# Multivariable calculus

Vaibhav Venkat

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# Chapter 1

# Derivatives

## 1.1 Derivatives in several variables

### 1.1.1 Partial Derivatives

**Definition 1.1.** Partial derivative A partial derivative is differentiating a function with respect to only one variable.

$$\frac{\partial f}{\partial a_i} = \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

**Remark.** Just imagine every other $a_k$ where $k \in \{1, \ldots, n\}$ is a constant.

**Example.** Let

$$f(x, y) = x^2 + x^3y^2 + y^{78}$$

$$\frac{\partial f}{\partial x} = 2x + 3x^2y^2 + 0$$

◇

**Example.** Let

$$f(a, b) = a\sin(b) + b^2$$

$$\frac{\partial f}{\partial b} = a\cos(b) + 2b$$

◇

### 1.1.2 Partial derivatives in $\mathbb{R}^n$

Just evaluate the derivative at each value

**Example.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$f(x, y) = (x^2 y, \cos(y))$$

$$\frac{\partial f}{\partial y} = (x^2, -\sin(y))$$

$$\frac{\partial f}{\partial x} = (2xy, 0)$$

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### 1.1.3 The derivative in several variables

Before we start differentiating, it is helpful to note that the previous definition(s) of derivatives will not work for all dimensions, thus, we come up with a generalization

**Definition 1.2.** Derivative (Alternate) $f$ is diffentiable at $\alpha$ with the derivative $\beta$ if and only if

$$\lim_{h \to 0} \frac{(f(a+h) - f(a)) - (\beta\alpha)}{h}$$

### 1.1.4 Jacobian Matrix

**Definition 1.3.** Jacobian Matrix Let $\mathbf{S}$ be a subset of $\mathbb{R}^n$.
The Jacobian matrix of $f : S \to \mathbb{R}^m$ is an $m \times n$ matrix of the partial derivatives of $f$ at $\alpha$

$$\left|\mathbf{J} f(\alpha)\right| = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\alpha) & \dots & \frac{\partial f_1}{\partial x_n}(\alpha) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\alpha) & \dots & \frac{\partial f_m}{\partial x_n}(\alpha) \end{pmatrix}$$

**Example.** Let $f(x, y) = (x^3 y, 2x^2 y^2, xy)$

$$\begin{pmatrix} 3x^2 y & x^3 \\ 4xy^2 & 4x^2 y \\ y & x \end{pmatrix}$$

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Because of this, we have to update our derivative definition

**Definition 1.4.** Derivative Let **S** $\subset$R$^n$ and let $f : S \to \mathbb{R}^m$. Let $\alpha$ be a point in **S**. If $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$, such that

$$\lim_{\vec{h} \to 0} \frac{(f(a+h) - f(a)) - \mathbf{L}(\vec{h})}{\vec{h}} = \vec{0}$$

Then $f$ is differentiable at $\alpha$, **L** is the unique derivative of $f$, which is denoted $f'$