Algorithms-II
Autumn 2022
Tutorial- 2

1. Suppose that a maximum flow is already computed in a graph with integral capacities. Then the capacity of exactly one edge is increased by 1. Which of the following is true about the complexity of finding the new maximum flow?
   a) O(1), as the max flow will not change
   b) O(1), as the capacity of only 1 edge is increased by 1
   c) O(V+E)
   d) O(VE)

2. Let G be a network with source s, sink t, and integer capacities. Prove or disprove the following statements:
   a) If all capacities are even then there is a maximum flow f such that f(e) is even for all edges e.
   b) If all capacities are odd then there is a maximum flow f such that f(e) is odd for all edges e.

3. In some country there are n cities and m bidirectional roads between them. Each city has an army. Army of the i-th city consists of a_i soldiers. Now soldiers roam. After roaming each soldier has to either stay in his city or go to the one of neighboring cities by moving along at most one road. Check if it is possible that after roaming there will be exactly b_i soldiers in the i-th city.

4. Prove that every k-regular bipartite graph has a perfect matching (prove using notions of max flow).

5. The edge connectivity of an undirected connected graph is the minimum number of edges that has to be removed from the graph to make it disconnected. Given algorithm to find the edge-connectivity of a graph.

6. A number k of trucking companies, $c_1,...,c_k$, want to use a common road system, which is modeled as a directed graph, for delivering goods from source locations to a common target location. Each trucking company c_i has its own source location, modeled as a vertex s_i in the graph, and the common target location is another vertex t. (All these k +1 vertices are distinct.) The trucking companies want to share the road system for delivering their goods, but they want to avoid getting in each other's way while driving. Thus, they want to paths in the graph, one connecting each source s_i to the target t, such that no two trucks use a common road. We assume that there is no problem if trucks of different companies pass through a common vertex. Design an algorithm for the companies to use to determine k such paths, if possible, and otherwise return "impossible".

7. Consider a large task T that can be decomposed into n subtasks T_1, T_2, …T_n. The subtasks are to be performed on 2 machines M_1 and M_2. A subtask can be done on any of the two machines, but will incur different costs on the two machines. Specifically, subtask T_i incurs a cost C_i if it is done on machine M_1, and cost D_i if it is done on machine M_2. Also, some pairs of these subtasks need to communicate among themselves. If such a pair of subtasks T_i and T_j are run on the same machine the communication cost incurred is 0, otherwise, they incur a communication cost of P_{ij}. If a pair of subtasks do not need to communicate, they incur no communication cost irrespective of which machines they are run on. Find an assignment of the subtasks to the two machines so that the total cost of executing T is minimized. (Hint: Reduce the problem of finding the minimum cost of executing T to that of finding a minimum cut in a graph that you construct. First show the construction of the graph clearly stating what are the nodes and edges, and then argue why the minimum cut in this graph will give the minimum cost).

8. Suppose that instead of a single capacity c(u,v) on each edge, each edge has a pair (l(u,v), c(u,v)) and the capacity constraint is changed such that the l(u,v) <= f(u,v) <= c(u,v) (So each edge must carry some minimum flow on it). Can you find a feasible flow on this network? If yes, can you then find the maximum flow?