a^T-1} \mathbb{P}\{Z_a^t \text{ and } (u_a^t = m)\} = \sum_{m=0}^{\bar{u}_a^T-1} \sum_{t=0}^{T-1} \mathbb{P}\{Z_a^t \text{ and } (u_a^t = m)\} \\
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&\leq \sum_{m=0}^{\bar{u}_a^T-1} 1 = \bar{u}_a^T.
\end{aligned}
$$

We have used the fact that for $0 \leq i < j \leq t-1$, $(Z_a^i, (u_a^i = m))$ and $(Z_a^j, (u_a^j = m))$ are mutually exclusive.

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&\leq \sum_{t=0}^{T-1} \sum_{x=\bar{u}_a^T}^{t} \sum_{y=1}^{t} \mathbb{P}\left\{\hat{p}_a(x) + \sqrt{\frac{2}{x}\ln(t)} \geq \hat{p}_\star(y) + \sqrt{\frac{2}{y}\ln(t)}\right\} \text{ where}
\end{aligned}
$$

$\hat{p}_a(x)$ is the empirical mean of the first x pulls of arm a, and $\hat{p}_\star(y)$ is the empirical mean of the first y pulls of arm $\star$.

Step 4.2: Bounding B

- Fix $x \in \{\bar{u}_a^T, \bar{u}_a^T + 1, \ldots, t\}$ and $y \in \{1, 2, \ldots, t\}$.

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$$\hat{p}_a(x) + \sqrt{\frac{2}{x}\ln(t)} \geq \hat{p}_\star(y) + \sqrt{\frac{2}{y}\ln(t)}$$

$$\implies \left(\hat{p}_a(x) + \sqrt{\frac{2}{x}\ln(t)} \geq p_\star\right) \text{ or } \left(\hat{p}_\star(y) + \sqrt{\frac{2}{y}\ln(t)} < p_\star\right).$$

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Fact: If $\alpha > \beta$, then $\alpha \geq \gamma$ or $\beta < \gamma$. Holds for arbitrary α, β, γ!

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2. Since $x \geq \bar{u}_a^T$, we have $\sqrt{\frac{2}{x}\ln(t)} \leq \sqrt{\frac{2}{\bar{u}_a^T}\ln(t)} \leq \frac{\Delta_a}{2}$, and so

$$\hat{p}_a(x) + \sqrt{\frac{2}{x}\ln(t)} \geq p_\star \implies \hat{p}_a(x) \geq p_a + \frac{\Delta_a}{2}.$$

Step 4.3: Bounding B

Continuing from Step 4.1, using the two results from Step 4.2, and invoking Hoeffding's Inequality:

$$B \leq \sum_{t=0}^{T-1} \sum_{x=\bar{u}_a^T}^{t} \sum_{y=1}^{t} \mathbb{P}\left\{\hat{p}_a(x) + \sqrt{\frac{2}{x}\ln(t)} \geq \hat{p}_\star(y) + \sqrt{\frac{2}{y}\ln(t)}\right\}$$

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We are done!

Summary of Proof

- To upper-bound regret, upper-bound the number of pulls of each sub-optimal arm a.
- Give each such arm a $\bar{u}_a^T$ pulls for free.
- Beyond $\bar{u}_a^T$ pulls, arm a's UCB will have width at most $\Delta_a/2$.
- If a continues to be pulled beyond $\bar{u}_a^T$ pulls, either its empirical mean has deviated by more than $\Delta_a/2$ from its true mean, or $\star$'s UCB has fallen below its true mean.
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- KL-UCB uses the KL inequality, and avoids the naive "union bound" to transform the random number pulls to a constant.

Multi-armed Bandits

1. Concentration bounds
2. Analysis of UCB
3. Understanding Thompson Sampling
4. Other bandit problems

Thompson Sampling (Thompson, 1933)

- At time t, arm *a* has s_a^t successes (1's) and f_a^t failures (0's).

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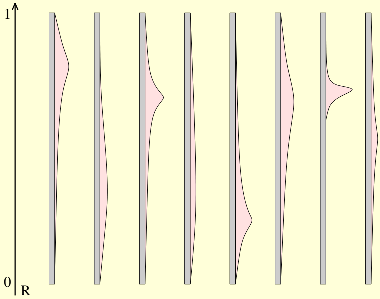

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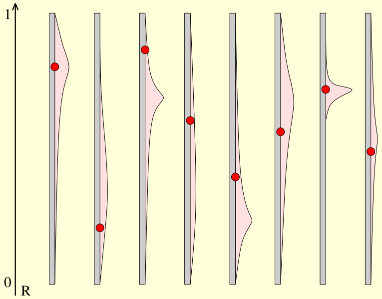

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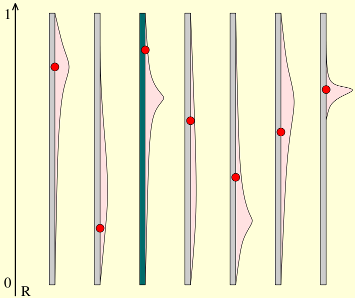

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$$Belief_m = \mathbb{P}\{w|e_1, e_2, \ldots, e_m\}$$

Bayesian Inference

$$\begin{aligned}
Belief_{m+1}(w) &= \mathbb{P}\{w|e_1, e_2, \dots, e_{m+1}\} \\
&= \frac{\mathbb{P}\{e_1, e_2, \dots, e_{m+1}|w\}\mathbb{P}\{w\}}{\mathbb{P}\{e_1, e_2, \dots, e_{m+1}\}} \\
&= \frac{\mathbb{P}\{e_1, e_2, \dots, e_m|w\}\mathbb{P}\{e_{m+1}|w\}\mathbb{P}\{w\}}{\mathbb{P}\{e_1, e_2, \dots, e_{m+1}\}} \\
&= \frac{\mathbb{P}\{e_1, e_2, \dots, e_m, w\}\mathbb{P}\{e_{m+1}|w\}}{\mathbb{P}\{e_1, e_2, \dots, e_{m+1}\}} \\
&= \frac{\mathbb{P}\{w|e_1, e_2, \dots, e_m\}\mathbb{P}\{e_1, e_2, \dots, e_m\}\mathbb{P}\{e_{m+1}|w\}}{\mathbb{P}\{e_1, e_2, \dots, e_{m+1}\}} \\
&= \frac{Belief_m(w)\mathbb{P}\{e_{m+1}|w\}}{\sum_{w' \in W} Belief_m(w')\mathbb{P}\{e_{m+1}|w'\}}.
\end{aligned}$$

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- We achieve exactly that by taking

$$Belief_m(x) = Beta_{s+1,f+1}(x)dx$$

when the first m pulls yield s 1's and f 0's!

Principle of Selecting Arm to Pull

- We have a belief distribution for each arm's mean.
- Together, these distributions represent a belief distribution over bandit instances.
- We sample a bandit instance I from the joint belief distribution, and
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- Alternative interpretation: the probability with which we pick an arm is our belief that it is optimal. For example, if $A = \{1, 2\}$, the probability of pulling 1 is $\mathbb{P}\{x_1^t > x_2^t\} =$

$$\int_{x_1=0}^{1} \int_{x_2=0}^{x_1} Beta_{s_1^t+1, f_1^t+1,}(x_1) Beta_{s_2^t+1, f_2^t+1,}(x_2) dx_2 dx_1.$$

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- What if the arms' (true) means vary over time?
  - Nonstationary setting, seen for example, in on-line ads.
  - Approach depends on nature of drift/change in rewards.
  - In practice, one might only trust most recent data from arms.
  - In practice, the set of arms can itself change over time!

Other Bandit Problems

- Pure exploration.
  - Separate "testing" and "live" phases.
  - In testing phase, rewards don't matter.
  - PAC formulation: W.p. at least $1 - \delta$, must return an ϵ-optimal arm, while incurring a small number of pulls.
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- Limited number of feedback stages.
  - Suppose you are given budget T, but your algorithm can look at history only $s < T$ times?
  - UCB, Thompson Sampling, etc. are fully sequential ($s = T$).
  - How to manage with fewer "stages" s?

Other Bandit Problems

- What if the number of arms is large (thousands, millions)?
  - If arms can be described using features, mean reward is often treated as a (linear) function of these features.
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- What if the rewards aren’t from a fixed random process?
  - Adversarial bandits make no assumption on the rewards.
  - Possible to show sub-linear regret when compared against playing a single arm for the entire run.
  - Necessary to use a randomised algorithm.