CS 747, Autumn 2020: Week 2, Lecture 1

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Autumn 2020

Multi-armed Bandits

1. Achieving sub-linear regret
2. A lower bound on regret
3. UCB, KL-UCB algorithms
4. Thompson Sampling algorithm
5. Summary and outlook

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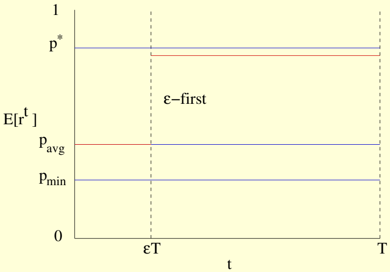

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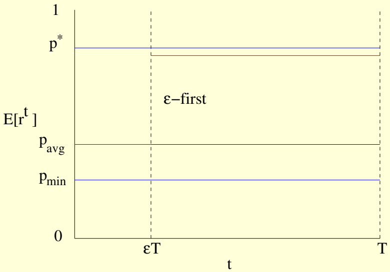

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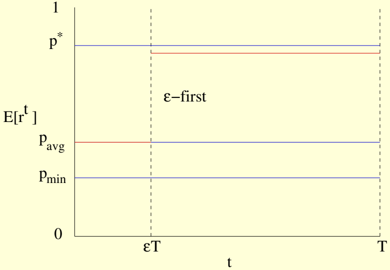

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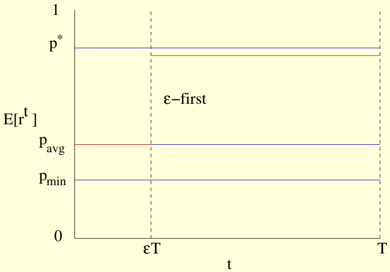

$$R_T = Tp^\star - \sum_{t=0}^{T-1} \mathbb{E}[r^t] = Tp^\star - \sum_{t=0}^{\epsilon T-1} \mathbb{E}[r^t] - \sum_{t=\epsilon T}^{T-1} \mathbb{E}[r^t]$$

$$= Tp^\star - \epsilon Tp_{\text{avg}} - \sum_{t=\epsilon T}^{T-1} \mathbb{E}[r^t] \geq Tp^\star - \epsilon Tp_{\text{avg}} - (T - \epsilon T)p^\star$$

$$= \epsilon(p^\star - p_{\text{avg}})T = \Omega(T).$$

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- ϵ-greedy: On each step
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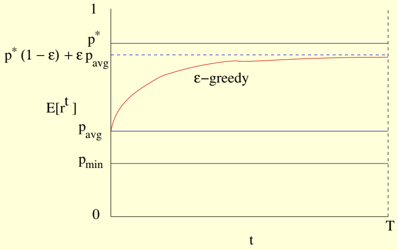

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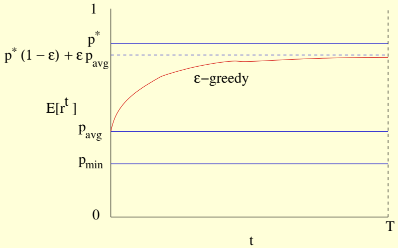

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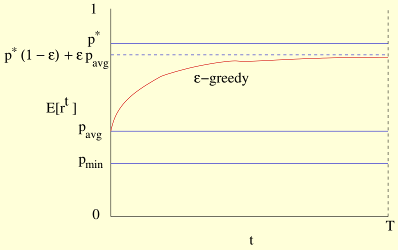

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\begin{aligned}
R_T &= Tp^\star - \sum_{t=0}^{T-1} \mathbb{E}[r^t] \\
&\geq Tp^\star - \sum_{t=0}^{T-1} ((\epsilon)p_{\text{avg}} + (1-\epsilon)p^\star) \\
&= \epsilon(p^\star - p_{\text{avg}})T = \Omega(T).
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- On the contrary, suppose we start exploiting after pulling each arm a finite U times.
- With probability $(1 - p^{\star})^{U} > 0$, an optimal arm will have empirical mean 0.
- A non-optimal arm may thereafter be "exploited" for ever, giving linear regret.

How to achieve Sub-linear Regret?

C2. **Greed in the Limit**. Let *exploit*(T) denote the number of pulls that are greedy w.r.t. the empirical mean up to horizon T. For sub-linear regret, we need

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- **Result.** An algorithm L achieves sub-linear regret on all instances $I \in \bar{\mathcal{I}}$ if and only if it satisfies C1 and C2 on all $I \in \bar{\mathcal{I}}$. In short: "GLIE" $\iff$ sub-linear regret.

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  What happened when we took $\epsilon_t = \epsilon$? What will happen by taking $\epsilon_t = \frac{1}{(t+1)^2}$?

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A Lower Bound on Regret

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- Paraphrasing Lai and Robbins (1985; see Theorem 2).

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$$R_T(L, I) = o(T^\alpha).$$

Then, for every bandit instance $I \in \bar{\mathcal{I}}$, as $T \to \infty$:

$$\frac{R_T(L, I)}{\ln(T)} \geq \sum_{a: p_a(I) \neq p^\star(I)} \frac{p^*(I) - p_a(I)}{KL(p_a(I), p^\star(I))},$$

where for $x, y \in [0, 1)$, $KL(x, y) \stackrel{\text{def}}{=} x \ln \frac{x}{y} + (1 - x) \ln \frac{1-x}{1-y}$.

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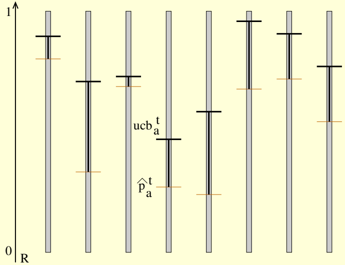
Upper Confidence Bounds

- UCB (Auer et al., 2002)
  - At time t, for every arm a, define $\text{ucb}_a^t = \hat{p}_a^t + \sqrt{\frac{2\ln(t)}{u_a^t}}$.
  - $\hat{p}_a^t$ is the empirical mean of rewards from arm a.
  - u_a^t the number of times a has been sampled at time t.



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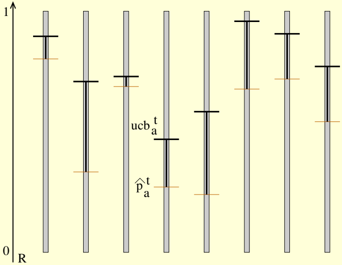

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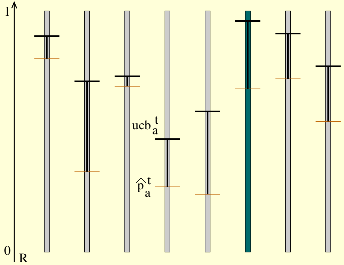

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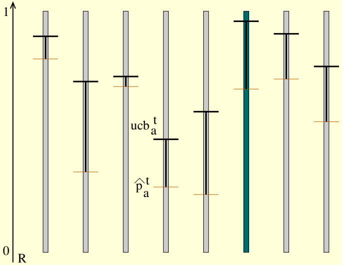

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- Achieves regret of $O(\log(T))$: optimal dependence on T.

KL-UCB (Garivier and Cappé, 2011)

- Identical to UCB algorithm on previous slide, except for a different definition of the upper confidence bound.

  ucb-kl$_a^t = \max\{q \in [\hat{p}_a^t, 1]$ such that
  $u_a^t KL(\hat{p}_a^t, q) \leq \ln(t) + c \ln(\ln(t))\}$, where $c \geq 3$.

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  Easy to compute ucb-kl_a^t numerically (for example through binary search).

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  Regret of KL-UCB asymptotically matches Lai and Robbins' lower bound!

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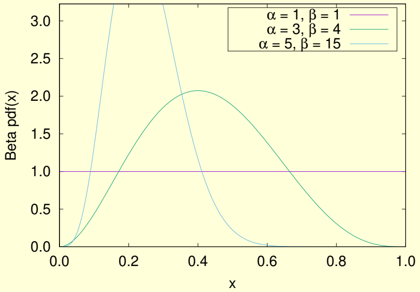
Before Moving on . . . The Beta Distribution

- Beta(α, β) defined on $[0, 1]$.
  Two parameters: α and β.

$$\text{Mean} = \frac{\alpha}{\alpha + \beta}; \quad \text{Variance} = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$



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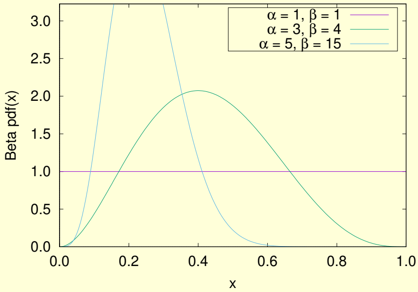

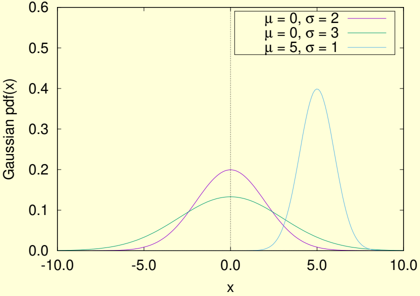

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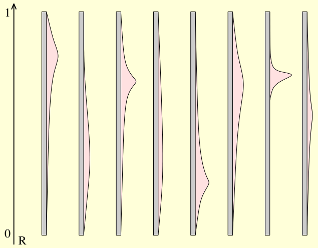

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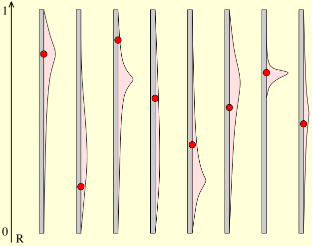

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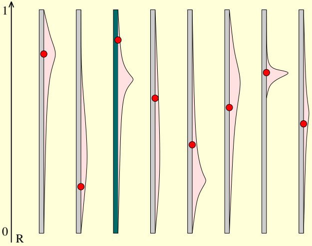

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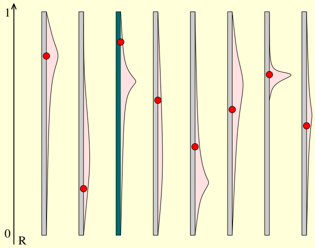

- Achieves optimal regret (Kaufmann et al., 2012); is excellent in practice (Chapelle and Li, 2011).

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Summary

- We desire low, sub-linear regret on all bandit instances.
- Possible if and only if algorithm satisfies GLIE conditions.
- If an algorithm gives sub-polynomial regret on all instances, it must give super-logarithmic regret on all instances (Lai and Robbins, 1985).

- UCB algorithm achieves logarithmic dependence on T.
- KL-UCB also improves the accompanying constant, thereby matching the lower bound (asymptotically).
- Thompson Sampling, a qualitatively different randomised algorithm, also matches regret lower bound.
- UCB, KL-UCB, Thompson Sampling all examples of optimism in the face of uncertainty principle.

- Next week: concentration inequalities, analysis of UCB, KL-UCB, Thompson Sampling, other bandit formulations.