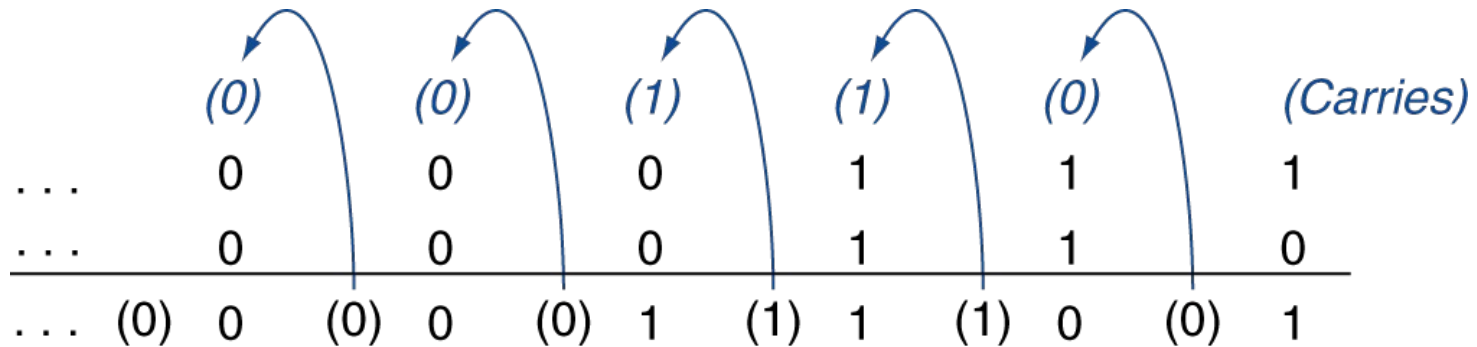# Chapter 3

Arithmetic for Computers

Arithmetic for Computers

- Operations on integers
  - Addition and subtraction
  - Multiplication and division
  - Dealing with overflow
- Floating-point real numbers
  - Representation and operations

Integer Addition

- Example: 7 + 6

- Overflow if result out of range
  - Adding +ve and –ve operands, no overflow
  - Adding two +ve operands
    - Overflow if result sign is 1
  - Adding two –ve operands
    - Overflow if result sign is 0

Integer Subtraction

- Add negation of second operand

- Example: 7 – 6 = 7 + (–6)

  +7:   0000 0000 … 0000 0111
  –6:   1111 1111 … 1111 1010
  +1:   0000 0000 … 0000 0001

- Overflow if result out of range

  - Subtracting two +ve or two –ve operands, no overflow

  - Subtracting +ve from –ve operand

    - Overflow if result sign is 0

  - Subtracting –ve from +ve operand

    - Overflow if result sign is 1
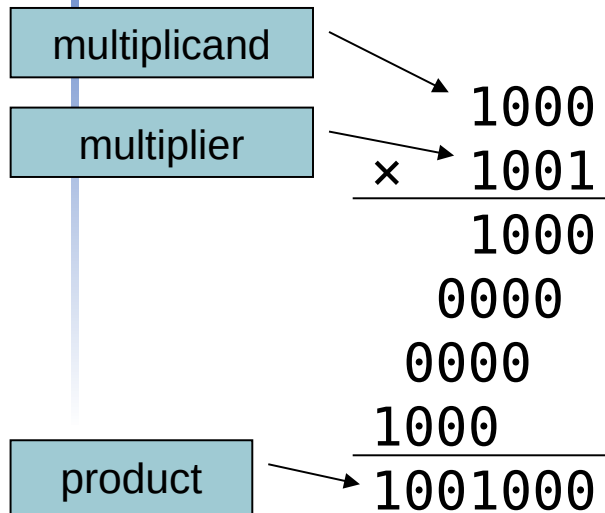
Dealing with Overflow

- Some languages (e.g., C) ignore overflow
  - Use MIPS `addu`, `addui`, `subu` instructions
- Other languages (e.g., Ada, Fortran) require raising an exception
  - Use MIPS `add`, `addi`, `sub` instructions
  - On overflow, invoke exception handler
    - Save PC in exception program counter (EPC) register
    - Jump to predefined handler address
    - `mfc0` (move from coprocessor reg) instruction can retrieve EPC value, to return after corrective action
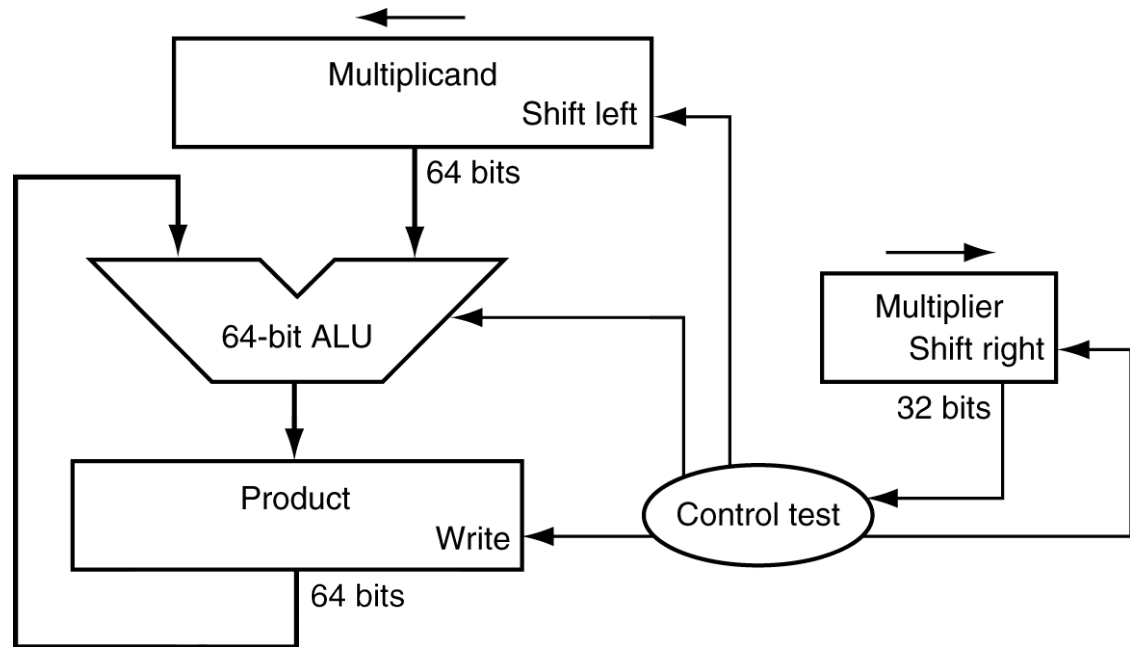
Arithmetic for Multimedia

- Graphics and media processing operates on vectors of 8-bit and 16-bit data
    - Use 64-bit adder, with partitioned carry chain
        - Operate on 8×8-bit, 4×16-bit, or 2×32-bit vectors
    - SIMD (single-instruction, multiple-data)
- Saturating operations
    - On overflow, result is largest representable value
        - c.f. 2s-complement modulo arithmetic
    - E.g., clipping in audio, saturation in video

Multiplication

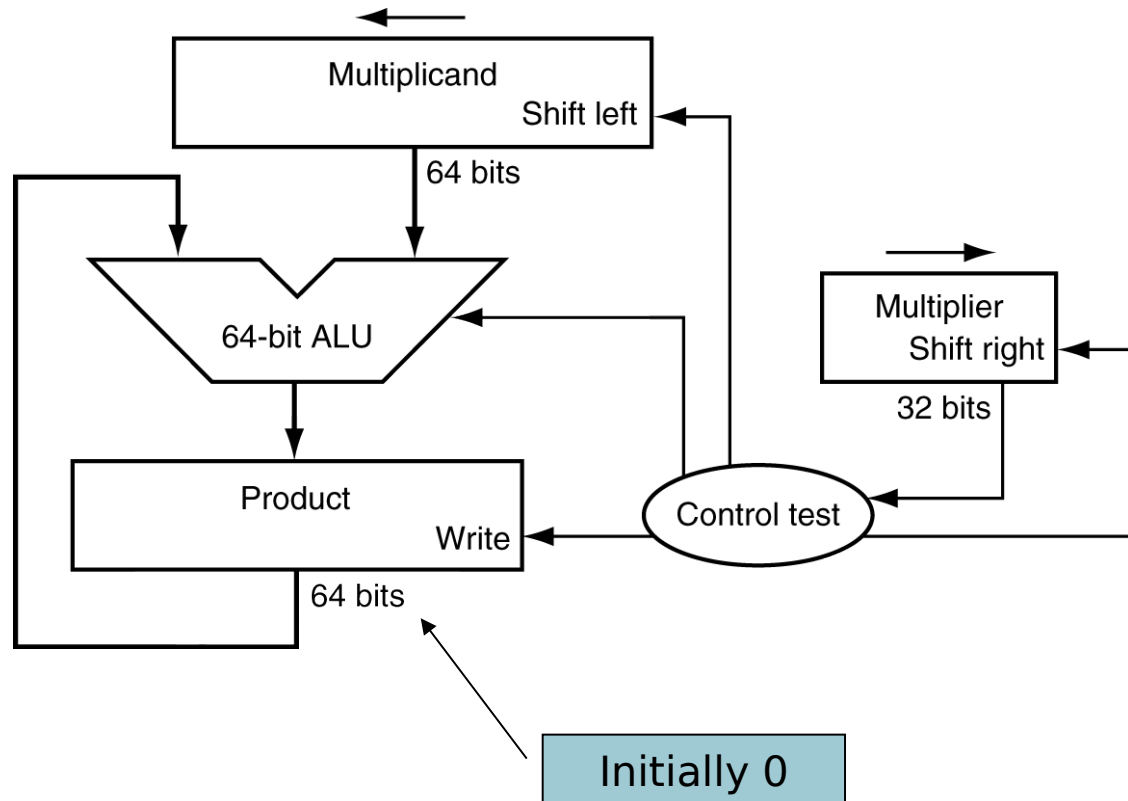- ## Start with long-multiplication approach

multiplicand

multiplier

product

```
        1000
    ×   1001
        1000
       0000
      0000
     1000
    1001000
```

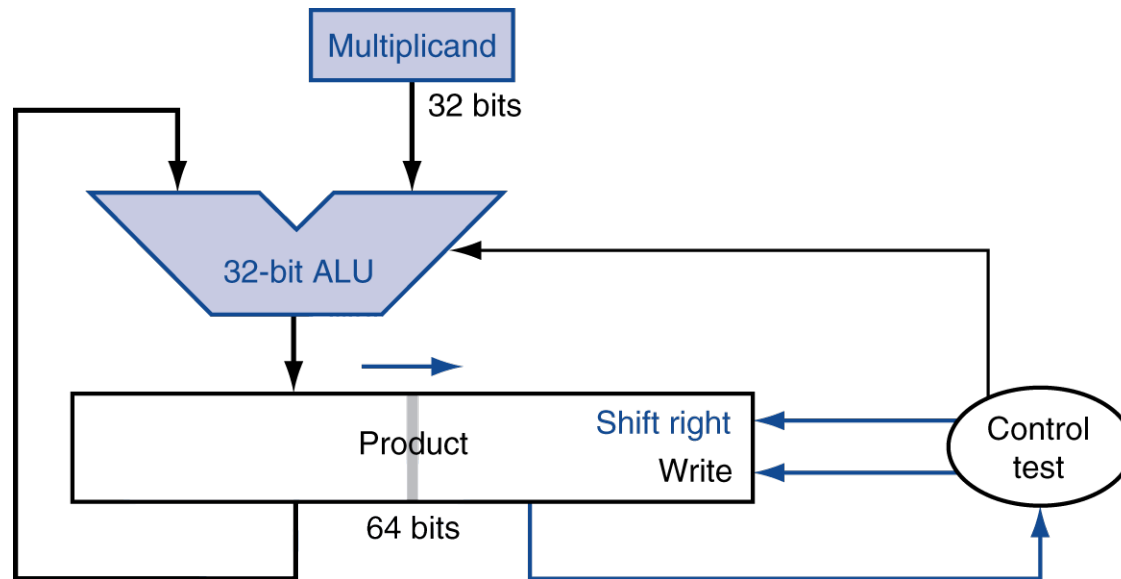Length of product is the sum of operand lengths

Multiplication Hardware
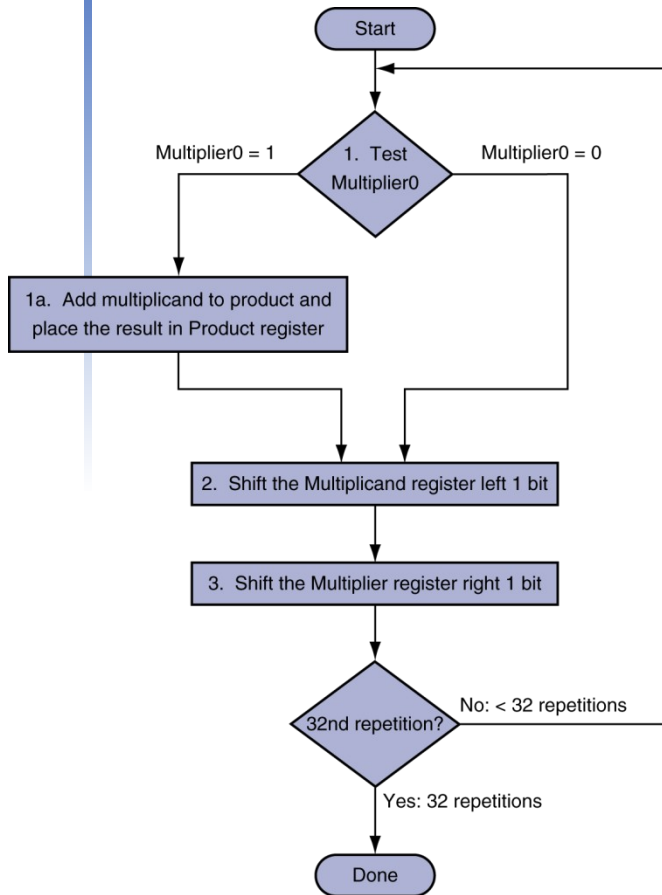
Optimized Multiplier

- Perform steps in parallel: add/shift

- One cycle per partial-product addition
  - That's ok, if frequency of multiplications is low

Faster Multiplier

- Uses multiple adders
  - Cost/performance tradeoff
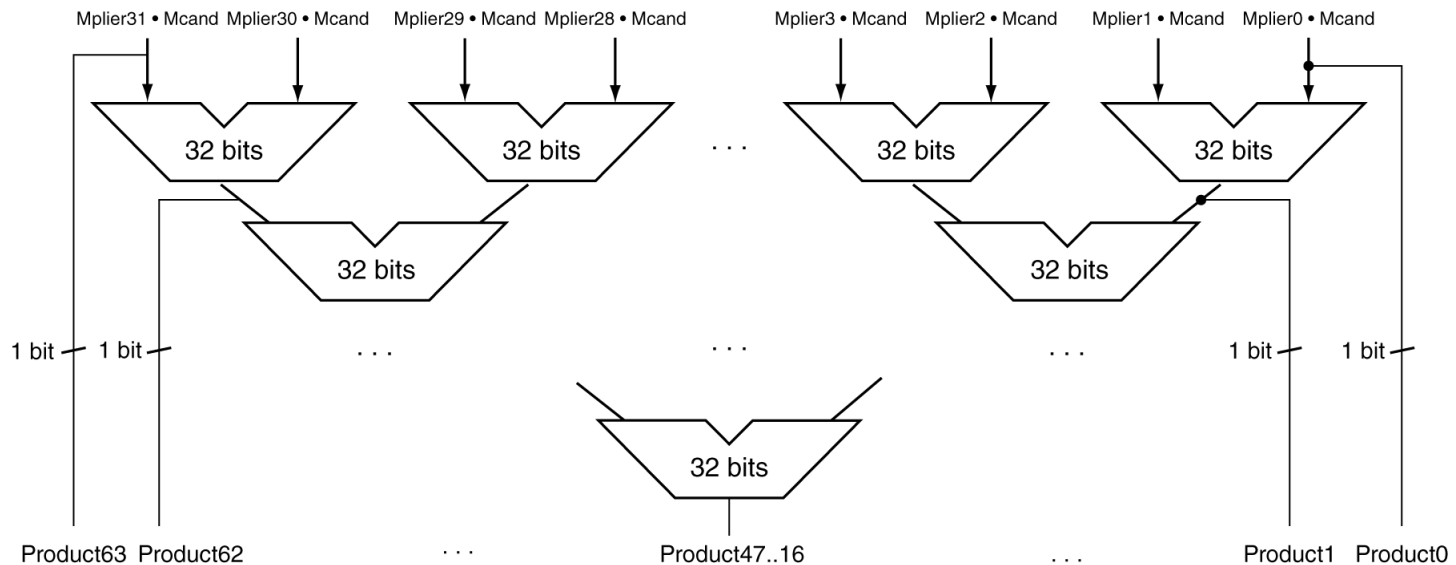
- Can be pipelined
  - Several multiplication performed in parallel

MIPS Multiplication

- Two 32-bit registers for product
  - HI: most-significant 32 bits
  - LO: least-significant 32-bits
- Instructions
  - `mult rs, rt / multu rs, rt`
    - 64-bit product in HI/LO
  - `mfhi rd / mflo rd`
    - Move from HI/LO to rd
    - Can test HI value to see if product overflows 32 bits
  - `mul rd, rs, rt`
    - Least-significant 32 bits of product –> rd

Division

- Check for 0 divisor
- Long division approach
  - If divisor ≤ dividend bits
    - 1 bit in quotient, subtract
  - Otherwise
    - 0 bit in quotient, bring down next dividend bit
- Restoring division
  - Do the subtract, and if remainder goes < 0, add divisor back
- Signed division
  - Divide using absolute values
  - Adjust sign of quotient and remainder as required

quotient

dividend

```
              1001
      1000 )1001010
           -1000
              10
             101
            1010
           -1000
              10
```

divisor

remainder

n-bit operands yield *n*-bit quotient and remainder

Division Hardware

Start

1. Subtract the Divisor register from the Remainder register and place the result in the Remainder register

Test Remainder

Remainder ≥ 0   Remainder < 0

2a. Shift the Quotient register to the left, setting the new rightmost bit to 1

2b. Restore the original value by adding the Divisor register to the Remainder register and placing the sum in the Remainder register. Also shift the Quotient register to the left, setting the new least significant bit to 0

3. Shift the Divisor register right 1 bit

33rd repetition?

No: < 33 repetitions

Yes: 33 repetitions

Done

Initially divisor in left half

Divisor   Shift right

64 bits

64-bit ALU

Quotient   Shift left

32 bits

Remainder   Write

Control test

64 bits

Initially dividend

Optimized Divider

- One cycle per partial-remainder subtraction
- Looks a lot like a multiplier!
  - Same hardware can be used for both

Faster Division

- Can't use parallel hardware as in multiplier
    - Subtraction is conditional on sign of remainder
- Faster dividers (e.g. SRT devision) generate multiple quotient bits per step
    - Still require multiple steps

MIPS Division

- Use HI/LO registers for result
  - HI: 32-bit remainder
  - LO: 32-bit quotient
- Instructions
  - `div rs, rt  /  divu rs, rt`
  - No overflow or divide-by-0 checking
    - Software must perform checks if required
  - Use `mfhi, mflo` to access result

Floating Point

- Representation for non-integral numbers
  - Including very small and very large numbers
- Like scientific notation
  - -2.34×10^{56} ← normalized
  - $+0.002 \times 10^{-4}$ ← not normalized
  - $+987.02 \times 10^{9}$
- In binary
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$
- Types `float` and `double` in C

Floating Point Standard

- Defined by IEEE Std 754-1985

- Developed in response to divergence of representations

  - Portability issues for scientific code

- Now almost universally adopted

- Two representations

  - Single precision (32-bit)

  - Double precision (64-bit)

IEEE Floating-Point Format

	single: 8 bits double: 11 bits	single: 23 bits double: 52 bits
S	Exponent	Fraction

$$x = (-1)^S \times (1 + Fraction) \times 2^{(Exponent - Bias)}$$

- S: sign bit (0 ⇒ non-negative, 1 ⇒ negative)
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the "1." restored
- Exponent: excess representation: actual exponent + Bias
  - Ensures exponent is unsigned
  - Single: Bias = 127; Double: Bias = 1203

Single-Precision Range

- Exponents 00000000 and 11111111 reserved

- Smallest value

  - Exponent: 00000001
    $\Rightarrow$ actual exponent = $1 - 127 = -126$

  - Fraction: 000…00 $\Rightarrow$ significand = 1.0

  - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$

- Largest value

  - exponent: 11111110
    $\Rightarrow$ actual exponent = $254 - 127 = +127$

  - Fraction: 111…11 $\Rightarrow$ significand $\approx$ 2.0

  - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

Double-Precision Range

- Exponents 0000…00 and 1111…11 reserved

- Smallest value
  - Exponent: 00000000001
    $\Rightarrow$ actual exponent = 1 – 1023 = –1022
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$

- Largest value
  - Exponent: 11111111110
    $\Rightarrow$ actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 $\Rightarrow$ significand $\approx$ 2.0
  - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

Floating-Point Precision

- Relative precision
  - all fraction bits are significant
  - Single: approx 2^{-23}
    - Equivalent to $23 \times \log_{10} 2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
  - Double: approx 2^{-52}
    - Equivalent to $52 \times \log_{10} 2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision

Floating-Point Example

- Represent –0.75
    - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
    - S = 1
    - Fraction = $1000\ldots00_2$
    - Exponent = –1 + Bias
        - Single: $-1 + 127 = 126 = 01111110_2$
        - Double: $-1 + 1023 = 1022 = 01111111110_2$
- Single: $1011111101000\ldots00$
- Double: $1011111111101000\ldots00$

Floating-Point Example

- What number is represented by the single-precision float

  110000001101000…00

  - S = 1

  - Fraction = $01000…00_2$

  - Fxponent = 10000001_2 = 129

- $x = (-1)^1 \times (1 + 01_2) \times 2^{(129-127)}$

  $= (-1) \times 1.25 \times 2^2$

  $= -5.0$

Floating-Point Addition

- Consider a 4-digit decimal example
    - $9.999 \times 10^1 + 1.610 \times 10^{-1}$
- 1. Align decimal points
    - Shift number with smaller exponent
    - $9.999 \times 10^1 + 0.016 \times 10^1$
- 2. Add significands
    - $9.999 \times 10^1 + 0.016 \times 10^1 = 10.015 \times 10^1$
- 3. Normalize result & check for over/underflow
    - 1.0015×10^2
- 4. Round and renormalize if necessary
    - 1.002×10^2

Floating-Point Addition

- Now consider a 4-digit binary example
  - $1.000_2 \times 2^{-1} + -1.110_2 \times 2^{-2}$ (0.5 + –0.4375)
- 1. Align binary points
  - Shift number with smaller exponent
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1}$
- 2. Add significands
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1} = 0.001_2 \times 2^{-1}$
- 3. Normalize result & check for over/underflow
  - $1.000_2 \times 2^{-4}$, with no over/underflow
- 4. Round and renormalize if necessary
  - $1.000_2 \times 2^{-4}$ (no change) = 0.0625

FP Adder Hardware

- Much more complex than integer adder
- Doing it in one clock cycle would take too long
  - Much longer than integer operations
  - Slower clock would penalize all instructions
- FP adder usually takes several cycles
  - Can be pipelined

FP Adder Hardware
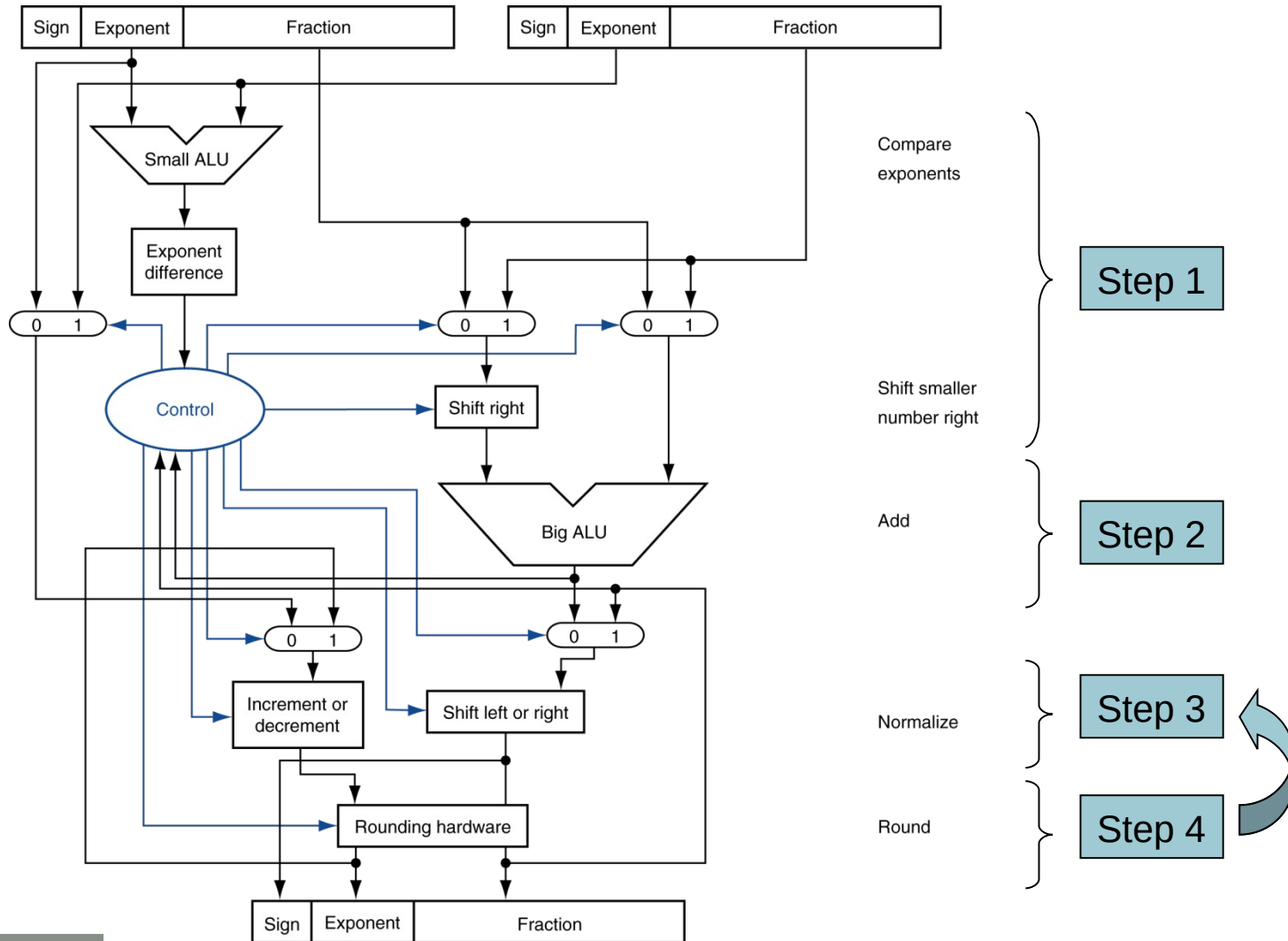
FP Arithmetic Hardware

- FP multiplier is of similar complexity to FP adder
    - But uses a multiplier for significands instead of an adder
- FP arithmetic hardware usually does
    - Addition, subtraction, multiplication, division, reciprocal, square-root
    - FP ↔ integer conversion
- Operations usually takes several cycles
    - Can be pipelined

FP Instructions in MIPS

- FP hardware is coprocessor 1
  - Adjunct processor that extends the ISA
- Separate FP registers
  - 32 single-precision: $f0, $f1, … $f31
  - Paired for double-precision: $f0/$f1, $f2/$f3, …
    - Release 2 of MIPs ISA supports 32 × 64-bit FP reg's
- FP instructions operate only on FP registers
  - Programs generally don't do integer ops on FP data, or vice versa
  - More registers with minimal code-size impact
- FP load and store instructions
  - `lwc1, ldc1, swc1, sdc1`
    - e.g., `ldc1 $f8, 32($sp)`

FP Instructions in MIPS

- Single-precision arithmetic
  - `add.s`, `sub.s`, `mul.s`, `div.s`
    - e.g., `add.s $f0, $f1, $f6`
- Double-precision arithmetic
  - `add.d`, `sub.d`, `mul.d`, `div.d`
    - e.g., `mul.d $f4, $f4, $f6`
- Single- and double-precision comparison
  - `c.xx.s`, `c.xx.d` (*xx* is `eq`, `lt`, `le`, …)
  - Sets or clears FP condition-code bit
    - e.g. `c.lt.s $f3, $f4`
- Branch on FP condition code true or false
  - `bc1t`, `bc1f`
    - e.g., `bc1t TargetLabel`

FP Example: °F to °C

- C code:

```
float f2c (float fahr) {
    return ((5.0/9.0)*(fahr - 32.0));
}
```

  - fahr in $f12, result in $f0, literals in global memory space

- Compiled MIPS code:

```
f2c: lwc1  $f16, const5($gp)
     lwc2  $f18, const9($gp)
     div.s $f16, $f16, $f18
     lwc1  $f18, const32($gp)
     sub.s $f18, $f12, $f18
     mul.s $f0,  $f16, $f18
     jr    $ra
```

FP Example: Array Multiplication

- X = X + Y × Z
  - All 32 × 32 matrices, 64-bit double-precision elements
- C code:

```
void mm (double x[][],
          double y[][], double z[][]) {
  int i, j, k;
  for (i = 0; i! = 32; i = i + 1)
    for (j = 0; j! = 32; j = j + 1)
      for (k = 0; k! = 32; k = k + 1)
        x[i][j] = x[i][j]
                      + y[i][k] * z[k][j];
}
```

  - Addresses of x, y, z in $a0, $a1, $a2, and
    i, j, k in $s0, $s1, $s2

FP Example: Array Multiplication

- ## MIPS code:

```
        li    $t1, 32          # $t1 = 32 (row size/loop end)
        li    $s0, 0           # i = 0; initialize 1st for loop
L1:     li    $s1, 0           # j = 0; restart 2nd for loop
L2:     li    $s2, 0           # k = 0; restart 3rd for loop
        sll   $t2, $s0, 5      # $t2 = i * 32 (size of row of x)
        addu  $t2, $t2, $s1    # $t2 = i * size(row) + j
        sll   $t2, $t2, 3      # $t2 = byte offset of [i][j]
        addu  $t2, $a0, $t2    # $t2 = byte address of x[i][j]
        l.d   $f4, 0($t2)      # $f4 = 8 bytes of x[i][j]
L3:     sll   $t0, $s2, 5      # $t0 = k * 32 (size of row of z)
        addu  $t0, $t0, $s1    # $t0 = k * size(row) + j
        sll   $t0, $t0, 3      # $t0 = byte offset of [k][j]
        addu  $t0, $a2, $t0    # $t0 = byte address of z[k][j]
        l.d   $f16, 0($t0)     # $f16 = 8 bytes of z[k][j]
```

…

FP Example: Array Multiplication

…

```
    sll   $t0, $s0, 5         # $t0 = i*32 (size of row of y)
    addu  $t0, $t0, $s2       # $t0 = i*size(row) + k
    sll   $t0, $t0, 3         # $t0 = byte offset of [i][k]
    addu  $t0, $a1, $t0       # $t0 = byte address of y[i][k]
    l.d   $f18, 0($t0)        # $f18 = 8 bytes of y[i][k]
    mul.d $f16, $f18, $f16    # $f16 = y[i][k] * z[k][j]
    add.d $f4, $f4, $f16      # f4=x[i][j] + y[i][k]*z[k][j]
    addiu $s2, $s2, 1         # $k k + 1
    bne   $s2, $t1, L3        # if (k != 32) go to L3
    s.d   $f4, 0($t2)         # x[i][j] = $f4
    addiu $s1, $s1, 1         # $j = j + 1
    bne   $s1, $t1, L2        # if (j != 32) go to L2
    addiu $s0, $s0, 1         # $i = i + 1
    bne   $s0, $t1, L1        # if (i != 32) go to L1
```

Accurate Arithmetic

- IEEE Std 754 specifies additional rounding control
    - Extra bits of precision (guard, round, sticky)
    - Choice of rounding modes
    - Allows programmer to fine-tune numerical behavior of a computation
- Not all FP units implement all options
    - Most programming languages and FP libraries just use defaults
- Trade-off between hardware complexity, performance, and market requirements

Subword Parallellism

- Graphics and audio applications can take advantage of performing simultaneous operations on short vectors
    - Example: 128-bit adder:
        - Sixteen 8-bit adds
        - Eight 16-bit adds
        - Four 32-bit adds
- Also called data-level parallelism, vector parallelism, or Single Instruction, Multiple Data (SIMD)

x86 FP Architecture

- Originally based on 8087 FP coprocessor
  - 8 × 80-bit extended-precision registers
  - Used as a push-down stack
  - Registers indexed from TOS: ST(0), ST(1), …
- FP values are 32-bit or 64 in memory
  - Converted on load/store of memory operand
  - Integer operands can also be converted on load/store
- Very difficult to generate and optimize code
  - Result: poor FP performance

x86 FP Instructions

Data transfer	Arithmetic	Compare	Transcendental
FILD mem/ST(i) FISTP mem/ST(i) FLDPI FLD1 FLDZ	FIADDP mem/ST(i) FISUBRP mem/ST(i) FIMULP mem/ST(i) FIDIVRP mem/ST(i) FSQRT FABS FRNDINT	FICOMP FIUCOMP FSTSW AX/mem	FPATAN F2XMI FCOS FPTAN FPREM FPSIN FYL2X

- Optional variations
  - I: integer operand
  - P: pop operand from stack
  - R: reverse operand order
  - But not all combinations allowed

Streaming SIMD Extension 2 (SSE2)

- Adds 4 × 128-bit registers
  - Extended to 8 registers in AMD64/EM64T
- Can be used for multiple FP operands
  - 2 × 64-bit double precision
  - 4 × 32-bit double precision
  - Instructions operate on them simultaneously
    - Single-Instruction Multiple-Data

Matrix Multiply

- Unoptimized code:

```
1. void dgemm (int n, double* A, double* B, double* C)
2. {
3.   for (int i = 0; i < n; ++i)
4.     for (int j = 0; j < n; ++j)
5.     {
6.       double cij = C[i+j*n]; /* cij = C[i][j] */
7.       for(int k = 0; k < n; k++ )
8.         cij += A[i+k*n] * B[k+j*n]; /* cij += A[i][k]*B[k][j] */
9.       C[i+j*n] = cij; /* C[i][j] = cij */
10.    }
11. }
```

Matrix Multiply

- x86 assembly code:

```
1. vmovsd (%r10),%xmm0   # Load 1 element of C into %xmm0

2. mov %rsi,%rcx         # register %rcx = %rsi

3. xor %eax,%eax         # register %eax = 0

4. vmovsd (%rcx),%xmm1   # Load 1 element of B into %xmm1

5. add %r9,%rcx          # register %rcx = %rcx + %r9

6. vmulsd (%r8,%rax,8),%xmm1,%xmm1 # Multiply %xmm1,
   element of A

7. add $0x1,%rax         # register %rax = %rax + 1

8. cmp %eax,%edi         # compare %eax to %edi

9. vaddsd %xmm1,%xmm0,%xmm0 # Add %xmm1, %xmm0

10. jg 30 <dgemm+0x30>   # jump if %eax > %edi

11. add $0x1,%r11d       # register %r11 = %r11 + 1

12. vmovsd %xmm0,(%r10)  # Store %xmm0 into C element
```

Matrix Multiply

- ## Optimized C code:

```
1. #include <x86intrin.h>
2. void dgemm (int n, double* A, double* B, double* C)
3. {
4.  for ( int i = 0; i < n; i+=4 )
5.   for ( int j = 0; j < n; j++ ) {
6.     __m256d c0 = _mm256_load_pd(C+i+j*n); /* c0 = C[i][j] */
7.     for( int k = 0; k < n; k++ )
8.      c0 = _mm256_add_pd(c0, /* c0 += A[i][k]*B[k][j] */
9.               _mm256_mul_pd(_mm256_load_pd(A+i+k*n),
10.              _mm256_broadcast_sd(B+k+j*n)));
11.  _mm256_store_pd(C+i+j*n, c0); /* C[i][j] = c0 */
12.  }
13. }
```

Matrix Multiply

- Optimized x86 assembly code:

```
1. vmovapd (%r11),%ymm0          # Load 4 elements of C into %ymm0
2. mov %rbx,%rcx                 # register %rcx = %rbx
3. xor %eax,%eax                 # register %eax = 0
4. vbroadcastsd (%rax,%r8,1),%ymm1 # Make 4 copies of B element
5. add $0x8,%rax                 # register %rax = %rax + 8
6. vmulpd (%rcx),%ymm1,%ymm1     # Parallel mul %ymm1,4 A elements
7. add %r9,%rcx                  # register %rcx = %rcx + %r9
8. cmp %r10,%rax                 # compare %r10 to %rax
9. vaddpd %ymm1,%ymm0,%ymm0      # Parallel add %ymm1, %ymm0
10. jne 50 <dgemm+0x50>          # jump if not %r10 != %rax
11. add $0x1,%esi                # register % esi = % esi + 1
12. vmovapd %ymm0,(%r11)         # Store %ymm0 into 4 C elements
```

Right Shift and Division

- Left shift by *i* places multiplies an integer by 2^i

- Right shift divides by 2^i?
  - Only for unsigned integers

- For signed integers
  - Arithmetic right shift: replicate the sign bit
  - e.g., –5 / 4
    - $11111011_2 >> 2 = 11111110_2 = –2$
    - Rounds toward –∞
  - c.f. $11111011_2 >>> 2 = 00111110_2 = +62$

Associativity

- Parallel programs may interleave operations in unexpected orders
  - Assumptions of associativity may fail

		(x+y)+z	x+(y+z)
x	-1.50E+38		-1.50E+38
y	1.50E+38	0.00E+00	
z	1.0	1.0	1.50E+38
		1.00E+00	0.00E+00

- Need to validate parallel programs under varying degrees of parallelism

Who Cares About FP Accuracy?

- Important for scientific code
  - But for everyday consumer use?
    - "My bank balance is out by 0.0002¢!" ☹

- The Intel Pentium FDIV bug
  - The market expects accuracy
  - See Colwell, *The Pentium Chronicles*

Concluding Remarks

- Bits have no inherent meaning
  - Interpretation depends on the instructions applied
- Computer representations of numbers
  - Finite range and precision
  - Need to account for this in programs

Concluding Remarks

- ISAs support arithmetic
    - Signed and unsigned integers
    - Floating-point approximation to reals
- Bounded range and precision
    - Operations can overflow and underflow
- MIPS ISA
    - Core instructions: 54 most frequently used
        - 100% of SPECINT, 97% of SPECFP
    - Other instructions: less frequent