CS 747, Autumn 2020: Week 4, Lecture 1

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Autumn 2020

Markov Decision Problems

1. Definitions
   - Markov Decision Problem
   - Policy
   - Value Function
2. MDP planning
3. Alternative formulations
4. Applications
5. Policy Evaluation

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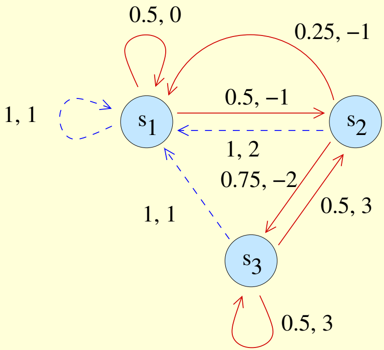
Markov Decision Problems (MDPs)



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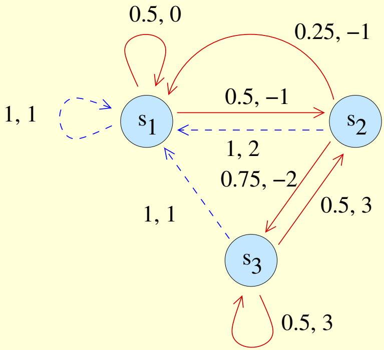

An MDP $M = (S, A, T, R, \gamma)$ has these elements.

S: a set of states.

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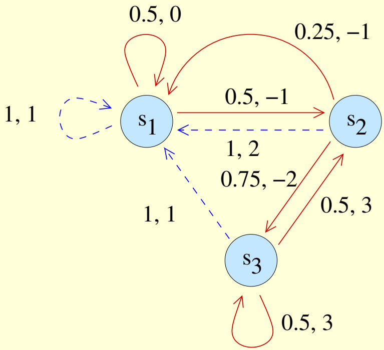

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Let us assume $S = \{s_1, s_2, \ldots, s_n\}$, and hence $|S| = n$.

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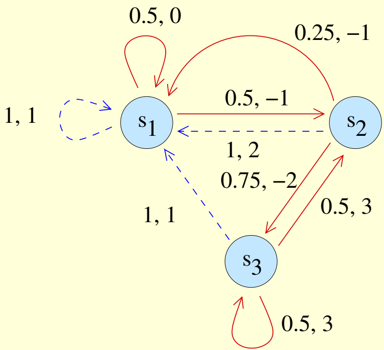

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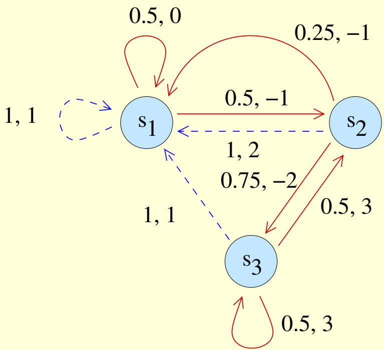


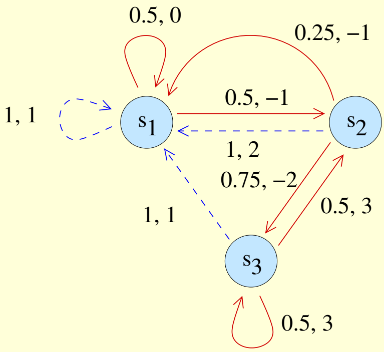
An MDP $M = (S, A, T, R, \gamma)$ has these elements.

A: a set of actions.

Let us assume $A = \{a_1, a_2, \ldots, a_k\}$, and hence $|A| = k$.

Here $A = \{\text{RED}, \text{BLUE}\}$.

Markov Decision Problems (MDPs)



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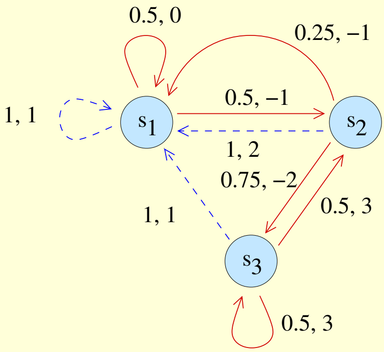

An MDP $M = (S, A, T, R, \gamma)$ has these elements.

T: a transition function.

- For $s, s' \in S, a \in A$: $T(s, a, s')$ is the probability of reaching s' by starting at s and taking action a.
- Thus, $T(s, a, \cdot)$ is a probability distribution over S.

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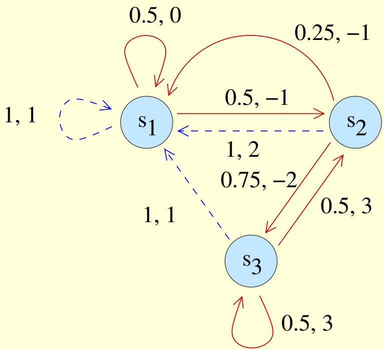

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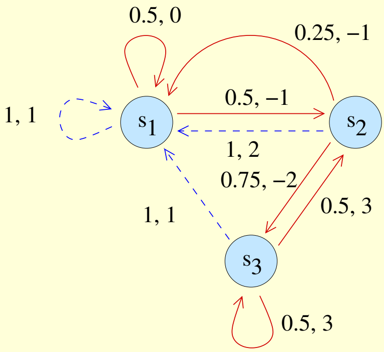

An MDP $M = (S, A, T, R, \gamma)$ has these elements.

R: a reward function.

- For $s, s' \in S, a \in A$: $R(s, a, s')$ is the (numeric) reward for reaching s' by starting at s and taking action a.
- Assume rewards are from $[-R_{\max}, R_{\max}]$ for some $R_{\max} \geq 0$.

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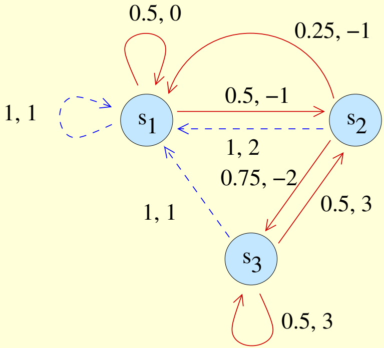

An MDP $M = (S, A, T, R, \gamma)$ has these elements.

γ, a discount factor—coming up shortly.

Agent-Environment Interaction

$t = 0$

Agent is born in some state s^0, takes action a^0.

Environment generates and provides the agent

next state $s^1 \sim T(s^0, a^0, \cdot)$ and

reward $r^0 = R(s^0, a^0, s^1)$.

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$t = 1$

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Resulting trajectory: $s^0, a^0, r^0, s^1, a^1, r^1, s^2, \ldots$.

Describing the Agent's Behaviour

$$\begin{tikzcd}[column sep=huge] \text{Agent} \arrow[r, shift left=2, "a^t"] & \text{Environment} \arrow[l, shift left=2, "{r^t, s^{t+1}}"] \end{tikzcd}$$

Describing the Agent's Behaviour

$$\text{Agent} \underset{r^t, s^{t+1}}{\overset{a^t}{\rightleftarrows}} \text{Environment}$$

- How does the agent pick a^t?

Describing the Agent's Behaviour

$$\xrightarrow{\quad a^t \quad}$$

Agent Environment

$$\xleftarrow{\quad r^t, s^{t+1} \quad}$$

- How does the agent pick a^t?
  In principle, it can decide by looking at the preceding history

$$s^0, a^0, r^0, s^1, a^1, r^1, s^2, \ldots, s^t.$$

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- In other words, the agent follows a policy $\pi : S \to A$.

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  Observe that π is Markovian, deterministic, and stationary.

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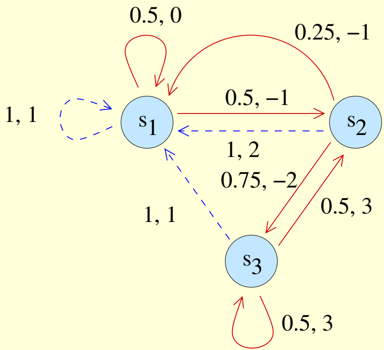
  Observe that π is Markovian, deterministic, and stationary.

  We will justify this choice in due course!

Illustration: Policy



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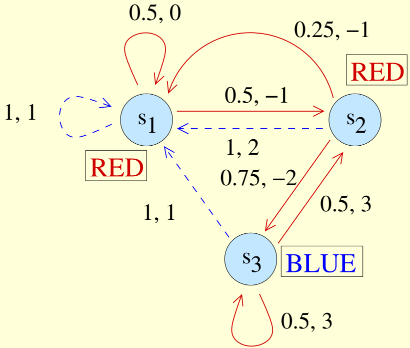

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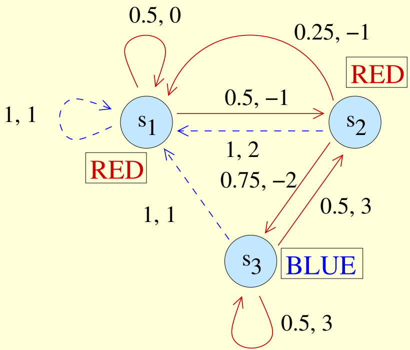

- Illustrated policy π such that

$$\pi(s_1) = \text{RED}; \pi(s_2) = \text{RED}; \pi(s_3) = \text{BLUE}.$$

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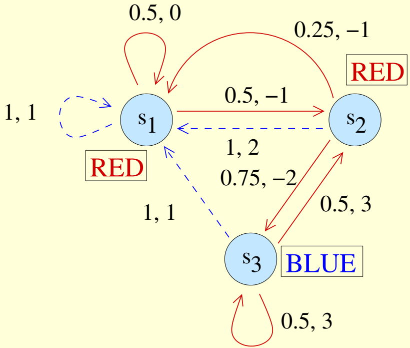

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What happens by "following" π, starting at s_1?

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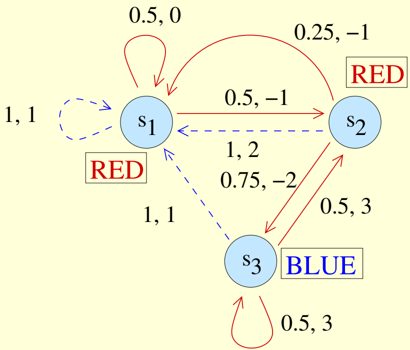

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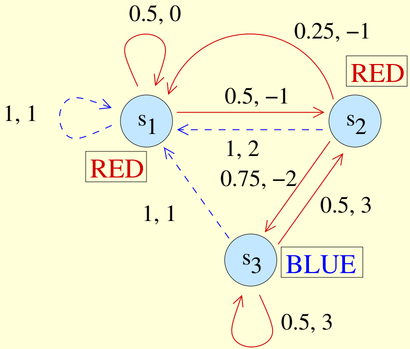
What happens by "following" π, starting at s_1?

  - s_1, RED, s_1, RED, s_2, RED, s_3, BLUE, s_1, ...
  - s_1, RED, s_2, RED, s_1, RED, s_1, RED, s_1, . . .

Illustration: Policy



- Let Π denote the set of all policies.

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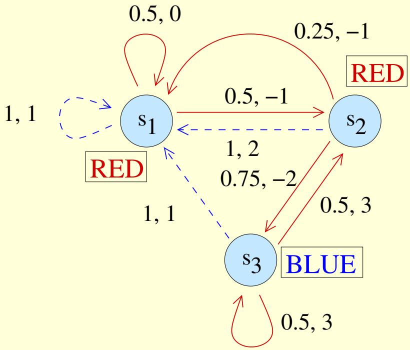

- Let Π denote the set of all policies.
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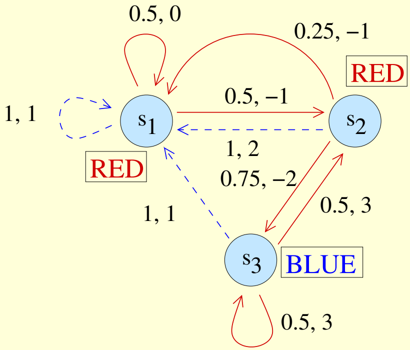

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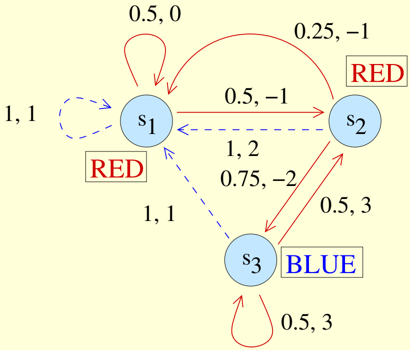

- Let Π denote the set of all policies.
- What is $|\Pi|$? k^n.
- Which $\pi \in \Pi$ is a "good" policy?

State Values for Policy π

- For $s \in S$, $V^{\pi}(s) \stackrel{\text{def}}{=} \mathbb{E}_{\pi}\left[r^0 + \quad r^1 + \quad\quad r^2 + \quad\quad r^3 + \ldots | s^0 = s\right]$

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- For $s \in S$, $V^{\pi}(s) \stackrel{\text{def}}{=} \mathbb{E}_{\pi}\left[r^0 + \gamma r^1 + \gamma^2 r^2 + \gamma^3 r^3 + \ldots | s^0 = s\right]$ where $\gamma \in [0, 1)$ is a discount factor.

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- γ is an element of the MDP. Larger γ, farther "lookahead".

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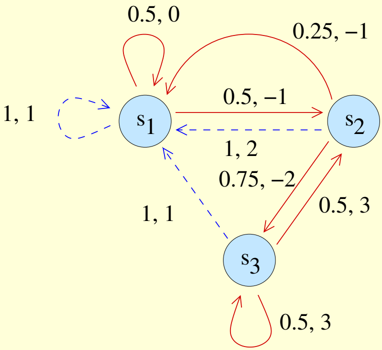

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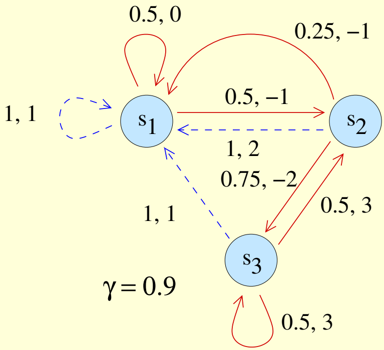

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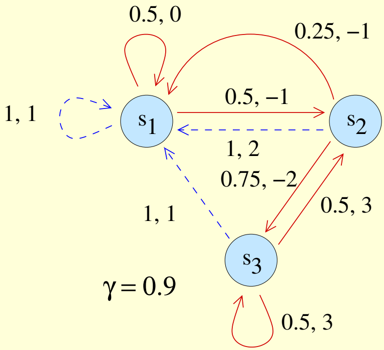

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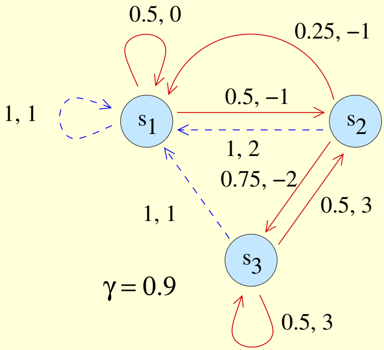

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  $V^{\pi} : S \to \mathbb{R}$ is the Value Function of π.

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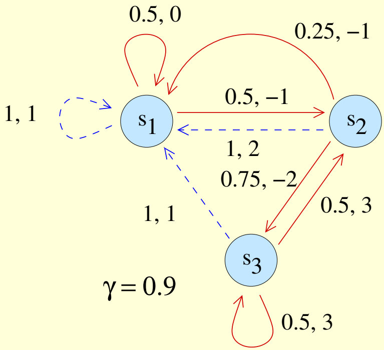

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Optimal Policies

- Here are value functions from our example MDP.

π	$V^{\pi}(s_1)$	$V^{\pi}(s_2)$	$V^{\pi}(s_3)$
RRR	4.45	6.55	10.82
RRB	-5.61	-5.75	-4.05
RBR	2.76	4.48	9.12
RBB	2.76	4.48	3.48
BRR	10.0	9.34	13.10
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Which policy would you prefer?

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Which policy would you prefer?

Every MDP is guaranteed to have an optimal policy $\pi^{\star}$ s.t.

$$\forall \pi \in \Pi, \forall s \in S : V^{\pi^{\star}}(s) \geq V^{\pi}(s).$$

MDP Planning

MDP Planning problem: Given $M = (S, A, T, R, \gamma)$, find a policy $\pi^\star$ from the set of all policies Π such that $\forall s \in S, \forall \pi \in \Pi$: $V^{\pi^\star}(s) \geq V^{\pi}(s)$.

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- Every MDP is guaranteed to have a deterministic, Markovian, stationary optimal policy.
- An MDP can have more than one optimal policy.
- However, the value function of every optimal policy is the same, unique "optimal value function" $V^\star$.

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- Sometimes there is a reward $R(s')$ given on reaching next state s', regardless of start state s and action a.

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- It is relatively straightforward to handle all these variations.

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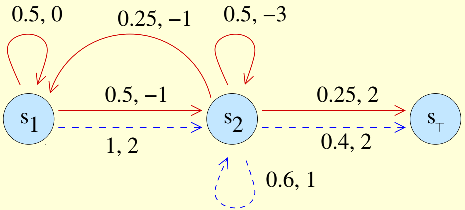

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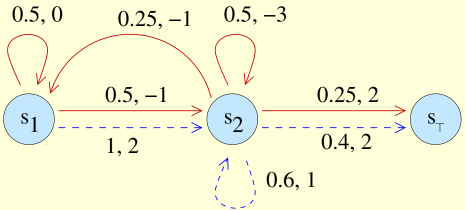

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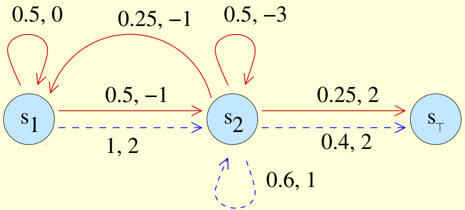

- Additionally, from every non-terminal state and for every policy, there is a non-zero probability of reaching the terminal state in a finite number of steps.
- Hence, trajectories or episodes almost surely terminate after a finite number of steps.

Definition of Values

- We defined $V^{\pi}(s)$ as an **Infinite discounted reward**:
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- **Average reward** (withholding some technical details):
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Markov Decision Problems

1. Definitions
   - Markov Decision Problem
   - Policy
   - Value Function
2. MDP planning
3. Alternative formulations
4. Applications
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Controlling a Helicopter (Ng *et al.*, 2003)

- Episodic or continuing task? What are S, A, T, R, γ?

[1]

1. `https://www.publicdomainpictures.net/pictures/20000/velka/police-helicopter-8712919948643Mk.jpg`.

Succeeding at Chess

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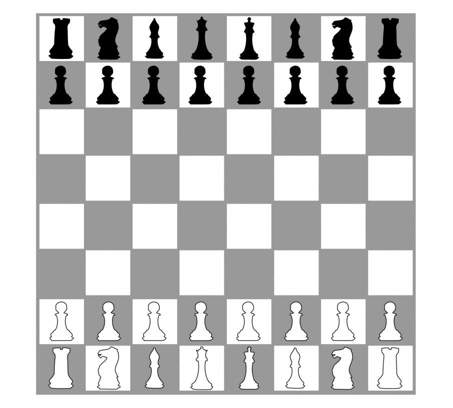

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1. https://www.publicdomainpictures.net/pictures/80000/velka/chess-board-and-pieces.jpg.

Preventing Forest Fires (Lauer *et al.*, 2017)

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- Single state. k actions.
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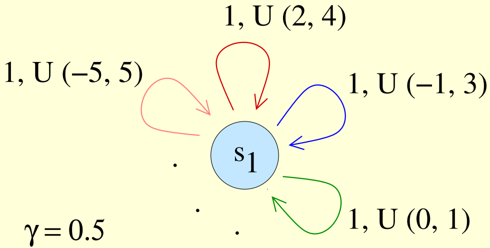

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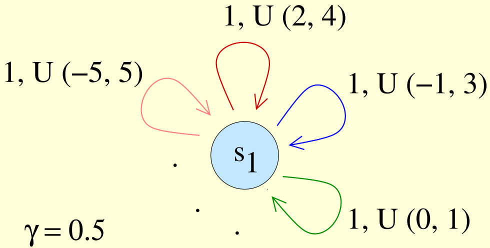

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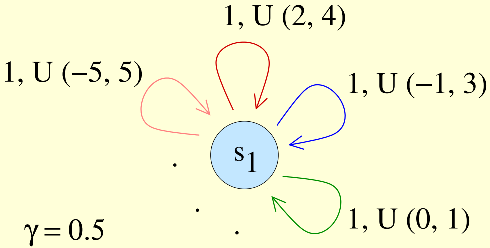

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Structure of State Values

Let us investigate state values. For $\pi \in \Pi, s \in S$:

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- This approach needs poly$(n, k) \cdot k^n$ arithmetic operations. We hope to be more efficient (wait for next week).

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- We will find use for Q^{π} and $Q^{\star}$ next week.

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