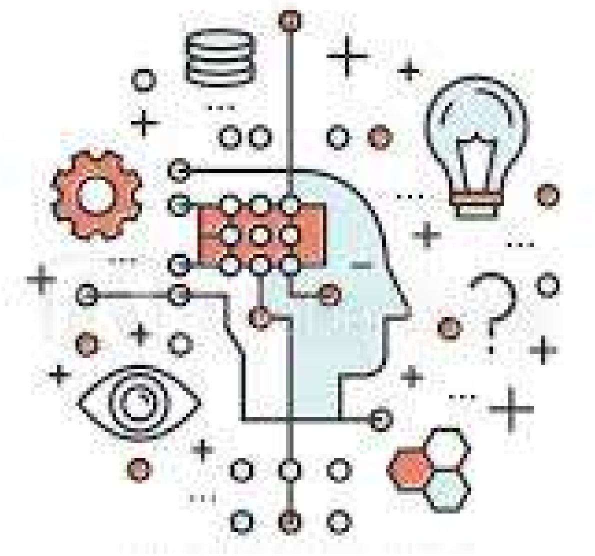
CS 321 SOFT COMPUTING



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Course Objectives

The course enables the students to:

- To understand the concept of fuzzy logic and controllers.
- To understand the various architectures of ANN and its learning methods.
- To learn about basic concepts of genetic algorithm and its operators.
- To understand the Artificial Neural Networks.
- To understand the Genetic Algorithms.

Course Outcomes

Upon completing the course, you will be able to learn following parameters:

- Solve numerical on Fuzzy sets and Fuzzy Reasoning .
- Develop Fuzzy Inference System (FIS).
- Solve problems on Genetic Algorithms.
- Explain concepts of neural networks.
- Develop neural networks models for various applications.

Syllabus

MODULE – I

- **Fuzzy Set Theory:** Basic Definition and Terminology, Set Theoretic Operations, Fuzzy types and levels, MF Formulation and Parameterization, MF of two dimensions, Fuzzy Union, Intersection and Complement, Fuzzy Number, Fuzzy measure. (8L)

MODULE – II

- **Fuzzy Logic:** Fuzzy Rules and Fuzzy Reasoning: Extension Principles and Fuzzy Relations, Fuzzy IF THEN Rules, Defuzzification, FuzzyReasoning. Fuzzy Inference System: Introduction, Mamdani Fuzzy Models, Other Variants, Sugeno Fuzzy Models, Tsukamoto Fuzzy Models. (8L)

Syllabus

MODULE – III

- **Fundamentals of Genetic Algorithms:** Basic Concepts, Creation of Offsprings, Encoding, Fitness Functions, Reproduction, Genetic Modelling: Inheritance Operators, Cross over, Inversion and detection, Mutation operator, Bitwise operators. (8L)

Syllabus

MODULE – IV

- **Introduction to Artificial Neural Networks:** What is a Neural Network? Human Brain, Models of Neuron, Neural Network viewed as Directed Graphs, Feedback, Network Architecture, Knowledge Representation, Learning processes:(Error correction, Memory-Based, Hebbian, Competitive, Boltzman, Supervised, Unsupervised), Memory, Adaptation. (8L)

MODULE – V

- Perceptrons, Adaline, Back Propagation Algorithm, Methods of Speeding, Convolution Networks, Radical Basis Function Networks, Covers Theorem, Interpolation Learning, The Hopfield Network. (8L)

Books

Text Book :

- Jang J.S.R., Sun C.T. and Mizutani E., “Neuro-Fuzzy and Soft Computing”, PHI/Pearson Education, New Delhi 2004. (T1)
- Rajasekaran S. & Vijayalakshmi G.A. Pai, PHI, New Delhi 2003. (T2)
- Ross T. J., “Fuzzy Logic with Engineering Applications.”, TMH, New York, 1997. (T3)
- Haykins Simon, ”Neural Networks: A Comprehensive Foundation, Pearson Education,2002. (T4)

Reference Book:

- Ray K.S. ,”Soft Computing and Its application”, Vol 1, Apple Academic Press, 2015. (R1)
- Lee K.H. ,”First Course on Fuzzy Theory and App.”, Adv in Soft Computing Springer, 2005.(R2)
- Zimmermann H.Z. ,”Fuzzy Set Theory and its App “ , 4th Edition, Springer Science, 2001.(R3)

Marks Distribution

- Continuous Internal Assessment: 50
  - Mid Semester examination: 25
  - Two quizzes: 20 (2×10)
  - Teacher's Assessment: 5
- Semester End Examination: 50

Module - I

Fuzzy Set Theory

- Basic Definition and Terminology
- Set Theoretic Operations
- Fuzzy types and levels
- MF Formulation and Parameterization
    - MF of two dimensions
- Fuzzy Union, Intersection and Complement
- Fuzzy Number
- Fuzzy measure

1. Introduction

- **Soft Computing** is an emerging approach to computing which parallels the remarkable ability of the human mind to reason and learn in an environment of uncertainty and impression. (Lotfi A. Zadeh)
- Soft computing consists of several computing techniques including fuzzy set theory, neural network, optimization methodologies like genetic algorithms, simulated annealing, etc.

Fuzzy Set Theory

- What is a classical set?

$$\boxed{A = \{1, 2, 3, 4, 5, 6\} \qquad or \qquad A = \{x \mid 1 \leq x \leq 6\}}$$

- Can you design a set of tall persons / temperature / weight ?

- There is a sharp transition between inclusion and exclusion.
- Fuzzy set is without a crisp boundary.
- The change between two sets is gradual which is characterized by use of a membership function.

1. Introduction

Basic Terminology

- X: Universe of discourse (space of objects)
- x: an element that belongs to X $(x \in X)$
- In a crisp set, if $A \subseteq X$, either $x \in A$ or $x \notin A$ which can be indicated as $(x, 1)$ or $(x, 0)$ respectively. (Degree of membership)
- A fuzzy set is defined as $\boxed{A = \{(x, \mu_A(x)) \mid x \in X\}}$, where $\mu_A(x)$ is called the membership function (MF) for the fuzzy set A.
- The MF maps each elements of X to a membership grade between 0 and 1.
- Fuzzy set may be discrete or continuous / ordered or unordered.

Let X = {San Francisco, Boston, Los Angeles} be the set of cities one may choose to live in. The fuzzy set C = "desirable city to live in" may be described as follows:

$$C = \{(\text{San Francisco}, 0.9), (\text{Boston}, 0.8), (\text{Los Angeles}, 0.6)\}.$$

Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of numbers of children a family may choose to have. Then the fuzzy set A = "sensible number of children in a family" may be described as follows:

$$A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}.$$

1. Introduction

Let $X = R^+$ be the set of possible ages for human beings. Then the fuzzy set $B =$ "about 50 years old" may be expressed as

$$B = \{(x, \mu_B(x) | x \in X\},$$

where

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}.$$

- The specification of MF is subjective and nonrandom.
- Fuzzy set can be written as:

$$A = \begin{cases} \sum_{x_i \in X} \mu_A(x_i)/x_i, & \text{if } X \text{ is a collection of discrete objects.} \\ \int_X \mu_A(x)/x, & \text{if } X \text{ is a continuous space (usually the real line } R\text{).} \end{cases}$$

1. Introduction

- **Linguistic Variable:** Ex: Age, Temperature, Height, Weight

- **Linguistic values:** A linguistic variable can assume different linguistic values. Ex: young, teenage, middle ages, old.

- **Support:** The support of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$.

$$support\ (A) = \{x \mid \mu_A(x) > 0\}$$

- **Core:** The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$.

$$core(A) = \{x \mid \mu_A(x) = 1\}$$

1. Introduction

- **Normality:** A fuzzy set is normal if its core is nonempty.

- **Crossover point:** A crossover point of a fuzzy set is a point $x \in X$ at which $\mu_A(x) = 0.5$.

$$crossover(A) = \{\, x \mid \mu_A(x) = 0.5\}$$

- **Fuzzy singleton:** A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton.

- α **– cut:** The α – cut or α – level set of a fuzzy set A is a crisp set defined by:

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$$

- **Strong** α **– cut** or strong α – level set are defined by:

$$A'_\alpha = \{x \mid \mu_A(x) > \alpha\}$$

Set Theoretical Operations

- **Containment / subset:** Fuzzy set A is contained in fuzzy set B, or A is a subset of B, or A is smaller than or equal to B if and only if $\mu_A(x) \leq \mu_B(x)$.

$$A \nsubseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$$

- **Union (Disjunction):** The union of two fuzzy sets A and B is defined as $C = A \cup B$ and the MF are related as:

$$\mu_C(x) = \max\big(\mu_A(x), \mu_B(x)\big) = \mu_A(x) \vee \mu_B(x)$$

- **Intersection (Conjunction):** The intersection of two fuzzy sets A and B is defined as $C = A \cap B$ and the MF are related as:

$$\mu_C(x) = \min\big(\mu_A(x), \mu_B(x)\big) = \mu_A(x) \wedge \mu_B(x)$$

Set Theoretical Operations

- **Complement (negation):** The complement of fuzzy set A, denoted by $\bar{A}$ is defined as:

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

- **Difference:** The difference of two fuzzy sets A and B, denoted as $A - B$ is given as:

$$\mu_{A-B}(x) = \mu_A(x) \wedge \mu_{\bar{B}}(x) = \min(\mu_A(x), \mu_{\bar{B}}(x))$$

- **Cartesian product:** Let A and B be fuzzy sets in X and Y respectively. The Cartesian product denoted by $A \times B$, is a fuzzy set in the product space $X \times Y$ with the MF

$$\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y))$$

- **Cartesian co-product:** $A + B$ is a fuzzy set with membership function:

$$\mu_{A+B}(x, y) = \max(\mu_A(x), \mu_B(y))$$

Set Theoretical Operations

- **Algebraic sum:** The algebraic sum C of two fuzzy sets A and B is defined as:

$$\mu_C(x) = \mu_A(x) + \mu_B(x) - \mu_{A(x)} . \mu_B(x)$$

- **Algebraic product:** The algebraic product C of two fuzzy sets A and B is defined as:

$$\mu_C(x) = \mu_A(x) . \mu_B(x)$$

- **Bounded sum:** The bounded sum of two fuzzy sets A and B is defined as:

$$\mu_{A \oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$$

- **Bounded difference:** The bounded difference of two fuzzy sets A and B is defined as:

$$\mu_{A \odot B}(x) = \mathrm{m}ax\{0, \mu_A(x) - \mu_B(x)\}$$

Properties of fuzzy set examples

Fuzzy Set Operations

$A = \{(2, 0.8), (4, 0.3), (7, 0.5)\}$ $B = \{(2, 0.5), (4, 0.2), (7, 0.7)\}$

① Algebraic sum: $\mu_C(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$

$\mu_C(2) = 0.8 + 0.5 - (0.8 \times 0.5)) = 0.9$

$\mu_C(4) = 0.3 + 0.2 - (0.3 \times 0.2) = 0.44$

$\mu_C(7) = 0.5 + 0.7 - (0.5 \times 0.7) = 0.85.$

$C = \{(2, 0.9), (4, 0.44), (7, 0.85)\}$

② Algebraic product: $\mu_C(x) = \mu_A(x) \cdot \mu_B(x).$

$\mu_C(2) = 0.8 \times 0.5 = 0.4$

$\mu_C(4) = 0.3 \times 0.2 = 0.06.$

$\mu_C(7) = 0.5 \times 0.7 = 0.35$

$C = \{(2, 0.4), (4, 0.06), (7, 0.35)\}.$

Properties of fuzzy set examples

(3) Bounded sum: $\mu_C(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$.

$\mu_C(2) = \min\{1, 1.3\} = 1$

$\mu_C(4) = \min\{1, 0.5\} = 0.5$.

$\mu_C(7) = \min\{1, 1.2\} = 1$

$C = \{(2,1), (4, 0.5), (7,1)\}$.

(4) Bounded difference: $\mu_C(x) = \max\{0, \mu_A(x) - \mu_B(x)\}$

$\mu_C(2) = \max\{(0, 0.3)\} = 0.3$

$\mu_C(4) = \max\{0, 0.1\} = 0.1$

$\mu_C(7) = \max\{0, -0.2\} = 0$

$C = \{(2, 0.3), (4, 0.1), (7, 0)\}$.

Properties of Fuzzy Sets

Let A, B and C be three fuzzy sets in a universe of discourse U.

- **Idempotent law:** $A \cup A = A$, $A \cap A = A$
- **Commutative law:** $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Associative law:** $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
- **Absorption law:** $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- **Distribution law:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Involution law:** $\bar{\bar{A}} = A$
- **De Morgan's law:** $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$
- **Identity law:** $A \cup \emptyset = A$, $A \cup U = U$, $A \cap \emptyset = \emptyset$, $A \cap U = A$ ($\emptyset$ is an empty fuzzy set with $\mu_{\emptyset}(x) = 0$)
- **Complement law:** $A \cup \bar{A} \neq U$, $A \cap \bar{A} \neq \emptyset$

Operations on fuzzy set examples

Properties of Fuzzy Set Examples.

Consider two fuzzy sets

$A = \{(2, 0.8), (4, 0.3), (7, 0.5)\}$ $B = \{(2, 0.5), (4, 0.2), (7, 0.7)\}$

① Idempotent law: $A \cup A = A$, $A \cap A = A$

$A \cup A = \{(4, 0.3), (2, 0.8), (7, 0.5)\} = A$.

Similarly for $A \cap A = A$.

② Commutative law: $A \cup B = B \cup A$, $A \cap B = B \cap A$.

$A \cup B = \{(2, 0.8), (4, 0.3), (7, 0.7)\}$.

$B \cup A = \{(2, 0.8), (4, 0.3), (7, 0.7)\}$.

$\Rightarrow A \cup B = B \cup A$.

Similarly for intersection.

$A \cup (B \cup C)$

Operations on fuzzy set examples

Similarly

(3) <u>Associative law</u>: $(A \cup B) \cup C = A \cup (B \cup C)$.

Let $C = \{(2, 0.1), (4, 0.9), (7, 0.3)\}$.

$A \cup B = \{(2, 0.8), (4, 0.3), (7, 0.7)\}$.

$(A \cup B) \cup C = \{(2, 0.8), (4, 0.9), (7, 0.7)\}$. — (I)

$B \cup C = \{(2, 0.5), (4, 0.9), (7, 0.7)\}$

$A \cup (B \cup C) = \{(2, 0.8), (4, 0.9), (7, 0.7)\}$ — (II).

From (I) & (II) $(A \cup B) \cup C = A \cup (B \cup C)$.

Properties of fuzzy set examples

④ Absorption law: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

$$A \cap B = \{(2, 0.5), (4, 0.2), (7, 0.5)\}$$

$$A \cup (A \cap B) = \{(2, 0.8), (4, 0.3), (7, 0.5)\} = A$$

⑤ Distribution law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$B \cap C = \{(2, 0.1), (4, 0.2), (7, 0.3)\}.$$

$$A \cup (B \cap C) = \{(2, 0.8), (4, 0.3), (7, 0.5)\} \text{ — (i)}$$

$$A \cup B = \{(2, 0.8), (4, 0.3), (7, 0.7)\}.$$

$$A \cup C = \{(2, 0.8), (4, 0.9), (7, 0.5)\}.$$

$$(A \cup B) \cap (A \cup C) = \{(2, 0.8), (4, 0.3), (7, 0.5)\} \text{ — (ii)}.$$

From (i) & (ii) $\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Properties of fuzzy set examples

(6) Involution law: $\bar{\bar{A}} = A$.

$$\bar{A} = \{(2, 0.2), (4, 0.7), (7, 0.5)\}$$

$$\bar{\bar{A}} = \{(2, 0.8), (4, 0.3), (7, 0.5)\} = A.$$

(7) De Morgan's law: $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$$A \cup B = \{(2, 0.8), (4, 0.3), (7, 0.7)\}.$$

$$\overline{A \cup B} = \{(2, 0.2), (4, 0.7), (7, 0.3)\}. \quad \text{— (i)}$$

$$\bar{A} \cap \bar{B} = \{(2, 0.2), (4, 0.7), (7, 0.3)\} \quad \text{— (ii)}.$$

From (i) & (ii) $\Rightarrow$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

(9) Complement law: $A \cap \bar{A} \neq \phi$

$$A = \{(2, 0.8), (4, 0.3), (7, 0.5)\}.$$

$$\bar{A} = \{(2, 0.2), (4, 0.7), (7, 0.5)\}$$

$$A \cap \bar{A} = \{(2, 0.2), (4, 0.3), (7, 0.5)\} \neq \phi.$$

Introduction

- **Height:** The height of a fuzzy set A is the largest membership grade of an element in A.

$$height\ (A) = \ max_X(\mu_A(X))$$

- **Convexity:** A fuzzy set A is convex if and only if for any $x_1, x_2 \in X$ and any $\lambda \in [0, 1]$,

$$\mu_A(\lambda\, x_1 + (1-\lambda)x_2) \geq \min\ \{\mu_A(x_1), \mu_A(x_2)\}$$

- **Fuzzy numbers:** A fuzzy number A is a fuzzy set that satisfies the conditions for normality and convexity.

- **Bandwidth:** For a normal and convex fuzzy set, the bandwidth or width is defined as the distance between the two unique / extreme crossover points.

$$width(A) = |x_2 - x_1|\ where\, \mu_A(x_1) = \mu_A(x_2) = 0.5$$

Introduction

- **Symmetry:** A fuzzy set A is symmetric if its MF is symmetric around a certain point $x = c$, namely,

$$\mu_A(c + x) = \mu_A(c - x)\ \forall\, x \in X$$

- **Open left, open right and closed fuzzy sets:** A fuzzy set is
  - Open left if $\lim_{x\to-\infty} \mu_A(x) = 1$ and $\lim_{x\to+\infty} \mu_A(x) = 0$;
  - Open right if $\lim_{x\to-\infty} \mu_A(x) = 0$ and $\lim_{x\to+\infty} \mu_A(x) = 1$;
  - Closed if $\lim_{x\to-\infty} \mu_A(x) = \lim_{x\to+\infty} \mu_A(x) = 0$;

Membership functions formulation

- **Member functions (MF)** defines the fuzziness in a fuzzy set.
- We know that a fuzzy set A in the universe of discourse X can be defined as a set of ordered pairs:

$$A = \{(x, \mu_A(x) \mid x \in X\}$$

- μ_A is called the membership function of A.
- It maps X to the membership space M, i.e. $\mu_A : X \rightarrow M$.
- The membership values ranges in the interval $[0, 1]$.

MF of One Dimension

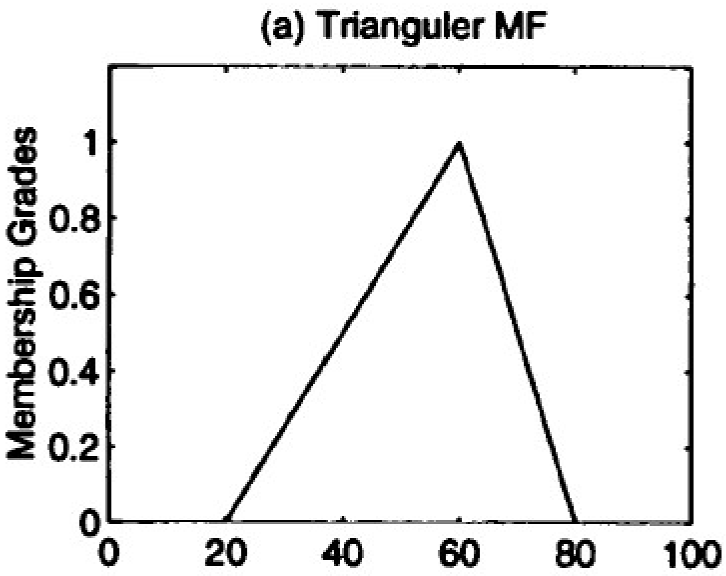
- Triangular MFs

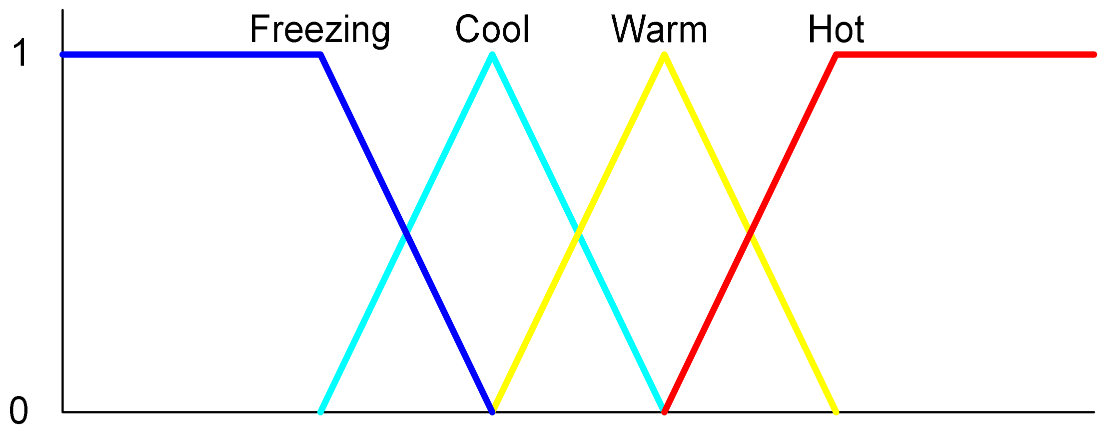
A triangular MF is specified by three parameters $\{a, b, c\}$ as follows:

$$\text{triangle}(x; a, b, c) = \begin{cases} 0, & x \leq a. \\ \dfrac{x-a}{b-a}, & a \leq x \leq b. \\ \dfrac{c-x}{c-b}, & b \leq x \leq c. \\ 0, & c \leq x. \end{cases}$$

Membership functions formulation

- Triangular MFs



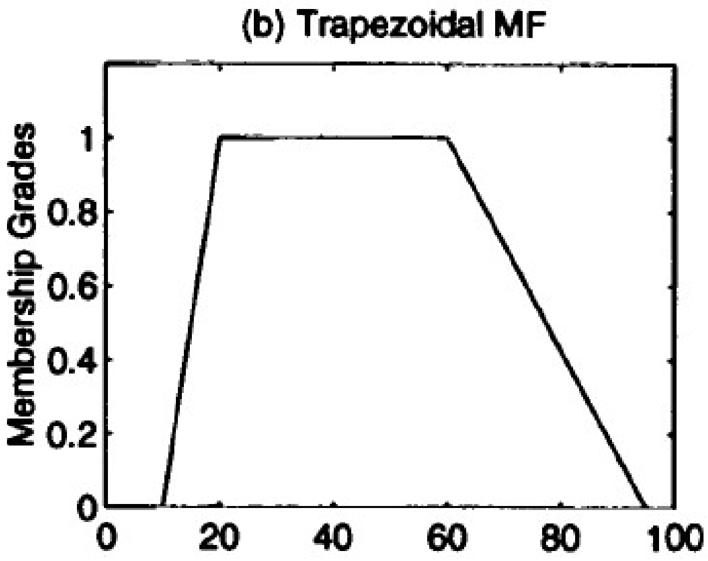


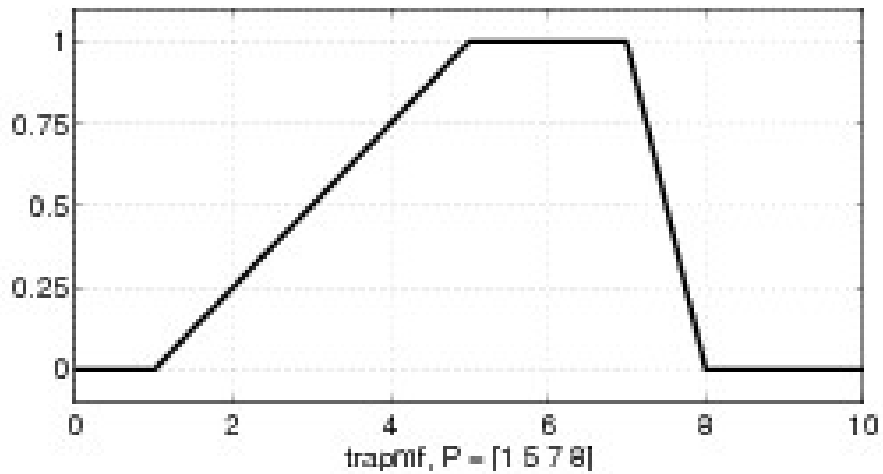
Membership functions formulation

- Trapezoidal MFs

A **trapezoidal MF** is specified by four parameters $\{a, b, c, d\}$ as follows:

$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0, & x \leq a. \\ \dfrac{x-a}{b-a}, & a \leq x \leq b. \\ 1, & b \leq x \leq c. \\ \dfrac{d-x}{d-c}, & c \leq x \leq d. \\ 0, & d \leq x. \end{cases}$$

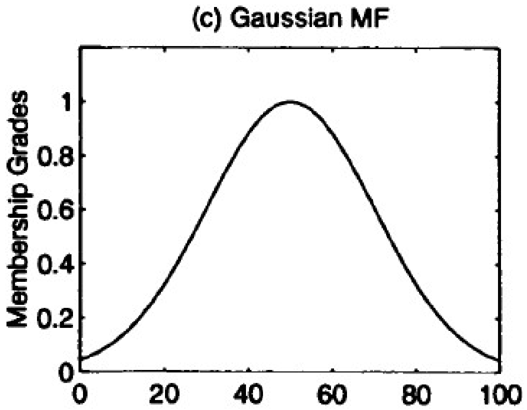




Membership functions formulation

- Gaussian MFs

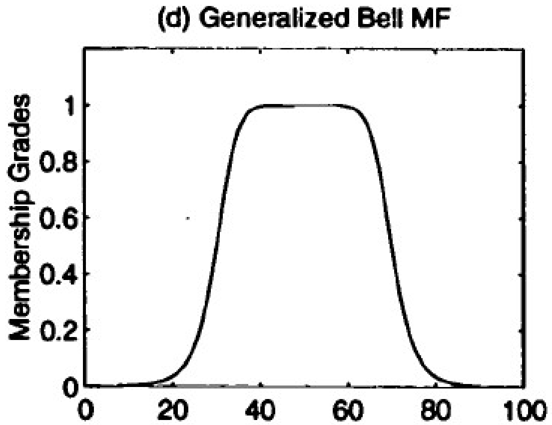
A **Gaussian MF** is specified by two parameters $\{c, \sigma\}$:



$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$

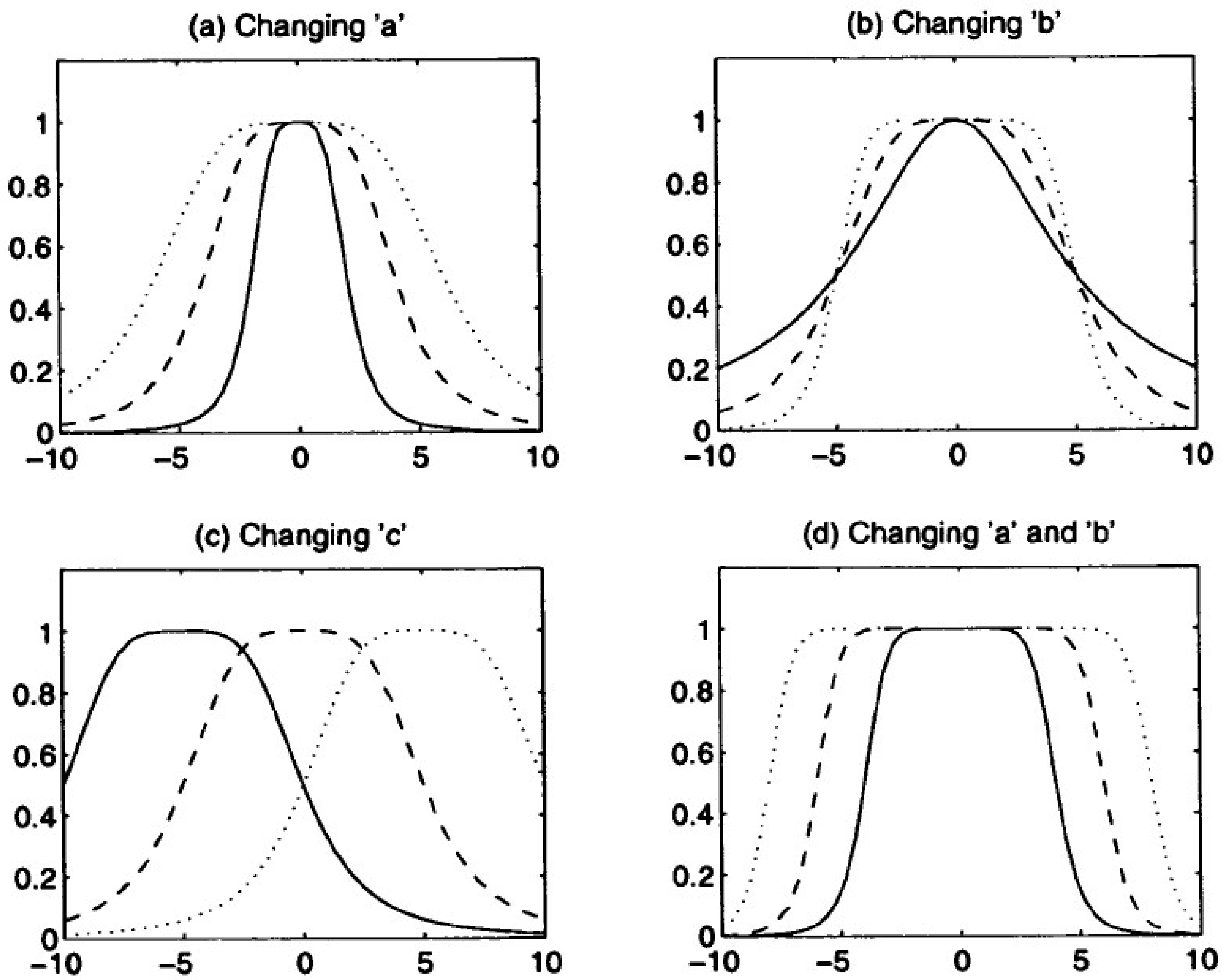
- Generalised bell function / Cauchy MF

A **generalized bell MF** (or **bell MF**) is specified by three parameters $\{a, b, c\}$:



$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}},$$

Membership functions formulation



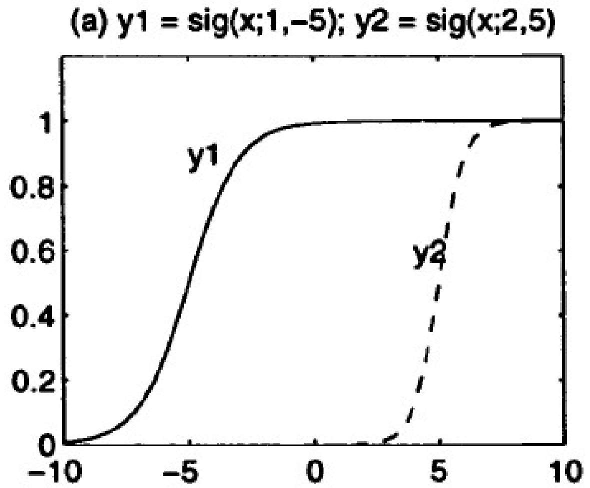
Membership functions formulation

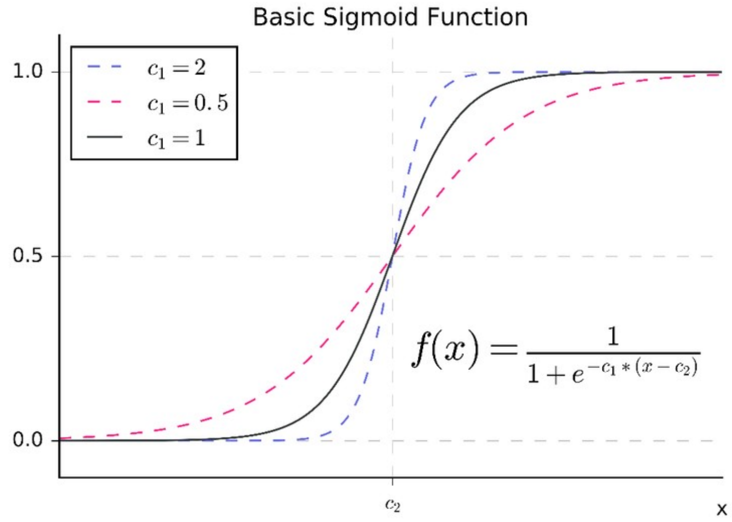
- Sigmoidal MFs

A **sigmoidal MF** is defined by

$$\text{sig}(x; a, c) = \frac{1}{1 + \exp[-a(x - c)]},$$

where a controls the slope at the crossover point $x = c$.





Membership functions formulation

MF of Two Dimensions

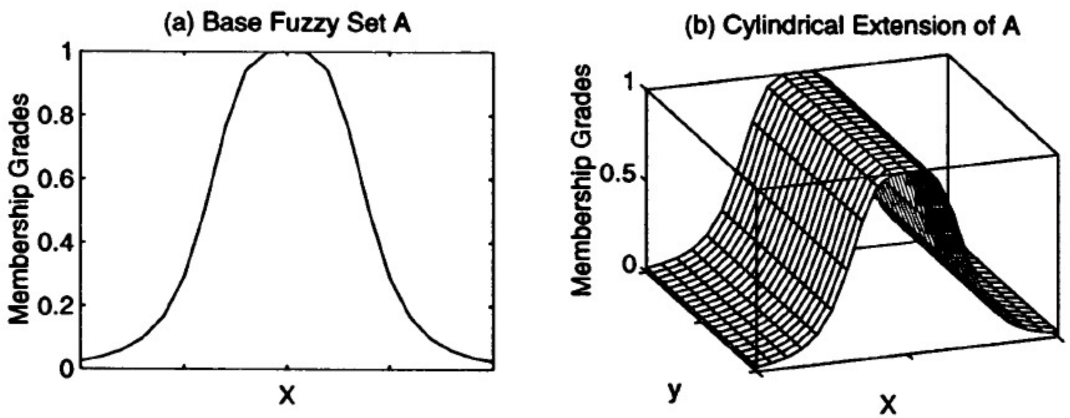
- Sometimes it is needed to use two MFs on different inputs (different Universe of Discourse).
- A one dimensional MF is extended to two dimensional using the extension principle.

Cylindrical extension

- If A is a fuzzy set in X, then its cylindrical extension in $X \times Y$ is a fuzzy set $c(A)$ defined by:

If A is a fuzzy set in X, then its **cylindrical extension** in $X \times Y$ is a fuzzy set $c(A)$ defined by

$$c(A) = \int_{X \times Y} \mu_A(x)/(x, y).$$



Membership functions formulation

- **Projection of a fuzzy set**

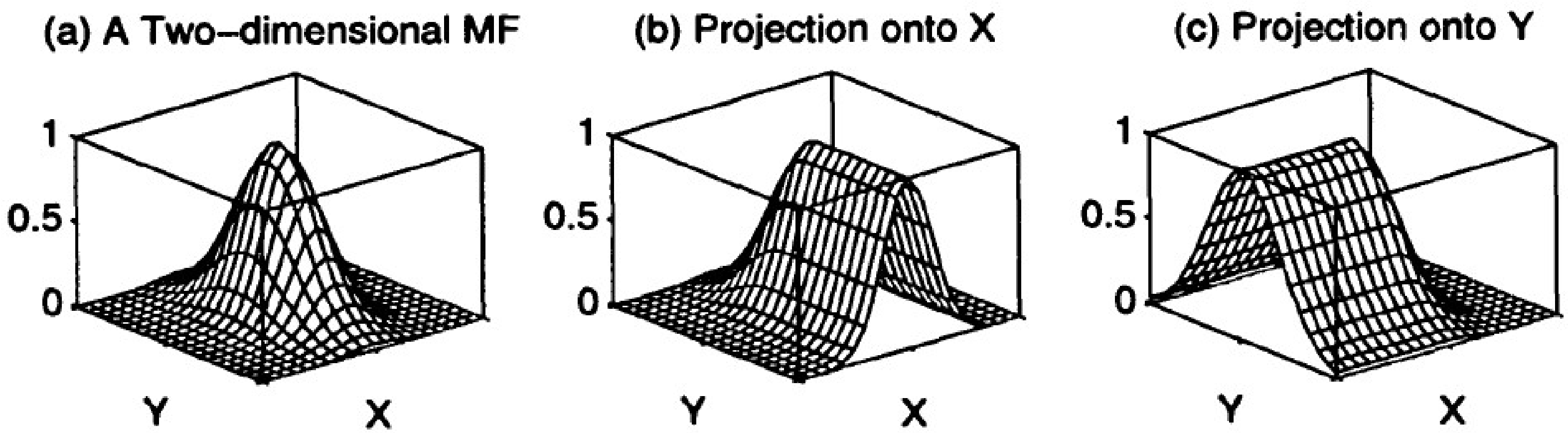
Let R be a two-dimensional fuzzy set on $X \times Y$. Then the **projections** of R onto X and Y are defined as

$$R_X = \int_X [\max_y \mu_R(x, y)]/x$$

and

$$R_Y = \int_Y [\max_x \mu_R(x, y)]/y,$$

respectively.



Fuzzy Operators

T – Norm

- It is an operator defined in the form $t: [0,1] \times [0,1] \rightarrow [0,1]$ and satisfies the following properties:
    1. $t(x,0) = 0, t(x,1) = t(1,x) = x$ [boundary condition]
    2. $t(x,y) = t(y,x)$ [Commutative]
    3. $if\ x \leq x', y \leq\ y'\ \Rightarrow t(x,y) \leq t(x',y')$ [monotonic]
    4. $t(t(x,y),z) = t(x,t(y,z))$ [Associativity]

- The four T – norm operators are:
    1. Intersection
    2. Algebraic product
    3. Bounded product
    4. Drastic product

Fuzzy Operators

T – conorm (S – Norm)

- It is an operator defined in the form $s: [0,1] \times [0,1] \rightarrow [0,1]$ and satisfies the following properties:
  1. $s(x,0) = x, s(x,1) = s(1,x) = 1$ [boundary condition]
  2. $s(x,y) = s(y,x)$ [Commutative]
  3. $if\ x \leq x', y \leq\ y'\ \Rightarrow s(x,y) \leq s(x',y')$ [monotonic]
  4. $s(s(x,y),z) = s(x,s(y,z))$ [Associativity]

- The four S – norm operators are:
  1. Union
  2. Algebraic sum
  3. Bounded sum
  4. Drastic sum

Fuzzy Measures

- Fuzzy measure explains the imprecision or ambiguity in the assignment of an element a to two or more crisp sets.
- A value between [0, 1] is assigned to each possible crisp set to which the element might belong.
- This value represents the degree of evidence or certainty or belief of the element's membership in the set.
- It is defined by a function $g: P(X) \rightarrow [0, 1]$ which assigns a crisp subset of a universe of discourse X a number in the unit interval [0, 1], where $P(X)$ is power set of X.
- The function g satisfies the axioms:
  - Boundary Conditions: $g(\emptyset) = 0; g(X) = 1$
  - Monotonic: if $A \subseteq B \subseteq X$, then $g(A) \leq g(B) \leq g(X)$
- The g values of individual element of a set X represented as $g(x_1), g(x_2), \dots g(x_n)$ or $g^1, g^2, \dots\dots, g^n$ are called singletons or densities.
- The g values of all possible subsets of X is called a lattice.

Fuzzy Measures

- g values can be calculated using Sugeno λ method defined as:

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A).\, g_\lambda\,(B)$$

- Solving it for the value of λ, we get

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda g^{i})\text{, where } \lambda \in [-1, \propto]$$

- We solve the equation to estimate the parameter λ and subsequently find the g values for different combinations of $x \in X$