Problem 1 :

1. Min - Cost Flow :

consider the graph $G(V, E, d, w)$,

where $d(v)$ = demand of the node,

$W(e)$ = cost of the edge

$c(e)$ = Capacity of edge e

Demand constraint : $f^{in}(v) - f^{out}(v) = d(v)$

{Here f is the value of flow}.

Variable : f_e : The value of flow on edge $e \in E$

(Note: Cost of flow on edge e = $f_e . w_e$)

Objective :- $\min \sum_{e \in E} f_e . w_e$

subject to :

$$f^{in}(v) - f^{out}(v) = d(v)$$

$$0 \leq f_e \leq c_e, \forall e \in E$$

This is an LP since objective fns & constraints are both linear fns.

Multigraph Reduction :

- If an edge (u, v) already exists in the graph, then we add 3 edges and 2 nodes: Nodes :- u^+, v^-

  Edges :- (u, u^+), (u^+, v^-), (v^-, v)

- The new nodes each have demand $0 \Rightarrow f^{in}(v) = f^{out}(v)$
- The cost are as follows :- $W(u, u^+) = 0$

  $W(u^+, v^-) = W(u, v)$ {cost of edge u, v}

  $W(v^{-1}, v) = 0$

- The Capacities are as follows :- $c(u, u^+) = \infty$ (or $c(u, v)$)

  $c(u^+, v^-) = c(u, v)$

  $c(v^{-1}, v) = \infty$ [or $c(u, v)$)

Note this transformation is efficient, since the reduced graph has extra k nodes and $2k$ edges; where k is the number of multiple paths existing in the graph. for a small k

Equivalence of the 2 problems:

($\Leftarrow$) The reduced graph satisfies all condition of the min-cost flow problem:

1) For edge (U^+, V^-); let the capacity be c then a flow flow $f \le c$ flows between U and V because demand (U^+) & demand $(V^-) = 0$

Thus the flow through the path U, U^+, V^-, V^+ would be the same even if it had been a single edge (U, V)

2) Cost of edge (U, U^+) & $(V^-, V) = 0$

$\Rightarrow$ cost of path $(U, U^+, V^-, V) =$ cost (U, V)

3) The direction of flow is maintained by the way the graph is constructed

The above 3 points imply that the solution to the min-cost flow of the reduced graph is same as that of the original graph.

Or generally, if there exists a feasible flow of cost W' in the reduced graph, then there exists a feasible flow of cost W in graph G, such that $W = W'$.

($\Rightarrow$) If there exists a flow of cost W in G then there exists a feasible flow of cost W' in G', such that $W = W'$. Construction ensures this implication because of 0 cost edges and augmented nodes with 0 demand.

Problem 2:

Decision Version of the problem:- Given a ~~decis~~ graph G and an target value k, an instance is said to belong to DS(D) iff G contains a dominating set of size atmost k. {DS(D) = decision problem for dominant set}

To show NP-completeness:

1. DS(D) $\in$ NP

NP is the set of decision problems that have an efficient certifier. (small)

For a certifier given, equal to the size of the target value that gives yes as an answer for the decision problem DS(D), we can use a polynomial-time algorithm to check if the certifier is a dominant set or not.

Algorithm:

For each node $u \in G$, check if either u or one of its neighbours belongs to the certifier set.

Time complexity

Let no. of nodes in $G = n$ & size of certifier $= p$

$\therefore$ Time Complexity $= O(np)$ where p is small (poly. in n).

2. DS(D) $\leq_p$ Y where Y is NP-complete.
Y $\leq_p$ DS(D)

Reduction Vertex-cover decision problem to dominating set decision problem:

Let G be the input graph for vertex cover problem, then we can modify it to graph G' as mentioned in the hint: {For an edge $e = (u, v)$, introduce a new node x_e and 2 edges (u, x_e) and (x_e, v)}

Thus, ~~we can s~~ if we can show that G has a vertex cover of size k iff G' has a dominant set of size k, this will prove that DS(D) is np-complete.

Equivalence of the 2 decision problems:

① Let a set A be the vertex cover of G.

Then, for any edge (u,v),

- either only u is in A
- only v is in A
- Both u & v are in A.

This ensures that edge $e=(u,v)$ is covered by the vertex cover.

A is the dominating set for G'.

Pf:-

- $x_e \notin A$ but either $u \in A$ or $v \in A$ and $(x_e, u) \in E$, $(x_e, v) \in E$
- For the original graph G, if any of the nodes say $u \notin A$ then $v \in A$, $(u, v) \in E$

⇒ if A is a vertex cover of G then G' has a dominating set of same size.

② Let a set B be the dominant set of G'. then G has a vertex cover of size less than or equal to the size of B.

- If $x_e \notin B$ then either u or v or both lie in B
- If $x_e \in B$ then either of node u or v can replace x_e in the dominating set still ensuring that all nodes are dominated.

The above 2 points ensure that atleast one node of each edge of the graph $G \in B'$ (The modified dominating set).

B' is the vertex cover for graph G. Hence given a dominating set of size K' of Graph G', there exists a vertex cover of G of size atmost K'.

PROBLEM 3

(a) Assignment Problem :

Variables : $x_{ij} = \begin{cases} 1 & \text{if person } i \text{ is assigned object } j \\ 0 & \text{o/w.} \end{cases}$

The problem statement implies that :-

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \& \quad \sum_{j=1}^{n} x_{ij} = 1$$

Hence, the LP is :-

$$\max \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \; ; \forall j$$

$$\sum_{j=1}^{n} x_{ij} = 1 \; ; \forall i$$

$$x_{ij} \in \{0, 1\}$$

This is an Integer Problem since all constraints are linear or integer-valued. and objective function is linear.

This can be relaxed to an LP because the integer constraints can be converted to linear constraints :-

$$\max \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_{ij}$$

s.t.

$$\sum_{i=1}^{n} x_{ij} = 1 \; ; \forall j \in \{1, \ldots, n\}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \; , \forall i \in \{1, \ldots, n\}$$

$$x_{ij} \geq 0 \quad \forall i, j \in \{1, \ldots, n\}, \{1, \ldots, n\}$$

2. <u>Uncapacitated facility location</u>: $F = \{\tilde{f}_1, \ldots, \tilde{f}_m\}$, $D = \{\tilde{d}_1, \ldots, \tilde{d}_n\}$

Variables:- $x_i = \begin{cases} 1 & \text{if } \tilde{f}_i \text{ is opened} \\ 0 & \text{o/w.} \end{cases}$

$y_{ij} = \begin{cases} 1 & \text{if } \tilde{f}_i \text{ is assigned to } \tilde{d}_j \\ 0 & \text{o/w.} \end{cases}$

Prob. statement implies that :- $\sum_{i=1}^{m} y_{ij} = 1$ for all $j \in \{1, \ldots, n\}$

<u>Hence optimisation</u>:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij} \cdot c_{ij} + \sum_{i=1}^{m} x_i \cdot f_i$$

subject to:-

$$\sum_{i=1}^{m} y_{ij} = 1 \; ; \forall j \in \{1, \ldots, n\}.$$

$$x_i - y_{ij} \geq 0 \quad \forall i \in \{1, \ldots, m\} \ \& \ \forall j \in \{1, \ldots, n\}$$

$$x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, m\}$$

$$y_{ij} \in \{0, 1\} \ \forall i \in \{1, \ldots, m\} \ \& \ \forall j \in \{1, \ldots, n\}$$

<u>Relaxed Linear Problem</u>:-

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij} c_{ij} + \sum_{i=1}^{m} x_i \cdot f_i$$

subject to

$$\sum_{i=1}^{m} y_{ij} = 1 \quad ; \forall j \in \{1, \ldots n\}$$

$$x_i - y_{ij} \geq 0 \quad ; \forall i \in \{1, \ldots, m\} \ \& \ \forall j \in \{1, \ldots, n\}$$

$$y_{ij} \geq 0 \quad ; \forall i \in \{1, \ldots m\} \ \& \ \forall j \in \{1, \ldots, n\}$$

$$x_{ij} \geq 0 \quad ; \forall i \in \{1, \ldots, m\}.$$

This is an LP because objective fn and constraints are all linear fns.

(c) Bin Packing:

As given in the hint, an m-tuple $(t_1, \ldots, t_m)$ is a config if $\sum t_i s_i \leq 1$

Let $T_1, T_2, \ldots, T_N$ be the complete enumeration of all possible configurations. Then a config T_i has T_{ij} peices of size s_j.

Variable = $x_i = \begin{cases} 1 & \text{if bin } i \text{ used} \\ 0 & \text{o/w} \end{cases}$

Let the number of pieces of size $s_j = u_j$

Constraint :- $\sum_{i=1}^{N} T_{ij} = u_j$

Optimisation Problem:

$$\min \sum_{i=1}^{N} x_i$$

subject to:

$$\sum_{i=1}^{N} T_{ij} = u_j \quad ; \forall j \in \{1, \ldots, m_i\} \ \& \ \forall i \in \{1, \ldots N\}$$

$$\sum_{j=1}^{m_i} T_{ij} s_j \leq 1 \quad \forall i \in \{1, \ldots, N\}$$

$$x_i \in \{0, 1\} \quad \forall i$$

Relaxation:

$$\min_{x_i \geq 0} \sum_{i=1}^{N} x_i$$

s.t.

$$\sum_{i=1}^{N} T_{ij} = u_j \quad ; \forall j = 1, \ldots, m_i \ \& \ \forall i = 1, \ldots, N$$

$$\sum_{j=1}^{m_i} T_{ij} s_j \leq 1 \quad \forall i \in \{1, \ldots, N\}$$

Problem 4:

(a) $W_{ij} = \min_{t \in T_i} \{ p_j^t - p_i^t \}$

Constraints :-

$v_i - p_i^t \geq v_j - p_j^t \quad ; \forall i \in T_i$
$\quad \& \forall j \in V$

This constraint holds for all $i \in V$

$\Rightarrow \quad v_i - p_i^t \geq v_j - p_j^t \quad ; \forall i \in T_i$
$\quad \forall i \in V \ \& \ \forall j \in V$

$\Rightarrow \quad p_j^t - p_i^t \geq v_j - v_i \quad ; \forall i \in T_i, \ \forall i, j \in V$

$\Rightarrow \quad \min_{t \in T_i} \{ p_j^t - p_i^t \} \geq v_j - v_i \quad ; \forall i \in V \ \& \ \forall j \in V$

$\Rightarrow \quad \boxed{v_j - v_i \leq w_{ij} \quad ; \forall i \in V \ \& \ \forall j \in V}$

b) v' and v'' are feasible solutions to the constraints

$$\Rightarrow \quad v'_j - v'_i \leq w_{ij} \quad \forall ij$$

$$\text{and } v''_j - v''_i \leq w_{ij} \quad \forall i,j$$

Let h define $(v' \vee v'') = (\max\{v_1', v_1''\}, \ldots, \max\{v_n', v_n''\})$
$= (h_1, h_2, \ldots, h_n)$

There can be 4 cases:

Case 1. $v_j' \geq v_j''$ & $v_i' \geq v_i''$

and $v_j' - v_i' \leq w_{ij}$ (as given); $\forall i,j$

$\Rightarrow h_j - h_i \leq w_{ij} \quad \forall i,j$

Case 2. $v_j'' \geq v_j'$ & $v_i'' \geq v_i'$

and $v_j'' - v_i'' \leq w_{ij}$ (given) ; $\forall i,j$

$\Rightarrow h_j - h_i \leq w_{ij} \quad \forall i,j$

Case 3. $v_j' \geq v_j''$ & $v_i'' > v_i'$

$v_j' - v_i'' \leq v_j' - v_i' \leq w_{ij} \quad \forall i,j$

$\Rightarrow h_j - h_i \leq w_{ij} \quad \forall i,j$

Case 3. $v_j'' \geq v_j'$ & $v_i' \geq v_i''$

$v_j'' - v_i' \leq v_j'' - v_i'' \leq w_{ij} \quad \forall i,j$

$\Rightarrow h_j - h_i \leq w_{ij} \quad \forall i,j$

Hence h is also a valid solution for the constraints

(c) <u>Variables</u>: $v_i \quad \forall i \in \{1, \dots, n\}$

Constraints: $v_j - v_i \le w_{ij} \quad \forall ij \in \{1, \dots, n\}$

<u>Linear Program</u>:

(Assuming that $v_i \ge 0$)

$$\max_{v_i \ge 0} \sum_{i=1}^{n} v_i$$

subject to:

$$v_j - v_i \le w_{ij} \quad \forall i \in \{1, \dots, n\}$$
$$\& \ \forall j \in V$$

This is an LP because both objective functions and constraints are linear.

Reformulating the LP:

$$\min_{v_i \ge 0} - \sum_{i=1}^{n} v_i$$

s.t.

$$v_j - v_i \le w_{ij} \quad \forall i \in V \ \& \ \forall j \in V$$

<u>Dual Formulation</u>:

$$\max_{\lambda_{ij} \ge 0} \min_{v_i \ge 0} - \sum_{i=1}^{n} v_i + \sum_{i \in V} \sum_{j \in V} \lambda_{ij} (v_j - v_i - w_{ij})$$

$$\max_{\substack{\lambda_{kl} \ge 0 \\ \forall k \in V \\ \& l \in V}} \min_{\substack{v_i \ge 0 \\ i \in V}} \sum_{i \in V} v_i \left(-1 + \sum_{k \in V} \lambda_{ki} - \sum_{l \in V} \lambda_{il}\right) + (-1) \sum_{k \in V} \sum_{l \in V} \lambda_{kl} \cancel{w_{ij}} w_{kl}$$

$$\max_{\substack{\lambda_{kl} \ge 0 \\ \forall k \in V \\ \& l \in V}} - \sum_{k \in V} \sum_{l \in V} \lambda_{kl} w_{kl}$$

s.t.

$$v_i \left(\sum_{k \in V} \lambda_{ki} - \sum_{l \in V} \lambda_{il} - 1 \right) \ge 0$$

$$\forall i \in V$$

Hence, the dual problem is :-

$$\min_{\substack{\lambda_{kl} \geq 0 \\ \forall k \in V \,\&\, l \in V}} \sum_{k \in V} \sum_{l \in V} \lambda_{kl} W_{kl}$$

Subject to:-

$$v_i \left(\sum_{k \in V} \lambda_{ki} - \sum_{l \in V} \lambda_{il} - 1 \right) \geq 0 \, ;$$

$$\forall i \in V$$

(d) <u>Min-cut Problem</u> :

max

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