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2020349

SC Assignment - 4 :

Points :- $\{(-2,15), (0,-1), (1,0), (3,-2)\}$.

Q1 (a)

Using Monomial basis, let the polynomial of degree 3 be $a + bx + cx^2 + dx^3$.

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_2^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

On solving above,

$a = -1$,
$b = -0.53$
$c = 2.26$.
$d = -0.73$

So the polynomial is $-1 + (-0.53)x + 2.26x^2 - 0.73x^3$

(b) For lagrange basis,

$$P(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2$$

$$+ \dots + y_4 \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \right]$$

$$P(x) = \frac{15(x-0)(x-1)(x-3)}{(-2)(-3)(-5)} - 1\frac{(x+2)(x-1)(x-3)}{(2)(-1)(-3)} + 0 + \frac{-2(x+2)(x)(x-1)}{(5)(3)(2)}$$

$$= \frac{-1}{30} \left[15(x)(x-1)(x-3) + 5(x-1)(x+2)(x-3) + 2x(x-1)(x+2) \right].$$

$$= \frac{-1}{30} \left[22x^3 - 68x^2 + 16x + 30 \right]$$

$$P(x) = -0.73x^3 + 2.26x^2 - 0.53x - 1.$$

(c) for newton basis, $a + bx + cx^2 + dx^3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 - x_1 & 0 & 0 \\ 1 & (x_3 - x_1) & (x_3 - x_1)(x_3 - x_2) & 0 \\ 1 & (x_4 - x_1) & (x_4 - x_1)(x_4 - x_2) & (x_4 - x_1)(x_4 - x_2)(x_4 - x_3) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 5 & 15 & 30 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$\Rightarrow a = 15$

$b = -8$

$c = 3$

$d = \frac{-22}{30}$

$(x^3 + x^2 - 2x)$

$P(x) = 15 - 8(x+2) + 3(x+2)(x) - \frac{22}{30}(x+2)(x)(x-1)$

$P(x) = 15 - 8x - 16 + 3x^2 + 6x - \frac{22}{30}(x^2 + 2x)(x-1)$

$P(x) = -1 - 8x + 6x + 3x^2 - \frac{22}{30}(x^3 + x^2 - 2x)$

$P(x) = -0.73x^3 + 2.26x^2 - 0.53x - 1$

(d) The polynomial obtained from monomial basis is

$-0.73x^2 + 2.26x^2 - 0.53x - 1$.

From Lagrange basis $= -0.73x^2 + 2.26x^2 - 0.53x - 1$

from Newton basis $= -0.73x^2 + 2.26x^2 - 0.53x - 1$.

Hence the polynomial in all basis remains the same.

with $a_0 = -1$, $a_1 = -0.53$ $a_2 = 2.26$ $a_3 = -0.73$.

Q2 For $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$.

Midpoint Rule :-

$$\int_a^b f(t)\,dt \approx (b-a)\, f\left(\frac{a+b}{2}\right).$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx \approx (1-0)\, f\left(\frac{1}{2}\right)$$

$$\approx \frac{1}{1+\frac{1}{4}}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{4}{5}$$

Trapezoid Rule

$$\int_a^b f(t)\,dt \approx \left(\frac{b-a}{2}\right)\left[f(a)+f(b)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1-0}{2}\left[f(0)+f(1)\right]$$

$$= \frac{1}{2}\left[1+\frac{1}{2}\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3}{4}$$

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Simpson's Rule

$$\Rightarrow \int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$$

$$= \frac{1}{6}\left[1+4\times\frac{4}{5}+\frac{1}{2}\right]$$

$$= \frac{1}{6}\left[1+\frac{16}{5}+\frac{1}{2}\right]$$

$$= \frac{1}{6}\times\frac{10+32+5}{10}$$

$$= \frac{47}{60}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{47}{60}$$

Gaussian 2pt :- we know in $(-1,1)$ gaussian point $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. so applying change limits

$$\int_a^b f(x)\,dx = \frac{b-a}{\beta-\alpha}\sum_{i=1}^{n} w_i f\left(\frac{(b-a)x_i + a\beta - b\alpha}{\beta-\alpha}\right)$$

taking $\alpha=-1$, $\beta=1$, $w_1=w_2=1$ & $x_1=\frac{1}{\sqrt{3}}$ & $x_2=\frac{-1}{\sqrt{3}}$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{2}\left[\cancel{\tfrac{1}{1}}\ f\left(\frac{x_1+1}{2}\right) + f\left(\frac{x_2+1}{2}\right)\right]$$

$$\frac{1}{2}\left[f\left(\frac{1+\frac{1}{\sqrt{3}}}{2}\right) + f\left(\frac{1-\frac{1}{\sqrt{3}}}{2}\right)\right.$$

$$= \frac{1}{2}\left[f\left(\frac{1+\sqrt{3}}{2\sqrt{3}}\right) + f\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)\right].$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{2}\left[\frac{1}{1+\frac{(1+\sqrt{3})^2}{12}}\right] + \frac{1}{2}\left[\frac{1}{1+\frac{(1-\sqrt{3})^2}{12}}\right]$$

$$= \frac{1}{2}\frac{\times 12}{[12+4+2\sqrt{3}]} + \frac{1}{2}\frac{\times 12}{[12+4-2\sqrt{3}]}$$

$$= \frac{6}{16+2\sqrt{3}} + \frac{6}{16-2\sqrt{3}}$$

$$= \frac{96-12\sqrt{3}+96+12\sqrt{3}}{256-12}$$

$$\int_0^1 \frac{1}{1+x^2} = \frac{192}{244} = 0.786.$$

for $\int_0^1 \sqrt{x} \log x \, dx = -\frac{4}{9}$

Midpoint Rule

$$\int_0^1 \sqrt{x} \log x \, dx = (1-0) f\left(\tfrac{1}{2}\right)$$

$$= \sqrt{\tfrac{1}{2}} \log\left(\tfrac{1}{2}\right)$$

$$= \frac{-\log 2}{\sqrt{2}}$$

Trapezoid Rule

$$\int_0^1 \sqrt{x} \log x \, dx = \frac{1}{2}\left[f(0) + f(1) \right]$$

$$= \frac{1}{2}\left[0 + 0 \right]$$

$$\int_0^1 \sqrt{x} \log x \, dx = 0$$

Simpson's Rule

$$\int_0^1 \sqrt{x} \log x \, dx = \frac{1}{6}\left[f(0) + 4 f\left(\tfrac{1}{2}\right) + f(1) \right]$$

$$= \frac{1}{6}\left[0 + 4 \times \frac{1}{\sqrt{2}} \log\left(\tfrac{1}{2}\right) + \sqrt{1} \log 1 \right]$$

$$= \frac{1}{6}\left[2\sqrt{2} \times -\log(2) \right]$$

$$\int_0^1 \sqrt{x} \log x \, dx = \frac{-\log(2)\sqrt{2}}{3}$$

Using Gaussian :-
we use change of limits to -1 to 1 from 0 - 1

$$\int_0^1 \sqrt{x}\log x\,dx = \frac{1}{2}\left[f\left(\frac{\frac{1}{\sqrt{3}}+1}{2}\right) + f\left(\frac{-\frac{1}{\sqrt{3}}+1}{2}\right)\right]$$

$$= \frac{1}{2}\left[f\left(\frac{1+\sqrt{3}}{2\sqrt{3}}\right) + f\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{1+\sqrt{3}}{2\sqrt{3}}}\log\left(\frac{1+\sqrt{3}}{2\sqrt{3}}\right) + \sqrt{\frac{\sqrt{3}-1}{2\sqrt{3}}}\log\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{0.78}\log(0.78) + \sqrt{0.21}\log(0.21)\right]$$

$$= \frac{1}{2}\left[0.88 \times -0.24 + 0.45 \times -1.56\right]$$

$$= -\frac{1}{2}\left[0.2112 + 0.702\right]$$

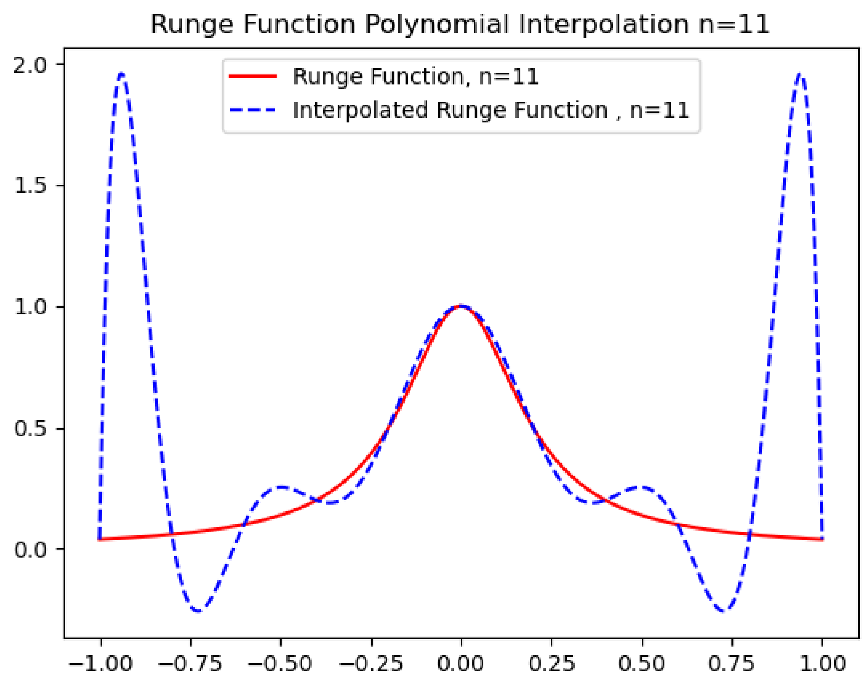
$$\int_0^1 \sqrt{x}\log x\,dx = -0.4566$$

Q3

Interpolating Runge's Function using polynomial interpolation

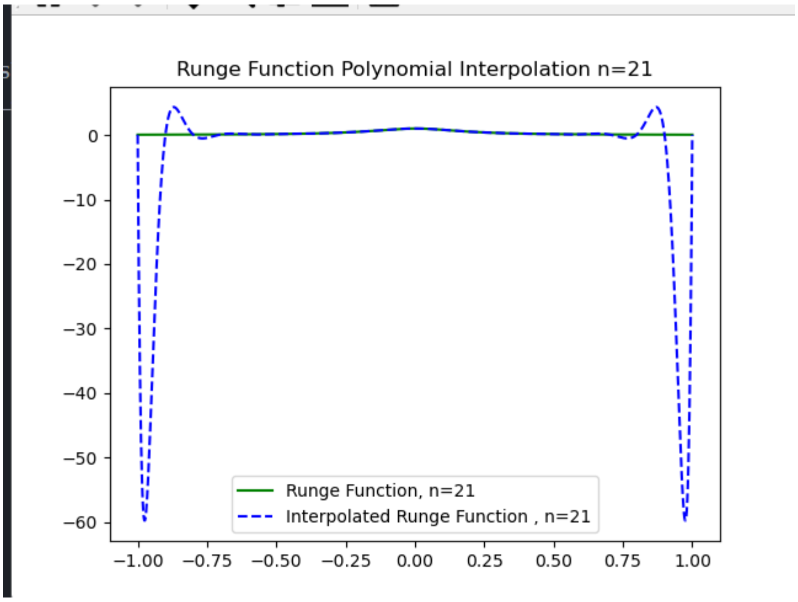
a) **Using n=11 equispaced nodes**

We observe that the polynomial interpolation almost correctly interpolates the Runge's function accurately except at the edges where the interpolation function does not work very well due to Runge's phenomena.



b) **Using n=21 equispaced nodes**

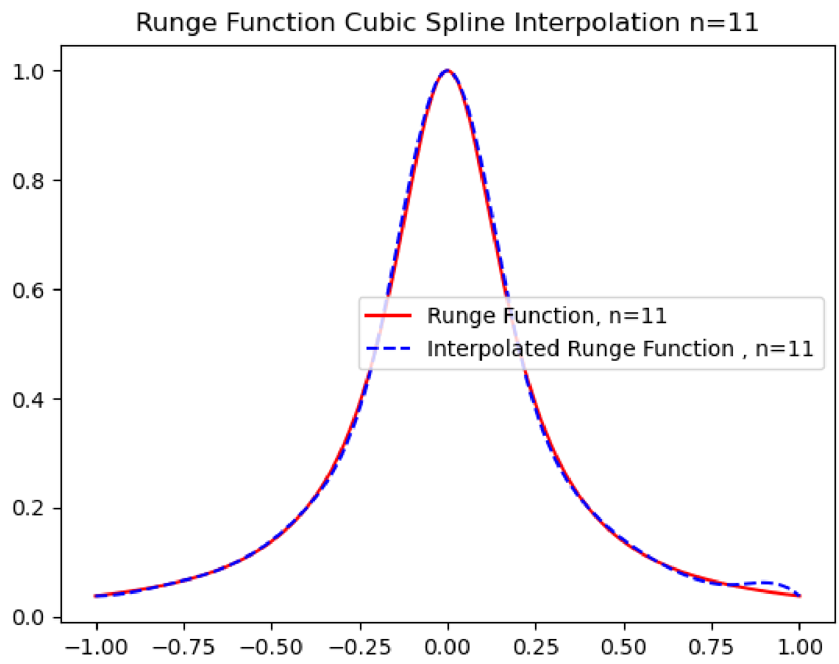
We observe that the polynomial interpolation almost correctly interpolates the Runge's function accurately except at the edges where the interpolation function does not work very well due to Runge's phenomena.



Interpolating Runge's Function using Cubic Spline interpolation

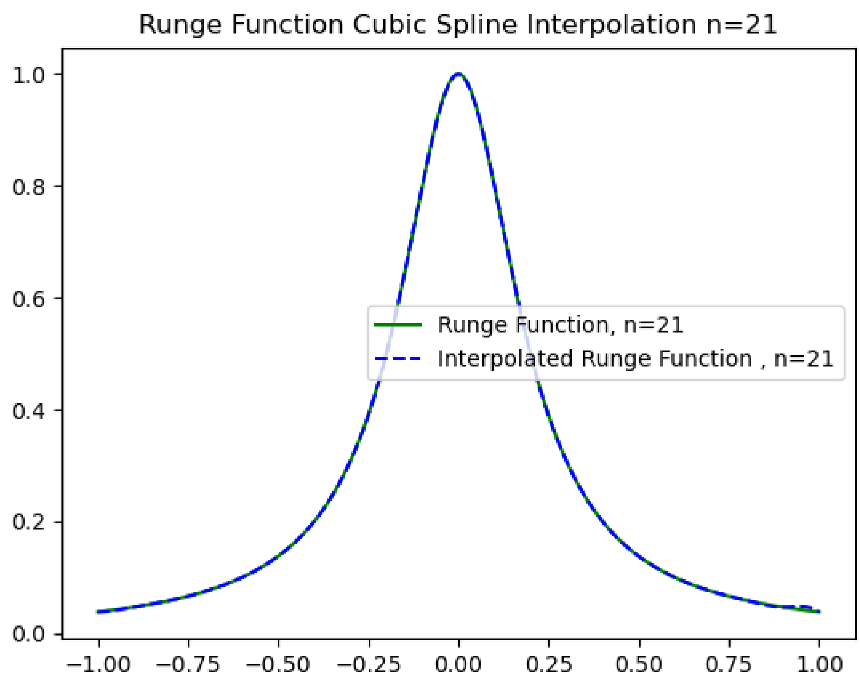
a) Using n=11 equispaced nodes

We observe that the cubic spline interpolation almost correctly interpolates the Runge's function accurately.



b) Using n=21 equispaced nodes

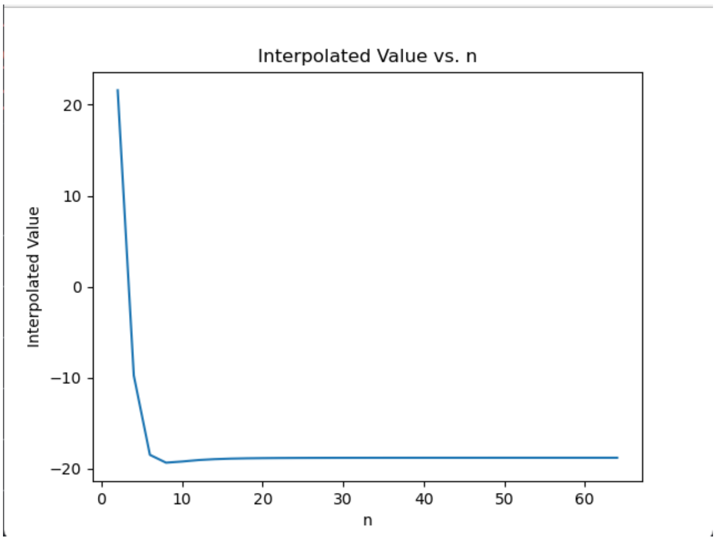
We observe that the polynomial interpolation almost correctly interpolates the Runge's function accurately.



Q4

The composite gaussian quadrature approximates the given integrals fairly closely.

The graph showing interpolated values vs n (number of intervals) is as follows.



The interpolated values and the relative errors are as follows:

n	Interpolated Value	Relative Error
2	21.5897	2.14849
4	-9.76123	0.480738
6	-18.479	0.0169866
8	-19.3402	0.0288255
10	-19.2121	0.022014
12	-19.0546	0.0136321
14	-18.9544	0.00830496
16	-18.8958	0.00518879
18	-18.8614	0.00335629
20	-18.8406	0.00224774
22	-18.8275	0.00155417
24	-18.8191	0.0011056
26	-18.8135	0.000806451
44	-18.8002	9.90686e-05
46	-18.7999	8.2926e-05
48	-18.7996	6.99395e-05
50	-18.7994	5.93973e-05

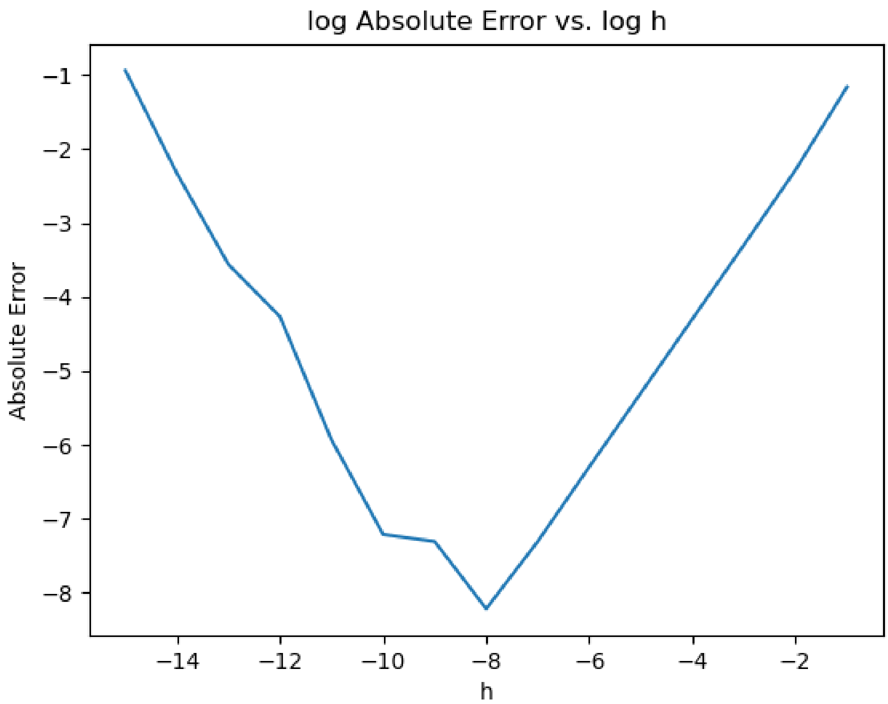
50	-18.7994	5.93973e-05
52	-18.7993	5.07679e-05
54	-18.7991	4.36497e-05
56	-18.799	3.77361e-05
58	-18.7989	3.27907e-05
60	-18.7988	2.86294e-05
62	-18.7988	2.51076e-05
64	-18.7987	2.21111e-05

The closest value approximation to actual value i.e., -18.798 is -18.7987 and relative error is 2.21 x 10^-5 at best approximation which is fairly closely 0. So the approximation works very well.

Problem 5

We take x_0=1 and plot the following log(absolute error) vs log(h).

The smallest absolute error becomes 10^{-8} which is fairly small and hence the error almost tends to 0 but not exactly zero. It is clear from the graph that till $\log_{10}(h)$ = -8 the relative error decreases and after the the value of h further gets more negative and 10^h gets smaller as h becomes larger the relative error again starts to increase. It is because of the fact that as 10^h get smaller the floating point roundoff starts to happen and the deviation x0+h starts getting rounded off giving incorrect results.



log(h)	log(absolute error)
−1	−1.1586
−2	−2.28175
−3	−3.29502
−4	−4.29635
−5	−5.29649
−6	−6.29649
−7	−7.30632
−8	−8.21493
−9	−7.30614
−10	−7.21037
−11	−5.93114
−12	−4.26489
−13	−3.55849
−14	−2.32631
−15	−0.936518