DATE
PAGE

# CS747 : Weekly Quiz 2

VIBHAV AGGARWAL
190050128

Ans Let $ucb_1^t$ and $ucb_2^t$ denote the upper confidence bounds for the first and second arms respectively.

We have, $$ucb_a^t = \hat{p}_a^t + \sqrt{\frac{2\ln(t)}{u_a^t}}$$

WLOG, let us assume that after t steps, either $ucb_1^t > ucb_2^t$ or $ucb_1^t = ucb_2^t$ and $1^{st}$ arm gets pulled while breaking the tie.

Therefore, now we need to prove that after some time T, arm 2 will get pulled where T is finite.
In fact, let us assume that T is the first time instant after t where arm 2 gets pulled.

So at time t + T, the ucb values will be :-

$$ucb_1^{t+T} = \hat{p}_1^{t+T} + \sqrt{\frac{2\ln(t+T)}{u_1^t + T}}$$

$$ucb_2^{t+T} = \hat{p}_2^t + \sqrt{\frac{2\ln(t+T)}{u_2^t}}$$

Essentially, we need to prove that there exists such T such that $ucb_2^{t+T} > ucb_1^{t+T}$.

DATE
PAGE

Lemma: There exists a $T$ such that the following holds:-

$$1 + \sqrt{\frac{2\ln(t+T)}{u_1^t + T}} < \sqrt{\frac{2\ln(t+T)}{u_2^t}}$$

Proof:

Since $\ln(t+T) < t+T$,

$$\text{LHS} = 1 + \sqrt{\frac{2\ln(t+T)}{u_1^t + T}} < 1 + \sqrt{\frac{2(t+T)}{u_1^t + T}}$$

Also, $\frac{t+T}{u_1^t + T} < \max\left(1, \frac{t}{u_1^t}\right)$

$$\Rightarrow \text{LHS} < 1 + \sqrt{2\max\left(1, \frac{t}{u_1^t}\right)} = \alpha \text{ (say)}$$

Setting $T = \left\lceil \exp\left(\frac{u_2^t \alpha^2}{2}\right) - t \right\rceil$, we get RHS $\geq \alpha$

This proves the lemma.

Finally, we have:

$$\begin{aligned} ucb_2^{t+T} &= p_2^t + \sqrt{\frac{2\ln(t+T)}{u_2^t}} \\ &\geq \sqrt{\frac{2\ln(t+T)}{u_2^t}} && [\text{since } p_2^t \geq 0] \\ &> 1 + \sqrt{\frac{2\ln(t+T)}{u_1^t + T}} && \text{for some } T \\ &\geq ucb_1^{t+T} && [\text{since } p_1^{t+T} \leq 1] \end{aligned}$$

This completes the proof.