Multivariable calculus

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Chapter 1

Derivatives

1.1 Derivatives in several variables

1.1.1 Partial Derivatives

Definition 1.1. Partial derivative A partial derivative is differentiating a function with respect to only one variable.

$$\frac{\partial f}{\partial a_i} = \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

Remark. Just imagine every other a_k where $k \in \{1, \ldots, n\}$ is a constant.

Example. Let

$$f(x, y) = x^2 + x^3y^2 + y^{78}$$

$$\frac{\partial f}{\partial x} = 2x + 3x^2y^2 + 0$$

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Example. Let

$$f(a, b) = a\sin(b) + b^2$$

$$\frac{\partial f}{\partial b} = a\cos(b) + 2b$$

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1.1.2 Partial derivatives in $\mathbb{R}^n$

Just evaluate the derivative at each value

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$f(x, y) = (x^2 y, \cos(y))$$

$$\frac{\partial f}{\partial y} = (x^2, -\sin(y))$$

$$\frac{\partial f}{\partial x} = (2xy, 0)$$

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1.1.3 The derivative in several variables

Before we start differentiating, it is helpful to note that the previous definition(s) of derivatives will not work for all dimensions, thus, we come up with a generalization

Definition 1.2. Derivative (Alternate) f is diffentiable at α with the derivative β if and only if

$$\lim_{h \to 0} \frac{(f(a+h) - f(a)) - (\beta\alpha)}{h}$$

1.1.4 Jacobian Matrix

Definition 1.3. Jacobian Matrix Let $\mathbf{S}$ be a subset of $\mathbb{R}^n$.
The Jacobian matrix of $f : S \to \mathbb{R}^m$ is an $m \times n$ matrix of the partial derivatives of f at α

$$\left|\mathbf{J}f(\alpha)\right| = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\alpha) & \dots & \frac{\partial f_1}{\partial x_n}(\alpha) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\alpha) & \dots & \frac{\partial f_m}{\partial x_n}(\alpha) \end{pmatrix}$$

Example. Let $f(x, y) = (x^3 y, 2x^2 y^2, xy)$

$$\begin{pmatrix} 3x^2 y & x^3 \\ 4xy^2 & 4x^2 y \\ y & x \end{pmatrix}$$

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Because of this, we have to update our derivative definition

Definition 1.4. Derivative Let $\mathbf{S} \subset \mathbb{R}^n$ and let $f : \mathbf{S} \to \mathbb{R}^m$. Let α be a point in $\mathbf{S}$. If $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$, such that

$$\lim_{\vec{h} \to 0} \frac{(f(a+h) - f(a)) - \mathbf{L}(\vec{h})}{\vec{h}} = \vec{0}$$

Then f is differentiable at α, $\mathbf{L}$ is the unique derivative of f, which is denoted $\big[Df(\alpha)\big]$

Theorem 1.5. Let $\mathbf{S} \subset \mathbb{R}^n$ and let $f : \mathbf{S} \to \mathbb{R}^m$. If f is differentiable at α, then all first order partial derivatives of f at α exist, and $\big[Df(\alpha)\big]$ is $\big[Jf(\alpha)\big]$

Proof. First, note that the derivative must be L, so we need to prove that

$$L(\vec{s_i}) = \vec{D_i} f(a)$$

Now, we can make the limit of a random scalar c approach 0, which means $c\vec{s_i}$ approaches 0

$$\lim_{c\vec{s_i} \to 0} \frac{f(a + c\vec{s_i}) - f(a) - L(c\vec{s_i})}{|cs_i|} = 0$$

$|c\vec{s_i}| = |c||\vec{s_i}|$ and $|\vec{s_i}|$ is equal to 1 (standard basis) so it is just $|c|$. Also since its approaching 0, c can be positive or negative

$$\lim_{c\vec{s_i} \to 0} \frac{f(a + c\vec{s_i}) - f(a) - L(c\vec{s_i})}{c} = 0$$

Since L is a linear transformation, $L(c\vec{s_i}) = cL(\vec{s_i})$

$$\lim_{c\vec{s_i} \to 0} \frac{f(a + c\vec{s_i}) - f(a)}{c} - L(\vec{s_i}) - = 0$$

This is the exact formula for the partial derivative $\square$

Definition 1.6. gradient

$$grad = \begin{pmatrix} D_1 f(\alpha) \\ \vdots \\ D_n f(\alpha) \end{pmatrix}$$

1.1.5 Directional derivatives

Definition 1.7. Directional derivatives Directional derivatives are simply a generalization of partial dervatives

$$\lim_{h\to 0} \frac{f(a+h\vec{v}) - f(a)}{h}$$

Proposition 1.8. If $U \subset \mathbb{R}^n$, and $f : U \to \mathbb{R}^m$ is differentiable at a point $\alpha \in U$, then all directional derivatives of f at α exist, and the directional derivative is equal to $\vec{v}\left[Df(a)\right]$

Proof. Let $g(\vec{h}) = (f(a+\vec{h}) - f(a)) - \vec{h}\left[Df(a)\right]$ Subsititute $h\vec{v}$ for $\vec{h}$ in g

$$\frac{g(h\vec{v})}{1} = f(a+h\vec{v}) - f(a) - h\vec{v}\left[Df(a)\right]$$

Since we know that f is differentiable at α

$$\lim_{h\to 0} \frac{g(h\vec{v})}{h|\vec{v}|} = 0$$

So divide everything by h (Note that the $|\vec{v}|$ cancels out on the LHS)

$$\frac{|\vec{v}|g(h\vec{v})}{h|\vec{v}|} = \frac{f(a+h\vec{v}) - f(a)}{h} - \vec{v}\left[Df(a)\right]$$

Since the LHS approaches 0 as $h \to 0$

$$\frac{f(a+h\vec{v}) - f(a)}{h} = \vec{v}\left[Df(a)\right]$$

□

Example. Let $S \subset \mathbb{R}^2$ and $f : S \to \mathbb{R}^3$ where $f(x, y) = (x + y, y^2 x, x^3)$. Find the directional derivative at $(3, 4)$ with the vector $\vec{v} = (2, 1)$
First, find the matrix of the derivative

$$\big(Df(x, y)\big) = \begin{pmatrix} 1 & 1 \\ y^2 & 2yx \\ 3x^2 & 0 \end{pmatrix}$$

So

$$\big(Df(3, 4)\big) = \begin{pmatrix} 1 & 1 \\ 16 & 24 \\ 27 & 0 \end{pmatrix}$$

Then, multiply by $\vec{v}$

$$\begin{pmatrix} 1 & 1 \\ 16 & 24 \\ 27 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 56 \\ 54 \end{pmatrix}$$

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Remark. While finding the Jacobian matrix is faster and easier, in cases where we are dealing with complex spaces and fields, it is best to use the limit definition in **1.7** and **1.4** as it allows you to compute the derivative safely.