ubhior Agarwal

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Q1 (a) Given eq^n :- $x_1^2 + 2x_2^2 + 3x_3^2 + (x_1 - x_2 + x_3 - 1)^2 +$
$(-x_1 - 4x_2 + 2)^2$

So we can write this in form of least linear squares, individually,

Minimise $x_1^2 + 2x_2^2 + 3x_3^2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now for remaing part
$(x_1 - x_2 + x_3 - 1)^2 + (-x_1 - 4x_2 + 2)^2 \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Now we can jointly write the sol^n of the given eq^n by making $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ -1 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

which gives the required A & b for least linear square problem for given.

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(b) Minimise, $(-6x_2+4)^2 + (-4x_1+3x_2-1)^2 + (x_1+8x_2-3)^2$

So we can formulate the least linear square as follows

$$\underbrace{\begin{bmatrix} 0 & -6 \\ -4 & 3 \\ 1 & 8 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}}_{b}$$

So A & b highlighted above give the A & b for least linear square.

(c) Minimise $2(-6x_2+4)^2 + 3(-4x_1+3x_2-1)^2 + 4(x_1+8x_2-3)^2$.

$\Rightarrow (-6\sqrt{2}x_2 + 4\sqrt{2})^2 + (-4\sqrt{3}x_1 + 3\sqrt{3}x_2 - \sqrt{3})^2 + (2x_1+16x_2-6)^2$

Now writing above in form of least linear square,

$$\underbrace{\begin{bmatrix} 0 & -6\sqrt{2} \\ -4\sqrt{3} & 3\sqrt{3} \\ 2 & 16 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -4\sqrt{2} \\ \sqrt{3} \\ 6 \end{bmatrix}}_{b}$$

Hence we get A & b for linear least square as required.

$(1\times n)$ $(n\times n)$

(d) Minimise $x^T x + \|Bx - d\|_2^2$, $B \in R^{p\times n}$ & $d \in R^p$

We can write above as $\|x\|_2^2 + \|Bx - d\|^2$ as $x^T x = \|x\|_2^2$

Now writing least linear square for both,

$\Rightarrow$ For $\|x\|^2 \Rightarrow Ix = 0$ where $I \in n\times n$ & $x \in R^n$, $0 \in R^n$

For $\|Bx - d\|_2^2 \Rightarrow Bx = d$ where $B \in R^{p\times n}$, $x \in R^n$ & $d \in R^p$

So now combing both we get $A = \begin{bmatrix} B \\ I \end{bmatrix}$ & $b = \begin{bmatrix} d \\ 0 \end{bmatrix}$

here $A \in R^{(p+n)\times n}$, $b \in R^{p+n}$, $I \in R^{n\times n}$, $0 \in R^n$

Hence, $A = \begin{bmatrix} B \\ I \end{bmatrix}$, $b = \begin{bmatrix} d \\ 0 \end{bmatrix}$.

(e) Minimise $x^T D x + \|Bx - d\|_2^2$, $D \in R^{n\times n}$ & is a diagonal matrix with positive entries.

So for $x^T D x$, let $D = \begin{bmatrix} d_1 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & \cdots \\ \vdots & & & \\ 0 & \cdots & & d_n \end{bmatrix}_{n\times n}$

Now $x \in R^n$,

$$\underset{1\times n}{[x_1 \ x_2 \cdots x_n]} \underset{n\times n}{\begin{bmatrix} d_1 & 0 & 0 & \cdots \\ 0 & d_2 & \cdots & \\ 0 & \cdots & & d_n \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x^T D x$$

$$x^T D x = \begin{bmatrix} x_1 d_1 & x_2 d_2 & \cdots & x_n d_n \end{bmatrix}_{1\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$x^T D x = d_1 x_1^2 + d_2 x_2^2 + \cdots d_n x_n^2$$

Minimising $x^T D x$ is equivalent to mimising

$$(\sqrt{d_1}\, x_1)^2 + (\sqrt{d_2}\, x_2)^2 + \cdots + (\sqrt{d_n}\, x_n)^2$$

So formulating above as least linear square

$$\underbrace{\begin{bmatrix} \sqrt{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{d_2} & 0 & 0 & \cdots & 0 \\ \vdots \\ \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{d_n} \end{bmatrix}_{n\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$A_1 = \begin{bmatrix} \sqrt{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{d_2} & \cdots & & \vdots \\ \vdots & 0 & & & \\ \vdots & 0 & \cdots & & \sqrt{d_n} \\ 0 & & & & \end{bmatrix}_{n\times n}, \quad b = \vec{0},\ b \in R^n, \quad d_1, d_2, d_n \in D$$

So on combing both eqn

, $A = \begin{bmatrix} B \\ A_1 \end{bmatrix}$, ~~A~~ $B \in R^{p \times n}$, $A_1 \in R^{n \times n}$

So $A \in R^{(p+n) \times n}$, $b = \begin{bmatrix} d \\ 0 \end{bmatrix}$ where $d \in R^{p}$

, $0 \in R^{n}$ so $b \in R^{p+n}$

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Q2 Given System :- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

we can formulate it as a linear least square system and here,

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ & $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ & $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

we know neccesary condn for minima at x is

$A^T A x = A^T b$ so,

$$\Rightarrow \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

$\Rightarrow x_2 = 1$ so, $x_1 = 2-1 = 1$

so, $\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \end{aligned}}$ hence the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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Now Euclidean norm of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\|x\| = \sqrt{x_1^2 + x_2^2}$

$\|x\| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Now for the sufficiency condition, x is the minima if $A^T A$ is +ve definite.

i.e. We need to show $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is +ve definite.

So, to prove ~~A~~ $A^T A$ is +ve definite.

$x^T (A^T A) x$ ~~$x^T(A^TA)x$~~ $> 0 \quad \forall x \in R^4$. & $x \neq 0$

$$\underset{1\times 2}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{1\times 2}{\begin{bmatrix} (x_1 + x_2) & (x_1 + 2x_2) \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$= x_1(x_1 + x_2) + x_2(x_1 + 2x_2)$$
$$= x_1^2 + x_1 x_2 + x_1 x_2 + 2x_2^2.$$
$$= x_1^2 + 2x_1 x_2 + x_2^2 + x_2^2.$$
$$= (x_1 + x_2)^2 + x_2^2 \quad > 0 \ \forall x_1, x_2 > 0$$

so if $x_1 \neq 0$ & $x_2 \neq 0$ i.e. $x \neq 0$ then,
$(x_1 + x_2)^2 + x_2^2 > 0$

So A is +ve definite which completes the sufficiency condⁿ

Now, $r = b - Ax = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

So $r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence the $\|r\|_2 = \sqrt{0^2+0^2+1} = 1$.

$$\boxed{\|r\|_2 = 1}$$

Q3 (a) Given: $A \in \mathbb{R}^{m \times n}$, $m \geq n$ & A has linearly independent columns ie $\text{Rank}(A) = n$ as $m \geq n$.

To Show: $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$ is non singular & $\in \mathbb{R}^{(m+n) \times (m+n)}$

Since A is full rank, ~~so we~~ $m \geq n$, so we know

$A^T A \in \mathbb{R}^{n \times n}$ is non singular. & hence invertible

So, $(A^T A) B = B(A^T A) = I$ where $B \in \mathbb{R}^{n \times n}$ & is inverse of $A^T A$

Let us take a $Y \in \mathbb{R}^{n \times m}$, $y_i = (A^T A)^{-1} A^T e_i$, e_i is the i^{th} standard basis of $\mathbb{R}^m$, $y_i \in \mathbb{R}^m$

also similiarly assume, $X \in \mathbb{R}^{m \times m}$ where ~~$x_i = e_i$~~ $x_i = e_i - A(A^T A)^{-1} A^T e_i$. ~~$e_i \in \mathbb{R}^n$~~. $e_i \in \mathbb{R}^n$.

Now, take $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ as the vector & it $\in \mathbb{R}^{n+m}$.

we now have $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_i \\ y_i \end{bmatrix}}_{(n+n) \times 1} = \begin{bmatrix} I x_i + A y_i \\ A^T x_i \end{bmatrix}$, $I \in \mathbb{R}^{m \times m}$.

Now,

$x_i = e_i - A(A^T A)^{-1} A^T e_i$

$A^T x_i = A^T e_i - (A^T A)(A^T A)^{-1} A^T e_i$

$A^T x_i = A^T e_i - A^T e_i \Rightarrow \boxed{A^T x_i = 0}$, $x_i + A y_i = e_i$

so, $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_i + A y_i & e_i \\ \cdot & 0 \end{bmatrix}$

So, $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + Ay \\ A^T x \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

So, finally,

$$\cancel{\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + A \\ \; \end{bmatrix}}$$

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x & AB \\ y & -B \end{bmatrix} = \begin{bmatrix} x + Ay & AB - AB \\ A^T x & (A^T A) B \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$ (as B is inverse of A^TA)

Which clearly form I_{m+n}

So $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} B' = I$ which shows $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$

is invertible.

(b) Let x' be the solⁿ of least square $Ax \simeq b$ from normal eqⁿ,

$x' = (A^TA)^{-1} A^T b$

Now we know for minimising residual i.e $\|b - Ax'\|_2$ residual must be orthogonal to A's range space

i.e $A^T(b - Ax') = 0$ —— (1)

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therefore, set $\hat{x} = b - Ax'$ & $\hat{y} = x'$, we get,

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \hat{x} + A\hat{y} \\ A^T \hat{x} \end{bmatrix}$$

$$= \begin{bmatrix} b - A\cancel{x'} + \cancel{A}\hat{x} \\ A^T (b - Ax') \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 0 \end{bmatrix} \quad \text{from} - (i)$$

This completes the forward direction of proof.
Now conversely let, assume.

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$I\hat{x} + A\hat{y} = b \Rightarrow \hat{x} + A\hat{y} = b$ — (ii)
$A^T\hat{x} = 0$ — (iii)

Now from (ii) $\Rightarrow \hat{x} = b - A\hat{y}$ & (iii) implies,

$$A^T\hat{x} = 0$$
ie $$A^T\hat{x} = A^T(b - A\hat{y})$$
$$= A^T b - A^T A\hat{y}$$

$$A^T b - A^T A\hat{y} = 0 \Rightarrow A^T b = A^T A\hat{y}$$

Now A^TA is non singular as $m \geq n$ & Rank $= n$,

$\hat{y} = (A^TA)^{-1} A^T b$ — (iv)

(iv) Clearly is the sol^n of $Ax' = b$,
So, $\hat{y} = x'$

& from (ii) $\hat{x} = b - A\hat{y} = b - Ax'$

$\hat{x} = b - Ax'$

This completes our converse proof & hence the whole proof that $\hat{x} = b - Ax'$, $\hat{y} = x'$ where x' is sol^n to $Ax' = b$

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Q4 (b) To find :- $\|b\|_2$

Given :- $[A \; b] = QR$

ie $[A \; b] = [q_1 \cdots q_{n+1}] \begin{bmatrix} R_{11} & - & - & R_{1n+1} \\ R_{21} & & & R_{2n+1} \\ \vdots & & & \vdots \\ R_{n+1,1} & & & R_{n+1,n+1} \end{bmatrix}$

so,

$$b = Q \begin{bmatrix} R_{1,n+1} \\ R_{2n+1} \\ \vdots \\ R_{n+1,n+1} \end{bmatrix}_{n+1 \times 1}$$

$$b = R_{1,n+1} q_1 + R_{2,n+1} q_2 + \cdots + R_{n+1,n+1} q_{n+1}$$

$$b = \sum_{i=1}^{n+1} R_{i,n+1} q_i$$

we know if $\langle q_1, q_2 \cdots q_n \rangle$ is a list of orthogonal vectors & $a_1, a_2, \cdots a_n$ is a scalar, then,

$$\|a_1 e_1 + a_2 e_2 + \cdots a_n e_n\|_2 = \sqrt{|a_1|^2 + |a_2|^2 + \cdots |a_n|^2}$$

So

$$\|b\|_2 = \left\| \sum_{i=1}^{n+1} R_{i,n+1} \cdot q_i \right\|_2$$

$$= \sqrt{|R_{1,n+1}|^2 + |R_{2,n+1}|^2 + \cdots + |R_{n+1,n+1}|^2}$$

$$\boxed{\|b\|_2 = \sqrt{\sum_{i=1}^{n+1} |R_{i,n+1}|^2}}$$

Q4 (b) Let $\hat{x}$ is the sol^n of least linear square problem.
Now the normal eq^n is

$$A^T A \hat{x} = A^T b$$

$\Rightarrow$ As given $A = QR$ (By QR factorisation) —(1)

$\Rightarrow A^T A \hat{x} = A^T b \Rightarrow (R^T Q^T) Q R \hat{x} = R^T Q^T b$

Multiplying by pseudo inverse R^+ on both sides.
R^+ is guaranteed to exist as R is a triangular matrix.

$$R^+ (R^T Q^T) Q R \hat{x} = R^+ R^T Q^T b$$

$\Rightarrow Q^T Q R \hat{x} = Q^T b$ (as $R^+ R^T = I$)

$\Rightarrow \| Q^T Q R \hat{x} \| = \| Q^T b \|$

$\Rightarrow \| Q R \hat{x} \| = \| b \|$ as $\| Q x \| = Q$ when Q is orthogonal

Now from (1) $\| A \hat{x} \| = \| b \|$

$\Rightarrow \| A \hat{x} \| = \sqrt{\sum_{i=1}^{n+1} |R_{i,n+1}|^2}$ (from part A).

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Q5 (a) $\frac{e^{\alpha t_i+\beta}}{1+e^{\alpha t_i+\beta}} \approx y_i$, $i=1 --, m$

Now, we can formulate it as linear least square below,

- $e^{\alpha t_i+\beta} = y_i + y_i e^{\alpha t_i+\beta}$

$\Rightarrow e^{\alpha t_i+\beta}[1-y_i] = y_i$

$\Rightarrow e^{\alpha t_i+\beta} = \frac{y_i}{1-y_i}$

$\Rightarrow \log(e^{\alpha t_i+\beta}) = \log\left(\frac{y_i}{1-y_i}\right)$

$\Rightarrow \alpha t_i+\beta = \log y_i - \log(1-y_i)$

Taking $\log y_i - \log(1-y_i) = b_i$ — (i)

~~$\alpha t_i + \beta$~~ $\alpha t_i + \beta = b_i$ — (ii)

we are doing (i) as can store (i) using y_i given as output in a vector & get a linear least square as below.

$\beta + \alpha t_i = b_i$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ | & \\ 1 & t_m \end{bmatrix}}_{A \quad m\times 2} \underbrace{\begin{bmatrix} \beta \\ \alpha \end{bmatrix}}_{x \quad 2\times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ | \\ b_m \end{bmatrix}}_{b \quad m\times 1}$$

Now

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$ for solving the

above eq^n using least linear squares.

Q6 (a) Here the given A is $A = \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}$, $b = \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$

we know for $Ax \simeq b$, the normal eq^n is.

$(A^TA)x = A^Tb$

$$\cancel{\begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}}\ [$$

$$\begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}_{3\times 2} x = \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} \cancel{-10^{-k}} + \cancel{10^{-k}} + 10^{-2k} \\ \cancel{-10^{-k}} + \cancel{10^{-k}} - 10^{-2k} \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_1 \cdot 10^{-2k} + x_2 = 10^{-2k}$$
$$x_1 + x_2 + x_2 \cdot 10^{-2k} = -10^{-2k}$$

$\Rightarrow \quad x_1 10^{-2K} - x_2 10^{-2K} = 2\cdot 10^{-2K}$

$\Rightarrow \quad x_1 - x_2 = 2$

$x_1 = 2 + x_2$

$\Rightarrow \quad 2 + 2x_2 + x_2 \cdot 10^{-2K} = -10^{-2K}$

$x_2 \left[2 + 10^{-2K}\right] = -10^{-2K} - 2$

$\Rightarrow \quad x_2 = \dfrac{-\cancel{[2+10^{-2K}]}}{\cancel{2+10^{-2K}}}$

$\boxed{x_2 = -1}$

$\Rightarrow \boxed{x_1 = 1}$

So $x_1 = 1$ & $x_2 = -1$ for all values of K theoretically.

- But for solving in computer we know ϵ_m for a system is $\approx 10^{-16}$. So when $K > 8$, $10^{-2K} \leq 10^{-16}$. Hence $1 + 10^{-2K} = 1$ in FP for $K > 8$. So in Python we get a singular matrix error for solving this as $A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is singular $\forall K > 8$

Vibhor Agarwal

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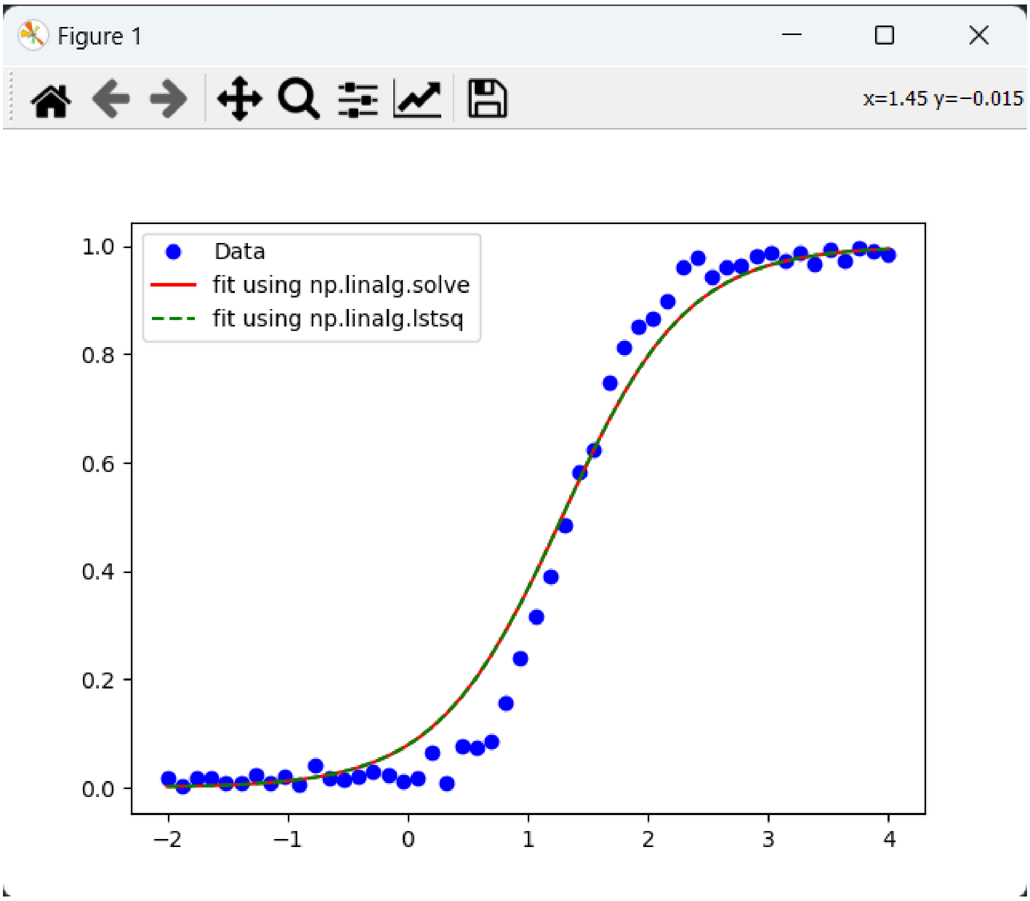
SC-Assignment 2

Q5

The curve fit obtained using np.linalg.lstsq and np.linalg.solve is as follows. We observe that the curves obtained using both the methods are identical to each other and fit the data almost perfectly given that we were given just 50 samples to train on.



The errors obtained using both the functions are also identical as np.lstsq internally uses the normal equation to solve linear least square system.

```
Error using np.linalg.solve:  6.980329177231975
Error using np.linalg.lstsq:  6.980329177231975
```

Q6

Part B

The solution obtained using the QR factorisation and then solving are as follows. We find even for large values of K the solution is very close to the actual solution i.e., x1= 1 and x2= -1.

```
k =  6
Solution using QR:  [ 1. -1.]
k =  7
Solution using QR:  [ 1. -1.]
k =  8
Solution using QR:  [ 1.00000002 -1.00000002]
k =  9
Solution using QR:  [ 1.00000001 -1.00000001]
k =  10
Solution using QR:  [ 1.0000007 -1.0000007]
k =  11
Solution using QR:  [ 1.00001169 -1.00001169]
k =  12
Solution using QR:  [ 0.99998488 -0.99998488]
k =  13
Solution using QR:  [ 0.99944487 -0.99944487]
k =  14
Solution using QR:  [ 1.00202995 -1.00202995]
k =  15
Solution using QR:  [ 0.89212348 -0.89212348]
```

Part C

We find that for k>8 the matrix A^TA has become singular and so the equation can't be solved in case of singular matrix. The answer obtained theoretically should be x_1=1 and x_2=-1. However practically we get singular matrix error for k>=8. This is because of present of 1+ 10^{-2k} in the matrix. Now for k>=8, 10^{-2k} becomes smaller than or equal to 10^{-16}. Now this value is smaller than the machine epsilon it becomes 1. This makes the matrix singular and we get singular matrix error on solving instead of getting (1,-1) as the required solution. Now the QR decomposition produces nearly accurate solution and does not suffer from the singular matrix problem. It is because it doesn't use inverse of A^TA for solution of but rather the inverse of R which being an upper triangular matrix can never be singular and hence avoids the singular matrix error.

```
k =  6
Solution using Normal:  [ 0.99991111 -0.99991111]
k =  7
Solution using Normal:  [ 1.00079992 -1.00079992]
k =  8
Error Singular Matrix
k =  9
Error Singular Matrix
k =  10
Error Singular Matrix
k =  11
Error Singular Matrix
k =  12
Error Singular Matrix
k =  13
Error Singular Matrix
k =  14
Error Singular Matrix
k =  15
Error Singular Matrix
```

Q7

The results obtained on denoising is as follows.

The curve in the blue represents the actual data. The subsequent colour curves mentioned in the legend represents the denoised signal.

- We observe that curve smoothening is best for lambda =100. At this value the curve doesn't deviate from the original curve much and produces a decently smoothened out signal.
- The lambda =1 also does smoothen and its better than doing no smoothening at all.
- But for lambda =10000 the resulting smoothened signal is very much deviated from the original signal. Hence this signal although being very smooth is of no use.
- This shows that on increasing lambda till a certain limit the curve gets smoother and doesn't deviate from the original data. But for very large values ok K the curve produced is very different from the actual data so is of no significance.