1. States of a Model System

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October 9, 2017

- Mechanics tells us the meaning of work; thermal physics tells us the meaning of heat.
- Three new quantities in thermal physics but do not appear in ordinary mechanics: entropy, temperature, and free energy.
- The **multiplicity** (degeneracy) of an energy level: number of quantum states with very nearly the same energy.
- A qunatum state of the system is a state of all particles; quantum states of a one-particle system are called **orbitals**.
- The energy of an orbital of a free particle with 3 positive integral quantum numbers n_x, n_y, n_z:

$$E = \frac{\hbar}{2M}(\frac{\pi}{L})^2(n_x^2 + n_y^2 + n_z^2)$$

- Binary model system: in a distinct site, an elementary magnet either points up or down, corresponding to magnetic moments $\pm m$.
- Listing every distinct state of the system:

$$(\uparrow_1 + \downarrow_1)(\uparrow_2 + \downarrow_2)\cdots(\uparrow_N + \downarrow_N) \tag{1}$$

- Total magnetic moment (M) of N magnets each of magnetic moment m:

$$M = Nm, (N-2)m, (N-4)m, \cdots, -Nm$$

- # of states with magnets up: $N_\uparrow = \frac{1}{2}N + s$ (s is an integer)
- # of states with magnets down: $N_\uparrow = \frac{1}{2}N - s$ (s is an integer)
- **spin excess** (the difference): $N_\uparrow - N_\downarrow = 2s$
- **multiplicity function**: $g(N,s)$, # of states having $N_\uparrow = \frac{1}{2}N + s$ magnets up and $N_\uparrow = \frac{1}{2}N - s$ magnets down, for a system of N magnets:

$$g(N,s) = \frac{N!}{(\frac{1}{2}N+s)!(\frac{1}{2}N-s)!} = \frac{N!}{N_\uparrow! N_\downarrow!} \tag{2}$$

- $g(N, s)$ is the number of states having the same value of s.
- then eq. (1) can be expressed as (similar to binomial expansion):

$$(\uparrow + \downarrow)^N = \sum_{s=-\frac{1}{2}N}^{\frac{1}{2}N} g(N, s) \uparrow^{\frac{1}{2}N+s} \downarrow^{\frac{1}{2}N-s}$$

- The total number of states is given by:

$$\sum_{s=-\frac{1}{2}N}^{\frac{1}{2}N} g(N, s) = 2^N \tag{3}$$

- To work with big N, we use log to simplify calculation. From eq. (2):

$$log(N, s) = logN! - log(\frac{1}{2}N + s)! - log(\frac{1}{2}N - s)! = logN! - logN_{\uparrow}! - logN_{\downarrow}! \tag{4}$$

- Using **Stirling approximation** we can rearrange eq. (4) to evaluate the logarithms (derivation shown in KK p.19). In the limit $s/N \ll 1$, with $N \gg 1$, we have the **Gaussian approximation** as a distribution of values of s:

$$g(N, s) \cong g(N, 0) exp(-2s^2/N)$$

where

$$g(N, 0) \cong (2/\pi N)^{1/2} 2^N$$

- **average value** of a function $f(s)$ over a probability distribution function $P(s)$:

$$< f >= \sum_s f(s)P(s)$$

where distribution function is normalized to unity:

$$\sum_s P(s) = 1$$

- To normalize eq. (2) using its property eq. (3), we define:

$$P(s) = g(N, s)/2^N$$

- Some mean values: $< s^2 >= \frac{1}{4}N$, $< (2s)^2 >= N$ (mean square spin excess)
- **fractional fluctuation** in 2s:

$$\mathscr{F} \equiv \frac{< (2s)^2 >^{1/2}}{N} = \frac{1}{\sqrt{N}}$$

The fractional fluctuation is smaller as N becomes larger, which means that the central peak of the distribution function becomes relatively more sharply defined as the size of the system increases.