CSORW4231 HOMEWORK 2

Due Mon, Feb 20

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Problem 1. Exercise 4.3-7 (Page 87).

Solution.
If set $T(n) \leq cn^{\log_3 4}$, for substitution: $\exists\ n > 0,\ c > 0, \forall\ n > n_0$, s.t.

$$\begin{aligned} T(n) &= 4T(\frac{n}{3}) + n \\ &\leq 4(c(\frac{n}{3})^{\log_3 4}) + n \\ &\leq cn^{\log_3 4} + n \\ &\nleq cn^{\log_3 4} \end{aligned}$$

Failed.

Let $T(n) \leq c_1 n^{\log_3 4} - bn$, $\exists\ n > 0,\ c_1 > 0, \forall\ n > n_0$,

$$\begin{aligned} T(n) &= 4T(\frac{n}{3}) + n \\ &\leq 4(c_1(\frac{n}{3})^{\log_3 4}) - \frac{4b}{3}n + n \\ &\leq c_1 n^{\log_3 4} - \frac{4b-3}{3}n \\ &\leq c_1 n^{\log_3 4} - bn \end{aligned}$$

Let $b \geq 3$, the assumption holds.

□

Problem 2. Exercise 4.4-4 and 4.4-5 (Page 93) and Problem 4-3. (a), (c), (e), (j) (Page 108).

Solution.

4.4-4 Draw recursion tree: (Figure 1)

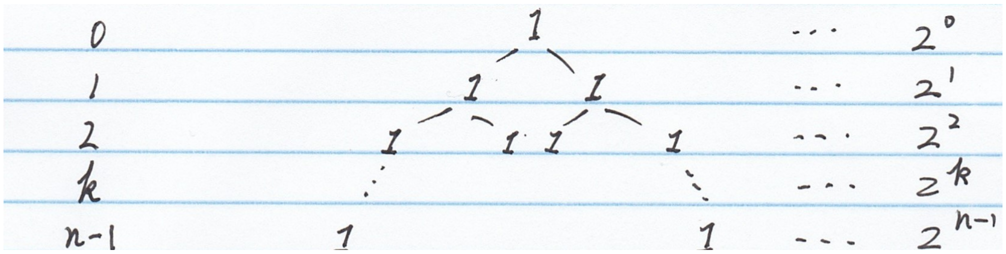

Figure 1: Recursion Tree of 4.4-4

So

$$T(n) = \sum_{k=0}^{n-1} 2^k = \frac{2^n - 1}{2 - 1} = 2^n - 1 = O(2^n)$$

Substitution verification: Let $T(n) \le 2^n - 1$,

$$T(n) \le 2 \cdot (2^{n-1} - 1) + 1 = 2^n - 1$$

4.4-5 Let $R(n) = T(\frac{n}{2}) + n$, rename $n = 2^m$, s.t., $S(m) = T(n) = T(2^m)$, change variables:

$$\begin{aligned}
S(m) &= T(n) \\
&= T(n-1) + R(n) \\
&= T(n-2) + R(n-1) + R(n) \\
&= R(1) + R(2) + \cdots + R(n-1) + R(n) \\
&= \sum_{i=1}^{n} (T(\frac{i}{2}) + i) \\
&= \sum_{i=1}^{n} T(\frac{i}{2}) + \frac{n(n+1)}{2} \\
&\le nT(\frac{n}{2}) + (n+1)T(\frac{n}{2}) \qquad \leftarrow \qquad (T(\frac{n}{2}) \ge T(\frac{n-1}{2}),\ T(\frac{n}{2}) \ge \frac{n}{2}) \\
&\le 2(n+1)T(\frac{n}{2}) \\
&= 2(2^m + 1)S(m-1) \\
&\le 2 \cdot 2^{m+1} S(m-1)
\end{aligned}$$

Let $\lg(S(m)) = Q(m)$, s.t.,

$$\begin{aligned}
Q(m) &\le 2 \cdot 2^{m+1} S(m-1) \\
&= \lg(2 \cdot 2^{m+1} S(m-1)) \\
&= \lg 2 + \lg 2^{m+1} + \lg(S(m-1)) \\
&= 1 + m + 1 + Q(m-1) \\
&= Q(m-1) + m + 2
\end{aligned}$$

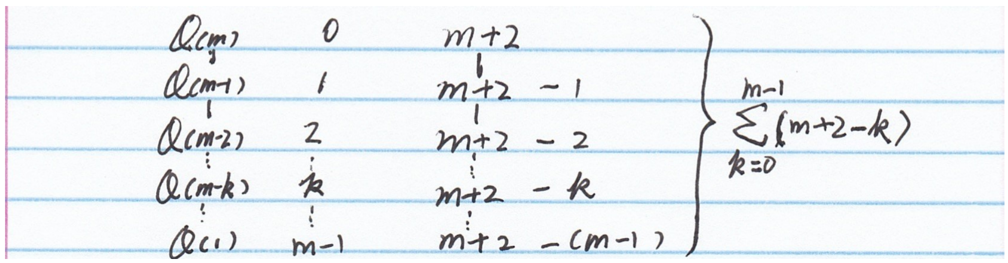

Figure 2: Recursion Tree of 4.4-5

Draw recursion tree: (Figure 2)

Guess $Q(m) = O(m^2)$, which can be verified by substitution: $\exists\ c > \frac{1}{2},\ \forall\ m \geq \frac{c+2}{2c-1}$, s.t., $Q(m) \leq cm^2$

$$
\begin{aligned}
Q(m) &\leq Q(m-1) + m + 2 \\
&= c(m-1)^2 + m + 2 \\
&\leq cm^2
\end{aligned}
$$

The $S(m)$ can be calculated from $Q(m)$:

$$S(m) = O(2^{m^2})$$

Then,

$$T(n) = S(\lg n) = O(2^{\lg^2 n})$$

4-3 a. By Master theorem $a = 4,\ b = 3,\ \epsilon > 0,\ f(n) = O(n \lg n) = O(n^{\log_3 4 - \epsilon})$, which applies case 1:

$$T(n) = \Theta(n^{\log_3 4})$$

4-3 c. By Master theorem $a = 4,\ b = 2,\ \epsilon > 0,\ f(n) = n^2\sqrt{n} = \Omega(n^{2.5}) = \Omega(n^{\log_2 4 + \epsilon})$, which applies case 3:

$$T(n) = \Theta(n^{2.5})$$

4-3 e. Draw recursion tree: (Figure 3)

Get

$$T(n) = \sum_{k=0}^{\lg n - 1} \frac{n}{\lg \frac{n}{2^k}} = O(n \lg\lg n)$$

4-3 j. Draw recursion tree: (Figure 4)

So totally $(\lg\lg n)$ levels, and each level has (n) actions, s.t.,

$$T(n) = O(n \lg\lg n)$$

$\square$

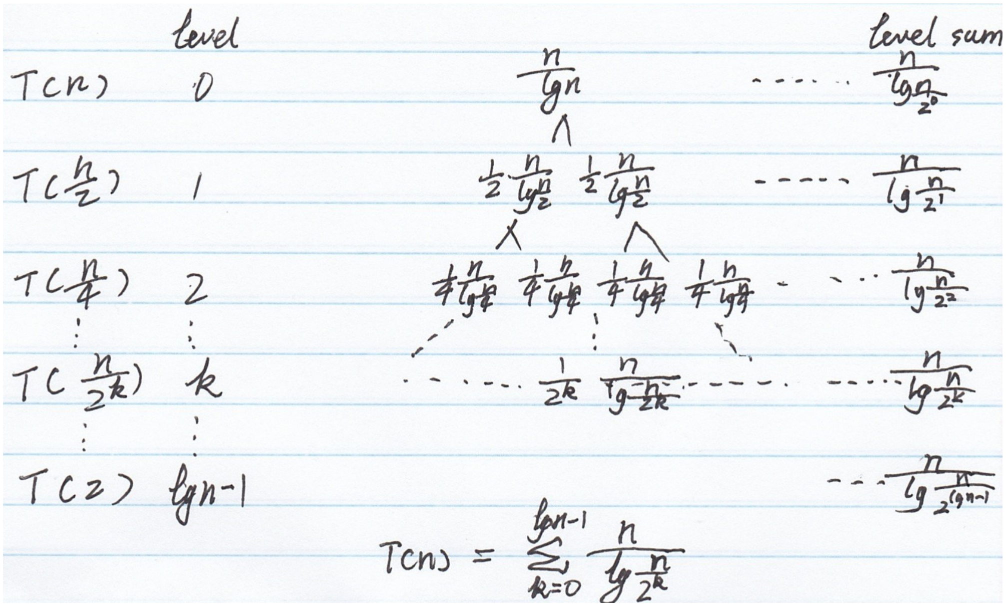

Figure 3: Recursion Tree of 4-3 e.

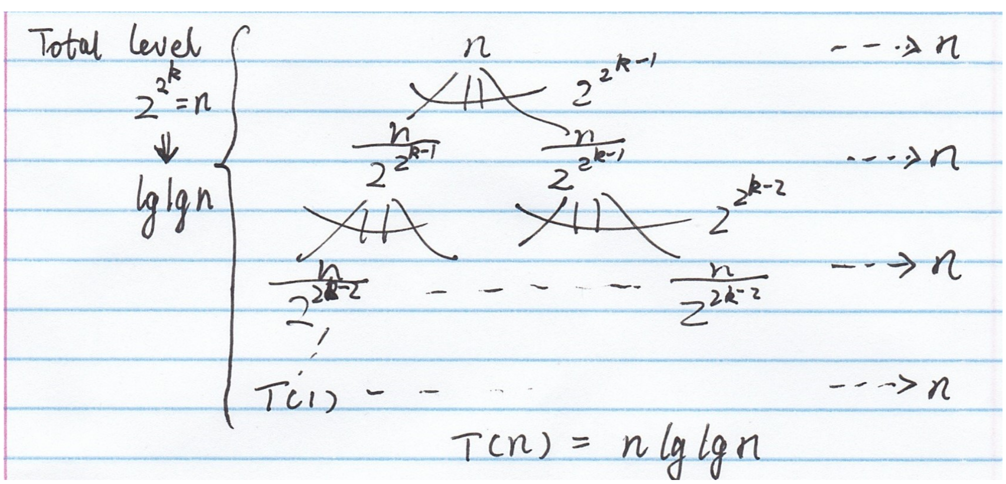

Figure 4: Recursion Tree of 4-3 j.

Problem 3. Problem 4-5 (b): Chip testing.

Solution.

Pair Pair all the chips into $\lfloor \frac{n}{2} \rfloor$ pairs. May result in 1 unpaired chip if n is odd.

Test Test each pair.
For the pairs of "both are good", remain one, discard the other one.
For other pairs, discard.
For the unpaired one, remain.

Repeat Repeat above for the remained chips, until only one remained.
The one is a good one.

Analysis Among all the pairs, there are 3 combinations for chip conditions:

$$x = number\ of \text{ pair: "good + good"}$$
$$y = number\ of \text{ pair: "good + bad"}$$
$$z = number\ of \text{ pair: "bad + bad"}$$

Except for 1 unpaired chip, if any.
The good chips number g(n), bad chips number b(n) s.t.,

$$\begin{cases} g(n) = 2x + y & (1) \\ b(n) = 2z + y & (2) \end{cases}$$

By our algorithm, only $x + z$ (even) or $x + z + 1$ (odd) are remained.
Apparently, $x + z \leq \frac{1}{2}(2x + 2y + 2z)$ (even) or $x + z + 1 \leq \lceil \frac{1}{2}(2x + 2y + 2z + 1) \rceil$(odd)
Which means by $\lfloor \frac{n}{2} \rfloor$ pairwise tests, the size of problem halved.
Because at the beginning of the problem, good chips are more than $\frac{n}{2}$, s.t.,
For even, equation $(1) - (2)$, remained $x > z$, good chips are still more than $\frac{n}{2}$ in the new problem.
For odd, 1 unpaired is good, $g(n) + 1 > b(n)$, $x + 1 > z$, good chips are still more than $\frac{n}{2}$ in the new problem.
For odd, 1 unpaired is bad, $g(n) > b(n) + 1$, $x > z + 1$, good chips are still more than $\frac{n}{2}$ in the new problem.

□

Problem 4. Exercise 5.2-5 (Page 122) and Exercise 5.4-6 (Page 142): Indicator random variables.

Solution.

5.2-5 For $1 \le i \le j \le n$, let $X_{i,j}$ be an event of pair $i,\ j$, as an inversion of array A,

$$X_{i,j} = \begin{cases} 1, & A[i] > A[j] \\ 0, & A[i] < A[j] \end{cases}$$

Calculate the expectation of total number of inversions X

$$\begin{aligned}
E\left[X\right] &= E\left[\sum_{i,j} X_{i,j}\right] \\
&= \sum_{i,j} E\left[X_{i,j}\right] \\
&= \sum_{i,j} Pr\{X_{i,j} = 1\} \\
&= \sum_{i,j} \frac{1}{2} \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 1 \\
&= \frac{1}{2}\sum_{i=1}^{n-1}(n-i) \\
&= \frac{1}{2}\left(n(n-1) - \frac{n(n-1)}{2}\right) \\
&= \frac{n(n-1)}{4}
\end{aligned}$$

5.4-6 Let X_i be an event of blank bin after n times tosses, Y_j be an event of a bin with exactly one ball after n times tosses., s.t.,

$$\begin{aligned}
Pr\{X_i = 1\} &= \left(1 - \frac{1}{n}\right)^n \\
Pr\{Y_j = 1\} &= \frac{1}{n}\binom{n}{1}\left(1 - \frac{1}{n}\right)^{n-1} = \left(1 - \frac{1}{n}\right)^{n-1}
\end{aligned}$$

For the event numbers X, Y, the expectations are:

$$
\begin{aligned}
E\left[X\right] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E\left[X_i\right] \\
&= \sum_{i=1}^{n} Pr\{X_i = 1\} \\
&= \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^{n} \\
&= n\left(\frac{n-1}{n}\right)^{n}
\end{aligned}
$$

$$
\begin{aligned}
E\left[Y\right] &= E\left[\sum_{i=1}^{n} Y_j\right] \\
&= \sum_{j=1}^{n} E\left[Y_j\right] \\
&= \sum_{j=1}^{n} Pr\{Y_j = 1\} \\
&= \sum_{j=1}^{n} \left(1 - \frac{1}{n}\right)^{n-1} \\
&= n\left(\frac{n-1}{n}\right)^{n-1}
\end{aligned}
$$

$\square$

Problem 5. Problem 7-3 (Page 187): Alternative quicksort analysis.

Solution.

a. Pivot is randomly picked from n length array, so the probability is $\frac{1}{n}$, s.t.,

$$E[X_i] = Pr\{X_i = 1\} = \frac{1}{n}$$

b. If the pivot is at q, then the array will be separated as subarrays with the size of $q-1$ and $n-q$. And this action of partition takes $O(n)$. This is how $E\big[T(n)\big]$ comes to be.

c. Rewrite the equation with explanations:

$$\begin{aligned}
E\big[T(n)\big] &= E\left[\sum_{q=1}^{n} X_q\left(T(q-1)+T(n-q)+\Theta(n)\right)\right] \\
&= \sum_{q=1}^{n} E\Big[X_q\left(T(q-1)+T(n-q)+\Theta(n)\right)\Big] && \leftarrow \quad \text{linearity of expectation} \\
&= \sum_{q=1}^{n} E\big[X_q\big]\cdot E\Big[\big(T(q-1)+T(n-q)+\Theta(n)\big)\Big] && \leftarrow \quad X_q \text{ independence} \\
&= \sum_{q=1}^{n} E\big[X_q\big]\cdot\Big(E\big[T(q-1)\big]+E\big[T(n-q)\big]+E\big[\Theta(n)\big]\Big) \\
&= \sum_{q=1}^{n} \frac{1}{n}\Big(E\big[T(q-1)\big]+E\big[T(n-q)\big]+E\big[\Theta(n)\big]\Big) && \leftarrow \quad \text{from } \mathbf{a.} \\
&= \frac{1}{n}\left(\sum_{q=1}^{n} E\big[T(q-1)\big]+\sum_{q=1}^{n} E\big[T(n-q)\big]+\sum_{q=1}^{n} E\big[\Theta(n)\big]\right) \\
&= \frac{1}{n}\left(\sum_{q=0}^{n-1} E\big[T(q)\big]+\sum_{q=0}^{n-1} E\big[T(q)\big]+n\Theta(n)\right) && \leftarrow \quad q' = (q-1) \text{ or } (n-q) \\
&= \frac{2}{n}\sum_{q=2}^{n-1} E\big[T(q)\big]+\frac{2}{n}\Big(E\big[T(0)\big]+E\big[T(1)\big]\Big)+\Theta(n) \\
&= \frac{2}{n}\sum_{q=2}^{n-1} E\big[T(q)\big]+\Theta(n) && \leftarrow \quad T(0)=T(1)=\Theta(1)
\end{aligned}$$

d. Proof

$$
\begin{aligned}
\sum_{k=2}^{n-1} k \lg k &= \sum_{k=2}^{\lceil \frac{n}{2} \rceil - 1} k \lg k + \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} k \lg k \\
&\le \sum_{k=2}^{\lceil \frac{n}{2} \rceil - 1} k \lg \frac{n}{2} + \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} k \lg n \\
&= \lg \frac{n}{2} \sum_{k=2}^{\lceil \frac{n}{2} \rceil - 1} k + \lg n \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} k \\
&= \lg \frac{n}{2} \left(\frac{\frac{n}{2}(\frac{n}{2} - 1)}{2} \right) + \lg n \left(\frac{n(n-1)}{2} - \frac{\frac{n}{2}(\frac{n}{2} - 1)}{2} \right) \\
&\le \left(\lg \frac{n}{2} \right) \cdot \frac{n^2}{8} + (\lg n) \left(\frac{n^2}{2} - \frac{n^2}{8} \right) \\
&= \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2
\end{aligned}
$$

e. Guess $E\left[T(q)\right] \le cq \lg q, \forall\ 2 \le q < n,\ a > 0, c > 0, n_0 > 0$, s.t.,

$$
\begin{aligned}
E\left[T(n)\right] &= \frac{2}{n} \sum_{q=2}^{n-1} E\left[T(q)\right] + \Theta(n) \\
&\le \frac{2}{n} \sum_{q=2}^{n-1} cq \lg q + an \\
&= \frac{2}{n} c \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + an \\
&= cn \lg n - \left(\frac{c}{4} - a \right) n \\
&\le cn \lg n \qquad \Leftarrow \qquad \forall\ n > n_0,\ \exists\ c \ge 4a
\end{aligned}
$$

$\square$