COMS 4771 Machine Learning
Problem Set #1

Jun Hu - jh3846@columbia.edu
Discussants:

July 12, 2017

Problem 1

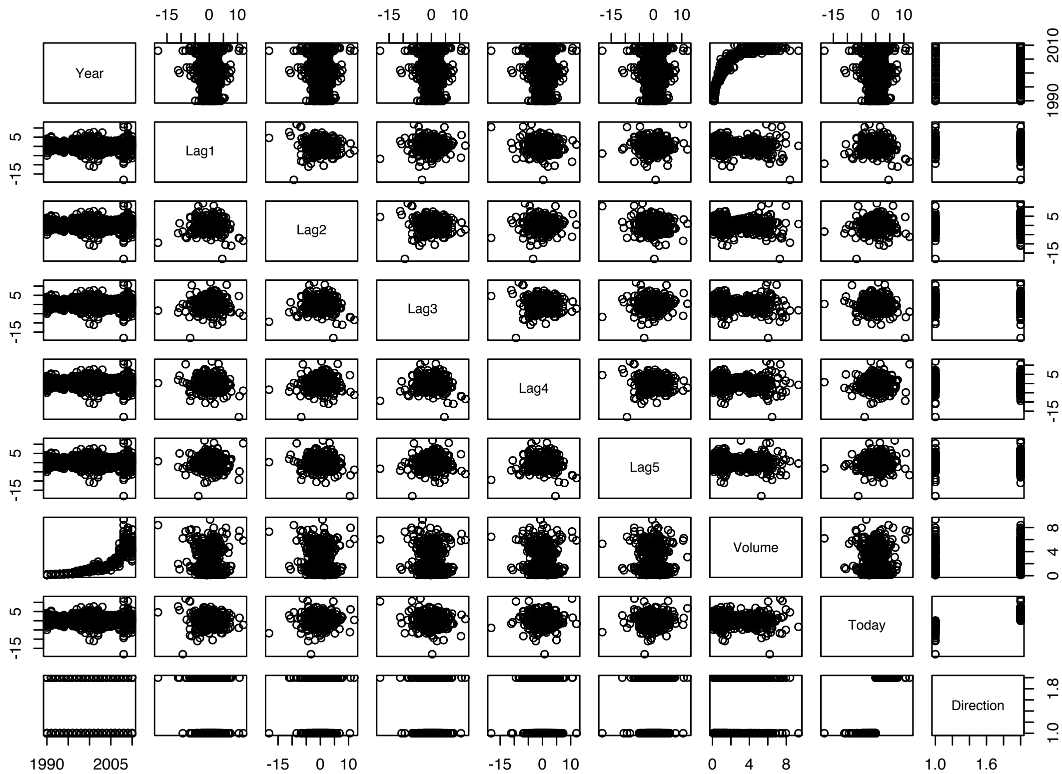
(a) First of all, to numerically summarize, execute:

```r
library(ISLR)
summary(Weekly)
```

```
      Year           Lag1               Lag2               Lag3               Lag4
 Min.   :1990   Min.   :-18.1950   Min.   :-18.1950   Min.   :-18.1950   Min.   :-18.1950
 1st Qu.:1995   1st Qu.: -1.1540   1st Qu.: -1.1540   1st Qu.: -1.1580   1st Qu.: -1.1580
 Median :2000   Median :  0.2410   Median :  0.2410   Median :  0.2410   Median :  0.2380
 Mean   :2000   Mean   :  0.1506   Mean   :  0.1511   Mean   :  0.1472   Mean   :  0.1458
 3rd Qu.:2005   3rd Qu.:  1.4050   3rd Qu.:  1.4090   3rd Qu.:  1.4090   3rd Qu.:  1.4090
 Max.   :2010   Max.   : 12.0260   Max.   : 12.0260   Max.   : 12.0260   Max.   : 12.0260
      Lag5              Volume            Today          Direction
 Min.   :-18.1950   Min.   :0.08747   Min.   :-18.1950   Down:484
 1st Qu.: -1.1660   1st Qu.:0.33202   1st Qu.: -1.1540   Up  :605
 Median :  0.2340   Median :1.00268   Median :  0.2410
 Mean   :  0.1399   Mean   :1.57462   Mean   :  0.1499
 3rd Qu.:  1.4050   3rd Qu.:2.05373   3rd Qu.:  1.4050
 Max.   : 12.0260   Max.   :9.32821   Max.   : 12.0260
```

To graphically summarize, execute:

```r
pairs(Weekly)
```



To observe all pairwise correlations, execute:

```
cor(Weekly[ ,  −9])
```

```
              Year         Lag1        Lag2        Lag3         Lag4         Lag5      Volume
Year    1.00000000 -0.032289274 -0.03339001 -0.03000649 -0.031127923 -0.030519101  0.84194162
Lag1   -0.03228927  1.000000000 -0.07485305  0.05863568 -0.071273876 -0.008183096 -0.06495131
Lag2   -0.03339001 -0.074853051  1.00000000 -0.07572091  0.058381535 -0.072499482 -0.08551314
Lag3   -0.03000649  0.058635682 -0.07572091  1.00000000 -0.075395865  0.060657175 -0.06928771
Lag4   -0.03112792 -0.071273876  0.05838153 -0.07539587  1.000000000 -0.075675027 -0.06107462
Lag5   -0.03051910 -0.008183096 -0.07249948  0.06065717 -0.075675027  1.000000000 -0.05851741
Volume  0.84194162 -0.064951313 -0.08551314 -0.06928771 -0.061074617 -0.058517414  1.00000000
Today  -0.03245989 -0.075031842  0.05916672 -0.07124364 -0.007825873  0.011012698 -0.03307778
              Today
Year   -0.032459894
Lag1   -0.075031842
Lag2    0.059166717
Lag3   -0.071243639
Lag4   -0.007825873
Lag5    0.011012698
Volume -0.033077783
Today   1.000000000
```

We found that there exists a higher correlation of 0.84 between Year and Volume, while others are pretty low. Also from the plot we can recognize there is an increasingly correlated pattern between Year and Volume.

(b) Execute:

```
glm.fit = glm(Direction~., data=Weekly[, c(2:7, 9)], family=binomial)
summary(glm.fit)
```

```
Call:
glm(formula = Direction ~ ., family = binomial, data = Weekly[,
    c(2:7, 9)])

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.6949  -1.2565   0.9913   1.0849   1.4579

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  0.26686    0.08593   3.106   0.0019 **
Lag1        -0.04127    0.02641  -1.563   0.1181
Lag2         0.05844    0.02686   2.175   0.0296 *
Lag3        -0.01606    0.02666  -0.602   0.5469
Lag4        -0.02779    0.02646  -1.050   0.2937
Lag5        -0.01447    0.02638  -0.549   0.5833
Volume      -0.02274    0.03690  -0.616   0.5377
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1496.2  on 1088  degrees of freedom
Residual deviance: 1486.4  on 1082  degrees of freedom
AIC: 1500.4

Number of Fisher Scoring iterations: 4
```

From the summary, it seems Lag2 has statistical significance.

(c) Execute:

```
glm.probs = predict(glm.fit, type = "response")
glm.pred = ifelse(glm.probs>0.5, "Up", "Down")
table(glm.pred, Weekly$Direction)
```

```
glm.pred Down  Up
    Down   54  48
    Up    430 557
```

The correct prediction is: accuracy $= \frac{54+557}{54+430+48+557} = 56.11\%$
So the incorrect prediction is: error rate $= 1 - 56.11\% = 43.89\%$

These numbers tell us the overall accuracy and error of predictions.

The correct market "Down" rate: is: recall $= \frac{54}{54+430} = 11.16\%$
So the incorrect market "Down" rate is: miss $= 1 - 11.16\% = 88.84\%$
The correct market "Up" rate is: specificity $= \frac{557}{48+557} = 92.07\%$
So the incorrect market "Up" rate is: miss $= 1 - 92.07\% = 7.93\%$
These numbers tell us when the market goes up, the model can predict correctly in a very high rate, but quite the contrary when the market goes down.

(d) Execute:

```
attach(Weekly)
train = (Year<2009)
Weekly.test = Weekly[!train,]
glm.fit = glm(Direction~Lag2, data=Weekly, family=binomial, subset=train)
glm.probs = predict(glm.fit, Weekly.test, type="response")
glm.pred = ifelse(glm.probs>0.5, "Up", "Down")
Direction.test = Direction[!train]
table(glm.pred, Direction.test)
```

```
        Direction.test
glm.pred Down Up
    Down    9  5
    Up     34 56
```

The overall correct prediction can be simply calculated by executing:

```
mean(glm.pred==Direction.test)
```

```
[1] 0.625
```

So the overall accuracy = 62.5%.

(e) Execute:

```
library(MASS)
lda.fit = lda(Direction~Lag2, data=Weekly, subset=train)
lda.pred = predict(lda.fit, Weekly.test)
table(lda.pred$class, Direction.test)
```

```
      Direction.test
       Down Up
  Down    9  5
  Up     34 56
```

To get overall correct prediction, execute:

```
mean(lda.pred$class==Direction.test)
```

```
[1] 0.625
```

The accuracy = 62.5%

(f) Execute(fix result using 'set.seed(23)'):

```
library(class)
train.X = as.matrix(Lag2[train])
test.X = as.matrix(Lag2[!train])
Direction.train = Direction[train]
set.seed(23)
knn.pred = knn(train.X, test.X, Direction.train, k=1)
table(knn.pred, Direction.test)
```

```
          Direction.test
 knn.pred Down Up
     Down   21 29
     Up     22 32
```

To get overall correct prediction, execute:

```
mean(knn.pred==Direction.test)
```

```
[1] 0.5096154
```

The accuracy = 50.96%

(g) The logistic regression model and linear discriminant analysis model appear to provide the best results of the data.

(h) By using the same train and test sets as above, we can change the predictors and k values to optimize the correct predictions as below:

1. logistic regression
Change predictors to $Lag2 + (Lag1)^2$ – execute the code:

```
glm.fit = glm(Direction~Lag2+I(Lag1^2), data=Weekly, family=binomial,
    subset=train)
glm.probs = predict(glm.fit, Weekly.test, type="response")
glm.pred = ifelse(glm.probs>0.5, "Up", "Down")
table(glm.pred, Direction.test)
```

```
         Direction.test
glm.pred Down Up
    Down    8  2
    Up     35 59
```

To calculate the current correct prediction:

```
mean(glm.pred==Direction.test)
```

```
[1] 0.6442308
```

Accuracy = 64.42%

2. linear discriminant analysis
Change predictors to $Lag2 + (Lag1)^2$ – execute the code:

```
lda.fit = lda(Direction~Lag2+I(Lag1^2), data=Weekly, subset=train)
lda.pred = predict(lda.fit, Weekly.test)
table(lda.pred$class, Direction.test)
```

```
      Direction.test
       Down Up
  Down    8  2
  Up     35 59
```

To calculate the current correct prediction:

```
mean(lda.pred$class==Direction.test)
```

```
[1] 0.6442308
```

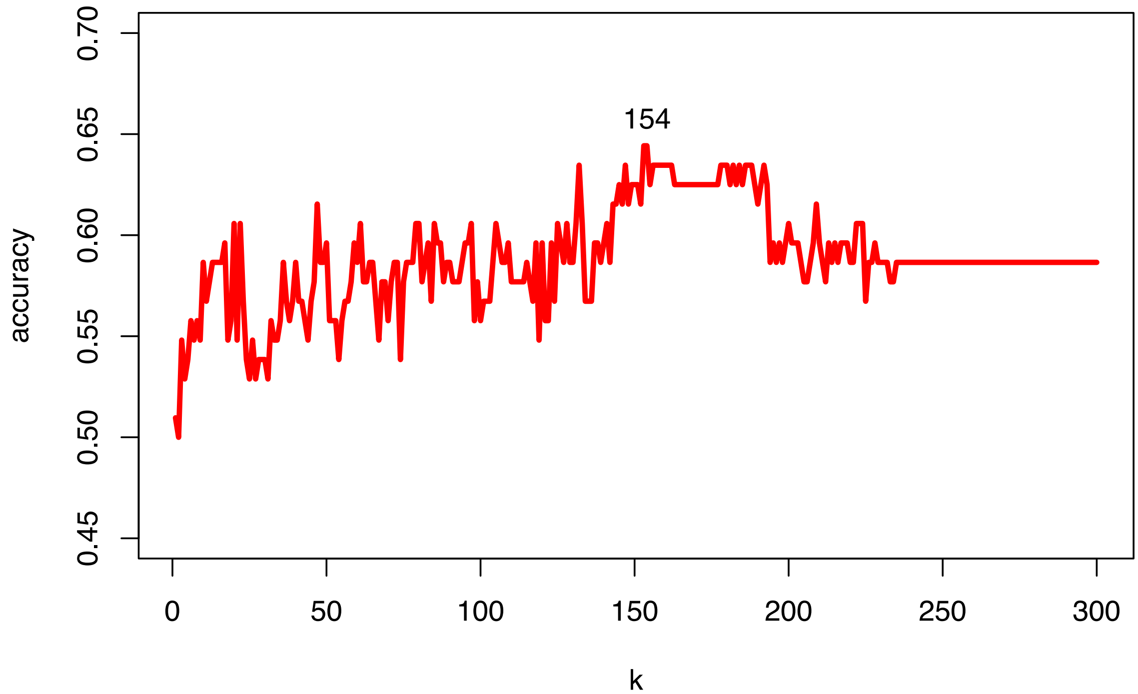
Accuracy = 64.42%

3. KNN
Test k from 1 to 300 (fix results by using 'set.seed(23)'):

```
k_increasement = data.frame(k=1:300, accuracy=NA)
for (i in 1:300){
  set.seed(23)
  knn.pred = knn(train.X, test.X, Direction.train, k=i)
  table(knn.pred, Direction.test)
  k_increasement$accuracy[i] = mean(knn.pred==Direction.test)
}
```

To calculate the current correct prediction and display as a plot, finally identify the value of k that obtain the best accuracy:

```
plot(x=k_increasement$k, y=k_increasement$accuracy, type='l', xlab="k",
    ylab="accuracy", ylim=c(0.45, 0.7), lwd=3, col='red')
identify(x=k_increasement$k, y=k_increasement$accuracy, plot=T)
```



The best correct prediction is at $k = 154$, and the this correct prediction can be retrieved as:

```
k_increasement[154,]
```

```
      k  accuracy
154 154 0.6442308
```

Accuracy = 64.42%

Problem 2

$$
\begin{aligned}
\mathbb{E}[X] &= \int xp(x)\mathrm{d}x \\
&= \iint xp(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint x\left(p(x \mid y)p(y)\right)\mathrm{d}x\mathrm{d}y \\
&= \int \left(\int x[p(x \mid y)\mathrm{d}x\right) p(y)\mathrm{d}y \\
&= \int \left(\mathbb{E}_x[x \mid y]\right) p(y)\mathrm{d}y \\
&= \mathbb{E}_y[\mathbb{E}_x[x \mid y]]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var} &= \mathbb{E}[x^2] - \mathbb{E}[x]^2 \\
&= \mathbb{E}_y[\mathbb{E}_x[x^2 \mid y]] - (\mathbb{E}_y[\mathbb{E}_x[x \mid y]])^2 \\
&= \mathbb{E}_y[\mathbb{E}_x[x^2 \mid y] - \mathbb{E}_x[x \mid y]^2] + \mathbb{E}_y[\mathbb{E}_x[x \mid y]^2] - (\mathbb{E}_y[\mathbb{E}_x[x \mid y]])^2 \\
&= \mathbb{E}_y[\mathrm{var}[x \mid y]] + \mathrm{var}_y[\mathbb{E}_x[x \mid y]]
\end{aligned}
$$

Problem 3

Because $y = x + z$, and x and y having Gaussian distributions, which means we have the distributions as below:

$$p(x) = \mathcal{N}(x \mid \mu_x, \Sigma_x) \tag{1}$$
$$p(y \mid x) = \mathcal{N}(y \mid \mu_z + x, \Sigma_z) \tag{2}$$

Compare (1) and (2) to (2.99) and (2.100), we can have:

$$\mu = \nu_x, \Lambda^{-1} = \Sigma_x, A = 1, b = \mu_z, L^{-1} = \Sigma_z \tag{3}$$

Because of (2.109) and (2.110), we can have:

$$p(y \mid x) = \mathcal{N}(\mathbb{E}[y], \mathrm{cov}[y]) \tag{4}$$
$$= \mathcal{N}(A\mu + b, L^{-1} + A\Lambda^{-1}A^T) \tag{5}$$

Use (3) in (5), we can find the marginal distribution $p(y)$:

$$p(y) = \mathcal{N}(y \mid \mu_x + \mu_z, \Sigma_x + \Sigma_z)$$

Problem 4

The data are linearly separable and the decision boundary is 0, which means separate these two classes by the hyperplane with corresponding $\sigma = 0.5$, that is $w^T\phi = 0$, which implies w goes to infinite. And logistic sigmoid becomes infinite steep. Or we see any other logistic function which decreases less sharply would give a lower maximum likelihood. However, whenever a posterior probability reaches 1, any function which passes through the data points would give the same likelihood. As there is no closed-form for the maximum likelihood, when we trying to estimate the optimal w, there will be infinite magnitude.

Problem 5

Since this is a binary classification problem, generally for data point $\{x_i, t_i\}$, if $t_i = 1$ for p, then $t_i = 0$ for $1 - p$, then we can have the likelihood (6) and its "log" form (7):

$$\prod_{i=1}^{N} p(x_i)^{t_i}(1 - p(x_i)^{1-t_i} \tag{6}$$

$$\sum_{i=1}^{N} t_i \log p(x_i) + (1 - t_i)\log(1 - p(x_i)) \tag{7}$$

In this specified problem, having instead a value π_n representing the probability that $t_n = 1$, which means t_i is substituted by π_i. Moreover, given the probability model $p(1 \mid \phi)$, we can have that $p(x_i)$ is given by $p(t_i = 1 \mid \phi(x_i))$. That is:

$$t_i \longrightarrow \pi_i \tag{8}$$

$$p(x_i) \longrightarrow p(t_i = 1 \mid \phi(x_i)) \tag{9}$$

Use (8) and (9) in (7), we can have the log likelihood for this specified data:

$$\sum_{i=1}^{N} \pi_i \log p(t_i = 1 \mid \phi(x_i)) + (1 - \pi_i)\log(1 - p(t_i = 1 \mid \phi(x_i)))$$