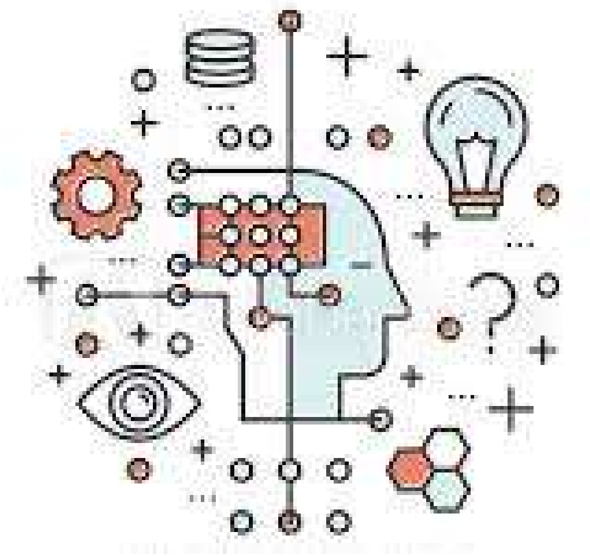
CS 321 SOFT COMPUTING



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Module - II

Fuzzy Logic

Fuzzy Rules and Reasoning

- Introduction
- Extension Principles and Fuzzy Relations
- Fuzzy IF THEN Rules
- Fuzzy Reasoning

Fuzzy Inference System

- Introduction
- Mamdani Fuzzy Models
- Sugeno Fuzzy Models
- Tsukamoto Fuzzy Models
- Other consideration

Fuzzy Rules and Reasoning – Extension Principle

- Fuzzy rules and fuzzy reasoning are the backbone of fuzzy inference system.
- Suppose that function f is a mapping from an n-dimensional Cartesian product space $X_1 \times X_2 \times \cdots X_n$ to a one-dimensional universe Y such that $y = f(x_1, \ldots, x_n)$, and suppose $A_1, \ldots, A_n$ are n fuzzy sets in $X_1, \ldots, X_n$, respectively. Then the extension principle asserts that the fuzzy set B induced by the mapping f is defined by

$$\mu_B(y) = \begin{cases} \max\limits_{(x_1,\ldots,x_n),\, (x_1,\ldots,x_n)=f^{-1}(y)} [\min_i \ \mu_{A_i}(x_i)], & \text{if } f^{-1}(y) \neq \emptyset. \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Fuzzy Rules and Reasoning – Fuzzy Relations

- Let X and Y be two universe of discourse. Then
$$R = \{((x, y), \mu_R(x, y)) \mid (x, y) \in X \times Y\}$$
is a binary fuzzy relation in $X \times Y$.
- Fuzzy relation example
- **Max – min composition**

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively. The **max-min composition** of $\mathcal{R}_1$ and $\mathcal{R}_2$ is a fuzzy set defined by

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{[(x, z), \max_y \min(\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(y, z))] | x \in X, y \in Y, z \in Z\},$$

or, equivalently,

$$\begin{aligned} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) &= \max_y \min[\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(y, z)] \\ &= \vee_y [\mu_{\mathcal{R}_1}(x, y) \wedge \mu_{\mathcal{R}_2}(y, z)], \end{aligned}$$

Fuzzy Rules and Reasoning – Fuzzy Relations

- **Max – product composition**

Assuming the same notation as used in the definition of **max-min composition**, we can define **max-product composition** as follows:

$$\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) = \max_y [\mu_{\mathcal{R}_1}(x, y)\mu_{\mathcal{R}_2}(y, z)].$$

- **Properties of Fuzzy Relations**
- Let X, Y, Z and W be three fuzzy sets and R, S and T be binary relations defined on $X \times Y, Y \times Z$ and $Z \times W$ respectively, then the following properties are satisfied:
  - Associativity: $R \circ (S \circ T) = (R \circ S) \circ T \circ$
  - Distributivity over intersection: $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
  - Weak Distributivity over intersection: $R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$
  - Monotonicity: $S \subseteq T \Rightarrow R \circ S \subseteq R \circ T$

Fuzzy IF – THEN rules

Linguistic Variables

- Principle of incompatibility: As the complexity of a system increases, our ability to make precise and yet significant statements about its behaviour diminishes until a threshold is reached beyond which precision and significance become almost mutually exclusive characteristics.

- A **linguistic variable** is characterized by a quintuple $(x, T(x), X, G, M)$ in which x is the name of the variable; $T(x)$ is the **term set** of x—that is, the set of its **linguistic values** or **linguistic terms**; X is the universe of discourse; G is a **syntactic rule** which generates the terms in $T(x)$; and M is a **semantic rule** which associates with each linguistic value A its meaning $M(A)$, where $M(A)$ denotes a fuzzy set in X.

- If age is interpreted as a linguistic variable, then its term set $T(age)$ could be

$$
\begin{aligned}
T(age) = \{ & \textit{young, not young, very young, not very young, \ldots,} \\
& \textit{middle aged, not middle aged, \ldots,} \\
& \textit{old, not old, very old, more or less old, not very old, \ldots,} \\
& \textit{not very young and not very old, \ldots} \},
\end{aligned}
$$

Fuzzy IF – THEN rules

- Let A be a linguistic value characterized by a fuzzy set with membership function $\mu_A(\cdot)$. Then A^k is interpreted as a modified version of the original linguistic value expressed as

$$A^k = \int_X [\mu_A(x)]^k / x.$$

In particular, the operation of **concentration** is defined as

$$\mathrm{CON}(A) = A^2,$$

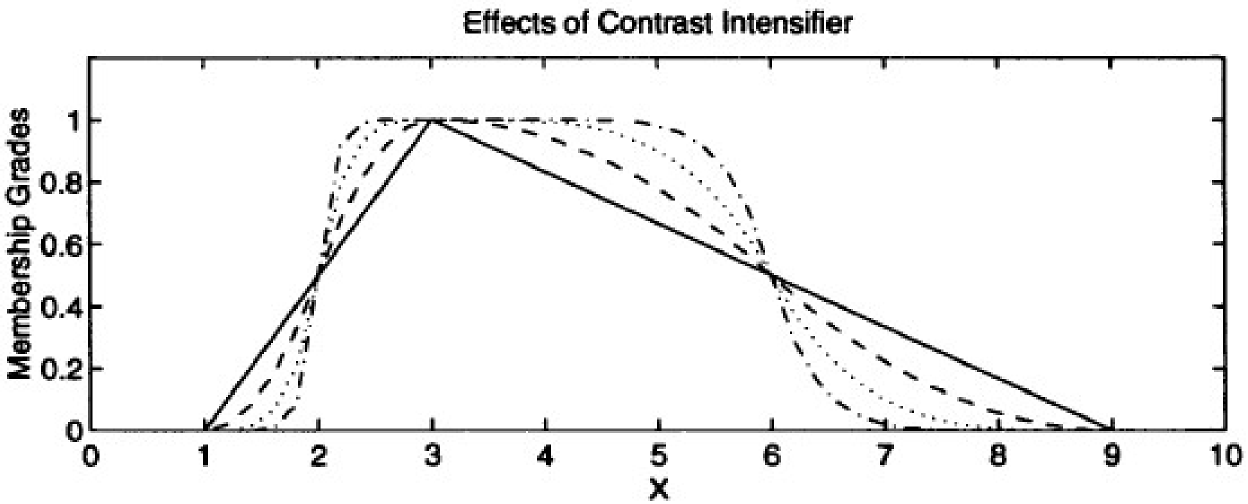
while that of **dilation** is expressed by

$$\mathrm{DIL}(A) = A^{0.5}.$$

- The operation of **contrast intensification** on a linguistic value A is defined by

$$\mathrm{INT}(A) = \begin{cases} 2A^2, & \text{for } 0 \le \mu_A(x) \le 0.5, \\ \neg 2(\neg A)^2, & \text{for } 0.5 \le \mu_A(x) \le 1. \end{cases}$$

Fuzzy IF – THEN rules

- 

- A term set $T = t_1, \ldots, t_n$ of a linguistic variable x on the universe X is **orthogonal** if it fulfills the following property:

$$\sum_{i=1}^{n} \mu_{t_i}(x) = 1, \ \forall x \in X,$$

where the t_i's are convex and normal fuzzy sets defined on X and these fuzzy sets make up the term set T.

Fuzzy IF – THEN rules

- Many of the fuzzy applications are based on fuzzy if-then rules.
- It expresses what happens if a fuzzy set is true.
- The fuzzy sets and fuzzy rules combine to form a fuzzy system.
- Human knowledge builds fuzzy rules.
- A fuzzy if-then rule (fuzzy rule, fuzzy implication, fuzzy conditional statement) has the form:

If x is A then y is B

where A and B are linguistic values defined by fuzzy sets on universes of discourse X and Y.

- *"x is A"* is called the antecedent or premise
- *"y is B"* is called the consequence or conclusion.

Fuzzy IF – THEN rules

- There are two ways to interpret a fuzzy if – then rule
  - *A* coupled with *B*
  - *A* entails *B*

- A fuzzy if – then rule can be defined as a fuzzy relation *R* on the space $X \times Y$ as,

$$R = A \rightarrow B = A \times B = \int_{X\times Y} \mu_R(x,y)/(x,y)\,, \forall\, x,y \in X \times Y \text{ (continuous)}$$
$$R = A \rightarrow B = A \times B = \sum_{X\times Y} \mu_R(x,y)/(x,y)\,, \forall\, x,y \in X \times Y \text{ (discrete)}$$

Fuzzy IF – THEN rules

A coupled with B

$$R = A \to B = A \times B = \int_{X\times Y} (T[\mu_A(x), \mu_B(y)])/(x, y)\,, \forall\, x, y \in X \times Y \text{ (continuous)}$$

$$R = A \to B = A \times B = \sum_{X\times Y} (T[\mu_A(x), \mu_B(y)])/(x, y)\,, \forall\, x, y \in X \times Y \text{ (discrete)}$$

- Hence there are four fuzzy relations which can be defined using the four T-norm operators.
  - A coupled with B using minimum $T-$ norm operator.
  - A coupled with B using algebraic product $T-$ norm operator.
  - A coupled with B using bounded product $T-$ norm operator.
  - A coupled with B using drastic product $T-$ norm operator.

Fuzzy IF – THEN rules

A coupled with B

- $R_m = A \times B = \int_{X\times Y} \mu_A(x) \wedge \mu_B(y)/(x,y)$, or $f_c(a,b) = a \wedge b$. This relation, which was proposed by Mamdani [3], results from using the min operator for conjunction.
- $R_p = A \times B = \int_{X\times Y} \mu_A(x)\mu_B(y)/(x,y)$, or $f_p(a,b) = ab$. Proposed by Larsen [2], this relation is based on using the algebraic product operator for conjunction.
- $R_{bp} = A \times B = \int_{X\times Y} \mu_A(x) \odot \mu_B(y)/(x,y) = \int_{X\times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1)/(x,y)$, or $f_{bp}(a,b) = 0 \vee (a+b-1)$. This formula employs the bounded product operator for conjunction.
- $R_{dp} = A \times B = \int_{X\times Y} \mu_A(x) \hat{\cdot}\, \mu_B(y)/(x,y)$, or

$$f(a,b) = a \hat{\cdot}\, b = \begin{cases} a & \text{if } b = 1. \\ b & \text{if } a = 1. \\ 0 & \text{otherwise.} \end{cases}$$

This formula uses the drastic product operator for conjunction.

Fuzzy IF – THEN rules

A entails B

- This can be defined using four different forms:

- Material implication:

$$R = A \to B = \neg A \cup B.$$

- Propositional calculus:

$$R = A \to B = \neg A \cup (A \cap B).$$

- Extended propositional calculus:

$$R = A \to B = (\neg A \cap \neg B) \cup B.$$

- Generalization of modus ponens:

$$\mu_R(x, y) = \sup\{c \mid \mu_A(x) \tilde{*} c \le \mu_B(y) \text{ and } 0 \le c \le 1\},$$

where $R = A \to B$ and $\tilde{*}$ is a T-norm operator.

Fuzzy IF – THEN rules

A entails B

- Material Implication:
  - $R_a = \neg A \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y))/(x, y)$, or $f_a(a, b) = 1 \wedge (1 - a + b)$.
  - Also called Zadeh's arithmetic rule
- Propositional Calculus:
  - $R_{mm} = \neg A \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))/(x, y)$, or $f_m(a, b) = (1 - a) \vee (a \wedge b)$.
  - Zadeh's min-max rule
- Extended propositional calculus
  - Boolean fuzzy implication.
- Generalization of modus ponens:
  - Goguen's fuzzy implication
  - $R_{\triangle} = \int_{X \times Y} (\mu_A(x) \tilde{<} \mu_B(y))/(x, y)$, where

$$a \tilde{<} b = \begin{cases} 1 & \text{if } a \leq b. \\ b/a & \text{if } a > b. \end{cases}$$

Fuzzy IF – THEN rules

Example: Let high speed (S_{HIGH}) is characterized by a fuzzy set with the universe of discourse $S = \{20, 25, 30, 45, 50\}$ and high brake pressure (P_{HIGH}) is characterized by a fuzzy set with the universe of discourse $P = \{1, 2, 3, 4\}$ as:

$$S_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (45, 0.8), (50, 1.0)\}$$

$$P_{HIGH} = \{(1, 0.4), (2, 0.6), (3, 0.7), (4, 0.8)\}.$$

Q. Determine the implication relation for the fuzzy rule "$S_{HIGH} \rightarrow P_{HIGH}$" using the interpretation $A \rightarrow B$ as A entails B.

Fuzzy Reasoning / Approximate Reasoning

Let A, A', and B be fuzzy sets of X, X, and Y, respectively. Assume that the fuzzy implication $A \rightarrow B$ is expressed as a fuzzy relation R on $X \times Y$. Then the fuzzy set B induced by "x is A'" and the fuzzy rule "if x is A then y is B" is defined by

$$\begin{aligned}\mu_{B'}(y) &= \max_x \min[\mu_{A'}(x), \mu_R(x, y)] \\ &= \vee_x [\mu_{A'}(x) \wedge \mu_R(x, y)],\end{aligned}$$

or, equivalently,

$$B' = A' \circ R = A' \circ (A \rightarrow B).$$

premise 1 (fact):	x is A,
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B.

premise 1 (fact):	x is A',
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B',

Fuzzy Reasoning / Approximate Reasoning

- **Generalized Modus Ponens (GMP)**

If x is A Then y is B

x is A'

---

y is B'

- **Generalized Modus Tollens (GMT)**

If x is A Then y is B

y is B'

---

x is A'

- To compute the MF *A'* or *B',* the max-min composition of fuzzy sets *B'* and *A'* with *R(x,y)* is used as:

$$B' = A' \circ R(x,y) \qquad \mu_B(y) = max[min(\mu_{A'}(x), \mu_R(x,y))]$$

$$A' = B' \circ R(x,y) \qquad \mu_A(x) = max[min(\mu_{B'}(y), \mu_R(x,y))]$$

Fuzzy Reasoning / Approximate Reasoning

- **Single Rule with Single Antecedent**

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x(\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y).\end{aligned}$$

- **Single Rule with Multiple Antecedent**

If *x* is *A* and *y* is *B* then *z* in *C*

premise 1 (fact):	x is A' and y is B',
premise 2 (rule):	if x is A and y is B then z is C,
consequence (conclusion):	z is C'.

The fuzzy rule in premise 2 can be put into the simpler form "$A \times B \rightarrow C$."

The resulting C' is expressed as

$$C' = (A' \times B') \circ (A \times B \rightarrow C).$$

$$\begin{aligned}\mu_{C'}(z) &= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\ &= \vee_{x,y}\{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\ &= \underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\ &= \underbrace{(w_1 \wedge w_2)}_{\text{firing strength}} \wedge \mu_C(z),\end{aligned}$$

Fuzzy Reasoning / Approximate Reasoning

- **Multiple Rule with Multiple Antecedent**

premise 1 (fact):	x is A' and y is B',
premise 2 (rule 1):	if x is A_1 and y is B_1 then z is C_1,
premise 3 (rule 2):	if x is A_2 and y is B_2 then z is C_2,
consequence (conclusion):	z is C',

$$R_1 = A_1 \times B_1 \rightarrow C_1 \text{ and } R_2 = A_2 \times B_2 \rightarrow C_2.$$

$$\begin{aligned} C' &= (A' \times B') \circ (R_1 \cup R_2) \\ &= [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2] \\ &= C_1' \cup C_2', \end{aligned}$$

Fuzzy Reasoning / Approximate Reasoning

Degrees of compatibility Compare the known facts with the antecedents of fuzzy rules to find the degrees of compatibility with respect to each antecedent MF.

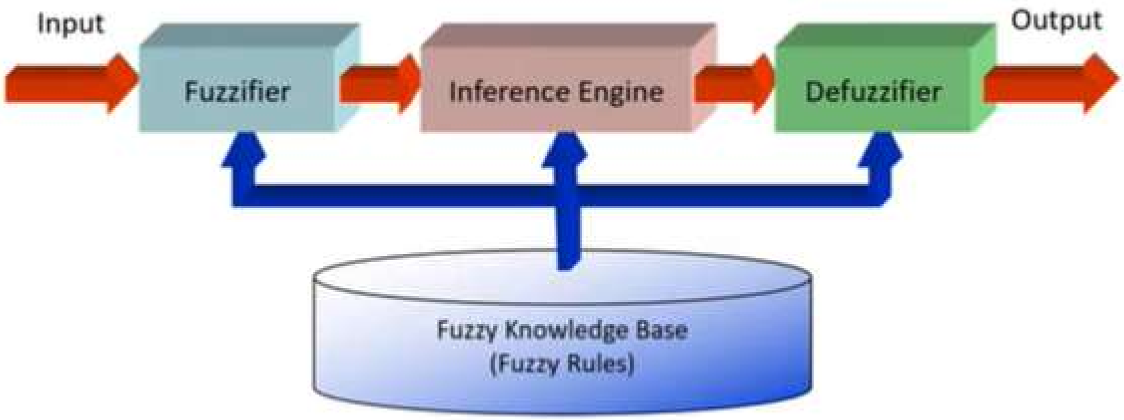
Firing strength Combine degrees of compatibility with respect to antecedent MFs in a rule using fuzzy AND or OR operators to form a firing strength that indicates the degree to which the antecedent part of the rule is satisfied.

Qualified (induced) consequent MFs Apply the firing strength to the consequent MF of a rule to generate a qualified consequent MF. (The qualified consequent MFs represent how the firing strength gets propagated and used in a fuzzy implication statement.)

Overall output MF Aggregate all the qualified consequent MFs to obtain an overall output MF.

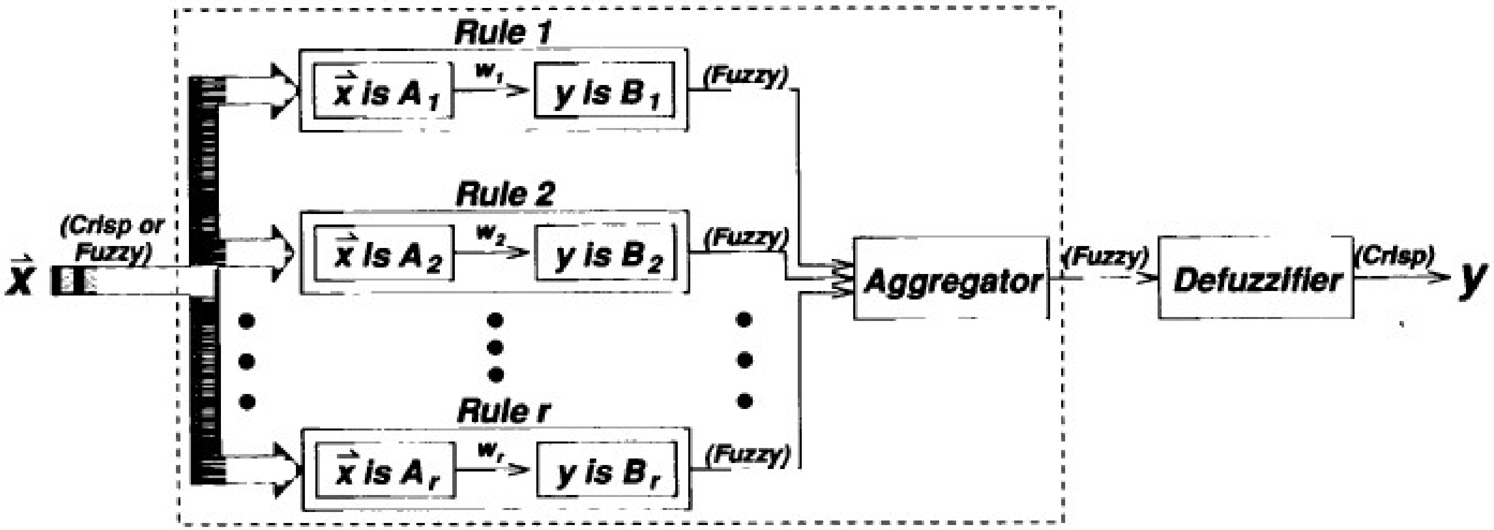
Fuzzy Inference System

- A FIS is based on the concept of fuzzy set theory, fuzzy if –then rules and fuzzy reasoning.
- Also knows as fuzzy-rule based system, fuzzy expert system, fuzzy model, fuzzy associative memory, fuzzy logic controller, fuzzy system.
- It is a mapping between a given input to output using fuzzy concepts.
- It consists of: a rule base, a database, reasoning mechanism, defuzzifier.



Fuzzy Inference System

- Fuzzifier: Converts crisp input into linguistic variables using the membership functions stored in the fuzzy knowledge base.
- Inference engine: It takes the fuzzy value and comparable to this fuzzy value it produces output in terms of a fuzzy value.



- Fuzzy knowledgebase: It is a database containing a number of fuzzy if-then rules.
- Defuzzifier: It converts the fuzzy values to a crisp value. Methods: Centroid of Area (COA), Bisector of Area (BOA), Mean of maximum (MOM), Smallest of maximum (SOM), Largest of Maximum (LOM).

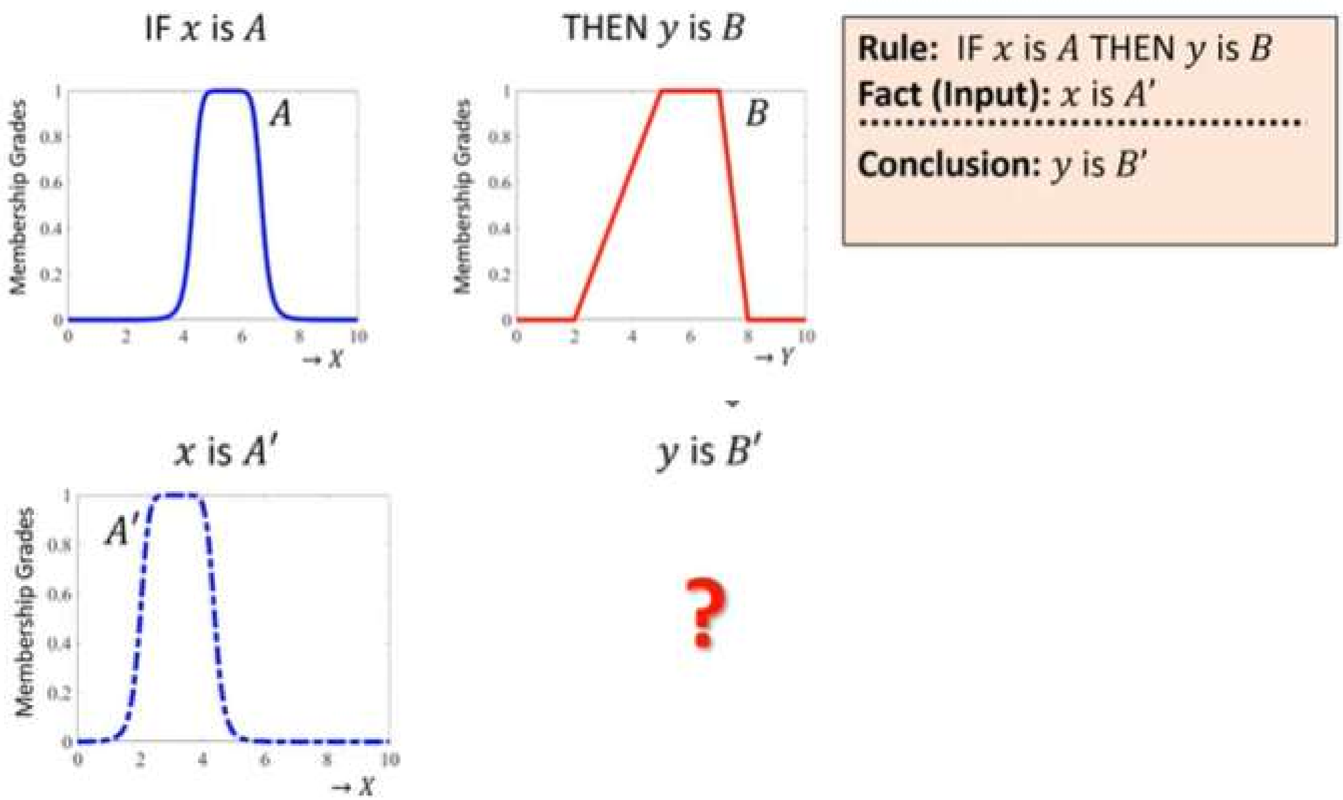
Mamdani Fuzzy Model

- Was developed in 1975 by Prof. E. H. Mamdani.
- It was used to control a steam engine and boiler combination by using linguistic rules.
- We generally use Mamdani model if both the antecedent and consequent are fuzzy.
- Steps:
  - Fuzzification of input.
  - Rule evaluation:
  - Apply the antecedents of the fuzzy rules.
  - If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation that is applied to the consequent part.
  - Aggregation of the output of fuzzy rules if more than one rule is fired.
  - Defuzzification.
- Mamdani fuzzy model using Max-Min and Max-Product composition

Mamdani Fuzzy Model

I. Single rule with single antecedent

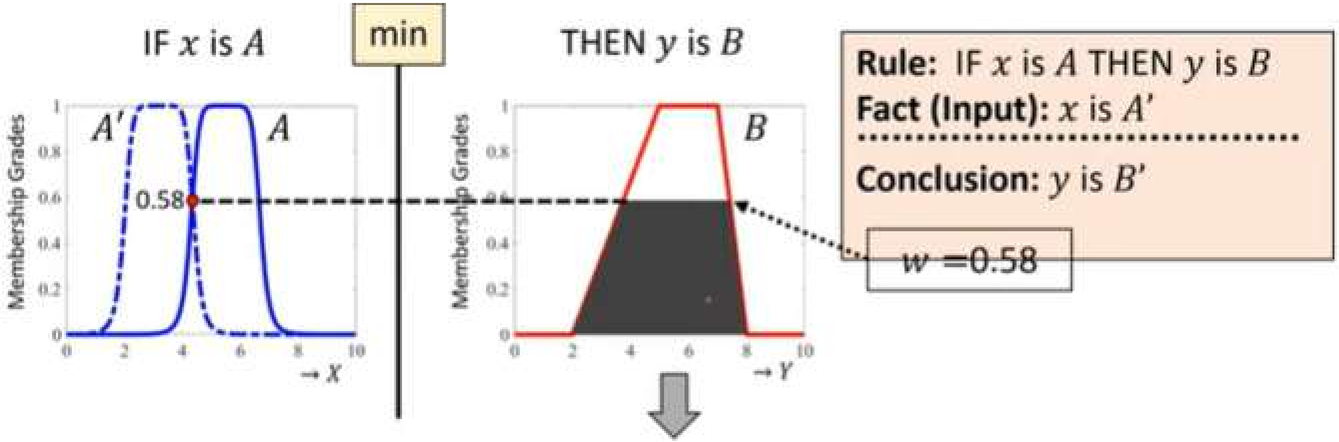
- Using Max-min composition



Mamdani Fuzzy Model

I. Single rule with single antecedent

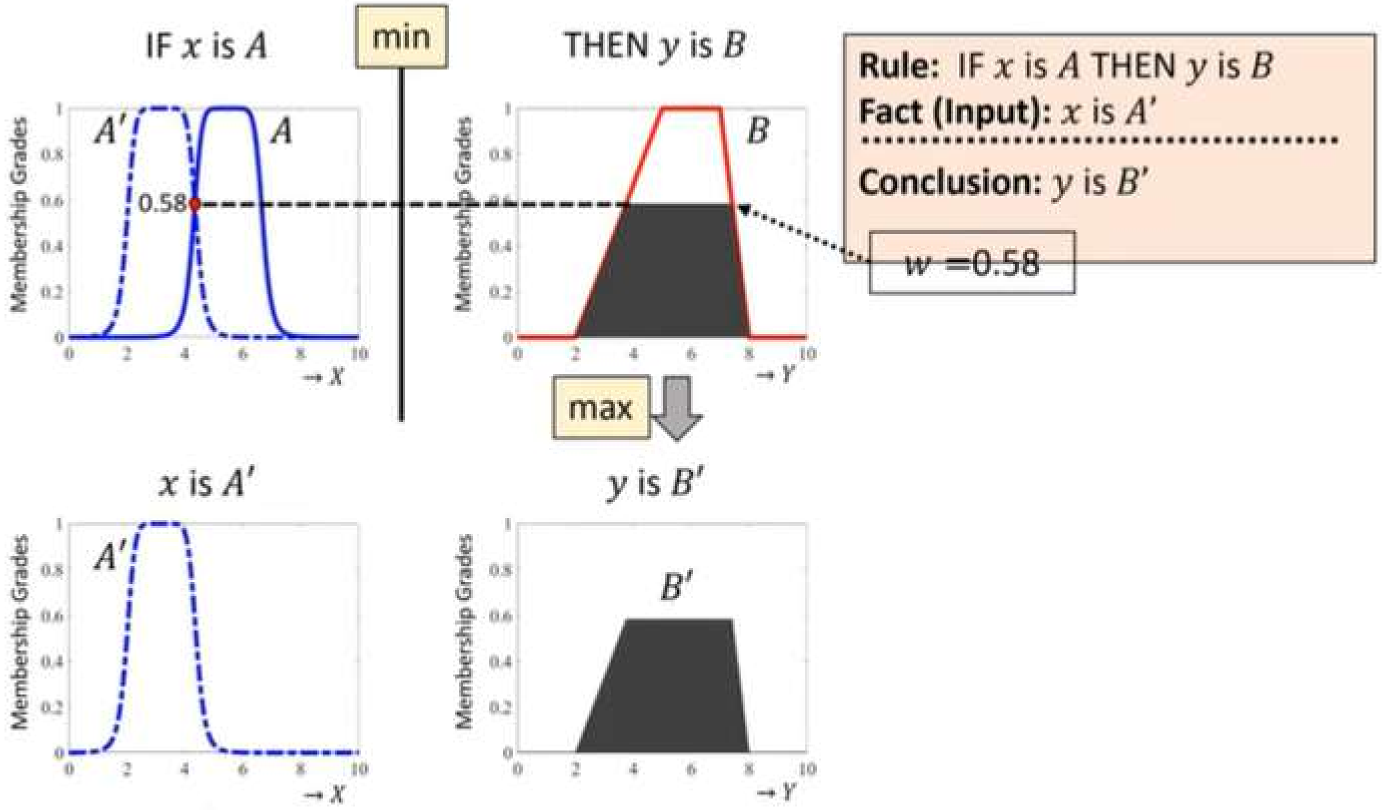
- Using Max-min composition



Mamdani Fuzzy Model

I. Single rule with single antecedent

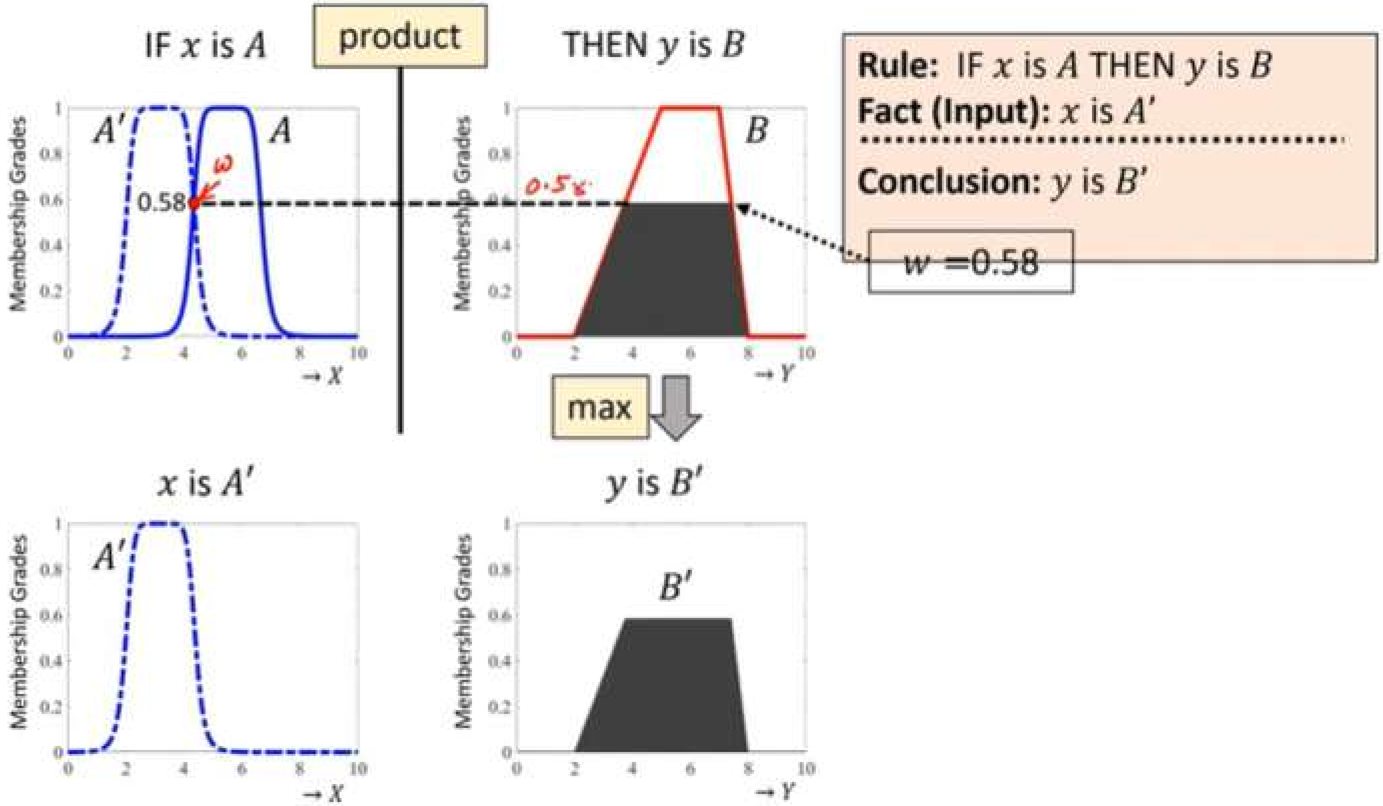
- Using Max-min composition



Mamdani Fuzzy Model

I. Single rule with single antecedent

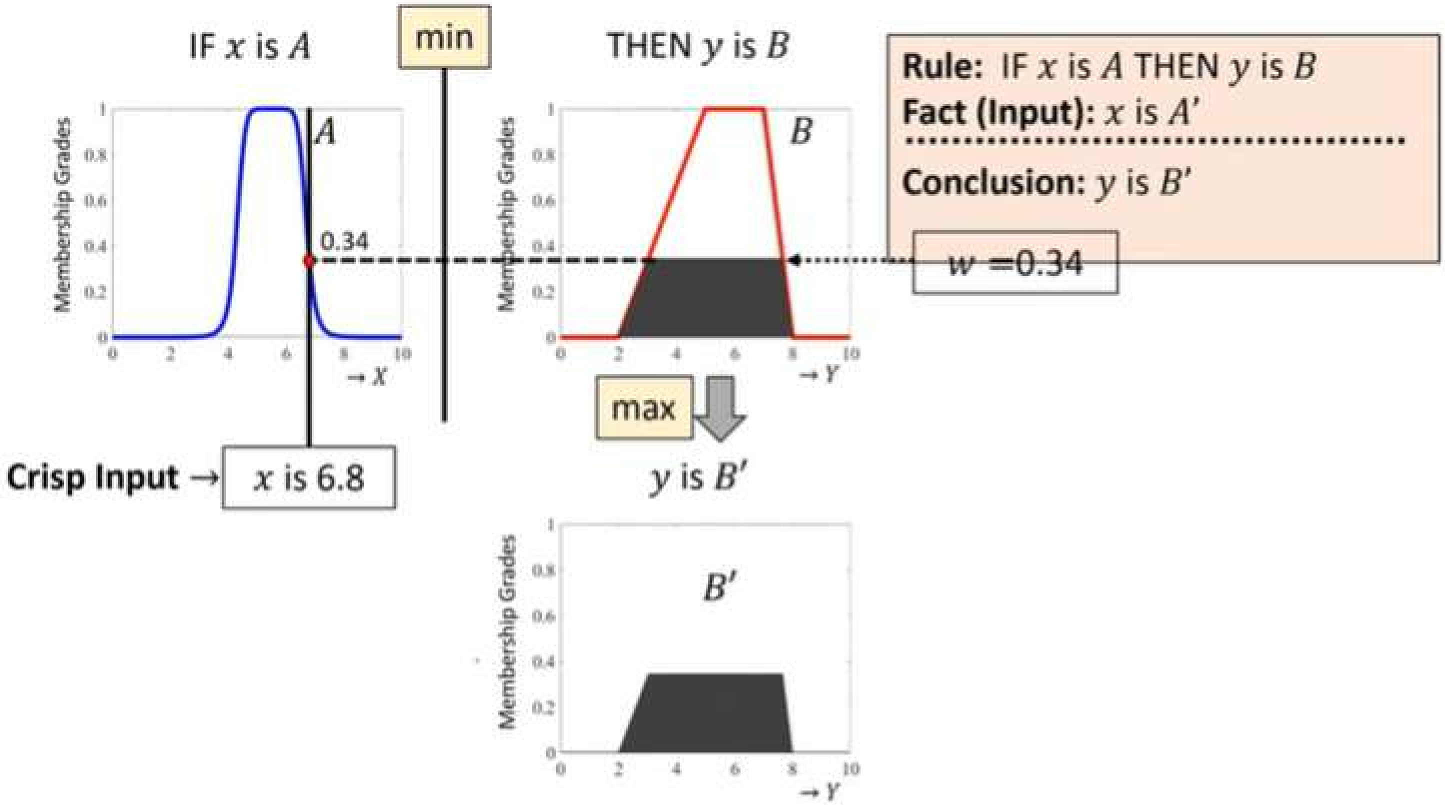
- Using Max-product composition



Mamdani Fuzzy Model

I. Single rule with single antecedent

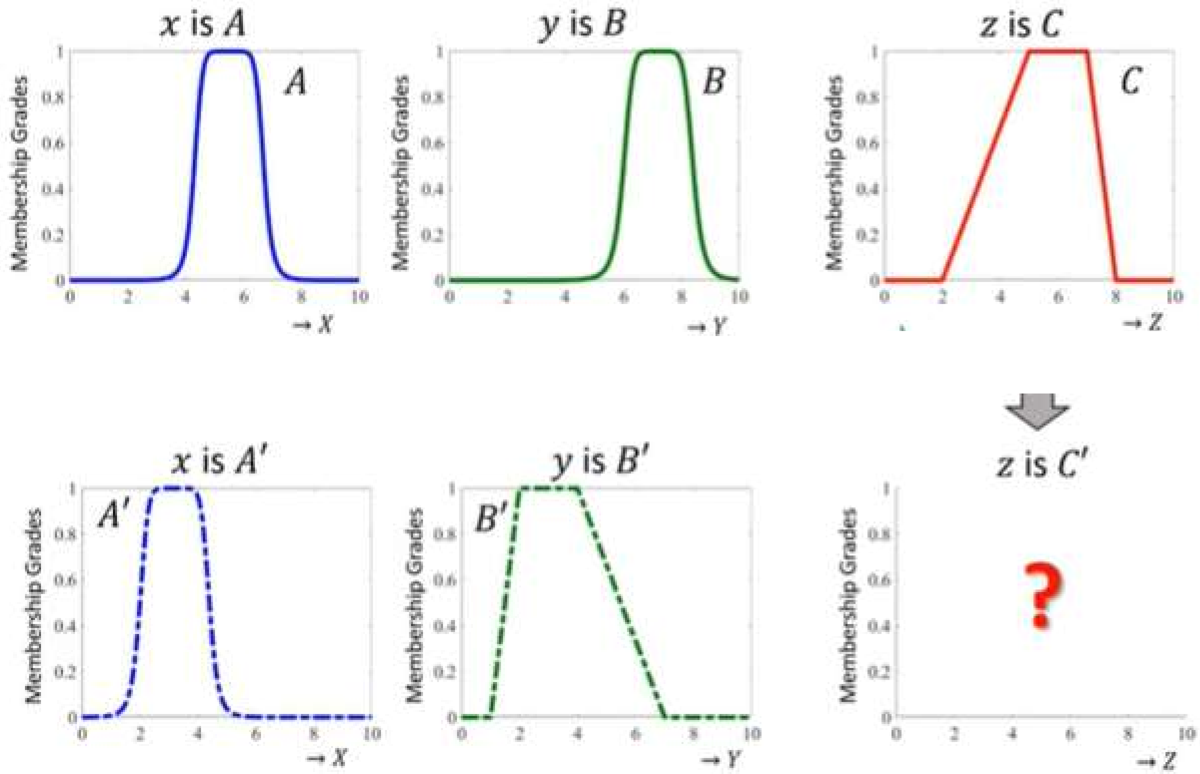
- Using Max-min / Max-product composition (using crisp input)



Mamdani Fuzzy Model

II. Single rule with multiple antecedent

- Using Max-min composition

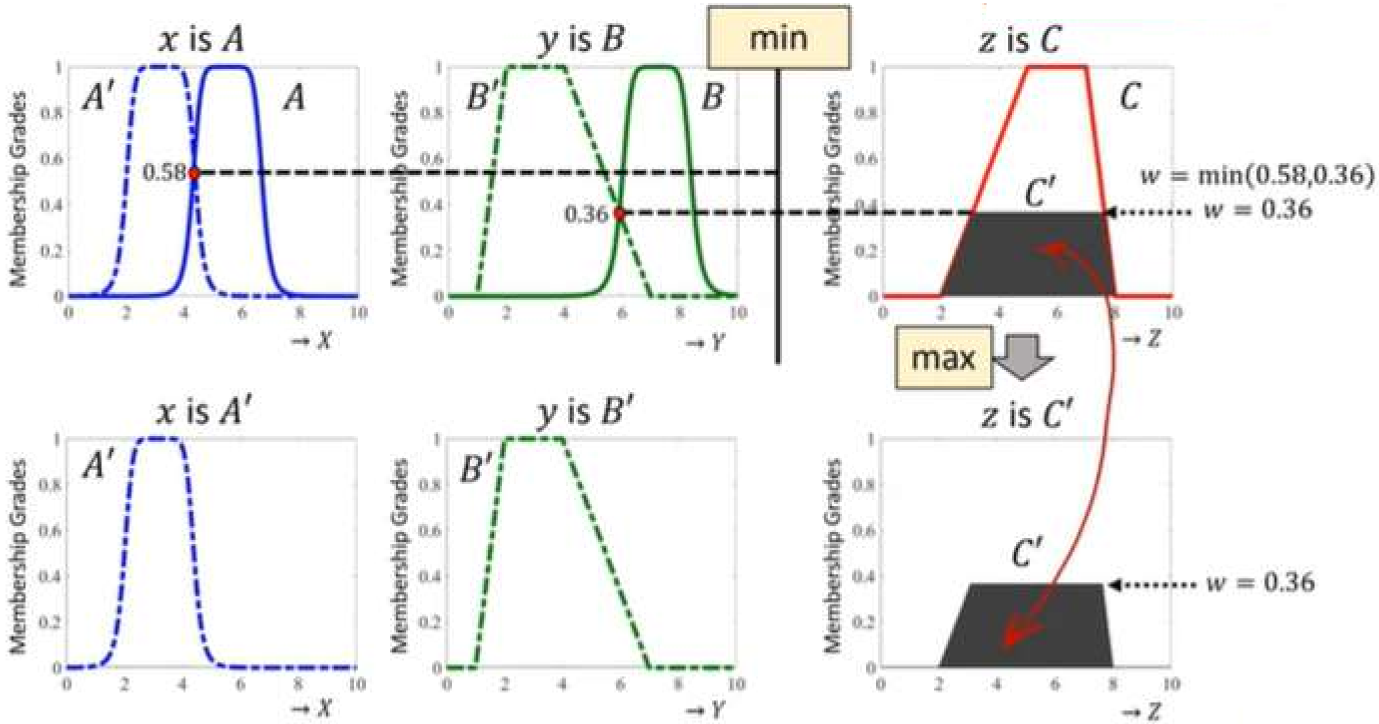


Rule: IF x is A and y is B THEN z is C
Fact (Input): x is A' and y is B'
....
Conclusion: z is C'

Mamdani Fuzzy Model

II. Single rule with multiple antecedent

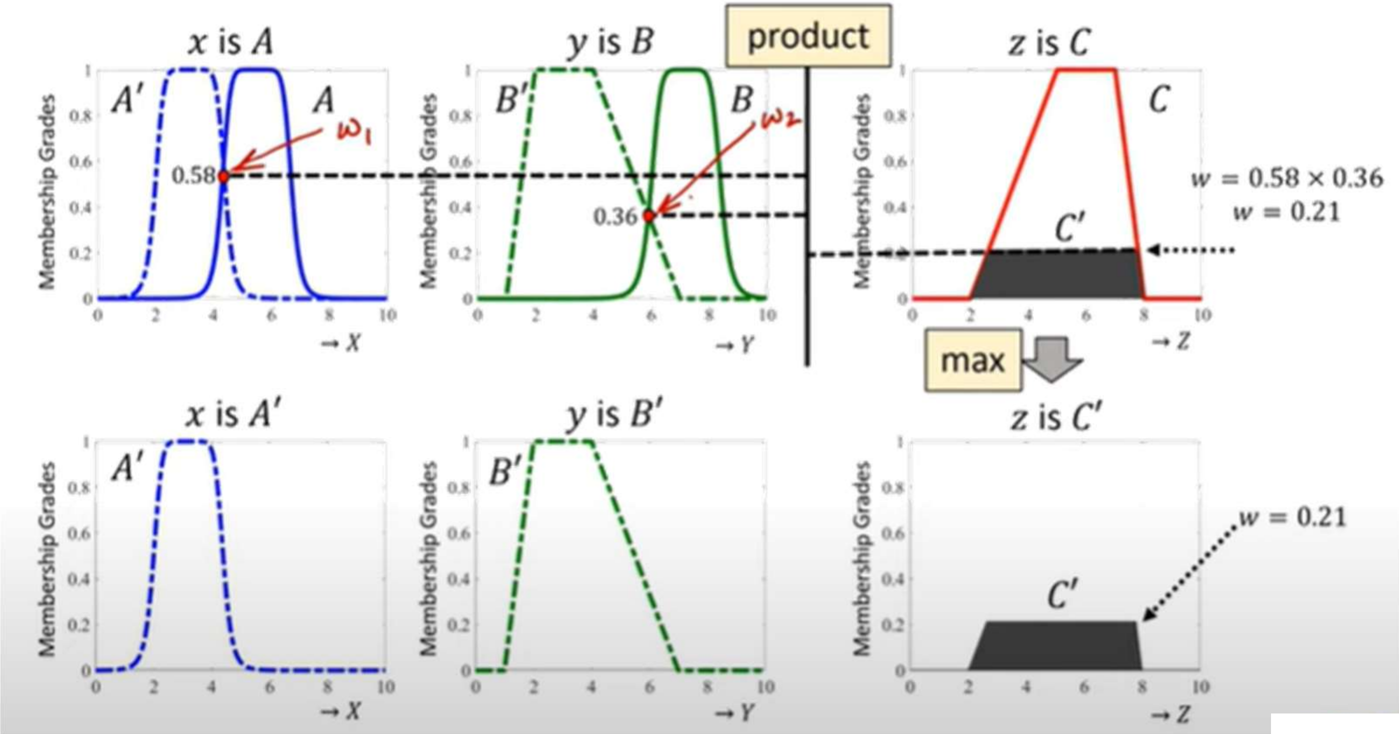
- Using Max-min composition



Mamdani Fuzzy Model

II. Single rule with multiple antecedent

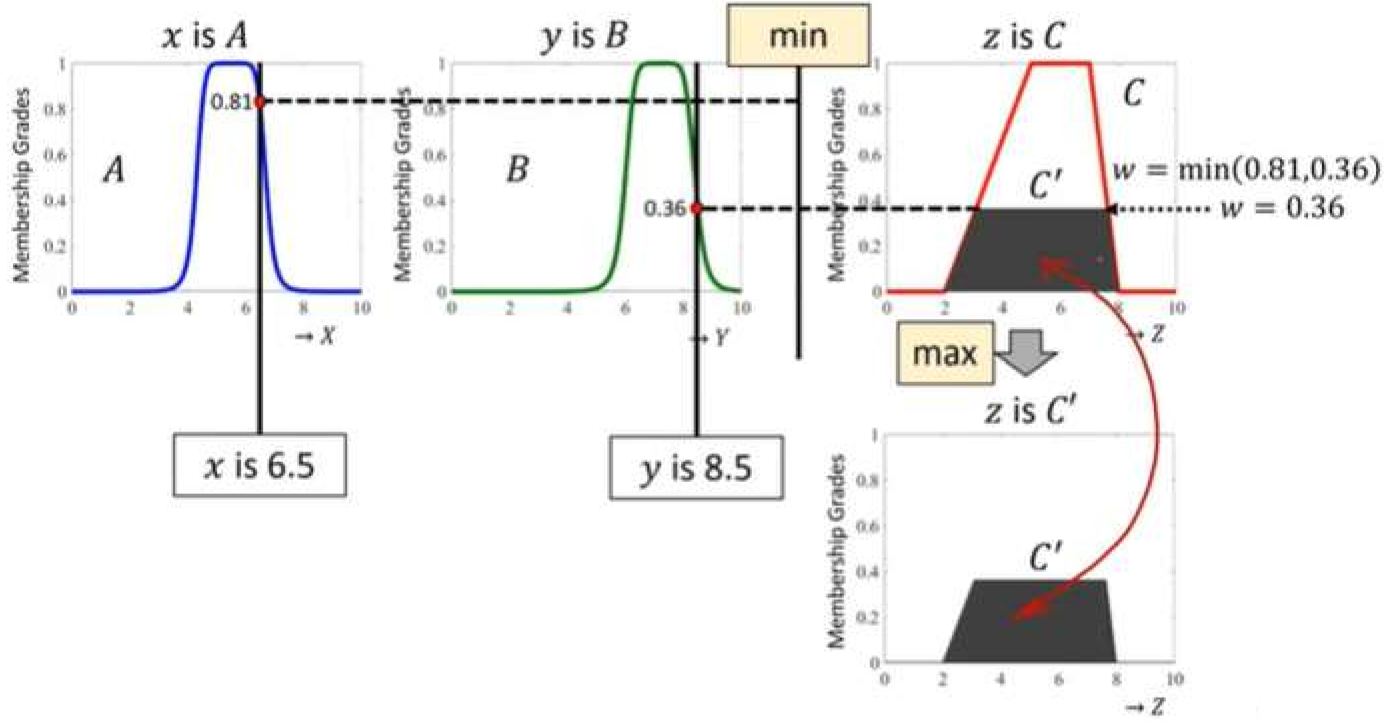
- Using Max-product composition



Mamdani Fuzzy Model

II. Single rule with multiple antecedent

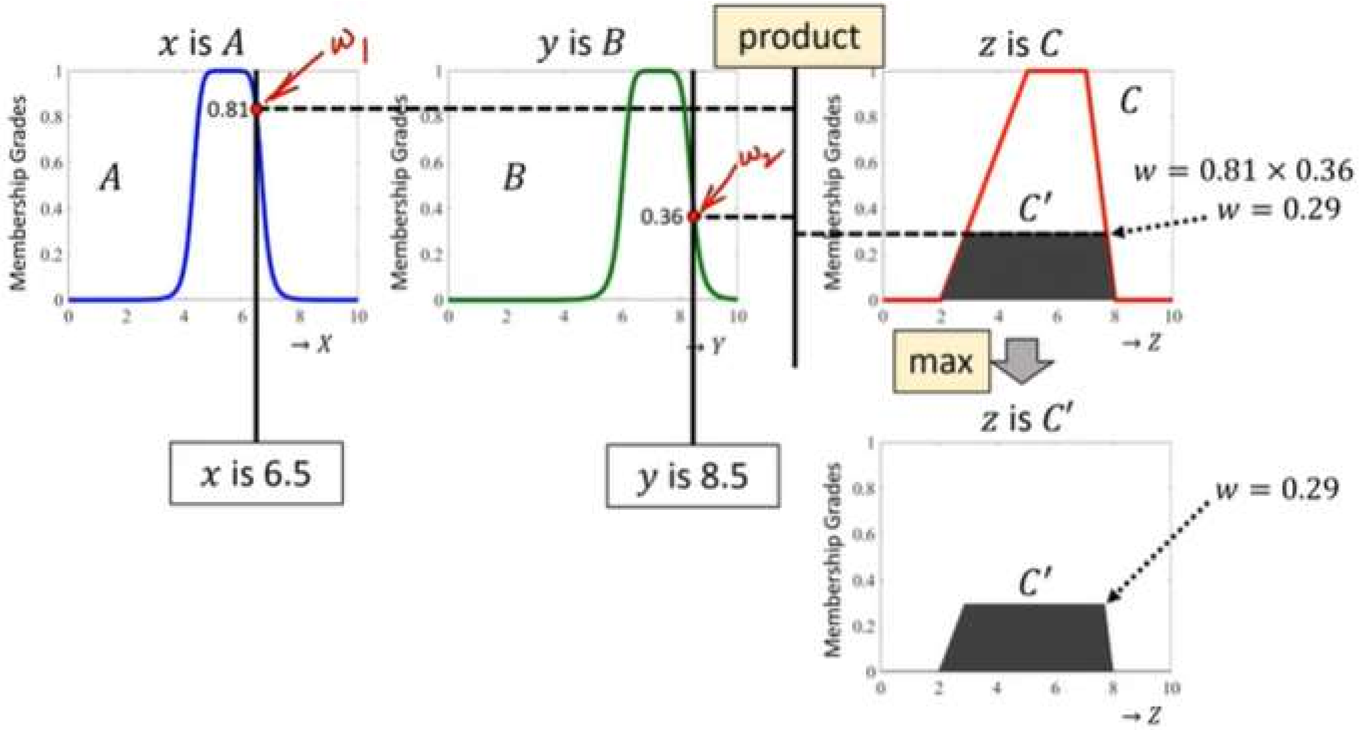
- Using Max-min composition (crisp input)



Mamdani Fuzzy Model

II. Single rule with multiple antecedent

- Using Max-product composition (crisp input)



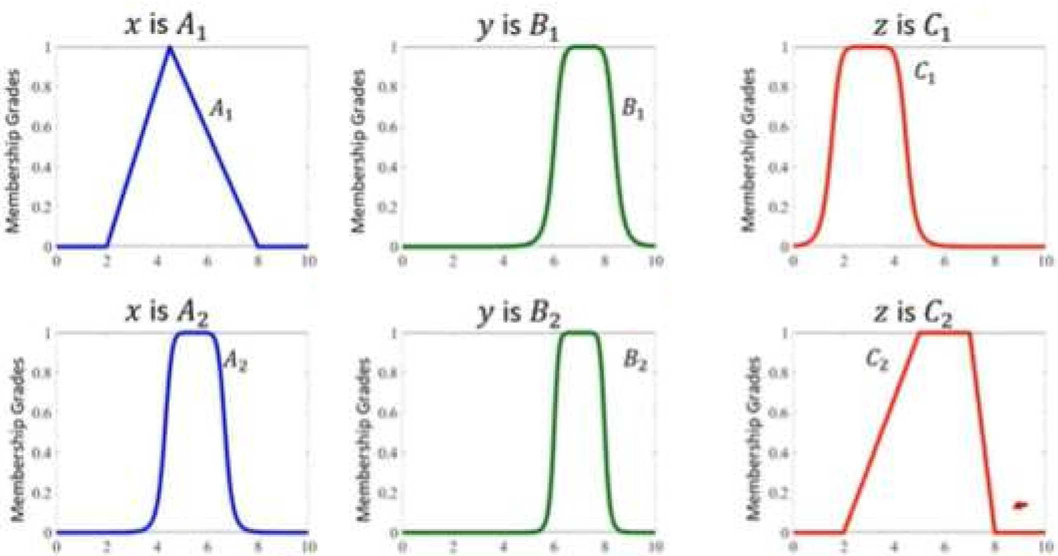
Mamdani Fuzzy Model

III. Multiple rule with multiple antecedent

- Using Max-min composition

Rule 1: IF x is $A1$ and y is $B1$ THEN z is $C1$
Rule 2: if x is $A2$ and y is $B2$ THEN z is $C2$
Fact (Input): x is A' and y is B'

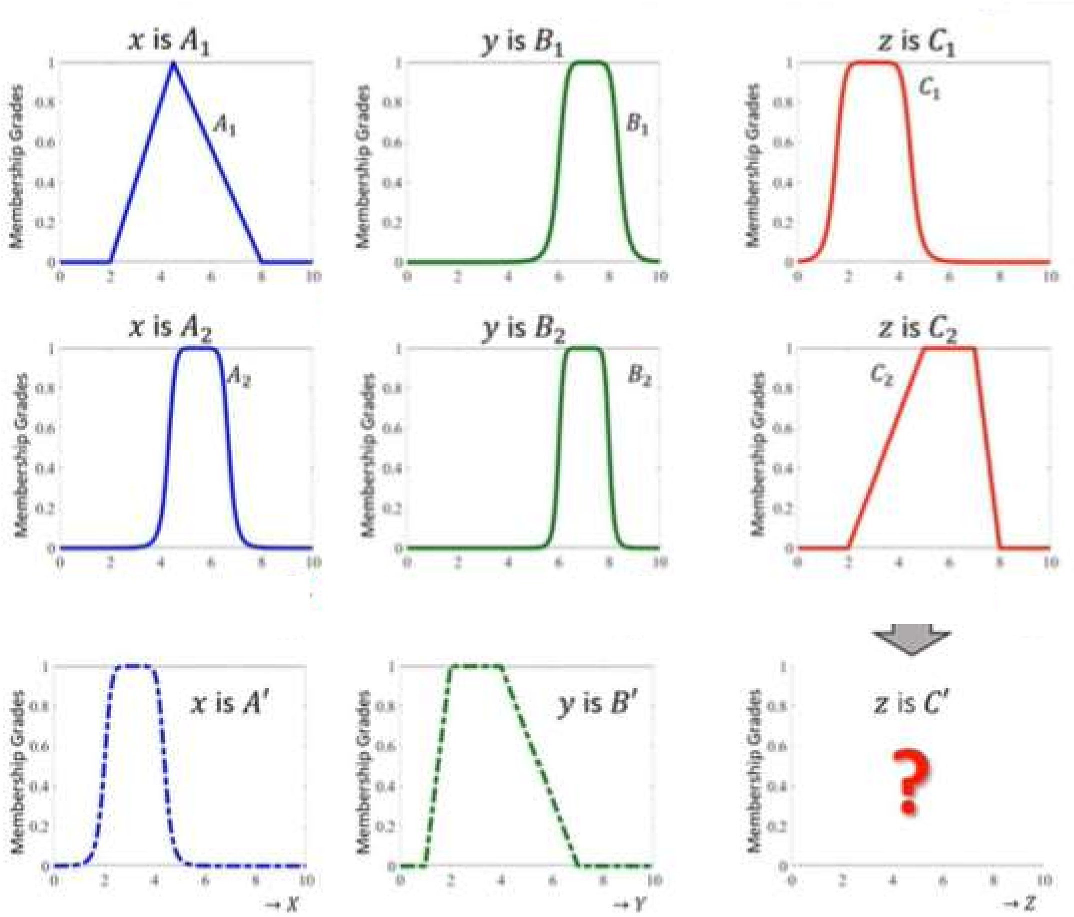
Conclusion: z is C'



Mamdani Fuzzy Model

III. Multiple rule with multiple antecedent

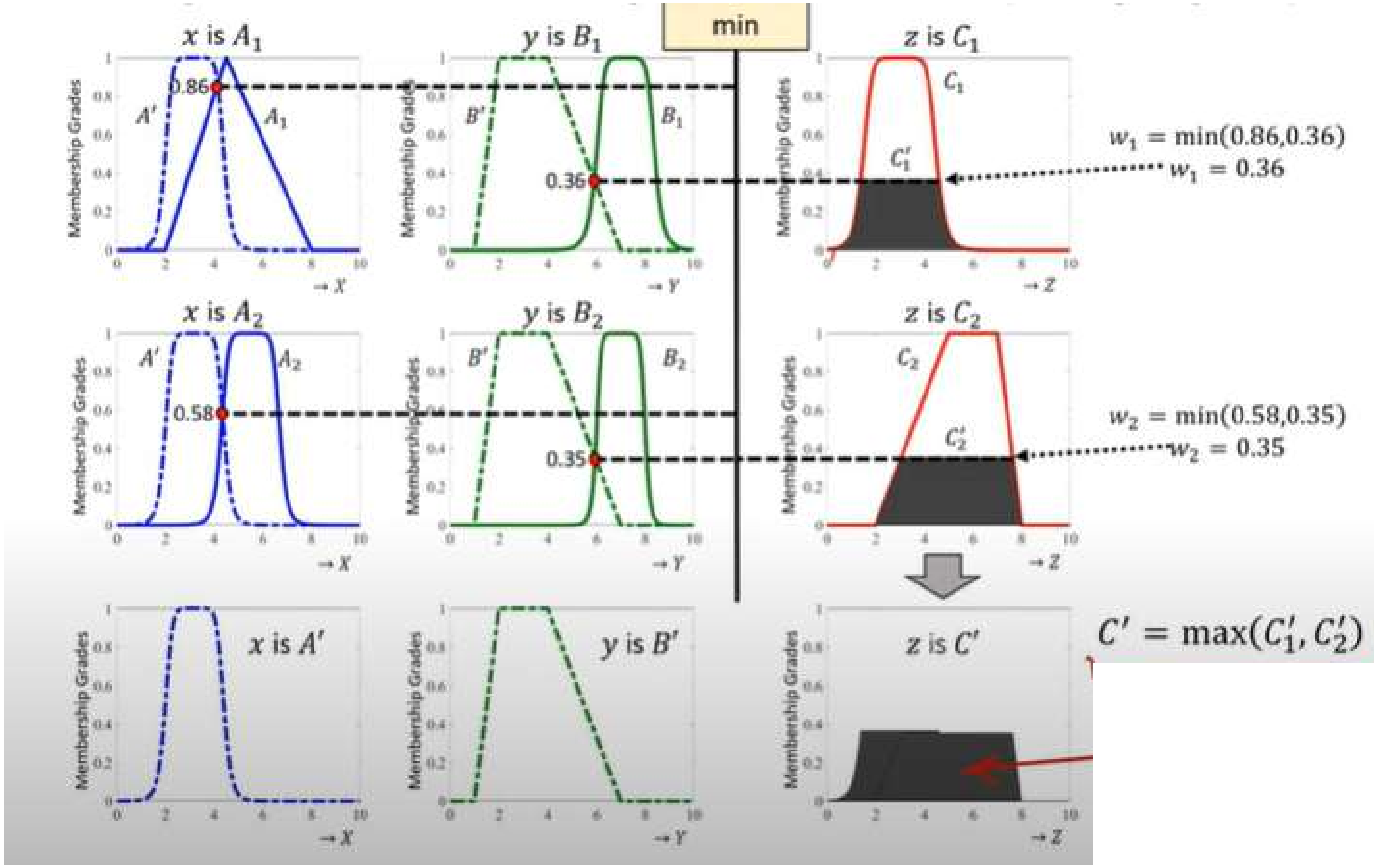
- Using Max-min composition



Mamdani Fuzzy Model

III. Multiple rule with multiple antecedent

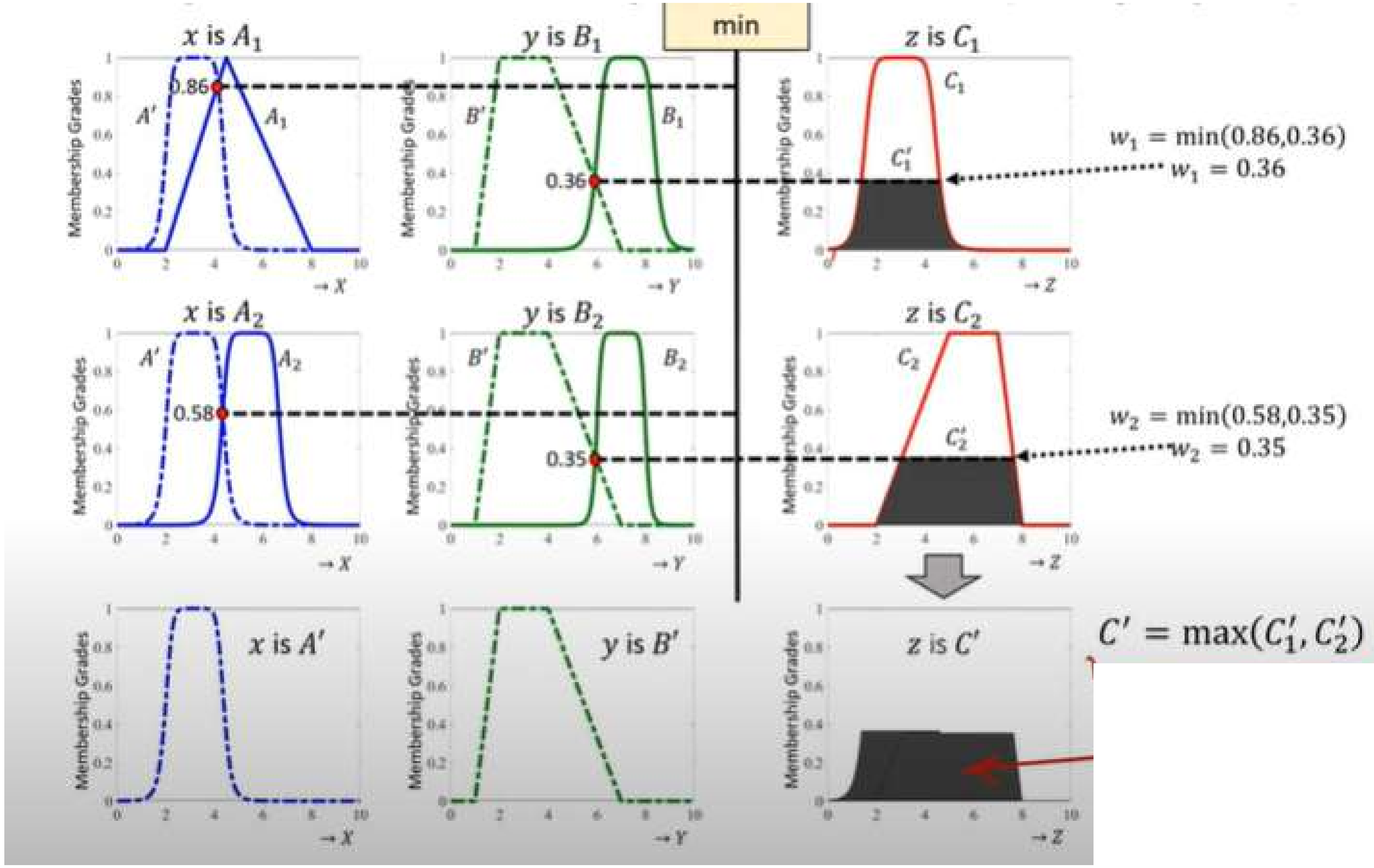
- Using Max-min composition



Mamdani Fuzzy Model

III. Multiple rule with multiple antecedent

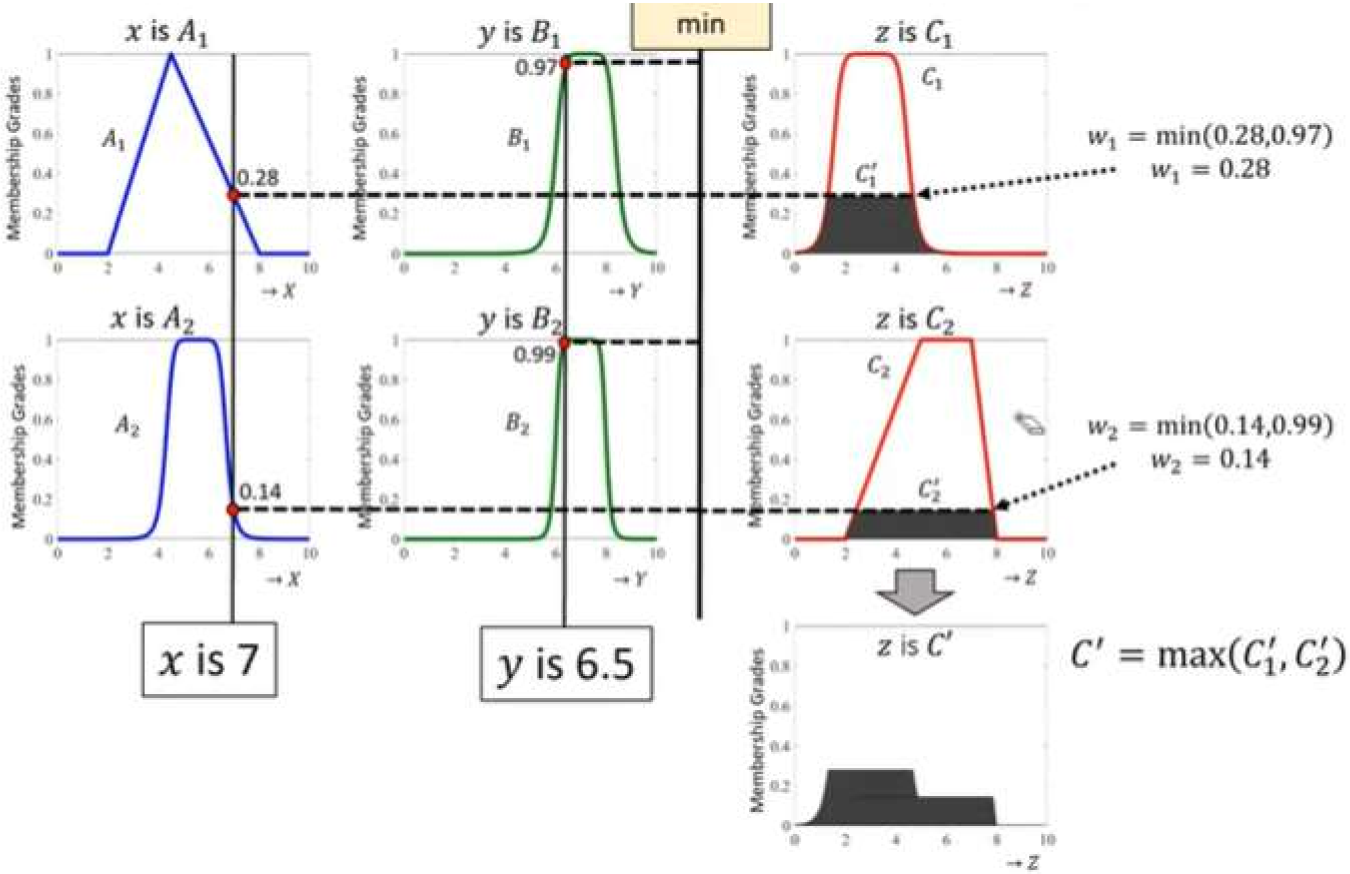
- Using Max-product composition



Mamdani Fuzzy Model

III. Multiple rule with multiple antecedent

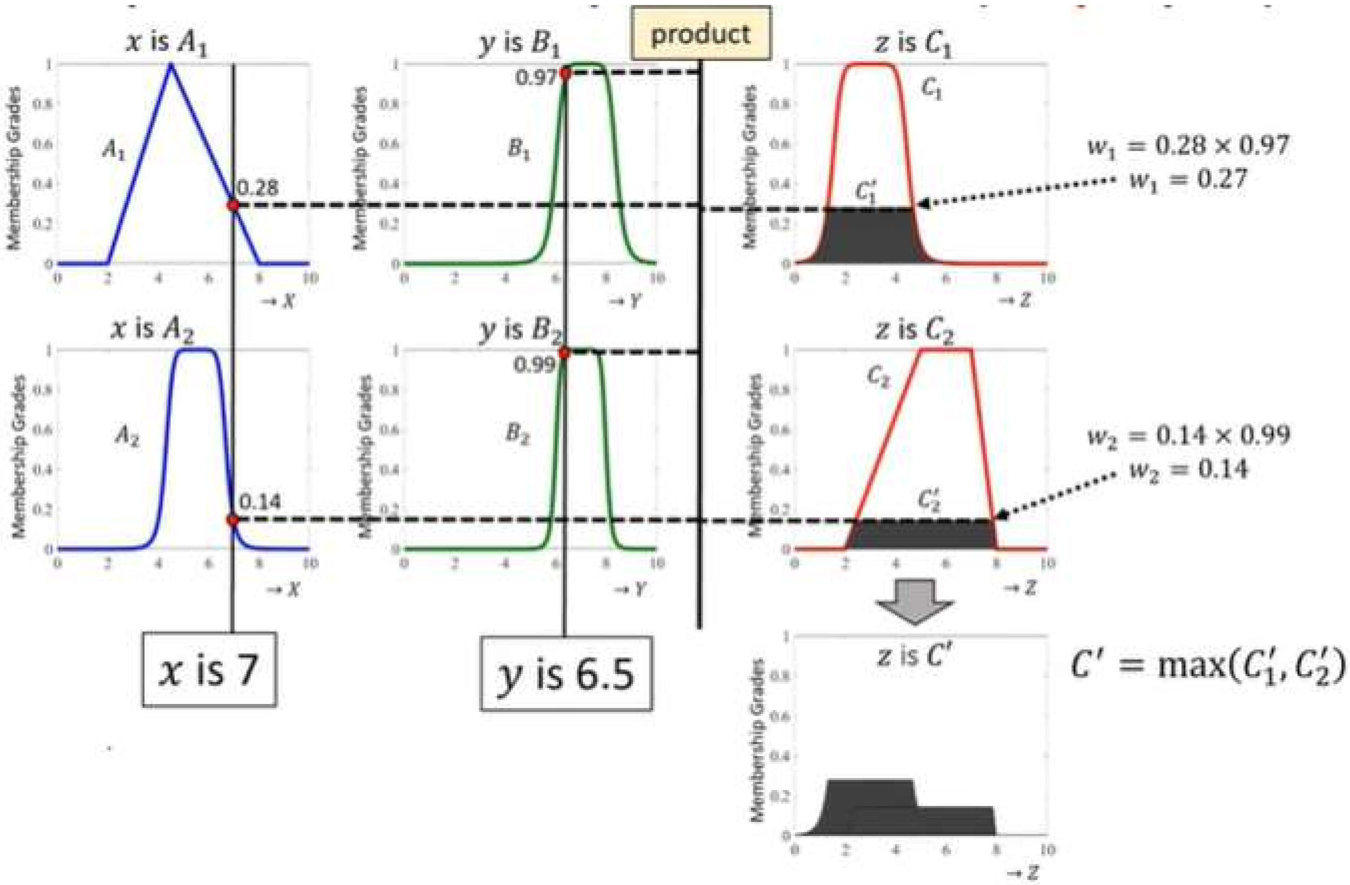
- Using Max-min composition (crisp input)



Mamdani Fuzzy Model

III. Multiple rule with multiple antecedent

- Using Max-product composition (crisp input)

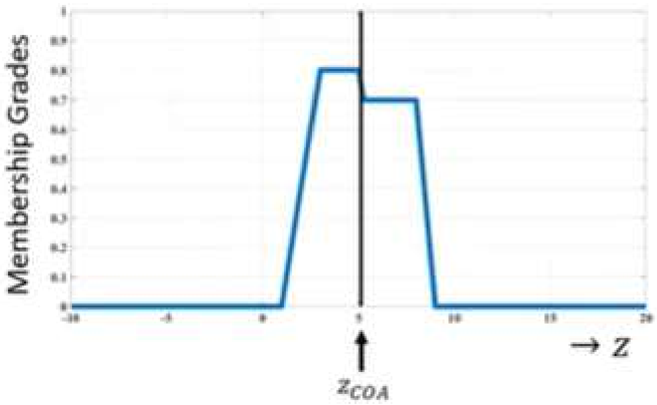


Mamdani Fuzzy Model

Defuzzification

- Centroid of Area (COA)

$$z_{\text{COA}} = \frac{\int_Z \mu_A(z) z \, dz}{\int_Z \mu_A(z) \, dz},$$



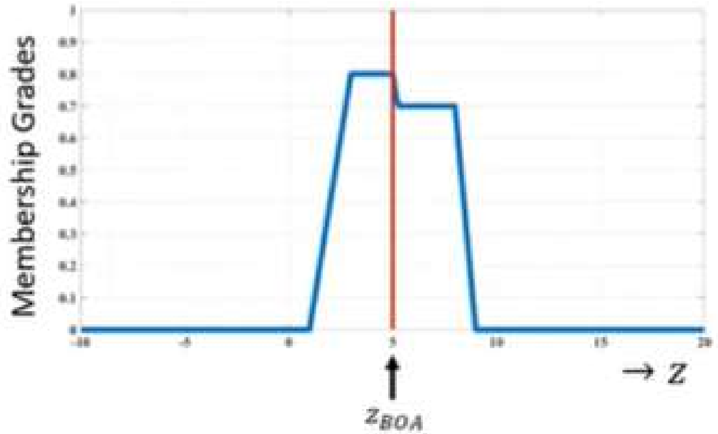
Mamdani Fuzzy Model

Defuzzification

- Bisector of Area (BOA)

$$\int_{\alpha}^{z_{\text{BOA}}} \mu_A(z)\, dz = \int_{z_{\text{BOA}}}^{\beta} \mu_A(z)\, dz,$$

where $\alpha = \min\{z|z \in Z\}$ and $\beta = \max\{z|z \in Z\}$



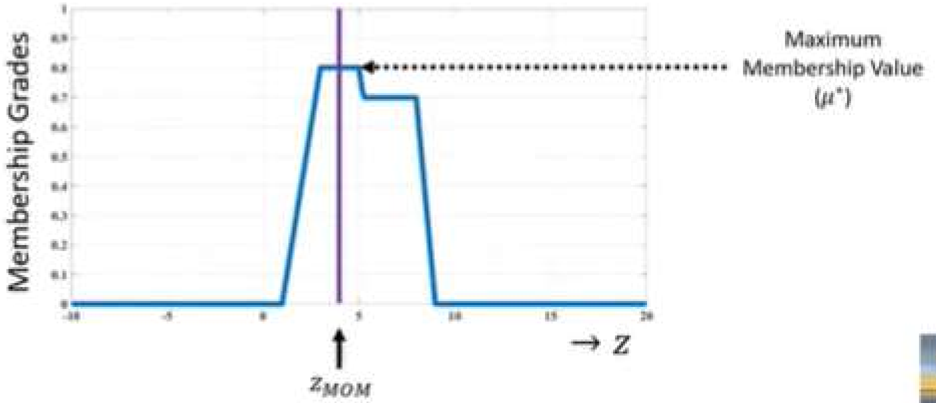
Mamdani Fuzzy Model

Defuzzification

- Mean of Maximum (MOM)

$$z_{\mathrm{MOM}} = \frac{\int_{Z'} z\, dz}{\int_{Z'} dz},$$

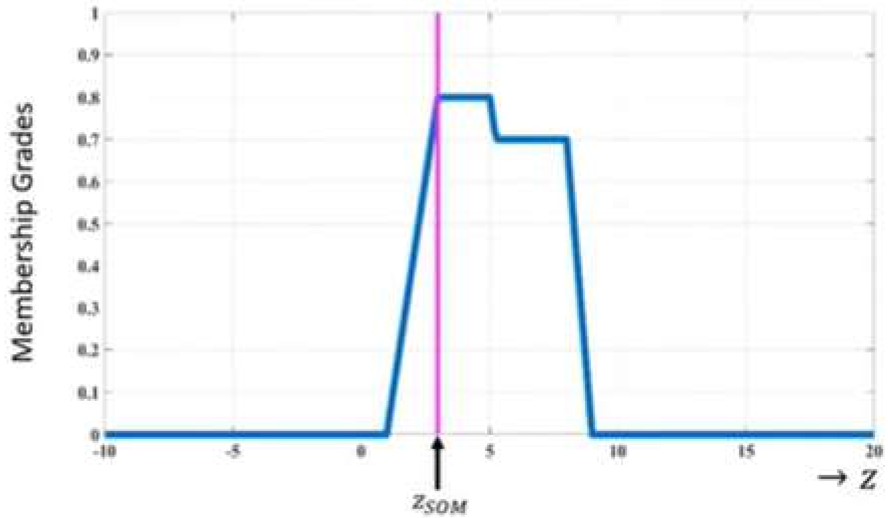
where $Z' = \{z \mid \mu_A(z) = \mu^*\}$



Mamdani Fuzzy Model

Defuzzification

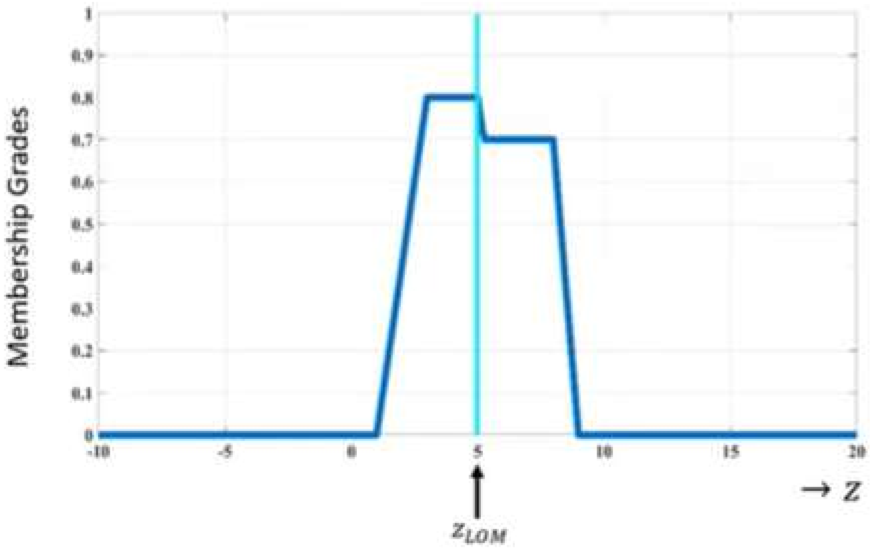
- Smallest of Maximum (SOM)



Mamdani Fuzzy Model

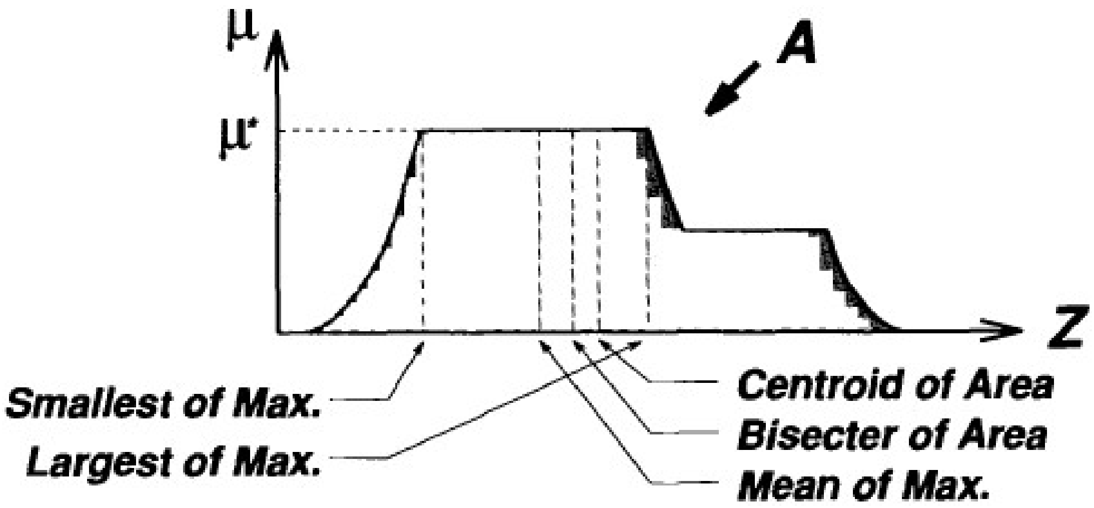
Defuzzification

- Largest of Maximum (LOM)



Mamdani Fuzzy Model

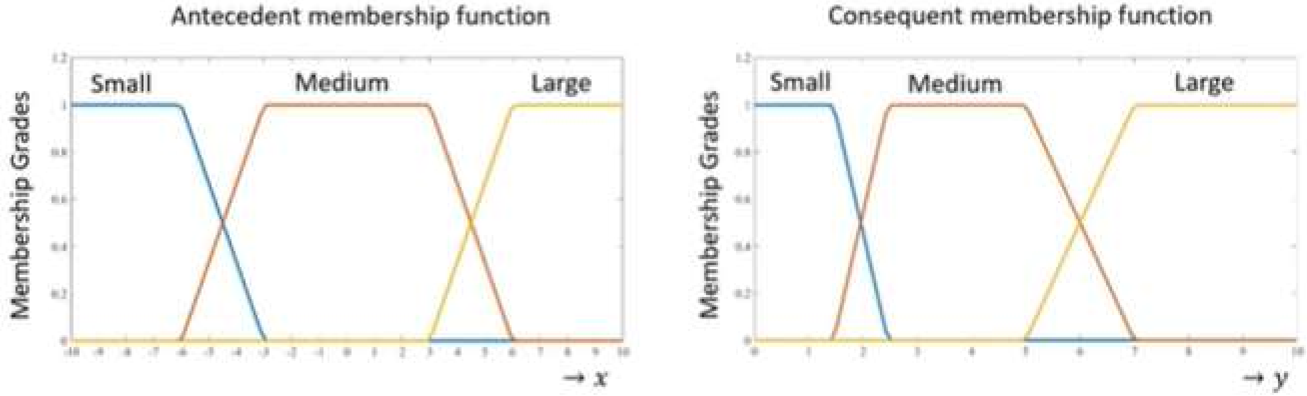
Defuzzification



Mamdani Fuzzy Model Example

Multiple Rules Single Antecedent

An example of a single-input single-output (SISO) Mamdani fuzzy model is shown below for antecedent and consequent membership functions with universe of discourse $X \in [-10,10]$ and $Y \in [0,10]$, respectively $\forall x \in X, y \in Y$.



The SISO Mamdani fuzzy model with three rules can be expressed as:

Rule 1: IF x is **small** THEN y is **small**

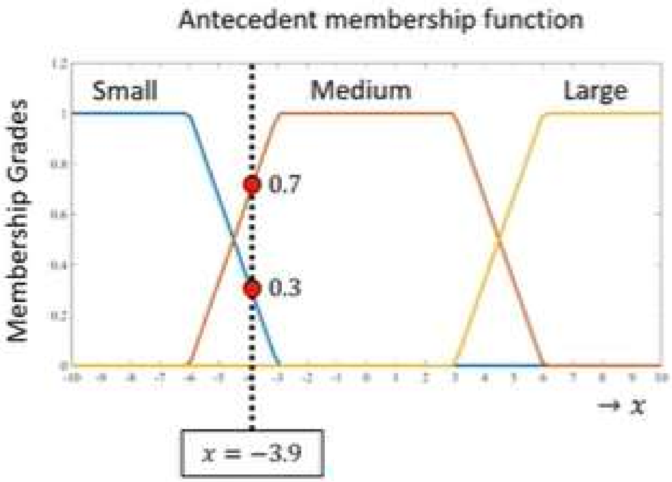
Rule 2: IF x is **medium** THEN y is **medium**

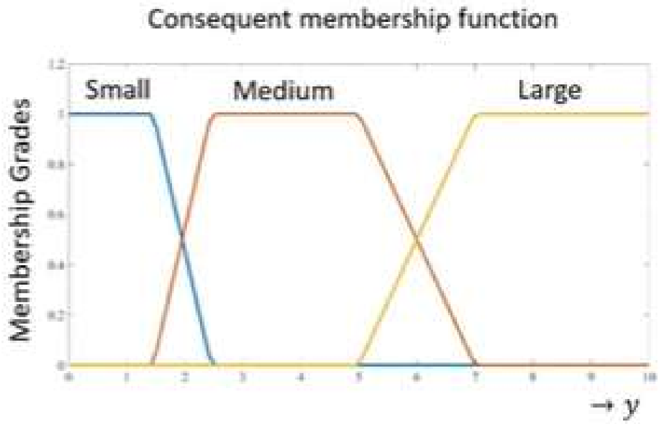
Rule 3: IF x is **large** THEN y is **large**

What will the output y for input $x = -3.9$?

Mamdani Fuzzy Model Example

Multiple Rules Single Antecedent





For input $x = -3.9$;

$$\mu(x = -3.9)\Big|_{\text{Small}} = 0.3$$

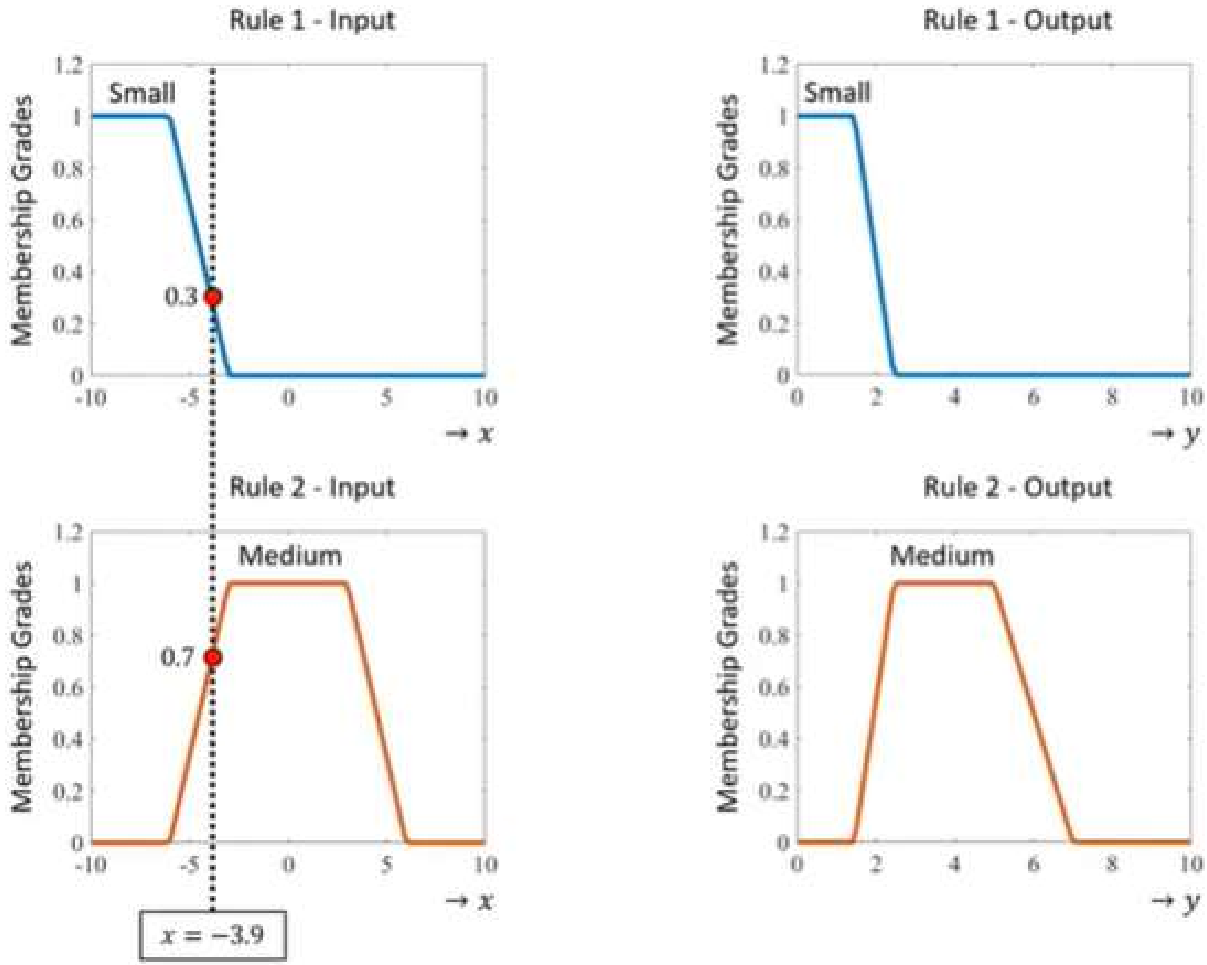
$$\mu(x = -3.9)\Big|_{\text{Medium}} = 0.7$$

Given Rules

Rules	Input	Output
Rule 1	Small	Small
Rule 2	Medium	Medium
Rule 3	Large	Large

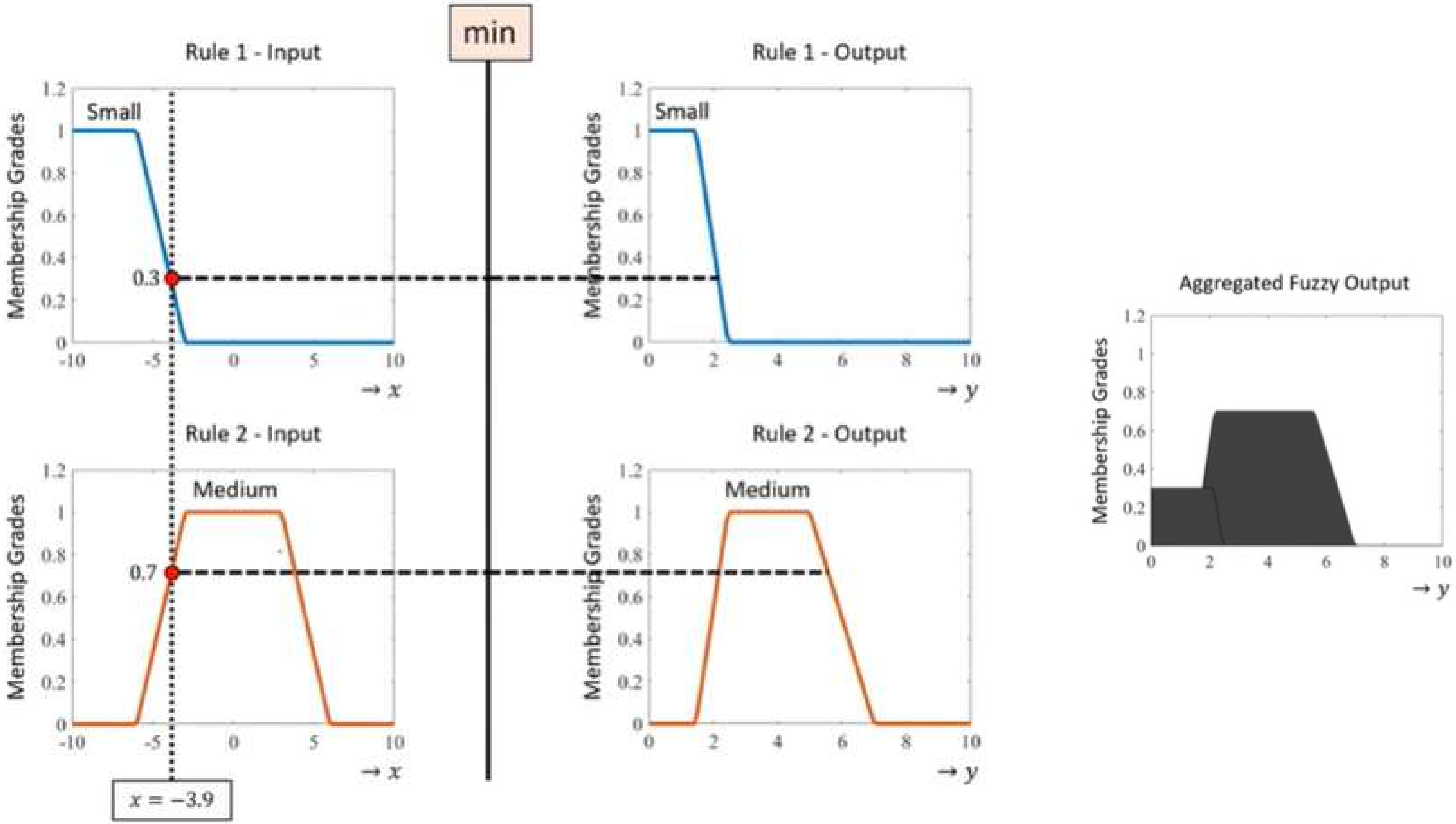
Mamdani Fuzzy Model Example

Multiple Rules Single Antecedent



Mamdani Fuzzy Model Example

Multiple Rules Single Antecedent



Mamdani Fuzzy Model Example

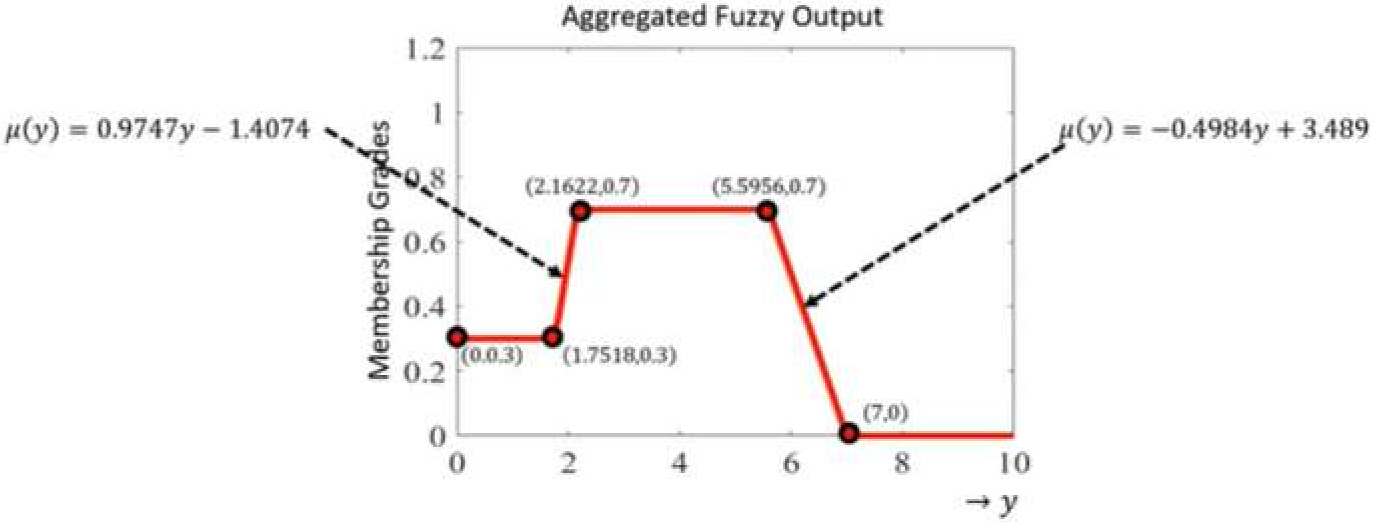
Multiple Rules Single Antecedent

Defuzzification using Centroid of Area (COA):

$$y_{COA} = \frac{\int_{y \in Y} \mu(y) y \, dy}{\int_{y \in Y} \mu(y) \, dy}$$

$$y_{COA} = \frac{\int_0^{1.7518}(0.3)y\,dy + \int_{1.7518}^{2.1622}(0.9747y - 1.4074)y\,dy + \int_{2.1622}^{5.5956}(0.7)\,y\,dy + \int_{5.5956}^{7}(-0.4984y + 3.489)\,y\,dy}{\int_0^{1.7518}(0.3)\,dy + \int_{1.7518}^{2.1622}(0.9747y - 1.4074)\,dy + \int_{2.1622}^{5.5956}(0.7)\,dy + \int_{5.5956}^{7}(-0.4984y + 3.489)\,dy}$$

$$y_{COA}\Big|_{x=-3.9} = 3.6327$$



Mamdani Fuzzy Model Example

Multiple Rules Single Antecedent

Defuzzification using Bisector of Area (BOA):

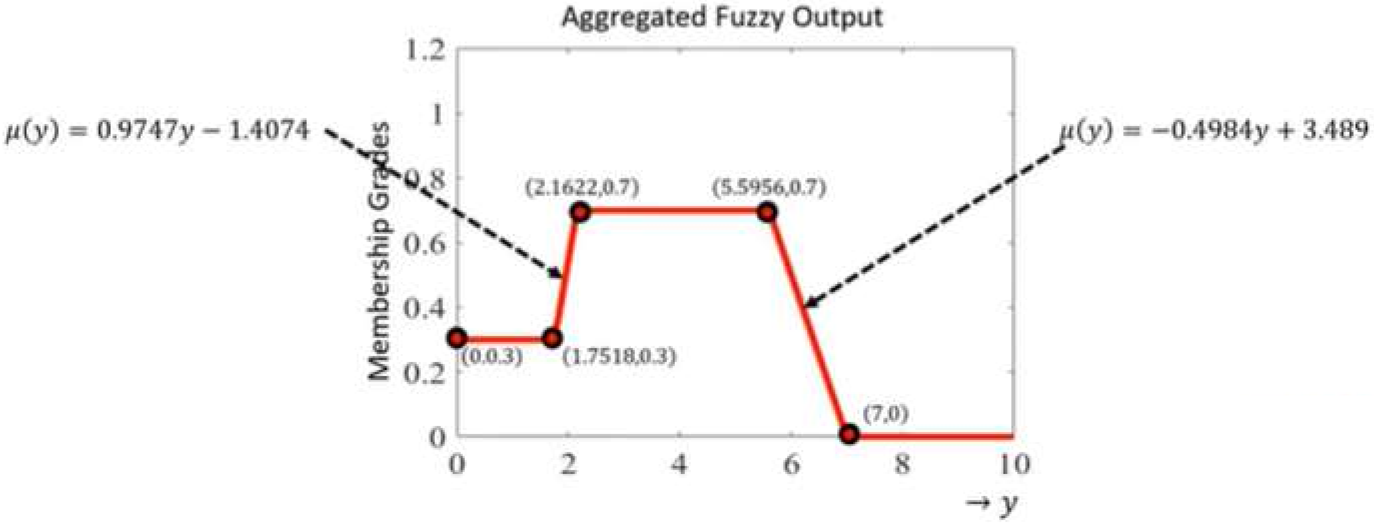
$$\int_{\alpha}^{y_{BOA}} \mu(y)\,dy = \int_{y_{BOA}}^{\beta} \mu(y)\,dy$$

where, $\alpha = \min\{y|y \in Y\}$ and $\beta = \max\{y|y \in Y\}$.

$\alpha = 0$ and $\beta = 7$

$$\int_{\alpha=0}^{1.7518} (0.3)\,dy + \int_{1.7518}^{2.1622} (0.9747y - 1.4074)\,dy + \int_{2.1622}^{y_{BOA}} (0.7)\,dy = \int_{y_{BOA}}^{5.5956} (0.7)\,dy + \int_{5.5956}^{\beta=7} (-0.4984y + 3.489)\,dy$$

$$y_{BOA}\Big|_{x=-3.9} = 3.7087$$



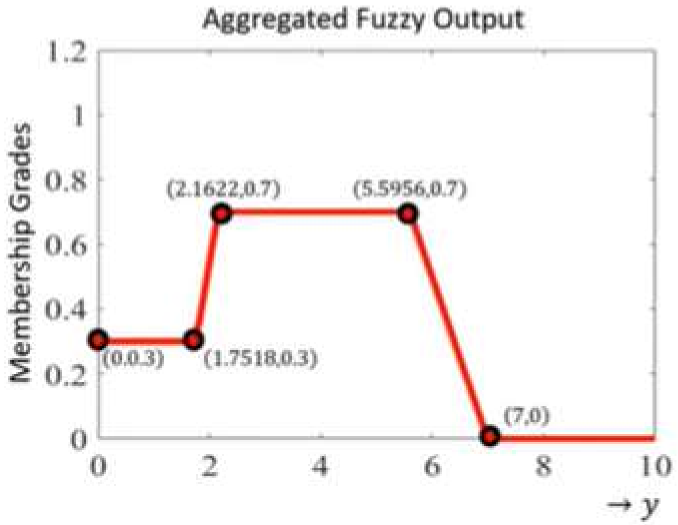
Mamdani Fuzzy Model Example

Multiple Rules Single Antecedent

Defuzzification using Mean of Maximum (MOM):

$$y_{MOM} = \frac{2.1622 + 5.5956}{2}$$

$$y_{MOM}\Big|_{x=-3.9} = 3.8789$$



Mamdani Fuzzy Model Example

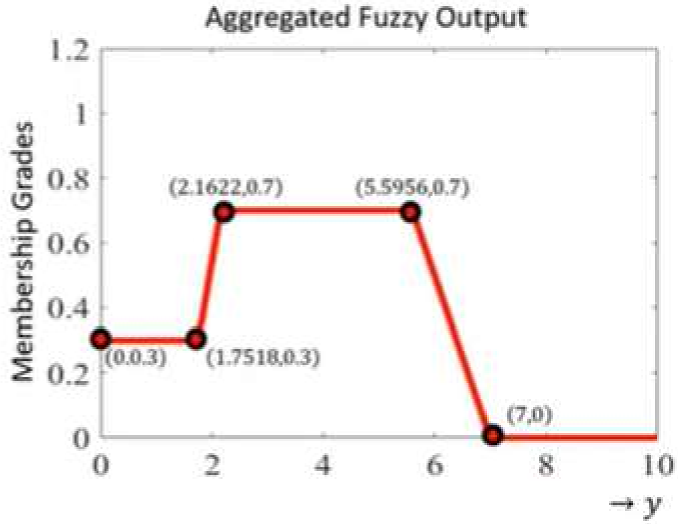
Multiple Rules Single Antecedent

Defuzzification using Smallest of Maximum (SOM):

$$y_{SOM}\Big|_{x=-3.9} = 2.1622$$

Defuzzification using Largest of Maximum (LOM):

$$y_{LOM}\Big|_{x=-3.9} = 5.5956$$



Sugeno Fuzzy Model

- Also known as the TSK model named after Takagi, Sugeno and Kang
- A typical fuzzy rule in a Sugeno system has the form

$$\text{if } x \text{ is } A \text{ and } y \text{ is } B \text{ then } z = f(x,\, y),$$

where A and B are fuzzy sets in the antecedent, while $z = f(x, y)$ is a crisp function in the consequent.

- When *f(x, y)* is a first order polynomial, the FIS is called first-order Sugeno fuzzy model.
- When *f(x, y)* is a constant, the FIS is called a zero order Sugeno fuzzy model which is like a special case of the Mamdani fuzzy model and Tsukamoto fuzzy model.
- Here each rule has a crisp output, the overall output is obtained via weighted average.
- The firing strength of i^{th}-rule is defined by,

$$w_i = \mu_{A_i}(x) \wedge \mu_{B_i}(y)$$

- The overall output is taken as the weighted average of each rule's output as follows:

$$z^* = \frac{\sum_{i=1}^{n} w_i \times z_i}{\sum_{i=1}^{n} w_i}$$

where z_i is a polynomial in the input variables x and y.

Sugeno Fuzzy Model

Zero Order TSK Fuzzy Model

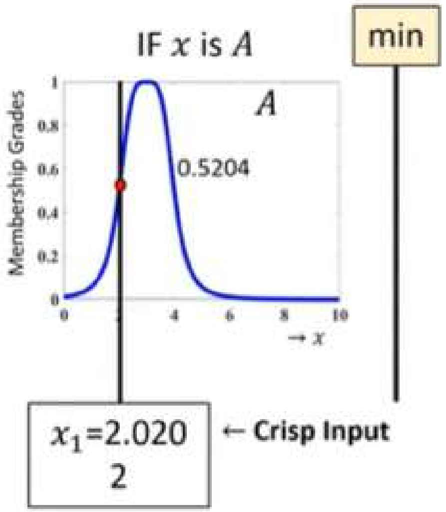
- *f(x, y) = p_0*
- **Single Rule with Single Antecedent**

Rule: IF x is A THEN $y = f(\cdot) = p_0$
Fact (Input): $x = x_1$

....

Conclusion: $y = p_0$

Let us assume, $f(\cdot) = p_0 = 3.737$ which is a constant value.



$\boxed{p_0 = 3.737}$

$w_1 = 0.5204$

$y_1 = 3.737$

$$y^* = \frac{w_1 \times y_1}{w_1} = y_1$$

$$y^* = 3.737$$

Sugeno Fuzzy Model

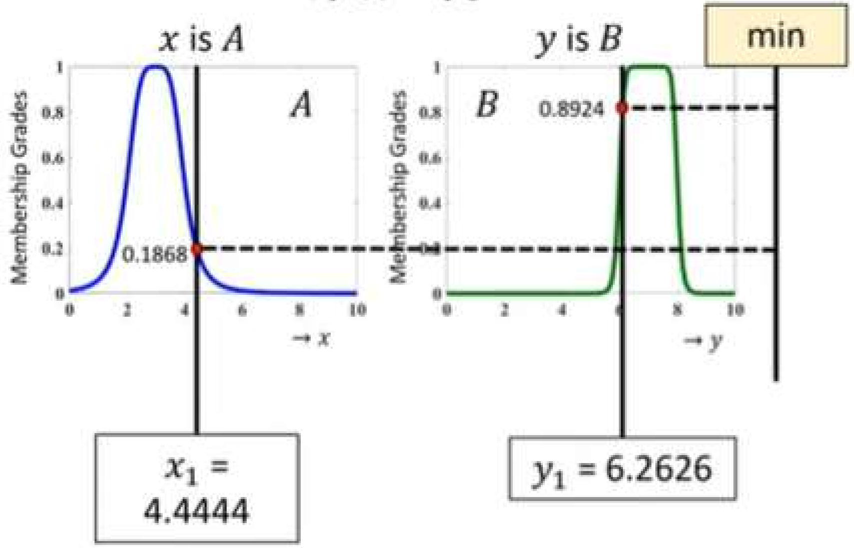
- **Single Rule with Multiple Antecedent**

Rule: IF x is A AND y is B THEN $z = f(\cdot) = p_0$
Fact (Input): $x = x_1$ AND $y = y_1$

....

Conclusion: $z = p_0$

Let us assume, $f(\cdot) = p_0 = 3.737$ which is a constant value.



$p_0 = 3.737$

w_1
$= \min(0.1868, 0.8924)$
$w_1 = 0.1868$
$z_1 = 3.737$

$$z^* = \frac{w_1 \times z_1}{w_1} = z_1$$

$z^* = 3.737$

Sugeno Fuzzy Model

- **Multiple Rule with Multiple Antecedent**

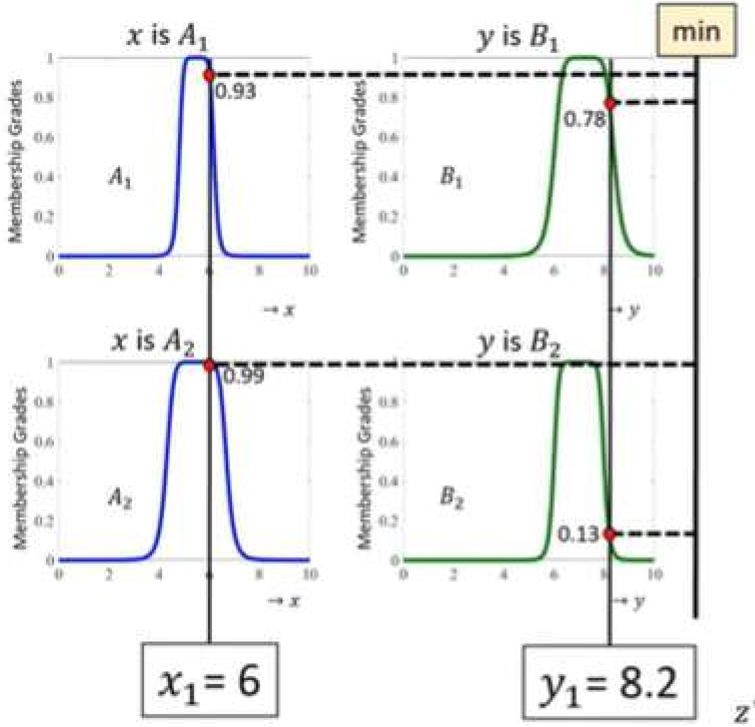
Rule 1: IF x is A_1 AND y is B_1 THEN $z = f(\cdot) = p_0^1$
Rule 2: IF x is A_2 AND y is B_2 THEN $z = f(\cdot) = p_0^2$
Fact (Input): $x = x_1$ AND $y = y_1$

....

Conclusion: z is $z^* = \frac{w_1 \times p_0^1 + w_2 \times p_0^2}{w_1 + w_2}$

Sugeno Fuzzy Model

- **Multiple Rule with Multiple Antecedent**



$$p_0^1 = 8$$

$$w_1 = \min(0.93, 0.78)$$
$$w_1 = 0.78$$

$$z_1 = 8$$

$$p_0^2 = 6$$

$$w_2 = \min(0.99, 0.13)$$
$$w_2 = 0.13$$

$$z_2 = 6$$

$$z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2}$$

$$z^* = \frac{0.78 \times 8 + 0.13 \times 6}{0.78 + 0.13} = 7.7143$$

Sugeno Fuzzy Model

First Order TSK Fuzzy Model

- $f(x, y) = p_0 + p_1x + p_2y$

For first-order TSK fuzzy model,

$$z_i = f(x, y) = p_0 + p_1x + p_2y$$

The firing strength of i^{th}-rule is defined by,

$$w_i = \mu_{A_i}(x) \wedge \mu_{B_i}(y)$$

The overall output is taken as the weighted average of each rule's output as follows:

$$z^* = \frac{\sum_{i=1}^{n} w_i \times z_i}{\sum_{i=1}^{n} w_i}$$

where z_i is a polynomial in the input variables x and y.

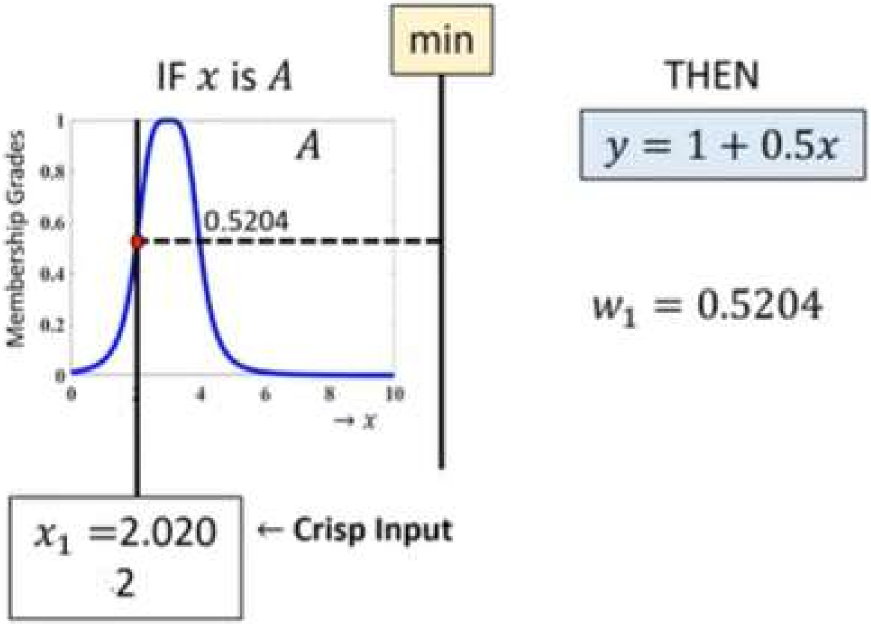
Sugeno Fuzzy Model

- **Single Rule with Single Antecedent**

Rule: IF x is A THEN $y = f(x) = p_0 + p_1 x$
Fact (Input): $x = x_1$

....

Conclusion: $y = y^* = p_0 + p_1 x_1$



If $x_1 = 2.0202 \Rightarrow y_1 = 1 + 0.5 \times 2.0202 = 2.0101$

$$\Rightarrow y^* = \frac{w_1 \times y_1}{w_1} = y_1 = 2.0101$$

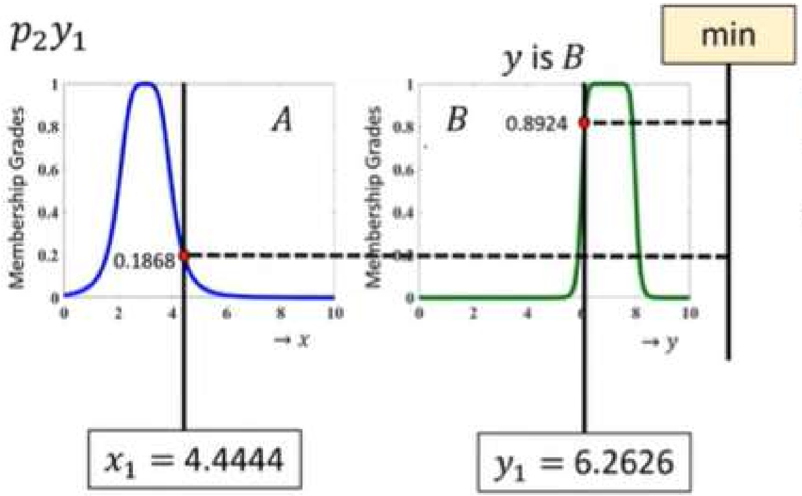
Sugeno Fuzzy Model

- **Single Rule with Multiple Antecedent**

Rule: IF x is A AND y is B THEN $z = f(x, y) = p_0 + p_1x + p_2y$
Fact (Input): $x = x_1$ AND $y = y_1$

....

Conclusion: $z = z^* = p_0 + p_1x_1 + p_2y_1$



$z = 0.1 + 0.5x + 0.2y$

$w_1 = \min(0.1868, 0.8924)$
$w_1 = 0.1868$

$z_1 = 0.1 + 0.5 \times 4.4444 + 0.2 \times 6.2626$
$= 3.5747$

$$z^* = \frac{w_1 \times z_1}{w_1} = z_1$$

$z^* = 3.5747$

Sugeno Fuzzy Model

- **Multiple Rule with Multiple Antecedent**

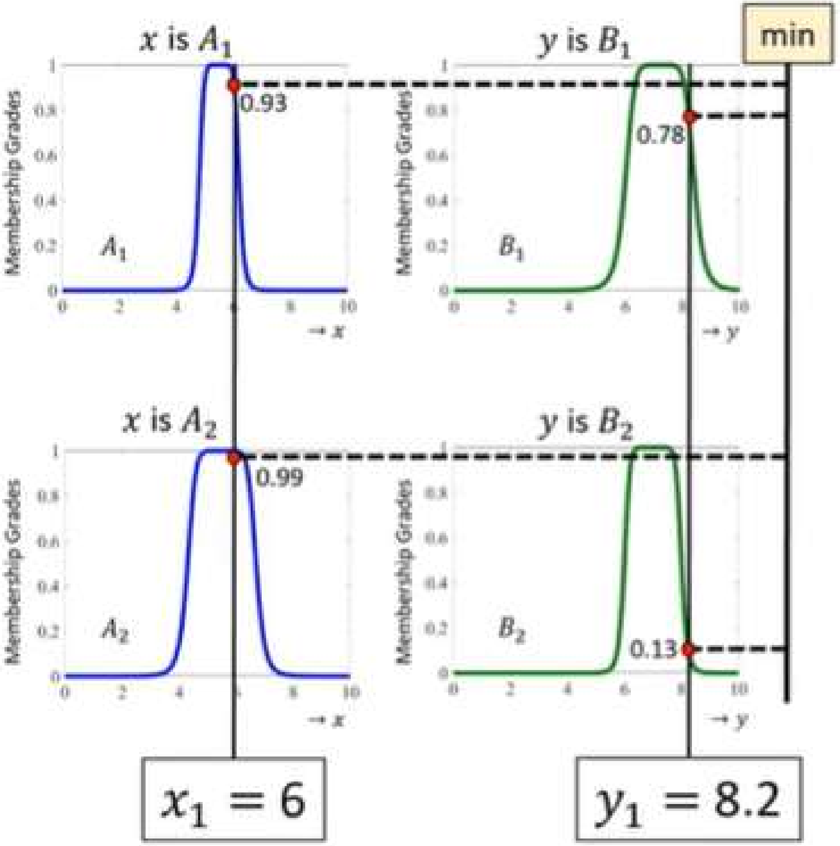
Rule 1: IF x is A_1 AND y is B_1 THEN $z = z_1 = p_0^1 + p_1^1 x + p_2^1 y$
Rule 2: IF x is A_2 AND y is B_2 THEN $z = z_2 = p_0^2 + p_1^2 x + p_2^2 y$
Fact (Input): $x = x_1$ AND $y = y_1$

....

Conclusion: z is $z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2}$

Sugeno Fuzzy Model

- **Multiple Rule with Multiple Antecedent**



$$z_1 = 0.1 + 0.5x + 0.2y$$

$$w_1 = \min(0.93, 0.78)$$
$$w_1 = 0.78$$

$$z_1 = 0.1 + 0.5 \times 6 + 0.2 \times 8.2$$
$$z_1 = 4.74$$

$$z_2 = 0.02 + 0.25x + 0.1y$$

$$w_2 = \min(0.99, 0.13)$$
$$w_2 = 0.13$$

$$z_2 = 0.02 + 0.25 \times 6 + 0.1 \times 8.2$$
$$z_2 = 2.34$$

$$z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2} = \frac{0.78 \times 4.74 + 0.13 \times 2.34}{0.78 + 0.13} \quad \Rightarrow \quad z^* = 4.3971$$

Sugeno (TSK) Fuzzy Model Example

The rule base of a first-order TSK fuzzy model is defined as:

Rule 1: IF A is **LOW** **OR** B is **HIGH** THEN the output $z = 10 + 0.2x + 0.3y$

Rule 2: IF A is **LOW** **OR** B is **MEDIUM** THEN the output $z = x + 2y$

Rule 3: IF A is **MEDIUM** **OR** B is **LOW** THEN the output $z = 2x + y$

Rule 4: IF A is **HIGH** **OR** B is **HIGH** THEN the output $z = 0.1x + 0.03y$

The membership function of inputs A and B with the universe of discourse X and Y, respectively are given as below:

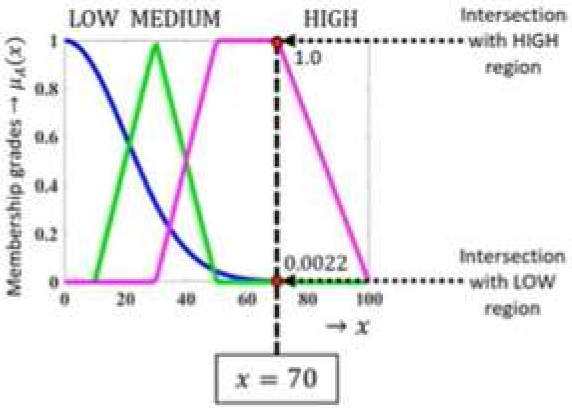
$\mu_{A_{\mathrm{LOW}}}(x) = gaussmf(x; 0{,}20)$
$\mu_{A_{\mathrm{MEDIUM}}}(x) = trimf(x; 10{,}30{,}50)$
$\mu_{A_{\mathrm{HIGH}}}(x) = trapmf(x; 30{,}50{,}70{,}100)$

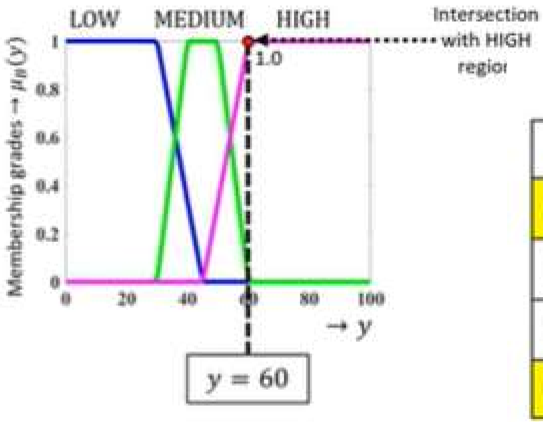
$\mu_{B_{\mathrm{LOW}}}(y) = trapmf(y; 0{,}0{,}30{,}45)$
$\mu_{B_{\mathrm{MEDIUM}}}(y) = trapmf(y; 30{,}40{,}50{,}60)$
$\mu_{B_{\mathrm{HIGH}}}(y) = trapmf(y; 45{,}60\ 100{,}100)$

Calculate the output z through pictorial representation for inputs $x = 70$ **OR** $y = 60$.

(The universe of discourse of inputs X and Y are 0 to 100 $\forall x \in X, y \in Y$).

Sugeno (TSK) Fuzzy Model Example

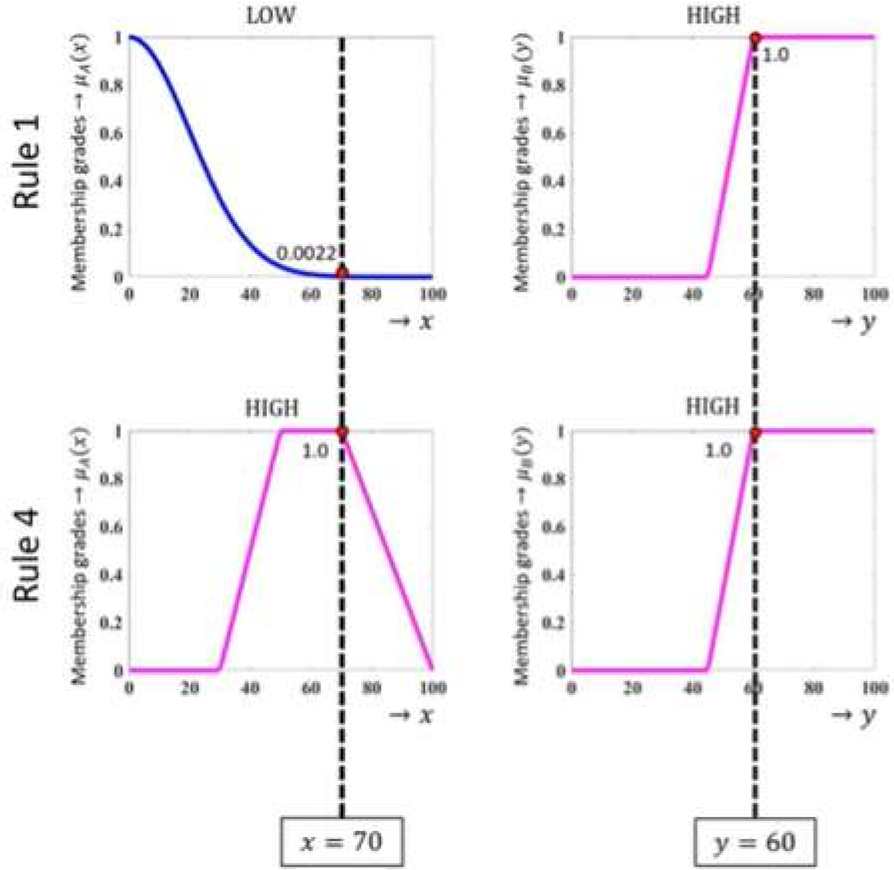




Given Rules

OR	Input A	Input B	Output z
Rule 1	LOW	HIGH	$10 + 0.2x + 0.3y$
Rule 2	LOW	MEDIUM	$x + 2y$
Rule 3	MEDIUM	LOW	$2x + y$
Rule 4	HIGH	HIGH	$0.1x + 0.03y$

Sugeno (TSK) Fuzzy Model Example



$$z = 10 + 0.2x + 0.3y$$

$$w_1 = \max(0.0022, 1.0) = 1.0$$

$$z_1 = 10 + 0.2 \times 70 + 0.3 \times 60$$
$$z_1 = 42$$

$$z = 0.1x + 0.03y$$

$$w_2 = \max(1.0, 1.0) = 1.0$$

$$z_1 = 0.1 \times 70 + 0.03 \times 60$$
$$z_1 = 8.8$$

Sugeno (TSK) Fuzzy Model Example

$$w_1 = 1$$

$$w_2 = 1$$

$$z_1 = 42$$

$$z_2 = 8.8$$

$$z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2}$$

$$z^* = \frac{1 \times 42 + 1 \times 8.8}{1 + 1}$$

$$z^* = \frac{42 + 8.8}{2} = \frac{50.8}{2}$$

$$z^* = 25.4$$

$$\boxed{z = 10 + 0.2x + 0.3y}$$

$$w_1 = \max(0.0022, 1.0) = 1.0$$

$$z_1 = 10 + 0.2 \times 70 + 0.3 \times 60$$
$$z_1 = 42$$

$$\boxed{z = 0.1x + 0.03y}$$

$$w_2 = \max(1.0, 1.0) = 1.0$$

$$z_1 = 0.1 \times 70 + 0.03 \times 60$$
$$z_1 = 8.8$$

Tsukamoto Fuzzy Model

- In Tsukamoto fuzzy model, the consequent of each fuzzy if – then rue is represented by a fuzzy set with a monotonically (increasing / decreasing) membership function.
- As a result, the inferred output of each rule is a crisp value corresponding to the firing strength w of that rule. The overall output is taken as the weighted average of the output of each rule.
- Tsukamoto fuzzy model avoids the time consuming process of defuzzification.
- A fuzzy rule base with n rules for input membership functions A_i and B_i with the universe of discourse X and Y, respectively and the output membership function C_i with the universe of discourse Z is defined as,

$$\text{IF } x \text{ is } A_i \text{ AND/OR } y \text{ is } B_i \text{ THEN } z \text{ is } C_i$$

where $i = 1,2,3,\ldots,n$ and fuzzy sets A_i, B_i, and C_i are expressed as,

$$A_i = \int_{x\in X} \mu_{A_i}(x)/x\,; \quad B_i = \int_{y\in Y} \mu_{B_i}(y)/y\,; \quad C_i = \int_{z\in Z} \mu_{C_i}(z)/z$$

Tsukamoto Fuzzy Model

- The firing strength of i^{th} rule is defined by:

$$w_i = \mu_{A_i}(x) \wedge \mu_{B_i}(y)$$

- The overall output is taken as the weighted average of the output of each rule as follows:

$$z^* = \frac{\sum_{i=1}^{n} w_i \times z_i}{\sum_{i=1}^{n} w_i}$$

where z_i is the output of each rule induced by the firing strength w_i and the output membership function C_i.

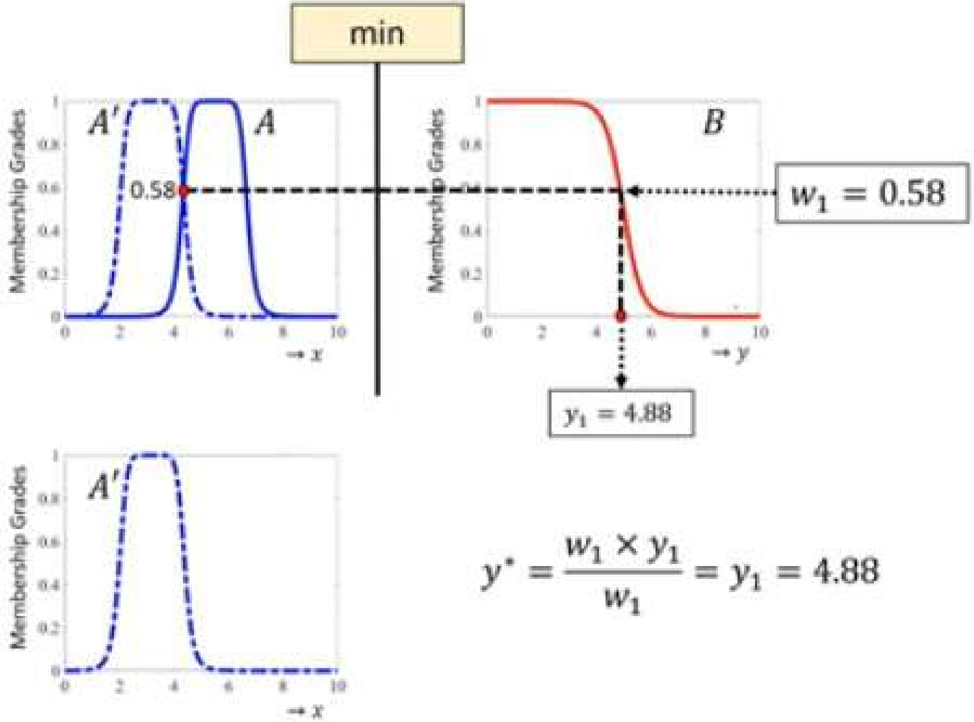
- Since the reasoning mechanism of the Tsukamoto fuzzy model doesn't follow strictly the compositional rule of inference, the output is always crisp even when the inputs are fuzzy.

Tsukamoto Fuzzy Model

- **Single Rule with Single Antecedent (Fuzzy Input)**

Rule: IF x is A THEN y is B
Fact (Input): x is A'

Conclusion: y is $y^* = \frac{w_1 \times y_1}{w_1} = y_1$



$$y^* = \frac{w_1 \times y_1}{w_1} = y_1 = 4.88$$

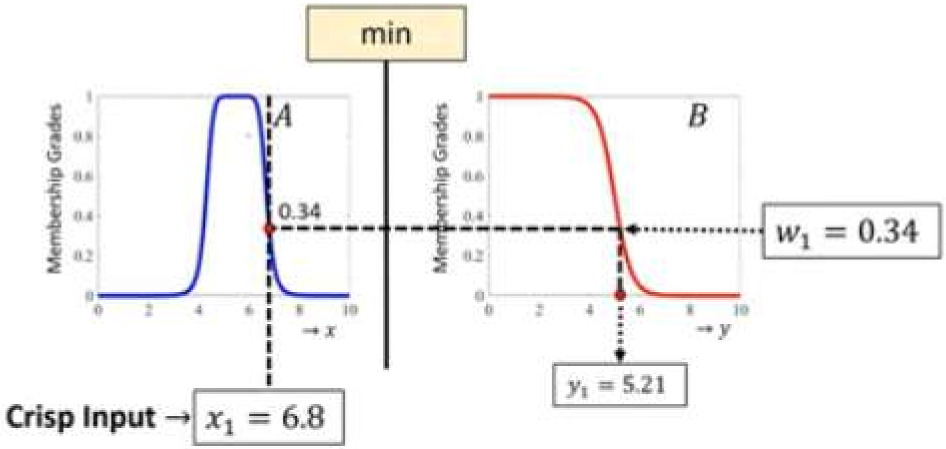
Tsukamoto Fuzzy Model

- **Single Rule with Single Antecedent (Crisp Input)**

Rule: IF x is A THEN y is B

Fact (Input): $x = x_1$

Conclusion: y is $y^* = \frac{w_1 \times y_1}{w_1} = y_1$



$$y^* = \frac{w_1 \times y_1}{w_1} = y_1 = 5.21$$

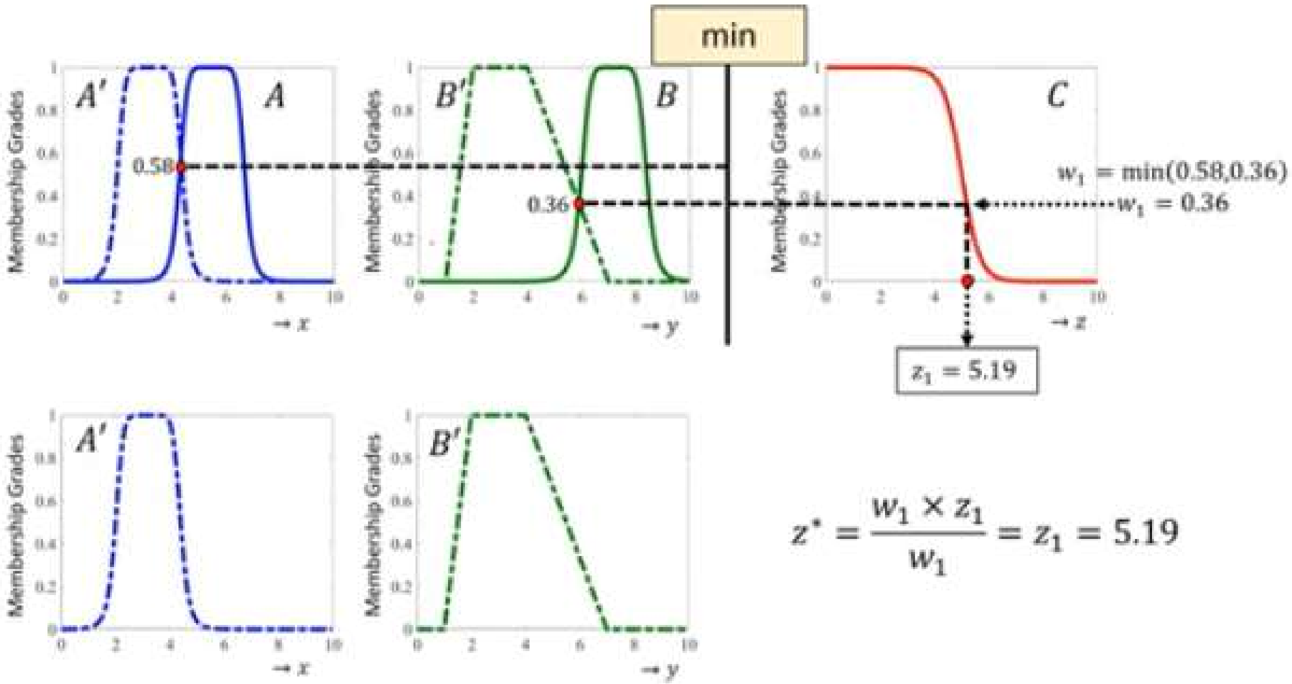
Tsukamoto Fuzzy Model

- **Single Rule with Multiple Antecedent (Fuzzy Input)**

Rule: IF x is A AND y is B THEN z is C
Fact (Input): x is A' AND y is B'

....

Conclusion: z is $z^* = \frac{w_1 \times z_1}{w_1} = z_1$



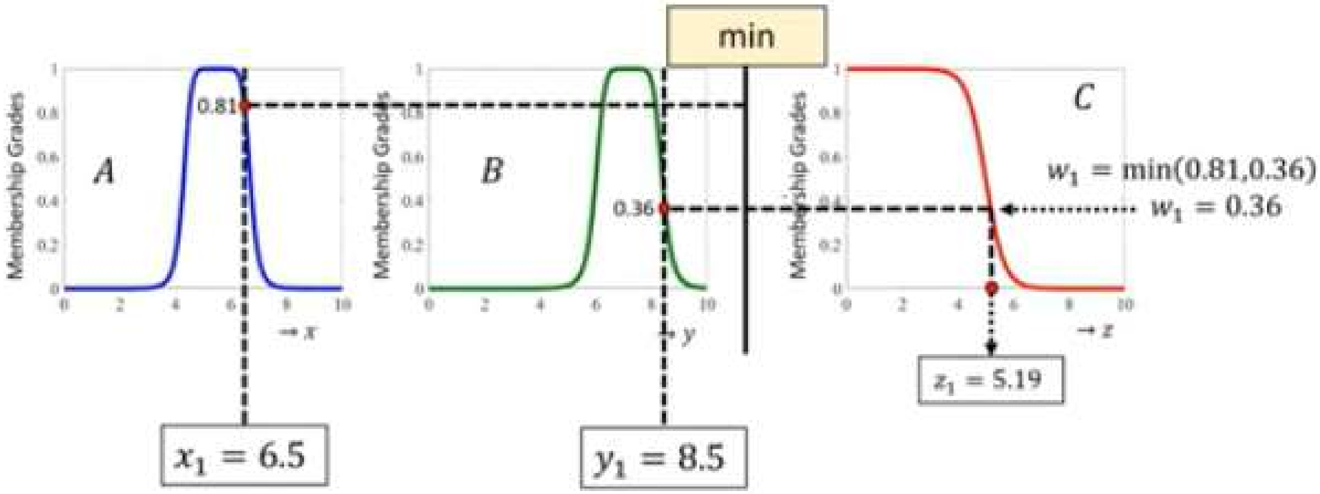
$$z^* = \frac{w_1 \times z_1}{w_1} = z_1 = 5.19$$

Tsukamoto Fuzzy Model

- **Single Rule with Multiple Antecedent (Crisp Input)**

Rule: IF x is A AND y is B THEN z is C
Fact (Input): $x = x_1$ AND $y = y_1$

Conclusion: z is $z^* = \frac{w_1 \times z_1}{w_1} = z_1$



$$z^* = \frac{w_1 \times z_1}{w_1} = z_1 = 5.19$$

Tsukamoto Fuzzy Model

- **Multiple Rule with Multiple Antecedent (Fuzzy Input)**

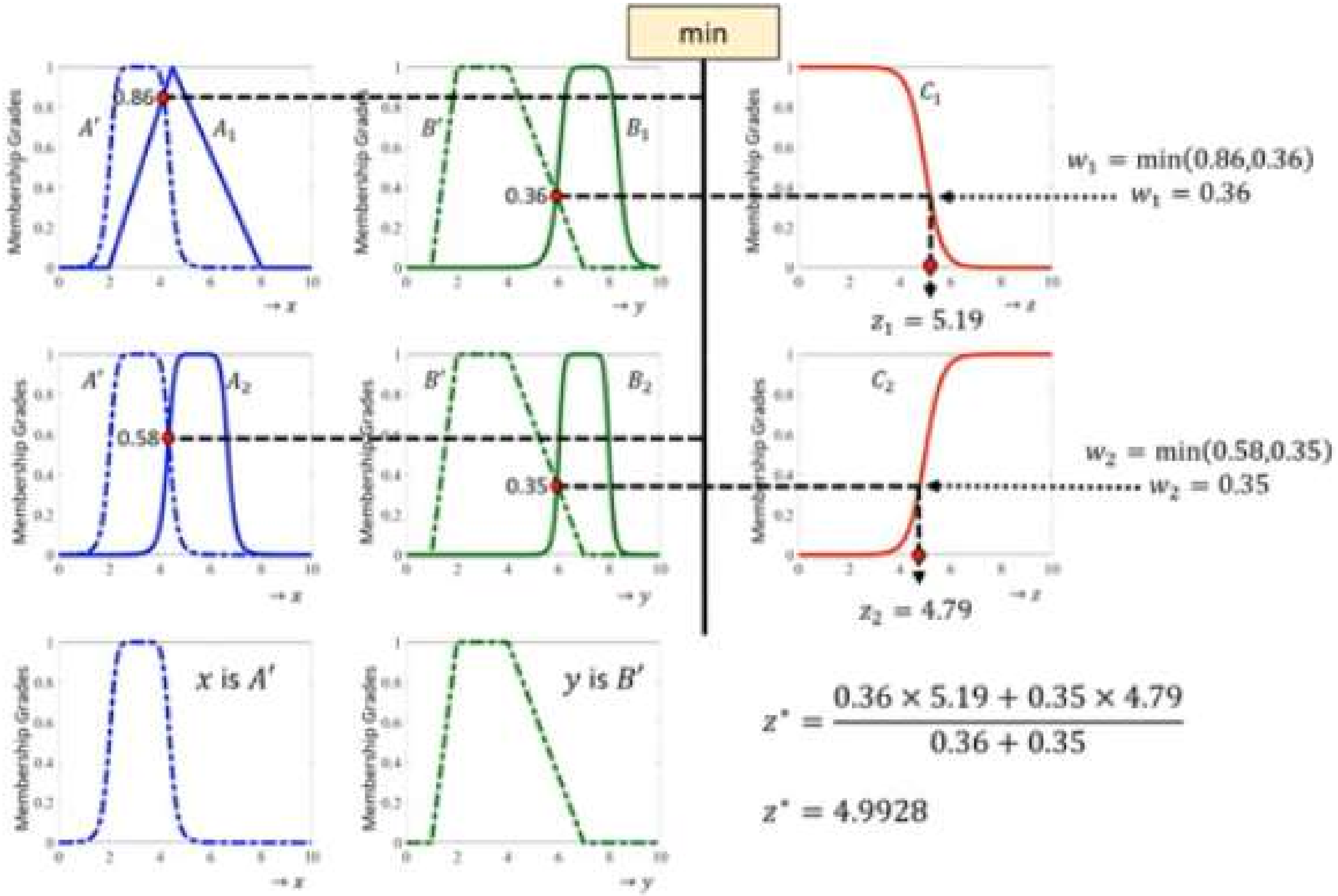
Rule 1: IF x is A_1 AND y is B_1 THEN z is C_1
Rule 2: if x is A_2 AND y is B_2 THEN z is C_2
Fact (Input): x is A' AND y is B'

....

Conclusion: z is $z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2}$

Tsukamoto Fuzzy Model

- **Multiple Rule with Multiple Antecedent (Fuzzy Input)**



Tsukamoto Fuzzy Model

- **Multiple Rule with Multiple Antecedent (Crisp Input)**

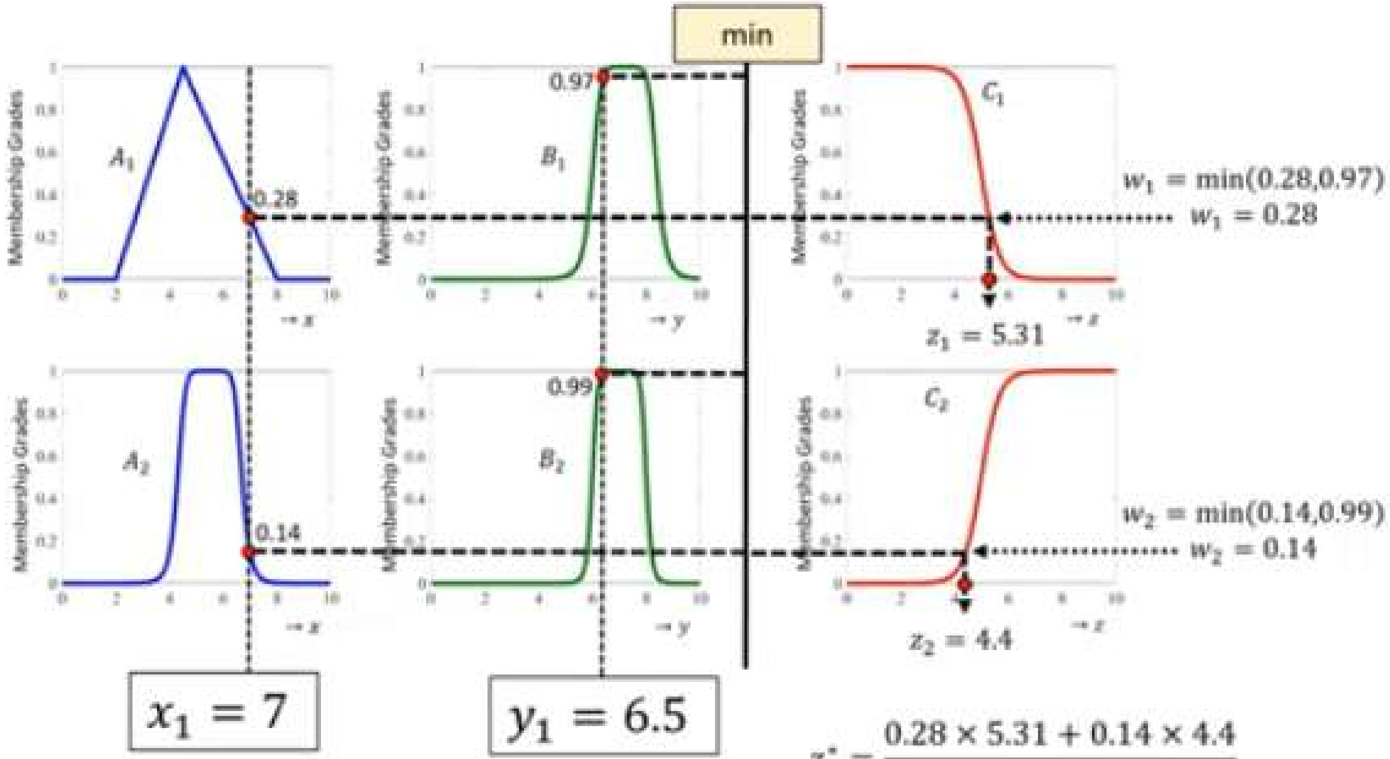
Rule 1: IF x is A_1 AND y is B_1 THEN z is C_1
Rule 2: if x is A_2 AND y is B_2 THEN z is C_2
Fact (Input): $x = x_1$ AND $y = y_1$

....

Conclusion: z is $z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2}$

Tsukamoto Fuzzy Model

- **Multiple Rule with Multiple Antecedent (Crisp Input)**



$$z^* = \frac{0.28 \times 5.31 + 0.14 \times 4.4}{0.28 + 0.14}$$

$$z^* = 5.0067$$

Tsukamoto Fuzzy Model Example

The rule base of a Tsukamoto fuzzy model is defined as:

Rule 1: IF A is LOW **AND** B is MEDIUM THEN the output C is LOW

Rule 2: IF A MEDIUM **AND** B is HIGH THEN the output C is HIGH

Rule 3: IF A is HIGH **AND** B is HIGH THEN the output C is HIGH

The membership function of inputs A, B and output C with the universe of discourse X, Y, and Z, respectively are given as below:

$$\mu_{A_{LOW}}(x) = trapmf(x; 0,0,20,40)$$
$$\mu_{A_{MEDIUM}}(x) = trapmf(x; 20,40,60,80)$$
$$\mu_{A_{HIGH}}(x) = trapmf(x; 60,80,100,100)$$

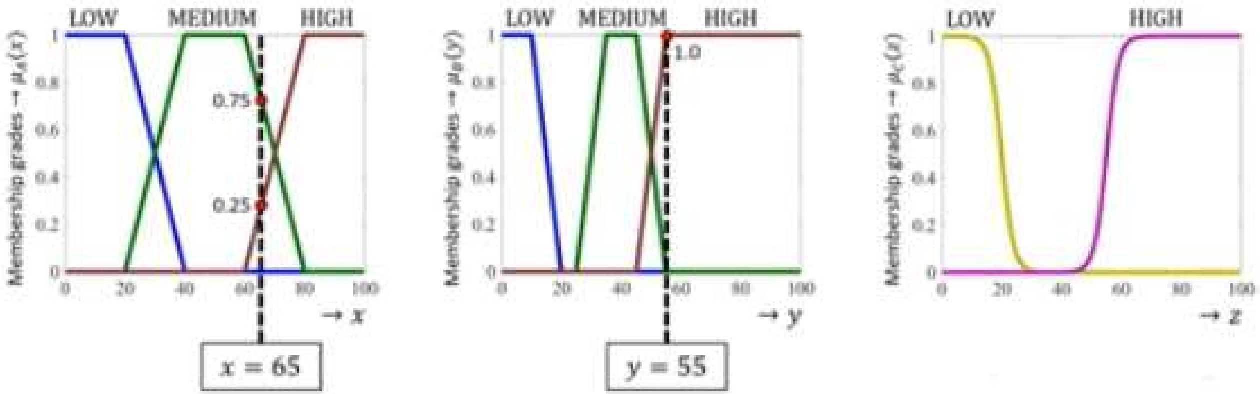
$$\mu_{B_{LOW}}(y) = trapmf(y; 0,0,10,20)$$
$$\mu_{B_{MEDIUM}}(y) = trapmf(y; 25,35,45,55)$$
$$\mu_{B_{HIGH}}(y) = trapmf(y; 45,55,100,100)$$

$$\mu_{C_{LOW}}(z) = sigmf(z; -0.5,20)$$
$$\mu_{C_{HIGH}}(z) = sigmf(z; 0.5,55)$$

Find the crisp output z for inputs $x = 65$ **AND** $y = 55$.

(The universe of discourse of inputs and outputs X, Y, and Z are 0 to 100 $\forall x \in X, y \in Y, z \in Z$).

Tsukamoto Fuzzy Model Example



$$\mu_{A_{LOW}}(x=65) = \max\left\{\min\left[\frac{65-0}{0-0}, 1, \frac{40-65}{40-20}\right], 0\right\} = \max\left\{\min\left[\infty, 1, \frac{-25}{20}\right], 0\right\} = 0$$

$$\mu_{A_{MEDIUM}}(x=65) = \max\left\{\min\left[\frac{65-20}{40-20}, 1, \frac{80-65}{80-60}\right], 0\right\} = \max\left\{\min\left[\frac{45}{20}, 1, \frac{15}{20}\right], 0\right\} = \frac{15}{20} = 0.75$$

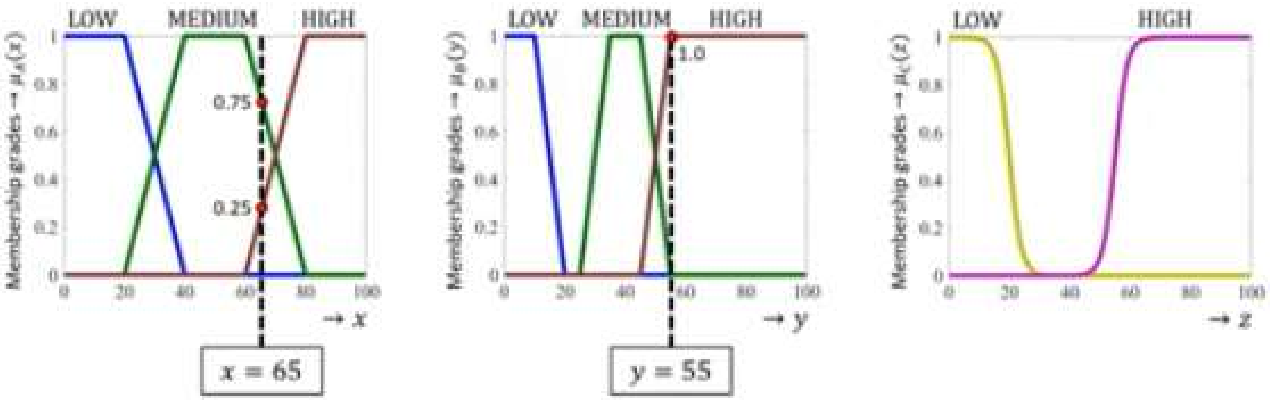
$$\mu_{A_{HIGH}}(x=65) = \max\left\{\min\left[\frac{65-60}{80-60}, 1, \frac{100-65}{100-100}\right], 0\right\} = \max\left\{\min\left[\frac{5}{20}, 1, \infty\right], 0\right\} = 0.25$$

$$\mu_{B_{LOW}}(y=55) = \max\left\{\min\left[\frac{55-0}{0-0}, 1, \frac{20-55}{20-10}\right], 0\right\} = \max\left\{\min\left[\infty, 1, \frac{-35}{10}\right], 0\right\} = 0$$

$$\mu_{B_{MEDIUM}}(y=55) = \max\left\{\min\left[\frac{55-25}{35-25}, 1, \frac{55-55}{55-45}\right], 0\right\} = \max\left\{\min\left[\frac{30}{10}, 1, 0\right], 0\right\} = 0$$

$$\mu_{B_{HIGH}}(y=55) = \max\left\{\min\left[\frac{55-45}{55-45}, 1, \frac{100-55}{100-100}\right], 0\right\} = \max\left\{\min\left[\frac{10}{10}, 1, \infty\right], 0\right\} = 1$$

Tsukamoto Fuzzy Model Example



The intersection of given input value $x = 65$ with input fuzzy regions of $\mu_A(x)$ will be:

$\mu_{A_{\text{MEDIUM}}}(x = 65) = 0.75$

$\mu_{A_{\text{HIGH}}}(x = 65) = 0.25$

The intersection of given input value $y = 55$ with input fuzzy regions of $\mu_B(y)$ will be:

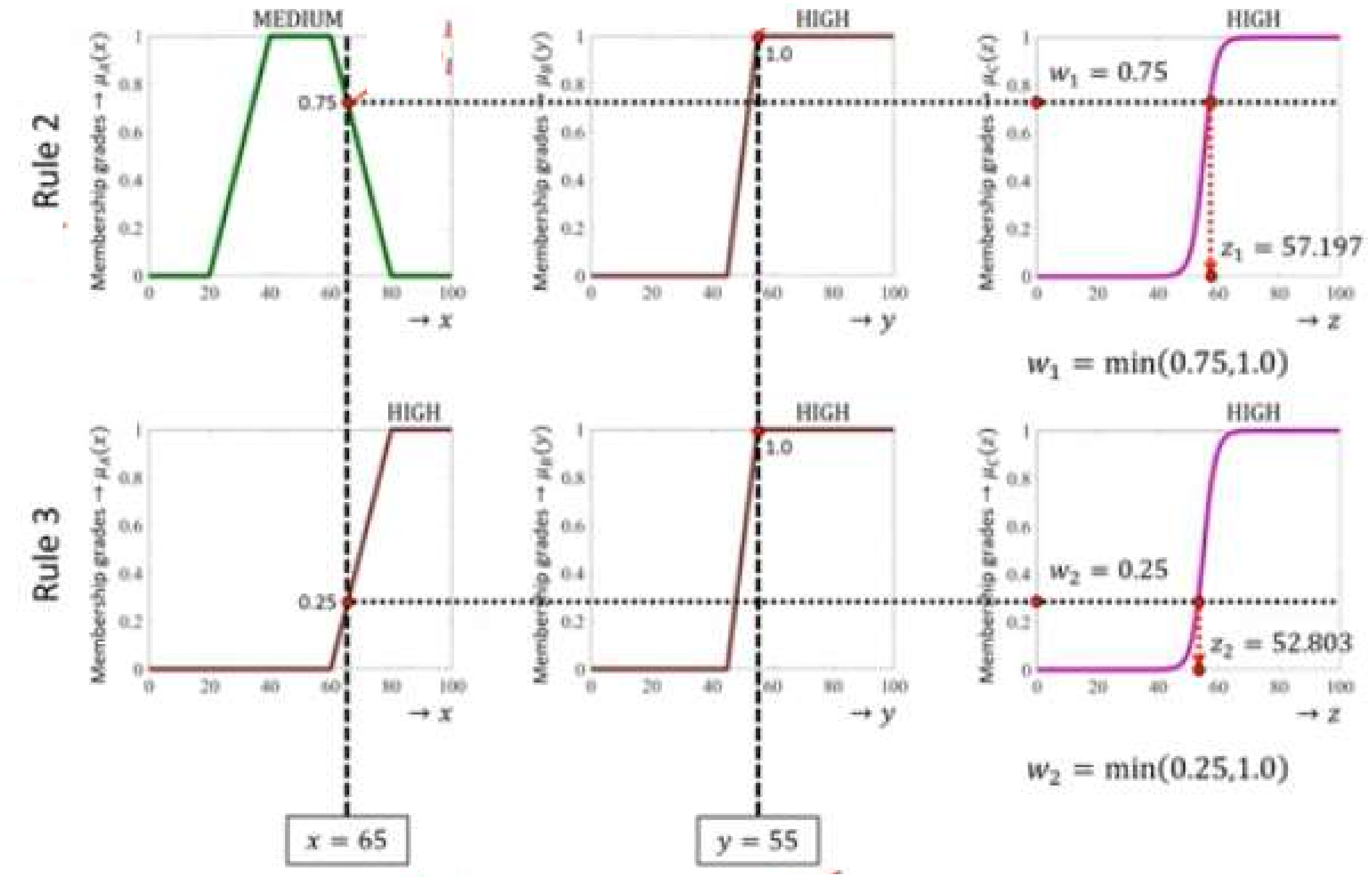
$\mu_{B_{\text{HIGH}}}(y = 55) = 1$

The combination of fuzzy rules obtained are:

Input A	Input B	Output C
MEDIUM	HIGH	HIGH
HIGH	HIGH	HIGH

AND	Input A	Input B	Output C
Rule 1	LOW	MEDIUM	LOW
Rule 2	MEDIUM	HIGH	HIGH
Rule 3	HIGH	HIGH	HIGH

Tsukamoto Fuzzy Model Example



Tsukamoto Fuzzy Model Example

$$w_1 = \mu_{C_{\mathrm{HIGH}}}(z_1) = 0.75$$

$$w_2 = \mu_{C_{\mathrm{HIGH}}}(z_2) = 0.25$$

$$z_1 = 57.197$$

$$z_2 = 52.803$$

$$z^* = \frac{w_1 \times z_1 + w_2 \times z_2}{w_1 + w_2}$$

$$z^* = \frac{0.75 \times 57.197 + 0.25 \times 52.803}{0.75 + 0.25}$$

$$z^* = \frac{56.0985}{1}$$

$$z^* = 56.0985$$